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A11 + A12

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Tutorial 1

20BCE10077

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1

$$x' = 2x - y \qquad x(0) = 1$$
$$y' = -x + 2y \qquad y(0) = 0$$

Since eq^n is in form of differential equation :

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$$

$$X' = AX \quad \Rightarrow \boxed{X' = AX}$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ &

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

for eigen values, $|A - \lambda I| = 0$

2

$$\Rightarrow \left| \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\Rightarrow 4 + \lambda^2 - 4\lambda - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 3\lambda + 3 = 0$$

$$\lambda(\lambda-1) - 3(\lambda-1) = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$$\boxed{\lambda = 3, 1}$$

For $\lambda = 3$,

$$[A - \lambda I]\, x = 0$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to \cancel{R_1 - R_2}\ R_2 - R_1$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank = 1, $n = 2$, $n - r = 1$

1 unknown will be given arbitary value.

$-x_1 - y_1 = 0$ (3)

let $x_1 = k_1 \Rightarrow y = -k_1$

$\Rightarrow \quad X_1 = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

for $\lambda = 1$

$$[A - \lambda I]X = 0$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Rank $= 1$, $n = 2$, $n - r = 1$

1 unknown will be given arbitary value.

$x_1 - y_1 = 0$

let $x_1 = k_2$, $\Rightarrow y_1 = k_2$

$$X_2 = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

general solution is $X = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{1t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{1t} \\ -c_1 e^{3t} + c_2 e^{1t} \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 e^{3t} + c_2 e^{t} \\ -c_1 e^{3t} + c_2 e^{t} \end{bmatrix}$$

4

$x = c_1 e^{3t} + c_2 e^{t}$

$y = -c_1 e^{3t} + c_2 e^{t}$

$x(0) = 1 \Rightarrow 1 = c_1 + c_2$ — ①

$y(0) = 0 \Rightarrow 0 = -c_1 + c_2$ — ②

$\rightarrow c_2 = c_1$

$c_1 \Rightarrow 1/2$

Solving ① & ②

$2c_2 = 1 \Rightarrow c_2 = 1/2$

therefore $c_1 = +1/2$

$x = \frac{e^{3t}}{2} + \frac{e^{t}}{2}$

$y = -\frac{1}{2}(e^{3t}) + \frac{1}{2}e^{t}$

$\boxed{x = \frac{e^{3t}}{2} + \frac{e^{t}}{2}}$

$\boxed{y = \frac{e^{t}}{2} - \frac{e^{3t}}{2}}$

2

$$A = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix} \qquad D = 3A^2 + A^{-1}$$

The eigen values are given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ ~~5~~ solving the determinant

$$(5-\lambda)(-6-\lambda) + 18 = 0$$

$$\Rightarrow -30 - 5\lambda + 6\lambda + \lambda^2 + 18 = 0$$

$$\lambda^2 + \lambda - 12 = 0$$

$$\lambda^2 + 4\lambda - 3\lambda - 12 = 0, \quad \lambda(\lambda+4) - 3(\lambda+4) = 0$$

$$\lambda - 3 = 0, \quad \lambda + 4 = 0, \quad \boxed{\lambda = 3, -4}$$

Eigen values of $A = 3, -4$

So eigen values of $A^2 = 9, 16$

& eigen values of $A^{-1} = \frac{1}{3}, \frac{1}{4}$

~~So, the~~

so the eigen values of D

$\lambda = 3 \Rightarrow 3 \cdot 9 + \frac{1}{3} \Rightarrow 82/3$

$\lambda = -4 \Rightarrow 3 \cdot 16 - \frac{1}{4} \Rightarrow 191/4$

Eigen Values of matrix $D = \overset{\lambda = 3}{82/3}, \overset{\lambda=-4}{\frac{191}{4}}$

$= \boxed{\frac{82}{3}, \frac{191}{4}}$

3

$x' = -y$
$y' = -x$

Since equation is in form of differential,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{X'} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X}$$

$X' = AX$,

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ & $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

eigen values of A are given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & -1 \\ -1 & 0-\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0$$

solving the determinant,

$$(-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda+1)(\lambda-1) = 0 \quad , \quad \boxed{\lambda = -1, 1}$$

$\boxed{\text{eigen values of } A = -1, 1}$

Eigen vectors are given by

$$[A - \lambda I] X = 0$$

for $\begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

for lamda $\lambda = -1$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank = 1 , $n = 2$, $n - r = 1$,

1 unknown will be given arbitary value.

$$x_1 - x_2 = 0$$

let $x_1 = k_1 \Rightarrow x_2 \Rightarrow k_1$

$$X_1 = k\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for $\lambda = 1$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

, rank = 1 , $n = r$

$n - r = 1$

1 unknown will be given arbitary value.

$$-x_1 - x_2 = 0$$

let $x_1 = k_2 \Rightarrow x_2 = -k_2$

$$X_2 = k_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

General solution is

$$X = c_1 e^{-1t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{1t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} c_1 e^{-t} + c_2 e^{t} \\ c_1 e^{-t} - c_2 e^{t} \end{bmatrix} //$$

$$\Rightarrow \quad x(t) = c_1 e^{-t} + c_2 e^{t}$$
$$y(t) = c_1 e^{-t} - c_2 e^{t}$$