

1 1.11

Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below.

Let L be the set of all lower bounds of B . $L \subset S$.

Because B is bounded below, L is not empty.

Take $b \in B$. For all $l \in L$, l is a lower bound of B , so $l \leq b$ holds. This means b is an upper bound of L . Because B is not empty, $b \in B$ exists and L is bounded above.

Due to the least-upper-bound property of S , $\sup L$ exists: $\sup L$ is an upper bound of L , and any $\gamma < \sup L$ is not an upper bound of L .

For any $b \in B$, b is an upper bound of L , so $\sup L \leq b$. This means $\sup L$ is a lower bound of B , and $\sup L \in L$.

Because $\sup L$ is an upper bound of L , any $l \in L$ satisfies $l \leq \sup L$.

So, $\sup L$ is a lower bound of B , and any $l > \sup L$ is not a lower bound of B . This means $\sup L = \inf B$.

2 2.23

By definition, E^c is closed if and only if any limit point of E^c is not a point of E . This is equivalent to its contrapositive: any $p \in E$ is not a limit point of E^c . By the definition of a limit point, this means that every $p \in E$ has at least one neighborhood $N_r(p)$ such that every $q \in N_r(p)$ satisfies $q \in E$ as long as $p \neq q$. The constraint “as long as $p \neq q$ ” is actually unnecessary, because p itself also satisfies $p \in E$. So, this can be restated as every $p \in E$ has at least one neighborhood $N_r(p)$ such that $N_r(p) \subset E$. This is exactly the definition of an open set.