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III. Bundles and connect°

1) Principal bundle.

noted $P \xrightarrow{\pi} M$ or $P(M,G)$

A G-principal bundle P over M, or a principal bundle P over M with struct grp G is as follows:

→ P and M are diff manifolds with a surjective smooth map, the project° $\pi : P \to M$, $p \mapsto \pi(p) = x$.
So that $\pi^{-1}(x)$ is a submanifold of P called the fiber over $x \in M$. → P is a Fiber bundle!

→ G is a Lie grp (connected) acting smoothly on the right on P, ie there is a smooth right G-act° : $P \times G \to P$, $(p,g) \mapsto pg$ or noted $R_g : P \to P$, $p \mapsto R_g p \equiv pg$, $\forall g \in G$.

It commute with the project° : $\boxed{\pi \circ R_g = \pi}$ ie $\pi(pg) = \pi(p) = x$.

↳ Means that G acts only along fibers over $x \in M$.

It is required that :
- $\forall p,q \in \pi^{-1}(x)$, $\exists g \in G$ s.t $q = pg$: act° of G on $\pi^{-1}(x)$ transitive
- $\forall p \in \pi^{-1}(x)$, $p = pg$ iff $g = e$: the act° of G on $\pi^{-1}(x)$ free

(Said otherwise $\pi^{-1}(x)$ is an orbit of G, and defining the little grp / stabilizer grp of a point $p \in P$ as $\mathrm{Stab}(p) \equiv \{ g \in G \mid pg = p \}$, any p has trivial stabilizer.)

↳ This means that $\pi^{-1}(x)$ is homeomorphic/diffeomorphic to G as a manifold. So we note $\pi^{-1}(x) \equiv G_x$. G_x diffeo to G but do not have a grp struct.

→ Bundle chart : given $\{U_i\}_{i \in I \subset \mathbb{R}}$ a covering of M, a bundle chart is a diffeo

$$\phi_i : \pi^{-1}(U_i) \subset P \longrightarrow U_i \times G \qquad \text{s.t } \pi \circ \phi_i^{-1} = \mathrm{proj}_{U_i}$$
$$p \longmapsto \phi_i(p) = (x = \pi(p) \, ; \, g) \qquad \text{ie } \pi \circ \phi_i^{-1}(x,g) = x \checkmark$$

The set $\{U_i, \phi_i\}$ is a bundle atlas.

Rmk that it means that a bundle is locally trivial, meaning that domains $\pi^{-1}(U_i)$ above any $U_i \subset M$ can be seen as a simple cartesian product $U_i \times G$.

A bundle is trivial if it can be shown globally isomorphic to $M \times G$; note $P \simeq M \times G$.

NB : R_g compatible with ϕ_i in that $R_{g'} \phi_i^{-1}(x,g) \equiv \phi_i^{-1}(x,g)g' = \phi_i^{-1}(x, gg')$ ✓

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→ A local sect° is a smooth map $\sigma : \mathcal{U} \subset M \longrightarrow \pi^{-1}(\mathcal{U}) \subset P$ s.t $\pi \circ \sigma = id_{\mathcal{U}}$. $\forall x \in \mathcal{U}$, $\sigma(x) \in G_x$.

Given a sect° σ one can have a bundle chart ϕ_σ via $\phi_\sigma^{-1}(x,e) \equiv \sigma(x)$

so that indeed $\forall p \in \pi^{-1}(\mathcal{U})$ is reached from $\sigma(x)$: $p = \sigma(x) g = \phi_\sigma^{-1}(x,e) g = \phi_\sigma^{-1}(x,g)$

and has thus coordinates (x,g). So σ is often called trivializing sect°.

Rmk: If P has a global sect° $\sigma : M \to P$, then $\phi_\sigma^{-1} : M \times G \longrightarrow P \Rightarrow P$ trivial!

→ As a manifold P has an algebra of smooth fct° $C^\infty(P)$

Given a rep (ρ, V) for G, we also define set of smooth V-valued fct° $C^\infty(P,V)$

and the subset of ρ-equivariant such fct° $C^\infty_{eq}(P,V) = C^\infty(P,\rho)$, ie fct° which

are well-behaved under R_g : $R_g^* f = \rho(g^{-1}) f$ ie $(f \circ R_g)(p) = f(pg) = \rho(g^{-1}) f(p)$.

1.1 Associated (vect) bundles and gauge grp

E, A vector bundle is a G-bundle with typical fiber a rep space V for G via ρ.

G acts on the left on E :
$$\begin{aligned} G \times E &\to E \\ (g, v) &\mapsto \rho(g) v \end{aligned} \qquad . \quad \begin{aligned} \pi_E : E &\to M \\ v &\mapsto \pi_E(v) = x \end{aligned}$$

The fiber over x is $V_x = \pi_E^{-1}(x)$.

A sect° of E $s : \mathcal{U} \longrightarrow \pi_E^{-1}(\mathcal{U}) \subset E$ is seen via some bundle chart to be the graph

of a V-valued fct° on $\mathcal{U} \subset M$. Indeed $\phi(s(x)) = (x, v(x))$, with $\phi : \pi_E^{-1}(\mathcal{U}) \subset E \to \mathcal{U} \times V$.

→ Given a principal bundle $P(M,G)$, and a rep (ρ, V) of G, the associated bundle to P is

as follow : - define act° of G on $P \times V$ as
$$\begin{aligned} (P \times V) \times G &\longrightarrow P \times V \\ \{(p,v), g\} &\longmapsto (pg\,;\, \rho(g^{-1}) v) \end{aligned}$$

Rmk that this is a right act°.

- declare an equivalence relat° $(p;v) \sim_G (pg\,;\,\rho(g^{-1})v)$ so that classes of equiv in $(P \times V)$ are noted $[p,g]$.

- The associated vect bundle is then $E \equiv P \times V / \sim_G = \begin{vmatrix} P \times_G V \\ P \times_\rho V \end{vmatrix}$.

$[p,v] \in E$ and $\pi_E([p,v]) = \pi(p) = x \in M$.

E is trivial if P is trivial. Sect° $s : \mathcal{U} \to E$ are noted $\Gamma(E)$.

NB : There is an isomorphism $\Gamma(E) \simeq C^\infty(P, \rho)$.

Indeed given $f \in C^\infty(P,\rho)$ the map $\begin{array}{l} P \longrightarrow P \times V \\ p \longmapsto (p, f(p)) \\ pg \longmapsto (pg, f(pg)) = (pg, \rho(g^{-1}) f(p)) \end{array}$ allows to def a sect° $s \in \Gamma(E)$ by dividing out by r_G.

So from now on we will call f a sect° of E instead of s.

→ P as a manifold has a diffeo grp $\mathrm{Diff}(P)$. In general it does not preserve fibrat°.

The subgrp of $\mathrm{Diff}(P)$ preserving fibrat° of P is the grp of vertical automorphisms :

$\begin{array}{l} \varphi : P \to P \\ p \mapsto \varphi(p) \end{array}$ s.t $\begin{array}{l} R_g \circ \varphi = \varphi \circ R_g \\ \varphi(pg) = \varphi(p) g \end{array}$ and $\begin{array}{l} \pi \circ \varphi = \pi \\ \pi(\varphi(p)) = \pi(p) \end{array}$. Noted $\mathrm{Aut}_v(P)$

$\hookrightarrow$ $\mathrm{Aut}_v(P) \equiv \{ \varphi : P \to P \in \mathrm{Diff}(P) \mid \underset{②}{\varphi(pg) = \varphi(p) g} ,\ \underset{①}{\pi \circ \varphi(p) = \pi(p)} \}$

Means that φ move points along fibers in a way compatible with right G-act° on P.

Can see that φ is induced by smooth map $\gamma : P \to G$ because one should have

$\varphi(p) = p\gamma(p) \in G_{\pi(p)}$ on account of ①. Then on account of ② :

$$\left.\begin{array}{l} \varphi(pg) = pg\,\gamma(pg) \\ \quad = \varphi(p) g = p\gamma(p) g \end{array}\right\} \to R_g^* \gamma(p) = \gamma(pg) = g^{-1}\gamma(p) g .$$

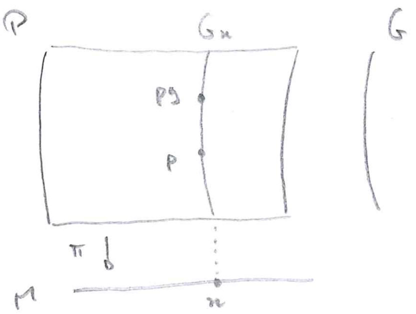
$\hookrightarrow$ γ is then a conjugat° - equiv G-valued fct°. The set of such fct° is called

the gauge grp, noted $\mathcal{G} \equiv \{ \gamma : \mathcal{U} \to G \mid R_g^* \gamma = g^{-1} \gamma g \}$

rmk : $\mathcal{G} \simeq \Gamma(P \times_{\mathrm{conj}} G)$ NB : Gauge grp is G-valued maps ⊕ equiv !

The isomorphism btwn $\mathrm{Aut}_v(P)$ and $\mathcal{G}$ is simply : $\varphi(p) = p\gamma(p)$ $\forall p \in P$.

Both are infinite dim grps.



NB : $\mathrm{Aut}_v(P)$ grp under comp so :

$$\Phi^* \varphi(p) \equiv \varphi \circ \Phi(p) = \varphi(p\alpha(p)) = \begin{cases} p\alpha(p)\, \gamma(p\alpha(p)) \\ \varphi(p)\alpha(p) = p\gamma(p)\alpha(p) \end{cases}$$

$\hookrightarrow$ $(R^*_{\alpha(p)} \gamma)(p) \equiv \gamma(p\alpha(p)) = \alpha(p)^{-1} \gamma(p) \alpha(p)$

ie gauge grp acts on itself via $\gamma^\alpha \equiv R^*_\alpha \gamma = \alpha^{-1} \gamma \alpha$.

1.2 Tgt bundle, Vert bundle, diff forms

We still only use diff struct on P as a manifold.

→ P has a tgt bundle $TP = \bigcup_{p \in P} T_p P$

- The right act° has push forward $R_{g*} : TP \longrightarrow TP$ sends $X_p \in T_p P$ to $(R_{g*}X)_{pg} \in T_{pg}P$.

  A vect field $X \in \Gamma(TP)$ st $R_{g*} X_p = X_{pg}$ is called right-invariant.

  Notice that for such vect field : $\pi_* R_{g*} X_p = \begin{cases} (\pi \circ R_g)_* X_p = \pi_* X_p & \in T_{x=\pi(p)} M \\ \pi_* X_{pg} & \in T_x M \end{cases}$

  So X is "the same" along fibers G_x

  and is thus send to / projected to the same vect field on M : $\underbrace{\pi_* X_{pg} = \pi_* X_p}$ $\in T_x M$.

  So only right-inv vect fields project to well-def vect fields on M. def projectable vect field!

  Note set of right-inv vect fields $\Gamma^G(TP) \subset \Gamma(TP)$. Clearly a sub Lie-alg.

- The project° has push forward $\pi_* : TP \longrightarrow TM$ sends $X_p \in T_p P$ to $(\pi_* X)_{\pi(p)} \in T_x M$.

  It is a surject° and is kernel is non empty : element in $\ker \pi_*$ are vertical vectors (fields)

  They form obviously a vector subspace of TP, but also $\pi_* [X, Y] = [\pi_* X, \pi_* Y] = 0$

  $\forall X, Y \in \ker \pi_*$, so they form a sub Lie-algebra : noted $VP = \bigcup_{p \in P} V_p P \subset TP$.

  Explicitely one can see how a vertical vector (field) is generated by $R_{g(t)}$ on P :

  Given $X \in \text{Lie}\, G$, $g(t) \equiv \exp(tX)$ is a smooth curve in G, and $R_{g(t)}\, p = p\, g(t)$

  is a curve in the fiber $G_{\pi(p)}$. The vect tgt to it is $X^v_p \equiv \frac{d}{dt}\Big|_{t=0} p \exp(tX)$

  Or given $f \in C^\infty(P)$, it acts via : $X^v_p f(p) \equiv \frac{d}{dt}\Big|_{t=0} f(p\exp(tX))$

  This allows to see that $R_{g*} X^v_p = \frac{d}{dt}\Big|_{t=0} R_g(p \exp(tX)) = \frac{d}{dt}\Big|_{t=0} p \exp(tX) g$

  $$= \frac{d}{dt}\Big|_{t=0} pg\, g^{-1} \exp(tX) g = \frac{d}{dt}\Big|_{t=0} pg \exp(t g^{-1} X g) \equiv (Ad_{g^{-1}} X)^v_{pg}.$$

  ↳ $R_{g*} X^v = (Ad_{g^{-1}} X)^v$

  It is clear that $\pi_* X^v_p \equiv \frac{d}{dt}\Big|_{t=0} \pi(p\exp(tX)) = \frac{d}{dt}\Big|_{t=0} \pi(p) = 0$.

  And also $\pi_* R_{g*} X^v_p = \frac{d}{dt}\Big|_{t=0} \pi(pg\, e^{Ad_{g^{-1}} X t}) = \frac{d}{dt}\Big|_{t=0} \pi(pg) = 0$.

  } So vert vectors are projectable with trivial proj.

  $\Rightarrow \Gamma(VP) \subset \Gamma^G(TP)$

  Fact : $X \to X^v$ Lie alg morph, i.e. $[X, Y]^v = [X^v, Y^v]$, see postit!

Proof 1^v Lie-alg morph: actn of X^v on $\varphi \in C^\infty(P,\rho)$

is $X^v_p \varphi(p) \equiv \frac{d}{d\tau}\Big|_{\tau=0} \varphi(p \exp \tau X) = \frac{d}{d\tau}\Big|_{\tau=0} \rho(\exp -\tau X)\varphi(p) \equiv -\rho_*(X)\varphi(p)$

↳ notimed equiv of φ ✓

$$[X^v, Y^v]_p \varphi(p) = X^v(-\rho_*(Y)\varphi)(p) - Y^v(-\rho_*(X)\varphi)(p)$$
$$= \rho_*(Y)\rho_*(X)\varphi(p) - \rho_*(X)\rho_*(Y)\varphi(p)$$
$$= \rho_*([Y,X])\varphi(p) = -\rho_*([X,Y])\varphi(p) \checkmark$$

$[X^v, Y^v]$ acts as a vert vect field generated by $[X,Y]$:

↳ $[X^v, Y^v] = [X,Y]^v$ ✓

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- $\text{Aut}_v(P)$ acts by pushforward, $\psi_*: TP \to TP$ sends $X_p \in T_pP$ to $\psi_* X_p \in T_{\psi(p)}P = T_{p\gamma(p)}P$

Given $X \in \Gamma(TP)$ with flow $\phi(\tau)$, ie $\phi(0) = p$ and $X_p \equiv \frac{d}{d\tau}\big|_{\tau=0} \phi(\tau)$
$X_p \in T_pP$

By def : $\psi_* X_p \equiv \frac{d}{d\tau}\big|_{\tau=0} \psi(\phi(\tau)) = \frac{d}{d\tau}\big|_{\tau=0} \phi(\tau)\gamma(\phi(\tau)) = \frac{d}{d\tau}\big|_{\tau=0} \phi(\tau)\cdot\gamma(p) + p\,\frac{d}{d\tau}\big|_{\tau=0} \gamma(\phi(\tau))$

$$= R^*_{\gamma(p)*} X_p + p\,\frac{d}{d\tau}\Big|_{\tau=0} \gamma(\phi(\tau)) .$$

Rmk that given ext der on P : d_P, and $\gamma : P \to G$ smooth fct° we have by def $d\gamma_p(X_p) = X_p(\gamma)(p) \equiv \frac{d}{d\tau}\big|_{\tau=0} \gamma(\phi(\tau))$ has value in $T_{\gamma(p)}G$.

So $[\gamma^{-1}d\gamma]_p(X_p) = \gamma(p)^{-1} d\gamma_p(X_p) = \gamma(p)^{-1}\frac{d}{d\tau}\big|_{\tau=0} \gamma(\phi(\tau)) = \frac{d}{d\tau}\big|_{\tau=0} \underbrace{\gamma(p)^{-1}\gamma(\phi(\tau))}_{=e \text{ in } \tau=0} \in T_eG = \text{Lie}\,G$!

our 2nd term is : $p\frac{d}{d\tau}\big|_{\tau=0} \gamma(\phi(\tau)) = p\gamma(p)\cdot\gamma(p)^{-1}\frac{d}{d\tau}\big|_{\tau=0}\gamma(\phi(\tau)) = p\gamma(p)\{\underbrace{[\gamma^{-1}d\gamma]_p(X_p)}_{\in\, \text{Lie}\,G}\}$

$$= \frac{d}{ds}\Big|_{s=0} p\gamma(p)\exp\{s[\gamma^{-1}d\gamma]_p(X_p)\}$$

$$\equiv \{[\gamma^{-1}d\gamma]_p(X_p)\}^v_{p\gamma(p)}$$ the def of a vertical vect field !

↳ $\psi_* X_p = R^*_{\gamma(p)*} X_p + \{[\gamma^{-1}d\gamma]_p(X_p)\}^v_{p\gamma(p)}$ (I)

→ As a diff manifold P has a DG-alg $\Omega(P) = \bigoplus_{k=0}^{\dim P} \Omega^k(P)$, where the wedge product the inner derivative and the exterior / DeRham deriv are def as before. Also since $C^\infty(M)$ is a $\Gamma(TP)$-~~alg~~ Module/alg and $\Gamma(TP)$ is a Lie-alg, d is given by the Koszul formula. Its DeRham complex is $(\Omega(P), d)$ and its d-cohomology is def the usual way.

1 - Given rep (ρ, V) of G we also have V-valued diff forms on P : $\Omega(P,V) = \Omega(P)\otimes V$
merely a Graded-VS, but i and d still def.
Subspace of which is ρ-equivariant V-valued diff forms : $\Omega_{eq}(P,V) \equiv \Omega(P,\rho) \subset \Omega(P,V)$
↳ $\alpha \in \Omega^k(P,\rho)$ is s.t $R_g^*\alpha = \rho(g^{-1})\alpha$ ~ also said pseudotensorial of type (ρ, V)

- Subspace of $\Omega(P)$ is <u>horizontal forms</u>, note $\Omega_{hor}(P)$: vanishes if evaluated on VP.
  i.e $\alpha \in \Omega^n_{hor}(P)$ iff $\alpha(X_1 \dots X_n) = 0$. if $X_i \in \Gamma(VP)$. also said <u>semi-basic</u>

- Set of forms both horizontal and pseudo-tensorial are said <u>tensorial</u> ~~(basic)~~. Note $\Omega_{hor}(P,\rho)$ or $\Omega_{tens}(P,V)$. (*)

- If $V = A$ is also an alg then $\Omega(P,A)$ is a DG-alg where $\wedge$ is defined and the previous nomenclature still applies.

  Ex: $A = \text{Lie}\,G$, $\rho = Ad$ or ad $\rightsquigarrow$ $\Omega_{tens}(P, \text{Lie}\,G) \Big| \begin{array}{l} \subset \Omega_{hor}(P, \text{Lie}\,G) \\ \subset \Omega_{eq}(P, \text{Lie}\,G) \end{array}$

- <u>Some pullbacks</u> :

(1)

  - $R^*_g : \Omega_{pg}(P) \longrightarrow \Omega_p(P)$
    $\alpha_{pg} \longmapsto R^*_g \alpha_{pg} = (R^*_g \alpha)_p$

  - $\pi^* : \Omega_x(M) \longrightarrow \Omega_{\pi^{-1}(x)}(P)$
    $\beta_x \longmapsto (\pi^*\beta)_{\pi^{-1}(x)}$

  - $\sigma^* : \Omega_{\sigma(x)}(P) \longrightarrow \Omega_x(M)$
    $\alpha_{\sigma(x)} \longmapsto (\sigma^*\alpha)_x$.

(*) rmq : $\beta \in \Omega_x(M)$, $\pi^*\beta \in \Omega(P)$ is s.t $R^*_g \pi^*\beta = (\pi \circ R_g)^*\beta = \pi^*\beta$
ie $\pi^*\beta$ is righ-inv, or of trivial equivariance! And $\pi^*\beta \in \Omega_{hor}(P)$!
$\hookrightarrow$ $\alpha \in \Omega(P)$ s.t $\Big| \begin{array}{l} R^*_g\alpha = \alpha \\ \& \alpha \in \Omega_{hor}(P) \end{array}$ are called <u>basic</u>, noted $\Omega_{basic}(P) \subset \Omega_{eq}(P,V)$, because they are in the image of π^*.

1.3 Gauge trsf (active)

The act° of $\mathcal{G}$, the gauge grp, on diff forms is def via act° of $Aut_v(P)$.
Thus the <u>gauge trsf</u> of $\alpha \in \Omega(P)$ is : $\alpha^\gamma \equiv \psi^*\alpha$ for $\gamma \in \mathcal{G}$ and $\psi \in Aut_v(P)$ with $\psi(p) = p\gamma(p)$. Said <u>active</u> GT since move on P!

Remark that on $\alpha \in \Omega^1_{hor}(P,\rho)$ ie tensorial of type (ρ, V) one has:

$$\alpha^\gamma_p(X_p) \equiv (\varphi^*\alpha)_p(X_p) = \alpha_{\varphi(p)}(\varphi_* X_p) = \alpha_{p\gamma(p)}\left(R_{\gamma(p)*}X_p + \{[\gamma^{-1}d\gamma]_p(X_p)\}^v_{p\gamma(p)}\right)$$

$$= \alpha_{p\gamma(p)}(R_{\gamma(p)*}X_p) \quad \text{since } \alpha \text{ is horizontal.}$$

$$= R^*_{\gamma(p)}\alpha_{p\gamma(p)}(X_p) = \rho(\gamma(p)^{-1})\,\alpha_p(X_p) \quad \text{since } \alpha \text{ is } \rho\text{-equiv}: R^*_g\alpha = \rho(g^{-1})\alpha .$$

↳ $\boxed{\alpha^\gamma = \rho(\gamma^{-1})\alpha}$

So the GT is read-off the equiv of α, or determined by it only!
This wouldn't be the case for only pseudo-tensorial forms, because the vertical part of $\varphi_* X_p$ would need to be evaluated ...

NB: by repeating GT one sees that the act° of $\mathcal{G}$ is a right-act°!

2] Connection

From the def of a bundle each T_pP $\forall p \in P$ has a canonical subspace V_pP isomorphic (as a Lie alg) to Lie G, and generated by the right act° of G on P. Furthermore this space is in the kernel of π_*. We thus have the short exact sequence of VS & Lie Alg:

$$\circledast \qquad 0 \longrightarrow \text{Lie}\,G \xrightarrow{\;|^v\;} T_pP \xrightarrow{\;\pi_*\;} T_{\pi(p)=x}M \longrightarrow 0$$

with $\text{Im}\, |^v = \ker \pi_*$.
$|^v$ inj and π_* surj.

So that $T_xM \simeq T_pP / V_pP$.

But, there is no canonical way to choose a complement to V_pP in T_pP, ie to write $X_p \in T_pP$ as the sum of a vert part and an "horizontal" (non-vertical) part, or to lift a vector $Y_x \in T_xM$ to ~~T_pP~~ an element with no vertical part in T_pP to complement V_pP: no canonical splitting of $\circledast$!
→ No way to compare points in 2 ≠ fibers over ≠ points of M!
→ We need to make a choice for such | complement / splitting at each $p \in P$.

↳ since there is 2 ways to split the SES, we have 2 ways to describe our choice of complement!

2.1 Connect° : geometric def

An Ehresmann connect° is a choice of smooth distribut° $p \longmapsto H_pP$ $\forall p \in P$ s.t :

- $T_pP = H_pP \oplus V_pP$, and $X_p = X_p^h + X_p^v$.
- $R_{g*} H_pP = H_{pg}P$, an horiz vect stays horiz under right act° of G .

Since $V_pP = \ker \pi_*$, the restrict° $\pi_* : H_pP \longrightarrow T_{\pi(p)}M$ is a VS isomorphism.

Then given $X \in \Gamma(TM)$ a vect field with flow $c(\tau)$ in M, there is a unique R_g-inv vect field $\bar{X}^h \in \Gamma(TP)$ with flow $\bar{c}^h(\tau)$ s.t $\bar{X}^h_p \in H_pP$ $\forall p \in P$ and $\pi_* \bar{X}^h_p = X_x \in T_xM$ and $\pi \circ \bar{c}^h(\tau) = c(\tau)$. $\bar{X}^h / \bar{c}^h(\tau)$ is the horizontal lift of $X / c(\tau)$.

Update : Consider M with $TM = \bigcup_{x \in M} T_xM$, a smooth k-dim distrib is a smooth map $\forall x$; $x \longmapsto D_x \subset T_xM$ where D_x k-dim sub vector space of T_xM.

The distrib $D = \bigcup_{x \in M} D_x$ is said integrable or involutive iff $\forall X_x, Y_x \in D_x \subset T_xM$ $[X_x, Y_x] \in D_x$. It means that there are k-dim submanifolds $N \subset M$ s.t $T_xN = D_x$ $\forall x \in N$. N is called an integral submanifold of D, or a leaf, and M is said foliated by the N's , or to be a foliation.

↳ Rmk : VP is a dim G- integrable distribut° whose leaf are the fibers G_x , and P is foliated by the fibers !

In general $HP = \bigcup_{p \in P} H_pP$ is not integrable. When it is, the leaf $\tilde{M} \subset P$ are s.t $\tilde{M} \xrightarrow{\pi} M$ is a diffeo so that $P = M \times H$ is trivial !

NB : the horiz lift operat° $|^h : T_xM \longrightarrow H_pP \subset T_pP$ is s.t $\boxed{\pi_* \circ |^h = id_{T_xM}}$
This is a splitting of the SES !

$$0 \longrightarrow \mathrm{Lie}\, G \xrightarrow{|^v} T_pP \underset{|^h}{\overset{\pi_*}{\rightleftarrows}} T_xM \qquad \text{so that} \quad T_pP = V_pP \oplus H_pP$$
$$= \mathrm{Lie}\, G|^v \oplus T_xM|^h \; !$$

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→ The name "connect°" stems from the fact that it allows to "connect" fibers at different points of M in the following way:

Consider x_0 and $x_1 \in M$ linked by a curve $c(\tau)$ s.t $c(0) = x_0$ and $c(1) = x_1$.
Fix a point $p_0 \in \pi^{-1}(x_0)$. To which point in $\pi^{-1}(x_1)$ does it correspond? Without connect° there's no telling, the quest° makes no sense. But with a connect° there is a unique horiz lift $c^h(\tau)$ of $c(\tau)$ s.t $c^h(0) = p_0$ and $c^h(1) = p_1 \in \pi^{-1}(x_1)$!
↳ p_1 is thus called the parallel trsp of p_0 along $c(\tau)$! The lift $c^h(\tau)$ connects the fibers $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$, and varying p_0 in $\pi^{-1}(x_0)$ this procedure gives an isomorphism $\pi^{-1}(x_0) \approx \pi^{-1}(x_1)$ ✓

→ The tgt vect fd X^h along $c^h(\tau)$ (lift of X tgt vect field of $c(\tau)$) is horiz: $X^h_{c^h(\tau)} \in T_{c^h(\tau)}P$ by def.

Now given $\varphi \in C^\infty(P,\rho) = C^\infty_{eq}(P,V)$, which is equiv to a sect° s_φ of an assoc vect bundle $E = P \times_\rho V$, the covariant derivative of φ along $X \in \Gamma(TM)$ is def as: $\ulcorner D_X\varphi \equiv X^h(\varphi) \lrcorner$ ie $D_X\varphi(p) \equiv X^h_p(\varphi)(p) \equiv \frac{d}{d\tau}\Big|_{\tau=0} \varphi(c^h(\tau))$, $c^h(0) = p$.

Notice that $D_X\varphi \in C^\infty(P,\rho)$, justifying the name of the diff operator.
Indeed ~~$D_X\varphi(pg) = X^h$~~ $(R_g^*\varphi)(p) = \varphi \circ R_g(p) = \varphi(pg) = \rho(g^{-1})\varphi(p)$.
and $(R_g^* D_X\varphi)(p) = D_X\varphi \circ R_g(p) \equiv X^h_{pg}(\varphi)(pg) \equiv \frac{d}{d\tau}\Big|_{\tau=0} \varphi(c^h(\tau)g) = \rho(g^{-1}) \frac{d}{d\tau}\Big|_{\tau=0} \varphi(c^h(\tau))$
↳ $\ulcorner R_g^* D_X\varphi = \rho(g^{-1}) D_X\varphi \lrcorner$ ↳ φ is said //-trsported if $\ulcorner D_X\varphi = 0 \lrcorner$ ✓

↳ So the cov der along X is an operator $D_X : C^\infty(P,\rho) \longrightarrow C^\infty(P,\rho)$.
↳ And in generalizat° of Riemann geom the connect° is a map:

$D : \Gamma(TM) \times C^\infty(P,\rho) \longrightarrow C^\infty(P,\rho)$ s.t it is

linear in 1st arg:
- $D_{X+X'}\varphi = D_X\varphi + D_{X'}\varphi$, for $X, X' \in \Gamma(TM)$
- $D_{fX}\varphi = f^h D_X\varphi$, for $f \in C^\infty(M)$

addit and Leibniz like in 2nd arg:
- $D_X(\varphi + \varphi') = D_X\varphi + D_X\varphi'$, for $\varphi, \varphi' \in C^\infty(P,\rho)$
- $D_X(f\varphi) = X^h(f)\cdot\varphi + f\, D_X\varphi$, for $f(pg) = f(p)$ a R_g-inv fct°

Ex: check it ✓

So, giving the connect° as an horiz distrib is equiv to giving the connect° as the differential operator D acting on $C^\infty(P,\rho) \simeq \Gamma(E)$.

↳ Can analogy with Riem geom be pushed further?

- On (M,∇) torsion def as
$$T : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(X,Y) \longmapsto T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y]$$

In the context of (P,D) there is no analogue not° because neither $D_\varphi X$ nor $[X,\varphi]$ make sens!

- Curvature of ∇ is def as
$$R : \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \longrightarrow \Gamma(TM)$$
$$(X,Y,Z) \longmapsto R(X,Y)Z \equiv \left([\nabla_X,\nabla_Y] - \nabla_{[X,Y]}\right)Z \,.$$

It generalizes for (P,D) as
$$\Omega : \Gamma(TM)\times\Gamma(TM)\times C^\infty(P,\rho) \longrightarrow C^\infty(P,\rho)$$
$$(X,Y,\varphi) \longmapsto \boxed{\Omega(X,Y)\varphi \equiv \left([D_X,D_Y] - D_{[X,Y]}\right)\varphi} \,.$$

That is $\boxed{\Omega(X,Y)\varphi \equiv \left([X^h,Y^h] - [X,Y]^h\right)\varphi}$.

Clearly Ω, the curvature of the connect° operator D, measures failure of $|^h$ (horit lift) to be a Lie-alg morphism, ie the non-integrability of the distribut° HP.

Since $\pi_* \circ |^h = id_{TM}$, $\pi_*[X,Y]^h = [X,Y]$. Also, $\pi_*[X^h,Y^h] \overset{*}{=} [\pi_* X^h, \pi_* Y^h] \overset{*}{=} [X,Y]$. So $[X^h,Y^h]$ and $[X,Y]^h$ have same project° on M.

This means that $[X^h,Y^h] - [X,Y]^h \in \Gamma(VP)$ ($\to \ker \pi_*$!), and since the act° of a vert vect on $\varphi \in C^\infty(P,\rho)$ is $X^v(\varphi) = -\rho_*(X)\varphi$, it means that $\Omega(X,Y)$ has values in $Lie\,G$! (the elemt generating vert part of $[X^h,Y^h]$) $\Rightarrow$ Ω is antisym in X,Y, linear and $Lie\,G$-valued: the curvature Ω is a $Lie\,G$-valued 2-form on M; $\Omega \in \Omega^2(M, Lie\,G)$ ✓

↳ should then write $\rho_*(\Omega(X,Y))\varphi$.

NB: If the curvature vanishes, $\Omega = 0$, then the connect° D is said flat and the horiz distribut° HP is integrable ($|^h$ is a Lie-alg morphism) and the principal bundle P is trivial: $P = M \times H$.

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2.2 Connect°: algebraic def

Update: Let ω be a V-valued 1-form on M, $\omega \in \Omega^1(M,V)$, V a vect space with kernel of cst rank k, ie $\dim(\ker \omega_x) = k \ \forall x \in M$, so that at any $x \in M$ $\ker \omega_x \subset T_x M$.

↳ $\ker \omega$ is a smooth k-dim distribution on M.

Frobenius Thm: This distrib is integrable iff $d\omega = 0$ on $\ker \omega$.

Indeed d given by Koszul formula so for $X, Y \in \ker \omega$:

$$d\omega(X,Y) = X\underbrace{\omega(Y)}_{0} - Y\underbrace{\omega(X)}_{0} - \omega([X,Y])$$

so $d\omega(X,Y) = 0 \iff [X,Y] \in \ker \omega$, ie $\ker \omega$ integrable.

Instead of the previous geometric def, we can caracterize an Ehresmann connect° via diff forms: An Ehresmann connect° 1-form is a Lie G-valued 1-form on P $\omega \in \Omega^1(P, \mathrm{Lie}\,G)$ satisfying:

- $\omega_p(X^v_p) = X \in \mathrm{Lie}\,G$, X^v the fondamental Vert vector generated by $X \in \mathrm{Lie}\,G$.
- $R^*_g \omega_{pg} = Ad_{g^{-1}} \omega_p$, ω pseudo tensorial of type $(Ad, \mathrm{Lie}\,G)$, $\omega \in \Omega^1_{eq}(P, \mathrm{Lie}\,G)$

The horiz subspace at p is then $H_p P \equiv \ker \omega_p$.

- The 2nd condit° implies that if $X_p \in \ker \omega_p$, $R^*_g \omega_{pg}(X_p) = \omega_{pg}(R_{g*} X_p) = Ad_{g^{-1}} \omega_p(X_p) = 0$. So if $X_p \in \ker \omega_p$ then $R_{g*} X_p \in \ker \omega_p$, which is equivalent to the 2nd cond def geometric Connect° ✓
- The First Cond establish isom $V_p P \simeq \mathrm{Lie}\,G$, thus a surject° $T_p P \to \mathrm{Lie}\,G$ that allows to define horiz part of any $X \in T_p P$: $\omega_p(X_p) \in \mathrm{Lie}\,G$, so $\omega_p(X_p)|^v \in V_p P$ and $X^h_p \equiv X_p - \omega_p(X_p)|^v$, and by def $X^h_p \in \ker \omega_p$ ✓
  This is equiv to the 1st cond def geometric connect° ✓

Rmq: X^h_p thus defined is R_g-inv iff X_p is R_g-inv.
But given $X_x \in T_x M$, horiz lift X^h_p is def as R_g-inv as before!

NB : the connect° 1-form $\omega_p : T_pP \to \text{Lie}\,G$ is s.t $\ulcorner \omega \circ |^v = id_{\text{Lie}G} \lrcorner$

This is a splitting of the SES !

$$0 \longrightarrow \text{Lie}\,G \xrightarrow{|^v} T_pP \xrightarrow{\pi_*} T_xM \longrightarrow 0$$

with ω_p going back from T_pP to $\text{Lie}\,G$ and $|^h$ going back from T_xM to T_pP.

So $T_pP = V_pP \oplus H_pP$
$= \text{Lie}G|^v \oplus T_xM|^h$

NB : As said before choice of connect° (geom / algebraic) is not unique.

A principal bundle with a choice of connect° 1-form is noted (P,ω), but another choice ω' on P would be possible : (P,ω').

↳ The space of connect° 1-forms on P is noted $\mathcal{C}(P)$.

Rmk that it is an affine space. Indeed given $\omega,\omega' \in \mathcal{C}(P)$

$\omega+\omega' \notin \mathcal{C}(P)$ because $(\omega+\omega')(X^v) = \omega(X^v) + \omega'(X^v) = 2X \neq X$!

But $\omega-\omega'$ is a Lie G-valued Ad-equiv horizontal 1-form !

ie $\omega-\omega' = \varphi \in \Omega^1_{hor}(P,\text{Ad}) = \Omega^1_{tens}(P,\text{Lie}\,G)$, tensorial type $(\text{Ad},\text{Lie}G)$

↳ Said otherwise if $\omega \in \mathcal{C}(P)$, for $\varphi \in \Omega^1_{tens}(P,\text{Lie}G)$, $\omega+\varphi \in \mathcal{C}(P)$

$\Rightarrow$ $\mathcal{C}(P)$ is an affine space modeled on the vectors space $\Omega^1_{tens}(P,\text{Lie}\,G)$!

Rmk : $\lambda\omega + \lambda'\omega' \in \mathcal{C}(P)$ iff $\lambda' = 1-\lambda$! $\rightsquigarrow$ $\omega_\lambda \equiv \lambda\omega + (1-\lambda)\omega' \in \mathcal{C}(P)$ ✓

→ The integrability of $H_pP = \ker \omega_p$ is controlled by : $d\omega(X^h,Y^h)$

By def of X^h, Y^h : $d\omega(X^h,Y^h) = X^h \cdot \underbrace{\omega(Y^h)}_{0} - Y^h \cdot \underbrace{\omega(X^h)}_{0} - \omega([X^h,Y^h])$

Since ω_p project on Lie G, $\omega([X^h,Y^h])$ is the elemt of Lie G generating ~~...~~ the vertical part of $[X^h,Y^h]$! So, up to sign, this is the curvature of the connect° !

↳ The curvature 2-form of the connect° 1-form will thus be defined as :

$\ulcorner \Omega \equiv d\omega \circ |^h \lrcorner$ ie $\Omega(X,Y) = d\omega(X^h,Y^h)$ $\forall\, X,Y \in \Gamma(TP)$

By def it is horizontal, $\Omega \in \Omega^2_{hor}(P,\text{Lie}G)$

Also : $R_g^*\Omega(X,Y) = R_g^*\, d\omega(X^h,Y^h) = d\, R_g^*\omega(X^h,Y^h) = d\, \text{Ad}_{g^{-1}}\omega(X^h,Y^h) = \text{Ad}_{g^{-1}}\, d\omega(X^h,Y^h)$
$= \text{Ad}_{g^{-1}}\Omega(X,Y)$ → ie Ω is Ad-equiv, $\Omega \in \Omega^2_{eq}(P,\text{Lie}G)$

Therefore Ω is tensorial of type $(\text{Ad},\text{Lie}G)$, $\Omega \in \Omega^2_{tens}(P,\text{Lie}G)$

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→ ~~[illegible]~~ Cartan Structure Equat°

The curvature 2-form is given by Cartan struct Eq :

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega] = d\omega + \omega\wedge\omega \,.$$

Proven by evaluat° on (X^v,Y^v) ; (X^v,Y^h) and (X^h,Y^h).
Knowing that $[X^v,Y^v] = [X,Y]^v$ and $[X^v,Y^h] \in HP$.

- $$(d\omega+\omega\wedge\omega)(X^v,Y^v) = d\omega(X^v,Y^v) + \omega(X^v)\omega(Y^v) - \omega(Y^v)\omega(X^v)$$
$$= X^v\underbrace{\omega(Y^v)}_{Y} - Y^v\underbrace{\omega(X^v)}_{X} - \omega(\underbrace{[X^v,Y^v]}) + [X,Y] \,.$$
$$= \underbrace{X^v Y}_{0} - \underbrace{Y^v X}_{0} - \omega([X,Y]^v) + [X,Y] = 0 = \Omega(X^v,Y^v) \checkmark$$

- $$(d\omega+\omega\wedge\omega)(X^v,Y^h) = X^v\cancel{\omega(Y^h)}^{\,0} - \underbrace{Y^h\underbrace{\omega(X^v)}_{X}}_{0} - \underbrace{\omega(\underbrace{[X^v,Y^h]}_{\in HP})}_{0} + \omega(X^v)\cancel{\omega(Y^h)}^{\,0} - \cancel{\omega(Y^h)}^{\,0}\omega(X^v)$$
$$= 0$$
$$= \Omega(X^v,Y^h) \checkmark$$

- $$(d\omega+\omega\wedge\omega)(X^h,Y^h) = X^h\cancel{\omega(Y^h)}^{\,0} - Y^h\cancel{\omega(X^h)}^{\,0} - \omega([X^h,Y^h]) + \cancel{\omega(X^h)}^{\,0}\omega(Y^h) - \cancel{\omega(Y^h)}^{\,0}\omega(X^h)$$
$$= -\omega([X^h,Y^h]) = \Omega(X^h,Y^h) \checkmark$$

In practical computat° the curvature as given by Cartan struct eq is what is used.

NB : Notice the analogy with the Maurer-Cartan eq for the MC 1-form ϖ
↳ $\varpi \in \Omega^1_{eq}(G, \mathrm{Lie}\,G)$ ie $R_g^*\varpi = \mathrm{Ad}_{g^{-1}}\varpi$

And $d\varpi + \tfrac{1}{2}[\varpi,\varpi] = d\varpi + \varpi\wedge\varpi = 0$.

MC form looks much like a connect° 1-form on G ! But a flat one.
This observat° will be made precise in chap 3 when we consider Cartan geometry

2.3 Exterior covariant derivative

The def of Ω from ω suggests to generalize the operat° on any form α which is equivariant, or pseudotensorial of type (ρ, V).

- So given rep (ρ,V) of G, on $\Omega_{eq}(P,V) \equiv \Omega(P,\rho)$ define the ext cov deriv D by : $\ulcorner D\alpha \equiv d\alpha \circ |^h \lrcorner$, $\alpha \in \Omega^k(P,\rho)$.

  ie $\quad D\alpha(X_1 \dots X_k) \equiv d\alpha(X_1^h \dots X_k^h)$.

  By def $D\alpha \in \Omega_{hor}(P,V)$ and, as for Ω, it is easily shown that $D\alpha \in \Omega(P,\rho)$. So $D : \Omega^k(P,\rho) \longrightarrow \Omega^{k+1}(P,\rho)$ preserve equivariance and even $D\alpha \in \Omega_{tens}(P,V)$. Generalize if $V=\mathcal{A}$ so that ?

- Clearly D is a derivat° of $\Omega(P,\mathcal{A})$: $\ulcorner D(\alpha\wedge\beta) = D\alpha\wedge\beta - (-)^k\alpha\wedge D\beta \lrcorner$

- On $\Omega_{tens}(P,V)$, the ext cov der can be expressed via ω :

  For $\beta \in \Omega_{tens}(P,V)$, $\ulcorner D\beta = d\beta + \rho_*(\omega)\wedge\beta \lrcorner$

  Proven on $\beta \in \Omega^1_{tens}(P,V)$ evaluated on (X^v,Y^v) ; (X^v,Y^h) and (X^h,Y^h).

  - $(d\beta + \rho_*(\omega)\beta)(X^v,Y^v) = X^v\cancel{\beta(Y^v)}^{\to 0} - Y^v\cancel{\beta(X^v)}^{\to 0} - \beta(\underbrace{[X^v,Y^v]}_{[X,Y]^v}) + \rho_*(\omega(X^v))\cancel{\beta(Y^v)}^{\to 0} - \rho_*(\omega(Y^v))\cancel{\beta(X^v)}^{\to 0}$

    $= 0 = D\beta(X^v,Y^v)$ ✓

  - $(d\beta + \rho_*(\omega)\beta)(X^v,Y^h) = X^v\beta(Y^h) - Y^h\cancel{\beta(X^v)}^{\to 0} - \beta([X^v,Y^h]) + \rho_*(\omega(X^v))\beta(Y^h) - \rho_*(\omega(\cancel{Y^h}))\beta(\cancel{X^v})$

    $= X^v\beta(Y^h) - \beta([X^v,Y^h]) + \rho_*(X)\beta(Y^h)$ ⊛

  Rmk that : $[L_X\alpha](Y) = [(i_X d + d i_X)\alpha](Y) = d\alpha(X,Y) + d(\alpha(X))(Y)$
  $= d\alpha(X,Y) + Y\cdot\alpha(X) = X\cdot\alpha(Y) - \alpha([X,Y])$

  so that $\quad X\cdot\alpha(Y) = [L_X\alpha](Y) + \alpha([X,Y])$.

  So ⊛ $= [L_{X^v}\beta](Y) + \rho_*(X)\beta(Y^h)$.

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To go forward we need to give a new def of Lie derivative on forms:
In the way it is def on forms $L_X \equiv i_X d + d i_X$ it should ~~consistent with~~ acts on 0-form as: $L_X f = i_X df + d\underbrace{i_X f}_{0} = df(X) = X(f)$

but $X(f)$ is def as: $X(f) = \frac{d}{d\tau}\Big|_{\tau=0} f(\phi_t) = \frac{d}{d\tau}\Big|_{\tau=0} \phi_t^* f$ for ϕ_t flow of X.

So we will have: $\boxed{L_X \alpha = \frac{d}{d\tau}\Big|_{\tau=0} \phi_t^* \alpha}$ for $\alpha \in \Omega(P)$.

↳ Here we need: $L_{X^v} \beta = \frac{d}{d\tau}\Big|_{\tau=0} R^*_{g(\tau)} \beta$ for $g(\tau) = \exp \tau X$, $X \in \text{Lie}\, G$

$$= \frac{d}{d\tau}\Big|_{\tau=0} \rho_*(g(\tau)^{-1})\beta \equiv -\rho_*(X)\beta \checkmark$$

That is finally: $(d\beta + \rho_*(\omega)\beta)(X^v, Y^h) = 0 = D\beta(X^v, Y^h)$ ✓

- $(d\beta + \rho_*(\omega)\beta)(X^h, Y^h) = d\beta(X^h, Y^h) + \rho_*(\cancelto{0}{\omega(X^h)})\beta(Y^h) - \rho_*(\cancelto{0}{\omega(Y^h)})\beta(X^h)$

$$\equiv D\beta(X^h, Y^h) \checkmark$$

■

NB: * For $\varphi \in \Omega^0_{eq}(P,\rho) = \Omega^0(P,\rho) = C^\infty_{eq}(P,V) = C^\infty(P,\rho)$

Its ext cov deriv is $D\varphi \in \Omega^1_{tens}(P,V)$ so that $D\varphi(X) \equiv d\varphi(X^h) = X^h(\varphi)$
that is the cov deriv of φ as def geometrically! ⤳ Noted "$D_X\varphi$".
↳ So we recover, and generalize, the notion.

* By the way, since $\iota_X \varphi = \varphi(X) = 0$ $\forall X \in \Gamma(TP)$, and for $X \in \Gamma(HP)$
in particular $\varphi \in \Omega^0_{tens}(P,V)$ so its ext cov deriv is:
$\boxed{D\varphi = d\varphi + \rho_*(\omega)\varphi}$ (In physics "Minimal coupling")

φ is said covariantly cst (~ // transported) if $D\varphi = 0$.
Idem, $\alpha \in \Omega_{eq}(P,V)$ is cov cst if $D\alpha = 0$.

Rmk: The Cartan struct eq $\Omega = D\omega = d\omega + \frac{1}{2}[\omega,\omega]$ must not be thought of as an applicat° of $D = d + \rho_*(\omega)$ on ω with $\rho_* = ad$ since ω is not tensorial!
The factor $\frac{1}{2}$ should signal this fact ✓

NB : Ω is tensorial of type $(Ad, \mathrm{Lie}\,G)$ so its cov deriv is

$$D\Omega = d\Omega + \rho_*(\omega)\Omega = d\Omega + [\omega, \Omega] \qquad (\rho_* = ad).$$

$$= d\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right) + [\omega, d\omega + \tfrac{1}{2}[\omega,\omega]]$$

$$= \underbrace{\tfrac{1}{2}[d\omega,\omega] - \tfrac{1}{2}[\omega,d\omega]}_{-[\omega,d\omega]} + [\omega, d\omega] + \tfrac{1}{2}\cancelto{0}{[\omega,[\omega,\omega]]}$$

$$= \quad -[\omega, d\omega] \quad + [\omega, d\omega] = 0$$

↳ This result is the Bianchi identity : $\ulcorner D\Omega = 0 \lrcorner$.

NB : $D : \Omega_{tens}(P,V) \longrightarrow \Omega_{tens}(P,V)$, so for $\beta \in \Omega_{tens}(P,V)$:

$$D^2\beta \equiv DD\beta = d(D\beta) + \rho_*(\omega)\wedge D\beta = d(d\beta + \rho_*(\omega)\wedge\beta) + \rho_*(\omega)\wedge(d\beta + \rho_*(\omega)\wedge\beta)$$

$$= d\rho_*(\omega)\wedge\beta - \rho_*(\omega)\wedge d\beta + \rho_*(\omega)\wedge d\beta + \rho_*(\omega)\wedge\rho_*(\omega)\wedge\beta.$$

$$= \rho_*(d\omega + \omega\wedge\omega)\wedge\beta = \rho_*(\Omega)\wedge\beta.$$

↳ $\ulcorner D^2\beta = \rho_*(\Omega)\beta \lrcorner$.

Rmk that for $\beta = \varphi \in \Omega^0_{tens}(P,\rho) = C^\infty(P,\rho)$ this is : $D^2\varphi \in \Omega^2_{tens}(P,\rho)$

So $D^2\varphi(X,Y) = \rho_*(\Omega(X,Y))\varphi$

$$= \rho_*(-d\omega(X^h,Y^h))\varphi = -\rho_*(\omega([X^h,Y^h]))\varphi.$$

ie $\omega([X^h,Y^h])$ is the Lie alg elem^t generating $[X^h,Y^h]$, which acts on φ.

↳ This is indeed, up to sign, the "geometric" def of the curvature given p-g-!

2.4 | Act° of the gauge grp.

As we have seen the $\mathrm{Aut}_v(P)/\mathcal{G}$ grp acts on $\Omega(P)$ by pullback, and in particular on $\Omega_{tens}(P,V)$ the result is read-off the ρ-equiv property !

So Given $\psi \in \mathrm{Aut}_v(P) \simeq \gamma \in \mathcal{G}$:

- The gauge trsf of $\varphi \in \Omega^0_{tens}(P,V) = C^\infty(P,V) \simeq s \in E = P\times_\rho V$

  is : $\varphi^\gamma \equiv \psi^*\varphi = \rho(\gamma^{-1})\varphi$ ✓

- The GT of $D\varphi \in \Omega^1_{tens}(P,V)$: $(D\varphi)^\gamma = \psi^* D\varphi = \rho(\gamma^{-1}) D\varphi$
- The gauge trsf of $\Omega \in \Omega_{tens}(P, \text{Lie}\,G)$ is : $\Omega^\gamma \equiv \psi^*\Omega = \text{Ad}_{\gamma^{-1}}\Omega$
- Also true for GT of element of $\mathcal{G}$, as special case 1st point. $= \gamma^{-1}\Omega\gamma$

But the connect° is only equivariant, pseudotensorial of type $(\text{Ad}, \text{Lie}\,G)$.
So to find its GT a computat° is necessary. Given $X_p \in T_pP$:

$$
\begin{aligned}
(\psi^*\omega_{\psi(p)})(X_p) &= \omega_{\psi(p)}(\psi_* X_p) && \text{(I) p-4-} \\
&= \omega_{p\gamma(p)}\left(R_{\gamma(p)*}X_p + \{[\gamma^{-1}d\gamma]_p(X_p)\}^v_{p\gamma(p)}\right) && 1^{st}\text{ cond of }\omega \\
&= R^*_{\gamma(p)}\omega_{p\gamma(p)}(X_p) + [\gamma^{-1}d\gamma]_p(X_p) && 2^{nd}\text{ cond, equiv of }\omega. \\
&= \text{Ad}_{\gamma^{-1}(p)}\omega_p(X_p) + [\gamma^{-1}d\gamma]_p(X_p) \\
&= \left(\text{Ad}_{\gamma(p)^{-1}}\omega_p + \gamma^{-1}d\gamma_p\right)(X_p)
\end{aligned}
$$

$$
\hookrightarrow \boxed{\begin{aligned}\omega^\gamma \equiv \psi^*\omega &= \text{Ad}_{\gamma^{-1}}\omega + \gamma^{-1}d\gamma \\ &= \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma\end{aligned}}
$$

⇝ "$\gamma^{-1}d\gamma$" is the "pseudotensorial" part!
Notice again this is a right-act° !

<u>NB</u>: $\text{Aut}_v(P)$ acts in a way compatible with $\wedge, d$ so it preserves the Cartan Struct Eq and the writing of D on tensorial forms in term of ω.

This can be seen in that :

-
$$
\begin{aligned}
d\omega^\gamma + \tfrac{1}{2}[\omega^\gamma,\omega^\gamma] &= d(\gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma) + \tfrac{1}{2}[\gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma, \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma] \\
&= d\gamma^{-1}\omega\gamma + \gamma^{-1}d\omega\gamma - \gamma^{-1}\omega d\gamma + \tfrac{1}{2}[\gamma^{-1}\omega\gamma, \gamma^{-1}\omega\gamma] + d\gamma^{-1}d\gamma \\
&\quad + \tfrac{1}{2}[\gamma^{-1}\omega\gamma, \gamma^{-1}d\gamma] + \tfrac{1}{2}[\gamma^{-1}d\gamma, \gamma^{-1}\omega\gamma] + \tfrac{1}{2}[\gamma^{-1}d\gamma, \gamma^{-1}d\gamma] \\
&= \gamma^{-1}d\omega\gamma + \tfrac{1}{2}\gamma^{-1}[\omega,\omega]\gamma + d\gamma^{-1}d\gamma + [\gamma^{-1}\omega\gamma, \gamma^{-1}d\gamma] \\
&\quad \underbrace{-\gamma^{-1}d\gamma\gamma^{-1}\omega\gamma - \gamma^{-1}\omega d\gamma}_{-[\gamma^{-1}\omega\gamma,\, \gamma^{-1}d\gamma]} + \underbrace{\gamma^{-1}d\gamma\wedge\gamma^{-1}d\gamma}_{-d\gamma^{-1}d\gamma} \\
&= \gamma^{-1}(d\omega + \tfrac{1}{2}[\omega,\omega])\gamma = \gamma^{-1}\Omega\gamma \equiv \Omega^\gamma
\end{aligned}
$$

-
$$
\begin{aligned}
d\varphi^\gamma + \rho_*(\omega^\gamma)\varphi^\gamma &= d(\rho(\gamma^{-1})\varphi) + \rho_*(\gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma)\rho(\gamma^{-1})\varphi \\
&= d\rho(\gamma^{-1})\varphi + \rho(\gamma^{-1})d\varphi + \rho(\gamma^{-1})\rho_*(\omega)\rho(\gamma)\rho(\gamma^{-1})\varphi + \rho_*(\gamma^{-1}d\gamma\gamma^{-1})\varphi \\
&= \rho_*(\cancel{d\gamma^{-1}})\varphi + \rho(\gamma^{-1})[d\varphi + \rho_*(\omega)\varphi] + \rho_*(\cancel{\gamma^{-1}d\gamma}\gamma^{-1})\varphi.
\end{aligned}
$$

ie $d\varphi^r + \rho_*(\omega^r)\varphi^r = \rho(r^{-1})D\varphi = D^r\varphi^r = (D\varphi)^r$ ✓

⇝ Related to gauge principle in physics!

3] Local descript°

By local we mean descript° of object on P as seen from an open set $U \subset M$.
For this we need a local sect° $\sigma : U \subset M \to \pi^{-1}(U) \subset P$ via which we can pullback object from $\pi^{-1}(U) \subset P$ to $U \subset M$! Rmk: σ ~ choice of gauge / local gauge!

- The pullback of ω is: $A \equiv \sigma^*\omega \in \Omega^1(M, \mathrm{Lie}\,G)$

  Given $\{T_a\}$ basis of Lie G and $\{x^\mu\}$ coord syst on U, we have:

  $\boxed{A = A_\mu dx^\mu = A^a_\mu T_a dx^\mu}$. The components A^a_μ is the Yang-Mills potential ✓

- The pullback of Ω is: $F \equiv \sigma^*\Omega \in \Omega^2(M, \mathrm{Lie}G)$

  Explicitly: $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} F^a_{\mu\nu} T_a dx^\mu \wedge dx^\nu$.

  The pullback is comp with $\wedge, d$ so we have Cartan Struct Eq: $F = dA + \frac{1}{2}[A,A]$
  $= dA + A\wedge A$

  ie $F = \left(\partial_\mu A_\nu + \frac{1}{2}[A_\mu, A_\nu]\right) dx^\mu \wedge dx^\nu$
  $= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right) dx^\mu \wedge dx^\nu$

  ↳ $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]}$. Further; $[A_\mu, A_\nu] = [A^a_\mu T_a, A^b_\nu T_b]$
  $= A^a_\mu A^b_\nu [T_a, T_b]$
  $= A^a_\mu A^b_\nu \underbrace{f^c_{ab}}$ T_c

  other notat° for struct cst of Lie G!

  ↳ $\boxed{F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + A^a_\mu A^b_\nu f^c_{ab}}$

  This is the YM Field Strength ✓

- The pullback of φ is: $\phi \equiv \sigma^*\varphi \in C^\infty(M, V) = \Omega^0(M, V)$.

  Explicitly $\phi : U \subset M \to V$, $\phi(x) = \varphi(\sigma(x))$ it is a matter field in physics!

- The pullback of $D\varphi$ is: $D\phi \equiv \sigma^* D\varphi \in \Omega^1(M, V)$

  Also $\sigma^* D\varphi = \sigma^*(d\varphi + \rho_*(\omega)\varphi) = d\sigma^*\varphi + \rho_*(\sigma^*\omega)\sigma^*\varphi$

  So $\boxed{D\phi = d\phi + \rho_*(A)\phi}$

  Suppose $\rho_*(T_a) = T^i{}_j$ because $\{e_i\}$ basis of V, so that $\phi = \phi^i e_i$.
  $= \rho_*(T_a)^i{}_j$

  ↳ And $\rho_*(T_a)^i{}_j e_k \equiv \delta^i_k e_j$.

UMONS

so $\quad D\phi = D_\nu \phi\, dx^\nu = \{D_\nu \phi^i\} e_i\, dx^\nu$

and $\quad d\phi + \rho_*(A)\phi = \{\partial_\nu \phi + \rho_*(A_\nu)\phi\} dx^\nu$

$$= \{\partial_\nu \phi^i e_i + \rho_*(A^a_\nu t_a)\phi^k e_k\} dx^\nu$$
$$= \{\partial_\nu \phi^i e_i + A^i_{j\nu}\, \rho_*(t_a)^j_i\, \phi^k e_k\} dx^\nu$$
$$= \{\partial_\nu \phi^i + A^i_{j\nu}\phi^j\} e_i\, dx^\nu$$

$\hookrightarrow \quad \boxed{D_\nu \phi^i = \partial_\nu \phi^i + A^i_{j\nu}\phi^j}$. This is minimal coupling of matter field ϕ with the YM potential A ✓

3.1 Local descript° active GT

Define the local gauge grp $\mathcal{G}_{loc} = \{\gamma_{loc} \equiv \sigma^*\gamma = \gamma\circ\sigma \mid \gamma \in \mathcal{G}\}$ over $U \subset M$.

So the local GT of $\varphi, D\varphi, A$ and F are:

$\phi^{\gamma_{loc}} \equiv \sigma^*(\varphi^\gamma) = \sigma^*(\rho(\gamma^{-1})\varphi) = \rho[(\sigma^*\gamma)^{-1}]\sigma^*\varphi = \rho(\gamma^{-1}_{loc})\phi$ ✓

so idem : $\begin{cases} (D\phi)^{\gamma_{loc}} \equiv \sigma^*(D\varphi^\gamma) = \rho(\gamma^{-1}_{loc})D\phi \\ F^{\gamma_{loc}} \equiv \sigma^*(\Omega^\gamma) = Ad_{\gamma^{-1}_{loc}} F = \gamma^{-1}_{loc} F \gamma_{loc} \\ A^{\gamma_{loc}} \equiv \sigma^*(\omega^\gamma) = Ad_{\gamma^{-1}_{loc}} A + \gamma^{-1}_{loc} d\gamma_{loc} \end{cases}$ ✓

Also : $F^{\gamma_{loc}} = dA^{\gamma_{loc}} + \frac{1}{2}[A^{\gamma_{loc}}, A^{\gamma_{loc}}]$

$(D\phi)^{\gamma_{loc}} = D^{\gamma_{loc}}\phi^{\gamma_{loc}} = d\phi^{\gamma_{loc}} + \rho_*(A^{\gamma_{loc}})\phi^{\gamma_{loc}}$ ✓

NB : $\sigma^*\varphi^* = (\varphi\circ\sigma)^*$ and $\varphi\circ\sigma(x) = \varphi(\sigma(x)) = \sigma(x)\gamma(\sigma(x)) = \sigma(x)\gamma_{loc}(x)$!

So pullback of gauge trsformed objects on P is equiv to pullback of non gauge trsformed objects on P by new section $\sigma' = \sigma\gamma_{loc}$ over the same $U \subset M$, with $\gamma_{loc} : U \to G \in \mathcal{G}_{loc}$.

$\hookrightarrow$ { One observer choose σ and sees ω as A.
Another obs choose $\sigma' = \sigma\gamma_{loc}$ and sees ω as $A^{\gamma_{loc}}$.

$\Rightarrow$ 2 viewpoint of same global (on P) object !

This we prove in the following

3.2 Change of local descript° : Passive GT

Given another open set $\mathcal{U}' \subset M$ on which one chooses a sect° $\sigma' : \mathcal{U}' \to \pi^{-1}(\mathcal{U}') \subset M$

Suppose $\mathcal{U}$ and $\mathcal{U}'$ overlap : $\mathcal{U} \cap \mathcal{U}' \neq \phi$, then $\forall x \in \mathcal{U} \cap \mathcal{U}'$ $\sigma(x)$ and $\sigma'(x)$ belong to the same fiber $\pi^{-1}(x) \subset \pi^{-1}(\mathcal{U} \cap \mathcal{U}') \subset P$.

So $\exists$ $g : \mathcal{U} \cap \mathcal{U}' \to G$, called a transit° funct°, s.t : $\sigma'(x) = \sigma(x) g(x)$.

[Ex: given $\mathcal{U}_i, \mathcal{U}_j, \mathcal{U}_k$ with overlaps, work out the compatibility condit° on the transit° funct° g_{ij}, g_{jk}, g_{ik}.

- Given $X \in \Gamma(TM)$ with flow $c(\tau)$, $X_x \in T_x M$ and $c(0) = x$, we have the pushforward $\sigma_* X_x \in T_{\sigma(x)} P \subset TP|_{\pi^{-1}(\mathcal{U})}$ $\forall x \in \mathcal{U} \subset M$.

  Then also $\sigma'_* X_x \in T_{\sigma'(x)} P = T_{\sigma(x) g(x)} P$ $\forall x \in \mathcal{U} \cap \mathcal{U}'$.

  Explicitly :
$$\sigma'_* X_x \equiv \frac{d}{d\tau}\Big|_{\tau=0} \sigma'(c(\tau)) = \frac{d}{d\tau}\Big|_{\tau=0} \sigma(c(\tau))\, g(c(\tau))$$
$$= \frac{d}{d\tau}\Big|_{\tau=0} \sigma(c(\tau))\, g(x) + \sigma(x) \frac{d}{d\tau}\Big|_{\tau=0} g(c(\tau))$$
$$= \sigma_* X_x \cdot g(x) + \sigma(x)\, g(x) \cdot g(x)^{-1} \frac{d}{d\tau}\Big|_{\tau=0} g(c(\tau))$$
$$= R_{g(x)*}(\sigma_* X_x) + \underbrace{\sigma'(x) \underbrace{[g^{-1} dg]_x (X_x)}_{\in \mathrm{Lie}\, G}}_{\frac{d}{d\tau}\big|_{\tau=0} \sigma'(x) \exp(\tau [g^{-1} dg]_x (X_x))}$$

$$\boxed{\sigma'_* X_x \equiv R_{g(x)*}(\sigma_* X_x) + \{[g^{-1} dg]_x (X_x)\}^{v}_{\sigma'(x)}}$$

- Change of local descript° of tensorial forms on P :

  Given $\alpha \in \Omega_{\mathrm{tens}}(P, V)$, its pullback on $\left|\begin{array}{l} \mathcal{U} \text{ is } a_x \equiv \sigma^* \alpha_{\sigma(x)} \\ \mathcal{U}' \text{ is } a'_x \equiv \sigma'^* \alpha_{\sigma'(x)} \end{array}\right.$

  How are they related on $\mathcal{U} \cap \mathcal{U}'$?

Given $X_x \in T_xM$, $\forall x \in U \cap U'$:

$$a'_x(X_x) = \sigma'^* \alpha_{\sigma'(x)}(X_x) = \alpha_{\sigma'(x)}(\sigma'_* X_x)$$

$$= \alpha_{\sigma'(x)}(R_{g(x)*}\, \sigma_* X_x) \quad \text{since } \alpha \text{ is horiz!}$$

$$= R^*_{g(x)}\, \alpha_{\sigma'(x)}(\sigma_* X_x)$$

$$= \rho(g(x)^{-1})\, \alpha_{\sigma(x)}(\sigma_* X_x) \quad \text{since } \alpha \text{ is } \rho\text{-equiv!}$$

$$= \rho(g(x)^{-1})\, \sigma^* \alpha_{\sigma(x)}(X_x) = \rho(g(x)^{-1})\, a_x(X_x).$$

$\hookrightarrow$ $a' = \rho(g^{-1})\, a$ on $U \cap U' \subset M$ ✓

$\Rightarrow$ Then for $\varphi, D\varphi \in \Omega_{tens}(P, V)$ and $\Omega \in \Omega_{tens}(P, \mathrm{Lie}\,G)$ we have:

$$\phi' = \rho(g^{-1})\phi \;;\; (D\phi)' = \rho(g^{-1})\, D\phi \;;\; F' = \mathrm{Ad}_{g^{-1}} F = g^{-1} F g.$$

These are passive gauge trsf, different local descript° of same global objects!

- Change of local descript° of the connect° form:

$A \equiv \sigma^*\omega$ on U, $A' \equiv \sigma'^*\omega$ on U'. Then for $X_x \in T_xM$, $\forall x \in U \cap U'$:

$$A'_x(X_x) = \sigma'^* \omega_{\sigma'(x)}(X_x) = \omega_{\sigma'(x)}(\sigma'_* X_x)$$

$$= \omega_{\sigma'(x)}\left(R_{g(x)*}\, \sigma_* X_x + \{[g^{-1}dg]_x(X_x)\}^v_{\sigma'(x)}\right)$$

$$= R^*_{g(x)}\, \omega_{\sigma'(x)}(\sigma_* X_x) + [g^{-1}dg]_x(X_x) \quad \text{since } \omega(X^v) = X \in \mathrm{Lie}\,G$$

$$= \mathrm{Ad}_{g(x)^{-1}}\, \omega_{\sigma(x)}(\sigma_* X_x) + [g^{-1}dg]_x(X_x) \quad \text{since } \omega \text{ is Ad-equiv}$$

$$= \mathrm{Ad}_{g(x)^{-1}}\, \sigma^* \omega_{\sigma(x)}(X_x) + [g^{-1}dg]_x(X_x)$$

$$= \mathrm{Ad}_{g(x)^{-1}}\, A_x(X_x) + [g^{-1}dg]_x(X_x)$$

$\hookrightarrow$ $A' = \mathrm{Ad}_{g(x)^{-1}} A + g^{-1}dg = g^{-1} A g + g^{-1}dg$

This is the passive GT of the local representative of ω / YM gauge potential.

Rmk: This implies $F' = dA' + \frac{1}{2}[A', A']$ and $(D\phi)' = D'\phi' \equiv d\phi' + \rho_*(A')\phi'$.

4) Physics : Lagrangian

Physical theory = Choice of Lagrangian (not totally true of course) : $\mathcal{L}$.

$\mathcal{L}$ is a functionnal on the space of field $\mathcal{F} = \{A^a_\mu, \phi^i\}$.

$\mathcal{L} : \mathcal{F} \longrightarrow \mathbb{R}$
$\{A^a_\mu, \phi^i\} \longmapsto \mathcal{L}(A^a_\mu, \phi^i)$

. Then built act° functionnal : $S[A^a_\mu, \phi^i] = \int \mathcal{L}(A^a_\mu, \phi^i) d^n x$ with $m = \dim M$.

Via variationnal principle $\delta S = 0$ obtain field eq for A^a_μ and ϕ^i (Euler-Lagrange eq).

Quantum : Go from $\mathcal{L}$ to Hamiltonian, then canonical quantizat°.
Or built Q-zat° $Z[A^a_\mu, \phi^i] = \int \delta A^a_\mu \delta\phi^i \exp \frac{i}{\hbar} S[A^a_\mu, \phi^i]$, Feynman path int.

To constrain the choice of $\mathcal{L}$, can appeal to symetry principles,

i) GR principle require $\mathcal{L}$ to be inv under coord changes on $M \Leftrightarrow$ equiv $\mathcal{L}$ to be Diff(M) - inv ! The latter has deeper meaning regarding "reality" of M.
(This is judged strong principle, but not entirely trivial why !
↳ Kretschmann (Erich) object° 1917.)

ii) In the same spirit, $\mathcal{L}$ required to be indep of choice of local sect° σ / gauge.
$\Leftrightarrow$ equiv $\mathcal{L}$ to be $\mathrm{Aut}_V(P) \simeq \mathcal{G}$ invariant, (Again latter stronger meaning but even less clear why $\rightarrow$ Generalized Kretschmann object° !)
This is the gauge principle ✓

Taken together $\mathcal{L}$ is required $\mathrm{Diff}(M) \times \mathcal{G}$ -invariant. Strongly restrict class of admissible theory already !

→ Requirent of GR i) is met if $L = \mathcal{L}\, d^n m$ is a diff form, of max degree on M (L is a m-form). But it should be $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ - valued, and the basic field are $A, F \in \Omega(M, \mathrm{Lie}\,G)$ and $\phi, D\phi \in \Omega(M, V)$.
For A, F it is easy, use Killing form $K(X,Y) \propto \mathrm{Tr}(XY)$ for $X, Y \in \mathrm{Lie}\,G$.
For $\phi, D\phi$, need to suppose V has non-degen bilin form $\langle\, ,\rangle : V \times V \longrightarrow \mathbb{K}$.
~~So can built e.g~~ In general might need Hodge op $* : \Omega^k(M) \longrightarrow \Omega^{m-k}(M)$ so that $\forall\, \alpha \in \Omega^k(M)$, $\alpha \wedge * \alpha$ is a m-form ✓

Ex : $L(A) = \mathrm{Tr}(F \wedge * F) + m^2 \, \mathrm{Tr}(A \wedge * A)$ (1)
$L(\phi) = \langle D\phi, * D\phi \rangle + m^2 \langle \phi, * \phi \rangle$ (2)

1st terms are kinematic
2nd terms are mass terms (hence "m^2").

If one doesn't use $*$ to build L, the theory is said topological.

Ex : In $\dim M = 3$, $L(A) = \mathrm{Tr}\left[A \wedge F + \frac{1}{3} A \wedge A \wedge A\right]$ (3) is a Chern-Simons Lagrangian/Th.
In $\dim M = 4$, if $*F = \pm F$ the field strength is said self/anti self dual
and $L(A) = \mathrm{Tr}(F \wedge F)$ (4) describe Yang-Mills instantons.

→ Requirement of ii), the gage principle is met if :

- First for $\Omega(M,V)$, the rep through which they trsf is unitary.
  Given $\langle , \rangle$ on V, its unitary grp $U(V)$ is subgrp of $GL(V)$ s.t $\forall\, u, v \in V$
  $\forall\, U \in U(V)$: $\langle Uu, Uv \rangle = \langle u, v \rangle$.
  ↗ see postit for ρ_* !
  So a unitary rep of G is $\rho : G \to U(V)$, so that $\langle \rho(g)u, \rho(g)v \rangle = \langle u, v \rangle$ ✓

  ↳ If so we have for (2) : $L(\phi)^\gamma \equiv L(\phi^\gamma) = L(\phi)$ ✓

- Also, L should involve field trsforming tensorielly under $\mathcal{G}$, ie local rep of tensorial forms !

  ↳ ok for (2), but in (1) only $\mathrm{Tr}(F \wedge * F)$ is admissible.
  Indeed $\mathrm{Tr}(F^\gamma \wedge * F^\gamma) = \mathrm{Tr}(\gamma^{-1} F \gamma \wedge * \gamma^{-1} F \gamma) = \mathrm{Tr}(\gamma^{-1} F \wedge * F \gamma) = \mathrm{Tr}(F \wedge * F)$
  du to cyclicity of Tr. ✓
  But due to non tensoriality of A : $\mathrm{Tr}(A^\gamma \wedge * A^\gamma) \neq \mathrm{Tr}(A \wedge * A)$ ✗
  Also (3) is ok, but CS (4) is not. ↳ mass term for A YM potential forbidden !

Ex : Prototypical gage th Lagrangian ; $L(A, \phi) = \mathrm{Tr}(F \wedge * F) + \langle D\phi, * D\phi \rangle + m^2 \langle \phi, * \phi \rangle$.
(Coupling of YM pot A with a scalar field ϕ)

NB : Absence of mass for A at odd with phenomenology of nuclear interact°, since suggest A is long range, not confined to nuclear short distances.

Rmk : Without choice of L, there is an eq that F satisfies always ("kinematically" vs "dynamically" for an eq stemming from L) : Bianchi identity $\boxed{DF = 0}$

Rmk: For ρ uni-rep, $\frac{d}{d\tau}\Big|_{\tau=0} \langle \rho(\exp \tau X)u, \rho(\exp\tau X)v\rangle$

- $= \frac{d}{d\tau}\Big|_{\tau=0} \langle u, v\rangle = 0$, for $X \in \mathrm{Lie}\,G$.

- $= \langle \frac{d}{d\tau}\Big|_{\tau=0} \rho(\exp\tau X)u, v\rangle + \langle u, \frac{d}{d\tau}\Big|_{\tau=0} \rho(\exp\tau X)v\rangle$

  $\equiv \langle \rho_*(X)u, v\rangle + \langle u, \rho_*(X)v\rangle$

$\rightarrow$ ⌜ $\langle \rho_*(X)u, v\rangle + \langle u, \rho_*(X)v\rangle = 0$ ⌟

4.1) Field Eq for A

Let's consider just the ex of YM eq without source (free).

The Lagrangian of a free YM potential, uncoupled to any matter field, is just:

$L_{YM}(A) = \mathrm{Tr}(F \wedge * F)$. So $S_{YM}[A] = \int L_{YM} = \int \mathrm{Tr}(F \wedge * F)$.

We want $\delta S = 0 \ \forall \delta A$. Notice $\delta F = \delta(dA + \frac{1}{2}[A,A]) = d\delta A + [A, \delta A] = D(\delta A)$.

So $\delta S_{YM} = \int \mathrm{Tr}(\delta F \wedge * F) + \mathrm{Tr}(F \wedge * \delta F) = 2\int \mathrm{Tr}(\delta F \wedge * F) = 2 \int \mathrm{Tr}(D\delta A \wedge * F)$

by int by part: $= -2 \int \mathrm{Tr}(\delta A \wedge D * F) \overset{!}{=} 0$ qlq soit δA.

We obtain the YM eq: $\boxed{D * F = 0}$

This is a $(m-1)$-form, to obtain the component version it is better to consider the dualized eq: $* D * F = 0$

$$\Rightarrow \boxed{\begin{array}{l} (-)^{s+(m-1)} \sqrt{|g|}^{-1} D_\sigma \left(F_{\mu\nu} g^{\mu\sigma} g^{\nu\rho} \sqrt{|g|} \right) g_{\rho r} \, dx^r . \\ \Rightarrow D_\sigma \left(F_{\mu\nu} g^{\mu\sigma} g^{\nu\rho} \sqrt{|g|} \right) = 0 . \end{array}}$$

↳ in flat spacetime: $D^\mu F_{\mu\nu} = 0$, for abelian gauge theory: $\partial^\mu F_{\mu\nu} = 0$ Maxwell Eq! (in vacuum)

Bianchi being $dF = 0 \rightsquigarrow \partial_{[\rho} F_{\mu\nu]} = 0$, other half of Maxwell Eq ✓

So half Maxwell's Eq are non-dynamical!

4.2) Field Eq for ϕ (coupled to A)

The Lagrangian is $L_\phi(\phi, A) = \int \langle D\phi, * D\phi \rangle - m^2 \langle \phi, * \phi \rangle$, the act° $S_\phi[\phi, A] = \int L(\phi, A)$

$$\delta S_\phi = \delta \int \langle D\phi, * D\phi \rangle - m^2 \langle \phi, * \phi \rangle = \delta \int - \langle \phi, D * \phi \rangle - m^2 \langle \phi, * \phi \rangle$$

$$= - \int \langle \delta\phi, D * D\phi + m^2 * \phi \rangle = 0 \quad \text{for any } \delta\phi$$

Obtain Klein-Gordon Eq: $\boxed{D * D\phi + m^2 * \phi = 0}$

This is a m-form, in comp: $\boxed{D_\nu \left(\sqrt{|g|} g^{\mu\nu} D_\mu \phi \right) + m^2 \phi \sqrt{|g|} = 0}$

↳ In flat spacetime: $D^\mu D_\mu \phi + m^2 \phi = 0$ ✓

For free ϕ (no A): $\partial^\mu \partial_\mu \phi + m^2 \phi = 0$ ie $\Box \phi + m^2 \phi = 0$ ✓

NB: Importance $k = \mathrm{Tr}$ and $\langle , \rangle$ to be non-degenerate to be able to extract field Eq!

↳ non-deg $g \Leftrightarrow \{ g(x,y) = 0 \ \forall y \Rightarrow x = 0 \}$

4.1-2 | YM eq sourced by scalar field

Consider $L(A,\phi) = \mathrm{Tr}(F\wedge *F) + \langle D\phi, * D\phi\rangle + m^2 \langle \phi, *\phi\rangle$, $S = \int L(A,\phi)$

$$\delta_A \langle D\phi, * D\phi\rangle = \langle \rho_*(\delta A)\phi, * D\phi\rangle + \langle D\phi, * \rho_*(\delta A)\phi\rangle$$
$$= \langle \rho_*(\delta A)\phi, * D\phi\rangle - \langle * \rho_*(\delta A) D\phi, \phi\rangle .$$

By the way : $\langle \phi, \psi\rangle = \mathrm{Tr}(|\psi\rangle\langle\phi|)$

[$\{e_i\}$ basis of V, $\langle e_i, e_j\rangle = \delta_{ij}$ and $|e_i\rangle \otimes \langle e_i| = |e_i\rangle\langle e_i| = id_V$]

So note : $\langle \rho_*(\delta A)\phi, * D\phi\rangle = - \langle \phi, \rho_*(\delta A) * D\phi\rangle = - \mathrm{Tr}(|\rho_*(\delta A) * D\phi\rangle\langle\phi|)$
$$\equiv - \mathrm{Tr}(\delta A\, |* D\phi\rangle\langle\phi|) .$$

and : $- \langle * \rho_*(\delta A) D\phi, \phi\rangle = \mathrm{Tr}(\delta A\, |\phi\rangle\langle * D\phi|)$.

Then the eq for A is given by :

$$\delta_A S = \int \mathrm{Tr}\left\{ \delta A \left(D*F + |\phi\rangle\langle * D\phi| - |* D\phi\rangle\langle\phi| \right)\right\} = 0 \quad \forall\, \delta A .$$

$\hookrightarrow$ $D*F = |*D\phi\rangle\langle\phi| - |\phi\rangle\langle *D\phi| \equiv J$ last eq define the current of ϕ, J

$D*F = J$ is the sourced YM eq, eq btwn $(n-1)$-forms .

To have it in comp, easier to take the Hodge dual :

$*D*F = *J$ eq btwn 1-forms .

$\hookrightarrow$ $\frac{1}{\sqrt{|g|}} D_\sigma\left(F_{\mu\nu}\, g^{\mu\sigma} g^{\nu\rho} \sqrt{|g|}\right) = |D_\nu\phi\rangle\langle\phi| g^{\nu\rho} - |\phi\rangle\langle D_\nu\phi| g^{\nu\rho} = J^\rho$ ✓

For Maxwell in flat spacetime this is : $\partial_\mu F^{\mu\nu} = D^\nu\phi^\dagger\phi - \phi^\dagger D^\nu\phi = J^\nu$

The eq for ϕ is the same as before .

4.3 Spont Sym Breaking and mass of YM potential

As said mass term for A forbidden by gauge sym!

In 1964 Brout-Englert; Higgs; Hagen-Guralnik-Kibble proposed SSB mechanism.

Let's see how it works on simple abelian model, a $U(1)$-gauge theory.

Lagrangian is : $L(A,\phi) = F^\dagger \wedge * F + D\phi^\dagger * D\phi - V(\phi)\,\mathrm{vol}$, $(\mathrm{vol} = *1)$.

ϕ is embedded in the potential $V(\phi) = \mu^2 \phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$, $\lambda > 0$ and $\mu^2 \in \mathbb{R}$.

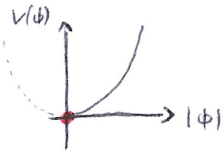
The vacuum of the theory is determined by field conf of minimal energy : $V(\phi_{min}) \leqslant V(\phi)$

$$\frac{\delta V(\phi)}{\delta \phi^\dagger} = \mu^2\phi + 2\lambda\,\phi\phi^\dagger\phi = \phi(\mu^2 + 2\lambda\,\phi^\dagger\phi) = 0 \qquad \hookrightarrow \phi_{min} = \underline{VEV}.$$

$\hookrightarrow$ either $\phi_0 = 0$ or ϕ_0 s.t $\phi_0^\dagger\phi_0 = |\phi_0|^2 = \frac{-\mu^2}{2\lambda} \leadsto |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}}$.

The theory has 2 phases :

- $\mu^2 > 0$: only solut° is $\phi_0 = 0$, potential is



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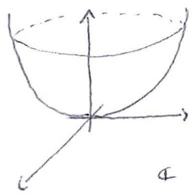


ϕ_0 is a $U(1)$-inv solut° : $\phi_0^\gamma = e^{i\gamma(x)}\phi_0 = 0 = \phi_0$ ✓

$\rightarrow$ Perturb exp around vacuum : $\phi = \phi_0 + H = H$, the Lagrangian is then

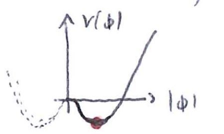
$L(A,H) = F^\dagger \wedge * F + DH^\dagger * DH$, Theory of coupled massless A and H ✓

- $\mu^2 < 0$: beside unique $\phi_0 = 0$ there is a manifold of solut° ϕ_0 s.t $|\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} = v > 0$.

by the way $V(\phi_0) = -\frac{\mu^4}{2\lambda} < 0$ so the $[\phi_0]$ determine the vacuum!

$\Rightarrow$ Not 1 vacuum conf, but a continuum of vacua ϕ_0 : degeneracy of vacuum!

Potential is



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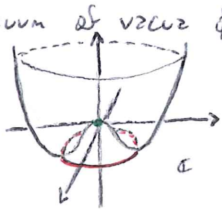
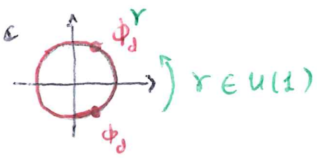


When ϕ "chooses" one vacuum in $[\phi_0]$ this breaks $U(1)$-sym because

$U(1)$ freely and transitively on $[\phi_0]$, ie it trsf one vacuum in another :



$\Rightarrow$ Spont select° of a vacuum = SSB !

→ Perturb expansion : $\phi = \bar{\Phi} + H$, then :

$$V(\phi) = (\mu^2 v^2 + \lambda v^4) - 2\mu^2 H^+ H + (\text{self coupling } H^3 \text{ and } H^4)$$

$$D\phi^+ * D\phi = DH^+ * DH + v^2 A^+ * A .$$

So the Lagrangian becomes :

$$L(A,H) = F^+ \wedge * F + \underline{v^2 A^+ * A} + DH^+ * DH - \underline{2\mu^2 H^+ H} + (\text{self coupling of } H) .$$

Notice that a mass term for H and A is generated by VEV of ϕ/H !

So the interact° carried by A can be short range ✓

4.4 | Too early, but still : Dirac spinors .

Just not° dropping : Given V endowed non degen bilin form $\langle\, ,\, \rangle$, it Clifford Alg is generated by elmt of V s.t : $u \otimes v + v \otimes u = 2\langle u,v\rangle \, id_V$

often noted $\{u,v\} = 2\langle u,v\rangle \, id_V$

Given $\{e_i\}$ basis of V : $\{e_i, e_j\} = 2\langle e_i, e_j\rangle \, id_V = 2 g_{ij} \, id_V$

→ Admit possible to have Clifford alg on $T_x M$ generated by (local) Dirac gamma matrices $\{\gamma_\mu\}$ s.t: $\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \, id_{\mathbb{C}^4}$, represented/acting on spinors $\psi \,(\in \mathbb{C}^4$ usually$)$.

(Really need to def soldering form, as seen in Cartan geom, Chap 4).

This new ingredient allows to buit Gamma 1-form, a Clifford-alg valued 1-form : $\gamma \equiv \gamma_\mu \, dx^\mu$.

So that we have the Dirac operator : $\not{D}\psi \equiv \gamma \wedge * D\psi$, a m-form .

It is then possible with spinors to have a Lagrangian with one deriv only !

↳ $L_D(A,\psi) = \langle \psi, \not{D}\psi\rangle - m^2 \langle \psi, *\psi\rangle$, $S_D \equiv \int L_D$.

$\delta_\psi S_D = \langle \delta\psi, \not{D}\psi - m^2 * \psi\rangle = 0 \quad \forall\, \delta\psi \implies$ Dirac Eq : $\ulcorner \not{D}\psi - m^2 * \psi = 0 \lrcorner$

In comp : $\sqrt{|g|}\,(\gamma^\mu D_\mu \psi - m^2 \psi)\, dx^m = 0$

↳ $\ulcorner (\gamma^\mu D_\mu - m^2)\psi = 0 \lrcorner$

Dirac current def via : $\delta_A S_D = \int \mathrm{Tr}(\delta A \wedge J)$, J $(n-1)$-form !

$$\delta_A S_D = \int \langle \psi, \gamma \wedge \rho_*(\delta A)\psi\rangle = \int \langle \psi, \rho_*(\delta A) \wedge * \gamma\psi\rangle = \int \mathrm{Tr}\big(|\rho_*(\delta A) \wedge * \gamma \psi\rangle\langle \psi|\big)$$

$$= \int \mathrm{Tr}\big(\delta A \wedge |*\gamma\psi\rangle\langle\psi|\big) \to \ulcorner J = |*\gamma\psi\rangle\langle\psi| \lrcorner$$

↳ $*J = J_\mu \, dx^\mu = \frac{(-)^{s+(n-1)}}{\sqrt{|g|}} |\gamma_\mu \psi\rangle\langle\psi| \, dx^\mu \qquad \ulcorner J_\mu = \frac{(-)^{s+(n-1)}}{\sqrt{|g|}} |\gamma_\mu\psi\rangle\langle\psi| \lrcorner$

→ For YM coupled to spinor : $\mathcal{L}(A,\psi) = \mathrm{Tr}(F\wedge *F) + \langle \psi, \not{D}\psi - m*\psi\rangle$

Dirac eq the same but YM eq is : $\boxed{D*F = J}$

or $*D*F = *J$ $\rightsquigarrow$ $\boxed{D_\sigma(F_{\beta\nu}\, g^{\beta\sigma} g^{\nu\rho})\, g_{\rho\mu} = |\gamma_\mu\psi\rangle\langle\psi|}$

Notice, as in scalar ϕ case, that the current is not gage-inv but trsf like YM eq of course.

In Maxwell F is inv, so should be the current so $J_\mu = \langle\psi, \gamma_\mu\psi\rangle = \psi^\dagger \gamma_0 \gamma_\mu \psi$,

γ_0 is used to play the role of $\langle\,,\rangle$ (rigorously a mistake).

4.5 | Gage-fixing

- The act° of $\mathcal{G}$ on any one object on P, $\chi = \{\varphi, \omega\}$ is it gage orbit $\mathcal{O}(\chi) = \{\chi^\gamma \mid \gamma \in \mathcal{G}\}$. There might be 2 object χ, χ' of the same type not related by gauge trsf : χ, χ' are s.t $\nexists\, \gamma \in \mathcal{G}$ s.t $\chi' = \chi^\gamma$

$\Leftrightarrow$ χ, χ' belong to $\neq$ gage orbits : $\mathcal{O}(\chi) \neq \mathcal{O}(\chi')$.

The space of type χ objects is then fibered by act° of $\mathcal{G}$ into non intersecting orbits !

Ex : $\mathcal{E}(P)$, For $\omega \in \mathcal{E}(P)$; $\mathcal{O}(\omega) = \{\omega^\gamma \mid \gamma \in \mathcal{G}\}$

There are gage non-equiv connect° on P so $\mathcal{G}$ acts, on the right, on $\mathcal{E}(P)$ which is then fibered into gage orbits.

Gage-equiv classes are parametrised by the quotient : $\mathcal{E}(P)/\mathcal{G}$ → "Moduli space" noted $\mathcal{M}$.

↳ Can think of $\mathcal{E}(P)$ as an ∞-dim principal bundle over $\mathcal{M}$ with struct grp $\mathcal{G}$: $\mathcal{E}(P) \xrightarrow{\mathcal{G}} \mathcal{M}$ or $\mathcal{E}(P) = \mathcal{E}(P)[\mathcal{M}, \mathcal{G}]$.

Same with space of sect° φ : $\mathcal{S}(\varphi)$ ✓

NB : Most often than not $\mathcal{E}(P)/\mathcal{G} = \mathcal{M}$ is ill-def, not even a manifold so need to introduce further considerat° and restrict° to make this picture work.

- Consider : if physics is gage-inv, then it is not that easy to read the physical content of a gage theory from $\mathcal{L}$ of from field eq !

As in GR, despite being coord-inv, need to choose coord syst to compute stuff → Need to check that result do not depend on this choice !

In the same manner in gauge th. need to choose a local gauge/sect° σ to compute, but then check that nothing (physical) depend on that choice.
From the equivalent global viewpoint a choice of gauge would be the select° of a single representative in each gauge orbit!

↳ would be global sect° $\Sigma : \mathcal{M} \longrightarrow \mathcal{C}(P)$ (or $\mathcal{S}(\varphi)$)

Possible only if the bundle $\mathcal{C}(P)$ is trivial!

Spoiler: generically it is not in non-abelian theories!

- Gribov ambiguity: 1978, construct counter ex via "Coulomb gauge".
- Singer: 1978, prove no-go thm for bundle over 4-sphere and compact non-ab Lie grp.

→ Only local gauge are possible.

NB: "explicitly" a gauge fixing is achieved by starting with some ω and saying that we choose $\gamma \in \mathcal{G}$ s.t some cond $f(\omega^\gamma)=0$ is fulfilled.

This is solved for γ as a fct° (fctionnal) of ω: $\gamma = \gamma(\omega)$.

But, mind that such "field dep gauge trsf" is indeed a GT, ie

$\gamma(\omega) \in \mathcal{G}$ iff is has the required equiv/gauge trsf, that is:

$$R_g^* \gamma(\omega) = g^{-1}\gamma(\omega) g \iff \gamma(\omega)^g \equiv \gamma(\omega^g) = g^{-1}\gamma(\omega) g.$$

If this is not the case | $\gamma(\omega)$ is not an elemt of $\mathcal{G}$!
| $f(\omega^\gamma)=0$ is not a gauge-fixing!

Rmk: In many instances, a GF does not fully fix the gauge, but a residual gauge sym remains.

↳ Like the Lorentz gauge in EM that still allows harmonic GT!