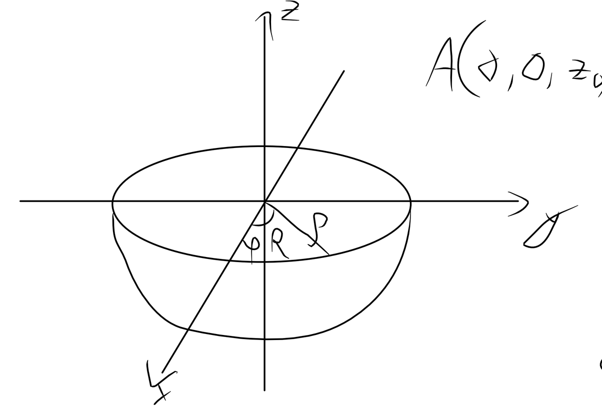


$A(0,0,z_0)$

Obręcz

$$d\phi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{\lambda R}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}}\,d\varphi$$

$$dq = \lambda\, dl = \lambda R\, d\varphi \qquad \lambda = \frac{Q}{2\pi R}$$

$$\phi = \int_0^{2\pi}\frac{\lambda R}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}}\,d\varphi = \frac{2\pi\frac{Q}{2\pi R}R}{4\pi\varepsilon_0\sqrt{R^2+z^2}} =$$

współrzędna punktu

$$= \frac{Q}{4\pi\varepsilon_0\sqrt{R^2+z^2}}$$

Dysk

sfery

dyski

$$Q \to dq = \sigma\, dS = \sigma\, 2\pi\rho\, d\rho \qquad \sigma = \frac{Q}{\pi R^2}$$

$$\phi = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{\rho\, d\rho}{\sqrt{\rho^2+z^2}} = \frac{\sigma}{4\varepsilon_0}\int_{z_0^2}^{R^2+z_0^2}\frac{du}{\sqrt{u}} = \frac{\sigma}{4\varepsilon_0}\left[2\sqrt{u}\right]_{z_0^2}^{R^2+z_0^2} =$$

$$\left\{\begin{aligned} \rho^2+z^2 &= u \\ 2\rho\, d\rho &= du \end{aligned}\right\}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z_0^2} - |z|\right) = \frac{Q}{2\pi R^2\varepsilon_0}\left(\sqrt{R^2+z^2} - |z|\right)$$

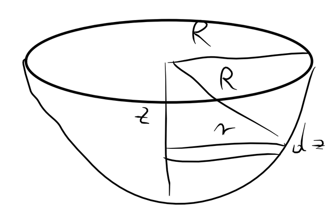
Półkula: $\rho_0 = \gamma$

$$Q \to dq = \gamma\cdot\pi r^2\, dz$$

$$d\phi = \frac{\gamma\,\cancel{\pi r^2}}{2\cancel{\pi r^2}\varepsilon_0}\left(\sqrt{r^2+(z-z_0)^2} - |z-z_0|\right)dz$$

$$\phi = \frac{\gamma}{2\varepsilon_0}\int_{-R}^{0}\left(\sqrt{R^2-z^2+(z-z_0)^2} - |z-z_0|\right)dz =$$

$$= \frac{\gamma}{2\varepsilon_0}\int_{-R}^{0}\left(\sqrt{R^2+z_0^2-2z_0z} - |z-z_0|\right)dz =$$



$$R^2 = z^2 + r^2$$

$$r^2 = R^2 - z^2$$

Wcześniej z oznaczało odległość od punktu, zmieniono je na $z - z_0$.

$$= \frac{\gamma}{2\varepsilon_0}\left[\int_{|R+z_0|}^{\sqrt{R^2+z_0^2}}\left(-\frac{1}{z_0}u^2\right)du - \int_{-R-z_0}^{-z_0}|w|\,dw\right] = \frac{\gamma}{2\varepsilon_0}\left[\frac{1}{3z_0}\left(|R+z_0|^3 - (R^2+z_0^2)^{\frac{3}{2}}\right) + \frac{1}{2}\left(\mathrm{sgn}(z_0)z_0^2 - \mathrm{sgn}(R+z_0)(R+z_0)^2\right)\right]$$

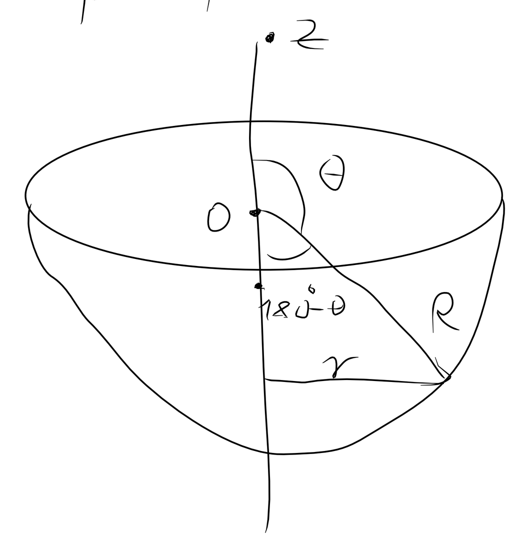
$$u = \sqrt{R^2+z_0^2-2z_0z}$$

$$du = \frac{-z_0\, dz}{\sqrt{R^2+z_0^2-2z_0z}} = \frac{-z_0}{u}dz$$

$$w = z - z_0$$

$$dw = dz$$

pół sfera:



$$Q \to dq = \sigma\cdot 2\pi r R\, d\theta = 2\pi\sigma R^2\sin\theta\, d\theta$$

$$r = R\sin(\pi-\theta) = R\sin\theta$$

$$d\phi = \frac{dq}{4\pi\varepsilon_0\sqrt{R^2+z^2}} = \frac{2\pi\sigma R^2}{4\pi\varepsilon_0\sqrt{R^2\sin^2\theta + R^2\cos^2\theta + z^2 - 2zR\cos\theta}}\sin\theta\, d\theta =$$

$$R \to R\sin\theta$$

$$z \to z_0 - R\cos\theta$$

$$= \frac{\sigma R^2}{2\varepsilon_0}\frac{\sin\theta\, d\theta}{\sqrt{R^2+z_0^2-2z_0R\cos\theta}}$$

$$\phi = \frac{\sigma R^2}{2\varepsilon_0}\int_{\frac{\pi}{2}}^{\pi}\frac{\sin\theta}{\sqrt{R^2+z_0^2-2z_0R\cos\theta}}\,d\theta = \frac{\sigma R}{4\varepsilon_0 z}\int_{R^2+z_0^2}^{(R+z_0)^2}\frac{du}{\sqrt{u}} = \frac{\sigma R}{2\varepsilon_0 z_0}\left(|R+z_0| - \sqrt{R^2+z_0^2}\right) =$$

$$= \frac{Q}{4\pi R z_0\varepsilon_0}\left(|R+z_0| - \sqrt{R^2+z_0^2}\right)$$

$$\left\{\begin{aligned} u &= R^2+z_0^2-2z_0R\cos\theta \\ du &= 2z_0R\sin\theta\, d\theta \\ \tfrac{\pi}{2} &\to R^2+z_0^2 \\ \pi &\to R^2+z_0^2+2z_0R = (R+z_0)^2 \end{aligned}\right\} \qquad \sigma = \frac{Q}{2\pi R^2}$$

półkula $\rho_0 = \gamma$

$$Q \to dq = \gamma\cdot 2\pi r^2\, dr$$

$$d\phi = \frac{dq}{4\pi r z_0\varepsilon_0}\left(|r+z_0| - \sqrt{r^2+z_0^2}\right) = \frac{\gamma r}{2z_0\varepsilon_0}\left(|r+z_0| - \sqrt{r^2+z_0^2}\right)dr$$

$$\phi = \frac{\gamma}{2\varepsilon_0 z_0}\left(\underbrace{\int_0^R r|r+z_0|\,dr}_{I_1} - \underbrace{\int_0^R r\sqrt{r^2+z_0^2}\,dr}_{I_2}\right) = \frac{\gamma}{2\varepsilon_0 z_0}\left(\frac{R^3}{3} + \frac{R^2z_0}{2} - \frac{1}{3}\sqrt{R^2+z_0^2}^{\,3} + \frac{1}{3}|z_0|^3\right)$$

$$I_1 = \int_0^R r|r+z_0|\,dr = \int_0^R (r^2+rz_0)\,dr = \frac{R^3}{3} + \frac{R^2z_0}{2}$$

dla $z_0 > 0$

$$I_2 = \frac{1}{2}\int_{z_0^2}^{R^2+z_0^2}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_{z_0^2}^{R^2+z_0^2} = \frac{1}{3}(R^2+z_0^2)^{\frac{3}{2}} - \frac{1}{3}|z_0|^3$$

$$r^2 + z_0^2 = u$$

$$2r\,dr = du$$

Dla $z_0 > 0$

$$\frac{\cancel{\gamma}}{\cancel{2\varepsilon_0}}\left[\frac{1}{3\cancel{z_0}}\left(|R+z_0|^3 - (R^2+z_0^2)^{\frac{3}{2}}\right) + \frac{1}{2}{}^{z_0}\left(\mathrm{sgn}(z_0)z_0^2 - \mathrm{sgn}(R+z_0)(R+z_0)^2\right)\right] = \frac{\cancel{\gamma}}{\cancel{2\varepsilon_0 z_0}}\left(\frac{R^3}{3} + \frac{R^2z_0}{2} - \frac{1}{3}\sqrt{R^2+z_0^2}^{\,3} + \frac{1}{3}|z_0|^3\right)$$

$$\frac{1}{3}|R+z_0|^3 \cancel{- \frac{1}{3}(R^2+z_0^2)^3} + \frac{1}{2}z_0\left(z_0^2 - (R+z_0)^2\right) = \frac{R^3}{3} + \frac{R^2z_0}{2} \cancel{- \frac{1}{3}\sqrt{R^2+z_0^2}} + \frac{1}{3}|z_0|^3$$

$$\cancel{\frac{1}{3}R^3} + \cancel{R^2z_0} + \cancel{Rz_0^2} + \cancel{\frac{1}{3}z_0^3} + \cancel{\frac{1}{2}z_0^3} - \cancel{\frac{1}{2}R^2z_0} - \cancel{Rz_0^2} - \cancel{\frac{1}{2}z_0^3} = \cancel{\frac{R^3}{3}} + \cancel{\frac{R^2z_0}{2}} + \cancel{\frac{1}{3}z_0^3}$$

$$0 = 0$$

więc jest to ta sama funkcja (dla $z_0 > 0$)