Real Analysis Homework 13

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1 Problem 6.7.1

1. Let X and Z be normed vector spaces and U an open subset of X.

   (a) Assume that U is convex: for all $x, y \in U$ we have that $(1t)x + ty \in U$ for all $t \in (0, 1)$. Assume that $f : \bar{U} \to Z$ is a continuous function and that f is Gateaux differentiable on U and that there exists some $M > 0$ such that

   $$\|\partial f(x, v)\| \leq M \text{ for all } x \in U \text{ and } v \in X, \|v\| = 1 \ .$$

   Show: f is Lipschitz continuous on $\bar{U}$ and M is a Lipschitz constant for f on $\bar{U}$.

Solution:

Proof. Define $x_t = (1 - t)x + ty$ for all $x, y \in U$ and $t \in (0, 1)$. Since, U is convex, $x_t \in U$. By Theorem 6.29 part (a), there exists $t \in (0, 1)$ such that, for $z \in Z$ and $x, y \in \bar{U}$

$$\|f(y) - f(x) - z\| \leq \|\partial f(x, y - x) - z\| \ .$$

For $z = 0$,

$$\|f(y) - f(x)\| \leq \|\partial f(x, y - x)\| \ .$$

Then, by Lemma 6.18,

$$\partial f(x_t, y - x) = \frac{\|y - x\|}{\|y - x\|} \partial f(x_t, y - x) = \|y - x\| \partial f\left(x_t, \frac{y - x}{\|y - x\|}\right) \ .$$

This gives us,

$$\begin{aligned}
\|f(y) - f(x)\| &\leq \|\partial f(x, y - x)\| \\
&= \left\| \|y - x\| \partial f\left(x_t, \frac{y - x}{\|y - x\|}\right) \right\| \\
&\leq \left\| \partial f\left(x_t, \frac{y - x}{\|y - x\|}\right) \right\| \|y - x\| \\
&\leq M \|y - x\| \ .
\end{aligned}$$

Therefore, f is Lipschitz continuous on $\bar{U}$ and M is a Lipschitz constant for f on $\bar{U}$. ■

2. (b) Assume that $f : U \to Z$ is Frechet differentiable on U and that $Df : U \to \mathscr{L}(X, Z)$ is continuous. Show: f is locally Lipschitz continuous on U. Actually for every compact subset K of U there exists an open set V such that $K \subseteq V \subseteq U$ and f is Lipschitz continuous on V .

Solution:

Proof. Let $\varepsilon = 1$. By continuity definition 3.1, there exists some r_1 such that, for $y \in U_{r_1}(x)$, with $x \in U$,

$$\|Df(x) - Df(y)\| < 1 \ .$$

Then, since U is open, there exists some r_2 such that $U_{r_2} \in U$. Let $\delta = \min\{r_1, r_2\}$. Then, for all $y_1 \in U_\delta(x)$,

$$\begin{aligned} \|Df(x)\| &= \|Df(x) - Df(y_1) + Df(y_1)\| \\ &\leq \|Df(x) - Df(y_1)\| + \|Df(y_1)\| \\ &\leq 1 + \|Df(y_1)\| \ . \end{aligned}$$

Note, that $U_\delta(x)$ is a convex open set. Set $V = U_\delta(x)$ and $M = 1 + \|Df(y)\|$. Then, for $v \in X$ with $\|v\| = 1$ and $y_2 \in V$, $\|Df(y_2)\| < M$ and

$$\begin{aligned} \|\partial f(y_2, v)\| &= \|Df(y_2)v\| \leq \|Df(y_2)\|\|v\| \\ &= \|Df(y_2)\| < M \ . \end{aligned}$$

Finally, by theorem 6.23, since f is Frechet differentiable on on U, f is also Gateaux differentiable on U. Therefore, from part (a) shown above, f is Lipschitz continuous on V, an open subset of U. ■

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