Carleton University
Department of Systems and Computer Engineering
SYSC5606 Introduction to Mobile Communications
Summer 2015
Question Set 3
Posted: June 13, 2015

Question 1

(a) In a DS-CDMA multiple user system, how many simultaneous users can be supported such that an average bit error rate less than 10^{-3} is maintained for each user? Assume all users employ power control such that the received power of each user is maintained at an average $E_b/N_0 = 10$ dB, and assume each user has a PN code that is produced from an 7-bit shift register. Mention any other assumptions that you make.

(b) How many users can be supported if each uses a different orthogonal spreading code of length 64?

Solution 1

(a) Here $E_b/N_0 \neq \infty$ so must use

$$P_e = Q\left(\frac{1}{\sqrt{\frac{K-1}{3N} + \frac{N_0}{2E_b}}}\right)$$

where $N = 2^m - 1 = 2^7 - 1 = 127$. We need to find K such that $P_e = 10^{-3}$. From the table we can see that $Q(3.1) \simeq 10^{-3}$.

$$\begin{aligned}
3.1 &= \frac{1}{\sqrt{\frac{K-1}{3\times 127} + \frac{1}{2\times 10}}} \\
9.61 &= \frac{1}{\frac{K-1}{381} + \frac{1}{20}} \\
9.61 &= \frac{381 \times 20}{20 \times (K-1) + 381} \\
9.61 \times (20 \times (K-1) + 381) &= 7620 \\
9.61 \times (20 \times (K-1)) &= 3958.59 \\
(K-1) &= 20.59 \\
K &= 21.59
\end{aligned}$$

So 21 users can be supported. Additional assumptions (in addition to perfect power control) are

1. No out of cell interference
2. Fading and shading neglected
3. Spreading codes are random

(b) If codes are *orthogonal*, the first term in the denominator above is zero, and

$$\begin{aligned} BER &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ &= Q\left(\sqrt{20}\right) \\ &= Q\,(4.47) \\ &= 3.8721 \times 10^{-6} \end{aligned}$$

Maximum number of users is the maximum number of orthogonal codewords of length 64 which is 64.

Question 2

The output of a channel, sampled at time kT is

$$y_k = h_0 a_k + h_1 a_{k-1} + v_k$$

where the date symbols $a_k = \pm 1$ with equal probability, and are uncorrelated, and the v_k are independent, zero-mean sampled of noise with variance of σ^2. A linear equalizer with 3 tap coefficients is used to process the channel's output (equalizer output at time k is $z_k = \sum_{n=0}^{2} w_n * y_{k-n}$). Its 3 tap coefficients are required to minimize the mean squared value of the error between the kth equalizer output sample and the $k-1$ st data symbols a_{k-1}.

a) Derive expressions for the optimum tap coefficients and minimum mean squared error as functions of the above parameters.

b) Determined the coefficients and minimum MSE for $h_0 = 1+j$, $h_1 = 0.3-0.1j$, and $\sigma^2 = 0.01$.

Solution 2

(a) From above equation, the equalizer input vector at time k is

$$\mathbf{y}_k = \mathbf{H}\mathbf{a}_k + \mathbf{v}_k$$

where

$$\mathbf{y}_k = \begin{bmatrix} y_k \\ y_{k-1} \\ y_{k-2} \end{bmatrix}, \mathbf{H} = \begin{bmatrix} h_0 & h_1 & 0 & 0 \\ 0 & h_0 & h_1 & 0 \\ 0 & 0 & h_0 & h_1 \end{bmatrix}, \mathbf{a}_k = \begin{bmatrix} a_k \\ a_{k-1} \\ a_{k-2} \\ a_{k-3} \end{bmatrix}, and\ \mathbf{v}_k = \begin{bmatrix} v_k \\ v_{k-1} \\ v_{k-2} \end{bmatrix}$$

$$MMSE = \mathbb{E}[|e_k|^2] = \mathbb{E}[a_k^2] + \mathbf{w}_k^H \mathbb{E}[\mathbf{y}_k \mathbf{y}_k^H]\mathbf{w}_k - 2\mathbb{E}[a_k \mathbf{y}_k^H]\mathbf{w}_k$$

Let

$$\mathbf{R} = \mathbb{E}[\mathbf{y}_k \mathbf{y}_k^H] = \begin{bmatrix} |y_k|^2 & y_k y_{k-1}^* & y_k y_{k-2}^* \\ y_{k-1} y_k^* & |y_{k-1}|^2 & y_{k-1} y_{k-2}^* \\ y_{k-2} y_k^* & y_{k-2} y_{k-1}^* & |y_{k-2}|^2 \end{bmatrix}$$

$$\mathbf{p} = \mathbb{E}[\mathbf{y}_k a_{k-1}^*]$$

Then,

$$MMSE = \mathbb{E}[a_k^2] + \mathbf{w}_k^H \mathbf{R} \mathbf{w}_k - 2\mathbf{p}^H \mathbf{w}_k$$

The filter $\mathbf{w}$ that optimizes MMSE expression can be shown to be

$$\hat{\mathbf{w}} = \mathbf{R}^{-1}\mathbf{p}$$

Let us first find $\mathbf{R}$ in terms of h_0, h_1 and σ^2. Compute each components of $\mathbf{R}$

$$\begin{aligned}
\mathbb{E}[|y_k|^2] &= \mathbb{E}[(h_0 a_k + h_1 a_{k-1} + v_k)(h_0 a_k + h_1 a_{k-1} + v_k)^*] \\
&= \mathbb{E}[|h_0 a_k|^2 + h_1 a_{k-1} h_0^* a_k^* + v_k h_0^* a_k^* \\
&+ h_0 a_k h_1^* a_{k-1}^* + |h_1 a_{k-1}|^2 + v_k h_1^* a_{k-1}^* \\
&+ h_0 a_k v_k^* + h_1 a_{k-1} v_k^* + |v_k|^2] \\
&= |h_0|^2 \mathbb{E}[|a_k|^2] + h_1 h_0^* \mathbb{E}[a_{k-1} a_k^*] + h_0^* \mathbb{E}[v_k a_k^*] \\
&+ h_0 h_1^* \mathbb{E}[a_k a_{k-1}^*] + |h_1|^2 \mathbb{E}|a_{k-1}|^2] + h_1^* \mathbb{E}[v_k a_{k-1}^*] \\
&+ h_0 \mathbb{E}[a_k v_k^*] + h_1 \mathbb{E}[a_{k-1} v_k^*] + \mathbb{E}[|v_k|^2]
\end{aligned}$$

Using the fact that $a_k = \pm 1$ with equal probability and are uncorrelated then $\mathbb{E}[|a_k|^2] = 1$, for any k, $\mathbb{E}[a_i a_j^*] - \mathbb{E}[a_i]\mathbb{E}[a_j^*] = 0$ for all $i \neq j$ results in $\mathbb{E}[a_i a_j^*] = 0$ for all $i \neq j$ since $\mathbb{E}[a_i] = 0$ and $\mathbb{E}[a_j^*] = 0$. v_k are independent and zero mean noise $\mathbb{E}[v_k] = 0$, $\mathbb{E}[|v_k|^2] = \sigma^2$, $\mathbb{E}[a_i v_k^*] = 0$ for all i's. Therefore $\mathbb{E}[a_{k-1} a_k^*] = 0$, $\mathbb{E}[v_k a_k^*] = 0$, $\mathbb{E}[a_k a_{k-1}^*] = 0$, $\mathbb{E}[v_k a_{k-1}^*] = 0$, $\mathbb{E}[a_k v_k^*] = 0$ and $\mathbb{E}[a_{k-1} v_k^*] = 0$. Hence,

$$\mathbb{E}[|y_k|^2] = |h_0|^2 + |h_1|^2 + \sigma^2$$

Similarly we can show that $\mathbb{E}[y_k y_{k-1}^*] = h_1 h_0^*$, $\mathbb{E}[y_k y_{k-2}^*] = 0$, $\mathbb{E}[y_{k-1} y_k^*] = h_1^* h_0$, $\mathbb{E}[y_{k-1} y_{k-2}^*] = h_1 h_0^*$, $\mathbb{E}[y_{k-2} y_k^*] = 0$ and $\mathbb{E}[y_{k-2} y_{k-1}^*] = h_1^* h_0$. Let $r_0 = |h_0|^2 + |h_1|^2 + \sigma^2$ and $r_1 = h_1^* h_0$ then,

$$\mathbf{R} = \mathbb{E}[\mathbf{y}_k \mathbf{y}_k^H] = \begin{bmatrix} r_0 & r_1 & 0 \\ r_1^* & r_0 & r_1 \\ 0 & r_1^* & r_0 \end{bmatrix}$$

Using matrix inversion rule we can show that

$$\mathbf{R}^{-1} = \frac{1}{r_0^3 - 2r_0|r_1|^2} \begin{bmatrix} (r_0^2 - |r_1|^2) & -r_0 r_1 & r_1^2 \\ -r_0 r_1 * & r_0^2 & -r_0 r_1 \\ (r_1^*)^2 & -r_0 r_1^* & (r_0^2 - |r_1|^2) \end{bmatrix}$$

and

$$\begin{aligned}
\mathbf{p} = \mathbb{E}[\mathbf{y}_k a_{k-1}^*] = \mathbb{E} & \begin{bmatrix} h_0 a_k a_{k-1}^* + h_1 a_{k-1} a_{k-1}^* + v_k a_{k-1}^* \\ h_0 a_{k-1} a_{k-1}^* + h_1 a_{k-2} a_{k-1}^* + v_{k-1} a_{k-1}^* \\ h_0 a_{k-2} a_{k-1}^* + h_1 a_{k-3} a_{k-1}^* + v_{k-2} a_{k-1}^* \end{bmatrix} \\
&= \begin{bmatrix} h_0 \mathbb{E}[a_k a_{k-1}^*] + h_1 \mathbb{E}[|a_{k-1}|^2] + \mathbb{E}[v_k a_{k-1}^*] \\ h_0 \mathbb{E}[|a_{k-1}|^2] + h_1 \mathbb{E}[a_{k-2} a_{k-1}^*] + \mathbb{E} v_{k-1} a_{k-1}^*] \\ h_0 \mathbb{E}[a_{k-2} a_{k-1}^*] + h_1 \mathbb{E}[a_{k-3} a_{k-1}^*] + \mathbb{E}[v_{k-2} a_{k-1}^*] \end{bmatrix} \\
&= \begin{bmatrix} h_1 \\ h_0 \\ 0 \end{bmatrix}
\end{aligned}$$

Hence,

$$\begin{aligned}
\hat{\mathbf{w}} &= \mathbf{R}^{-1}\mathbf{p} \\
&= \frac{1}{r_0^3 - 2r_0|r_1|^2} \begin{bmatrix} (r_0^2 - |r_1|^2) & -r_0 r_1 & r_1^2 \\ -r_0 r_1 * & r_0^2 & -r_0 r_1 \\ (r_1^*)^2 & -r_0 r_1^* & (r_0^2 - |r_1|^2) \end{bmatrix} \begin{bmatrix} h_1 \\ h_0 \\ 0 \end{bmatrix} \\
&= \frac{1}{r_0^2 - 2|r_1|^2} \begin{bmatrix} h_1(r_0 - \frac{|r_1|^2}{r_0}) - r_1 h_0 \\ -h_1 r_1^* + h_0 r_0 \\ \frac{h_1 (r_1^*)^2}{r_0} - h_0 r_1^* \end{bmatrix}
\end{aligned}$$

We can simplify MMSE using above rules and submitting estimated filter $\hat{\mathbf{w}}$

$$\begin{aligned}
MMSE &= 1 + \mathbf{w}_k^H \mathbf{R} \mathbf{w}_k - 2\mathbf{p}^H \mathbf{w}_k \\
&= 1 + \mathbf{p}^H \mathbf{R}^{-1} \mathbf{R} \mathbf{R}^{-1} \mathbf{p} - 2\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \\
&= 1 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}
\end{aligned}$$

(b) Substituting the numerical values,

$$\hat{\mathbf{w}} = \begin{bmatrix} 0.0011 - 0.0004i \\ 0.4960 + 0.4960i \\ 0.0470 - 0.1411i \end{bmatrix}$$

$$MMSE = 0.0075$$

Question 3

In an omnidirectional (single-cell, single-sector) CDMA cellular system, $E_b/N_0 = 20$ dB is required for each user. If 100 users, each with a baseband data rate of 13 kbps, are to be accommodated, determine the minimum channel bit rate of the spread spectrum chip sequence. Ignore voice activity considerations.

Solution 3

Using equation 9.29

$$\frac{E_b}{N_0} = \frac{W/R}{N-1}$$

where W is total RF bandwidth, R is information bit rate, and N is the number of users. Therefore the channel bit rate of the spread spectrum chip sequence W is

$$\begin{aligned} W &= \frac{E_b}{N_0} \times R \times (N-1) \\ &= 100 \times 13 \times 10^3 \times (100-1) \\ &= 1.287 \times 10^8 \; chips/sec \\ &= 128.7 \times \; Mchips/sec \end{aligned}$$

Question 4 (Bonus)

Compare $\bar{\gamma}/\Gamma$ (selection diversity) with $\bar{\gamma_M}/\Gamma$ (maximal ratio combining) for one to six branches. Specifically, compare how the average SNR increases for each diversity scheme as a new branch is added. Does this make sense? What is the average SNR improvement offered by 6-branch maximal ratio combining as compared to 6-branch selection diversity? If $\gamma/\Gamma = 0.01$, determine the probability that the received signal will be below this threshold for maximal ratio combining and selection diversity (assume 6 branch are used). How does this compare with a single Rayleigh fading channel with the same threshold?

Solution 4

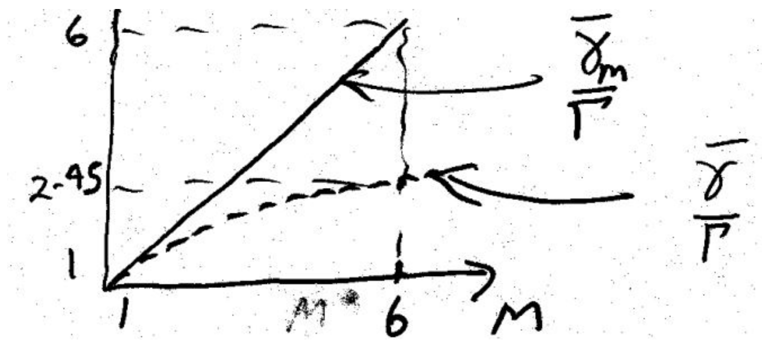
For maximal ratio combining with equal average SNR per branch can be shown to be (equation 7.70)

$$\frac{\bar{\gamma_M}}{\Gamma} = M$$

where Γ is average SNR per branch and M number of branches. For selection diversity using equation 7.62

$$\frac{\bar{\gamma}}{\Gamma} = \sum_{k=1}^{M} \frac{1}{k}$$

The maximal ratio combiner, the average SNR $\bar{\gamma}$ increases linearly with M. Diversity improvement for selection diversity increases more slowly with M than for maximal ratio combining. Therefore, maximal ratio combining requires less branches than the selection diversity to achieve a specific average SNR improvement.



For 6-branch maximal ratio combining, the average SNR improvement is

$$\begin{aligned} 10\log_{10}\bar{\gamma_M}/\Gamma &= 10\log_{10} 6 \\ &= 7.78\, dB \end{aligned}$$

For 6-branch selection diversity, the average SNR improvement is

$$\begin{aligned} 10\log_{10}\bar{\gamma}/\Gamma &= 10\log_{10}\sum_{k=1}^{M}\frac{1}{k} \\ &= 10\log_{10} 2.45 \\ &= 3.89\, dB \end{aligned}$$

If $\gamma/\Gamma = 0.01$, for 6-branch maximal ratio combining,

$$\begin{aligned} p(\gamma_M \leq \gamma) &= 1 - e^{-\gamma/\Gamma}\sum_{k=1}^{6}\frac{(\gamma/\Gamma)^{k-1}}{(k-1)!} \\ &= 1 - e^{-0.01}\sum_{k=1}^{6}\frac{(0.01)^{k-1}}{(k-1)!} \\ &= 1.37 \times 10^{-15} \end{aligned}$$

For 6-branch selection diversity,

$$\begin{aligned} p(\gamma_M \leq \gamma) &= (1 - e^{-\gamma/\Gamma})^M \\ &= (1 - e^{-0.01})^6 \\ &= 9.7 \times 10^{-13} \end{aligned}$$

For a single Rayleigh fading channel,

$$
\begin{aligned}
p(\gamma_i \leq \gamma) &= 1 - e^{-\gamma/\Gamma} \\
&= 1 - e^{-0.01} \\
&= 9.95 \times 10^{-3}
\end{aligned}
$$