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> with(Physics):with(DifferentialGeometry):with(plots):with
(PDEtools):with(Tensor):
> #The sign convention for the Ricci tensor of the DifferentialGeometry and Physics
    packages is the same followed in MTW.
>
> #Manifold definition:
> DGsetup([t, r, u, v], M, verbose)
    The following coordinates have been protected:
        [t, r, u, v]
    The following vector fields have been defined and protected:
        [ '*'(D_t), '*'(D_r), '*'(D_u), '*'(D_v) ]
    The following differential 1-forms have been defined and protected:
        [ '*'(dt), '*'(dr), '*'(du), '*'(dv) ]
        frame name: M

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(1)

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M >
M > g1 := evalDG( f(r)*dt &t dt + ((1 - u*chi(r)*cos(int(-tau
    (r), r) + v))^2)*dr &t dr + du &t du + (u^2)*dv &t dv)
g1:= (f(r) dt) dt + ( (u chi(r) cos( - (int(tau(r) dr) + v) - 1)^2 dr) dr + ( '*'(du) ) du
    + (u^2 dv) dv

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M > g1i:= InverseMetric(g1)
g1i:= ( 1/f(r) D_t ) D_t
    + ( 1 / ( chi(r)^2 cos( - (int(tau(r) dr) + v) )^2 u^2 - 2 u chi(r) cos( - (int(tau(r) dr) + v) + 1
    D_r ) D_r + ( '*'(D_u) ) D_u + ( 1/u^2 D_v ) D_v

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M > #Christoffel symbols of the second kind:
M > C1 := Christoffel(g1, "SecondKind")
C1:= ( ( ( d/dr f(r) / (2 f(r)) D_t ) dt ) dr + ( ( d/dr f(r) / (2 f(r)) D_t ) dr ) dt
    - ( ( ( ( d/dr f(r) ) / ( 2 ( chi(r)^2 cos( - (int(tau(r) dr) + v) )^2 u^2 - 2 u chi(r) cos(
    - (int(tau(r) dr) + v) + 1 ) ) D_r ) dt ) dt )
    + ( ( u ( tau(r) sin( - (int(tau(r) dr) + v) ) chi(r) + ( d/dr chi(r) ) cos( - (int(tau(r) dr) + v) ) )
    / ( u chi(r) cos( - (int(tau(r) dr) + v) ) - 1

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$$\begin{aligned}
& D_- r \Big) dr \Big) dr + \left(\left(\frac{\chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right)}{u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1} D_- r \right) dr \right) du \\
& - \left(\left(\left(\frac{u \chi(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right)}{u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1} D_- r \right) dr \right) dv \right) \\
& + \left(\left(\frac{\chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right)}{u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1} D_- r \right) du \right) dr \\
& - \left(\left(\left(\frac{u \chi(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right)}{u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1} D_- r \right) dv \right) dr \right) - \left(\left(\left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1 \right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) D_- u \right) dr \right) dr \\
& - \left(\left(u D_- u \right) dv \right) dv \\
& + \left(\left(\frac{\left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1 \right) \chi(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right)}{u} D_- v \right) \right. \\
& \left. dr \right) dr + \left(\left(\frac{1}{u} D_- v \right) du \right) dv + \left(\left(\frac{1}{u} D_- v \right) dv \right) du
\end{aligned}$$

M > #Ricci tensor:

M > **R1:=RicciTensor(C1)**

$$\begin{aligned}
R1 := & \left(\left(2 \tau(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right) \left(\frac{d}{dr} f(r) \right) f(r) \chi(r) u + 2 \left(\frac{d}{dr} \chi(r) \right) \left(\frac{d}{dr} f(r) \right) f(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) u + \left(\frac{d}{dr} f(r) \right)^2 \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) u \right. \right. \\
& - 2 f(r) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) \left(\frac{d^2}{dr^2} f(r) \right) u - \left(\frac{d}{dr} f(r) \right)^2 + 2 \left(\frac{d^2}{dr^2} f(r) \right) f(r) \Big) \Bigg/ \left(4 f(r) \left(\chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^3 u^3 - 3 \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 u^2 + 3 u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1 \right) \right) dt \Big) dt \\
& + \left(\left(2 \tau(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right) \left(\frac{d}{dr} f(r) \right) f(r) \chi(r) u + 2 \left(\frac{d}{dr} \chi(r) \right) \left(\frac{d}{dr} f(r) \right) f(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) u + \left(\frac{d}{dr} f(r) \right)^2 \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) u \right. \right.
\end{aligned} \tag{5}$$

$$\begin{aligned}
& -2 f(r) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) \left(\frac{d^2}{dr^2} f(r)\right) u - \left(\frac{d}{dr} f(r)\right)^2 + 2 \left(\frac{d^2}{dr^2} f(r)\right) f(r) \Bigg/ \left(4 f(r)^2 \left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1\right)\right) dr \\
& + \left(\frac{\left(\frac{d}{dr} f(r)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right)}{2 f(r) \left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1\right)} dr\right) du \\
& - \left(\left(\frac{\left(\frac{d}{dr} f(r)\right) u \chi(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right)}{2 f(r) \left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1\right)} dr\right) dv\right) \\
& + \left(\frac{\left(\frac{d}{dr} f(r)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right)}{2 f(r) \left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1\right)} du\right) dr \\
& - \left(\left(\frac{\left(\frac{d}{dr} f(r)\right) u \chi(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right)}{2 f(r) \left(u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 1\right)} dv\right) dr\right)
\end{aligned}$$

M >

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M > #4-potential vector:

M > **A := DGzip**(([At, Ar, Au, Av])(t, r, u, v), [D_t, D_r, D_u, D_v], "plus")

A:= At(t, r, u, v) D_t + Ar(t, r, u, v) D_r + Au(t, r, u, v) D_u + Av(t, r, u, v) D_v

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M > #Covariant derivative of the 4-potential:

M > **Dc:=CovariantDerivative(A, C1):**

M > #The second covariant derivative of the 4-potential:

M > **Dc2:=CovariantDerivative(Dc, C1):**

M > #Contraction of the indices of the covariant derivatives to get the Laplacian:

M > **L:=ContractIndices(g1i, Dc2, [[1, 2], [2,3]]):**

M >

M > #####"Mass term" of the Maxwell equations, i.e, contraction of the Ricci tensor with the 4-potential:

M > #Rising one index of the Ricci tensor:

M > **R1up:=ContractIndices(g1i, R1, [[1, 1]]):**

M > #Contraction of the Ricci tensor with the four-potential:

M > **Mass:=ContractIndices(A, R1up, [[1,2]]):**

M > #Maxwell equations!!!!!!!!!!!!!!:

M > **Me:= L − Mass**

$$\begin{aligned}
Me := & \left(-2 \left(\frac{\partial^2}{\partial t^2} At(t, r, u, v) \right) f(r) u^2 - 2 \left(\frac{\partial^2}{\partial u^2} At(t, r, u, v) \right) u^2 f(r)^2 - 2 \left(\frac{\partial^2}{\partial r^2} \right. \right. \\
& At(t, r, u, v) \left. \right) f(r)^2 u^2 + 2 \left(\frac{\partial^2}{\partial u^2} At(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) \right. \\
& \left. + v \right)^3 f(r)^2 u^5 + 2 \left(\frac{\partial^2}{\partial t^2} At(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 f(r) u^5 \\
& + 2 \left(\frac{\partial^2}{\partial v^2} At(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 f(r)^2 u^3 - 6 \left(\frac{\partial^2}{\partial u^2} At(t, \right. \\
& r, u, v) \left. \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r)^2 u^4 - 6 \left(\frac{\partial^2}{\partial t^2} At(t, r, u, \right. \\
& v) \left. \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r) u^4 - 6 \left(\frac{\partial^2}{\partial v^2} At(t, r, u, \right. \\
& v) \left. \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r)^2 u^2 + 6 \left(\frac{\partial^2}{\partial u^2} At(t, r, u, v) \right) \chi(r) \cos \left(- \left(\right. \right. \\
& \left. \int \tau(r) dr \right) + v \left. \right) f(r)^2 u^3 + 2 \left(\frac{\partial^2}{\partial r^2} At(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) \right. \\
& \left. + v \right) f(r)^2 u^3 + 6 \left(\frac{\partial^2}{\partial t^2} At(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u^3 \\
& + 6 \left(\frac{\partial^2}{\partial v^2} At(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r)^2 u - 2 \left(\frac{\partial^2}{\partial v^2} At(t, r, \right. \\
& u, v) \left. \right) f(r)^2 - 2 At(t, r, u, v) \left(\frac{d}{dr} f(r) \right) \chi(r) f(r) \tau(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) u^3 \\
& + 2 At(t, r, u, v) \left(\frac{d^2}{dr^2} f(r) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u^3 + At(t, r, u, \right. \\
& v) \left(\frac{d}{dr} f(r) \right)^2 u^2 - 2 \left(\frac{\partial}{\partial u} At(t, r, u, v) \right) f(r)^2 u - 2 \left(\frac{\partial}{\partial t} Ar(t, r, u, v) \right) \left(\frac{d}{dr} \right. \\
& f(r) \left. \right) u^2 - 3 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial r} At(t, r, u, v) \right) f(r) u^2 - 2 At(t, r, u, v) \left(\frac{d^2}{dr^2} \right. \\
& f(r) \left. \right) f(r) u^2 + 4 \left(\frac{\partial}{\partial u} At(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 f(r)^2 u^4 \\
& + 2 \left(\frac{\partial}{\partial t} Ar(t, r, u, v) \right) \left(\frac{d}{dr} f(r) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 u^5 - 10 \left(\frac{\partial}{\partial u} \right. \\
& At(t, r, u, v) \left. \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r)^2 u^3 - 6 \left(\frac{\partial}{\partial t} Ar(t, r, u, \right. \\
& v) \left. \right) \left(\frac{d}{dr} f(r) \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 u^4 - At(t, r, u, v) \left(\frac{d}{dr} \right.
\end{aligned}
\tag{7}$$

$$\begin{aligned}
& f(r)^2 \chi(r) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) u^3 - 2 \left(\frac{\partial}{\partial r} At(t, r, u, v)\right) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) f(r)^2 \left(\frac{d}{dr} \chi(r)\right) u^3 + 8 \left(\frac{\partial}{\partial u} At(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) f(r)^2 u^2 + 6 \left(\frac{\partial}{\partial t} Ar(t, r, u, v)\right) \left(\frac{d}{dr} f(r)\right) \chi(r) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) u^3 - 2 \left(\frac{\partial}{\partial v} At(t, r, u, v)\right) \chi(r) f(r)^2 \sin\left(-\left(\int \tau(r) \, dr\right) + v\right) u - 2 \left(\frac{\partial}{\partial v} At(t, r, u, v)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^2 f(r)^2 \sin\left(-\left(\int \tau(r) \, dr\right) + v\right) u^3 + 4 \left(\frac{\partial}{\partial v} At(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) f(r)^2 \sin\left(-\left(\int \tau(r) \, dr\right) + v\right) u^2 - 2 \left(\frac{\partial}{\partial r} At(t, r, u, v)\right) \chi(r) f(r)^2 \tau(r) \sin\left(-\left(\int \tau(r) \, dr\right) + v\right) u^3 - 2 At(t, r, u, v) \left(\frac{d}{dr} f(r)\right) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) f(r) \left(\frac{d}{dr} \chi(r)\right) u^3 + 3 \left(\frac{d}{dr} f(r)\right) \left(\frac{\partial}{\partial r} At(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) f(r) u^3 \Bigg/ \left(2 u^2 f(r)^2 \left(\chi(r)^3 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^3 u^3 - 3 \chi(r)^2 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^2 u^2 + 3 u \chi(r) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) - 1\right)\right) D_- t + \left(2 \left(\frac{\partial^2}{\partial u^2} Ar(t, r, u, v)\right) f(r)^2 u^2 + 2 \left(\frac{\partial^2}{\partial t^2} Ar(t, r, u, v)\right) f(r) u^2 + 2 f(r)^2 u^2 \left(\frac{\partial^2}{\partial r^2} Ar(t, r, u, v)\right) + 8 f(r)^2 u^5 \left(\frac{\partial}{\partial u} Ar(t, r, u, v)\right) \chi(r)^4 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^4 + 2 f(r)^2 u^4 Av(t, r, u, v) \chi(r)^2 \tau(r) + 2 f(r)^2 u^3 Av(t, r, u, v) \sin\left(-\left(\int \tau(r) \, dr\right) + v\right) \left(\frac{d}{dr} \chi(r)\right) - 26 f(r)^2 u^4 \left(\frac{\partial}{\partial u} Ar(t, r, u, v)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^3 - 2 f(r)^2 u^4 Ar(t, r, u, v) \chi(r)^2 \tau(r)^2 - 2 f(r)^2 u^4 Ar(t, r, u, v) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^2 \left(\frac{d}{dr} \chi(r)\right)^2 + 4 f(r)^2 u^3 \left(\frac{\partial}{\partial r} Au(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^2 + 30 f(r)^2 u^3 \left(\frac{\partial}{\partial u} Ar(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) \, dr\right) + v\right)^2 - 2 f(r)^2 u^3 \left(\frac{\partial}{\partial r} Ar(t, r, u, v)\right) \cos\left(-\left(\int \tau(r) \, dr\right) + v\right) \left(\frac{d}{dr} \chi(r)\right) \right)
\end{aligned}$$

$$\begin{aligned}
& -2 f(r)^2 u^2 A u(t, r, u, v) \cos\left(-\left(\int \tau(r) dr\right) + v\right) \left(\frac{d}{dr} \chi(r)\right) + 2 f(r)^2 u^4 \left(\frac{\partial^2}{\partial r^2} A r(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 - 2 f(r)^2 u^3 \left(\frac{d^2}{dr^2} \chi(r)\right) A r(t, r, u, v) \cos\left(-\left(\int \tau(r) dr\right) + v\right) - 4 f(r)^2 u^3 \left(\frac{\partial^2}{\partial r^2} A r(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) + 2 \left(\frac{\partial^2}{\partial v^2} A r(t, r, u, v)\right) \chi(r)^4 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^4 f(r)^2 u^4 \\
& - 8 \left(\frac{\partial^2}{\partial v^2} A r(t, r, u, v)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^3 f(r)^2 u^3 + 12 \left(\frac{\partial^2}{\partial v^2} A r(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 f(r)^2 u^2 - 8 \left(\frac{\partial^2}{\partial v^2} A r(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) f(r)^2 u + 2 \left(\frac{\partial^2}{\partial u^2} A r(t, r, u, v)\right) \chi(r)^4 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^4 f(r)^2 u^6 - 8 \left(\frac{\partial^2}{\partial u^2} A r(t, r, u, v)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^3 f(r)^2 u^5 + 12 \left(\frac{\partial^2}{\partial u^2} A r(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 f(r)^2 u^4 - 8 \left(\frac{\partial^2}{\partial u^2} A r(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) f(r)^2 u^3 + 2 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r)^4 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^4 f(r) u^6 - 8 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^3 f(r) u^5 + 12 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 f(r) u^4 - 8 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) f(r) u^3 + 2 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r)^4 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^4 f(r) u^6 - 8 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^3 f(r) u^5 + 12 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 f(r) u^4 - 8 \left(\frac{\partial^2}{\partial t^2} A r(t, r, u, v)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) f(r) u^3 + 2 \left(\frac{\partial^2}{\partial v^2} A r(t, r, u, v)\right) f(r)^2 + 2 f(r)^2 u^4 \left(\frac{d}{dr} \tau(r)\right) A r(t, r, u, v) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right) \sin\left(-\left(\int \tau(r) dr\right) + v\right) + 2 f(r)^2 u^4 \left(\frac{\partial}{\partial r} A r(t, r, u, v)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right) \tau(r) \sin\left(-\left(\int \tau(r) dr\right) + v\right) + \left(\frac{d^2}{dr^2} f(r)\right) A r(t, r, u, v) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + v\right)^2 f(r) u^4 - 2 \left(\frac{d^2}{dr^2} f(r)\right) A r(t, r, u, v) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + v\right) f(r) u^3
\end{aligned}$$

$$\begin{aligned}
& + 2 f(r)^2 u^4 \left(\frac{d^2}{dr^2} \chi(r) \right) Ar(t, r, u, v) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 \\
& + 2 f(r)^2 u^4 \left(\frac{\partial}{\partial r} Ar(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 \left(\frac{d}{dr} \chi(r) \right) \\
& + 2 f(r)^2 u^3 Ar(t, r, u, v) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) \tau(r)^2 - 6 f(r)^2 u^4 \left(\frac{\partial}{\partial v} \right. \\
& Ar(t, r, u, v) \left. \right) \chi(r)^4 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 \sin \left(- \left(\int \tau(r) dr \right) + v \right) \\
& - 4 f(r)^2 u^4 \left(\frac{\partial}{\partial r} Av(t, r, u, v) \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right) \sin \left(- \left(\int \tau(r) dr \right) \right. \\
& + v \left. \right) - 2 f(r)^2 u^3 \left(\frac{d}{dr} \tau(r) \right) Ar(t, r, u, v) \chi(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) \\
& - 4 f(r)^2 u^3 Ar(t, r, u, v) \tau(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) \left(\frac{d}{dr} \chi(r) \right) \\
& - 2 f(r)^2 u^3 Av(t, r, u, v) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) \tau(r) - 2 f(r)^2 u^3 \left(\frac{\partial}{\partial r} \right. \\
& Ar(t, r, u, v) \left. \right) \chi(r) \tau(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) - 2 f(r)^2 u^2 Au(t, r, u, \\
& v) \chi(r) \tau(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) - 18 f(r)^2 u^2 \left(\frac{\partial}{\partial v} Ar(t, r, u, v) \right) \chi(r)^2 \cos \left(\right. \\
& - \left(\int \tau(r) dr \right) + v \left. \right) \sin \left(- \left(\int \tau(r) dr \right) + v \right) + 18 f(r)^2 u^3 \left(\frac{\partial}{\partial v} Ar(t, r, u, \right. \\
& v) \left. \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 \sin \left(- \left(\int \tau(r) dr \right) + v \right) - 2 \left(\frac{\partial}{\partial t} At(t, r, u, \right. \\
& v) \left. \right) \left(\frac{d}{dr} f(r) \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r) u^4 + \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial r} Ar(t, \right. \\
& r, u, v) \left. \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r) u^4 + 4 \left(\frac{\partial}{\partial t} At(t, r, u, v) \right) \left(\frac{d}{dr} \right. \\
& f(r) \left. \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u^3 - 2 \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial r} Ar(t, r, u, \right. \\
& v) \left. \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u^3 - 2 \left(\frac{\partial}{\partial t} At(t, r, u, v) \right) \left(\frac{d}{dr} \right. \\
& f(r) \left. \right) f(r) u^2 + \left(\frac{d}{dr} f(r) \right) \left(\frac{\partial}{\partial r} Ar(t, r, u, v) \right) f(r) u^2 + \left(\frac{d^2}{dr^2} f(r) \right) Ar(t, r, \\
& u, v) f(r) u^2 + 2 \left(\frac{\partial}{\partial u} Ar(t, r, u, v) \right) f(r)^2 u - \left(\frac{d}{dr} f(r) \right)^2 Ar(t, r, u, v) u^2
\end{aligned}$$

$$\begin{aligned}
& + 4 f(r)^2 u^3 \left(\frac{\partial}{\partial r} A v(t, r, u, v) \right) \chi(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) - 4 f(r)^2 u^2 \left(\frac{\partial}{\partial r} \right. \\
& A u(t, r, u, v) \left. \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) - 14 f(r)^2 u^2 \left(\frac{\partial}{\partial u} A r(t, r, u, \right. \\
& v) \left. \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) + 6 f(r)^2 u \left(\frac{\partial}{\partial v} A r(t, r, u, v) \right) \chi(r) \sin \left(- \left(\int \tau(r) dr \right) + v \right) \\
& - \left(\frac{d}{dr} f(r) \right)^2 A r(t, r, u, v) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 u^4 \\
& + 2 \left(\frac{d}{dr} f(r) \right)^2 A r(t, r, u, v) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) u^3 \Bigg) / \\
& \left(2 u^2 f(r)^2 \left(\chi(r)^4 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^4 u^4 - 4 \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 u^3 \right. \right. \\
& + 6 \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 u^2 - 4 u \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) + 1 \left. \right) \Bigg) D_- r + \left(-2 \left(\frac{\partial^2}{\partial u^2} A u(t, r, u, v) \right) u^2 f(r) - 2 \left(\frac{\partial^2}{\partial r^2} A u(t, r, u, \right. \right. \\
& v) \left. \right) f(r) u^2 - \left(\frac{\partial}{\partial r} A u(t, r, u, v) \right) \left(\frac{d}{dr} f(r) \right) u^2 + 4 \left(\frac{\partial}{\partial v} A v(t, r, u, v) \right) f(r) u \\
& - 2 \left(\frac{\partial}{\partial u} A u(t, r, u, v) \right) f(r) u + 2 \left(\frac{\partial^2}{\partial u^2} A u(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) \right. \right. \\
& dr) + v) \left. \right)^3 f(r) u^5 + 2 \left(\frac{\partial^2}{\partial v^2} A u(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 f(r) u^3 \\
& - 6 \left(\frac{\partial^2}{\partial u^2} A u(t, r, u, v) \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r) u^4 - 6 \left(\frac{\partial^2}{\partial v^2} A u(t, \right. \\
& r, u, v) \left. \right) \chi(r)^2 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^2 f(r) u^2 + 6 \left(\frac{\partial^2}{\partial u^2} A u(t, r, u, \right. \\
& v) \left. \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u^3 + 2 \left(\frac{\partial^2}{\partial r^2} A u(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u \\
& + 6 \left(\frac{\partial^2}{\partial v^2} A u(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u^3 + 6 \left(\frac{\partial^2}{\partial v^2} A u(t, r, u, v) \right) \chi(r) \cos \left(- \left(\int \tau(r) dr \right) + v \right) f(r) u \\
& - 2 \left(\frac{\partial^2}{\partial t^2} A u(t, r, u, v) \right) u^2 - 2 \left(\frac{\partial^2}{\partial v^2} A u(t, r, u, v) \right) f(r) - 4 \left(\frac{\partial}{\partial v} A v(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 f(r) u^4 + 4 \left(\frac{\partial}{\partial u} A u(t, r, u, \right. \\
& v) \left. \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 f(r) u^4 - 2 \left(\frac{d}{dr} f(r) \right) A r(t, r, u, \\
& v) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 u^4 - 4 \left(\frac{\partial}{\partial r} A r(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 u^4 - 4 \left(\frac{\partial}{\partial r} A r(t, r, u, v) \right) \chi(r)^3 \cos \left(- \left(\int \tau(r) dr \right) + v \right)^3 u^4
\end{aligned}$$

$$\begin{aligned} & \left(\int \tau(r) \, dr \right) + v \Big)^3 f(r) u^4 - 4 Au(t, r, u, v) \chi(r)^3 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^3 f(r) u^3 \\ & + 12 \left(\frac{\partial}{\partial v} Av(t, r, u, v) \right) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 f(r) u^3 - 10 \left(\frac{\partial}{\partial u} Au(t, \right. \\ & r, u, v) \Big) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 f(r) u^3 + 4 \left(\frac{d}{dr} f(r) \right) Ar(t, r, u, \\ & v) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 u^3 + 8 \left(\frac{\partial}{\partial r} Ar(t, r, u, v) \right) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) \\ & + v \Big)^2 f(r) u^3 + 8 Au(t, r, u, v) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 f(r) u^2 \\ & + \left(\frac{\partial}{\partial r} Au(t, r, u, v) \right) \left(\frac{d}{dr} f(r) \right) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u^3 - 2 \left(\frac{\partial}{\partial r} \right. \\ & Au(t, r, u, v) \Big) \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big) f(r) \left(\frac{d}{dr} \chi(r) \right) u^3 - 12 \left(\frac{\partial}{\partial v} Av(t, r, u, \right. \\ & v) \Big) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big) f(r) u^2 + 8 \left(\frac{\partial}{\partial u} Au(t, r, u, v) \right) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) \\ & + v \Big) f(r) u^2 - 2 \left(\frac{d}{dr} f(r) \right) Ar(t, r, u, v) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) \\ & + v \Big) u^2 - 2 Ar(t, r, u, v) \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big) f(r) \left(\frac{d}{dr} \chi(r) \right) u^2 + 2 Av(t, \\ & r, u, v) \chi(r) f(r) \sin \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u^2 - 4 \left(\frac{\partial}{\partial r} Ar(t, r, u, v) \right) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) \\ & + v \Big) f(r) u^2 - 2 \left(\frac{\partial}{\partial v} Au(t, r, u, v) \right) \chi(r) f(r) \sin \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u \\ & - 6 Au(t, r, u, v) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big) f(r) u - 4 Ar(t, r, u, \\ & v) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^3 f(r) \left(\frac{d}{dr} \chi(r) \right) u^4 + 4 Av(t, r, u, \\ & v) \chi(r)^3 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 f(r) \sin \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u^4 - 2 \left(\frac{\partial}{\partial v} Au(t, \right. \\ & r, u, v) \Big) \chi(r)^3 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 f(r) \sin \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u^3 + 6 Ar(t, \\ & r, u, v) \chi(r) \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big)^2 f(r) \left(\frac{d}{dr} \chi(r) \right) u^3 - 6 Av(t, r, u, \\ & v) \chi(r)^2 \cos \Big(- \left(\int \tau(r) \, dr \right) + v \Big) f(r) \sin \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u^3 - 2 \left(\frac{\partial}{\partial r} Au(t, r, \right. \\ & u, v) \Big) \chi(r) f(r) \tau(r) \sin \Big(- \left(\int \tau(r) \, dr \right) + v \Big) u^3 + 4 \left(\frac{\partial}{\partial v} Au(t, r, u, \right. \end{aligned}$$

$$\begin{aligned}
& \nu) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^2 - 2 Ar(t, r, u, \\
& \nu) \chi(r) f(r) \tau(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^2 - 4 Ar(t, r, u, \nu) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) \tau(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^4 + 6 Ar(t, r, u, \nu) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) \tau(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^3 + 2 \left(\frac{\partial^2}{\partial t^2} Au(t, r, u, \nu)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 u^5 - 6 \left(\frac{\partial^2}{\partial t^2} Au(t, r, u, \nu)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 u^4 + 6 \left(\frac{\partial^2}{\partial t^2} Au(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) u^3 + 2 Au(t, r, u, \nu) f(r) \Bigg/ \left(2 f(r) u^2 \left(\chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 u^3 - 3 \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 u^2 + 3 u \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) - 1\right) \right) D_- u + \left(-2 \left(\frac{\partial^2}{\partial u^2} Av(t, r, u, \nu)\right) f(r) u^3 - 2 \left(\frac{\partial^2}{\partial r^2} Av(t, r, u, \nu)\right) f(r) u^3 - 2 \left(\frac{\partial^2}{\partial v^2} Av(t, r, u, \nu)\right) f(r) u + 4 Ar(t, r, u, \nu) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) \left(\frac{d}{dr} \chi(r)\right) u^4 - 6 Ar(t, r, u, \nu) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) \left(\frac{d}{dr} \chi(r)\right) u^3 - 4 \left(\frac{\partial}{\partial v} Au(t, r, u, \nu)\right) f(r) + 2 \left(\frac{\partial^2}{\partial u^2} Av(t, r, u, \nu)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 f(r) u^6 + 2 \left(\frac{\partial^2}{\partial v^2} Av(t, r, u, \nu)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 f(r) u^4 - 6 \left(\frac{\partial^2}{\partial u^2} Av(t, r, u, \nu)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) u^5 - 6 \left(\frac{\partial^2}{\partial v^2} Av(t, r, u, \nu)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) u^3 + 6 \left(\frac{\partial^2}{\partial u^2} Av(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u^4 + 2 \left(\frac{\partial^2}{\partial r^2} Av(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u^4 + 6 \left(\frac{\partial^2}{\partial v^2} Av(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u^2 + 2 Av(t, r, u, \nu) \chi(r)^2 f(r) u^3 - 2 \left(\frac{\partial^2}{\partial t^2} Av(t, r, u, \nu)\right) u^3 + 8 \left(\frac{\partial}{\partial u} Av(t, r, u, \nu)\right)
\end{aligned}$$

$$\begin{aligned}
& \nu) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 f(r) u^5 + 4 A\nu(t, r, u, \nu) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 f(r) u^4 + 4 \left(\frac{\partial}{\partial \nu} A u(t, r, u, \nu)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 f(r) u^3 - 22 \left(\frac{\partial}{\partial u} A\nu(t, r, u, \nu)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) u^4 \\
& - 2 A\nu(t, r, u, \nu) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u^4 - 6 A\nu(t, r, u, \nu) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) u^3 - 12 \left(\frac{\partial}{\partial \nu} A u(t, r, u, \nu)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) u^2 + \left(\frac{d}{dr} f(r)\right) \left(\frac{\partial}{\partial r} A\nu(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) u^4 - 2 A r(t, r, u, \nu) \chi(r)^2 f(r) \tau(r) u^3 - 2 \left(\frac{\partial}{\partial r} A\nu(t, r, u, \nu)\right) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) \left(\frac{d}{dr} \chi(r)\right) u^4 + 20 \left(\frac{\partial}{\partial u} A\nu(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u^3 - 2 \left(\frac{\partial}{\partial \nu} A\nu(t, r, u, \nu)\right) \chi(r) f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^2 \\
& + 2 \left(\frac{d}{dr} f(r)\right) A r(t, r, u, \nu) \chi(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^2 + 2 A r(t, r, u, \nu) f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) \left(\frac{d}{dr} \chi(r)\right) u^2 + 2 A\nu(t, r, u, \nu) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u^2 + 4 \left(\frac{\partial}{\partial r} A r(t, r, u, \nu)\right) \chi(r) f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^2 \\
& + 12 \left(\frac{\partial}{\partial \nu} A u(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) f(r) u - 2 A u(t, r, u, \nu) \chi(r) f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u - \left(\frac{d}{dr} f(r)\right) \left(\frac{\partial}{\partial r} A\nu(t, r, u, \nu)\right) u^3 \\
& - 6 \left(\frac{\partial}{\partial u} A\nu(t, r, u, \nu)\right) f(r) u^2 + 2 \left(\frac{\partial^2}{\partial t^2} A\nu(t, r, u, \nu)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 u^6 - 6 \left(\frac{\partial^2}{\partial t^2} A\nu(t, r, u, \nu)\right) \chi(r)^2 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 u^5 + 6 \left(\frac{\partial^2}{\partial t^2} A\nu(t, r, u, \nu)\right) \chi(r) \cos\left(-\left(\int \tau(r) dr\right) + \nu\right) u^4 - 4 A r(t, r, u, \nu) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^3 f(r) \tau(r) u^4 - 2 \left(\frac{\partial}{\partial \nu} A\nu(t, r, u, \nu)\right) \chi(r)^3 \cos\left(-\left(\int \tau(r) dr\right) + \nu\right)^2 f(r) \sin\left(-\left(\int \tau(r) dr\right) + \nu\right) u^4 + 2 \left(\frac{d}{dr} f(r)\right) A r(t, r, u, \nu)
\end{aligned}$$

