Plasma Physics

Matthew Kunz

July 18, 2016

1 Lecture 1 - Magnetohydrodynamics (MHD)

The officially provided notes is at here.

	SW (1 au)	ICM ($\sim$ 100 kpc)	GC (0.1 pc)	ISM (warm)
T	10 eV	8 keV	2 keV	1 eV
n	$10\,\mathrm{cm}^{-3}$	$5\times10^{-3}\,\mathrm{cm}^{-3}$	$100\,\mathrm{cm}^{-3}$	$1\,\mathrm{cm}^{-3}$
B	$100\,\mu$G	$1\,\mu$G	1 mG	$5\,\mu$G
$v_{th,i}$	40 km/s	1000 km/s	600 km/s	10 km/s
v_A	70 km/s	30 km/s	200 km/s	10 km/s
$\beta_i = v_{th,i}^2/v_A^2$	$\sim 0.3-1$	$\sim 10^3$	~ 100	~ 1
λ_{mfp}	$\sim 0.1-1$ au	$\sim 0.1-10$ kpc	~ 0.01 pc	$\sim 10^{-7}$ pc
ρ_i	$\sim 10^{-7}$ au	~ 1 npc	~ 1 ppc	$\sim 10^{-11}$ pc

Table 1: Scales

You can't teach MHD in 1.5 hours, so we will just highlight something useful to most of us. The first thing we consider is what is called a Plasma. This is a difficult question to answer. Many different people identify themselves as plasma physicists. The above table is everything that can come into the umbrella of plasma physics. Below is a diagram (from the notes) that encompasses 35 orders of magnitudes in scale:

The one thing that is important is the line on the graph, which is the plasma parameter which is defined as:

$$g = n\lambda_D^3 = 1 \tag{1.1}$$

where λ_D is the Debye length that is defined by the screening of the charge in a plasma where

$$\phi \sim \frac{1}{r}e^{-r/\lambda_D} \tag{1.2}$$

The plasma parameter can be defined in various ways, and when we can treat plasma as having collective behavior when $g \gg 1$:

$$g \sim \frac{\mathrm{KE}}{\mathrm{PE}} \sim \frac{T}{e/\lambda_D} \sim \frac{\lambda_{mfp}}{\lambda_D} \gg 1 \tag{1.3}$$

In the table these are basically things that are interesting to us. SW refer to the solar wind, which we can measure with spacecrafts. Other interesting media we will see in following lectures are ICM (Intergalactic medium), GC (Galactic center), and ISM (Inter-stellar medium). The nice thing about the ISM is that everything is about 1.

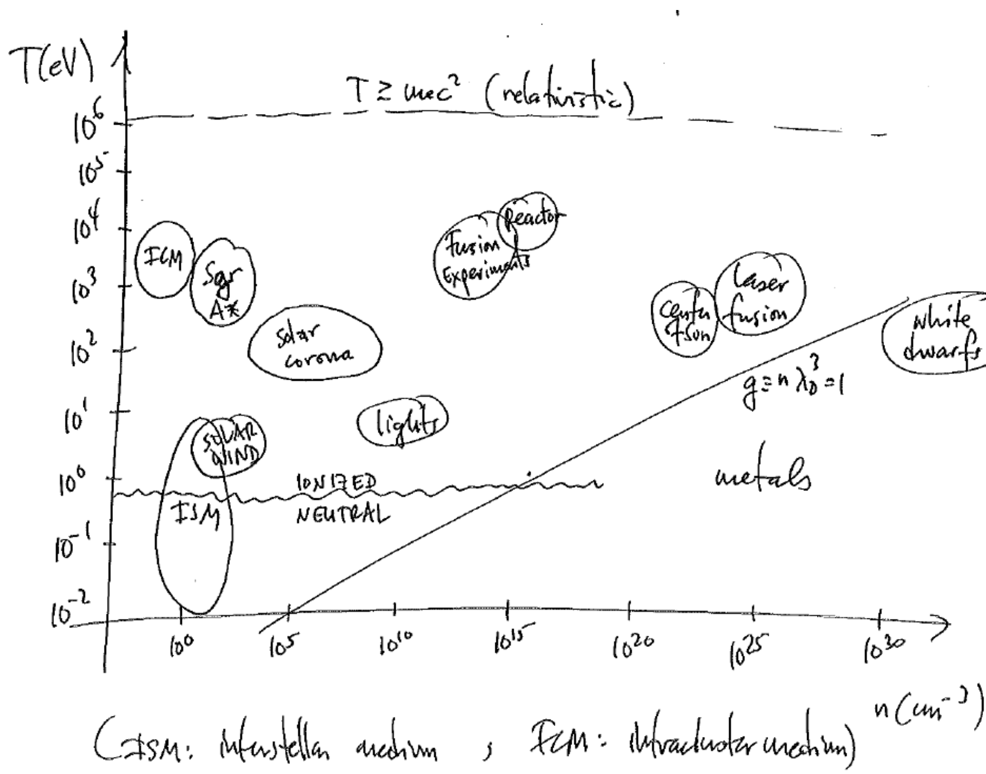

Figure 1.1: Scales of plasma physics

To get an idea of what the units mean, the Earth magnetic field is about 0.5 G. The thermal speed $v_{th,i}$ is a measure of the random motion of particles

$$v_{th,i} = \left(\frac{2T}{m_i}\right)^{1/2}, \quad f \sim e^{-v^2/v_{th}^2} \tag{1.4}$$

One more speed which is the Alfven speed

$$v_A = \frac{B}{\sqrt{4\pi mn}} \tag{1.5}$$

This is the speed that a disturbance propagates along the magnetic field. The beta parameter β_i is the ratio of the two velocities.

Now that we have defined what is a plasma, we need to talk about what is a fluid. This is where the mean free path comes in $\lambda_{mfp} \sim v_{th}\tau_{\rm col}$. This characterizes how often the particles scatter. When this path is small like ISM the medium behaves like a fluid because the distribution function looks very much like a Maxwellian. The larger this scale gets you start to question whether this medium behaves like a fluid.

The last length scale is called ρ_i which is $v_{th,i}/\Omega_i$ which is the thermal speed divided by the gyrofrequency of the ion. This is the size of a particle's gyration in a magnetic field. This scale is tiny compared to most of other scales, which is characteristic of astrophysical plasmas. This means the particles know about the magnetic field.

One more thing is the typical gyro-frequencies of particles. The table gives a nice impression of the separation of scales for astrophysical situations. In a typical plasma we should have $\lambda_{\text{Debye}} \ll \rho_i \ll \lambda_{mfp}$

Now we will just write down the equations for plasma. Lets fix some notations first. ϱ is different from ρ, where the later means charge density, and former means $\Sigma_s m_s n_s$.

The MHD equations

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0 \tag{1.6}$$

$$\frac{dM}{dt} = -\int \nabla \cdot (\varrho \mathbf{u})\, dV = -\oint \varrho \mathrm{u} \cdot d\mathbf{S} \tag{1.7}$$

$$\varrho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \mathbf{g}\varrho + \frac{\mathbf{J} \times \mathbf{B}}{c} \tag{1.8}$$

The third equation is basically Newton's second law. The left hand side is the comoving derivative which we can also write as $D\mathbf{u}/Dt$. The third term in the rhs is what makes MHD MHD, because we have Lorentz force. This is implicitly non-relativistic, $v/c \ll 1$.

Lets talk a bit about the Lorentz force. We can rewrite it as

$$\frac{\mathbf{J} \times \mathbf{B}}{c} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} = \frac{\mathbf{B} \cdot \nabla \cdot \mathbf{B}}{4\pi} - \nabla \frac{B^2}{8\pi} \tag{1.9}$$

the second term is like a pressure of the magnetic field.

One last equation we want to write down is the entropy equation

$$\frac{Ds}{Dt} = \frac{P}{\gamma - 1} \frac{D \ln P e^{-\gamma}}{Dt} = \dot{Q} \tag{1.10}$$

If we have a incompressible fluid $\nabla \cdot \mathbf{u} = 0$ then it implies $T = \text{const}$ in a fluid element.

One difficulty of the MHD arises from the vector derivative $\mathbf{u} \cdot \nabla \mathbf{u}$. It is very easy to drop terms when doing vector analysis. In cylindrical coordinates for example, when we decompose the terms we will get the Coriolis force and the centrifugal force. The details are sketched in the notes.

Now we have Maxwell equations:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.11}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}) \tag{1.12}$$

the last term is a way to abstract the electric field where we could take $\mathbf{E} = \eta \mathbf{J}$, which is just Ohm's law. The last term is a way to characterize resistive diffusion in the fluid. This term is not important when the magnetic Reynolds number is high.

The first term can be expanded using vector analysis

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{u} \tag{1.13}$$

There is also a last term but it is zero because $\nabla \cdot \mathbf{B} = 0$. The first term is called "stretching", the second "advection", and the third "compression".

One thing we can do is take this form of the induction equation, and the continuity equation and combine them. We then have

$$\frac{D}{Dt} \frac{\mathbf{B}}{\varrho} = \frac{\mathbf{B}}{\varrho} \cdot \nabla \mathbf{u} \tag{1.14}$$

This means that if we have a point $\mathbf{x}$ moving with velocity $\mathbf{u}$, and another point $\mathbf{x}+\boldsymbol{\xi}$ moving with $\mathbf{u}(\mathbf{x}+\boldsymbol{\xi})$ then we have

$$\frac{D\boldsymbol{\xi}}{Dt} \approx \boldsymbol{\xi} \cdot \nabla \mathbf{u} \tag{1.15}$$

What this means is that fluid elements that lie on field lines initially will remain on the field lines. This is called the Lindquist theorem, or "flux freezing".

We also have the Alfven theorem which states that, if we have some surface S and there are some magnetic field lines that threads the surface with $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$. If the above equation is true then the flux does not change with the flow of the fluid: $d\Phi/dt = 0$. The magnetic field is "frozen" into the fluid and if the fluid moves, the lines moves with it.

Now lets come back to the Lorentz force. Remember we have

$$\frac{\mathbf{J}\times\mathbf{B}}{c} = \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi} - \nabla\frac{B^2}{8\pi} = -\nabla\cdot\left[\frac{B^2}{8\pi}\mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi}\right] = -\nabla\mathbf{M} \tag{1.16}$$

where $\mathbf{M}$ is called the Maxwell stress tensor. The first term in the bracket is diagonal, and the second term is the thing that introduces anisotropy to the system by Lorentz force.

People usually say the first term is pressure and second term is tension but that is not exactly true. Lets define $\hat{b} = \mathbf{B}/B$. We can write the Lorentz force as

$$\mathbf{F} = \frac{B^2}{4\pi}\hat{b}\cdot\nabla\hat{b} - (\mathbf{I} - \hat{b}\hat{b})\cdot\nabla\frac{B^2}{8\pi} \tag{1.17}$$

The first term is apparently perpendicular to the field line, and can be thought of as the curvature force, as in the curved field line wants to be straightened. The second term creates pressure towards the region where magnetic field is small, wanting to spread out the field lines to be more even. This Lorentz force is the fundamental difference between MHD and hydro, because magnetic field doesn't want to be messed with, and it will direct the fluid.

An important part of MHD is the linear theory. We expand the quantities we are interested in around an "equilibrium":

$$\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}, \quad \varrho = \varrho_0 + \delta\varrho, \quad \mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}, \quad P = P_0 + \delta P \tag{1.18}$$

we use δ to denote Eulerian perturbation, and Δ to be Lagrangian perturbation, where the former means the perturbation in lab frame, and the latter means a change in the comoving frame of the fluid. Lets use Eulerian perturbations for now for simplicity.

We have some equilibrium state denoted by suffix zero, and introduce some perturbation to it, and in linear theory we just drop all the quadratic terms in perturbation: $\delta^2 \longrightarrow 0$. In a homogeneous stationary plasma, after we drop the quadratic terms we replace the perturbation with

$$\delta \sim e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} \tag{1.19}$$

and then just do linear algebra.

One thing we can get out of this is called the Alfven wave. This is when $\mathbf{k} = k_{\|}\hat{b}_0$. There is no change in $B_{\|}$ and ϱ. The dispersion relation is very simple

$$\omega = \pm k_{\|} v_A \tag{1.20}$$

This is the fundamental building block of plasma perturbations.

We can also have compression waves in the fluid, which is like sound wave

$$\omega = \pm k_{\|} c_s, \quad c_s = \left(\frac{\gamma P_0}{\varrho_0}\right)^{1/2} \tag{1.21}$$

where c_s is the speed of sound in the fluid.

We can also have a whole class of magnetic-sonic waves. Let $\mathbf{k} = k_{\|}\hat{b}_0 + \mathbf{k}_{\perp}$. In regions with high P and high B, we have fast wave, and in regions with small B we have slow wave. The details of these waves are in the notes.

There is also an entropy mode where $\omega = 0$. That is all the mode we want to talk about in MHD.

There is one more section which is called the Reduced MHD, but we are going to skip it because of time. It is a useful technique for doing asymptotic expansions.

When we wrote down all the equations we didn't talk about what was the velocity. In MHD it is technically defined as

$$\mathbf{u} = \frac{\sum_s m_s n_s \mathbf{u}_s}{\sum_s m_s n_s} \tag{1.22}$$

which basically means center-of-mass velocity. In the limit of $n_i/n_n \ll 1$ the fluid velocity is governed by the neutral particle velocity which may not make sense in MHD because neutral particles are not coupled to the magnetic field. That is why we need to talk about multi-fluid MHD.

If we derive everything from the Vlasov equation, we can see that we have continuity equation for every species

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0 \tag{1.23}$$

and force equation for every species

$$m_s n_s \frac{D\mathbf{u}_s}{Dt_s} = -\nabla P_s + q_s n_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c}\right) + \mathbf{R}_s \tag{1.24}$$

where the last term is the friction between species which add up to zero when we sum all species. We also have the entropy equation for each species.

When we write down the induction equation for multi-fluid plasma, what should we use as the plasma velocity $\mathbf{u}$? Suppose there is some frame where the magnetic field is frozen, such that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u}_f \times \mathbf{B}) \tag{1.25}$$

for some $\mathbf{u}_f$. We can write it as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [(\mathbf{u}_f - \mathbf{u}_e) \times \mathbf{B} + (\mathbf{u}_e - \mathbf{u}_i) \times \mathbf{B} + (\mathbf{u}_i - \mathbf{u}_n) \times \mathbf{B} + u_n \times \mathbf{B}] \tag{1.26}$$

The three terms can be associated with different things. $u_f - u_e$ is what we call Ohmic dissipation, which is like collisional resistivity. $u_i - u_n$ is associated with Ambipolar diffusion, and $u_i - u_e$ is associated with Hall effect. The Hall effect is related to nothing but the current $\mathbf{J} = en_i(\mathbf{u}_i - \mathbf{u}_e)$. Because $\mathbf{J} = \nabla \times \mathbf{B}$ this introduces wave dispersion in MHD.

Lets come back to ambipolar diffusion. When do we expect u_i and u_n to differ? This is related to the collision time scale between ions and neutral particles

$$\frac{u_i - u_n}{\tau_{ni}} \sim \frac{1}{\varrho_n}\left(\frac{\mathbf{J} \times \mathbf{B}}{c}\right) \tag{1.27}$$

This is a kind of anisotropic diffusion for the magnetic field. It is useful to re-derive the linear theory with these kind of drift speed between different species of particles, and see what happens to different kinds of waves.

Where can we use these MHD? In solar wind people have used reduced MHD but that's about it, because solar wind the fluid is not very collisional. In ICM the collision is not quite strong enough. In GC the mean free path is about the Bondi radius, especially if we keep going in near the black hole it is worse. ISM has a cold and a warm phase, but both are pretty good for MHD.

2 Lecture 2 - Kinetics

Lets start with a recap. MHD is a fluid theory, where λ_{Debye} is infinitesimal, meaning the fluid is quasi neutral. Another thing is that the ion gyro path is infinitesimal. The third quantity is the mean free path which is infinitesimal. MHD is a fluid theory of infinitesimal fluid elements subject to the Lorentz force.

What if we want to relax some of the above assumptions? Suppose we are dealing with ICM, SW, GC, etc. which are not in the ideal MHD regime. There are a number of different ways to relax the assumptions:

- Tracing particle motion, guiding center motion. Most of the kinetic discussions start with particle motion.
- Adiabatic invariance, pressure anisotropy. Deviations from Maxwell distribution.
- Braginskii-MHD and "CGL" equations (monograph from 1965).
- What if collisions are not strong enough? We need to give up MHD as a theory:
- Vlasov-Landau kinetic equation. Dealing with distribution functions directly.
- Ordering parameters. For the system of interest, what are the small parameters so that we can expand our system of equations in terms of these small parameters?
- Kinetic MHD:Barnes & Landau damping, Firehose & mirror instabilities.
- Preview of gyrokinetics. This is an advanced tool that astrophysics should pick up in their toolset. A simple derivation is in the notes.

2.1 Particle Motion

There are two types of particle motion in EM field. If we have a particle at position $\mathbf{r}$, which is gyrating along some $\mathbf{B}$ field. It makes a lot of sense to define another vector $\mathbf{R}$ which is the center of the Larmor motion of the particle. The motion of the particle can be decomposed into the motion of the guiding center and the gyration itself. Let $\mathbf{r} = \mathbf{R} + \rho$. We have

$$\rho = -\frac{\mathbf{v}\times\hat{b}}{\Omega}, \quad \hat{b} = \frac{\mathbf{B}}{B}, \quad \Omega = \frac{qB}{mc} \tag{2.1}$$

If B and E are uniform and time independent, we can easily write down how the guiding center evolves

$$\begin{aligned}\dot{\mathbf{R}} &= \dot{\mathbf{r}} - \dot{\rho} \\ &= \mathbf{v} + \frac{q}{m}\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right] \times \frac{\hat{b}}{\Omega} \\ &= v_\parallel \hat{b} + c\frac{\mathbf{E}\times\mathbf{B}}{B^2}\end{aligned} \tag{2.2}$$

The second term is called the "E cross B" drift. To get an idea of what it does, consider a particle in a B field which simply gyrates in a circle. If we add E field to the system. We know that $\rho = v_\perp/\Omega$. As particle gyrates it gets accelerated so ρ increases, but when it comes back it decelerates. So on average its gyration is drifting in the $\mathbf{E}\times\mathbf{B}$ direction.

The important thing about this drift is that it is species-independent. This means that this drift introduces no current. For general force $\mathbf{F} \neq q\mathbf{E}$, we can generalize the formula for the drift to be

$$\dot{\mathbf{R}}_{\mathrm{dirft}} = \frac{c\mathbf{F}\times\mathbf{B}}{qB^2} \tag{2.3}$$

Now lets make the magnetic field non-uniform but still static. Lets repeat the whole calculation. The total derivative is with respect to the particle trajectory $d/dt \to \partial/\partial t + \mathbf{v}\cdot\nabla$

$$\begin{aligned}\dot{\mathbf{R}} &= \dot{\mathbf{r}} - \dot{\rho} \\ &= v_\parallel \hat{b} + \frac{c\mathbf{E}\times\mathbf{B}}{B^2} + \mathbf{v}\times\frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right)\end{aligned} \tag{2.4}$$

The last term is new. It will give two things: grad-B drift and the curvature drift. From the mathematics view it is just vector calculus, but we can reason it from the physics point of view. We associate a force to when B is non-uniform, and a force to when B has some curvature. When B is not uniform, remember

$$\mathbf{F} = \nabla\left(\mu\cdot\mathbf{B}\right), \quad \mu = -\frac{1}{2}\frac{mv_\perp^2}{B}\hat{b} \tag{2.5}$$

If we put this into the formula we get

$$\hat{\mathbf{v}}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\hat{b}\times\nabla\ln\mathbf{B} \tag{2.6}$$

This means that the particle drifts perpendicular to the direction where there is gradient of B. The curvature drift is similar with

$$\hat{\mathbf{v}}_{\mathrm{curv}} = \frac{mv_\perp^2}{r_c}\hat{r}_c \tag{2.7}$$

2.2 Adiabatic Invariance

In plasma physics adiabatic invariance is associated with some periodic motion that particles undergo. One kind of such motion is simply the gyration around B fields. Another kind of periodic motion is associated with the drifts, for example a particle bouncing in a magnetic bottle. We will now talk about this kind of motion in detail. This is similar to the Poincare invariant in kinematics $I = \oint \mathbf{p}\cdot d\mathbf{q}$.

We wrote down the magnetic moment of the particle $\mu = mv_\perp^2/2B$. This is actually the first adiabatic invariant. If for example a particle is gyrating in a uniform B field, and we very slowly decrease the magnetic field. We can see that the perpendicular velocity of the particle will adjust and decrease as well, so that this quantity is conserved. Another way to look at it is that magnetic flux in the Larmor orbit is conserved.

One can show that μ is conserved to the precision of $e^{-\Omega/\omega}$ where ω is the rate of change of the magnetic field. We call this conservation "to all orders" since it is exponential.

Another conserved quantity is $J = \oint mv_\parallel d\ell_B$ where ℓ_B is the path along B and $v_\parallel$ is the guiding center motion parallel to the magnetic field. Consider a magnetic bottle. Since energy of the particle is conserved and also μ is conserved, so $v_\parallel^2$ will decrease in regions of large B, and particles will eventually reflect. In this kind of periodic mirroring motion, J is conserved.

2.3 Anisotropy

If we look at the magnetic moment of a species of particles and take the expectation value

$$\langle \mu \rangle = \frac{\int \mu f d^3V}{\int f d^3V} = \frac{\int \frac{1}{2}\frac{mv_\perp^2}{B} f d^3V}{n} = \frac{P_\perp}{nB} = \frac{T_\perp}{B} \tag{2.8}$$

One can do the same thing with J:

$$\left\langle J^2 \right\rangle = \frac{\int m^2 v_\parallel^2 \ell_B^2 f d^3V}{\int f d^3V} = \frac{T_\parallel B^2}{n^2} \tag{2.9}$$

where we choose $\ell = B/n$. These are the two most important quantities when one goes to weakly collisional systems. This means that if we adjust B or n then the perpendicular/parallel motion of the particles will collectively adjust.

In terms of a fluid we can write this in a nicer way

$$\frac{D}{Dt}\left(\frac{P_\perp}{nB}\right) \approx 0, \quad \frac{D}{Dt}\left(\frac{P_\parallel B^2}{n^3}\right) \approx 0 \tag{2.10}$$

These are called double adiabatic laws and replaces the equation of entropy. A nice paper discussing this is (Chew, Goldberger, & Low 1956). The total P can be split such that $P = 2P_\perp/3 + P_\parallel/3$.

We now want to introduce a right hand side to these equations. We take the difference of the equations we get

$$\left[\frac{D}{Dt} + \frac{D(\ln Bn^{-7/3})}{Dt}\right](P_\perp - P_\parallel) = 3P\frac{D\ln Bn^{-2/3}}{Dt} + \text{heat flows} + \text{cooling/heating} - \nu(P_\perp - P_\parallel) \tag{2.11}$$

This is one version of the so-called CGL equations.

One distribution function that exhibits this anisotropy is the "bi-Maxwellian" distribution

$$\frac{f}{n} = \frac{e^{-v_\perp^2/v_{th\perp}^2} e^{-v_\parallel^2/v_{th\parallel}^2}}{\pi^{3/2} v_{th\perp}^2 v_{th\parallel}} \tag{2.12}$$

Consider the limit where the frequency of time changes $\omega \sim D/Dt \ll \nu_{\text{coll}} \ll \Omega$. This limit is good for ICM, GC, etc. The entire left hand side of the above equation can be dropped, and the largest term is the ν term on the right.

This limit is considered in (Braginskii 1965). We can rewrite the equation as

$$P_\perp - P_\parallel = \frac{3P}{\nu}\frac{D\ln}{Dt}\frac{B}{n^{2/3}} = \frac{3P}{\nu}\left(\hat{b}\hat{b} - \frac{1}{3}\mathbf{I}\right) : \nabla\mathbf{u} \tag{2.13}$$

where $\mathbf{ab} : \nabla\mathbf{c} = a_i b_j \partial c_j / \partial x_i$.

Lets try to understand how this connects to our previous discussions. One way to include this anisotropy in our fluid equations is to promote P to a tensor:

$$\mathbf{P} = P_\perp\left(\mathbf{I} - \hat{b}\hat{b}\right) + P_\parallel \hat{b}\hat{b} \tag{2.14}$$

The equation can be closed because of our previous equation. This new pressure term introduces anisotropy viscosity. It is also called the "Braginskii viscosity".

What this equation says is that there is viscosity, but the viscosity only acts along the magnetic field lines, because with the ordering it is incredibly difficult to communicate between the field lines due to particles restrained to gyrate around the field lines in very small orbits.

This is already very different from MHD, where the only anisotropy is due to the Lorentz force. Now if there is heat flux in the system it is subject to the same constraint of flowing along the magnetic field:

$$\mathbf{Q} = -\chi\hat{b}\hat{b}\cdot\nabla T \tag{2.15}$$

Notice the appearance of the tensor $\hat{b}\hat{b}$. This is called anisotropic conduction.

2.4 Collision Not Strong Enough

We picked up the first order correction to MHD which is pressure anisotropy. Let us now talk about when collision is not enough and we need to consider kinetics. However kinetics is a very deep subject because one doubles the dimensions of the problem by introducing distribution functions.

The kinetic equation is a statistical description of the plasma. The problem about simulating a system of particles directly is that we have too many particles ($\sim 10^{28}$). Another problem is that it is very sensitive to initial conditions because particle trajectories are chaotic. The third problem is that we are not interested in the individual particles but only some collective quantities, so why trace them individually in the first place?

We instead describe a large number of particles using their collective statistical properties which is their distribution in the phase space

$$\int f(t,\mathbf{x},\mathbf{v})\,d^3v = n \tag{2.16}$$

$$\int f\mathbf{v}\,d^3v = n\mathbf{u} \tag{2.17}$$

$$\int f\frac{1}{2}mv_\perp^2\,d^3v = P_\perp \tag{2.18}$$

The equation for the distribution function is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{a}\cdot\frac{\partial f}{\partial\mathbf{v}} = C[f] \tag{2.19}$$

This is called the Vlasov equation where the right hand side is a functional describing the collisional terms. It has 3 spatial and 3 velocity dimensions, so solving it is very expensive. In addition this is coupled to Maxwell equations, which we need to solve simultaneously.

This equation is extremely rich, even when doing linear theory. What happens if we don't want to solve this? We can remove some degrees of freedom, reduce dimensions, or find other approximations. One way is to find small parameters $\epsilon \ll 1$. One possible one is that $\epsilon \sim \omega/\Omega$, meaning the rate of change is small compared to gyro frequency. It could be $\epsilon \sim \rho/\ell$, $\epsilon \sim v/c$, etc. These are starting points of kinetic MHD, and gyrokinetics. The simple derivations of these are in the notes.

In kinetic MHD, we have $\rho/\ell \sim \epsilon$ where ℓ is any scale in the problem. However in gyrokinetics we have $\rho/\rho_\parallel \sim \epsilon$ but $\rho/\rho_\perp \sim 1$. This is an important distinction between them.

2.5 Instabilities

We will briefly cover some instabilities. First one is the Firehose instability. It is basically Alfven waves when

$$P_\parallel > P_\perp + \frac{B^2}{4\pi} \tag{2.20}$$

When this happens the magnetic field will develop wild oscillations.

Another is the Mirror instability which is associated with the growth of slow modes. If we have a series of magnetic mirrors and if we squeeze the fields the particles will move away from that point. When

$$P_\perp > P_\parallel + \frac{B^2}{8\pi} \tag{2.21}$$

the mirrors will grow and become unstable.