A new optimal algorithm for Level-Synchronous Parallel BFS in the Cilk model

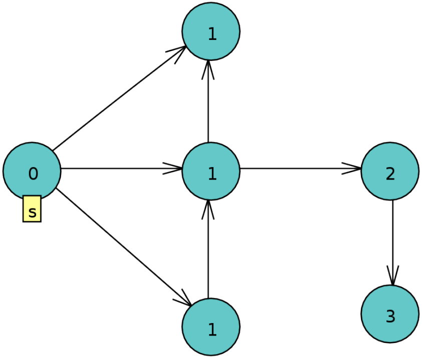
Yonatan R. Fogel

Stony Brook University

May 18, 2013

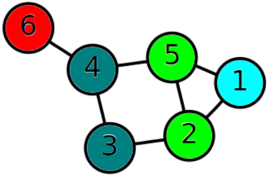
BFS

- Given
  - Graph $G = (\mathbb{V}, \mathbb{E})$ with diameter D and max out-degree Δ
  - Source vertex $s \in \mathbb{V}$
- Calculate
  - $Dist_{u \in \mathbb{V}}$ = length of the shortest path from s to u in G



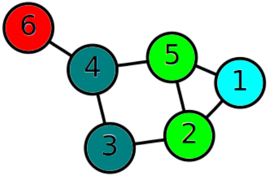
Parallel BFS

- The runtime for serial BFS is $T_s = \Theta\left(|\mathbb{V}| + |\mathbb{E}|\right)$
- For large graphs, calculating *Dist* can be slow
- We can speed up BFS by processing edges and/or nodes in parallel
- Our new TBN-BFS achieves near-perfect linear speadup when $p \ll \frac{|V|+|E|}{D\log(|V|+|E|)}$



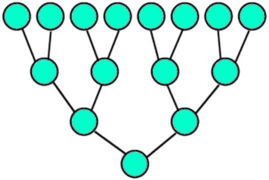
Level-Synchronous BFS

- All nodes at distance d from s are processed before any nodes at distance $d' > d$
- Level-Synchronous BFS has lower bounds of
  - $T_p = \Omega\left(\frac{|\mathbb{V}|+|\mathbb{E}|}{p} + D\right)$
  - $T_1, W_p = \Omega\left(T_s\right) = \Omega\left(|\mathbb{V}| + |\mathbb{E}|\right)$
- Non Level-Synchronous BFS algorithms
  - Distinguished-BFS [Ullman and Yannakakis, 1990]



Cilk Model

- Large shared memory
- Consistent caches between cores
- Synchronizing (or launching) n tasks takes $T_\infty = \Theta(\log n)$ time
- Cilk+ has this model with the addition of randomized work stealing [Frigo et al., 2009]
- A parallel algorithm has the following properties
  - $T_p =$ running time using p cores
  - $T_1 =$ running time using 1 core
  - $W_p =$ work done using p cores



Non Cilk Model

- Level Synchronous BFS algorithms not using the Cilk Model
  - Cray-BFS uses the Cray MTA-2 hardware model [Bader and Madduri, 2006]
  - Block-Queue-BFS uses the Intel Mic hardware model [Saule and Catalyurek, 2012]
  - Blue-BFS uses the BlueGene/L hardware model [Yoo et al., 2005]

Level-Synchronous Parallel BFS in the Cilk Model

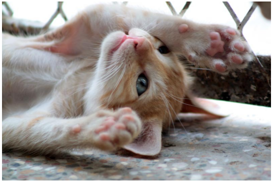
- In the Cilk Model, Level-Synchronous BFS has a lower bound of $T_p = \Omega\left(\frac{|\mathbb{V}|+|\mathbb{E}|}{p} + D \log p\right)$
- MIT-Bag-BFS uses penants, bags, and reducer hyperobjects [Leiserson and Schardl, 2010]
- Our new TBN-BFS algorithm
  - Runs on standard hardware implemented using Cilk+
  - Can do its own scheduling deterministically
  - Achieves the lower bound for T_p, T_1, W_p in the worst case independently of Δ (scale-free)

Serial-BFS

1. for each vertex $u \in \mathbb{V}$
2. $Dist_u \leftarrow \infty$
3. $Dist_s \leftarrow 0$
4. $Q \leftarrow \emptyset$
5. $\textsc{Enqueue}(Q, s)$
6. while $Q \neq \emptyset$ do
7. $u \leftarrow \textsc{Dequeue}(Q)$ {Potential source of
8.   for each vertex v in $\Gamma(u)$ do parallelism: lines 7–8}
9.   if $Dist_v = \infty$ then {Potential issues for
10. $Dist_v \leftarrow Dist_u + 1$ parallelism: lines 9–11}
11. $\textsc{Enqueue}(Q, v)$

TBN-BFS - Summary

- $T_1 = \mathcal{O}\left(|\mathbb{V}| + |\mathbb{E}|\right)$ (optimal)
- $W_p = \mathcal{O}\left(|\mathbb{V}| + |\mathbb{E}|\right)$ (optimal)
- $T_p = \mathcal{O}\left(\frac{|\mathbb{V}|+|\mathbb{E}|}{p} + D \log p\right)$ (optimal)
- Scale-free
- Runs on standard hardware implemented using Cilk+
- Can do its own scheduling deterministically



TBN-BFS - High Level

- To get optimal T_1, T_p the algorithm is
  1. For each level $\ell \in [0 \dots D)$
     1.1 Prepare to split work
     1.2 Split work and process edges
     1.3 Deduplicate vertexes and combine queues
- To get optimal W_p modify the algorithm to
  1. Reduce search space
  2. Dynamically choose number of cores

TBN-BFS - Prepare to Split Work

1. Receive input $Q_{in} \subseteq \mathbb{V}$ where
   1.1 Each vertex in Q_{in} is unique
   1.2 $\forall u \in Q_{in} : Dist_u = \ell$
   1.3 $\forall u \in Q_{in} : |\Gamma(u)| > 0$
2. Generate $OutDegrees[0 \leq i < |Q_{in}|] = |\Gamma(Q_{in}[i])|$ in parallel
3. Perform a parallel prefix sum on $OutDegrees$

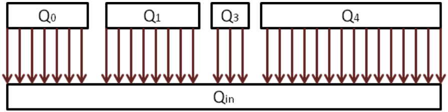
$Q_{in} =$	1	3	2	4	$\dots$	$\dots$
$OutDegrees_{before} =$	5	7	1	2	$\dots$	$\dots$
$OutDegrees_{after} =$	5	12	13	15	$\dots$	m_ℓ

TBN-BFS - Split Work and Process Edges

1. Each core i in parallel
   1.1 searches *OutDegrees* for $1 + \left\lfloor \frac{i\ m_\ell}{p} \right\rfloor$ to find starting edge
   $\left\{ \mathcal{O}\left(\log \frac{n_\ell}{p} + \log p \right) \text{ work} \right\}$
   1.2 processes $\left\lfloor \frac{m_\ell}{p} \right\rfloor$ consecutive edges
      1.2.1 $Q_i \leftarrow \emptyset$
      1.2.2 for each edge (u, v)
      1.2.3 if $Dist_v = \infty$ then {Benign race condition}
      1.2.4 $Dist_v \leftarrow Dist_u + 1$ {Maintains $Q_i n$ invariant 2}
      1.2.5 $Owner_v \leftarrow i$
      1.2.6 if $|\Gamma(v)| > 0$ {Maintains $Q_i n$ invariant 3}
      1.2.7 $\textsc{Enqueue}(Q_i, v)$

TBN-BFS - Deduplicate Vertexes and Combine Queues

1. Each core i in parallel
   1.1 $Q_i \leftarrow \{u \in Q_i : Owner_u = i\}$ $\{$Remove duplicate vertexes$\}$
   1.2 $Size_i \leftarrow |Q_i|$
2. Perform a parallel prefix sum on *Size*
3. Each core i copies its queue back into Q_{in} at offset $Size_{i-1}$ $\{Size_{-1}$ is treated as $0\}$

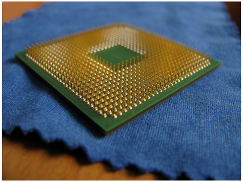


TBN-BFS - Reduce Search Space

1. $N \equiv |OutDegrees|$
2. $OutDegree_k \equiv OutDegree[\left\lfloor \frac{(k+1)\ N}{p} \right\rfloor : \left\lfloor \frac{k\ N}{p} \right\rfloor]$
3. Invariant: each core i first edge is in $OutDegree_k$ for exactly one k
   3.1 For each $0 \leq k < p$ in parallel
      3.1.1 Find cores $[f_k : \ell_k)$ whose first degree is in $OutDegree_k$ $\{\text{Each range } [f_k : \ell_k) \text{ is pairwise disjoint, } \sum_k([f_k : \ell_k]) = [0 : p)\}$
      3.1.2 Assign k in parallel to $SubList_{i \in [f:\ell)}$
4. Core i searches for first degree in $OutDegree_{SubList_i}$ instead of $OutDegree$
5. W_p reduces from $\mathcal{O}(|\mathbb{V}| + |\mathbb{E}| + Dp \log p)$ to $\mathcal{O}(|\mathbb{V}| + |\mathbb{E}| + Dp)$

TBN-BFS - Dynamically Choose Number of Cores

- In "Prepare to Split Work", right after the parallel prefix sum, $p_\ell \leftarrow \text{MIN}(m_\ell, p)$
- Use at most p_ℓ cores until next time m_ℓ is calculated
- This ensures the $\mathcal{O}(p_\ell)$ work every level is $\mathcal{O}(m_\ell)$
- W_p reduces from $\mathcal{O}(|\mathbb{V}| + |\mathbb{E}| + Dp)$ to $\mathcal{O}(|\mathbb{V}| + |\mathbb{E}| + D) = \mathcal{O}(|\mathbb{V}| + |\mathbb{E}|)$ (optimal)



Conclusion

- BFS is a hot topic recently
- Our new TBN-BFS algorithm
  - $T_1 = \mathcal{O}\left(|\mathbb{V}| + |\mathbb{E}|\right)$ (optimal)
  - $W_p = \mathcal{O}\left(|\mathbb{V}| + |\mathbb{E}|\right)$ (optimal)
  - $T_p = \mathcal{O}\left(\frac{|\mathbb{V}|+|\mathbb{E}|}{p} + D \log p\right)$ (optimal)
  - Scale-free
  - Runs on standard hardware implemented using Cilk+
  - Can do its own scheduling deterministically
  - Preliminary experiments show merit

Future Work

- Optimize TBN-BFS for the PRAM model
  - TBN-BFS runs in same time for PRAM but is not asymptotically optimal
  - Try using approximate parallel prefix sum [Goldberg and Zwick, 1995]
- Modify TBN-BFS to remove false sharing
  - Examine using an $\mathcal{O}(n)$ sort algorithm for levels where $m_\ell = \Omega\left(n^{\frac{1}{c}}\right)$
  - Try to distribute work by cacheline
- Examine non level-synchronous approaches
- Implement and optimize TBN-BFS
- Run experiments comparing TBN-BFS to existing algorithms

References I

Bader, D. A. and Madduri, K. (2006).
Designing multithreaded algorithms for breadth-first search and st-connectivity on the cray mta-2.
In *Proceedings of the 2006 International Conference on Parallel Processing*, ICPP '06, pages 523–530, Washington, DC, USA. IEEE Computer Society.

Frigo, M., Halpern, P., Leiserson, C. E., and Lewin-Berlin, S. (2009).
Reducers and other cilk++ hyperobjects.
In *Proceedings of the twenty-first annual symposium on Parallelism in algorithms and architectures*, SPAA '09, pages 79–90, New York, NY, USA. ACM.

References II

Goldberg, T. and Zwick, U. (1995).
Optimal deterministic approximate parallel prefix sums and their applications.
In *In Proc. Israel Symp. on Theory and Computing Systems (ISTCS'95*, pages 220–228.

Leiserson, C. E. and Schardl, T. B. (2010).
A work-efficient parallel breadth-first search algorithm (or how to cope with the nondeterminism of reducers).
In *Proceedings of the 22nd ACM symposium on Parallelism in algorithms and architectures*, SPAA '10, pages 303–314, New York, NY, USA. ACM.

References III

Saule, E. and Catalyurek, U. V. (2012).
An early evaluation of the scalability of graph algorithms on the intel mic architecture.
In *Proceedings of the 2012 IEEE 26th International Parallel and Distributed Processing Symposium Workshops & PhD Forum*, IPDPSW '12, pages 1629–1639, Washington, DC, USA. IEEE Computer Society.

Ullman, J. and Yannakakis, M. (1990).
High-probability parallel transitive closure algorithms.
In *Proceedings of the second annual ACM symposium on Parallel algorithms and architectures*, SPAA '90, pages 200–209, New York, NY, USA. ACM.

References IV

Yoo, A., Chow, E., Henderson, K., McLendon, W., Hendrickson, B., and Catalyurek, U. (2005).
A scalable distributed parallel breadth-first search algorithm on bluegene/l.
In *Proceedings of the 2005 ACM/IEEE conference on Supercomputing*, SC '05, pages 25–, Washington, DC, USA. IEEE Computer Society.

TBN-BFS - Reduce Search Space

1. $N \leftarrow |OutDegrees|$
2. Each core i
   - 2.1 $FirstDegree \leftarrow OutDegrees[\left\lfloor \frac{iN}{p} \right\rfloor - 1]$
   - 2.2 $FirstDegreeNext \leftarrow OutDegrees[\left\lfloor \frac{(i+1)N}{p} \right\rfloor - 1]$
   - 2.3 $FirstCore \leftarrow \left\lceil \frac{p \ \ FirstDegree}{m_\ell} \right\rceil$
   - 2.4 $LastCore \leftarrow \left\lceil \frac{p \ \ FirstDegreeNext}{m_\ell} \right\rceil$
   - 2.5 parallel for $j \leftarrow FirstCore$ to $LastCore$
   - 2.6 $\quad SubList_j \leftarrow i$
3. Using $SubList_i$, core i can search only n_ℓ / p indexes
4. W_p reduces from $\mathcal{O}\left(|\mathbb{V}| + |\mathbb{E}| + Dp \log p\right)$ to $\mathcal{O}\left(|\mathbb{V}| + |\mathbb{E}| + Dp\right)$