Tutorial 1

Applied Linear Algebra

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1

Proving existence of Inverse

To prove linear transformation is One-One

Let $(x_1, y_1, z_1) \in R^3$

Let $(x_2, y_2, z_2) \in R^3$

$$T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$$

$$(2x_1, x_1+y_1+z_1, -y_1) = (2x_2, x_2+y_2+z_2, -y_2)$$

On comparison:

$$2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$$-y_1 = -y_2 \Rightarrow y_1 = y_2$$

$$x_1+y_1+z_1 = x_2+y_2+z_2 \Rightarrow z_1 = z_2$$

Thus $(x_1, y_1, z_1) = (x_2, y_2, z_2)$

To prove linear transformation $(U \to V)$ is onto

let $V = (a, b, c) \in V$ (i.e R^3)

such that

$$T(x, y, z) = (a, b, c)$$

On comparison :

$2x = a \rightarrow x = a/_2$ — ①

$-y = c \rightarrow y = c$ — ②

$x+y+z = b \rightarrow z = b+c-a/_2$ — ③

On observing ①, ② & ③, we can conclude that

$(x, y, z) \in U$ (i.e R^3)

Thus linear transformation is onto

We can now conclude that inverse of given linear transformation exists.

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1

Standard Basis of $R^3 = B$

$B = \{(0,0,1), (1,0,0), (0,1,0)\}$

$T(0,0,1) = (0,1,0)$

$T(1,0,0) = (2,1,0)$

$T(0,1,0) = (0,1,-1)$

Then matrix representation of given linear transformation w.r.to B is :

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$\det(A) = 2(0+1) = 2 \neq 0$

Thus, inverse of A exists as it is linearly independent.

Calculating inverse of A :

Augmented matrix of A

$$C = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

~~$R_2 \to R_2 - R_1$~~

$$R_1 \to R_1/2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1 & 1 \end{array}\right]$$

$$R_2 \to R_2 - R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1/2 & 1 & 1 \end{array}\right]$$

$$A^{-1} = 1/2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ -1 & 2 & 2 \end{bmatrix}$$

Using $T^{-1}(a,b,c) = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Thus inverse of given linear transformation is

$$T^{'}(a,b,c) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\Rightarrow \tfrac{1}{2}\begin{bmatrix} a & -2c & -a+2b+2c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{a}{2} & -c & \frac{-a+2b+2c}{2} \end{bmatrix}$$

// Ans

Thus

$$T^{-1} : R^3 - R^3$$

$$T^{'}(x,y,z) = \left(\frac{x}{2}, -y, \frac{-x+2y+2z}{2}\right)$$

2

$B = \{(1,0), (0,1)\}$

$C = \{(1,2), (2,1)\}$

R^3

Transition matrix

Solution:

To find Transition matrix

$(1,0) = \alpha(1,2) + B(2,1)$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 0 \end{array}\right] \to R_2 \to R_2 - 2R_1 \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & -2 \end{array}\right]$$

$R_2 \to R_2/-3 \quad R_1 \to R_1 - 2R_2$

$1 - \frac{4}{3}$

$$\left[\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \end{array}\right]$$

$\boxed{\alpha = -1/3}$ $\boxed{B = 2/3}$

$(0,1) = \alpha(1,2) + B(2,1)$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 0 \end{array}\right]$$

after reducing it to identity matrix

$\boxed{\alpha = 2/3}$ $\boxed{B = -1/3}$

$$P = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \rightarrow$$ Transition Matrix from B coordinates to C coordinates.

For any $x \in R^2$

$$[x]_C = P[x]_B$$

finding Transition matrix $C \rightarrow B$

$$(1,2) = \alpha(1,0) + B(0,1)$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \boxed{\alpha = 1 \quad B = 2}$$

$$(2,1) = \alpha(1,0) + \beta(0,1)$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

already in form of identity matrix

$\therefore \quad \alpha = 2 \ , \ B = 1$

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow$$ Transition matrix from C coordinates to B coordinates.

3

6

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (4x+3y,\ x+y)$

$B = \{(1,0),\ (0,1)\}$ $\quad C = \{(1,2),\ (2,1)\}$

Solution:

finding transition matrix:

$(1,0) = \alpha(1,2) + \beta(2,1)$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$R_2 \to R_2 / -3 \quad R_1 \to R_1 - 2R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix}$

$\Rightarrow \boxed{\alpha = -1/3},\ \boxed{\beta = 2/3}$

$(0,1) = \alpha(1,2) + \beta(2,1)$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$R_2 \to R_2 / -3 \quad R_1 \to R_1 - 2R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$

$\Rightarrow \boxed{\alpha = 2/3}\ \boxed{\beta = -1/3}$

~~P =~~

Transition matrix $\rightarrow$ $P = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$

$T(1,0) = (4,1) = 4(1,0) + 1(0,1)$

$T(0,1) = (3,1) = 3(1,0) + 1(0,1)$

$A_{BB} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ $\leftarrow$ matrix Representation w.r.t. B.

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To find matrix Representation with respect to C (7)

$T(1,2) = (10,3) = \alpha(1,2) + \beta(2,1)$

$P^{-1} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 10 \\ 3 \end{bmatrix} = \begin{bmatrix} -10/3 + 3\times\frac{2}{3} \\ \frac{20}{3} - 3\times\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{10}{3} + 2 \\ \frac{20}{3} - 1 \end{bmatrix}$

$= \begin{bmatrix} -4/3 \\ 17/3 \end{bmatrix}$

$T(2,1) = (11,3) = \alpha(1,2) + \beta(2,1)$

$\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 11 \\ 3 \end{bmatrix} = \begin{bmatrix} -11/3 + 3\times\frac{2}{3} \\ \frac{22}{3} - 3\times\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{11}{3} + 2 \\ \frac{22}{3} - 1 \end{bmatrix}$

$= \begin{bmatrix} -\frac{5}{3} \\ 19/3 \end{bmatrix}$

$A_{CC} = \begin{bmatrix} -4/3 & -5/3 \\ 17/3 & 19/3 \end{bmatrix}$ ← matrix Representation with respect to C.

Finding P^{-1}

$\left[\begin{array}{cc|cc} -1/3 & 2/3 & 1 & 0 \\ 2/3 & -1/3 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 + 2R_1$

$$\left[\begin{array}{cc|cc} -1/3 & 2/3 & 1 & 0 \\ 0 & -\frac{1}{3}+\frac{4}{3} & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} -1/3 & 2/3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$R_1 \to R_1 - \frac{2}{3}R_2$

$1 - \frac{2}{3} \times 2$

$1 - \frac{4}{3}$

$3 - \frac{4}{3}$ -11

$$\left[\begin{array}{cc|cc} -1/3 & 0 & -1/3 & -2/3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$R_1 \to 3R_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\therefore \quad P^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Now, we know that

$$A_{BB} = P^{-1} A_{CC} P$$

$$\therefore \quad = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4/3 & -5/3 \\ 17/3 & 19/3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$-\frac{6}{9}$ $\frac{2}{3}$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{9} - \frac{10}{9} & -\frac{8}{9} + \frac{5}{9} \\ -\frac{17}{9} + \frac{14 \times 2}{9} & \frac{2 \times 17}{9} - \frac{19}{9} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2/3 & -1/3 \\ 7/3 & 5/3 \end{bmatrix}$$

$$\begin{bmatrix} -2/3 + \frac{14}{3} & -\frac{1}{3} + \frac{10}{3} \\ -\frac{4}{3} + \frac{7}{3} & -\frac{2}{3} + \frac{5}{3} \end{bmatrix}$$

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7/21

$$\Rightarrow \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \Rightarrow A_{BB}$$

Hence verified.

$\therefore$ A_{CC} is similar to A_{BB}