Problem Set 5 Solution

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: [1, Exercise 4.2]
The probability distribution of the discrete random variable X is

$$f(x) = \binom{3}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{3-x}, \; x = 0, 1, 2, 3.$$

Find the mean of x.
Solution:

$$f(x) = \left(\frac{3!}{x!(3-x)!}\right)\left(\frac{1}{4^x}\right)\left(\frac{3^{3-x}}{4^{3-x}}\right) = \left(\frac{3!}{x!(3-x)!}\right)\left(\frac{1}{4^x}\right)\left(\frac{3^3 4^x}{3^x 4^3}\right)$$

$$= \frac{162}{(64)(x!)(3-x)!(3^x)} = \frac{81}{(32)(x!)(3-x)!(3^x)}$$

$$E(X) = \sum_{x=0}^{3} x f(x) = (0)\left(\frac{27}{64}\right) + (1)\left(\frac{27}{64}\right) + (2)\left(\frac{9}{64}\right) + (3)\left(\frac{1}{64}\right) = \frac{3}{4}$$

Problem 2: [1, Exercise 4.11]
The density function of coded measurements of the pitch diameter of threads of a fitting is

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 < x < 1, \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected value of X.
Solution:

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^1 x f(x) dx = \int_0^1 \frac{4}{\pi(1+x^2)} x dx$$

$$= \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2} dx = \frac{4}{\pi} * \frac{ln(x^2+1)}{2}\Bigg|_0^1 = \frac{2ln2}{\pi} = \frac{ln4}{\pi}$$

Problem 3: [1, Exercise 4.12] If a dealers profit, in units of 5000, on a new automobile can be looked upon as a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{elsewhere} \end{cases}$$

Find the average profit per automobile.
Solution:

$$E(X) = \int_0^1 xf(x)dx = \int_0^1 2x(1-x)dx$$
$$= 2\int_0^1 (x-x^2)dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

Therefore:

$$Average\,profit \;=\; Expection * Total\;units = (1/3)(5000) = \$1,667.77$$

Problem 4: [1, Exercise 4.50]

For a laboratory assignment, if the equipment is working, the density function of the observed outcome X is

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise} \end{cases}$$

Find the variance and standard deviation of X.
Solution:

$$E(X) = \int_0^1 x(fx)dx = \int_0^1 2x(1-x)dx = 2\int_0^1 (x-x^2)dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

$$E(X^2) = \int_0^1 xf(x)dx = \int_0^1 2x^2(1-x)dx = 2\int_0^1 (x^2-x^3)dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}$$

$$\Rightarrow \sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

$$\sigma = \sqrt{Var(X)} = \sqrt{1/18} = 0.2357$$

Problem 5: [1, Exercise 4.52]

Random variables X and Y follow a joint distribution

$$f(x,y) = \begin{cases} 2, & 0 < x \leq y < 1, \\ 0, & \text{otherwise} \end{cases}$$

Determine the correlation coefficient between X and Y.
Solution:

$$f(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_{y=x}^{1} 2dy = 2y\,\Big|_x^1 = 2(1-x)$$

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = 2\int_{x=0}^{1} x(1-x)dx = 2\int_{x=0}^{1} (x-x^2)dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = 2\int_{x=0}^{1} x^2(1-x)dx = 2\int_{x=0}^{1}(x^2-x^3)dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$f(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_{x=0}^{y} 2dx = 2y\Big|_0^y = 2y$$

$$E(Y) = \int_{-\infty}^{+\infty} yf(y)dy = 2\int_{y=0}^{1} y^2dy = 2\left(\frac{y^3}{3}\right)\Big|_0^1 = \frac{2}{3}$$

$$E(Y^2) = \int_{-\infty}^{+\infty} y^2 f(y)dy = 2\int_{y=0}^{1} y^3dy = \frac{y^4}{2}\Big|_0^1 = \frac{1}{2}$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = 2\int_{x=0}^{y}\int_{y=0}^{1} xydxdy$$

$$= \int_{y=0}^{1} x^2y\Big|_{x=0}^{y} dy = \int_{y=0}^{1} y^3dy = \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{36}$$

$$\Rightarrow Cor(X,Y) = p(X,Y) = \frac{Cov(X,Y)}{\sigma_x\sigma_y}$$

$$= \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{1/36}{\sqrt{1/18}\sqrt{1/18}} = \frac{1/36}{1/18} = \frac{1}{2}$$

Problem 6: [1, Exercise 4.34]
Let X be a random variable with the following probability distribution:

x	-2	3	5
$f(x)$	0.3	0.2	0.5

Find the standard deviation of X.
Solution:

$$E(X) = \mu = \sum_x xf(x) = (-2)(0.3) + (3)(0.2) + (5)(0.5) = 2.5$$

$$E(X^2) = \sum_x x^2f(x) = (-2)^2(0.3) + (3)^2(0.2) + (5)^2(0.5) = 15.5$$

$$\sigma^2 = E(X^2) - [E(X)]^2 = 15.5 - (2.5)^2 = 9.25$$

$$\Rightarrow \sigma = \sqrt{9.25} = 3.041$$

Problem 7:[1, Exercise 4.36] Suppose that the probabilities are $0.4, 0.3, 0.2$, and 0.1,

respectively, that $0, 1, 2$, or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures striking this subdivision.
Solution:

$$x = \{0, 1, 2, 3\}$$

$$f(x) = \{0.4, 0.3, 0.2, 0.1\}$$

$$E(X) = \mu = \sum_x xf(x) = (0)(0.4) + (1)(0.3) + (2)(0.2) + (3)(0.1) = 1$$

$$E(X^2) = \sum_x x^2 f(x) = (0^2)(0.4) + (1^2)(0.3) + (2^2)(0.2) + (3^2)(0.1) = 2$$

$$\Rightarrow \sigma^2 = E(X^2) - [E(X)]^2 = 2^2 - 1^2 = 1$$

Problem 8: [1, Exercise 4.54]
Using [1, Theorem 4.5] and [1, Corollary 4.6], find the mean and variance of the random variable $Z = 5X + 3$, where X has the probability distribution of Problem 7.
Solution:
From Prob. 7, we have:

$$\mu = 1 \; and \; \sigma^2 = 1$$

$$E(aX + b) = aE(X) + b$$

$$\Rightarrow \mu_Z = \mu_{5X+3} = E(5X + 3) = 5E(X) + 3 = 5\mu + 3 = (5)(1) + 3 = 8$$

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2$$

$$\Rightarrow \sigma^2_Z = \sigma^2_{5X+3} = (5^2)(1^2) = 25$$

Problem 9: [1, Exercise 4.62]
If X and Y are independent random variables with variances $\sigma^2_X = 5$ and $\sigma^2_Y = 3$, find the variance of the random variable $Z = -2X + 4Y - 3$.
Solution:

$$\sigma^2_Z = \sigma^2_{-2X+4Y-3} = (-2^2)\sigma^2_X + (4^2)\sigma^2_Y = (4)(5) + (16)(3) = 68$$

Problem 10: [2, Section 4.1, Exercise 2]
If an integer between 1 and 100 is to be chosen at random, what is the expected value?
Solution:

$$x = \{1, 2, ..., 100\}$$

$$f(x) = \frac{1}{100}$$

$$\Rightarrow E(X) = \sum_x xf(x) = \frac{1}{100}(1 + 2 + ... + 100) = \frac{1}{100}\frac{(100)(101)}{2} = 50.5$$

Problem 11: [2, Section 4.1, Exercise 8]
Suppose that X and Y have a continuous joint distribution for which the joint p.d.f. is as follows:

$$f(x,y) = \begin{cases} 12y^2, & 0 \le y \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of $E(XY)$.
Solution:

$$E(XY) = \int_{x=0}^{1}\int_{y=0}^{x} xy12y^2 dxdy = 12\int_{x=0}^{1}\int_{y=0}^{x} xy^3 dxdy$$

$$= 12\int_{x=0}^{1} \frac{xy^4}{4}\Big|_{y=0}^{x} dx = 3\int_{x=0}^{1} x^5 dx = \frac{x^6}{2}\Big|_0^1 = \frac{1}{2}$$

Problem 12: [2, Section 4.2, Exercise 8]
Suppose that a class contains 10 boys and 15 girls, and suppose that 8 students are selected at random from the class without replacement. Let X denote the number of boys that are selected, and let Y denote the number of girls that are selected. Find $E(X - Y)$.
Solution:

$$E(X) = \sum_x xf(x) = 8\left(\frac{10}{25}\right) = \frac{16}{5}$$

$$Y = 8 - X \Rightarrow E(Y) = 8 - E(X) = 8 - \frac{16}{5} = \frac{24}{5}$$

$$\Rightarrow E(X - Y) = E(X) - E(Y) = \frac{16}{5} - \frac{24}{5} = -\frac{8}{5}$$

Problem 13: [2, Section 4.3, Exercise 4]
Suppose that X is a random variable for which $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. Show that $E[X(X-1)] = \mu(\mu-1) + \sigma^2$.
Solution:

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X) = E(X^2) - \mu$$

$$= Var(X) + [E(X)]^2 - \mu = \sigma^2 + \mu^2 - \mu = \mu(\mu-1) + \sigma^2$$

Problem 14: [2, Section 4.4, Exercise 11]
Suppose that X is a random variable for which the m.g.f. is as follows:

$$\phi(t) = \frac{1}{5}e^t + \frac{2}{5}e^{4t} + \frac{2}{5}e^{8t}, \text{ for } -\infty < t < \infty.$$

Find the the probability distribution of X.
Solution:
If X can take only a finite number of values $x_1, x_2, ..., x_k$ with probabilities $p_1, p_2, ..., p_k$, respectively, the m.g.f of X will be:

$$\psi(t) = p_1 e^{tx_1} + p_2 e^{tx_2} + ... + p_k e^{tx_k}$$

By matching this expression for $\psi(t)$ with the given expression, it can be seen that X can only take 3 values 1,4 and 8.
Therefore:

$$f(1) = \frac{1}{5}, f(4) = \frac{2}{5}, f(8) = \frac{2}{5}$$

References

[1] Walpole, R. E., Myers, R. H., Myers, S. L. and Ye, K., *Probability & Statistics for Engineers & Scientists*, 9th ed., MA, USA: Prentice-Hall, 2012.

[2] DeGroot, M. H. and Schervish, M. J., *Probability and Statistics*, 4th ed., MA, USA: Pearson Education, Inc., 2012.

[3] Murray, R. S., John, J. S. and R, A. Srinivasan, *Probability and Statistics*, 3rd ed., USA: McGraw-Hill, 2009.