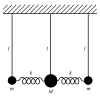
10.2

Saturday, January 28, 2023 8:41 PM

**Zadanie 2**

Znaleźć w przybliżeniu harmonicznym częstości i postaci drgań normalnych układu trzech wahadeł matematycznych o masach $m$, $M$, $m$ i jednakowej długości $l$, połączonych nieważkimi sprężynami o stałej sprężystości $k$. Wahadła drgają w płaszczyźnie rysunku. W położeniu równowagi układu sprężyny mają swoją długość swobodną. W chwili początkowej lewemu wahadłu nadano prędkość $v_0$.




**Odpowiedź:**

$\omega_1^2 = \frac{g}{l}$, $\omega_2^2 = \frac{k}{m} + \frac{g}{l}$, $\omega_3^2 = \frac{k}{m} + \frac{g}{l} + \frac{2k}{M}$

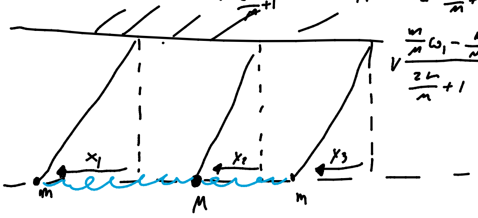
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} A_1 \sin(\omega_1 t) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} A_2 \sin(\omega_2 t) + \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix} A_3 \sin(\omega_3 t)$$

$A_1 = v\frac{m/M}{2m/M+1}$, $A_2 = v/2$, $A_3 = \frac{v}{2}\frac{1}{2m/M+1}$,

$$A_1\omega_1 - \frac{2m}{M}\omega_3 A_3$$

$$v\frac{m/M}{\frac{2m}{M}+1}\,\omega_1 - \frac{2m}{M}\omega_3 \frac{v}{2}\frac{1}{\frac{2m}{M}+1} =$$

$$v\frac{\frac{m}{M}\omega_1 - \frac{m}{M}\omega_3}{\frac{2m}{M}+1}$$




$$\begin{cases} m\ddot{x}_1 = -m\frac{g}{l}x_1 - k(x_1 - x_2) = -\left(m\frac{g}{l} + k\right)x_1 + kx_2 \\ M\ddot{x}_2 = -M\frac{g}{l}x_2 - k(x_2 - x_1) - k(x_2 - x_3) = kx_1 - \left(M\frac{g}{l} + 2k\right)x_2 + kx_3 \\ m\ddot{x}_3 = -m\frac{g}{l}x_3 - k(x_3 - x_2) = kx_2 - \left(m\frac{g}{l} + k\right)x_3 \end{cases}$$

$$\Rightarrow \frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\left(\frac{g}{l} + \frac{k}{m}\right) & \frac{k}{m} & 0 \\ \frac{k}{M} & -\left(\frac{g}{l} + \frac{2k}{M}\right) & \frac{k}{M} \\ 0 & \frac{k}{m} & -\left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \det\begin{pmatrix} \omega^2 - \left(\frac{g}{l} + \frac{k}{m}\right) & \frac{k}{m} & 0 \\ \frac{k}{M} & \omega^2 - \left(\frac{g}{l} + \frac{2k}{M}\right) & \frac{k}{M} \\ 0 & \frac{M}{m} & \omega^2 - \left(\frac{g}{l} + \frac{k}{m}\right) \end{pmatrix} = 0$$

$$\Rightarrow \left(\omega^2 - \left(\frac{g}{l} + \frac{k}{m}\right)\right)^2\left(\omega^2 - \left(\frac{g}{l} + \frac{2k}{M}\right)\right) - \frac{2k^2}{mM}\left(\omega^2 - \left(\frac{g}{l} + \frac{k}{m}\right)\right) = 0$$

$$\Rightarrow \omega^2 - \left(\frac{g}{l} + \frac{k}{m}\right) = 0 \ \vee \ \left(\omega^2 - \left(\frac{g}{l} + \frac{k}{m}\right)\right)\left(\omega^2 - \left(\frac{g}{l} + \frac{2k}{M}\right)\right) - \frac{2k^2}{mM} = 0$$

$$\Rightarrow \omega^4 - \left(\frac{2g}{l} + k\left(\frac{1}{m} + \frac{2}{M}\right)\right)\omega^2 + \frac{g^2}{l^2} + \frac{2gk}{lM} + \frac{gk}{lm} + \cancel{\frac{2k^2}{mM}} - \cancel{\frac{2k^2}{mM}} = 0$$

$$\Rightarrow \omega^2 - \left(\frac{2g}{l} + k\left(\frac{1}{m} + \frac{2}{M}\right)\right)\omega^2 + \frac{g^2}{l} + 2\frac{gk}{lM} + \frac{gk}{lm}$$

$$\Delta = \frac{4g^2}{l^2} + \frac{4gk}{lm} + \frac{8gk}{lM} + \frac{k^2}{m^2} + \frac{4k^2}{mM} + \frac{4k^2}{M^2} - 4\frac{g^2}{l^2} - 8\frac{gk}{lM} - 4\frac{gk}{lm} =$$

$$= k^2\left(\frac{1}{m} + \frac{2}{M}\right)^2$$

$$\Rightarrow \omega_{1/2}^2 = \frac{\frac{2g}{l} + \frac{k}{m} + \frac{2k}{M} \pm \frac{k}{m} \pm \frac{2k}{M}}{2}$$

$$= \frac{\frac{2g}{l} + \frac{k}{m} + \frac{2k}{M} + \frac{k}{m} + \frac{2k}{M}}{} \quad \frac{g}{} + \frac{k}{} + \frac{2k}{}$$

$$\omega_2^2 = \frac{\qquad}{2}$$

$$\omega_3^2 = \frac{\frac{2g}{l}+\frac{k}{m}+\frac{2k}{M}+\frac{k}{m}+\frac{2k}{M}}{2} = \frac{g}{l}+\frac{k}{m}+\frac{2k}{M}$$

$$\omega_1^2 = \frac{g}{l}$$

$$\omega_2^2 = \frac{g}{l}+\frac{k}{m}$$

$$\begin{pmatrix} \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) & \frac{k}{m} & 0 \\ \frac{k}{M} & \omega^2-\left(\frac{g}{l}+\frac{2k}{M}\right) & \frac{k}{M} \\ 0 & \frac{k}{m} & \omega^2-\left(\frac{g}{l}+\frac{k}{m}\right) \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

I $\omega_1^2 = \frac{g}{l}$

$$-A_1 + A_2 = 0 \Rightarrow A_1 = A_2$$

$$A_1 - 2A_2 + A_3 = 0 \Rightarrow A_1 = A_3 \Rightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

II $\omega_2^2 = \frac{g}{l}+\frac{k}{m}$ :

$$A_2 = 0$$

$$\frac{k}{M}A_1 - \frac{2k}{M}A_2 + \frac{k}{m}A_2 + \frac{k}{M}A_3 = 0 \Rightarrow A_1 = -A_3$$

$$\Rightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

III $\omega_3^2 = \frac{g}{l}+\frac{k}{m}+\frac{2k}{M}$

$$\frac{2k}{M}A_1 + \frac{k}{m}A_2 = 0 \Rightarrow A_2 = -\frac{2m}{M}A_1$$

$$\frac{k}{M}A_1 + \frac{k}{m}A_2 + \frac{k}{M}A_3 = 0 \Rightarrow A_3 = A_1$$

$$\Rightarrow \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = A\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}(t) = A_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\cos(\omega_1 t+\varphi_1) + A_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\cos(\omega_2 t+\varphi_2)$$

$$+ A_2\begin{pmatrix} 1 \\ -\frac{2k}{M} \end{pmatrix}\cos(\omega_2 t+\varphi_3)$$

$$+ A_3 \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix} \cos(\omega_2 t + \varphi_3)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \varphi_1 = \varphi_2 = \varphi_3 = -\frac{\pi}{2}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}(0) = \begin{pmatrix} V \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} \omega_1 A_1 + A_2\omega_2 + A_3\omega_3 = V \\ \omega_1 A_1 + A_3\omega_3 \cdot \left(-\frac{2m}{M}\right) = 0 \\ \omega_1 A_1 - A_2\omega_2 + \omega_3 A_3 = 0 \end{array}$$

$$\left(\begin{array}{ccc|c} \omega_1 & \omega_2 & \omega_3 & V \\ \omega_1 & 0 & -\frac{2m}{M}\omega_3 & 0 \\ \omega_1 & -\omega_2 & \omega_3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 2\omega_2 & 0 & V \\ \omega_1 & 0 & -\frac{2m}{M}\omega_3 & 0 \\ \omega_1 & -\omega_2 & \omega_3 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 0 & \omega_2 & 0 & \frac{V}{2} \\ \omega_1 & 0 & -\frac{2m}{M}\omega_3 & 0 \\ 0 & -\omega_2 & \omega_3\left(\frac{2m}{M}+1\right) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & \omega_2 & 0 & \frac{V}{2} \\ \omega_1 & 0 & -\frac{2m}{M}\omega_3 & 0 \\ 0 & 0 & \omega_3\left(\frac{2m}{M}+1\right) & \frac{V}{2} \end{array}\right)$$

$$\Rightarrow A_2 = \frac{V}{2\omega_2}$$

$$A_3 = \frac{V}{2} \cdot \frac{1}{\omega_3\left(\frac{2m}{M}+1\right)} \wedge \omega_1 A_1 = \frac{2m}{M}\omega_3 A_3$$

$$\Rightarrow A_1 = \frac{2m}{M} \cdot \frac{V}{2} \cdot \frac{1}{\omega_1\left(\frac{2m}{M}+1\right)}$$