Solution for Problem Set 7

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: [1, Exercise 8.2]
The lengths of time, in minutes, that 10 patients waited in a doctors office before receiving treatment were recorded as follows: 5, 11, 9, 5, 10, 15, 6, 10, 5, and 10. Treating the data as a random sample,find
(a) the mean;
(b) the median;
(c) the mode.
Solution:
(a)

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{10}(5 + 11 + 9 + 5 + 10 + 15 + 6 + 10 + 5 + 10) = 8.6(minutes)$$

(b)
n=10 $\Rightarrow$ n is even.
Arranged list (from min to max): 5, 5, 5, 6, 9, 10, 10, 10, 11, 15
Therefore:

$$\overline{x} = \frac{1}{2}\left(x_{n/2} + x_{n/2+1}\right) = \frac{1}{2}\left(x_5 + x_6\right) = \frac{1}{2}(9 + 10) = 9.5(minutes)$$

(c)
Mode are the values of the sample that most occur.
Therefore, mode are 5 and 10 minutes.

Problem 2: [1, Exercise 8.9]
Consider the data in Problem 1, find
(a) the range;
(b) the standard deviation.
Solution:
(a)

$$R = X_{max} - X_{min} = 15 - 5 = 10$$

(b)

$$S^2 = \frac{n\sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2}{n(n-1)} = \frac{(10)(838) - 86^2}{(10)(9)} = 10.933$$

$$\Rightarrow S = \sqrt{S^2} = \sqrt{10.933} = 3.307$$

Problem 3: [1, Exercise 8.12]
The tar contents of 8 brands of cigarettes selected at random from the latest list released by the Federal Trade Commission are as follows: 7.3, 8.6, 10.4, 16.1, 12.2, 15.1, 14.5, and 9.3 milligrams. Calculate
(a) the mean;
(b) the variance.
Solution:
(a)

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{8}(7.3 + 8.6 + 10.4 + 16.1 + 12.2 + 15.1 + 14.5 + 9.3) = 11.69(milligrams)$$

(b)

$$S^2 = \frac{n\sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2}{n(n-1)} = \frac{(8)(1168.21) - 93.5^2}{(8)(7)} = 10.776$$

Problem 4: [1, Exercise 8.13]
The grade-point averages of 20 college seniors selected at random from a graduating class are as follows:

3.2 1.9 2.7 2.4 2.8
2.9 3.8 3.0 2.5 3.3
1.8 2.5 3.7 2.8 2.0
3.2 2.3 2.1 2.5 1.9

Calculate the standard deviation.
Solution:

$$S^2 = \frac{n\sum_{i=1}^{n} X_i^2 - (\sum_{i=1}^{n} X_i)^2}{n(n-1)} = \frac{(2)(148.55) - (53.3)^2}{(20)(19)} = 0.342$$

$$\Rightarrow S = \sqrt{S^2} = \sqrt{0.342} = 0.585$$

Problem 5: [1, Exercise 8.19]
A certain type of thread is manufactured with a mean tensile strength of 78.3 kilograms and a standard deviation of 5.6 kilograms. How is the variance of the sample mean changed when the sample size is

(a) increased from 64 to 196?
(b) decreased from 784 to 49?
Solution:
(a)
For $n = 64 : \sigma_{\overline{X}} = \frac{\sigma^2}{n} = \frac{(5.6)^2}{64} = 0.49$

For $n = 196 : \sigma_{\overline{X}} = \dfrac{\sigma^2}{n} = \dfrac{(5.6)^2}{196} = 0.16$

Therefore, the variance of the sample mean is decreased from 0.49 to 0.16 when the sample size is increased from 64 to 196.

(b)

For $n = 748 : \sigma_{\overline{X}} = \dfrac{\sigma^2}{n} = \dfrac{(5.6)^2}{748} = 0.04$

For $n = 49 : \sigma_{\overline{X}} = \dfrac{\sigma^2}{n} = \dfrac{(5.6)^2}{49} = 0.64$

Therefore, the variance of the sample mean is increased from 0.04 to 0.64 when the sample size is increased from 748 to 49.

Problem 6: [1, Exercise 8.22]

The heights of 1000 students are approximately normally distributed with a mean of 174.5 centimeters and a standard deviation of 6.9 centimeters. Suppose 200 random samples of size 25 are drawn from this population and the means recorded to the nearest tenth of a centimeter. Determine

(a) the mean and standard deviation of the sampling distribution of $\overline{X}$;

(b) the number of sample means that fall between 172.5 and 175.8 centimeters inclusive;

(c) the number of sample means falling below 172.0 centimeters

Solution:

(a)

$$\mu_{\overline{X}} = \mu = 174.5$$
$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{6.9}{\sqrt{25}} = 1.38$$

(b)

$$z_1 = \frac{\overline{x}_1 - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{172.5 - 174.5}{1.38} = -1.45$$
$$z_2 = \frac{\overline{x}_2 - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{175.8 - 174.5}{1.38} = 0.94$$
$$\Rightarrow P(172.5 < \overline{X} < 175.8) = P(-1.45 < Z < 0.94)$$
$$= P(Z < 0.94) - P(Z < -1.45) = 0.8264 - 0.0735 = 0.7529$$

Therefore, the number of sample means between 172.5 and 175.8 centimeters inclusive is:

$$(200)(0.7529) = 151$$

(c)

$$z = \frac{\overline{x} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{172.0 - 174.5}{1.38} = -1.81$$
$$\Rightarrow P(\overline{X} < 172.0) = P(Z < -1.81) = 0.0351$$

Therefore, the number of sample means fall below 172.0 centimeters is:

$$(200)(0.0351) = 7$$

Problem 7: [1, Exercise 8.25]
The average life of a bread-making machine is 7 years, with a standard deviation of 1 year. Assuming that the lives of these machines follow approximately a normal distribution, find
(a) the probability that the mean life of a random sample of 9 such machines falls between 6.4 and 7.2 years;
(b) the value of x to the right of which 15% of the means computed from random samples of size 9 would fall.
Solution:
(a)

$$\mu_{\overline{X}} = \mu = 7$$
$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$
$$z_1 = \frac{\overline{x}_1 - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{6.4 - 7}{1/3} = -1.8$$
$$z_2 = \frac{\overline{x}_2 - \mu_{\overline{X}}}{\sigma_{\overline{X}}} = \frac{7.2 - 7}{1/3} = 0.6$$
$$P(6.4 < \overline{X} < 7.2) = P(-1.8 < Z < 0.6) = P(Z < 0.6) - P(Z < -1.8)$$
$$= 0.7257 - 0.0359 = 0.6898$$

(b)
The value of x to right of which 15% of the means $\Rightarrow$ Find z(85%).

$$z(85\%) = z(0.85) = 1.04$$

$$z = \frac{\overline{x} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} \Rightarrow \overline{x} = (z)(\sigma_{\overline{X}}) + \mu_{\overline{X}} = (1.04)(1/3) + 7 = 7.35$$

Problem 8: [1, Exercise 8.30]
The mean score for freshmen on an aptitude test at a certain college is 540, with a standard deviation of 50. Assume the means to be measured to any degree of accuracy. What is the probability that two groups selected at random, consisting of 32 and 50 students,respectively, will differ in their mean scores by
(a) more than 20 points?
(b) an amount between 5 and 10 points?
Solution:

$$\overline{X} = 540$$
$$\sigma = 50$$

$$n_1 = 32, n_2 = 50$$
$$\mu_1 - \mu_2 = 0$$
$$\sigma_{\overline{X_1}-\overline{X_2}} = \sigma\sqrt{1/n_1 + 1/n_2} = 50\sqrt{1/32 + 1/50} = 11.319$$

(a)

$$z_1 = -20/11.319 = -1.77$$
$$z_2 = 20/11.319 = 1.77$$
$$\Rightarrow P(|\overline{X_1} - \overline{X_2}| > 20) = 2P(Z < -1.77) = (2)(0.0384) = 0.0768$$

(b)

$$z_1 = 5/11.319 = 0.44$$
$$z_2 = 10/11.319 = 0.88$$
$$\Rightarrow P(-10 < \overline{X_1} - \overline{X_2} < -5) + P(5 < \overline{X_1} - \overline{X_2} < 10) = 2P(5 < |\overline{X_1} - \overline{X_2}| < 10)$$
$$= 2P(0.44 < Z < 0.88) = 2[P(Z < 0.88) - P(Z < 0.44)]$$
$$= 2(0.8106 - 0.6700) = 0.2812$$

Problem 9: [1, Exercise 8.41]
Assume the sample variances to be continuous measurements. Find the probability that a random sample of 25 observations, from a normal population with variance σ^2 = 6, will have a sample variance S^2.
(a) greater than 9.1;
(b) between 3.462 and 10.745
Solution:

$$n = 25$$
$$\sigma^2 = 6$$

(a)

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$
$$P(S^2 > 9.1) = P\left(\frac{(n-1)S^2}{\sigma^2} > \frac{(24)(9.1)}{6}\right) = P(\chi^2 > 36.4) = 0.05$$

(b)

$$P(3.462 < S^2 < 10.745) = P\left(\frac{(24)(3.462)}{6} < \frac{(n-1)S^2}{\sigma^2} < \frac{(24)(10.745)}{6}\right)$$
$$= P(13.848 < \chi^2 < 42.980) = P(\chi^2 < 42.980) - P(\chi^2 < 13.848) = P(\chi^2 > 13.848) - P(\chi^2 > 42.980)$$
$$= 0.95 - 0.01 = 0.94$$

Problem 10: [1, Exercise 8.67]
The breaking strength X of a certain rivet used in a machine engine has a mean 5000 psi and standard deviation 400 psi. A random sample of 36 rivets is taken. Consider the distribution of $\overline{X}$, the sample mean breaking strength.
(a) What is the probability that the sample mean falls between 4800 psi and 5200 psi?
(b) What sample n would be necessary in order to have

$$P(4900 < \overline{X} < 5100) = 0.99$$

Solution:

$$\mu = 5000\,psi$$
$$\sigma = 400\,psi$$
$$n = 36$$

(a)

$$\begin{aligned} P(4800 < \overline{X} < 5200) &= P\left(\frac{4800-5000}{400/\sqrt{36}} < Z < \frac{5200-5000}{400/\sqrt{36}}\right) \\ &= P(-3 < Z < 3) = P(Z < 3) - P(Z < -3) \\ &= 0.9987 - 0.0013 = 0.9974 \end{aligned}$$

(b)

$$\begin{aligned} P(4900 < \overline{X} < 5100) = 0.99 &\Rightarrow P(z_1 < Z < z_2) = 0.99 \\ &\Rightarrow P(Z < z) = 0.995 \Rightarrow z = 2.575 \\ &\Rightarrow \frac{5100-5000}{400/\sqrt{n}} < 2.575 \Rightarrow n \geq 107 \end{aligned}$$

Note: The value n can be affected by the z values picked (2.57 or 2.58).

References

[1] Walpole, R. E., Myers, R. H., Myers, S. L. and Ye, K., *Probability & Statistics for Engineers & Scientists*, 9th ed., MA, USA: Prentice-Hall, 2012.

[2] DeGroot, M. H. and Schervish, M. J., *Probability and Statistics*, 4th ed., MA, USA: Pearson Education, Inc., 2012.

[3] Murray, R. S., John, J. S. and R, A. Srinivasan, *Probability and Statistics*, 3rd ed., USA: McGraw-Hill, 2009.