DIAGONALIZATION

- **Example 2:** Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

- **Solution:** The standard formula for the inverse of a 2×2 matrix yields $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$
- Then, by associativity of matrix multiplication,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

DIAGONALIZATION

- Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1}$$
$$= PDD^2P^{-1} = PD^3P^{-1}$$

- In general, for $k \geq 1$,

- $A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$
$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

- A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal, matrix D.

THE DIAGONALIZATION THEOREM

- **Theorem 5:** An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

  In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P and n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

  In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

THE DIAGONALIZATION THEOREM

- **Proof:** First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$AP = A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] \quad (1)$$

while

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n] \quad (2)$$

- Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P, we have $AP = PD$.

THE DIAGONALIZATION THEOREM

- In this case, equations (1) and (2) imply that

$$[A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n] \qquad (3)$$

- Equating columns, we find that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \ldots, A\mathbf{v}_n = \lambda_n\mathbf{v}_n \qquad (4)$$

- Since P is invertible, its columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ must be linearly independent.

- Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \ldots, \lambda_n$ are eigenvalues and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are corresponding eigenvectors.

THE DIAGONALIZATION THEOREM

- This argument proves the "only if " parts of the first and second statements, along with the third statement, of the theorem.
- Finally, given any n eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ to construct D.
- By equations (1)–(3), $AP = PD$.
- This is true without any condition on the eigenvectors.

- If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$.

DIAGONALIZING MATRICES

- **Example 3:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

- **Solution:** There are four steps to implement the description in Theorem 5.
- *Step 1.* ***Find the eigenvalues of A.***
- Here, the characteristic equation turns out to involve a cubic polynomial that can be factored:

DIAGONALIZING MATRICES

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

- The eigenvalues are $\lambda = 1$ and $\lambda = -2$.
- *Step 2.* ***Find three linearly independent eigenvectors of A.***
- *Three* vectors are needed because A is a 3×3 matrix.
- This is a critical step.
- If it fails, then Theorem 5 says that A cannot be diagonalized.

DIAGONALIZING MATRICES

- Basis for $\lambda = 1$: $\mathrm{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
- Basis for $\lambda = -2$: $\mathrm{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathrm{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.
- *Step 3.* ***Construct P from the vectors in step 2.***
- The order of the vectors is unimportant.
- Using the order chosen in step 2, form

DIAGONALIZING MATRICES

$$P = [\mathrm{v}_1 \quad \mathrm{v}_2 \quad \mathrm{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- *Step 4.* ***Construct D from the corresponding eigenvalues.***
- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P.
- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

DIAGONALIZING MATRICES

- To avoid computing P^{-1}, simply verify that $AP = PD$.
- Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

DIAGONALIZING MATRICES

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
- **Proof:** Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A.
- Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1.
- Hence A is diagonalizable, by Theorem 5.
- It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.
- The 3 x 3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- If an $n \times n$ matrix A has n distinct eigenvalues, with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if $P = [\mathrm{v}_1 \quad \cdots \quad \mathrm{v}_2]$, then P is automatically invertible because its columns are linearly independent, by Theorem 2.

- When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows.

MATRICES WHOSE EIGENVALUES ARE NOT DISTINCT

- **Theorem 7:** Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.
  a. For $1 \leq k \leq p$,the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
  b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k
  c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to $\mathcal{B}_k$ for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.