UMONS

II. Differential forms on a manifold

0) Let's try to be as systematic as possible. For that we recall some definit°.

- (Finite dim) $\mathbb{k}$-vector space : $\mathbb{k}$ is a field ($\mathbb{R}$ or $\mathbb{C}$), E is an additive abelian grp not $(E,+)$. It is a VS if $\exists$ scalar multiplicat° s.t $\forall a,b,1 \in \mathbb{k}$ and $u,v \in E$;
  - ~~$\forall a \in \mathbb{k}$ and u,v~~ $a(u+v) = au + av$
  - $(a+b)u = au + bu$.
  - $(ab)u = a(bu)$
  - $1u = u$. Note $(E,+,\cdot)$.

Given a basis $\{e_i\}_{i=1...n}$, $n = \dim E$ (ie set of lin indep elmt of E), $\forall u \in E$ one has $u = u^i e_i$ with $\{u^i\} \in \mathbb{k}^n$. So a basis realise iso $E \simeq \mathbb{k}^n$.

- Normed $\mathbb{k}$-VS : E is endowed with a map $\| \, \| : E \to \mathbb{R}^+$ s.t
  - $\|u+v\| \leqslant \|u\| + \|v\|$ subadditive.
  - $\|\lambda u\| = |\lambda| \, \|u\|$ abs homogeneous. } semi-norme.
  - positive def : $\|u\| = 0 \Rightarrow u = 0$ / - not pos def : $\|u\| = 0 \not\Rightarrow u = 0$.
    $\|u\| \geqslant 0$

- Inner product space : E endowed with a ~~bilinear~~ sesqui-linear form $g : E \times E \to \mathbb{k}$, also noted $\langle \, , \rangle$ s.t :
  - lin 1st arg : $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$, $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ — means lin up to automorphism of $\mathbb{k}$.
  - conj symetric : $\langle u, v \rangle = \overline{\langle v, u \rangle}$ / equiv ; semi-lin on 2nd arg $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$, $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
  - pos def : $\langle u, u \rangle \geqslant 0$, $\langle u, u \rangle = 0 \Rightarrow u = 0$ / - not pos def : nondegenerate : $\langle u, u \rangle = 0 \not\Rightarrow u = 0$ but $\langle u, v \rangle = 0 \; \forall v \Rightarrow u = 0$.
    (ex : Minkowski !) ↳ ⚠ affine space ! ie diff 2 elemt = vect.

Rmq : $\langle u, u \rangle^{1/2} \equiv \|u\|$ inner prod can def a norm. But not all norms come from inner prod. So Inner prod spaces $\subset$ Normed VS.

- **Affine space** : Set A and VS, E, with right act° of grp struct of E on A :

$$\begin{aligned} A \times E &\longrightarrow A \\ (a,u) &\longmapsto a+u \end{aligned} \quad \text{where} \begin{cases} a+0=a \\ (a+u)+v = a+(u+v) \end{cases}$$

and ;
$$\left.\begin{aligned} \forall a \in A :\ & E \longrightarrow A \\ & u \longmapsto a+u \\ \forall u \in E :\ & A \longrightarrow A \\ & a \longmapsto a+u \end{aligned}\right\} \text{ are bij}$$

$\forall a,b \in A \ \exists!\, u \in E$ s.t : $b = a+u$

note : $b-a = u$.

Termin : A is said modeled on E

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- <u>graded $\mathbb{k}$-VS</u> : E decomposes as $E = \bigoplus_{k \in \mathbb{N}} E^k$ for E^k $\mathbb{k}$-vect subspace.

  k is the degree of an elem^t in E^k, called "homogeneous of deg k".

  <u>NB</u> : by conv $E^0 = \mathbb{k}$!

- <u>Algebra</u> : E a $\mathbb{k}$-VS endowed with bilinear map $\circ : E \times E \to E$ the "product" s.t

  $\forall A, B, C \in E$ and $\forall \alpha, \beta \in \mathbb{k}$:
  - $(\alpha A) \circ (\beta B) = \alpha\beta (A \circ B)$
  - $(A + B) \circ C = A \circ C + B \circ C$
  - $A \circ (B + C) = A \circ B + A \circ C$.

  $\hookrightarrow$ $\underbrace{(E, +, \circ)}_{\mathcal{A}}$ is a $\mathbb{k}$-algebra.

  → If $A \circ B = B \circ A$, $\mathcal{A}$ is a <u>commutative $\mathbb{k}$-alg</u>.

  → If E is a graded $\mathbb{k}$-VS so that the product is s.t :

  $\left.\begin{array}{l} \forall A \in E^p, \deg(A) = p \\ \forall B \in E^q, \deg(B) = q \end{array}\right\}$ then $A \circ B \in E^{p+q}$, $\deg(A \circ B) = p + q$

  then $\mathcal{A}$ is a <u>graded $\mathbb{k}$-alg</u>.

  If furthermore $A \circ B = (-)^{pq} B \circ A$, $\mathcal{A}$ is a <u>graded commutative $\mathbb{k}$-alg</u>.

- <u>Differential algebra</u> : $\mathcal{A}$ is endowed with a map $d : \mathcal{A} \to \mathcal{A}$ which is

  $\mathbb{k}$-linear $\left\{\begin{array}{l} d(A + B) = dA + dB \\ d(\lambda A) = \lambda dA \end{array}\right.$

  and satisfies Leibnitz rule : $d(A \circ B) = dA \circ B + A \circ dB$.

  $\hookrightarrow$ $\mathcal{A}$ is a <u>$\mathbb{k}$-differential alg</u>.

Such a map is called a <u>derivat°</u> of the algebra. The set of derivat° of an alg $\mathcal{A}$ is noted $\mathrm{Der}(\mathcal{A})$, and is clearly a $\mathbb{k}$-vectorspace !

<u>Rmq</u> : def the bracket operat° $[a, b] \equiv a \circ b - b \circ a$, then $\forall a \in \mathcal{A}$ it is easily shown that $[a, \,] : \mathcal{A} \to \mathcal{A}$ is a derivat°. Such a derivat° is called <u>inner</u>.

The set of inner derivat° is noted $\mathrm{In}(\mathcal{A})$.

The set of outer derivat° is $\mathrm{Out}(\mathcal{A}) = \mathrm{Der}(\mathcal{A}) / \mathrm{In}(\mathcal{A})$.

Rmq: A Lie algebra is an alg $\mathcal{A}$ with further requirement on its product $[\ ,\ ] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. In additio of bilinearity one requires:

- $[A,B] = -[B,A]$, anticommutativity.
- Jacobi identity: $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$.

Notice that a Lie Alg is automatically a diff alg since $\forall A \in \mathcal{A}$ it has the inner derivat° $[A,\ ] : \mathcal{A} \to \mathcal{A}$ for which the Leibniz rule is enforced by (equiv to) the Jacobi identity.

Rmq: Given an alg $\mathcal{A}$, def a bracket $[a,b] \equiv a \circ b - b \circ a$ automatically turn it into a Lie alg! → [By the way $\mathrm{Der}(\mathcal{A})$ is a Lie Alg with: $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$]

→ If $\mathcal{A}$ is a graded (commutative) $\mathbb{K}$-alg, and has a derivat° s.t:

$$\begin{cases} \forall A \in \mathcal{A}^p, \deg(A) = p : \quad dA \in \mathcal{A}^{p+|d|}, \deg(dA) = p + |d| \\ d(A \circ B) = dA \circ B + (-)^{p*|d|} A \circ dB \qquad \text{"graded (super) Leibniz rule"}. \end{cases}$$

↳ $\mathcal{A}$ is a graded (comm) diff alg and d is said homogenous of degree $|d|$.
(homogeneous derivat° of odd degree are called anti-derivat°).
If furthermore d is nilpotent: $d^2 = 0$, then $\mathcal{A}$ is called a diff graded alg! or DG-alg ✓

Ex: A graded Lie Alg is s.t:

- $[\mathcal{A}^p, \mathcal{A}^q] \in \mathcal{A}^{p+q}$
- $\forall A \in \mathcal{A}^p, B \in \mathcal{A}^q : [A,B] = -(-)^{pq}[B,A]$
- graded Jacobi identity: $(-)^{pr}[A,[B,C]] + (-)^{pq}[B,[C,A]] + (-)^{qr}[C,[A,B]] = 0$

it is a inner graded diff alg ✓

⊛ Given a graded diff alg, def the graded (super)-commutator (or bracket):
$\forall A \in \mathcal{A}^p$ and $\forall B \in \mathcal{A}^q$: $[A,B] \equiv A \circ B - (-)^{pq} B \circ A$
turn $\mathcal{A}$ into a graded Lie Alg ✓

⊛ The set of derivat° of a graded diff $\mathcal{A}$ $\mathrm{Der}(\mathcal{A})$ is a graded $\mathbb{K}$-VS and also a graded Lie-alg under the bracket $[d_1, d_2] \equiv d_1 \circ d_2 - (-)^{|d_1||d_2|} d_2 \circ d_1$

1) Exterior algebra.

- Dual space: E a $\mathbb{K}$-vect space, the set of linear maps from E to $\mathbb{K}$ is the dual of E noted $E^* = \mathcal{L}(E,\mathbb{K})$.

By def $\forall \omega \in E^*$:
- $\omega : E \longrightarrow \mathbb{K}$, $X \longmapsto \omega(X)$ "ω eats a X, spits a $\mathbb{K}$".
- $\forall \lambda \in \mathbb{K}$: $\omega(\lambda X) = \lambda \omega(X)$
- $\forall X, Y \in E$: $\omega(X+Y) = \omega(X) + \omega(Y)$.

E^* is also a $\mathbb{K}$-vect space under pointwise additᵒ and scalar multiplicatᵒ:
- $\forall \lambda \in \mathbb{K}$: $(\lambda\omega)(X) \equiv \lambda\omega(X)$
- $\forall \alpha, \beta \in E^*$: $(\alpha+\beta)(X) \equiv \alpha(X) + \beta(X)$.

$\dim E^* = \dim E$ (if $\dim < \infty$) and given a basis $\{e_i\}$ of E, the dual basis of E^* is $\{e^i\}$ or $\{e^{*i}\}$. It is s.t: $e^i(X) = X^i$, the i^{th} comp of X in the basis $\{e_i\}$

ie: $e^i(X) = e^i(X^j e_j) = X^j e^i(e_j) \overset{!}{=} X^i \Rightarrow e^i(e_j) = \delta^i_j$.

Rmq: The evaluatᵒ of ω on X defines the interior product ι_X

$$\forall X \in E : \iota_X : E^* \longrightarrow \mathbb{K}, \quad \omega \longmapsto \iota_X\omega \equiv \omega(X)$$

It is clearly linear ✓

Matrix notatᵒ: $\left.\begin{array}{l}\omega \in E^* \\ X \in E\end{array}\right\}$ $\omega(X) = \omega_i e^i(X^j e_j) = \omega_i X^j e^i(e_j) = \omega_i X^i \in \mathbb{K}$.

↳ Say that the n-uple $\{X^i\}$ is written as a column vect $\begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}$ and the n-uple $\{\omega_i\}$ is written as a row vect $(\omega_1 \dots \omega_n)$

The qtity $\omega(X) = \omega_i X^i$ is seen as matrix multiplicatᵒ: $(\omega_1 \dots \omega_n)\begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix} \in \mathbb{K}$.

→ If E is an inner product space, with nondegenerate g, then

$\forall X \in E$ the map $g(X, \;) : E \longrightarrow \mathbb{K}$, $Y \longmapsto g(X,Y)$ is in $E^* = \mathcal{L}(E,\mathbb{K})$

In comp: $X = X^i e_i$, $Y = Y^j e_j$ and $g = g_{ij}\, e^i \otimes e^j$ with $g_{ij} \overset{(-)}{=} g_{ji}$ if not real.

so $g(X,Y) = g_{ij}\, e^i(X)\, e^j(Y) = g_{ij} X^i Y^j$

Thus $\underbrace{g(X, \;)}_{\alpha} = \underbrace{g_{ij} X^i}_{\alpha_j} e^j \in E^*$.

Terminology : $g(X,\ )$, can be noted X^* is the dual of X.

and $g_{ij} X^j$, noted often X_i are the | dual comp of X
| comp of the dual of X.

↳ in physics this is "lowering the index".

→ E^* has a dual nondegenerate metric $g^* : E^* \times E^* \longrightarrow \mathbb{K}$.
$$(\alpha, \beta) \longmapsto g^*(\alpha,\beta)$$

Given $\{e^i\}$ basis of E^* : $\alpha = \alpha_i e^i$, $\beta = \beta_j e^j$ and $g^* = g^{*ij} e_i \otimes e_j$.

so $g^*(\alpha,\beta) = g^{*ij} \alpha_i \beta_j$

The map $g^*(\alpha,\ ) : E^* \longrightarrow \mathbb{K}$ is like evaluating β on an element
$\beta \longmapsto g^*(\alpha,\beta)$ of E with comp $g^{*ij}\alpha_j$

This elemt is the dual of α and $g^{*ij}\alpha_j$ are the | dual comp of α
| comp of the dual of α.

↳ in physics this is "raising the index".

→ The link btwn g and g^* is as follow : given $X \in E$ and $\beta \in E^*$,

def $\alpha = g(X,\ ) \in E^*$. It is required that $\boxed{g^*(\alpha,\beta) = \beta(X)}$.

That is the commutativity of the diagram :

$$\begin{tikzcd}
\underset{E \times E^*}{(X,\beta)} \arrow[rr, "e"] \arrow[dr, "g \times id"'] & & \mathbb{K} \ni \beta(X) = g^*(\alpha,\beta) \\
& \underset{(\alpha,\beta)}{E^* \times E^*} \arrow[ur, "g^*"'] &
\end{tikzcd}$$

In comp that means :

$$g^{*ij}\alpha_i \beta_j = \beta_j X^j$$

$$↳ g^{*ij} g_{ik} X^k \beta_j = \beta_j X^j \quad \Rightarrow \boxed{g^{*ij} g_{ik} = \delta^j_k}$$

So the comp of the dual metric g^* are simply the inverses of the comp of the metric g!

↳ $g^{*ij} = (g_{ij})^{-1}$. Often the star is dropped and comp of g^* are written g^{ij},

which are called by abuse "inverse metric".

- Generalizat° : Let $\Lambda^k(E, \mathbb{K})$ be the space of alternating multilin $\mathbb{K}$-valued maps on E.

ie $\forall X_1 \dots X_k \in E$, $\omega \in \Lambda^k(E, \mathbb{K})$ is s.t : - $\omega(X_1 \dots X_k)$ is linear in each arg.

- $\omega(X_1 \dots X_i \dots X_j \dots X_k) = -\omega(X_1 \dots X_j \dots X_i \dots X_k)$.

Say that ω is an exterior form of degree k / a k-ext form.

~~Endow Λ^k~~

NB : $\forall k$, $\Lambda^k(E, \mathbb{K})$ is a $\mathbb{K}$-VS so $\Lambda(E, \mathbb{K}) \equiv \bigoplus_{k \in \mathbb{N}_0}^{n} \Lambda^k(E, \mathbb{K})$ is a graded $\mathbb{K}$-VS !

and $\Lambda^0(E, \mathbb{K}) = \mathbb{K}$.

→ Endow $\Lambda(E, \mathbb{K})$ with a bilin map

$$\begin{aligned} \wedge : \Lambda(E, \mathbb{K}) \times \Lambda(E, \mathbb{K}) &\longrightarrow \Lambda(E, \mathbb{K}) \\ (\alpha \ , \ \beta) &\longmapsto \alpha \wedge \beta \end{aligned}$$

the "exterior product" s.t :

- $\forall \alpha$ p-form, β q-form $\Rightarrow \alpha \wedge \beta$ $p+q$-form $\left\{ \begin{array}{l} \alpha \in \Lambda^p(E, \mathbb{K}) \\ \beta \in \Lambda^q(E, \mathbb{K}) \end{array} \right. \Rightarrow \alpha \wedge \beta \in \Lambda^{p+q}(E, \mathbb{K})$.

(explicitely : $(\alpha \wedge \beta)(X_1 \dots X_p, X_{p+1} \dots X_{p+q})$ (*)

$$= \frac{1}{p!q!} \sum_{\sigma} \operatorname{sign}(\sigma)\, \alpha(X_{\sigma(1)} \dots X_{\sigma(p)})\, \beta(X_{\sigma(p+1)} \dots X_{\sigma(p+q)})$$).

- $\boxed{\alpha \wedge \beta = (-)^{pq} \beta \wedge \alpha}$ (follow from (*)).

This product turn $\Lambda(E, \mathbb{K})$ into a graded $\mathbb{K}$-alg, and even a graded comm $\mathbb{K}$-alg !

Rmq : $\alpha \wedge \alpha = 0$ for α odd-form.

Ex : α, β two 1-forms, $\alpha \wedge \beta$ is a 2-form.

$\forall X, Y \in E$: $\left. \begin{array}{l} (\alpha \wedge \beta)(X, Y) \equiv \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \\ (\beta \wedge \alpha)(X, Y) \equiv \beta(X)\alpha(Y) - \beta(Y)\alpha(X). \end{array} \right\} \alpha \wedge \beta = (-)^{1 \times 1} \beta \wedge \alpha$.

Terminology : the ext product is also called wedge product.

- Generalizat° : Let $\Lambda^k(E,\mathbb{K})$ be the space of alternating multilin $\mathbb{K}$-valued maps on E.

ie $\forall X_1 \dots X_k \in E$, $\omega \in \Lambda^k(E,\mathbb{K})$ is s.t : - $\omega(X_1 \dots X_k)$ is linear in each arg.

- $\omega(X_1 \dots X_i \dots X_j \dots X_k) = -\omega(X_1 \dots X_j \dots X_i \dots X_k)$.

Say that ω is an exterior form of degree k /

~~Endow Λ^k~~

NB : $\forall k$, $\Lambda^k(E,\mathbb{K})$ is a $\mathbb{K}$-VS so $\Lambda(E,\mathbb{K}$ … VS !

and $\Lambda^0(E,\mathbb{K}) = \mathbb{K}$.

→ Endow $\Lambda(E,\mathbb{K})$ with a bilin map $\wedge : \Lambda$ …

the "exterior product" s.t :

⚠

Rmq : if $X_j = \lambda X_i$ colinear vect → $\omega(X_1 \dots X_i \dots X_j \dots) \equiv 0$

So in n-dim E, a $(n+1)$-ext form would eat $n+1$ vect with 2 necessary colinear, so would be $\equiv 0$!

Idem for any k-form, $k > n$!

⇒ There is no ext form of degree > dim E ✓

- $\forall\, \alpha$ p-form, β q-form ⇒ $\alpha \wedge \beta$ $p+q$-form $\left\{ \begin{array}{l} \alpha \in \Lambda^p(E,\mathbb{K}) \\ \beta \in \Lambda^q(E,\mathbb{K}) \end{array} \right.$ ⇒ $\alpha \wedge \beta \in \Lambda^{p+q}(E,\mathbb{K})$.

(explicitely : $(\alpha \wedge \beta)(X_1 \dots X_p, X_{p+1} \dots X_{p+q})$ (*)

$$= \frac{1}{p!q!} \sum_{\sigma} \text{sign}(\sigma)\, \alpha(X_{\sigma(1)} \dots X_{\sigma(p)})\, \beta(X_{\sigma(p+1)} \dots X_{\sigma(p+q)})$$).

- $\boxed{\alpha \wedge \beta = (-)^{pq}\, \beta \wedge \alpha}$ (follow from (*)).

This product turn $\Lambda(E,\mathbb{K})$ into a graded $\mathbb{K}$-alg, and even a graded comm $\mathbb{K}$-alg !

Rmq : $\alpha \wedge \alpha = 0$ for α odd-form.

Ex : α, β two 1-forms, $\alpha \wedge \beta$ is a 2-form.

$\forall X, Y \in E$: $\left. \begin{array}{l} (\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \\ (\beta \wedge \alpha)(X,Y) = \beta(X)\alpha(Y) - \beta(Y)\alpha(X). \end{array} \right\}$ $\alpha \wedge \beta = (-)^{1\times 1}\, \beta \wedge \alpha$.

Terminology : the ext product is also called wedge product.

→ Given $\{e^i\}$ basis of $E^* = \Lambda^1(E,\mathbb{K})$; $\{e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_k}\}$ is a basis for $\Lambda^k(E,\mathbb{K})$.

Any $\alpha \in \Lambda^k(E,\mathbb{K})$ is written as ; $\alpha = \frac{1}{k!}\alpha_{i_1\ldots i_k}\, e^{i_1}\wedge\ldots\wedge e^{i_k}$ with the coef, or component, $\alpha_{i_1\ldots i_k}$ is a totally antisym qtty/tensor.

→ If $\dim E = n$, and since $e^{i_1}\wedge\ldots\wedge e^{i_k} \equiv 0$ if $i_m = i_n$, we have that :

(1) $e^{i_1}\wedge\ldots\wedge e^{i_k} \equiv 0$ if $k>n$

ie there is no ext form of degree greater than $\dim E$.

A n-form is said to be of Max degree, or to be a top Form. ✓
or a volume form.

(2) there are $\binom{n}{k}$ possible different $e^{i_1}\wedge\ldots\wedge e^{i_k}$, $k\leq n$.

So $\dim \Lambda^k(E,\mathbb{K}) = \binom{n}{k}$.

And $\dim \Lambda(E,\mathbb{K}) = \sum_{k\in\mathbb{N}} \dim\Lambda^k(E,\mathbb{K}) = \sum_{k=0}^{n}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}1^k 1^{n-k}$

$= (1+1)^n = 2^n$ ✓

→ Hodge duality : since $\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$ there is a vect space iso $\Lambda^k(E,\mathbb{K}) \simeq \Lambda^{n-k}(E,\mathbb{K})$

Given a nondegenerate g on E / g^* on $E^* = \Lambda^1(E,\mathbb{K})$, the iso is a duality called the Hodge duality noted $*: \Lambda^k(E,\mathbb{K}) \longrightarrow \Lambda^{n-k}(E,\mathbb{K})$

that can be written explicitely as acting on the basis of $\Lambda^k(E,\mathbb{K})$;

$$*\,(e^{i_1}\wedge\ldots\wedge e^{i_k}) = \frac{|\det g|^{1/2}}{(n-k)!}\,\varepsilon_{j_1\ldots j_n}\, g^{i_1j_1}\ldots g^{i_kj_k}\, e^{j_{k+1}}\wedge\ldots\wedge e^{j_n} \quad (1)$$

So that if $\alpha = \frac{1}{k!}\alpha_{i_1\ldots i_k}\, e^{i_1}\wedge\ldots\wedge e^{i_k} \in \Lambda^k(E,\mathbb{K})$

and $*\alpha = \frac{1}{(n-k)!}(*\alpha)_{j_{k+1}\ldots j_n}\, e^{j_{k+1}}\wedge\ldots\wedge e^{j_n} \in \Lambda^{n-k}(E,\mathbb{K})$

$$(*\alpha)_{j_{k+1}\ldots j_n} = \frac{|\det g|^{1/2}}{k!}\,\alpha^{j_1\ldots j_k}\,\varepsilon_{j_1\ldots j_k j_{k+1}\ldots j_n}$$ via (1) ($\to$ dual tensor to $\alpha_{i_1\ldots i_k}$)

Rmq : $*^2 \equiv *\circ * : \Lambda^k(E,\mathbb{K}) \longrightarrow \Lambda^k(E,\mathbb{K})$

$\alpha \longmapsto **\alpha = (-)^{k(n-k)+s}\alpha$ where (r,s) signature of g.

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→ The interior product is s.t $\forall X \in E$; $\iota_X : \Lambda^k(E,\mathbb{K}) \longrightarrow \Lambda^{k-1}(E,\mathbb{K})$
$\omega \longmapsto \iota_X\omega$

ie $(\iota_X\omega)(X_1, \dots X_{k-1}) \equiv \omega(X, X_1 \dots X_{k-1})$ alternating on last $k-1$ arg.

It is obviously a linear map, homog of degree -1. And of course $\iota_X \Lambda^0(E,\mathbb{K}) = 0$.

But furthermore one shows that : $\iota_X(\alpha \wedge \beta) = \iota_X\alpha \wedge \beta + (-)^{|\alpha|} \alpha \wedge \iota_X \beta$.

↳ ι_X is a derivat° of $\Lambda(E,\mathbb{K})$, $\forall X \in E$, which is then a graded comm diff alg !

Rmq: by def $\iota_X \circ \iota_Y + \iota_Y \circ \iota_X \equiv 0$ so $\iota_X^2 \equiv \iota_X \circ \iota_X = 0$ → so comm DG-alg

The int prod, also called inner derivative, is a nilpotent operator.

Notice that the bracket def via the wedge product vanishes identically due to graded commutativity. Said otherwise $\Lambda(E,\mathbb{K})$ is an abelian graded Lie alg.

- Generalizat° bis : $\Lambda^1(E,\mathbb{K}) = E^* = \mathcal{L}(E,\mathbb{K})$ set of lin maps $E \longrightarrow \mathbb{K}$ $(= \Lambda^0(E,\mathbb{K}))$
$\Lambda^k(E,\mathbb{K})$ set of alt multilin maps $E^k \longrightarrow \mathbb{K}$.

$\mathbb{K}$ is a field, ie an alg where mult/prod is commutative and $\exists$ inverse a^{-1} $\forall a \in \mathcal{A}$.
Generalizat° = weakening of requirement.

- So we could replace $\mathbb{K}$ by a vect space V and study/have started with the set of lin maps $E \longrightarrow V$ (VS morphisms) : $\mathcal{L}(E,V) = \Lambda^1(E,V)$, then think of set of Alt multilin maps $E^k \longrightarrow V$: $\Lambda^k(E,V)$, up to the graded vect space of V-valued ext forms : $\Lambda(E,V) = \bigoplus_{k \in \mathbb{N}}^{n} \Lambda^k(E,V)$.
But notice that in this case the wedge product could not be defined since it needs a product in V ($\mathbb{K}$ previously) to be given. A priori no such prod $\exists$ if V is merely a vect space.
- Then richer generalizat° is replacing the fields $\mathbb{K}$ by algebras $\mathcal{A}$ ✓
Start with lin maps $E \longrightarrow \mathcal{A}$: $\mathcal{L}(E,\mathcal{A}) = \Lambda^1(E,\mathcal{A})$
Then alt lin maps $E^k \longrightarrow \mathcal{A}$: $\Lambda^k(E,\mathcal{A})$, call it a $\mathcal{A}$-valued k-ext form.
Up to graded vect space of $\mathcal{A}$-valued ext forms : $\Lambda(E,\mathcal{A}) = \bigoplus_{k \in \mathbb{N}} \Lambda^k(E,\mathcal{A})$.

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Can be endowed with a wedge product. If $a, b \in \mathcal{A}$ with prod $a \circ b \in \mathcal{A}$, then
$$\wedge : \Lambda(E,\mathcal{A}) \times \Lambda(E,\mathcal{A}) \longrightarrow \Lambda(E,\mathcal{A}) \qquad \text{for } \alpha \in \Lambda^p(E,\mathcal{A})$$
$$(\alpha, \beta) \longmapsto \alpha \wedge \beta \qquad \beta \in \Lambda^q(E,\mathcal{A})$$

is explicitely : $(\alpha\wedge\beta)(X_1 \dots X_{p+q}) = \frac{1}{p!q!} \sum_\sigma \operatorname{sign}(\sigma)\, \alpha(X_{\sigma(1)} \dots X_{\sigma(p)}) \circ \beta(X_{\sigma(p+1)} \dots X_{\sigma(p+q)})$.

NB : The product in $\mathcal{A}$, often implicit and noted written, is not necessarily commutative !
So a priori we do not have graded commutativity $\alpha\wedge\beta \neq (-)^{pq} \beta\wedge\alpha$.

↳ This means that $\Lambda(E,\mathcal{A})$ is a graded $\mathbb{K}$-algebra ✓

Rmq : It also mean that $\alpha \wedge \alpha \neq 0$ for α odd form.
And that under the graded bracket $[\alpha,\beta] \equiv \alpha\wedge\beta - (-)^{pq} \beta\wedge\alpha$
$\Lambda(E,\mathcal{A})$ is a graded Lie alg ✓ ↳ For α odd : $[\alpha,\alpha] = 2\,\alpha\wedge\alpha$!

The inner deriv is as before so $\Lambda(E,\mathcal{A})$ is already a graded diff $\mathbb{K}$-alg ✓
DG-alg

→ Given a basis (or generators) of $\mathcal{A}$: $\{\tau_a\}_{a=1\dots \ell}$ with $\ell = \dim \mathcal{A}$, and the basis $\{e^i\}$ of $E^* = \Lambda^1(E,\mathbb{K})$.

A 1-form $\alpha \in \Lambda^1(E,\mathcal{A})$ is written as : $\alpha = \alpha_i e^i = \alpha^a{}_i\, \tau_a \otimes e^i$.

A k-form $\alpha \in \Lambda^k(E,\mathcal{A})$ is : $\alpha = \frac{1}{k!}\alpha_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}$,
$= \frac{1}{k!}\alpha^a{}_{i_1 \dots i_k}\, \tau_a \otimes e^{i_1} \wedge \dots \wedge e^{i_k}$.

Often the tensor product will be omitted.

We still have $\dim \Lambda^k(E,\mathcal{A}) = \binom{n}{k}$ and $\dim \Lambda(E,\mathcal{A}) = 2^n$ ✓

The Hodge op $* : \Lambda^k(E,\mathcal{A}) \longrightarrow \Lambda^{n-k}(E,\mathcal{A})$ is still def and as before ✓
(if ∃ nondegenerate g on E)

2| Differential forms

- **Differential of a funct°** : Let f in the set of continuous $\mathbb{K}$-valued fct° on E (a normed vect space) noted $C^0(E,\mathbb{K})$ $\left(\subset C^0(E)\right)$

Rmq: $\mathcal{L}(E,\mathbb{K}) \subset C^0(E,\mathbb{K})$.

Def: The differential of f at $a \in E$ is the unique map $L \in \mathcal{L}(E,\mathbb{K})$ such that

$$\lim_{h\to 0} \frac{f(a+h)-f(a)-L(h)}{\|h\|} = 0$$

Notation : L is variously noted $Df(a)$, $df(a)$ or df_a. $L(h) = Df(a)(h) = df_a(h)$.

→ One says that f is differentiable on E if f is diff $\forall a \in E$.

Rmq: $\forall a \in E$, $Df(a) \in \mathcal{L}(E,\mathbb{K}) = E^* = \Lambda^1(E,\mathbb{K})$ is an exterior 1-form.

→ If f is diff on E, then on E we have the map $Df : E \longrightarrow \mathcal{L}(E,\mathbb{K})$.
If Df is cont on E, $Df \in C^0(E)$, then f is said to be of class 1 : $f \in C^1(E)$.

→ If Df is diff on E, then on E we have the map $DDf \equiv D^2 f : E \longrightarrow \mathcal{L}(E,\mathcal{L}(E,\mathbb{K})) = \mathcal{L}^2(E,\mathbb{K})$
where $\mathcal{L}^2(E,\mathbb{K})$ bilin maps $E\times E \longrightarrow \mathbb{K}$.

ie $D^2 f(a) : E\times E \longrightarrow \mathbb{K}$

If $D^2 f$ is cont on E, then Df is class $C^1(E)$ $\Leftrightarrow$ say f is class $C^2(E)$, twice diff !

→ Iterat° : f is $C^k(E)$ if Df is $C^{k-1}(E)$. (↳ Schwarz : $D^2 f(a)$ sym !)

f is $C^\infty(E)$ if it is $C^k(E)$ $\forall k \geq 1$. It is said "smooth".

Ex : $E = \mathbb{R} = \mathbb{K}$ $(= E^*)$ so $L(h) = Lh$, $h \in E = \mathbb{R} = E^* = \mathcal{L}(\mathbb{R},\mathbb{R}) \ni L$. $\|h\| = h$.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)-Lh}{h} = 0 \quad \Leftrightarrow \quad L = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Usually noted $L = f'(a)$ $\left(= Df(a)\right)$. And $f' = Df : E = \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{R},\mathbb{R}) = \mathbb{R}$, real fct°
$x \longmapsto f'(x)$

And $D^2 f : E = \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{R},\mathcal{L}(\mathbb{R},\mathbb{R})) = \mathcal{L}(\mathbb{R},\mathbb{R}) = \mathbb{R}$. still real fct°.
f'' $\quad x \longmapsto f''(x)$

Def: derivative of f at a in the directᵒ of $v \in E$ is $\lim\limits_{t\to 0} \dfrac{f(a+tv)-f(a)}{t}$.

the i^{th} partial derivative of f at a is if $v = e_i$, noted $\partial_i f(a)$ or $\dfrac{\partial f}{\partial x^i}(a)$:

$$\partial_i f(a) \equiv \lim_{t\to 0} \frac{f(a+te_i)-f(a)}{t}.$$

Fact: If f der in a, all partial der are defined: $\partial_i f(a)\ \exists\ \forall\, i \in [1 \dots \dim(E)]$.

and it happens that $\partial_i f(a) = Df(a)(e_i)$.

[proof: take $h = te_i$ in def of $L = Df(a)$. $L(te_i) = t\,L(e_i)$ and $\|te_i\| = |t|\,\|e_i\| = |t|$.

↳ def becomes $\lim\limits_{t\to 0} \dfrac{f(a+te_i)-f(a)}{|t|} = L(e_i) = Df(a)(e_i) = \partial_i f(a)$

Thus $Df(a)(h) = Df(a)(h^i e_i) = h^i Df(a)(e_i) = h^i \partial_i f(a)$.

NB: all this generalize easily starting with fctᵒ valued in a vectorspace (or an algebra A) V, so that $f(a) = f^k(a)\,\bar e_k$ with $\{\bar e_k\}$ basis of V (A). $\left(f : E \to V,\ a \mapsto f(a)\right)$

In particular $Df(a)(h) = Df^k(a)(h)\,\bar e_k$, with $Df^k(a)(h) = h^i \partial_i f^k(a)$.

• **Differential Forms:** Consider $f \in C^{r+1}(E)$, write its differential at $a \in E$ as df_a from now on.

We've noticed that $df_a \in \mathcal{L}(E,\mathbb{K}) = \Lambda^1(E,\mathbb{K})$ is an exterior 1-form.

We will say that the map $df : E \longrightarrow \Lambda^1(E,\mathbb{K})$, $x \longmapsto df_x$ is a differential 1-form of class C^r

↳ signal dependence on x

→ Generally the vect space of diff 1-forms $\omega : E \longrightarrow \Lambda^1(E,\mathbb{K})$, $x \longmapsto \omega_x$ of class C^r on E is noted $\Omega^1(E)$.

→ We generalize by def diff k-forms as mappings $\omega : E \longrightarrow \Lambda^k(E,\mathbb{K})$, $x \longmapsto \omega_x$, constituting a VS noted $\Omega^k(E)$.

In particular 0-forms are just functᵒ, $\omega : E \longrightarrow \Lambda^0(E,\mathbb{K}) = \mathbb{K}$.

Then $\boxed{\Omega(E) \equiv \bigoplus_{k \in \mathbb{N}} \Omega^k(E)}$ is the graded vect space of diff forms.

Endowed with the wedge product, defined pointwisely, it is the graded (comm!) alg of diff forms.

Thanks to the inner derivative ι_X, it is already a graded comm diff alg. (comm DG-alg)

UMONS

→ As with exterior forms this can be further generalized.

Consider ~~[illegible]~~ mappings $\omega : E \longrightarrow \Lambda^k(E,\mathcal{A})$, $x \longmapsto \omega_x$, noted $\Omega^k(E,\mathcal{A})$, they are $\mathcal{A}$-valued diff k-forms.

So that $\boxed{\Omega(E,\mathcal{A}) \equiv \bigoplus_{k\in\mathbb{N}} \Omega(E,\mathcal{A})}$ is the $\overset{\text{DG-alg}}{\text{graded diff alg}}$ of $\mathcal{A}$-valued $_{\text{diff}}$ forms.

Here again we have inner deriv, and the wedge prod is no longer graded commutative.
(So $[\alpha,\beta] \equiv \alpha\wedge\beta - (-)^{pq}\beta\wedge\alpha$ turn $\Omega(E,\mathcal{A})$ into a graded Lie alg)

→ Hodge star op $*: \Omega^k(E,\mathcal{A}) \longrightarrow \Omega^{n-k}(E,\mathcal{A})$ still def and behaves as before if $\exists$ nondeg g on E.

- The exterior derivative: we introduce on $\Omega(E,\mathcal{A})$ a derivat° of degree 1, the ext derivative noted d.

ie : $d: \Omega^k(E,\mathcal{A}) \longrightarrow \Omega^{k+1}(E,\mathcal{A})$ is a linear operator and for $\alpha \in \Omega^p(E,\mathcal{A})$ and $\beta \in \Omega(E,\mathcal{A})$ we have $\boxed{d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-)^p\alpha\wedge d\beta}$.

It is required nilpotent $\boxed{d^2=0}$. With this $\Omega(E,\mathcal{A})$ is again a DG-alg.

→ For a fct° $f \in \Omega^0(E,\mathcal{A})$, $df \in \Omega^1(E,\mathcal{A})$.
$f \in \Omega^0(E)$, $df \in \Omega^1(E)$. We want it to coincide with the differential $Df = df$ seen before! ↳ df has value in $\Lambda^1(E,\mathbb{R}) = E^*$

Given $\{e^i\}$ basis of $E^* = \Lambda^1(E,\mathbb{R})$: $df = \partial_i f\, e^i$, with $\partial_i f : E \longrightarrow \mathbb{R}$ (or $\mathcal{A}$) i^{th} partial der of f

Since $d^2=0$:
$$d^2 f = d(\partial_i f)\wedge e^i + (-)^0\partial_i f\wedge de^i$$
$$= \underbrace{\partial_j\partial_i f\, e^j\wedge e^i}_{=0 \text{ since } 1^{st} \text{ sym and } 2^{nd} \text{ antisym}} + \partial_i f\wedge de^i \Rightarrow \underline{de^i = 0}$$

Notat° : $e^i \equiv dx^i$, so that $de^i = d^2x^i = 0$ is intuitive. And coherent with historical notat° for diff of a funct° and the integrat° measure (see ahead).

↳ So for $f \in \underset{\Omega^0(E,\mathcal{A})}{\Omega^0(E)}$, $df = \partial_i f\, dx^i \in \underset{\Omega^1(E,\mathcal{A})}{\Omega^1(E)}$. Evaluated at x : $df_x = \partial_i f(x)\, dx^i \in \Lambda^1(E,\mathcal{A})$.

→ More generally a diff 1-form $\omega \in \Omega^1(E)$, $\omega : E \longrightarrow \Lambda^1(E,\mathbb{R})$ (or $\mathcal{A}$) will be written

$\omega = \omega_i\, dx^i$ with $\omega_i : E \longrightarrow \mathbb{R}$ (or $\mathcal{A}$) $\left(\text{ie } \omega_i \in \underset{\Omega^0(E,\mathcal{A})}{\Omega^0(E)}\right)$. And $\omega_x = \omega_i(x)\, dx^i \in \Lambda^1(E,\mathcal{A})$
$= \omega^a{}_i\, \tau_a\, dx^i$ $\qquad = \omega^a{}_i(x)\, dx^i$ (τ_a)

UMONS

→ The ext derivative of a 1-form is then : $d\omega = d\omega_j \wedge dx^j = \partial_i \omega_j \, dx^i \wedge dx^j$, $i<j$

or $d\omega = \frac{1}{2} \partial_{[i}\omega_{j]} \, dx^i \wedge dx^j$. $\in \Omega^2(E,\mathcal{A})$.

→ A diff k-form is $\omega = \frac{1}{k!}\omega_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$. $\in \Omega^k(E,\mathcal{A})$.

and $d\omega = \frac{1}{k!} \partial_i \omega_{i_1 \dots i_k} \, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$ $\in \Omega^k(E,\mathcal{A})$.

$= \frac{1}{(k+1)!} \partial_{[i}\omega_{i_1 \dots i_k]} \, dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

__Terminology:__ For $\alpha \in \Omega^p(E,\mathcal{A})$ and $\beta \in \Omega^q(E,\mathcal{A})$ one says that

- α is closed if $d\alpha = 0$ (in the ker of d) p-cocycle $\in Z^p(E,d)$
- α is exact if $\alpha = d\beta$ (in the image of d) p-coboundary $\in B^p(E,d)$

Since $d^2=0$ all exact forms are closed. But not all closed forms ~~are~~ are necessarily exact !?

↳ Let U subset of E and consider $\Omega^p(U,\mathbb{R})$.

Poincaré lemma : If U is star-shaped, ie $\exists\, x_0 \in U$ s.t $\forall x \in U$ the line joining x to x_0 is contained in U, then a closed form on U is exact :

$\alpha \in \Omega^p(U,\mathbb{R})$, $d\alpha = 0 \Rightarrow \alpha = d\beta$.

So on topologically nontrivial $U \subset E$ closed forms might not be exact.

The set $H^p(U,d) = Z^p(U,d) / B^p(U,d)$ of closed but non exact p-forms on $U \subset E$ is the

↳ (classes, up to coboundaries!)

(abelian/additive) p^{th} - cohomology grp of U.

And $H(U,d) = \bigoplus_{p=0}^{n} H^p(U,d)$ is the d-cohomology of ~~E~~ $U \subset E$.

Rmq: E as a vect space is star-shaped, topologically trivial so its cohomology is trivial : all closed forms are exact , $H^p(E,d) \equiv 0$.

NB : whenever there is a nilpotent operator on an algebra $\delta : \mathcal{A} \to \mathcal{A}$, $\delta^2 = 0$, a cohomology for this operator may arise ! → See just below !

Some digressions

- What we just saw is a special instance of a very general not°. Let's seize the opportunity to define it.

But first let's def a not° that is needed now and soon again:

Module over an algebra: A module M over an alg A – or a A-Module – is like a VS E over a field $\mathbb{K}$ – or a $\mathbb{K}$-Vect space –

M is an abelian/additive grp and there is an act° (a left one) $A \times M \longrightarrow M$, $(a, m) \longmapsto a \circ m$

s.t:
- $a\circ(m+n) = a\circ m + a\circ n$
- $(a+b)\circ m = a\circ m + b\circ m$
- $(ab)\circ m = a\circ(b\circ m)$

This is a left A-Module.

Similarly one def a right A-Module, with right act° $M \times A \longrightarrow M$, $(m, a) \longmapsto m\circ a$.

A A-bimodule is a left and right A-Module.

- Chain and cochain complexes

A sequence $\cdots \leftarrow C_{n-1} \xleftarrow{g_n} C_n \xleftarrow{g_{n+1}} C_{n+1} \leftarrow \cdots$ of abelian grps or modules C_n with morphisms $g_n : C_n \longrightarrow C_{n-1}$ s.t $g_n \circ g_{n+1} \equiv 0$, or $g^2 = 0$, is a chain complex

and elements of any C_n is a k-chain.

A k-chain c s.t $g_n(c) = 0$ is a k-cycle, or cycle. Space/subgrp of cycle is noted $Z_n = \ker(g_n) \subset C_n$. Element of Z_n are said closed.

A k-chain c s.t $c = g_{n+1}(b)$ for $b \in C_{n+1}$ is a k-boundary, or boundary. Subgrp of boundaries noted $B_n = \mathrm{im}(g_{n+1}) \subset Z_n \subset C_n$. Element of B_n are said exact.

Set of cycles that are not boundaries, or closed but non-exact element, is noted:

$H_n \equiv Z_n / B_n$ and called the n^{th}-homology grp.

$H \equiv \bigoplus_{k \in \mathbb{N}} H_n$ is the homology of the chain complex (C_n, g_n).

Similarly a sequence $\dots \longrightarrow C^{n-1} \xrightarrow{f_{n-1}} C^n \xrightarrow{f_{n+1}} C^{n+1} \longrightarrow \dots$ of ab grps of modules, with morphisms $f_n : C^n \to C^{n+1}$ s.t $f_{n+1} \circ f_n \equiv 0$, or $f^2 = 0$ is a cochain complex and elem^t of C^n are k-cochains.

A k-cochain c s.t $f_n(c) = 0$ is a k-cocycle or cocycle. Whole set is noted $Z^n = \ker(f_n) \subset C^n$. Elem^t of Z^n are said closed.

A k-cochain c s.t $c = f_{n-1}(b)$ for $b \in C^{n-1}$ is a k-coboundary, or coboundary. $B^n = \mathrm{im}(f_{n-1}) \subset Z^n \subset C^n$. Elem^t of B^n are said exact.

Set of cocycles that aren't coboundaries, or closed but non-exact elem^t, is noted: $H^n = Z^n / B^n$ and called the k^{th}-cohomology grp.

$H = \bigoplus_{k \in \mathbb{N}} H^n$ is the cohomology of the cochain complex (C^n, f_n)

or the ~~g~~ f-cohomology of $C^\cdot = \bigoplus_{k \in \mathbb{N}} C^n$.

- Exact sequences : It is a cochain complex s.t $\ker(f_{n+1}) = \mathrm{im}(f_n)$. (Such a cochain complex is called acyclic)

ie exact sequences are cochain complexes with vanishing cohomology.
[since in chain complexes in general $\ker(f_{n+1}) \supset \mathrm{im}(f_n)$]

- Short exact sequences (SES) :

  - The sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact at A iff $f : A \to B$ is injective.
  - The sequence $B \xrightarrow{g} C \longrightarrow 0$ is exact at C iff $g : B \to C$ is surjective.
  - The sequence $0 \longrightarrow A \xrightarrow{h} B \longrightarrow 0$ is exact iff $h : A \to B$ is an i/o.

Any SES is $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, f inj, g surj

and $C \simeq B / f(A)$

If $\exists\ h : C \to B$ s.t $g \circ h = id$ on C
or $\exists\ k : B \to A$ s.t $k \circ f = id$ on A } the SES is split and $B \cong f(A) \oplus h(C)$

Ex : in chap 3, fiber bundles give rise to SES and connect° are means to split it
↳ geom : h is horiz lift/dist while algebraic : k is connect° form !

Ex : Deriv of A gives rise to SES : $0 \to \mathrm{In}(A) \to \mathrm{Der}(A) \to \mathrm{Out}(A) \to 0$

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Coming back to diff forms, notice that we have the cochain complex :

$$\dots \longrightarrow \Omega^{k-1}(U,A) \xrightarrow{d_{k-1}} \Omega^k(U,A) \xrightarrow{d_k} \Omega^{k+1}(U,A) \longrightarrow \dots$$

with $d_{k-1} \circ d_k \equiv 0$ or $d^2 = 0$ ($U \subset E$ open subset) : De Rham Complex.

Diff forms are cochains, closed ones are cocycles $\in Z^k(U,d)$, exact ones are coboundaries $\in B^k(U,d)$, closed non-exact ones $\in H^k(U,d)$.

$H(U,d) = \bigoplus_{k \in \mathbb{N}} H^k(U,d)$ is the d-cohomology of the De Rham complex.

- Koszul formula :

→ On the assumption that ~~E is upgraded to~~ we have a Lie algebra L ; with bracket/product

$[\,,\,] : L \times L \longrightarrow L$, $\quad [X,Y] = -[Y,X]$ ①

$(X,Y) \longmapsto [X,Y]$ $\quad [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$. ②

and that M is a L-Module, ie a module s.t the action $L \times M \to M$, $(X,m) \mapsto X \cdot m$

satisfies $[X,Y] \cdot m = X \cdot (Y \cdot m) - Y \cdot (X \cdot m)$.

We consider diff forms $\omega \in \Omega(L,M)$, ie $\omega \in \Omega^k(L,M)$ if $\omega : E \longrightarrow \Lambda^k(L,M)$.

They form a cochain complex : $\dots \xrightarrow{d} \Omega^{k-1}(L,M) \xrightarrow{d} \Omega^k(L,M) \xrightarrow{d} \Omega^{k+1}(L,M) \xrightarrow{d} \dots$

where the differential d, s.t $d^2 \equiv 0$ is given explicitely by the Koszul formula :

(K) $\left\{ \forall \omega \in \Omega^k(L,M) ; \; d\omega(X_0,\dots X_k) \equiv \sum_{i=0}^{k} (-)^i X_i \cdot \omega(X_0 \dots \overset{i}{\vee} \dots X_k) + \sum_{i<j} (-)^{i+j} \omega([X_i,X_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots X_k) \right.$

Ex : Try on $\omega \in \Omega^1(L,M)$, $d\omega(X,Y) = ?$

↳ $d^2\omega(X,Y,Z) = d(d\omega)(X,Y,Z) = ?$

Verify that $d^2 = 0$ requires ① and ②.

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→ to turn $\Omega(L, M)$ into a DG-algebra, need to replace M by A so that the wedge prod is well-def: $\Omega(L, A)$, A remains a L-module.
(A being a priori non commutative, the wedge prod is not graded-commutative and the graded commutator can be defined: $[\alpha, \beta] = \alpha\wedge\beta - (-)^{pq}\beta\wedge\alpha$).

Need also to ask that L acts as derivat° on A, ie that $\forall X \in L$ and $a, b \in A$ $X(ab) = X(a)b + aX(b)$. This ensure that d, as defined by (K) satisfies $\left[d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-)^p \alpha\wedge d\beta \right]$ for $\alpha\in\Omega^p(L, A)$.

→ Said otherwise the exterior derivative d on the DG-algebra $\Omega(L, A)$ is given explicitely by the Koszul formula (K)!

- Lie derivative: Given A we saw that $Der(A)$ is a Lie alg with bracket
$[d_1 ; d_2] = d_1\circ d_2 - (-)^{|d_1||d_2|} d_2\circ d_1$.

Given the DG-alg $\Omega(L, A)$ we have the inner derivat° $i_X : \Omega^n(L, A) \longrightarrow \Omega^{n-1}(L, A)$, deg -1
$\forall X \in L$.
and the ext derivative $d : \Omega^n(L, A) \longrightarrow \Omega^{n+1}(L, A)$, deg 1

We can thus form $[i_X, d] : \Omega^n(L, A) \longrightarrow \Omega^n(L, A)$ of deg 0! It is noted L_X $\forall X\in L$. It is called the Lie derivative!

Since $L_X \in Der(\Omega(L, A))$ and is deg 0: $L_X(\alpha\wedge\beta) = L_X\alpha\wedge\beta + \alpha\wedge L_X\beta$ ✓

Rmq: i_X, d and L_X form a closed Lie alg under the graded bracket since;

- $[d, d] = d^2 + d^2 = 0$
- $[i_X, i_Y] = i_X i_Y + i_Y i_X = 0$
- $[i_X, d] = L_X$; $[L_X, d] = 0$

And one can show that given $[L_X, i_Y]$,
$[L_X, L_Y] = [L_X, i_Y]d + d[L_X, i_Y]$
By the way one shows that(*):

- $[L_X, i_Y] = i_{[X,Y]}$ so,
- $[L_X, L_Y] = L_{[X,Y]}$ ⟶ NB: L, the Lie der is a morphism of Lie algebras!

(*) Verify on fct° (0-forms) and 1-forms (which generates $\Omega(L, A)$).

- Pullback and integrat°

Consider
$$\begin{tikzcd} F \arrow[r, "f"] & \mathbb{K} \\ E \arrow[u, "\varphi"] \arrow[ur, "?"'] & \end{tikzcd}$$

E, F VS and f smooth.

The map from E to $\mathbb{K}$, defined by the commutativity of this diagram, is called the pullback by φ of f to E.

Noted $\varphi^* f : E \longrightarrow \mathbb{K}$

Explicitely : $\varphi^* f = f \circ \varphi$.

$\rightarrow$ It is possible to def pullback by φ to E of diff forms on F.

$\{e^i\}$ basis of $\Lambda^1(F, \mathbb{K}) = F^*$

$\{\bar{e}^j\}$ basis of $\Lambda^1(E, \mathbb{K}) = E^*$

then $\boxed{(\varphi^* e^j) \equiv e^j \circ d\varphi = d\varphi^j = \partial_i \varphi^j \bar{e}^i}$

So that for $\omega = \omega_j e^j \in \Omega^1(F)$ has pullback :

$$\varphi^* \omega = \varphi^* \omega_j \, \varphi^* e^j = \omega_j \circ \varphi \;\; \partial_i \varphi^j \bar{e}^i \quad \in \Omega^1(E)$$

$$\hookrightarrow (\varphi^* \omega)_x = \omega_j(\varphi(x)) \, \partial_i \varphi^j(x) \, \bar{e}^i \quad \forall x \in E$$

Rmq : φ^* goes in the reverse direct° of φ ; $\varphi : E \longrightarrow F$ while $\varphi^* : F^* \longrightarrow E^*$.

Generalizing for a k-form $\omega = \frac{1}{k!} \omega_{i_1 \dots i_k} e^{i_1} \wedge \dots \wedge e^{i_k} \in \Omega^k(F)$.

$$\varphi^* \omega = \frac{1}{k!} \omega_{i_1 \dots i_k} \circ \varphi \;\; \underbrace{\varphi^* e^{i_1} \wedge \dots \wedge e^{i_k}}$$

$$\partial_{j_1} \varphi^{i_1} \bar{e}^{j_1} \wedge \dots \wedge \partial_{j_k} \varphi^{i_k} \bar{e}^{j_k}$$

$$= \underbrace{\partial_{j_1} \varphi^{i_1} \dots \partial_{j_k} \varphi^{i_k} \; \underbrace{\bar{e}^{j_1} \wedge \dots \wedge \bar{e}^{j_k}}_{\varepsilon^{j_1 \dots j_k} \, \bar{e}^1 \dots \bar{e}^k}}$$

$$\det[\partial_j \varphi^i] \; \underbrace{\varepsilon^{i_1 \dots i_k} \, \bar{e}^1 \dots \bar{e}^k}_{\bar{e}^{i_1} \wedge \dots \wedge \bar{e}^{i_k}}$$

$$\boxed{\varphi^* \omega = \frac{1}{k!} \omega_{i_1 \dots i_k} \circ \varphi \;\; \det[\partial_j \varphi^i] \; \bar{e}^{i_1} \wedge \dots \wedge \bar{e}^{i_k}}$$

$$\hookrightarrow (\varphi^* \omega)_x = \frac{1}{k!} \omega_{i_1 \dots i_k}(\varphi(x)) \, \det[\partial_j \varphi^i(x)] \; \bar{e}^{i_1} \wedge \dots \wedge \bar{e}^{i_k}.$$

→ <u>Composit° of pullbacks:</u>

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & F \arrow[r, "\psi"] & G \end{tikzcd} \qquad \begin{tikzcd} \overset{\Omega(G)}{G^*} \arrow[r, "\psi^*"] \arrow[rr, bend right, "(\psi\circ\varphi)^*"'] & \overset{\Omega(F)}{F^*} \arrow[r, "\varphi^*"] & \overset{\Omega(E)}{E^*} \end{tikzcd}$$

The 2nd diagram implies : $\ulcorner (\psi\circ\varphi)^* = \varphi^*\circ\psi^* \lrcorner$

Indeed $(\psi\circ\varphi)^* f = f\circ(\psi\circ\varphi) = \varphi^*(f\circ\psi) = \varphi^*\circ\psi^* f$ ✓

and $\varphi^*(\psi^* e^j)_x = \varphi^*(\partial_i\psi^j(x)\, \bar{e}^i) = (\varphi^*\, \partial_i\psi^j)(x)\ \varphi^*\bar{e}^i$

$$= \underbrace{\partial_i\psi^j(\varphi(x))\ \partial_k\varphi^i(x)}\ \bar{e}^k$$

$$= \partial_k(\psi\circ\varphi)^j(x)\ \bar{e}^k \equiv (\psi\circ\varphi)^* e^j \quad ✓$$

↳ So $\forall\ \omega \in \Omega^k(G)$; $(\psi\circ\varphi)^* = \varphi^*\circ\psi^*\omega \in \Omega^k(E)$.

↳ General result on composit° of pullbacks : $(\psi\circ\varphi)^* = \varphi^*\circ\psi^*$.

→ <u>Naturality of pullback</u> : Consider $(\Omega(E), d_E)$ and $(\Omega(F), d_F)$.

The following diagram commute

ie $d_E(\varphi^*\omega) = \varphi^* d_F\omega$,

for $\omega \in \Omega(F)$.

$$\begin{tikzcd} \Omega(F) \arrow[r, "d_F"] \arrow[d, "\varphi^*"'] & \Omega(F) \arrow[d, "\varphi^*"] \\ \Omega(E) \arrow[r, "d_E"'] & \Omega(E) \end{tikzcd}$$

Indeed for a funct° f : $d_E(\varphi^* f) = d_E(f\circ\varphi) = \partial_j(f\circ\varphi)\,\bar{e}^j = \partial_i f\circ\varphi\ \partial_j\varphi^i\ \bar{e}^j$

$$= \varphi^*\,\partial_i f\ \ \varphi^* e^i = \varphi^*(\partial_i f\, e^i) = \varphi^* d_F f \quad ✓$$

For a 1-form : $\varphi^*\omega = \omega_j\circ\varphi\ \ \partial_i\varphi^j\ \bar{e}^i$

$$d_E(\varphi^*\omega) = \Big\{ d_E(\omega_j\circ\varphi)\ \partial_i\varphi^j + \omega_j\circ\varphi\ \underbrace{d_E(\partial_i\varphi^j)}_{\partial_k\partial_i\varphi^j\ \bar{e}^k\wedge\bar{e}^i\ \equiv\ 0.} \Big\}\ \bar{e}^i$$

$$= \partial_m\omega_j\circ\varphi\ \ \partial_k\varphi^m\ \bar{e}^k \wedge \partial_i\varphi^j\ \bar{e}^i$$

$$= \varphi^*(\partial_m\omega_j)\ \ \varphi^* e^m \wedge \varphi^* e^j = \varphi^* d_F\omega \quad ✓$$

⇒ Relat° true of any k-form ✓

→ **Intégrat°** : k-forms are integrable on k-dim domains Ω.

0-form (funct°) f int on 0-dim $\Omega = \{x\}$, a point : $\int_x f = f(x)$.

1-form ω int on 1-dim $\Omega = \gamma$, a curve : $\int \omega = \int \omega_i e^i = \int \omega_1 dt$.

$i=1$ e^1 note dt

2-form ω int on 2-dim Ω, a surface :

↳ $\int \omega = \int \frac{1}{2} \omega_{ij} e^i \wedge e^j = \int \frac{1}{2}(\omega_{12} - \omega_{21}) \varepsilon^{12} dx^1 dx^2 = \iint \omega_{12} dx^1 dx^2 = \iint \omega_{12} dx dy$.

$i = \{1,2\}$ Note $e^1 = dx^1 = dx$
$e^2 = dx^2 = dy$

Generalizat° : k-form int on k-dim domain $\int \omega = \underbrace{\int \cdots \int}_{k} \omega_{1 \dots n} dx^1 \dots dx^n$.

⊛ Let $\Omega \subset E$, $\omega \in \Omega^k(F)$ and $\varphi : E \to F$, then : $\boxed{\int_\Omega \varphi^* \omega = \int_{\varphi(\Omega)} \omega}$

NB : All that was said up to now is valid in the cases

- $E = F$ and φ is a coordinate change $y = \varphi(x)$.
- $E = \mathbb{R}^p$, F is n-dim $p < n$. $\varphi(E)$ subspace of F, φ a parametrizat° of this subspace.

(Try ex of 1-form ω, $\Omega = [a,b]$ a segment in $E = \mathbb{R}$, $\varphi : [a,b] \to F$, $\varphi(\Omega) = \gamma$ curve in F)

Stokes Thm : Ω a k-dim bounded domain with boundary $\partial\Omega$. ω a $(k-1)$-form.

Then $$\int_\Omega d\omega = \int_{\partial\Omega} \omega .$$

$k=1$: ω 0-form, $d\omega = \partial_1 \omega e^1 = \partial_t \omega dt$ → $\int_a^b \partial_t \omega dt = \omega(b) - \omega(a)$

$\Omega = [a,b] \subset \mathbb{R}$, $\partial\Omega = \{a,b\}$

Fund Thm of calculus !

$k=2$: ω 1-form $\omega = \omega_1 e^1 + \omega_2 e^2 = \omega_x dx + \omega_y dy$, Ω 2d, $\partial\Omega$ closed curve.

$d\omega = \frac{1}{2}(\partial_1 \omega_2 - \partial_2 \omega_1) e^1 \wedge e^2 = (\partial_1 \omega_2 - \partial_2 \omega_1) e^1 e^2 = (\partial_x \omega_y - \partial_y \omega_x) dx dy$

↳ $\iint_\Omega (\partial_x \omega_y - \partial_y \omega_x) dx dy = \oint_{\partial\Omega} \omega_x dx + \omega_y dy$ Green Thm !

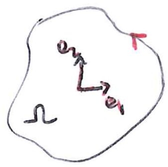
→ orientat° of domains : Forms are int on oriented domains (most often).
Orientat° on a domain induces orientat° on its boundary.

- a 1D domain γ is oriented if it is given a sense of parcour:

$\Omega = \gamma \qquad \partial\Omega = \{a, b\}$

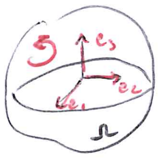
Rmq: only 2 possible orientat° on $\Omega / \partial\Omega$!

- a 2D domain is oriented via prescript° of order on basis.



$\partial\Omega = \gamma$ closed curve.

- 3D domain, idem



$\partial\Omega$ = closed surface.

NB: $\partial^2 = \partial \circ \partial \equiv 0$, nilpotent operator.

define a cell or k-simplex as a k-dim domain: $c \in C_k$ the set of all such domain.

$\partial : C_k \longrightarrow C_{k-1}$, This is a chain complex (C_k, ∂):

$$\dots \; C_{k-1} \xleftarrow{\;\partial\;} C_k \xleftarrow{\;\partial\;} C_{k+1} \longleftarrow \dots$$

Boundaryless cells, $\partial c = 0$, are cycle, ∂-closed $\in Z_k$

Boundaries of a cell, $c = \partial b$, are ∂-exact $\in B_k$

Boundaryless cells that are not boundaries $\in H_k = Z_k / B_k$ = simplicial homology!

↳ informs on non-triviality of topology of domains!

Rmq: Integrat° associate domains to forms to produce number $\mathbb{R}$: $\int_\Omega : \Omega^k(E) \longrightarrow \mathbb{R}$.
So int puts domain and forms in "duality" as one can write $\langle \omega, \Omega \rangle \equiv \int_\Omega \omega$ for Ω k-dim domain and ω a k-form.

From this viewpoint the ext derivative and the boundary operator are adjoint wrt this duality, since Stokes Thm is written: $\langle d\omega, \Omega \rangle = \langle \omega, \partial\Omega \rangle$
for ω a (k-1)-form.