A new optimal algorithm for parallel level synchronous BFS in the Cilk model

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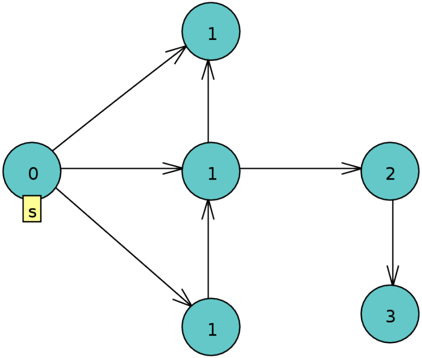
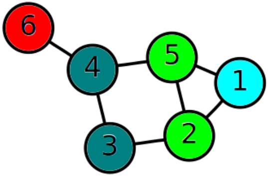
BFS

- Given
  - Graph $G = (\mathbb{V}, \mathbb{E})$ with diameter D and max out-degree Δ
  - Source vertex $s \in \mathbb{V}$
- Calculate
  - $Dist_{u \in \mathbb{V}}$ = length of the shortest path from s to u in G



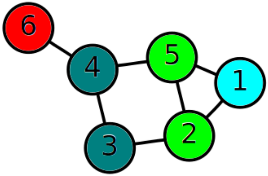
Parallel BFS

- The runtime for serial BFS is $T_s = \Theta\left(|\mathbb{V}| + |\mathbb{E}|\right)$
- For large graphs, calculating *Dist* can be slow
- We can speed up BFS by processing edges and/or nodes in parallel
- Our new TBN-BFS achieves near-perfect linear speadup when $p \ll \frac{|V|+|E|}{D\log(|V|+|E|)}$



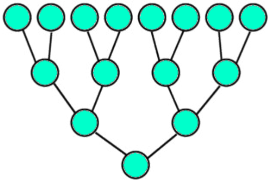
Level-Synchronous BFS

- All nodes at distance d from s are processed before any nodes at distance $d' > d$
- Level-Synchronous BFS has lower bounds of
  - $T_p = \Omega\left(\frac{|\mathbb{V}|+|\mathbb{E}|}{p} + D\right)$
  - $T_1, W_p = \Omega\left(T_s\right) = \Omega\left(|\mathbb{V}| + |\mathbb{E}|\right)$
- Non Level-Synchronous BFS algorithms
  - Distinguished-BFS [Ullman and Yannakakis, 1990]



Cilk Model

- Large shared memory
- Consistent caches between cores
- Synchronizing (or launching) n tasks takes $T_\infty = \Theta(\log n)$ time
- Cilk+ has this model with the addition of randomized work stealing [Frigo et al., 2009]
- BFS algorithms not using the Cilk Model
  - Cray-BFS uses the Cray MTA-2 hardware model [Bader and Madduri, 2006]
  - Block-Queue-BFS uses the Intel Mic model [Saule and Catalyurek, 2012]



Level-Synchronous Parallel BFS in the Cilk Model

- In the Cilk Model, Level-Synchronous BFS has a lower bound of $T_p = \Omega\left(\frac{|\mathbb{V}|+|\mathbb{E}|}{p} + D \log p\right)$
- MIT-Bag-BFS uses penants, bags, and reducer hyperobjects [Leiserson and Schardl, 2010]
- TBN-BFS algorithm
  - Runs on standard hardware implemented using Cilk+
  - Can do its own scheduling deterministically
  - Achieves the lower bound for T_p, T_1, W_p in the worst case independently of Δ (scale-free)

Serial-BFS

for each vertex $u \in \mathbb{V}$

  $Dist_u \leftarrow \infty$

$Dist_s \leftarrow 0$

$Q \leftarrow \emptyset$

$\textsc{Enqueue}(\ Q,\ s\ )$

while $Q \neq \emptyset$ do

  $u \leftarrow \textsc{Dequeue}(\ Q\ )$

  for each vertex v in $\Gamma(u)$ do

    if $Dist_v = \infty$ then

      $Dist_v \leftarrow Dist_u + 1$

      $\textsc{Enqueue}(\ Q, v\ )$

Terminology

- $n = |\mathbb{V}|$ the number of nodes in a graph
- $m = |\mathbb{E}|$ the number of edges in a graph
- $D = \mathcal{O}(n)$ the diameter of a graph
- $\Gamma_u =$ the set of vertexes adjacent to u
- $T_s =$ the running time for a serial algorithm
- $T_P =$ the running time for a parallel algorithm running on P cores
- $T_\infty =$ the running time for a parallel algorithm running on infinite cores
- $T_1 = \Omega(T_s) =$ the running time for a parallel algorithm running on one core
- $W_P = \Omega(T_s) =$ the total work done by a parallel algorithm running on P cores (excluding idle time)
    - Reducing W_P can reduce energy use [Albers and Antoniadis,]

Motivation for (P)BFS

BFS is used for

- Path Finding
  - Video Games
  - Google Maps
- Analyzing social networks
- Designing and analyzing VLSI
- Task scheduling
- As a primitive in other algorithms

Existing Algorithms

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- distinguished-bfs [Ullman and Yannakakis, 1990] contracts the graph to make it dense
- cray-bfs [Bader and Madduri, 2006] uses fast hardware mutexes and atomic increments

General Approaches

- Assumes PRAM model
- Specialized for specific hardware [Yoo et al., 2005]
  - GPU
  - CRAY (hardware mutex every 64 bits, atomic add) [Bader and Madduri, 2006]
- Uses atomic instructions [Saule and Catalyurek, 2012]
- Specialized for sparse (or dense) graphs only
- Specialized for bounded out-degree (not scale-free) [Leiserson and Schardl, 2010]
- T_1 or T_p is not asymptotically optimal
- Room for energy efficiency improvements (non-optimal W_P)
- Offloads some work to scheduler
  - Work-stealing (randomized) gives high probability bounds [Leiserson and Schardl, 2010]
- Non level-synchronous [Ullman and Yannakakis, 1990]

Clean BFS - Properties

- Recall $T_s = \mathcal{O}(n + m)$
- $T_1 = \mathcal{O}(n + m)$ (optimal)
- $W_P = \mathcal{O}(n + m)$ (optimal)
- $T_P = \mathcal{O}(n/p + m/p + D \log P)$ (optimal)
- Scale-free
- Deterministic worst case bounds

Clean BFS - High Level

For each level $\ell \in [0 \ldots D)$

1. To get optimal T_1, T_P
   1.1 Prepare to Split Work
   1.2 Split Work and Process Edges
   1.3 Dedup Vertexes and Combine Queues
2. To get optimal W_P
   2.1 Reduce Search Space
   2.2 Dynamically Choose Number of Cores

Clean BFS - Prepare to Split Work

1. One input queue $Q_{in} \subseteq \mathbb{V}$
   1.1 Each vertex in Q_{in} is unique
   1.2 $\forall u(u \in Q_{in} \Rightarrow Dist_u = \ell)$
   1.3 $\forall u(u \in Q_{in} \Rightarrow |\Gamma_u| > 0$
2. Generate $OutDegrees[0 \leq i < |Q_{in}|] = |\Gamma(Q_{in}[i])|$ in parallel
3. Perform a parallel prefix sum on $OutDegrees$

$Q_{in} =$	1	3	2	4	...	...
$OutDegrees_{before} =$	1	3	2	4	...	...
$OutDegrees_{after} =$	1	4	6	10	...	m_ℓ

Clean BFS - Split Work and Process Edges

1. Each core i processes m_ℓ / P edges
   1.1 searches *OutDegrees* for $1 + \left\lfloor \frac{i\ m_\ell}{P} \right\rfloor$ to find starting edge
   1.2 does $\mathcal{O}\left(\log \frac{n_\ell}{P} + \log P\right)$ work
   1.3 processes $\left\lfloor \frac{m_\ell}{P} \right\rfloor$ consecutive edges
      1.3.1 $Q_i \leftarrow \emptyset$
      1.3.2 for each edge (u, v)
      1.3.3 if $Dist_v = \infty$ then
      1.3.4 $Dist_v \leftarrow Dist_u + 1$
      1.3.5 $Owner_v \leftarrow i$
      1.3.6 $\textsc{Enqueue}(\ Q_i, v\ )$
   1.4 Benign race conditions

Clean BFS - Dedup Vertexes and Combine Queues

1. $Size_{-1} = 0$
2. Each core i uses $Owner$ to ensure each vertex lives in at most one output queue
   2.1 $Q_i \leftarrow \{u \in Q_i : Owner_u = i\}$
   2.2 $Size_i \leftarrow |Q_i|$
3. Perform a parallel prefix sum on $Size$
4. Each core i copies its queue back into Q_{in} at offset $Size_{i-1}$

Clean BFS - Reduce Search Space

1. $N \leftarrow |OutDegrees|$
2. Each core i
   2.1 $FirstDegree \leftarrow OutDegrees[\lfloor \frac{iN}{P} \rfloor - 1]$
   2.2 $FirstDegreeNext \leftarrow OutDegrees[\left\lfloor \frac{(i+1)N}{P} \right\rfloor - 1]$
   2.3 $FirstCore \leftarrow \left\lceil \frac{P \;\; FirstDegree}{m_\ell} \right\rceil$
   2.4 $LastCore \leftarrow \left\lceil \frac{P \;\; FirstDegreeNext}{m_\ell} \right\rceil$
   2.5 parallel for $j \leftarrow FirstCore$ to $LastCore$
   2.6 $\quad SubList_j \leftarrow i$
3. Using $SubList_i$, core i can search only n_ℓ / p indexes
4. W_P reduces from $\mathcal{O}\left(n + m + DP \log P\right)$ to $\mathcal{O}\left(n + m + DP\right)$

Clean BFS - Dynamically Choose Number of Cores

- In "Prepare to Split Work", right after the parallel prefix sum, $P_\ell \leftarrow \textsc{min}(\ m_\ell, P\ )$
- Use at most P_ℓ cores until next time m_ℓ is calculated
- This ensures the $\mathcal{O}(P_\ell)$ work every level is $\mathcal{O}(m_\ell)$
- W_P reduces from $\mathcal{O}(n + m + DP)$ to $\mathcal{O}(n + m + D) = \mathcal{O}(n + m)$ (optimal)

Conclusions

- Clean BFS has optimal T_1, T_P, W_P in our computation model
- TODO MORE?

Future Work

- Optimize Clean BFS for the PRAM model
  - Clean BFS runs in same time for PRAM but is not asymptotically optimal
  - Try using approximate parallel prefix sum [Goldberg and Zwick, 1995]
- Modify Clean BFS to remove false sharing
  - Examine using an $\mathcal{O}(n)$ sort algorithm for levels where $m_\ell = \Omega n^{\frac{1}{c}}$
  - Try to distribute work by cacheline
- Examine non level-synchronous approaches

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