Solution for Problem Set 3

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

March 24, 2015

Problem 1: [3, Exercise 2.38]
A coin is tossed three times. If X is a random variable giving the number of heads that arise, construct a table showing the probability distribution of X.
Solution:

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

Problem 2: [3, Exercise 2.39]
An urn holds 5 white and 3 black marbles. If 2 marbles are to be drawn at random without replacement and X denotes the number of white marbles, find the probability distribution for X.
Solution:

x	0	1	2
$f(x)$	3/28	15/28	5/14

Problem 3: [3, Exercise 2.40]
Work Problem 2 if the marbles are to be drawn with replacement.
Solution:

x	0	1	2
$f(x)$	9/64	15/32	25/64

Problem 4: [3, Exercise 2.41]
Let Z be a random variable giving the number of heads minus the number of tails in 2 tosses of a fair coin. Find the probability distribution of Z. Compare with the results of Examples 2.1 and 2.2.
Solution:

x	-2	0	2
$f(x)$	1/4	1/2	1/4

Note:
x: random variables
$f(x)$: probability functions

Problem 5: [3, Exercise 2.42]
Let X be a random variable giving the number of aces in a random draw of 4 cards from an ordinary deck of 52 cards. Construct a table showing the probability distribution of X.
Solution:

x	0	1	2	3	4
$f(x)$	$\frac{C_4^0 C_{48}^4}{C_{52}^4}$	$\frac{C_4^1 C_{48}^3}{C_{52}^4}$	$\frac{C_4^2 C_{48}^2}{C_{52}^4}$	$\frac{C_4^3 C_{48}^1}{C_{52}^4}$	$\frac{C_4^4 C_{48}^0}{C_{52}^4}$

Problem 6: [3, Exercise 2.43]
The probability function of a random variable X is shown in the following table. Construct a table giving the distribution function of X

x	1	2	3
$f(x)$	1/2	1/3	1/6

Solution:

x	1	2	3
$F(x)$	1/2	5/6	1

Problem 7: [3, Exercise 2.44]
Obtain the distribution for (a) Problem 1, (b) Problem 2, (c) Problem 3.
Solution:
a)

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1/8 & 0 \leq x < 1 \\ 1/2 & 1 \leq x < 2 \\ 7/8 & 2 \leq x < 3 \\ 1 & 3 \leq x < \infty \end{cases}$$

b)

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 3/28 & 0 \leq x < 1 \\ 18/28 & 1 \leq x < 2 \\ 1 & 2 \leq x < \infty \end{cases}$$

c)

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 9/64 & 0 \leq x < 1 \\ 39/64 & 1 \leq x < 2 \\ 1 & 2 \leq x < \infty \end{cases}$$

Problem 8: [3, Exercise 2.45]
Obtain the distribution function for (a) Problem 4, (b) Problem 5.
Solution:
a)

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 1/4 & -2 \leq x < 0 \\ 3/4 & 0 \leq x < 2 \\ 1 & 2 \leq x < \infty \end{cases}$$

b)

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ C_{48}^4/C_{52}^4 & 0 \leq x < 1 \\ C_{48}^4 + C_4^1 C_{48}^3/C_{52}^4 & 1 \leq x < 2 \\ C_{48}^4 + C_4^1 C_{48}^3 + C_4^2 C_{48}^2/C_{52}^4 & 2 \leq x < 3 \\ C_{48}^4 + C_4^1 C_{48}^3 + C_4^2 C_{48}^2 + C_4^3 C_{48}^1/C_{52}^4 & 3 \leq x < 4 \\ 1 & 4 \leq x < \infty \end{cases}$$

Problem 9: [3, Exercise 2.46]
The following table shows the distribution function of a random variable X. Determine (a) the probability function, (b) P($1 \leq X \leq 3$), (c) P($X \geq 2$), (d) P($X < 3$), (e) P($X > 1.4$).

x	1	2	3	4
$F(x)$	1/8	3/8	3/4	1

Solution:
a)

x	1	2	3	4
$f(x)$	1/8	1/4	3/8	1/4

b)

$$P(1 \leq x \leq 3) = 3/4$$

c)

$$P(x \geq 2) = 7/8$$

d)

$$P(x < 3) = 3/8$$

e)

$$P(x > 1.4) = 7/8$$

Problem 10: [3, Exercise 2.47]
A random variable X has density function:

$$f(x) = \begin{cases} ce^{-3x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Find (a) the constant c, (b) $P(1 < X < 2)$, (c) $P(X \geq 3)$, (d) $P(X < 1)$.
Solution:
a)

$$\int_{-\infty}^{\infty} ce^{-3x} dx = -\frac{c}{3}e^{-3x} \Big|_0^{+\infty}$$
$$= \frac{c}{3} - \lim_{x \to +\infty} \frac{c}{3}e^{-3x} = \frac{c}{3} = 1 \Rightarrow c = 3$$

b)

$$P(1 < X < 2) = \int_1^2 3e^{-3x} dx = e^{-3x} \Big|_1^2 = e^{-3} - e^{-6}$$

c)

$$P(X \geq 3) = \int_3^{+\infty} 3e^{-3x} dx = -e^{-3x} \Big|_3^{+\infty} = e^{-9}$$

d)

$$P(X < 1) = 1 - P(X \geq 1) = 1 - \int_1^{+\infty} 3e^{-3x} dx = 1 - \lim_{x \to +\infty} e^{-3x} + e^{-3} = 1 - e^{-3}$$

Problem 11: [3, Exercise 2.48]
Find the distribution function for the random variable of Problem 10. Graph the density and distribution functions, describing the relationship between them.
Solution:

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & x \leq 0 \end{cases} \Rightarrow F(x) = \begin{cases} 1 - e^{-3x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Problem 12: [3, Exercise 2.49]
A random variable X has density function

$$f(x) = \begin{cases} cx^2 & 1 \leq x \leq 2 \\ cx & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) the constant c, (b) $P(X > 2)$, (c) $P(1/2 < X < 3/2)$.
Solution:

$$f(x) = \begin{cases} cx^2 & 1 \le x \le 2 \\ cx & 2 < x < 3 \\ 0 & \text{otherwise} \end{cases} \Rightarrow F(x) = \begin{cases} 0 & x \le 1 \\ \frac{c}{3}x^3 & 1 \le x \le 2 \\ \\ \frac{c}{2}x^2 & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

a)

$$x < 1 \Rightarrow F(x) = 0$$
$$1 \le x \le 2 \Rightarrow F(x) = \frac{cx^3}{3} = \frac{c}{3}(2^3 - 1)$$
$$2 < x < 3 \Rightarrow F(x) = \frac{cx^2}{2} = \frac{c}{2}(3^2 - 2^2)$$
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx$$
$$= \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx = \frac{c}{3}(2^3 - 1) + \frac{c}{2}(3^2 - 2^2) = \frac{29}{6}c = 1 \Rightarrow c = \frac{6}{29}$$

b)

$$P(X > 2) = P(2 < X < 3) + P(3 < X)$$
$$= \frac{1}{2} * \frac{6}{29}(3^2 - 2^2) + 0)$$
$$= 1 - P(X - 2) = 1 - \frac{6}{29} * \frac{1}{3}(2^2 - 1) = \frac{15}{29}$$

c)

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = P\left(\frac{1}{2} < X < 1\right) + P\left(1 < X < \frac{3}{2}\right)$$
$$= 0 + \frac{6}{29} * \frac{1}{3}\left(\left(\frac{3}{2}\right)^3 - 1^3\right) = \frac{19}{116}$$

Problem 13: [3, Exercise 2.50]
Find the distribution function for the random variable X of Problem 12.
Solution:

$$F(x) = \begin{cases} 0 & x \le 1 \\ \dfrac{cx^3}{3} + k1 & 1 \le x \le 2 \\ \\ \dfrac{cx^2}{2} + k2 & 2 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$$

$$(1): \frac{c}{3}x^3 + k1 = \frac{c}{2}x^2 + k2 \Rightarrow \frac{6/29}{3} * 2^3 + k1 = \frac{6/29}{2} * 2^2 + k2$$

$$\Rightarrow \frac{16}{29} + k1 = \frac{12}{29} + k2 \Rightarrow k2 = k1 + \frac{4}{29}$$

$$(2): \frac{c}{3}x^3 + k1 + \frac{c}{2}x^2 + k2 = 1 \Rightarrow \frac{6/29}{3} * 1^3 + k1 + \frac{6/29}{2} * 3^2 + k2 = 1$$

$$\Rightarrow \frac{2}{29} + k1 + \frac{27}{29} + k2 = 1 \Rightarrow 1 + k1 + k2 = 1 \Rightarrow k1 + k2 = 0 \Rightarrow k1 = -k2$$

From (1) and (2):

$$k1 = -\left(k1 + \frac{4}{29}\right) \Rightarrow k1 = -\frac{2}{29}\,; k2 = \frac{2}{29}$$

Therefore:

$$F(x) = \begin{cases} 0 & x \leq 1 \\ \dfrac{2x^3 - 2}{29} & 1 \leq x \leq 2 \\ \\ \dfrac{3x^2 + 2}{29} & 2 \leq x \leq 3 \\ 1 & x \geq 3 \end{cases}$$

Problem 14: [3, Exercise 2.51]

The distribution function of a random variable X is given by

$$F(x) = \begin{cases} cx^3 & 0 \leq x < 3 \\ 1 & x \geq 3 \\ 0 & x < 0 \end{cases}$$

If $P(X = 3) = 0$, find (a) the constant c, (b) the density function, (c) $P(X > 1)$, (d) $P(1 < X < 2)$.

Solution:

$$F(x) = \begin{cases} cx^3 & 0 \leq x < 3 \\ 1 & x \geq 3 \\ 0 & x < 0 \end{cases} \Rightarrow f(x) = \begin{cases} 3cx^2 & 0 \leq x < 3 \\ 0 & otherwise \end{cases}$$

a)

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^3 cx^3 dx = 1 \Rightarrow cx^3 \Big|_0^3 = 1 \Rightarrow c = \frac{1}{27}$$

b)

$$f(x) = \begin{cases} \frac{1}{9}x^2 & 0 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

c)

$$P(X > 1) = \int_1^3 f(x)dx = \frac{1}{27}(3^3 - 1) = \frac{26}{27}$$

d)

$$P(1 < X < 2) = \int_1^2 f(x)dx = \frac{1}{27}(2^3 - 1) = \frac{7}{27}$$

Problem 15: [3, Exercise 2.52]
Can the function

$$F(x) = \begin{cases} c(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

be a distribution function? Explain.
Solution:

$$F(x) = \begin{cases} c(1-x^2) & 0 \le x \le 1 \\ 0 & otherwise \end{cases} \Rightarrow f(x) = \begin{cases} -2cx & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow c(1-x^2)\Big|_0^1 = 1$$
$$\Leftrightarrow c - c(1^2 - 0) = 1 \Leftrightarrow c - c = 1 \Leftrightarrow 0 = 1$$

Therefore, the given function can't be a distribution function.

Problem 16: [3, Exercise 2.53]
Let X be a random variable having density function

$$f(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) the value of the constant c, (b) $P(\frac{1}{2} < X < \frac{3}{2})$, (c) $P(X > 1)$, (d) the distribution function.
Solution:
a)

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^2 cxdx = 1 \Rightarrow \frac{cx^2}{2}\Big|_0^2 = 1$$
$$\Leftrightarrow c\left(\frac{2^2}{2} - 0\right) = 1 \Leftrightarrow 2c = 1 \Leftrightarrow c = \frac{1}{2}$$

b)

$$P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \frac{cx^2}{2}\Big|_{1/2}^{3/2} = \frac{x^2}{4}\Big|_{1/2}^{3/2} = \frac{1}{4}\left(\left(\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right) = \frac{1}{2}$$

c)

$$P(X > 1) = \frac{x^2}{4}\Big|_1^2 = \frac{1}{4}\left(2^2 - 1^2\right) = \frac{3}{4}$$

d)

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

Problem 17: [2, Exercise 3, section 3.1]
Suppose that two balanced dice are rolled, and let X denote the absolute value of the difference between the two numbers that appear. Determine and sketch the p.f. of X.
Solution:
There are total 36 possible outcomes when 2 dice are rolled because each die has 6 sides (from 1 to 6).

- X = 0 for 6 outcomes
- X = 1 for 10 outcomes
- X = 2 for 8 outcomes
- X = 3 for 6 outcomes
- X = 4 for 4 outcomes
- X = 5 for 2 outcomes

Therefore, the p.f. $f(x)$ is as follows:

x	0	1	2	3	4	5
$f(x)$	3/18	5/18	4/18	3/18	2/18	1/18

Problem 18: [2, Exercise 4, section 3.1]
Suppose that a fair coin is tossed 10 times independently. Determine the p.f. of the number of heads that will be obtained.
Solution:
For x = 0,1,...,10, the probability of obtaining exactly x heads is:

$$\binom{10}{x}\left(\frac{1}{2}\right)^{10}$$

Problem 19: [2, Exercise 5, section 3.1]
Suppose that a box contains seven red balls and three blue balls. If five balls are selected at random, without replacement, determine the p.f. of the number of red balls that will be obtained.
Solution:
For x = 2,3,4,5, the probability that obtaining x red balls is:

$$\binom{7}{x}\binom{3}{5-x}\Big/\binom{10}{5}$$

Problem 20: [2, Exercise 6, section 3.1]
Suppose that a random variable X has the binomial distribution with parameters n =

15 and p = 0.5. Find P($X < 6$).
Solution:
The desired probability is the sum of the entries for k = 0,1,2,3,4 and 5 in that part of the table of binomial probabilities corresponding to n = 15 and p = 0.5. The sum is:

$$0.1509$$

Problem 21: [2, Exercise 4, section 3.2]
Suppose that the p.d.f. of a random variable X is as follows:

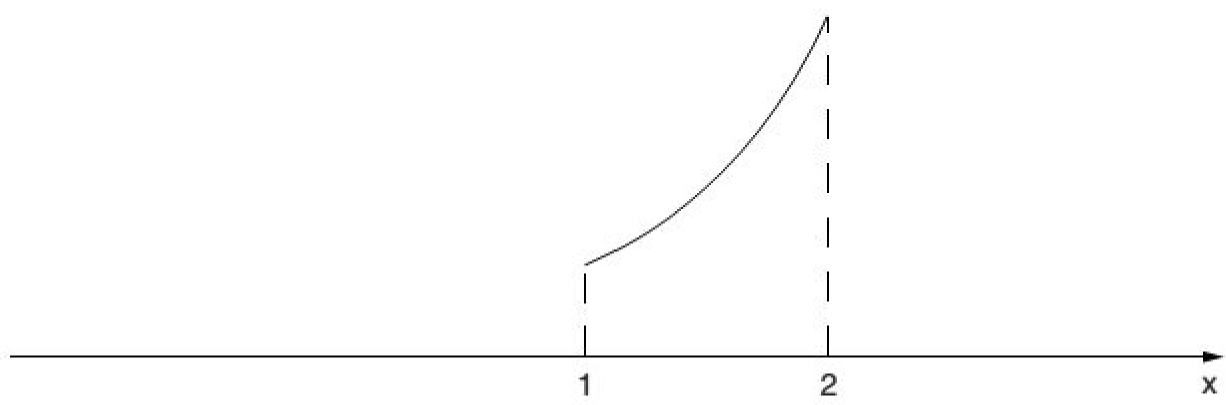
$$f(x) = \begin{cases} cx^2 & 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the value of the constant c and sketch the p.d.f.
Solution:
We have:

$$\int_{-\infty}^{\infty} f(x)dx = \int_1^2 cx^2 dx = \frac{cx^3}{3}\Big|_1^2 = \frac{7}{3}c = 1$$

$$\Rightarrow c = \frac{3}{7}$$

Therefore, the p.d.f is as follows:



b. Find the value of P(X > 3/2).
Solution:

$$\int_{3/2}^{2} f(x)dx = \int_{3/2}^{2} cx^2 dx = \frac{cx^3}{3}\Big|_{3/2}^{2} = \frac{x^3}{7}\Big|_{3/2}^{2} = \frac{37}{56}$$

Problem 22: [2, Exercise 5, section 3.2]
Suppose that the p.d.f. of a random variable X is as follows:

$$f(x) = \begin{cases} 1/8x & 0 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the value of t such that $\mathrm{P}(X \leq t) = 1/4$.
Solution:

$$\int_0^t \frac{1}{8} x dx = \frac{1}{16} x^2 \Big|_0^t = \frac{1}{4} \Rightarrow \frac{t^2}{16} = \frac{1}{4} \Rightarrow t = 2$$

b. Find the value of t such that $\mathrm{P}(X \geq t) = 1/2$.
Solution:

$$\int_t^4 \frac{1}{8} x dx = \frac{1}{16} x^2 \Big|_t^4 = \frac{1}{2} \Rightarrow 1 - \frac{t^2}{16} = \frac{1}{2} \Rightarrow t = \sqrt{8}$$

Problem 23: [1, Exercise 3.5]
Determine the value c so that each of the following functions can serve as a probability distribution of the discrete random variable X :
a. $f(x) = c(x^2 + 4)$, for $x = 0, 1, 2, 3$;
Solution:

$$\sum_{x=0}^{3} c(x^2 + 4) = 30c = 1$$

$$\Rightarrow c = \frac{1}{30}$$

b. $f(x) = c\binom{2}{x}\binom{3}{3-x}$, for $x = 0, 1, 2$.
Solution:

$$\sum_{x=0}^{2} c\binom{2}{x}\binom{3}{3-x} = c\left[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}\right] = 10c = 1$$

$$\Rightarrow c = \frac{1}{10}$$

Problem 24: [1, Exercise 3.30]
Measurements of scientific systems are always subject to variation, some more than others. There are many structures for measurement error, and statisticians spend a great deal of time modeling these errors. Suppose the measurement error X of a certain physical quantity is decided by the density function

$$f(x) = \begin{cases} k(3 - x^2) & -1 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a) Determine k that renders f(x) a valid density function.
Solution:

$$1 = k \int_{-1}^{1} (3 - x^2) dx = k\left(3x - \frac{x^3}{3}\right) \Big|_{-1}^{1} = \frac{16}{3} k$$

$$\Rightarrow k = \frac{3}{16}$$

b) Find the probability that a random error in measurement is less than 1/2.
Solution:

For $-1 \leq x \leq 1$:

$$F(x) = \frac{3}{16}\int_{-1}^{x}(3-t^2)d(t) = \left(3t - \frac{1}{3}t^3\right)\Big|_{-1}^{x} = \frac{1}{2} + \frac{9}{16}x - \frac{x^3}{16}$$

$$\Rightarrow P(X < \frac{1}{2}) = \frac{1}{2} - \left(\frac{9}{16}\right)\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^3 = \frac{99}{128}$$

c) For this particular measurement, it is undesirable if the magnitude of the error (i.e., $|x|$) exceeds 0.8. What is the probability that this occurs?
Solution:

$$P(|X| < 0.8) = P(X < -0.8) + P(X > 0.8) = F(-0.8) + 1 - F(0.8)$$

$$= 1 + \left(\frac{1}{2} - \frac{9}{16}0.8 + \frac{1}{16}0.8^3\right) - \left(\frac{1}{2} + \frac{9}{16}0.8 - \frac{1}{16}0.8^3\right) = 0.164$$

References

[1] Walpole, R. E., Myers, R. H., Myers, S. L. and Ye, K., *Probability & Statistics for Engineers & Scientists*, 9th ed., MA, USA: Prentice-Hall, 2012.

[2] DeGroot, M. H. and Schervish, M. J., *Probability and Statistics*, 4th ed., MA, USA: Pearson Education, Inc., 2012.

[3] Murray, R. S., John, J. S. and R, A. Srinivasan, *Probability and Statistics*, 3rd ed., USA: McGraw-Hill, 2009.