

Numerical Methods for PDEs

Homework 6

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Problem 1

In the modified PDE for the Lax-Wendroff method for $u_t + cu_x = 0$, derive the coefficient $\beta = \frac{ch^2}{6}(r^2 - 1)$ of numerical dispersion in $u_t + cu_x = \beta u_{xxx}$.

Solution: We start by Taylor expanding

$$\begin{aligned} u_j^{n+1} &= u_j^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \mathcal{O}(\Delta t^4), \\ u_{j\pm 1}^n &= u_j^n \pm \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} \pm \frac{\Delta x^3}{6} u_{xxx} + \mathcal{O}(\Delta x^4), \end{aligned}$$

and substituting them (neglecting the higher order terms) into the Lax-Wendroff scheme,

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{c\Delta t}{2\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{c^2\Delta t^2}{2\Delta x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n), \\ u_j^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} &= u_j^n - \frac{c\Delta t}{2\Delta x} \left(2\Delta x u_x + \frac{\Delta x^3}{3} u_{xxx} \right) + \frac{c^2\Delta t^2}{2\Delta x^2} \Delta x^2 u_{xx}. \end{aligned}$$

Doing some algebraic manipulations we reach,

$$u_t + cu_x = -\frac{\Delta t}{2} u_{tt} - c \frac{\Delta x^2}{6} u_{xxx} + \frac{1}{2} c^2 \Delta t u_{xx} - \frac{\Delta t^2}{6} u_{ttt}.$$

Using the original PDE we obtain that

$$\begin{aligned} u_t &= -cu_x \Rightarrow u_{tt} = c^2 u_{xx}, \\ &\Rightarrow u_{ttt} = -c^3 u_{xxx}. \end{aligned}$$

Hence,

$$\begin{aligned} u_t + cu_x &= -\frac{\Delta t}{2} c^2 u_{xx} - c \frac{\Delta x^2}{6} u_{xxx} + \frac{1}{2} c^2 \Delta t u_{xx} + \frac{\Delta t^2}{6} c^3 u_{xxx}, \\ &= -c \frac{\Delta x^2}{6} u_{xxx} + \frac{\Delta t^2}{6} c^3 u_{xxx}, \\ &= c \frac{\Delta x^2}{6} \left(\frac{c^2 \Delta t^2}{\Delta x^2} - 1 \right) u_{xxx}, \\ &= c \frac{\Delta x^2}{6} (r^2 - 1) u_{xxx}, \\ &= \beta u_{xxx}. \end{aligned}$$

Problem 2

Show that the two-step Lax-Wendroff method reduces to the original Lax-Wendroff scheme for $u_t + Au_x = 0$.

Solution: We start with the first step of the method

$$\begin{aligned} w_{i+1/2}^{n+1/2} &= \frac{1}{2} (w_i^n + w_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} A (w_{i+1}^n - w_i^n), \\ w_{i-1/2}^{n+1/2} &= \frac{1}{2} (w_i^n + w_{i-1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} A (w_i^n - w_{i-1}^n), \end{aligned}$$

and plug the terms into the second step,

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\Delta x} A (w_{i+1/2}^{n+1/2} - w_{i-1/2}^{n+1/2}), \\ &= w_i^n - \frac{\Delta t}{\Delta x} A \left[\frac{1}{2} (w_i^n + w_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} A (w_{i+1}^n - w_i^n) - \frac{1}{2} (w_i^n + w_{i-1}^n) + \frac{1}{2} \frac{\Delta t}{\Delta x} A (w_i^n - w_{i-1}^n) \right], \\ &= w_i^n - \frac{1}{2} A \frac{\Delta t}{\Delta x} (w_{i+1}^n - w_{i-1}^n) + \frac{1}{2} A^2 \frac{\Delta t^2}{\Delta x^2} (w_{i+1}^n - 2w_i^n + w_{i-1}^n), \end{aligned}$$

and we obtain the desired the original Lax-Wendroff scheme for $u_t + Au_x = 0$.

Problem 3

Show that Burgers' equation $u_t + uu_x = \nu u_{xx}$ with $u(x, t = 0) = u_l$, $x < 0$ and $u(x, t = 0) = u_r < u_l$, $x > 0$ has a traveling wave solution of the form $u(x, t) = w(x - st)$ by deriving an ODE for w and showing that the ODE is solved by

$$w(y) = u_r + \frac{1}{2}(u_l - u_r) \left[1 - \tanh \frac{(u_l - u_r)y}{4\nu} \right]$$

where $s = (u_l + u_r)/2$.

Solution:

Problem 4

Prove that the Lax-Friedrichs (LF) method is *positivity-preserving* (i.e., if $u_j^n > 0$ for all j , then $u_j^{n+1} > 0$ for all j) for Burgers' equation $u_t + (u^2/2)_x = 0$. The LF discretization of Burgers' equation is

$$u_j^{n+1} = \frac{1}{2} (u_{j-1}^n + u_{j+1}^n) - \frac{\Delta t}{4\Delta x} ((u_{j+1}^n)^2 - (u_{j-1}^n)^2)$$

Hint: For stability, $\Delta t \leq \Delta x / \max_i \{|u_i^n|\}$. *Note:* Lax-Friedrichs is also positivity-preserving for gas dynamics, unlike upwind schemes.

Solution:

Problem 5

Show that the upwind discretization of Burgers' equation $u_t + uu_x = 0$ for $u(x, t) > 0$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n (u_j^n - u_{j-1}^n)$$

is nonconservative, while the upwind discretization of Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ for $u(x, t) > 0$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x} ((u_j^n)^2 - (u_{j-1}^n)^2)$$

is conservative, by showing whether or not the scheme can be put into the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} (F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}).$$

Solution:
