極座標で表された点 $r\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$ を θ 方向に $\mathrm{d}\theta$ だけ回転させたもの

$$\begin{aligned}\begin{pmatrix}\cos\mathrm{d}\theta & 0 & \sin\mathrm{d}\theta\\ 0 & 1 & 0\\ -\sin\mathrm{d}\theta & 0 & \cos\mathrm{d}\theta\end{pmatrix} r\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} &= \begin{pmatrix}1 & 0 & \mathrm{d}\theta\\ 0 & 1 & 0\\ -\mathrm{d}\theta & 0 & 1\end{pmatrix} r\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}\\ &= r\begin{pmatrix}\sin\theta\cos\phi+\mathrm{d}\theta\cos\theta\\ \sin\theta\sin\phi\\ \cos\theta-\mathrm{d}\theta\sin\theta\cos\phi\end{pmatrix}\end{aligned}$$

が $r\begin{pmatrix}\sin\theta'\cos\phi'\\ \sin\theta'\sin\phi'\\ \cos\theta'\end{pmatrix}$ となるように θ',ϕ' を定める．すると，

$$\cos\theta' = \cos\theta - \mathrm{d}\theta\sin\theta\cos\phi$$

より

$$\begin{aligned}\sin^2\theta' &= 1-\cos^2\theta'\\ &= 1-\cos^2\theta + 2\mathrm{d}\theta\sin\theta\cos\theta\cos\phi\\ &= \sin^2\theta + 2\mathrm{d}\theta\sin\theta\cos\theta\cos\phi\end{aligned}$$

$$\therefore \sin\theta' = \sin\theta + \mathrm{d}\theta\cos\theta\cos\phi$$

であり，

$$\begin{aligned}\sin\phi' &= \frac{\sin\theta'\sin\phi'}{\sin\theta'}\\ &= \frac{\sin\theta\sin\phi}{\sin\theta+\mathrm{d}\theta\cos\theta\cos\phi}\\ &= \frac{\sin\theta\sin\phi}{\sin\theta}\left(1-\frac{\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\\ &= \sin\phi - \frac{\mathrm{d}\theta\cos\theta\sin\phi\cos\phi}{\sin\theta}\end{aligned}$$

$$\begin{aligned}\cos\phi' &= \frac{\sin\theta'\cos\phi'}{\sin\theta'}\\ &= \frac{\sin\theta\cos\phi+\mathrm{d}\theta\cos\theta}{\sin\theta+\mathrm{d}\theta\cos\theta\cos\phi}\\ &= \frac{\sin\theta\cos\phi+\mathrm{d}\theta\cos\theta}{\sin\theta}\left(1-\frac{\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\\ &= \cos\phi + \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta}\end{aligned}$$

となる．ここで，

$$\begin{aligned}\frac{\partial}{\partial\theta}\sin\theta' &= \cos\theta - \mathrm{d}\theta\sin\theta\cos\phi\\ &= \cos\theta'\end{aligned}$$

より

$$
\begin{aligned}
\frac{\partial\theta'}{\partial\theta} &= \frac{1}{\cos\theta'}\frac{\partial}{\partial\theta}\sin\theta' \\
&= 1
\end{aligned}
$$

であり，

$$
\frac{\partial}{\partial\theta}\sin\phi' = \frac{\mathrm{d}\theta\sin\phi\cos\phi}{\sin^2\theta}
$$

より

$$
\begin{aligned}
\frac{\partial\phi'}{\partial\theta} &= \frac{1}{\cos\phi'}\frac{\partial}{\partial\theta}\sin\phi' \\
&= \frac{\frac{\mathrm{d}\theta\sin\phi\cos\phi}{\sin^2\theta}}{\cos\phi + \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta}} \\
&= \frac{\mathrm{d}\theta\sin\phi}{\sin^2\theta}\left(1 - \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta\cos\phi}\right) \\
&= \frac{\mathrm{d}\theta\sin\phi}{\sin^2\theta}
\end{aligned}
$$

であるから，

$$
\begin{aligned}
\frac{\partial}{\partial\theta} &= \frac{\partial\theta'}{\partial\theta}\frac{\partial}{\partial\theta'} + \frac{\partial\phi'}{\partial\theta}\frac{\partial}{\partial\phi'} \\
&= \frac{\partial}{\partial\theta'} + \frac{\mathrm{d}\theta\sin\phi}{\sin^2\theta}\frac{\partial}{\partial\phi'}
\end{aligned}
$$

である．同様に

$$
\frac{\partial}{\partial\phi}\sin\theta' = -\mathrm{d}\theta\cos\theta\sin\phi
$$

より

$$
\begin{aligned}
\frac{\partial\theta'}{\partial\phi} &= \frac{1}{\cos\theta'}\frac{\partial}{\partial\phi}\sin\theta' \\
&= \frac{-\mathrm{d}\theta\cos\theta\sin\phi}{\cos\theta - \mathrm{d}\theta\sin\theta\cos\phi} \\
&= -\mathrm{d}\theta\sin\phi\left(1 + \frac{\mathrm{d}\theta\sin\theta\cos\phi}{\cos\theta}\right) \\
&= -\mathrm{d}\theta\sin\phi
\end{aligned}
$$

であり，

$$
\frac{\partial}{\partial\phi}\sin\phi' = \cos\phi - \frac{\mathrm{d}\theta\cos\theta\cos^2\phi}{\sin\theta} + \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta}
$$

より

$$
\begin{aligned}
\frac{\partial\phi'}{\partial\phi} &= \frac{1}{\cos\phi'}\frac{\partial}{\partial\phi}\sin\phi' \\
&= \frac{\cos\phi - \frac{\mathrm{d}\theta\cos\theta\cos^2\phi}{\sin\theta} + \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta}}{\cos\phi + \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta}} \\
&= 1 - \frac{\frac{\mathrm{d}\theta\cos\theta\cos^2\phi}{\sin\theta}}{\cos\phi + \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta}} \\
&= 1 - \frac{\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\left(1 - \frac{\mathrm{d}\theta\cos\theta\sin^2\phi}{\sin\theta\cos\phi}\right) \\
&= 1 - \frac{\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}
\end{aligned}
$$

であるから,

$$
\begin{aligned}
\frac{\partial}{\partial\phi} &= \frac{\partial\theta'}{\partial\phi'}\frac{\partial}{\partial\theta'} + \frac{\partial\phi'}{\partial\phi}\frac{\partial}{\partial\phi'} \\
&= -\mathrm{d}\theta\sin\phi\frac{\partial}{\partial\theta'} + \left(1 - \frac{\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\phi'}
\end{aligned}
$$

これらを用いて

$$
\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) = \sin\theta\cos\theta\frac{\partial Y}{\partial\theta'} - \frac{\mathrm{d}\theta\cos\theta\sin\phi}{\sin\theta}\frac{\partial Y}{\partial\phi'} + \sin^2\theta\frac{\partial^2 Y}{\partial\theta'^2} + 2\mathrm{d}\theta\sin\phi\frac{\partial^2 Y}{\partial\theta'\partial\phi'}
$$

$$
\frac{\partial^2 Y}{\partial\phi^2} = -\mathrm{d}\theta\cos\phi\frac{\partial Y}{\partial\theta'} + \frac{\mathrm{d}\theta\cos\theta\sin\phi}{\sin\theta}\frac{\partial Y}{\partial\phi'} - 2\mathrm{d}\theta\sin\phi\frac{\partial^2 Y}{\partial\theta'\partial\phi'} + \left(1 - \frac{2\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial^2 Y}{\partial\phi'^2}
$$

$$
\begin{aligned}
&\therefore \frac{1}{\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2}\right) \\
&= \frac{1}{\sin^2\theta}\left((\sin\theta\cos\theta - \mathrm{d}\theta\cos\phi)\frac{\partial Y}{\partial\theta'} + \sin^2\theta\frac{\partial^2 Y}{\partial\theta'^2} + \left(1 - \frac{2\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial^2 Y}{\partial\phi'^2}\right) \\
&= \frac{\sin\theta\cos\theta - \mathrm{d}\theta\cos\phi}{\sin^2\theta}\frac{\partial Y}{\partial\theta'} + \frac{\partial^2 Y}{\partial\theta'^2} + \frac{1}{\sin^2\theta}\left(1 - \frac{2\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial^2 Y}{\partial\phi'^2}
\end{aligned}
$$

一方で

$$
\begin{aligned}
&\frac{1}{\sin^2\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial Y}{\partial\theta'}\right) + \frac{\partial^2 Y}{\partial\phi'^2}\right) \\
&= \frac{\cos\theta'}{\sin\theta'}\frac{\partial Y}{\partial\theta'} + \frac{\partial^2 Y}{\partial\theta'^2} + \frac{1}{\sin^2\theta'}\frac{\partial^2 Y}{\partial\phi'^2} \\
&= \frac{\cos\theta - \mathrm{d}\theta\sin\theta\cos\phi}{\sin\theta + \mathrm{d}\theta\cos\theta\cos\phi}\frac{\partial Y}{\partial\theta'} + \frac{\partial^2 Y}{\partial\theta'^2} + \frac{1}{\sin^2\theta + 2\mathrm{d}\theta\sin\theta\cos\theta\cos\phi}\frac{\partial^2 Y}{\partial\phi'^2} \\
&= \frac{\cos\theta - \mathrm{d}\theta\sin\theta\cos\phi}{\sin\theta}\left(1 - \frac{\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial Y}{\partial\theta'} + \frac{\partial^2 Y}{\partial\theta'^2} + \frac{1}{\sin^2\theta}\left(1 - \frac{2\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial^2 Y}{\partial\phi'^2} \\
&= \frac{\sin\theta\cos\theta - \mathrm{d}\theta\cos\phi}{\sin^2\theta}\frac{\partial Y}{\partial\theta'} + \frac{\partial^2 Y}{\partial\theta'^2} + \frac{1}{\sin^2\theta}\left(1 - \frac{2\mathrm{d}\theta\cos\theta\cos\phi}{\sin\theta}\right)\frac{\partial^2 Y}{\partial\phi'^2}
\end{aligned}
$$

であるから，

$$\frac{1}{\sin^2\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\phi^2}\right)=\frac{1}{\sin^2\theta'}\left(\sin\theta'\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial Y}{\partial\theta'}\right)+\frac{\partial^2 Y}{\partial\phi'^2}\right)$$

よって，微分方程式

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{\partial^2 Y}{\partial\phi^2}=-l(l+1)\sin^2\theta Y$$

の解 Y は，微分方程式

$$\sin\theta'\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial Y}{\partial\theta'}\right)+\frac{\partial^2 Y}{\partial\phi'^2}=-l(l+1)\sin^2\theta' Y$$

も満たす．