can you give a more compolete description of your proposed algorithm for PopperSAT?

PopperSAT (for your fragment: finite additivity + full product axiom on normal ψ + "abnormal ⇒ all conditionals = 1", quantifier-free polynomial (in)equalities over $P(\varphi \mid \psi)$) can be implemented as a **finite outer search** over which conditioning events are normal (and at what "rank"), coupled with an inner **QF_NRA** check that looks very close to PrSAT's stochastic truth-table encoding.[1] [2] [3]

1) Semantic target (what we search for)

Work over the finite space of classical valuations W for n atoms, so $|W| = 2^n$, exactly as in the stochastic truth-table setup used by PrSAT. [1]
Use a finite **lexicographic probability system** (LPS) $(\mu_1, \ldots, \mu_K)$, where each μ_ℓ is an ordinary finitely additive probability measure on W, and define $P(\varphi \mid \psi)$ by selecting the first layer where $\mu_\ell(\psi) > 0$.[4] [3]
In this representation, when ψ is normal at the selected layer, $P(- \mid \psi)$ is literally a ratio inside μ_ℓ, so finite additivity and the Popper product axiom hold "for free"; when ψ is abnormal, Popper-style abnormality makes $P(- \mid \psi)$ trivial, and under your convention this is $P(\varphi \mid \psi) = 1$ for all φ.[2] [3]

2) Inputs and preprocessing

Input: a conjunction of polynomial equalities/inequalities over symbols $P(\varphi \mid \psi)$, where φ, ψ are propositional formulas over n atoms.
Precompute, for every propositional formula θ that occurs in the input (and every $\varphi \wedge \psi$), the set of worlds $S_\theta = \{w \in W : w \models \theta\}$ (a bitmask is convenient).

Let $\Psi = \{\psi_1, \ldots, \psi_m\}$ be the set of distinct conditioning formulas appearing in the input.

3) Inner SMT encoding (division-free, conjunctive)

Choose a layer bound K. A safe, simple choice is $K = m$ (you only need enough layers to distinguish the finitely many conditioning contexts that actually appear).
Introduce reals $w_{\ell,i}$ for $\ell = 1..K$, $i = 1..|W|$, intended as $\mu_\ell(\{w_i\})$, with:

- $w_{\ell,i} \geq 0$
- $\sum_i w_{\ell,i} = 1$ for each ℓ

This is exactly the PrSAT "stochastic truth-table" move, just replicated across layers.[1]

For each conditional term $P(\varphi \mid \psi_j)$ that appears, introduce a real variable $p_{\varphi|\psi_j}$ with $0 \le p_{\varphi|\psi_j} \le 1$.[2]

Define (as abbreviations you expand into sums) for each j, ℓ:

- $d_{j,\ell} := \sum_{i \in S_{\psi_j}} w_{\ell,i}$ (denominator mass of ψ_j at layer ℓ)
- $n_{j,\ell}(\varphi) := \sum_{i \in S_{\varphi \wedge \psi_j}} w_{\ell,i}$ (numerator mass)

Then, given a choice of which layer is "active" for ψ_j, you add **only polynomial** constraints, no division:

- Normal at layer k_j: $n_{j,k_j}(\varphi) = p_{\varphi|\psi_j} \cdot d_{j,k_j}$
- Abnormal: $p_{\varphi|\psi_j} = 1$ (your convention)[2]

Finally, translate the user's polynomial constraints by replacing each $P(\varphi \mid \psi_j)$ with $p_{\varphi|\psi_j}$.

This inner problem is a conjunction of polynomial (in)equalities over reals (QF_NRA), in the same spirit as PrSAT's translation for Kolmogorov conditionals.[1]

4) Outer search (the only "non-PrSAT" part)

You still need to connect each ψ_j to either:

- an **active layer index** $k_j \in \{1, \dots, K\}$, meaning "ψ_j is normal first at layer k_j", or
- **abnormal**, meaning it has zero mass at every layer.

Encode that by branching externally (multiple Z3 calls), so each call stays purely conjunctive:

If ψ_j is normal first at k_j, assert:

- $d_{j,\ell} = 0$ for all $\ell < k_j$
- $d_{j,k_j} > 0$

If ψ_j is abnormal, assert:

- $d_{j,\ell} = 0$ for all $\ell \le K$

This "first positive layer" mechanism is exactly what makes LPSs correspond to Popper functions on finite spaces.[3]

Practical branching strategy

Naively you have $(K+1)^m$ branches, but you can prune heavily by using logical relations between conditioning events before calling Z3:

- If $\psi \models \chi$, then first(ψ) cannot be *later* than first(χ) (since any layer giving ψ positive mass also gives χ positive mass).
- If ψ is unsatisfiable, force it abnormal immediately.
- Merge syntactically different but logically equivalent ψ's.

Implementation-wise, do a DFS that incrementally assigns layer/abnormal statuses and uses Z3 `push/pop` to prune early when partial assignments already make the QF_NRA constraints inconsistent.

5) Soundness/completeness intuition (for your axioms)

Soundness: any satisfying assignment to the $w_{\ell,i}$ defines K finitely additive measures, and the "first layer" constraints make each ψ either normal (ratio-defined) or abnormal (constant-1), matching your convention.[3] [2]
On normal ψ, conditionals are computed as ratios in one underlying probability measure, so the finite additivity and Popper product axiom behavior comes along automatically, rather than being separately axiomatized in SMT.[2]
Completeness (finite case): for the finite W you care about, LPS/Popper representation results ensure you're not losing models by searching in this form.[4] [3]

6) Output (model extraction)

If Z3 returns `sat` on some branch, you read off:

- The layers $\mu_\ell(w_i) = w_{\ell,i}$
- For each ψ_j, whether it's abnormal or its first layer k_j
- Conditional values $P(\varphi \mid \psi_j) = p_{\varphi|\psi_j}$

This gives you a concrete Popper model in essentially the same "stochastic truth-table" style as PrSAT, just with multiple layers.

If you want, I can take your earlier $n = 2$ example and show the exact PopperSAT SMT-LIB generated for each of the four (A normal/abnormal)×(B normal/abnormal) branches, so you can see the full end-to-end pipeline in the simplest nontrivial case.

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3. https://www.cs.cornell.edu/home/halpern/papers/lex.pdf
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5. https://plato.stanford.edu/ARCHIVES/WIN2009/entries/reasoning-defeasible/suppl4.html
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