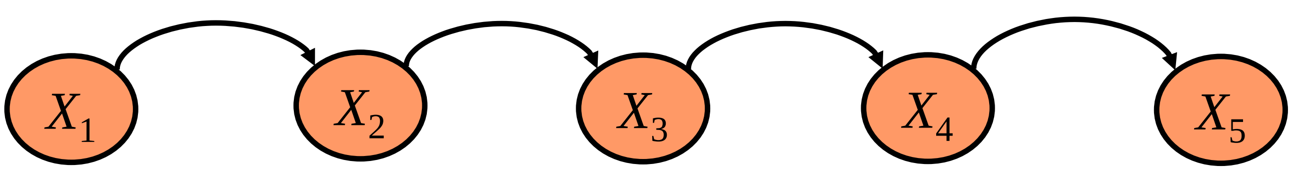
Markov chains

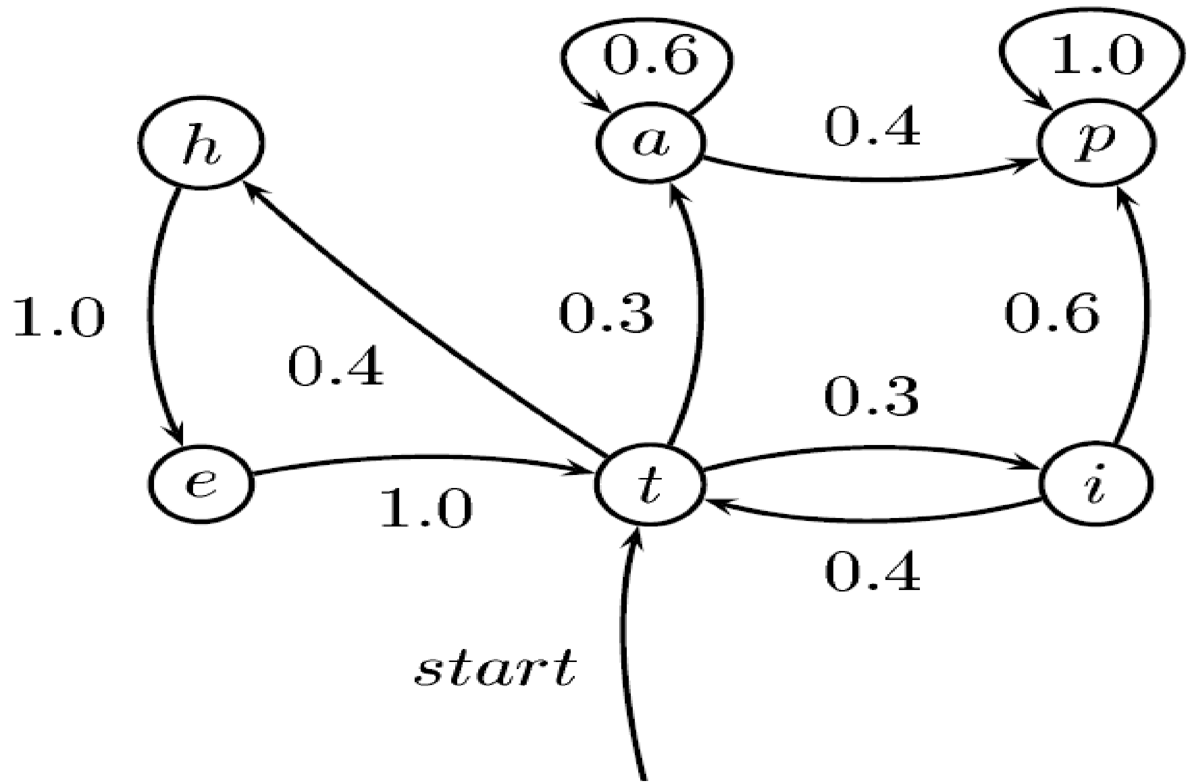
- To model complex interactions between components, use other kinds of models like Markov chains or more generally state space models.
- Many examples of dependencies among system components have been observed in practice and captured by Markov models.

Markov Process

- ***Markov process*** is a simple stochastic process in which the distribution of future states depends only on the present state and not on how it arrived in the present state.



Markov Models
– A finite state representation



Markov Property

- Many systems in real world have the property that given present state, the past states have no influence on the future. This property is called ***Markov property.***

Components of Stochastic Processes

The state space of a stochastic process is
the set of all values that the X_n's can take.

(we will be concerned with
stochastic processes with a finite # of states)

Time: T (n = 0, 1, 2, . . .)

State: S (υ-dimensional vector, $\mathbf{s} = (s_1, s_2, \ldots, s_\upsilon)$)

In general, there are m states,

$\mathbf{s}^1, \mathbf{s}^2, \ldots, \mathbf{s}^m$ or $\mathbf{s}^0, \mathbf{s}^1, \ldots, \mathbf{s}^{m-1}$

Also, X_n takes one of m values, so $X_n \leftrightarrow \mathbf{s}$.

Markov Chain Definition

A stochastic process $\{ X_n \}$ is called a Markov chain if

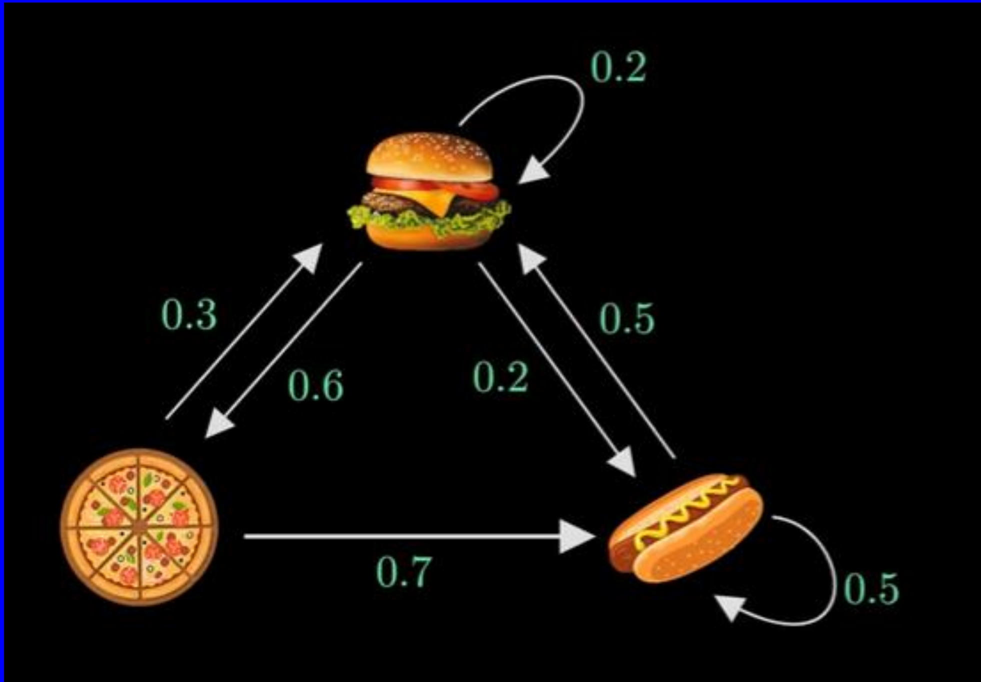
$$P\{ X_{n+1} = j \mid X_0 = k_0, \ldots, X_{n-1} = k_{n-1}, X_n = i \}$$

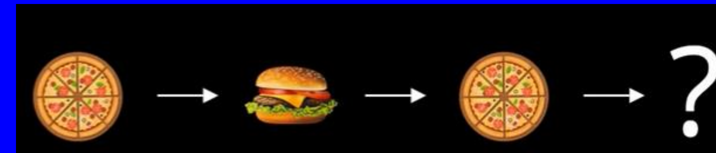
$$= P\{ X_{n+1} = j \mid X_n = i \}$$ ← transition probabilities

for every $i, j, k_0, \ldots, k_{n-1}$ and for every n.

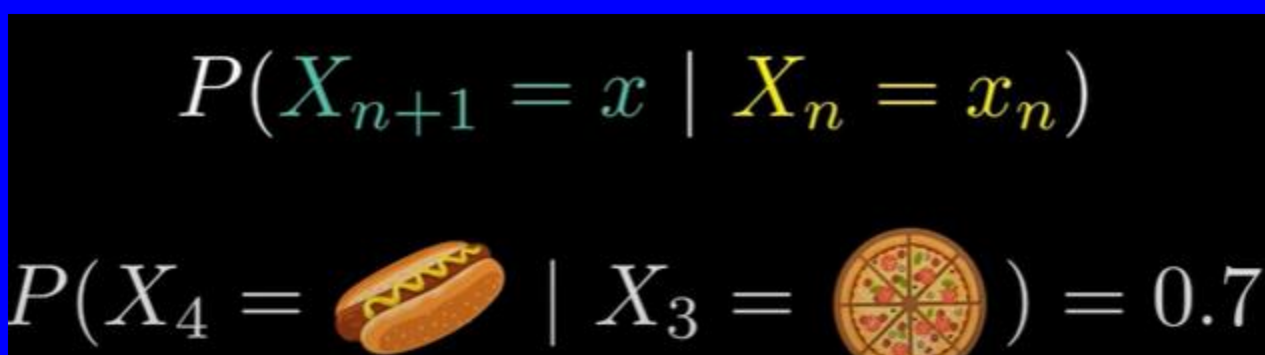
Discrete time means $n \in N = \{0, 1, 2, \ldots\}$.

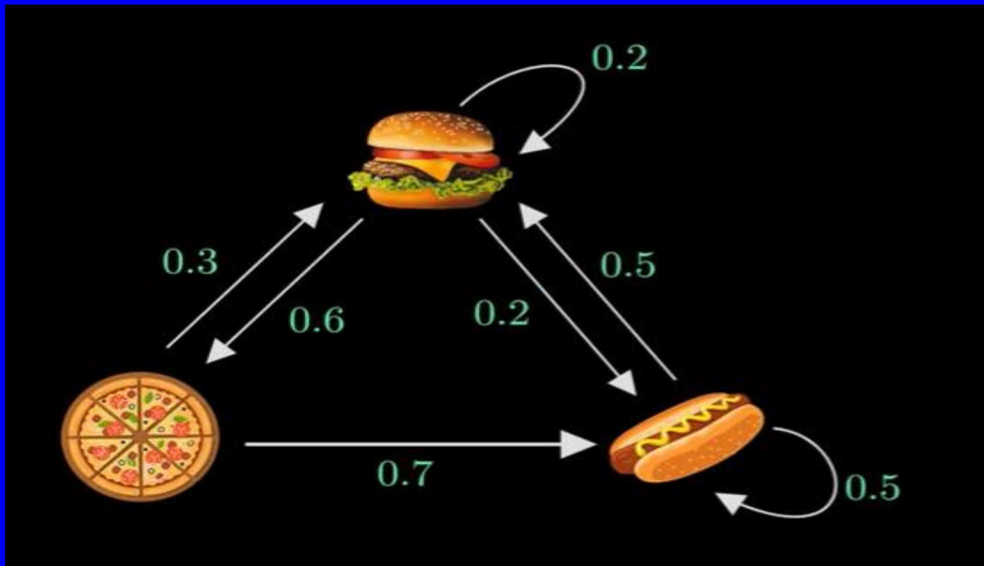
The future behavior of the system depends only on the current state i and not on any of the previous states.

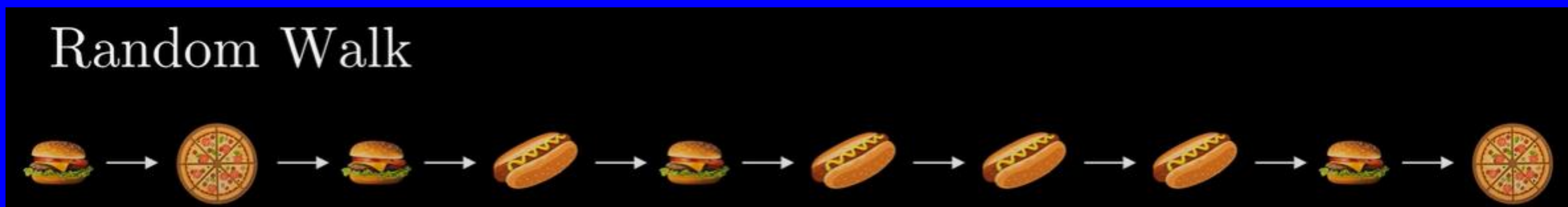

0.2
0.3
0.6
0.2
0.5
0.7
0.5

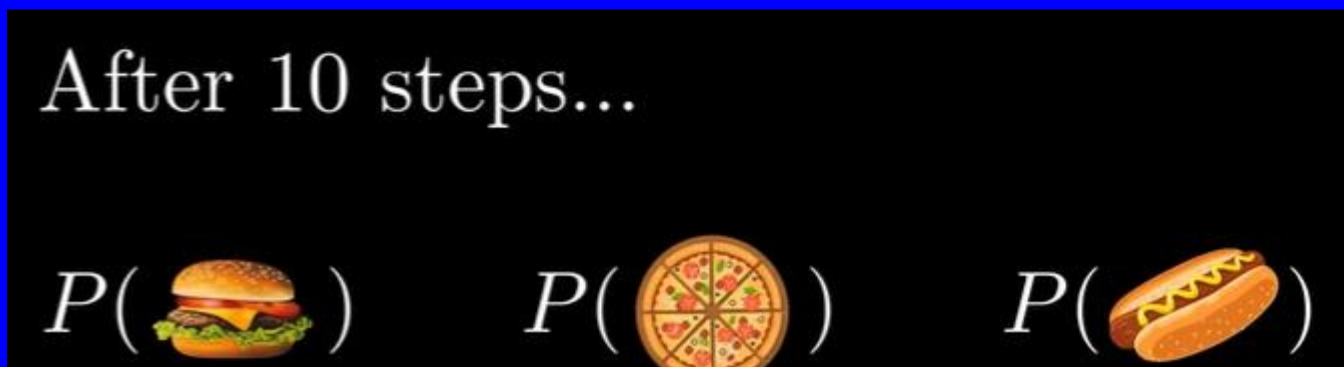

?


$P(X_4 =$ | $X_1 =$, $X_2 =$, $X_3 =$)


$P(X_{n+1} = x \mid X_n = x_n)$
$P(X_4 =$ | $X_3 =$) = 0.7


0.2
0.3
0.6
0.2
0.5
0.7
0.5


Random Walk

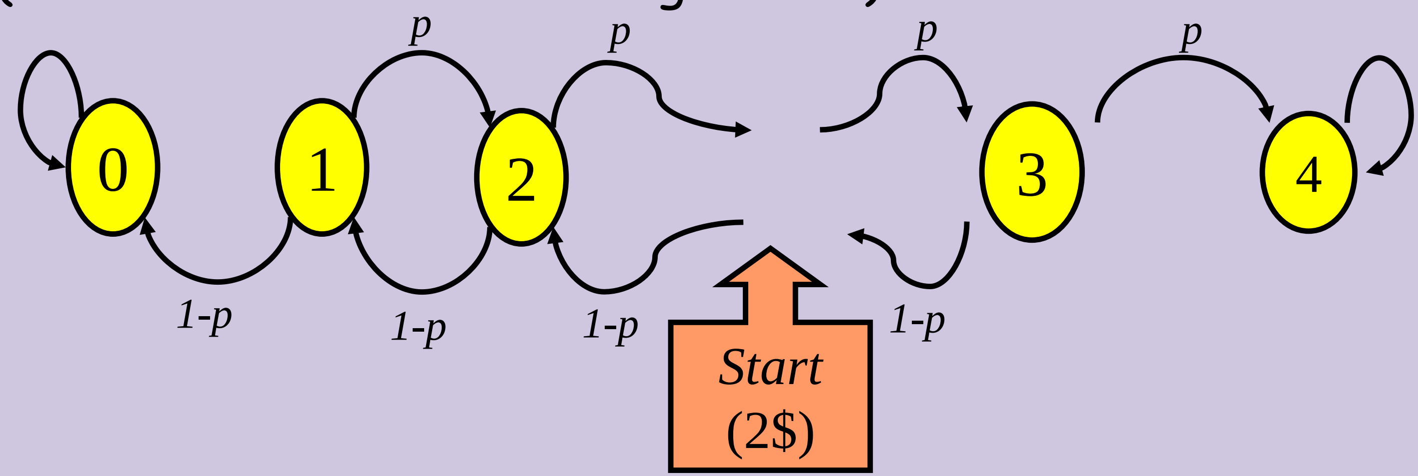

After 10 steps...
P( )
P( )
P( )

Markov Process

Gambler's Example

- Gambler starts with \$2
- At each play we have one of the following:
  - Gambler wins \$1 with probability p
  - Gambler looses \$1 with probability $1\text{-}p$
- Game ends when gambler goes broke, or gains a fortune of \$4

  (Both 0 and 4 are absorbing states)



Gambler's Ruin

At time 0 I have X_0 = \$2, and each day I make a \$1 bet. I win with probability p and lose with probability $1-p$. I'll quit if I ever obtain \$4 or if I lose all my money.

State space is $\boldsymbol{S} = \{\,0,\ 1,\ 2,\ 3,\ 4\,\}$

Let X_n = amount of money I have after the bet on day n.

$$\text{So, } X_1 = \begin{cases} 3 \text{ with probabilty } p \\ 1 \text{ with probabilty } 1-p \end{cases}$$

If $X_n = 4$, then $X_{n+1} = X_{n+2} = \cdots = 4$.

If $X_n = 0$, then $X_{n+1} = X_{n+2} = \cdots = 0$.

Stationary Transition Probabilities

$$P\{X_{n+1} = j \mid X_n = i\} = P\{X_1 = j \mid X_0 = i\} \text{ for all } n$$

(They don't change over time)

We will only consider stationary Markov chains.

The one-step transition matrix for a Markov chain with states $\boldsymbol{S} = \{0, 1, 2\}$ is

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$$

where $p_{ij} = \Pr\{X_1 = j \mid X_0 = i\}$

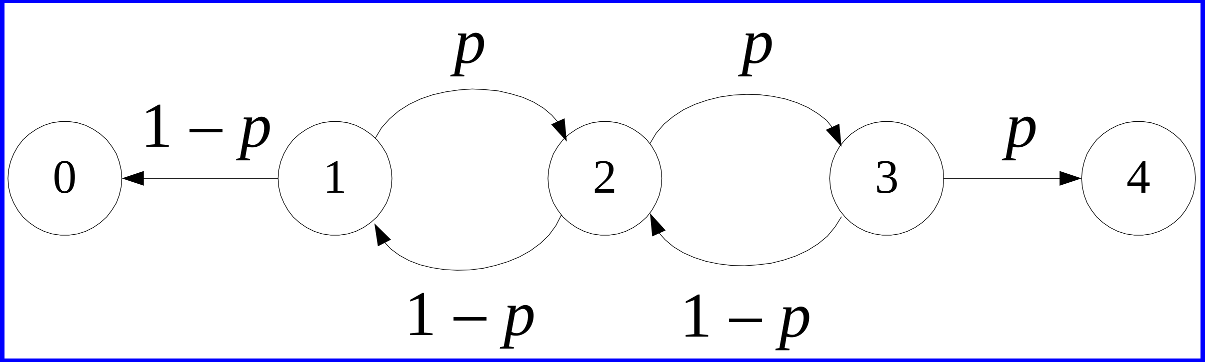
The Transition Matrix

Gambler's Ruin Example

	0	1	2	3	4
0	1	0	0	0	0
1	1-p	0	p	0	0
2	0	1-p	0	p	0
3	0	0	1-p	0	p
4	0	0	0	0	1

Gambler's Ruin Revisited for $p = 0.75$

State-transition network



State-transition matrix

	0	1	2	3	4
0	1	0	0	0	0
1	0.25	0	0.75	0	0
2	0	0.25	0	0.75	0
3	0	0	0.25	0	0.75
4	0	0	0	0	1

The Transition Matrix

Transition Matrix

$$\begin{array}{ccc} & \overbrace{\longleftarrow \text{list all states} \longrightarrow}^{X_{t+1}} & \\ X_t \left\{ \begin{array}{c} \text{list} \\ \text{all} \\ \text{states} \end{array} \right. \updownarrow & \begin{pmatrix} \texttt{insert} \\ \texttt{probabilities} \\ p_{ij} \end{pmatrix} & \begin{array}{l} \longleftarrow \text{rows add to 1} \\ \longleftarrow \text{rows add to 1} \end{array} \end{array}$$

- the ROWS represent NOW, or FROM (X_t);
- the COLUMNS represent NEXT, or TO (X_{t+1});
- entry (i, j) is the CONDITIONAL probability that NEXT = j, given that NOW = i: the probability of going FROM state i TO state j

$$\boxed{p_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i).}$$

The Transition Matrix

Notes:

1. The transition matrix P must list all possible states in the state space S.
2. P is a square matrix (N × N), because X_{t+1} and X_t both take values in the same state space S (of size N).
3. The rows of P should each sum to 1:

$$\sum_{j=1}^{N} p_{ij} = \sum_{j=1}^{N} \mathbb{P}(X_{t+1} = j \mid X_t = i) = \sum_{j=1}^{N} \mathbb{P}_{\{X_t = i\}}(X_{t+1} = j) = 1.$$

4. The columns of P do not in general sum to 1

Example: Three boys A, B and C, are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is as likely to throw the ball to B as to A. Find the transition matrix.

Example: Company A has 40% market share in the local markets for its cosmetics, while the other two companies, B and C, have equal share each on 1st January 2018. A study by the market research company has disclosed the following data for every year.

- Company A retains 70% of its customers and gain 5% from company B and 10% from company C,
- Company B retains 90% of its customers and gains 14% from company A and 5% from company C, and
- Company C retains 85% of its customers and gains 16% from Company A and 5% from company B.

What You Should Know About Markov Chains

- How to define states of a discrete time process.
- How to construct a state-transition matrix.

Next Problems

- How to calculate the probability of the states.
- How to calculate the probability after n-steps.
- How to find the probability of the chain.

Multi-steps Transition

Probability of reaching state j from state i after exactly n steps

Multi-steps Transition

The t-step transition probabilitie

Let $\{X_0, X_1, X_2, \ldots.\}$ be a Markov chain with state space S = $\{1, 2, \ldots, N\}$. Recall that the elements of the transition matrix P are defined as:

$$(P)_{ij} = p_{ij} = \mathbb{P}(X_1 = j \mid X_0 = i) = \mathbb{P}(X_{n+1} = j \mid X_n = i) \quad \text{for any } n.$$

p_{ij} is the probability of making a transition FROM state i TO state j in a SINGLE step.

Multi-steps Transition

The t-step transition probabilitie

The two-step transition probabilities are therefore given by the matrix P^2:

$$\boxed{\mathbb{P}(X_2 = j \mid X_0 = i) = \mathbb{P}(X_{n+2} = j \mid X_n = i) = \left(P^2\right)_{ij} \quad \text{for any } n.}$$

3-step transitions:

The three-step transition probabilities are therefore given by the matrix P^3

$$\boxed{\mathbb{P}(X_3 = j \mid X_0 = i) = \mathbb{P}(X_{n+3} = j \mid X_n = i) = \left(P^3\right)_{ij} \quad \text{for any } n.}$$

Multi-steps Transition

The t-step transition probabilitie

General case: t-step transitions: The above working extends to show that the t-step transition probabilities are given by the matrix P^t for any t:

$$\boxed{\mathbb{P}(X_t = j \mid X_0 = i) = \mathbb{P}(X_{n+t} = j \mid X_n = i) = \left(P^t\right)_{ij} \quad \text{for any } n.}$$

Multi-step Transition

Theorem: Let $\{X_0, X_1, X_2, \ldots .\}$ be a Markov chain with $N \times N$ transition matrix P. If the probability distribution of X_0 is given by the $1{\times}N$ row vector α^T, then the probability distribution of X_t is given by the $1 \times N$ row vector $\alpha^T P^t$.

That is: $X_0 = \alpha^T \quad \Rightarrow X_t = \alpha^T P^t$

$$\alpha = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_n \end{Bmatrix} = \begin{pmatrix} \mathbb{P}(X_0 = 1) \\ \mathbb{P}(X_0 = 2) \\ \vdots \\ \mathbb{P}(X_0 = N) \end{pmatrix}$$

Trajectory Probability

Let $X_0 = \alpha^T$. The probability of the trajectory $s_0, s_1, s_2, \ldots s_t$ is:

$$\boxed{= \; p_{s_{t-1},s_t} \times p_{s_{t-2},s_{t-1}} \times \ldots \times p_{s_0,s_1} \times} \boxed{\alpha_{s0}}$$

Proof:

$$
\begin{aligned}
& \mathbb{P}(X_0 = s_0, X_1 = s_1, \ldots, X_t = s_t) \\
&= \mathbb{P}(X_t = s_t \,|\, X_{t-1} = s_{t-1}, \ldots, X_0 = s_0) \times \mathbb{P}(X_{t-1} = s_{t-1}, \ldots, X_0 = s_0) \\
&= \mathbb{P}(X_t = s_t \,|\, X_{t-1} = s_{t-1}) \times \mathbb{P}(X_{t-1} = s_{t-1}, \ldots, X_0 = s_0) \quad \text{(Markov Property)} \\
&= p_{s_{t-1},s_t} \mathbb{P}(X_{t-1} = s_{t-1} \,|\, X_{t-2} = s_{t-2}, \ldots, X_0 = s_0) \times \mathbb{P}(X_{t-2} = s_{t-2}, \ldots, X_0 = s_0) \\
& \quad \vdots \\
&= p_{s_{t-1},s_t} \times p_{s_{t-2},s_{t-1}} \times \ldots \times p_{s_0,s_1} \times \mathbb{P}(X_0 = s_0) \\
&= p_{s_{t-1},s_t} \times p_{s_{t-2},s_{t-1}} \times \ldots \times p_{s_0,s_1} \times \alpha_{s0}
\end{aligned}
$$

Markov Process

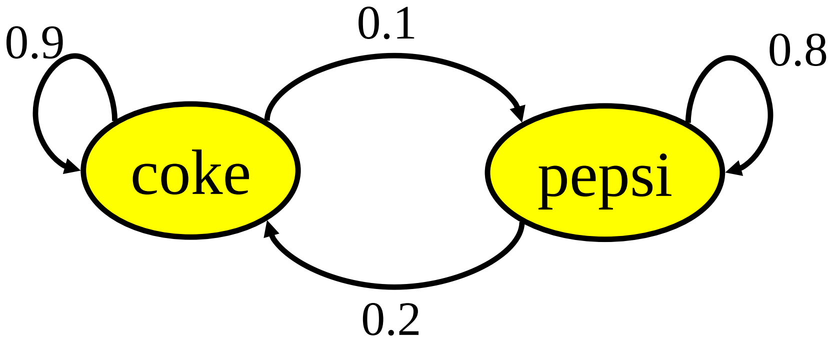
Coke vs. Pepsi Example

- Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will also be Coke.

- If a person's last cola purchase was Pepsi, there is an 80% chance that his next cola purchase will also be Pepsi.

transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$



Markov Process

Coke vs. Pepsi Example (cont)

Given that a person is currently a Pepsi purchaser, what is the probability that he will purchase Coke two purchases from now?

P[Pepsi→?→Coke] =

P[Pepsi→Coke→Coke] + P[Pepsi→ Pepsi →Coke] =

0.2 * 0.9 + 0.8 * 0.2 = 0.34

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

Pepsi → ? ? → Coke

Markov Process

Coke vs. Pepsi Example (cont)

Given that a person is currently a Coke purchaser, what is the probability that he will purchase Pepsi **three** purchases from now?

$$P^3 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

Markov Process

Coke vs. Pepsi Example (cont)

- Assume each person makes one cola purchase per week
- Suppose 60% of all people now drink Coke, and 40% drink Pepsi
- What fraction of people will be drinking Coke three weeks from now?

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \qquad P^3 = \begin{bmatrix} 0.781 & 0.219 \\ 0.438 & 0.562 \end{bmatrix}$$

$$\mathrm{P}[X_3=\text{Coke}] = 0.6 * 0.781 + 0.4 * 0.438 = 0.6438$$

X_i - the distribution in week i

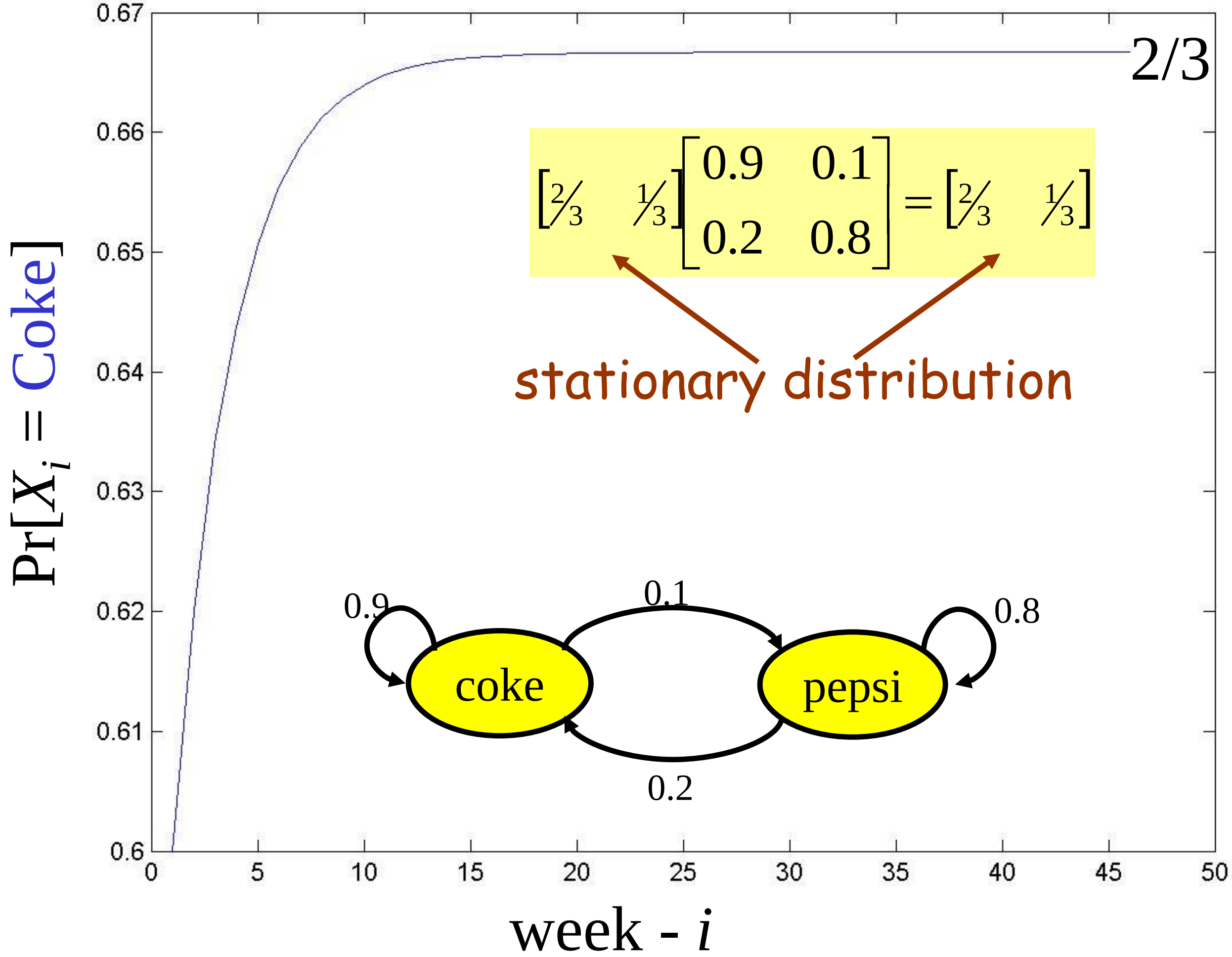
$X_0=(0.6,0.4)$ - initial distribution

$X_3= X_0 * P^3 =(0.6438,0.3562)$

Markov Process

Coke vs. Pepsi Example (cont)

Simulation:



- How to calculate the probability of the states.

Example 0: A man either uses his car or takes a bus or a train to work each day. The TPM of the Markov chain with these three states 1 (Car), 2 (Bus), 3 (Train) is

$$P = \begin{array}{c c} & \begin{array}{ccc} C & B & T \end{array} \\ \begin{array}{c} C \\ B \\ T \end{array} & \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$. Calculate $P(X_2 = 3)$.

A 70% chance that a person uses Car to go to work $\boxed{\text{at the first day.}}$

Similarly, there is a 20%, 10% chances that a person use bus and Train to go to work at the first day.

What is the probability that on the 2nd day, a man use TRAIN to go to work?

- How to calculate the probability of the states and the probability after n-steps.

Example 0: A man either uses his car or takes a bus or a train to work each day. The TPM of the Markov chain with these three states 1 (Car), 2 (Bus), 3 (Train) is

$$P = \begin{array}{c} \\ C \\ B \\ T \end{array}\begin{array}{c} \begin{array}{ccc} C & B & T \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$. Calculate $P(X_2 = 3)$.

Example 1: The TPM of the Markov chain with three states 1, 2, 3 is

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$. Calculate

(i) $P(X_2 = 3)$.

Example 2

- Three boys A, B and C, are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C, but C is likely to throw the ball ti B as to A.

- If the initial probability distribution of three states A, B, and C is 0,3; 0,4; and 0,3 respectively. Find:

(i) Transition matrix.

(ii) $P(X_2=B)$;

(iii) $P(X_3=B; X_2=C; X_1=B; X_0=A)$;

(iv) The distribution of the ball after two round.

- How to find the probability of the chain.

$P(X_3=2, X_1=3, X_0=2)$

Interpretation:

- Started from State 2
- After 1 time period, it move to State 3
- The after 2 time period, it move to State 2 again

Example 2: The TPM of the Markov chain with three states 1, 2, 3 is

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$.

Calculate

(i) $P(X_2 = 1)$

(ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

Example: Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with transition matrix P as:

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{pmatrix}$$

and initial distribution: $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.2 \end{pmatrix}$

a) Find $\Pr(X_2 = 3 \mid X_0 = 1)$; $\Pr(X_2 = 1 \mid X_0 = 2)$.
b) Find $\Pr(X_3 = 1 \mid X_0 = 1)$
c) Find $\Pr(X_{10} = 1 \mid X_8 = 1, X_7 = 1)$
d) Find $\Pr(X_0=1, X_2 = 2, X_3 = 3)$
e) Find $\Pr(X_9 = 2 \mid X_{10} = 3, X_8 = 1)$
f) Find $\Pr(X_3 = 2)$

2. Consider the Markov chain with three states, S={1;2;3}, that has the following transition matrix.

$$P = \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{4} & \tfrac{1}{4} \\ \tfrac{1}{3} & 0 & \tfrac{2}{3} \\ \tfrac{1}{2} & \tfrac{1}{2} & 0 \end{bmatrix}$$

a. Draw the state transition diagram for this chain.

b. If we know $P(X_0=1)=P(X_0=2)=1/4$. Find $P(X_0=3; X_1=2; X_2=1)$.

2. A company circulates for first time in the market a new product, say K. The market's research shows that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will return to K next week. Find the market's share for the K two weeks after its first circulation, provided that the market's conditions remain unchanged.

3. In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale.

(i). Find the probability that the grandson of a man from Harvard went to Harvard.

(ii). Modify the above by assuming that the son of a Harvard man always went to Harvard. Again, find the probability that the grandson of a man from Harvard went to Harvard.

3. In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale.

(i). Find the probability that the grandson of a man from Harvard went to Harvard.

(ii). Modify the above by assuming that the son of a Harvard man always went to Harvard. Again, find the probability that the grandson of a man from Harvard went to Harvard.

4. Assume that a man's profession can be classified as professinal, skilled labourer or unskilled labour. Assume that, of the sons of professional men, 80 percent are professional, 10 percen are skilled labourers, and 10 percent are unskilled labourer. In the case of sons of skilled labourers, 60 percent are skilled laboured, 20 percent are professinal and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourer of the sons are unskilled labourers, and 25 percent each are in the other two categories. Set up the matrix of transition probabilities. Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.

Conditional vs. Unconditional Probabilities

Let state space $\boldsymbol{S} = \{1, 2, \ldots, m\}$.

Let $p_{ij}^{(n)}$ be conditional n-step transition probability → $\mathrm{P}^{(n)}$.

Let $\mathbf{q}(n) = \big(q_1(n), \ldots, q_m(n)\big)$ be vector of all unconditional probabilities for all m states after n transitions.

Perform the following calculations:

$$\mathbf{q}(n) = \mathbf{q}(0)\mathbf{P}^{(n)} \quad \text{or} \quad \mathbf{q}(n) = \mathbf{q}(n-1)\mathbf{P}$$

where $\mathbf{q}(0)$ is initial unconditional probability.

The components of $\mathbf{q}(n)$ are called the transient probabilities.

Transition Probabilities for n Steps

Property 1: Let $\{X_n : n = 0, 1, \ldots\}$ be a Markov chain with state space $\boldsymbol{S}$ and state-transition matrix $\mathbf{P}$. Then for i and $j \in \boldsymbol{S}$, and $n = 1, 2, \ldots$

$$\mathrm{P}\{X_n = j \mid X_0 = i\} = p_{ij}^{(n)}$$

where the right-hand side represents the ij^{th} element of the matrix $\mathbf{P}^{(n)}$.

Steady-State Probabilities

Property 2: Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ is the m-dimensional row vector of steady-state (unconditional) probabilities for the state space $\boldsymbol{S} = \{1,\ldots,m\}$. To find steady-state probabilities, solve linear system:

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}, \quad \textstyle\sum_{j=1,m} \pi_j = 1, \quad \pi_j \geq 0, \quad j = 1,\ldots,m$$

Consider three big markets A, B, C and TPM corresponding to them is given as below

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

At present, its market shares are 30%, 40% and 30% respectively.

After 1 time period, its marker shares will be:

$\pi_1 = \pi_0 P = [0.37 \quad 0.44 \quad 0.19]$

After 2 time period, its marker shares will be:

$\pi_2 = \pi_1 P = \pi_0 P^2 = [0.399 \quad 0.408 \quad 0.193]$

After 1 time period, its marker shares will be:

$$\pi_1 = \pi_0 P = [0.37 \quad 0.44 \quad 0.19]$$

After 2 time period, its marker shares will be:

$$\pi_2 = \pi_1 P = \pi_0 P^2 = [0.399 \quad 0.408 \quad 0.193]$$

After 3 time period, its marker shares will be:

$$\pi_3 = \pi_2 P = \pi_0 P^3 = [0.413 \quad 0.3996 \quad 0.1991]$$

After 7 time period, its marker shares will be:

$$\pi_7 = \pi_6 P = \pi_0 P^7 = [0.4 \quad 0.4 \quad 0.2]$$

After 12 time period, its marker shares will be:

$$\pi_{12} = \pi_{11} P = \pi_0 P^{12} = [0.4 \quad 0.4 \quad 0.2]$$

Brand Switching Example →

We approximate $q_i(0)$ by dividing total customers using brand i in week 27 by total sample size of 500:

$$\mathbf{q}(0) = (125/500, 230/500, 145/500) = (0.25, 0.46, 0.29)$$

To predict market shares for, say, week 29 (that is, 2 weeks into the future), we simply apply equation with $n = 2$:

$$\mathbf{q}(2) = \mathbf{q}(0)\mathbf{P}^{(2)}$$

$$\mathbf{q}(2) = (0.25, 0.46, 0.29)\begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}^2$$

$$= (0.327, 0.406, 0.267)$$

= expected market share from brands 1, 2, 3

Steady-State Probabilities

Brand switching example:

$$(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_2, \pi_3) \begin{bmatrix} 0.90 & 0.07 & 0.03 \\ 0.02 & 0.82 & 0.16 \\ 0.20 & 0.12 & 0.68 \end{bmatrix}$$

$$\pi_1 + \pi_2 + \pi_2 = 1, \;\; \pi_1 \geq 0, \;\; \pi_2 \geq 0, \;\; \pi_3 \geq 0$$

Steady-State Equations for Brand Switching Example

$$\pi_1 = 0.90\pi_1 + 0.02\pi_2 + 0.20\pi_3$$

$$\pi_2 = 0.07\pi_1 + 0.82\pi_2 + 0.12\pi_3$$

$$\pi_3 = 0.03\pi_1 + 0.16\pi_2 + 0.68\pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 \geq 0,\ \ \pi_2 \geq 0,\ \ \pi_3 \geq 0$$

Total of 4 equations in 3 unknowns

➔ Discard 3rd equation and solve the remaining system to get :

$$\pi_1 = 0.474,\ \ \pi_2 = 0.321,\ \ \pi_3 = 0.205$$

➔ Recall: $q_1(0) = 0.25,\ \ q_2(0) = 0.46,\ \ q_3(0) = 0.29$

Comments on Steady-State Results

1. Steady-state predictions are never achieved in actuality due to a combination of

   (i) errors in estimating **P**

   (ii) changes in **P** over time

   (iii) changes in the nature of dependence relationships among the states.

2. Nevertheless, the use of steady-state values is an important diagnostic tool for the decision maker.

3. Steady-state probabilities might not exist unless the Markov chain is ergodic.

Existence of Steady-State Probabilities

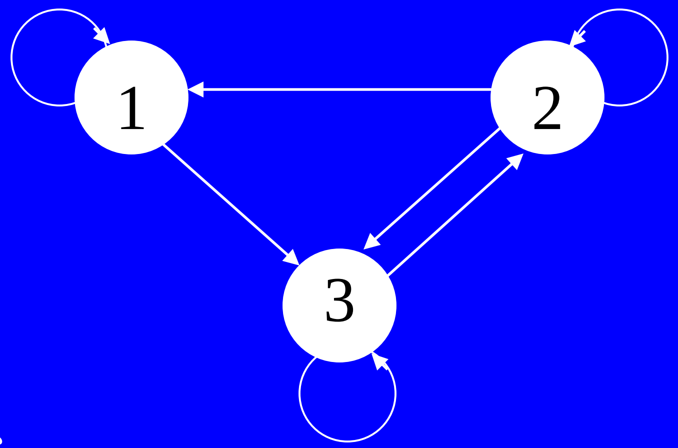
A Markov chain is ergodic if it is aperiodic and allows the attainment of any future state from any initial state after one or more transitions. If these conditions hold, then

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)}$$

For example,

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0 & 0.9 & 0.1 \end{bmatrix}$$

State-transition network



Conclusion: chain is ergodic.

Example 1: Given that a person last cola purchase was COKE, there is a 90% chance that his next cola purchase will also be COKE. If a person last cola purchase was PEPSI, there is a 80% chance that his next cola purchase will also be PEPSI. The present market share of the COKE and PEPSI is 55% and 45% respectively. Construct the TPM. In the long run, what is the market share of such cola?