IV. Cartan Geometry

Some Complement on grps and Lie alg

- Consider Lie H subalg of Lie G. The coset $\mathrm{Lie}\,G/_{\mathrm{Lie}\,H}$ is a vect space, with element written: $X + \mathrm{Lie}\,H = [X]$.

  Sum is $[X] + [Y] = (X + \mathrm{Lie}\,H) + (Y + \mathrm{Lie}\,H) = X + Y + \mathrm{Lie}\,H = [X+Y]$.

  Scalar mult is $\lambda[X] = \lambda(X + \mathrm{Lie}\,H) = \lambda X + \lambda \mathrm{Lie}\,H = \lambda X + \mathrm{Lie}\,H = [\lambda X]$.

  - $\mathrm{Lie}\,G/_{\mathrm{Lie}\,H}$ is a rep space for H via Ad:

    $\mathrm{Ad}_h[X] = h[X]h^{-1} = h(X + \mathrm{Lie}\,H)h^{-1} = hXh^{-1} + \mathrm{Lie}\,H = [h^{*}Xh^{-1}]$ ✓

  - $\mathrm{Lie}\,G/_{\mathrm{Lie}\,H}$ is also a rep space for Lie H via ad:

    $\mathrm{ad}_Y[X] = [Y,[X]] = [Y, X + \mathrm{Lie}\,H] = [Y,X] + [Y, \mathrm{Lie}\,H] = [Y,X] + \mathrm{Lie}\,H = \big[[Y,X]\big] = [\mathrm{ad}_Y X]$

- H is a normal subgrp of G if: for $h \in H$, $g^{-1}hg \in H$ $\forall g \in G$. Noted $H \triangleleft G$.
  Lie H is an ideal of Lie G if: for $X \in \mathrm{Lie}\,H$, $[X,Y] \in \mathrm{Lie}\,H$ $\forall Y \in \mathrm{Lie}\,G$.

  ↳ If $H \triangleleft G$, Lie H is an ideal of Lie G. If Lie H ideal, $\mathrm{Lie}\,G/_{\mathrm{Lie}\,H}$ is a Lie alg!

- A vect space is a (Lie G, H)-module if: V is both a Lie G-module via $\bar{\rho}_*$ and a rep space for H via ρ s.t the induced rep ρ_* for Lie H coincide with $\bar{\rho}_*$.

  ↳ Automatic if $H \subset G$, and V a rep space for G.

- A morphism of grps is a map $\varphi : G \longrightarrow G'$, $g \longmapsto \varphi(g)$ s.t $\varphi(ab) = \varphi(a)\varphi(b)$,

  on the left product in G, on the right product in G'.

  From this is follows that $\varphi(e) = e'$, for $e \in G$ and $e' \in G'$ neutral, and $\varphi(a^{-1}) = \varphi(a)^{-1}$

  ↳ grp automorphism $\mathrm{Aut}(G) = \{\varphi : G \to G \mid \varphi \text{ a grp morph}\}$.

- <u>Semidirect product</u>: Given H and K, and a grp morphism $\varphi : H \longrightarrow \mathrm{Aut}(K)$, a new grp is formed by def the product of pairs $(h,k) \in H \times K$

$$(h_1,k_1)\cdot(h_2,k_2) \equiv \left(h_1h_2,\ k_1\,\varphi(h_1)[k_2]\right) \quad , \text{ note } \ulcorner H \ltimes K \lrcorner$$

The neutral elmt is (e,e'), the inverse of (h,k) is $(h^{-1}, \varphi(h^{-1})[k^{-1}])$.

$(h,k)(h,k)^{-1}$ is clear

$$(h,k)^{-1}(h,k) = (h^{-1}, \varphi(h^{-1})[k^{-1}])(h,k) = (hh^{-1}, \varphi(h^{-1})[k^{-1}]\,\varphi(h^{-1})[k])$$
$$= (e, \varphi(h^{-1})[k^{-1}k]) = (e, \varphi(h^{-1})[e']) = (e,e') \ \checkmark$$

<u>Rmk</u>: $K \triangleleft (H \ltimes K)$ indeed $(h^{-1},\varphi(h^{-1})[k^{-1}])(e,\ell)(h,k) = (h^{-1},\varphi(h^{-1})[k^{-1}]\varphi(h^{-1})[\ell])(h,k)$

$$= (h^{-1},\varphi(h^{-1})[k^{-1}\ell])(h,k) = (h^{-1}h, \varphi(h^{-1})[k^{-1}\ell]\varphi(h^{-1})[k])$$
$$= (e, \varphi(h^{-1})[k^{-1}\ell k]) \in (e,K) \simeq K \ \checkmark$$

Noted that $H \simeq (H,e')$ and $K \simeq (e,K)$.

<u>NB</u>: If K is abelian, $k^{-1}\ell k = \ell$, so the above result is $(e, \varphi(h^{-1})[\ell])$.

<u>Ex</u>: $ISO(n) = Eucl(n) \equiv SO(n) \ltimes \mathbb{R}^n$, $\mathbb{R}^n$ abelian (Poincaré $n \to (1,n-1)$).

$(\Lambda,t)\cdot(\Lambda',t') \equiv (\Lambda\Lambda', t+\Lambda t')$, $\quad \varphi : SO \longrightarrow \mathrm{Aut}(\mathbb{R}^n)$, $s \longmapsto \Lambda = \Lambda(s)$ natural matrix $(n\times n)$ rep of SO.

$(\Lambda,t)^{-1} = (\Lambda^{-1}, \Lambda^{-1}(-t)) = (\Lambda^{-1}, -\Lambda^{-1}t)$

$\mathbb{R}^n \triangleleft (ISO(n))$: $\quad (\Lambda^{-1},-\Lambda^{-1}t)\cdot(e,T)\cdot(\Lambda,t) = (\Lambda^{-1}, -\Lambda^{-1}t + \Lambda^{-1}T)\cdot(\Lambda,t)$

$$= (\Lambda^{-1}, \Lambda^{-1}(-t+T))\cdot(\Lambda,t) = (\Lambda^{-1}\Lambda, \Lambda^{-1}(-t+T) + \Lambda^{-1}t)$$
$$= (e, \Lambda^{-1}T)$$

<u>NB</u>: There is a different way to rep this (very often used!).

Use the grp morphism $ISO(n) \longrightarrow GL(n+1)$ ie $(\Lambda,t) \longmapsto \begin{pmatrix}\Lambda & t\\ 0 & 1\end{pmatrix}$

$(e,0) \longmapsto \begin{pmatrix}id & 0\\ 0 & 1\end{pmatrix}$ neutral, $\begin{pmatrix}\Lambda & t\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}\Lambda^{-1} & -\Lambda^{-1}t\\ 0 & 1\end{pmatrix}$ inverse.

Because the semidirect prod is given simply by matrix multiplicat° !

$$\begin{pmatrix}\Lambda & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Lambda' & t'\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\Lambda\Lambda' & \Lambda t'+t\\ 0 & 1\end{pmatrix} \ \checkmark$$

And clearly:

$$\begin{pmatrix}\Lambda^{-1} & -\Lambda^{-1}t\\ 0 & 1\end{pmatrix}\begin{pmatrix}e & T\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Lambda & t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\Lambda^{-1} & \Lambda^{-1}T - \Lambda^{-1}t\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Lambda & t\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\Lambda^{-1}\Lambda & \Lambda^{-1}T - \Lambda^{-1}t + \Lambda^{-1}t\\ 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}e & \Lambda^{-1}T\\ 0 & 1\end{pmatrix} \ \checkmark$$

1] Cartan connect°

- Consider $P(M,H)$ a H-principal bundle, and $\mathrm{Lie}\,G \supset \mathrm{Lie}\,H$ s.t $\dim \mathrm{Lie}\,G = \dim P$.

  A Cartan connect° on P is a $\mathrm{Lie}\,G$-valued 1-form s.t

  - $\varpi_p(X^v_p) = X \in \mathrm{Lie}\,H$
  - $R^*_h\, \varpi_{ph} = \mathrm{Ad}_{h^{-1}}\, \varpi_p$, ϖ pseudotensorial of type $(\mathrm{Ad}, \mathrm{Lie}\,G)$, $\varpi \in \Omega^1_{eq}(P, \mathrm{Lie}\,G)$
  - $\varpi_p : T_pP \longrightarrow \mathrm{Lie}\,G$ is a linear isomorphism $\forall\, p \in P$.

  Last requirement (Cartan condit°) means P is a parallelizable manifold.

- Let's define the curvature via the Cartan struct eq : $\boxed{\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi]}$.

  Then, because of equiv of ϖ we have : $\boxed{R^*_h\, \Omega = \mathrm{Ad}_{h^{-1}}\, \Omega}$, ie $\Omega \in \Omega^2_{eq}(P, \mathrm{Lie}\,G)$.

  By the way Ω is horizontal : $\Omega(X^v, Y) = 0$.

  Indeed, $d\varpi(X^v, Y) + [\varpi(X^v), \varpi(Y)] = X^v[\varpi(Y)] - \overset{\to 0}{Y[\varpi(X^v)]} - \varpi([X^v, Y]) + [X, \varpi(Y)]$.

  but remember : $L_{X^v}\varpi\,(Y) = (i_{X^v}d + d\,i_{X^v})\varpi\,(Y) = d\varpi(X^v, Y) + d(i_{X^v}\varpi)(Y)$

  $$= X^v[\varpi(Y)] - Y[\cancel{\varpi(X^v)}] - \varpi([X^v, Y]) + Y[\cancel{\varpi(X^v)}]$$

  So $\Omega(X^v, Y) = L_{X^v}\varpi\,(Y) + [X, \varpi(Y)] = \left(L_{X^v}\varpi + [X, \varpi]\right)(Y)$.

  but $L_{X^v}\varpi = \frac{d}{d\tau}\Big|_{\tau=0} R^*_{\exp \tau X}\,\varpi = \frac{d}{d\tau}\Big|_{\tau=0} \mathrm{Ad}_{e^{-\tau X}}\,\varpi = \frac{d}{d\tau}\Big|_{\tau=0} e^{-\tau X}\varpi\, e^{\tau X} = -X\varpi + \varpi X$

  ie $\boxed{L_{X^v}\varpi = -[X, \varpi]}$ which is the infinitesimal version of ϖ's equivariance !

  (and a relat° true also for a principal connect°) $\to$ $\Omega(X^v, Y) = 0$ ✓

  $\Rightarrow$ Ω is tensorial of type $(\mathrm{Ad}, \mathrm{Lie}\,G)$ ie $\Omega \in \Omega^2_{tens}(P, \mathrm{Lie}\,G)$.

NB : From the very def we have the Bianchi identity : $\boxed{d\Omega + [\varpi, \Omega] = 0}$.

Rmk : A Cartan geometry (P, ϖ) / connect° ϖ is said flat iff $\Omega = 0$.

- The gauge grp $\mathcal{H} \equiv \{ \gamma : P \to H \mid R_h^* \gamma = h^{-1}\gamma h \} \simeq \mathrm{Aut}_v(P)$ acts as usual:

$$\varpi^\gamma \equiv \psi^* \varpi = \mathrm{Ad}_{\gamma^{-1}} \varpi + \gamma^{-1} d\gamma \qquad \text{and} \qquad \Omega^\gamma \equiv \psi^* \Omega = \mathrm{Ad}_{\gamma^{-1}} \Omega \quad ✓ \qquad \text{Active GT}.$$

Locally, on $U \subset M$, given $\sigma : U \to \pi^{-1}(U) \subset P$, def $\bar{A} \equiv \sigma^* \varpi \in \Omega^1(M, \mathrm{Lie}G)$ and $\bar{F} \equiv \sigma^* \Omega \in \Omega^2(M, \mathrm{Lie}G)$. Given another σ' on U, or σ' on U' s.t $U' \cap U \neq \emptyset$, so that $\sigma'(x) = \sigma(x) h(x)$ for some $h : U \cap U' \to H$: $\bar{A}' \equiv \sigma'^* \varpi$ and $\bar{F}' \equiv \sigma'^* \Omega$ and

$$\bar{A}' = \mathrm{Ad}_{h^{-1}} \bar{A} + h^{-1} dh \quad , \quad \bar{F}' = \mathrm{Ad}_{h^{-1}} \bar{F} \quad ✓ \qquad \text{Passive GT}.$$

- Consider the projectᵒ $\tau : \mathrm{Lie}G \to \mathrm{Lie}G / \mathrm{Lie}H$, $\Theta = \tau(\Omega)$ is the torsion of ϖ.

If Ω is LieH-valued, ie if $\Theta = 0$, ϖ is said torsion free.

NB: since LieH is a subalgebra, torsion freeness is an Ad_H-inv property / a $\mathcal{H}$ gauge-inv property, ie it is true of a whole gauge orbit.
Or, also, remains true upon gluing over any $U' \cap U$: if $\bar{A}$ is torsion free, so is $\bar{A}'$.

1.1 Soldering form. (New!)

The Cartan connectᵒ allows a new constructᵒ establishing that $TM \simeq P \times_{\mathrm{Ad}_H} \mathrm{Lie}G / \mathrm{Lie}H$.
It is as follows. First we show that thx to ϖ we have:

$$C^\infty_{eq}(P, \mathrm{Lie}G) \simeq \Gamma^H(P) \qquad \text{ie iso btwn equiv LieG-valued fctᵒ and } R_H\text{-inv vect fields on } P.$$

(i) $\quad \varphi \longmapsto \varpi^{-1}(\varphi)$

(ii) $\quad \varpi(X^H) \longleftarrow X^H$

Indeed

(ii) $\quad \varphi(p) \equiv \varpi_p(X^H_p)$ for $X^H \in \Gamma^H(P)$.

Then $\varphi(ph) \equiv \varpi_{ph}(X^H_{ph}) \underset{R_H\text{-inv}}{=} \varpi_{ph}(R_{h*} X^H_p) = R_h^* \varpi_{ph}(X^H_p) \underset{\text{Ad-equiv of } \varpi}{=} \mathrm{Ad}_{h^{-1}} \varpi_p(X^H_p) = \mathrm{Ad}_{h^{-1}} \varphi(p)$

So indeed $\varphi = \varpi(X^H)$ is Ad-equiv LieG-valued fctᵒ ✓

(ii) $X_p \equiv \varpi_p^{-1}(\varphi(p))$ for $\varphi \in C^\infty_{eq}(P, \mathrm{Lie}\,G)$.

Then $X_{ph} \equiv \varpi_{ph}^{-1}(\varphi(ph)) = \varpi_{ph}^{-1}(Ad_{h^{-1}}\varphi(p))$.

Now ϖ satisfies: $R_h^* \circ \varpi_p = \varpi_{ph} \circ R_{h*} = Ad_{h^{-1}} \circ \varpi_p \to \varpi_{ph} = Ad_{h^{-1}} \circ \varpi_p \circ R_{h^{-1}*}$

So $X_{ph} = R_{h*} \circ \varpi_p^{-1} \circ Ad_h(Ad_{h^{-1}}\varphi(p)) = R_{h*}\varpi_p^{-1}(\varphi(p)) \equiv R_{h*}X_p$, ie $X \in \Gamma^H(P)$ ✓

Now $X^H \in \Gamma^H(P)$ projects on a well-def $X \in \Gamma(TM)$: $X = \pi_* X^H$.

By the way via the above constructo $C^\infty_{eq}(P, \mathrm{Lie}H) \simeq \Gamma(VP)$ (because 1st prop of ϖ!)

and $\Gamma(VP) = \ker \pi_*$! So we have $C^\infty_{eq}(P, \mathrm{Lie}G/_{\mathrm{Lie}H}) \simeq \Gamma(TM)$ ✓

In other words we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
C^\infty_{eq}(P, \mathrm{Lie}H) & \xrightarrow{i} & C^\infty_{eq}(P, \mathrm{Lie}G) & \xrightarrow{\tau} & C^\infty_{eq}(P, \mathrm{Lie}G/_{\mathrm{Lie}H}) \simeq C^\infty_{eq}(P, \mathrm{Lie}G)/_{C^\infty_{eq}(P, \mathrm{Lie}H)} \\
\wr\!\!\wr\ \varpi_{|VP} & & \wr\!\!\wr\ \varpi & \overset{\theta}{\nearrow} & \wr\!\!\wr\ e \uparrow \\
\Gamma(VP) & \longrightarrow & \Gamma^G(P) & \xrightarrow{\pi_*} & \Gamma(TM)
\end{array}$$

Finally we know from general considerato that $C^\infty_{eq}(P, \mathrm{Lie}G/_{\mathrm{Lie}H}) \simeq \Gamma(P \times_{Ad_H} \mathrm{Lie}G/_{\mathrm{Lie}H})$

So that indeed: $\Gamma(TM) \simeq \Gamma(P \times_{Ad_H} \mathrm{Lie}G/_{\mathrm{Lie}H})$ and $\boxed{TM \simeq P \times_{Ad_H} \mathrm{Lie}G/_{\mathrm{Lie}H}}$ ✓

↳ This is called a soldering: $\mathrm{Lie}G/_{\mathrm{Lie}H}$ is "soldered" on T_xM!

This is done via the soldering form | $\theta_p \equiv \tau \circ \varpi_p : T_pP \to \mathrm{Lie}G/_{\mathrm{Lie}H}$ ✓

NB: $\theta \in \Omega^1_{tens}(P, \mathrm{Lie}G/_{\mathrm{Lie}H})$. | $e_x : T_xM \to \mathrm{Lie}G/_{\mathrm{Lie}H}$ ✓

Related, in view of the diagram: $\theta_p(X_p) = e_x(\pi_* X_p)$ for $X_p \in T_pP$.

Given $\sigma : U \subset M \to P$: $(\sigma^*\theta_p)(X_x) = \theta_p(\sigma_* X_x)$ for $X_x \in T_xM$.

$$= e_x(\pi_*\sigma_* X_x)$$
$$= e_x((\pi\circ\sigma)_* X_x) = e_x(id_{M*} X_x) = e_x(X_x).$$

e is then local rep of soldering: it is a $\mathrm{Lie}G/_{\mathrm{Lie}H}$-valued 1-form on $U \subset M$; $e \in \Omega^1(U, \mathrm{Lie}G/_{\mathrm{Lie}H})$

So given $\{x^\mu\}$ on U: $e = e_\mu dx^\mu = e^a{}_\mu \tau_a dx^\mu$ for $\{\tau_a\}$ basis of $\mathrm{Lie}G/_{\mathrm{Lie}H}$.

Can also write: $e^a = e^a{}_\mu dx^\mu$.

⇒ This is the "moving frame" of Cartan, or "tetrad"/"Vierbein"/"Vielbein" of GR!

1.2 Reductive geometry

A Cartan geometry is reductive iff there is a Ad_H-inv splitting $\mathrm{Lie}\,G = \mathrm{Lie}\,H + \mathfrak{m}$, ie $Ad_H \mathfrak{m} \subset \mathfrak{m}$. In this case we have a clean splitting $\varpi = \omega + \theta$, and also $\bar{\Omega} = \Omega + \Theta$. The LieH-valued part ω is clearly a principal connect° on P.

So HP can be defined the usual way: $H_pP = \ker \omega_p \quad \forall p \in P$.

There is a cov deriv on $\Omega_{\text{tens}}(P,V)$, expressed via ω: $D = d + \rho_*(\omega)$

↳ In particular $D\theta = d\theta + ad_\omega \theta = d\theta + [\omega, \theta]$ this is Θ if $\mathfrak{m}$ abelian or if $[\mathfrak{m},\mathfrak{m}] \in \mathrm{Lie}H$.

↳ $D\bar{\Omega} = d\bar{\Omega} + [\omega, \bar{\Omega}]$, by Bianchi: $D\bar{\Omega} = -[\theta, \bar{\Omega}]$

If $\Theta = 0$ this is $D\Omega = -[\theta, \Omega] \Rightarrow \begin{cases} D\Omega = 0 & \in \mathrm{Lie}H \\ [\theta, \Omega] = 0 & \in \mathfrak{m} \end{cases}$ ⇝ diff Bianchi id on Ω ⇝ algebraic Bianchi id ✓

Rmk: In general Ω is not necessarily the curvature of ω!

↳ $\Omega \neq d\omega + \frac{1}{2}[\omega,\omega]$. It is if $\mathfrak{m}$ is abelian or if $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{m}$ ($\mathfrak{m}$ subalg!) → GR

It may be that a term $\frac{1}{2}[\theta,\theta]$ contribute to Ω. → de Sitter gravity.

→ If $\mathfrak{m}$ is endowed with an (Ad_H-inv) non-degenerate bilin form $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$

then $g \equiv \eta \circ e : T_xM \times T_xM \longrightarrow \mathbb{R}$ is a metric on TM!

$(X,Y) \longmapsto g(X,Y) \equiv \eta(e(X), e(Y))$

in components this is: $g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu e^b{}_\nu$ ✓

Rmk: At a point $p \in P$ the diagram is

$$\begin{tikzcd} \mathrm{Lie}H \arrow[r, hook] & \mathrm{Lie}G \arrow[r, "\tau"] & \mathrm{Lie}G/_{\mathrm{Lie}H} = \mathfrak{m} \arrow[l, bend right, "\Sigma"'] \\ V_pP \arrow[u, "\omega = \omega_{p|V_p}\ \wr"] \arrow[r, hook] & T_pP \arrow[u, "\wr\ \varpi_p = \omega + \theta"] \arrow[r] & T_xM \arrow[u, "\wr\ e \sim \theta"] \end{tikzcd}$$

A reductive case means a sect° Σ for τ

so that $\mathrm{Lie}G/_{\mathrm{Lie}H} = \mathfrak{m}$, $\Sigma : \mathfrak{m} \to \mathrm{Lie}G$ and $\tau \circ \Sigma = id_{\mathfrak{m}} \Rightarrow \mathrm{Lie}G = \mathrm{Lie}H + \mathfrak{m}$.

↳ splitting of the exact top row!

↳ Induce splitting of bottom row, that is a connect° ω ✓

1.3 | Parabolic geometry

- It is another wide and important class of Cartan geometry.
  $\mathrm{Lie}\,G = \mathfrak{g}$ is a graded Lie alg: $\mathfrak{g} = \bigoplus_{i=-k}^{k} \mathfrak{g}_i = \underbrace{\mathfrak{g}_{-k} + \dots + \mathfrak{g}_{-1}}_{\mathfrak{m}} + \underbrace{\mathfrak{g}_0 + \mathfrak{g}_1 + \dots \mathfrak{g}_k}_{\mathrm{Lie}H = \mathfrak{h}}$.

  with $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$.

  Rmk: $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $|i+j| > k$, in particular $\mathfrak{g}_{-k}$ and $\mathfrak{g}_k$ are abelian.

  $\rightarrow$ $\mathrm{Lie}H = \mathfrak{h}$ and $\mathfrak{m}$ are subalgebras.
  But the geom is not reductive since $[\mathfrak{g}_{-i}, \mathfrak{g}_j] \subset \mathfrak{h}$ if $j \geq i$.

- Yet the Cartan connect° splits: $\varpi = \theta + \omega = \bigoplus_{i=-k}^{-1} \theta_i + \bigoplus_{i=0}^{k} \omega_i = \theta_{-k} + \dots + \theta_{-1} + \omega_0 + \dots + \omega_k$.
  and ω is a principal connect° on $P(M,H)$.
  $\bar{\Omega}$ splits in the same way: $\bar{\Omega} = \Theta + \Omega = \bigoplus_{i=-k}^{-1} \Theta_i + \bigoplus_{i=0}^{k} \Omega_i$.
  $\hookrightarrow$ but in general cannot say that Ω_i is the curvature of ω_i!

Ex: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, projective and conformal Cartan geometries are of this type.

$\hookrightarrow$ $\varpi = \theta + \omega_0 + \omega_1$

$\bar{\Omega} = \Theta + \Omega_0 + \Omega_1$

$$\begin{cases} \Theta = d\theta + [\omega_0, \theta] & \rightsquigarrow \text{Torsion } \checkmark \\ \Omega_0 = d\omega_0 + \tfrac{1}{2}[\omega_0, \omega_0] + [\theta, \omega_1] & \rightsquigarrow \Omega_0 \text{ not curv of } \omega_0! \\ \Omega_1 = d\omega_1 + [\omega_0, \omega_1] \end{cases}$$

- Again if non-deg bilin form is given on $\mathfrak{m}$ a metric on M (TM) is induced via θ: $g(X,Y) = \eta(e(X), e(Y))$.

  $\hookrightarrow$ It is not necessary H-inv, so that there may be gauge transf of g!
  Notably this is the case in conformal geometry.

Rmk: Any $\mathfrak{g}_{\bar{k}} = \bigoplus_{i=k} \mathfrak{g}_i \subset \mathfrak{h}$ is a subalgebra, to which correspond $K \subset H$ subgrps.
So that $P(M,H)$ is "multiply fibered": $P \xrightarrow{H} M$, but also $P \xrightarrow{K} P'$ with P' a bundle also if $\mathrm{Lie}K = \mathfrak{g}_{\bar{k}}$.

1.4) Klein geometry

Consider a Lie grp G with subgrp H, G/H is an homogeneous space : Klein geometry. G is a H-bundle over G/H, and the Maurer-Cartan form is exactly a flat Cartan connect°!

- $\varpi_G(X^v) = X \in \mathrm{Lie}\,H$ (as a special case of $\varpi_G(X_g) = X_e \in \mathrm{Lie}\,G$ for X_g left-inv!)
- $R_h^* \varpi_G = Ad_{h^{-1}} \varpi_G$ (as a special case of $R_g^* \varpi_G = Ad_{g^{-1}} \varpi_G$)
- $\varpi_G : T_g G \to T_e G = \mathrm{Lie}\,G$ linear isomorphism ✓.

So a Klein geometry is the flat limit of a Cartan geometry of type $(\mathrm{Lie}\,G, H)$! Rmk then that flatness in the sense of Cartan is not flatness in the sense of Riemann ✓

1.5) (Pseudo) Riemannian Geometry

Considering $\mathrm{Lie}\,G = \mathfrak{so}(r,s) \ltimes \mathbb{R}^{r,s}$ and $H = SO(r,s)$: $\varpi = \omega + \theta$ and $\Omega = \Theta + R$ with $\Theta = d\theta + \omega\theta$ and $R = d\omega + \frac{1}{2}[\omega,\omega]$ the torsion and Riemann tensor.

Given η on $\mathbb{R}^{r,s}$, $SO(r,s)$-inv metric : a Riemannian metric on M is $g(X,Y) = \eta(e(X), e(Y))$

If $\Theta = 0$ then ω is solved for θ : $\omega = \omega(\theta)$ is the Levi-Civita connect°.

$\Rightarrow$ This is the "tetrad"/Cartan formulat° of pseudo-Riemannian geometry.

- $(r,s) = (1, n-1)$: Lorentz geom
- $(r,s) = (0, n)$: Riemann geom.

In the case $\Omega = 0$, $M = \mathbb{R}^{r,s}$!

NB : This means that Cartan geom generalize both Klein and Riemann geom!

$$\begin{tikzcd}
\frac{ISO(r,s)}{SO(r,s)} \simeq \mathbb{R}^{r,s} \ \text{(Euclid/Minkowski) flat globally} \arrow[r] \arrow[d] & M \ \text{(Riemann) locally } \mathbb{R}^{r,s} \simeq T_xM \text{ curved} \arrow[d] \\
\text{globally curved } \ G/H \ \text{(Klein)} \arrow[r] & (P, \varpi) \ \text{Cartan.}
\end{tikzcd}$$

2] Physics