Problem Set 6 Solution

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: [3, Exercise 3.75]
Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find (a) $\mathrm{Var}(X)$, (b) $\mathrm{Var}(Y)$, (c) σ_X, (d) σ_Y, (e) σ_{XY}, (f) ρ.
Solution:

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} xf(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} x(x+y)dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (x^2+xy)dxdy = \int_{y=0}^{1}\left(\frac{x^3}{3}+\frac{x^2y}{2}\right)\Big|_{x=0}^{1} dy = \int_{y=0}^{1}\left(\frac{1}{3}+\frac{y}{2}\right)dy$$

$$= \frac{y}{3}+\frac{y^2}{4}\Big|_{y=0}^{1} = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2f(x)dx = \int_{-\infty}^{+\infty} x^2f(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} x^2(x+y)dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (x^3+x^2y)dxdy = \int_{y=0}^{1}\left(\frac{x^4}{4}+\frac{x^3y}{3}\right)\Big|_{x=0}^{1} dy = \int_{y=0}^{1}\left(\frac{1}{4}+\frac{y}{3}\right)dy$$

$$= \frac{y}{4}+\frac{y^2}{6}\Big|_{y=0}^{1} = \frac{1}{4}+\frac{1}{6} = \frac{5}{12}$$

$$E(Y) = \int_{-\infty}^{+\infty} yf(y)dy = \int_{-\infty}^{+\infty} yf(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} y(x+y)dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (y^2+xy)dxdy = \int_{x=0}^{1}\left(\frac{y^3}{3}+\frac{xy^2}{2}\right)\Big|_{y=0}^{1} dx = \int_{x=0}^{1}\left(\frac{1}{3}+\frac{x}{2}\right)dx$$

$$= \frac{x}{3}+\frac{x^2}{4}\Big|_{x=0}^{1} = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

$$E(Y^2) = \int_{-\infty}^{+\infty} y^2f(y)dy = \int_{-\infty}^{+\infty} y^2f(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} y^2(x+y)dxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}(y^3+xy^2)dxdy = \int_{x=0}^{1}\left(\frac{y^4}{4}+\frac{xy^3}{3}\right)\Big|_{y=0}^{1} dx = \int_{x=0}^{1}\left(\frac{1}{4}+\frac{x}{3}\right)dx$$

$$= \frac{x}{4}+\frac{x^2}{6}\Big|_{x=0}^{1} = \frac{1}{4}+\frac{1}{6} = \frac{5}{12}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} xy(x+y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1}(x^2y+xy^2)dxdy$$

$$= \int_{y=0}^{1}\left(\frac{x^3y}{3}+\frac{x^2y^2}{2}\right)\Big|_{x=0}^{1} = \int_{y=0}^{1}\left(\frac{y}{3}+\frac{y^2}{2}\right)dy = \frac{y^2}{6}+\frac{y^3}{6}\Big|_{y=0}^{1} = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

a)

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

b)

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

c)

$$\sigma_X = \sqrt{Var(X)} = \sqrt{\frac{11}{144}} = \frac{\sqrt{11}}{12}$$

d)

$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{\frac{11}{144}} = \frac{\sqrt{11}}{12}$$

e)

$$\sigma_{XY} = Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144}$$

f)

$$p(X,Y) = Cor(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = -\frac{1/144}{\sqrt{11}/12\sqrt{11}/12} = -\frac{1}{11}$$

Problem 2: [3, Exercise 3.76]
Work Problem 1 if the joint density function is

$$f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

$$E(X) = \int_{-\infty}^{+\infty} xf(x,y)dxdy = \int_{x=0}^{+\infty}\int_{y=0}^{+\infty} xe^{-(x+y)}dxdy = 1$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2f(x,y)dxdy = \int_{x=0}^{+\infty}\int_{y=0}^{+\infty} x^2e^{-(x+y)}dxdy = 2$$

$$E(Y) = \int_{-\infty}^{+\infty} yf(x,y)dxdy = \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} ye^{-(x+y)}dxdy = 1$$
$$E(Y^2) = \int_{-\infty}^{+\infty} x^2 f(x,y)dxdy = \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} y^2 e^{-(x+y)}dxdy = 2$$
$$E(XY) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x,y)dxdy = \int_{x=0}^{+\infty} \int_{y=0}^{+\infty} xye^{-(x+y)}dxdy = 1$$

a)
$$Var(X) = E(X^2) - [E(X)]^2 = 2 - 1^2 = 1$$

b)
$$Var(Y) = E(Y^2) - [E(Y)]^2 = 2 - 1^2 = 1$$

c)
$$\sigma_X = \sqrt{Var(X)} = \sqrt{1} = 1$$

d)
$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{1} = 1$$

e)
$$\sigma_{XY} = E(XY) - E(X)E(Y) = 1 - 1 = 0$$

f)
$$p(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0}{1 * 1} = 0$$

Problem 3: [3, Exercise 3.79]
Find (a) the covariance, (b) The correlation coefficient of two random variables X and Y if $E(X) = 2$, $E(Y) = 3$, $E(XY) = 10$, $E(X^2) = 9$, $E(Y^2) = 16$.
Solution:
a)
$$Cov(X,Y) = \sigma_{XY} = E(XY) - E(X)E(Y) = 10 - 2 * 3 = 4$$
b)
$$Var(X) = E(X^2) - [E(X)]^2 = 9 - 2^2 = 5 \Rightarrow \sigma_X = \sqrt{5}$$
$$Var(Y) = E(Y^2) - [E(Y)]^2 = 16 - 3^2 = 7 \Rightarrow \sigma_Y = \sqrt{7}$$
$$Cor(X,Y) = p(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{4}{\sqrt{5}\sqrt{7}} = \frac{4}{\sqrt{35}}$$

Problem 4: [3, Exercise 3.80]
The correlation coefficient of two random variables X and Y is $-\frac{1}{4}$ while their variances

are 3 and 5. Find the covariance.
Solution:

$$p(X,Y) = -\frac{1}{4}$$

$$Var(X) = \sigma_X^2 = 3 \Rightarrow \sigma_X = \sqrt{3}$$

$$Var(Y) = \sigma_Y^2 = 5 \Rightarrow \sigma_Y = \sqrt{5}$$

$$p(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sqrt{3}\sqrt{5}} = -\frac{1}{4} \Rightarrow \sigma_{XY} = -\frac{\sqrt{15}}{4}$$

Problem 5: [3, Exercise 3.81]
Let X and Y have joint density function

$$f(x,y) = \begin{cases} x+y & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the conditional expectation of (a) Y given X, (b) X given Y.
Solution:
a)

$$f_1(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_{y=0}^{1} (x+y)dy = xy + \frac{y^2}{2}\Big|_{y=0}^{1} = x + \frac{1}{2}$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{x+y}{x+1/2} = \frac{2(x+y)}{2x+1}$$

$$E(Y|X) = \int_{-\infty}^{+\infty} yf(y|x)dy = \int_{y=0}^{1} \frac{2y(x+y)}{2x+1}dy$$

$$= \frac{xy^2}{2x+1} + \frac{2/3y^3}{2x+1}\Big|_{y=0}^{1} = \frac{x+2/3}{2x+1} = \frac{3x+2}{6x+3}$$

b)

$$f_2(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_{x=0}^{1} (x+y)dx = \frac{x^2}{2} + xy\Big|_{x=0}^{1} = y + \frac{1}{2}$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{x+y}{y+1/2} = \frac{2(x+y)}{2y+1}$$

$$E(X|Y) = \int_{-\infty}^{+\infty} xf(x|y)dx = \int_{x=0}^{1} \frac{2x(x+y)}{2y+1}dx$$

$$= \frac{x^2y}{2y+1} + \frac{2/3x^3}{2y+1}\Big|_{x=0}^{1} = \frac{y+2/3}{2y+1} = \frac{3y+2}{6y+3}$$

Problem 6: [3, Exercise 3.82]
Work Problem 5 if

$$f(x,y) = \begin{cases} 2e^{-(x+2y)} & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Solution:
a)

$$f_1(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_{y=0}^{1} 2e^{-(x+2y)}dy = e^{-x}$$

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{2e^{-(x+2y)}}{e^{-x}} = 2e^{-2y}$$

$$E(Y|X) = \int_{-\infty}^{+\infty} yf(y|x)dy = \int_{y=0}^{\infty} 2ye^{-2y}dy = \frac{1}{2}$$

b)

$$f_2(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_{x=0}^{+\infty} 2e^{-(x+2y)}dx = 2e^{-2y}$$

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{2e^{-(x+2y)}}{2e^{-2y}} = e^{-x}$$

$$E(X|Y) = \int_{-\infty}^{+\infty} xf(x|y)dx = \int_{x=0}^{+\infty} xe^{-x}dx = 1$$

Problem 7: [3, Exercise 3.83]
Let X and Y have the joint probability function given in below table. Find the conditional expectation of (a) Y given X, (b) X given Y.

X \ Y	0	1	2
0	1/18	1/9	1/6
1	1/9	1/18	1/9
2	1/6	1/6	1/18

Solution:
a)

$$P(Y|X=0) = \frac{P(X=0,Y)}{P(X=0)}$$

$$P(X=0) = \sum_{y} P(X=0,Y) = \frac{1}{18} + \frac{1}{9} + \frac{1}{6} = \frac{1}{3}$$

$$P(Y=0|X=0) = \frac{P(X=0,Y=0)}{P(X=0)} = \frac{1/18}{1/3} = \frac{1}{6}$$

$$P(Y=1|X=0) = \frac{P(X=0,Y=1)}{P(X=0)} = \frac{1/9}{1/3} = \frac{1}{3}$$

$$P(Y=2|X=0) = \frac{P(X=0,Y=2)}{P(X=0)} = \frac{1/6}{1/3} = \frac{1}{2}$$

$$E(Y|X=0) = \sum_y y * P(Y|X=0) = 0 * \frac{1}{6} + 1 * \frac{1}{3} + 2 * \frac{1}{2} = \frac{4}{3}$$

Similarly:

$$E(Y|X=1) = \sum_y y * P(Y|X=1) = 0 * \frac{2}{5} + 1 * \frac{1}{5} + 2 * \frac{2}{5} = 1$$

$$E(Y|X=2) = \sum_y y * P(Y|X=2) = 0 * \frac{3}{7} + 1 * \frac{3}{7} + 2 * \frac{1}{7} = \frac{5}{7}$$

X	0	1	2
$E(Y\|X)$	4/3	1	5/7

b)

$$P(X|Y=0) = \frac{P(X,Y=0)}{P(Y=0)}$$

$$P(Y=0) = \sum_x P(X,Y=0) = \frac{1}{18} + \frac{1}{9} + \frac{1}{6} = \frac{1}{3}$$

$$P(X=0|Y=0) = \frac{P(X=0,Y=0)}{P(Y=0)} = \frac{1/18}{1/3} = \frac{1}{6}$$

$$P(X=1|Y=0) = \frac{P(X=1,Y=0)}{P(Y=0)} = \frac{1/9}{1/3} = \frac{1}{3}$$

$$P(X=2|Y=0) = \frac{P(X=2,Y=0)}{P(Y=0)} = \frac{1/6}{1/3} = \frac{1}{2}$$

$$E(X|Y=0) = \sum_x x * P(X|Y=0) = 0 * \frac{1}{6} + 1 * \frac{1}{3} + 2 * \frac{1}{2} = \frac{4}{3}$$

Similarly:

$$E(X|Y=1) = \sum_x x * P(X|Y=1) = 0 * \frac{1}{3} + 1 * \frac{1}{6} + 2 * \frac{1}{2} = \frac{7}{6}$$

$$E(X|Y=2) = \sum_x x * P(X|Y=2) = 0 * \frac{1}{2} + 1 * \frac{1}{3} + 2 * \frac{1}{6} = \frac{2}{3}$$

Y	0	1	2
$E(X\|Y)$	4/3	7/6	2/3

Problem 8: [3, Exercise 3.84]
Find the conditional variance of (a) Y given X, (b) X given Y for the distribution of Problem 5.
Solution:

$$E(X|Y) = \frac{3y+2}{6y+3}$$
$$E(Y|X) = \frac{3x+2}{6x+3}$$
$$f(x|y) = \frac{2(x+y)}{2y+1}$$
$$f(y|x) = \frac{2(x+y)}{2x+1}$$

a)

$$Var(Y|X) = E(Y^2|X) - [E(Y|X)]^2$$
$$= \int_{-\infty}^{+\infty} y^2 f(y|x)dy - \frac{(3x+2)^2}{(6x+3)^2} = \int_{y=0}^{1} y^2 \frac{2(x+y)}{2x+1} dy - \frac{9x^2+12x+4}{(6x+3)^2}$$
$$= \frac{2/4y^4 + 2/3xy^3}{2x+1}\Bigg|_{y=0}^{1} - \frac{9x^2+12x+4}{(6x+3)^2} = \frac{1/2+2/3x}{2x+1} - \frac{9x^2+12x+4}{9(2x+1)^2}$$
$$= \frac{6x^2+6x+1}{18(2x+1)^2}$$

b)

$$Var(X|Y) = E(X^2|Y) - [E(X|Y)]^2$$
$$= \int_{-\infty}^{+\infty} x^2 f(x|y)dx - \frac{(3y+2)^2}{(6y+3)^2} = \int_{x=0}^{1} x^2 \frac{2(x+y)}{2y+1} dy - \frac{9y^2+12y+4}{(6y+3)^2}$$
$$= \frac{2/4x^4 + 2/3yx^3}{2y+1}\Bigg|_{x=0}^{1} - \frac{9y^2+12y+4}{(6y+3)^2} = \frac{1/2+2/3y}{2y+1} - \frac{9y^2+12y+4}{9(2y+1)^2}$$
$$= \frac{6y^2+6y+1}{18(2y+1)^2}$$

Problem 9: [3, Exercise 3.85]
Work Problem 8 for the distribution of Problem 6.
Solution:

$$E(X|Y) = 1$$
$$E(Y|X) = \frac{1}{2}$$
$$f(x|y) = e^{-x}$$
$$f(y|x) = 2e^{-2y}$$

a)

$$Var(Y|X) = E(Y^2|X) - [E(Y|X)]^2$$
$$= \int_{-\infty}^{+\infty} y^2 f(y|x)dy - \left(\frac{1}{2}\right)^2 = \int_{y=0}^{+\infty} y^2 2e^{-2y}dy - \frac{1}{4}$$
$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

b)

$$Var(X|Y) = E(X^2|Y) - [E(X|Y)]^2$$
$$= \int_{-\infty}^{+\infty} x^2 f(x|y)dx - 1^2 = \int_{x=0}^{+\infty} x^2 e^{-x}dx - 1$$
$$= 2 - 1 = 1$$

Problem 10: [3, Exercise 3.86]
Work Problem 8 for the distribution of the below table.

X \ Y	0	1	2
0	1/18	1/9	1/6
1	1/9	1/18	1/9
2	1/6	1/6	1/18

Solution:
a)

X	0	1	2
$Var(Y\|X)$	5/9	4/5	24/49

b)

Y	0	1	2
$Var(X\|Y)$	5/9	29/36	5/9

Problem 11: [3, Exercise 3.105]
Let X be a random variable that can take on the values 2, 1, and 3 with respective probabilities 1/3, 1/6, and 1/2. Find (a) the mean, (b) the variance, (c) the moment generating function, (d) the characteristic function, (e) the third moment about the mean.
Solution:

x	2	1	3
$f(x)$	1/3	1/6	1/2

a)

$$E(X) = \sum_x xf(x) = 2 * \frac{1}{3} + 1 * \frac{1}{6} + 3 * \frac{1}{2} = \frac{7}{3}$$

b)

$$E(X^2) = \sum_x x^2 f(x) = 2^2 * \frac{1}{3} + 1^2 * \frac{1}{6} + 3^2 * \frac{1}{2} = 6$$

$$Var(X) = E(X^2) - [E(X)]^2 = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}$$

c)

$$M_x(t) = E(e^{tx}) = \sum_x e^{tx} f(x) = \frac{1}{3}e^{2t} + \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} = \frac{e^t + 2e^{2t} + 3e^{3t}}{6}$$

d)

$$\varphi_x(t) = E(e^{itx}) = \frac{e^{i\omega} + 2e^{2i\omega} + 3e^{3i\omega}}{6}$$

e)

$$\begin{aligned}
\mu_3 &= E[(X-\mu)]^3 = E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3) \\
&= \mu_3' - 3\mu_2'\mu + 3\mu^3 - \mu^3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 \\
\mu_3' &= E(X^3) = 2^3 * \frac{1}{3} + 1^3 * \frac{1}{6} + 3^3 * \frac{1}{2} = \frac{49}{3} \\
\Rightarrow \mu_3 &= \frac{49}{3} - 3 * 6 * \frac{7}{3} + 2\left(\frac{7}{3}\right)^3 = -\frac{7}{27}
\end{aligned}$$

Problem 12: [3, Exercise 3.106]
Work Problem 11 if X has density function

$$f(x) = \begin{cases} c(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is an appropriate constant.
Solution:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{+\infty} f(x)dx = \int_{x=0}^{1} c(1-x)dx = cx - \frac{cx^2}{2}\Bigg|_{x=0}^{1} \\
&= c - \frac{c}{2} = \frac{c}{2} = 1 \Rightarrow c = 2
\end{aligned}$$

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a)

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{x=0}^{1} 2x(1-x)dx = x^2 - \frac{2x^3}{3}\Bigg|_{x=0}^{1} = 1 - \frac{2}{3} = \frac{1}{3}$$

b)

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{x=0}^{1} 2x^2(1-x)dx = \frac{2x^3}{3} - \frac{x^4}{2}\Bigg|_{x=0}^{1} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

c)

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx = \int_{x=0}^{1} e^{tX} 2(1-x)dx = \frac{2(e^t - 1 - t)}{t^2}$$

d)

$$\varphi_x(t) = E(e^{itx}) = \frac{-2(e^{i\omega} - 1 - i\omega)}{\omega^2}$$

e)

$$\mu_3 = E(X-\mu)^3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$$

$$\mu_3' = E(X^3) = \int_{x=0}^{1} 2x^3(1-x)dx = \frac{x^4}{2} - \frac{2x^5}{5}\Bigg|_{x=0}^{1} = \frac{1}{10}$$

$$\Rightarrow \mu_3 = \frac{1}{10} - 3 * \frac{1}{6} * \frac{1}{3} + 2\left(\frac{1}{3}\right)^3 = \frac{1}{135}$$

Problem 13: [3, Exercise 3.109]
Let X and Y have joint density function

$$f(x,y) = \begin{cases} cxy & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find (a) $E(X^2 + Y^2)$, (b) $E(\sqrt{X^2 + Y^2})$.
Solution:

$$F(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} cxydxdy$$

$$= \int_{y=0}^{1} \frac{c}{2}x^2y\Bigg|_{x=0}^{1} dy = \int_{y=0}^{1} \frac{c}{2}ydy = \frac{c}{4}y^2\Bigg|_{Y=0}^{1} = \frac{c}{4} = 1 \Rightarrow c = 4$$

a)

$$E(X^2 + Y^2] = \int_{x=0}^{1}\int_{y=0}^{1} (x^2+y^2)f(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1} 4(x^2+y^2)xydxdy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (4x^3y + 4xy^3)dxdy = \int_{y=0}^{1} x^4y + 2x^2y^3\Bigg|_{x=0}^{1} dy$$

$$= \int_{y=0}^{1} (y + 2y^3)dy = \frac{y^2}{2} + \frac{1}{2}y^4\Bigg|_{y=0}^{1} = \frac{1}{2} + \frac{1}{2} = 1$$

b)

$$E(\sqrt{(X^2+Y^2)} = \int_{x=0}^{1}\int_{y=0}^{1}\sqrt{(x^2+y^2)}f(x,y)dxdy = \int_{x=0}^{1}\int_{y=0}^{1}4\sqrt{(x^2+y^2)}xydxdy$$

$$= 2\int_{y=0}^{1}\frac{2}{3}y(x^2+y^2)^{3/2}\Big|_{x=0}^{1} dy = \int_{y=0}^{1}\frac{4}{3}y\left[\left(1+y^2\right)^{3/2}-\left(y^2\right)^{3/2}\right]dy$$

$$= \frac{2}{3}*\frac{2}{5}(1+y^2)^{5/2}-\frac{4}{15}y^5\Big|_{y=0}^{1} = \frac{4}{15}*4\sqrt{2}-\frac{8}{15} = \frac{8(2\sqrt{2}-1)}{15}$$

References

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[3] Murray, R. S., John, J. S. and R, A. Srinivasan, *Probability and Statistics*, 3rd ed., USA: McGraw-Hill, 2009.