```
> with(Physics):with(DifferentialGeometry):with(plots):with
  (PDEtools):with(Tensor):
> #The sign convention for the Ricci tensor of the DifferentialGeometry and Physics
      packages is the same followed in MTW.
>
> #Manifold definition:
> DGsetup([t, r, u, v], M, verbose)
```

The following coordinates have been protected:

$$[t, r, u, v]$$

The following vector fields have been defined and protected:

$$[\text{`*`}(D\_t), \text{`*`}(D\_r), \text{`*`}(D\_u), \text{`*`}(D\_v)]$$

The following differential 1-forms have been defined and protected:

$$[\text{`*`}(dt), \text{`*`}(dr), \text{`*`}(du), \text{`*`}(dv)]$$

frame name: M **(1)**

```
M >
M > g1 := evalDG( f(r)*dt &t dt +  g(r,u,v)*dr &t dr +  du &t du
    + (u^2)*dv &t dv)
```

$$g1 := (f(r)\,dt)\,dt + (g(r,u,v)\,dr)\,dr + (\text{`*`}(du))\,du + (u^2\,dv)\,dv \quad \textbf{(2)}$$

```
M > g1i:= InverseMetric(g1)
```

$$g1i := \left(\frac{1}{f(r)}\,D\_t\right)D\_t + \left(\frac{1}{g(r,u,v)}\,D\_r\right)D\_r + (\text{`*`}(D\_u))\,D\_u + \left(\frac{1}{u^2}\,D\_v\right)D\_v \quad \textbf{(3)}$$

```
M > #Christoffel symbols of the second kind:
M > C1 := Christoffel(g1, "SecondKind")
```

$$C1 := \left(\left(\frac{\frac{d}{dr}f(r)}{2\,f(r)}\,D\_t\right)dt\right)dr + \left(\left(\frac{\frac{d}{dr}f(r)}{2\,f(r)}\,D\_t\right)dr\right)dt \quad \textbf{(4)}$$

$$- \left(\left(\left(\frac{\frac{d}{dr}f(r)}{2\,g(r,u,v)}\,D\_r\right)dt\right)dt\right) + \left(\left(\frac{\frac{\partial}{\partial r}g(r,u,v)}{2\,g(r,u,v)}\,D\_r\right)dr\right)dr$$

$$+ \left(\left(\frac{\frac{\partial}{\partial u}g(r,u,v)}{2\,g(r,u,v)}\,D\_r\right)dr\right)du + \left(\left(\frac{\frac{\partial}{\partial v}g(r,u,v)}{2\,g(r,u,v)}\,D\_r\right)dr\right)dv$$

$$+ \left(\left(\frac{\frac{\partial}{\partial u}g(r,u,v)}{2\,g(r,u,v)}\,D\_r\right)du\right)dr + \left(\left(\frac{\frac{\partial}{\partial v}g(r,u,v)}{2\,g(r,u,v)}\,D\_r\right)dv\right)dr$$

$$- \left(\left(\left(\frac{\left(\frac{\partial}{\partial u}g(r,u,v)\right)}{2}\,D\_u\right)dr\right)dr\right) - (((u\,D\_u)\,dv)\,dv)$$

$$- \left(\left(\left(\frac{\frac{\partial}{\partial v}g(r,u,v)}{2\,u^2}\,D\_v\right)dr\right)dr\right) + \left(\left(\frac{1}{u}\,D\_v\right)du\right)dv + \left(\left(\frac{1}{u}\,D\_v\right)dv\right)du$$

M > #Ricci tensor:

M > R1:=RicciTensor(C1)

$$R1 := -\left(\left(\frac{1}{4\,g(r,u,v)^2\,f(r)}\left(2\left(\frac{d^2}{dr^2}\,f(r)\right)g(r,u,v)\,f(r)-\left(\frac{d}{dr}\,f(r)\right)^2 g(r,u,v)\right.\right.\right. \tag{5}$$

$$\left.\left.\left.-\left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)\right)dt\right)dt-\left(\left(\left(\left(\right.\right.\right.\right.$$

$$-\frac{-\frac{\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)u\,g(r,u,v)}{2}+\frac{\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)^2}{4}}{g(r,u,v)\,u^2}+\frac{\frac{\partial^2}{\partial v^2}\,g(r,u,v)}{2\,u^2}$$

$$-\frac{\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)^2}{4\,g(r,u,v)}+\frac{\left(\frac{\partial^2}{\partial u^2}\,g(r,u,v)\right)}{2}-\frac{1}{f(r)^2\,g(r,u,v)}\left(\right.$$

$$-\frac{\left(\frac{d^2}{dr^2}\,f(r)\right)g(r,u,v)\,f(r)}{2}+\frac{\left(\frac{d}{dr}\,f(r)\right)^2 g(r,u,v)}{4}$$

$$\left.\left.+\frac{\left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)}{4}\right)\right)dr\Bigg)dr\Bigg)$$

$$+\left(\frac{\left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}\,dr\right)du$$

$$+\left(\frac{\left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}\,dr\right)dv$$

$$+\left(\frac{\left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}\,du\right)dr$$

$$-\left(\left(\frac{2\left(\frac{\partial^2}{\partial u^2}\,g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)^2}{4\,g(r,u,v)^2}\,du\right)du\right)$$

$$-\left(\left(\frac{1}{4\,g(r,u,v)^2\,u}\left(2\left(\frac{\partial^2}{\partial u\partial v}\,g(r,u,v)\right)g(r,u,v)\,u-\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\partial}{\partial u}\right.\right.\right.$$

$$\left.\left.\left.g(r,u,v)\right)u-2\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)g(r,u,v)\right)du\right)dv\Bigg)$$

$$+\left(\frac{\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial v}g(r,u,v)\right)}{4f(r)g(r,u,v)}dv\right)dr-\left(\left(\frac{1}{4g(r,u,v)^2u}\left(2\left(\frac{\partial^2}{\partial u\partial v}g(r,u,v)\right)g(r,u,v)u-\left(\frac{\partial}{\partial v}g(r,u,v)\right)\left(\frac{\partial}{\partial u}g(r,u,v)\right)u-2\left(\frac{\partial}{\partial v}g(r,u,v)\right)g(r,u,v)\right)dv\right)du\right)$$

$$-\left(\left(\frac{1}{4g(r,u,v)^2}\left(2\left(\frac{\partial}{\partial u}g(r,u,v)\right)ug(r,u,v)+2\left(\frac{\partial^2}{\partial v^2}g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial v}g(r,u,v)\right)^2\right)dv\right)dv\right)$$

```
M >
M >
M > #4-potential vector:
M > A := DGzip([At, Ar, Au, Av])(t, r, u, v), [D_t, D_r, D_u,
    D_v], "plus")
```

$$A := At(t,r,u,v)\,D\_t+Ar(t,r,u,v)\,D\_r+Au(t,r,u,v)\,D\_u+Av(t,r,u,v)\,D\_v \qquad (6)$$

```
M > #Covariant derivative of the 4-potential:
M > Dc:=CovariantDerivative(A, C1):
M > #The second covariant derivative of the 4-potential:
M > Dc2:=CovariantDerivative(Dc, C1):
M > #Contraction of the indices of the covariant derivatives to get the Laplacian:
M > L:=ContractIndices(g1i, Dc2, [[1, 2], [2,3]]):
M >
M > ####"Mass term" of the Maxwell equations, i.e, contraction of the Ricci tensor
        with the 4-potential:
M > #Rising one index of the Ricci tensor:
M > R1up:=ContractIndices(g1i, R1, [[1, 1]]):
M > #Contraction of the Ricci tensor with the four-potential:
M > Mass:=ContractIndices(A, R1up, [[1,2]]):
```

```
M > #Maxwell equations!!!!!!!!!!!!!!!!:
M > Me:= L &minus Mass
```

$$Me := \frac{1}{2g(r,u,v)^2u^2f(r)^2}\left(2\left(\frac{\partial^2}{\partial u^2}At(t,r,u,v)\right)g(r,u,v)^2u^2f(r)^2+\left(\frac{\partial}{\partial u}g(r,u,v)\right)\left(\frac{\partial}{\partial u}At(t,r,u,v)\right)u^2g(r,u,v)f(r)^2+2\left(\frac{\partial^2}{\partial t^2}At(t,r,u,v)\right)g(r,u,v)^2f(r)u^2-\left(\frac{\partial}{\partial r}g(r,u,v)\right)f(r)^2\left(\frac{\partial}{\partial r}At(t,r,u,v)\right)u^2-\left(\frac{\partial}{\partial r}g(r,u,\right. \qquad (7)$$

$$
\begin{aligned}
&\nu)\Big)\, f(r)\, At(t, r, u, \nu)\left(\frac{\mathrm{d}}{\mathrm{d}r}\, f(r)\right) u^2 + 2\left(\frac{\partial}{\partial t}\, Ar(t, r, u, \nu)\right) g(r, u, \nu)^2 \left(\frac{\mathrm{d}}{\mathrm{d}r}\right. \\
&\left. f(r)\right) u^2 + 2\left(\frac{\partial}{\partial u}\, At(t, r, u, \nu)\right) g(r, u, \nu)^2 f(r)^2 u + 2\left(\frac{\partial^2}{\partial r^2}\, At(t, r, u,\right. \\
&\left.\nu)\right) g(r, u, \nu)\, f(r)^2 u^2 + 2\, g(r, u, \nu)\, f(r)\, At(t, r, u, \nu)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\, f(r)\right) u^2 + 3\, g(r, \\
&u, \nu)\, f(r)\left(\frac{\partial}{\partial r}\, At(t, r, u, \nu)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\, f(r)\right) u^2 - g(r, u, \nu)\, At(t, r, u, \nu)\left(\frac{\mathrm{d}}{\mathrm{d}r}\right. \\
&\left. f(r)\right)^2 u^2 + 2\left(\frac{\partial^2}{\partial \nu^2}\, At(t, r, u, \nu)\right) g(r, u, \nu)^2 f(r)^2 + \left(\frac{\partial}{\partial \nu}\, g(r, u, \nu)\right)\left(\frac{\partial}{\partial \nu}\right. \\
&\left. At(t, r, u, \nu)\right) g(r, u, \nu)\, f(r)^2\Big) D\_t + \frac{1}{2\, u^2\, g(r, u, \nu)^3\, f(r)^2}\Big(2\left(\frac{\partial}{\partial u}\, Ar(t, r, \right. \\
&\left. u, \nu)\right) g(r, u, \nu)^3\, f(r)^2\, u - Ar(t, r, u, \nu)\left(\frac{\partial}{\partial r}\, g(r, u, \nu)\right)^2 f(r)^2\, u^2 - Ar(t, r, u, \\
&\nu)\, g(r, u, \nu)^2 \left(\frac{\mathrm{d}}{\mathrm{d}r}\, f(r)\right)^2 u^2 + 3\left(\frac{\partial}{\partial \nu}\, Ar(t, r, u, \nu)\right)\left(\frac{\partial}{\partial \nu}\, g(r, u, \nu)\right) g(r, \\
&u, \nu)^2\, f(r)^2 - Ar(t, r, u, \nu)\left(\frac{\partial}{\partial \nu}\, g(r, u, \nu)\right)^2 g(r, u, \nu)\, f(r)^2 - \left(\frac{\partial}{\partial r}\, g(r, u, \right. \\
&\left.\nu)\right) Av(t, r, u, \nu)\left(\frac{\partial}{\partial \nu}\, g(r, u, \nu)\right) f(r)^2\, u^2 + \left(\frac{\partial}{\partial r}\, Ar(t, r, u, \nu)\right) g(r, u, \\
&\nu)^2\, f(r)\left(\frac{\mathrm{d}}{\mathrm{d}r}\, f(r)\right) u^2 + 2\left(\frac{\partial}{\partial u}\, g(r, u, \nu)\right)\left(\frac{\partial}{\partial r}\, Au(t, r, u, \nu)\right) g(r, u, \\
&\nu)\, f(r)^2\, u^2 + 2\left(\frac{\partial}{\partial \nu}\, g(r, u, \nu)\right)\left(\frac{\partial}{\partial r}\, Av(t, r, u, \nu)\right) g(r, u, \nu)\, f(r)^2\, u^2 \\
&- 2\, g(r, u, \nu)^2\, f(r)\left(\frac{\mathrm{d}}{\mathrm{d}r}\, f(r)\right)\left(\frac{\partial}{\partial t}\, At(t, r, u, \nu)\right) u^2 + 2\left(\frac{\partial^2}{\partial u^2}\, Ar(t, r, u, \right. \\
&\left.\nu)\right) g(r, u, \nu)^3\, f(r)^2\, u^2 + 2\left(\frac{\partial^2}{\partial t^2}\, Ar(t, r, u, \nu)\right) g(r, u, \nu)^3\, f(r)\, u^2 + 2\left(\frac{\partial^2}{\partial r^2}\right. \\
&\left. Ar(t, r, u, \nu)\right) g(r, u, \nu)^2\, f(r)^2\, u^2 + 2\, Ar(t, r, u, \nu)\, g(r, u, \nu)^2\, f(r)^2\left(\frac{\partial^2}{\partial \nu^2}\, g(r, \right. \\
&\left. u, \nu)\right) + \left(\frac{\partial^2}{\partial r \partial u}\, g(r, u, \nu)\right) Au(t, r, u, \nu)\, g(r, u, \nu)\, f(r)^2\, u^2 + \left(\frac{\partial^2}{\partial r \partial \nu}\, g(r, u, \right. \\
&\left.\nu)\right) Av(t, r, u, \nu)\, g(r, u, \nu)\, f(r)^2\, u^2 + Ar(t, r, u, \nu)\, g(r, u, \nu)^2\, f(r)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\right. \\
&\left. f(r)\right) u^2 + 2\left(\frac{\partial^2}{\partial u^2}\, g(r, u, \nu)\right) Ar(t, r, u, \nu)\, g(r, u, \nu)^2\, f(r)^2\, u^2 + \left(\frac{\partial^2}{\partial r^2}\, g(r, u, \right.
\end{aligned}
$$

$$
\begin{aligned}
&\nu)\Big)\,Ar(t,r,u,\nu)\,g(r,u,\nu)\,f(r)^2\,u^2+3\left(\frac{\partial}{\partial u}\,Ar(t,r,u,\nu)\right)\left(\frac{\partial}{\partial u}\,g(r,u,\right.\\
&\left.\nu)\right)g(r,u,\nu)^2\,f(r)^2\,u^2-Ar(t,r,u,\nu)\left(\frac{\partial}{\partial u}\,g(r,u,\nu)\right)^2 g(r,u,\nu)\,f(r)^2\,u^2\\
&+2\,Ar(t,r,u,\nu)\left(\frac{\partial}{\partial u}\,g(r,u,\nu)\right)g(r,u,\nu)^2\,f(r)^2\,u+\left(\frac{\partial}{\partial r}\,g(r,u,\nu)\right)\left(\frac{\partial}{\partial r}\right.\\
&\left.Ar(t,r,u,\nu)\right)g(r,u,\nu)\,f(r)^2\,u^2-\left(\frac{\partial}{\partial r}\,g(r,u,\nu)\right)Au(t,r,u,\nu)\left(\frac{\partial}{\partial u}\,g(r,u,\right.\\
&\left.\nu)\right)f(r)^2\,u^2+2\,g(r,u,\nu)^3\,f(r)^2\left(\frac{\partial^2}{\partial\nu^2}\,Ar(t,r,u,\nu)\right)\Big)\,D\_r\\
&-\left(\frac{1}{2\,u^2\,g(r,u,\nu)^2\,f(r)}\left(-\left(\frac{\partial^2}{\partial u^2}\,g(r,u,\nu)\right)Au(t,r,u,\nu)\,g(r,u,\nu)\,f(r)\,u^2\right.\right.\\
&-\left(\frac{\partial^2}{\partial u\partial\nu}\,g(r,u,\nu)\right)A\nu(t,r,u,\nu)\,g(r,u,\nu)\,f(r)\,u^2-2\left(\frac{\partial^2}{\partial u^2}\,Au(t,r,u,\right.\\
&\left.\nu)\right)u^2\,g(r,u,\nu)^2\,f(r)+Ar(t,r,u,\nu)\left(\frac{\partial^2}{\partial r\partial u}\,g(r,u,\nu)\right)g(r,u,\nu)\,f(r)\,u^2\\
&+Ar(t,r,u,\nu)\left(\frac{\partial}{\partial u}\,g(r,u,\nu)\right)g(r,u,\nu)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)u^2+2\left(\frac{\partial}{\partial r}\,Ar(t,r,u,\right.\\
&\left.\nu)\right)\left(\frac{\partial}{\partial u}\,g(r,u,\nu)\right)g(r,u,\nu)\,f(r)\,u^2+Au(t,r,u,\nu)\left(\frac{\partial}{\partial u}\,g(r,u,\nu)\right)^2 f(r)\,u^2\\
&+\left(\frac{\partial}{\partial u}\,g(r,u,\nu)\right)A\nu(t,r,u,\nu)\left(\frac{\partial}{\partial\nu}\,g(r,u,\nu)\right)f(r)\,u^2-\left(\frac{\partial}{\partial u}\,g(r,u,\right.\\
&\left.\nu)\right)g(r,u,\nu)\,f(r)\left(\frac{\partial}{\partial u}\,Au(t,r,u,\nu)\right)u^2+4\left(\frac{\partial}{\partial\nu}\,A\nu(t,r,u,\nu)\right)g(r,u,\\
&\nu)^2\,f(r)\,u-2\left(\frac{\partial^2}{\partial t^2}\,Au(t,r,u,\nu)\right)g(r,u,\nu)^2\,u^2+\left(\frac{\partial}{\partial r}\,g(r,u,\nu)\right)\left(\frac{\partial}{\partial r}\,Au(t,\right.\\
&\left.r,u,\nu)\right)f(r)\,u^2-\left(\frac{\partial}{\partial r}\,Au(t,r,u,\nu)\right)g(r,u,\nu)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)u^2+2\,A\nu(t,r,u,\\
&\nu)\left(\frac{\partial}{\partial\nu}\,g(r,u,\nu)\right)g(r,u,\nu)\,f(r)\,u-2\,g(r,u,\nu)^2\,f(r)\left(\frac{\partial}{\partial u}\,Au(t,r,u,\nu)\right)u\\
&-2\,g(r,u,\nu)\,f(r)\left(\frac{\partial^2}{\partial r^2}\,Au(t,r,u,\nu)\right)u^2+2\,Au(t,r,u,\nu)\,g(r,u,\nu)^2\,f(r)\\
&-\left(\frac{\partial}{\partial\nu}\,g(r,u,\nu)\right)\left(\frac{\partial}{\partial\nu}\,Au(t,r,u,\nu)\right)g(r,u,\nu)\,f(r)-2\,g(r,u,\nu)^2\,f(r)\left(\frac{\partial^2}{\partial\nu^2}\right.\\
&\left.Au(t,r,u,\nu)\right)\Big)\,D\_u\Big)+\frac{1}{2\,u^3\,g(r,u,\nu)^2\,f(r)}\left(2\left(\frac{\partial^2}{\partial u^2}\,A\nu(t,r,u,\right.\right.
\end{aligned}
$$

$$\left(\frac{\partial}{\partial u} Av(t, r, u, v) \right) g(r, u, v)^2 f(r) u^3 + \left(\frac{\partial}{\partial u} Av(t, r, u, v) \right) \left(\frac{\partial}{\partial u} g(r, u, v) \right) g(r, u, v) f(r) u^3 + 6 \left(\frac{\partial}{\partial u} Av(t, r, u, v) \right) g(r, u, v)^2 f(r) u^2 + 2 \left(\frac{\partial^2}{\partial t^2} Av(t, r, u, v) \right) g(r, u, v)^2 u^3 - \left(\frac{\partial}{\partial r} g(r, u, v) \right) \left(\frac{\partial}{\partial r} Av(t, r, u, v) \right) f(r) u^3 + 2 \left(\frac{\partial}{\partial u} g(r, u, v) \right) Av(t, r, u, v) g(r, u, v) f(r) u^2 + \left(\frac{\partial}{\partial r} Av(t, r, u, v) \right) g(r, u, v) \left(\frac{\mathrm{d}}{\mathrm{d}r} f(r) \right) u^3 + 2 g(r, u, v) f(r) \left(\frac{\partial^2}{\partial r^2} Av(t, r, u, v) \right) u^3 + \left(\frac{\partial}{\partial v} Av(t, r, u, v) \right) \left(\frac{\partial}{\partial v} g(r, u, v) \right) g(r, u, v) f(r) u + \left(\frac{\partial^2}{\partial u \partial v} g(r, u, v) \right) Au(t, r, u, v) g(r, u, v) f(r) u - Ar(t, r, u, v) \left(\frac{\partial^2}{\partial r \partial v} g(r, u, v) \right) g(r, u, v) f(r) u - Ar(t, r, u, v) \left(\frac{\partial}{\partial v} g(r, u, v) \right) g(r, u, v) \left(\frac{\mathrm{d}}{\mathrm{d}r} f(r) \right) u - 2 \left(\frac{\partial}{\partial r} Ar(t, r, u, v) \right) \left(\frac{\partial}{\partial v} g(r, u, v) \right) g(r, u, v) f(r) u - Au(t, r, u, v) \left(\frac{\partial}{\partial u} g(r, u, v) \right) \left(\frac{\partial}{\partial v} g(r, u, v) \right) f(r) u - Av(t, r, u, v) \left(\frac{\partial}{\partial v} g(r, u, v) \right)^2 f(r) u + Av(t, r, u, v) g(r, u, v) f(r) \left(\frac{\partial^2}{\partial v^2} g(r, u, v) \right) u + 2 g(r, u, v)^2 f(r) \left(\frac{\partial^2}{\partial v^2} Av(t, r, u, v) \right) u + 4 \left(\frac{\partial}{\partial v} Au(t, r, u, v) \right) g(r, u, v)^2 f(r) \Bigg) D\_v$$

M >

M >