FACULTY OF INFORMATION TECHNOLOGY

Fall,2017

MAT 207- LINEAR ALGEBRA

Lecture 4 – Matrix Algebra

Content

Matrix Factorizations

Matrix Factorizations

- A ***factorization*** of a matrix A is an equation that expresses A as a product of two or more matrices.

- Whereas matrix multiplication involves a ***synthesis*** of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an ***analysis*** of data.

THE LU FACTORIZATION

- The LU factorization, described on the next few slides, is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$Ax = b_1, \;\; Ax = b_2, \ldots , \;\; Ax = bp \qquad\qquad (1)$$

- When A is invertible, one could compute A^{-1} and then compute $A^{-1}b_1$, $A^{-1}b_2$, and so on.

THE LU FACTORIZATION

- At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.
- Then A can be written in the form $A = LU$, were L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.
- For instance, see Fig. 1 below. Such a factorization is called an **LU factorization** of A. The matrix L is invertible and is called a unit lower triangular matrix.

THE LU FACTORIZATION

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

- Before studying how to construct L and U, we should look at why they are so useful. When $A = LU$, the equation $Ax = b$ can be written as $L(Ux) = b$.

THE LU FACTORIZATION

- Writing y for Ux, we can find x by solving the pair of equations

$$\boxed{\begin{aligned} Ly &= b \\ Ux &= y \end{aligned}}$$

- First solve $Ly = b$ for y, and then solve $Ux = y$ for x. See Fig. 2 on the next slide. Each equation is easy to solve because L and U are triangular.

THE LU FACTORIZATION

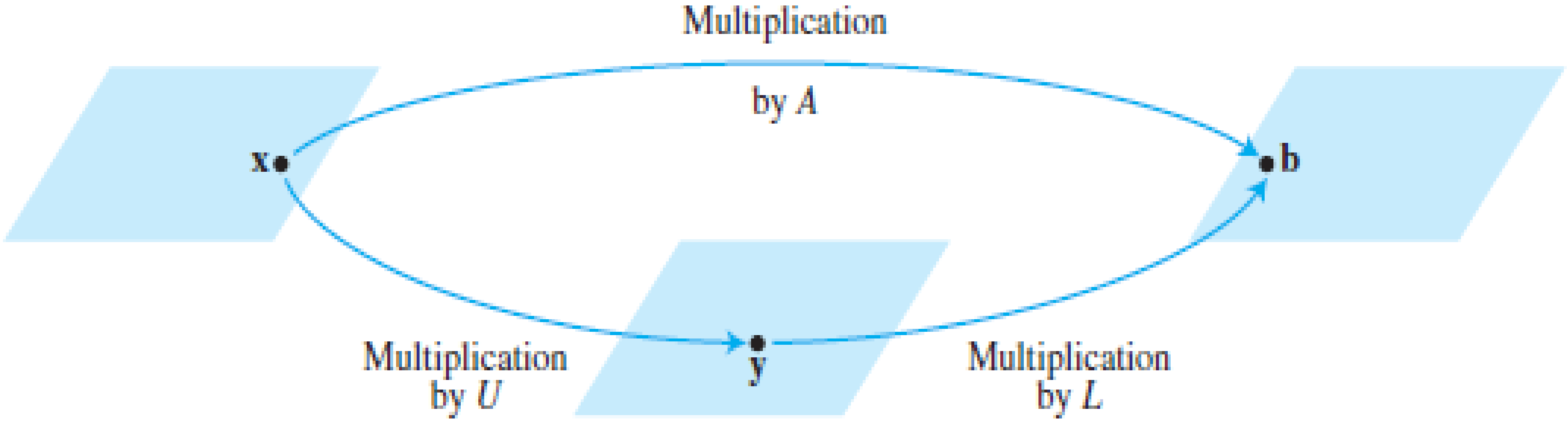

FIGURE 2 Factorization of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

THE LU FACTORIZATION

- **Example 1** It can be verified that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

- Use this factorization of *A* to solve *Ax*=*b*, where $\mathrm{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

THE LU FACTORIZATION

- Then, for $Ux = y$, the “backward” phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

$$[L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \quad \mathbf{y}]$$

- Then, for $Ux = y$, the “backward” phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.

THE LU FACTORIZATION

- For instance, creating the zeros in column 4 of $[U \quad y]$ requires 1 division in row 4 and 3 multiplication-addition pairs to add multiples of row 4 to the rows above.

$$[U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

AN LU FACTORIZATION ALGORITHM

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.
- In this case, there exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that

$$E_p \ldots E_1 A = U$$

- Then

$$A = (Ep \ldots E_1)^{-1} U = LU \quad (3)$$

- Where

$$L = (E_p \ldots E_1)^{-1}$$

AN LU FACTORIZATION ALGORITHM

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

- Note that row operations in equation (3), which reduce A to U, also reduce the L in equation (4) to I, because $E_p \ldots E_1 L = (E_p \ldots E_1)(E_p \ldots E_1)^{-1} = I$. This observation is the key to *constructing* L.

AN LU FACTORIZATION ALGORITHM

- **Algorithm for an LU Factorization**

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I.

AN LU FACTORIZATION ALGORITHM

- **Example 2** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

Solution Since A has four rows, L should be 4×4. The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

AN LU FACTORIZATION ALGORITHM

- Compare the first columns of A and L. *The row operations that create zeros in the first column of A will also create zeros in the first column of L.*
- To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, *highlight the entries* in each matrix that are used to determine the sequence of row operations that transform A onto U.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

AN LU FACTORIZATION ALGORITHM

- The highlighted entries above determine the row reduction of A to U. At each pivot column, divide the highlighted entries by the pivot and place the result onto L:

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

- An easy calculation verifies that this L and U satisfy $LU = A$.

SUBSPACES OF $\mathbb{R}^n$

SUBSPACES OF $\mathbb{R}^n$

- **Definition**: A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
  a) The zero vector is in H.
  b) For each $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

SUBSPACES OF $\mathbb{R}^n$

- A plane through the origin is the standard way to visualize the subspace in Example 1 on the next slide. See Fig. 1 below:

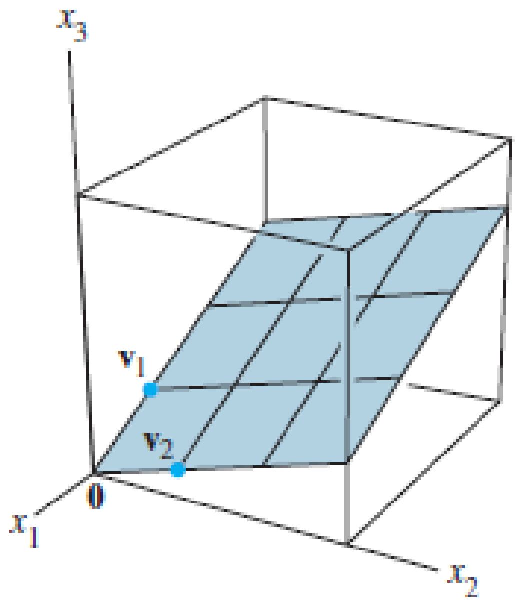

FIGURE 1

Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ as a plane through the origin.

SUBSPACES OF $\mathbb{R}^n$

- **Example 1** If v_1 and v_2 are in $\mathbb{R}^n$ and H = Span{v_1, v_2}, then H is a subspace of $\mathbb{R}^n$. To verify this statement, note that the zero vector is in H (because $0v_1 + 0v_2$ is a linear combination of v_1 and v_2).
- Now take two arbitrary vectors in H, say,

$$u = s_1v_1 + s_2v_2 \quad \text{and} \quad v = t_1v_1 + t_2v_2$$

- Then

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

- which shows that u + v is a linear combination of v_1 and v_2 and hence is in H. Also, for any scalar c, the vector cu is in H, because $cu = c(s_1v_1 + s_2v_2) = cs_1(v_1) + cs_2(v_2)$.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **column space** of a matrix A is the set Col A of all linear combinations of the columns of A.

- **Example 4** Let $A=\begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and b = $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether b is in the column space of A.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Solution:** The vector **b** is a linear combination of the columns of A if and only if **b** can be written as A**x** for some x, that is, if and only if the equation Ax = b has a solution.
- Row reducing the augmented matrix $[A \quad \mathbf{b}]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- We conclude that Ax = b is consistent and **b** is in Col A.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- **Definition:** The **null space** of a matrix A is the set Nul A of all solutions of the homogenous equation Ax = 0.
- **Theorem 12:** The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system Ax = 0 of m homogenous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

- **Proof:** The zero vector is in Nul A (because $A0 = 0$). To show that Nul A satisfies that other two properties required for a subspace, take any **u** and **v** in Nul A.

COLUMN SPACE AND NULL SPACE OF A MATRIX

- That is, suppose $A\mathbf{u} = 0$ and $A\mathbf{v} = 0$. Then, by a property of matrix multiplication,

$$A(u + v) = Au + Av = 0 + 0 = 0$$

- Thus $\mathbf{u} + \mathbf{v}$ satisfies $A = 0$, and so $\mathbf{u} + \mathbf{v}$ is in Nul A. Also, for any scalar c, $\mathrm{A}(\mathrm{c}\mathbf{u}) = c(\mathrm{A}\mathbf{u}) = \mathbf{x}c(0) = 0$, which shows that $c\mathbf{u}$ is in Nul A.

BASIS FOR A SUBSPACE

- **Definition:** A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

- **Example 5** The columns of an invertible $n \times n$ matrix form a basis for all of because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

BASIS FOR A SUBSPACE

- One such matrix is the $n \times n$ identity matrix. Its columns are denoted by e_1, . . . , e_n:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix},$$

- The set $\{e_1, \ldots, e_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See Fig. 3 on the next slide.

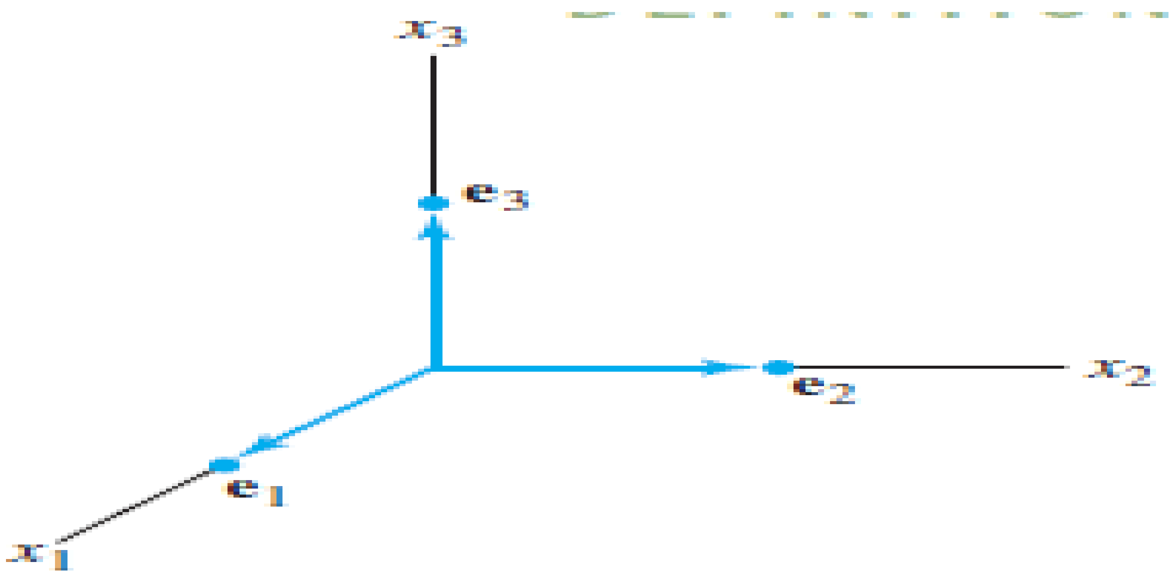

FIGURE 3
The standard basis for $\mathbb{R}^3$.

Theorem 13: The pivot columns of a matrix A form a basis for the column space of A.

Dimension And Rank

COORDINATE SYSTEMS

- **Definition**: Suppose the set $\beta = \{b_1, \ldots, bp\}$ is a basis for a subspace H. For each x in H, the **coordinates of x relative to the basis** β are the weights $c_1, \ldots, c_p$ such that $x = c_1 b_1 + \cdots + c_p b_p$, and the vector in $\mathbb{R}^p$

$$[x]_\beta = \begin{bmatrix} c_1 \\ . \\ . \\ . \\ c_p \end{bmatrix}$$

- is called the **coordinate vector of x (relative to β)** or the β-**coordinate vector of x.**

COORDINATE SYSTEMS

Example 1 Let $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\beta = \{v_1, v_2\}$. Then β is a basis for H = Span $\{v_1, v_2\}$ because v_1 and v_2 are linearly independent. Determine if x is in H, and if it is, find the coordinate vector of x relative to β.

- **Solution** If x is in H, then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

COORDINATE SYSTEMS

- The scalars c_1 and c_2, if they exist, are the β-coordinates of $\mathbf{x}$. Row operations show that

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[x]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The basis β determines a "coordinate system" on H, which can be visualized by the grid shown in Fig. 1

COORDINATE SYSTEMS

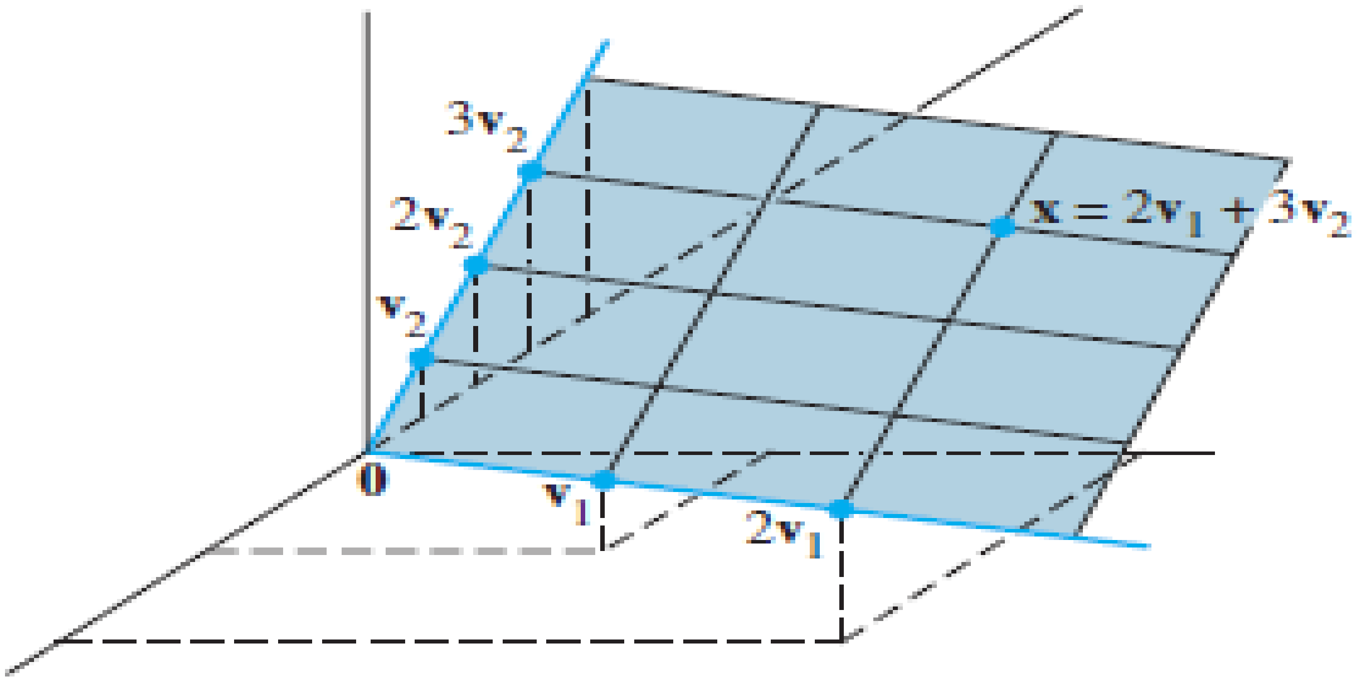

FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

THE DIMESION OF A SUBSPACE

- **Definition**: The **dimension** of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace $\{0\}$ is defined to be zero

- **Definition:** The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

THE DIMESION OF A SUBSPACE

- **Example 3** Determine the rank of the matrix

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

Solution Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

- The matrix A has 3 pivot columns, so rank $A = 3$.

THE DIMESION OF A SUBSPACE

- **Theorem 14** If a matrix A has n columns, then rank A + dim Nul$A = n$.

- **Theorem 15** Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

RANK AND THE INVERTIBLE MATRIX THEOREM

- **The Invertible Theorem (continued)** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. dim Col $A = n$

  p. rank $A = n$

  q. Nul $A = \{0\}$

  r. dim Nul $A = 0$

Thank you for listening