Mid Term Examination

Differential & Difference Equation

(1)

MAT2001

(A11 + A12)

19/04/2021

20BCE10077

Fiza Siddiqui

Class No - 0419

$y_1 = x_1 + 0.5x_2$

$y_2 = 0.5x_1 + x_2$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Ax = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$y = Ax$ & $y = \lambda x$

$\therefore \quad Ax = \lambda x$

Eigen value $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (0.5)^2 = 0$$

$= \lambda^2 + 1 - 2\lambda - 0.25 = 0$

$= \lambda^2 - 2\lambda + 0.75 = 0$

$100\lambda^2 - 200\lambda + 75 = 0$

2

$4\lambda^2 - 8\lambda + 3 = 0$

$\Rightarrow 4\lambda^2 - 6\lambda - 2\lambda + 3 = 0$

$\Rightarrow 2\lambda(2\lambda - 3) - 1(2\lambda - 3) = 0$

$\Rightarrow (2\lambda -)(2\lambda - 3) = 0$

Therefore, $\boxed{\lambda = 1/2 \,, \; 3/2}$

Eigen vector are given by

$[A - \lambda I][x] = \vec{0}$

for $\lambda = 1/2$

$$\begin{bmatrix} 1 - \cancel{3}/2 & 0.5 \\ 0.5 & 1 - \cancel{3}/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{0}1/2 & 0.5 \\ 0.5 & \cancel{0}1/2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1/2 & 0.5 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$n = 2$
$r = 1$
$n - r = 1$

1 unknown will be given arbitary value

$\frac{1}{2}x_1 + 0.5x_2 = 0$

$x_1 + x_2 = 0$

let $\boxed{x_1 = k_1}$

$\boxed{x_2 = -k_1}$

$$x_1 = k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3

for $\lambda = 3/2$

$$\begin{bmatrix} 1-3/2 & 0.5 \\ 0.5 & 1-3/2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & 0.5 \\ 0.5 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} -1/2 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$n = 2$
$r = 1$
$n - r = 1$

1 unknown will be given arbitary value

$$-\frac{1}{2} x_1 + 0.5 x_2 = 0$$

$$-x_1 + x_2 = 0$$

$x_1 = k_1$

$x_2 = k_1$

$$x_2 = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The principal membrane has coordinates $\cos\phi$, $\sin\phi$

$x_2 \to 45^\circ$ $\quad x_1 — 135^\circ$

4

$\therefore$ if we stretch

The principal membrane has co-ordinates

$\cos\phi$, $\sin\phi$ $\quad \boxed{\tan\phi}$

$x_2 \to 45^\circ$ $\quad x_1 \to 135^\circ$

$\therefore$ if we stretch it, the membrane will

stretch by $1/2$, $3/2$ factor

$$z_1 = \frac{1}{2}\cos\phi \qquad z_2 = \frac{3}{2}\sin\phi$$

$$\sin^2\phi + \cos^2\phi = 1$$

$$\frac{z_1^2}{(1/2)^2} + \frac{z_2^2}{(3/2)^2} = 1$$

elipse

Ques 2

$$A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \qquad P^{-1}AP = D$$

solution :

To find eigen values

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -3 & -3 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

determinant

$$(4-\lambda)\{-(2-\lambda)(2-\lambda)+3\} + 3\{3(2-\lambda)-3\}$$
$$-3\{3(-1)(-2-\lambda)\} = 0$$

$$\Rightarrow (4-\lambda)\{(-2-\lambda)(2-\lambda)+3\} + 3(6-3\lambda-3) - 3\{3-2-\lambda\} = 0$$

$$(4-\lambda)\{-4+\cancel{2\lambda}-\cancel{2\lambda}+\lambda^2+3\} + 3\{3-3\lambda\} - 3(1-\lambda) = 0$$

$$(4-\lambda)\{\lambda^2-1\} + \cancel{3\times 9(1-\lambda)} - 9\{(\lambda-1)\} + 3(\lambda-1) = 0$$

$$(4-\lambda)\{(\lambda+1)(\lambda-1)\} - 9(\lambda-1) + 3(\lambda-1) = 0$$

$$(\lambda-1)\{(4-\lambda)(\lambda+1) - 9 + 3\} = 0$$

$$\{\lambda-1\}\left[\{4\lambda+4-\lambda^2-\lambda\}-6\right]=0$$

$$(\lambda-1)\{-\lambda^2-2+3\lambda\}=0$$

$$(\lambda-1)(\lambda^2-3\lambda+2)=0$$

$$(\lambda-1)(\lambda^2-\lambda-2\lambda+2)=0$$

$$(\lambda-1)(\lambda(\lambda-1)-2(\lambda-1))=0$$

$$(\lambda-1)(\lambda-1)(\lambda+2)=0$$

$$\lambda=1,1,2$$

~~Value~~ Eigen values = $\underline{\underline{1,1,2}}$

For eigen vectors - $[A-\lambda I]\vec{x}=\vec{0}$

for $\lambda=1$

$$\begin{bmatrix}3 & -3 & -3\\ 3 & -3 & -3\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_3 \to 3R_3+R_2$

$$\begin{bmatrix}3 & -3 & -3\\ 3 & -3 & -3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank $= 1$ $\qquad n - r = 2$

$n = 3$

Therefore two unknowns will be given arbitrary value

Let $\boxed{x_2 = k_1}$ & $\boxed{x_3 = k_2}$

$x_1 - x_2 - x_3 = 0$

$x_1 - k_1 - k_2 = 0$

$\boxed{x_1 = k_1 + k_2}$ $\qquad x = k_1 + k_2$

Thus $\cancel{X = \begin{bmatrix} k_1 \\ k_2 \\ k_1 - k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$

$$X_1 = \begin{bmatrix} k_1 + k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 2$

$$\begin{bmatrix} 2 & -3 & -3 \\ 3 & -4 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(R_1 \to \frac{R_1}{2}\right) \quad A \sim \begin{bmatrix} 1 & -\frac{3}{2} & -\frac{3}{2} \\ 3 & -4 & -3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 1/2 & 3/2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + R_1 \qquad , \qquad R_2 \to 2R_2$$

$$\begin{bmatrix} 1 & -3/2 & -3/2 \\ 0 & 1/2 & 3/2 \\ 0 & -1/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \quad \begin{bmatrix} 1 & -3/2 & -3/2 \\ 0 & 1 & 3 \\ 0 & -1/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \to R_1 + \frac{3R_2}{2}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & -1/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{R_2}{2}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank} = 2 \qquad n = 3$$

$$n - r = 1$$

1 unknown will be given arbitary value

let $x_1 = k_1$

$R_3 \to R_3 + \frac{R_2}{2}$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank = 2

n = 3

n - r = 1 , 1 unknown will be given arbitary value

let $x_3 = k_1$

$x_1 + 3x_3 = 0$ $\qquad$ $x_2 + 3x_3 = 0$

$\boxed{x_1 = -3k_1}$ $\qquad$ $\boxed{x_2 = -3k_1}$

$$X_2 = \begin{bmatrix} -3k_1 \\ -3k_1 \\ k_1 \end{bmatrix} = k \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

P = modal matrix

$$P = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} \leftarrow \text{modal matrix}$$

Determinant of P = $|P| = 1(3) + 1(1) - 3(0)$

$= \cancel{3} = \cancel{2}\ 4$

4

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ ← diagonal matrix

adj of P =

$a_{11} = +3$, $a_{12} = 1$, $a_{13} = 1$

$a_{21} = -4$, $a_{22} = +1$, $a_{23} = -1$

$a_{31} = -3$, $a_{32} = 0$, $a_{33} = -1$

$$\text{adj } P = \begin{bmatrix} +3 & -4 & -3 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 3 & -4 & -3 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$P^{-1}AP = D$

$$\begin{bmatrix} 3 & -4 & -3 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ Hence Proved. ~~that~~

$\underline{3}$

$v(t) = V_0 \cos \frac{1}{2} \pi t$

$$f(t) = \begin{cases} 0 & -\frac{1}{2} < x \leq \frac{1}{2} \\ V_0 \cos \frac{1}{2} \pi t , & \end{cases}$$

$\cancel{P = 2l} \quad = \quad \omega = \frac{1}{2}\pi ,$

$a_0 = \frac{1}{1/2} \int_0^{1/2} \cancel{(f(t)\,dt}$

here $l = \frac{\pi}{\omega} = \frac{\pi}{\pi} 2 = 2$

$$F(t) = \begin{cases} 0 & -2 < t < -1 \\ V_0 \cos \frac{1}{2}\pi t , & -1 < t < 1 \\ 0 , & 1 < t < 2 \end{cases}$$

Here f, even function, thus $b_n = 0$ (f, even)

$a_0 = \frac{2}{2L} \int_0^L f(t)\,dt = \frac{2}{2} \int_0^1 V_0 \cos(\frac{1}{2}\pi t)\,dt$

$= \cancel{\frac{V_0}{0}} \left[\frac{V_0 \sin(\frac{1}{2}\pi t)}{\frac{1}{2}\pi} \right]_0^1$

$$= \frac{2V_0}{\pi}\left[\sin \pi/2 - \sin 0\right]$$

$$= \frac{2V_0}{\pi}$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{\cancel{1}}\int_0^1 V_0\cos\left(\tfrac{1}{2}\pi t\right)\cos\left(\frac{n\pi t}{2}\right)dt$$

$$= 2V_0\int_0^1 \cos\left(\frac{\pi t}{2}+\frac{n\pi t}{2}\right)+\cos\left(\frac{\pi t}{2}-\frac{n\pi t}{2}\right)$$

$$= 2V_0\left[\frac{\sin(\pi/2+n\pi/2)}{\pi/2+n\pi/2}+\frac{\sin(\pi/2-n\pi/2)t}{\pi/2-n\pi/2}\right]_0^1$$

$$= \frac{2V_0}{\cancel{\pi}}\left[\frac{\sin(\pi/2+n\pi/2}{\pi/2+n\pi/2}+\frac{\sin(\pi/2-n\pi/2)}{\pi/2-n\pi/2}\right]$$

$$= \frac{4V_0}{\pi}\left[\cos\left(\frac{n\pi}{L}\right)\left[\frac{1}{1+n}+\frac{1}{1-n}\right]\right.$$

$$\hookrightarrow \frac{2}{1-n^2}$$

$$= \frac{4V_0}{\pi(1-n^2)}\cos\left(\frac{n\pi}{L}\right)$$

$$a_1 = \frac{2}{1}\int_0^1 f(t)\cos\frac{\pi t}{1}\,dt$$

$$= v_0\int_0^1 \frac{1+\cos 2\pi t}{2}\,dt$$

$$= \frac{v_0}{2}\left[t + \frac{\sin \pi t}{\pi}\right]_0^1$$

$$= \frac{v_0}{2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

$$f(t) = \frac{2v_0}{\pi} + \frac{v_0}{2}\cos(n\,t/_2) +$$

$$\frac{4v_0}{\pi(1-n^2)}\sum_{n=2}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi t}{2}\right)}{(1-n^2)}.$$

$f(x) = e^x \quad 0 < x < 2$

a

$(0, 2)$

cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{a}$$

$$a_0 = \frac{2}{a} \int_0^a f(x)\, dx$$

$$= \frac{2}{2} \int_0^2 e^x\, dx = \int_0^2 e^x\, dx$$

$$= \left[e^x \right]_0^2 = e^2 - 1$$

$$\boxed{a_0 = e^2 - 1}$$

$$a_n = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a}\, dx$$

$$= \frac{2}{2} \int_0^2 e^x \cos \frac{n\pi x}{2}\, dx = \int_0^2 e^x \cos \frac{n\pi x}{2}\, dx$$

$$= \left[\frac{e^x}{1 + \left(\frac{n\pi}{2}\right)^2} \left(1 \cdot \cos \frac{n\pi x}{2} + \frac{n\pi}{2} \sin \frac{n\pi}{2} x \right) \right]_0^2$$

$\{$ formula : $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

$$\left[\frac{e^2}{1+\left(\frac{n\pi}{2}\right)^2}\left(\cos n\pi + \underbrace{\frac{n\pi}{2}\sin n\pi}_{=0}\right) - \frac{e^0}{1+(n\pi/2)^2}\left(\cos 0 + \underbrace{\frac{n\pi}{2}\sin 0}_{=0}\right)\right]$$

$$\Rightarrow \frac{e^2}{1+\left(\frac{n\pi}{2}\right)^2}(-1)^n - \frac{1}{1+(n\pi/2)^2}$$

$$\boxed{a_n = \frac{\left((-1)^n e^2 - 1\right)}{1+\left(\frac{n\pi}{2}\right)^2}}$$

$$f(x) = \frac{e^2-1}{2} + \sum_{\infty}^{n=1} \frac{(-1)^n e^2 - 1}{1+\left(\frac{n\pi}{2}\right)^2} \cos\frac{n\pi x}{2}$$

$$= \frac{e^2-1}{2} + \sum_{n=1}^{\infty} \frac{4\left((-1)^n e^2 - 1\right)}{4+(n\pi)^2} \cos\frac{n\pi x}{2}$$

$$f(x) = \frac{e^2-1}{2} + 4\left(\cos\frac{\pi x}{2}\,\frac{(-e^2-1)}{4+\pi^2} + \frac{\cos\pi x\,(e^2-1)}{4+4\pi^2} \cdots\cdots\right)$$

Q5

$$f(x) = \begin{cases} \frac{1}{2\pi} & -\infty < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}^{-1}\{f(x)\} = \int_{-\infty}^{\infty} F(\alpha) e^{i\alpha x} \, d\alpha$$

$$= \int_{-\infty}^{0} \frac{1}{2\pi} e^{i\alpha x} \, d\alpha + \int_{0}^{\infty} 0 \, d\alpha$$

$$\Rightarrow \int_{-\infty}^{0} \frac{1}{2\pi} e^{i\alpha x} \, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{i\alpha x} \, d\alpha = \frac{1}{2\pi} \left[\frac{e^{i\alpha x}}{ix} \right]_{-\infty}^{0}$$

$$= \frac{1}{2\pi} \left[\frac{e^{ix(0)}}{ix} - \frac{e^{ix(-\infty)}}{ix} \right]$$

$$= \frac{1}{2\pi} \left(\frac{1}{ix} - 0 \right) =$$

$$\mathcal{F}^{-1}\{f(x)\} = \frac{1}{2\pi i x} \Rightarrow f(x) = \frac{1}{2\pi i x}$$

$$f(x) = \frac{1}{2\pi i x}$$