Problem Set 4

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: [3, Exercise 2.54]
The joint probability function of two discrete random variables X and Y is given by $f(x, y) = cxy$ for $x = 1, 2, 3$ and $y = 1, 2, 3$, and equals zero otherwise. Find (a) the constant c, (b) $P(X = 2, Y = 3)$, (c) $P(1 \leq X \leq 2, Y \leq 2)$, (d) $P(X \geq 2)$, (e) $P(Y < 2)$, (f) $P(X = 1)$, (g) $P(Y = 3)$.
Solution:
a)

X \ Y	1	2	3	Total ⇩
1	c	2c	3c	6c
2	2c	4c	6c	12c
3	3c	6c	9c	18c
Total ⇨	6c	12c	18c	36c

From the table, we have:

$$\sum_x \sum_y f(x, y) = 1 \Rightarrow 36c = 1 \Rightarrow c = \frac{1}{36}$$

b)

$$P(X = 2, Y = 3) = 6c = \frac{6}{36} = \frac{1}{6}$$

c)

$$P(1 \leq X \leq 2, Y \leq 2) = P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 2, Y = 1) + P(X = 2, Y = 2)$$

$$= c + 2c + 2c + 4c = 9c = \frac{9}{36} = \frac{1}{4}$$

d)

$$P(X \geq 2) = 1 - P(X < 2) = 1 - P(X = 1) = 1 - 6 * \frac{1}{36} = \frac{5}{6}$$

Or:

$$P(X \geq 2) = P(X = 2) + P(X = 3) = 12c + 18c = 30c = \frac{30}{36} = \frac{5}{6}$$

e)

$$P(Y < 2) = P(Y = 1) = 6c = \frac{6}{36} = \frac{1}{6}$$

f)

$$P(X = 1) = 6c = \frac{6}{36} = \frac{1}{6}$$

g)

$$P(Y = 3) = 18c = \frac{18}{36} = \frac{1}{2}$$

Problem 2: [3, Exercise 2.55]
Find the marginal probability functions of (a) X and (b) Y for the random variables of Problem 1. (c) Determine whether X and Y are independent.
Solution:
a)Marginal probability function of X:

$$P(X = x) = f(x) = \begin{cases} 6c = 1/6 & x = 1 \\ 12c = 1/3 & x = 2 \\ 18c = 1/2 & x = 3 \end{cases} \Rightarrow f_1(x) = \begin{cases} x/6 & x = 1, 2, 3 \\ 0 & otherwise \end{cases}$$

b)Marginal probability function of Y:

$$P(Y = y) = f(y) = \begin{cases} 6c = 1/6 & y = 1 \\ 12c = 1/3 & y = 2 \\ 18c = 1/2 & y = 3 \end{cases} \Rightarrow f_2(y) = \begin{cases} y/6 & y = 1, 2, 3 \\ 0 & otherwise \end{cases}$$

c)If X and Y are independent:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$
$$\Rightarrow f(x, y) = f_1(x)f_2(y)$$

We have:

$$f(x, y) = cxy = \frac{1}{36} * xy = \frac{xy}{36}$$
$$f_1(x)f_2(y) = \frac{x}{6} * \frac{y}{6} = \frac{xy}{36}$$
$$\Rightarrow f(x, y) = \frac{xy}{36} = f_1(x)f_2(y)$$

Therefore:

$$X \; and \, Y \; are \; independent$$

Problem 3: [3, Exercise 2.56]
Let X and Y be continuous random variables having joint density function

$$f(x,y) = \begin{cases} c(x^2+y^2) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine (a) the constant c, (b) $\mathrm{P}(X < \frac{1}{2}, Y > \frac{1}{2})$, (c) $\mathrm{P}(\frac{1}{4} < X < \frac{3}{4})$, (d) $\mathrm{P}(Y < \frac{1}{2})$, (e) whether X and Y are independent.
Solution:
a)

$$F(x,y) = \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} f(x,y)dxdy = 1 \Rightarrow \int_{x=0}^{1}\int_{y=0}^{1} c(x^2+y^2)dxdy = 1$$

$$= c\int_{y=0}^{1}\left(\frac{x^3}{3}+xy^2\right)\Big|_{x=0}^{1} dy = c\int_{y=0}^{1}\left(\frac{1}{3}+y^2\right)dy$$

$$= c\left(\frac{1}{3}y+\frac{y^3}{3}\right)\Big|_0^1 = c\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{2}{3}c = 1 \Rightarrow c = \frac{3}{2}$$

b)

$$P\left(X<\frac{1}{2}, Y>\frac{1}{2}\right) = \int_{x=0}^{1/2}\int_{y=1/2}^{1}\frac{3}{2}(x^2+y^2)dxdy$$

$$= \int_{x=0}^{1/2}\frac{3}{2}\left(x^2y+\frac{y^3}{3}\right)\Big|_{y=1/2}^{1} dx = \frac{3}{2}\int_{x=0}^{1/2}\left(x^2+\frac{1}{3}-\frac{1}{2}x^2-\frac{1}{24}\right)dx$$

$$= \frac{3}{2}\int_{x=0}^{1/2}\left(\frac{1}{2}x^2+\frac{7}{24}\right)dx = \frac{3}{2}\left(\frac{1}{6}x^3+\frac{7}{24}x\right)\Big|_0^{1/2} = \frac{1}{4}$$

c)

$$P\left(\frac{1}{4}<X<\frac{3}{4}\right) = \int_{x=1/4}^{3/4}\int_{y=0}^{1}\frac{3}{2}(x^2+y^2)dxdy$$

$$= \frac{3}{2}\int_{x=1/4}^{3/4}\left(x^2y+\frac{y^3}{3}\right)\Big|_{y=0}^{1} dx = \frac{3}{2}\int_{x=1/4}^{3/4}\left(x^2+\frac{1}{3}\right)dx = \frac{3}{2}\left(\frac{x^3}{3}+\frac{1}{3}x\right)\Big|_{1/4}^{3/4} = \frac{29}{64}$$

d)

$$P\left(Y<\frac{1}{2}\right) = \int_{x=0}^{1}\int_{y=0}^{1/2}\frac{3}{2}(x^2+y^2)dxdy = \frac{3}{2}\int_{y=0}^{1/2}\frac{x^3}{3}+y^2x\Big|_{x=0}^{1} dy$$

$$= \frac{3}{2}\int_{y=0}^{1/2}\left(\frac{1}{3}+y^2\right)dy = \frac{3}{2}\left(\frac{1}{3}y+\frac{y^3}{3}\right)\Big|_0^{1/2} = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{8}\right) = \frac{5}{16}$$

e)If X and Y are independent:

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y)$$

$$\Rightarrow F(x,y) = F_1(x)F_2(y)$$

We have:

$$F_1(x) = \int_0^x \int_0^1 \frac{3}{2}(x^2+y^2)dxdy = \frac{3}{2}\int_0^x \left(x^2y + \frac{y^3}{3}\right)\Big|_0^1 dx$$
$$= \frac{3}{2}\int_0^x \left(x^2 + \frac{1}{3}\right) dx = \frac{3}{2} * \frac{1}{3}(x^3+x) = \frac{1}{2}(x^3+x)$$
$$F_2(y) = \int_0^1 \int_0^y \frac{3}{2}(x^2+y^2)dxdy = \frac{3}{2}\int_0^y \frac{x^3}{3} + y^2x \,\Big|_0^1 dy$$
$$= \frac{3}{2}\int_0^y \left(\frac{1}{3} + y^2\right) dy = \frac{3}{2} * \frac{1}{3}(y^3+y) = \frac{1}{2}(y^3+y)$$
$$F(x,y) = \int_0^x \int_0^y \frac{3}{2}(x^2+y^2)dxdy = \frac{3}{2}\int_0^x x^2y + \frac{y^3}{3}\,\Big|_0^y dx = \frac{3}{2}\int_0^x \left(x^2y + \frac{y^3}{3}\right) dx$$
$$= \frac{3}{2}\left(x^2y + \frac{y^3}{3}\right)\Big|_0^x = \frac{3}{2}\left(\frac{x^3}{3}y + \frac{y^3}{3}x\right) = \frac{1}{2}(x^3y + xy^3)$$
$$\Rightarrow F(x,y) = \frac{1}{2}(x^3y+xy^3) \neq \frac{1}{4}(x^3+x)(y^3+y) = F_1(x)F_2(y)$$

Therefore:

$$X \; and \, Y \; are \; not \; independent$$

Problem 4: [3, Exercise 2.57]
Find the marginal distribution functions (a) of X and (b) of Y for the density function of Problem 3.
Solution:
a)

$$F_1(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2}\,(x^3+x) & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

b)

$$F_2(y) = \begin{cases} 0 & y \le 0 \\ \frac{1}{2}\,(y^3+x) & 0 \le y \le 1 \\ 1 & y \ge 1 \end{cases}$$

Problem 5: [3, Exercise 2.58]
Find the conditional probability function (a) of X given Y, (b) of Y given X, for the distribution of Problem 1.
Solution:
a)

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{xy/36}{y/6} = \frac{x}{6} = f_1(x)$$

b)

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{xy/36}{x/6} = \frac{y}{6} = f_2(y)$$

Problem 6: [3, Exercise 2.59]
Let

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional density function of (a) X given Y, (b) Y given X.
Solution:

$$f_1(x) = \int_0^1 (x+y)dy = xy + \frac{y^2}{2}\bigg|_0^1 = x + \frac{1}{2}$$

$$f_2(y) = \int_0^1 (x+y)dx = xy + \frac{x^2}{2}\bigg|_0^1 = y + \frac{1}{2}$$

a)

$$f(x|y) = \begin{cases} \dfrac{f(x,y)}{f_2(y)} = \dfrac{x+y}{y+1/2} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{other x}, 0 \le y \le 1 \end{cases}$$

b)

$$f(y|x) = \begin{cases} \dfrac{f(x,y)}{f_1(x)} = \dfrac{x+y}{x+1/2} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & 0 \le x \le 1, \text{other y} \end{cases}$$

Problem 7: [3, Exercise 2.60]
Find the conditional density function of (a) X given Y, (b) Y given X, for the distribution of Problem 3.
Solution:

$$f_1(x) = F_1'(x) = \frac{1}{2}(x^3 + x)' = \frac{1}{2}(3x^2 + 1)$$

$$f_2(y) = F_2'(y) = \frac{1}{2}(y^3 + y)' = \frac{1}{2}(3y^2 + 1)$$

a)

$$f(x|y) = \begin{cases} \dfrac{f(x,y)}{f_2(y)} = \dfrac{3/2(x^2+y^2)}{1/2(3y^2+1)} = \dfrac{x^2+y^2}{y^2+1/3} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{other x}, 0 \le y \le 1 \end{cases}$$

b)

$$f(y|x) = \begin{cases} \dfrac{f(x,y)}{f_1(x)} = \dfrac{3/2(x^2+y^2)}{1/2(3x^2+1)} = \dfrac{x^2+y^2}{x^2+1/3} & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & 0 \le x \le 1, \text{other y} \end{cases}$$

Problem 8: [3, Exercise 2.61]
Let

$$f(x,y) = \begin{cases} e^{-(x+y)} & x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

be the joint density function of X and Y. Find the conditional density function of (a) X given Y, (b) Y given X.
Solution:

$$f_1(x) = \int_0^{+\infty} e^{-(x+y)} dy = -e^{-x}e^{-y} \Big|_0^{+\infty}$$

$$= -e^{-x}\left(-e^{-0} + \lim_{y\to+\infty} e^{-y}\right) = e^{-x}$$

$$f_2(y) = \int_0^{+\infty} e^{-(x+y)} dx = -e^{-y}e^{-x} \Big|_0^{+\infty}$$

$$= -e^{-y}\left(-e^{-0} + \lim_{x\to+\infty} e^{-x}\right) = e^{-y}$$

a)

$$f(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{e^{-(x+y)}}{e^{-y}} = e^{-x} = f_1(x)$$

$$f(x|y) = \begin{cases} e^{-x} & x \geq 0, y \geq 0 \\ 0 & x < 0, y \geq 0 \end{cases}$$

b)

$$f(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{e^{-(x+y)}}{e^{-x}} = e^{-y} = f_2(y)$$

$$f(y|x) = \begin{cases} e^{-y} & x \geq 0, y \geq 0 \\ 0 & x \geq 0, y < 0 \end{cases}$$

Problem 9: [3, Exercise 2.81]
Suppose that $f(x) = c/3^x,\ x = 1, 2, ...,$ is the probability function for a random variable X. (a) Determine c. (b) Find the distribution function. (c) Graph the probability function and the distribution function. (d) Find $P(2 \leq X < 5)$. (e) Find $P(X \geq 3)$.
Solution:

$$f(x) = \frac{c}{3^x} = c\left(\frac{1}{3}\right)^x$$

a)

$$f(x) = c\left[\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + ... + \left(\frac{1}{3}\right)^n\right] = \frac{2}{3}\left(1 + ... + \frac{1}{3^{n-1}}\right)$$

$$= \frac{c}{3} * \left(\frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}}\right) = \frac{c}{3} * \frac{3}{2}\left(1 - \frac{1}{3^n}\right) = \frac{c}{2} = 1 \Rightarrow c = 2$$

b)

$$F(x) = 2\left[\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + ... + \left(\frac{1}{3}\right)^y\right] = \frac{2}{3}\left(1 + ... + \frac{1}{3^{y-1}}\right)$$

$$= \frac{2}{3} * \left(\frac{1 - \frac{1}{3^y}}{1 - \frac{1}{3}}\right) = \frac{2}{3} * \frac{3}{2}\left(1 - \frac{1}{3^y}\right) = 1 - \frac{1}{3^y} = 1 - 3^{-y}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - 3^{-y} & y \le x < y + 1; y = 1, 2, 3... \end{cases}$$

d)

$$P(2 \le x < 5) = P(X = 4) - P(X = 2) = (1 - 3^{-4}) - (1 - 3^{-2}) = 3^{-2} - 3^{-4} = \frac{26}{81}$$

e)

$$P(X \ge 3) = 1 - P(X < 3) = 1 - P(X = 2) = 1 - (1 - 3^{-2}) = 3^{-2} = \frac{1}{9}$$

Problem 10: [3, Exercise 2.82]
Suppose that

$$f(x) = \begin{cases} cxe^{-2x} & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

is the density function for a random variable X. (a) Determine c. (b) Find the distribution function. (c) Graph the density function and the distribution function. (d) Find $P(X \ge 1)$. (e) Find $P(2 \le X < 3)$.
Solution:
a)

$$F(x) = \int_{-\infty}^{+\infty} f(x)dx = 1$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{+\infty} cxe^{-2x}dx = c\int_{0}^{+\infty} xe^{-2x}dx$$

$$= c\left(-\frac{1}{2}xe^{-2x} + \int -\frac{1}{2}e^{-2x}dx\right)\Bigg|_0^{+\infty}$$

$$= \frac{c}{2}\left(xe^{-2x} - \frac{1}{2}e^{-2x}\right)\Bigg|_0^{+\infty} = \frac{c}{4}(2xe^{-2x} - e^{-2x})\Bigg|_0^{+\infty}$$

$$= \frac{c}{4}e^{-2x}(2x - 1)\Bigg|_0^{+\infty} = \frac{c}{4}\left(\lim_{x\to+\infty}\frac{2x-1}{e^{2x}} - 1\right)$$

$$= \frac{c}{4}\left(\lim_{x\to+\infty}\frac{2}{2e^x} - 1\right) (by\ L'Hospital's\ Rule)$$

$$= \frac{c}{4}\left(\lim_{x\to+\infty} e^{-2x} - 1\right) = \frac{c}{4} = 1 \Rightarrow c = 4$$

b)

$$F(x) = \int 4xe^{-2x}dx = 4\int xe^{-2x}dx$$

$$= 4\left(\frac{-xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}dx\right) = 2\left(-xe^{-2x} - \frac{1}{2}e^{-2x} + k\right)$$

$$F(0) = 0 - \frac{1}{2}e^0 + 1 = -\frac{1}{2} + k = 0 \Rightarrow k = \frac{1}{2}$$

$$\Rightarrow F(x) = 2\left(-xe^{-2x} - \frac{1}{2}e^{-2x} + \frac{1}{2}\right) = 1 - e^{-2x}(2x+1)$$

$$F(x) = \begin{cases} 1 - e^{-2x}(2x+1) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

d)

$$P(X \geq\ 1) = 1 - P(X < 1) = 1 - (1 - e^{-2*1}(2*1+1)) = 3e^{-2}$$

e)

$$P(2 \leq X \leq 3) = P(X < 3) - P(X < 2)$$
$$= [1 - e^{-6}(3*2+1)] - [1 - e^{-4}(2*2+1)] = 5e^{-4} - 7e^{-6}$$

Problem 11: [3, Exercise 2.83]
The probability function of a random variable X is given by

$$f(x) = \begin{cases} 2p & x = 1 \\ p & x = 2 \\ 4p & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

where p is a constant. Find (a) $P(0 \leq X < 3)$, (b) $P(X > 1)$.
Solution:

$$f(x) = \begin{cases} 2p & x = 1 \\ p & x = 2 \\ 4p & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow 2p + p + 4p = 1 \Rightarrow 7p = 1 \Rightarrow p = \frac{1}{7}$$

a)

$$P(0 \leq X < 3] = P(X = 0) + P(X = 1) + P(X = 2) = 0 + 2*\frac{1}{7} + \frac{1}{7} = \frac{3}{7}$$

b)

$$P(X > 1) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left(0 + 2*\frac{1}{7}\right) = \frac{5}{7}$$

Problem 12: [3, Exercise 2.84]
(a) Prove that for a suitable constant c,

$$F(x) = \begin{cases} 0 & x \geq 0 \\ c(1-e^{-x})^2, & x > 0 \end{cases}$$

is the distribution function for a random variable X, and find this c. (b) Determine $P(1 < X < 2)$.
Solution:
a)

$$F(x) = \int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{+\infty} f(x)dx = c(1-e^{-x})^2 \Big|_{0}^{+\infty} = 1$$
$$= c\left[\lim_{x\to+\infty}(1-e^{-x})^2 - 0\right] = c \Rightarrow c = 1$$

b)

$$P(1 < X < 2) = F(2) - F(1) = (1-e^{-2})^2 - (1-e^{-1})^2 = e^{-4} - 3e^{-2} + 2e^{-1}$$

Problem 13: [3, Exercise 2.86]
Two independent random variables, X and Y, have respective density functions

$$f(x) = \begin{cases} c_1 e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$g(y) = \begin{cases} c_2 y e^{-3y} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

Find (a)c_1 and c_2,(b) $P(X+Y > 1)$,(c) $P(1 < X < 2, Y \geq 1)$,(d)$P(1 < X < 2)$,(e)$P(Y \geq 1)$.
Solution:
a)

$$c_1 = 2, c_2 = 9$$

b)

$$P(X+Y > 1) = 9e^{-2} - 14e^{-3}$$

c)

$$P(1 < X < 2, Y \geq 1) = 4e^{-5} - 4e^{-7}$$

d)

$$P(1 < X < 2) = e^{-2} - e^{-4}$$

e)

$$P(Y \geq 1) = 4e^{-3}$$

Problem 14: [3, Exercise 2.88]
Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} c(2x+y) & 0<x<1, 0<y<2 \\ 0 & otherwise \end{cases}$$

Find (a) the constant c, (b) $P(X > \frac{1}{2}, Y < \frac{3}{2})$, (c) the (marginal) density function of X, (d) the (marginal) density function of Y.
Solution:
a)

$$F(x,y) = 1 = \int_{x=0}^{1}\int_{y=0}^{2} c(2x+y)dxdy = \int_{x=0}^{1} 2xy + \frac{y^2}{2}\Bigg|_{y=0}^{2} dx$$

$$= c\int_{x=0}^{1}(4x+2)dx = c(2x^3+2x)\Bigg|_{0}^{1} = 4c = 1 \Rightarrow c = \frac{1}{4}$$

b)

$$\left(P > \frac{1}{2}, Y < \frac{3}{2}\right) = \frac{1}{4}\int_{x=1/2}^{1}\int_{y=0}^{3/2}(2x+y)dxdy = \frac{1}{4}\int_{x=1/2}^{1} 2xy + \frac{y^2}{2}\Bigg|_{y=0}^{3/2} dx$$

$$= \frac{1}{4}\int_{x=1/2}^{1}\left(3x+\frac{9}{8}\right)dx = \frac{1}{4}\left(\frac{3x^2}{2}+\frac{9x}{8}\right)\Bigg|_{1/2}^{1} = \frac{1}{4}\left(\frac{3}{2}+\frac{9}{8}-\frac{3*1/2^2}{2}-\frac{9}{8}*\frac{1}{2}\right) = \frac{27}{64}$$

c)

$$f_1(x) = \int_0^2 \frac{1}{4}(2x+y)dy = \frac{1}{4}\left(2xy+\frac{y^2}{2}\right)\Bigg|_0^2 = \frac{1}{4}\left(2x*2+\frac{4}{2}\right) = x+\frac{1}{2}$$

$$f_1(x) = \begin{cases} x+\frac{1}{2} & 0<x<1 \\ 0 & otherwise \end{cases}$$

d)

$$f_2(y) = \int_0^1 \frac{1}{4}(2x+y)dx = \frac{1}{4}(x^2+xy)\Bigg|_0^1 = \frac{1}{4}(y+1)$$

$$f_2(y) = \begin{cases} \frac{1}{4}(y+1) & 0<y<2 \\ 0 & otherwise \end{cases}$$

Problem 15: [3, Exercise 2.91]

$$f(x,y) = \begin{cases} 1/y & 0<x<y, 0<y<1 \\ 0 & otherwise \end{cases}$$

(a) Determine whether X and Y are independent, (b) Find $P(X > \frac{1}{2})$. (c) Find $P(X < \frac{1}{2}, Y > \frac{1}{3})$. (d) Find $P(X+Y > \frac{1}{2})$.

Solution:

a)

$$X \; and \, Y \; are \; not \; independent$$

b)

$$P\left(X > \frac{1}{2}\right) = \frac{1}{2}\left(1 - ln2\right)$$

c)

$$P\left(X < \frac{1}{2}, Y > \frac{1}{3}\right) = \frac{1}{6} + \frac{1}{2} ln2$$

d)

$$P\left(X + Y > \frac{1}{2}\right) = \frac{1}{2} ln2$$

References

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