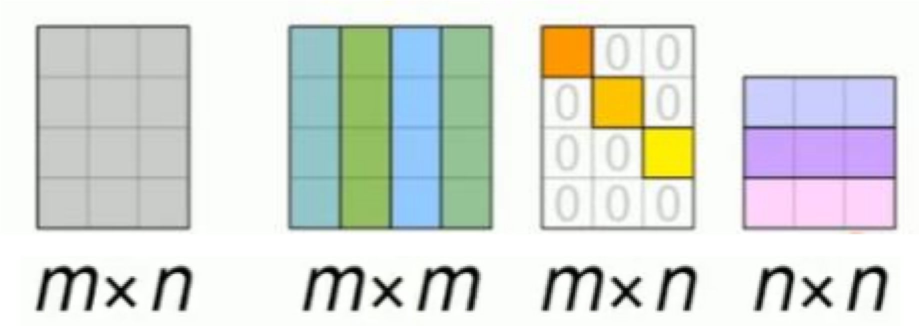
THE SINGULAR VALUE DECOMPOSITION

- **Theorem: The Singular Value Decomposition** Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ diagonal matrix Σ whose diagonal entries are non-negative (the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$), and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

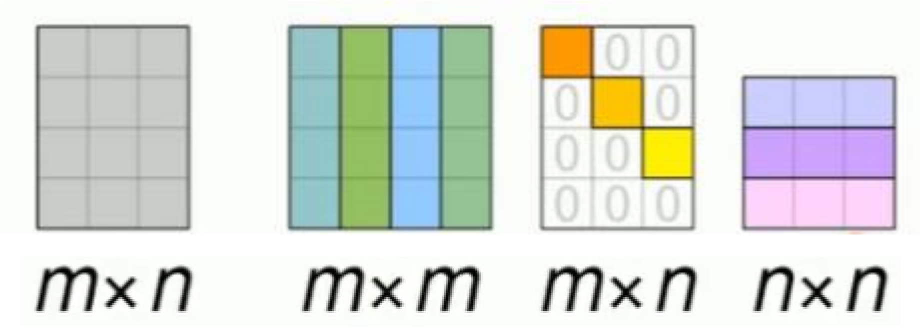
$$A = U\Sigma V^T$$



THE SINGULAR VALUE DECOMPOSITION

- The columns of U in such a decomposition are called **left singular vectors** of A, and the columns of V are called **right singular vectors** of A.
- The diagonal entries of Σ are called the **singular values** of A

$$A = U\Sigma V^T$$



THE SINGULAR VALUE DECOMPOSITION

- **Proof** Let λ_i and v_i be as in Theorem , so that $\{Av_1, \ldots, Av_r\}$ is an orthogonal basis for Col A.
- Normalize each Av_i to obtain an orthonormal basis $\{u_1, \ldots, u_r\}$, where

$$u_i = \frac{1}{\|Av_i\|} Av_i = \frac{1}{\sigma_1} Av_i$$

(*)

- And

$$Av_i = \sigma_i u_i \qquad (1 \le i \le r)$$

- Now extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_m\}$ of $\mathbb{R}^m$, and let

$$U = [u_1 \ u_2 \ \ldots \ um] \ \text{ and } V = [v_1 \ v_2 \ \ldots \ vm]$$

- By construction, U and V are orthogonal matrices.

THE SINGULAR VALUE DECOMPOSITION

- Also, from (*),

$$AV = [Ax_1 \ \ldots \ Avr \ \ 0 \ \ldots \ 0] = [\sigma_1 u_1 \ldots \sigma_r u_r \ \ 0 \ldots 0]$$

- Let D be the diagonal matrix with diagonal entries $\sigma_1,\ldots,\sigma_r$, and let Σ be as follow. Then

$$U\Sigma = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m\,]\left[\begin{array}{cccc|c} \sigma_1 & & & 0 & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$$

$$= [\,\sigma_1\mathbf{u}_1 \quad \cdots \quad \sigma_r\mathbf{u}_r \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\,]$$

$$= AV$$

- Since V is an orthogonal matrix, $U\Sigma V^T = AVVT = A$.

THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution A construction can be divided into three steps.

- ***Step 1. Find an orthogonal diagonalization of A^TA***. That is, find the eigenvalues of A^TA and a corresponding orthonormal set of eigenvectors
- ***Step 2. Set up V and Σ***. Arrange the eigenvalues of A^TA in decreasing order.
- ***Step 3. Construct U***. When A has rank r, the first r columns of U are the normalized vectors obtained from $Av_1, \ldots, Av_r$.

THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - ***Step 1. Find an orthogonal diagonalization of $A^T A$.***

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{\mathit{det}(A^T A - \lambda I) = 0}$$

The eigenvalues are $\lambda = 2$, $\lambda = 1$, and $\lambda = 0$

THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- ***Step 1. Find an orthogonal diagonalization of A^TA.***

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ **$det(A^TA - \lambda I) = 0$**

  - Basis for $\lambda = 2$: $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

  - Basis for $\lambda = 1$: $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and Basis for $\lambda = 0$: $v_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

THE SINGULAR VALUE DECOMPOSITION

- ***Step 2. Set up V and Σ***. Arrange the eigenvalues of A^TA in **decreasing order**. The corresponding unit eigenvectors, v_1, v_2, and v_3, are the right singular vectors of A.

$$V = [v_1 \;\; v_2 \;\; v_3] = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

- The square roots of the eigenvalues are the singular values:

$$\sigma_1 = \sqrt{2}, \qquad \sigma_2 = 1, \qquad \sigma_3 = 0$$

THE SINGULAR VALUE DECOMPOSITION

- The matrix Σ is

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- ***Step 3*. *Construct U*.** When A has rank r, the first r columns of U are the normalized vectors obtained from $Av_1, \ldots, Av_r$.

$$A = U\Sigma V^T \qquad AV = U\Sigma$$

THE SINGULAR VALUE DECOMPOSITION

- Thus

$$u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Note that $\{u_1, u_2\}$ is already a basis for $\mathbb{R}^2$. Thus no additional vectors are needed for U, and $U = [u_1 \ \ u_2]$. The singular value decomposition of A is

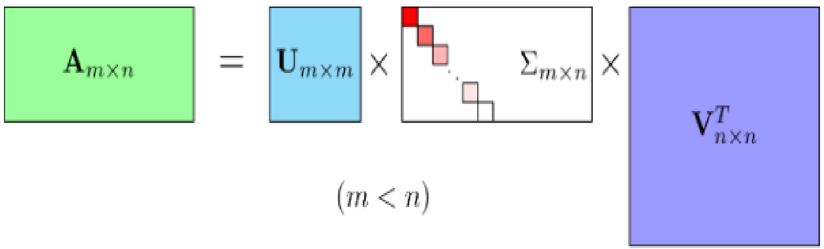
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

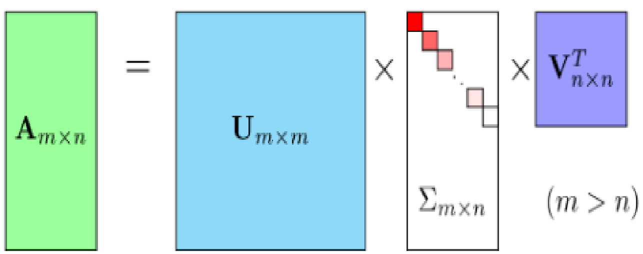
THE SINGULAR VALUE DECOMPOSITION

- **Example** Construct a singular value decomposition of

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

The Singular Value Decomposition

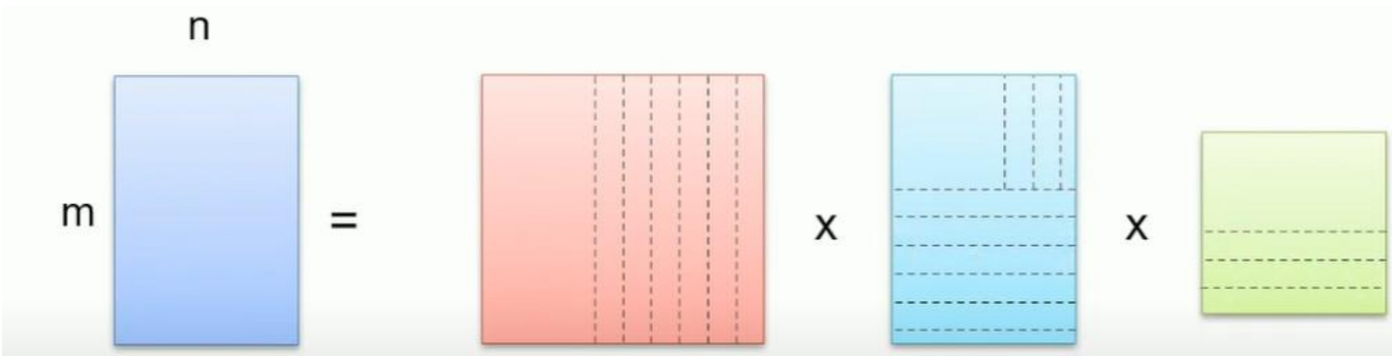


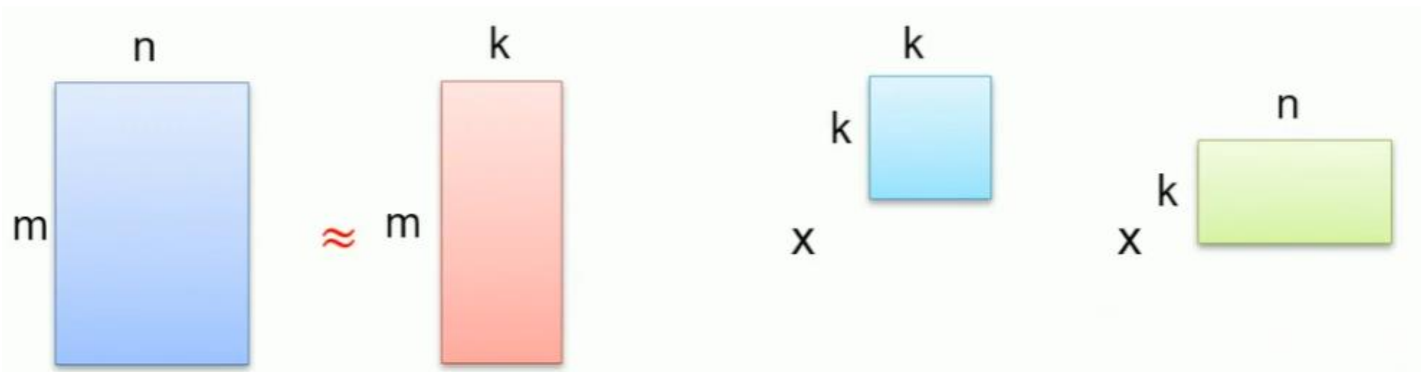


$$\mathbf{A} \approx \mathbf{A}_k = \mathbf{U}_k \Sigma_k (\mathbf{V}_k)^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^2 + \cdots + \sigma_k \mathbf{u}_k \mathbf{v} k^T$$

$$||\mathbf{A} - \mathbf{A}_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$

Image Compression





$$nk + k + km$$

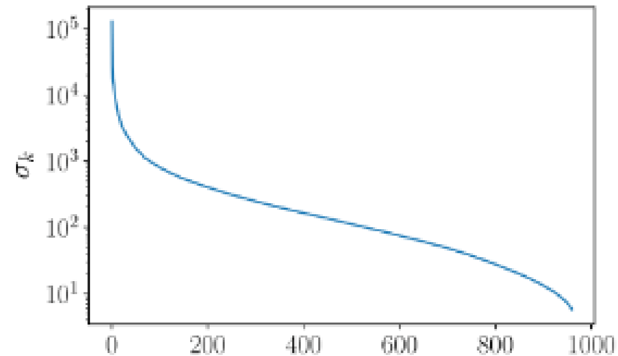
$$nm$$

$$\frac{nk+k+km}{nm}$$

The Singular Value Decomposition

$$\mathbf{A} \approx \mathbf{A}_k = \mathbf{U}_k \Sigma_k (\mathbf{V}_k)^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^2 + \cdots + \sigma_k \mathbf{u}_k \mathbf{v} k^T$$

$$||\mathbf{A} - \mathbf{A}_k||_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$



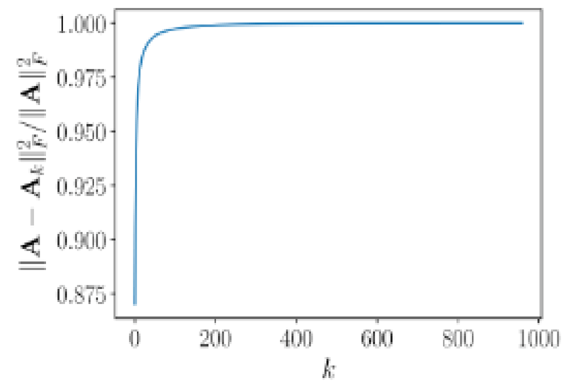
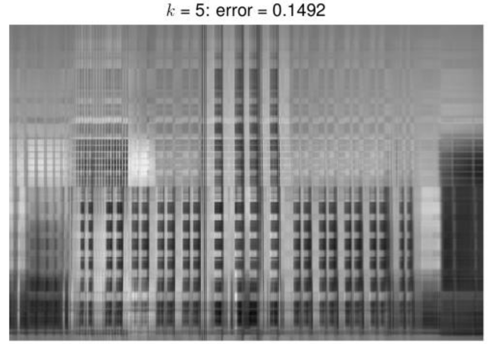
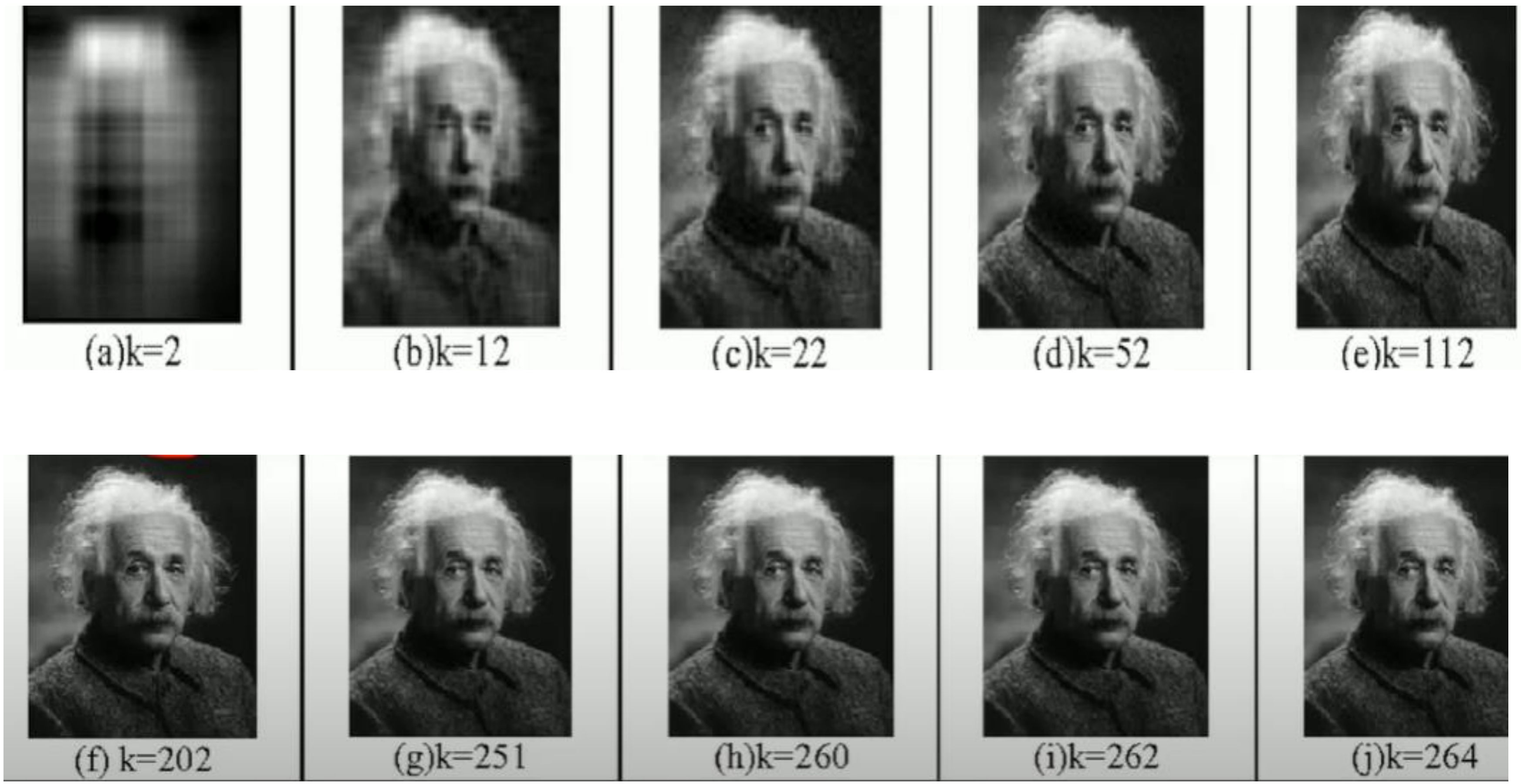


Image Compression

$960 \times 1440.$

k = 5: error = 0.1492




(a)k=2
(b)k=12
(c)k=22
(d)k=52
(e)k=112
(f) k=202
(g)k=251
(h)k=260
(i)k=262
(j)k=264