Fall,2017

MAT 207- LINEAR ALGEBRA

Lecture 5 – Determinants

Content

Introduction to Determinants

Introduction to Determinants

- **Definition**: For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det A = a_{11} \det A_{11} - a_{12} det A_{12} + \cdots + (-1)^{1+n} a^{1n} det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} Det\, A_{1j}$$

Introductions to Determinants

- **Example 1** Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute det$A$ = a_{11}detA_{11} - a_{12}detA_{12} + a_{13}detA_{13}:

$$detA = 1 \cdot det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 1(0 – 2) – 5(0 – 0) + 0(-4 – 0) = -2$$

Introductions to Determinants

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets.
- Thus the calculation in Example 1 can be written as

$$detA = 1 \cdot \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} = \cdots = -2$$

Introductions to Determinants

- Given $A = [a_{ij}]$, the ***(i, j)*-cofactor** of A is the number C_{ij} given by

$$Cij = (-1)^{i\,+\,j}\, detA_{ij} \qquad (4)$$

- $detA = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1nC1n}$
- This formula is called **a cofactor expansion across the first row** of A.

Introductions to Determinants

- **Theorem 1:** The determinant of an $n \times n$ matrix A can be computed by a cofactor across any row or down any column. The expansion across the i th row using the cofactors in (4) is

$$detA = ai_1C_{i1} + + \cdots + ainCin$$

- The cofactor expansion down the j th column is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Introductions to Determinants

- **Example 2** Use a cofactor expansion across the third row to compute det A, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

- **Solution** Compute

$$detA = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= (-1)^{3+1}a_{31}detA_{31} + (-1)^{3+2}a_{32}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33}$$

$$= \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$$

= 0 + 2(-1) + 0 = -2

Introductions to Determinants

- **Theorem 2:** If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & * & * \\ 0 & b & * \\ * & 0 & c \end{vmatrix} = \text{abc}$$

Properties of Determinants

Properties of Determinants

- **Theorem 3:** Let A be a square matrix

  a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

  b) If two rows of A are interchanged to produce B, then $\det B = - \det A$.

  a) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$

Properties of Determinants

- **Example 1** Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

- **Solution** The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries. The first two row replacements in column 1 do not change the determinant:

Properties of Determinants

- $detA = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$

- An interchange of rows 2 and 3 reverses the sign of the determinant, so

$$detA = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = 15$$

Properties of Determinants

- **Theorem 4**: A square matrix A is invertible if and only if det $A \neq 0$.
- **Example 3** Compute det A, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$
- **Solution** Add 2 times row 1 to row 3 to obtain

$$detA = det\begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0$$

because the second and third rows of the second matrix are equal.

Properties of Determinants

- **Theorem 5:** If A is a $n \times n$ matrix, then $\det A^{\mathrm{T}} = \det A$.

➡ See Proof in textbook

- **Theorem 6:** If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det A)$.

- **Example 5** Verify Theorem 6 for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

Properties of Determinants

- **Solution**

$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}$$

- and

$$\det AB = 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45$$

Since det A = 9 and det B = 5,

$$(\det A)(detB) = 9 \cdot 5 = 45 = \det AB$$

Crammer’s rule, Volume and Linear Transformation

Crammer's Rule

- **Theorem 7**: Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax=b$ has entries given by

$$x_i = \frac{detA_i(b)}{detA}, \quad i = 1, 2, \ldots, n \qquad (1)$$

- **Proof** Denote the columns of A by $a_1, \ldots, a_n$ and the columns of the $n \times n$ identity matrix I by $e_1, \ldots, e_n$. If $Ax = b$, the definition of matrix multiplication shows that

$$A.Ii(x) = A[e_1 \ \ldots \ x \ \ldots \ en] = A[e_1 \ \ldots \ Ax \ \ldots \ Ae_n]$$

$$= [a_1 \ \ldots \ b \ \ldots a_n] = A_i(b)$$

Crammer’s Rule

- By the multiplicative property of determinants,

$$(detA)\left(detI_i(x)\right) = detAi(b)$$

- The second determinant on the left is simply x_{i}. Hence $(detA)\cdot xi = detAi(b)$. This proves (1) because A is invertible and det $A \neq 0$.

- **Example 1** Use Cramer’s rule to solve the system

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$$

Crammer's Rule

- **Solution** View the system as $A\mathrm{x} = b$. Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

- Since $\det A = 2$, the system has a unique solution. By Cramer's rule,

$$x_1 = \frac{detA_1(b)}{detA} = \frac{24+16}{2} = 20$$

$$x_2 = \frac{detA_2(b)}{detA} = \frac{24+30}{2} = 27$$

A FORMULA FOR A^{-1}

- **Theorem 8**: Let A be an invertible $n \times n$ matrix. Then
$$A^{-1} = \frac{1}{detA} adjA$$
- **Example 3** Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.
- **Solution** The nine cofactors are

$$C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \quad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$
$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \quad C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, \quad C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$
$$C_{31} = +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 4, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

A FORMULA FOR A^{-1}

- The adjugate matrix is the *transpose* of the matrix of cofactors. Thus

$$adjA = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

- We could compute det A directly,
- $(adjA) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = 14I$

A FORMULA FOR A^{-1}

- Since (adj A)A = 14I, Theorem 8 shows that det A = 14 and

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$$

DETERMINANTS AS AREA OR VOLUME

- **Theorem 9:** If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is [det A]. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is |det A|.

- **Proof** The theorem is obviously true for any 2×2 diagonal matrix:

$$\left|det\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad| = \begin{Bmatrix} area\ of \\ rectangle \end{Bmatrix}$$

- See Fig. 1 on the next slide.

DETERMINANTS AS AREA OR VOLUME

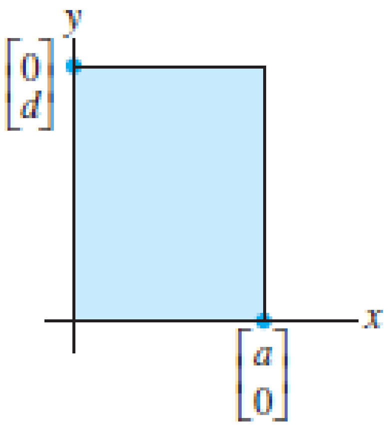

FIGURE 1
Area $= |ad|$.

- It will suffice to show that any 2×2 matrix $A = [a_1 \;\; a_2]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$.

DETERMINANTS AS AREA OR VOLUME

- Let a_1 and a_2 be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by a_1 and a_2 equals the area of the parallelogram determined by a_1 and a_2+ca_1.

- To prove this statement, we may assume that a_2 is not a multiple of a_1, for otherwise the two parallelograms would be degenerate and have zero area.

- If L is the line through 0 and a_1, then $a_2 + L$ is the line through a_2 parallel to L, and $a_2 + \mathrm{c}a_1$ is on this line. See Fig. 2 on the next slide.

DETERMINANTS AS AREA OR VOLUME

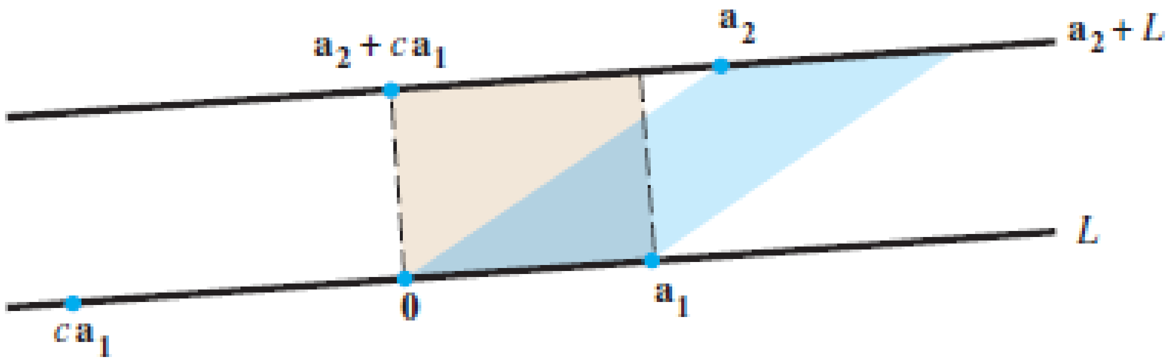

FIGURE 2 Two parallelograms of equal area.

- The points a_2 and $a_2 + ca_1$ have the same perpendicular distance to L. Hence the two parallelograms in Fig. 2 have the same area, since they share the base from 0 to a_1.

DETERMINANTS AS AREA OR VOLUME

- The proof for $\mathbb{R}^3$ is similar. The theorem is obviously true for a 3×3 diagonal matrix. See Fig. 3 below:

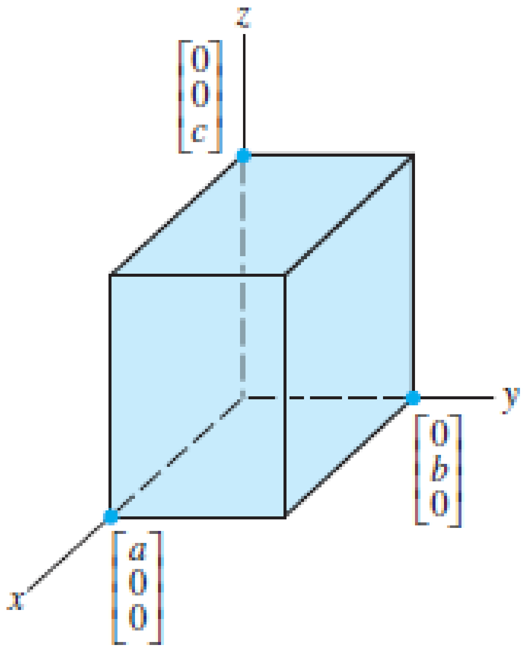

FIGURE 3
Volume $= |abc|$.

DETERMINANTS AS AREA OR VOLUME

- And any 3×3 matrix A can be transformed into a diagonal matrix using column operations that do not change $|\det A|$.
- A parallelepiped is shown in Fig. 4 below as a shaded box with two sloping sides.

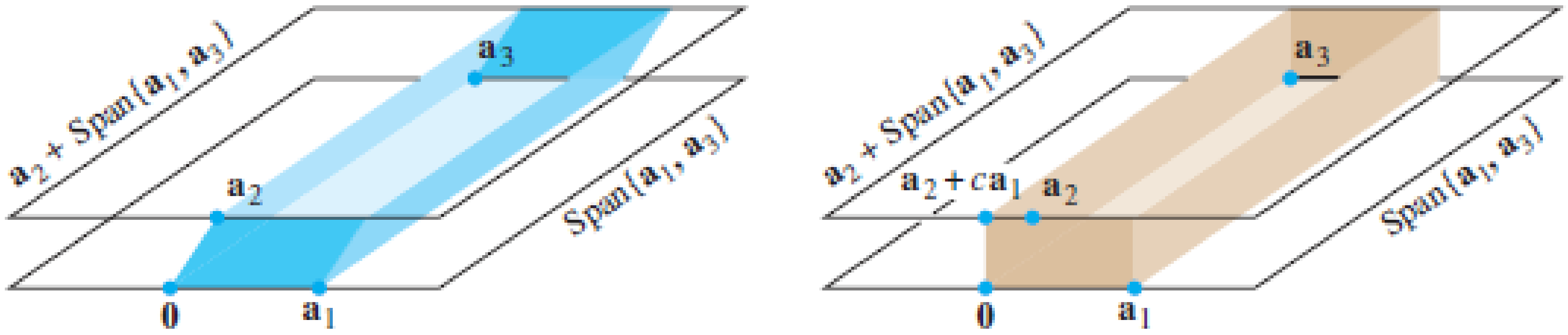

FIGURE 4 Two parallelepipeds of equal volume.

DETERMINANTS AS AREA OR VOLUME

- Its volume is the area of the base in the plane Span$\{a_1, a_3\}$ times the altitude of a_2 above Span$\{a_1, a_3\}$. Any vector $a_2 + ca_1$ lies in the plan Span$\{a_1, a_3\}$, which is parallel to Span$\{a_1, a_3\}$.

- Hence the volume of the parallelepiped is unchanged when $[a_1 \;\; a_2 \;\; a_3]$ is changed to $[a_1 \quad a_2 + ca_1 \quad a_3]$.

DETERMINANTS AS AREA OR VOLUME

- **Example 4** Calculate the area of the parallelogram determined by the points (-2, -2), (0, 3), (4, -1), and (6, 4). See Fig. 5(a) below:

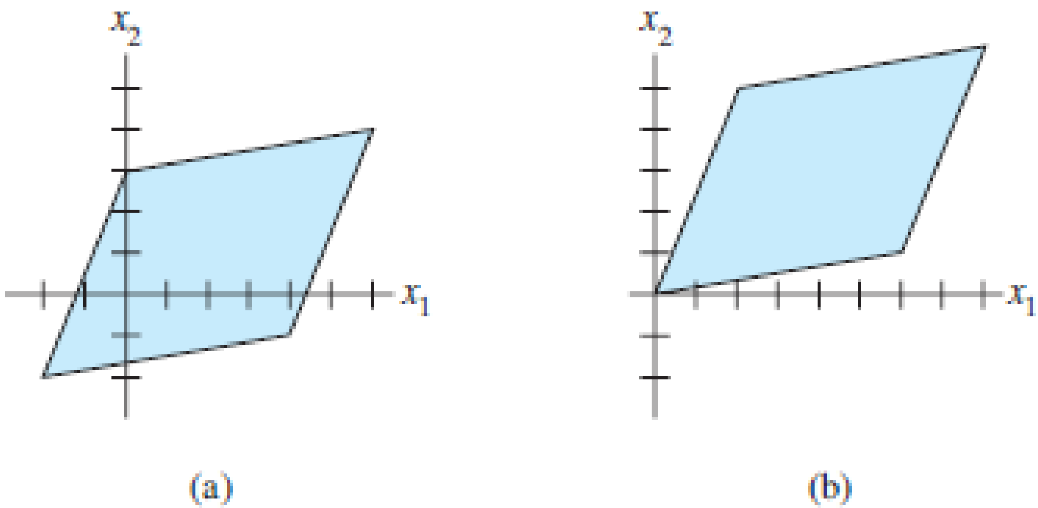

FIGURE 5 Translating a parallelogram does not change its area.

DETERMINANTS AS AREA OR VOLUME

- **Solution** First translate the parallelogram to one having the origin as a vertex. For example, subtract the vertex (-2, -2) from each of the four vertices.
- The new parallelogram has the same area, and its vertices are (0, 0), (2, 5), (6, 1), and (8, 6). See Fig. 5(b) on the previous slide.
- This parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

- Since $|\det A| = |-28|$, the area of the parallelogram is 28.

LINEAR TRANSFORMATIONS

- **Theorem 10**: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |detA| \cdot \{area\ of\ S\} \qquad (5)$$

- If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |detA| \cdot \{volume\ of\ S\} \qquad (6)$$

- See proof in textbook

LINEAR TRANSFORMATIONS

- **Example 5** Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

- **Solution** We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, because if $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and x = $A\mathbf{u}$, then

$$u_1 = \frac{x_1}{a} \text{ and } u_2 = \frac{x_2}{b}$$

LINEAR TRANSFORMATIONS

- It follows that **u** is in the unit disk, with $u_2^1 + u_2^2 \leq 1,$ if any only if **x** is in E, with $(x_1/a)^2 + (x_2/b)^2 \leq 1$. By generalization of Theorem 10,

$$\{area\ of\ ellipse\} = \{area\ of\ T(D)\}$$

$$= |detA| \cdot \{area\ of\ D\}$$

$$= ab \cdot \pi(1)^2 = \mu ab$$

Thank you for listening