Solution for Problem Set 10

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: (Test of means and proportions using normal distribution)
The mean lifetime of a sample of 100 fluorescent light bulbs produced by a company is computed to be 1570 hours with a standard deviation of 120 hours. If is the mean lifetime of all the bulbs produced by the company, test the hypothesis $\mu = 1600$ hours against the alternative hypothesis $\mu \neq 1600$ hours, using a level of significance of:
(a) 0.05
(b) 0.01.
(c) Find the P value of the test.
Solution:

$$n = 100$$
$$\overline{X} = 1570$$
$$\sigma = 120$$
$$\mu = 1600$$

We need to decide between these two hypotheses:
- Null hypothesis $(H_0) : \mu = 1600$ (hours)
- Alternative hypothesis $(H_1) : \mu \neq 1600$ (hours)
$\Rightarrow$ One Sample with Two-Tailed Hypothesis Testing
n > 30; σ is known $\Rightarrow$ z-test

$\alpha_1 = 0.05$ for both tails
$\Rightarrow \alpha_1/2 = 0.05/2 = 0.025$ for each tail
$\Rightarrow$ Critical values: $z_{\alpha_1/2} = z_{0.025} = -1.96; -z_{\alpha_1/2} = -z_{0.025} = 1.96$
We get the following decision rule:
- Reject H_0 if z score of the sample mean $\overline{X}$ is outside the range -1.96 to 1.96
- Accept H_0 otherwise

$\alpha_2 = 0.01$ for both tails
$\Rightarrow \alpha_2/2 = 0.01/2 = 0.005$ for each tail
$\Rightarrow$ Critical values: $z_{\alpha_2/2} = z_{0.005} = -2.58; -z_{\alpha_2/2} = z_{0.005} = 2.58$
We get the following decision rule:
- Reject H_0 if z score of the sample mean $\overline{X}$ is outside the range -2.58 to 2.58
- Accept H_0 otherwise

Apply the formula to calculate z score:

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{1570 - 1600}{120/\sqrt{100}} = -2.5$$

a)z = -2.5 $\Rightarrow$ z is outside the range -1.96 to 1.96
Therefore, we reject the hypothesis H_0 at 0.05 level of significance.
b)z = -2.5 $\Rightarrow$ z is inside the range -2.58 to 2.58
Therefore, we accept the hypothesis H_0 at 0.01 level of significance.
c)

$$P = 2p(z \leq -2.58) = 2 * 0.0062 = 0.0124$$

Problem 2: (Test of means and proportions using normal distribution)
The breaking strengths of cables produced by a manufacturer have mean 1800 lb and standard deviation 100 lb. By a new technique in the manufacturing process it is claimed that the breaking strength can be increased. To test this claim, a sample of 50 cables is tested, and it is found that the mean breaking strength is 1850 lb.
(a) Can we support the claim at a 0.01 level of significance?
(b) What is the P value of the test?
Solution:

$$n = 50$$
$$\overline{X} = 1850$$
$$\sigma = 100$$
$$\mu = 1800$$

We need to decide between these two hypotheses:
- Null hypothesis $(H_0) : \mu = 1800$ (lb) and there is no change on breaking strength
- Alternative hypothesis $(H_1) : \mu > 1800$ (lb) and there is a change on breaking strength
$\Rightarrow$ One Sample with One-Tailed Hypothesis Testing
n > 30; σ is known $\Rightarrow$ z-test

a) $\alpha = 0.01$ for one tail only
$\Rightarrow$ Critical value: $-z_\alpha = -z_{0.01} = 2.33$
We get the following decision rule:
- Reject H_0 if z score of the sample mean $\overline{X}$ is greater than 2.33
- Accept H_0 otherwise

Apply the formula to calculate z score:

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{18500 - 1800}{100/\sqrt{50}} = 3.55$$

z = 3.55 $\Rightarrow$ z > 2.33
Therefore, we reject the hypothesis H_0 at 0.01 level of significance.
The result is highly significant and the claim will be supported.
b)

$$P = p(z \geq 3.55) = p(z \leq -3.55) = 0.0002$$

Problem 3: (Test involving differences of means and proportions)
An examination was given to two classes consisting of 40 and 50 students, respectively. In the first class the mean grade was 74 with a standard deviation of 8, while in the second class the mean grade was 78 with a standard deviation of 7. Is there a significant difference between the performance of the two classes at a level of significance of:
(a) 0.05
(b) 0.01
(c) What is the P value of the test?
Solution:

$$n_1 = 40, n_2 = 50$$
$$\overline{X}_1 = 74, \overline{X}_2 = 78$$
$$\sigma_1 = 78, \sigma_2 = 7$$

We need to decide between these two hypotheses:
- Null hypothesis $(H_0) : \mu_{\overline{X}_1} = \mu_{\overline{X}_2}$ and the difference is barely between two classes due to chance
- Alternative hypothesis $(H_1) : \mu_{\overline{X}_1} \neq \mu_{\overline{X}_2}$ and the difference is significant between two classes

$\Rightarrow$ Two Samples with Two-Tailed Hypothesis Testing
$n_1, n_2 > 30$; σ_1, σ_2 are known $\Rightarrow$ z-test

$\alpha_1 = 0.05$ for both tails
$\Rightarrow \alpha_1/2 = 0.05/2 = 0.025$ for each tail
$\Rightarrow$ Critical values: $z_{\alpha_1/2} = z_{0.025} = -1.96; -z_{\alpha_1/2} = -z_{0.025} = 1.96$
We get the following decision rule:
- Reject H_0 if z score of the sample mean $\overline{X}$ is outside the range -1.96 to 1.96
- Accept H_0 otherwise

$\alpha_2 = 0.01$ for both tails
$\Rightarrow \alpha_2/2 = 0.01/2 = 0.005$ for each tail
$\Rightarrow$ Critical values: $z_{\alpha_2/2} = z_{0.005} = -2.58; -z_{\alpha_2/2} = -z_{0.005} = 2.58$
We get the following decision rule:
- Reject H_0 if z score of the sample mean $\overline{X}$ is outside the range -2.58 to 2.58
- Accept H_0 otherwise

The mean and standard deviation of the difference in mean are given by:

$$\mu_{\overline{X}_1 - \overline{X}_2} = 0$$

$$\sigma_{\overline{X}_1 - \overline{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{8^2}{40} + \frac{7^2}{50}} = 1.606$$

Apply the formula to calculate z score:

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sigma_{\overline{X}_1 - \overline{X}_2}} = \frac{74 - 78}{1.606} = -2.49$$

a)z = -2.49 $\Rightarrow$ z is outside the range -1.96 to 1.96
Therefore, we reject the hypothesis H_0 at 0.05 level of significance.
There is a significance difference in performance between two classes.
b)z = -2.49 $\Rightarrow$ z is inside the range -2.58 to 2.58
Therefore, we accept the hypothesis H_0 at 0.01 level of significance.
There is no significance difference in performance between two classes.
c)

$$P = 2p(z \leq -2.49) = 2 * 0.0064 = 0.0128$$

Problem 4: (Test involving differences of means and proportions)
The mean height of 50 male students who showed above-average participation in college athletics was 68.2 inches with a standard deviation of 2.5 inches, while 50 male students who showed no interest in such participation had a mean height of 67.5 inches with a standard deviation of 2.8 inches.
(a) Test the hypothesis that male students who participate in college athletics are taller than other male students.
(b) What is the P value of the test?
Solution:

$$n_1 = 50, n_2 = 50$$

$$\overline{X}_1 = 68.2, \overline{X}_2 = 67.5$$

$$\sigma_1 = 2.5, \sigma_2 = 2.8$$

Null hypothesis $(H_0) : \mu_{\overline{X}_1} = \mu_{\overline{X}_2}$ and there is no difference between two mean heights
Alternative hypothesis $(H_1) : \mu_{\overline{X}_1} > \mu_{\overline{X}_2}$ and the mean height of the first group is greater than the mean height of the second group
$\Rightarrow$ Two Samples with One-Tailed Hypothesis Testing
$n_1, n_2 > 30$; σ_1, σ_2 are known $\Rightarrow$ z-test

a) $\alpha = 0.05$ for one tail only
(Note: Take a small random value of α if it is not given such as 0.05 or 0.01)
$\Rightarrow$ Critical value: $-z_\alpha = -z_{0.05} = 1.645$
We get the following decision rule:
- Reject H_0 if z score of the sample mean $\overline{X}$ is greater than 1.645
- Accept H_0 otherwise

The mean and standard deviation of the difference in mean are given by:

$$\mu_{\overline{X}_1 - \overline{X}_2} = 0$$

$$\sigma_{\overline{X}_1 - \overline{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{(2.5)^2}{50} + \frac{(2.8)^2}{50}} = 0.53$$

Apply the formula to calculate z score:

$$z = \frac{\overline{X}_1 - \overline{X}_2}{\sigma_{\overline{X}_1 - \overline{X}_2}} = \frac{68.2 - 67.5}{0.53} = 1.32$$

z = 1.32 $\Rightarrow$ z < 1.645
Therefore, we accept the hypothesis H_0 at 0.05 level of significance.
There is no difference between the mean height.
b)

$$P = p(z \geq 1.32) = p(Z \leq -1.32) = 0.0934$$

Problem 5: (Test involving student's t distribution)
In the past a machine has produced washers having a mean thickness of 0.050 inch. To determine whether the machine is in proper working order a sample of 10 washers is chosen for which the mean thickness is 0.053 inch and the standard deviation is 0.003 inch. Test the hypothesis that the machine is in proper working order using a level of significance of:
(a) 0.05
(b) 0.01
(c) Find the P value of the test.
Solution:

$$n = 10$$
$$\overline{X} = 0.053$$
$$\mu = 0.050$$
$$S = 0.003$$

We need to decide between these two hypotheses:
- Null hypothesis (H_0) : $\mu = 0.050$ and the machine is in proper working order
- Alternative hypothesis (H_1) : $\mu \neq 0.050$ and the machine is not in proper working order
$\Rightarrow$ One Sample with Two-Tailed Hypothesis Testing
n < 30; σ is unknown $\Rightarrow$ t-test

υ = n - 1 = 10 - 1 = 9 (degrees of freedom)

$\alpha_1 = 0.05$ for both tails
$\Rightarrow \alpha_1/2 = 0.05/2 = 0.025$ for each tail
$\Rightarrow$ Critical values: $-t_{\alpha_1/2} = -t_{0.025} = -2.26; t_{\alpha_1/2} = t_{0.025} = 2.26$
We get the following decision rule:
- Reject H_0 if t score of the sample mean $\overline{X}$ is outside the range -2.26 to 2.26
- Accept H_0 otherwise

$\alpha_2 = 0.01$ for both tails
$\Rightarrow \alpha_2/2 = 0.01/2 = 0.005$ for each tail
$\Rightarrow$ Critical values: $-t_{\alpha_2/2} = -t_{0.005} = -3.25; t_{\alpha_2/2} = t_{0.005} = 3.25$
We get the following decision rule:
- Reject H_0 if t score of the sample mean $\overline{X}$ is outside the range -3.25 to 3.25
- Accept H_0 otherwise

Apply the formula to calculate t score:

$$t = \frac{\overline{X} - \mu}{S}\sqrt{n-1} = \frac{0.053 - 0.050}{0.003}\sqrt{10-1} = 3.00$$

a)t = 3.00 $\Rightarrow$ t is outside the range -2.26 to 2.26
Therefore, we reject the hypothesis H_0 at 0.05 level of significance.
The machine is not in proper working order.
b)t = 3.00 $\Rightarrow$ t is inside the range -3.25 to 3.25
Therefore, we accept the hypothesis H_0 at 0.01 level of significance.
The machine is in proper working order.
c)

$$t = 3.00\, with\, \upsilon = 9\, (dof)$$

$$\Rightarrow 0.005 < p < 0.01 \Rightarrow 0.01 < 2p < 0.02$$

$$P = 2p(t \geq 3.00)$$

$$\Rightarrow 0.01 < P < 0.02$$

$$P = 0.015\, (using\, R\, software)$$

Problem 6: (Test involving student's t distribution)
A test of the breaking strengths of 6 ropes manufactured by a company showed a mean breaking strength of 7750 lb and a standard deviation of 145 lb, whereas the manufacturer claimed a mean breaking strength of 8000 lb. Can we support the manufacturer's claim at a level of significance of:
(a) 0.05
(b) 0.01
(c) What is the P value of the test?
Solution:

$$n = 6$$

$$\overline{X} = 7750$$

$$\mu = 8000$$

$$S = 145$$

We need to decide between these two hypotheses:
- Null hypothesis $(H_0) : \mu = 8000$ (lb) and the manufacturer's claim is justified
- Alternative hypothesis $(H_1) : \mu < 8000$ (lb) and the manufacturer's claim is not justified
$\Rightarrow$ One Sample with One-Tailed Hypothesis Testing
n < 30; σ is unknown $\Rightarrow$ t-test

υ = n - 1 = 6 - 1 = 5 (degrees of freedom)

$\alpha_1 = 0.05$ for one tail only
$\Rightarrow$ Critical value: $-t_{\alpha_1} = -t_{0.05} = -2.01$
We get the following decision rule:
- Reject H_0 if t score of the sample mean $\overline{X}$ is smaller than -2.01
- Accept H_0 otherwise

$\alpha_2 = 0.01$ for one tail only

$\Rightarrow$ Critical value: $-t_{\alpha_2} = -t_{0.01} = -3.36$
We get the following decision rule:
- Reject H_0 if t score of the sample mean $\overline{X}$ is smaller than -3.36
- Accept H_0 otherwise

Apply the formula to calculate t score:

$$t = \frac{\overline{X} - \mu}{S}\sqrt{n-1} = \frac{7750 - 8000}{145}\sqrt{6-1} = -3.86$$

a)t = -3.86 $\Rightarrow t < -2.01$
Therefore, we reject the hypothesis H_0 at 0.05 level of significance.
The manufacturer's claim is not justified.
b)t = -3.86 $\Rightarrow t < -3.36$
Therefore, we reject the hypothesis H_0 at 0.01 level of significance.
The manufacturer's claim is not justified.
c)

$$t = -3.86\,with\,\upsilon = 5\,(dof)$$
$$\Rightarrow 0.005 < p < 0.01$$
$$P = p(t \leq -3.86)$$
$$\Rightarrow 0.005 < P < 0.01$$
$$P = 0.006\,(using\,R\,software)$$

Problem 7: (Test involving student's t distribution)
The IQs (intelligence quotients) of 16 students from one area of a city showed a mean of 107 with a standard deviation of 10, while the IQs of 14 students from another area of the city showed a mean of 112 with a standard deviation of 8. Is there a significant difference between the IQs of the two groups at a:
(a) 0.01,
(b) 0.05 level of significance
(c) What is the P value of the test?
Solution:

$$n_1 = 16, n_2 = 14$$
$$\overline{X}_1 = 107, \overline{X}_2 = 112$$
$$S_1 = 10, S_2 = 8$$

We need to decide between these two hypotheses:
- Null hypothesis (H_0) : $\mu_1 = \mu_2$ and there is essentially no difference between two groups
- Alternative hypothesis (H_1) : $\mu_1 \neq \mu_2$ and there is a significant difference between two groups
$\Rightarrow$ Two Samples with Two-Tailed Hypothesis Testing
$n_1, n_2 < 30$; σ_1, σ_2 are unknown $\Rightarrow$ t-test

$\upsilon = n_1 + n_2$ - 2 = 16 + 14 - 2 = 28 (degrees of freedom)

$\alpha_1 = 0.01$ for both tails
$\Rightarrow \alpha_1/2 = 0.01/2 = 0.005$ for each tail
$\Rightarrow$ Critical values: $-t_{\alpha_1/2} = -t_{0.005} = -2.76; t_{\alpha_1/2} = t_{0.005} = 2.76$
We get the following decision rule:
- Reject H_0 if t score of the sample mean $\overline{X}$ is outside the range -2.76 to 2.76
- Accept H_0 otherwise

$\alpha_2 = 0.05$ for both tails
$\Rightarrow \alpha_2/2 = 0.05/2 = 0.025$ for each tail
$\Rightarrow$ Critical values: $-t_{\alpha_2/2} = -t_{0.025} = -2.05; t_{\alpha_2/2} = t_{0.025} = 2.05$
We get the following decision rule:
- Reject H_0 if t score of the sample mean $\overline{X}$ is outside the range -2.05 to 2.05
- Accept H_0 otherwise

The standard deviation of the difference in mean is given by:

$$\sigma = \sqrt{\frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{16(10)^2 + 14(8)^2}{16 + 14 - 2}} = 9.44$$

Apply the formula to calculate t score:

$$t = \frac{\overline{X}_1 - \overline{X}_2}{\sigma\sqrt{1/n_1 + 1/n_2}} = \frac{107 - 112}{9.44\sqrt{1/16 + 1/14}} = -1.45$$

a)t = -1.45 $\Rightarrow$ t is inside the range -2.76 to 2.76
Therefore, we accept the hypothesis H_0 at 0.01 level of significance.
There is essentially no difference between the IQs of two groups.
b)t = -1.45 $\Rightarrow$ t is inside the range -2.05 to 2.05
Therefore, we accept the hypothesis H_0 at 0.05 level of significance.
There is essentially no difference between the IQs of two groups.
c)

$$t = -1.45 \, with \, v = 28 \, (dof)$$

$$\Rightarrow 0.05 < p < 0.1 \Rightarrow 0.1 < 2p < 0.2$$

$$P = 2p(t \leq -1.45)$$

$$\Rightarrow 0.1 < P < 0.2$$

$$P = 0.158 \, (using \, R \, software)$$

Problem 8: (Test involving the chi-square distribution)
In the past the standard deviation of weights of certain 40.0 oz packages filled by a machine was 0.25 oz. A random sample of 20 packages showed a standard deviation of 0.32 oz. Is the apparent increase in variability significant at the:
(a) 0.05
(b) 0.01 level of significance

(c) What is the P value of the test?
Solution:

$$n = 20$$

$$S = 0.32$$

$$\sigma = 0.25$$

We need to decide between these two hypotheses:
- Null hypothesis $(H_0) : \sigma_1 = 0.25$ (oz) and the observed result is due to chance
- Alternative hypothesis $(H_1) : \sigma_1 > 0.25$ (oz) and the variability has increased

$\Rightarrow$ One Sample with One-Tailed Hypothesis Testing
$n < 30$; σ is known $\Rightarrow \chi^2$-test

υ = n - 1 = 20 - 1 = 19 (degrees of freedom)

$\alpha_1 = 0.05$ for one tail only
$\Rightarrow$ Critical value: $\chi^2(\alpha_1) = \chi^2(0.05) = 30.1$

We get the following decision rule:
- Reject H_0 if χ^2 value for the sample is greater than 30.1
- Accept H_0 otherwise

$\alpha_2 = 0.01$ for one tail only
$\Rightarrow$ Critical value: $\chi^2(\alpha_2) = \chi^2(0.01) = 36.2$

We get the following decision rule:
- Reject H_0 if χ^2 value for the sample is greater than 36.2
- Accept H_0 otherwise

Apply the formula to calculate the χ^2 value:

$$\chi^2 = \frac{nS^2}{\sigma^2} = \frac{20(0.32)^2}{(0.25)^2} = 32.8$$

a)$\chi^2 = 32.8 \Rightarrow \chi^2 > 30.1$
Therefore, we reject the hypothesis H_0 at 0.05 level of significance.
The variability has increased.
b)$\chi^2 = 32.8 \Rightarrow \chi^2 < 36.2$
Therefore, we accept the hypothesis H_0 at 0.01 level of significance.
The observed result is due to chance.
c)

$$P = p(\chi^2 \geq 32.8) = 0.025$$