Assignment 3

A-11

22/08/21

20BCE10077

MAT 3002

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Given: $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

& $B = \{ \underbrace{\sin(x)}_{U_1}, \underbrace{\sin(2x)}_{U_2}, \underbrace{\sin(4x)}_{U_3} \}$

Inner product of U_1 & U_2:

$\langle U_1, U_2\rangle \Rightarrow \langle \sin x, \sin 2x\rangle$

$\Rightarrow \int_{-\pi}^{\pi} \sin x \sin 2x\, dx \Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \sin x \sin 2x \cdot 2\, dx$

$\Rightarrow \frac{1}{2}\int_{-\pi}^{+\pi} (\cos x - \cos 3x)\, dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right]_{-\pi}^{\pi}$

$= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right]_{-\pi}^{\pi} \cancel{\sin \pi} = \frac{1}{2} \cdot 0$

$= 0$

Inner product of U_2 & U_3:

$\langle U_2, U_3\rangle = \langle \sin 2x, \sin 4x\rangle$

$= \int_{-\pi}^{\pi} \sin 2x \sin 4x\, dx$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\sin 4x \sin 2x \; dx$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \cos(2x) - \cos(6x) \cdot dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 6x}{6}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \cdot 0$$

$$= 0$$

Inner Product of v_1 & v_3:

$$\langle v_1, v_3 \rangle = \langle \sin x, \sin 4x \rangle$$

$$= \int_{-\pi}^{\pi} \sin x \sin 4x \; dx \Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \sin x \sin 4x \cdot 2 \; dx$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \cos 3x - \cos 5x \; dx$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 5x}{5}\right]_{-\pi}^{\pi}$$

$$\Rightarrow \frac{1}{2} \cdot 0$$

$$\Rightarrow 0$$

Set B is orthogonal as

$\langle v_1, v_2 \rangle = 0$

$\langle v_2, v_3 \rangle = 0$

$\langle v_1, v_3 \rangle = 0$

Hence Proved.

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$$B = \{(1,2,-2),\ (4,3,2),\ (1,2,1)\} \quad R^3$$

Solution:

$$B = \{ \underbrace{(1,2,-2)}_{w_1},\ \underbrace{(4,3,2)}_{w_2},\ \underbrace{(1,2,1)}_{w_3} \}$$

Using Gram-Schmidt method:

- $V_1 = w_1$
  $= (1,2,-2)$

- $V_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$$= (4,3,2) - \frac{\langle (4,3,2)(1,2,-2) \rangle}{\langle (1,2,-2)(1,2,-2) \rangle} (1,2,-2)$$

$$= (4,3,2) - \frac{6}{9}(1,2,-2) \Rightarrow (4,3,2) - \left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$$

$\rightarrow V_2 = (10/3,\ 5/3,\ 10/3)$

- $V_3 = w_3 - \dfrac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \dfrac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

$$\Rightarrow (1,2,1) - \frac{\langle (1,2,1)(1,2,-2) \rangle}{\langle (1,2,-2)(1,2,-2) \rangle}(1,2,-2) - \frac{\langle (1,2,1)(10/3, 5/3, 10/3) \rangle}{\langle 10/3, 5/3, 10/3)(10/3, 5/3, 10/3) \rangle}(10/3, 5/3, 10/3)$$

$$\Rightarrow (1,2,1) - \frac{3}{9}(1,2,-2) - \frac{30/3}{225/9}(10/3,\ 5/3,\ 10/3)$$

$\Rightarrow (1,2,1) - \frac{1}{3}(1,2,-2) - \frac{10}{25}(10/3, 8/3, 10/3)$

$\Rightarrow (1,2,1) - \frac{1}{3}(1,2,-2) - \frac{1}{3}(4,2,4)$

$\Rightarrow (1,2,1) - \frac{1}{3}(5, 4, 2)$

$\Rightarrow \left(1 - \frac{5}{3}, 2 - \frac{5}{3}, 1 - 2/3\right)$

$V_3 \Rightarrow (-2/3, 2/3, 1/3)$

Orthogonal Basis $(\bar{B})$: $\{(1,2,-2), (10/3, 8/3, 10/3), (-2/3, 2/3, 1/3)\}$

Coordinates of $(1,1,1)$:

i) $\alpha_1 = \frac{\langle (1,1,1)(1,2,-2) \rangle}{\langle (1,2,-2)(1,2,-2) \rangle} = \frac{1+2-2}{1+4+4}$

$\boxed{\alpha_1 = 1/9}$

ii) $\alpha_2 = \frac{\langle (1,1,1)(10/3, 8/3, 10/3) \rangle}{\langle (10/3, 8/3, 10/3)(10/3, 8/3, 10/3) \rangle} = \frac{25/3}{225/9}$

$\boxed{\alpha_2 = 1/3}$

iii) $\alpha_3 = \frac{\langle (1,1,1)(-2/3, 2/3, 1/3) \rangle}{\langle (-2/3, 2/3, 1/3)(-2/3, 2/3, 1/3) \rangle} = \frac{1/3}{9/9}$

$\boxed{\alpha_3 = 1/3}$ Coordinates: $(1/9, 1/3, 1/3)$

$W = \{(x, y, z) (x + y - 2z = 0)\}$ of R^3

Solution:

$$w = \{(x, 2z - x, z) \mid x, z \in R\}$$
$$= x(1, -1, 0) + z(0, 2, 1)$$

Thus

$$B = \text{basis of } W = \{(1, -1, 0), (0, 2, 1)\}$$

But, B is not orthogonal

• Converting to orthogonal:

• $v_1 = w_1 = (1, -1, 0)$

• $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$$= (0, 2, 1) - \frac{\langle (0, 2, 1)(1, -1, 0) \rangle}{\langle (1, -1, 0)(1, -1, 0) \rangle} (1, -1, 0)$$

$$= (0, 2, 1) - \frac{(-2)}{(2)} (1, -1, 0) \Rightarrow (0, 2, 1) + (1, -1, 0)$$

$$v_2 = (1, 1, 1)$$

Thus, $\bar{B} = \{(1, -1, 0), (1, 1, 1)\}$ is orthogonal Basis of W

& $\langle (1, -1, 0)(1, 1, 1) \rangle = 0$

· Extending $\bar{B}$ to basis of R^3

Checking for linearity independency for $(1,0,0)$

$$B' = \{(1,-1,0), (1,1,1), (1,0,0)\}$$

$$\alpha(1,-1,0) + \beta(1,1,1) + \gamma(1,0,0) = (0,0,0)$$

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ -\alpha+\beta &= 0 \\ \beta &= 0 \end{aligned}\right\} \alpha = \beta = \gamma = 0$$

Thus $B' = \{(1,-1,0), (1,1,1), (1,0,0)\}$ forms basis R^3

Converting B' into orthogonal:

$v_1 = 1,-1,0$

$v_2 = 1,1,1$

$$v_3 = (1,0,0) - \frac{\langle (1,0,0)(1,-1,0)\rangle}{\langle (1,-1,0)(1,-1,0)\rangle}(1,-1,0)$$

$$- \frac{\langle (1,0,0)(1,1,1)\rangle}{\langle (1,1,1)(1,1,1)\rangle}(1,1,1)$$

$$= (1,0,0) - \frac{1}{2}(1,-1,0) - \frac{1}{3}(1,1,1)$$

$$v_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

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$B'' = \{ (1,-1,0), (1,1,1), (\frac{1}{6}, \frac{1}{6}, -1/3) \}$ → orthogonal basis of R^3

where $\langle (1,-1,0)(1,1,1) \rangle = 0$

$\langle (1,1,1)(1/6, 1/6, -1/3) \rangle = 0$

$\langle (1,-1,0)(1/6, 1/6, -1/3) \rangle = 0$

Orthogonal Complement:

$$w^{\perp} = \text{span}(1/6, 1/6, -1/3)$$

$$= \text{span}(1,1,-2)$$

$$= \{ (x,y,z) \mid x-y+z=0 \}$$

$$= \{ (a, b, -a-b) \mid a, b \in R \}$$

$$= a(1,0,-1) + b(0,1,-1)$$

- **Basis of $w^{\perp}$**

$$C' = \{ (1,0,-1), (0,1,-1) \}$$

4th

$W = \{ (x, y, z) \mid x - y - z = 0 \}$ of R^3

$x = (2, 3, 5)$ as $x = p + q$

Solution:

$W = \{(y + z, y, z) \mid y, z \in R^3\}$

$= y(1,1,0) + z(1,0,1)$

$B =$ basis of w $= \{(1,1,0), (1,0,1)\}$

However, B is not orthogonal

Converting to orthogonal:

$v_1 = w_1 = (1, 0, 1)$

$v_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$= (1,1,0) - \dfrac{\langle (1,1,0)(1,0,1) \rangle}{\langle (1,0,1)(1,0,1)} (1,0,1)$

$\Rightarrow (1,1,0) - \frac{1}{2}(1, 0, 1)$

$\Rightarrow \left(\frac{1}{2}, 1, -\frac{1}{2} \right) \times 2$

$v_2 = (1, 2, -1)$

$B = \{(1,0,1),(1,2,-1)\}$

• Converting into orthonormal

$$B' = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)\right\}$$

Calculating P where $(x = 2,3,5)$

$$c_1 = \left\langle (2,3,5)\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\rangle = \frac{2+5}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$c_2 = \left\langle (2,3,5)\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)\right\rangle = \frac{2+6-5}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$

Thus $p = c_1 u_1 + c_2 u_2$

$$= \frac{7}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \frac{3}{\sqrt{6}}\left(\frac{1}{6}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$$= (7/2, 0, 7/2) + (3/6, 6/6, -3/6)$$

$$= (7/2, 0, 7/2) + (1/2, 1, -1/2) \Rightarrow \left(\frac{7+1}{2}, 1, \frac{7-1}{2}\right)$$

$P = (4,1,3)$

• Calculation of q

$q + p = x = (2,3,5)$

$q = x - p = (4,1,3) - (2,3,5)$

$= (2,-2,-2)$

Now, for verification:

① $P = (4,1,3)$

since $4-1-3=0$, we can say $P \in W$

② $\langle p, q\rangle = \langle (4,1,3)(2,-2,-2)\rangle$

$= 8-2-6 = 8-8 = 0$

Hence Proved.

5th

$B = \{(3/5, 4/5, 0), (-4/5, 3/5, 0), (0, 0, 1)\}$

$C = \{(3/7, -6/7, -2/7), (2/7, 3/7, -6/7), (6/7, 2/7, 3/7)\}$

Calculating coordinates:

i) coordinates of $(3/5, 4/5, 0)$ with respect to C

Since, C is a orthonormal coordinates can be given by:

i) $\alpha_1 = \langle (\frac{3}{5}, \frac{4}{5}, 0) (\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7}) \rangle$

$= \frac{9}{35} + \frac{(-24)}{35} = -15/35$

ii) $\alpha_2 = \langle (\frac{3}{5}, \frac{4}{5}, 0), (\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}) \rangle$

$= \frac{6}{35} + \frac{12}{35} - \frac{18}{35}$

iii) $\alpha_3 = \langle (\frac{3}{5}, \frac{4}{5}, 10), (\frac{6}{7}, \frac{2}{7}, \frac{3}{7}) \rangle$

$= \frac{18}{35} + \frac{8}{35} = \frac{26}{35}$

ii) coordinates of $(-\frac{4}{5}, \frac{3}{5}, 0)$ with Respect to C

Since C is orthonormal basis, coordinates can be given by

i) $\alpha_1 = \langle (-\frac{4}{5}, \frac{3}{5}, 0) (\frac{3}{7}, -\frac{6}{7}, \frac{2}{7}) \rangle$

$= \frac{-12}{35} + \frac{(-18)}{35} = \frac{-30}{35}$

ii) $\alpha_2 = \langle (\frac{-4}{5}, \frac{3}{5}, 0), (\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}) \rangle$

$= \frac{-8+9}{35} = 1/35$

iii) $\alpha_3 = \langle (-\frac{4}{5}, \frac{2}{5}, 0)(\frac{6}{7}, 2/7, \frac{3}{7}) \rangle$

$= \frac{-24+6}{35} = \frac{-18}{35}$

iii) coordinates of $(0,0,1)$ w.r.t C

Since C is orthonormal basis, coordinates can be given by:

i) $\alpha_1 = \langle (0,0,1)(\frac{3}{7}, -\frac{6}{7}, \frac{2}{7}) \rangle = 2/7$

ii) $\alpha_2 = \langle (0,0,1)(\frac{2}{7}, \frac{5}{7}, \frac{-6}{7}) \rangle = -6/7$

iii) $\alpha_3 = \langle (0,0,1)(\frac{6}{7}, \frac{3}{7}, \frac{3}{7}) \rangle = 3/7$

Transition matrix

Thus, transition matrix from B coordinates to coordinates is given by

$$P = \begin{bmatrix} -15/35 & -30/35 & 2/7 \\ 18/35 & 1/35 & -6/7 \\ 26/35 & -18/35 & 3/7 \end{bmatrix}$$

P^T (P Transpose) $= \begin{bmatrix} -15/35 & 18/35 & 26/35 \\ -30/35 & 1/35 & -18/35 \\ 2/7 & -6/7 & 3/7 \end{bmatrix}$

Now,

$$PP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, ~~it~~ proved that transition matrix from B coordinates to C coordinates is an orthogonal matrix.

Q6

$$A = \begin{bmatrix} 4 & -3 \\ 11 & -2 \\ 5 & -10 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 15 \\ -20 \end{bmatrix} \qquad (1)$$

To find QR factorisation of A. And least square solⁿ

Solution:

column vectors of $A = \{ \underbrace{(4,11,5)}_{u_1}, \underbrace{(-3,-2,-10)}_{u_2} \}$

Checking if the vectors are orthogonal or not;

$\langle u_1, u_2 \rangle \Rightarrow \langle (4,11,5),(-3,-2,-10) \rangle \neq 0$

Therefore column vectors of A are not orthogonal.

→ Converting to orthogonal set:

$v_1 = (4,11,5)$

$$v_2 = (-3,-2,-10) - \frac{\langle (-3,-2,-10)(4,11,5) \rangle}{\langle (4,11,5)(4,11,5) \rangle} (4,11,5)$$

$$v_2 = (-3,-2,-10) + \frac{84}{162}(4,11,5)$$

$$= (-3,-2,-10) + \frac{14}{27}(4,11,5)$$

$$v_2 = \left(-\frac{25}{27}, \frac{100}{27}, -\frac{200}{27} \right)$$

Orthogonal Set: $\{ (4,11,5), (-25/27, 100/27, -200/27) \}$

Orthonormal Set: $\left\{ \dfrac{v_1}{\|v_1\|}, \dfrac{v_2}{\|v_2\|} \right\}$

$$\Rightarrow \left\{ \frac{1}{\sqrt{162}}(4, 11, 5), \ \frac{27}{225}(-25, 100, -200) \right\}$$

Orthonormal Set: $\left\{ \left(\dfrac{4}{\sqrt{162}}, \dfrac{11}{\sqrt{162}}, \dfrac{5}{\sqrt{162}}\right), \left(-1/9, 4/9, -8/9\right) \right\}$

Therefore <u>Q is:</u>

$$Q = \begin{bmatrix} 4/\sqrt{162} & -1/9 \\ 11/\sqrt{162} & 4/9 \\ 5/\sqrt{162} & -8/9 \end{bmatrix}$$

<u>Calculating R:</u>

$$R = Q^T A =$$

$$= \underset{2\times 3}{\begin{bmatrix} 4/\sqrt{162} & 11/\sqrt{162} & 5/\sqrt{162} \\ -1/9 & 4/9 & -8/9 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 4 & -3 \\ 11 & -2 \\ 5 & -10 \end{bmatrix}}$$

$$R = \begin{bmatrix} 162/\sqrt{162} & -84/\sqrt{162} \\ 0 & 75/9 \end{bmatrix}$$

$$R = \begin{bmatrix} 162/\sqrt{162} & -84/\sqrt{162} \\ 0 & 25/3 \end{bmatrix}$$

An upper triangular Matrix.

Verifying

$$QR = \begin{bmatrix} 4/\sqrt{162} & -1/9 \\ 11/\sqrt{162} & 4/9 \\ 5/\sqrt{162} & -8/9 \end{bmatrix}_{3\times 2} \begin{bmatrix} 162/\sqrt{162} & -84/\sqrt{162} \\ 0 & 25/3 \end{bmatrix}_{2\times 2}$$

$$\Rightarrow \begin{bmatrix} \frac{4}{\sqrt{162}} \times \frac{162}{\sqrt{162}} & \frac{-84\times 4}{162} - \frac{25}{9\times 3} \\ \frac{11}{\sqrt{162}} \times \frac{162}{\sqrt{162}} & \frac{-84\times 11}{162} + \frac{25\times 4}{9\times 3} \\ \frac{5}{\sqrt{162}} \times \frac{162}{\sqrt{162}} & \frac{-84\times 5}{162} - \frac{8\times 25}{9\times 3} \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -486/162 \\ 4 & -324/162 \\ 5 & -1620/162 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & -3 \\ 11 & -2 \\ 5 & -10 \end{bmatrix} = A$$

Therefore, A can be factorised in Q & R.

$$\therefore \quad Q = \begin{bmatrix} 4/\sqrt{162} & -1/9 \\ 11/\sqrt{162} & 4/9 \\ 5/\sqrt{162} & -8/9 \end{bmatrix} \qquad R = \begin{bmatrix} 162/\sqrt{162} & -84/\sqrt{162} \\ 0 & 25/3 \end{bmatrix}$$

Least Square Solution:

$Ax = b$

$\vec{x} = R^{-1} Q^{T} b$ R - inverse

Q - Transpose

(4)

$$R = \begin{bmatrix} 162/\sqrt{162} & -84/\sqrt{162} \\ 0 & 25/3 \end{bmatrix} \qquad Q = \begin{bmatrix} 4/\sqrt{162} & -1/9 \\ 11/\sqrt{162} & 4/9 \\ 5/\sqrt{162} & -8/9 \end{bmatrix}$$

$$|R| = \frac{162}{\sqrt{162}} \times \frac{25}{3} = \frac{150}{\sqrt{2}}$$

$$Q^T = \begin{bmatrix} 4/\sqrt{162} & 11/\sqrt{162} & 5/\sqrt{162} \\ -1/9 & 4/9 & -8/9 \end{bmatrix}$$

$|R| = 150/\sqrt{2}$

$R^T \Rightarrow$

$$\vec{R} = \begin{bmatrix} 25/3 & 0 \\ 84/\sqrt{162} & 162/\sqrt{162} \end{bmatrix}^T \Rightarrow \begin{bmatrix} 25/3 & 84/\sqrt{162} \\ 0 & 162/\sqrt{162} \end{bmatrix}$$

Therefore, $R^{-1} = \dfrac{\sqrt{2}}{150}\begin{bmatrix} 25/3 & 84/\sqrt{162} \\ 0 & 162/\sqrt{162} \end{bmatrix}$

$$\vec{x} = R^{-1} Q^T b$$

$$= \frac{\sqrt{2}}{150}\begin{bmatrix} 25/3 & 84/\sqrt{162} \\ 0 & 162/\sqrt{162} \end{bmatrix}_{2\times 2}\begin{bmatrix} 4/\sqrt{162} & 11/\sqrt{162} & 5/\sqrt{162} \\ -1/9 & 4/9 & -8/9 \end{bmatrix}_{2\times 3}\begin{bmatrix} 0 \\ 15 \\ -20 \end{bmatrix}_{3\times 1}$$

$$\Rightarrow \frac{\sqrt{2}}{150}\begin{bmatrix} 25/3 & 84/\sqrt{162} \\ 0 & 162/\sqrt{162} \end{bmatrix}\begin{bmatrix} (11\times 15 - 5\times 20)/\sqrt{162} \\ (4\times 15 + 8\times 20)/9 \end{bmatrix}$$

$$\Rightarrow \frac{\sqrt{2}}{150}\begin{bmatrix} 25/3 & 84/\sqrt{162} \\ 0 & 162/\sqrt{162} \end{bmatrix}\begin{bmatrix} 65/\sqrt{162} \\ 220/9 \end{bmatrix} = \begin{bmatrix} \frac{25\times 65}{3\sqrt{162}} + \frac{84}{\sqrt{162}} \times \frac{220}{9} \\ \frac{162\times 220}{\sqrt{162}\times 9} \end{bmatrix}$$

$$\Rightarrow \frac{\sqrt{2}}{150}\begin{bmatrix} (25\times 65 + 28\times 220)/3\sqrt{162} \\ 440/\sqrt{2} \end{bmatrix} \Rightarrow \frac{\sqrt{2}}{150}\begin{bmatrix} 7785/3\sqrt{162} \\ 440/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow \frac{\sqrt{2}}{150}\begin{bmatrix} 865/3\sqrt{2} \\ 440/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{865}{150} \times \frac{\sqrt{2}}{3\sqrt{2}} \\ \frac{440}{\sqrt{2}} \times \frac{\sqrt{2}}{150} \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 173/90 \\ 44/15 \end{bmatrix}$$

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