SRM UNIVERSITY

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Department of Mathematics

Sub Title: STATISTICS AND NUMERICAL METHODS

Sub Code: MA1006A

Unit -III - SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

FORMULA:

PROBLEMS BASED ON NEWTON'S METHOD

1. METHOD OF TANGENTS OR NEWTON RAPHSON METHOD

The first approximation to the root of f(x) = 0 by the method of Tangents :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$, n = 0,1,2,3,….

(i) If f(x) = $\sqrt{a}$ then $x_{n+1} = \left(\frac{1}{2}\right)\left[x_n + \frac{a}{x_n}\right]$, n = 0, 1, 2, 3…

(ii) If f(x) = $\frac{1}{a}$ then $x_{n+1} = (x_n)[2 - ax_n]$, n = 0, 1, 2, 3…

(iii) If f(x) = $\frac{1}{\sqrt{a}}$ then $x_{n+1} = \frac{1}{2}(x_n)[3 - ax_n{}^2]$, n = 0, 1, 2, 3…

(iv) If f(x) = $a^{\left(\frac{1}{p}\right)}$ then $x_{n+1} = \frac{(p-1)x_n{}^p + a}{px_n{}^{p-1}}$, n = 0,1,2,3,….

PART- A

1) To fit the straight line y = mx +c, to n observations the normal equations are

a) $a\sum x^2 + b\sum x = \sum xy$, $a\sum x + nb = \sum y$

b) $a\sum x^2 + b\sum x = \sum y$, $a\sum x + nb = \sum xy$

c) $a\sum x^2 + b\sum x = \sum xy^2$, $a\sum x + nb = \sum y$ **Ans (a)**

d) None of these

2) To fit the parabola y = $ax^2 + bx + c$, to n observations the normal equations are

a) $a\sum x^4 + b\sum x^3 + c\sum x^2 = \sum x^2 y$, $a\sum x^3 + b\sum x^2 + c\sum x = \sum xy$, $a\sum x^2 + b\sum x + nc = \sum y$

b) $a\sum x^4 + b\sum x^3 + c\sum x^2 = \sum x^2 y$, $a\sum x^3 + b\sum x^2 + c\sum x = \sum xy$

c) $a\sum x^2 + b\sum x = \sum xy$, $a\sum x + nb = \sum y$ **Ans (a)**

d) None of these

3) The value of $\log_{10} e$ is ?

a) 0 b) 1 c) d) None of these **Ans (c)**

4) The method which gives a unique set of values to the constants in the equation of the curve is

a) Principle of least squares b) Method of group averages

c) Gauss elimination d) None of these **Ans (a)**

5) When we fit a straight line by the method of least squares, the error is

a) $E = \sum x^2 - a\sum xy - b\sum y$

b) $E = \sum y^2 - a\sum x^2 y - b\sum xy$ **Ans (c)**

c) $E = \sum y^2 - a\sum xy - b\sum y$

d) None of these

6) When we fit a parabola by the method of least squares, the error is

a) $E = \sum x^2 - a\sum xy - b\sum y - c\sum x$

b) $E = \sum y^2 - a\sum x^2 y - b\sum xy - c\sum y$ **Ans (b)**

c) $E = \sum y^2 - a\sum xy - b\sum y$

d) None of these

7) The Iterative formula to find $\sqrt{N}$ using Newton Raphson formula is

a) $x_{n+1} = x_n (2 - N x_n)$ b) $x_{n+1} = \frac{1}{2}(x_n + \frac{N}{x_n})$

c) $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{N x_n})$ d) $x_{n+1} = \frac{1}{k} [(k-1) x_n + \frac{N}{x_n^{k-1}}]$ **Ans (b)**

8) The condition of convergence of the iteration method for $x = Ø(x)$ is

a) $| Ø'(x) | > 1$ b) $| Ø'(x) | < 1$ c) $Ø'(x) > 0$ d) $| Ø''(x) | < 1$ **Ans (b)**

9) The Iterative formula to find $1/N$ using Newton Raphson formula is

a) $x_{n+1} = x_n (2 - N x_n)$ b) $x_{n+1} = \frac{1}{2}(x_n + \frac{N}{x_n})$

c) $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{N x_n})$ d) $x_{n+1} = \frac{1}{k} [(k-1) x_n + \frac{N}{x_n^{k-1}}]$ **Ans (a)**

10) The Iterative formula to find $\sqrt[k]{N}$ using Newton Raphson formula is

a) $x_{n+1} = x_n (2 - N x_n)$ b) $x_{n+1} = \frac{1}{2}(x_n + \frac{N}{x_n})$

c) $x_{n+1} = \frac{1}{2}(x_n + \frac{1}{N x_n})$ d) $x_{n+1} = \frac{1}{k} [(k-1) x_n + \frac{N}{x_n^{k-1}}]$ **Ans (d)**

11) The order of convergence of Newton Raphson method is

a) 1 b) 1.62 c) 3 d) 2 **Ans (d)**

12) The rate of convergence of iteration method is

a) Linear b) Quadratic c) Cubic d) None of these **Ans (b)**

13)The method which is also called as method of tangent is

a) Bisection method b) Newton raphson method

c) Iteration method d) Secant method. **Ans (b)**

14) The condition of convergence of the Newton Raphson method is

a) $|f(x)f''(x)| < |f'(x)|^2$ b) $|f(x)f''(x)| > |f'(x)|^2$

c) $|f'(x)|^2 > 0$ d) $|f(x)f''(x)| = |f'(x)|^2$ **Ans (a)**

15) As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is known as

a) Gauss Jordan method b) Gauss Seidel method

c) Jacobi's method d) Gauss Elimination method **Ans (b)**

16) The condition of convergence of Jacobi and Gauss Seidel iterative methods is

a) Physically dominant b) Diagonally dominant

c) Non-singular d) Singular. **Ans (b)**

17) In solving simultaneous linear equations by Gauss Elimination method, the co-efficient matrix is reduced to

a) Identity matrix b) Upper triangular matrix

c) Symmetric matrix d) Orthogonal matrix **Ans (b)**

18) Gauss Seidal method converges ________ than the Jacobi iterative method.

a) Faster b) Slower c) Weaker d) None of these **Ans (a)**

19) Which method converges faster of all the iterative methods?

a) Gauss Jordan method b) Gauss elimination method

c) Gauss Seidal method d) Jacobi iterative method **Ans (c)**

20) Which of the following is not an iterative method?

a) Gauss Seidal b) Gauss Jacobi

c) Gauss Elimination d) N-R Method **Ans (c)**

PART –B

1. If an approximate value of the root of the equation $x^x = 1000$ is 4.5, find a better approximation of the root by Newton's method.

$$f(x) = x\log_e x - \log_e 1000$$

=> f'(x) = 1+ $\log_e x$,

Approximation is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$x_1 = 4 + \frac{0.0605}{1.6532}$ = 4.5366.

2. Write Newton's formula to find the cube root of N.

Let x = $\sqrt[3]{N}$

$x^3 - N = 0$

=> $f(x) = x^3 - N$ and $f'(x) = 3x^2$,

By Newton's method,

$$x_{n+1} = x_n - \frac{(x_n)^3 - N}{3(x_n)^2}$$

$$=> x_{n+1} = \frac{2(x_n)^3 + N}{3(x_n)^2}$$

3. Establish an iteration formula to find the reciprocal of a positive number N by Newton's method

Let $N = \frac{1}{x}$

$=>$ f(x) = $\frac{1}{x} - N$ and f'(x) = $\frac{1}{x^2}$

$$=> x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{-\left(\frac{1}{x_n^2}\right)}$$

$=> x_{n+1} = x_n\,(2 - Nx_n)$, n = 0,1,2,3,….

4. Find the first approximation to the root of $x^3 + 3x - 1 = 0$ lying between 0 and 1 by Newton's method.

Let f(x) = $x^3 + 3x - 1$,

f'(x) = $3x^2 + 3$

$$=> x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$=> x_1 = 0 - \left(-\frac{1}{3}\right) = 0.3333$$

5. By Gauss elimination, solve x + y = 2 , 2x + 3y = 5

The augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad R2 \rightarrow R_2 - 2R_1$$

$= > x+y = 2$ & $y = 1$

$= > x = 1$ & $y = 1$

6. Solve 3x+ 2y = 4 , 2x – 3y = 7 by Gauss elimination method .

The augmented matrix is

$$[A,B] = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -3 & 7 \end{bmatrix}$$

$$\approx \begin{bmatrix} 3 & 2 & 4 \\ 0 & -13 & 13 \end{bmatrix} \quad R_2 \to 3R_2 - 2R_1$$

$= > 3x+2y = 4$ & $-13y = 13$

$= > x = 2$ & $y = -1$

7.. Solve 3x+ 2y = 4 , 2x – 3y = 7 by Gauss Jordan method.

The augmented matrix is

$$[A,B] = \begin{bmatrix} 3 & 2 & 4 \\ 2 & -3 & 7 \end{bmatrix}$$

$$\approx \begin{bmatrix} 3 & 2 & 4 \\ 0 & -13 & 13 \end{bmatrix} \quad R_2 \to 3R_2 - 2R_1$$

$$\approx \begin{bmatrix} 3 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix} \quad R_2 \to \frac{R_2}{-13} \quad , \quad R_1 \to R_1 - 2R_2$$

$$\approx \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad R_1 \to \frac{R_1}{3}$$

$= > x = 2$ & $y = -1$

8. . Solve x+y = 2 , 2x+ 3y = 5 by Gauss Jordan method .

The augmented matrix is

$$[A,B] = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

$\approx \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

$\approx \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $R_1 \to R_1 - R_2$

=> x = 1 & y = 1

9. Check whether Gauss seidel method can be used to solve 2x-3y+2 0z= 35 ,20x+y-2z = 17 , 3x+20y-z =-18 is less number of iteration ? If possible, solve them?

The given system is not diagonally dominant.

Gauss seidel method does not suit.

But rearrangement of equations as 20x+y-2z= 17, 3x+20y-z =-18 and 2x-3y+20z = 25 produces a diagonally dominant system, for which Gauss seidel method can be applied.

For solution, we express $x = \frac{1}{20}[17 - y + 2z]$: $y = \frac{1}{20}[-18 - 3x + z]$

and z= $\frac{1}{20}[25 - 2x + 3y]$

10. . Find an iterative formula to find $\sqrt{N}$, N is positive number.

The square root of a positive number N is the root of the equation.

$x^2 - N = 0$ ie, f(x) = $x^2 - N$

=> f'(x) = 2x

By Newton's algorithm $x_{k+1} = x_k - \frac{(x_k)^2 - N}{2x_k}$

=> $x_{k+1} = \frac{2(x_k)^2 - (x_k)^2 + N}{2x_k}$

Therefore $x_{k+1} = \frac{(x_k)^2 + N}{2x_k}$

PART-C

1. Using Newton-Raphson's method, solve $x \log_{10} x$ = 12.34, start with x_0 = 10

Given f(x) = $x \log_{10} x$ - 12.34.

Therefore we can take theinitial approximation to the root x_0 = 10.

f(x) = $x \log_{10} x$ - 12.34,

f'(x) = $\log_{10} x + x \frac{1}{x} \log_{10} e = \log_{10} x$+ 0.4343

(Since $\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$)

Newton's Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, n = 0,1,2,3,….-------(1)

Putting n = 0 in (1) , we get the first approximation $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$=10 - \frac{10 \log_{10} 10 - 12.34}{\log_{10} 10 + 0.4343} = 10 + \frac{2.34}{1.4343}$

Putting n = 1 in (1) , we get the second approximation

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 11.631 - \frac{11.631 \log_{10} 11.631 - 12.34}{\log_{10} 11.631 + 0.4343} = 11.631 - \frac{0.0541919}{1.4999} = 11.594$

Putting n = 2 in (1) , we get the third approximation

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 11.594 - \frac{11.594 \log_{10} 11.594 - 12.34}{\log_{10} 11.594 + 0.4343} = 11.594$

Hence the real root of f(x) = 0, correct to three decimal places is 11.594

2. Find a root of $x \log_{10} x$ - 1.2 = 0 by Newton-Raphson's method correct to three decimal places.

SOLUTION : Given f(x) = $x \log_{10} x$– 1.2 ,

f'(x) = $\log_{10} x + x \frac{1}{x} \log_{10} e = \log_{10} x$+ 0.4343

Now f(2) = - 0.59794 (-ve) and f(3) = 0.23136 (+ve) .

Hence the root lies between 2 and 3. Also we have the magnitude of f(3) is less than f(2).

Therefore we can assume that $x_0 = 3$, is the initial approximation to the root.

(Since $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$)

Newton's Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, n = 0,1,2,3,....-------(1)

The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3\log_{10}3 - 1.2}{\log_{10}3 + 0.4343} = 2.746$$

The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.746 - \frac{2.746\log_{10}2.746 - 1.2}{\log_{10}2.746 + 0.4343} = 2.741$$

The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.741 - \frac{2.741\log_{10}2.741 - 1.2}{\log_{10}2.741 + 0.4343} = 2.741$$

Hence the real root of f(x) = 0, correct to three decimal places is 2.741

3. Solve 10 x + y + z = 12 ,2x + 10 y + z = 13 , x + y + 5z = 7 by Gauss Jordan method.

Solution : The given system of equations in augmented matrix form is

$$[A\ ;\ B] = \begin{pmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix} \quad R_1' \to \frac{R_1}{10}$$

$$= \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 0 & \frac{49}{5} & \frac{4}{5} & \frac{106}{10} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{58}{10} \end{pmatrix} \quad R_2' \to R_2 - 2\,R_1' \;\&\; R_3' \to R_3 - R_1'$$

$$= \begin{pmatrix} 1 & \frac{1}{10} & \frac{1}{10} & \frac{12}{10} \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & \frac{9}{10} & \frac{49}{10} & \frac{58}{10} \end{pmatrix} \quad R_2'' \to R'_2 \div \frac{49}{5}$$

$$= \begin{pmatrix} 1 & 0 & 0.0918 & 1.0918 \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 0 & 4.8265 & 4.8265 \end{pmatrix} \quad R_1'' \to R'_1 - \frac{1}{10}\ R_2' \ \ \&\ \ R_3'' \to R'_3 - \frac{9}{10} R_2''$$

$$= \begin{pmatrix} 1 & 0 & 0.0918 & 1.0918 \\ 0 & 1 & \frac{4}{49} & \frac{53}{49} \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad R_3''' \to R''_3 \div 4.8265$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad R_1''' \to R''_1 - 0.0918\ R_3''' \ \ \&\ \ R_2''' \to R''_2 - \frac{4}{49} R_3'''$$

The matrix finally reduces to the form given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

=> x = 1 , y = 1 & z = 1

4. Find the inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ using Gauss Jordan method .

<u>SOLUTION :</u> Let A = $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ and X = $\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

so that, AX = I

Augmented Matrix = $[A; I] = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{-3}{2} & 1 & 0 \\ 0 & \frac{7}{2} & \frac{17}{2} & \frac{-1}{2} & 0 & 1 \end{pmatrix} R_2{}' \to R_2 - \frac{3}{2}R_1 \,\&\, R_3{}' \to R_3 - \frac{1}{2}R_1$$

$$= \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & \frac{-3}{2} & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{pmatrix} R_3{}'' \to \ R_3{}' - 7R_2$$

$$= \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 6 & \frac{-17}{4} & \frac{3}{4} \\ 0 & 0 & -2 & 10 & -7 & 1 \end{pmatrix} R_2{}'' \to \ R_2{}' + \frac{3}{4}R_3{}''$$

$$= \begin{pmatrix} 2 & 1 & 0 & 6 & \frac{-7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 6 & \frac{-17}{4} & \frac{3}{4} \\ 0 & 0 & -2 & 10 & -7 & 1 \end{pmatrix} R_1{}'' \to \ R_1{}' + \frac{1}{2}R_3{}''$$

$$= \begin{pmatrix} 2 & 0 & 0 & -6 & 5 & -1 \\ 0 & \frac{1}{2} & 0 & 6 & \frac{-17}{4} & \frac{3}{4} \\ 0 & 0 & -2 & 10 & -7 & 1 \end{pmatrix} R_1{}''' \to \ R_1{}'' - \ 2R_2{}''$$

$$= \begin{pmatrix} 1 & 0 & 0 & -3 & \frac{5}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & 12 & \frac{-17}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -5 & \frac{7}{2} & \frac{-1}{2} \end{pmatrix} R_1{}'^{v} \to \frac{R_1{}'''}{2}, \quad R_2{}'^{v} = 2\ R_2{}''' \,\&\, R_3{}'^{v} \to \frac{-R_3{}'''}{2}$$

Hence, the inverse of the given matrix is $\begin{pmatrix} -3 & \frac{5}{2} & \frac{-1}{2} \\ 12 & \frac{-17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & \frac{-1}{2} \end{pmatrix}$

5. Using Gauss Jordan method, find the inverse of the matrix $\begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$

SOLUTION : Let A = $\begin{pmatrix}2&2&3\\2&1&1\\1&3&5\end{pmatrix}$ and X = $\begin{pmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\\x_{31}&x_{32}&x_{33}\end{pmatrix}$ be the inverse of A.

so that, AX = I

Augmented Matrix = $[A;I] = \begin{pmatrix}1&3&51&0&0\\2&1&10&1&0\\2&2&30&0&1\end{pmatrix}$

$$= \begin{pmatrix}1&3&5\;1&0&0\\0&-5&-9-2&1&0\\0&-4&-7-2&0&1\end{pmatrix} R_2{}' \rightarrow R_2 - 2R_1 \text{\&}\ R_3{}' \rightarrow R_3 - 2R_1$$

$$= \begin{pmatrix}1&3&5\;\;1&0&0\\0&1&\frac{9}{5}\;\;\frac{2}{5}&\frac{-1}{5}&0\\0&-4&-7-2&0&1\end{pmatrix} R_2{}'' \rightarrow \frac{R_2{}'}{-5}$$

$$= \begin{pmatrix}1&3&5\,1&0&0\\0&1&\frac{9}{5}\,\frac{2}{5}&\frac{-1}{5}&0\\0&0&\frac{1}{5}\frac{-2}{5}&\frac{-4}{5}&1\end{pmatrix} R_3{}'' \rightarrow\ R_3{}' + 4R_2{}''$$

$$= \begin{pmatrix}1&3&5\,1&0&0\\0&1&\frac{9}{5}\,\frac{2}{5}&\frac{-1}{5}&0\\0&0&1-2&-4&5\end{pmatrix} R_3{}''' \rightarrow\ 5R_3{}''$$

$$= \begin{pmatrix}1&3&0\,11&20&-25\\0&1&0\;4&7&-9\\0&0&1-2&-4&5\end{pmatrix} R_1{}'' \rightarrow\ R_1{}' - 5R_3{}''',\quad R_2{}''' \rightarrow\ R_2{}'' - \frac{9}{5}R_3{}'''$$

$$= \begin{pmatrix}1&0&0-1&-1&2\\0&1&0\;4&7&-9\\0&0&1-2&-4&5\end{pmatrix} R_1{}''' \rightarrow\ R_1{}'' - 3R_2{}''',$$

Hence, the inverse of the given matrix is $\begin{pmatrix} -1 & -1 & 2 \\ 4 & 7 & -9 \\ -2 & -4 & 5 \end{pmatrix}$

6. Using Gauss Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$

<u>SOLUTION :</u> Let A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ and X = $\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.so that,

AX = I

Augmented Matrix = $[A; I] = \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{pmatrix} R_2{}' \to R_2 - R_1 \;\&\; R_3{}' \to R_3 + 2R_1$$

$$= \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{pmatrix} R_2{}'' \to R_3{}' + R_2{}'$$

$$= \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{pmatrix} R_2{}'' \to \frac{1}{2}R_2{}' \;\&\; R_3{}'' \to \frac{-1}{4}R_3{}'$$

$$= \begin{pmatrix} 1 & 1 & 0 & \frac{7}{4} & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{pmatrix} R_1{}'' \to R_1{}' - 3R_3{}'' \;\&\; R_2{}''' \to R_2{}'' + 3R_3{}'''$$

$$= \begin{pmatrix} 1 & 0 & 0 & 3 & 1 & 3 \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} R_1{}''' \to R_1{}'' - R_2{}'',$$

Hence, the inverse of the given matrix is $\begin{pmatrix} 3 & 1 & 3 \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$

8. Solve the following equation using Jacobi's iteration method

20x + y – 2z = 17, 3x + 20y – z = -18 & 2x -3y + 20z = 25

<u>SOLUTION:</u>

The given equation can be written as

x = $\frac{1}{20}(17 - y + 2z)$

y = $\frac{1}{20}(-18 - 3x + z)$ ----------- (1)

z = $\frac{1}{20}(25 - 2x + 3y)$

First approximation

$x_1 = \frac{1}{20}(17 - y_0 + 2z_0)$

$y_1 = \frac{1}{20}(-18 - 3x_0 + z_0)$ -------------- (2)

$z_1 = \frac{1}{20}(25 - 2\,x_0 + 3y_0)$

Putting $x_0 = y_0 = z_0 = 0$ in (2), we get $x_1 = \frac{17}{20}$= 0.85, $y_1 = \frac{-18}{20}$= -0.9 & $z_1 = \frac{25}{20}$ = 1.25

Second approximation

$x_2 = \frac{1}{20}(17 - y_1 + 2z_1)$

$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1)$ --------------(3)

$z_2 = \frac{1}{20}(25 - 2\,x_1 + 3y_1)$

Putting x_1= 0.85, y_1= -0.9 & z_1 = 1.25 in (3), we get $x_2 = 1.02$, $y_2 = -0.965$ & $z_2 = 1.03$

Third approximation

$x_3 = \frac{1}{20}(17 - y_2 + 2z_2)$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) \qquad \text{--------------(4)}$$

$z_3 = \frac{1}{20}(25 - 2\,x_2 + 3y_2)$

Putting $x_2 = 1.02$, $y_2 = -0.965$ & $z_2 = 1.03$ in (4),

we get $x_3 = 1.00125$, $y_3 = -1.0015$ & $z_3 = 1.0032$

Fourth approximation

$x_4 = \frac{1}{20}(17 - y_3 + 2z_3)$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) \qquad \text{--------------(5)}$$

$z_4 = \frac{1}{20}(25 - 2\,x_3 + 3y_3)$

Putting $x_3 = 1.00125$, $y_3 = -1.0015$ & $z_3 = 1.0032$ in (5),

we get $x_4 = 1.00039$, $y_4 = -1.00003$ & $z_4 = 0.99965$

From 3rd and 4th approximation, we get approximately x =1, y= -1 & z = 1.

9. Solve the following equation using Gauss Seidal iteration method

20x - y – 2z = 17, 3x + 20y – z = -18 & 2x -3y + 20z = 25

<u>SOLUTION:</u>

The given system is 20x - y – 2z = 17---(1) , 3x + 20y – z = -18 ----(2) & 2x -3y + 20z = 25 ----(3)

Clearly the coefficient matrix $\begin{pmatrix} 20 & -1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{pmatrix}$ is diagonally dominant. Hence we can apply Gauss seidal method without any difficulty. From (1) , (2) & (3) we get x = $\frac{1}{20}(17 + y + 2z)$

y = $-\frac{1}{20}(18 + 3x - z)$

z = $\frac{1}{20}(25 - 2x + 3y)$

First iteration

$x_1 = \frac{1}{20}(17 + y_0 + 2z_0)$ --------------(4)

$y_1 = \frac{-1}{20}(18 + 3x_1 + z_0)$ --------------(5)

$z_1 = \frac{1}{20}(25 - 2\,x_1 + 3y_1)$ --------------(6)

Putting $y_0 = z_0 = 0$ in (4), we get $x_1 = \frac{17}{20}$= 0.85,

Putting x_1= 0.85, $z_0 = 0$ in (5), we get $y_1 = \frac{-1}{20}(18 + 3(0.85) + 0)$ = -1.0275

Putting x_1= 0.85, y_1= -1.0275 in (6), we get $z_1 = \frac{1}{20}(25 - 2\,(0.85) + 3(-1.0275))$ = 1.010875

Therefore in the first iteration , we get x_1= 0.85, y_1 = -1.0275 &$z_1 = 1.010875$

Second iteration

$x_2 = \frac{1}{20}(17 + y_1 + 2z_1)$ -------------(7)

$y_2 = \frac{-1}{20}(18 + 3x_2 + z_1)$ --------------(8)

$z_2 = \frac{1}{20}(25 - 2\,x_2 + 3y_2)$ --------------(9)

Putting y_1= -1.0275& z_1= 1.010875 in (7),

we get $x_2 = \frac{1}{20}(17 - 1.0275 + 2(1.010875))$ = 0.8997125

Putting x_2= 0.8997125, $z_1 = 1.0108750$ in (8), we get y_2 =-0.984413

Putting x_2= 0.8997125 ,y_2 $=$-0.984413 in (9), we get z_2= 1.012366 Therefore in the second iteration , we get x_2= 0.8997125 , y_2 $=$-0.984413 &z_2= 1.012366

Third iteration

$x_3 = \frac{1}{20}(17 + y_2 + 2z_2)$ -------------(10)

$y_3 = \frac{-1}{20}(18 + 3x_3 + z_2)$ --------------(11)

$z_3 = \frac{1}{20}(25 - 2\,x_3 + 3y_3)$ --------------(12)

Putting y_2 $=$-0.984413, z_2= 1.012366 in (10), we get x_3= 0.0.90202

Puttingx_3= 0.0.90202 ,z_2= 1.012366 in (11), we get y_3 $=$-0.98468

Puttingx_3= 0.0.90202 ,y_3 $=$-0.98468in (12), we get z_3= 1.0120 Therefore in the Third iteration , we get x_3= 0.0.90202 , y_3 $=$-0.98468 &z_3= 1.0120

Fourth iteration

$x_4 = \frac{1}{20}(17 + y_3 + 2z_3)$ -------------(13)

$y_4 = \frac{-1}{20}(18 + 3x_4 + z_3)$ --------------(14)

$z_4 = \frac{1}{20}(25 - 2\,x_4 + 3y_4)$ --------------(15)

Putting y_3 $=$-0.98468 &z_3= 1.0120 in (13), we get x_4= 0.90197

Puttingx_4= 0.90197 $\&z_3 = 1.0120$ in (14), we get y_4 $=$-0.98469

Putting$x_4 = 0.90197$,$y_4 = -0.98469$ in (15), we get z_4= 1.0121 Therefore in the Fourthiteration , we get $x_4 = 0.90197$,$y_4 = -0.98469$&z_4= 1.0121

Fifth iteration

$x_5 = \frac{1}{20}(17 + y_4 + 2z_4)$ -------------(16)

$y_5 = \frac{-1}{20}(18 + 3x_5 + z_4)$ --------------(17)

$z_5 = \frac{1}{20}(25 - 2\,x_5 + 3y_5)$ --------------(18)

Putting $y_4 = -0.98469$&z_4= 1.0121 in (16), we get x_5= 0.90197

Puttingx_5= 0.90197 $\&z_4 = 1.0120$ in (17), we get y_5 = -0.98469

Putting$x_5 = 0.90197$,$y_5 = -0.98469$ in (18), we get z_5= 1.0121 Therefore in the Fifthiteration , we get $x_5 = 0.90197$,$y_5 = -0.98469$&z_5= 1.0121

In the fourth and fifth iterations, the value are equal.

Therefore,we get x= 0.90197 , $y = -0.98469$& z = 1.0121

10. Using Gauss Seidal iteration method,solve the following equation with x = 1, y= -2, z=3

x +3y +52z = 173.61, x - 27y + 2z = 71.31&41x -2y + 3z = 65.46

SOLUTION:

The given system is 20x - y – 2z = 17---(1) , 3x + 20y – z = -18 ----(2) & 2x -3y + 20z = 25 ----(3)

Clearly the coefficient matrix $\begin{pmatrix} 20 & -1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{pmatrix}$ is diagonally dominant. Hence we can apply Gauss seidal method without any difficulty. From (1) , (2) & (3) we get

x = $\frac{1}{20}(17 + y + 2z)$

y = $-\frac{1}{20}(18 + 3x - z)$

z = $\frac{1}{20}(25 - 2x + 3y)$

First iteration

$x_1 = \frac{1}{20}(17 + y_0 + 2z_0)$ --------------(4)

$y_1 = \frac{-1}{20}(18 + 3x_1 + z_0)$ --------------(5)

$z_1 = \frac{1}{20}(25 - 2\,x_1 + 3y_1)$ --------------(6)

Putting $y_0 = z_0 = 0$ in (4), we get $x_1 = \frac{17}{20}$= 0.85,

Putting x_1= 0.85, $z_0 = 0$ in (5), we get $y_1 = \frac{-1}{20}(18 + 3(0.85) + 0)$ = -1.0275

Putting x_1= 0.85, y_1= -1.0275 in (6), we get $z_1 = \frac{1}{20}(25 - 2\,(0.85) + 3(-1.0275))$ = 1.010875

Therefore in the first iteration , we get x_1= 0.85, y_1 = -1.0275 &$z_1 = 1.010875$

Second iteration

$x_2 = \frac{1}{20}(17 + y_1 + 2z_1)$ -------------(7)

$y_2 = \frac{-1}{20}(18 + 3x_2 + z_1)$ --------------(8)

$z_2 = \frac{1}{20}(25 - 2\,x_2 + 3y_2)$ --------------(9)

Putting y_1= -1.0275 & z_1= 1.010875 in (7),

we get $x_2 = \frac{1}{20}(17 - 1.0275 + 2(1.010875))$ = 0.8997125

Putting x_2= 0.8997125, $z_1 = 1.0108750$ in (8), we get $y_2 =$ -0.984413

Putting x_2= 0.8997125 ,$y_2 =$ -0.984413 in (9), we get z_2= 1.012366 Therefore in the second iteration , we get x_2= 0.8997125 , $y_2 =$ -0.984413 &z_2= 1.012366

Third iteration

$x_3 = \frac{1}{20}(17 + y_2 + 2z_2)$ -------------(10)

$y_3 = \frac{-1}{20}(18 + 3x_3 + z_2)$ --------------(11)

$z_3 = \frac{1}{20}(25 - 2\,x_3 + 3y_3)$ --------------(12)

Putting $y_2 =$ -0.984413, z_2= 1.012366 in (10), we get x_3= 0.0.90202

Puttingx_3= 0.0.90202 ,z_2= 1.012366 in (11), we get y_3 = -0.98468

Puttingx_3= 0.0.90202 ,y_3 = -0.98468in (12), we get z_3= 1.0120 Therefore in the Third iteration , we get x_3= 0.0.90202 , y_3 = -0.98468 &z_3= 1.0120

Fourth iteration

$x_4 = \frac{1}{20}(17 + y_3 + 2z_3)$ -------------(13)

$y_4 = \frac{-1}{20}(18 + 3x_4 + z_3)$ --------------(14)

$z_4 = \frac{1}{20}(25 - 2\,x_4 + 3y_4)$ --------------(15)

Putting y_3 = -0.98468 &z_3= 1.0120 in (13), we get x_4= 0.90197

Puttingx_4= 0.90197 &z_3 = 1.0120 in (14), we get y_4 = -0.98469

Putting$x_4 = 0.90197$,$y_4 = -0.98469$ in (15), we get z_4= 1.0121 Therefore in the Fourth iteration , we get $x_4 = 0.90197$,$y_4 = -0.98469$&z_4= 1.0121

Fifth iteration

$x_5 = \frac{1}{20}(17 + y_4 + 2z_4)$ -------------(16)

$y_5 = \frac{-1}{20}(18 + 3x_5 + z_4)$ --------------(17)

$z_5 = \frac{1}{20}(25 - 2\,x_5 + 3y_5)$ --------------(18)

Putting $y_4 = -0.98469$&z_4= 1.0121 in (16), we get x_5= 0.90197

Puttingx_5= 0.90197 &z_4 = 1.0120 in (17), we get y_5 = -0.98469

Putting$x_5 = 0.90197$,$y_5 = -0.98469$ in (18),

we get z_5= 1.0121

Therefore in the Fifth iteration , we get $x_5 = 0.90197$,$y_5 = -0.98469$&z_5= 1.0121

In the fourth and fifth iterations, the value are equal.

Therefore, we get x$= 0.90197$, $y = -0.98469$& z = 1.0121

11. Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ by power method.

<u>SOLUTION:</u>

Let the initial eigen vector be $X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Now, $X_1 = AX_0 = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 13 \end{pmatrix}$ = 13 $\begin{pmatrix} \frac{3}{13} \\ \frac{9}{13} \\ 1 \end{pmatrix}$ = 13 $\begin{pmatrix} 0.231 \\ 0.692 \\ 1 \end{pmatrix}$ = 13 $X_1{}'$

$$X_2 = AX_1{}' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.231 \\ 0.692 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.307 \\ 6.077 \\ 12.537 \end{pmatrix} = 12.537 \begin{pmatrix} 0.104 \\ 0.485 \\ 1 \end{pmatrix} = 12.537\, X_2{}'$$

$$X_3 = AX_2{}' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.104 \\ 0.485 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.559 \\ 5.282 \\ 11.836 \end{pmatrix} = 11.836 \begin{pmatrix} 0.047 \\ 0.446 \\ 1 \end{pmatrix} = 11.836 X_3{}'$$

$$X_4 = AX_3{}' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.047 \\ 0.446 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.385 \\ 5.033 \\ 11.737 \end{pmatrix} = 11.737 \begin{pmatrix} 0.032 \\ 0.429 \\ 1 \end{pmatrix} = 11.737 X_4{}'$$

$$X_5 = AX_4{}' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.032 \\ 0.429 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.319 \\ 4.954 \\ 11.684 \end{pmatrix} = 11.684 \begin{pmatrix} 0.027 \\ 0.423 \\ 1 \end{pmatrix} = 11.684\, X_5{}'$$

$$X_6 = AX_5{}' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.027 \\ 0.423 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.296 \\ 4.907 \\ 11.665 \end{pmatrix} = 11.665 \begin{pmatrix} 0.025 \\ 0.421 \\ 1 \end{pmatrix} = 11.665\, X_6{}'$$

$$X_7 = AX_6' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.025 \\ 0.421 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.288 \\ 4.917 \\ 11.659 \end{pmatrix} = 11.659\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix} = 11.659\, X_7'$$

$$X_8 = AX_7' = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.291 \\ 4.919 \\ 11.663 \end{pmatrix} = 11.663\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix} = 11.663\, X$$

Therefore, the dominant eigen vector is X = $\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix}$

To find the dominant eigen value, we have to solve AX = λX, for λ

$$\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 0.025 \\ 0.422 \\ 1 \end{pmatrix} \Rightarrow \lambda = 11.663$$