Spring, 2023

Probability & Statistics

Lecture 1: Probability Theory

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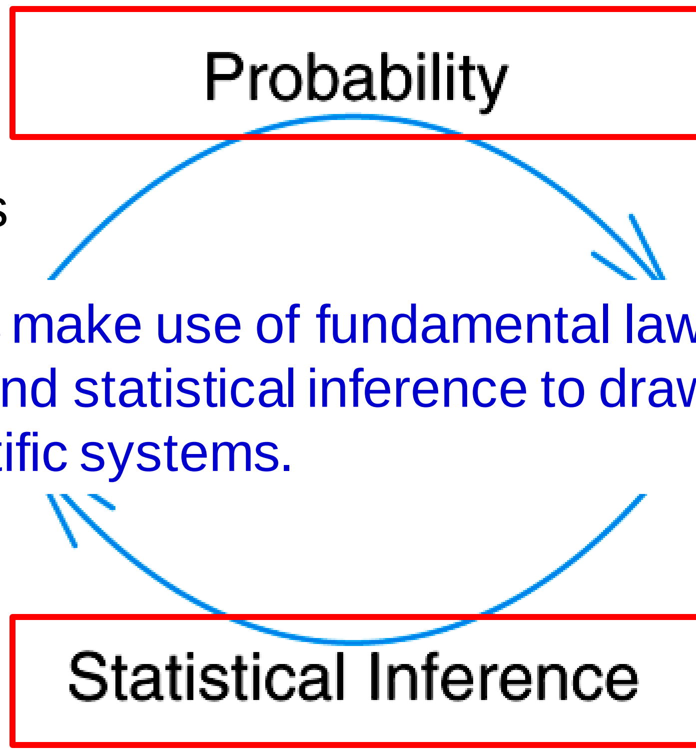
Contents

- Fundamental relationship between probability and inferential statistics

Population: all elements



Samples: Elements that have some specific criteria.

Statisticians make use of fundamental laws of probability and statistical inference to draw conclusions about scientific systems.

making scientific judgments in the face of **uncertainty and variation**

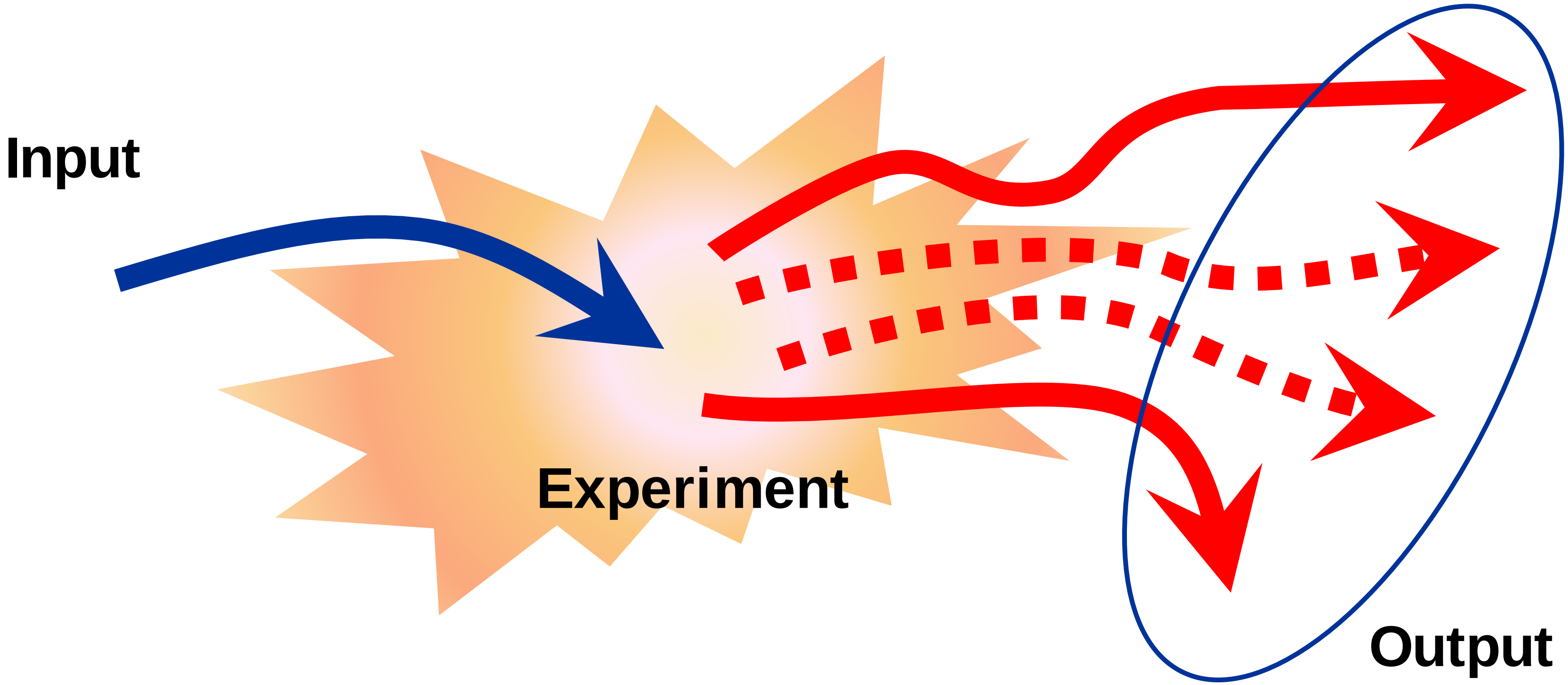
INTRODUCTION TO PROBABILITY

1. Introduction
2. Sample Spaces
3. Probability Values
4. Problems

Introduction

- Probability theory is a branch of mathematics that has been developed to deal with **uncertainty.**
- Today, probability theory is recognized as one of the most interesting and also one of the most useful areas of mathematics. It provides the basis for the science of statistical inference through experimentation and data analysis—an area of crucial importance in an increasingly quantitative world.
- Through its applications to problems such as the assessment of system reliability, the interpretation of measurement accuracy, and the maintenance of suitable quality controls, probability theory is particularly relevant to the engineering sciences today.

Sample Spaces



- An experiment can be thought of as any process or procedure for which more than one outcome is possible.

Sample Spaces (contd.)

- The goal of probability theory is to provide a mathematical structure for understanding or explaining the chances or likelihoods of the various outcomes actually occurring.
- Sample Space $\mathcal{S}$ of an experiment is a set consiting of all possible experimental outcomes.

Sample Spaces (contd.)

Example 1: A manager supervises the operation of three power plants, plant X, plant Y, and plant Z. At any given time, each of the three plants can be classified as either generating electricity (1) or being idle (0). With the notation (0, 1, 0) used to represent the situation where plant Y is generating electricity but plants X and Z are both idle, the sample space for the status of the three plants at a particular point in time is:

Venn diagram

	S
(0, 0, 0)	(1, 0, 0)
(0, 0, 1)	(1, 0, 1)
(0, 1, 0)	(1, 1, 0)
(0, 1, 1)	(1, 1, 1)

This box contains 8 individual outcomes

Sample space for power plant example

Sample Spaces (contd.)

Games of chances involve the toss of a coin, roll of a dice, or the use of a pack of cards.

- The toss of a coin have sample space $\mathcal{S} = \{\text{head}, \text{tail}\}$
- A usual six-sided die have a sample space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

Example 2: Sample space for rolling two dice

$\mathcal{S}$

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

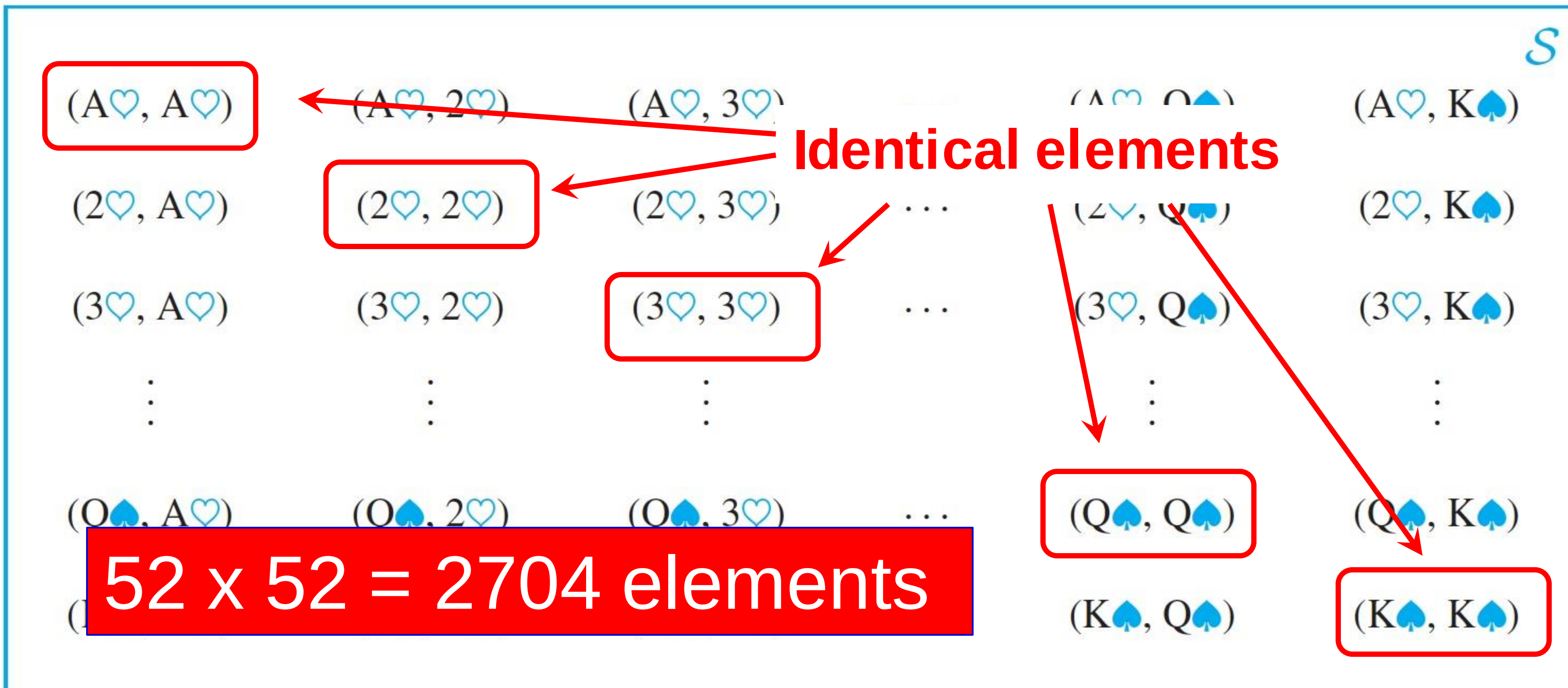
Sample Spaces (contd.)

Example 3: If a card is chosen from an ordinary pack of 52 playing cards, the sample space consists of the 52 individual cards is

$\mathcal{S}$

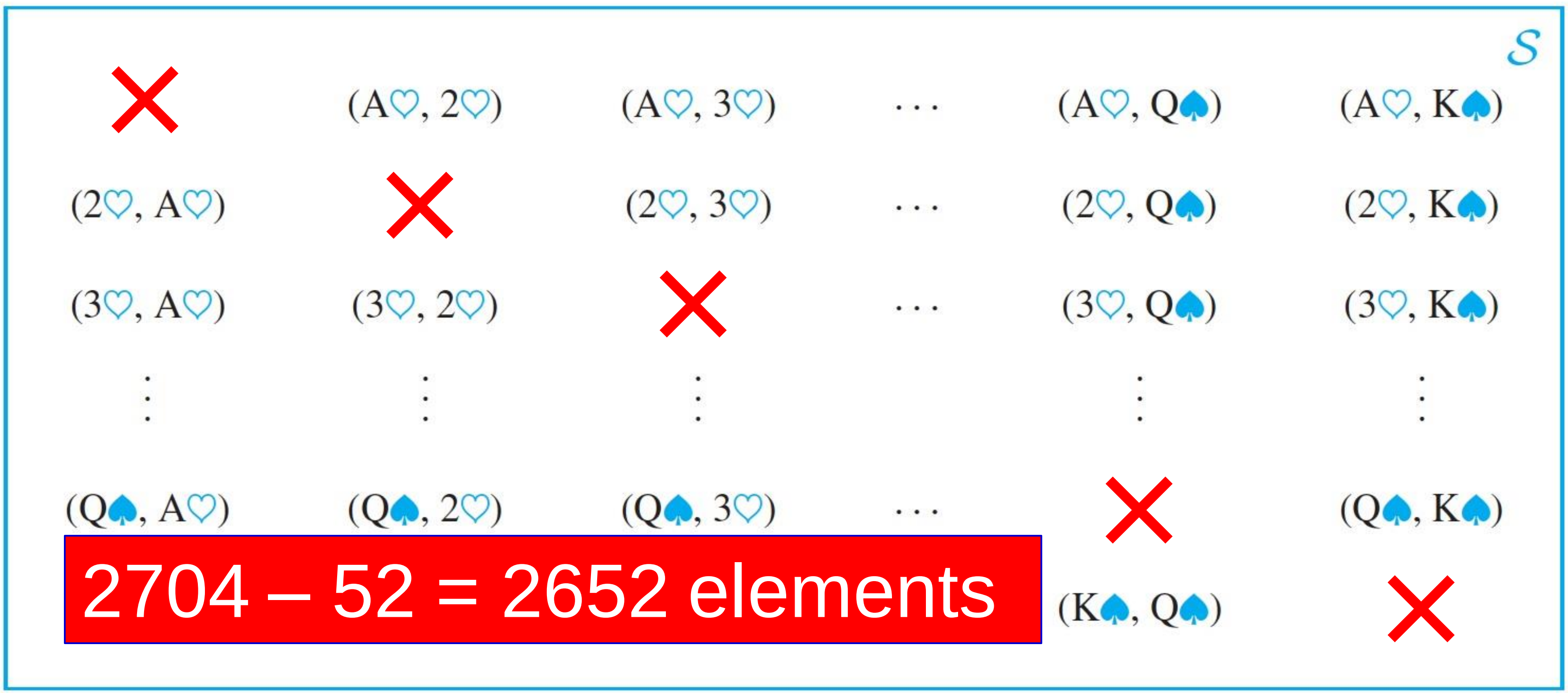
A♡	2♡	3♡	4♡	5♡	6♡	7♡	8♡	9♡	10♡	J♡	Q♡	K♡
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A◇	2◇	3◇	4◇	5◇	6◇	7◇	8◇	9◇	10◇	J◇	Q◇	K◇
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

Example 4: If two cards are drawn, then it is necessary to consider whether they are drawn. If the drawing is performed ***with replacement***, so that the initial card drawn is returned to the pack and the second drawing is from a full pack of 52 cards



Sample Spaces (contd.)

If two cards are drawn ***without replacement***, so that the second card is drawn from a reduced pack of 51 cards, then the sample space will be a subset of that above



Probability Values

Probabilities: A set of **probability values** for an experiment with a sample space $\mathcal{S} = \{O_1, O_2, \ldots, O_n\}$ consists of some probabilities $p_1, p2, \ldots, p_n$ such that

$$0 \leq p_1 \leq 1, 0 \leq p2 \leq 1, \ldots, 0 \leq p_n \leq 1$$

and
$$p_1 + p2 + \ldots + p_n = 1$$

The probability of outcome O_i occuring is said to be p_i, and this is written as $P(O_i) = p_i$.

- If two outcomes has identical probability values assigned to them, then they can be thought of as being equally likely to occur.
- If one outcome has a larger probability value assigned to it than another outcome, then the first outcome can be thought of as being more likely to occur.
- If a particular outcome has a probability value of one, then it is certain to occur. There is no uncertainty in the experiment.

Probability Values (contd.)

Example 5: Suppose that the machine breakdowns occur with probability values of P(electrical) = 0.2, P(mechanical) = 0.5, and P(misuse) = 0.3. This is a valid probability assignment since

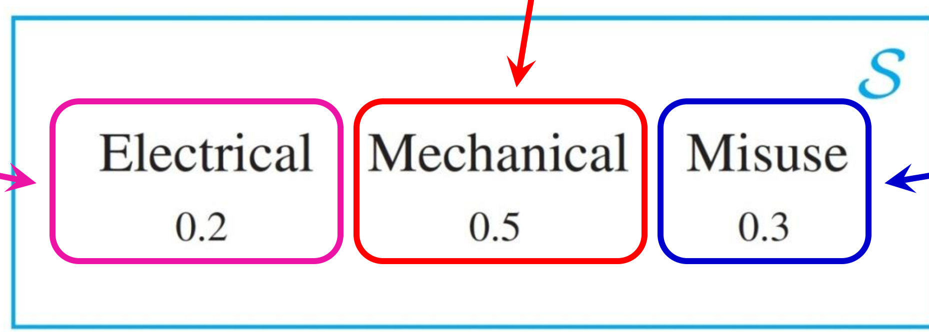
$$0.2, 0.3, 0.5 \in [0, 1]; \;\; 0.2 + 0.3 + 0.5 = 1$$

This does not mean that of the next two machine breakdowns, exactly one will be for mechanical reasons, or that in the next ten machine breakdowns, exactly five will be for mechanical reasons.

It mean that, in the long run

50% of the breakdowns

20% of the breakdowns



30% of the breakdowns

Probability Values (contd.)

For a coin toss, the probabilities will be given by

$$P(head) = p,\ P(tail) = 1 - p$$

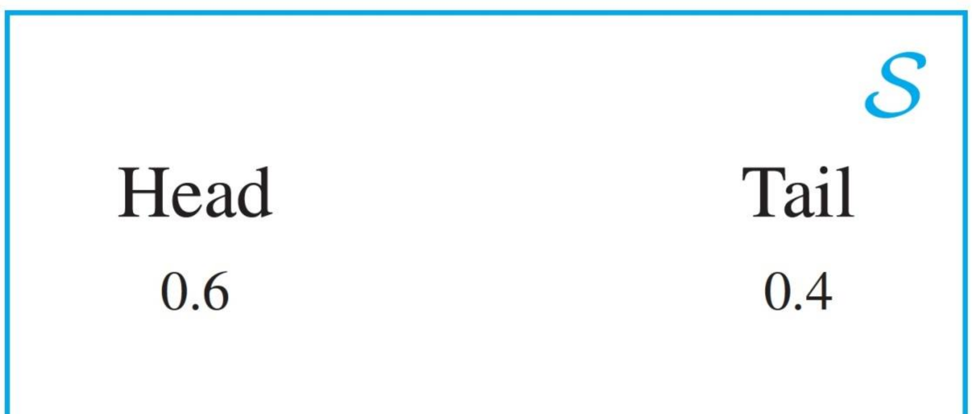
for some value of p with $0 \leq p \leq 1$. A fair coin will have $p = 0.5$ so that

$$P(head) = 0.5,\ P(tail) = 0.5$$

with the two outcomes being equally likely. A biased coin will have $p \neq 0.5$.

Example 6: if $p = 0.6$ then $P(head) = 0.6,\ P(tail) = 0.4$

The coin toss is more likely to record a head.



Probability values for a biased coin

Probability Values (contd.)

A fair die will have each of six outcomes equally likely, with each being assigned the same probability.

						S
1	2	3	4	5	6	
1/6	1/6	1/6	1/6	1/6	1/6	

A biased die will have different probability for each side

						S
1	2	3	4	5	6	
0.10	0.15	0.15	0.15	0.15	0.30	

If two die are thrown and each of the 36 outcomes are equally likely (as will be the case with two fair dice that are shaken properly), the probability value of each outcome will necessarily be 1/36.

$\mathcal{S}$

(1, 1) 1/36	(1, 2) 1/36	(1, 3) 1/36	(1, 4) 1/36	(1, 5) 1/36	(1, 6) 1/36
(2, 1) 1/36	(2, 2) 1/36	(2, 3) 1/36	(2, 4) 1/36	(2, 5) 1/36	(2, 6) 1/36
(3, 1) 1/36	(3, 2) 1/36	(3, 3) 1/36	(3, 4) 1/36	(3, 5) 1/36	(3, 6) 1/36
(4, 1) 1/36	(4, 2) 1/36	(4, 3) 1/36	(4, 4) 1/36	(4, 5) 1/36	(4, 6) 1/36
(5, 1) 1/36	(5, 2) 1/36	(5, 3) 1/36	(5, 4) 1/36	(5, 5) 1/36	(5, 6) 1/36
(6, 1) 1/36	(6, 2) 1/36	(6, 3) 1/36	(6, 4) 1/36	(6, 5) 1/36	(6, 6) 1/36

Probability Values (contd.)

If a card is drawn at random from a pack of cards, then there are 52 possible outcomes in the sample space, and each one is equally likely so that each would be assigned a probability value of 1/52.

$\mathcal{S}$

A♡	2♡	3♡	4♡	5♡	6♡	7♡	8♡	9♡	10♡	J♡	Q♡	K♡
1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52
A♢	2♢	3♢	4♢	5♢	6♢	7♢	8♢	9♢	10♢	J♢	Q♢	K♢
1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52

Probability Values (contd.)

If two cards are drawn with replacement, and if both the cards can be assumed to be chosen at random through suitable shuffling of the pack before and between the drawings, then each of the 52×52 = 2704 elements of the sample space will be equally likely and hence should each be assigned a probability value of 1/2704.

S

(A♡, A♡) 1/2704	(A♡, 2♡) 1/2704	(A♡, 3♡) 1/2704	· · · · · ·	(A♡, Q♠) 1/2704	(A♡, K♠) 1/2704
(2♡, A♡) 1/2704	(2♡, 2♡) 1/2704	(2♡, 3♡) 1/2704	· · · · · ·	(2♡, Q♠) 1/2704	(2♡, K♠) 1/2704
(3♡, A♡) 1/2704	(3♡, 2♡) 1/2704	(3♡, 3♡) 1/2704	· · · · · ·	(3♡, Q♠) 1/2704	(3♡, K♠) 1/2704
⋮	⋮	⋮		⋮	⋮
(Q♠, A♡) 1/2704	(Q♠, 2♡) 1/2704	(Q♠, 3♡) 1/2704	· · · · · ·	(Q♠, Q♠) 1/2704	(Q♠, K♠) 1/2704
(K♠, A♡) 1/2704	(K♠, 2♡) 1/2704	(K♠, 3♡) 1/2704	· · · · · ·	(K♠, Q♠) 1/2704	(K♠, K♠) 1/2704

Probability Values (contd.)

If the drawing is performed without replacement but again at random, then the sample space has only 2652 elements and each would have a probability of 1/2652.

$\mathcal{S}$

	(A♡, 2♡) 1/2652	(A♡, 3♡) 1/2652	⋯ ⋯	(A♡, Q♠) 1/2652	(A♡, K♠) 1/2652
(2♡, A♡) 1/2652		(2♡, 3♡) 1/2652	⋯ ⋯	(2♡, Q♠) 1/2652	(2♡, K♠) 1/2652
(3♡, A♡) 1/2652	(3♡, 2♡) 1/2652		⋯ ⋯	(3♡, Q♠) 1/2652	(3♡, K♠) 1/2652
⋮	⋮	⋮		⋮	⋮
(Q♠, A♡) 1/2652	(Q♠, 2♡) 1/2652	(Q♠, 3♡) 1/2652	⋯ ⋯		(Q♠, K♠) 1/2652
(K♠, A♡) 1/2652	(K♠, 2♡) 1/2652	(K♠, 3♡) 1/2652	⋯ ⋯	(K♠, Q♠) 1/2652	

Problems

- **Problem 1:** What is the sample space for the number of aces in a hand of 13 playing cards?
- Solution: $\mathcal{S} = \{0, 1, 2, 3, 4\}$
- **Problem 2:** A bag contains balls that are either red or blue and either dull or shiny. What is the sample space when a ball is chosen from the bag?
- Solution:

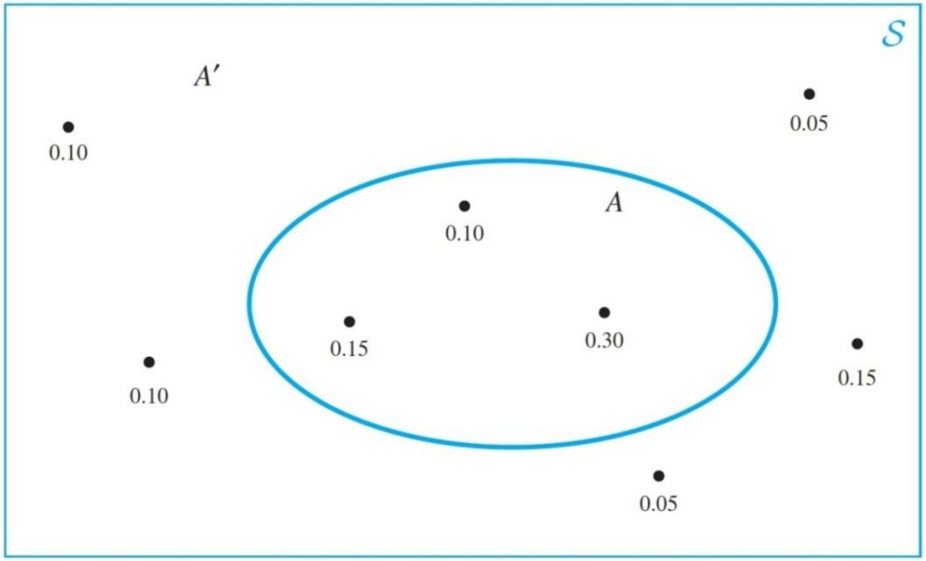
$$\mathcal{S} = \{(\text{red,shiny}), (\text{red,dull}), (\text{blue,shiny}), (\text{blue,dull})\}$$

EVENTS

1. Events and Complements
2. Examples of Events
3. Problems

Events and Complements

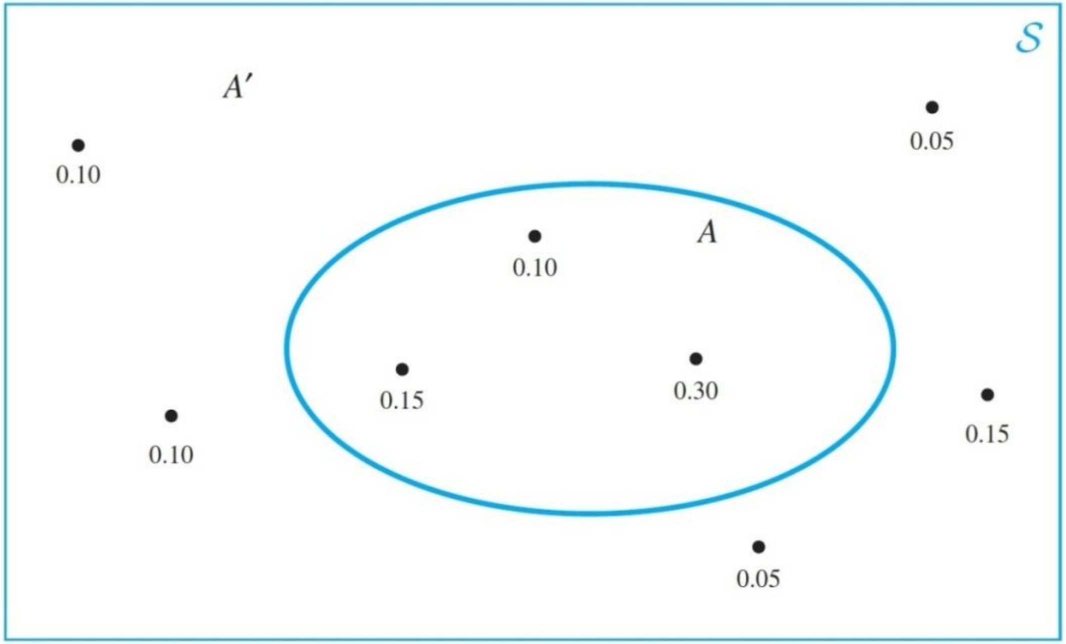
- **Events**: An event A is a subset of the sample space $\mathcal{S}$. It collects outcomes of particular interest. The probability of an event A, $P(A)$, is obtained by summing the probabilities of the outcomes contained within the event A.



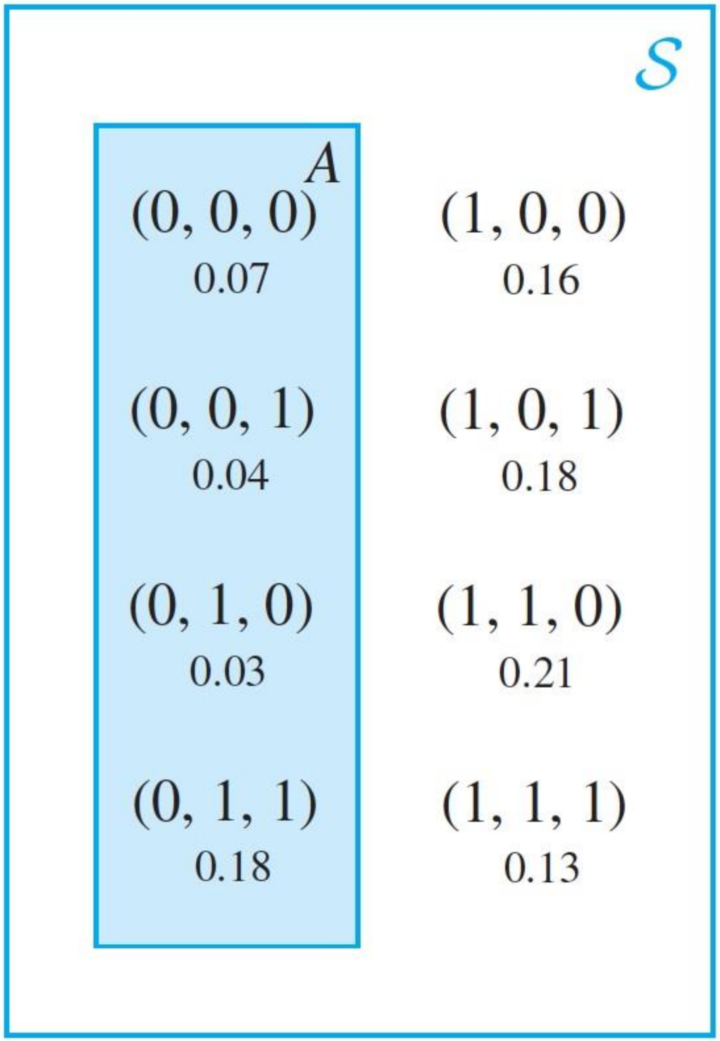
$$P(A) = 0.10 + 0.15 + 0.30 = 0.55$$

- **Complements**: The event A', the complement of an event A, is the event consisting of everything in the sample space $\mathcal{S}$ that is not contained within the event A. In all cases

$$P(A) + P(A') = 1$$



Events and Complements (contd.)



Event A: Plant X idle

Example 7: the probability that all three plants are idle is

$$P((0,0,0)) = 0.07$$

and the probability that only plant X is idle is

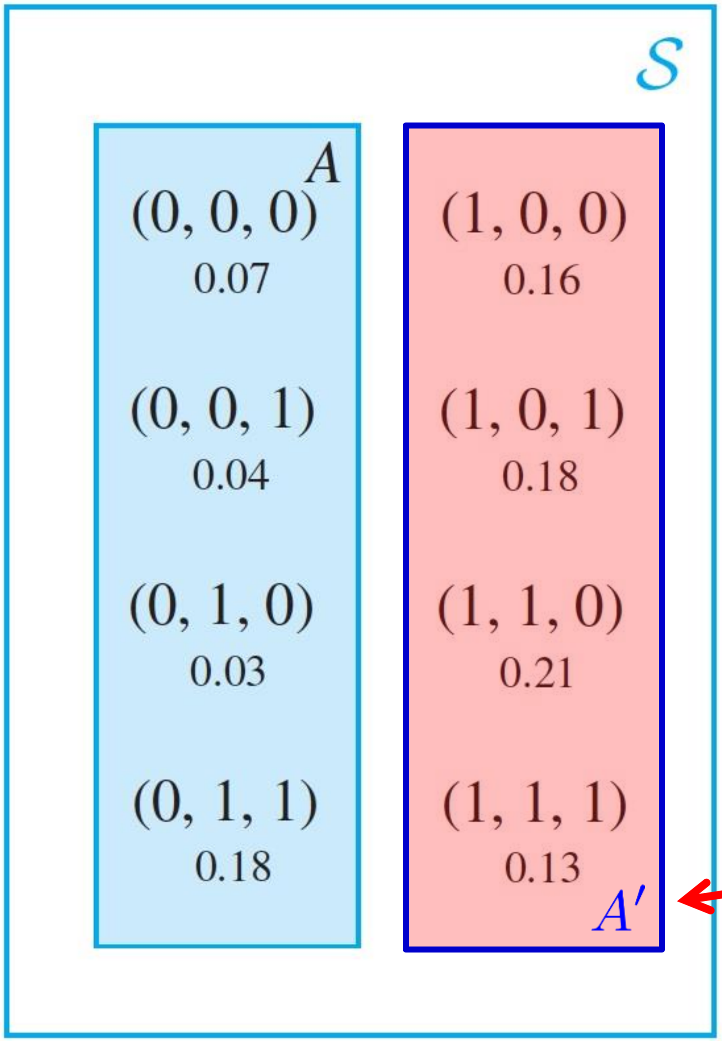
$$P((0,1,1)) = 0.18$$

The event that plant X is idle is given by

$$A = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1)\}$$

and it has a probability of

$$\begin{aligned} P(A) = &\ P((0,0,0)) + P((0,0,1)) \\ + &\ P((0,1,0)) + P((0,1,1)) \\ = &\ 0.07 + 0.04 + 0.03 + 0.18 \\ = &\ 0.32 \end{aligned}$$



Event A: Plant X idle

The complement of this event is

$$A' = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

which corresponds to plant X generating electricity, and it has a probability of

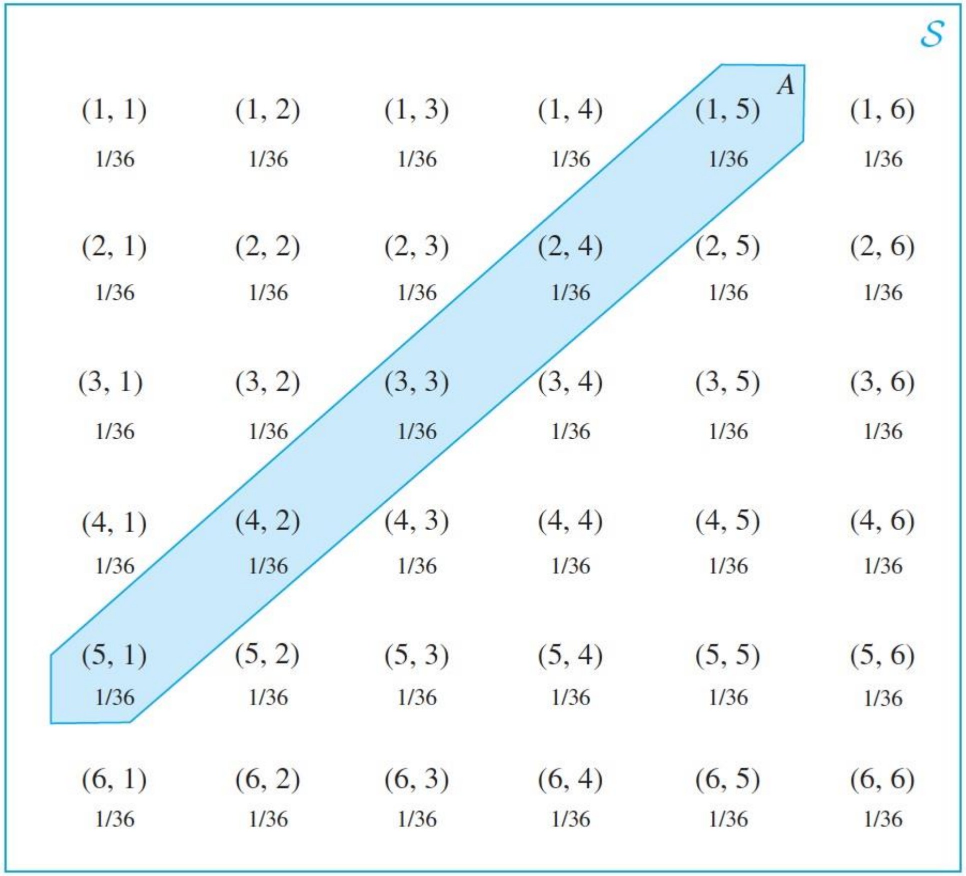
$$P(A') = 1 - P(A) = 1 - 0.32 = 0.68$$

Event A': Plant X generates electricity

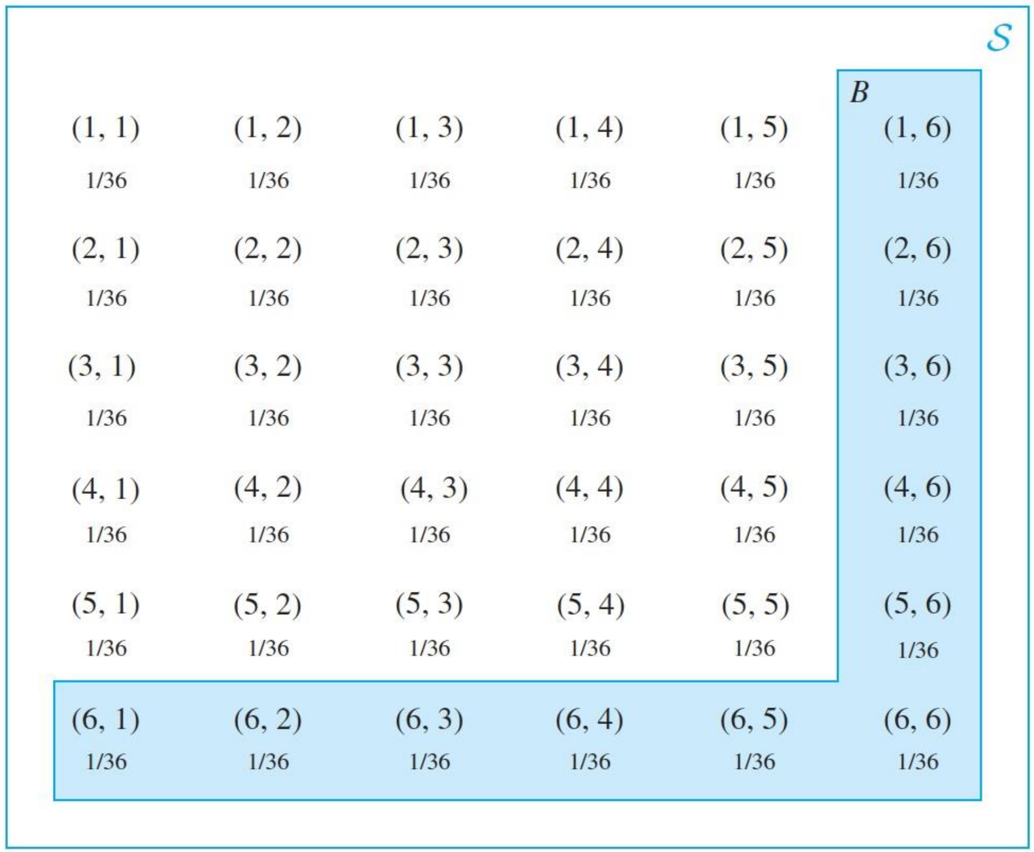
Events and Complements (contd.)

Example 8: The event that the sum of the scores of two dice is equal to 6. This event is given by

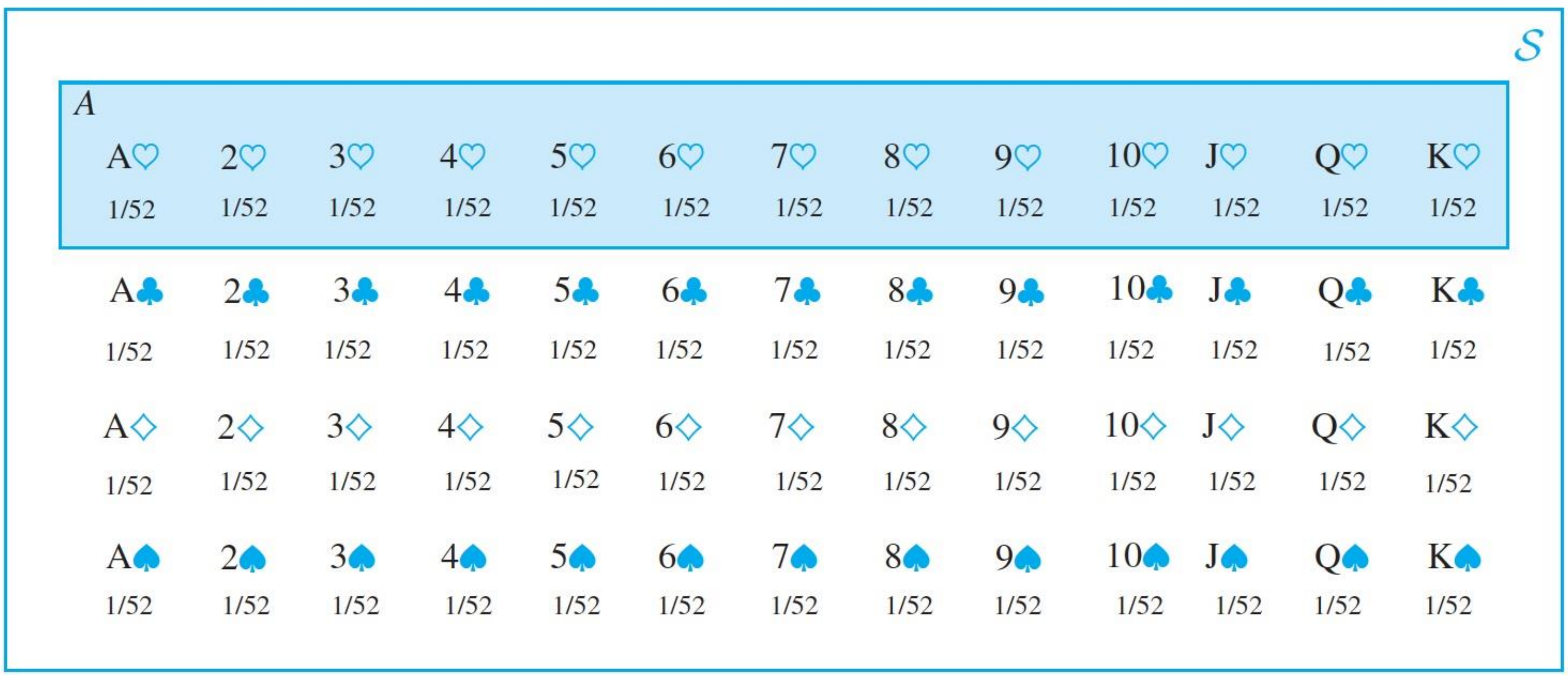
$$A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \longrightarrow P(A) = \frac{5}{36} \approx 14\%$$



Example 9: the event that at least one of the two dice records a 6, which is seen to have a probability of 11/36. The complement of this event is the event that neither die records a 6, with a probability of $1 - 11/36 = 25/36$.

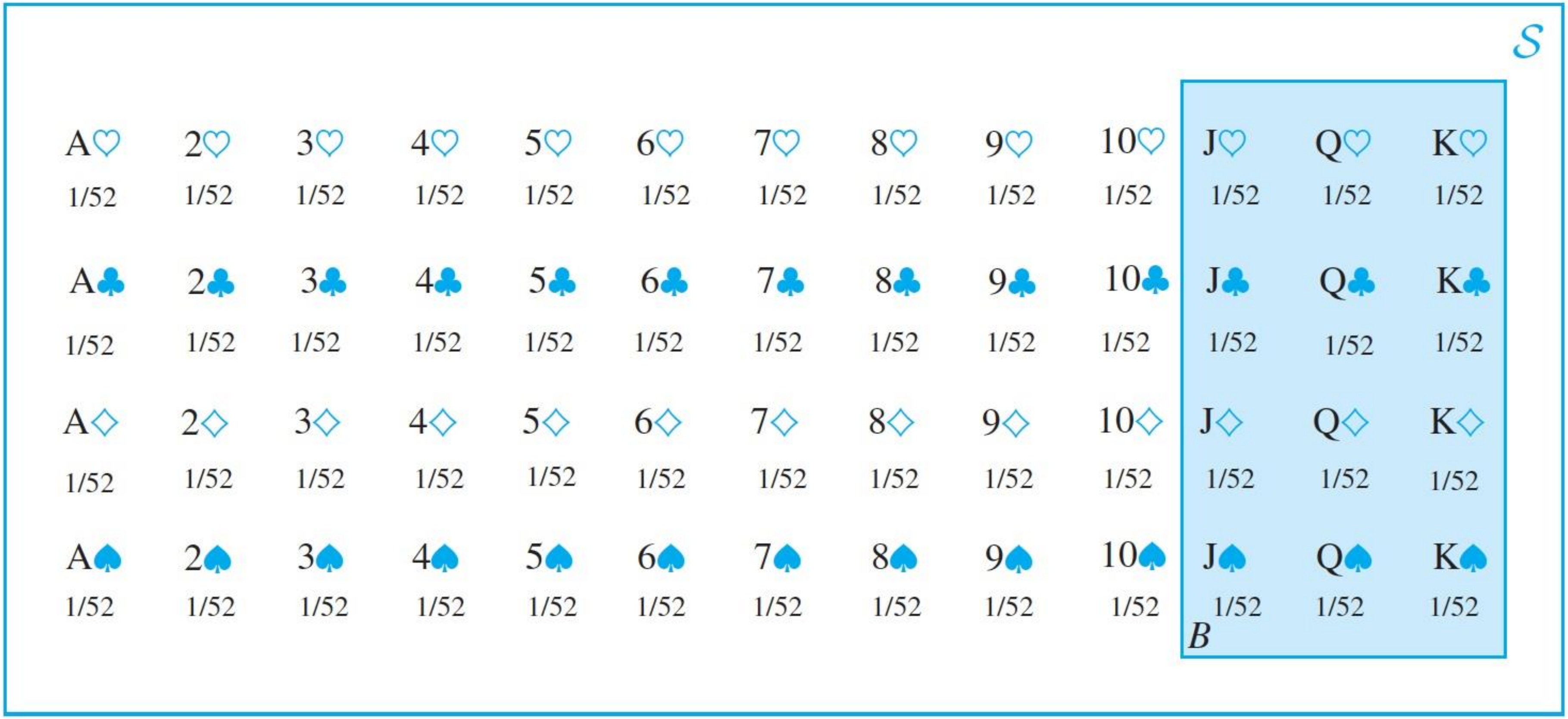


Example 10: The event that a card drawn from a pack of cards belongs to the heart suit. This event consists of the 13 outcomes corresponding to the 13 cards in the heart suit. If the drawing is done at random, with each of the 52 possible outcomes being equally likely with a probability of 1/52, then the probability of drawing a heart is clearly 13/52 = 1/4.

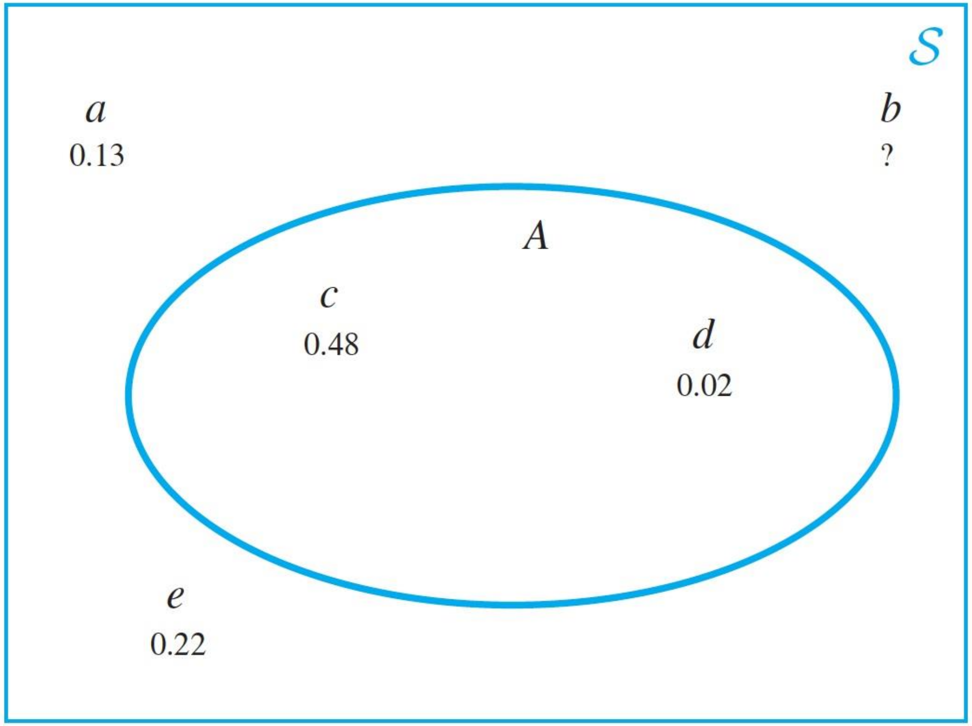


Events and Complements (contd.)

Example 11: The event that a picture card (jack, queen, or king) is drawn, with a probability of 12/52 = 3/13.



Problem 3: Consider the sample space in the following figure with outcomes *a, b, c, d, e.* Calculate $P(b), P(A), P(A')$?



Solution:

$$P(b) = 0.15$$

because

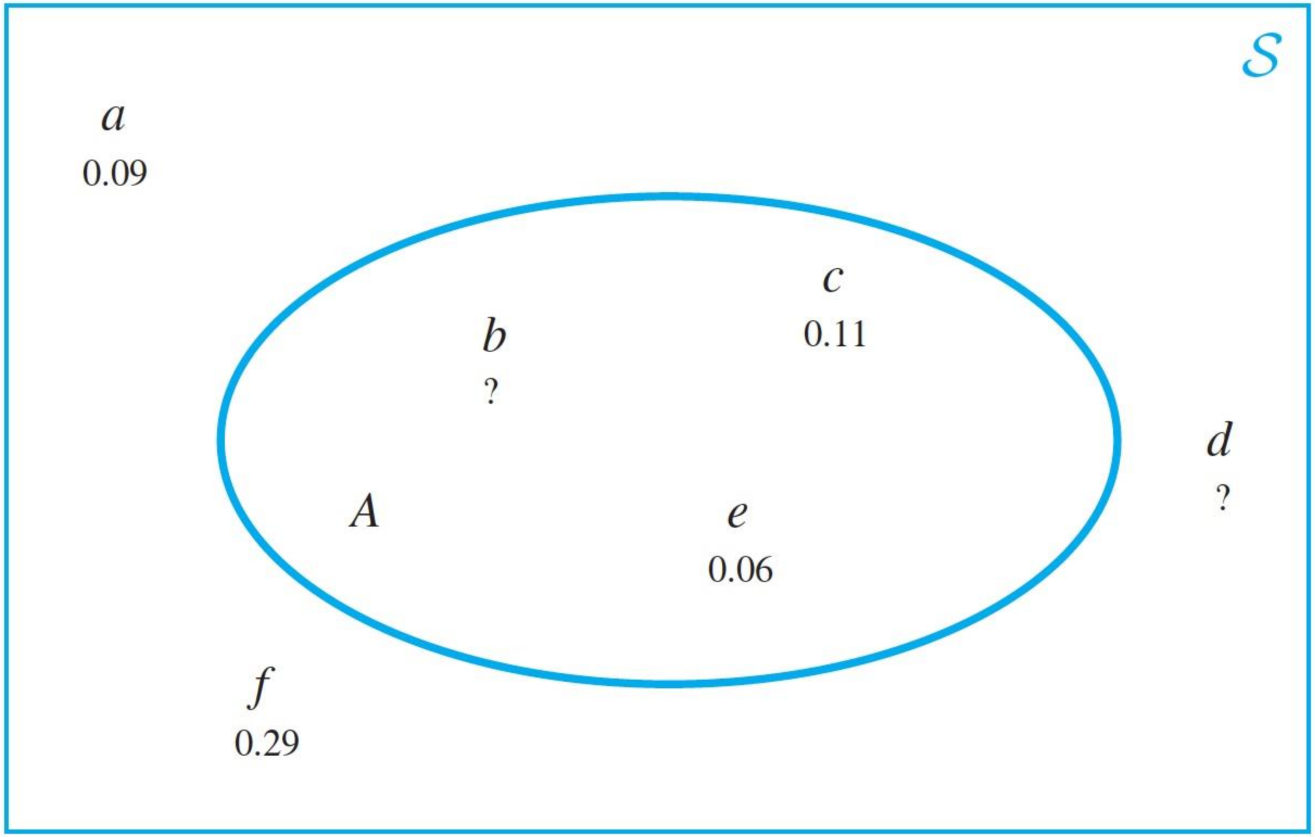
$$\sum(P(a), P(b), P(c), P(d)) = 1$$

and

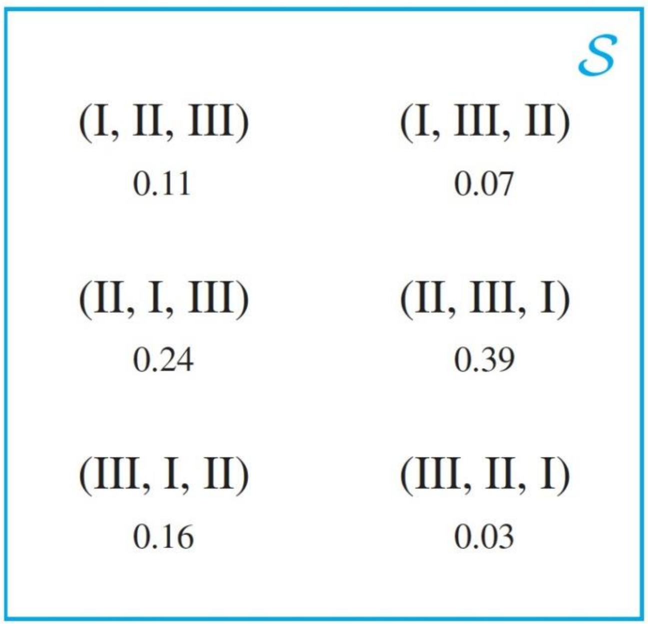
$$\begin{aligned} P(A') = & \quad P(a) + P(e) + P(b) \\ = & \quad 0.50 \end{aligned}$$

$$P(A) = 1 - P(A') = 0.50$$

Problem 4: Consider the sample space in the following figure with outcomes *a, b, c, d, e, f.* Calculate $P(b), P(d), P(A')$? If $P(A) = 0.27$.



Problem 5: Three types of batteries are being tested, type I, type II, and type III. The outcome (I, II, III) denotes that the battery of type I fails first, the battery of type II next, and the battery of type III lasts the longest. See figure.



What is the probability that
(a) the type I battery lasts longest?
(b) the type I battery lasts shortest?
(c) the type I battery lasts longer than the type II battery?

Solution:

(a) $P((\text{III,II,I})) + P((\text{II,III,I})) = 0.42$

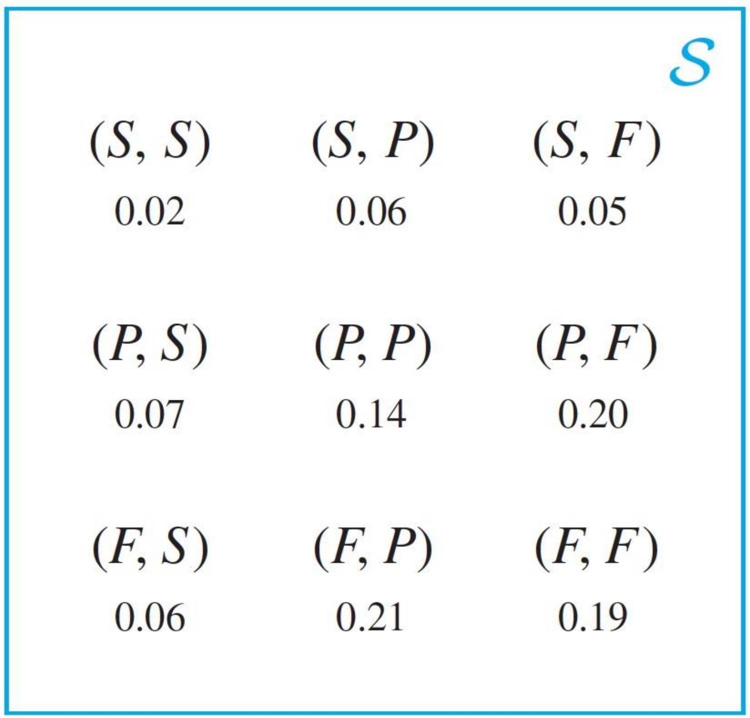
(b) $P((\text{I,II,III})) + P((\text{I,III,II})) = 0.18$

(c) $P((\text{II,I,III})) + P((\text{II,III,I})) + P((\text{III,II,I})) = 0.66$

Problems

Problem 6: A factory has two assembly lines, each of which is shut down (S), at partial capacity (P), or at full capacity (F).

The sample space is given in the figure, where, for example, (S, P) denotes that the first assembly line is shut down and the second one is operating at partial capacity.



What is the probability that

(a) both assembly lines are shut down?

(b) neither assembly line is shut down?

Solution:

(a) P(both assembly lines are shut down) = 0.02.

(b) P(neither assembly line is shut down) = 0.74.

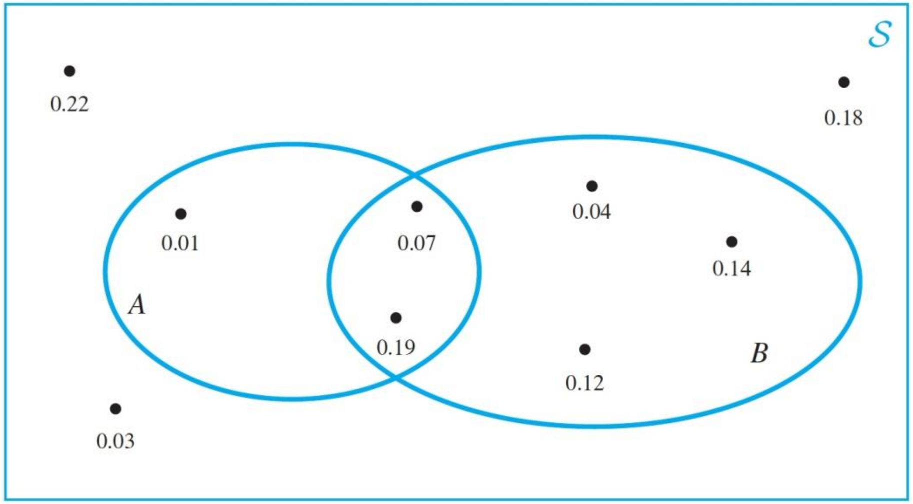
COMBINATIONS OF EVENTS

1. Intersections of Events
2. Unions of Events
3. Examples of Intersections and Unions
4. Combinations of Three or More Events
5. Problems

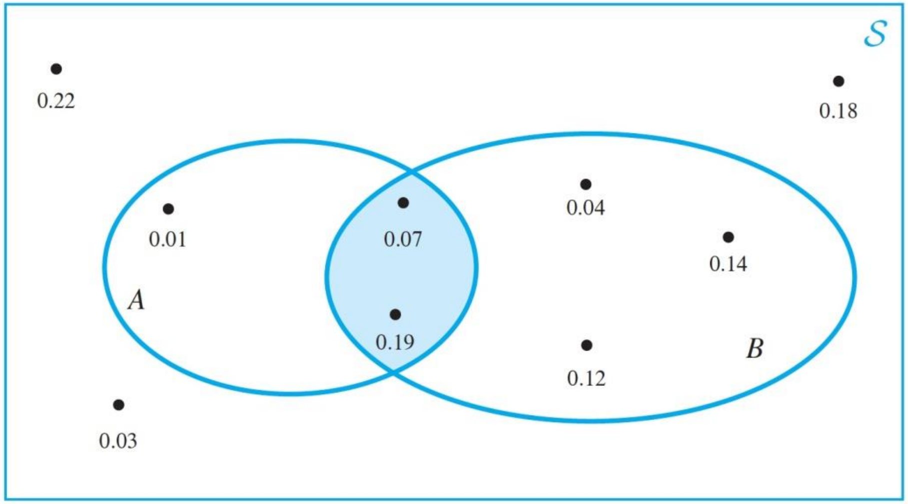
Intersections of Events

The event $A \cap B$ is the **intersection of the events** A and B and consists of the outcomes that are contained within both events A and B. The probability of this event, $P(A \cap B)$, is the probability that both events A and B occur simultaneously.

Events A and B



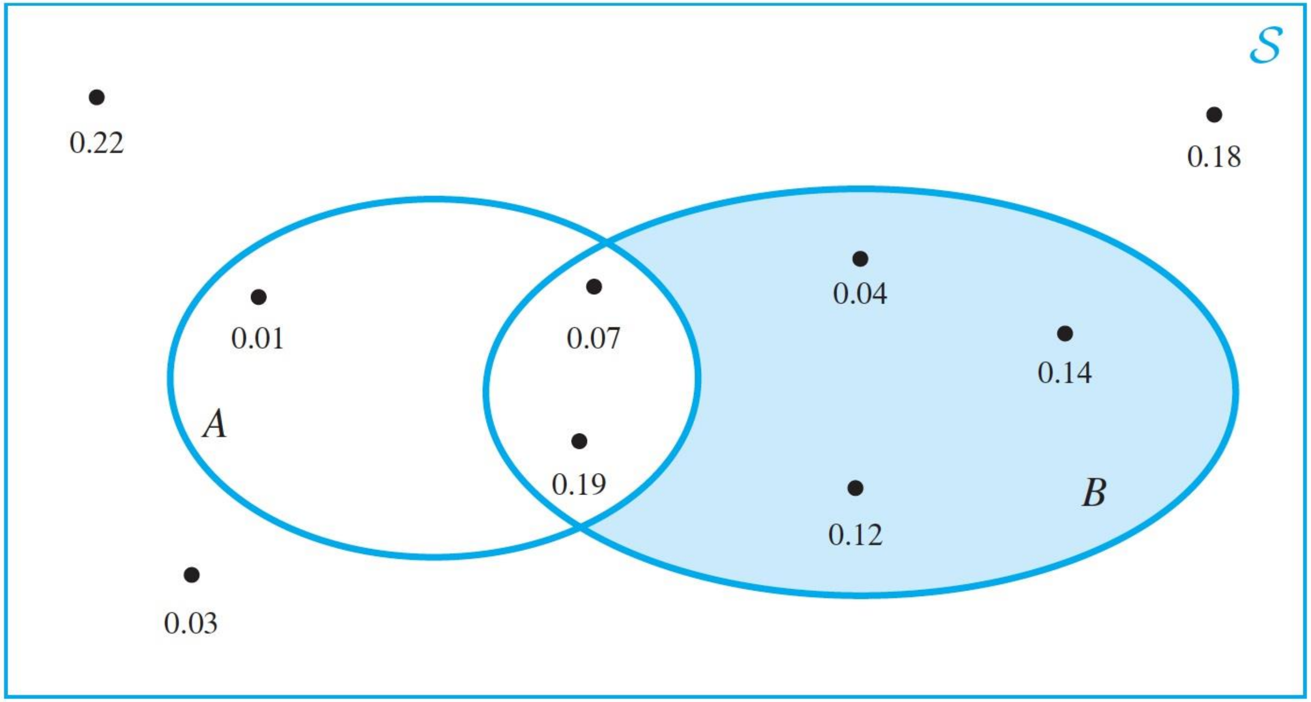
The event A ∩ B



$$P(A \cap B) = 0.26$$

Intersections of Events (contd.)

The event $A' \cap B$ illustrated in the figure. This event consists of the three outcomes that are contained within event B but that are not contained within event A. It has a probability of $P(A' \cap B) = 0.30$

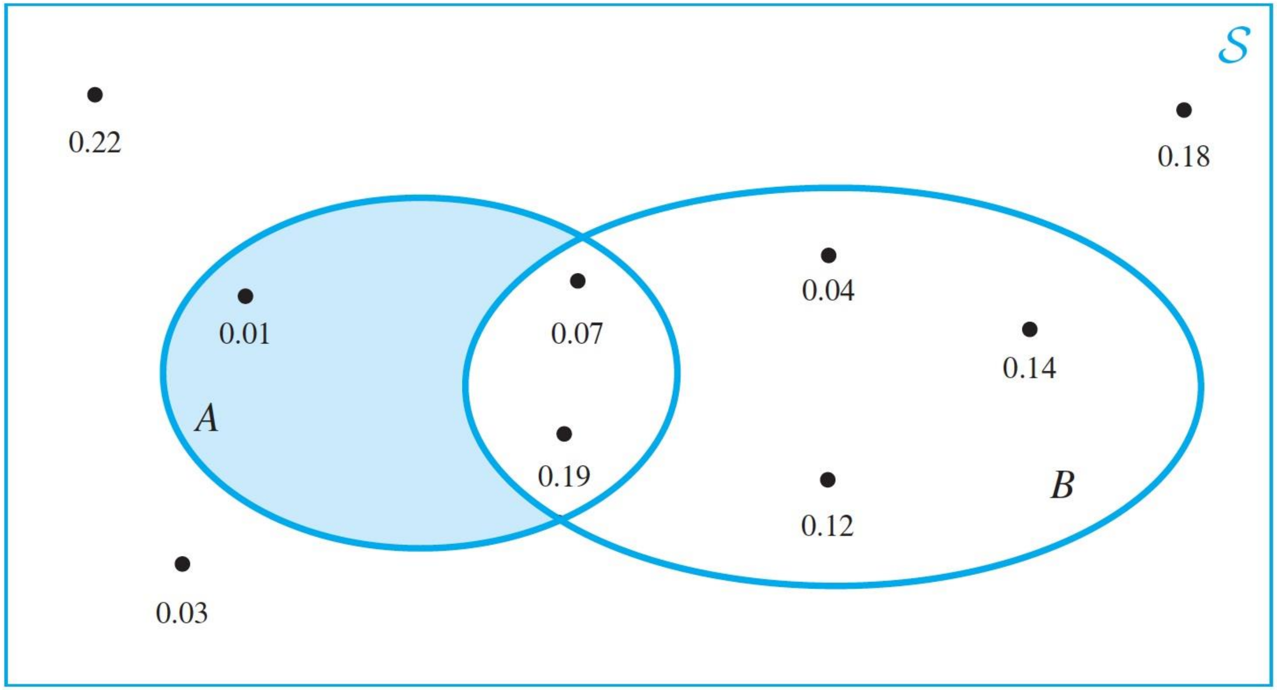


The event $A' \cap B$

The event $A \cap B'$, which has a probability of

$$P(A \cap B') = 0.01$$

This is the probability that event A occurs but event B does not

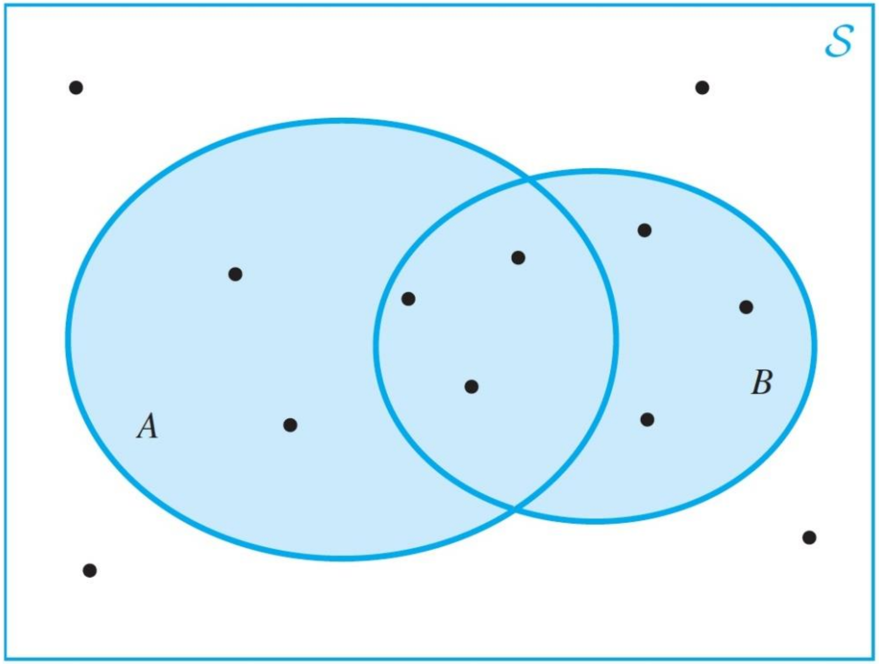


The event $A \cap B'$

Unions of Events

The event $A \cup B$ is the **union of events** A and B and consists of the outcomes that are contained within at least one of the events A and B. The probability of this event,

$P(A \cup B)$, is the probability that at least one of the events A and B occurs.



The event $P(A \cup B)$

Notice that the outcomes in the event $A \cup B$ can be classified into three kinds. They are

1. in event A, *but not in event B*
2. in event B, *but not in event A*
3. in both events *A and B*

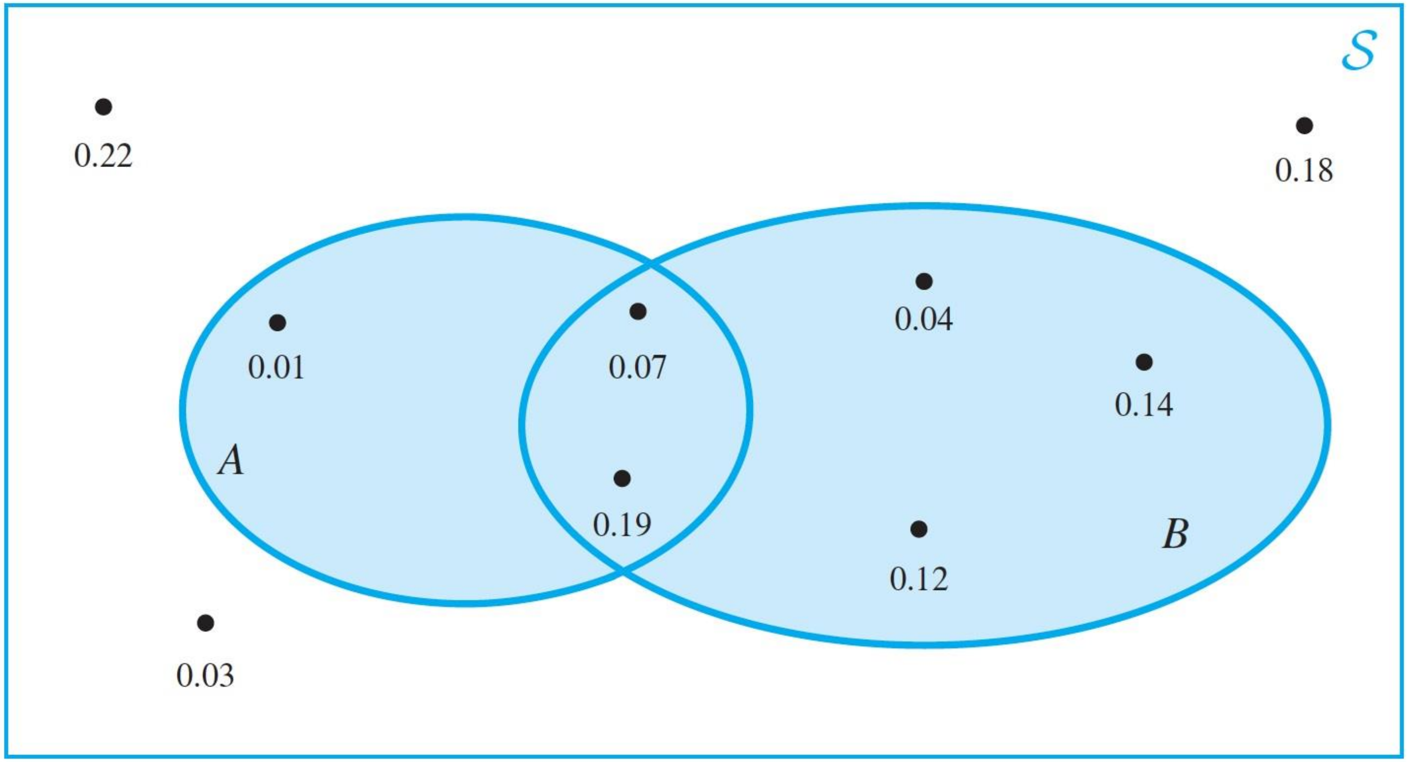
$$
\begin{aligned}
P(A \cup B) = &\quad P(A' \cap B) + P(A \cap B') \\
+ &\quad P(A \cap B)
\end{aligned}
$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Unions of Events (contd.)

- **Example 12:** The event $A \cup B$ consists of the six outcomes illustrated in the Figure, and it has a probability of

$$P(A \cup B) = 0.01 + 0.07 + 0.19 + 0.04 + 0.12 + 0.14 = 0.57$$

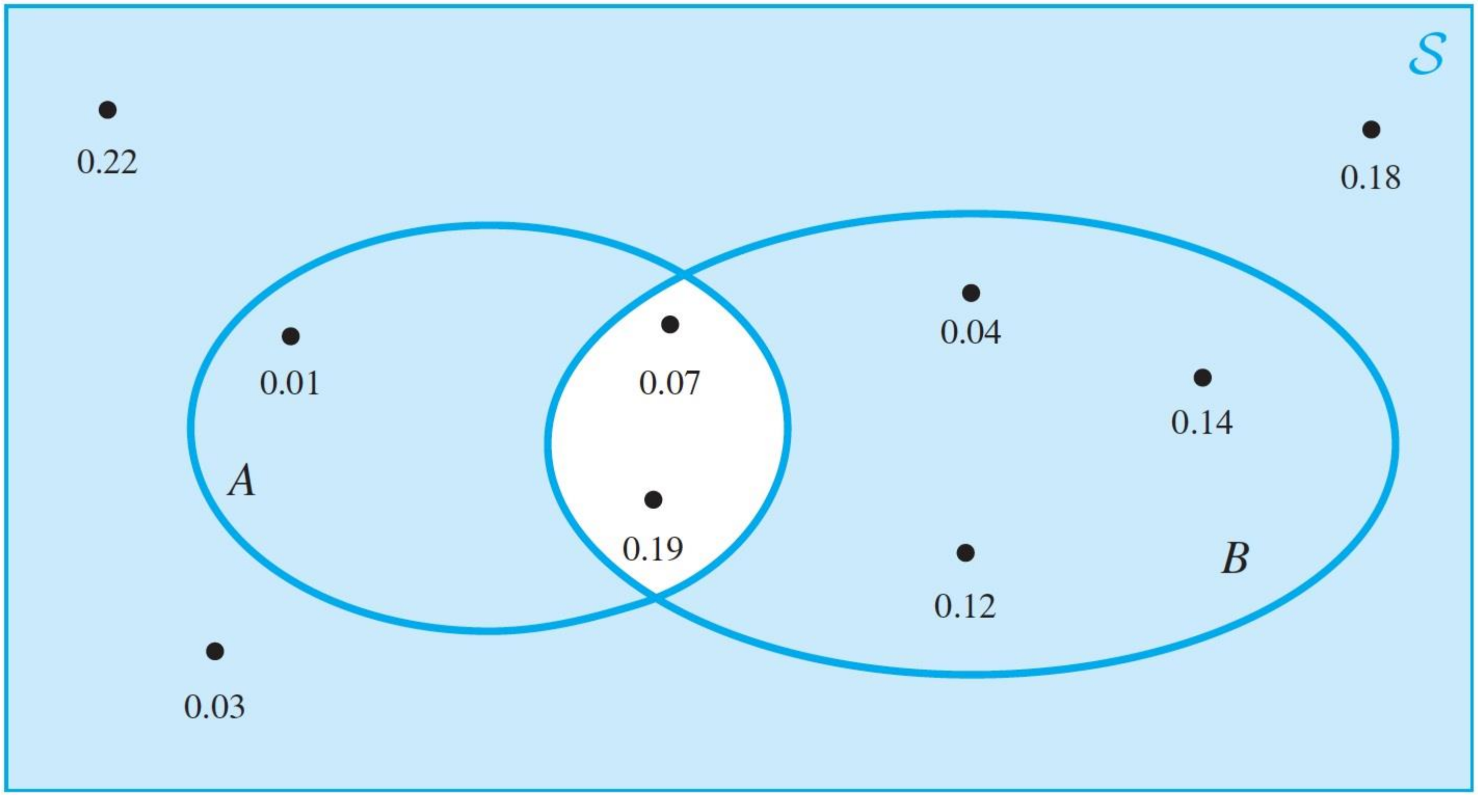


The event $P(A \cup B)$

Unions of Events (contd.)

- **Example 13:** The event $A' \cup B'$ consists of the seven outcomes illustrated in the Figure, and it has a probability of

$$P(A' \cup B') = 0.01 + 0.22 + 0.18 + 0.03 + 0.04 + 0.14 + 0.12 = 0.74$$



The event $P(A' \cup B')$

Examples of Intersections and Unions

Example 14: The quality check has two components, the first being an evaluation of the quality of the picture obtained on the television set, and the second being an evaluation of the appearance of the television set, which looks for scratches or other visible deformities on the appliance. Each of the two evaluations is graded as Perfect, Good, Satisfactory, or Fail. The 16 outcomes are illustrated in the Figure together with a set of probability values, where the notation (P, G), for example, means that an appliance has a Perfect picture and a Good appearance.

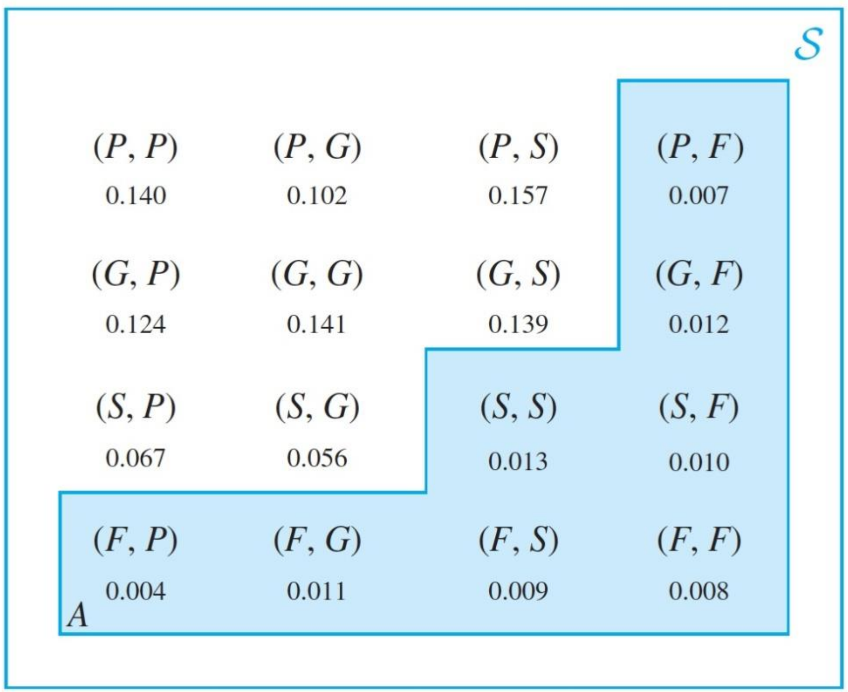
$\mathcal{S}$

(P, P) 0.140	(P, G) 0.102	(P, S) 0.157	(P, F) 0.007
(G, P) 0.124	(G, G) 0.141	(G, S) 0.139	(G, F) 0.012
(S, P) 0.067	(S, G) 0.056	(S, S) 0.013	(S, F) 0.010
(F, P) 0.004	(F, G) 0.011	(F, S) 0.009	(F, F) 0.008

Decision: An appliance that fails on either of the two evaluations will not be shipped. Furthermore, that appliances that score an evaluation of *Satisfactory* on both accounts will also not be shipped.

An initial question of interest concerns the probability that an appliance cannot be shipped. This event A, say, consists of the outcomes

$$P(A) = \{(F,P),(F,G),(F,S),(S,S),(F,F),(S,F),(G,F),(P,F)\}$$



The probability that an appliance cannot be shipped is then

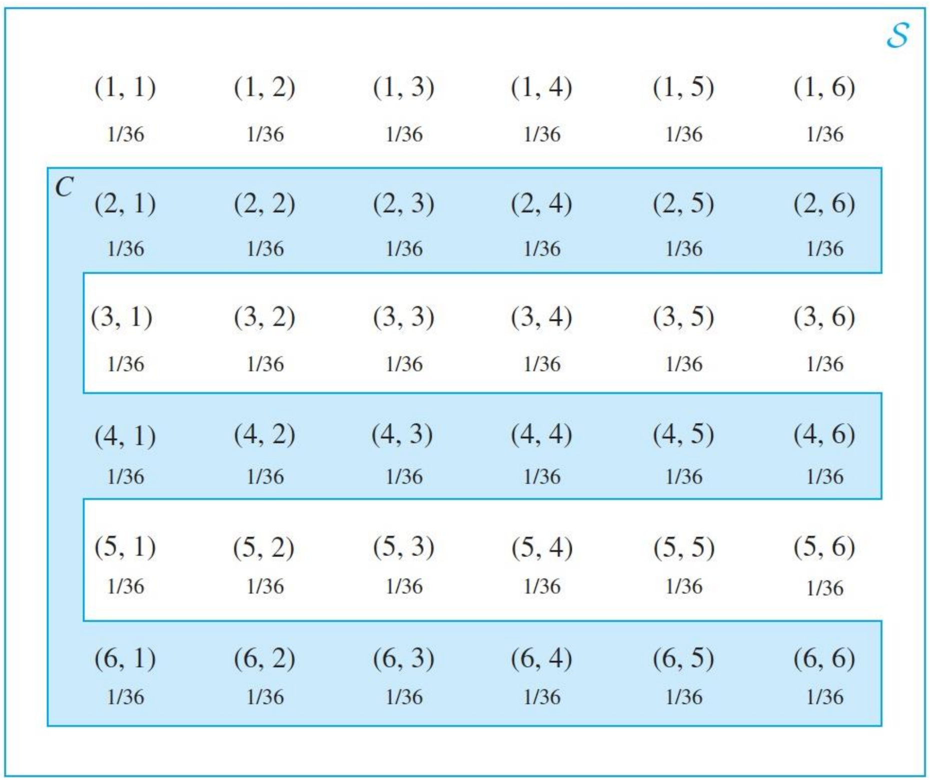
$$\begin{aligned} P(A) = \; & P((F,P)) + P((F,G)) + P((F,S)) \\ + \; & P((S,S)) + P((F,F)) + P((S,F)) \\ + \; & P((G,F)) + P((P,F)) \\ = \; & 0.074 \end{aligned}$$

In the long run about 7.4% of the television sets will fail the quality check.

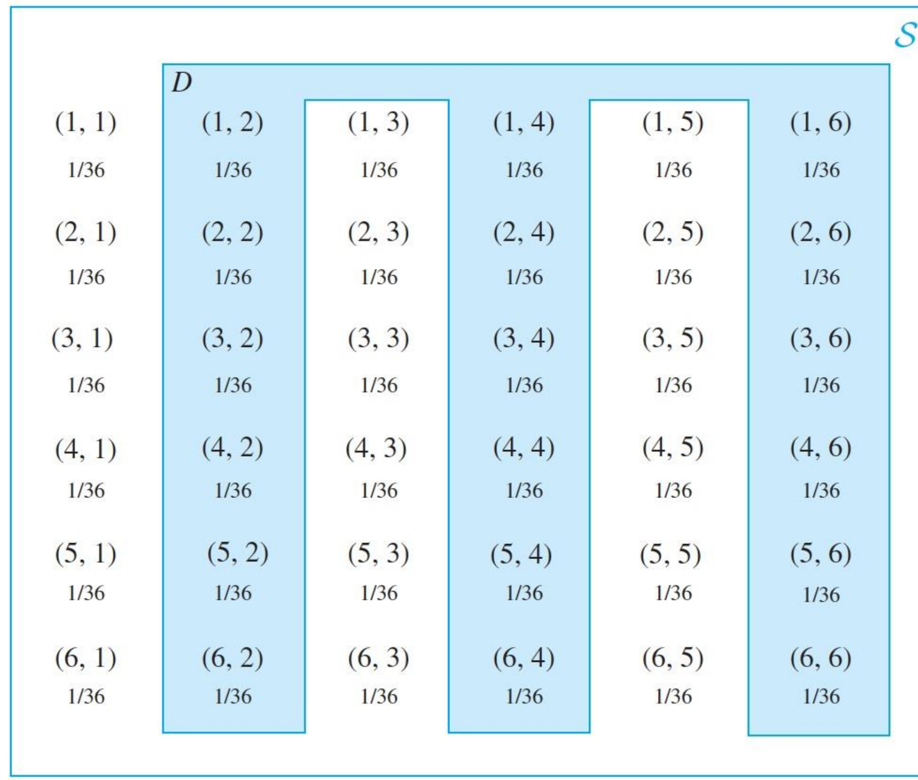
Examples of Intersections and Unions (contd.)

Example 15: If one die is red and the other is blue,

The figure illustrates the event *D, say, that an* even score is obtained on the red die

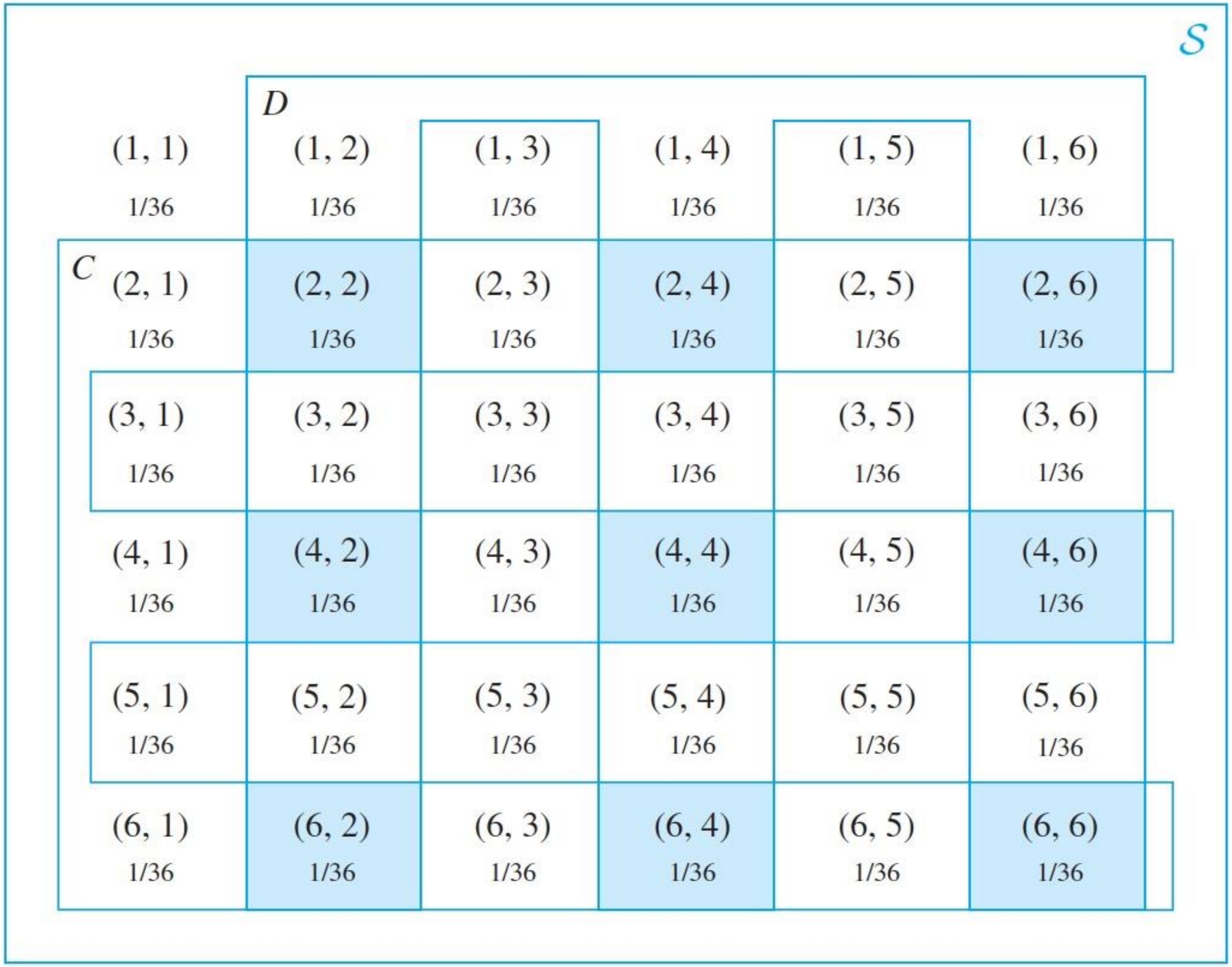


The figure illustrates the event *D, say, that an* even score is obtained on the blue die



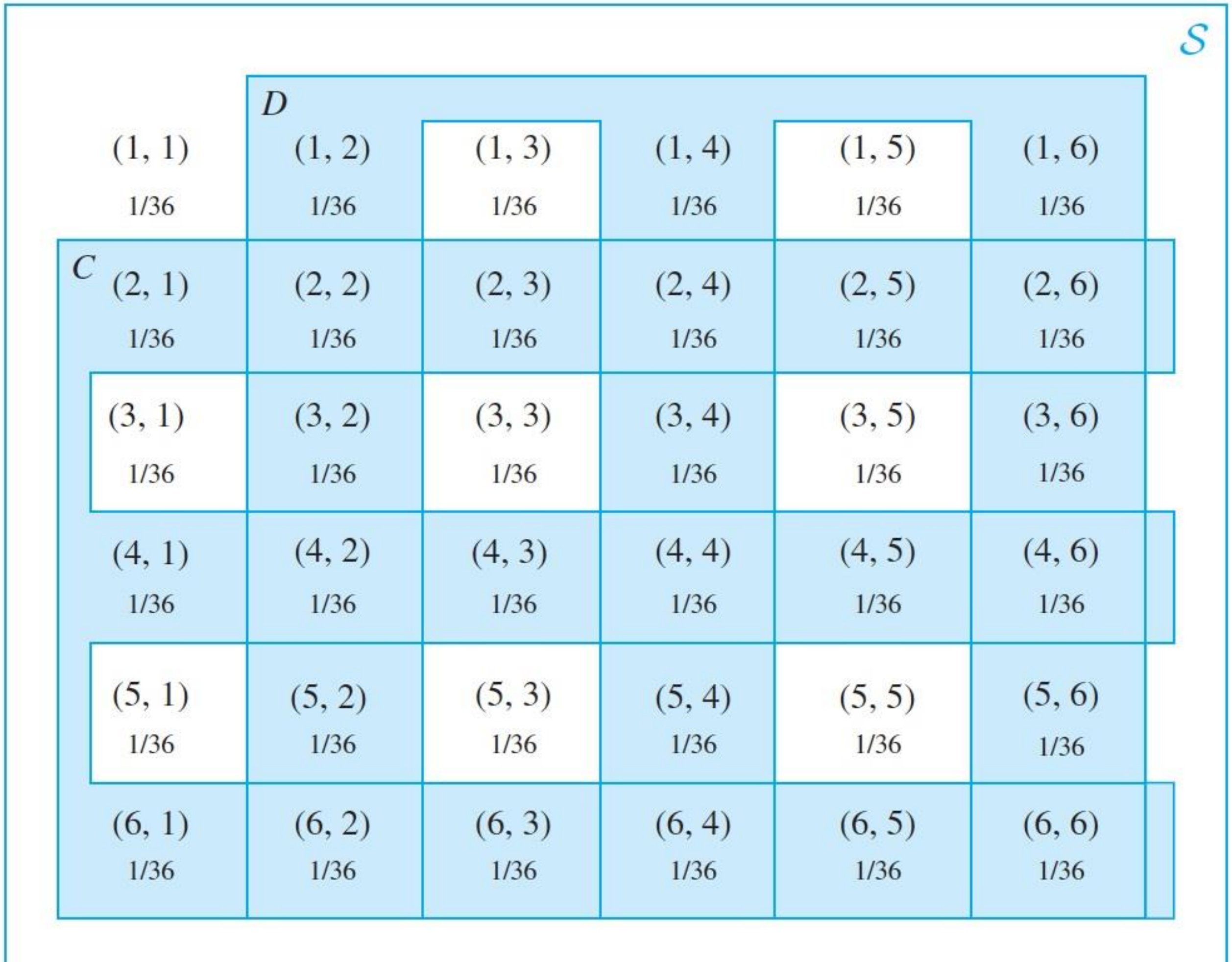
Examples of Intersections and Unions (contd.)

The figure illustrates the event $C \cap D$, which is the event that both dice have even scores.



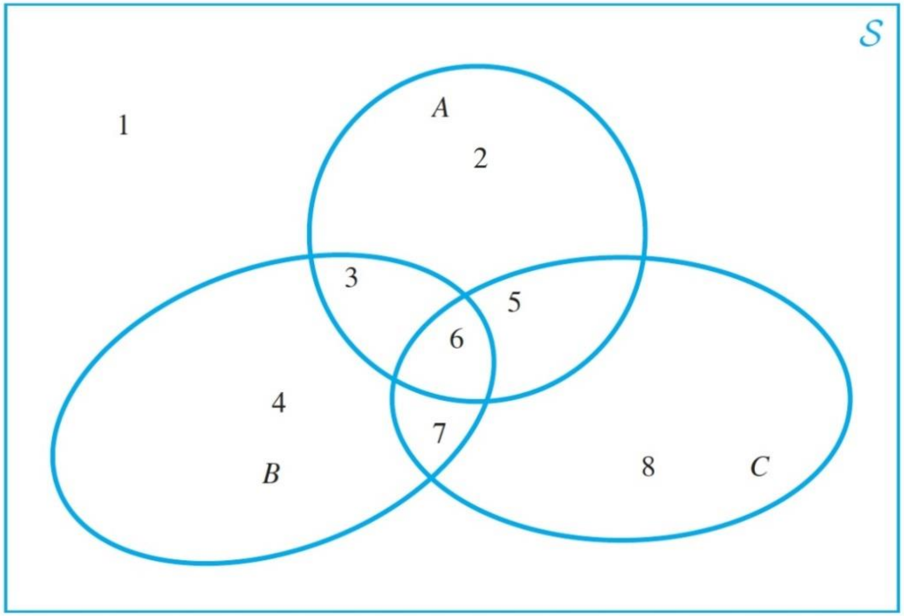
Examples of Intersections and Unions (contd.)

The figure illustrates the event $C \cup D$, the event that at least one die has an even score.



Combination of Three or More Events

- Intersections and unions can be extended in an obvious manner to three or more events. The Figure illustrates how three events *A*, *B*, and *C* can divide a sample space into eight distinct and separate regions.
- The event *A*, for example, is composed of the regions 2, 3, 5, and 6, and
- The event $A \cap B$ is composed of the regions 3 and 6.
- The event $A \cap B \cap C$, the intersection of the events *A*, *B*, and *C*, consists of the outcomes that are simultaneously contained within all three events *A*, *B*, and *C*. It corresponds to region 6.



- The event $A \cup B \cup C$, the union of the events *A*, *B*, and *C*, consists of the outcomes that are in at least one of the three events *A*, *B*, and *C*. It corresponds to all of the regions except for region 1.
- Hence region 1 can be referred to as $(A \cup B \cup C)'$ since it is the complement of the event $A \cup B \cup C$.

- **Problem 7:** Consider the sample space and events in the Figure. Calculate the probabilities of the events:

  (a) B

  (b) $B \cap C$

  (c) $A \cup C$

  (d) $A \cap B \cap C$

  (e) $A \cup B \cup C$

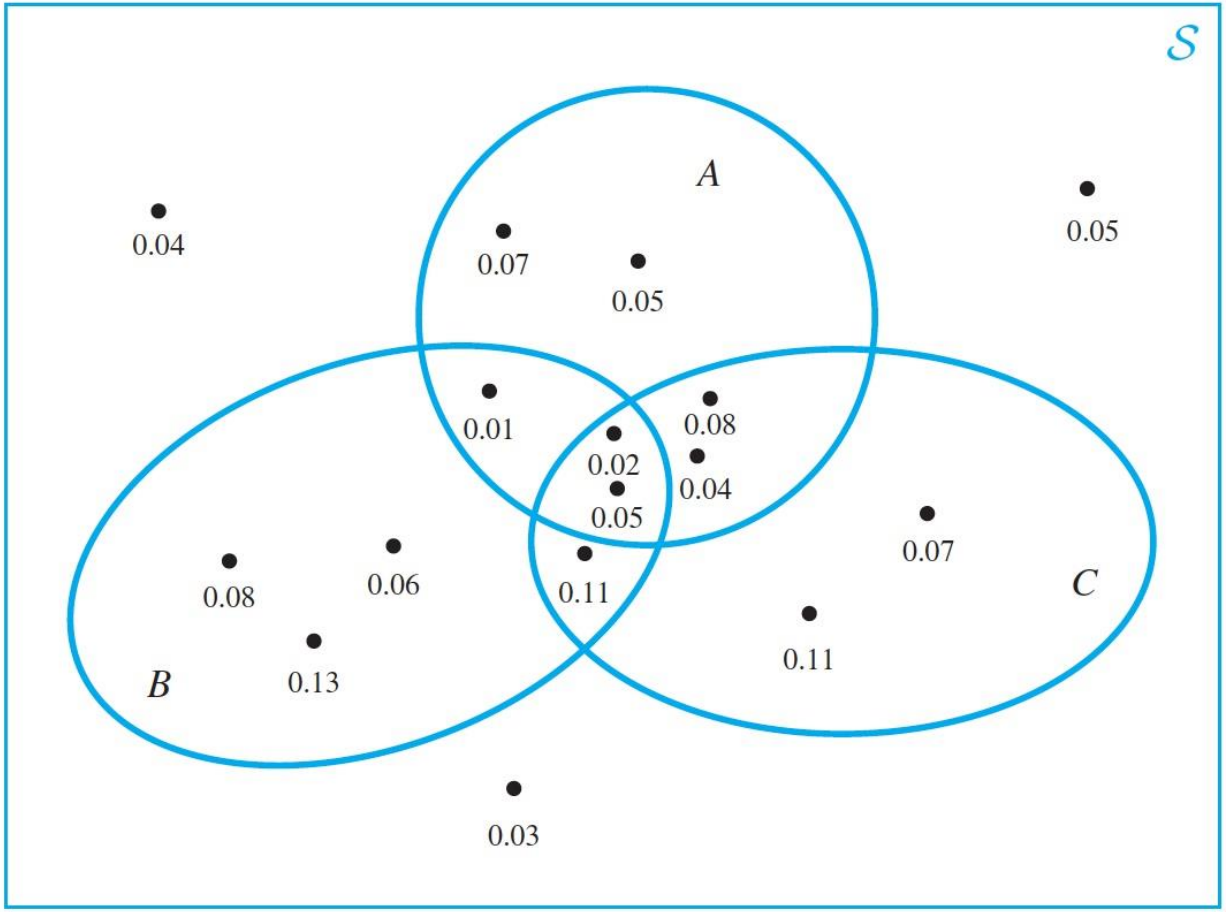
  (f) $A \cap B$

  (g) $B \cup C$

  (h) $A \cup (B \cap C)$

  (i) $(A \cup B) \cap C$

  (j) $(A \cup C)$



AXIOMS AND THEOREMS

Axioms and Basic Theorems

- Axiom 1: For every event A, $P(A) \geq 0$
- Axiom 2: $P(\mathcal{S}) = 1$
- Axiom 3: For every infinite sequence of disjoint events $A_1, A_2, \ldots,$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

- Definition 1: A probability measure, or simply a probability, on a sample space $\mathcal{S}$ is a specification of numbers $P(A)$ for all events A that satisfy Axioms 1,2, and 3.
- Theorem 1: $P(\emptyset) = 0$
- Theorem 2: For every finite sequence of *n* disjoint events $A_1, A_2, \ldots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Axioms and Basic Theorems (contd.)

- Theorem 3: For every event A, $P(A') = 1 - P(A)$
- Theorem 4: If $A \subset B$, then $P(A) \leq P(B)$
- Theorem 5: For every event A, $0 \leq P(A) \leq 1$
- Theorem 6: For every two events A, B

$$P(A \cap B') = P(A) - P(A \cap B)$$

- Theorem 7: For every two events A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Theorem 8 (Bonferroni Inequality): For all events $A_1, A_2, \ldots, A_n$

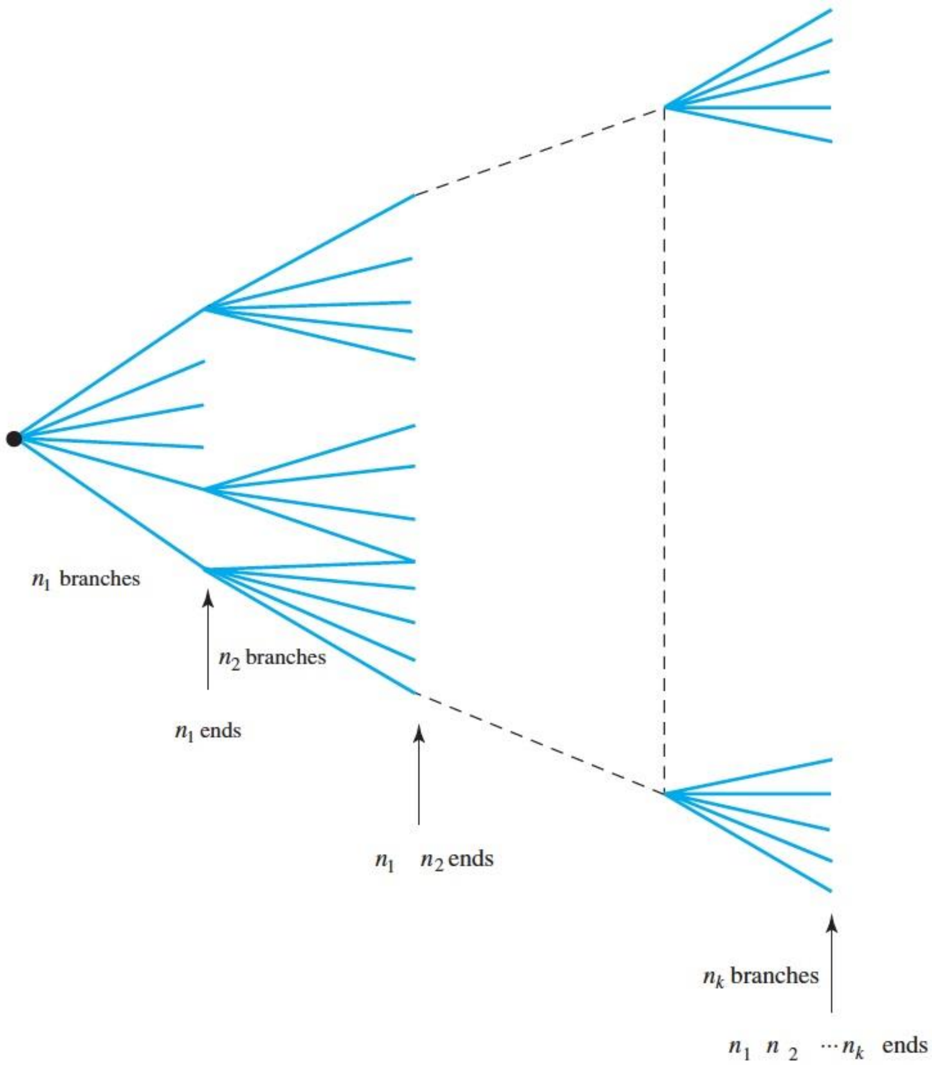
$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \text{ and } P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i')$$

- **Note:** Probability Zero does not mean impossible.

COUNTING TECHNIQUES

1. Multiplication Rule
2. Permutations and Combinations
3. Problems

Multiplication Rule

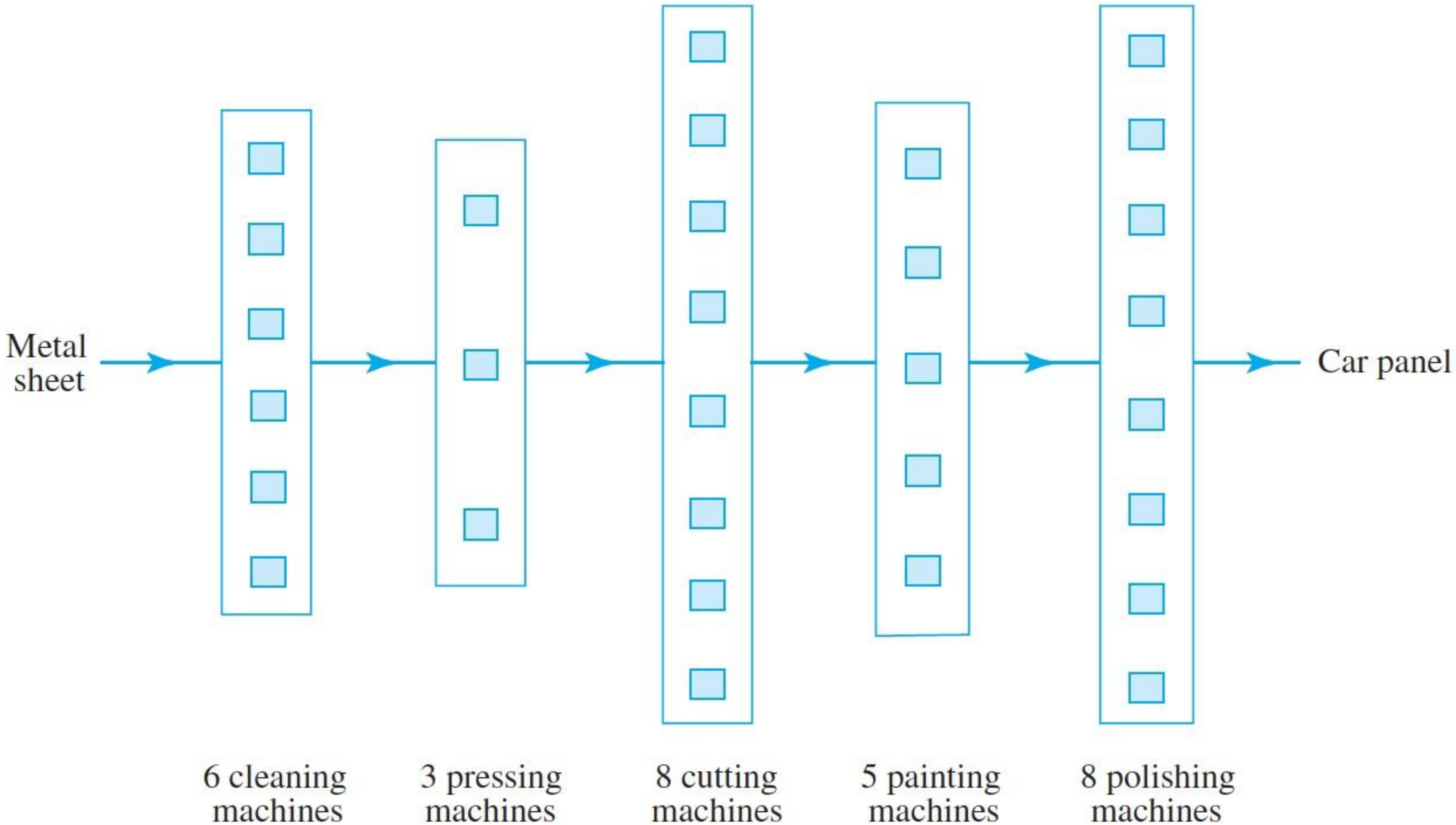


If an experiment consists of k components for which the number of possible outcomes are $n_1, \ldots, n_k$, then the total number of experimental outcomes (the size of the sample space) is equal to

$$n_1 \times n_2 \times \ldots \times n_k$$

Multiplication Rule (contd.)

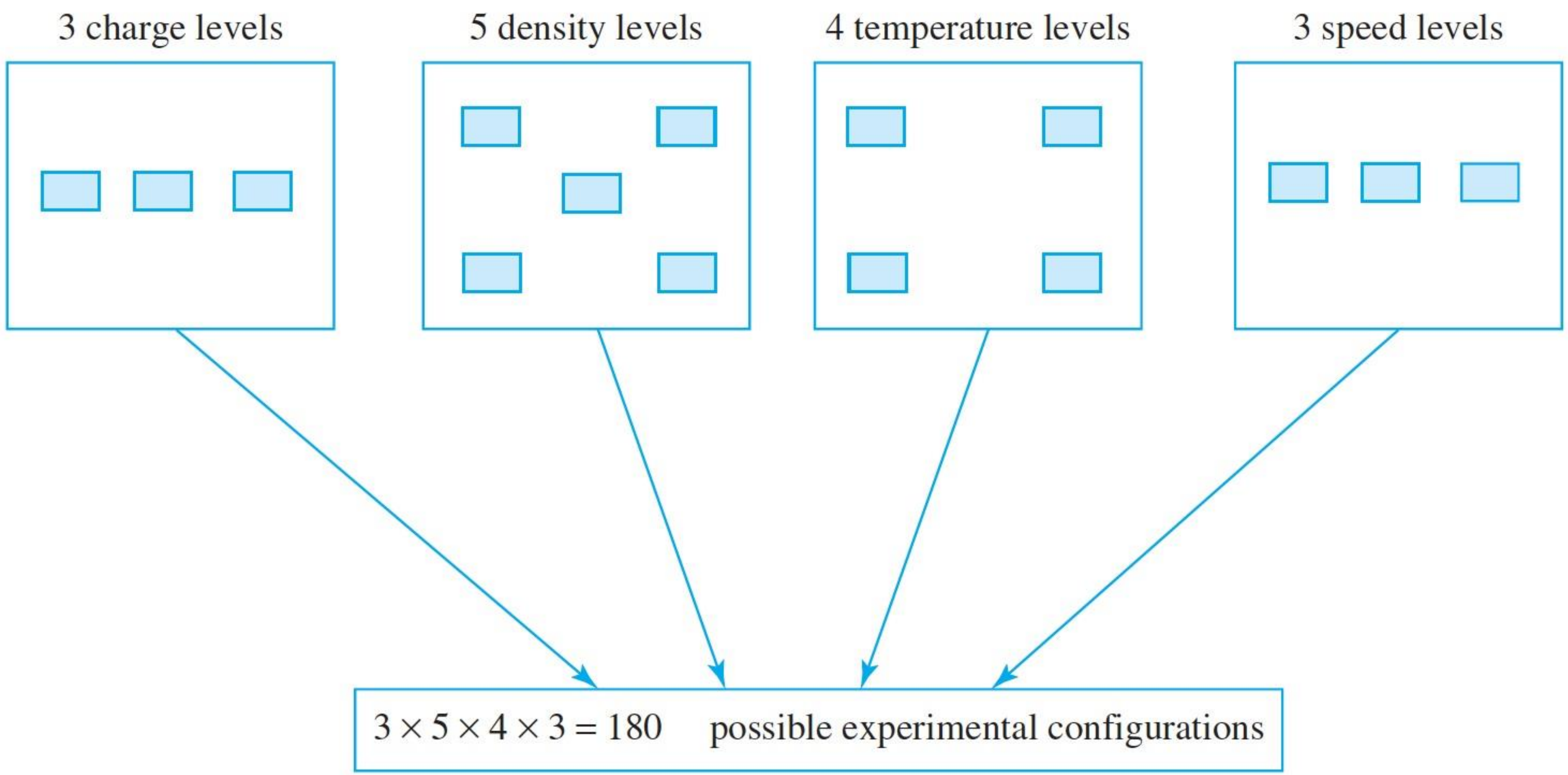
Example 16: Manufacturing process for car side panels



Total number of pathways is $6 \times 3 \times 8 \times 5 \times 8 = 5760$.

Multiplication Rule (contd.)

Example 17: Experimental configurations for fiber coatings



Permutations and Combinations

To calculate how many ways a series of distinguishable k objects can be drawn from a pool of n objects. If the drawings are performed with replacement, then the k drawings are identical events, each with

n possible outcomes,

and the multiplication rule shows that there are n^k possible ways to draw the k objects.

If the drawings are made without replacement, then the outcome is said to be a **permutation** of k objects from the original n objects. If only one object is chosen, then clearly there are only n possible outcomes. If two objects are chosen, then there will be

$n(n-1)$ possible outcomes,

since there are n possibilities for the first choice and then only $n-1$ possibilities for the second choice. More generally, if k objects are chosen, there will be

$n(n-1)(n-2) \cdot \cdot \cdot (n-k+1)$ possible outcomes,

which is obtained by multiplying together the number of choices at each drawing.

Permutations and Combinations (contd.)

Factorials: If *n* is a positive integer, the quantity *n!* called *"n factorial"* is defined to be

$$P_n = n! = n(n-1)(n-2)\dots 1$$

The quantity 0! is taken to be equal to 1.

Permutations: A permutation of *k* objects from *n* objects ($n \geq k$) is an ordered sequence of *k* objects selected without replacement from the group of *n* objects. The number of possible permutations of *k* objects from *n* objects is

$$P_{n,k} = n(n-1)(n-2)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

Combinations: A combination of *k objects from n objects* (*n* ≥ *k*) *is an unordered collection of k* objects selected without replacement from the group of *n* objects*.* The number of possible combinations of *k* objects from *n* objects is

$$C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Note that

$$C_{n,1} = \binom{n}{1} = \frac{n!}{(n-1)!1!} = n \qquad C_{n,2} = \frac{n(n-1)}{2}$$

$$C_{n,(n-1)} = \binom{n}{n-1} = \frac{n!}{1!(n-1)!} = n \qquad C_{n,n} = 1$$

Example 19: Suppose that in the taste test, each participant samples eight products and is asked to select the three best products, but not in any particular order.

The number of possible answers to the test is then

$$C_{8,3} = \binom{8}{3} = \frac{8!}{3!5!} = 56$$

Summary

1. Introduction to Probability
2. Events
3. Combinations of Events
4. Axioms and Basic Theorems
5. Counting Techniques