

## 1 1.11

Suppose  $S$  is an ordered set with the least-upper-bound property,  $B \subset S$ ,  $B$  is not empty, and  $B$  is bounded below.

Let  $L$  be the set of all lower bounds of  $B$ .  $L \subset S$ .

Because  $B$  is bounded below,  $L$  is not empty.

Take  $b \in B$ . For all  $l \in L$ ,  $l$  is a lower bound of  $B$ , so  $l \leq b$  holds. This means  $b$  is an upper bound of  $L$ . Because  $B$  is not empty,  $b \in B$  exists and  $L$  is bounded above.

Due to the least-upper-bound property of  $S$ ,  $\sup L$  exists:  $\sup L$  is an upper bound of  $L$ , and any  $\gamma < \sup L$  is not an upper bound of  $L$ .

For any  $b \in B$ ,  $b$  is an upper bound of  $L$ , so  $\sup L \leq b$ . This means  $\sup L$  is a lower bound of  $B$ , and  $\sup L \in L$ .

Because  $\sup L$  is an upper bound of  $L$ , any  $l \in L$  satisfies  $l \leq \sup L$ .

So,  $\sup L$  is a lower bound of  $B$ , and any  $l > \sup L$  is not a lower bound of  $B$ . This means  $\sup L = \inf B$ .