Fall,2017

MAT 207- LINEAR ALGEBRA

Lecture 6 – Vector Space

Content

Coordinate Systems

THE UNIQUE REPRESENTATION THEOREM

- **Theorem 7:** Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for vector space *V*. Then for each **x** in *V*, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \qquad (1)$$

- **Proof:** Since $\mathcal{B}$ spans *V*, there exist scalars such that (1) holds.
- Suppose **x** also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

for scalars $d_1, \ldots, d_n$.

THE UNIQUE REPRESENTATION THEOREM

- Then, subtracting, we have

$$0 = \mathrm{x} - \mathrm{x} = (c_1 - d_1)\mathrm{b}_1 + \ldots + (c_n - d_n)\mathrm{b}_n \qquad (2)$$

- Since B is linearly independent, the weights in (2) must all be zero. That is, for $1 \le j \le n$.

- **Definition:** Suppose B $= \{\mathrm{b}_1, \ldots, \mathrm{b}_n\}$ is a basis for *V* and **x** is in *V*. **The coordinates of x relative to the basis B** (or the **B-coordinate of x**) are the weights $c_1, \ldots, c_n$ such that $\mathrm{x} = c_1\mathrm{b}_1 + \ldots + c_n\mathrm{b}_n$

THE UNIQUE REPRESENTATION THEOREM

- If $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of **x**, then the vector in $\mathbb{R}^n$

$$[\mathrm{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x**.

COORDINATES IN $\mathbb{R}^n$

- **Example 1:** Let $\mathrm{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathrm{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathrm{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and

  $\mathcal{B} = \{\mathrm{b}_1, \mathrm{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.
- **Solution:** The $\mathcal{B}$-coordinate c_1, c_2 of $\mathbf{x}$ satisfy

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

COORDINATES IN $\mathbb{R}^n$

- Or
- $$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \qquad (3)$$
- This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left.
- In any case, the solution is $c_1 = 3$, $c_2 = 2$.
- Thus $\mathrm{x} = 3\mathrm{b}_1 + 2\mathrm{b}_2$ and
- $$[\mathrm{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

COORDINATES IN $\mathbb{R}^n$

- The matrix in (3) changes the $\mathcal{B}$-coordinates of a vector **x** into the standard coordinates for **x**.
- An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $\mathcal{B} = \{\mathrm{b}_1, \ldots, \mathrm{b}_n\}$.
- Let $P_{\mathcal{B}} = \begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \cdots & \mathrm{b}_n \end{bmatrix}$
- Then the vector equation

$$\mathrm{x} = c_1\mathrm{b}_1 + c_2\mathrm{b}_2 + \ldots + c_n\mathrm{b}_n$$

is equivalent to

$$\mathrm{x} = P_{\mathrm{B}}[\mathrm{x}]_{\mathrm{B}} \qquad (4)$$

COORDINATES IN $\mathbb{R}^n$

-
- $P_{\mathcal{B}}$ is called the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.
- Left-multiplication by $P_{\mathcal{B}}$ transforms the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ into $\mathbf{x}$.
- Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem).

THE COORDINATE MAPPING

- **Theorem 8:** Let $\mathcal{B} = \{\mathrm{b}_1, \ldots, \mathrm{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathrm{x} \mapsto [\mathrm{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.
- **Proof:** Take two typical vectors in V, say,

$$\mathrm{u} = c_1\mathrm{b}_1 + \ldots + c_n\mathrm{b}_n$$
$$\mathrm{w} = d_1\mathrm{b}_1 + \ldots + d_n\mathrm{b}_n$$

- Then, using vector operations,

$$\mathrm{u} + \mathrm{v} = (c_1 + d_1)\mathrm{b}_1 + \ldots + (c_n + d_n)\mathrm{b}_n$$

THE COORDINATE MAPPING

- It follows that

$$[\mathbf{u}+\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{w}]_{\mathcal{B}}$$

- So the coordinate mapping preserves addition.
- If r is any scalar, then

$$r\mathbf{u} = r(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = (rc_1)\mathbf{b}_1 + \ldots + (rc_n)\mathbf{b}_n$$

THE COORDINATE MAPPING

- So

$$[r\mathbf{u}]_B = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r[\mathbf{u}]_B$$

- Thus the coordinate mapping also preserves scalar multiplication and hence is a linear transformation.
- The linearity of the coordinate mapping extends to linear combinations.
- If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then
- $$[c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p]_B = c_1[\mathbf{u}_1]_B + \ldots + c_p[\mathbf{u}_p]_B \qquad (5)$$

THE COORDINATE MAPPING

-
- In words, (5) says that the $\mathcal{B}$-coordinate vector of a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ is the *same* linear combination of their coordinate vectors.
- The coordinate mapping in Theorem 8 is an important example of an *isomorphism* from V onto $\mathbb{R}^n$.
- In general, a one-to-one linear transformation from a vector space V onto a vector space W is called an **isomorphism** from V onto W.

THE COORDINATE MAPPING

- **Example 7:** Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$

  and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$
  Determine if **x** is in *H*, and if it is, find the coordinate vector of **x** relative to $\mathcal{B}$.

THE COORDINATE MAPPING

- **Solution:** If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1\begin{bmatrix}3\\6\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}3\\12\\7\end{bmatrix}$$

- The scalars c_1 and c_2, if they exist, are the B-coordinates of $\mathbf{x}$.

THE COORDINATE MAPPING

- Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Thus $c_1 = 2$, $c_2 = 3$ and $[\mathbf{x}]_B$

The Dimension of a Vector Subspace

DIMENSION OF A VECTOR SPACE

Theorem 9: If a vector space *V* has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in *V* containing more than *n* vectors must be linearly dependent.

Theorem 10: If a vector space *V* has a basis of *n* vectors, then every basis of *V* must consist of exactly *n* vectors.

Definition: If *V* is spanned by a finite set, then *V* is said to be **finite-dimensional**, and the **dimension** of *V*, written as dim *V*, is the number of vectors in a basis for *V*. The dimension of the zero vector space **{0}** is defined to be zero. If *V* is not spanned by a finite set, then *V* is said to be **infinite-dimensional**

DIMENSION OF A VECTOR SPACE

- **Example 3:** Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a,b,c,d \text{ in } \mathbb{R} \right\}$$

- H is the set of all linear combinations of the vectors

DIMENSION OF A VECTOR SPACE

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

- Clearly, $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, but $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$.
- By the Spanning Set Theorem, we may discard $\mathbf{v}_3$ and still have a set that spans H.

DIMENSION OF A VECTOR SPACE

- Finally, $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
- So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent and hence is a basis for H.
- Thus dim H = 3

- **Theorem 11:** Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

THE BASIS THEOREM

- **Theorem 12:** Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.
- **Proof:** By Theorem 11, a linearly independent set S of p elements can be extended to a basis for V.
- But that basis must contain exactly p elements, since dim .
- So S must already be a basis for V.

THE BASIS THEOREM

- Now suppose that S has p elements and spans V.
- Since V is nonzero, the Spanning Set Theorem implies that a subset S' of S is a basis of V.
- Since $\dim V = p$, S' must contain p vectors.
- Hence $S = S'$.

THE DIMENSIONS OF NUL *A* AND COL *A*

- Let *A* be an $m \times n$ matrix, and suppose the equation $A\mathrm{x} = 0$ has *k* free variables.

- A spanning set for Nul *A* will produce exactly *k* linearly independent vectors—say, $\mathrm{u}_1, \ldots, \mathrm{u}_k$ one for each free variable.

- So $\{\mathrm{u}_1, \ldots, \mathrm{u}_k\}$ is a basis for Nul *A*, and the number of free variables determines the size of the basis.

THE DIMENSIONS OF NUL *A* AND COL *A*

- Thus, the dimension of Nul *A* is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col *A* is the number of pivot columns in *A*.

- **Example 5:** Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

THE DIMENSIONS OF NUL *A* AND COL *A*

- **Solution:** Row reduce the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- There are three free variable—x_2, x_4 and x_5.
- Hence the dimension of Nul A is 3.
- Also dim Col $A = 2$ because A has two pivot columns.

Rank

THE ROW SPACE

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$.
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A.
- Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$.
- Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

THE ROW SPACE

Theorem 13: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Example 2:** Find bases for the row space, the column space, and the null space of the matrix

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

THE ROW SPACE

- **Solution:** To find bases for the row space and the column space, row reduce A to an echelon form:

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B).

THE ROW SPACE

- Thus

  Basis for Row $A: \{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$
- For the column space, observe from *B* that the pivots are in columns 1, 2, and 4.
- Hence columns 1, 2, and 4 of *A* (not *B*) form a basis for Col *A*:

Basis for Col $A: \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

THE ROW SPACE

- Notice that any echelon form of A provides (in its nonzero rows) a basis for Row A and also identifies the pivot columns of A for Col A.
- However, for Nul A, we need the *reduced echelon form*.
- Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THE ROW SPACE

- The equation $Ax = 0$ is equivalent to $Cx = 0$, that is,

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

- So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables.
- The calculations show that

THE ROW SPACE

- Basis for Nul $A: \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

- Observe that, unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.

The Rank Theorem

- **Definition:** The **rank** of A is the dimension of the column space of A.
- Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T.
- The dimension of the null space is sometimes called the **nullity** of A.
- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank}\, A + \dim \text{Nul}\, A = n$$

The Rank Theorem

- **Example 3:**
  a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?
  b. Could a 6×9 matrix have a two-dimensional null space?
- **Solution:**
  a. Since A has 9 columns, $(\text{rank } A)+2=9$, and hence rank . $A=7$
  b. No. If a 6×9 matrix, call it B, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem

THE INVERTIBLE MATRIX THEOREM (CONTINUED)

-
  - But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of Col B cannot exceed 6; that is, rank B cannot exceed 6.

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

  m. The columns of A form a basis of $\mathbb{R}^n$.

  n. Col $A = \mathbb{R}^n$

  o. Dim Col $A = n$

  p. Rank $A = n$

  q. Nul $A = \{0\}$

  r. Dim Nul $A = 0$

Change Of Basis

Change Of Basis

- **Example 1** Consider two bases $\beta = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for a vector space V, such that

$$b_1 = 4c_1 + c_2 \quad \text{and} \quad b_2 = -6c_1 + c_2 \quad (1)$$

- Suppose $x = 3b_1 + b_2$ (2)

- That is, suppose $[x]_\beta = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_C$.

Change Of Basis

- **Solution** Apply the coordinate mapping determined by C to $\mathbf{x}$ in (2). Since the coordinate mapping is a linear transformation,

$$\begin{aligned}[x]_C &= [3\mathrm{b}_1 + \mathrm{b}_2]_C \\ &= [3\mathrm{b}_1]_C + [\mathrm{b}_2]_C\end{aligned}$$

- We can write the vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[x]_C = [\mathrm{b}_1]_C \quad [\mathrm{b}_2]_C \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad (3)$$

Change Of Basis

- This formula gives $[x]_C$, once we know the columns of the matrix. From (1),

$$[\mathrm{b}_1]_C = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } [\mathrm{b}_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

- Thus, (3) provides the solution:

$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

- The *C*-coordinates of **x** match those of the **x** in Fig. 1, as seen on the next slide.

Change Of Basis

- **Theorem 15**: Let $\beta = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_2\}$ for a vector space V. Then there is a unique n × n matrix $\underset{c \leftarrow \beta}{P}$ such that

$$[x]_C = \underset{c \leftarrow \beta}{P} [x]_\beta$$

- The columns of $\underset{c \leftarrow \beta}{P}$ are the C-coordinate vectors of the vectors in the basis β. That is,

$$\underset{c \leftarrow \beta}{P} = [\ [b_1]_C \quad [b_2]_C \quad \ldots \quad [b_n]_C]$$

Change Of Basis

- The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.** Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.
- Figure 2 below illustrates the change-of-coordinates equation (4).

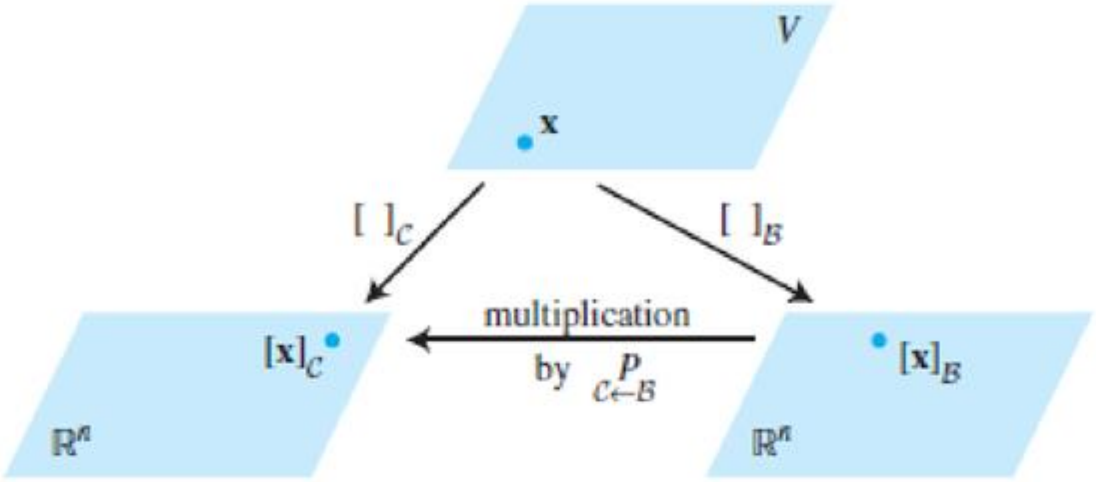

FIGURE 2 Two coordinate systems for V.

Change Of Basis

- **Example 2** Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ and consider the bases for $\mathbb{R}^n$ given by $\beta = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$. Find the change-of-coordinates matrix from β to C.

- **Solution** The matrix $\beta \overset{P}{\leftarrow} C$ involves the C-coordinate vectors of b_1 and b_2. Let $[b_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[b_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[c_1 \quad c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1 \quad \text{and} \quad [c_1 \quad c_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_2$$

Change Of Basis

- To solve both systems simultaneously, augment the coefficient matrix with b_1 and b_2, and row reduce:

$$[c_1 \quad c_2 \vdots b_1 \quad b_2] = \begin{bmatrix} 1 & 3 & \vdots & -9 & -5 \\ -4 & -5 & \vdots & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 6 & 4 \\ 0 & 1 & \vdots & -5 & -3 \end{bmatrix}$$

- Thus

$$[\mathrm{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \text{ and } [\mathrm{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

- The desired change-of-coordinates matrix is therefore

$$\underset{c \leftarrow \beta}{P} = [[b_1]_c \quad [b_2]_c] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Thank you for listening