Fall,2017

MAT 207- LINEAR ALGEBRA

Lecture 6 – Vector Space

Content

Vector Space and Subspace

VECTOR SPACES AND SUBSPACES

- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition and multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in V and for all scalars c and d.

  1. The sum of **u** and **v**, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
  2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
  3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
  4. There is a zero vector 0 in V such that

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VECTOR SPACES AND SUBSPACES

5. For each **u** in *V*, there is a vector $-\mathbf{u}$ in *V* such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of **u** by *c*, denoted by *c***u**, is in *V*.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Using these axioms, we can show that the zero vector in Axiom 4 is unique, and the vector **-u** , called the **negative** of **u**, in Axiom 5 is unique for each **u** in *V*.

VECTOR SPACES AND SUBSPACES

- **Definition:** A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The zero vector of V is in H.
  b. H is closed under vector addition .That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

SUBSPACES

- Properties (a), (b), and (c) guarantee that a subspace H of V is itself a *vector space*, under the vector space operations already defined in V.

- Every subspace is a vector space.

- Conversely, every vector space is a subspace (of itself and possibly of other larger spaces

A SUBSPACE SPANNED BY A SET

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{\mathbf{0}\}$.

- As the term **linear combination** refers to any sum of scalar multiples of vectors, and Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_p$.

A SUBSPACE SPANNED BY A SET

- **Example 10:** Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

  Show that H is a subspace of V.

- **Solution:** The zero vector is in H, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
- To show that H is closed under vector addition, take two arbitrary vectors in H, say,

$$\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 \quad \text{and} \quad \mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad .$$

- By Axioms 2, 3, and 8 for the vector space V,

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \\ &= (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2 \end{aligned}$$

A SUBSPACE SPANNED BY A SET

- **So** $\mathbf{u} + \mathbf{w}$ is in H.

- Furthermore, if c is any scalar, then by Axioms 7 and 9,

$$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$

  which shows that $c\mathbf{u}$ is in H and H is closed under scalar multiplication.

- Thus H is a subspace of V.

A SUBSPACE SPANNED BY A SET

- **Theorem 1:** If $\mathbf{v}_1,\ldots,\mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is a subspace of V.

- We call Span $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$.

- Give any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in H such that

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Null Space, Column Space and Linear Transformation

NULL SPACE OF A MATRIX

- **Definition:** The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

- **Theorem 2:** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

NULL SPACE OF A MATRIX

- **Proof:** Nul A is a subset of $\mathbb{R}^n$ because A has n columns.
- We need to show that Nul A satisfies the three properties of a subspace
- **0** is in Null A.
- Next, let **u** and **v** represent any two vectors in Nul A.
- Then **Au = 0** and **Av = 0**
- To show that **u+v** is in **Nul** ***A***, we must show that **A(u+v) = 0**

.

- Using a property of matrix multiplication, compute

- Thus **u+v** is in Nul A, and Nul A is closed under vector addition.

NULL SPACE OF A MATRIX

- Finally, if c is any scalar, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(0) = 0$$

  which shows that $c\mathbf{u}$ is in Nul A.
- Thus Nul A is a subspace of $\mathbb{R}^n$.

- **An Explicit Description of Nul *A***
- There is no obvious relation between vectors in Nul A and the entries in A.
- We say that Nul A is defined *implicitly*, because it is defined by a condition that must be checked.

NULL SPACE OF A MATRIX

- No explicit list or description of the elements in Nul A is given.
- *Solving* the equation $Ax = 0$ amounts to producing an *explicit* description of Nul A.
- **Example 3:** Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

NULL SPACE OF A MATRIX

- **Solution:** The first step is to find the general solution of $Ax = 0$ in terms of free variables.

- Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

NULL SPACE OF A MATRIX

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5,\ x_3 = -2x_4 + 2x_5$ with x_2, x_4, and x_5 free.
- Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

NULL SPACE OF A MATRIX

- Every linear combination of **u**, **v**, and **w** is an element of Nul A.
- Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for Nul A.

- **Definition:** The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$$

COLUMN SPACE OF A MATRIX

- **Theorem 3:** The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.
- A typical vector in Col A can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

- The notation $A\mathbf{x}$ for vectors in Col A also shows that Col A is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.
- The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

COLUMN SPACE OF A MATRIX

Example 7: Let $A=\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\mathrm{u}=\begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathrm{v}=\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

a. Determine if **u** is in Nul A. Could **u** be in Col A?
b. Determine if **v** is in Col A. Could **v** be in Nul A?

COLUMN SPACE OF A MATRIX

- **Solution:**
  a. An explicit description of Nul A is not needed here. Simply compute the product $A\mathbf{u}$.

$$A\mathrm{u} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

COLUMN SPACE OF A MATRIX

- **u** is *not* a solution of $\mathbf{Ax} = \mathbf{0}$, so **u** is not in Nul *A*.
- Also, with four entries, **u** could not possibly be in Col *A*, since Col *A* is a subspace of $\mathbb{R}^3$.

b. Reduce $\begin{bmatrix} A & \mathrm{v} \end{bmatrix}$ to an echelon form.

$$\begin{bmatrix} A & \mathrm{v} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

c. The equation $\mathbf{Ax} = \mathbf{b}$ is consistent, so **v** is in Col *A*.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- **Definition:** A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that
  i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and
  ii. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

KERNEL AND RANGE OF A LINEAR TRANSFORMATION

- The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that (the zero vector in W).

- The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

- The kernel of T is a subspace of V.

- The range of T is a subspace of W.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN MATRIX *A*

Null A	**Column A**
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$
2. Nul A is implicitly defined; *i.e.*, you are given only a condition Ax = 0 that vectors in Nul A must satisfy.	2. Col A is explicitly defined; *i.e.*, you are told how to build vectors in Col A.

CONTRAST BETWEEN NUL *A* AND COL *A* FOR AN MATRIX *A*

3. It takes time to find vectors in Nul *A*. Row operations on are required.	3. It is easy to find vectors in Col *A*. The columns of a are displayed; others are formed from them.
4. There is no obvious relation between Nul *A* and the entries in *A*.	4. There is an obvious relation between Col *A* and the entries in *A*, since each column of *A* is in Col *A*.

CONTRAST BETWEEN NUL A AND COL *A* FOR AN MATRIX *A*

5. A typical vector **v** in Nul *A* has the property that $Av = 0$.	5. A typical vector **v** in Col *A* has the property that the equation is consistent.
6. Given a specific vector **v**, it is easy to tell if **v** is in Nul *A*. Just compare *A***v**	6. Given a specific vector **v**, it may take time to tell if v is in Col *A*. Row operations on $\begin{bmatrix} A & v \end{bmatrix}$ are required.

Linear Independent Sets and Bases

LINEAR INDEPENDENT SETS; BASES

- An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0 \qquad (1)$$

  has *only* the trivial solution, .

- The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \ldots, c_p$, *not all zero*, such that (1) holds.

- In such a case, (1) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

LINEAR INDEPENDENT SETS; BASES

- **Theorem 4:** An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some $\mathbf{v}_j$ (with) is a linear combination of the preceding vectors, .

- **Definition:** Let *H* be a subspace of a vector space *V*. An indexed set of vectors $\mathcal{B} = \{b_1, \ldots, b_p\}$ in *V* is a basis for *H* if
  (i) $\mathcal{B}$ is a linearly independent set, and
  (ii) The subspace spanned by $\mathcal{B}$ coincides with *H*; that is,

LINEAR INDEPENDENT SETS; BASES

- The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself.

- Thus a basis of V is a linearly independent set that spans V.

- When $H \neq V$, condition (ii) includes the requirement that each of the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ must belong to H, because Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \ldots, \mathbf{b}_p$.

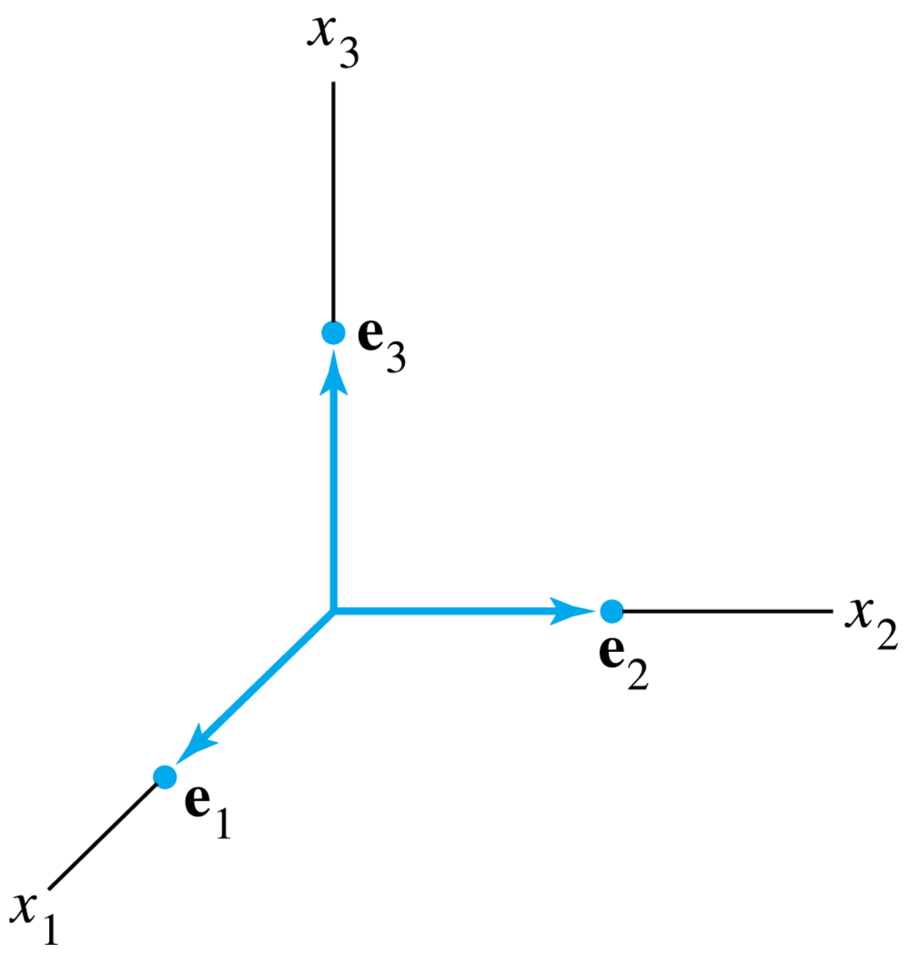
Standard Basis

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$\mathrm{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathrm{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathrm{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See the following figure.

Standard Basis



The standard basis for $\mathbb{R}^3$.

THE SPANNING SET THEOREM

- **Theorem 5:** Let $S = \{v_1, \ldots, v_p\}$ be a set in V, and let
- . $H = \text{Span}\{v_1, \ldots, v_p\}$

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{0\}$, some subset of S is a basis for H.

THE SPANNING SET THEOREM

- **Proof:**
  a. By rearranging the list of vectors in S, if necessary, we may suppose that $\mathbf{v}_p$ is a linear combination of $\mathrm{v}_1,\ldots,\mathrm{v}_{p-1}$—say,

$$\mathrm{v}_p = a_1\mathrm{v}_1 + \ldots + a_{p-1}\mathrm{v}_{p-1} \qquad (3)$$

- Given any $\mathbf{x}$ in H, we may write

$$\mathrm{x} = c_1\mathrm{v}_1 + \ldots + c_{p-1}\mathrm{v}_{p-1} + c_p\mathrm{v}_p \qquad (4)$$

for suitable scalars $c_1, \ldots, c_p$.

- Substituting the expression for $\mathbf{v}_p$ from (3) into (4), it is easy to

see that $\mathbf{x}$ is a linear combination of $\mathrm{v}_1,\ldots\mathrm{v}_{p-1}$.

THE SPANNING SET THEOREM

- Thus $\{\mathrm{v}_1, \ldots, \mathrm{v}_{p-1}\}$ spans H, because **x** was an arbitrary element of H.

b, Try to do it Yourself

THE SPANNING SET THEOREM

- **Example 7: Let** $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ **and** $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Then find a basis for the subspace H.

- **Solution:** Every vector in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because
- $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$$

THE SPANNING SET THEOREM

- Now let **x** be any vector in *H*—say, $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.
- Since $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, we may substitute

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2 \end{aligned}$$

- Thus **x** is in Span $\{\mathbf{v}_1, \mathbf{v}_2\}$, so every vector in *H* already belongs to Span $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- We conclude that *H* and Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ are actually the set of vectors.
- It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of *H* since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

BASIS FOR COL B

- **Example 8:** Find a basis for Col B, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution:** Each nonpivot column of B is a linear combination of the pivot columns.
- In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.
- By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span Col B.

BASIS FOR COL *B*

- Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Since $\mathbf{b}_1 \neq \mathbf{0}$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent. (Theorem 4).

- Thus S is a basis for Col B.

BASES FOR NUL A AND COL A

- **Theorem 6:** The pivot columns of a matrix A form a basis for Col A.
- Proof : see textbook
- **Warning:**
- The pivot columns of a matrix A are evident when A has been reduced only to echelon form.
- But, be careful to use the pivot columns of A itself for the basis of Col A.
- Row operations can change the column space of a matrix.
- The columns of an echelon form B of A are often not in the column space of A.

Thank you for listening