1 Magnetars

1.1 September 23

Magnetars are pulsars with luminous X-ray emission and regular bursting behavior. The energy output is larger than the spin down power, which can only come from the magnetic field.

The spectrum of magnetars vary, but usually it consists of a soft component centered around $1\,\mathrm{keV}$, and a hard component in the MeV range. The soft component can usually be fit with a quasi-thermal spectrum with two blackbodies, but the hard X-ray emission in particular is non-thermal and has to be of magnetospheric origin.

The question is then, how is the energy released. If one simply work out the energistics, the total energy output of a magnetar is usually larger than or comparable to the total magnetic energy of the spin-down dipole field. So either the efficiency is very high, or the magnetic energy in the magnetosphere is replenished somehow. To be more precise, the dipole component of the magnetic field is not enough to provide the energy content of the bursts. Therefore people expect that there is much higher magnetic field inside the star, in configurations more convoluted than the dipole, for example a toroidally dominated configuration. There is some discussion about the maximum magnetic field you can have in a magnetar.

The surroundings of a magnetar is filled with plasma. Since the field is so high (even the spindown dipole field), it is easy to fill the magnetosphere. If the crust is deformed by some differential rotation, there will be twist of magnetic field lines building up in the magnetosphere. Assuming there is B_ϕ in the magnetosphere already, and to zeroth order $E_\parallel$ is zero, then the twist will live forever and there will simply be enough electric current to sustain $\nabla \times B$. In this limit there will be no dissipation and no radiation. To get energy dissipation and observable emission, one needs nonzero $\boldsymbol{E} \cdot \boldsymbol{j}$.

A better way to quantify the nonzero $\boldsymbol{E}$ is the voltage along the field lines, which is basically

$$\Phi_e = \int \boldsymbol{E} \cdot d\boldsymbol{l} \tag{1.1}$$

where the integral is along the magnetic field line. The question is how the twist of magnetic field will evolve in time and what kind of radiation will it emit.

The dynamics of the magnetosphere is govern by the following (simple) equations:

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \tag{1.2}$$

$$\frac{1}{c}\frac{\partial \psi}{\partial t} = \frac{\partial \Phi_\parallel}{\partial t} \tag{1.3}$$

where ψ is the twist. The twist is defined as the accumulative angle that the field line builds up:

$$\psi = \int \frac{B_\phi d\boldsymbol{l}}{Br\sin\theta} = \int d\phi \tag{1.4}$$

which is simply geometry. Now we let current flow along the magnetic field line, which must be maintained by a longitudinal voltage given by (1.1). If we consider to neighboring field lines and draw a closed loop by connecting the end points, then the rate of change of magnetic flux through the loop will be the circulation of $\boldsymbol{E}$ field along the loop. In the limit the two field lines are infinitely close, the voltage between end points is negligible, so

$$\frac{d\delta\Phi}{dt} = \int E_\parallel dl_1 - \int E_\parallel dl_2 \tag{1.5}$$

1

The point is that, this time derivative of the differential flux $\delta\Phi$ only depends on B_ϕ. One can see that by (some argument which we will talk about again next time). In the end what we can find is that

$$\delta\Phi_e = \frac{\dot{\psi}(f)}{2\pi}\delta f \tag{1.6}$$

1.1.1 Discussion about the derivation

There are two ways of deriving equation (1.6). Both start with considering a thin loop formed by points P_1, P_2 very close to each other on the star as footpoints of magnetic fieldlines, and the other footpoints Q_1 and Q_2 on the surface of the other hemisphere. The loop $\overline{P_1P_2Q_2Q_1}$ is what we want to integrate $E_\parallel$ along.

The first way is outlined in Andrei's review (which I failed to outline). The other way is what we discussed a lot about after the lecture. The problem is about the definition of the surface over which magnetic flux $\delta\Phi$ is computed. Since the magnetic field lines are moving with time, the flux surface bound by the field lines change, therefore the total derivative $d\delta\Phi/dt$ has two contributions, one from varying B field, and the other from changing of the area itself.

If one considers a simple configuration of some B field over an area, which is moved by $v\Delta t$, then we can find the expression for $d\Phi/dt$ by the following:

$$\Phi_t = \int \boldsymbol{B}_t \cdot d\boldsymbol{S}_t \tag{1.7}$$

$$\Phi_{t+\Delta t} = \int \boldsymbol{B}_{t+\Delta t} \cdot d\boldsymbol{S}_{t+\Delta t} \tag{1.8}$$

Now in order to form a closed surface, we have S_t, $S_{t+\Delta t}$, and the loop formed by the boundaries of S_t and $S_{t+\Delta t}$, with width $v\Delta t$. Integrating $\boldsymbol{B}_t$ over this surface will give zero flux. So we have

$$0 = \int \boldsymbol{B}_t \cdot d\boldsymbol{S}_{t+\Delta t} - \int \boldsymbol{B}_t \cdot d\boldsymbol{S}_t + \oint \boldsymbol{B}_t \cdot (d\boldsymbol{l} \times \boldsymbol{v}\Delta t) \tag{1.9}$$

where the last line integral is over the boundary of the area. Now we can see that the first two terms are almost $\Phi_{t+\Delta t} - \Phi_t$, only missing a $\Delta\boldsymbol{B}$ term, so we have

$$\Phi_{t+\Delta t} - \Phi_t = \int \Delta\boldsymbol{B} \cdot d\boldsymbol{S} - \oint \boldsymbol{B} \cdot (d\boldsymbol{l} \times \boldsymbol{v}\Delta t) \tag{1.10}$$

Note that we now omit the subscript t on the right hand side, because everything is evaluated at time t. Dividing by Δt and using Stoke's Theorem we get

$$\frac{d\Phi}{dt} = \int \left[\frac{\partial\boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B})\right] d\boldsymbol{S} \tag{1.11}$$

Now consider our problem of flux over the loop $\overline{P_1P_2Q_2Q_1}$. Since the boundaries $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$ are along magnetic field lines, we can define the surface to be exactly along magnetic field lines. Therefore by construction Φ is identically zero. Since the field lines are moving, the time derivative of this Φ is given

2

by equation (1.11). So by construction we have $d\Phi/dt = 0$:

$$\frac{d\Phi}{dt} = 0 = \int \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \right] \cdot d\boldsymbol{S}$$
$$= - \int \nabla \times (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{S} \tag{1.12}$$
$$= - \oint (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$$

In other words, the "comoving electric field" integrates to zero along the loop.

If one proceeds with the last line, we have

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = - \oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} \tag{1.13}$$

Consider the left hand side. If we make the assumption that the stellar surface is a good conductor, inside which $\boldsymbol{E}$ vanishes, then the integral of $\boldsymbol{E}$ along the loop becomes

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = \int_{P_2 Q_2} \boldsymbol{E} \cdot d\boldsymbol{l} - \int_{P_1 Q_1} \boldsymbol{E} \cdot d\boldsymbol{l} = \int E_{\parallel} dl_2 - \int E_{\parallel} dl_2 = \delta \Phi_e \tag{1.14}$$

which is the difference of the voltage drops along the adjacent field lines. The right hand side of equation (1.13) can be written as $- \oint (\boldsymbol{B} \times d\boldsymbol{l}) \cdot \boldsymbol{v}$, which obviously vanishes along the field lines, and is nonzero only on the small segments on the stellar surface. If we further assume that the points P_1 and P_2 are fixed on the stellar surface, with only their other footpoints Q_1 and Q_2 moving due to evolving magnetic field, and the evolution of the footpoints is evolution of the twist ψ, then the right hand side of equation (1.13) can be written as

$$- \oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = - \int_{Q_1 Q_2} (\boldsymbol{B} \times d\boldsymbol{l}) \cdot \boldsymbol{v} = \int_{Q_2 Q_1} (r_{\perp} \dot{\psi} B_p) dl \tag{1.15}$$

Finally we identify the poloidal flux $\delta f = \int 2\pi r_{\perp} B_p dl$, so that we have

$$\delta \Phi_e = \frac{\dot{\psi} \delta f}{2\pi} \tag{1.16}$$

which is exactly the result (1.6) Andrei obtained.

A good extension of this formalism is that, if we assume finite conductivity inside the star, then it is very easy to modify the end result (1.16). We just need to add a resistive term to the integral of $\boldsymbol{E}$, which modifies $\delta \Phi_e$.

1.2   September 28

To extend the discussion about the untwist problem. Apart from the discussion above, there is the original Andrei's formalism. The way is to consider the magnetic flux only across the meridional plane. The toroidal flux function across the same magnetic loop $\overline{P_1 P_2 Q_2 Q_1}$ is

$$F = \int B_{\phi} dl ds = \int \frac{B_{\phi} dl \delta f}{2\pi r_{\perp} B_p} = \frac{\psi \delta f}{2\pi} \tag{1.17}$$

where dl is along the field line and ds is across the field line. This result is a geometrical identity

$$\psi = \frac{1}{2\pi}\frac{\partial F}{\partial f} \tag{1.18}$$

To get the original result we take the derivative of this equation with respect to time

$$\dot{\psi} = \frac{1}{2\pi}\partial_f\partial_t F = \frac{1}{2\pi}\partial_f\frac{dF}{dt} \tag{1.19}$$

The last equality is because F is a function of f and t, but the time evolution of F is due to the breathing of the field lines, so the time derivative of F is simply the total time derivative. Now we have

$$\frac{dF}{dt} = \oint \boldsymbol{E}'_p \cdot d\boldsymbol{l}_p = \oint \boldsymbol{E} \cdot d\boldsymbol{l} = \delta\Phi_e \tag{1.20}$$

where E' means the electric field in the frame comoving with the magnetic field lines.

A crucial argument to this method is that, to justify the last equation that our integration of $E_\parallel$ is correct, we need $E'_\phi = 0$. This is true because if we decompose the magnetic field into poloidal and toroidal components

$$\boldsymbol{B} = \frac{\nabla f \times \boldsymbol{e}_\phi}{2\pi} + B_\phi \boldsymbol{e}_\phi \tag{1.21}$$

by the "frozen-in condition" we have

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{1.22}$$

therefore for the $\boldsymbol{v}_p$ of the field line in the poloidal component we can find E'_ϕ such that

$$\partial_t \boldsymbol{E}'_\phi = \partial_t \boldsymbol{E} + \boldsymbol{v}_p \times \boldsymbol{B}_p = 0 \tag{1.23}$$

1.2.1 Discharge

The next logical step is to figure out what controls the voltage drop. That is connected to the discharge mechanism. The equation (1.16) tells us that (filling the missing c back)

$$\dot{\psi} = 2\pi c \frac{\partial \Phi_e}{\partial f} \tag{1.24}$$

Therefore the larger the voltage, the faster the untwisting. (Is this really true? It's only the derivative of voltage entering the equation.)

Consider axisymmetric configuration, the total poloidal current I over a field line bundle is the circulation of B_ϕ. The first question is that whether this current is possible to be sustained by particles extracted from the surface. The picture by Thompson and Duncan is that the particles supplied from the surface is sufficient to carry the current. To work out this problem one needs to consider gravity.

The minimum number density of particles to conduct a certain current is given by $n_{min} = j/ec$. The voltage produced by missing this amount of charge is given by

$$\Phi_e \sim 4\pi e n_{min} R^2 = 4\pi R^2 \frac{j}{c} \tag{1.25}$$

4

This is loosely applying the Maxwell equation

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1.26}$$

by assuming that $\boldsymbol{B}$ is extremely large, and that $\nabla \times \boldsymbol{B}$ is given. The plasma has to figure out a way to sustain this curl of $\boldsymbol{B}$, which is the current the field demands, and we call it $\boldsymbol{j}_B = n_{min}ec$. If we have vacuum instead of this amount of charge, and we want to find the characteristic electric field associated to this missing amount of charge is given by equation (1.25).

Another way (better way) to look at the equation (1.26) is

$$\frac{\partial E_\parallel}{\partial t} = 4\pi(j_B - j) \tag{1.27}$$

so it says if we don't have the desired j, the electric field grows. Let's look at the time scale associated with the missing j. Over a characteristic time τ, assuming initially we have some plasma but with $v = 0$, we have

$$\frac{E_\parallel}{\tau} \sim 4\pi(j_B - j) \tag{1.28}$$

$$\tau e E_\parallel \sim mv \sim mj/ne \tag{1.29}$$

putting these equations together we have

$$\tau^2 \sim \frac{mj}{4\pi e^2 n(j_B - j)} = \frac{1}{\omega_p^2}\frac{j}{j_B - j} \tag{1.30}$$

So the time scale is comparable to the plasma time scale, which is very short. The morale is that the plasma responds to missing current very quickly, so electric field develops very quickly in response to any mismatch of j_B and j. If we estimate the plasma frequency

$$\omega_p^2 = \frac{4\pi e^2 n}{m_e} \sim \frac{4\pi e j}{m_e c} \sim \frac{eB}{m_e R}, \quad \omega_p \sim 10^{13}B_{15}^{1/2}\,\text{rad/s} \tag{1.31}$$

which is much smaller compared to the light crossing time of the system.

If we look for a steady state where some $E_\parallel$ overcomes gravity to pull enough charges from the star to conduct the current, we will find that it is actually impossible. The continuity equations for the charges, together with divergence of B being zero, we have $\nabla \cdot \boldsymbol{j}_\pm = \nabla \cdot \boldsymbol{B} = 0$ because the $\partial_t(n_\pm e)$ term in the continuity equation is zero. Because j is parallel to B, let $\boldsymbol{j} = \alpha\boldsymbol{B}$, we have

$$\nabla \cdot \boldsymbol{j}_\pm = (\boldsymbol{B} \cdot \nabla)\alpha = \frac{\partial}{\partial l}\frac{j_\pm}{B} \tag{1.32}$$

so α is constant along the field line. This means that $n_\pm v_\pm/B$ is constant along field lines, so if $n_+ = n_-$ in the atmosphere, we have

$$\frac{j_+}{j_-} = -\frac{v_+}{v_-} = \text{const} \tag{1.33}$$

The argument is that, if there is some $E_\parallel$ that pulls out the particles from the surface self-consistently, then you can't have v_+/v_- to be constant everywhere. For example v_+ is zero at one footpoint, with v_- nonzero, but the opposite is true at the other footpoint.

5

1.3 September 30

1.3.1 About annihilation

At the end of last meeting there was a question about the possibility of pair annihilation in the current bundle. Let's estimate the particle density in more detail here. Consider a twisted dipole magnetosphere. On a given dipole field line the r and θ are related by $r = R_{\max} \sin^2 \theta$. Given a twist ψ, the current flowing will be (assuming ψ is relatively small, $\psi < 1$, so that the global field is still approximately dipole)

$$j = \frac{c\psi B}{4\pi R_{\max}}, \quad n_e = \frac{\psi B}{4\pi e R_{\max}} \tag{1.34}$$

The flux function for dipole field is simply $f = f_{\max} \sin^2 \theta$. This is relatively simple to compute since one can just integrate the B field flux over the surface up to angle θ.

If we write $u = f/f_{\max} = \sin^2 \theta$, which is essentially rescaled f, then we can write

$$j = \frac{RB}{2\pi\mu} \frac{\partial I}{\partial u}, \quad I = \frac{\psi c\mu u^2}{4R^2} \tag{1.35}$$

From equation (1.34) we can get a numerical value of then particle density:

$$n_e = \psi \times 10^{16} \frac{B}{B_Q} \left(\frac{R_6}{R_{\max}} \right) \text{cm}^{-3} \tag{1.36}$$

where B_Q is the critical field $B_Q = 4.4 \times 10^{13}$ G.

Now we consider pair annihilation. An electron-positron pair can collide and turn into a pair of photons. The cross section will be inversely proportional to their relative velocity, and the rate of annihilation will be proportional to the cross section times v, so the rate of annihilation in weak field ($B/B_Q \ll 1$) and vacuum will be

$$\mathcal{R}_0^{(2)} = \sigma_{\text{ann}} v \sim 1.5 \times 10^{-14} \text{cm}^3/\text{s} \sim 0.1\sigma_T c \tag{1.37}$$

However, when B is not far smaller than B_Q, the formula needs to be changed. If $B \gtrsim 2B_Q$, we have

$$\mathcal{R}^{(2)} = \frac{1.74\mathcal{R}_0}{b^3} \exp\left(-\frac{2}{b} \right) \tag{1.38}$$

where $b = B/B_Q$. So in strong field this rate is suppressed by a coefficient of b^{-3}. This was first done by Wunner in 1979. There are a few more papers, e.g. (Kaminker et. al. 1987).

In strong field there is an additional process, which is 1-photon annihilation, which is one pair annihilating into a single photon. Normally this is not allowed due to conservation of energy and momentum, but the magnetic field can take the extra momentum. Wunner reported the result for the ratio between 1-photon and 2-photon annihilation:

$$\frac{\mathcal{R}^{(1)}}{\mathcal{R}^{(2)}} = 496b^2 \tag{1.39}$$

So 1-photon annihilation is much more efficient than the 2-photon annihilation in strong field. The overall rate is dominated by 1-photon annihilation, and it scales with b^{-1} when $B/B_Q > 1$.

The most important region for pair annihilation is when $b \gtrsim 1$ because when B is too large the rate is suppressed. The change of $n_\pm$ due to annihilation can be estimated as:

$$\frac{\Delta n_\pm}{n_\pm} \sim \frac{r}{c} \frac{\dot{n}_{\text{ann}}}{n_\pm} \sim \frac{r}{c} \mathcal{R} n_e \sim \frac{0.1\psi}{R_7/R_{max}} \tag{1.40}$$

where n_e is given by equation (1.36).

1.3.2 Double layer

Consider a 1D box with conductor at both ends. If one pushes some current j at the ends of the box, and we have plenty of plasma in the box, then electric field will be induced and pushes the plasma to conduct the current j. However for a neutron star, originally we don't have much plasma around the star, except in a very thin atmosphere near the surface. The scale height of the atmosphere should be

$$h \sim \frac{kT}{gm_p} \sim 1\,\mathrm{cm} \tag{1.41}$$

where g is the surface gravity of the neutron star $g \sim GM/R^2 \sim 10^{14} cm/s^2$. This gravity is very weak. The electric field that one needs to offset this gravity is orders of magnitude less than B. As discussed last time, there is no self-consistent solution for a steady electric field to help conduct the current required by the boundary condition. The result is that the system will relax to what is called a "double layer".

Let's assume the current is conducted by negative and positive charge $j = j_- + j_+$, and that $j_+ = j_-$. So we have

$$n_\pm = \frac{j_\pm}{ev_\pm} = \frac{j}{2ev_\pm} \tag{1.42}$$

Near the anode $v_- \approx c$ while $v_+ \sim 0$ which means $n_+ \gg n_-$, so the charge density is $\rho \sim en_+$. High density translates to high electric field due to Gauss's law $4\pi\rho = \partial_z E$, and the charges will quickly accelerate and charge density quickly decreases. Near the surface we have $E_z \sim \rho z \sim jz/v$. The work done by the electric field will be

$$zEe \sim \frac{mv^2}{2}, \quad v(z) \sim \sqrt[3]{\frac{ejz^2}{m}} \tag{1.43}$$

So velocity is proportional to $z^{2/3}$, and $\rho \sim en_+ \sim z^{-2/3}$. This holds up to the point where $v \sim c$. At this point v stops growing, and n_+ should match n_- so that $\rho \sim 0$. For relativistic case we can find that

$$\gamma \sim \frac{ejz^2}{mc^3} \tag{1.44}$$

The thickness of the acceleration layer, which is the z position where v becomes c can be found by

$$z^2 \sim \frac{mc^2ce}{e^2j} \sim c^2\frac{m}{n_0e^2} \sim c^2/\omega_p^2 \tag{1.45}$$

where n_0 is a characteristic number density in the layer. The solution is that opposite charges are concentrated at the two ends of the box, and provide the acceleration electric field in between. The charges are highly concentrated in a small scale comparable to the plasma skin depth. There is huge electric field in between, which can be estimated as

$$E \sim 4\pi\rho z \sim \frac{4\pi j}{\omega_p} \tag{1.46}$$

we can find the potential corresponding to this electric field

$$\frac{e\Phi_e}{mc^2} = \frac{4\pi jeR}{\omega_p mc^2} = \frac{\omega_p}{c}R = \frac{R}{\lambda_p} \tag{1.47}$$

This voltage is huge. We estimated $\omega_p \sim 10^{13}\,\mathrm{rad/s}$, so that this voltage is around 10^9. This is the double layer solution found by Carlqvist in 1982.

Note that if we turn off gravity then with any finite temperature we don't have this structure. At any given temperature then scale height of the atmosphere is infinite, so the particles can freely stream out from the star to conduct the current. Even with gravity, if it is weak enough, in other words if the density at the top of the scale height is sufficient to conduct the current

$$\frac{n_{\text{atm}}}{n_0} = \frac{n_{\text{atm}}^0 \exp(-z/h)}{j/ec} > 1 \tag{1.48}$$

then it is possible to conduct the current without forming the double layer.

1.4 October 5

We established that if the current can't be conducted, a double layer will be formed with large electric field. The electric field will accelerate particles which induces pair creation.

1.4.1 Resonant scattering

In magnetars the dominant pair creation process is resonant scattering, as opposed to curvature radiation in standard pulsars. In the former case an electron moving at Lorentz factor γ scatters a seed photon at energy $\hbar\omega$ to energy $\gamma^2\hbar\omega$, which then convert to an electron-positron pair. In the later the particles move along curved field lines and emit curvature radiation at energy $\hbar c \gamma^3 / R_c$. The reason that curvature radiation is subdominant (or straight negligible) in magnetars is that γ doesn't reach high enough values. For curvature radiation to be high enough energy to produce pairs, γ needs to be on the order of $\gtrsim 10^6$, whereas for resonant scattering the Lorentz factor required is much lower, at the order of $\sim 10^3$.

The physics of resonant scattering is simply the interaction of an oscillating particle in magnetic field interacting with radiation. Consider an oscillator under external harmonic force, the equation of motion is

$$\ddot{x} + \omega_0^2 x = \frac{eE}{m} \tag{1.49}$$

using Fourier transform we can find

$$x = \frac{eE}{m(\omega_0^2 - \omega^2)}, \quad \ddot{x} = \frac{eE\omega^2}{m(\omega_0^2 - \omega^2)} \tag{1.50}$$

In this kind of classic scattering cross section, we have near resonance

$$\frac{d\sigma}{d\Omega'} = r_e^2 \frac{\omega^2}{(\omega - \omega_B)^2} \left| \boldsymbol{e}^* \cdot \boldsymbol{e}_- \right|^2 \left| \boldsymbol{e}'^* \cdot \boldsymbol{e}_- \right|^2 \tag{1.51}$$

where r_e is the classical electron radius and $\omega_B = eB/m_e c$. The factor r_e^2 enters because this is to some extent like a Thomson scattering with incoming and outgoing photons. $\boldsymbol{e}$ and $\boldsymbol{e}'$ are the polarization vectors of the incoming and outgoing waves, and $\boldsymbol{e}_-$ is the vector corresponding to the electron orbit direction.

This equation (1.51) is singular when $\omega = \omega_B$ which is exactly the case we are interested in. The way to view what is happening at resonance is to look at quantum mechanically. We can think of the scattering as a two-step process: the electron first absorbs a photon, and then spontaneously emits a photon. The classical cross section needs to be modified by a factor of Γ which is related to the lifetime of the electron in the excited state:

$$\frac{\omega^2}{(\omega - \omega_B)^2} \longrightarrow \frac{\omega^2}{(\omega - \omega_B)^2 + (\Gamma/2)^2} \tag{1.52}$$

If we define $B_Q = m_e^2 c^3/\hbar e$ as the critical field, the energy difference between the lowest and second lowest Landau levels is $\hbar\omega_B$ when $B \ll B_Q$, and it becomes

$$\left[\left(1 + 2\frac{B}{B_Q}\right)^{1/2} - 1\right] m_e c^2 \tag{1.53}$$

The correction Γ will be given by

$$\frac{\Gamma}{\omega_B} = \frac{4}{3}\alpha\frac{B}{B_Q} \tag{1.54}$$

which was found in (Herold et. al. 1982). When $\hbar\omega_B$ becomes comparable to $m_e c^2$ the formula (1.52) needs to be changed, which was considered in (Herold 1979).

If we plug equation (1.54) into equation (1.52) and take the limit $\Gamma \to 0$, we get

$$\frac{1}{(\omega - \omega_B)^2 + (\Gamma/2)^2} \longrightarrow \frac{2\pi}{\Gamma}\delta(\omega - \omega_B) \tag{1.55}$$

so obviously the cross section peaks at $\omega = \omega_B$.

If we define $\mu = \cos\theta$ and $\mu' = \cos\theta'$ where θ and θ' are angles of incoming and outgoing photons with respect to the magnetic field, then the cross section should be written as (if the electron is at rest)

$$\frac{d\sigma}{d\mu'} = 3\pi^2 r_e c\delta(\omega - \omega_B)\left|\boldsymbol{e}^* \cdot \boldsymbol{e}_\pm\right|^2 \left|\boldsymbol{e}'^* \cdot \boldsymbol{e}_\pm\right|^2 \tag{1.56}$$

We can define two linear polarizations for a photon in the magnetic field, which have slightly different propagation speeds. Given $\boldsymbol{B}$ and $\boldsymbol{k}$ vectors, we can have $\boldsymbol{E}$ vector in the plane of $\boldsymbol{B}$-$\boldsymbol{k}$, which we call "O" mode, or parallel mode, and $\boldsymbol{E}$ perpendicular to the plane of $\boldsymbol{B}$-$\boldsymbol{k}$, which we call "E" mode, or perpendicular mode. The refraction indices of these two modes are different:

$$N_\perp = 1 + \frac{2\alpha}{45\pi}\left(\frac{B}{B_Q}\right)^2 \sin^2\theta \tag{1.57}$$

$$N_\parallel = 1 + \frac{7\alpha}{90\pi}\left(\frac{B}{B_Q}\right)^2 \sin^2\theta \tag{1.58}$$

If we integrate equation (1.56) over θ' we can find the total cross section

$$\sigma_{\text{tot}} = \sum_{e'} \int_{-1}^{1} \frac{d\sigma}{d\mu'}d\mu' = 2\pi^2 r_e c\xi\delta(\omega - \omega_B) \tag{1.59}$$

where ξ is 1 for perpendicular mode and ξ is equal to μ^2 for parallel mode. This is due to pure geometry and come from the $\boldsymbol{e} \cdot \boldsymbol{e}_\pm$ factor. This final result applies not only for $B \ll B_Q$ but also for B comparable to B_Q. If one wants we can convolute this result with some frequency distribution, and we can find a rate for resonant scattering

$$\dot{n}_{\text{res}} \sim n_e n_{\text{ph}}\sigma_{\text{eff}}^{\text{res}} c \tag{1.60}$$

where

$$\sigma_{\text{eff}}^{\text{res}} \approx \frac{2\pi^2 r_e c}{\omega_B} \tag{1.61}$$

compare this with Thomson cross section

$$\sigma_T \approx \frac{8\pi}{3}r_e^2 \tag{1.62}$$

so for B larger than critical field this resonant cross section is larger than the normal Thomson cross section.