#MCF: The Physics of Magnetic Confinement

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1 Lecture 1

We are going to talk about magnetic confinement in fusion. This is our only talk which is not on Astrophysics in this school, so lets start by talking about how this differ from astrophysical plasmas. The first difference is that, for the sun as an example, it has a great mass and the gravitational force can provide confinement for the plasma. However for controlled fusion we don't have such luxury, and we need magnetic or hydro means to confine the plasma.

The way we do fusion is to heat up hydrogen to the extent such that at the tail of the distribution the atoms have just enough energy to tunnel through the Coulomb barrier to undergo fusion. Therefore our goal is to confine hot plasma. We use a toroidal structure and try to achieve an equilibrium state, which we will talk about subsequently.

We are not going to talk too much about how to heat up the plasma. In a short sentence we just put current into the plasma and due to resistivity it will heat up.

1.1 Single Particle Confinement

The basic concept of magnetic confinement is simple. With magnetic field particles will stream along the magnetic field lines while gyrating at radius

$$\rho = v_\perp \Omega \propto v_{\rm th}/B \sim \sqrt{\beta} \tag{1.1}$$

However this only works with 2 dimensions and particles can still stream along the field line.

One idea to overcome this is to use a magnetic mirror, briefly covered in Matt Kunz's lecture. This is what happens in Van Allen radiation belt around Earth. Since magnetic moment is an adiabatic invariant in slowly changing inhomogeneous magnetic field, particles will bounce off regions with higher magnetic field. Since energy is conserved

$$\frac{mv_\parallel^2}{2} = E - \mu B \tag{1.2}$$

therefore at bounce point $E = \mu B_{\rm max}$. However this method of confinement can lose a lot of plasma when it is hot.

Another solution is to use the field line topology to confine the plasma. The "Hairy ball theorem" tells that the only topology with confined trajectories is a torus. Once we confine ourselves to torus, there are three possibilities: closed lines, surfaces, or toroidal annuli. The third idea is not very appealing especially when there are many particles, since particles on different field lines will interact.

If we have a circular magnetic field in a plane, particles gyrating in the magnetic field will drift vertically due to grad-B drift. If we put E field to prevent this movement, the particles will drift outwards due to E cross B drift. So it does not work.

The solution is to twist the magnetic field by introducing poloidal B_θ. This is the Tokamak. We can use a toroidal coil to provide the toroidal magnetic field and use a coil inside and outside to make poloidal magnetic field. In this kind of configuration, if the particles sample the entire toroidal surface, then there is no net drift.

However there is a complication. The field strength changes over the particle trajectory and some particles might get trapped. At this point lets introduce some quantities

$$\psi_p = \oint \mathbf{B} \cdot d\mathbf{a}^\theta, \quad \psi_t = \oint \mathbf{B} \cdot d\mathbf{a}^\zeta \tag{1.3}$$

where ζ labels the coordinate along the torus long direction, and θ labels the short direction (orthorgonal).

1.2 Magnetic Topology

Tamm's Theorem states that no average radial drift in axisymmetric torus. If the confining field is axisymmetric, the canonical angular momentum is conserved

$$p_\zeta = Rmv_\zeta + \frac{ZeRA_\zeta}{c} = \text{const} \tag{1.4}$$

Trapped particles in an axisymmetric torus will close in their orbits and precess along the field lines.

There are other symmetries that also prevent average radial drift. Lets consider

$$\psi_* = \psi_p - \frac{I(\psi_p)v_\parallel}{\Omega} \tag{1.5}$$

where $I \sim B/R$ is small. We want to argue that

$$\frac{d\psi_*}{dt} = \frac{2v_\parallel^2 + v_\perp^2}{\Omega B^2} \left(\mathbf{B} \times \nabla\psi_p \cdot \nabla B - I\mathbf{B} \cdot \nabla B \right) \tag{1.6}$$

To summarize, the magnetic field topology will be

$$2\pi\mathbf{B} = \nabla\psi_t \times \nabla\vartheta + \nabla\zeta \times \nabla\psi_p(\psi_t, \vartheta, \zeta) \tag{1.7}$$

The trajectory of the field line is given by

$$\frac{dx^i}{d\tau} = \mathbf{B} \cdot \nabla x^i = B^i \tag{1.8}$$

We want to choose ζ as time-like cooordinate:

$$\frac{d\psi_t}{d\zeta} = -\frac{\partial\psi_p}{\partial\vartheta}, \quad \frac{d\vartheta}{d\zeta} = \frac{\partial\psi_p}{\partial\psi_t} = 1/q \tag{1.9}$$

which is the pitch of the magnetic field.

These two equations can be identified as Hamilton equations of this system where

$$\psi_p \leftrightarrow H, \quad \zeta \leftrightarrow t \tag{1.10}$$

where θ and ψ_t are position and momentum. This Hamiltonian system has one degree of freedom which allows 1D trajectories that are closed lines, and 2D trajectories that wraps around the torus.

Now we have two possibilities, tokamaks and stellarators. Both are topologically tori, but Tokamaks are easier to construct with confined drift orbits. However in stellarators the confining poloidal field is generated externally and it is harder to confine the orbits. But this system has more flexibility in shaping.

1.3 Plasma Equilibrium

The force on the plasma is

$$\frac{\partial(m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot \mathbf{P}_s - e_s n_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c}\right) = \mathbf{F}_s \tag{1.11}$$

Since we want to find equilibrium without external forces, both the first term on lhs and the rhs are zero. Poisson's equation tells us that

$$\nabla^2 \Phi = -4\pi \sum_s e_s n_s, \quad \sum_s e_s n_s \sim e n_e \left(\frac{\lambda_D}{L}\right)^2 \tag{1.12}$$

As long as the plasma is quasineutral, with density variation much smaller than the length scale of the system, then the static electric field can be neglected. Near thermal equilibrium we have $\nabla \cdot \mathbf{P}_s = \nabla p_s$. So we have after summing over all species

$$c\nabla p = \mathbf{J} \times \mathbf{B} \tag{1.13}$$

What this does tell us is that in a collisionless limit particles propagate parallel to the field lines, and in the collisional limit we have sound waves across the field lines.

Another consequence of quasineutrality is $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$, so

$$\nabla \cdot \mathbf{J} \sim \frac{\rho}{\tau} \sim \frac{J}{L}\left(\frac{v_{th,e}^2}{c}\right)^2 \frac{1}{\Omega_e \tau} \ll \frac{J}{L} \tag{1.14}$$

Therefore mainly $\mathbf{J}_\perp$ enters the equation before. This means there has to be some parallel current $\nabla \cdot J_\parallel \neq 0$.

There is an alternative physical interpretation. Lets define the curvature

$$\kappa = \mathbf{b} \cdot \nabla \mathbf{b} = \frac{4\pi}{c} \frac{\mathbf{J} \times \mathbf{B}}{B^2} + \frac{\nabla_\perp B}{B} \tag{1.15}$$

If we use this to replace the J cross B term we have

$$\frac{B^2}{4\pi}\kappa = \nabla_\perp \left(p + \frac{B^2}{4\pi}\right) \tag{1.16}$$

This means if we want to stretch field lines it has some tension to oppose that, and it balances plasma pressure.

Lets look at a simplified version of axisymmetric equilibrium. We look at the radial component of Ampere's Law

$$\frac{4\pi}{c} J^r = \mathcal{J}\left(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta}\right), \quad \mathcal{J}^{-1} = \nabla r \cdot (\nabla\theta \times \nabla\zeta) \tag{1.17}$$

where r is some variable that labels the field lines. This tells us that B_ζ only depends on flux label r, so we can write B field as

$$\mathbf{B} = I(r)\nabla\zeta + \nabla\zeta \times \nabla\psi_p \tag{1.18}$$

We can plug this into our force balance to get

$$cp' = \mathcal{J}^{-1}B^\theta(qJ^\theta - J^\zeta) \tag{1.19}$$

and we can find some expressions for J^θ and J^ζ. We can find the Grad-Shafranov equation

$$R^2\nabla \cdot (R^{-2}\nabla\psi_p) = -I\frac{dI}{d\psi_p} - 4\pi R^2\frac{dp}{d\psi_p} \tag{1.20}$$

If we use cylindrical coordinates (R, Z) and define $\psi_p = u\sqrt{R}$, then we can find a Poisson equation for u, given $p(u)$, $I(u)$ and some boundary condition we can solve it iteratively.

Lets look at one way to solve it numerically. The idea is to map the constant flux surfaces conformally to a circle, and transform the Poisson equation accordingly. Now because this is a periodic domain we can solve the equation using spectral method. On a computer we can use FFT to quickly solve the equation using Fourier transform. However when we do the conformal map back to the original domain, there might be bunching of grid points or region with relatively few grid points, so one needs to be careful.

We can also use similar analysis to give Grad-Shafranov equation for the parallel current:

$$J_\parallel = \frac{-c}{4\pi}B\frac{dI}{d\psi_p} - \frac{cI}{B}\frac{dp}{d\psi_p} = \sum_s e_s \int d^3v\, v_\parallel F_s \tag{1.21}$$

The occurance of distribution function necessitates kinetic treatment.

1.4 Stability

In this kind of plasma there is a wide zoo of plasma instabilities. Due to the toroidal geometry it is difficult to decouple the modes, so it is a huge mess. Lets only talk a bit about the most unstable modes.

When we solve the equilibrium problem we saw there are a number of different forces on the plasma. We have field line “tension” which opposes the bending of magnetic field lines. The most dangerous mode is due to opposing the bending field line which has $k_\parallel \ll k_\perp$. There is the distinction of “rational” field lines which closes themselves, and “irrational” field lines which don’t. Only on rational field lines we can have $k_\parallel = 0$. Flute perturbations are unstable when the field line curves towards the plasma, but it its stable when the field line curvature is away from plasma.

Tokamak has good and bad curvature regions. Field lines will bend outwards in the bad curvature region to somewhat self-stabilize. If we have field-aligned perturbations at a given radius, but different pitch of magnetic field lines at neighboring radius it could also stabilize the perturbation.

Another kind of instability is kink instability. Kinks come from helical perturbation and is current driven. If we kink a magnetic field line bundle like a hose, the upper part of the kink has less magnetic pressure so it is even easier to kink upwards, which runs away. Again field line bending can stabilize this effect.

In theory if we want to find out whether the plasma is stable, we want to linearize the MHD equations and solve the linear problem as an eigenvalue problem. However in Tokamak plasmas all the modes couple, so it is not as easy to do the same thing. We use the so-called energy principle by looking at the energy and work of the displacements

$$\delta W(\xi^*, \xi) = -\frac{1}{2}\int d^3r \xi^* \cdot \mathbf{F}(\xi), \quad E(\xi^*, \xi) = \frac{1}{2}\int d^3r \rho_0 \dot{\xi}^* \cdot \dot{\xi} \tag{1.22}$$

and we use variational principle on

$$\Omega^2 = \frac{\delta W}{E}, \quad \delta\Omega^2 = 0 \tag{1.23}$$

If we expand the displacement into discretized modes, then we find

$$\delta W = \sum_n |a_n|^2 \omega_n^2 \tag{1.24}$$

When $\delta W < 0$ we have an instability, and if $\delta W \geq 0$ for all displacements then we have stability. The complication here is that in MHD we have a continuous spectrum of modes, so we need to do more work but the principle is similar.

In practice the problem is reduced to numerically minimizing the function

$$\delta W = \sum_n \sum_m a_n a_m \mathbf{F}(\xi_n, \xi_m) \tag{1.25}$$

I honestly don't understand what I'm writing...