```
> with(Physics):with(DifferentialGeometry):with(plots):with
  (PDEtools):with(Tensor):
> #The sign convention for the Ricci tensor of the DifferentialGeometry and Physics
      packages is the same followed in MTW.
>
> #Manifold definition:
> DGsetup([t, r, u, v], M, verbose)
```

The following coordinates have been protected:

$$[t, r, u, v]$$

The following vector fields have been defined and protected:

$$[\ `*`(D\_t),\ `*`(D\_r),\ `*`(D\_u),\ `*`(D\_v)\ ]$$

The following differential 1-forms have been defined and protected:

$$[\ `*`(dt),\ `*`(dr),\ `*`(du),\ `*`(dv)\ ]$$

frame name: M **(1)**

```
M >
M > g1 := evalDG( f(r)*dt &t dt +  g(r,u,v)*dr &t dr +  du &t du
    + dv &t dv)
```

$$g1 := (f(r)\ dt)\ dt + (g(r, u, v)\ dr)\ dr + (\ `*`(du))\ du + (\ `*`(dv))\ dv \quad \textbf{(2)}$$

```
M > g1i:= InverseMetric(g1)
```

$$g1i := \left(\frac{1}{f(r)}\ D\_t\right) D\_t + \left(\frac{1}{g(r, u, v)}\ D\_r\right) D\_r + (\ `*`(D\_u))\ D\_u + (\ `*`(D\_v))\ D\_v \quad \textbf{(3)}$$

```
M > #Christoffel symbols of the second kind:
M > C1 := Christoffel(g1, "SecondKind")
```

$$C1 := \left(\left(\frac{\frac{\mathrm{d}}{\mathrm{d}r} f(r)}{2 f(r)}\ D\_t\right) dt\right) dr + \left(\left(\frac{\frac{\mathrm{d}}{\mathrm{d}r} f(r)}{2 f(r)}\ D\_t\right) dr\right) dt \quad \textbf{(4)}$$

$$- \left(\left(\left(\frac{\frac{\mathrm{d}}{\mathrm{d}r} f(r)}{2 g(r, u, v)}\ D\_r\right) dt\right) dt\right) + \left(\left(\left(\frac{\frac{\partial}{\partial r} g(r, u, v)}{2 g(r, u, v)}\ D\_r\right) dr\right) dr\right)$$

$$+ \left(\left(\left(\frac{\frac{\partial}{\partial u} g(r, u, v)}{2 g(r, u, v)}\ D\_r\right) dr\right) du\right) + \left(\left(\left(\frac{\frac{\partial}{\partial v} g(r, u, v)}{2 g(r, u, v)}\ D\_r\right) dr\right) dv\right)$$

$$+ \left(\left(\left(\frac{\frac{\partial}{\partial u} g(r, u, v)}{2 g(r, u, v)}\ D\_r\right) du\right) dr\right) + \left(\left(\left(\frac{\frac{\partial}{\partial v} g(r, u, v)}{2 g(r, u, v)}\ D\_r\right) dv\right) dr\right)$$

$$- \left(\left(\left(\frac{\left(\frac{\partial}{\partial u} g(r, u, v)\right)}{2}\ D\_u\right) dr\right) dr\right) - \left(\left(\left(\frac{\left(\frac{\partial}{\partial v} g(r, u, v)\right)}{2}\ D\_v\right) dr\right) dr\right)$$

```
M > #Ricci tensor:
M > R1:=RicciTensor(C1)
```

$$R1 := -\left(\left(\frac{1}{4\,g(r,u,v)^2\,f(r)}\left(2\left(\frac{d^2}{dr^2}f(r)\right)g(r,u,v)\,f(r)-\left(\frac{d}{dr}f(r)\right)^2 g(r,u,v)\right.\right.\right. \tag{5}$$

$$\left.\left.\left.-\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial r}g(r,u,v)\right)f(r)\right)dt\right)dt\right)-\left(\left(\left(\left(-\frac{\left(\frac{\partial}{\partial v}g(r,u,v)\right)^2}{4\,g(r,u,v)}\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\frac{\left(\frac{\partial^2}{\partial v^2}g(r,u,v)\right)}{2}-\frac{\left(\frac{\partial}{\partial u}g(r,u,v)\right)^2}{4\,g(r,u,v)}+\frac{\left(\frac{\partial^2}{\partial u^2}g(r,u,v)\right)}{2}\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{f(r)^2\,g(r,u,v)}\left(-\frac{\left(\frac{d^2}{dr^2}f(r)\right)g(r,u,v)\,f(r)}{2}+\frac{\left(\frac{d}{dr}f(r)\right)^2 g(r,u,v)}{4}\right.\right.\right.$$

$$\left.\left.\left.\left.+\frac{\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial r}g(r,u,v)\right)f(r)}{4}\right)\right)dr\right)dr\right)$$

$$+\left(\frac{\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial u}g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}dr\right)du$$

$$+\left(\frac{\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial v}g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}dr\right)dv$$

$$+\left(\frac{\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial u}g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}du\right)dr$$

$$-\left(\left(\frac{2\left(\frac{\partial^2}{\partial u^2}g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial u}g(r,u,v)\right)^2}{4\,g(r,u,v)^2}du\right)du\right)$$

$$-\left(\left(\frac{2\left(\frac{\partial^2}{\partial u\partial v}g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial v}g(r,u,v)\right)\left(\frac{\partial}{\partial u}g(r,u,v)\right)}{4\,g(r,u,v)^2}du\right)\right.$$

$$\left.dv\right)+\left(\frac{\left(\frac{d}{dr}f(r)\right)\left(\frac{\partial}{\partial v}g(r,u,v)\right)}{4\,f(r)\,g(r,u,v)}dv\right)dr$$

$$-\left(\left(\frac{2\left(\frac{\partial^2}{\partial u\partial v}g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial v}g(r,u,v)\right)\left(\frac{\partial}{\partial u}g(r,u,v)\right)}{4\,g(r,u,v)^2}dv\right)\right.$$

$$du\Bigg) - \left(\left(\frac{2\left(\frac{\partial^2}{\partial v^2}\, g(r,u,v)\right) g(r,u,v) - \left(\frac{\partial}{\partial v}\, g(r,u,v)\right)^2}{4\, g(r,u,v)^2}\, dv\right) dv\right)$$

```
M >
M >
M > #4-potential vector:
M > A := DGzip(([At, Ar, Au, Av])(t, r, u, v), [D_t, D_r, D_u,
    D_v], "plus")
```

$$A := At(t,r,u,v)\, D\_t + Ar(t,r,u,v)\, D\_r + Au(t,r,u,v)\, D\_u + Av(t,r,u,v)\, D\_v \qquad (6)$$

```
M > #Covariant derivative of the 4-potential:
M > Dc:=CovariantDerivative(A, C1):
M > #The second covariant derivative of the 4-potential:
M > Dc2:=CovariantDerivative(Dc, C1):
M > #Contraction of the indices of the covariant derivatives to get the Laplacian:
M > L:=ContractIndices(g1i, Dc2, [[1, 2], [2,3]]):
M >
M > ####"Mass term" of the Maxwell equations, i.e, contraction of the Ricci tensor
      with the 4-potential:
M > #Rising one index of the Ricci tensor:
M > R1up:=ContractIndices(g1i, R1, [[1, 1]]):
M > #Contraction of the Ricci tensor with the four-potential:
M > Mass:=ContractIndices(A, R1up, [[1,2]]):
```

```
M > #Maxwell equations!!!!!!!!!!!!!!!!:
M > Me:= L &minus Mass
```

$$Me := \frac{1}{2\, g(r,u,v)^2\, f(r)^2}\left(\left(\frac{\partial}{\partial v}\, g(r,u,v)\right)\left(\frac{\partial}{\partial v}\, At(t,r,u,v)\right) f(r)^2\, g(r,u,v) \qquad (7)$$

$$+\left(\frac{\partial}{\partial u}\, g(r,u,v)\right)\left(\frac{\partial}{\partial u}\, At(t,r,u,v)\right) f(r)^2\, g(r,u,v) + 2\left(\frac{\partial^2}{\partial v^2}\, At(t,r,u,v)\right) g(r,u,v)^2\, f(r)^2 + 2\left(\frac{\partial^2}{\partial u^2}\, At(t,r,u,v)\right) g(r,u,v)^2\, f(r)^2 + 2\left(\frac{d^2}{dr^2} f(r)\right) g(r,u,v)\, f(r)\, At(t,r,u,v) - g(r,u,v)\left(\frac{d}{dr} f(r)\right)^2 At(t,r,u,v)$$

$$- f(r)\left(\frac{d}{dr} f(r)\right) At(t,r,u,v)\left(\frac{\partial}{\partial r}\, g(r,u,v)\right) + 2\left(\frac{\partial^2}{\partial r^2}\, At(t,r,u,v)\right) g(r,u,v)\, f(r)^2 + 3\, g(r,u,v)\, f(r)\left(\frac{d}{dr} f(r)\right)\left(\frac{\partial}{\partial r}\, At(t,r,u,v)\right) + 2\left(\frac{d}{dr} f(r)\right)\left(\frac{\partial}{\partial t}\, Ar(t,r,u,v)\right) g(r,u,v)^2 - f(r)^2\left(\frac{\partial}{\partial r}\, At(t,r,u,v)\right)\left(\frac{\partial}{\partial r}\, g(r,u,v)\right)$$

$$+2\left(\frac{\partial^2}{\partial t^2}\,At(t,r,u,v)\right)g(r,u,v)^2 f(r)\Big)D\_t+\frac{1}{2\,g(r,u,v)^3 f(r)^2}\Big(2\Big(\frac{\partial^2}{\partial v^2}$$

$$Ar(t,r,u,v)\Big)g(r,u,v)^3 f(r)^2+3\left(\frac{\partial}{\partial u}\,Ar(t,r,u,v)\right)\left(\frac{\partial}{\partial u}\,g(r,u,\right.$$

$$\left.v)\right)g(r,u,v)^2 f(r)^2-Ar(t,r,u,v)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)^2 g(r,u,v)\,f(r)^2-Ar(t,r,$$

$$u,v)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)^2 g(r,u,v)\,f(r)^2+2\,Ar(t,r,u,v)\,g(r,u,v)^2 f(r)^2\left(\frac{\partial^2}{\partial u^2}\right.$$

$$\left.g(r,u,v)\right)+2\,Ar(t,r,u,v)\,g(r,u,v)^2 f(r)^2\left(\frac{\partial^2}{\partial v^2}\,g(r,u,v)\right)+3\left(\frac{\partial}{\partial v}\,Ar(t,r,\right.$$

$$\left.u,v)\right)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)g(r,u,v)^2 f(r)^2+2\,g(r,u,v)^3 f(r)^2\left(\frac{\partial^2}{\partial u^2}\,Ar(t,r,u,\right.$$

$$\left.v)\right)+2\left(\frac{\partial^2}{\partial r^2}\,Ar(t,r,u,v)\right)g(r,u,v)^2 f(r)^2+\left(\frac{\partial^2}{\partial r\partial u}\,g(r,u,v)\right)Au(t,r,u,$$

$$v)\,g(r,u,v)\,f(r)^2-Au(t,r,u,v)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)^2$$

$$+\left(\frac{\partial^2}{\partial r^2}\,g(r,u,v)\right)Ar(t,r,u,v)\,g(r,u,v)\,f(r)^2+\left(\frac{\partial^2}{\partial r\partial v}\,g(r,u,v)\right)Av(t,r,$$

$$u,v)\,g(r,u,v)\,f(r)^2-Av(t,r,u,v)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)^2$$

$$+\left(\frac{\partial}{\partial r}\,Ar(t,r,u,v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)g(r,u,v)^2 f(r)+\left(\frac{\partial}{\partial r}\,Ar(t,r,u,v)\right)\left(\frac{\partial}{\partial r}\right.$$

$$\left.g(r,u,v)\right)g(r,u,v)\,f(r)^2+2\left(\frac{\partial}{\partial r}\,Au(t,r,u,v)\right)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)g(r,u,$$

$$v)\,f(r)^2+2\left(\frac{\partial}{\partial r}\,Av(t,r,u,v)\right)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)g(r,u,v)\,f(r)^2+Ar(t,r,u,$$

$$v)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\,f(r)\right)g(r,u,v)^2 f(r)-Ar(t,r,u,v)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)^2 g(r,u,v)^2-Ar(t,$$

$$r,u,v)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)^2 f(r)^2-2\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)\left(\frac{\partial}{\partial t}\,At(t,r,u,v)\right)g(r,u,$$

$$v)^2 f(r)+2\left(\frac{\partial^2}{\partial t^2}\,Ar(t,r,u,v)\right)g(r,u,v)^3 f(r)\Big)D\_r$$

$$+\frac{1}{2\,g(r,u,v)^2 f(r)}\Big(2\left(\frac{\partial^2}{\partial v^2}\,Au(t,r,u,v)\right)g(r,u,v)^2 f(r)+2\left(\frac{\partial^2}{\partial u^2}\,Au(t,r,\right.$$

$$\left.u,v)\right)g(r,u,v)^2 f(r)-\left(\frac{\partial^2}{\partial r\partial u}\,g(r,u,v)\right)Ar(t,r,u,v)\,g(r,u,v)\,f(r)-Au(t,$$

$$r, u, v)\left(\frac{\partial}{\partial u}\,g(r, u, v)\right)^2 f(r) + Au(t, r, u, v)\,g(r, u, v)\,f(r)\left(\frac{\partial^2}{\partial u^2}\,g(r, u, v)\right)$$
$$- Av(t, r, u, v)\left(\frac{\partial}{\partial v}\,g(r, u, v)\right)\left(\frac{\partial}{\partial u}\,g(r, u, v)\right) f(r) + Av(t, r, u, v)\,g(r, u,$$
$$v)\,f(r)\left(\frac{\partial^2}{\partial u \partial v}\,g(r, u, v)\right) - 2\left(\frac{\partial}{\partial r}\,Ar(t, r, u, v)\right)\left(\frac{\partial}{\partial u}\,g(r, u, v)\right) g(r, u,$$
$$v)\,f(r) - Ar(t, r, u, v)\left(\frac{\partial}{\partial u}\,g(r, u, v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right) g(r, u, v) + \left(\frac{\partial}{\partial v}\,g(r, u,$$
$$v)\right) g(r, u, v)\,f(r)\left(\frac{\partial}{\partial v}\,Au(t, r, u, v)\right) + \left(\frac{\partial}{\partial u}\,g(r, u, v)\right) g(r, u, v)\,f(r)\left(\frac{\partial}{\partial u}\right.$$
$$\left.Au(t, r, u, v)\right) + 2\left(\frac{\partial^2}{\partial t^2}\,Au(t, r, u, v)\right) g(r, u, v)^2 + \left(\frac{\partial}{\partial r}\,Au(t, r, u, v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\right.$$
$$\left.f(r)\right) g(r, u, v) - \left(\frac{\partial}{\partial r}\,Au(t, r, u, v)\right)\left(\frac{\partial}{\partial r}\,g(r, u, v)\right) f(r) + 2\,g(r, u,$$
$$v)\,f(r)\left(\frac{\partial^2}{\partial r^2}\,Au(t, r, u, v)\right)\Bigg) D\_u + \frac{1}{2\,g(r, u, v)^2\,f(r)}\Bigg(2\left(\frac{\partial^2}{\partial v^2}\,Av(t, r, u,$$
$$v)\right) g(r, u, v)^2\,f(r) + 2\left(\frac{\partial^2}{\partial u^2}\,Av(t, r, u, v)\right) g(r, u, v)^2\,f(r) - Au(t, r, u,$$
$$v)\left(\frac{\partial}{\partial v}\,g(r, u, v)\right)\left(\frac{\partial}{\partial u}\,g(r, u, v)\right) f(r) + Au(t, r, u, v)\,g(r, u, v)\,f(r)\left(\frac{\partial^2}{\partial u \partial v}\right.$$
$$\left.g(r, u, v)\right) - \left(\frac{\partial^2}{\partial r \partial v}\,g(r, u, v)\right) Ar(t, r, u, v)\,g(r, u, v)\,f(r) - Av(t, r, u,$$
$$v)\left(\frac{\partial}{\partial v}\,g(r, u, v)\right)^2 f(r) + Av(t, r, u, v)\,g(r, u, v)\,f(r)\left(\frac{\partial^2}{\partial v^2}\,g(r, u, v)\right)$$
$$- 2\left(\frac{\partial}{\partial r}\,Ar(t, r, u, v)\right)\left(\frac{\partial}{\partial v}\,g(r, u, v)\right) g(r, u, v)\,f(r) - Ar(t, r, u, v)\left(\frac{\partial}{\partial v}\right.$$
$$\left.g(r, u, v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right) g(r, u, v) + \left(\frac{\partial}{\partial v}\,g(r, u, v)\right) g(r, u, v)\,f(r)\left(\frac{\partial}{\partial v}\,Av(t, r,\right.$$
$$\left.u, v)\right) + \left(\frac{\partial}{\partial u}\,g(r, u, v)\right) g(r, u, v)\,f(r)\left(\frac{\partial}{\partial u}\,Av(t, r, u, v)\right) + 2\left(\frac{\partial^2}{\partial t^2}\,Av(t, r,\right.$$
$$\left.u, v)\right) g(r, u, v)^2 + \left(\frac{\partial}{\partial r}\,Av(t, r, u, v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right) g(r, u, v) - \left(\frac{\partial}{\partial r}\,Av(t, r, u,\right.$$
$$\left.v)\right)\left(\frac{\partial}{\partial r}\,g(r, u, v)\right) f(r) + 2\,g(r, u, v)\,f(r)\left(\frac{\partial^2}{\partial r^2}\,Av(t, r, u, v)\right)\Bigg) D\_v$$

M >

M >