Solution for Problem Set 8

HANU - Faculty of Information and Technology
MAT204: Probability & Statistics

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Problem 1: [1, Exercise 9.2]
An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a standard deviation of 40 hours. If a sample of 30 bulbs has an average life of 780 hours, find a 96% confidence interval for the population mean of all bulbs produced by this firm.
Solution:

$$n = 30, \overline{x} = 780, \sigma = 40$$

$$1 - \alpha = 96\% = 0.96 \Rightarrow \alpha = 0.04 \Rightarrow z_{\alpha/2} = z_{0.02} = 2.05$$

Therefore, a 96% confidence interval for the population mean can be calculated as:

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow 780 - (2.05)\frac{40}{\sqrt{30}} < \mu < 780 + (2.05)\frac{40}{\sqrt{30}}$$

$$\Rightarrow 765 < \mu < 795$$

Problem 2: [1, Exercise 9.5]
A random sample of 100 automobile owners in the state of Virginia shows that an automobile is driven on average 23,500 kilometers per year with a standard deviation of 3900 kilometers. Assume the distribution of measurements to be approximately normal.
(a) Construct a 99% confidence interval for the average number of kilometers an automobile is driven annually in Virginia.
(b) What can we assert with 99% confidence about the possible size of our error if we estimate the average number of kilometers driven by car owners in Virginia to be 23,500 kilometers per year?
Solution:

$$n = 100, \overline{x} = 23,500, \sigma = 3900$$

$$1 - \alpha = 99\% = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow z_{\alpha/2} = z_{0.005} = 2.575$$

(a) A 99% confidence interval for the population mean is:

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\Rightarrow 23,500 - (2.575)\frac{3900}{10} < \mu < 23,500 + (2.575)\frac{3900}{10}$$
$$\Rightarrow 22,496 < \mu < 24,504$$

(b) We can be 99% confident that the error will not exceed a specified amount e when the sample size is:

$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$$

Therefore:

$$e < z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = (2.575)\frac{3900}{10} = 1004$$

In another way, the possible size of our error must be smaller than 1004.

Problem 3: [1, Exercise 9.8]
An efficiency expert wishes to determine the average time that it takes to drill three holes in a certain metal clamp. How large a sample will she need to be 95% confident that her sample mean will be within 15 seconds of the true mean? Assume that it is known from previous studies that $\sigma = 40$ seconds.
Solution:

$$e = 15, \sigma = 40$$
$$1 - \alpha = 95\% = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow z_{\alpha/2} = z_{0.025} = 1.96$$
$$n = \left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$$
$$\Rightarrow n = \left(\frac{(1.96)(40)}{15}\right)^2 = 28$$

Problem 4: [1, Exercise 9.12]
A random sample of 10 chocolate energy bars of a certain brand has, on average, 230 calories per bar, with a standard deviation of 15 calories. Construct a 99% confidence interval for the true mean calorie content of this brand of energy bar. Assume that the distribution of the calorie content is approximately normal.
Solution:

$$n = 10, \overline{x} = 230, s = 15$$
$$v = n - 1 = 10 - 1 = 9$$

$$1-\alpha = 99\% = 0.99 \Rightarrow \alpha = 0.01 \Rightarrow t_{\alpha/2} = t_{0.005} = 3.25 with v = 9 (degrees of freedom)$$

A 99% confidence interval for the population mean is:

$$\overline{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

$$\Rightarrow 230 - (3.25)\frac{15}{\sqrt{10}} < \mu < 230 + (3.25)\frac{15}{\sqrt{10}}$$
$$\Rightarrow 214.58 < \mu < 245.42$$

Problem 5: [1, Exercise 9.19]
A random sample of 25 tablets of buffered aspirin contains, on average, 325.05 mg of aspirin per tablet, with a standard deviation of 0.5 mg. Find the 95% tolerance limits that will contain 90% of the tablet contents for this brand of buffered aspirin. Assume that the aspirin content is normally distributed.
Solution:

$$n = 25, \overline{x} = 325.05$$

$$s = 0.5, \gamma = 5\%$$

$$1 - \alpha = 90\% \,\&\, n = 25 \Rightarrow \; k = 2.208$$

$$(Look\; up\; the\; Tolerance\; Factors\; for\; Normal\; Distribution\; table)$$

So:

$$\overline{x} \pm ks \Rightarrow 325.05 \pm (2.208)(0.5) \Rightarrow (323.946, 326.151)$$

Therefore, we are 95% confident that the above tolerance interval will contain 90% of the aspirin contents for this brand of buffered aspirin.

Problem 6: [1, Exercise 9.35]
A random sample of size $n_1 = 25$, taken from a normal population with a standard deviation $\sigma_1 = 5$, has a mean $\overline{x}_1 = 80$. A second random sample of size $n_2 = 36$, taken from a different normal population with a standard deviation $\sigma_2 = 3$, has a mean $\overline{x}_2 = 75$. Find a 94% confidence interval for $\mu_1 - \mu_2$.
Solution:

$$n_1 = 25, n_2 = 36$$

$$\overline{x}_1 = 80, \overline{x}_2 = 75$$

$$\sigma_1 = 5, \sigma_2 = 3$$

$$\alpha = 0.06 \Rightarrow z_{\alpha/2} = z_{0.03} = 1.88$$

Therefore, a 94% confidence interval for $\mu_1 - \mu_2$ is:

$$(\overline{x}_1 - \overline{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\Rightarrow (80 - 75) - (1.88)\sqrt{\frac{25}{25} + \frac{9}{36}} < \mu_1 - \mu_2 < (80 - 75) + (1.88)\sqrt{\frac{25}{25} + \frac{9}{36}}$$
$$\Rightarrow 2.9 < \mu_1 - \mu_2 < 7.1$$

Problem 7: [1, Exercise 9.36]
Two kinds of thread are being compared for strength. Fifty pieces of each type of thread are tested under similar conditions. Brand A has an average tensile strength of 78.3 kilograms with a standard deviation of 5.6 kilograms, while brand B has an average tensile strength of 87.2 kilograms with a standard deviation of 6.3 kilograms. Construct a 95% confidence interval for the difference of the population means.
Solution:

$$n_1 = n_A = 50, n_2 = n_B = 50$$

$$\overline{x}_1 = \overline{x}_A = 78.3, \overline{x}_2 = \overline{x}_B = 87.2$$

$$\sigma_1 = \sigma_A = 5.6, \sigma_2 = \sigma_B = 6.3$$

$$1 - \alpha = 95\% = 0.95 \Rightarrow \alpha = 0.05 \Rightarrow z_{\alpha/2} = z_{0.025} = 1.96$$

Therefore, a 95% confidence interval for the difference of the population means is:

$$(\overline{x}_1 - \overline{x}_2) - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$(87.2 - 78.3) \pm 1.96\sqrt{\frac{5.62}{50} + \frac{6.32}{50}} = 8.9 \pm 2.34$$

$$\Rightarrow 6.56 < \mu_1 - \mu_2 < 11.24$$

Problem 8: [1, Exercise 9.43]
A taxi company is trying to decide whether to purchase brand A or brand B tires for its fleet of taxis. To estimate the difference in the two brands, an experiment is conducted using 12 of each brand. The tires are run until they wear out. The results are:
Brand A: $\overline{x}_1 = 36{,}300$ kilometers, $s_1 = 5000$ kilometers.
Brand B: $\overline{x}_2 = 38{,}100$ kilometers, $s_2 = 6100$ kilometers.
Compute a 95% confidence interval for $\mu_A - \mu_B$ assuming the populations to be approximately normally distributed. You may not assume that the variances are equal.
Solution:

$$n_1 = n_2 = n_A = n_B = 12$$

$$\overline{x}_1 = \overline{x}_A = 36,300, \overline{x}_2 = \overline{x}_B = 38,100$$

$$s_1 = s_A = 5,000, s_2 = s_B = 6,100$$

$$\upsilon = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{[(s_1^2/n_1)^2/(n_1 - 1)] + [(s_2^2/n_2)^2/(n_2 - 1)]}$$

$$= \frac{5000^2/12 + 6100^2/12}{[(5000^2/12)^2)(12 - 1)] + [(6100^2/12)^2)(12 - 1)]} = 21$$

$$\alpha = 0.05 \Rightarrow t_{\alpha/2} = t_{0.025} = 2.080\, with\, \upsilon =\ 21\, (degrees\, of\, freedom)$$

Therefore:

$$(\overline{x}_1 - \overline{x}_2) - t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

$$(36,300 - 38,100) \pm (2.080)\sqrt{\frac{5000^2}{12} + \frac{6100^2}{12}} = -1,800 \pm 4,736$$

$$\Rightarrow -6,536 < \mu_1 - \mu_2 < 2,936$$

Problem 9: [1, Exercise 9.48]
An automotive company is considering two types of batteries for its automobile. Sample information on battery life is collected for 20 batteries of type A and 20 batteries

of type B. The summary statistics are $\overline{x}_A = 32.91$, $\overline{x}_B = 30.47$, $s_A = 1.57$, and $s_B = 1.74$. Assume the data on each battery are normally distributed and assume $\sigma_A = \sigma_B$.
(a) Find a 95% confidence interval on $\mu_A - \mu_B$.
(b) Draw a conclusion from (a) that provides insight into whether A or B should be adopted.
Solution:

$$n_1 = n_2 = n_A = n_B = 20$$

$$\overline{x}_1 = \overline{x}_A = 32.91, \overline{x}_2 = \overline{x}_B = 30.47$$

$$s_1 = s_A = 1.57, s_2 = s_B = 1.74$$

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-2)s_2^2}{n_1+n_2-2}} = 1.657$$

$$\upsilon = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{[(s_1^2/n_1)^2/(n_1-1)] + [(s_2^2/n_2)^2/(n_2-1)]} = 38$$

(a)

$$\alpha = 0.05 \Rightarrow t_{\alpha/2} = t_{0.025} \approx 2.042 \, with \, \upsilon = 38 \, (degrees \, of \, freedom)$$

Therefore:

$$(\overline{x}_1 - \overline{x}_2) - t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$(32.91 - 30.47) \pm (2.042)(1.657)\sqrt{\frac{1}{20} + \frac{1}{20}} = 2.44 \pm 1.07$$

$$\Rightarrow 1.37 < \mu_A - \mu_B < 3.51$$

(b) Since it is apparent that type A battery has longer life, it should be adopted.

Problem 10: [1, Exercise 9.50]
Two levels (low and high) of insulin doses are given to two groups of diabetic rats to check the insulin-binding capacity, yielding the following data:
Low dose: $n_1 = 8$, $\overline{x}_1 = 1.98$, $s_1 = 0.51$
High dose: $n_2 = 13$, $\overline{x}_2 = 1.30$, $s_2 = 0.35$
Assume that the variances are equal. Give a 95% confidence interval for the difference in the true average insulin-binding capacity between the two samples.
Solution:

$$n_1 = 8, n_2 = 13$$

$$\overline{x}_1 = 1.98, \overline{x}_2 = 1.30$$

$$s_1 = 0.51, s_2 = 0.35$$

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-2)s_2^2}{n_1+n_2-2}} = 0.416$$

$$\upsilon = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{[(s_1^2/n_1)^2/(n_1-1)] + [(s_2^2/n_2)^2/(n_2-1)]} = 19$$

$$\alpha = 0.05 \Rightarrow t_{\alpha/2} = t_{0.025} = 2.093 \, with \, \upsilon = \; 19 \, (degrees \, of \, freedom)$$

Therefore:

$$(\overline{x}_1 - \overline{x}_2) - t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\overline{x}_1 - \overline{x}_2) + t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$(1.98 - 1.30) \pm (2.093)(0.416)\sqrt{\frac{1}{8} + \frac{1}{13}} = 0.68 \pm 0.39$$

$$\Rightarrow 0.29 < \mu_1 - \mu_2 < 1.07$$

References

[1] Walpole, R. E., Myers, R. H., Myers, S. L. and Ye, K., *Probability & Statistics for Engineers & Scientists*, 9th ed., MA, USA: Prentice-Hall, 2012.

[2] DeGroot, M. H. and Schervish, M. J., *Probability and Statistics*, 4th ed., MA, USA: Pearson Education, Inc., 2012.

[3] Murray, R. S., John, J. S. and R, A. Srinivasan, *Probability and Statistics*, 3rd ed., USA: McGraw-Hill, 2009.