Two Methods to Compute Survival Probability with Dirichlet Boundary Conditions

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Our agent obeys the following Langevin equation:

$$dX = \mu\, dt + \sqrt{2D}\, dW, \qquad X(0) = 0,$$

where X is the current belief state, μ is the "natural" drift towards a particular decision, D is the diffusion coefficient, and W is a Wiener process. The decision thresholds are set at $x = -L < 0$ and $x = H > 0$.

1 Computation using the Forward FPE

Suppose the agent begins at $X(0) = 0$. Then the probability that the agent has not decided/escaped at time t is the *survival probability*, given by

$$S(t) = \int_{-L}^{H} c(x, t|0, 0)\, dx,$$

where $c(x, t|0, 0)$ is the decision probability concentration given the initial conditions. It will be designated $c(x, t)$ from here on.

Now, $c(x, t)$ obeys the forward Fokker-Planck equation:

$$\begin{aligned} c_t &= -\mu c_x + D c_{xx}, \\ c(-L, t) &= c(H, t) = 0, \\ c(x, 0) &= \delta(x). \end{aligned} \tag{1}$$

A change of variables $x \mapsto x - \mu t$ transforms the PDE into a drift-free diffusion equation, from which it is apparent that the fundamental solution is

$$\Phi(x, t; -\mu) = \frac{1}{\sqrt{4\pi D t}} \exp\big(-(x - \mu t)^2/4Dt\big),$$

where noting the use of the parameter $-\mu$ in the forward FPE will be relevant later.

Now we wish to impose the boundary conditions, which we can do via an infinite series generated by method of images.

By inspection, we arrive at

$$c(x,t) = \Phi(x,t) + \sum_{n=1}^{\infty}(-1)^n \Big[e^{-\mu\xi_n^{(L,H)}D^{-1}} \Phi\left(x + 2\xi_n^{(L,H)},\, t\right) \dots \\ + e^{\mu\xi_n^{(H,L)}D^{-1}} \Phi\left(x - 2\xi_n^{(H,L)},\, t\right) \Big],$$

where

$$\xi_n^{(a,b)} = \left\lceil \frac{n}{2} \right\rceil a + \left\lfloor \frac{n}{2} \right\rfloor b.$$

This particular form isn't important for our purposes; let's denote the concentration by

$$c(x,t) = F\big(\Phi(x,t;-\mu)\big).$$

Then the survival probability is given by

$$S(t) = \int_{-L}^{H} F\big(\Phi(x,t;-\mu)\big)\, dx.$$

2 Computation using the Backward FPE

(See Gardiner section 5.2.7(a) for most of the derivation.)

Suppose the agent begins at $X(0) = x$. The decision probability concentration is $c(y,t|x,0)$, and we define the survival probability by

$$S(x,t) = \int_{-L}^{H} c(y,t|x,0)\, dy.$$

For such a time-homogeneous system, we have

$$c(y,t|x,0) = c(y,0|x,-t)$$

and hence the corresponding backward Fokker-Planck equation is

$$\partial_t c(y,t|x,0) = \mu\partial_x c(y,t|x,0) + D\partial_{xx} c(y,t|x,0).$$

Gardiner asserts that $S(x,t)$ therefore obeys the following backward Fokker-Planck equation:

$$\begin{aligned} S_t &= \mu S_x + DS_{xx}, \\ S(L,t) &= S(H,t) = 0, \\ S(x,0) &= \begin{cases} 1, & -L \le x \le H \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{2}$$

A change of variables $x \mapsto x + \mu t$ transforms the PDE into a drift-free diffusion equation, from which it is apparent that the fundamental solution is

$$\Phi(x,t;\mu) = \frac{1}{\sqrt{4\pi Dt}} \exp\big(-(x+\mu t)^2/4Dt\big).$$

Now we impose the boundary conditions. which gives the Green's function

$$G(x,t) = F\big(\Phi(x,t;\mu)\big),$$

where F is defined as before. Note the use of the parameter μ here, as opposed to $-\mu$ when using the forward method.

Convolving with the initial condition, we have

$$S(x,t) = \int_{-L}^{H} F\big(\Phi(x-y,t;\mu)\big)\,dy = \int_{-L}^{H} F\big(\Phi(y-x,t;-\mu)\big)\,dy,$$

and evaluating at $x=0$ to ensure $X(0)=0$, we have

$$S(0,t) = \int_{-L}^{H} F\big(\Phi(y,t;-\mu)\big)\,dy,$$

which is the same as $S(t)$ defined in section 1.