3) Manifolds

M Topological manifold : set of points s.t $\forall p \in M$ $\exists \mathcal{U}$ around p and a map $\phi : \mathcal{U} \longrightarrow \mathbb{R}^n$. $n = \dim M$. $\{\mathcal{U}, \phi\}$ called a local chart. or coordinate chart. $\phi(p) = \{x^\mu(p)\}$ also written $\phi^\mu(p)$.
$\{\mathcal{U}_i\}_{i \in I \subset \mathbb{R}}$ called a covering of M if $\bigcup_{i \in I} \mathcal{U}_i = M$.
$\{\mathcal{U}_i, \phi_i\}$ = atlas of M. And for $\mathcal{U}_i \cap \mathcal{U}_j \neq \phi$

$\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous.

A differentiable manifold is a topological manifold s.t $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are differentiable.
A smooth manifold is a diff manifold s.t $\phi_j \circ \phi_i^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth : C^∞-Manifold.
↳ it is possible to define differential calculus.

A funct° $f : M \longrightarrow \mathbb{R}$ is smooth iff $\tilde{f} = f \circ \phi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth.
Can write $f(p) = \tilde{f}(x^\mu)$. ↳ (For any $\{\mathcal{U}_i, \phi_i\}$ covering the domain of f).

The set of smooth fct° on M is noted $C^\infty(M)$. It is a commutative algebra under the pointwise product $(fg)(p) = f(p)\, g(p)$.

Tangent Vectors : most basic object one can define on diff Man.
Given a curve $\gamma : \mathbb{R} \longrightarrow M$, $\tau \longmapsto \gamma(\tau)$ s.t $\gamma(0) = p \in M$, the tgt vect at p is def as : $X_p \equiv \frac{d}{d\tau}\Big|_{\tau=0} \gamma(\tau) = \mathring{\gamma}(\tau)\Big|_{\tau=0}$.
Rmq that many curves can give rise to same vect. The set of all possible tgt vect at p (tgt to all possible curves passing through p) is called the tgt space at p, noted T_pM.

It is not evident, but it has a vect space struct. To see this it is useful to understand how vect acts as derivat° of smooth fct°.

Take $f \in C^\infty(M)$, $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$, $\tau \longmapsto f(\gamma(\tau))$ the tgt vect X_p at p to γ is seen as acting on f via : $\boxed{X_p(f)(p) \equiv \frac{d}{dt}\Big|_{\tau=0} f \circ \gamma(\tau)}$.

The structure of vect space is clearly seen since given γ_1 and γ_2 with tgt X_1, X_2, both at p, it is possible to add $f \circ \gamma_1$ and $f \circ \gamma_2$ and the def above gives :

$$\frac{d}{d\tau}\Big|_{\tau=0} [(f \circ \gamma_1) + (f \circ \gamma_2)](\tau) \equiv X_{1,p}(f)(p) + X_{2,p}(f)(p) = (X_1 + X_2)_p(f)(p) \checkmark$$

From the def it is also clear that X_p is a derivat° of funct° since given $f, g \in C^\infty(M)$

$$X_p(fg)(p) \equiv \frac{d}{d\tau}\Big|_{\tau=0} (fg) \circ \gamma(\tau) = \frac{d}{d\tau}\Big|_{\tau=0} f(\gamma(\tau))\, g(\gamma(\tau)) = \frac{d}{d\tau}\Big|_{\tau=0} f(\gamma(\tau)) \cdot g(p) + f(p) \frac{d}{d\tau}\Big|_{\tau=0} g(\gamma(\tau))$$

$$\equiv X_p(f)(p)\, g(p) + f(p)\, X_p(g)(p).$$

$\hookrightarrow$ $\boxed{X(fg) = X(f)g + fX(g)}$. $\checkmark$

$\rightarrow$ It is useful to have a coordinate rep of those. Write $\phi = x^\mu$ for the coord|chart funct°.

Define $x^\mu \circ \gamma : \mathbb{R} \longrightarrow M \longrightarrow \mathbb{R}^n$

$\tau \longmapsto \gamma(\tau) \longmapsto x^\mu \circ \gamma(\tau) \equiv \gamma^\mu(\tau)$.

Then $\frac{d}{d\tau}\Big|_{\tau=0} \gamma^\mu(\tau) = \dot{\gamma}^\mu(0) \equiv X^\mu(p) \in \mathbb{R}^n$.

And $X_p(f)(p) \equiv \frac{d}{d\tau}\Big|_{\tau=0} \tilde{f} \circ \gamma^\mu(\tau) = \frac{\partial}{\partial x^\mu} \tilde{f}(\gamma^\mu(\tau)) \frac{d}{d\tau}\gamma^\mu(\tau)\Big|_{\tau=0}$

$$= \dot{\gamma}^\mu(0) \frac{\partial}{\partial x^\mu} f(p) = X^\mu(p) \frac{\partial}{\partial x^\mu} f(p)$$

So $\boxed{X_p = X^\mu(p) \frac{\partial}{\partial x^\mu}}$ we see how it is a 1^{st}-order differential operator/a derivat°.

NB : if γ^i curve def direct° of increase of coord x^i, $i \in \{1 \dots n\}$, ie setting $\tau = x^i$

then $\dot{\gamma}^\mu(0) = \begin{cases} 1 & \mu = i \\ 0 & \mu \neq 0 \end{cases}$ So the tgt vect at p associated with coord syst $\{x^\mu\}$ are $\{\partial_\mu\}_p$ and form a basis for T_pM. (equiv of $\{e_i\}$ in E).

It is called a frame, or holonomic frame.

$\hookrightarrow$ dessin au tableau ?

Tgt bundle : The set of all tgt spaces at any $p \in M$; $TM \equiv \bigcup_{p\in M} T_pM$.

It is a manifold, smooth, with local coord ~~patches~~ $\{x^\mu_{(p)}, X^\mu_{(p)}\}$ so that locally, for $\mathcal{U} \subset M$; $TM|_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^n$.

Vect fields : Smooth maps $X : M \longrightarrow TM$, $p \longmapsto X_p$. Set of them noted $\mathfrak{X}(M)$ or $\Gamma(TM)$

latter notat° to mean that vect fields are sect° of the Tgt bundle.

Elements of $\Gamma(TM)$ acts as derivat° of degree 0 of the algebra $C^\infty(M)$
They belong to $Der(C^\infty(M))$. From the general case we know $Der(A)$ is a Lie alg for the bracket $[d_1, d_2] \equiv d_1 \circ d_2 - (-)^{|d_1||d_2|} d_2 \circ d_1$. So we have that bracket of vect fields is still a vect field : $[X,Y] \equiv X\circ Y - Y \circ X \in \Gamma(TM)$

↳ Vect fields form a Lie algebra, and $C^\infty(M)$ is a module for this Lie alg. ie $C^\infty(M)$ is a $\Gamma(TM)$-module.

Rmq: curve $\gamma(\tau)$ to which X is tgt $\forall \tau$ is the flow of X. integral curve.

Coordinate change :

$$x^\mu = \phi : M \longrightarrow \mathbb{R}^n \xrightarrow{\phi'} \mathbb{R}^n$$
$$p \longmapsto \underset{q}{x^\mu(p)} \longmapsto \underset{r}{y^\mu(x^\nu(p))}$$

n fct° of n variables (not always invertible!).

If so, Jacobian of coord change $\frac{\partial y^\mu}{\partial x^\nu}(p) \equiv G^\mu{}_\nu(p)$ is a $n\times n$ invertible matrix : so $\in GL_n(\mathbb{R})$

Rmq: not all $n\times n$ coef in $G^\mu{}_\nu$ are indep or arbitrary.
A funct° $\tilde{G}^\mu{}_\nu : M \longrightarrow GL_n(\mathbb{R})$, $p \longmapsto \tilde{G}^\mu{}_\nu(p)$ would have more d.o.f than a coord change.

So for later considerat°, keep in mind that GL-inv and coord inv are not identical.

↳ Clearly the notion of vect (field) is indep of coord choice. But it is important to know how its coord representat° changes.

Write $\gamma^\mu(\tau)$ in coord $\{x^\mu\}$ and $\gamma^{\mu'}(\tau)$ in coord $\{y^\mu\}$

Have $f(p) = \tilde{f}(x^\mu) = \tilde{f}'(y^\mu)$

So $f(\gamma(\tau)) = \tilde{f}(\gamma^\mu(\tau)) = \tilde{f}'(\gamma^{\mu'}(\tau))$.

Then : $X_p(f)(p) \equiv \frac{d}{d\tau}\Big|_{\tau=0} \tilde{f}'(\gamma^{\mu'}(\tau)) = \frac{\partial}{\partial y^\mu} \tilde{f}'(\gamma^{\mu'}(\tau)) \underbrace{\frac{d}{d\tau}\Big|_{\tau=0} \gamma^{\mu'}(\tau)}_{\equiv Y^\mu(p)} = Y^\mu(p) \frac{\partial}{\partial y^\mu} f(p).$

but also $= \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial}{\partial x^\nu} \tilde{f}(\gamma^\nu(\tau)) \cdot \frac{\partial y^\mu}{\partial x^\alpha}(\gamma^\alpha(\tau)) \frac{d}{d\tau}\gamma^\alpha(\tau)\Big|_{\tau=0}.$ ⊛ = since $\frac{\partial y^\mu}{\partial x^\sigma}\frac{\partial x^\nu}{\partial y^\mu} = \delta^\nu_\sigma$

$= \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial}{\partial x^\nu} f(p) \frac{\partial y^\mu}{\partial x^\alpha}(p) X^\alpha(p) = \frac{\partial y^\mu}{\partial x^\alpha} X^\alpha(p) \cdot \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} f(p).$

So as expected the frame trsf as $\frac{\partial}{\partial x^\mu} \longrightarrow \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} \equiv \frac{\partial}{\partial y^\mu}$ with the inv Jacobian, and the coord rep as $X^\mu \longrightarrow \frac{\partial y^\mu}{\partial x^\alpha} X^\alpha \equiv Y^\mu$.

But in both coord syst : $X = X^\mu \frac{\partial}{\partial x^\mu} = Y^\nu \frac{\partial}{\partial y^\nu}$ ⊛

<u>Frame bundle</u> : Given a (first) choice of coord syst $\{x^\mu\}$ on $U \subset M$, we have the associated frame $\{\partial_\mu\}$. Any other choice gives another frame $\{\bar{G}^\nu_\mu \partial_\nu\}$.

<u>Rmq</u> : a further change would be $\{\bar{G}'^\nu_\mu \bar{G}^\alpha_\nu \partial_\alpha\}$, notice this is a <u>right act°</u>.

A right act° is an operat° s.t $R_{ab} = R_b \circ R_a$. $\left[\begin{array}{l} R_G \partial = \bar{G}\partial \quad \text{then :} \\ R_{G'} R_G \partial = R_{GG'}\partial = (GG')\bar{}\,\partial = \bar{G}'\bar{G}\partial \checkmark \end{array}\right.$

The set of all frames at p, noted L_pM is the orbit of ∂_ν by $\bar{G}^\nu_\mu \in GL_n(\mathbb{R})$

The frame bundle is $LM = \bigcup_{p \in M} L_pM$. It is a manifold with local coordinates $\{x^\mu, G^\mu{}_\nu\}$ so that locally, for $U \subset M$; $LM|_U \simeq U \times GL_n(\mathbb{R})$. (⇒ Principal bundle)

↳ A point in LM is a point p of M with coord x^μ, <u>and</u> a frame $G_\mu{}^\nu \partial_\nu$ with coord $G_\mu{}^\nu$.

<u>NB</u> : the coord indep of X means that if $(\frac{\partial}{\partial x^\mu} ; X^\mu)$ is a descript° of X, so is $(\bar{G}^\mu_\nu \frac{\partial}{\partial x^\mu} ; G^\nu_\nu X^\nu) = (\frac{\partial}{\partial y^\nu} ; Y^\nu)$. First is a frame, point in LM, second is elect of $\mathbb{R}^n$.

A vect field is then equiv class $[\partial_\mu ; X^\mu] = [\bar{G}^\mu{}_\nu \partial_\mu ; G^\nu{}_\nu X^\nu]$ by act° of $GL_n(\mathbb{R})$

Note $TM = LM \times_{GL} \mathbb{R}^n \equiv LM \times \mathbb{R}^n / GL$, says TM vector bundle <u>associated</u> to LM via the act° of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$.

Pushforward: Let $\varphi : M \to N$, $p \mapsto \varphi(p)$ be a smooth map. $\{x^\mu\}$ around p, $\{y^\nu\}$ around $\varphi(p)$.
↳ $x^\mu(p)$ ↳ $\varphi^\nu(p) = y^\nu \circ \varphi(p)$

The associated pushforward is a smooth map $\varphi_* : TM \to TN$
Given $X_p \in T_pM$ and $f : N \to \mathbb{R}$ it is by def: $X_p \mapsto (\varphi_* X)_{\varphi(p)}$

$$(\varphi_* X)_{\varphi(p)}(f)(\varphi(p)) \equiv \frac{d}{d\tau}\Big|_{\tau=0} f(\varphi(\gamma(\tau))) = \frac{d}{d\tau}\Big|_{\tau=0} \tilde{f}(\varphi^\nu(\gamma(\tau))) = \frac{\partial}{\partial y^\nu}\tilde{f}(\varphi^\nu(\gamma(\tau)))\frac{d}{d\tau}\varphi^\nu(\gamma^\mu(\tau))\Big|_{\tau=0}$$

$$= \frac{\partial}{\partial y^\nu} f(\varphi(\gamma(\tau)))\frac{\partial \varphi^\nu}{\partial x^\mu}(\gamma(\tau))\frac{d}{d\tau}\gamma^\mu(\tau)\Big|_{\tau=0}.$$

$$= \frac{\partial \varphi^\nu}{\partial x^\mu}(p)\, X^\mu(p)\, \frac{\partial}{\partial y^\nu} f\,(\varphi(p)) \quad ✓$$

Rmq: if $\varphi : M \to M'$ is smooth and has smooth inverse it is called a diffeomorphism.
Set of all diffeo, noted Diff(M), is a grp under composit°: if $\varphi : M \to M$ diffeo and $\psi : M \to M$ also diffeo and $\operatorname{Im}\varphi \cap \operatorname{domain}\psi \neq \emptyset$, then $\psi \circ \varphi$ is a diffeo.
Notice that $\varphi_* X$ closely resemble the formula for coord change of the rep of X in the $\{y^\mu\}$ chart: $X_p(f)(p) = \frac{\partial y^\mu}{\partial x^\alpha} X^\alpha(p) \frac{\partial}{\partial y^\mu} f(p)$.

But these are really different things, the latter is a passive trsf: a relabelling of the same objects (p and X), the former is an active trsf: objects are dragged by φ on the manifold from one point to another.

↳ In GR physics, it is said that th/physics is coord-inv → but then it is also Diff(M) invariant: th/physics cannot say if things happen at p or $\varphi(p)$ on M!
Therefore points in M are not physical and M is not spacetime!
Spacetime is equiv class of M under Diff(M): M, M' which are diffeomorphic are physically indistinguishable.
Said otherwise, points of M aren't physical; relat° btwn fields/event labelled by points are.

→ Pushforward is a Lie alg morphism: $\varphi_*[X,Y]_p = [\varphi_* X, \varphi_* Y]_{\varphi(p)}$.

→ Composit° law :

$$\begin{array}{ccccc} M & \xrightarrow{\varphi} & N & \xrightarrow{\psi} & O \\ p & \longmapsto & \varphi(p) & \longmapsto & \psi\circ\varphi(p) = r \\ \{x^\mu\} & & \{y^\nu\} = q & & \{z^\alpha\} \end{array}$$

$$TM \xrightarrow{\varphi_*} TN \xrightarrow{\psi_*} TO \qquad (\psi\circ\varphi)_* = \psi_*\circ\varphi_* .$$

$$\left[(\psi\circ\varphi)_* X\right](f) \equiv \frac{d}{d\tau}\Big|_{\tau=0} f\left(\psi\circ\varphi\circ\gamma(\tau)\right) = \frac{\partial}{\partial z^\alpha} f\left(\psi\circ\varphi\circ\gamma(\tau)\right) \frac{d}{d\tau}(\psi\circ\varphi)^\alpha(\gamma(\tau))\Big|_{\tau=0}$$

$$= \frac{\partial}{\partial z^\alpha} f\left(\psi\circ\varphi\circ\gamma(\tau)\right) \frac{\partial \psi^\alpha}{\partial y^\nu} \frac{d}{d\tau} \varphi^\nu(\gamma(\tau))\Big|_{\tau=0}$$

$$= \frac{\partial}{\partial z^\alpha} f\left(\psi\circ\varphi\circ\gamma(\tau)\right) \frac{\partial \psi^\alpha}{\partial y^\nu}(\varphi\circ\gamma(\tau)) \frac{\partial \varphi^\nu}{\partial x^\mu}(\gamma(\tau)) \frac{d}{d\tau} \gamma^\mu(\tau)\Big|_{\tau=0} .$$

$$= \frac{\partial \psi^\alpha}{\partial y^\nu}(r) \frac{\partial \varphi^\nu}{\partial x^\mu}(q)\, X^\mu(p) \frac{\partial}{\partial z^\alpha} f(r) \quad ✓$$

Cotangent bundle : dual space to T_pM noted T_p^*M = space of linear maps $\omega : T_pM \longrightarrow \mathbb{R}$.
Since $\{\partial/\partial x^\mu\}$ basis T_pM, note $\{dx^\mu\}$ basis of T_p^*M so that $dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$
and a covector at p is $\omega_p = \omega_\mu(p)\, dx^\mu$. Also called ext 1-form.
Cotangent bundle is $T^*M = \bigcup_{p\in M} T_p^*M$. It is a manifold with local coord $\{x^\mu, \omega_\mu\}$ so that for $\mathcal{U} \subset M$; $T^*M|_{\mathcal{U}} \simeq \mathcal{U} \times \mathbb{R}^{n*}$ where $\mathbb{R}^{n*}$ is dual of $\mathbb{R}^n$, ie space of row vector.

↳ A covector field, or differential 1-form is a smooth map

$$\omega : M \longrightarrow T^*M, \qquad p \longmapsto \omega_p = \omega_\mu(p)\, dx^\mu$$

⇒ Given $\varphi : M \to N$, the pullback $\varphi^* : T^*N \longrightarrow T^*M$, $\omega_{\varphi(p)} \longmapsto (\varphi^*\omega)_p$ is defined as before, with same composit° law : $(\psi\circ\varphi)^* = \varphi^*\circ\psi^*$.

There is a duality btwn pullback and push forward : Let $X_p \in T_pM$ and $\omega_{\varphi(p)} \in T^*_{\varphi(p)}N$

We have $\left\lceil (\varphi^*\omega)_p(X_p) = \omega_{\varphi(p)}(\varphi_* X_{|\varphi(p)}) \right\rfloor$

$$= \omega_\alpha(\varphi(p)) \frac{\partial \varphi^\alpha}{\partial x^\nu} dx^\nu \left(X^\mu \frac{\partial}{\partial x^\mu}\right) \quad \text{see p15}$$

$$= \omega_\alpha(\varphi(p)) \frac{\partial \varphi^\alpha}{\partial x^\mu} X^\mu .$$

↳ p23 $= \omega_\alpha(\varphi(p))\, dy^\alpha \left(\frac{\partial \varphi^\nu}{\partial x^\mu} X^\mu \frac{\partial}{\partial y^\nu}\right)$

$$= \omega_\nu(\varphi(p)) \frac{\partial \varphi^\nu}{\partial x^\mu} X^\mu .$$

Rmk: tensor bundle $\mathcal{T}^r_s = \overset{r}{\otimes} TM \overset{s}{\otimes} T^*M$

tensor field = smooth map $T : M \to \mathcal{T}^r_s$

$p \mapsto T(p)^r_s \in \overset{r}{\otimes} T_pM \overset{s}{\otimes} T^*_pM$

Differential forms on a manifold :

T_pM is a vector space so we can define the ext algebra on it $\Lambda(T_pM, \mathbb{R})$ with inner prod, ext prod. From this can define maps from M to $\Lambda(T_pM, \mathbb{R})$, so we have the DG-alg of diff forms on M, noted $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$, where $\Omega^k(M)$ stands for $\Omega^k(\Gamma(TM), \mathbb{R})$. A diff form on M "eats" vectors fields and spits out real numbers.

$i_X : \Omega^k(M) \to \Omega^{k-1}(M)$, $\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$ known as "Cartan magic formula"

$d : \Omega^k(M) \to \Omega^{k+1}(M)$, i_X and $d \in Der(\Omega(M))$ so $L_X \equiv [i_X, d] \in Der(\Omega(M))$

Furthermore since $\omega(X_1 \dots X_k) \in C^\infty(M)$ for $\omega \in \Omega^k(M)$ and since $C^\infty(M)$ is a $\Gamma(TM)$-module, the exterior derivative is given by the Koszul formula.

The cochain complex $(\Omega(M), d)$: $\dots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \dots$ is known as the DeRham complex, d the DeRham derivative.

The DeRham cohomology $H(M,d) = \bigoplus_{k=0}^{\dim M} H^k(M,d)$ with $H^k(M,d) = Z^k(M,d) / B^k(M,d)$

where $Z^k(M,d)$ set of closed k-forms and $B^k(M,d)$ set of exact k-forms.

($\hookrightarrow$ Measure how far from an exact sequence DeRham complex is.)

$\omega \in \Omega^k(M)$ is a coord indep object, but given a coord syst $\{x^\mu\}$ on $\mathcal{U} \subset M$ a k-form on $\mathcal{U} \subset M$ is written $\omega = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \in \Omega^k(\mathcal{U}, \mathbb{R})$, with $\omega_{\mu_1 \dots \mu_k} \in C^\infty(\mathcal{U})$

evaluated at $p \in \mathcal{U}$: $\omega_p = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k}(p)\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$.

NB: $\Omega(M)$ is graded commutative; $\alpha \wedge \beta = (-)^{pq} \beta \wedge \alpha$ for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$.

$\to$ Generalises to algebra-valued diff forms : $\Omega(M, \mathcal{A}) \simeq \Omega(M) \otimes \mathcal{A}$, still a DG-alg.
It is a Lie alg for the bracket ; $[\alpha, \beta] \equiv \alpha \wedge \beta - (-)^{pq} \beta \wedge \alpha$.

→ Given $\varphi: M \to N$ smooth, the pullback generalizes $\varphi^*: \Omega(N) \to \Omega(M)$.

and we still have naturality of pullback : $d_M \circ \varphi^* = \varphi^* \circ d_N$

→ The integration of a k-form on M is as follow : given $U \subset M$ and $\omega \in \Omega^k(U)$

$\varphi = \phi^{-1}$ for $\phi: U \subset M \to \Omega \subset \mathbb{R}^k$ a coord syst on $U \subset M$ and Ω domain in $\mathbb{R}^k$.

$p \mapsto \phi(p) = x^\mu$

Then $\displaystyle\int_{U=\varphi(\Omega)} \omega \equiv \int_\Omega \varphi^* \omega$ ① This allow to def int on M as integral in $\mathbb{R}^k$.

Thus is also def $\int_U : \Omega^n(M) \to \mathbb{R}$ and duality $\langle \omega, U \rangle \stackrel{①}{=}$ with U seen as

elem^t of chain complex (C_n, ∂) and we still have Stokes Thm : $\langle d\omega, U \rangle = \langle \omega, \partial U \rangle$

Adding a new object for first time : connection.

A connect° is a map $\nabla: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

which is $C^\infty(M)$-linear in the first arg :
- $\nabla_{X+X'} Y = \nabla_X Y + \nabla_{X'} Y$.
- $\nabla_{fX} Y = f \nabla_X Y$, $f \in C^\infty(M)$.

and is additive and Leibniz-like on 2nd arg :
- $\nabla_X (Y+Y') = \nabla_X Y + \nabla_X Y'$
- $\nabla_X (fY) = X(f) Y + f \nabla_X Y$.

→ In a coord syst $\{x^\mu\}$ on $U \subset M$, ∇ is entirely determined by its act° on $\{\partial_\mu\}$:

define Christoffel symbols / Symbols of the connect° by $\boxed{\nabla_{\partial_\mu} \partial_\nu \equiv \Gamma^\alpha_{\mu\nu} \partial_\alpha}$.

So that $\nabla_X Y = \nabla_{X^\mu \partial_\mu}(Y^\nu \partial_\nu) = X^\mu \nabla_{\partial_\mu}(Y^\nu \partial_\nu) = X^\mu \{ \partial_\mu Y^\nu \partial_\nu + Y^\nu \underbrace{\nabla_{\partial_\mu} \partial_\nu}_{\Gamma^\alpha_{\mu\nu} \partial_\alpha} \}$

$= X^\mu \{ \partial_\mu Y^\nu + Y^\alpha \Gamma^\nu_{\mu\alpha} \} \partial_\nu$

$= X^\mu \{ \partial_\mu Y^\nu + \Gamma^\nu_{\mu\alpha} Y^\alpha \} \partial_\nu \equiv X^\mu \nabla_\mu Y^\nu \partial_\nu$. ✓

Under coord change $\{y^\mu\}$, $\partial_{\mu'} \equiv \frac{\partial}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial y^\mu}\partial_\nu$

Since $X = X^\mu \partial_\mu = X^{\mu'}\partial_{\mu'}$ is inv / coord indep, so is $\nabla_X Y$, but the coord representat° of symbols of connect° change:

$$\Gamma'^{\rho}_{\mu\nu}\,\partial_{\rho'} \equiv \nabla_{\partial_{\mu'}}\partial_{\nu'} = \nabla_{\frac{\partial x^\alpha}{\partial y^\mu}\partial_\alpha}\left(\frac{\partial x^\beta}{\partial y^\nu}\partial_\beta\right) = \frac{\partial x^\alpha}{\partial y^\mu}\nabla_{\partial_\alpha}\left(\frac{\partial x^\beta}{\partial y^\nu}\partial_\beta\right) = \frac{\partial x^\alpha}{\partial y^\mu}\partial_\alpha\left(\frac{\partial x^{\sigma}}{\partial y^\nu}\right)\partial_{\sigma} + \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\underbrace{\nabla_{\partial_\alpha}\partial_\beta}_{\Gamma^\sigma_{\alpha\beta}\partial_\sigma}$$

$$\Gamma'^{\rho}_{\mu\nu}\frac{\partial x^\sigma}{\partial y^\rho}\partial_\sigma = \underbrace{\frac{\partial x^\alpha}{\partial y^\mu}\partial_\alpha}_{\frac{\partial}{\partial y^\mu}}\left(\frac{\partial x^\sigma}{\partial y^\nu}\right)\partial_\sigma + \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\Gamma^\sigma_{\alpha\beta}\partial_\sigma$$

$$\hookrightarrow \boxed{\Gamma'^{\rho}_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\Gamma^\sigma_{\alpha\beta}\frac{\partial y^\rho}{\partial x^\sigma} + \frac{\partial^2 x^\sigma}{\partial y^\mu \partial y^\nu}\frac{\partial y^\rho}{\partial x^\sigma}}$$

→ parallel transport : consider a curve $\gamma : \mathbb{R} \to M$, $\tau \mapsto \gamma(\tau)$ with tgt vector $\dot\gamma(\tau)$

The // transp of Y along γ is by def : $\nabla_{\dot\gamma(\tau)} Y \in \Gamma(TM)$ s.t $\nabla_{\dot\gamma(\tau)} Y = 0$.
Y is // transported along γ if : $\nabla_{\dot\gamma(\tau)} Y = 0$.

A geodesic is an auto parallel curve, ie a curve s.t $\boxed{\nabla_{\dot\gamma(\tau)}\dot\gamma(\tau) \equiv 0}$. (1)
It is a curve of straightest path !

In a coord syst $\{x^\mu\}$ the curve is $\gamma^\mu(\tau)$ and $\dot\gamma(\tau) = \dot\gamma^\mu(\tau)\partial_\mu$, so that we have (1) as : $\dot\gamma^\mu\partial_\mu\dot\gamma^\nu + \Gamma^\nu_{\mu\rho}\dot\gamma^\mu\dot\gamma^\rho = 0$

or $\boxed{\ddot\gamma^\rho + \Gamma^\rho_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu = 0}$ rmk: Often coord of curve is $x^\mu(\tau)$.

→ Is it possible to define act° of ∇_X on $\Omega^1(M)$?

First need to def act° on $C^\infty(M)$: $\nabla_X f \equiv X(f)$

Then, since $\omega(Y) \in C^\infty(M)$: $\nabla_X(\omega(Y)) = X(\omega(Y)) = X(\omega_\nu Y^\nu) = X^\mu\partial_\mu(\omega_\nu Y^\nu)$
$= X^\mu(\partial_\mu\omega_\nu Y^\nu + \omega_\nu\partial_\mu Y^\nu)$.

Remark that $\omega(\nabla_X Y) = \omega_\nu(X^\mu\partial_\mu Y^\nu + \Gamma^\nu_{\mu\sigma}Y^\sigma X^\mu)$.

Could then require $(\nabla_X\omega)(Y) \equiv X^\mu(\partial_\mu\omega_\nu - \omega_\sigma\Gamma^\sigma_{\mu\nu})Y^\nu$

So that we would have $\nabla_X[\omega(Y)] = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)$!

So, define $\nabla : \Gamma(TM)\times\Omega^1(M) \longrightarrow \Omega^1(M)$ which is given in a coord syst by
$(X,\omega) \longmapsto \nabla_X\omega$

act° on $\{\partial_\mu\}$ and $\{dx^\mu\}$: $\nabla_{\partial_\mu} dx^\nu \equiv -\Gamma^\nu_{\mu\sigma}dx^\sigma$

So indeed $\nabla_X\omega = X^\mu\nabla_{\partial_\mu}(\omega_\nu dx^\nu) = X^\mu(\partial_\mu\omega_\nu dx^\nu + \omega_\nu \underbrace{\nabla_{\partial_\mu}dx^\nu}_{-\Gamma^\nu_{\mu\sigma}dx^\sigma})$

$= X^\mu(\underline{\partial_\mu\omega_\nu - \omega_\sigma\Gamma^\sigma_{\mu\nu}})dx^\nu = X^\mu \underline{\nabla_\mu\omega_\nu} dx^\nu$ ✓

↳ This generalizes to $\nabla_X : \mathcal{T}^r_s \longrightarrow \mathcal{T}^r_s$ acting as a derivat° that does not change tensorial order, ie : $\nabla_X(T\otimes S) = \nabla_X T\otimes S + T\otimes\nabla_X S$.

→ <u>Torsion</u> : smooth map $T : \Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM)$
$(X,Y) \longmapsto T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y]$

Clearly $T(X,Y) = -T(Y,X)$. It is def by act° on $\{\partial_\mu\}$, given a coord syst on $U\subset M$:

$T(\partial_\mu,\partial_\nu) \equiv \underline{T_{\mu\nu}{}^\sigma}\partial_\sigma = \nabla_{\partial_\mu}\partial_\nu - \nabla_{\partial_\nu}\partial_\mu - \cancelto{0}{[\partial_\mu,\partial_\nu]} = (\underline{\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}})\partial_\sigma$.

$T(X,Y) = 0 \implies \nabla_X Y - \nabla_Y X = [X,Y]$ or $\iff \Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ sym symbol of connect° !

→ <u>Curvature</u> : Smooth map $R : \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM)$
$(X,Y,Z) \longmapsto R(X,Y)Z \equiv ([\nabla_X,\nabla_Y] - \nabla_{[X,Y]})Z$.

Clearly antisym on X,Y. Given $\{x^\mu\}$ on $U\subset M$, R is given by its act° on $\{\partial_\mu\}$:

$R(\partial_\mu,\partial_\nu)\partial_\beta \equiv \underline{R_{\mu\nu\beta}{}^\sigma}\partial_\sigma = [\nabla_{\partial_\mu},\nabla_{\partial_\nu}]\partial_\beta - \cancelto{0}{[\partial_\mu,\partial_\nu]}\partial_\beta = \nabla_{\partial_\mu}\nabla_{\partial_\nu}\partial_\beta - \nabla_{\partial_\nu}\nabla_{\partial_\mu}\partial_\beta$

$= \nabla_{\partial_\mu}(\Gamma^\sigma_{\nu\beta}\partial_\sigma) - \nabla_{\partial_\nu}(\Gamma^\sigma_{\mu\beta}\partial_\sigma)$

$= \partial_\mu\Gamma^\sigma_{\nu\beta}\partial_\sigma + \Gamma^\sigma_{\nu\beta}\underbrace{\nabla_{\partial_\mu}\partial_\sigma}_{\Gamma^\tau_{\mu\sigma}\partial_\tau} - \partial_\nu\Gamma^\sigma_{\mu\beta}\partial_\sigma - \Gamma^\sigma_{\mu\beta}\underbrace{\nabla_{\partial_\nu}\partial_\sigma}_{\Gamma^\tau_{\nu\sigma}\partial_\tau}$

$= (\underline{\partial_\mu\Gamma^\sigma_{\nu\beta} - \partial_\nu\Gamma^\sigma_{\mu\beta} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\beta} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\beta}})\partial_\sigma$

Adding another object : Metric

The metric is a smooth $g : M \longrightarrow B(TM)$ into sym non degenerate bilin forms on T_pM with sign (r,s).
$p \longmapsto g_p$

That is $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$
$(X,Y) \longmapsto g_p(X,Y)$

Or $g : \Gamma(TM) \times \Gamma(TM) \longrightarrow C^\infty(M)$
$(X,Y) \longmapsto g(X,Y)$

- $s=0$: Riemannian Manifold
- $s\neq 0$: Pseudo-Riemannian Manifold
- $r=1,\ s\neq 0$: Lorentzian Manifold
  ↳ spacetime in Physics
  in GR $r=1, s=3$; $(+---)$

Norm of X def by $\|X\|^2 \equiv g(X,X)$.

→ In a coord syst $\{x^\mu\}$ on $U \subset M$: $g(\partial_\mu,\partial_\nu) \equiv g_{\mu\nu}$ so that $g(X,Y) = g_{\mu\nu} X^\mu Y^\nu$.

Under coord change $\{y^\mu\}$: $g^*(\partial_{\mu'},\partial_{\nu'}) = g'_{\mu\nu}$ also $= g\left(\frac{\partial x^\alpha}{\partial y^\mu}\partial_\alpha, \frac{\partial x^\beta}{\partial y^\nu}\partial_\beta\right) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} g(\partial_\alpha,\partial_\beta)$

That is : $\ulcorner\ g'_{\mu\nu}\dfrac{\partial y^\mu}{\partial x^\alpha}\dfrac{\partial y^\nu}{\partial x^\beta} = g_{\alpha\beta}\ \lrcorner$

→ Under Diff(M) : $\varphi \in$ Diff(M) is an isometry of g if $g_{\varphi(p)}(\varphi_* X, \varphi_* Y) = g_p(X,Y)$.

ie $g_{\mu\nu}(\varphi(p))\dfrac{\partial\varphi^\mu}{\partial x^\alpha}\dfrac{\partial\varphi^\nu}{\partial x^\beta} X^\alpha X^\beta = g_{\alpha\beta}(p) X^\alpha Y^\beta \Rightarrow \ulcorner\ g_{\mu\nu}(\varphi(p))\dfrac{\partial\varphi^\mu}{\partial x^\alpha}\dfrac{\partial\varphi^\nu}{\partial x^\beta} = g_{\alpha\beta}(p)\ \lrcorner$

NB : resemblance btwn coord change and isometric diff !

→ Thanks to g on TM we can define Hodge duality, $* : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ as before on p4'. Idem on $\Omega(M,A)$, $* : \Omega^k(M,A) \longrightarrow \Omega^{n-k}(M,A)$.

→ Also, g selects a special connect° ; the unique ∇ s.t :
- $T(X,Y) = 0 \quad \forall X,Y$
- $\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

It is the Levi-Civita connect°.

In component it means $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$, symetry of symbols of ∇. (1)

And $X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

$$X^\mu \partial_\mu(g_{\alpha\beta} X^\alpha Z^\beta) = g_{\alpha\beta} X^\mu(\cancel{\partial_\mu Y^\alpha} + \Gamma^\alpha_{\mu\nu} Y^\nu) Z^\beta + g_{\alpha\beta} Y^\alpha X^\mu(\cancel{\partial_\mu Z^\beta} + \Gamma^\beta_{\mu\nu} Z^\nu)$$

$$X^\mu \partial_\mu g_{\alpha\beta}\cdot Y^\alpha Z^\beta + \cancel{g_{\alpha\beta} X^\mu \partial_\mu Y^\alpha \cdot Z^\beta} + \cancel{g_{\alpha\beta} Y^\alpha X^\mu \partial_\mu Z^\beta}$$

So : $X^\mu\left\{\partial_\mu g_{\alpha\beta} Y^\alpha Z^\beta - g_{\nu\beta}\Gamma^\nu_{\mu\alpha} Y^\alpha Z^\beta - g_{\alpha\nu}\Gamma^\nu_{\mu\beta} Y^\alpha Z^\beta\right\} = 0$

That is $\partial_\mu g_{\alpha\beta} - g_{\nu\beta}\Gamma^\nu_{\mu\alpha} - g_{\alpha\nu}\Gamma^\nu_{\mu\beta} \equiv \nabla_\mu g_{\alpha\beta} = 0$ ✓ (2)

- Given a curve γ along which Y and Z are //-trsported we thus have:

$$\nabla_{\dot\gamma}\, g(Y,Z) = g(\underbrace{\nabla_{\dot\gamma} Y}_{0}, Z) + g(Y, \underbrace{\nabla_{\dot\gamma} Z}_{0}) = 0 \; !$$

ie the "scalar" prod (the angle) btwn 2 vect is preserved under //-trsp def by the LC connect°.

- In particular $\nabla_{\dot\gamma}\, g(X,X) = 0$ for $\nabla_{\dot\gamma} X = 0$. So the norm of //-trsprted vect is also preserved.
- Also for geodesic γ: $\nabla_{\dot\gamma}\dot\gamma = 0$ so $\dot\gamma\, g(\dot\gamma,\dot\gamma) = 2g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = 0$.

  ie tgt vect of a geodesic has cst norm.

- Solving (1) & (2), $\Gamma^\alpha_{\mu\nu}$ is written as funct° of $g_{\mu\nu}$: $\Gamma = \Gamma(g) \to \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$
- Given a curve $\gamma(\tau)$ an infinitesimal arc length is $g(\dot\gamma,\dot\gamma)^{1/2} d\tau = L(\gamma)$.

  The length of γ between two points p and q is $\int_p^q L(\gamma)$. The extremal curve γ s.t $\delta \int_p^q L(\gamma) = 0$ has equat° (Euler-Lagrange eq): $\ddot\gamma^\alpha + \Gamma^\alpha_{\mu\nu}\dot\gamma^\mu\dot\gamma^\nu = 0$ where the Γ's are those in term of g: $\Gamma = \Gamma(g)$.

↳ ie geodesics of LC connect° are extremal curves!

<u>Bilan</u>:

- Topological level: no diff calculus, ∃ chain complex (C_n, ∂); info on topology.
- Differential level: diff calc on M possible, TM, T^*M, cochain / De Rham complex $(\Omega(M), d)$, $\Omega(M)$ DG-alg (variational calc possible).

Gravitat° happens here!

- Connect° level: // trsp, geodesic, torsion, curvature.
- Metric level: Hodge duality on $\Omega(M)$, length / angle, causal struct on M, LC-connect°; geodesic = extremal curve.

4) Lie groups

A Lie grp is a grp which is also a diff manifold so that its composit° law

$G \times G \to G$ is smooth and s.t

- it is associative $(ab)c = a(bc) = abc$
- has neutral element (unit) e so that $ae = ea = a$.
- $\forall a \in G$, $\exists a'$ s.t $aa' = a'a = e$.

Rmk: As a manifold it is not necessarily connected.

If the comp law is s.t $ab = ba$, the grp is said abelian.

→ since G is a diff Man, its tangent bundle is defined T_gG at any $g \in G$: $TG = \bigcup_{g \in G} T_gG$

In particular $T_eG \equiv \text{Lie}\, G$ is the Lie alg of G.

→ The left translat° is the smooth map $L_g : G \to G$, $h \mapsto L_g h \equiv gh$ def $\forall g \in G$.

Its pushforward is $L_{g*} : T_hG \to T_{gh}G$, $X_h \mapsto L_{g*}X_h$

A vect is said left invariant if $L_{g*} X_h = X_{gh}$.

Its pullback is $L_g^* : \Omega_{gh}(G) \to \Omega_h(G)$

The right translat° is $R_g : G \to G$, $h \mapsto R_g h \equiv hg$ def $\forall g \in G$.

Its pushforward is $R_{g*} : T_hG \to T_{hg}G$, $X_h \mapsto R_{g*}X_h$, The pullback is $R_g^* : \Omega_{hg}(G) \to \Omega_h(G)$

NB: both L_g and R_g are diffeo of G, $\text{Diff}(G)$. They clearly commute: $L_h R_g = R_g L_h$.

→ The Maurer-Cartan form: $\forall g \in G$ def as $L_{g^{-1}*} : T_gG \to T_eG = \text{Lie}\, G$.

Noted $\varpi_g \equiv L_{g^{-1}*}$, $\varpi \in \Omega^1(G, \text{Lie}G)$ so that $\omega(X) \in C^\infty(G, \text{Lie}G)$ for $X \in \Gamma(TG)$.

The MC form is <u>left invariant</u> : $L_g^* \varpi_{gh}(X_h) = \varpi_{gh}(L_{g*}X_h) \equiv L_{(gh)^{-1}*}(L_{g*}X_h)$
$= L_{h^{-1}g^{-1}*}L_{g*}X_h = L_{h^{-1}*}X_h \equiv \varpi_h(X_h)$.

$\hookrightarrow \boxed{L_g^* \varpi_{gh} = \varpi_h}$

It is <u>Ad_g-equivariant</u> (right equiv) : $R_g^* \varpi_{hg}(X_h) = \varpi_{hg}(R_{g*}X_h) \equiv L_{(hg)^{-1}*}R_{g*}X_h$

$= L_{g^{-1}*}L_{h^{-1}*}R_{g*}X_h = L_{g^{-1}*}R_{g*}L_{h^{-1}*}X_h \equiv (L_{g^{-1}}R_g)_*\varpi_h(X_h) = g^{-1}\varpi_h(X_h)g$

$=: Ad_{g^{-1}}\varpi_h(X_h) \qquad \Rightarrow \boxed{R^*_g \varpi_{hg} = Ad_{g^{-1}}\varpi_h}$

<u>Rmq</u> : If X is left-inv then $\varpi_g(X_g) = L_{g^{-1}*}X_g = X_e \quad \forall g \in G$.
ie $\varpi(X)$ is a cst fct° on G.

→ The exterior derivative on $\Omega(G, \text{Lie}\, G)$ is given by the Koszul formula since indeed $\Gamma(TG)$ is a Lie-alg and $C^\infty(M, \text{Lie}\, G)$ is a $\Gamma(TG)$-module. So on X, Y left inv :

$$d\varpi(X,Y) = X\cdot\underbrace{\varpi(Y)}_{cst} - Y\cdot\underbrace{\varpi(X)}_{cst} - \varpi([X,Y]) = -\varpi([X,Y])$$

by the way $\forall g \in G \quad \varpi_g([X,Y]_g) \equiv L_{g^{-1}*}[X,Y]_g = [L_{g^{-1}*}X_g, L_{g^{-1}*}Y_g]_e$
since pushforward Lie-alg morphism. So $\varpi_g([X,Y]_g) = [\varpi_g(X_g), \varpi_g(Y_g)]$
where the bracket on right is in $\text{Lie}\, G$.

$\Rightarrow \quad d\varpi(X,Y) + [\varpi(X), \varpi(Y)] = 0$

In term of the graded commutator in $\Omega(G, \text{Lie}\, G)$ this is : $\boxed{d\varpi + \frac{1}{2}[\varpi,\varpi] = 0}$.
This is the <u>Maurer-Cartan eq</u>.

It is valid on any $X, Y \in \Gamma(TG)$, not just on left inv ones, since left-inv vect are basis of $T_gG \ \forall g \in G$. Indeed $\forall X_e \in \text{Lie}\, G$, $L_{h*}X_e$ is left-inv : $L_{g*}(L_{h*}X_e) = L_{gh*}X_e$
So given a basis $\{\partial_{e,i}\}$ of $\text{Lie}\, G$ $\{L_{g*}\partial_{e,i}\}$ is a basis of T_gG ✓

→ For $X \in \text{Lie}\, G = T_eG$, its flow is $c(\tau)$, $c(0) = e$ and $\frac{d}{d\tau}c(\tau)\big|_{\tau=0} = X$. So usually one notes $c(\tau) \equiv \exp \tau X$, as the curve in G generated by $X \in \text{Lie}\, G$.

→ Representat° of groups : Given G a Lie grp , a rep of G is a space (usually a VS) V and a grp morphism $\rho : G \to GL(V)$ s.t $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$.

For any $v \in V$, the act° of $g \in G$ is rep as $\rho(g)v \in V$.

As special applicat° of the pushforward, we have the induced representat° of the Lie algebra of G, $\rho_* : T_eG = \mathrm{Lie}\,G \to M_n(V)$ which is a Lie-alg homomorphism.

$$X \longmapsto \rho_*(X)$$

For any $v \in V$, the act° of Lie G is rep as $\rho_*(X)v \in V$.

- It happens that Lie G is a rep space for G, via the Ad-rep :

$$\mathrm{Ad} : G \to GL(\mathrm{Lie}\,G), \quad g \longmapsto \mathrm{Ad}_g$$, so that $\forall X \in \mathrm{Lie}\,G$; $\mathrm{Ad}_g X \equiv g X g^{-1}$ ✓

And indeed : $\mathrm{Ad}_{g_1 g_2} X \equiv g_1 g_2 X (g_1 g_2)^{-1} = g_1 g_2 X g_2^{-1} g_1 = \mathrm{Ad}_{g_1}(\mathrm{Ad}_{g_2} X)$

So $\mathrm{Ad}_{g_1 g_2} = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}$ ✓

- This induces a rep of Lie G on itself, via the ad-rep :

$$\mathrm{ad} : \mathrm{Lie}\,G \to M_n(\mathrm{Lie}\,G), \quad X \longmapsto \mathrm{ad}_X$$

Take $g(\tau) = \exp \tau X$, $g(0) = e = g(0)^{-1}$, $\frac{d}{d\tau}\Big|_{\tau=0} g(\tau) = 0$ and $\frac{d}{d\tau}\Big|_{\tau=0} g(\tau)^{-1} = -g(\tau)^{-1} \frac{d}{d\tau}\Big|_{\tau=0} g(\tau) \cdot g(\tau)^{-1}$

$$\mathrm{ad}_X(Y) \equiv \mathrm{Ad}_{g(\tau)*} Y = \frac{d}{d\tau}\Big|_{\tau=0} \mathrm{Ad}_{g(\tau)} Y$$

$$= \frac{d}{d\tau}\Big|_{\tau=0} g(\tau) Y g(\tau)^{-1} = \overbrace{\frac{d}{d\tau}\Big|_{\tau=0} g(\tau)}^{X} Y g(\tau)^{-1}\Big|_{\tau=0} + g(\tau) Y\Big|_{\tau=0} \overbrace{\frac{d}{d\tau}\Big|_{\tau=0} g(\tau)^{-1}} = XY - YX$$

↳ $\mathrm{ad}_X(Y) \equiv [X, Y]$ ✓

And indeed : $(\mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X) Z = [X,[Y,Z]] - [Y,[X,Z]] = [X,[Y,Z]] + [Y,[Z,X]]$

$= -[Z,[X,Y]]$ via Jacobi identity.

$= [[X,Y],Z] \equiv \mathrm{ad}_{[X,Y]}(Z)$.

So $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$ and ad is indeed a Lie alg-morphism ✓

→ Given a basis $\{e_i\}$ of Lie G, the struct coef of Lie G are def via :

$[e_i, e_j] = C^k_{ij}\, e_k$, so that $C^k_{ij} = -C^k_{ji}$. $\{C^k_{ij}\}$ caracterises the Lie alg of G.

→ Given (ρ, V) a rep of G, one can define a sym bilinear form on Lie G via :

$B(X,Y) \equiv \mathrm{Tr}\left[\rho_*(X)\rho_*(Y)\right] \quad \forall\ X, Y \in \mathrm{Lie}\,G$. ie $B : \mathrm{Lie}\,G \times \mathrm{Lie}\,G \longrightarrow \mathbb{R}$

The Killing form (def by Cartan) is the sym bilin form where $\rho_* \equiv \mathrm{ad}$!

$k(X,Y) \equiv \mathrm{Tr}\left(\mathrm{ad}_X \circ \mathrm{ad}_Y\right)$. ie $k : \mathrm{Lie}\,G \times \mathrm{Lie}\,G \longrightarrow \mathbb{R}$.

In a basis $\{e_i\}$ of Lie G : $\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}(e_k) = \mathrm{ad}_{e_i}([e_j, e_k]) = C^m_{jk}\ \mathrm{ad}_{e_i}(e_m)$

$= C^m_{jk}\, C^n_{im}\, e_n$.

↳ $\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}$ is a matrix sending e_k to e_n : Tr is thus taken over k and n.

Thus $k(X,Y) = k_{ij}\, X^i Y^j = C^m_{jn}\, C^n_{im}\, X^i Y^j$

since $k(e_i, e_j) \equiv \mathrm{Tr}(\mathrm{ad}_{e_i} \circ \mathrm{ad}_{e_j}) = C^m_{jn}\, C^n_{im}$ ✓

- In particular for matrix Lie alg : $X^I e_I = X^{ij} e_{ij}$ where $\{e_{ij}\}$ basis of matrices.
  A priori no constraint $\{e_{ij}\}$ basis of $M_n(\mathbb{R})$ = Lie alg of $GL_n(\mathbb{R})$.
  Given $[e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{li}\, e_{kj} = \underbrace{\left(\delta_{jk}\, \delta^m_i\, \delta^n_l - \delta_{li}\, \delta^m_k\, \delta^n_j\right)}_{C^{mn}_{ij,kl}} e_{mn}$

  One finds $K_{IJ} = C^M_{IN}\, C^N_{JM}$

  ie $k_{ij\,kl} = C^{gh}_{ij,mn}\, C^{mn}_{kl,gh} = 2n\, \delta_{il}\, \delta_{kj} - 2\, \delta_{ij}\, \delta_{kl}$ ✓

  So $k(X,Y) = k_{ijkl}\, X^{ij} Y^{kl} = 2n\, X^{ik} Y^{ki} - 2 X^{ii} Y^{kk} = 2n\, \mathrm{Tr}(XY) - 2\, \mathrm{Tr}(X)\mathrm{Tr}(Y)$. ✓

  For Special grps, ie with $\det = 1$, elmt of Lie alg is traceless so : $k(X,Y) \propto \mathrm{Tr}(XY)$ ✓