```
> with(Physics):with(DifferentialGeometry):with(plots):with(PDEtools):with(Tensor):
```

> *#The sign convention for the Ricci tensor of the DifferentialGeometry and Physics packages is the same followed in MTW.*

\>

> *#Manifold definition:*

```
> DGsetup([t, r, u, v], M, verbose)
```

The following coordinates have been protected:

$$[t, r, u, v]$$

The following vector fields have been defined and protected:

$$[\,`*`(D\_t),\ `*`(D\_r),\ `*`(D\_u),\ `*`(D\_v)\,]$$

The following differential 1-forms have been defined and protected:

$$[\,`*`(dt),\ `*`(dr),\ `*`(du),\ `*`(dv)\,]$$

frame name: M **(1)**

M >

> *#Cartesian coordinates in the transversal plane:*

```
M > g1 := evalDG( dt &t dt +  (1/f(r))*g(r,u,v)*dr &t dr +  (1/f(r))*du &t du + (1/f(r))*dv &t dv)
```

$$g1 := (\,`*`(dt)\,)\,dt + \left(\frac{g(r,u,v)}{f(r)}\,dr\right)dr + \left(\frac{1}{f(r)}\,du\right)du + \left(\frac{1}{f(r)}\,dv\right)dv \qquad \textbf{(2)}$$

```
M > g1i:= InverseMetric(g1)
```

$$g1i := (\,`*`(D\_t)\,)\,D\_t + \left(\frac{f(r)}{g(r,u,v)}\,D\_r\right)D\_r + (f(r)\,D\_u)\,D\_u + (f(r)\,D\_v)\,D\_v \qquad \textbf{(3)}$$

M > *#Christoffel symbols of the second kind:*

```
M > C1 := Christoffel(g1, "SecondKind")
```

$$\begin{aligned}
C1 := &-\left(\left(\left(\frac{g(r,u,v)\left(\frac{d}{dr}f(r)\right)-\left(\frac{\partial}{\partial r}g(r,u,v)\right)f(r)}{2\,f(r)\,g(r,u,v)}\,D_r\right)dr\right)dr\right) \qquad \textbf{(4)}\\
&+\left(\left(\frac{\frac{\partial}{\partial u}g(r,u,v)}{2\,g(r,u,v)}\,D_r\right)dr\right)du+\left(\left(\frac{\frac{\partial}{\partial v}g(r,u,v)}{2\,g(r,u,v)}\,D_r\right)dr\right)dv\\
&+\left(\left(\frac{\frac{\partial}{\partial u}g(r,u,v)}{2\,g(r,u,v)}\,D_r\right)du\right)dr+\left(\left(\frac{\frac{d}{dr}f(r)}{2\,f(r)\,g(r,u,v)}\,D_r\right)du\right)du\\
&+\left(\left(\frac{\frac{\partial}{\partial v}g(r,u,v)}{2\,g(r,u,v)}\,D_r\right)dv\right)dr+\left(\left(\frac{\frac{d}{dr}f(r)}{2\,f(r)\,g(r,u,v)}\,D_r\right)dv\right)dv\\
&-\left(\left(\left(\frac{\left(\frac{\partial}{\partial u}g(r,u,v)\right)}{2}\,D_u\right)dr\right)dr\right)-\left(\left(\left(\frac{\frac{d}{dr}f(r)}{2\,f(r)}\,D_u\right)dr\right)du\right)
\end{aligned}$$

$$-\left(\left(\left(\frac{\frac{d}{dr}\,f(r)}{2\,f(r)}\,D\_u\right)du\right)dr\right)-\left(\left(\left(\frac{\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)}{2}\,D\_v\right)dr\right)dr\right)$$

$$-\left(\left(\left(\frac{\frac{d}{dr}\,f(r)}{2\,f(r)}\,D\_v\right)dr\right)dv\right)-\left(\left(\left(\frac{\frac{d}{dr}\,f(r)}{2\,f(r)}\,D\_v\right)dv\right)dr\right)$$

M > *#Ricci tensor:*

M > R1:=RicciTensor(C1)

$$R1 := -\Bigg(\Bigg(\frac{1}{4\,f(r)^2\,g(r,u,v)}\Bigg(2\left(\frac{\partial^2}{\partial u^2}\,g(r,u,v)\right)f(r)^2\,g(r,u,v)+2\left(\frac{\partial^2}{\partial v^2}\,g(r,u,\right. \tag{5}$$

$$\left.v)\right)f(r)^2\,g(r,u,v)-\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)^2 f(r)^2-\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)^2 f(r)^2$$

$$-4\left(\frac{d^2}{dr^2}\,f(r)\right)f(r)\,g(r,u,v)+4\left(\frac{d}{dr}\,f(r)\right)^2 g(r,u,v)+2\left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial r}\right.$$

$$\left.g(r,u,v)\right)f(r)\Bigg)dr\Bigg)dr\Bigg)-\left(\left(\frac{\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)\left(\frac{d}{dr}\,f(r)\right)}{4\,g(r,u,v)\,f(r)}\,dr\right)du\right)$$

$$-\left(\left(\frac{\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{d}{dr}\,f(r)\right)}{4\,g(r,u,v)\,f(r)}\,dr\right)dv\right)$$

$$-\left(\left(\frac{\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)\left(\frac{d}{dr}\,f(r)\right)}{4\,g(r,u,v)\,f(r)}\,du\right)dr\right)$$

$$-\Bigg(\Bigg(\frac{1}{4\,f(r)^2\,g(r,u,v)^2}\Bigg(2\left(\frac{\partial^2}{\partial u^2}\,g(r,u,v)\right)f(r)^2\,g(r,u,v)-\left(\frac{\partial}{\partial u}\,g(r,u,\right.$$

$$\left.v)\right)^2 f(r)^2-2\left(\frac{d^2}{dr^2}\,f(r)\right)f(r)\,g(r,u,v)+3\left(\frac{d}{dr}\,f(r)\right)^2 g(r,u,v)+\left(\frac{d}{dr}\right.$$

$$\left.f(r)\right)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)\Bigg)du\Bigg)du\Bigg)$$

$$-\left(\left(\frac{2\left(\frac{\partial^2}{\partial u\partial v}\,g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)}{4\,g(r,u,v)^2}\,du\right)\right.$$

$$\left.dv\right)-\left(\left(\frac{\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{d}{dr}\,f(r)\right)}{4\,g(r,u,v)\,f(r)}\,dv\right)dr\right)$$

$$-\left(\left(\frac{2\left(\frac{\partial^2}{\partial u\partial v}\,g(r,u,v)\right)g(r,u,v)-\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)}{4\,g(r,u,v)^2}\,dv\right)\right.$$

$$du\Big) + \left(\frac{1}{4\,f(r)^2\,g(r,u,v)^2}\left(-2\left(\frac{\partial^2}{\partial v^2}\,g(r,u,v)\right)f(r)^2\,g(r,u,v) + \left(\frac{\partial}{\partial v}\,g(r,u,v)\right)^2 f(r)^2 + 2\left(\frac{d^2}{dr^2}\,f(r)\right)f(r)\,g(r,u,v) - 3\left(\frac{d}{dr}\,f(r)\right)^2 g(r,u,v) - \left(\frac{d}{dr}\,f(r)\right)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)\right)dv\right)dv$$

```
M >
M >
M > #4-potential vector:
M > A := DGzip(([At, Ar, Au, Av])(t, r, u, v), [D_t, D_r, D_u,
    D_v], "plus")
```

$$A := At(t,r,u,v)\,D\_t + Ar(t,r,u,v)\,D\_r + Au(t,r,u,v)\,D\_u + Av(t,r,u,v)\,D\_v \qquad \textbf{(6)}$$

```
M > #Covariant derivative of the 4-potential:
M > Dc:=CovariantDerivative(A, C1):
M > #The second covariant derivative of the 4-potential:
M > Dc2:=CovariantDerivative(Dc, C1):
M > #Contraction of the indices of the covariant derivatives to get the Laplacian:
M > L:=ContractIndices(g1i, Dc2, [[1, 2], [2,3]]):
M >
M > ####"Mass term" of the Maxwell equations, i.e, contraction of the Ricci tensor
       with the 4-potential:
M > #Rising one index of the Ricci tensor:
M > R1up:=ContractIndices(g1i, R1, [[1, 1]]):
M > #Contraction of the Ricci tensor with the four-potential:
M > Mass:=ContractIndices(A, R1up, [[1,2]]):
```

```
M > #Maxwell equations!!!!!!!!!!!!!!!!:
M > Me:= L &minus Mass
```

$$Me := \frac{1}{2\,g(r,u,v)^2}\left(\left(\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)\left(\frac{\partial}{\partial u}\,At(t,r,u,v)\right)f(r)\,g(r,u,v) + \left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\partial}{\partial v}\,At(t,r,u,v)\right)f(r)\,g(r,u,v) + 2\left(\frac{\partial^2}{\partial u^2}\,At(t,r,u,v)\right)f(r)\,g(r,u,v)^2 + 2\left(\frac{\partial^2}{\partial v^2}\,At(t,r,u,v)\right)f(r)\,g(r,u,v)^2 + 2\left(\frac{\partial^2}{\partial t^2}\,At(t,r,u,v)\right)g(r,u,v)^2 - \left(\frac{\partial}{\partial r}\,At(t,r,u,v)\right)f(r)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right) - \left(\frac{\partial}{\partial r}\,At(t,r,u,v)\right)g(r,u,v)\left(\frac{d}{dr}\,f(r)\right) + 2\left(\frac{\partial^2}{\partial r^2}\,At(t,r,u,v)\right)f(r)\,g(r,u,v)\right)D\_t \qquad \textbf{(7)}$$

$$+\frac{1}{2\,f(r)\,g(r,u,v)^{3}}\Bigg(2\left(\frac{\partial^{2}}{\partial r^{2}}\,Ar(t,r,u,v)\right)g(r,u,v)^{2}\,f(r)^{2}+2\left(\frac{\partial^{2}}{\partial u^{2}}\,Ar(t,r,u,v)\right)f(r)^{2}\,g(r,u,v)^{3}+2\left(\frac{\partial^{2}}{\partial t^{2}}\,Ar(t,r,u,v)\right)f(r)\,g(r,u,v)^{3}$$

$$+2\left(\frac{\partial^{2}}{\partial v^{2}}\,Ar(t,r,u,v)\right)f(r)^{2}\,g(r,u,v)^{3}+2\left(\frac{\partial}{\partial v}\,Av(t,r,u,v)\right)f(r)\,g(r,u,v)^{2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)$$

$$+2\left(\frac{\partial}{\partial u}\,Au(t,r,u,v)\right)f(r)\,g(r,u,v)^{2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)+3\left(\frac{\partial}{\partial u}\,Ar(t,r,u,v)\right)f(r)^{2}\,g(r,u,v)^{2}\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)$$

$$+3\left(\frac{\partial}{\partial v}\,Ar(t,r,u,v)\right)f(r)^{2}\,g(r,u,v)^{2}\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)-Av(t,r,u,v)\,f(r)\,g(r,u,v)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)$$

$$-Au(t,r,u,v)\,f(r)\,g(r,u,v)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)+2\,Ar(t,r,u,v)\left(\frac{\partial^{2}}{\partial u^{2}}\,g(r,u,v)\right)f(r)^{2}\,g(r,u,v)^{2}+2\,Ar(t,r,u,v)\left(\frac{\partial^{2}}{\partial v^{2}}\,g(r,u,v)\right)f(r)^{2}\,g(r,u,v)^{2}$$

$$+\left(\frac{\partial^{2}}{\partial r\partial v}\,g(r,u,v)\right)Av(t,r,u,v)\,g(r,u,v)\,f(r)^{2}+\left(\frac{\partial^{2}}{\partial r\partial u}\,g(r,u,v)\right)Au(t,r,u,v)\,g(r,u,v)\,f(r)^{2}-3\,Ar(t,r,u,v)\left(\frac{\mathrm{d}^{2}}{\mathrm{d}r^{2}}\,f(r)\right)f(r)\,g(r,u,v)^{2}$$

$$+Ar(t,r,u,v)\left(\frac{\partial^{2}}{\partial r^{2}}\,g(r,u,v)\right)f(r)^{2}\,g(r,u,v)-Ar(t,r,u,v)\,f(r)^{2}\,g(r,u,v)\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)^{2}-Ar(t,r,u,v)\,f(r)^{2}\,g(r,u,v)\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)^{2}$$

$$-\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)Av(t,r,u,v)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)^{2}+2\left(\frac{\partial}{\partial v}\,g(r,u,v)\right)\left(\frac{\partial}{\partial r}\,Av(t,r,u,v)\right)g(r,u,v)\,f(r)^{2}$$

$$-\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)Au(t,r,u,v)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)f(r)^{2}+2\left(\frac{\partial}{\partial u}\,g(r,u,v)\right)\left(\frac{\partial}{\partial r}\,Au(t,r,u,v)\right)g(r,u,v)\,f(r)^{2}$$

$$+\left(\frac{\partial}{\partial r}\,Ar(t,r,u,v)\right)f(r)^{2}\,g(r,u,v)\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)-3\left(\frac{\partial}{\partial r}\,Ar(t,r,u,v)\right)f(r)\,g(r,u,v)^{2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)-Ar(t,r,u,v)\,f(r)^{2}\left(\frac{\partial}{\partial r}\,g(r,u,v)\right)^{2}$$

$$+3\,Ar(t,r,u,v)\,g(r,u,v)^{2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\,f(r)\right)^{2}\Bigg)D\_r+\frac{1}{2\,f(r)\,g(r,u,v)^{2}}\Bigg(\Bigg(\Bigg(\frac{\partial}{\partial v}$$

$$g(r, u, v)\Big) \left(\frac{\partial}{\partial v} Au(t, r, u, v)\right) f(r)^2 g(r, u, v) + 2 \Big(\frac{\partial^2}{\partial u^2} Au(t, r, u,$$

$$v)\Big) f(r)^2 g(r, u, v)^2 + 2 \left(\frac{\partial^2}{\partial v^2} Au(t, r, u, v)\right) f(r)^2 g(r, u, v)^2 + \Big(\frac{\partial}{\partial u} Au(t, r,$$

$$u, v)\Big) f(r)^2 g(r, u, v) \left(\frac{\partial}{\partial u} g(r, u, v)\right) - 2 \left(\frac{\partial}{\partial u} Ar(t, r, u, v)\right) f(r)\, g(r, u,$$

$$v)^2 \left(\frac{d}{dr} f(r)\right) - \left(\frac{\partial^2}{\partial r \partial u} g(r, u, v)\right) Ar(t, r, u, v)\, g(r, u, v)\, f(r)^2 - 2 \Big(\frac{\partial}{\partial u} g(r,$$

$$u, v)\Big) \left(\frac{\partial}{\partial r} Ar(t, r, u, v)\right) g(r, u, v)\, f(r)^2 + Av(t, r, u, v) \Big(\frac{\partial^2}{\partial u \partial v} g(r, u,$$

$$v)\Big) f(r)^2 g(r, u, v) + Au(t, r, u, v) \left(\frac{\partial^2}{\partial u^2} g(r, u, v)\right) f(r)^2 g(r, u, v) - Av(t, r,$$

$$u, v)\, f(r)^2 \left(\frac{\partial}{\partial v} g(r, u, v)\right) \left(\frac{\partial}{\partial u} g(r, u, v)\right) - Au(t, r, u, v)\, f(r)^2 \Big(\frac{\partial}{\partial u} g(r,$$

$$u, v)\Big)^2 + Ar(t, r, u, v)\, f(r)\, g(r, u, v) \left(\frac{\partial}{\partial u} g(r, u, v)\right) \left(\frac{d}{dr} f(r)\right) + 2 \Big(\frac{\partial^2}{\partial t^2}$$

$$Au(t, r, u, v)\Big) g(r, u, v)^2 f(r) + 2 \left(\frac{\partial^2}{\partial r^2} Au(t, r, u, v)\right) f(r)^2 g(r, u, v) - \Big(\frac{\partial}{\partial r}$$

$$Au(t, r, u, v)\Big) f(r)^2 \left(\frac{\partial}{\partial r} g(r, u, v)\right) - 3 \left(\frac{d}{dr} f(r)\right) \left(\frac{\partial}{\partial r} Au(t, r, u, v)\right) g(r, u,$$

$$v)\, f(r) - 2 \left(\frac{d^2}{dr^2} f(r)\right) Au(t, r, u, v)\, g(r, u, v)\, f(r) + \left(\frac{d}{dr} f(r)\right) Au(t, r, u,$$

$$v) \left(\frac{\partial}{\partial r} g(r, u, v)\right) f(r) + 3 \left(\frac{d}{dr} f(r)\right)^2 Au(t, r, u, v)\, g(r, u, v)\Big)\, D\_u$$

$$+ \frac{1}{2 f(r)\, g(r, u, v)^2} \Big(2 \left(\frac{\partial^2}{\partial v^2} Av(t, r, u, v)\right) f(r)^2 g(r, u, v)^2 + \Big(\frac{\partial}{\partial u} g(r, u,$$

$$v)\Big) \left(\frac{\partial}{\partial u} Av(t, r, u, v)\right) f(r)^2 g(r, u, v) + 2 \left(\frac{\partial^2}{\partial u^2} Av(t, r, u, v)\right) f(r)^2 g(r,$$

$$u, v)^2 + \left(\frac{\partial}{\partial v} Av(t, r, u, v)\right) f(r)^2 g(r, u, v) \left(\frac{\partial}{\partial v} g(r, u, v)\right) - 2 \Big(\frac{\partial}{\partial v} Ar(t, r,$$

$$u, v)\Big) f(r)\, g(r, u, v)^2 \left(\frac{d}{dr} f(r)\right) - \left(\frac{\partial^2}{\partial r \partial v} g(r, u, v)\right) Ar(t, r, u, v)\, g(r, u,$$

$$v)\, f(r)^2 - 2 \left(\frac{\partial}{\partial v} g(r, u, v)\right) \left(\frac{\partial}{\partial r} Ar(t, r, u, v)\right) g(r, u, v)\, f(r)^2 + Av(t, r, u,$$

$$v) \left(\frac{\partial^2}{\partial v^2} g(r, u, v)\right) f(r)^2 g(r, u, v) + Au(t, r, u, v) \Big(\frac{\partial^2}{\partial u \partial v} g(r, u,$$

$$v)\Big) f(r)^2 g(r,u,v) - Av(t,r,u,v)\, f(r)^2 \left(\frac{\partial}{\partial v} g(r,u,v)\right)^2 - Au(t,r,u,$$

$$v)\, f(r)^2 \left(\frac{\partial}{\partial v} g(r,u,v)\right) \left(\frac{\partial}{\partial u} g(r,u,v)\right) + Ar(t,r,u,v)\, f(r)\, g(r,u,v) \left(\frac{\partial}{\partial v}\right.$$

$$\left. g(r,u,v)\right) \left(\frac{d}{dr} f(r)\right) + 2 \left(\frac{\partial^2}{\partial r^2} Av(t,r,u,v)\right) f(r)^2 g(r,u,v) + 2 \left(\frac{\partial^2}{\partial t^2} Av(t,\right.$$

$$\left. r,u,v)\right) g(r,u,v)^2 f(r) - \left(\frac{\partial}{\partial r} Av(t,r,u,v)\right) f(r)^2 \left(\frac{\partial}{\partial r} g(r,u,v)\right) - 3 \left(\frac{d}{dr}\right.$$

$$\left. f(r)\right) \left(\frac{\partial}{\partial r} Av(t,r,u,v)\right) g(r,u,v)\, f(r) - 2 \left(\frac{d^2}{dr^2} f(r)\right) Av(t,r,u,v)\, g(r,u,$$

$$v)\, f(r) + \left(\frac{d}{dr} f(r)\right) Av(t,r,u,v) \left(\frac{\partial}{\partial r} g(r,u,v)\right) f(r) + 3 \left(\frac{d}{dr} f(r)\right)^2 Av(t,$$

$$r,u,v)\, g(r,u,v)\Big)\, D\_v$$

M >

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E1 >