Solution for Problem Set 1

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: [3, Exercise 1.47]
Determine the probability p, or an estimate of it, for each of the following events:

a. A king, ace, jack of clubs, or queen of diamonds appears in drawing a single card from a well-shuffled ordinary deck of cards.
Solution :
Number of kings = 4
Number of aces = 4
Number of jack of clubs = 1
Number of queen of diamonds = 1

$$P(A) = \frac{4+4+1+1}{52} = \frac{10}{52} = \frac{5}{26}$$

b. The sum 8 appears in a single toss of a pair of fair dice.
Solution :
8 = 3 + 5 = 5 + 3
8 = 2 + 6 = 6 + 2
8 = 4 + 4

$$P(B) = \frac{2+2+1}{6*6} = \frac{5}{36}$$

c. A non defective bolt will be found next if out of 600 bolts already examined, 12 were defective.
Solution :

$$P(C) = \frac{600-12}{600} = 0.98$$

d. A 7 or 11 comes up in a single toss of a pair of fair dice.
Solution :
7 = 1 + 6 = 6 + 1
7 = 2 + 5 = 5 + 2
7 = 3 + 4 = 4 + 3
11 = 5 + 6 = 6 + 5

$$P(D) = \frac{2+2+2+2}{6*6} = \frac{8}{36} = \frac{2}{9}$$

e. At least 1 head appears in 3 tosses of a fair coin.

Solution :

$$P(E) = 1 - \left(\frac{1}{2} * \frac{1}{2} * \frac{1}{2}\right) = 1 - \frac{1}{8} = \frac{7}{8}$$

Problem 2: [3, Exercise 1.49]
A marble is drawn at random from a box containing 10 red, 30 white, 20 blue, and 15 orange marbles. Find the probability that it is

a. orange or red;

Solution :

$$P(A) = \frac{10 + 15}{10 + 30 + 20 + 15} = \frac{25}{75} = \frac{1}{3}$$

b. not red or blue;

Solution :

$$P(B) = 1 - \frac{10 + 20}{75} = \frac{3}{5}$$

c. not blue;

Solution :

$$P(C) = 1 - \frac{20}{75} = \frac{11}{15}$$

d. white;

Solution :

$$P(D) = \frac{30}{75} = \frac{2}{5}$$

e. red, white, or blue.

Solution :

$$P(E) = \frac{10 + 30 + 20}{75} = \frac{4}{5}$$

Problem 3: [3, Exercise 1.50]
Two marbles are drawn in succession from the box of Problem 2, replacement being made after each drawing. Find the probability that

a. both are white;

Solution :

$$P(A) = \frac{30 * 30}{75 * 75} = \frac{4}{25}$$

b. the first is red and the second is white;

$$P(B) = \frac{10 * 30}{75 * 75} = \frac{4}{75}$$

c. neither is orange;

$$P(C) = 1 - \frac{(15 * 15) + (15 * 60) + (60 * 15)}{75 * 75} = \frac{16}{25}$$

d. they are either red or white or both (red and white);

$$P(D) = \frac{(10*10)+(30*30)+(10*30)+(30*10)}{75*75} = \frac{64}{225}$$

e. the second is not blue;
Solution :

$$P(E) = 1 - \frac{20}{75} = \frac{11}{15}$$

f. the first is orange;
Solution :

$$P(F) = \frac{15}{75} = \frac{1}{5}$$

g. at least one is blue;
Solution :

$$P(G) = \frac{(20*55)+(55*25)+(20*20)}{75*75} = \frac{104}{225}$$

h. at most one is red;
Solution :

$$P(H) = 1 - \frac{10*10}{75*75} = \frac{221}{225}$$

i. the first is white but the second is not;
Solution :

$$P(I) = \frac{30*45}{75*75} = \frac{6}{25}$$

j. only one is red;
Solution :

$$P(J) = \frac{(10*65)+(65*10)}{75*75} = \frac{52}{225}$$

Problem 4: [3, Exercise 1.51]
Work Problem 3 with no replacement after each drawing.

a. both are white;
Solution :

$$P(A) = \frac{30*29}{75*74} = \frac{29}{185}$$

b. the first is red and the second is white;
Solution :

$$P(B) = \frac{10*30}{75*74} = \frac{2}{37}$$

c. neither is orange;
Solution :

$$P(C) = 1 - \frac{(15*14)+(15*60)+(60*15)}{75*74} = \frac{118}{185}$$

d. they are either red or white or both (red and white);
Solution :

$$P(D) = \frac{(10*9)+(30*29)+(10*30)+(30*10)}{75*74} = \frac{52}{185}$$

e. the second is not blue;
Solution :
$$P(E) = 1 - \frac{20}{75} = \frac{11}{15}$$

f. the first is orange;
Solution :
$$P(F) = \frac{15}{75} = \frac{1}{5}$$

g. at least one is blue;
Solution :
$$P(G) = \frac{(20*55)+(55*25)+(20*19)}{75*74} = \frac{86}{185}$$

h. at most one is red;
Solution :
$$P(H) = 1 - \frac{10*9}{75*74} = \frac{182}{185}$$

i. the first is white but the second is not;
Solution :
$$P(I) = \frac{30*45}{75*74} = \frac{9}{37}$$

j. only one is red;
Solution :
$$P(J) = \frac{(10*65)+(65*10)}{75*74} = \frac{26}{111}$$

Problem 5: [2, Exercise 7, Section 1.4]
Suppose that a number x is to be selected from the real line S, and let A, B, and C be the events represented by the following subsets of S, where the notation $\{x : ---\}$ denotes the set containing every point x for which the property presented following the colon is satisfied:
$$\begin{aligned} A &= \{x : 1 \le x \le 5\} \\ B &= \{x : 3 < x \le 7\} \\ C &= \{x : x \le 0\} \end{aligned}$$

Describe each of the following events as a set of real numbers:

a. A'
Solution:
$$A' = \{x : x < 1 \; or \; x > 5\}$$

b. $A \cup B$
Solution:
$$A \cup B = \{x : 1 \le x \le 7\}$$

c. $B \cap C'$
Solution:
$$B \cap C' = B = \{x : 3 < x \le 7\}$$

d. $A' \cap B' \cap C'$

Solution:

$$A' \cap B' \cap C' = \{x : 0 < x < 1 \; or \; x > 7\}$$

e. $(A \cup B) \cap C$

Solution:

$$(A \cup B) \cap C = \emptyset$$

Problem 6: [2, Section 1.4]
Prove the following Theorems:
<u>Theorem 1:</u> (De Morgan's Law) For every two sets A, B

$$(A \cup B)' = A' \cap B' \text{ and } (A \cap B)' = A' \cup B'$$

Solution:
$a)\, Prove : (A \cup B)' = A' \cap B'$
$Let\, P = (A \cup B)'\, and\, Q = A' \cap B'$
$Let\, x \in P \Rightarrow x \in (A \cup B)'$
$\Rightarrow x \notin (A \cup B)$
$\Rightarrow x \notin A\, and\, x \notin B$
$\Rightarrow x \in A'\, and\, x \in B'$
$\Rightarrow x \in A' \cap B'$
$\Rightarrow x \in Q$
$Therefore, P \subset Q\, (1)$

$Let\, y \in Q \Rightarrow y \in A' \cap B'$
$\Rightarrow y \in A' and\, y \in B'$
$\Rightarrow y \notin A\, and\, y \notin B$
$\Rightarrow y \notin (A \cup B)$
$\Rightarrow y \in (A \cup B)'$
$\Rightarrow y \in P$
$Therefore, Q \subset P\, (2)$

$From\, (1)\, and\, (2) : P = Q$
$\Rightarrow (A \cup B)' = A' \cup B'$

$b)\, Prove : (A \cap B)' = A' \cup B'$
$Let\, M = (A \cap B)'\, and\, N = A' \cup B'$
$Let\, x \in M \Rightarrow x \in (A \cap B)'$
$\Rightarrow x \notin (A \cap B)$
$\Rightarrow x \notin A\, and\, x \notin B$
$\Rightarrow x \in A'\, and\, x \in B'$
$\Rightarrow x \in A' \cup B'$
$\Rightarrow x \in M$
$Therefore, M \subset N\, (1)$

$Let\, y \in N \Rightarrow y \in A' \cup B'$
$\Rightarrow y \in A' and\, y \in B'$

$\Rightarrow y \notin A \, and \, y \notin B$
$\Rightarrow y \notin (A \cap B)$
$\Rightarrow y \in (A \cap B)'$
$\Rightarrow y \in M$
$Therefore, N \subset M$ (2)

$From$ (1) and (2) $: M = N$
$\Rightarrow (A \cap B)' = A' \cup B'$
<u>Theorem 2:</u> (Distributive Properties) For every three sets A, B, and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:
$a) Prove : A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$Let \, x \in A \cap (B \cup C)$
$If \, x \in A \cap (B \cup C) \, then \, x \in A \, and \, x \in (B \, or \, C)$
$\Rightarrow x \in A \, and \, \{x \in B \, or \, x \in C\}$
$\Rightarrow \{x \in A \, and \, x \in B\} \, or \, \{x \in A \, and \, x \in C\}$
$\Rightarrow x \in (A \cap B) \, or \, x \in (A \cap C)$
$\Rightarrow x \in (A \cap B) \cup (A \cap C)$
$Therefore, A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ (1)

$Let \, y \in (A \cup B) \cap (A \cup C)$
$If \, y \in (A \cap B) \cup (A \cap C) \, then \, y \in (A \cap B) \, and \, y \in (A \cap C)$
$\Rightarrow y \in (A \, and \, B) \, or \, (A \, and \, C)$
$\Rightarrow \{y \in A \, or \, y \in B\} \, or \, \{y \in A \, or \, y \in C\}$
$\Rightarrow y \in A \, and \, \{y \in B \, or \, y \in C\}$
$\Rightarrow y \in A \, and \, y \in (B \, or \, C)$
$\Rightarrow y \in A \cap \; (B \cup C)$
$Therefore, (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ (2)
$From$ (1) and (2) $:$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$b) Prove : A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$Let \, x \in A \cup (B \cap C)$
$If \, x \in A \cup (B \cap C) \, then \, x \in A \, or \, x \in (B \, and \, C)$
$\Rightarrow x \in A \, or \, \{x \in B \, and \, x \in C\}$
$\Rightarrow \{x \in A \, or \, x \in B\} \, and \, \{x \in A \, or \, x \in C\}$
$\Rightarrow x \in (A \, or \, B) \, and \, x \in (A \, or \, C)$
$\Rightarrow x \in (A \cup B) \cap x \in (A \cup C)$
$\Rightarrow x \in (A \cup B) \cap (A \cup C)$
$Therefore, A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ (1)

$Let \, y \in (A \cup B) \cap (A \cup C)$
$If \, y \in (A \cup B) \cap (A \cup C) \, then \, y \in (A \, or \, B) \, and \, y \in (A \, or \, C)$
$\Rightarrow \{y \in A \, or \, y \in B\} \, and \, \{y \in A \, or \, y \in C\}$
$\Rightarrow y \in A \, or \, \{y \in B \, and \, y \in C\}$
$\Rightarrow y \in A \, or \, \{y \in (B \, and \, C)\}$
$\Rightarrow y \in A \cup \; \{y \in (B \cap C)\}$

$\Rightarrow y \in A \cup (B \cap C)$
$Therefore, (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ (2)
$From$ (1) and (2) :
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Theorem 3: For every two sets A, and B, $A \cap B$ and $A \cap B'$ are disjoint and $A = (A \cap B) \cup (A \cap B')$. In addition, B and $A \cap B'$ are disjoint, and $A \cup B = B \cup (A \cap B')$.
Solution:

Problem 7: [2, Exercise 10, Section 1.4]
Three six-sided dice are rolled. The six sides of each die are numbered from 1 to 6. Let A be the event that the first die shows an even number, let B be the event that the second die shows an even number, and let C be the event that the third die shows an even number. Also, for each $i = 1, ..., 6$, let A_i be the event that the first die shows the number i, let B_i be the event that the second die shows the number i, and let C_i be the event that the third die shows the number i. Express each of the following events in terms of the named events described above:

a. The event that all three dice show even numbers.
Solution:

$$P(A)P(B)P(C)$$

b. The event that no die shows an even number.
Solution:

$$(1 - P(A))(1 - P(B))(1 - P(C))$$

c. The event that at least one die shows an odd number.
Solution:

$$1 - P(A)P(B)P(C)$$

d. The event that at most two dice show odd numbers.
Solution:

$$1 - ((1 - P(A))(1 - P(B))(1 - P(C))$$

e. The event that the sum of the three dice is no greater than 5.
Solution:
3 = 1+1+1
4 = 2+1+1
5 = 3+1+1 = 2+2+1

$$P(A_1)P(B_1)P(C_1) + P(A_2)P(B_1)P(C_1) + P(A_1)P(B_2)P(C_1)$$

$$+P(A_1)P(B_1)P(C_2) + P(A_3)P(B_1)P(C_1) + P(A_1)P(B_3)P(C_1)$$

$$+P(A_1)P(B_1)P(C_3)+P(A_2)P(B_2)P(C_1)+P(A_2)P(B_1)P(C_2)+P(A_1)P(B_2)P(C_2)$$

Problem 8: [2, Exercise 3, Section 1.5]
Consider two events A and B such that $P(A) = 1/3$ and $P(B) = 1/2$. Determine the value of $P(B \cap A')$ for each of the following conditions:

a. A and B are disjoint;

Solution:

$$P(B \cap A') = P(B) = \frac{1}{2}$$

b. $A \subset B$;

Solution:

$$P(B \cap A') = P(B) - P(A \cap B) = P(B) - P(A) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

c. $P(A \cap B) = \frac{1}{8}$.

Solution:

$$P(B \cap A') = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

Problem 9: [2, Exercise 4, Section 1.7]
If a man has six different sport-shirts and four different pairs of slacks, how many different combinations can he wear?
Solution:

$${}_6C_1 * {}_4C_1 = 24$$

Problem 10: [2, Exercise 10, Section 1.7]
A box contains 100 balls, of which r are red. Suppose that the balls are drawn from the box one at a time, at random, without replacement. Determine:

a. the probability that the first ball drawn will be red;

Solution:

$$P(A) = \frac{r}{100}$$

b. the probability that the 50^{th} ball drawn will be red;

Solution:

The number of ways to arrange (100-r) non-red ball(s) in 100 positions:

$$P^{100}_{100-r} = \frac{100!}{[100 - (100 - r)]!} = \frac{100!}{r!}$$

The number of ways to arrange (100-r) non-red ball(s) in 99 positions:

$$P^{99}_{100-r} = \frac{99!}{[99 - (100 - r)]!} = \frac{99!}{(r - 1)!}$$

Probability that the 50th balls drawn will be red:

$$P(B) = \frac{P^{99}_{100-r}}{P^{100}_{100-r}} = \frac{99!}{(r-1)!} * \frac{r!}{100!} = \frac{r}{100}$$

c. the probability that the last ball drawn will be red.

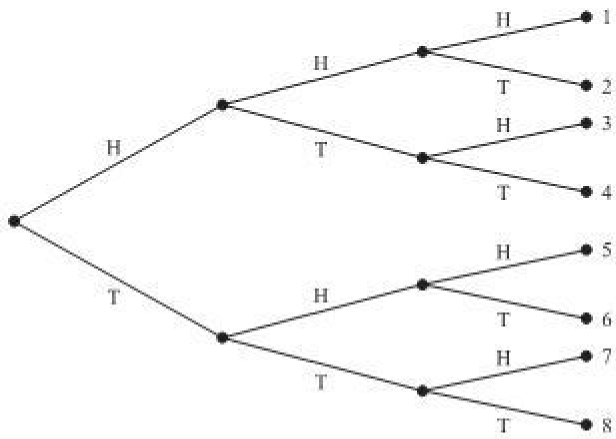
Solution:

$$Similar\,to\,Question\,b \Rightarrow P(C) = \frac{r}{100}$$

Problem 11: [2, Exercise 7, Section 1.8]
If k people are seated in a random manner in a row containing n seats $(n > k)$, what is the probability that the people will occupy k adjacent seats in the row?
Solution:

$$\frac{(n-k+1)*k!}{{}_nP_k} = \frac{n-k+1}{{}_nC_k}$$

Problem 12: [3, Exercise 1.59]
A coin is tossed 3 times. Use a tree diagram to determine the various possibilities that can arise.
Solution:



Problem 13: [3, Exercise 1.60]
Three cards are drawn at random (without replacement) from an ordinary deck of 52 cards. Find the number of ways in which one can draw

a. a diamond and a club and a heart in succession;
Solution:

$$13 * 13 * 13 = 2197$$

b. two hearts and then a club or a spade.
Solution:

$$13 * 12 * 13 + 13 * 12 * 13 = 13 * 12 * 26 = 4056$$

Problem 14: [3, Exercise 1.61]
In how many ways can 3 different coins be placed in 2 different purses?
Solution:

$${}_3C_0 + {}_3C_1 + {}_3C_2 + {}_3C_3 = 8$$

Problem 15: [3, Exercise 1.64]
In how many ways can 5 people be seated on a sofa if there are only 3 seats available?
Solution :

$${}_5P_3 = 60$$

Problem 16: [3, Exercise 1.65]
In how many ways can 7 books be arranged on a shelf if

a. any arrangement is possible;
Solution :

$$7! = 5040$$

b. 3 particular books must always stand together;
Solution :

$$5! * 3! = 720$$

c. two particular books must occupy the end.
Solution :

$$5! * 2! = 240$$

Problem 17: [3, Exercise 1.66]
How many numbers consisting of five different digits each can be made from the digits $1, 2, 3, ..., 9$ if

a. the numbers must be odd;
Solution :

$${}_5P_1 * {}_8P_4 = 8400$$

b. the first two digits of each number are even.
Solution :

$${}_4P_2 * {}_7P_3 = 2520$$

Problem 18: [3, Exercise 1.69]
In how many ways can 3 men and 3 women be seated at a round table if

a. no restriction is imposed;
Solution :

$$5! = 120$$

b. 2 particular women must not sit together;
Solution :

$$5! - 4! * 2! = 72$$

c. each woman is to be between 2 men.
Solution :

$$2 * 3! = 12$$

Problem 19: [3, Exercise 1.72]
In how many ways can 6 questions be selected out of 10?
Solution :

$${}_{10}C_6 = 210$$

Problem 20: [3, Exercise 1.73]
How many different committees of 3 men and 4 women can be formed from 8 men and 6 women?
Solution :

$$_8C_3 * {}_6C_4 = 840$$

Problem 21: [3, Exercise 1.74]
In how many ways can 2 men, 4 women, 3 boys and 3 girls be selected from 6 men, 8 women, 4 boys and 5 girls if

a. no restrictions are imposed;
Solution :

$$_6C_2 * {}_8C_4 * {}_4C_3 * {}_5C_3 = 42000$$

b. a particular man and woman must be selected.
Solution :

$$_5C_1 * {}_7C_3 * {}_4C_3 * {}_5C_3 = 7000$$

Problem 22: [3, Exercise 1.75]
In how many ways can a group of 10 people be divided into

a. two groups consisting of 7 and 3 people;
Solution:

$$_{10}C_7 * {}_3C_3 = 120$$

b. three groups consisting of 5, 3 and 2 people.
Solution:

$$_{10}C_5 * {}_5C_3 * {}_2C_2 = 2520$$

Problem 23: [3, Exercise 1.76]
From 5 statisticians and 6 economists, a committee of 3 statisticians and 2 economists is to be formed. How many different committees can be formed if

a. no restrictions are imposed;
Solution :

$$_5C_3 * {}_6C_2 = 150$$

b. two particular statisticians must be on the committee;
Solution :

$$_3C_1 * {}_6C_2 = 45$$

c. 1 particular economist cannot be on the committee.
Solution :

$$_5C_3 * {}_5C_2 = 100$$

Problem 24: [3, Exercise 1.86]
An urn contains 6 red and 8 blue marbles. Five marbles are drawn at random from it without replacement. Find the probability that 3 are red and 2 are blue.
Solution:

$$\frac{{}_6C_3 * {}_8C_2}{{}_{14}C_5}$$

Problem 25: [3, Exercise 1.87]

a. Find the probability of getting the sum 7 on at least 1 of 3 tosses of a pair of fair dice;
Solution:
7 = 3+4 = 4+3 = 2+5 = 5+2 = 1+6 = 6+1 $\Rightarrow$ 6 (ways)

$$P\,(sum\ 7) = \frac{6}{6*6} = \frac{1}{6}$$

$$\Rightarrow P\,(sum\ 7\ on\ at\ least\ 1\ of\ 3\ tosses) = 1 - \left(1 - \frac{1}{6}\right)^3 = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$$

b. How many tosses are needed in order that the probability in (a) be greater than 0.95.
Solution:

$$1 - \left(\frac{5}{6}\right)^n > 0.95 \Rightarrow \left(\frac{5}{6}\right)^n < 0.05$$

$$\Rightarrow n * ln\left(\frac{5}{6}\right) < ln(0.05) \Rightarrow n > 16.43 \Rightarrow n \geq 17\,(tosses) \Rightarrow at\ least\ 17\ tosses$$

Problem 26: [3, Exercise 1.90]
A sample space consists of 3 sample points with associated probabilities given by $2p, p^2$, and $4p - 1$. Find the value of p.
Solution :

$$2p + p^2 + 4p - 1 = 1\ (p > 0) \Rightarrow p^2 + 6p - 2 = 0$$
$$\Rightarrow \triangle' = 3^2 + 2 = 11 \Rightarrow p = \sqrt{11} - 3$$

Problem 27: [3, Exercise 1.91]
How many words can be made from 5 letters if

a. all letters are different;
Solution :

$$5! = 120$$

b. 2 letters are adjacent;
Solution :

$$\frac{5!}{2!} = 60$$

c. all letters are different but 2 particular letters cannot be adjacent.
Solution :

$$5! - 4! * 2 = 72$$

Problem 28: [3, Exercise 1.92]
Four integers are chosen at random between 0 and 9, inclusive. Find the probability that

a. they are all different;
Solution :

$$P(A) = \frac{9}{10} * \frac{8}{10} * \frac{7}{10} = \frac{63}{125}$$

b. not more than 2 are the same.
Solution :
Total numbers that 0 is the same $= {}_{10}C_4 * 4! = 5040$

Total numbers that 2 are same and 2 are different $= \dfrac{{}_{10}C_1 * {}_9C_2 * 4!}{2!} = 4320$

Total numbers that 2 are same and 2 others are same $= \dfrac{{}_{10}C_2 * 4!}{2!2!} = 270$

Total numbers that not more than 2 are the same $= 5040 + 4320 + 270 = 9630$

$$P(B) = \frac{9630}{10^4} = \frac{9630}{10000} = \frac{963}{1000}$$

Problem 29: [3, Exercise 1.93]
A pair of dice is tossed repeatedly. Find the probability that an 11 occurs for the first time on the 6th toss.
Solution :

$$P = \left(\frac{34}{36}\right)^5 * \left(\frac{2}{36}\right) = \frac{1,419,857}{34,012,224} \approx 0.04$$

Problem 30: [3, Exercise 1.94]
What is the least number of tosses needed in Problem 29 so that the probability of getting an 11 will be greater than

a. 0.5;
Solution :

$$P(A) = \left(\frac{34}{36}\right)^n < 0.5$$

$$\Rightarrow n * log\left(\frac{34}{36}\right) < log\,(0.5) \Rightarrow n \geq 13\,(tosses) \Rightarrow At\,least\,13\,tosses$$

b. 0.95.
Solution:

$$P(B) = \left(\frac{34}{36}\right)^n < 0.05$$

$$\Rightarrow n * log\left(\frac{34}{36}\right) < log\,(0.05) \Rightarrow n \geq 53\,(tosses) \Rightarrow At\,least\,53\,tosses$$

References

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[3] Murray, R. S., John, J. S. and R, A. Srinivasan, *Probability and Statistics*, 3rd ed., USA: McGraw-Hill, 2009.