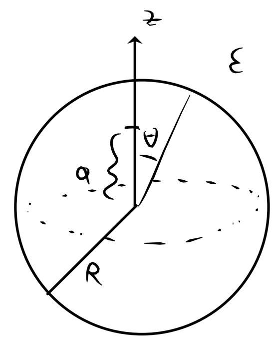
1.



symetria względem φ

nie wybucha w $r = 0$

$$\Phi(\bar r) = \begin{cases} \frac{q}{4\pi\varepsilon_0}\frac{1}{|\bar r-\bar a|} + \sum\limits_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & \text{dla } r<R \\ \frac{q}{4\pi\varepsilon_0\varepsilon}\frac{1}{|\bar r-\bar a|} + \sum\limits_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta) & \text{dla } r>R \end{cases}$$

nie wybucha w $r \to \infty$

Warunki brzegowe:

1) $\Phi(\bar R_-) = \Phi(\bar R_+)$

2) $\bar D_\perp(\bar R_-) = \bar D_\perp(R_+) \Rightarrow \left.\frac{\partial\Phi}{\partial r}\right|_{r=R_-} = \varepsilon \left.\frac{\partial\Phi}{\partial r}\right|_{r=R_+}$

1) $\Phi(\bar R_-) = \Phi(\bar R_+)$ $\quad R>a \Rightarrow \frac{1}{|\bar R-\bar a|} = \frac{1}{R}\sum\limits_{l=0}^{\infty}\left(\frac{a}{R}\right)^l P_l(\cos\theta)$

$$\sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\left(\frac{a}{R}\right)^l + A_l R^l\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\left(\frac{q}{4\pi\varepsilon_0\varepsilon}\frac{1}{R}\left(\frac{a}{R}\right)^l + B_l\frac{1}{R^{l+1}}\right)P_l(\cos\theta)$$

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\left(\frac{a}{R}\right)^l + A_l R^l = \frac{q}{4\pi\varepsilon_0\varepsilon}\frac{1}{R}\left(\frac{a}{R}\right)^l + B_l\frac{1}{R^{l+1}}$$

$$B_l = \frac{q}{4\pi\varepsilon_0}a^l\left(1-\frac{1}{\varepsilon}\right) + A_l R^{2l+1}$$

2) $$\left.\frac{\partial\Phi}{\partial r}\right|_{R_-} = \sum_{l=0}^{\infty}\left(\frac{-q}{4\pi\varepsilon_0}(l+1)\frac{1}{R^2}\left(\frac{a}{R}\right)^l + A_l l R^{l-1}\right)P_l(\cos\theta)$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{R_+} = \sum_{l=0}^{\infty}\left(\frac{-q}{4\pi\varepsilon_0\varepsilon}(l+1)\frac{1}{R^2}\left(\frac{a}{R}\right)^l - B_l(l+1)\frac{1}{R^{l+2}}\right)P_l(\cos\theta)$$

$$\cancel{\frac{-q}{4\pi\varepsilon_0}(l+1)\frac{1}{R^2}\left(\frac{a}{R}\right)^l} + A_l l R^{l-1} = \cancel{\frac{-q}{4\pi\varepsilon_0}(l+1)\frac{1}{R^2}\left(\frac{a}{R}\right)^l} - B_l\varepsilon(l+1)\frac{1}{R^{l+2}}$$

$$A_l l R^{l-1} = -B_l\varepsilon(l+1)\frac{1}{R^{l+2}}$$

$$A_l l R^{l-1} = -\left(\frac{q}{4\pi\varepsilon_0}a^l\left(1-\frac{1}{\varepsilon}\right) + A_l R^{2l+1}\right)\varepsilon(l+1)\frac{1}{R^{l+2}}$$

$$A_l l R^{l-1} = \frac{q}{4\pi\varepsilon_0}\frac{1}{R^2}\left(\frac{a}{R}\right)^l(1-\varepsilon)(l+1) - A_l(l+1)\varepsilon R^{l-1}$$

$$\left\{ A_l = \frac{q}{4\pi\varepsilon_0}\frac{a^l}{R^{2l+1}}\frac{(1-\varepsilon)(l+1)}{l+\varepsilon(l+1)} \right.$$

$$B_l = \frac{q}{4\pi\varepsilon_0}a^l\left(1-\frac{1}{\varepsilon}\right) + \frac{q}{4\pi\varepsilon_0}a^l\frac{(1-\varepsilon)(l+1)}{l+\varepsilon(l+1)}$$

$$\left\{ B_l = \frac{q}{4\pi\varepsilon_0}a^l\left(1-\frac{1}{\varepsilon}\right)\left(\frac{l}{l+\varepsilon(l+1)}\right) \right.$$

a)
$$\Phi(\bar r) = \begin{cases} \frac{q}{4\pi\varepsilon_0}\frac{1}{|\bar r-\bar a|} + \sum\limits_{l=0}^{\infty}\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\left(\frac{ar}{R^2}\right)^l\frac{(1-\varepsilon)(l+1)}{l+\varepsilon(l+1)}P_l(\cos\theta) & \text{dla } r<R \\ \frac{q}{4\pi\varepsilon_0\varepsilon}\frac{1}{|\bar r-\bar a|} + \sum\limits_{l=0}^{\infty}\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\left(\frac{a}{R}\right)^l\left(1-\frac{1}{\varepsilon}\right)\left(\frac{l}{l+\varepsilon(l+1)}\right)P_l(\cos\theta) & \text{dla } r>R \end{cases}$$

b) Przeżywa tylko wyraz $l=0$

$$\Phi(0) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} + \frac{1}{R}\left(\frac{1-\varepsilon}{\varepsilon}\right)\right)$$

c) $$\frac{-\sigma}{\varepsilon_0} = \left.\frac{\partial\Phi}{\partial r}\right|_{R_+} - \left.\frac{\partial\Phi}{\partial r}\right|_{R_-}$$

$$-\frac{\sigma}{\varepsilon_0} = \sum_{l=0}^{\infty}\left(\frac{-q}{4\pi\varepsilon_0\varepsilon}(l+1)\frac{1}{R^2}\left(\frac{a}{R}\right)^l - B_l(l+1)\frac{1}{R^{l+2}} + \frac{q}{4\pi\varepsilon_0}(l+1)\frac{1}{R^2}\left(\frac{a}{R}\right)^l - A_l l R^{l-1}\right)P_l(\cos\theta)$$

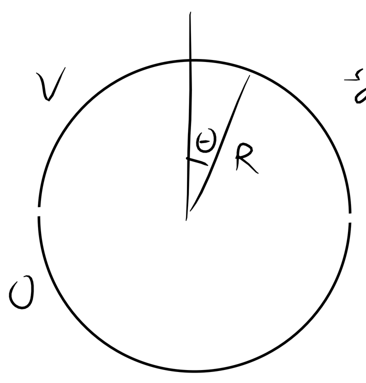
$$-\sigma = \frac{q}{4\pi R^2}\sum_{l=0}^{\infty}\left(-\frac{1}{\varepsilon}(l+1)\left(\frac{a}{R}\right)^l - (l+1)\left(\frac{a}{R}\right)^l\left(1-\frac{1}{\varepsilon}\right)\left(\frac{l}{l+\varepsilon(l+1)}\right) + (l+1)\left(\frac{a}{R}\right)^l - \left(\frac{a}{R}\right)^l l\frac{(1-\varepsilon)(l+1)}{l+\varepsilon(l+1)}\right)P_l(\cos\theta)$$

$$-Q = \int_0^{\pi}\int_0^{2\pi}\sigma R^2\sin\theta\, d\varphi\, d\theta = \frac{q}{2}\sum_{l=0}^{\infty}\int_0^{\pi}P_l(\cos\theta)\sin\theta\, d\theta =$$

$$= \frac{q}{2}\sum_{l=0}^{\infty}\int_{-1}^{1}P_l(\xi)P_0(\xi)\,d\xi = \frac{q}{2}\sum_{l=0}^{\infty}\frac{2}{2l+1}\delta_{l0} = q\,\Big|_{l=0}$$

$$-\frac{Q}{q} = 1-\frac{1}{\varepsilon} \xrightarrow{\varepsilon=5} \frac{Q}{q} = \frac{-4}{5}$$

2.



symetria względem φ

$$\begin{cases} \Phi(r>R,\theta) = \sum\limits_{l=0}^{\infty}A_l\frac{1}{r^{l+1}}P_l(\cos\theta) \\ \Phi\left(R_+, \theta\in\left[0;\frac{\pi}{2}\right[\right) = V \\ \Phi\left(R_+, \theta\in\left]\frac{\pi}{2};\pi\right]\right) = 0 \end{cases}$$

$$\int_0^{\pi}\Phi\, P_{l'}(\cos\theta)\sin\theta\, d\theta = V\int_0^{\frac{\pi}{2}}P_{l'}(\cos\theta)\sin\theta\, d\theta + \int_{\frac{\pi}{2}}^{\pi}0\, d\theta$$

$$\sum_{l=0}^{\infty}A_l\frac{1}{R^{l+1}}\frac{2}{2l+1}\delta_{ll'} = V\int_0^1 P_{l'}(x)\,dx$$

Dokładność do $\frac{1}{r^4} \Rightarrow l'\in[0;3]$

$l'=0$: $\quad 2A_0\frac{1}{R} = V \Rightarrow A_0 = \frac{1}{2}VR$

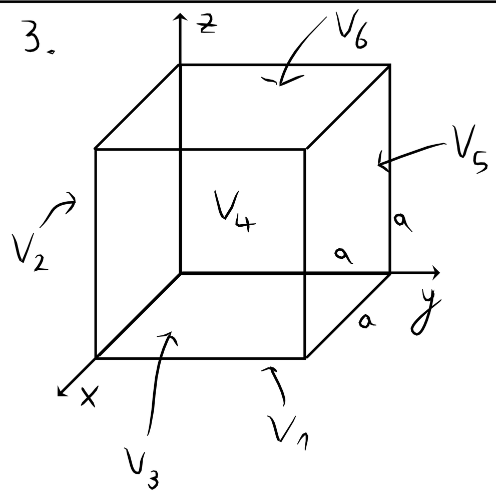
$l'=1$: $\quad \frac{2}{3}A_1\frac{1}{R^2} = V\int_0^1 x\,dx = \frac{1}{2}V \Rightarrow A_1 = \frac{3}{4}VR^2$

$l'=2$: $\quad \frac{2}{5}A_2\frac{1}{R^3} = V\int_0^1\frac{1}{2}(3x^2-1)\,dx = 0 \Rightarrow A_2 = 0$

$l'=3$ $\quad \frac{2}{7}A_3\frac{1}{R^4} = V\int_0^1\frac{1}{2}(5x^3-3x)\,dx = -\frac{1}{8}V \Rightarrow A_3 = -\frac{7}{16}VR^4$

$$\Phi(r\gg R,\theta) = V\left(\frac{1}{2}\frac{R}{r} + \frac{3}{4}\left(\frac{R}{r}\right)^2\cos(\theta) - \frac{7}{32}\left(\frac{R}{r}\right)^4\left(5\cos^3(\theta) - 3\cos(\theta)\right)\right)$$

3.



Osobno rozpatruję (V_1, V_6); (V_2, V_5); $(V_3; V_4)$

I (V_1, V_6) $\quad \phi(x,y,z) = A(x) + B(y) + C(z)$

$$0 = \Delta\Phi = \frac{1}{A}\frac{\partial^2 A}{\partial x^2} + \frac{1}{A}\frac{\partial^2 B}{\partial y^2} + \frac{1}{A}\frac{\partial^2 C}{\partial z^2} = \alpha+\beta+\gamma$$

$$\begin{cases} A(0)=0 & B(0)=0 & C(0)=V_1 \\ A(a)=0 & B(a)=0 & C(a)=V_2 \end{cases}$$

Warunki jednorodne $\Rightarrow \alpha,\beta<0 \Rightarrow \gamma>0$

$\frac{\partial^2 A}{\partial x^2} - \alpha A = 0 \qquad A = a_1\sin(\sqrt{-\alpha}\,x) + a_2\cos(\sqrt{-\alpha}\,x)$

$A(0)=0 \Rightarrow a_2=0 \qquad A(a)=0 \Rightarrow \sqrt{-\alpha}\,a = k\pi \qquad \sqrt{-\alpha} = \frac{k\pi}{a}$

$A(x) = a_1\sin\left(\frac{k\pi}{a}x\right) \qquad$ Podobnie $B(y) = b_1\sin\left(\frac{l\pi}{a}y\right)$

$\gamma>0 \qquad C = c_1\sinh(\sqrt{\gamma}\,z) + c_2\cosh(\sqrt{\gamma}\,z)$

$\sqrt{\gamma} = \frac{\pi}{a}\sqrt{k^2+l^2}$

$$\Phi(x,y,z) = \sum_{l,k=0}^{\infty}A_{kl}\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{a}y\right)\left(\cosh(\sqrt{\gamma}\,z) + B_{kl}\sinh(\sqrt{\gamma}\,z)\right)$$

$$\Phi(x,y,0) = V_1 = \sum_{l,k}^{\infty}A_{kl}\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{a}y\right) \quad \Big/ \int_0^a\int_0^a\sin\left(\frac{k'\pi}{a}x\right)\sin\left(\frac{l'\pi}{a}y\right)dx\,dy$$

$$V_1\int_0^a\int_0^a\sin\left(\frac{k'\pi}{a}x\right)\sin\left(\frac{l'\pi}{a}y\right)dx\,dy = A_{kl}\frac{a^2}{4}\delta_{kk'}\delta_{ll'}$$

$$A_{kl}\frac{a^2}{4} = V_1\frac{a^2}{\pi^2 kl}\left(1-\cos(k\pi)\right)\left(1-\cos(l\pi)\right)$$

$A_{kl}\neq 0$ dla $k\equiv l\equiv 1 \pmod 2$ tzn. nieparzystych

$$A_{kl} = \frac{16V_1}{\pi^2 kl}$$

$$\Phi(x,y,a) = V_6 = \sum_{\substack{k,l\equiv 1\\(\mathrm{mod}\,2)}}^{\infty}\frac{16V_1}{\pi^2 kl}\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{a}y\right)\left(\cosh\left(\sqrt{k^2+l^2}\,\pi\right) + B_{kl}\sinh\left(\sqrt{k^2+l^2}\,\pi\right)\right)$$

$$\frac{\pi^2 V_6}{16V_1} = \sum_{k,l\equiv 1}^{\infty}\frac{1}{kl}\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{a}y\right)\left(\cosh\sqrt{k^2+l^2}\,\pi + B_{kl}\sinh\left(\sqrt{k^2+l^2}\,\pi\right)\right) \quad \Big/ \int_0^a\int_0^a\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{a}y\right)dx\,dy$$

$$\frac{\cancel{\pi^2}V_6}{\cancel{16}_{4}V_1}\frac{\cancel{4}\cancel{a^2}}{\cancel{\pi^2}\cancel{kl}} = \frac{\cancel{a^2}}{\cancel{4}\cancel{kl}}\delta_{kk'}\delta_{ll'}\left(\cosh\left(\sqrt{k^2+l^2}\,\pi\right) + B_{kl}\sinh\left(\sqrt{k^2+l^2}\,\pi\right)\right)$$

$$\frac{V_6}{V_1}\operatorname{csch}\left(\sqrt{k^2+l^2}\,\pi\right) - \coth\left(\sqrt{k^2+l^2}\,\pi\right) = B_{kl} \qquad \text{dla } k,l \text{ - nieparzystych } \sqrt{k^2+l^2}\notin\mathbb{Z}$$

$$\Phi(x,y,z) = \sum_{l,k=0}^{\infty}\frac{16V_1}{\pi^2 kl}\sin\left(\frac{k\pi}{a}x\right)\sin\left(\frac{l\pi}{a}y\right)\left(\cosh\left(\frac{\pi}{a}\sqrt{k^2+l^2}\,z\right) + \left(\frac{V_6}{V_1}\operatorname{csch}\left(\sqrt{k^2+l^2}\,\pi\right) - \coth\left(\sqrt{k^2+l^2}\,\pi\right)\right)\sinh\left(\frac{\pi}{a}\sqrt{k^2+l^2}\,z\right)\right)$$

Podobnie dla (V_2, V_5) i (V_3, V_4)

Kładąc $x=x_3$ $y=x_2$ $z=x_1$ i biorąc σ - permutacje $\{1,2,3\}$

$$\Phi(x_1,x_2,x_3) = \sum_{\sigma}\sum_{\substack{l,k\equiv 1\\(\mathrm{mod}\,2)}}^{\infty}\frac{8V_{\sigma(1)}}{\pi^2 kl}\left(\sin\left(\frac{k\pi}{a}x_{\sigma(2)}\right)\sin\left(\frac{l\pi}{a}x_{\sigma(3)}\right)\left(\cosh\left(\sqrt{k^2+l^2}\frac{\pi}{a}x_{\sigma(1)}\right) + \left(\frac{V_{7-\sigma(1)}}{V_{\sigma(1)}}\operatorname{csch}\left(\sqrt{k^2+l^2}\,\pi\right) - \coth\left(\sqrt{k^2+l^2}\,\pi\right)\right)\sinh\left(\sqrt{k^2+l^2}\frac{\pi}{a}x_{\sigma(1)}\right)\right)\right)$$