#MCF: The Physics of Magnetic Confinement

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1 Lecture 1

We are going to talk about magnetic confinement in fusion. This is our only talk which is not on Astrophysics in this school, so lets start by talking about how this differ from astrophysical plasmas. The first difference is that, for the sun as an example, it has a great mass and the gravitational force can provide confinement for the plasma. However for controlled fusion we don't have such luxury, and we need magnetic or hydro means to confine the plasma.

The way we do fusion is to heat up hydrogen to the extent such that at the tail of the distribution the atoms have just enough energy to tunnel through the Coulomb barrier to undergo fusion. Therefore our goal is to confine hot plasma. We use a toroidal structure and try to achieve an equilibrium state, which we will talk about subsequently.

We are not going to talk too much about how to heat up the plasma. In a short sentence we just put current into the plasma and due to resistivity it will heat up.

1.1 Single Particle Confinement

The basic concept of magnetic confinement is simple. With magnetic field particles will stream along the magnetic field lines while gyrating at radius

$$\rho = v_\perp \Omega \propto v_{\text{th}}/B \sim \sqrt{\beta} \tag{1.1}$$

However this only works with 2 dimensions and particles can still stream along the field line.

One idea to overcome this is to use a magnetic mirror, briefly covered in Matt Kunz's lecture. This is what happens in Van Allen radiation belt around Earth. Since magnetic moment is an adiabatic invariant in slowly changing inhomogeneous magnetic field, particles will bounce off regions with higher magnetic field. Since energy is conserved

$$\frac{mv_\parallel^2}{2} = E - \mu B \tag{1.2}$$

therefore at bounce point $E = \mu B_{\text{max}}$. However this method of confinement can lose a lot of plasma when it is hot.

Another solution is to use the field line topology to confine the plasma. The "Hairy ball theorem" tells that the only topology with confined trajectories is a torus. Once we confine ourselves to torus, there are three possibilities: closed lines, surfaces, or toroidal annuli. The third idea is not very appealing especially when there are many particles, since particles on different field lines will interact.

If we have a circular magnetic field in a plane, particles gyrating in the magnetic field will drift vertically due to grad-B drift. If we put E field to prevent this movement, the particles will drift outwards due to E cross B drift. So it does not work.

The solution is to twist the magnetic field by introducing poloidal B_θ. This is the Tokamak. We can use a toroidal coil to provide the toroidal magnetic field and use a coil inside and outside to make poloidal magnetic field. In this kind of configuration, if the particles sample the entire toroidal surface, then there is no net drift.

However there is a complication. The field strength changes over the particle trajectory and some particles might get trapped. At this point lets introduce some quantities

$$\psi_p = \oint \mathbf{B} \cdot d\mathbf{a}^\theta, \quad \psi_t = \oint \mathbf{B} \cdot d\mathbf{a}^\zeta \tag{1.3}$$

where ζ labels the coordinate along the torus long direction, and θ labels the short direction (orthorgonal).

1.2 Magnetic Topology

Tamm's Theorem states that no average radial drift in axisymmetric torus. If the confining field is axisymmetric, the canonical angular momentum is conserved

$$p_\zeta = Rmv_\zeta + \frac{ZeRA_\zeta}{c} = \text{const} \tag{1.4}$$

Trapped particles in an axisymmetric torus will close in their orbits and precess along the field lines.

There are other symmetries that also prevent average radial drift. Lets consider

$$\psi_* = \psi_p - \frac{I(\psi_p)v_\parallel}{\Omega} \tag{1.5}$$

where $I \sim B/R$ is small. We want to argue that

$$\frac{d\psi_*}{dt} = \frac{2v_\parallel^2 + v_\perp^2}{\Omega B^2} \left(\mathbf{B} \times \nabla\psi_p \cdot \nabla B - I\mathbf{B} \cdot \nabla B \right) \tag{1.6}$$

To summarize, the magnetic field topology will be

$$2\pi\mathbf{B} = \nabla\psi_t \times \nabla\vartheta + \nabla\zeta \times \nabla\psi_p(\psi_t, \vartheta, \zeta) \tag{1.7}$$

The trajectory of the field line is given by

$$\frac{dx^i}{d\tau} = \mathbf{B} \cdot \nabla x^i = B^i \tag{1.8}$$

We want to choose ζ as time-like cooordinate:

$$\frac{d\psi_t}{d\zeta} = -\frac{\partial\psi_p}{\partial\vartheta}, \quad \frac{d\vartheta}{d\zeta} = \frac{\partial\psi_p}{\partial\psi_t} = 1/q \tag{1.9}$$

which is the pitch of the magnetic field.

These two equations can be identified as Hamilton equations of this system where

$$\psi_p \leftrightarrow H, \quad \zeta \leftrightarrow t \tag{1.10}$$

where θ and ψ_t are position and momentum. This Hamiltonian system has one degree of freedom which allows 1D trajectories that are closed lines, and 2D trajectories that wraps around the torus.

Now we have two possibilities, tokamaks and stellarators. Both are topologically tori, but Tokamaks are easier to construct with confined drift orbits. However in stellarators the confining poloidal field is generated externally and it is harder to confine the orbits. But this system has more flexibility in shaping.

1.3 Plasma Equilibrium

The force on the plasma is

$$\frac{\partial(m_s n_s \mathbf{u}_s)}{\partial t} + \nabla \cdot \mathbf{P}_s - e_s n_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c} \right) = \mathbf{F}_s \tag{1.11}$$

Since we want to find equilibrium without external forces, both the first term on lhs and the rhs are zero. Poisson's equation tells us that

$$\nabla^2 \Phi = -4\pi \sum_s e_s n_s, \quad \sum_s e_s n_s \sim e n_e \left(\frac{\lambda_D}{L} \right)^2 \tag{1.12}$$

As long as the plasma is quasineutral, with density variation much smaller than the length scale of the system, then the static electric field can be neglected. Near thermal equilibrium we have $\nabla \cdot \mathbf{P}_s = \nabla p_s$. So we have after summing over all species

$$c\nabla p = \mathbf{J} \times \mathbf{B} \tag{1.13}$$

What this does tell us is that in a collisionless limit particles propagate parallel to the field lines, and in the collisional limit we have sound waves across the field lines.

Another consequence of quasineutrality is $\nabla \cdot \mathbf{J} = -\partial \rho / \partial t$, so

$$\nabla \cdot \mathbf{J} \sim \frac{\rho}{\tau} \sim \frac{J}{L} \left(\frac{v_{th,e}^2}{c} \right)^2 \frac{1}{\Omega_e \tau} \ll \frac{J}{L} \tag{1.14}$$

Therefore mainly $\mathbf{J}_\perp$ enters the equation before. This means there has to be some parallel current $\nabla \cdot J_\parallel \neq 0$.

There is an alternative physical interpretation. Lets define the curvature

$$\kappa = \mathbf{b} \cdot \nabla \mathbf{b} = \frac{4\pi}{c} \frac{\mathbf{J} \times \mathbf{B}}{B^2} + \frac{\nabla_\perp B}{B} \tag{1.15}$$

If we use this to replace the J cross B term we have

$$\frac{B^2}{4\pi} \kappa = \nabla_\perp \left(p + \frac{B^2}{4\pi} \right) \tag{1.16}$$

This means if we want to stretch field lines it has some tension to oppose that, and it balances plasma pressure.

Lets look at a simplified version of axisymmetric equilibrium. We look at the radial component of Ampere's Law

$$\frac{4\pi}{c} J^r = \mathcal{J} \left(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \right), \quad \mathcal{J}^{-1} = \nabla r \cdot (\nabla \theta \times \nabla \zeta) \tag{1.17}$$

where r is some variable that labels the field lines. This tells us that B_ζ only depends on flux label r, so we can write B field as

$$\mathbf{B} = I(r)\nabla\zeta + \nabla\zeta \times \nabla\psi_p \tag{1.18}$$

We can plug this into our force balance to get

$$cp' = \mathcal{J}^{-1}B^\theta(qJ^\theta - J^\zeta) \tag{1.19}$$

and we can find some expressions for J^θ and J^ζ. We can find the Grad-Shafranov equation

$$R^2\nabla \cdot (R^{-2}\nabla\psi_p) = -I\frac{dI}{d\psi_p} - 4\pi R^2\frac{dp}{d\psi_p} \tag{1.20}$$

If we use cylindrical coordinates (R, Z) and define $\psi_p = u\sqrt{R}$, then we can find a Poisson equation for u, given $p(u)$, $I(u)$ and some boundary condition we can solve it iteratively.

Lets look at one way to solve it numerically. The idea is to map the constant flux surfaces conformally to a circle, and transform the Poisson equation accordingly. Now because this is a periodic domain we can solve the equation using spectral method. On a computer we can use FFT to quickly solve the equation using Fourier transform. However when we do the conformal map back to the original domain, there might be bunching of grid points or region with relatively few grid points, so one needs to be careful.

We can also use similar analysis to give Grad-Shafranov equation for the parallel current:

$$J_\parallel = \frac{-c}{4\pi}B\frac{dI}{d\psi_p} - \frac{cI}{B}\frac{dp}{d\psi_p} = \sum_s e_s \int d^3v\, v_\parallel F_s \tag{1.21}$$

The occurance of distribution function necessitates kinetic treatment.

1.4 Stability

In this kind of plasma there is a wide zoo of plasma instabilities. Due to the toroidal geometry it is difficult to decouple the modes, so it is a huge mess. Lets only talk a bit about the most unstable modes.

When we solve the equilibrium problem we saw there are a number of different forces on the plasma. We have field line "tension" which opposes the bending of magnetic field lines. The most dangerous mode is due to opposing the bending field line which has $k_\parallel \ll k_\perp$. There is the distinction of "rational" field lines which closes themselves, and "irrational" field lines which don't. Only on rational field lines we can have $k_\parallel = 0$. Flute perturbations are unstable when the field line curves towards the plasma, but it its stable when the field line curvature is away from plasma.

Tokamak has good and bad curvature regions. Field lines will bend outwards in the bad curvature region to somewhat self-stabilize. If we have field-aligned perturbations at a given radius, but different pitch of magnetic field lines at neighboring radius it could also stabilize the perturbation.

Another kind of instability is kink instability. Kinks come from helical perturbation and is current driven. If we kink a magnetic field line bundle like a hose, the upper part of the kink has less magnetic pressure so it is even easier to kink upwards, which runs away. Again field line bending can stabilize this effect.

In theory if we want to find out whether the plasma is stable, we want to linearize the MHD equations and solve the linear problem as an eigenvalue problem. However in Tokamak plasmas all the modes couple,

so it is not as easy to do the same thing. We use the so-called energy principle by looking at the energy and work of the displacements

$$\delta W(\xi^*, \xi) = -\frac{1}{2} \int d^3 r \xi^* \cdot \mathbf{F}(\xi), \quad E(\xi^*, \xi) = \frac{1}{2} \int d^3 r \rho_0 \dot{\xi}^* \cdot \dot{\xi} \tag{1.22}$$

and we use variational principle on

$$\Omega^2 = \frac{\delta W}{E}, \quad \delta \Omega^2 = 0 \tag{1.23}$$

If we expand the displacement into discretized modes, then we find

$$\delta W = \sum_n |a_n|^2 \omega_n^2 \tag{1.24}$$

When $\delta W < 0$ we have an instability, and if $\delta W \geq 0$ for all displacements then we have stability. The complication here is that in MHD we have a continuous spectrum of modes, so we need to do more work but the principle is similar.

In practice the problem is reduced to numerically minimizing the function

$$\delta W = \sum_n \sum_m a_n a_m \mathbf{F}(\xi_n, \xi_m) \tag{1.25}$$

I honestly don't understand what I'm writing...

2 Lecture 2

We talked about how to confine the plasma with magnetic field configuration. Today we are going to talk about the practical challenges and how to put it into computer.

2.1 Turbulence and Transport

Even when we have an MHD stable plasma in the tokamak, there are small turbulences in the system which is highly anisotropic.

The first question we want to ask is how good must the confinement be? We want to induce fusion, so we want to balance the power that goes into the fusion plasma, vs the rate the energy is transported out

$$\frac{p}{\tau_E} = \left(\frac{\partial p}{\partial t}\right)_{\text{transport}} = \mathcal{S}_{\text{fusion}} \tag{2.1}$$

The rate of fusion is $\mathcal{S}_{\text{fusion}} = n^2 \langle \sigma v \rangle E_{\text{fusion}}$. Plugging our scales in, we find that the confinement is on the order of $\tau_E \sim 1\,\text{s}$.

Without magnetic field, the average speed of particles at thermal speed would be $v_{th} \sim 10^5\,\text{m/s}$, so for practical devices the confinement time is just $t \sim 10^{-5}\,\text{s}$. We need to use the magnetic field to help prolong the confinement time by 5 orders of magnitude.

In a magnetic field, we could have transport between field lines just due to collisions, since collision can reverse the direction of its momentum and it will gyrate around a different field line. If we model the collision process as a random walk, we can get a diffusion equation, with diffusion coefficient

$$D \approx \frac{\ell^2}{2\tau} \tag{2.2}$$

Lets say the random walk step size $\ell \sim \rho$, and the time to be $\tau \sim \tau_C \sim \lambda_{mfp}/v_{th}$. Combining these we have

$$\tau_E \sim \left(\frac{a}{\rho}\right)^2 \left(\frac{a}{v_{th}}\right) \cdots \sim \left(\frac{\lambda_{mfp}}{a}\right) \tag{2.3}$$

This ratio is huge, so confinement time can be very long.

Remember in our toroidal magnetic field the particles undergo trapped orbits, which are about 100 times bigger than their gyro radius. This is called Neo-classical transport. But even with $1/100$ over the above time we can still have way longer time than 1s. Fusion would be solved if we only have this kind of transport.

But we don't. We have the Toroidal drift instability (ITG/ETG). If we have temperature gradient with magnetic field gradient, then the temperature will interfere with grad-B drift. At outer edge of the torus, ∇B aligns with temperature gradient, this will give arise to charge separation at the surface with hot and cold gradient, which will induce E cross B drift. In the inside of the torus we have opposite effect which stabilizes the plasma.

The instability growth rate must depend on temperature gradient since there is no instability when there is not gradient. The inside and outside of the torus will compete and when we have

$$\gamma(T') \gtrsim \frac{v_{th}}{qR} \tag{2.4}$$

then we have net instability. The temperature gradient has to exceed a critical value to sustain the instability.

This instability depends on charge separation via magnetic drifts, therefore growth rate is related to the rate at which charge separation occurs. The time rate of charge separation is about the time for particles to drift across the wavelength of the perturbation

$$\gamma_{k_\perp} \sim k_\perp v_d \sim \frac{v_{th}}{R} k_\perp \rho \tag{2.5}$$

However we can't take $k_\perp$ to be really large, because when the wavelength is smaller than the particle gyro radius, the particle trajectory will sample many wavelengths and smear out the effect.

Lets estimate the confine time in the presence of this turbulence. The random walk size is around the eddy size which is $\ell \sim \rho$. The average time would be $\tau \sim \rho/v_E \sim a/v_{th}$. This gives a confinement time $\tau_E \sim 1$.

Everything up to now has been fluid even though collisional mean free path in the plasma is long. What about kinetic effects? Significant number of particles travel slower than the sound speed, and some particles are trapped in bad curvature region and stay there indefinitely. Both these effects have a destabilizing effect on the plasma.

The diffusion coefficient depends strongly on the temperature of the plasma

$$D_{\text{eff}} \sim \frac{v_{th}}{a} \left(\frac{\rho}{a}\right)^2 p \propto T^{5/2} \tag{2.6}$$

This means that diffusion coef is large in core, and making it difficult to go above critical temperature gradient.

How do we solve this problem on the computer? There is a huge separation of scales of the problem, from electron and ion gyromotion at 10^{-12} s, to the confinement time scale $\tau_c \sim 1$ s. If we address this problem directly we need to resolve all the scales. In addition to this, since we need to model the distribution

function, we also need to sample the v space, which could form small scales as well. Particles on different Larmor orbits separated by turbulence wavelength will see different potential, so we have small scales in $v_\perp$ comparable to the scale of density fluctuations. In the end the effect of turbulence cascades into small scales in both space and momentum space.

If we estimate how many grid points we need to resolve the scales, 10^6 grid points for each spatial direction and 10 points for each of the velocity direction, then we need 10^{34} points, which is not possible.

What we do is taking the gyrokinetic approach. We average over the fast gryomotion and eliminate one degree of freedom of the particles, and eliminate a fast time scale. This saves about 10^{10} grid points. We can do better than this by doing gyrokinetics at multi-scale. If we take $f = F + \delta f$, where F defines the mean piece of the distribution function. We assume that

$$\nabla_\perp \ln F \sim L^{-1}, \quad \nabla_\perp \ln \delta f \sim \rho^{-1}, \quad \nabla_\parallel \ln \delta f \sim L^{-1} \tag{2.7}$$

And the mean profile evolves slowly compared to the turbulence. ρ/L is assumed to be small and it will be our expansion parameter. Doing this we can get the Gyrokinetic-Poisson system: ...

We solve the system by solving the detailed turbulence in only small parts of the box, and map out the region in between, by either taking average or using periodicity.