Notes on "Finite-Dimensional Vector Spaces" by Paul R. Halmos

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Each `\section` corresponds to the scope of one member's assignment, and each `\subsection` corresponds to one theorem or exercise in the textbook, specified in the format $m.n$ where m is the section number and n is the theorem/exercise number. If n is not given, we use $n = 1$ instead.

Contents

1 Toga (2022/09/19)

1.1 Exercise 1.1

(a) Since addition is commutative, $0+\alpha = \alpha+0$ holds. We also have $\alpha+0 = \alpha$ by definition, hence $0 + \alpha = \alpha$.

2 Mohehe (2022/09/27)

2.1 Exercise 1.1

(b) If $\alpha + \beta = \alpha + \gamma$, we have $\beta = \beta + 0 = 0 + \beta = (\alpha + (-\alpha)) + \beta = ((-\alpha) + \alpha) + \beta = (-\alpha) + (\alpha + \beta) = (-\alpha) + (\alpha + \gamma) = ((-\alpha) + \alpha) + \gamma = (\alpha + (-\alpha)) + \gamma = 0 + \gamma = \gamma + 0 = \gamma$ by definition. Therefore, $\beta = \gamma$ holds.

(c) We have $\alpha+(\beta-\alpha) = \alpha+(\beta+(-\alpha)) = \alpha+((-\alpha)+\beta) = (\alpha+(-\alpha))+\beta = 0 + \beta = \beta + 0 = \beta$ by definition. Therefore, $\alpha + (\beta - \alpha) = \beta$ holds.

(d) We have $\alpha 0 + \alpha 0 = \alpha(0 + 0) = \alpha 0 = \alpha 0 + 0$ by definition, hence $\alpha 0 = 0$ by Exercise 1(b). We also have $\alpha \cdot 0 = 0 \cdot \alpha$ by definition. Therefore, $\alpha \cdot 0 = 0 \cdot \alpha = 0$

(e) We have $\alpha+(-1)\alpha = 1\alpha+(-1)\alpha = (1+(-1))\alpha = 0\alpha = 0$ by definition and Exercise 1(d). Since the additive inverse is unique, we obtain $(-1)\alpha = -\alpha$.

(f) We have $(-\alpha)(-\beta) = ((-1)\alpha)((-1)\beta) = (\alpha(-1))((-1)\beta) = \alpha((-1)((-1)\beta)) = \alpha((-1)(-1)\beta)$ by Exercise 1(e) and definition. We also have $(-1)(-1) = 0 + (-1)(-1) = (1 + (-1)) + (-1)(-1) = 1 + (-1) + (-1)(-1) = 1 + (-1)((-1) + 1) = 1 + (-1)(1 + (-1)) = 1 + (-1)0 = 1 + 0 = 1$ by definition. By it and definition, $\alpha((-1)(-1)\beta) = \alpha(1\beta) = \alpha(\beta 1) = \alpha\beta$ holds. Therefore, $(-\alpha)(-\beta) = \alpha\beta$ holds.

(g) If $\alpha\beta = 0$, suppose $\alpha \neq 0$ and $\beta \neq 0$ hold. By supposition and definition, we have $0 = \alpha^{-1}0 = \alpha^{-1}(\alpha\beta) = (\alpha^{-1}\alpha)\beta = (\alpha\alpha^{-1})\beta = 1\beta = \beta 1 = \beta$, hence $\beta = 0$. However, this result contradicts supposition, "$\alpha \neq 0$ and $\beta \neq 0$". Therefore, if $\alpha\beta = 0$, then either $\alpha = 0$ or $\beta = 0$ (or both).

3 Mohehe (2022/09/19)

3.1 Exercise 1.2

(a) The set of positive integers is not a field since there is no additive inverse for 1.

(b) The set of integers is not a field since there is no multiplicative inverse for 2.

(c) There exists a bijective map φ from $\mathcal{N}$ (or $\mathcal{Z}$) to $\mathcal{Q}$ [1], where $\mathcal{Q}$ is a field [2]. We can make $\mathcal{N}$ a field by re-defining (i) addition by $a \oplus b = \varphi^{-1}(\varphi(a) + \varphi(b))$ and (ii) multiplication by $a \otimes b = \varphi^{-1}(\varphi(a)\varphi(b))$ for each $a, b \in \mathcal{N}$. Note that the additive and multiplicative identities become $\varphi^{-1}(0)$ and $\varphi^{-1}(1)$, respectively. For each $\alpha \in \mathcal{N}$, the additive inverse becomes $\varphi^{-1}(-\varphi(\alpha))$, and the multiplicative inverse becomes $\varphi^{-1}(1/\varphi(\alpha))$ if $\alpha \neq \varphi^{-1}(0)$.

Let $\alpha, \beta, \gamma, \alpha', \beta' \in \mathcal{N}$. Note that

1) $\alpha \oplus \beta = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)) = \varphi^{-1}(\varphi(\beta) + \varphi(\alpha)) = \beta \oplus \alpha$ holds.(addition is commutative)

2) $\alpha \oplus (\beta \oplus \gamma) = \alpha \oplus (\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(\varphi(\beta) + \varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha) + (\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}((\varphi(\alpha) + \varphi(\beta)) + \varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha) + \varphi(\beta))) + \varphi(\gamma)) = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)) \oplus \gamma = (\alpha \oplus \beta) \oplus \gamma$ holds.(addition is associative)

3) $\alpha \oplus \varphi^{-1}(0) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(0))) = \varphi^{-1}(\varphi(\alpha) + 0) = \varphi^{-1}(\varphi(\alpha)) = \alpha$ holds.(there exists additive identity, $\varphi^{-1}(0)$) If α' and β' are additive identity, we have $\alpha' = \alpha' \oplus \beta' = \beta' \oplus \alpha' = \beta'$ by 1) and the definition of additive identity.(additive identity is unique)

4) $-\varphi(\alpha) \in \mathcal{Q}$ holds by definition, so $\varphi^{-1}(-\varphi(\alpha)) \in \mathcal{N}$ holds. Therefore, $\alpha \oplus \varphi^{-1}(-\varphi(\alpha)) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(-\varphi(\alpha)))) = \varphi^{-1}(\varphi(\alpha) + (-\varphi(\alpha))) = \varphi^{-1}(0)$ holds.(for each α ($\alpha \in \mathcal{N}$), there exists additive inverse) For each α, if α' and β' are additive inverse, we have $\alpha' = \alpha' \oplus \varphi^{-1}(0) = \alpha' \oplus (\alpha \oplus \beta') = (\alpha' \oplus \alpha) \oplus \beta' = (\alpha \oplus \alpha') \oplus \beta' = \varphi^{-1}(0) \oplus \beta' = \beta \oplus \varphi^{-1}(0) = \beta'$ by 1), 2), 3) and the definition of additive inverse.(additive inverse is unique)

5) $\alpha \otimes \beta = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)) = \varphi^{-1}(\varphi(\beta)\varphi(\alpha)) = \beta \otimes \alpha$ holds.(multiplication is commutative)

6) $\alpha \otimes (\beta \otimes \gamma) = \alpha \otimes (\varphi^{-1}(\varphi(\beta)\varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(\varphi(\beta)\varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\beta)))\varphi(\gamma)) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)) \otimes \gamma = (\alpha \otimes \beta) \otimes \gamma$ holds.(multiplication is associative)

7) $\alpha \otimes \varphi^{-1}(1) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(1))) = \varphi^{-1}(\varphi(\alpha) \cdot 1) = \alpha$ holds.(there exists additive identity, $\varphi^{-1}(1)$) If α' and β' are additive identity, we have $\alpha' = \alpha' \otimes \beta' = \beta' \otimes \alpha' = \beta'$ by 5) and definition of multiplicative identity.(multiplicative identity is unique)

8) For each α ($\alpha \neq \varphi^{-1}(0)$), $(1/\varphi(\alpha)) \in \mathcal{Q}$ holds by definition, so $\varphi^{-1}(1/\varphi(\alpha)) \in \mathcal{N}$ holds. Therefore, $\alpha \otimes \varphi^{-1}(1/\varphi(\alpha)) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(1/\varphi(\alpha)))) = \varphi^{-1}(\varphi(\alpha)(1/\varphi(\alpha))) = \varphi^{-1}(1)$ holds.(for each α ($\alpha \in \mathcal{N}$), there exists multiplicative inverse) For each α ($\alpha \neq \varphi^{-1}(0)$), if α' and β' are multiplicative inverse, we have $\alpha' = \alpha' \otimes \varphi^{-1}(1) = \alpha' \otimes (\alpha \otimes \beta') = (\alpha' \otimes \alpha) \otimes \beta' = (\alpha \otimes \alpha') \otimes \beta' = \varphi^{-1}(1) \otimes \beta' = \beta' \otimes \varphi^{-1}(1) = \beta'$ by 5), 6), 7) and the definition of multiplicative inverse.(multiplicative inverse is unique)

9) $\alpha \otimes (\beta \oplus \gamma) = \alpha \otimes (\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(\varphi(\beta) + \varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha)(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta) + \varphi(\alpha)\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\beta))) + \varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha \otimes \beta) + \varphi(\alpha \otimes \gamma)) = \alpha \otimes \beta \oplus \alpha \otimes \gamma$ holds.(distributive law stands)

4 Mohehe (2022/10/8)

4.1 Exercise 1.3

For two integers a and b, we denote by $a \,\%\, b$ the remainder after dividing a by b, and write $b \mid a$ if and only if $a \,\%\, b = 0$. For clarity, we denote the ordinary sum and product of two integers a and b by $a +_{\mathcal{Z}} b$ and $a \cdot_{\mathcal{Z}} b$, respectively. Note that $\alpha + \beta = (\alpha +_{\mathcal{Z}} \beta) \,\%\, m$ and $\alpha\beta = (\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m$ for $\alpha, \beta \in \mathcal{Z}_m$.

(a) Let $\alpha, \beta, \gamma \in \mathcal{Z}_m$, $k \in \mathcal{Z}$

1' Proof : if m is a prime, $\mathcal{Z}_m$ is a field.
Suppose m is a prime,

1) $\alpha + \beta = (\alpha +_{\mathcal{Z}} \beta) \,\%\, m = (\beta +_{\mathcal{Z}} \alpha) \,\%\, m = \beta + \alpha$ (addition is commutative)
2) Since $\alpha +_{\mathcal{Z}} (\beta + \gamma) = \alpha +_{\mathcal{Z}} (\beta +_{\mathcal{Z}} \gamma) \,\%\, m \equiv \alpha +_{\mathcal{Z}} (\beta +_{\mathcal{Z}} \gamma) = (\alpha +_{\mathcal{Z}} \beta) +_{\mathcal{Z}} \gamma \equiv (\alpha +_{\mathcal{Z}} \beta)\%m +_{\mathcal{Z}} \gamma = (\alpha + \beta) +_{\mathcal{Z}} \gamma \pmod m$ holds, $\alpha + (\beta + \gamma) = (\alpha +_{\mathcal{Z}} (\beta + \gamma))\%m = ((\alpha + \beta) +_{\mathcal{Z}} \gamma)\%m = (\alpha + \beta) + \gamma$ holds.(addition is associative)
3) $\alpha + 0 = (\alpha +_{\mathcal{Z}} 0)\%m = \alpha\%m = \alpha$ (there exists additive identity) By it and 1), if β and γ are additive identity, $\beta = \beta + \gamma = \gamma + \beta = \gamma$ (additive identity is unique)
4) If $\alpha +_{\mathcal{Z}} \beta = m$, $\alpha + \beta = (\alpha +_{\mathcal{Z}} \beta) \,\%\, m = m \,\%\, m = 0$ (there exists additive inverse)
5) $\alpha\beta = (\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m = (\beta \cdot_{\mathcal{Z}} \alpha) \,\%\, m = \beta\alpha$ (multiplication is commutative)
6) Since $\alpha \cdot_{\mathcal{Z}} (\beta\gamma) = \alpha \cdot_{\mathcal{Z}} ((\beta \cdot_{\mathcal{Z}} \gamma) \,\%\, m) \equiv \alpha \cdot_{\mathcal{Z}} (\beta \cdot_{\mathcal{Z}} \gamma) = (\alpha \cdot_{\mathcal{Z}} \beta) \cdot_{\mathcal{Z}} \gamma \equiv ((\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m) \cdot_{\mathcal{Z}} \gamma = (\alpha\beta) \cdot_{\mathcal{Z}} \gamma \pmod m$ holds, $\alpha(\beta\gamma) = (\alpha \cdot_{\mathcal{Z}} (\beta\gamma)) \,\%\, m = ((\alpha\beta) \cdot_{\mathcal{Z}} \gamma) \,\%\, m = (\alpha\beta)\gamma$ holds.(multiplication is associative)

7) $\alpha 1 = (\alpha_{\mathcal{Z}} 1) \,\%\, m = \alpha \,\%\, m = \alpha$ (there exists multiplicative identity) By it and 5), if β and γ are multiplicative identity, $\beta = \beta\gamma = \gamma\beta = \gamma$ (multiplicative identity is unique)

8) For all $\alpha(\alpha \neq 0)$, suppose there doesn't exist β that makes $\alpha\beta = 1$. There exist $\beta, \gamma \in \mathcal{Z}_m$ with $\beta \neq \gamma$ and $\alpha\beta = \alpha\gamma$, because β is any one from 0 to $m-1$ and $\alpha\beta$ is any one from 0 to $m-1$ except 1. Therefore, $(\alpha \cdot_{\mathcal{Z}} \beta +_{\mathcal{Z}} (-\alpha \cdot_{\mathcal{Z}} \gamma) =)\ \alpha \cdot_{\mathcal{Z}} (\beta +_{\mathcal{Z}} (-\gamma)) = km$ holds. The right side has divisor m, but it contradicts that the left side doesn't have divisor of m except 1, because $0 < \alpha < (m-1)$ and $((-m) < (\beta +_{\mathcal{Z}} (-\gamma)) < 0$ or $0 < (\beta +_{\mathcal{Z}} (-\gamma)) < m)$ holds. Thus, there exists β that makes $\alpha\beta = 1$.(there exists maltiplicative inverse)

A brief proof: Since each $\alpha \in \mathcal{Z}_m \setminus \{0\}$ is coprime to m, there exist integers x and y such that $\alpha \cdot_{\mathcal{Z}} x +_{\mathcal{Z}} m \cdot_{\mathcal{Z}} y = 1$ by [3]. Putting $x' = x \,\%\, m \in \mathcal{Z}_m$, we obtain $\alpha x' = (\alpha \cdot_{\mathcal{Z}} x) \,\%\, m = (\alpha \cdot_{\mathcal{Z}} x +_{\mathcal{Z}} m \cdot_{\mathcal{Z}} y) \,\%\, m = 1 \,\%\, m = 1$. Hence $x' = \alpha^{-1}$.

9) $\alpha(\beta + \gamma) = (\alpha \cdot_{\mathcal{Z}} (\beta + \gamma)) \,\%\, m = (\alpha \cdot_{\mathcal{Z}} ((\beta +_{\mathcal{Z}} \gamma) \,\%\, m)) \,\%\, m \equiv (\alpha \cdot_{\mathcal{Z}} (\beta +_{\mathcal{Z}} \gamma)) \,\%\, m = (\alpha \cdot_{\mathcal{Z}} \beta + \alpha \cdot_{\mathcal{Z}} \gamma) \,\%\, m \equiv ((\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m +_{\mathcal{Z}} (\alpha \cdot_{\mathcal{Z}} \gamma) \,\%\, m) \,\%\, m \equiv (\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m + (\alpha \cdot_{\mathcal{Z}} \gamma) \,\%\, m \equiv \alpha\beta + \alpha\gamma$ holds.(distributive law stands)

In conclusion, if m is a prime, $\mathcal{Z}_m$ is a field.

2' Proof : If $\mathcal{Z}_m$ is a field, m is a prime.
By contraposition, it is equivalent to prove "If m is not a prime, $\mathcal{Z}_m$ is not a field." We can show 1) to 7) and 9) in the same way as 1'. For each m, suppose there exist α and β that make $\alpha\beta = 1$. m is not a prime, so let p be one of prime factors of m and then we have $m = p \cdot_{\mathcal{Z}} p'$.$(p' \in \mathcal{Z}$ and $1 < p' < m)$
If $\alpha = p$, by $\alpha\beta = 1$ and $m = p \cdot_{\mathcal{Z}} p'$, we have $\alpha \cdot_{\mathcal{Z}} \beta = k \cdot_{\mathcal{Z}} m + 1 (k \in \mathcal{Z}) \Leftrightarrow p \cdot_{\mathcal{Z}} \beta = k \cdot_{\mathcal{Z}} p \cdot_{\mathcal{Z}} p' +_{\mathcal{Z}} 1 \Leftrightarrow (\beta +_{\mathcal{Z}} (-k \cdot_{\mathcal{Z}} p')) \cdot_{\mathcal{Z}} p = 1$. The right side is 1 but the left one is not 1 because of $1 < p$ and $\beta +_{\mathcal{Z}} (-k \cdot_{\mathcal{Z}} p') \in \mathcal{Z}$. Therefore It is contradicted. For each m, there doesn't exist α and β that make $\alpha\beta = 1$. In conclusion, "If m is not a prime, $\mathcal{Z}_m$ is not a field." and "If $\mathcal{Z}_m$ is a field, m is a prime."

Because of 1' and 2', $\mathcal{Z}_m$ is a field if and only if m is a prime.

(b) 4

(c) 5

4.2 Exercise 1.4

Define $\alpha_n = \overbrace{1 + \cdots + 1}^{n \text{ terms}}$ for $n \in \{1, 2, \ldots\}$. Then,

$$\alpha_m \alpha_n = \alpha_m (\overbrace{1 + \cdots + 1}^{n \text{ terms}})$$

$$
\begin{aligned}
&= \alpha_m((\overbrace{1+\cdots+1}^{(n-1)\text{ terms}})+1) \\
&= \alpha_m(\overbrace{1+\cdots+1}^{(n-1)\text{ terms}})+\alpha_m\cdot 1 && \text{(by distributive law)} \\
&= \alpha_m(\overbrace{1+\cdots+1}^{(n-1)\text{ terms}})+\alpha_m && \text{(by definition of multiplicative identity)} \\
&= \cdots \\
&= \alpha_m(1+1)+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \alpha_m\cdot 1+\alpha_m\cdot 1+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by distributive law)} \\
&= \overbrace{\alpha_m+\cdots+\alpha_m}^{n\text{ terms}} && \text{(by definition of multiplicative identity)} \\
&= \overbrace{1+\cdots+1}^{m\text{ terms}}+(\overbrace{1+\cdots+1}^{m\text{ terms}})+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \overbrace{1+\cdots+1}^{m\text{ terms}}+((\overbrace{1+\cdots+1}^{(m-1)\text{ terms}})+1)+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \overbrace{1+\cdots+1}^{m\text{ terms}}+(1+(\overbrace{1+\cdots+1}^{(m-1)\text{ terms}}))+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by commutative property)} \\
&= \overbrace{1+\cdots+1}^{(m+1)\text{ terms}}+(\overbrace{1+\cdots+1}^{(m-1)\text{ terms}})+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by associative property)} \\
&= \cdots \\
&= \overbrace{1+\cdots+1}^{(2m-2)\text{ terms}}+(1+1)+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \overbrace{1+\cdots+1}^{2m\text{ terms}}+\overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by associative property)} \\
&= \cdots \\
&= \overbrace{1+\cdots+1}^{m(n-1)\text{ terms}}+\alpha_m \\
&= \cdots \\
&= \overbrace{1+\cdots+1}^{(mn-2)\text{ terms}}+(1+1) \\
&= \overbrace{1+\cdots+1}^{mn\text{ terms}} && \text{(by associative property)} \\
&= \alpha_{mn}
\end{aligned}
$$

for all m, n.

Assume there exists an n with $\alpha_n = 0$ but $\alpha_k \neq 0$ for any $k < n$. It suffices to prove that n is a prime.
Suppose n is not any prime. Let p be one of prime factors of n and then we have $n = pp'(p' \in \mathcal{N}$ and $p' > 1)$. By $\alpha_m\alpha_n = \alpha_{mn}$ for all m and n, $\alpha_n = \alpha_p\alpha_{p'}$ holds. We have either $\alpha_p = 0$ or $\alpha_{p'} = 0$ (or both) because of $\alpha_n = 0$ and Exercise 1.1 (g). However, it is contradictory to $\alpha_p = 0$ and $\alpha_{p'} = 0$. Therefore, n is a prime.

4.3 Exercise 1.5

(a) For the followings, it is used that $\mathcal{Q}$ and $\mathcal{R}$ are fields. Note that $\sqrt{2} \in \mathcal{R}$ and $\sqrt{2} \notin \mathcal{Q}$
Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{Q}(\sqrt{2}) \subset \mathcal{R}$.
For all $\alpha_1, \alpha_2 \in \mathcal{Q}(\sqrt{2})$, $\alpha_1 + \alpha_2 \in \mathcal{Q}(\sqrt{2})$ and $\alpha_1\alpha_2 \in \mathcal{Q}(\sqrt{2})$ by the followings.
There exist $a, b, c, d \in \mathcal{Q}$, $\alpha_1 = a + b\sqrt{2}$ and $\alpha_2 = c + d\sqrt{2}$ hold.
We have $\alpha_1 + \alpha_2 = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$ and $(a + c), (b + d) \in \mathcal{Q}$, so $\alpha_1 + \alpha_2 \in \mathcal{Q}(\sqrt{2})$.
In addition, we have $\alpha_1\alpha_2 = (a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd)+(ad+bc)\sqrt{2}$ and $(ac + 2bd), (ad + bc) \in \mathcal{Q}$, so $\alpha_1\alpha_2 \in \mathcal{Q}(\sqrt{2})$

1) $\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1$ (addition is commutative)
2) $\alpha_1 + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + \alpha_3$ (addition is associative)
3) We have $0 = 0 + 0\sqrt{2} \in \mathcal{Q}(\sqrt{2})$ and $\alpha_1 + 0 = \alpha_1$
($\mathcal{Q}(\sqrt{2})$ has additive identity)
4) For all $\alpha_1 \in \mathcal{Q}(\sqrt{2})$, put $\alpha_1 = a + b\sqrt{2}$ with $a, b \in \mathcal{Q}$. There exists $\alpha_1' \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1' = (-a) + (-b)\sqrt{2}$. We have $\alpha_1 + \alpha_1' = a + b\sqrt{2} + (-a) + (-b)\sqrt{2} = a - a + b\sqrt{2} - b\sqrt{2} = 0$. Therefore, to every $\alpha_1 \in \mathcal{Q}(\sqrt{2})$, there corresponds $\alpha_1' \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1 + (-\alpha_1) = 0$
5) $\alpha_1\alpha_2 = \alpha_2\alpha_1$ (multiplication is commutative)
6) $\alpha_1(\alpha_2\alpha_3) = (\alpha_1\alpha_2)\alpha_3$ (multiplication is associative)
7) We have $1 = 1 + 0\sqrt{2} \in \mathcal{Q}(\sqrt{2})$ and $\alpha_1 \cdot 1 = \alpha_1$
($\mathcal{Q}(\sqrt{2})$ has multiplicative identity)
8) For all $\alpha_1 \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1 \neq 0$, put $\alpha_1 = a + b\sqrt{2}$ with $a, b \in \mathcal{Q}$. In this case, $a \neq 0$ or $b \neq 0$ holds by the followings.
"If $\alpha_1 = 0$, we have $\alpha_1 = a + b\sqrt{2} = 0 \Leftrightarrow a = -b\sqrt{2}$. Therefore, $a = b = 0$ by $a, b \in \mathcal{Q}$."
Let $\alpha_1'' = \dfrac{a}{a^2 - 2b^2} + \left(-\dfrac{b}{a^2 - 2b^2}\right)\sqrt{2} \in \mathcal{Q}(\sqrt{2})$. Note that we have $a^2 - 2b^2 = (a + b\sqrt{2})(a - b\sqrt{2})$ and $a, b \in \mathcal{Q}$ with ($a \neq 0$ or $b \neq 0$), so we have $a+b\sqrt{2} \neq 0$ and $a-b\sqrt{2} \neq 0$, and then $a^2-2b^2 \in \mathcal{Q}$ with $a^2-2b^2 \neq 0$. We have $\alpha_1\alpha_1'' = (a+b\sqrt{2})\left(\dfrac{a}{a^2 - 2b^2} + \left(-\dfrac{b}{a^2 - 2b^2}\right)\sqrt{2}\right) =$

$\dfrac{a^2 - ab\sqrt{2} + ab\sqrt{2} - 2b^2}{a^2 - 2b^2} = 1$. Therefore, to every $\alpha_1 \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1 \neq 0$, there exists $\alpha_1'' \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1\alpha_1'' = 1$

9) $\alpha_1(\alpha_2 + \alpha_3) = \alpha_1\alpha_2 + \alpha_1\alpha_3$ (distributive law stands)

from 1) to)9, $\mathcal{Q}(\sqrt{2})$ is a field.

(b) Let $\mathcal{Z}(\sqrt{2})$ be the set of all numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are integers. If $\mathcal{Z}(\sqrt{2})$ is a field, $2 = 2 + 0\sqrt{2}(\in \mathcal{Z}(\sqrt{2}))$ has multiplicative inverse. There exists $\exists\beta_1 = \{\alpha + \beta\sqrt{2} | \beta_1 \in \mathcal{Z}(\sqrt{2})\}$ with $2\beta_1 = 1 \iff \beta_1 = \frac{1}{2}$. However, $\frac{1}{2} \notin \mathcal{Z}(\sqrt{2})$ holds, so $\mathcal{Z}(\sqrt{2})$ is not a field.

Another way : Let $\mathcal{Z}(\sqrt{2})$ be the set of all numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are integers. $\mathcal{Z}(\sqrt{2})$ is not a field since there is no multiplicative inverse for $2 + \sqrt{2} \in \mathcal{Z}(\sqrt{2})$ by the followings.
Suppose there exists multiplicative inverse for $2 + \sqrt{2}$. There exists $\exists\beta_1 \in \mathcal{Z}(\sqrt{2})$ with $\beta_1 = \alpha + \beta\sqrt{2}$ and $(2 + \sqrt{2})\beta_1 = 1$ by supposition. Therefore, we have $(2+\sqrt{2})\beta_1 = (2+\sqrt{2})(\alpha+\beta\sqrt{2}) = 2(\alpha+\beta)+(\alpha+2\beta)\sqrt{2} = 1 \iff 2(\alpha+\beta)-1 = -(\alpha+2\beta)\sqrt{2} \implies 2(2(\alpha+\beta)^2 - 2(\alpha+\beta)) - 1 = 2(\alpha+2\beta)^2$. It is contradicted because the left is odd number and the right is even number. Therefore, $(2 + \sqrt{2})\beta_1 = 1$ is contradicted and then there is no multiplicative inverse for $2 + \sqrt{2}$.

4.4 Exercise 1.6

(a) Let P be such set of all polynomials with integer coefficients, $\mathrm{id} \in P$ be $\mathrm{id}(x) = x$ $(x \in \mathcal{R})$, and $I \in P$ be $I(x) = 1$ $(x \in \mathcal{R})$.

Suppose there exists $q \in P$ with $\mathrm{id} \cdot q = I$. Then, we have $\mathrm{id}(0) \cdot q(0) = 0$, and it is contradicted to supposition. Therefore, there does not exist $q \in P$ with $\mathrm{id} \cdot q = I$. In other words, id does not have the multiplicative inverse. In conclusion, the set of all polynomials with integer coefficients does not form a field.

(b) the set of all polynomials with real number coefficients does not form a field for the same reason.

4.5 Exercise 1.7

(a) Suppose $\mathfrak{F}$ is a field. Let $(\alpha, \beta) \in \mathfrak{F}$ with $\alpha, \beta \in \mathcal{R}$. Then, additive identity would be $(0, 0)$, because for all α, β, we have $\alpha + 0 = \alpha, \beta + 0 = \beta$. In additon, multiplicative identity would be $(1, 1)$, because for all α, β, we have $\alpha 1 = \alpha, \beta 1 = \beta$.

Here, think about $(0, 1)(\neq (0, 0))$. For all (α, β), we have $(0, 1)(\alpha, \beta) = (0, \beta)(\neq (1, 1))$. Therefore, $(0, 1)$ does not have multiplicative inverse. In conclusion, $\mathfrak{F}$ is not a field.

(b) Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \in \mathfrak{F}$ with $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathcal{R}$.

(A) (1) $(\alpha_1,\beta_1)+(\alpha_2,\beta_2) = (\alpha_1+\alpha_2,\beta_1+\beta_2) = (\alpha_2,\beta_2)+(\alpha_1,\beta_1)$.(addition is commutative)

(2) $(\alpha_1,\alpha_1)+((\alpha_2,\beta_2)+(\alpha_3,\beta_3)) = (\alpha_1+\alpha_2+\alpha_3,\beta_1+\beta_2+\beta_3) = ((\alpha_1,\beta_1)+(\alpha_2,\beta_2))+(\alpha_3,\beta_3)$.(addition is associative)

(3) There exists $(0,0) \in \mathfrak{F}$ such that $(\alpha_1,\beta_1)+(0,0) = (\alpha_1,\beta_1)$ for every (α_1,β_1). ($\mathfrak{F}$ has additive identity)

(4) For every (α_1,β_1), there exists $(-\alpha_1,-\beta_1) \in \mathfrak{F}$ such that $(\alpha_1,\beta_1)+(-\alpha_1,-\beta_1) = (0,0)$.

(B) (1) $(\alpha_1,\beta_1)(\alpha_2,\beta_2) = (\alpha_1\alpha_2-\beta_1\beta_2,\alpha_1\beta_2+\alpha_2\beta_1) = (\alpha_2,\beta_2)(\alpha_1,\beta_1)$. (multiplication is commutative)

(2) $((\alpha_1,\beta_1)(\alpha_2,\beta_2))(\alpha_3,\beta_3) = (\alpha_1\alpha_2\alpha_3-\beta_1\beta_2\alpha_3-\alpha_1\beta_2\beta_3-\beta_1\alpha_2\beta_3,\alpha_1\beta_2\alpha_3+\beta_1\alpha_2\alpha_3+\alpha_1\alpha_2\beta_3-\beta_1\beta_2\beta_3) = (\alpha_1,\beta_1)((\alpha_2,\beta_2)(\alpha_3,\beta_3))$.(multiplication is associative)

(3) There exists $(1,0) \in \mathfrak{F}$ such that $(\alpha_1,\beta_1)(1,0) = (\alpha_1,\beta_1)$ for every (α_1,β_1). ($\mathfrak{F}$ has multiplicative identity)

(4) For every $(\alpha_1,\beta_1)(\neq(0,0))$, there exists $\left(\frac{\alpha_1}{\alpha_1^2+\beta_1^2},-\frac{\beta_1}{\alpha_1^2+\beta_1^2}\right) \in \mathfrak{F}$ such that $(\alpha_1,\beta_1)\left(\frac{\alpha_1}{\alpha_1^2+\beta_1^2},-\frac{\beta_1}{\alpha_1^2+\beta_1^2}\right) = (1,0)$.

(C) $(\alpha_1,\beta_1)((\alpha_2,\beta_2)+(\alpha_3,\beta_3)) = (\alpha_1,\beta_1)(\alpha_2+\alpha_3,\beta_2+\beta_3) = (\alpha_1\alpha_2+\alpha_1\alpha_3-\beta_1\beta_2-\beta_1\beta_3,\alpha_1\beta_2+\alpha_1\beta_3+\beta_1\alpha_2+\beta_1\alpha_3) = (\alpha_1\alpha_2-\beta_1\beta_2,\alpha_1\beta_2+\beta_1\alpha_2)+(\alpha_1\alpha_3-\beta_1\beta_3,\alpha_1\beta_3+\alpha_3\beta_1) = (\alpha_1,\beta_1)(\alpha_2,\beta_2)+(\alpha_1,\beta_1)(\alpha_3,\beta_3)$.(distributive law stands)

(c) Let $\mathfrak{F}'$ be the set of all pairs of (α,β) of complex numbers.

(a) Suppose $\mathfrak{F}'$ is a field. Let $(\alpha,\beta) \in \mathfrak{F}'$ with $\alpha,\beta \in \mathcal{C}$. Then, additive identity would be $(0,0)$, because for all α,β, we have $\alpha+0=\alpha,\beta+0=\beta$. In additon, multiplicative identity would be $(1,1)$, because for all α,β, we have $\alpha1=\alpha,\beta1=\beta$.

Here, think about $(0,1)(\neq(0,0))$. For all (α,β), we have $(0,1)(\alpha,\beta) = (0,\beta)(\neq(1,1))$. Therefore, $(0,1)$ does not have multiplicative inverse. In conclusion, $\mathfrak{F}'$ is not a field.

(b) Suppose $\mathfrak{F}'$ is a field. Let $(\alpha,\beta) \in \mathfrak{F}'$ with $\alpha,\beta \in \mathcal{C}$. Then, additive identity would be $(0,0)$, because for all α,β, we have $\alpha+0=\alpha,\beta+0=\beta$. In additon, multiplicative identity would be $(1,0)$, because for all α,β, we have $\alpha1-\beta0=\alpha,\alpha0+\beta1=\beta$.

Here, think about $(i,1)(\neq(0,0))$. There exists (α,β) such that $(i,1)(\alpha,\beta) = (\alpha i-\beta,\beta i+\alpha) = (1,0)$. Then, we have $\alpha i-\beta=1$,and $\beta i+\alpha=0 \iff \alpha i-\beta=0$. It is contradicted. Therefore, $(i,1)$ does not have multiplicative inverse. In conclusion, $\mathfrak{F}'$ is not a field.

5 Johno (2022/11/27)

5.1 Exercise 4.1

(a) We have $0 + x = x + 0 = x$ by definition.

(b) It follows from definition that $0 + 0 = 0$, hence $0 = -0$ by the uniqueness of additive inverse.

(c) We can prove this as in Exercise 1.1 (d).

(d) The same as above.

(e) Let $\alpha x = 0$ hold. If $\alpha \neq 0$, then $x = 1x = (\frac{1}{\alpha}\alpha)x = \frac{1}{\alpha}(\alpha x) = \frac{1}{\alpha}0 = 0$ by definition and Exercise 2.1 (c).

(f) By definition and Exercise 2.1 (d), we have $1x+(-1)x = (1-1)x = 0x = 0$. Hence $-x = -(1x) = (-1)x$.

(g) It follows from definition that $y + (x - y) = y + (-y + x) = (y - y) + x = 0 + x = x + 0 = x$.

6 Mohehe (2022/11/31)

6.1 Exercise 4.2

We have $Z_p^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in Z_p\}$. Moreover, Z_p has p members. Therefore, the number of the vectors in this vector space is p^n.

6.2 Exercise 4.3

Suppose $\mathcal{V}$ is a vector space. If $x = (0, 1) \in \mathcal{V}$, we have $1(0, 1) = (0, 1)$ by definition of vector spaces, but we also have$1(0, 1) = (0, 0)$ by definition of $\mathcal{V}$. It is contradicted, so $\mathcal{V}$ is not a vector space.

6.3 Exercise 4.4

(a) For $(1, 0, 0) \in \mathcal{V}$ and $i \in \mathcal{C}$, we have $i(1, 0, 0) = (i, 0, 0) \notin \mathcal{V}$, because $i \notin \mathcal{R}$. Therefore $\mathcal{V}$ is not a vector space.

(b) Let $(0, a_2, a_3), (0, b_2, b_3), (0, c_2, c_3) \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{C}$.

   We have $(0, a_2, a_3)+(0, b_2, b_3) = (0, a_2+b_2, a_3+b_3) \in \mathcal{V}$ and $\alpha(0, a_2, a_3) = (0, \alpha a_2, \alpha a_3) \in \mathcal{V}$. Therefore $\mathcal{V}$ is closed under addition and scalar multiplication.

   (A) (1) $(0, a_2, a_3)+(0, b_2, b_3) = (0, a_2+b_2, a_3+b_3) = (0, b_2, b_3)+(0, a_2, a_3)$. (addition is commutative)

(2) $(0, a_2, a_3) + ((0, b_2, b_3) + (0, c_2, c_3)) = (0, a_2, a_3) + (0, b_2 + c_2, b_3 + c_3) = (0, a_2 + b_2 + c_2, a_3 + b_3 + c_3) = (0, a_2, a_3) + (0, b_2, b_3) + (0, c_2, c_3)$. (addition is associative)

(3) There exists $(0, 0, 0) \in \mathcal{V}$ such that $(0, a_2, a_3)+(0, 0, 0) = (0, a_2, a_3)$ for every $(0, a_2, a_3)$. ($\mathcal{V}$ has additive identity)

(4) For every $(0, a_2, a_3)$, there exists $(0, -a_2, -a_3)$ such that $(0, a_2, a_3)+ (0, -a_2, -a_3) = (0, 0, 0)$.

(B) (1) $\alpha(\beta(0, a_2, a_3)) = (0, \alpha\beta a_2, \alpha\beta a_3) = (\alpha\beta)(0, a_2, a_3)$. (multiplication by scalars is associative)

(2) We have $1(0, a_2, a_3) = (0, a_2, a_3)$ for $1 \in \mathcal{C}$ and for every $(0, a_2, a_3)$.

(C) (1) $\alpha((0, a_2, a_3) + (0, b_2, b_3)) = \alpha(0, a_2 + b_2, a_3 + b_3) = (0, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3) = (0, \alpha a_2, \alpha a_3) + (0, \alpha b_2, \alpha b_3) = \alpha(0, a_2, a_3) + \alpha(0, b_2, b_3)$.

(2) $(\alpha+\beta)(0, a_2, a_3) = (0, (\alpha+\beta)a_2, (\alpha+\beta)a_3) = (0, \alpha a_2+\beta a_2, \alpha a_3+ \beta a_3) = (0, \alpha a_2, \alpha a_3) + (0, \beta a_2, \beta a_3) = \alpha(0, a_2, a_3) + \beta(0, a_2, a_3)$.

In conclusion, $\mathcal{V}$ is a vector space.

(c) For $(0, 1, 1), (1, 0, 1) \in \mathcal{V}$, we have $(0, 1, 1)+(1, 0, 1) = (1, 1, 2) \notin \mathcal{V}$, because $1 \neq 0$. Therefore $\mathcal{V}$ is not a vector space.

(d) Let $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \in \mathcal{V}$ with $a_1+a_2 = b_1+b_2 = c_1+c_2 = 0$ and $\alpha, \beta \in \mathcal{C}$.

We have $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \in \mathcal{V}$ because $a_1+b_1+a_2+b_2 = 0$. Moreover, we have $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3) \in \mathcal{V}$ because $\alpha a_1 + \alpha a_2 = \alpha(a_1 + a_2) = 0$. Therefore $\mathcal{V}$ is closed under addition and scalar multiplication.

(A) (1) $(a_1, a_2, a_3)+(b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) = (b_1, b_2, b_3)+ (a_1, a_2, a_3)$. (addition is commutative)

(2) $(a_1, a_2, a_3)+((b_1, b_2, b_3)+(c_1, c_2, c_3)) = (a_1, a_2, a_3)+(b_1+b_1, b_2+ c_2, b_3+c_3) = (a_1+b_1+c_1, a_2+b_2+c_2, a_3+b_3+c_3) = (a_1, a_2, a_3)+ (b_1, b_2, b_3) + (c_1, c_2, c_3)$. (addition is associative)

(3) There exists $(0, 0, 0) \in \mathcal{V}$ such that $(a_1, a_2, a_3)+(0, 0, 0) = (a_1, a_2, a_3)$ for every (a_1, a_2, a_3). ($\mathcal{V}$ has additive identity)

(4) For every (a_1, a_2, a_3), there exists $(-a_1, -a_2, -a_3)$ such that $(a_1, a_2, a_3)+ (-a_1, -a_2, -a_3) = (0, 0, 0)$.

(B) (1) $\alpha(\beta(a_1, a_2, a_3)) = (\alpha\beta a_1, \alpha\beta a_2, \alpha\beta a_3) = (\alpha\beta)(a_1, a_2, a_3)$. (multiplication by scalars is associative)

(2) We have $1(a_1, a_2, a_3) = (a_1, a_2, a_3)$ for $1 \in \mathcal{C}$ and for every (a_1, a_2, a_3).

(C) (1) $\alpha((a_1, a_2, a_3)+(b_1, b_2, b_3)) = \alpha(a_1+b_1, a_2+b_2, a_3+b_3) = (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3) = (\alpha a_1, \alpha a_2, \alpha a_3) + (\alpha b_1, \alpha b_2, \alpha b_3) = \alpha(a_1, a_2, a_3) + \alpha(b_1, b_2, b_3)$.

(2) $(\alpha+\beta)(a_1,a_2,a_3) = ((\alpha+\beta)a_1,(\alpha+\beta)a_2,(\alpha+\beta)a_3) = (\alpha a_1 + \beta a_1, \alpha a_2+\beta a_2, \alpha a_3+\beta a_3) = (\alpha a_1,\alpha a_2,\alpha a_3)+(\beta a_1,\beta a_2,\beta a_3) = \alpha(a_1,a_2,a_3)+\beta(a_1,a_2,a_3)$.

(e) For $(1,0,0),(0,1,0) \in \mathcal{V}$, we have $(1,0,0)+(0,1,0) = (1,1,0) \notin \mathcal{V}$, because $1+1 \neq 0$. Therefore $\mathcal{V}$ is not a vector space.

6.4 Exercise 4.5

(a) For $t^3+t^2, -t^3 \in \mathcal{V}$, we have $t^3+t^2+(-t^3) = t^2 \notin \mathcal{V}$, because t^2 doesn't have degree 3. Therefore $\mathcal{V}$ is not a vector space.

(b) Let $x, y \in \mathcal{V}$, $\alpha \in \mathcal{C}$.

For $\mathcal{V}$ multiplication by scalars is distributive with respect to vector addition, because $\mathcal{V} \subset \mathcal{P}$. By it and $2x(0) = x(1)$, $2(x+y)(0) = 2(x(0)+y(0)) = 2x(0)+2y(0) = x(1)+y(1) = (x+y)(1)$. For $\mathcal{V}$ multiplication by scalars is associative, because $\mathcal{V} \subset \mathcal{P}$. By it and $2x(0) = x(1)$, $2(\alpha x)(0) = 2\alpha x(0) = \alpha 2x(0) = \alpha(2x(0)) = \alpha x(1)$ Therefore, $\mathcal{V}$ is closed. For $\mathcal{V}$ addition is associative and commutative, multiplication by scalar is associative, $1x = x$ $(1 \in \mathcal{C})$ for every vector x, multiplication by scalars is distributive with respect to vector addition, and multiplication by vectors is distributive with respect to scalar addition, because $\mathcal{V} \subset \mathcal{P}$. In addition, there exists $0 \in \mathcal{V}$ such that $x+0 = x$, and to every vector $x \in \mathcal{V}$ there corresponds a vector $-x \in \mathcal{V}$ such that $x+(-x) = 0$. In conclution, $\mathcal{V}$ is vector space.

(c) Let $x \in \mathcal{V}$ with $x \neq 0$. For $-1 \in \mathcal{C}$, $-1x = -x(\neq 0)$ by Exercise 2.1(f). However $-x \notin \mathcal{V}$ because $-x < 0$. Therefore $\mathcal{V}$ is not a vector space.

(d) Let $x, y \in \mathcal{V}$, $\alpha \in \mathcal{C}$.

We have $(x+y)(t) = x(t)+y(t) = x(1-t)+y(1-t) = (x+y)(1-t)$ and $(\alpha x)(t) = \alpha x(t) = \alpha x(1-t) = (\alpha t)(1-t)$ by $x(t) = x(1-t)$. Therefore $\mathcal{V}$ is closed. For $\mathcal{V}$, there exists $0 \in \mathcal{V}$ such that $x+0 = x$, and to every vector $x \in \mathcal{V}$ there corresponds a vector $-x \in \mathcal{V}$ such that $x+(-x) = 0$. Thus, similar to (d), $\mathcal{V}$ is vector space.

7 Johno (2022/12/26)

7.1 Section 6

Remark 7.1. *If $\{x_i\}$ is linearly independent, then a necessary and sufficient condition that x be a linear combination of $\{x_i\}$ is that the enlarged set, obtained by adjoining x to $\{x_i\}$, be linearly dependent.*

Proof. Suppose $x = \sum_i \alpha_i x_i$. Then $\sum_i(-\alpha_i)x_i + x = 0$, hence $\{x_i\} \cup \{x\}$ is dependent. Next, suppose $\{x_i\} \cup \{x\}$ is dependent. Then there exist a set $\{\alpha_i\} \cup \{\beta\}$ of scalars (not all zero) such that $\sum_i \alpha_i x_i + \beta x = 0$. By the independence of $\{x_i\}$, $\beta = 0$ implies that $\alpha_i = 0$ for all i, a contradiction. Hence $\beta \neq 0$ and we obtain $x = \sum_i (-\alpha_i/\beta)x_i$. $\square$

7.2 Exercise 7.1

(a) Since $x + y + z - u = 0$, the four vectors are dependent. It is obvious that $\{x, y, z\}$ is independent. Now let us consider the set $\{x, y, u\}$. If $\alpha x + \beta y + \gamma u = 0$, then the three equations: $\alpha + \gamma = 0$, $\beta + \gamma = 0$, $\gamma = 0$ hold, so that $\alpha = \beta = \gamma = 0$. The independence of $\{x, z, u\}$ and $\{y, z, u\}$ can be similarly proved.

(b) We can prove this in a similar way as in Exercise 7.1(a).

8 Mohehe (2022/1/02)

8.1 Exercise 7.2

Suppose when ξ be rational, vectors 1 and $\xi \in \mathcal{R}$ be linearly independent. Let $\alpha \neq 0 \in \mathcal{R}$, then $\alpha\xi \in \mathcal{R}$. $-\alpha\xi \cdot 1 + \alpha \cdot \xi = 0$ so vectors 1 and ξ be linearly dependent. It is contradicted to supposition, so ξ need to be irrational.

8.2 Exercise 7.3

Let $\alpha, \beta,$ and γ are scalars. $\alpha(x+y)+\beta(y+z)+\gamma(z+x) = 0 \iff (\gamma+\alpha)x+(\alpha+\beta)y+(\beta+\gamma)z = 0 \iff \gamma+\alpha = \alpha+\beta = \beta+\gamma = 0$ because $x, y,$ and z are linearly independent. In addition, $\gamma + \alpha = \alpha + \beta = \beta + \gamma = 0 \iff \alpha = \beta = \gamma = 0$ so $x + y, y + x,$ and $z + x$ are linearly independent.

References

[1] `https://proofwiki.org/wiki/Rational_Numbers_are_Countably_Infinite`

[2] `https://proofwiki.org/wiki/Rational_Numbers_form_Field`

[3] `https://proofwiki.org/wiki/Bezout%27s_Identity`