Fall,2019

MAT 207- LINEAR ALGEBRA

Lecture 2 - Linear equations in Linear Algebra

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Linear Independence

LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = 0$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0 \qquad (2)$$

LINEAR INDEPENDENCE

- **Example 1:** Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

LINEAR INDEPENDENCE

- **Solution:** We must determine if there is a nontrivial solution of the equation on the previous slide.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- x_1 and x_2 are basic variables, and x_3 is free.
- Each nonzero value of x_3 determines a nontrivial solution of (1).
- Hence, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent.

LINEAR INDEPENDENCE

- To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

- Thus, $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free.
- Choose any nonzero value for x_3—say, $x_3 = 5$.
- Then $x_1 = 10$ and $x_2 = -5$.

Linear Equation

- Substitute these values into equation (1) and obtain the equation below.

$$10v_1 - 5v_2 + 5v_3 = 0$$

- This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

LINEAR INDEPENDENCE OF MATRIX COLUMNS

- Suppose that we begin with a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ instead of a set of vectors.
- The matrix equation $A\mathbf{x} = 0$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = 0$$

- *Each linear dependence relation among the columns of A corresponds to a nontrivial solution of* $A\mathbf{x} = 0$

- The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = 0$ has *only* the trivial solution.

Set of One or Two Vector

- A set containing only one vector – say, **v** – is linearly independent if and only if **v** is not the zero vector.

- This is because the vector equation $x_1\mathrm{v} = 0$ has only the trivial solution when $\mathrm{v} \neq 0$.

- The zero vector is linearly dependent because $x_1 0 = 0$ has many nontrivial solutions.

Set of One or Two Vectors

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Set of One or Two Vectors

- **THEOREM 7**: **Characterization of Linearly Dependent Sets**

- An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in *S* is a linear combination of the others.
- In fact, if *S* is linearly dependent and $v_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

Set of One or Two Vectors

Example 4: Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$.

Describe the set spanned by **u** and **v**, and explain why a vector **w** is in Span {**u**, **v**} if and only if {**u**, **v**, **w**} is linearly dependent.

Solution: The vectors **u** and **v** are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$.

Set of One or Two Vector

- If **w** is a linear combination of **u** and **v**, then {**u**, **v**, **w**} is linearly dependent, by Theorem 7.

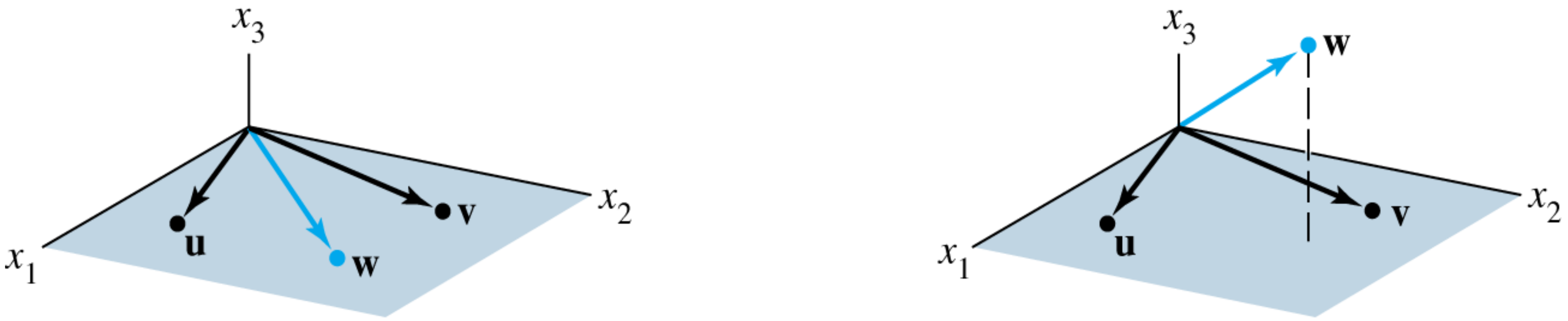
- Conversely, suppose that {**u**, **v**, **w**} is linearly dependent.

- By theorem 7, some vector in {**u**, **v**, **w**} is a linear combination of the preceding vectors (since $u \neq 0$).

- That vector must be **w**, since **v** is not a multiple of **u**.

Set of Two or More Vectors

- So **w** is in Span {**u**, **v**}. Fig. 2 below



Linearly dependent,
w in Span{**u**, **v**}

Linearly independent,
w *not* in Span{**u**, **v**}

- The set {**u**, **v**, **w**} will be linearly dependent if and only if **w** is in the plane spanned by **u** and **v**.

Set of Two or More Vectors

- **THEOREM 8:**

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if .

- **Example** : The vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ are linearly dependent by Theorem 8, because there are three vectors in the set and there are only two entries in each vector.

Set of Two or More Vectors

- **THEOREM 9:**

If a set $S = \{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Proof:

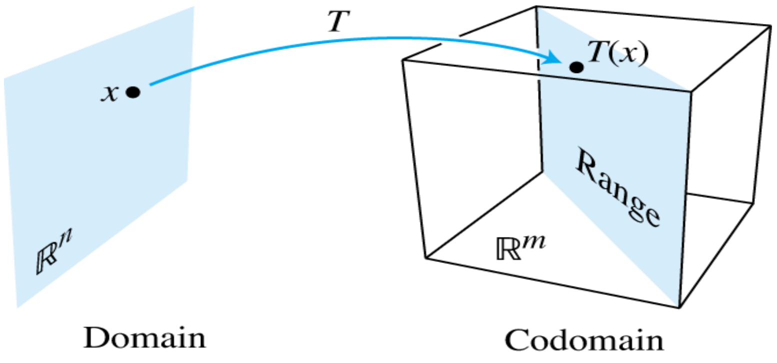
By renumbering the vectors, we may suppose $\mathrm{v}_1 = 0$

Then the equation $1\mathrm{v}_1 + 0\mathrm{v}_2 + \ldots + 0\mathrm{v}_p = 0$ shows that *S* in linearly dependent.

Introduction to Linear transformations

LINEAR TRANSFORMATIONS

- The transformation (map) $T\colon \mathbb{R}^n \rightarrow \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$.
- For **x** in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of **x** (under the action of T).
- The set of all images $T(\mathbf{x})$ is called the **range** of T



Domain, codomain, and range of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

MATRIX TRANSFORMATIONS

- For each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ is computed as $A\mathbf{x}$, where A is an $m \times n$ matrix.
- For simplicity, we denote such a *matrix transformation* by $\mathrm{x} \mapsto A\mathrm{x}$.
- Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries.
- The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

MATRIX TRANSFORMATIONS

- **Example 1:**

Let $A=\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathrm{u}=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b=\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, c$=\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$.

and define a transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathrm{x})=A\mathrm{x}$, so that

$$T(\mathrm{x})=A\mathrm{x}=\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_1-3x_2 \\ 3x_1+5x_2 \\ -x_1+7x_2 \end{bmatrix}$$

Linear transformation

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

Linear Transformations

Solution:

a. Compute $T(\mathrm{u}) = A\mathrm{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

b. Solve $T(\mathrm{x}) = \mathrm{b}$ for **x**. That is, solve $A\mathrm{x} = \mathrm{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \qquad (1)$$

Linear Transformations

- Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \quad (2)$$

- Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathrm{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$

- The image of this **x** under *T* is the given vector **b**.

Linear Transformations

c. Any **x** whose image under T is **b** must satisfy equation (1).

- From (2), it is clear that equation (1) has a unique solution.
- So there is exactly one **x** whose image is **b**.

d. The vector **c** is in the range of T if **c** is the image of some **x** in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some **x**.

- This is another way of asking if the system $A\mathbf{x} = \mathbf{c}$

  is consistent.

Linear Transformations

- To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

  - The third equation, $0 = -35$, shows that the system is inconsistent.
  - So **c** is *not* in the range of T.

Linear Transformations

- **Definition:** A transformation (or mapping) T is **linear** if:
  i. $T(\mathrm{u}+\mathrm{v}) = T(\mathrm{u}) + T(\mathrm{v})$ for all **u**, **v** in the domain of T;
  ii. $T(c\mathrm{u}) = cT(\mathrm{u})$ for all scalars c and all **u** in the domain of T.

These two properties lead to the following useful facts.

- If T is a linear transformation, then $T(0) = 0$ (3)
- and $T(c\mathrm{u} + d\mathrm{v}) = cT(\mathrm{u}) + dT(\mathrm{v})$. (4)

Generation case :

$$T(c_1\mathrm{v}_1 + \ldots + c_p\mathrm{v}_p) = c_1T(\mathrm{v}_1) + \ldots + c_pT(\mathrm{v}_p)$$

THE MATRIX OF A LINEAR TRANSFORMATION

THE MATRIX OF A LINEAR TRANSFORMATION

THEOREM 10 : Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix *A* such that

$$T(x) = Ax \quad \text{for all x in } \mathbb{R}^n$$

In fact, A is the m $\times$ n matrix whose j^{th} column is the vector $T(e_j)$, where e_j is the j^{th} column of the identity matrix in $\mathbb{R}^n$

$$A = [T(e_1)...T(e_n)]$$

The Matrix of Linear Transformations

- **Proof**: Write x= $I_n x = [e_1 \; \dots e_2]x = x_1 e_1 + \; \dots + x_n e_n$, and use the linearity of T to compute

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n)$$

$$= [T(e_1) \quad \dots \quad T(e_1)] \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = Ax$$

The Matrix of Linear Transformations

- **Example 2**:

Find the standard matrix A for the dilation transformation T(x)= 3x, for x in $\mathbb{R}^2$.

- **Solution**: Write

$$T(e_1)= e_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \; and \; T(e_2)= e_2= \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

EXISTENCE AND UNIQUENESS QUESTIONS

- **Definition**: A mapping T: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^n$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

- **Example 4**: Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$? Is T a one-to-one mapping?

EXISTENCE AND UNIQUENESS QUESTIONS

- **Solution**: Since *A* happens to be in echelon form, we can see at once that *A* has a pivot position in each row. By Theorem 4 in Section 1.4, for each **b** in $\mathbb{R}^3$, the equation Ax=b is consistent. In other words, the linear transformation T maps $\mathbb{R}^4$(its domain) onto $\mathbb{R}^3$.

- However, since the equation Ax=b has a free variable (because there are four variables and only three basic variables), each **b** is the image of more than one **x**. This is, *T* is *not* one-to-one.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Theorem 11**: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation T(x)=0 has only the trivial solution.
- **Theorem 12**: Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b) T is one-to-one if and only if the columns of A are linearly independent.

EXISTENCE AND UNIQUENESS QUESTIONS

- **Proof:**

a) By Theorem 4 in Section 1.4, the columns of *A* span $\mathbb{R}^m$ if and only if for each b in $\mathbb{R}^m$ the equation Ax=b is consistent—in other words, if and only if for every **b**, the equation T(x)=b has at least one solution. This is true if and only if *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

- The equations T(x)=0 and Ax=0 are the same except for notation. So, by Theorem 11, *T* is one-to-one if and only if Ax=0 has only the trivial solution. This happens if and only if the columns of *A* are linearly independent.

An Application of Linear Symstem

Polynomial Interpolation :

- The simplest example of such a problem is to find a linear polynomial

$$y = ax + b \quad (1)$$

- The graph of (1) is a line passes through two distinct point $(x_1\ , y_1)$

$(x_2\ , y_2)$, we must have :

$$y_1 = ax_1 + b \quad ; y_2 = ax_2 + b$$

- For Example the line y = ax + b passes through the points (2,1); (5,4)
- Then the equation of the line is : y = x - 1

Polynomial Interpolation

- **In general case** : of finding polynomial whose graph through n-distinct points (x_1 , y_1) , (x_2 , y_2) ,……., (x_n , y_n)
- We looking for a polynomial of the form

$$\boxed{p(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{n-1} x^{n-1}}$$

- it follows that the coordinates of the points must satisfy

$$\begin{array}{l} a_0 + a_1 x_1 + a_2 x_1{}^2 + \ldots\ldots + a_{n-1} x_1{}^{n-1} = y_1 \\ a_0 + a_1 x_2 + a_2 x_2{}^2 + \ldots\ldots + a_{n-1} x_2{}^{n-1} = y_2 \\ \ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots\ldots \\ a_0 + a_1 x_n + a_2 x_n{}^2 + \ldots\ldots + a_{n-1} x_n{}^{n-1} = y_n \end{array}$$

Polynomial Interpolation

- The augmented matrix for the system is

$$\begin{bmatrix} 1 & x_1 & {x_1}^2 & \ldots. & {x_1}^{\mathrm{n}-1} & y_1 \\ 1 & x_1 & {x_1}^2 & \ldots. & {x_2}^{\mathrm{n}-1} & y_2 \\ & & & & \ldots\ \ldots\ \ldots & \\ 1 & x_1 & {x_1}^2 & \ldots. & {x_n}^{\mathrm{n}-1} & y_n \end{bmatrix}$$

- Hence the interpolating polynomial can be found by reducing this matrix to reduced row echelon form (Gauss–Jordan elimination).

- **EXAMPLE :** Find a cubic polynomial whose graph passes through the points $(1, 3), (2, -2), (3, -5), (4, 0)$

- **SOLUTION**

Since there are four points, we will use an interpolating polynomial of degree $n = 3$. Denote this polynomial by

$x_1 = 1$, $x_1 = 2$, $x_1 = 3$, $x_1 = 4$ and $y_1 = 3$, $y_2 = -2$, $y_3 = -5$, $y_4 = 0$

Polynomial Interpolation

- the augmented matrix for the linear system in the unknowns $a0$, $a1$, $a2$, and $a3$ is
- $$\begin{bmatrix} 1 & x_1 & {x_1}^2 & \dots. & {x_1}^{n-1} & y_1 \\ 1 & x_1 & {x_1}^2 & \dots. & {x_2}^{n-1} & y_2 \\ & & & & \dots\dots\dots & \\ 1 & x_1 & {x_1}^2 & \dots. & {x_n}^{n-1} & y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

- the reduced row echelon form of this matrix is

Polynomial Interpolation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

- from which it follows that $a0 = 4$, $a1 = 3$, $a2 = -5$, $a3 = 1$. Thus, the interpolating polynomial is

$$p(x) = 4 + 3x - 5x^2 + x^3$$

Thank you for Listening