GR meets astrophysics

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1 Lecture 1

We want to make connection between GR and the plasma astrophysics. Why do we need this when we have gravitational waves now? There are a lot of limitations of gravitational waves. It is limited due to characteristics of signals and detectors. At higher frequency noise grows linearly, and sky location is very poor. Strong gravity effects in astrophysics are all tied to some analytical models, and are limited in scope. We also are interested in the EM connections to the gravitational wave signals.

Therefore we want to study mergers of compact objects within surrounding plasma. Two binary non-rotating black holes disturbs the spacetime enough that it launches jets from the surrounding plasma and allow the plasma to tap into the kinetic energy of the rotation.

We will be talking about the challenges and problems simulating and controlling dynamic space time. Let's go slowly from the beautiful theory of Einstein and put it into the computer to compute it dynamically.

We start with the theorem by Hadamard: any problem about a physical system should be well posed. By well posed (WP), we mean:

- Solution exists
- The solution to the problem is unique
- The solution must depend continuously on initial and boundary data

By the last statement we mean that the magnitude of solution in a general sense should

$$|u|_T \leq |u|_{t=0} K e^{\beta T} \tag{1.1}$$

where K and β should not depend on initial or boundary data. If any problem we are solving does not satisfy any of the statements we should throw it away immediately.

Lets talk about the sufficient and necessary conditions for WPness. If we can pose the problem in this form

$$u_{,t} = \sum A^i \partial_i u + \text{"Rest"} \tag{1.2}$$

then the sufficient condition is that the matrix A is diagonalizable, and the eigenvalues are real.

A simple example is $u_{,tt} = u_{,xx}$. If we define $f = u_{,x}$ and $g = u_{,t}$ then we can define the dynamic variables as u, f, g and it satisfy the sufficient condition.

Consider now this system

$$u_{,t} = u_{,x} + v_{,x}, \quad v_{,t} = u_{,x} \tag{1.3}$$

This system has A in a Jordan block form. It turns out that this system admits the solution in the form of

$$u = DF_1(t+x) \cdot t + F_2(t+x), \quad v = F_1(t+x) \tag{1.4}$$

the first part of u grows linearly with time. This is called a weakly hyperbolic system, and we now care about the "Rest" of the problem. If we couple this, and add u to $v_{,t}$, then the solution of v becomes

$$v = c_1 e^{c_1(t+x)} + c_2 e^{t\sqrt{c_2}} \tag{1.5}$$

and this solution is exponentially unstable to changes to initial condition, which is a badly posed problem. We especially want to avoid this in a numerical system because the initial conditions are never exact.

How do we make sure our complex problem is well posed when such a simple one can be bad? Lets consider the Einstein equation

$$G_{ab} = 8\pi T_{ab} \tag{1.6}$$

The left hand side part has form of $\partial^2 g$, whereas the right hand side depends on the fluid quantities $\mathcal{F}$ and g. We also have the constraint equation of the fluid $\nabla_a T^{ab} = 0$, which concerns $\partial\mathcal{F}$ and ∂g. For our purposes they are decoupled, so we are allowed to treat T_{ab} as the "Rest".

Lets say some words about gravity quickly. We have the metric tensor which describes the spacetime manifold g_{ab}. This determines a length

$$ds^2 = g_{ab} dx^a dx^b \tag{1.7}$$

Note that we use Einstein summation notation to add the indices a and b. Both indices take values $0, 1, 2, 3$ where 0 is for time part and others are for spatial part.

If we have a curved manifold, when we take derivatives we need to take into account the change of manifold in addition to the change of the field. We define the covariant derivative ∇_a which does exactly that. We have by definition

$$\nabla_a g_{bc} = 0 \tag{1.8}$$

The covariant derivative of a vector is defined as

$$\nabla_a v^b = \partial_a v^b + \Gamma^b{}_{ac} v^c \tag{1.9}$$

Some other definitions are

$$\nabla_a f = \partial_a f, \quad \nabla_a v_b = \partial_a v_b - \Gamma^c{}_{ab} v_c \tag{1.10}$$

Now we need to define what the Γ symbol is. It is called the Chrsitoffel symbol

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ae}(g_{be,c} + g_{ce,b} - g_{bc,e}) \tag{1.11}$$

Now we can define the so-called Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd} \Gamma^a{}_{ec} - \Gamma^e{}_{bc} \Gamma^a{}_{ed} \tag{1.12}$$

This curvature tensor describes the curvedness of the spacetime manifold.

If we have two world lines along the manifold, and the acceleration between them can be characterized by the Riemann tensor

$$a^a = -R^a{}_{cbd} \xi^c \xi^d L^b \tag{1.13}$$

where $\xi^c = dx^c/d\lambda\ L^b$ is the distance between the world line. A funny fact about the curved manifold is that derivatives depend on paths, and the order of taking them, and the Riemann tensor characterizes this:

$$[\nabla_b \nabla_c - \nabla_c \nabla_b]\, v_a = R^d{}_{abc} v_d \tag{1.14}$$

We finally define the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \quad R_{ab} = R^d{}_{adb}, \quad R = g^{ab} R_{ab} \tag{1.15}$$

Now lets get into how to solve these equations in the computer. In a sense our problem is ill formed from the get go because everything depends on our choice of coordinates. So we need to find a formulation.

The one we introduce now is called the ADM formulation. Given a spacetime we can foliate it with space-like slices Σ_{t_i}. A way to understand space-like hypersurfaces is that its norm at any place n^a is time-like: $n_a n^a = -1$. Here we implicitly choose our signature to be $(-1, 1, 1, 1)$ so that the Minkowski spacetime is $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. Time-like vectors are inside the lightcone and space-like vectors are outside.

We can now label these hypersurfaces with t equal to constant. Because the hypersurfaces have constant t, we can define $n_a = \nabla_a t = \partial_a t$. If we go from the hypersurface of t_1 to the next hypersurface of t_2. For a typical observer, he will not measure really time $t_2 - t_1$, but some time proportional to that:

$$\tau = \alpha(t_2 - t_1) \tag{1.16}$$

where α is called the lapse function. The vector that points from a point (t_1, x, y, z) to a point (t_2, x, y, z) is what we call ∂_t. However this may not coincide with the vector n_a we defined above, so we define a new vector

$$\beta^a = \partial_t^a - n^a \tag{1.17}$$

which is called the shift vector.

Now we can split up the metric tensor

$$ds^2 = -\alpha^2 dt^2 + h_{ij} \left(dx^i + \beta^i dt \right) \left(dx^j + \beta^j dt \right) \tag{1.18}$$

where h_{ij} will be the metric on the hypersurface.

We can find the relationship between the metric tensor of the individual hypersurfaces and the full metric:

$$h_{ab} = g_{ab} + n_a n_b \tag{1.19}$$

This metric has several properties:

- $h_a{}^b h_b{}^c = h_a{}^c$
- $h_{ab} n^a = 0$
- $h_{ab} S^b = S_a$

where S^b is a spacelike vector. So h_{ab} is like a projection operator onto the hypersurface.

We need one more concept so that we can foliate the spacetime. This is the so-called extrinsic curvature

$$K_{ab} = - \perp\perp \nabla_a n_b = -h_a{}^\alpha h_b{}^\beta \nabla_\alpha n_\beta \tag{1.20}$$

This is a useful quantity to measure the rate of change of the hypersurface when embedded into the whole manifold as one goes with the flow. It can be shown that

$$\partial_t h_{ab} = -2\alpha K_{ab} + \beta^l \partial_l h_{ab} + h_{al}\partial_b \beta^l + h_{bl}\partial_a \beta^l \tag{1.21}$$

Now we are equiped to look at numerical relativity. One takes the Einstein equations and take different projections. If we take a vector and dot it $v^a n_a = v_n$ to get the component along the direction of n. We can also dot it with h to get its projection to the hypersurface. If we dot the Einstein equation with $n^a n^b$ then we get

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{1.22}$$

Here ρ is the energy density that is measured by the observers that are moving normal to the spatial hypersurface, $\rho = T_{ab}n^a n^b$. ${}^{(3)}R$ is the intrinsic curvature of the hypersurface.

We can also imagine dotting the Einstein equation with $n_a h_c{}^b$:

$$D_b(K^{ab} - h^{ab}K) = 8\pi J^a \tag{1.23}$$

where D is the intrinsic covariant derivative along the hyper surface $D_b h_{ac} = 0$. Again J is the momentum measured by the same observers.

These two equations have no time derivatives and are called constraint equations. We have one final projection by $\perp\perp$

$$\begin{aligned} \partial_t K_{ab} &= \beta^l \partial_l K_{ab} + K_{al}\partial_b \beta^l + K_{bl}\partial_a \beta^l \\ &- D_a D_b \alpha + \alpha \left[{}^{(3)}R_{ab} + K K_{ab} - 2K_{ad}K^d{}_b - 8\pi \left(S_{ab} - \frac{h_{ab}}{2}(S-\rho) \right) \right] \end{aligned} \tag{1.24}$$

We also have

$$\partial_t h_{ab} = \beta^l \partial_l h_{ab} + h_{al}\partial_b \beta^l + h_{bl}\partial_a \beta^l - 2\alpha K_{ab} \tag{1.25}$$

From these we know that the spatial curvatures have second order time derivatives, which makes it into a hyperbolic system, but the rest unfortunately turns the system into weakly hyperbolic.

There is a whole bunch of choices of α and β, but lets first discuss the problem that this system is weakly hyperbolic. Lets take a step back and write down R_{ab}. Notice that taking the trace of the Einstein equation gives $R = -T$. So we can write

$$R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T \tag{1.26}$$

The Ricci tensor has a very provocative form

$$R_{ab} \longrightarrow g^{cd} g_{ab,cd} - 2\nabla_{(a}\Gamma_{b)} + \Gamma\Gamma + \partial g \partial g + \dots \tag{1.27}$$

Since we are interested in the hyperbolic part, the first term is perfect which is like the wave equation, but the second term is nasty. Due to human nature we want to throw away things we don't like, so lets try to choose α and β such that $\Gamma_b = 0$ then the second term is zero. Does this do the job?

Suppose we have the equation

$$\Box\phi = \sum A \partial_J \phi \partial_R \phi \tag{1.28}$$

which is actually what happens with the above equation for R_{ab} and it turns out it has nice properties. Another problem people found is that $\Gamma_b = 0$ gives $\nabla_a \nabla^a x^b = 0$ which means coordinates fly away at speed of light.

Now do we need to give up our coordinates to solve the problem? What we do is to put a source $\Gamma_b = H_b$, so that $-\nabla_a \nabla^a x^b = H^b$. However now H_b shows up in the equation, but if we have a hyperbolic system for H where $\nabla_a \nabla^a H_b =$ "something" then we can close the system.

The remaining challenge is to satisfy the constraints, since a small departure from the constraint due to numerics then we might go completely away from the solution. Therefore we need to find a way to bring the solution back to the constraint hypersurface when things go wrong.

Imagine at some point we choose the coordinates as discussed above, and there is a deviation $C^a = H^a + \Gamma^a$, then our system becomes

$$R_{ab} - \nabla_{(a} C_{b)} = 0 \tag{1.29}$$

we can substitute the constraint equation $\nabla^b G_{ab} = 0$ into the above equation we get

$$0 = \nabla^a \nabla_a C_b + C^a \nabla_{(b} C_{a)} \tag{1.30}$$

so we have a wave equation for C. In theory if we start from zero then it will remain zero. But in reality we do see problems.

Consider instead

$$0 = \nabla^a \nabla_a C_b + C^a \nabla_{(b} C_{a)} + \gamma_0 \left[t_{(a} C_{b)} - \frac{1}{2} g_{ab} t^c C_c \right] \tag{1.31}$$

where γ^0 is a numerical parameter and t a timelike vector field. If we do the same thing as above we have now

$$0 = \nabla^a \nabla_a C_b - 2\gamma_0 \nabla^a \left[t_{(a} C_{b)} \right] \tag{1.32}$$

This is now a damped wave equation and will damp away deviation from the constraint on a time scale determined by γ_0.