1 1.11

Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below.

Let L be the set of all lower bounds of B. $L \subset S$.

Because B is bounded below, L is not empty.

Take $b \in B$. For all $l \in L$, l is a lower bound of B, so $l \leq b$ holds. This means b is an upper bound of L. Because B is not empty, $b \in B$ exists and L is bounded above.

Due to the least-upper-bound property of S, $\sup L$ exists: $\sup L$ is an upper bound of L, and any $\gamma < \sup L$ is not an upper bound of L.

For any $b \in B$, b is an upper bound of L, so $\sup L \leq b$. This means $\sup L$ is a lower bound of B, and $\sup L \in L$.

Because $\sup L$ is an upper bound of L, any $l \in L$ satisfies $l \leq \sup L$.

So, $\sup L$ is a lower bound of B, and any $l > \sup L$ is not a lower bound of B. This means $\sup L = \inf B$.

2 2.23

By definition, E^{c} is closed if and only if any limit point of E^{c} is not a point of E. This is equivalent to its contrapositive: any $p \in E$ is not a limit point of E^{c}. By the definition of a limit point, this means that every $p \in E$ has at least one neighborhood $N_r(p)$ such that every $q \in N_r(p)$ satisfies $q \in E$ as long as $p \neq q$. The constraint "as long as $p \neq q$" is actually unnecessary, because p itself also satisfies $p \in E$. So, this can be restated as every $p \in E$ has at least one neighborhood $N_r(p)$ such that $N_r(p) \subset E$. This is exactly the definition of an open set.

3 2.34

Let (X, d) be a metric space and K a compact subset of X. Take $p \in K^{\mathrm{c}}$. For each $r > 0$, define $V_r = \{q \in X \mid d(p,q) > r\}$, then $V_r \subset V_{r'}$ if $r > r'$. Since $\bigcup_r V_r = X \setminus \{p\} \supset K$ and K is compact, we can choose $r_1, \ldots, r_n > 0$ with $K \subset \bigcup_{i=1}^n V_{r_i}$. Let $r_{\min} = \min\{r_1, \ldots, r_n\}$, then $K \subset V_{r_{\min}}$, hence $N_{r_{\min}}(p) \subset K^{\mathrm{c}}$. Therefore K^{c} is open and K is closed.