

# Notes on “Finite-Dimensional Vector Spaces”

by Paul R. Halmos

September 27, 2022

Each \section corresponds to the scope of one member’s assignment, and each \subsection corresponds to one theorem or exercise in the textbook, specified in the format  $m.n$  where  $m$  is the section number and  $n$  is the theorem/exercise number. If  $n$  is not given, we use  $n = 1$  instead.

## 1 Toga (2022/09/19)

### 1.1 Exercise 1.1

- (a) Since addition is commutative,  $0 + \alpha = \alpha + 0$  holds. We also have  $\alpha + 0 = \alpha$  by definition, hence  $0 + \alpha = \alpha$ .

## 2 Mohehe (2022/09/27)

### 2.1 Exercise 1.1

- (b) If  $\alpha + \beta = \alpha + \gamma$ , we have  $\beta = \beta + 0 = 0 + \beta = (\alpha + (-\alpha)) + \beta = ((-\alpha) + \alpha) + \beta = (-\alpha) + (\alpha + \beta) = (-\alpha) + (\alpha + \gamma) = ((-\alpha) + \alpha) + \gamma = (\alpha + (-\alpha)) + \gamma = 0 + \gamma = \gamma + 0 = \gamma$  by definition. Therefore,  $\beta = \gamma$  holds.
- (c) We have  $\alpha + (\beta - \alpha) = \alpha + (\beta + (-\alpha)) = \alpha + ((-\alpha) + \beta) = (\alpha + (-\alpha)) + \beta = 0 + \beta = \beta + 0 = \beta$  by definition. Therefore,  $\alpha + (\beta - \alpha) = \beta$  holds.
- (d) We have  $\alpha 0 + \alpha 0 = \alpha(0 + 0) = \alpha 0 = \alpha + 0$  by definition, hence  $\alpha 0 = 0$  by Exercise 1(b). We also have  $\alpha \cdot 0 = 0 \cdot \alpha$  by definition. Therefore,  $\alpha \cdot 0 = 0 \cdot \alpha = 0$ .
- (e) We have  $\alpha + (-1)\alpha = 1\alpha + (-1)\alpha = (1 + (-1))\alpha = 0\alpha = 0$  by definition and Exercise 1(d). Since the additive inverse is unique, we obtain  $(-1)\alpha = -\alpha$ .
- (f) We have  $(-\alpha)(-\beta) = ((-1)\alpha)((-1)\beta) = (\alpha(-1))((-1)\beta) = \alpha((-1)((-1)\beta)) = \alpha((-1)(-1)\beta)$  by Exercise 1(e) and definition. We also have  $(-1)(-1) = 0 + (-1)(-1) = (1 + (-1)) + (-1)(-1) = 1 + (-1) + (-1)(-1) = 1 +$

$(-1)((-1) + 1) = 1 + (-1)(1 + (-1)) = 1 + (-1)0 = 1 + 0 = 1$  by definition. By it and definition,  $\alpha((-1)(-1)\beta) = \alpha(1\beta) = \alpha(\beta 1) = \alpha\beta$  holds. Therefore,  $(-\alpha)(-\beta) = \alpha\beta$  holds.

- (g) If  $\alpha\beta = 0$ , suppose  $\alpha \neq 0$  and  $\beta \neq 0$  hold. By supposition and definition, we have  $0 = \alpha^{-1}0 = \alpha^{-1}(\alpha\beta) = (\alpha^{-1}\alpha)\beta = (\alpha\alpha^{-1})\beta = 1\beta = \beta 1 = \beta$ , hence  $\beta = 0$ . However, this result contradicts supposition, " $\alpha \neq 0$  and  $\beta \neq 0$ ". Therefore, if  $\alpha\beta = 0$ , then either  $\alpha = 0$  or  $\beta = 0$  (or both).

### 3 Joh (2022/09/19)

#### 3.1 Exercise 1.2

- (a) The set of positive integers is not a field since there is no additive inverse for 1.
- (b) The set of integers is not a field since there is no multiplicative inverse for 2.
- (c) There exists a bijective map  $\varphi$  from  $\mathbb{N}$  (or  $\mathbb{Z}$ ) to  $\mathbb{Q}[1]$ , where  $\mathbb{Q}$  is a field [2]. We can make  $\mathbb{N}$  a field by re-defining (i) addition by  $a \oplus b = \varphi^{-1}(\varphi(a) + \varphi(b))$  and (ii) multiplication by  $a \otimes b = \varphi^{-1}(\varphi(a)\varphi(b))$  for each  $a, b \in \mathbb{N}$ . Note that the additive and multiplicative identities become  $\varphi^{-1}(0)$  and  $\varphi^{-1}(1)$ , respectively. For each  $\alpha \in \mathbb{N}$ , the additive inverse becomes  $\varphi^{-1}(-\varphi(\alpha))$ , and the multiplicative inverse becomes  $\varphi^{-1}(1/\varphi(\alpha))$  if  $\alpha \neq \varphi^{-1}(0)$ .

Let  $\alpha, \beta, \gamma, \alpha', \beta' \in \mathbb{N}$ . Note that

1)  $\alpha \oplus \beta = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)) = \varphi^{-1}(\varphi(\beta) + \varphi(\alpha)) = \beta \oplus \alpha$  holds.(addition is commutative)

(from here, mohehe)

2)  $\alpha \oplus (\beta \oplus \gamma) = \alpha \oplus (\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(\varphi(\beta) + \varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha) + (\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}((\varphi(\alpha) + \varphi(\beta)) + \varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha) + \varphi(\beta))) + \varphi(\gamma)) = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)) \oplus \gamma = (\alpha \oplus \beta) \oplus \gamma$  holds.(addition is associative)

3)  $\alpha \oplus \varphi^{-1}(0) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(0))) = \varphi^{-1}(\varphi(\alpha) + 0) = \varphi^{-1}(\varphi(\alpha)) = \alpha$  holds.(there exists additive identity,  $\varphi^{-1}(0)$ ) If  $\alpha'$  and  $\beta'$  are additive identity, we have  $\alpha' = \alpha' \oplus \beta' = \beta' \oplus \alpha' = \beta'$  by 1) and the definition of additive identity.(additive identity is unique)

4)  $-\varphi(\alpha) \in \mathbb{Q}$  holds by definition, so  $\varphi^{-1}(-\varphi(\alpha)) \in \mathbb{N}$  holds. Therefore,  $\alpha \oplus \varphi^{-1}(-\varphi(\alpha)) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(-\varphi(\alpha)))) = \varphi^{-1}(\varphi(\alpha) + (-\varphi(\alpha))) = \varphi^{-1}(0)$  holds.(for each  $\alpha$  ( $\alpha \in \mathbb{N}$ ), there exists additive inverse) For each  $\alpha$ , if  $\alpha'$  and  $\beta'$  are additive inverse, we have  $\alpha' = \alpha' \oplus \varphi^{-1}(0) = \alpha' \oplus (\alpha \oplus \beta') = (\alpha' \oplus \alpha) \oplus \beta' = (\alpha \oplus \alpha') \oplus \beta' = \varphi^{-1}(0) \oplus \beta' = \beta \oplus \varphi^{-1}(0) = \beta'$  by 1), 2), 3) and the definition of additive inverse.(additive inverse is unique)

5)  $\alpha \otimes \beta = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)) = \varphi^{-1}(\varphi(\beta)\varphi(\alpha)) = \beta \otimes \alpha$  holds.(multiplication is commutative)

6)  $\alpha \otimes (\beta \otimes \gamma) = \alpha \otimes (\varphi^{-1}(\varphi(\beta)\varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(\varphi(\beta)\varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\beta)))\varphi(\gamma)) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)) \otimes \gamma = (\alpha \otimes \beta) \otimes \gamma$  holds.(multiplication is associative)  
7)  $\alpha \otimes \varphi^{-1}(1) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(1))) = \varphi^{-1}(\varphi(\alpha) \cdot 1) = \alpha$  holds.(there exists additive identity,  $\varphi^{-1}(1)$ ) If  $\alpha'$  and  $\beta'$  are additive identity, we have  $\alpha' = \alpha' \otimes \beta' = \beta' \otimes \alpha' = \beta'$  by 5) and definition of multiplicative identity.(multiplicative identity is unique)  
8) For each  $\alpha$  ( $\alpha \neq \varphi^{-1}(0)$ ),  $(1/\varphi(\alpha)) \in \mathbb{Q}$  holds by definition, so  $\varphi^{-1}(1/\varphi(\alpha)) \in \mathbb{N}$  holds. Therefore,  $\alpha \otimes \varphi^{-1}(1/\varphi(\alpha)) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(1/\varphi(\alpha)))) = \varphi^{-1}(\varphi(\alpha)(1/\varphi(\alpha))) = \varphi^{-1}(1)$  holds.(for each  $\alpha$  ( $\alpha \in \mathbb{N}$ ), there exists multiplicative inverse) For each  $\alpha$  ( $\alpha \neq \varphi^{-1}(0)$ ), if  $\alpha'$  and  $\beta'$  are multiplicative inverse, we have  $\alpha' = \alpha' \otimes \varphi^{-1}(1) = \alpha' \otimes (\alpha \otimes \beta') = (\alpha' \otimes \alpha) \otimes \beta' = (\alpha \otimes \alpha') \otimes \beta' = \varphi^{-1}(1) \otimes \beta' = \beta' \otimes \varphi^{-1}(1) = \beta'$  by 5), 6), 7) and the definition of multiplicative inverse.(multiplicative inverse is unique)  
9)  $\alpha \otimes (\beta \oplus \gamma) = \alpha \otimes (\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(\varphi(\beta) + \varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha)(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta) + \varphi(\alpha)\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\beta))) + \varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha \otimes \beta) + \varphi(\alpha \otimes \gamma)) = \alpha \otimes \beta \oplus \alpha \otimes \gamma$  holds.(distributive law stands)

## 4 Mohehe

### 4.1 Exercise 1.3

For two integers  $a$  and  $b$ , we denote by  $a \% b$  the remainder after dividing  $a$  by  $b$ , and write  $b \mid a$  if and only if  $a \% b = 0$ . For clarity, we denote the ordinary sum and product of two integers  $a$  and  $b$  by  $a +_{\mathbb{Z}} b$  and  $a \cdot_{\mathbb{Z}} b$ , respectively. Note that  $\alpha + \beta = (\alpha +_{\mathbb{Z}} \beta) \% m$  and  $\alpha\beta = (\alpha \cdot_{\mathbb{Z}} \beta) \% m$  for  $\alpha, \beta \in \mathbb{Z}_m$ .

- (a)
- (b)
- (c)

### 4.2 Exercise 1.4

Define  $x_0, x_1, \dots, \in \mathfrak{F}$  as below:

$$x_0 = 0, \tag{1}$$

$$x_n = x_{n-1} + 1 \quad (n > 0). \tag{2}$$

We have  $x_m x_n = x_{mn}$  for all  $m$  and  $n$ . Assume there exists an  $n$  with  $x_n = 0$  but  $x_k \neq 0$  for any  $k < n$ . It suffices to prove that  $n$  is a prime.

## References

- [1] [https://proofwiki.org/wiki/Rational\\_Numbers\\_are\\_Countably\\_Infinite](https://proofwiki.org/wiki/Rational_Numbers_are_Countably_Infinite)

[2] [https://proofwiki.org/wiki/Rational\\_Numbers\\_form\\_Field](https://proofwiki.org/wiki/Rational_Numbers_form_Field)