FALL, 2019

Lecture 1 - Linear equations in Linear Algebra

Contents

1.1 Symstems of Linear Equations

- A **linear equation** in the variables $x_1, x_2, \dots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where b and the coefficients $a_1, a_2, \dots, a_n$ are real or complex numbers

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables .

Linear Equation

- A system of linear equations has
  1. **no** solution, or
  2. exactly **one** solution, or
  3. infinitely **many** solutions.
- A system of linear equations is said to be **consistent** if it has either **one** solution or infinitely **many** solutions.
- A system is **inconsistent** if it has **no solution**

Linear Equation

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

- with the coefficients of each variable aligned in columns,

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

coefficient matrix **augmented matrix**

SOLVING SYSTEM OF EQUATIONS

- **Example 1:** Solve the given system of equations.

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{aligned}
$$

Solution : We consider the corresponding augmented matrix

SOLVING SYSTEM OF EQUATIONS

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \text{Ix4-3} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \text{II/2}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} \text{III+3II} \longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{II+4III}$$

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix} \text{I+2II} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x_1 = 29$

$x_2 = 16$

$x_3 = 3$

triangular form.

ELEMENTARY ROW OPERATIONS

- **Elementary row operations** include the following:
    1. **(Replacement)** Replace one row by the sum of itself and a multiple of another row.
    2. **(Interchange)** Interchange two rows.
    3. **(Scaling)** Multiply all entries in a row by a nonzero constant.

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- **Two fundamental questions** about a linear system are as follows:
    1. Is the system consistent; that is, does at least one solution ***exist***?
    2. If a solution exists, is it the *only* one; that is, is the solution ***unique***?
- **In Example 1** : (29,16,3) is a solution of the system . Thus the system is consistent.

1.2 ROW REDUCTION AND ECHELON FORM

Row Reduction and Echelon Form

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following **three properties**:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

PIVOT POSITION

- Definition : A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contain a pivot position.

PIVOT POSITION

- **Example 2: Row reduce the matrix** A below to echelon form, and locate the **pivot columns** of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Solution:** The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

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PIVOT POSITION

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Next pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

ROW REDUCTION ALGORITHM

- **Example:** Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

ROW REDUCTION ALGORITHM

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Piovt column

STEP 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position

ROW REDUCTION ALGORITHM

- Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead)

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

- **STEP 3**: Use row replacement operations to create zeros in all positions below the pivot.

ROW REDUCTION ALGORITHM

- **Step 4** : With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ \hline 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

- For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add- 3/2 times the "top" row to the row below.

ROW REDUCTION ALGORITHM

- When we cover the row containing the second pivot position for step 4, we are left with a new submatrix that has only one row.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ \hline 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 5 : Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

ROW REDUCTION ALGORITHM

- The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row scaled by } \frac{1}{2}$$

- Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \leftarrow \text{Row } 1 + (9) \times \text{row } 2$$

ROW REDUCTION ALGORITHM

- Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \longrightarrow \text{Row scaled by } \frac{1}{3}$$

- This is the reduced echelon form of the original matrix.

SOLUTIONS OF LINEAR SYSTEMS

- Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three variables because the augmented matrix has four columns. The associated system of equations is

SOLUTIONS OF LINEAR SYSTEMS

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &x_1 = 1 + 5x_3 \\ &x_2 = 4 - x_3 \\ &x_3 \text{ is free} \end{aligned}$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable, x_3, is called a **free variable**.
- For instance, when $x_3 = 0$, the solution is (1,4,0); when $x_3 = 1$, the solution is (6,3,1).

EXISTENCE AND UNIQUENESS THEOREM

- Existence and Uniqueness Theorem

- A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

  $[0 \;\ldots\; 0 \;\; b]$ with b nonzero.

- If a linear system is consistent, then the solution set contains either
  (i) a unique solution, when there are no free variables, or
  (ii) infinitely many solutions, when there is at least on free variable.

CONCLUSION

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.3 VECTOR EQUATION

VECTOR EQUATIONS

Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.
- An example of a vector with two entries is $\mathrm{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

where w_1 and w_2 are any real numbers.

- The set of all vectors with two entries is denoted by $\mathbb{R}^2$ (read "r-two").

VECTOR EQUATIONS

Example 1: Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$

Solution: $4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$, $(-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$ and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

Algebraic Properties of $\mathbb{R}^n$

- For all u, v, w in $\mathbb{R}^{\boldsymbol{n}}$ and all scalars c, d :

1. u + v = v + u
2. (u + v) + w = u + (v + w)
3. u + 0 = 0 + u
4. u + (-u) = -u + u
5. c(u + v) = cu + cv
6. (c +d)u = cu + du
7. c (du) = (cd) u
8. 1u = u

LINEAR COMBINATIONS

- Given vectors $v_1, v_2, \ldots, v_p,$ in R^n and given scalars $c_1, c_2, \ldots, c_p$, the vector y defined by

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

- Is called a linear combination of $v_1, v_2, \ldots, v_p,$ with weights $c_1, c_2, \ldots, c_p$
- The weights in linear combination can be real numbers, including zero.

LINEAR COMBINATIONS

Example : Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Determine whether **b** can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \qquad (1)$$

If vector equation (1) has a solution, find it.

LINEAR COMBINATIONS

- **Solution:** Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underset{\uparrow \atop \mathbf{a}_1}{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}} + x_2 \underset{\uparrow \atop \mathbf{a}_2}{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}} = \underset{\uparrow \atop \mathbf{b}}{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}$$

Which is the same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

LINEAR COMBINATIONS

And

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \qquad (2)$$

- That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \qquad (3)$$

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence **b** is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array} \end{array}$$

- Write this matrix in a way that identifies its columns.

$$\begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{b} \end{bmatrix} \qquad (4)$$

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LINEAR COMBINATIONS

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

  has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (5)$$

- In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS

- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

  with $c_1, \ldots, c_p$ scalars.

1.4 MATRIX EQUATION $Ax = b$

1.4 MATRIX EQUATION $A\mathbf{x} = \mathbf{b}$

- **Definition:** If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and x**, denoted by **$A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in x as weights;** that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

- Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

MATRIX EQUATION Ax = b

- **Example :**

For $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^m$, write the linear combination $3\mathrm{v}_1 - 5\mathrm{v}_2 + 7\mathrm{v}_3$ as a matrix times a vector.

Solution:

Place $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5 and 7 into a vector $\mathbf{x}$. That is

$$3\mathrm{v}_1 - 5\mathrm{v}_2 + 7\mathrm{v}_3 = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathrm{x}$$

MATRIX EQUATION Ax = b

- Now, write the system of linear equations as a vector equation involving a linear combination of vectors.
- For example, the following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \qquad (1)$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad (2)$$

MATRIX EQUATION Ax = b

- As in the example, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad (3)$$

- Equation (3) has the form Ax = b . Such an equation is called a **matrix equation,** to distinguish it from a vector equation such as shown in (2).

MATRIX EQUATION Ax = b

Theorem 3 :

If A is an $m \times n$ matrix, with columns $a_1, \ldots, a_n$, and if b is in $\mathbb{R}^n$, then the matrix equation $A\mathbf{x} = \mathbf{b}$

has the same solution set as the vector equation

$$x_1\boldsymbol{a_1} + x_1\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\boldsymbol{a_1} \quad \boldsymbol{a_2} \ \ldots \ \boldsymbol{a_n} \quad \mathbf{b}]$$

EXISTENCE OF SOLUTIONS

THEOREM 4 :

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution.
b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

PROPERTIES OF THE MATRIX-VECTOR PRODUCT $A\mathbf{x}$

Theorem:

If A is an (m x n) matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then

a. A (u + v) = Au + Av
b. A(cu) = c (Au)

1.5 SOLUTION SETS OF LINEAR SYSTEM

HOMOGENEOUS LINEAR SYSTEMS

- A system of linear equations is said to be **homogeneous** if it can be written in the form Ax = 0 , where A is an (m x n) matrix and 0 is the zero vector in $\mathbb{R}^m$.
- Such a system Ax = 0 *always* has at least one solution, namely, this zero solution is called the **trivial solution**.
- The homogenous equation $A\mathbf{x} = 0$, the important question is whether there exists a **nontrivial solution,** that is, a nonzero vector x that satisfies $A\mathbf{x} = 0$.

HOMOGENEOUS LINEAR SYSTEMS

- **Example:** Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

- **Solution:** Let *A* be the matrix of coefficients of the system and row reduce the augmented matrix $[\,A \quad 0\,]$ to echelon form:

HOMOGENEOUS LINEAR SYSTEMS

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - \frac{4}{3}x_3 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3 \mathrm{v}$$

PARAMETRIC VECTOR FORM

- The equation of the form $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (s, t in $\mathbb{R}$) is called a **parametric vector equation** of the plane.

- In Example 1, the equation $\mathbf{x} = x_3\mathbf{v}$ (with x_3 free), or $\mathbf{x} = t\mathbf{v}$ (with t in $\mathbb{R}$), is a parametric vector equation of a line.

- Whenever a solution set is described explicitly with vectors as in Example 1, we say that the solution is in **parametric vector form**.

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Example :** Describe all solutions of $Ax = b$, where
- $$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- **Solution:** Row operations on $[\,A \;\; b\,]$ produce

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - \frac{4}{3}x_3 &= -1 \\ x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

$$x_1 = -1 + \frac{4}{3}x_3$$
$$x_2 = 2$$
$\mathbf{x}_3$ is free

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- As a vector, the general solution of $A\mathrm{x} = \mathrm{b}$ has the form

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathrm{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathrm{v}}$$

SOLUTIONS OF NONHOMOGENEOUS SYSTEMS

- The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \qquad (t \text{ in } \mathbb{R}) \qquad (3)$$

  describes the solution set of nonhomogeneous systems in parametric vector form.

- On the other hand, the solution set of homogeneous systems has the parametric vector equation

$$\mathbf{x} = t\mathbf{v} \qquad (t \text{ in } \mathbb{R}) \qquad (4)$$

[with the same **v** that appears in (3)].

- Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector **p** to the solutions of $A\mathbf{x} = 0$.

WRITING A SOLUTION SET (OF A CONSISTENT SYSTEM) IN PARAMETRIC VECTOR FORM

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

THANK YOU FOR LISTENING