Fall,2019

MAT 207- LINEAR ALGEBRA

Lecture 3 – Matrix Algebra

Content

3.1 Matrix Algebra

Matrix operations

- If A is an $m \times n$ matrix—that is, a matrix with m rows and n columns—then the scalar entry in the i-th row and j-th column of A is denoted by a_{ij} and is called the (i, j)-entry of A.
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \text{Column} & & \phantom{a_{1n}} \\ & & j & & \end{array} \\
\begin{array}{c} \\ \\ \text{Row } i \\ \\ \\ \end{array} & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\
 & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}
$$

Matrix notation.

Matrix Operations

- The columns are denoted by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and the matrix A is written as

$$A = [a_1 \quad a_2 \quad \ldots \quad a_n]$$

- The **diagonal entries** in an $A = \left[a_{ij} \right]$ matrix are a_{11}, a_{22}, a_{33}, …, and they form the **main diagonal** of A.
- A **diagonal matrix** is a square $n \times m$ matrix whose nondiagonal entries are zero.
- An example is the $n \times n$ identity matrix, I_n.

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0.

SUMS AND SCALAR MULTIPLES

- If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.
- The sum $A + B$ is defined only when A and B are the same size.
- **Example 1**: Let $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix},$ and $C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$

Find $A + B$ and $A + C$

SUMS AND SCALAR MULTIPLES

Solution:

- $A+B=\begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$ **but** $A+C$ is not defined because A and C have different sizes.

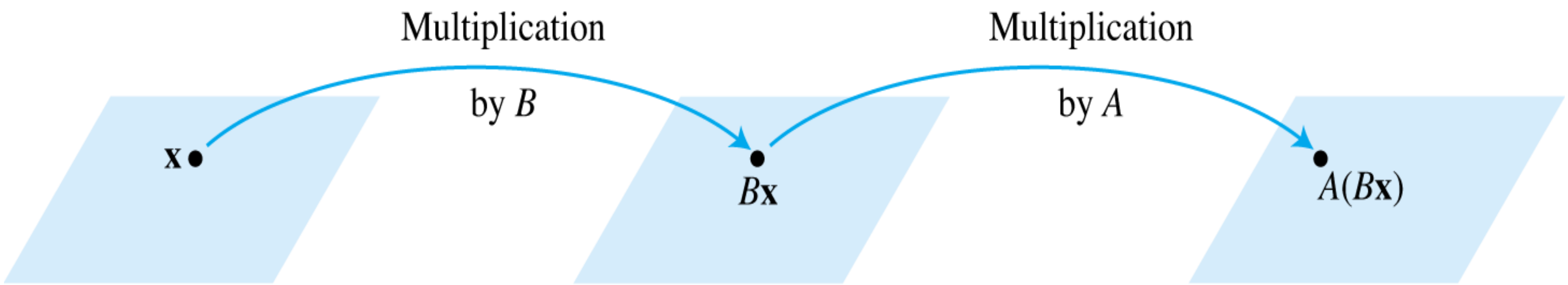
- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

SUMS AND SCALAR MULTIPLES

- **Theorem 1:** Let A, B, and C be matrices of the same size, and let r and s be scalars.
  a. $A+B=B+A$
  b. $(A+B)+C=A+(B+C)$
  c. $A+0=A$
  d. $r(A+B)=rA+rB$
  e. $(r+s)A=rA+sA$
  f. $r(sA)=(rs)A$

MATRIX MULTIPLICATION

- When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$. See the Fig. 2 below.



Multiplication by B and then A.

- Thus $A\,(B\mathbf{x})$ is produced from x by a *composition of mappings*—the linear transformations.

MATRIX MULTIPLICATION

- If A is $m \times n$, B is $n \times p$, and $\mathbf{x}$ is in R^p, denote the columns of B by $\mathbf{b}_1$, …, $\mathbf{b}_p$ and the entries in $\mathbf{x}$ by $\mathbf{x}_1$, …, $\mathbf{x}_p$.
- Then $$B\mathrm{x} = x_1\mathrm{b}_1 + \ldots + x_p\mathrm{b}_p$$
- By the linearity of multiplication by A,

$$A(B\mathrm{x}) = A(x_1\mathrm{b}_1) + \ldots + A(x_p\mathrm{b}_p)$$

$$= x_1 A\mathrm{b}_1 + \ldots + x_p A\mathrm{b}_p$$

- The vector $A(B\mathbf{x})$ is a linear combination of the vectors $A\mathbf{b}_1$, …, $A\mathbf{b}_p$, using the entries in $\mathbf{x}$ as weights.

MATRIX MULTIPLICATION

- In matrix notation, this linear combination is written as

$$A(B\mathrm{x}) = \begin{bmatrix} A\mathrm{b}_1 & A\mathrm{b}_2 & \cdots & A\mathrm{b}_p \end{bmatrix}\mathrm{x}$$

Example : Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

Solution : Write $B = \begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \end{bmatrix}$ and compute

MATRIX MULTIPLICATION

$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

$$AB = A\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Matrix multiplication

Row—column rule for computing *AB*

- If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots a_{i1}b_{\mathrm{i}j} + a_{in}b_{nj}$$

PROPERTIES OF MATRIX MULTIPLICATION

- **Theorem 2**: Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left distributive law)

c. $(B+C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

PROPERTIES OF MATRIX MULTIPLICATION

- If $AB = BA$, we say that A and B **commute** with one another.

- **Warnings:**
  1. In general, $AB \neq BA$.
  2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
  3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0 \; or \; B = 0$

POWERS OF A MATRIX

- If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A: $$A^k = \underbrace{A \cdots A}_{k}$$
- If A is nonzero and if $\mathbf{x}$ is in $\mathbb{R}^n$, then $A^k\mathbf{x}$ is the result of left-multiplying $\mathbf{x}$ by A repeatedly k times.
- If $k = 0$, then $A^0\mathbf{x}$ should be $\mathbf{x}$ itself.
- Thus A^0 is interpreted as the identity matrix.

THE TRANSPOSE OF A MATRIX

- Given an $m \times n$ matrix A, the **transpose** of A is the $m \times n$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem 3: Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$

3.2 The Inverse of Matrix

MATRIX OPERATIONS

- An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I$$

- where $I = I_n$, the $n \times n$ identity matrix.
- In this case, C is an **inverse** of A.
- In fact, C is uniquely determined by A, because if B were another inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

- This unique inverse is denoted by A^{-1} , so that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

MATRIX OPERATIONS

- **Theorem 4** : Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible

and $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A is not invertible

- The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc$$

MATRIX OPERATIONS

- **Theorem 5:**

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

MATRIX OPERATIONS

Theorem 6: An n x n matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

ALGORITHM FOR FINDING A^{-1}

- Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.
- **Example** : Find the inverse of matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

, if it exists.

- **Solution :**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

ALGORITHM FOR FINDING A^{-1}

- Theorem 6 shows, since $A \sim I$, that A is invertible, and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

- Now, check the final answer.

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.3 CHARACTERIZATIONS OF INVERTIBLE MATRICES

THE INVERTIBLE MATRIX THEOREM

- **Theorem 8:** Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
  a. A is an invertible matrix.
  b. A is row equivalent to the $n \times n$ identity matrix.
  c. A has n pivot positions.
  d. The equation $A\mathrm{x} = 0$ has only the trivial solution.
  e. The columns of A form a linearly independent set.

THE INVERTIBLE MATRIX THEOREM

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that.
k. There is an $n \times n$ matrix D such that.
l. A^T is an invertible matrix.

THE INVERTIBLE MATRIX THEOREM

- The proof will establish the "circle" of implications as shown in the following figure.

$$
\begin{tikzcd}
 & \text{(a)} \arrow[dr, Rightarrow] & \\
\text{(b)} \arrow[ur, Rightarrow] & & \text{(j)} \arrow[d, Rightarrow] \\
\text{(c)} \arrow[u, Rightarrow] & & \text{(d)} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

- If any one of these five statements is true, then so are the others.

THE INVERTIBLE MATRIX THEOREM

- **Example 1:** Use the Invertible Matrix Theorem to decide if A is invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

- **Solution :**

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

THE INVERTIBLE MATRIX THEOREM

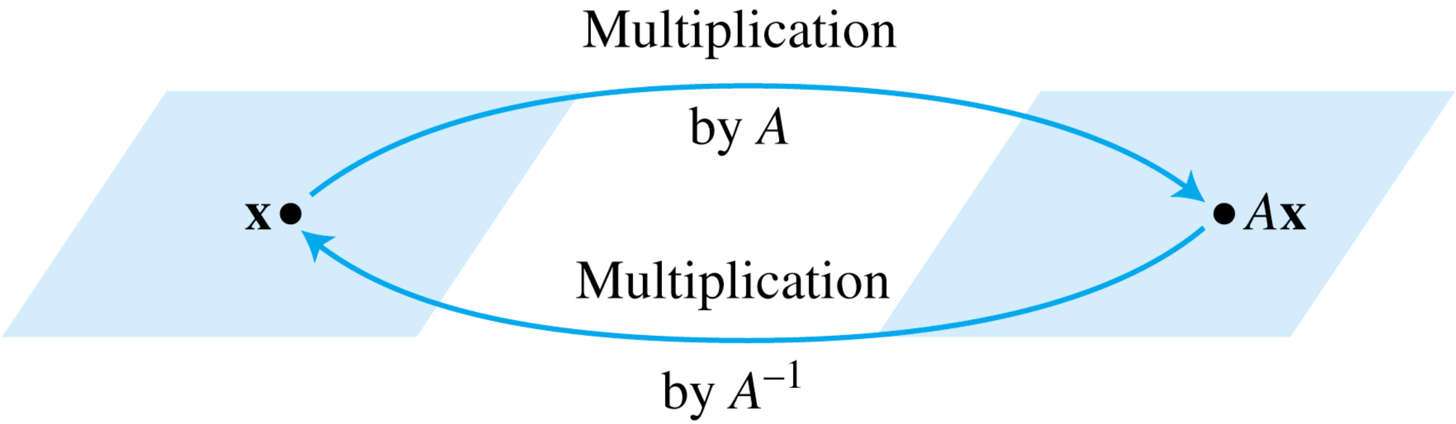
- So A has three pivot positions and hence is invertible, by the Invertible Matrix Theorem, statement (c).

- The Invertible Matrix Theorem *applies only to square matrices*

- For example, if the columns of a 4×3 matrix are linearly independent, we cannot use the Invertible Matrix Theorem to conclude anything about the existence or nonexistence of solutions of equation of the form $A\mathrm{x} = \mathrm{b}$

INVERTIBLE LINEAR TRANSFORMATIONS

- When a matrix A is invertible, the equation $A^{-1}A\mathbf{x} = \mathbf{x}$ can be viewed as a statement about linear transformations. See the following figure.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

INVERTIBLE LINEAR TRANSFORMATIONS

- A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (1)$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \qquad (2)$$

Theorem 9:

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying equation (1) and (2).

INVERTIBLE LINEAR TRANSFORMATIONS

- **Proof:** Suppose that T is invertible.
- Then (2) shows that T is onto $\mathbb{R}^n$, for if **b** is in $\mathbb{R}^n$ and $\mathrm{x} = S(\mathrm{b})$, then $T(\mathrm{x}) = T(S(\mathrm{b})) = \mathrm{b}$, so each **b** is in the range of T.
- Thus A is invertible, by the Invertible Matrix Theorem, statement (i).
- Conversely, suppose that A is invertible, and let $S(\mathrm{x}) = A^{-1}\mathrm{x}$. Then, S is a linear transformation, and S satisfies (1) and (2).
- For instance, $S(T(\mathrm{x})) = S(A\mathrm{x}) = A^{-1}(A\mathrm{x}) = \mathrm{x}$.
- Thus, T is invertible.

2.4 PARTITIONED MATRICES

PARTITIONED MATRICES

- **Example 1 : The matrix**

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

- Can also be written as the 2×3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

ADDITION AND SCALAR MULTIPLICATION

- If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A + B$.
- In this case, each block of $A + B$ is the (matrix) sum of the corresponding blocks of A and B.
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

MULTIPLICATION OF PARTITIONED MATRICES

- Partitioned matrices can be multiplied by the usual row—column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B.
- **Example: Let**

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- The 5 columns of A are partitioned into a set of 3 columns and then a set of 2 columns. The 5 rows of B are partitioned in the same way—into a set of 3 rows and then a set of 2 rows.

MULTIPLICATION OF PARTITIONED MATRICES

- . It can be shown that the ordinary product AB can be written as

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

- For instance

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

- Hence

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \end{bmatrix}$$

MULTIPLICATION OF PARTITIONED MATRICES

- **Theorem 10 :** Column—Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then

$$AB = \begin{bmatrix} \operatorname{col}_1(A) & \operatorname{col}_2(A) & \cdots & \operatorname{col}_n(A) \end{bmatrix} \begin{bmatrix} \operatorname{row}_1(B) \\ \operatorname{row}_2(B) \\ \vdots \\ \operatorname{row}_n(B) \end{bmatrix}$$

$$= \operatorname{col}_1(A)\operatorname{row}_1(B) + \cdots + \operatorname{col}_n(A)\operatorname{row}_n(B)$$

INVERSES OF PARTIONED MATRICES

- The next example illustrates calculations involving inverses and partitioned matrices.
- **Example 5** A matrix of the form $A = \begin{bmatrix} A11 & A12 \\ 0 & A22 \end{bmatrix}$
- Is said to be *block upper triangular*. Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible. Find a formula for A^{-1}.

INVERSES OF PARTIONED MATRICES

- **Solution** Denote A^{-1} by B and partition B so that

$$\begin{bmatrix} A11 & A12 \\ 0 & A22 \end{bmatrix} \begin{bmatrix} B11 & B12 \\ B21 & B22 \end{bmatrix} = \begin{bmatrix} Ip & 0 \\ 0 & Iq \end{bmatrix} \qquad (2)$$

This set is

$$A11B11 + A12B21 = Ip \qquad (3)$$

$$A11B11 + A12B22 = 0 \qquad (4)$$

$$A22B21 = 0 \qquad (5)$$

$$A22B22 = Iq \qquad (6)$$

INVERSES OF PARTIONED MATRICES

- By itself, equation (6) does not show that A_{22} is invertible. However, since A_{22} is square, **the Invertible Matrix Theorem** and (6) together show that A_{22} is invertible and $B22 = A_{22}^{-1}$.
- Next, left-multiply both sides of (5) by A_{22}^{-1} and obtain

$$B21 = A_{22}^{-1}0 = 0$$

- So that (3) simplifies to $A11B11 + 0 = Ip$
- Since A_{11} is square, this shows that A_{11} is invertible and $B22 = A_{22}^{-1}$. Finally, use these results with (4) to find that

$$A11B12 = -A12B22 = -A12A_{22}^{-1} \quad \text{and} \quad B12 = A_{11}^{-1}A12A_{22}^{-1}$$

INVERSES OF PARTIONED MATRICES

- Thus

$$A{-}1 = \begin{bmatrix} A11 & A12 \\ 0 & A22 \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & A_{11}^{-1} A12 A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

A **block diagonal matrix** is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.

Thank you for listening