Numerical calculation of vector light fields in a high NA microscope (Advanced Measurements and Laboratory Practice–3312032)

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1 Background

Light propagation theory in an optical microscopy is essential to understand the basic principle of an optical imaging system and its resolution. To increase optical resolving power, an objective of high numerical aperture (NA)[1] is generally used so that more light illuminates to (or is collected from) a sample under investigation. Through a high NA microscope objective, a light beam is tightly focused. The theoretical understanding of tight focusing [Zha09] requires knowledge of wave optics where vectorial behavior of the light field shall be taken into account.

In this work, it is not a requirement to go through all derivations in detail, instead you should concentrate on the *numerical evaluation of the vector fields in the focal region* (Section 5.2 in [Zha09] or 3.6 and 3.7 of the lecture note "Chapter 3: Propagation and focusing of optical fields"[2] in [OPT]). Nevertheless, you should familiarize briefly with the tight focusing process which consists of three main parts, as illustrated in Fig. 1.

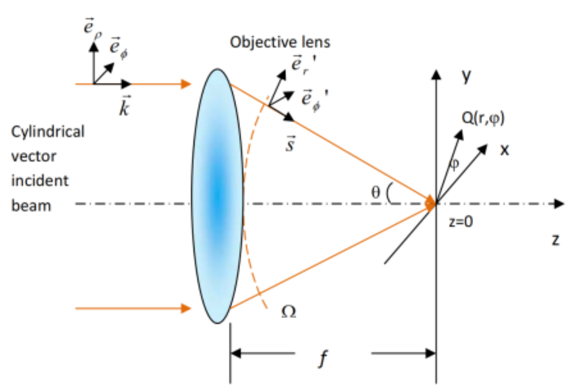


Fig. 1. Focusing of a vector beam with cylindrical symmetry [Zha09].

Before passing through the objective, the light beam is still treated precisely enough under paraxial approximation (Eqs. from 2.1 to 2.8 in [Zha09]). Then, refraction happens at the interface of the objective lens, and the refracted field is mapped onto an imaginary reference sphere with radius equal to the corresponding focal length (Eqs. from

[1] https://www.leica-microsystems.com/science-lab/collecting-light-the-importance-of-numerical-aperture-in-microscopy/
[2] https://www.photonics.ethz.ch/fileadmin/user_upload/Courses/NanoOptics/focus3.pdf

5.5 to 5.10 in [Zha09]). The introduction of such a reference sphere is possible for optical imaging systems where the objective lens is considered aplanatic. At last, the vector fields on the reference sphere, with spatially varying polarization states and amplitudes, propagate towards to the focus. In the focal region, the vector field is thus formed (Eq. 5.11 in [Zha09]), and in some cases the longitudinal field component (along the optical axis) can be significantly large.

2 Numerical calculation of the vector beams in focal region

2.1 Beam-like incident field

Regarding its axial symmetry in the optical imaging system, the paraxial incident beam is conveniently written in cylindrical coordinates (ρ, φ, z)

$$\vec{E}_i(\rho, \varphi) = l_0 P(\rho)[\cos\varphi_0 \vec{e}_\rho + \sin\varphi_0 \vec{e}_\varphi], \tag{1}$$

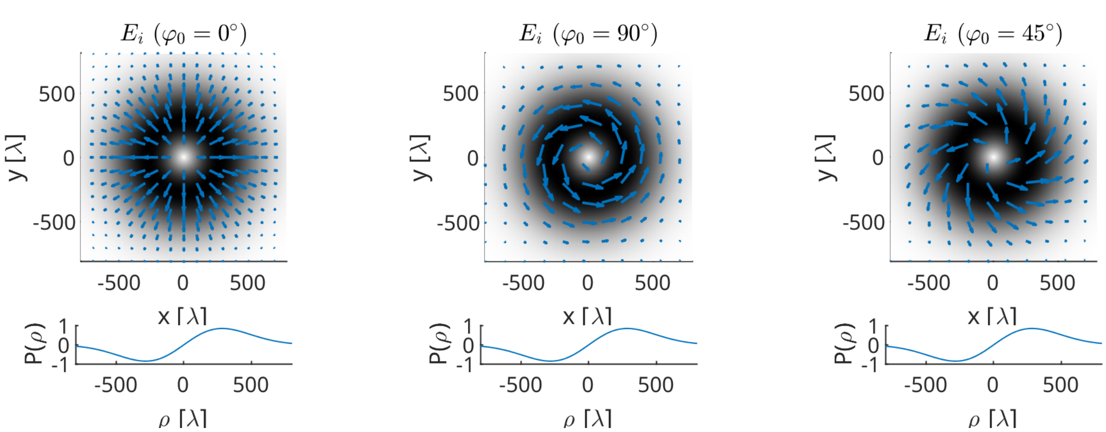
where

- l_0 denotes the peak field amplitude in the pupil plane,
- $P(\rho)$ is the field profile of radial dependence normalized to l_0 in the pupil plane,
- $\vec{e}_\rho$ and $\vec{e}_\varphi$ are the corresponding unit vectors of the cylindrical coordinate system.

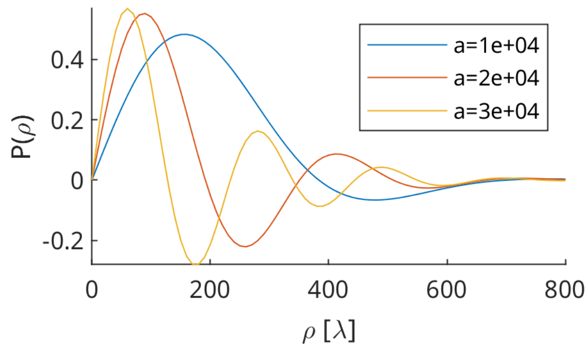
An example for the radial field profile or pupil function is $P(\rho) = \frac{\rho}{w} e^{-(\rho/w)^2}$, where ρ is the radius and w is the beam waist radius. It is indeed a modulated Gaussian profile with a null field in the center (i.e., on optical axis), shown in the grayscale colormap in the following figures. In deed, this is the field profile of a Laguerre-Gaussian beam.

The angle φ_0 is measured with respect to the radial unit vector, and it controls the ratio of radial to azimuthal field component. The following figures illustrate the incident vector field distributions in the waist plane (which is perpendicular to the optical axis) when the pupil function abovementioned is used. The wavelength used for plotting the figures is $\lambda = 1$ um, i.e., 10^{-6} m, the range of plotting are $x \in [-0.8, 0.8]$ mm, $y \in [-0.8, 0.8]$ mm as $800\lambda = 0.8 \times 10^{-3}$ m, and the beam waist radius is 0.4 um.

From left to right, you can see that the electric fields are polarized along the radial direction for $\varphi_0 = 0°$, along the azimuthal direction for $\varphi_0 = 90°$, and along the direction of $45°$ to either the radial or azimuthal unit vector.



Another example for the radial field profile is $P(\rho) = J_1(a\rho)e^{-(\rho/w)^2}$, where a is a scaling factor that controls the beam waist size and J_1 is the Bessel function of the first kind of the 1th-order. And this is a Bessel-Gaussian field profile. See how the radial field profile changes for different scaling factor a.



2.2 Refraction

After refracted by the high NA objective lens, the radial field profile in the pupil plane is mapped to an apodization function that is converging angle θ-dependent,

$$P(\rho) \mapsto P(\theta) = P(f \sin\theta)\sqrt{\cos\theta}. \tag{2}$$

Take the pupil function we used above, it is mapped to the following form

$$P(\theta) = P(f \sin\theta)\sqrt{\cos\theta} = \frac{2f \sin\theta}{w} e^{-(f \sin\theta/w)^2} \sqrt{\cos\theta}. \tag{3}$$

This function changes accordingly when another radial field profile $P(\rho)$ is used for the incident field $\vec{E}_i$. But, the essential process is to replace ρ with $f \sin\theta$.

And if the Bessel-Gaussian beam profile is used, the pupil function is then mapped to

$$P(\theta) = J_1(af \sin\theta)e^{-(f \sin\theta/w)^2} \sqrt{\cos\theta}. \tag{4}$$

In addition to the change in field amplitude, the polarization is changed as well during the refraction

$$\vec{e}_r' = \cos\theta \vec{e}_\rho + \sin \vec{e}_z, \tag{5}$$

$$\vec{e}_\varphi' = \vec{e}_\varphi. \tag{6}$$

Together, the refraction process yields the following field on the reference sphere

$$\vec{a}(\theta, \varphi) = l_0 f P(\theta)[\cos\varphi_0 \vec{e}_r' + \sin\varphi_0 \vec{e}_\varphi']. \tag{7}$$

2.3 Focal field

The electric in the focal region, so-called focal field, can be obtained through Richards-Wolf vector diffraction theory, i.e., integrating the vector field on the reference sphere [provided in Eq. (7)],

$$\vec{E}(r,\phi,z) = \frac{-ik}{2\pi}\int_0^{\theta_{\max}}\int_0^{2\pi}\vec{a}(\theta,\varphi)e^{ik(\vec{s}\cdot\vec{r})}\sin\theta d\varphi d\theta, \tag{8}$$

where the integration $\int_0^{\theta_{\max}}\int_0^{2\pi}[]\sin\theta d\varphi d\theta$ is performed over the reference sphere with the maximal angle $\theta_{\max}$ determined by the NA value of the objective lens, k denotes the wavenumber, $\vec{s}$ is the unit vector (as shown in Fig. 1) with its direction pointing into the focus and originating from an infinitesimal source in a tiny area $d\Omega = \sin\theta d\varphi d\theta$, and $\vec{r}$ is the radial vector of an observation point in the focal region and $r = \sqrt{x^2+y^2}$.

After some mathematical treatments, the focal field is rewritten in the following form

$$\begin{aligned}\vec{E}(r,\phi,z) = \frac{-ikfl_0}{2\pi}\int_0^{\theta_{\max}}\int_0^{2\pi}P(\theta)\sin\theta e^{ik[z\cos\theta+r\sin\cos(\varphi-\phi)]}\\ \times\left[\cos\varphi_0\begin{pmatrix}\cos\theta\cos(\varphi-\phi)\vec{e}_r\\ 0\vec{e}_\phi\\ \sin\theta\vec{e}_z\end{pmatrix}+\sin\varphi_0\begin{pmatrix}0\vec{e}_r\\ \cos(\varphi-\phi)\vec{e}_\phi\\ 0\vec{e}_z\end{pmatrix}\right]d\varphi d\theta,\end{aligned} \tag{9}$$

which can be further simplified as

$$\vec{E}(r,\phi,z) = E_r\vec{e}_r + E_\phi\vec{e}_\phi + E_z\vec{e}_z, \tag{10}$$

where

$$E_r(r,\phi,z) = kfl_0\cos\varphi_0\int_0^{\theta_{\max}}P(\theta)\sin\theta\cos\theta J_1(kr\sin\theta)e^{ikz\cos\theta}d\theta, \tag{11}$$

$$E_\phi(r,\phi,z) = kfl_0\sin\varphi_0\int_0^{\theta_{\max}}P(\theta)\sin\theta J_1(kr\sin\theta)e^{ikz\cos\theta}d\theta, \tag{12}$$

$$\text{and } E_z(r,\phi,z) = ikfl_0\cos\varphi_0\int_0^{\theta_{\max}}P(\theta)\sin^2\theta J_0(kr\sin\theta)e^{ikz\cos\theta}d\theta \tag{13}$$

are three field components expanded in cylindrical coordinates of the focal region, $J_n(x)$ is the Bessel function of the first kind of nth order.

2.4 Numerical integration over θ-anlge

It is recognized from Eqs. (11), (12) and (13), you should numerically evaluate the following three integrals,

$$I_r = \int_0^{\theta_{\max}} P(\theta) \sin\theta \cos\theta J_1(kr\sin\theta) e^{ikz\cos\theta} d\theta, \tag{14}$$

$$I_\phi = \int_0^{\theta_{\max}} P(\theta) \sin\theta J_1(kr\sin\theta) e^{ikz\cos\theta} d\theta, \tag{15}$$

$$I_z = \int_0^{\theta_{\max}} P(\theta) \sin^2\theta J_0(kr\sin\theta) e^{ikz\cos\theta} d\theta, \tag{16}$$

after which you should be able to calculate all three components of the vector beam in the focal region. There is no restriction on the choice of any programming language. As an example, however, Matlab provides a builtin function 'integral'[3] to calculate integration numerically.

Assume the objective has an NA of 0.9, the emersion medium is simply air, and the effective focal length is $f = 1$ mm, what is the distribution of each field component in the focal region including the xy-plane and rz-plane with r denoting the radial axis? The discussion can be made by using similar parameters for the waist radius, radial field profile, and wavelength, etc. Pay attention that the beam waist diameter should be appropriately choosen, i.e., smaller than the aperture diameter so that most of the incident light passes through the objective but large enough so that the most of the light is tightly focused.

References

[OPT] OPTICS: Nano-Optics 2018. https://www.photonics.ethz.ch/our-range/education/courses/nanooptics2/. 1

[Zha09] Qiwen Zhan. Cylindrical vector beams: from mathematical concepts to applications. *Advances in Optics and Photonics*, 1(1):1–57, January 2009. 1, 1

[3] https://www.mathworks.com/help/matlab/ref/integral.html