EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.
- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \qquad (*)$$

has a nontrivial solution.
- The set of *all* solutions of (*) is just the null space of the matrix $A - \lambda I$.
- So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ.
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

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- **Example 3:** Show that 7 is an eigenvalue of matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.
- **Solution:** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \qquad (1)$$

has a nontrivial solution.

- But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = \mathbf{0}$, or

$$(A - 7I)\mathbf{x} = \mathbf{0} \qquad (2)$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

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- The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions.
- To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.

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- **Example 4:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
- **Solution:** Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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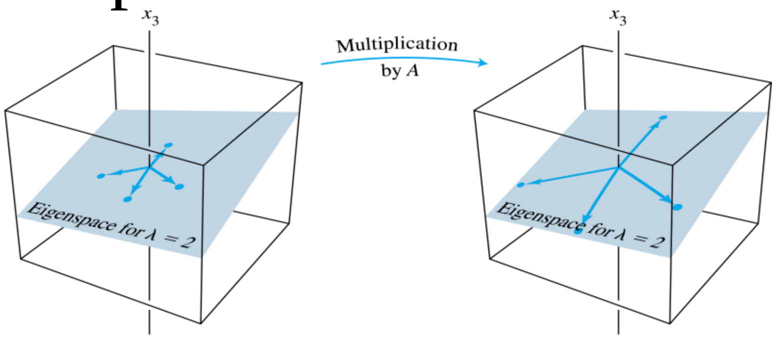
- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathbf{x}=\mathbf{0}$ has free variables.
- The general solution is
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \; x_2 \text{ and } x_3 \text{ free.}$$
- The eigenspace, shown in the following figure, is a two-dimensional subspace of $\mathbb{R}^3$. A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.



A acts as a dilation on the eigenspace.

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- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Proof:** For simplicity, consider the 3×3 case.
- If A is upper triangular, the $A - \lambda I$ has the form

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

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- The scalar λ is an eigenvalue of A if and only if the equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$ has a nontrivial solution, that is, if and only if the equation has a free variable.

- Because of the zero entries in $A-\lambda I$, it is easy to see that $(A-\lambda I)\mathbf{x}=\mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A-\lambda I$ is zero.

- This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A.

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- **Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.
- **Proof:** Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent.
- Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors.
- Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors.

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- Then there exist scalars $c_1, \ldots, c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} \qquad (5)$$

- Multiplying both sides of (5) by A and using the fact that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for each k, we obtain

$$c_1A\mathbf{v}_1 + \cdots + c_pA\mathbf{v}_p = A\mathbf{v}_{p+1}$$

$$c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p = \lambda_{p+1}\mathbf{v}_{p+1} \qquad (6)$$

- Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = 0 \qquad (7)$$

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- Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, the weights in (7) are all zero.
- But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct.
- Hence $c_i = 0$ for $i = 1, \ldots, p$.
- But then (5) says that $\mathrm{v}_{p+1} = 0$, which is impossible.
- Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent.

EIGENVECTORS AND DIFFERENCE EQUATIONS

- If A is an $n \times n$ matrix, then (8) is a *recursive* description of a sequence $\{x_k\}$ in $\mathbb{R}^n$.

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, (k = 0,1,2 \dots) \qquad (8)$$

- A **solution** of (8) is an explicit description of $\{x_k\}$ whose formula for each x_k does not depend directly on A or on the preceding terms in the sequence other than the initial term $\mathbf{x}_0$.

- The simplest way to build a solution of (8) is to take an eigenvector $\mathbf{x}_0$ and its corresponding eigenvalue λ and let

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0, (k = 1,2, \dots) \qquad (9)$$

- This sequence is a solution because

$$A\mathbf{x}_k = A(\lambda^k \mathbf{x}_0) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1} \mathbf{x}_0 = \mathbf{x}_{k+1}$$

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$$\mathbf{x}_k = \lambda^k \mathrm{x}_0, (k = 1,2, \ldots) \qquad (9)$$

- This sequence is a solution because

$$\begin{aligned} A\mathrm{x}_k &= A\left(\lambda^k \mathrm{x}_0\right) \\ &= \lambda^k (A\mathrm{x}_0) = \lambda^k (\lambda \mathrm{x}_0) \\ &= \lambda^{k+1} \mathrm{x}_0 = \mathrm{x}_{k+1} \end{aligned}$$