Notes on "Finite-Dimensional Vector Spaces" by Paul R. Halmos

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Each `\section` corresponds to the scope of one member's assignment, and each `\subsection` corresponds to one theorem or exercise in the textbook, specified in the format $m.n$ where m is the section number and n is the theorem/exercise number. If n is not given, we use $n = 1$ instead.

Contents

1 Toga (2022/09/19)

1.1 Exercise 1.1

(a) Since addition is commutative, $0+\alpha = \alpha+0$ holds. We also have $\alpha+0 = \alpha$ by definition, hence $0 + \alpha = \alpha$.

2 Mohehe (2022/09/27)

2.1 Exercise 1.1

(b) If $\alpha + \beta = \alpha + \gamma$, we have $\beta = \beta + 0 = 0 + \beta = (\alpha + (-\alpha)) + \beta = ((-\alpha) + \alpha) + \beta = (-\alpha) + (\alpha + \beta) = (-\alpha) + (\alpha + \gamma) = ((-\alpha) + \alpha) + \gamma = (\alpha + (-\alpha)) + \gamma = 0 + \gamma = \gamma + 0 = \gamma$ by definition. Therefore, $\beta = \gamma$ holds.

(c) We have $\alpha + (\beta - \alpha) = \alpha + (\beta + (-\alpha)) = \alpha + ((-\alpha) + \beta) = (\alpha + (-\alpha)) + \beta = 0 + \beta = \beta + 0 = \beta$ by definition. Therefore, $\alpha + (\beta - \alpha) = \beta$ holds.

(d) We have $\alpha 0 + \alpha 0 = \alpha(0 + 0) = \alpha 0 = \alpha + 0$ by definition, hence $\alpha 0 = 0$ by Exercise 1(b). We also have $\alpha \cdot 0 = 0 \cdot \alpha$ by definition. Therefore, $\alpha \cdot 0 = 0 \cdot \alpha = 0$

(e) We have $\alpha + (-1)\alpha = 1\alpha + (-1)\alpha = (1 + (-1))\alpha = 0\alpha = 0$ by definition and Exercise 1(d). Since the additive inverse is unique, we obtain $(-1)\alpha = -\alpha$.

(f) We have $(-\alpha)(-\beta) = ((-1)\alpha)((-1)\beta) = (\alpha(-1))((-1)\beta) = \alpha((-1)((-1)\beta)) = \alpha((-1)(-1)\beta)$ by Exercise 1(e) and definition. We also have $(-1)(-1) = 0 + (-1)(-1) = (1 + (-1)) + (-1)(-1) = 1 + (-1) + (-1)(-1) = 1 + (-1)((-1) + 1) = 1 + (-1)(1 + (-1)) = 1 + (-1)0 = 1 + 0 = 1$ by definition. By it and definition, $\alpha((-1)(-1)\beta) = \alpha(1\beta) = \alpha(\beta 1) = \alpha\beta$ holds. Therefore, $(-\alpha)(-\beta) = \alpha\beta$ holds.

(g) If $\alpha\beta = 0$, suppose $\alpha \neq 0$ and $\beta \neq 0$ hold. By supposition and definition, we have $0 = \alpha^{-1}0 = \alpha^{-1}(\alpha\beta) = (\alpha^{-1}\alpha)\beta = (\alpha\alpha^{-1})\beta = 1\beta = \beta 1 = \beta$, hence $\beta = 0$. However, this result contradicts supposition, "$\alpha \neq 0$ and $\beta \neq 0$". Therefore, if $\alpha\beta = 0$, then either $\alpha = 0$ or $\beta = 0$ (or both).

3 Mohehe (2022/09/19)

3.1 Exercise 1.2

(a) The set of positive integers is not a field since there is no additive inverse for 1.

(b) The set of integers is not a field since there is no multiplicative inverse for 2.

(c) There exists a bijective map φ from $\mathcal{N}$ (or $\mathcal{Z}$) to $\mathcal{Q}$ [1], where $\mathcal{Q}$ is a field [2]. We can make $\mathcal{N}$ a field by re-defining (i) addition by $a \oplus b = \varphi^{-1}(\varphi(a) + \varphi(b))$ and (ii) multiplication by $a \otimes b = \varphi^{-1}(\varphi(a)\varphi(b))$ for each $a, b \in \mathcal{N}$. Note that the additive and multiplicative identities become $\varphi^{-1}(0)$ and $\varphi^{-1}(1)$, respectively. For each $\alpha \in \mathcal{N}$, the additive inverse becomes $\varphi^{-1}(-\varphi(\alpha))$, and the multiplicative inverse becomes $\varphi^{-1}(1/\varphi(\alpha))$ if $\alpha \neq \varphi^{-1}(0)$.

Let $\alpha, \beta, \gamma, \alpha', \beta' \in \mathcal{N}$. Note that

1) $\alpha\oplus\beta = \varphi^{-1}(\varphi(\alpha)+\varphi(\beta)) = \varphi^{-1}(\varphi(\beta)+\varphi(\alpha)) = \beta\oplus\alpha$ holds.(addition is commutative)

2) $\alpha \oplus (\beta \oplus \gamma) = \alpha \oplus (\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))))) = \varphi^{-1}(\varphi(\alpha)+(\varphi(\beta)+\varphi(\gamma))) = \varphi^{-1}((\varphi(\alpha)+\varphi(\beta))+\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)+\varphi(\beta)))+\varphi(\gamma)) = \varphi^{-1}(\varphi(\alpha)+\varphi(\beta))\oplus\gamma = (\alpha\oplus\beta)\oplus\gamma$ holds.(addition is associative)

3) $\alpha\oplus\varphi^{-1}(0) = \varphi^{-1}(\varphi(\alpha)+\varphi(\varphi^{-1}(0))) = \varphi^{-1}(\varphi(\alpha)+0) = \varphi^{-1}(\varphi(\alpha)) = \alpha$ holds.(there exists additive identity, $\varphi^{-1}(0)$) If α' and β' are additive identity, we have $\alpha' = \alpha' \oplus \beta' = \beta' \oplus \alpha' = \beta'$ by 1) and the definition of additive identity.(additive identity is unique)

4) $-\varphi(\alpha) \in \mathcal{Q}$ holds by definition, so $\varphi^{-1}(-\varphi(\alpha)) \in \mathcal{N}$ holds. Therefore, $\alpha \oplus \varphi^{-1}(-\varphi(\alpha)) = \varphi^{-1}(\varphi(\alpha) + \varphi(\varphi^{-1}(-\varphi(\alpha)))) = \varphi^{-1}(\varphi(\alpha) + (-\varphi(\alpha))) = \varphi^{-1}(0)$ holds.(for each α ($\alpha \in \mathcal{N}$), there exists additive inverse) For each α, if α' and β' are additive inverse, we have $\alpha' = \alpha' \oplus \varphi^{-1}(0) = \alpha' \oplus (\alpha \oplus \beta') = (\alpha' \oplus \alpha) \oplus \beta' = (\alpha \oplus \alpha') \oplus \beta' = \varphi^{-1}(0) \oplus \beta' = \beta \oplus \varphi^{-1}(0) = \beta'$ by 1), 2), 3) and the definition of additive inverse.(additive inverse is unique)

5) $\alpha\otimes\beta = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)) = \varphi^{-1}(\varphi(\beta)\varphi(\alpha)) = \beta\otimes\alpha$ holds.(multiplication is commutative)

6) $\alpha\otimes(\beta\otimes\gamma) = \alpha\otimes(\varphi^{-1}(\varphi(\beta)\varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(\varphi(\beta)\varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta)\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\beta)))\varphi(\gamma)) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta))\otimes \gamma = (\alpha \otimes \beta) \otimes \gamma$ holds.(multiplication is associative)

7) $\alpha \otimes \varphi^{-1}(1) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(1))) = \varphi^{-1}(\varphi(\alpha) \cdot 1) = \alpha$ holds.(there exists additive identity, $\varphi^{-1}(1)$) If α' and β' are additive identity, we have $\alpha' = \alpha' \otimes \beta' = \beta' \otimes \alpha' = \beta'$ by 5) and definition of multiplicative identity.(multiplicative identity is unique)

8) For each α ($\alpha \neq \varphi^{-1}(0)$), $(1/\varphi(\alpha)) \in \mathcal{Q}$ holds by definition, so $\varphi^{-1}(1/\varphi(\alpha)) \in \mathcal{N}$ holds. Therefore, $\alpha\otimes\varphi^{-1}(1/\varphi(\alpha)) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(1/\varphi(\alpha)))) = \varphi^{-1}(\varphi(\alpha)(1/\varphi(\alpha))) = \varphi^{-1}(1)$ holds.(for each α ($\alpha \in \mathcal{N}$), there exists multiplicative inverse) For each α ($\alpha \neq \varphi^{-1}(0)$), if α' and β' are multiplicative inverse, we have $\alpha' = \alpha' \otimes \varphi^{-1}(1) = \alpha' \otimes (\alpha \otimes \beta') = (\alpha' \otimes \alpha) \otimes \beta' = (\alpha \otimes \alpha') \otimes \beta' = \varphi^{-1}(1) \otimes \beta' = \beta' \otimes \varphi^{-1}(1) = \beta'$ by 5), 6), 7) and the definition of multiplicative inverse.(multiplicative inverse is unique)

9) $\alpha \otimes (\beta \oplus \gamma) = \alpha \otimes (\varphi^{-1}(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\varphi^{-1}(\varphi(\beta) + \varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha)(\varphi(\beta) + \varphi(\gamma))) = \varphi^{-1}(\varphi(\alpha)\varphi(\beta) + \varphi(\alpha)\varphi(\gamma)) = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\beta)))+\varphi(\varphi^{-1}(\varphi(\alpha)\varphi(\gamma)))) = \varphi^{-1}(\varphi(\alpha\otimes\beta)+\varphi(\alpha\otimes \gamma)) = \alpha \otimes \beta \oplus \alpha \otimes \gamma$ holds.(distributive law stands)

4 Mohehe (2022/10/8)

4.1 Exercise 1.3

For two integers a and b, we denote by $a \,\%\, b$ the remainder after dividing a by b, and write $b \mid a$ if and only if $a \,\%\, b = 0$. For clarity, we denote the ordinary sum and product of two integers a and b by $a +_{\mathcal{Z}} b$ and $a \cdot_{\mathcal{Z}} b$, respectively. Note that $\alpha + \beta = (\alpha +_{\mathcal{Z}} \beta) \,\%\, m$ and $\alpha\beta = (\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m$ for $\alpha, \beta \in \mathcal{Z}_m$.

(a) Let $\alpha, \beta, \gamma \in \mathcal{Z}_m$, $k \in \mathcal{Z}$

1' Proof : if m is a prime, $\mathcal{Z}_m$ is a field.
Suppose m is a prime,

1) $\alpha + \beta = (\alpha +_{\mathcal{Z}} \beta) \,\%\, m = (\beta +_{\mathcal{Z}} \alpha) \,\%\, m = \beta + \alpha$ (addition is commutative)
2) Since $\alpha +_{\mathcal{Z}} (\beta + \gamma) = \alpha +_{\mathcal{Z}} (\beta +_{\mathcal{Z}} \gamma) \,\%\, m \equiv \alpha +_{\mathcal{Z}} (\beta +_{\mathcal{Z}} \gamma) = (\alpha +_{\mathcal{Z}} \beta) +_{\mathcal{Z}} \gamma \equiv (\alpha +_{\mathcal{Z}} \beta)\%m +_{\mathcal{Z}} \gamma = (\alpha + \beta) +_{\mathcal{Z}} \gamma \pmod{m}$ holds, $\alpha + (\beta + \gamma) = (\alpha +_{\mathcal{Z}} (\beta + \gamma))\%m = ((\alpha + \beta) +_{\mathcal{Z}} \gamma)\%m = (\alpha + \beta) + \gamma$ holds.(addition is associative)
3) $\alpha + 0 = (\alpha +_{\mathcal{Z}} 0)\%m = \alpha\%m = \alpha$ (there exists additive identity) By it and 1), if β and γ are additive identity, $\beta = \beta + \gamma = \gamma + \beta = \gamma$ (additive identity is unique)
4) If $\alpha +_{\mathcal{Z}} \beta = m$, $\alpha + \beta = (\alpha +_{\mathcal{Z}} \beta) \,\%\, m = m \,\%\, m = 0$ (there exists additive inverse)
5) $\alpha\beta = (\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m = (\beta \cdot_{\mathcal{Z}} \alpha) \,\%\, m = \beta\alpha$ (multiplication is commutative)
6) Since $\alpha \cdot_{\mathcal{Z}} (\beta\gamma) = \alpha \cdot_{\mathcal{Z}} ((\beta \cdot_{\mathcal{Z}} \gamma) \,\%\, m) \equiv \alpha \cdot_{\mathcal{Z}} (\beta \cdot_{\mathcal{Z}} \gamma) = (\alpha \cdot_{\mathcal{Z}} \beta) \cdot_{\mathcal{Z}} \gamma \equiv ((\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m) \cdot_{\mathcal{Z}} \gamma = (\alpha\beta) \cdot_{\mathcal{Z}} \gamma \pmod{m}$ holds, $\alpha(\beta\gamma) = (\alpha \cdot_{\mathcal{Z}} (\beta\gamma)) \,\%\, m = ((\alpha\beta) \cdot_{\mathcal{Z}} \gamma) \,\%\, m = (\alpha\beta)\gamma$ holds.(multiplication is associative)
7) $\alpha 1 = (\alpha_{\mathcal{Z}} 1) \,\%\, m = \alpha \,\%\, m = \alpha$ (there exists multiplicative identity) By it and 5), if β and γ are multiplicative identity, $\beta = \beta\gamma = \gamma\beta = \gamma$ (multiplicative identity is unique)
8) For all $\alpha(\alpha \neq 0)$, suppose there doesn't exist β that makes $\alpha\beta = 1$. There exist $\beta, \gamma \in \mathcal{Z}_m$ with $\beta \neq \gamma$ and $\alpha\beta = \alpha\gamma$, because β is any one from 0 to $m-1$ and $\alpha\beta$ is any one from 0 to $m-1$ except 1. Therefore, $(\alpha \cdot_{\mathcal{Z}} \beta +_{\mathcal{Z}} (-\alpha \cdot_{\mathcal{Z}} \gamma) =)\ \alpha \cdot_{\mathcal{Z}} (\beta +_{\mathcal{Z}} (-\gamma)) = km$ holds. The right side has divisor m, but it contradicts that the left side doesn't have divisor of m except 1, because $0 < \alpha < (m-1)$ and $((-m) < (\beta +_{\mathcal{Z}} (-\gamma)) < 0$ or $0 < (\beta +_{\mathcal{Z}} (-\gamma)) < m)$ holds. Thus, there exists β that makes $\alpha\beta = 1$.(there exists maltiplicative inverse)
A brief proof: Since each $\alpha \in \mathcal{Z}_m \setminus \{0\}$ is coprime to m, there exist integers x and y such that $\alpha \cdot_{\mathcal{Z}} x +_{\mathcal{Z}} m \cdot_{\mathcal{Z}} y = 1$ by [3]. Putting $x' = x \,\%\, m \in \mathcal{Z}_m$, we obtain $\alpha x' = (\alpha \cdot_{\mathcal{Z}} x) \,\%\, m = (\alpha \cdot_{\mathcal{Z}} x +_{\mathcal{Z}} m \cdot_{\mathcal{Z}} y) \,\%\, m = 1 \,\%\, m = 1$. Hence $x' = \alpha^{-1}$.

9) $\alpha(\beta+\gamma) = (\alpha \cdot_{\mathcal{Z}} (\beta+\gamma)) \,\%\, m = (\alpha \cdot_{\mathcal{Z}} ((\beta +_{\mathcal{Z}} \gamma) \,\%\, m)) \,\%\, m \equiv (\alpha \cdot_{\mathcal{Z}} (\beta +_{\mathcal{Z}} \gamma)) \,\%\, m = (\alpha \cdot_{\mathcal{Z}} \beta + \alpha \cdot_{\mathcal{Z}} \gamma) \,\%\, m \equiv ((\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m +_{\mathcal{Z}} (\alpha \cdot_{\mathcal{Z}} \gamma) \,\%\, m) \,\%\, m \equiv (\alpha \cdot_{\mathcal{Z}} \beta) \,\%\, m + (\alpha \cdot_{\mathcal{Z}} \gamma) \,\%\, m \equiv \alpha\beta + \alpha\gamma$ holds.(distributive law stands)

In conclusion, if m is a prime, $\mathcal{Z}_m$ is a field.

2' Proof : If $\mathcal{Z}_m$ is a field, m is a prime.
By contraposition, it is equivalent to prove "If m is not a prime, $\mathcal{Z}_m$ is not a field." We can show 1) to 7) and 9) in the same way as 1'. For each m, suppose there exist α and β that make $\alpha\beta = 1$. m is not a prime, so let p be one of prime factors of m and then we have $m = p \cdot_{\mathcal{Z}} p'$.($p' \in \mathcal{Z}$ and $1 < p' < m$)
If $\alpha = p$, by $\alpha\beta = 1$ and $m = p \cdot_{\mathcal{Z}} p'$, we have $\alpha \cdot_{\mathcal{Z}} \beta = k \cdot_{\mathcal{Z}} m + 1 (k \in \mathcal{Z}) \Leftrightarrow p \cdot_{\mathcal{Z}} \beta = k \cdot_{\mathcal{Z}} p \cdot_{\mathcal{Z}} p' +_{\mathcal{Z}} 1 \Leftrightarrow (\beta +_{\mathcal{Z}} (-k \cdot_{\mathcal{Z}} p')) \cdot_{\mathcal{Z}} p = 1$. The right side is 1 but the left one is not 1 because of $1 < p$ and $\beta +_{\mathcal{Z}} (-k \cdot_{\mathcal{Z}} p') \in \mathcal{Z}$. Therefore It is contradicted. For each m, there doesn't exist α and β that make $\alpha\beta = 1$. In conclusion, "If m is not a prime, $\mathcal{Z}_m$ is not a field." and "If $\mathcal{Z}_m$ is a field, m is a prime."

Because of 1' and 2', $\mathcal{Z}_m$ is a field if and only if m is a prime.

(b) 4

(c) 5

4.2 Exercise 1.4 (2022/10/24)

Define $\alpha_n = \overbrace{1 + \cdots + 1}^{n \text{ terms}}$ for $n \in \{1, 2, \ldots\}$. Then,

$$
\begin{aligned}
\alpha_m \alpha_n &= \alpha_m(\overbrace{1 + \cdots + 1}^{n \text{ terms}}) \\
&= \alpha_m((\overbrace{1 + \cdots + 1}^{(n-1) \text{ terms}}) + 1) \\
&= \alpha_m(\overbrace{1 + \cdots + 1}^{(n-1) \text{ terms}}) + \alpha_m \cdot 1 && \text{(by distributive law)} \\
&= \alpha_m(\overbrace{1 + \cdots + 1}^{(n-1) \text{ terms}}) + \alpha_m && \text{(by definition of multiplicative identity)} \\
&= \cdots \\
&= \alpha_m(1+1) + \overbrace{\alpha_m + \cdots + \alpha_m}^{(n-2) \text{ terms}} \\
&= \alpha_m \cdot 1 + \alpha_m \cdot 1 + \overbrace{\alpha_m + \cdots + \alpha_m}^{(n-2) \text{ terms}} && \text{(by distributive law)} \\
&= \overbrace{\alpha_m + \cdots + \alpha_m}^{n \text{ terms}} && \text{(by definition of multiplicative identity)}
\end{aligned}
$$

$$
\begin{aligned}
&= \overbrace{1+\cdots+1}^{m\text{ terms}} + (\overbrace{1+\cdots+1}^{m\text{ terms}}) + \overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \overbrace{1+\cdots+1}^{m\text{ terms}} + ((\overbrace{1+\cdots+1}^{(m-1)\text{ terms}}) + 1) + \overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \overbrace{1+\cdots+1}^{m\text{ terms}} + (1 + (\overbrace{1+\cdots+1}^{(m-1)\text{ terms}})) + \overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by commutative property)} \\
&= \overbrace{1+\cdots+1}^{(m+1)\text{ terms}} + (\overbrace{1+\cdots+1}^{(m-1)\text{ terms}}) + \overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by associative property)} \\
&= \cdots \\
&= \overbrace{1+\cdots+1}^{(2m-2)\text{ terms}} + (1+1) + \overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} \\
&= \overbrace{1+\cdots+1}^{2m\text{ terms}} + \overbrace{\alpha_m+\cdots+\alpha_m}^{(n-2)\text{ terms}} && \text{(by associative property)} \\
&= \cdots \\
&= \overbrace{1+\cdots+1}^{m(n-1)\text{ terms}} + \alpha_m \\
&= \cdots \\
&= \overbrace{1+\cdots+1}^{(mn-2)\text{ terms}} + (1+1) \\
&= \overbrace{1+\cdots+1}^{mn\text{ terms}} && \text{(by associative property)} \\
&= \alpha_{mn}
\end{aligned}
$$

for all m, n.

Assume there exists an n with $\alpha_n = 0$ but $\alpha_k \neq 0$ for any $k < n$. It suffices to prove that n is a prime.
Suppose n is not any prime. Let p be one of prime factors of n and then we have $n = pp'(p' \in \mathcal{N}$ and $p' > 1)$. By $\alpha_m \alpha_n = \alpha_{mn}$ for all m and n, $\alpha_n = \alpha_p \alpha_{p'}$ holds. We have either $\alpha_p = 0$ or $\alpha_{p'} = 0$ (or both) because of $\alpha_n = 0$ and Exercise 1.1 (g). However, it is contradictory to $\alpha_p = 0$ and $\alpha_{p'} = 0$. Therefore, n is a prime.

4.3 Exercise 1.5

(a) For the followings, it is used that $\mathcal{Q}$ and $\mathcal{R}$ are fields. Note that $\sqrt{2} \in \mathcal{R}$ and $\sqrt{2} \notin \mathcal{Q}$
Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{Q}(\sqrt{2}) \subset \mathcal{R}$.
For all $\alpha_1, \alpha_2 \in \mathcal{Q}(\sqrt{2})$, $\alpha_1 + \alpha_2 \in \mathcal{Q}(\sqrt{2})$ and $\alpha_1 \alpha_2 \in \mathcal{Q}(\sqrt{2})$ by the followings.
There exist $a, b, c, d \in \mathcal{Q}$, $\alpha_1 = a + b\sqrt{2}$ and $\alpha_2 = c + d\sqrt{2}$ hold.

We have $\alpha_1 + \alpha_2 = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$ and $(a + c), (b + d) \in \mathcal{Q}$, so $\alpha_1 + \alpha_2 \in \mathcal{Q}(\sqrt{2})$.
In addition, we have $\alpha_1\alpha_2 = (a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd)+(ad+bc)\sqrt{2}$ and $(ac + 2bd), (ad + bc) \in \mathcal{Q}$, so $\alpha_1\alpha_2 \in \mathcal{Q}(\sqrt{2})$

1) $\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1$ (addition is commutative)
2) $\alpha_1 + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + \alpha_3$ (addition is associative)
3) We have $0 = 0 + 0\sqrt{2} \in \mathcal{Q}(\sqrt{2})$ and $\alpha_1 + 0 = \alpha_1$
   ($\mathcal{Q}(\sqrt{2})$ has additive identity)
4) For all $\alpha_1 \in \mathcal{Q}(\sqrt{2})$, put $\alpha_1 = a + b\sqrt{2}$ with $a, b \in \mathcal{Q}$. There exists $\alpha_1' \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1' = (-a) + (-b)\sqrt{2}$. We have $\alpha_1 + \alpha_1' = a + b\sqrt{2} + (-a) + (-b)\sqrt{2} = a - a + b\sqrt{2} - b\sqrt{2} = 0$. Therefore, to every $\alpha_1 \in \mathcal{Q}(\sqrt{2})$, there corresponds $\alpha_1' \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1 + (-\alpha_1) = 0$
5) $\alpha_1\alpha_2 = \alpha_2\alpha_1$ (multiplication is commutative)
6) $\alpha_1(\alpha_2\alpha_3) = (\alpha_1\alpha_2)\alpha_3$ (multiplication is associative)
7) We have $1 = 1 + 0\sqrt{2} \in \mathcal{Q}(\sqrt{2})$ and $\alpha_1 \cdot 1 = \alpha_1$
   ($\mathcal{Q}(\sqrt{2})$ has multiplicative identity)
8) For all $\alpha_1 \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1 \neq 0$, put $\alpha_1 = a + b\sqrt{2}$ with $a, b \in \mathcal{Q}$. In this case, $a \neq 0$ or $b \neq 0$ holds by the followings.
   "If $\alpha_1 = 0$, we have $\alpha_1 = a + b\sqrt{2} = 0 \Leftrightarrow a = -b\sqrt{2}$. Therefore, $a = b = 0$ by $a, b \in \mathcal{Q}$."
   Let $\alpha_1'' = \dfrac{a}{a^2 - 2b^2} + \left(-\dfrac{b}{a^2 - 2b^2}\right)\sqrt{2} \in \mathcal{Q}(\sqrt{2})$. Note that we have $a^2 - 2b^2 = (a + b\sqrt{2})(a - b\sqrt{2})$ and $a, b \in \mathcal{Q}$ with ($a \neq 0$ or $b \neq 0$), so we have $a+b\sqrt{2} \neq 0$ and $a-b\sqrt{2} \neq 0$, and then $a^2-2b^2 \in \mathcal{Q}$ with $a^2 - 2b^2 \neq 0$. We have $\alpha_1\alpha_1'' = (a+b\sqrt{2})\left(\dfrac{a}{a^2 - 2b^2} + \left(-\dfrac{b}{a^2 - 2b^2}\right)\sqrt{2}\right) = \dfrac{a^2 - ab\sqrt{2} + ab\sqrt{2} - 2b^2}{a^2 - 2b^2} = 1$. Therefore, to every $\alpha_1 \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1 \neq 0$, there exists $\alpha_1'' \in \mathcal{Q}(\sqrt{2})$ with $\alpha_1\alpha_1'' = 1$
9) $\alpha_1(\alpha_2 + \alpha_3) = \alpha_1\alpha_2 + \alpha_1\alpha_3$ (distributive law stands)

from 1) to)9, $\mathcal{Q}(\sqrt{2})$ is a field.

(b) Let $\mathcal{Z}(\sqrt{2})$ be the set of all numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are integers. If $\mathcal{Z}(\sqrt{2})$ is a field, $2 = 2+0\sqrt{2}(\in \mathcal{Z}(\sqrt{2}))$ have multiplicative inverse. We have $2 \cdot \frac{1}{2} = 1$, but $\frac{1}{2} \notin \mathcal{Z}(\sqrt{2})$. Therefore, $\mathcal{Z}(\sqrt{2})$ is not a field.

Another way : Let $\mathcal{Z}(\sqrt{2})$ be the set of all numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are integers. $\mathcal{Z}(\sqrt{2})$ is not a field since there is no multiplicative inverse for $2 + \sqrt{2} \in \mathcal{Z}(\sqrt{2})$ by the followings.
Suppose there exists multiplicative inverse for $2 + \sqrt{2}$. There exists $\exists\beta_1 \in \mathcal{Z}(\sqrt{2})$ with $\beta_1 = \alpha + \beta\sqrt{2}$ and $(2 + \sqrt{2})\beta_1 = 1$ by supposition. Therefore,

we have $(2+\sqrt{2})\beta_1 = (2+\sqrt{2})(\alpha+\beta\sqrt{2}) = 2(\alpha+\beta)+(\alpha+2\beta)\sqrt{2} = 1 \Leftrightarrow 2(\alpha+\beta)-1 = -(\alpha+2\beta)\sqrt{2} \Leftarrow 2(2(\alpha+\beta)^2 - 2(\alpha+\beta)) - 1 = 2(\alpha+2\beta)^2$. It is contradicted because the left is odd number and the right is even number. Therefore, $(2+\sqrt{2})\beta_1 = 1$ is contradicted and then there is no multiplicative inverse for $2+\sqrt{2}$

References

[1] https://proofwiki.org/wiki/Rational_Numbers_are_Countably_Infinite

[2] https://proofwiki.org/wiki/Rational_Numbers_form_Field

[3] https://proofwiki.org/wiki/Bezout%27s_Identity