Real Analysis Homework 13

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November 20, 2017

1 Problem 6.4.1

1. Let I be a bounded interval and Z a Banach space. Let (f_n) be a sequence of differentiable functions $f_n : I \to Z$ such that $\sum_{n=1}^{\infty} f_n(x^o)$ converges in Z for some $x^o \in I$ and there exists a sequence of positive numbers (M_n) such that $||f_n'(x)|| \leq M_n$ for all $n \in \mathbb{N}$ and $x \in I$ and $\sum_{n=1}^{\infty} M_n$ converges in $\mathbb{R}$. Show that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly for $x \in I$ and provides a differentiable function $f : I \to Z$ such that $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$, with the convergence of the latter series being also uniform for $x \in I$.

Solution:

Proof. Let $(s_k(x))$ be a sequence of partial sums where $s_k : I \to Z$ is defined by

$$s_k(x) = \sum_{n=1}^{k} f_n(x) \ .$$

Therefore (s_k) is a sequence of differentiable functions since each element of the sequence is a sum of differentiable functions $f_n(x)$. Since $\sum_{n=1}^{\infty} f_n(x^o)$ converges in Z for some x^o in I, then $(s_k(x^o))$ converges in Z as $k \to \infty$ for some $x^o \in I$.

Now let $(s_k'(x))$ be a sequence of partial sums where $s_k' : I \to Z$ is defined by

$$s_k'(x) = \sum_{n=1}^{k} f_n'(x) \ .$$

Since there exists a sequence of positive numbers (M_n) such that $\sum_{n=1}^{\infty} M_n$ converges and $||f_n'(x)|| \leq M_n$ for all $x \in I$ and $n \in \mathbb{N}$, by the *Weierstraß Test*, the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly for all $x \in I$, therefore $(s_k'(x))$ converges as $k \to \infty$ uniformly for all $x \in I$ and provides a bounded function $f' : I \to Z$, i.e. $s_k'(x) \to f'(x)$.

Next, by *Theorem 6.16*, $(s_k(x))$ converges as $k \to \infty$ uniformly in $x \in I$ to a differentiable function $f : I \to Z$,

$$\begin{aligned} f(x) &= \lim_{k\to\infty} s_k(x) \\ &= \lim_{k\to\infty} \sum_{n=1}^{k} f_n(x) \\ &= \sum_{n=1}^{\infty} f_n(x) \ , \end{aligned}$$

for all $x \in I$ and

$$f'(x) = \lim_{k\to\infty} s_k'(x) = \lim_{k\to\infty} \sum_{n=1}^{k} f_n'(x) = \sum_{n=1}^{\infty} f_n'(x) \ ,$$

also for all $x \in I$. Note that the uniform convergence of the latter, for all $x \in I$, is proven by the *Weierstraß Test.* ∎

2 Problem 6.4.2

1. Show that $\sum_{n=1}^{\infty} \sin(2^{-n}t)$ converges uniformly in $t \in [a, a]$ for every $a > 0$ and provides a continuously differentiable function on $\mathbb{R}$. (A function is continuously differentiable if it is differentiable and its derivative is continuous.)

Solution:

Proof. Let $f_n = \sin(2^{-n}t)$ for all $n \in \mathbb{N}$ so that (f_n) is a sequence of differentiable functions $f_n : I \to \mathbb{R}$, with $I = [-a, a]$, for all $a \in \mathbb{R}$ with $a > 0$. Let $t_0 = 0$ ($t_0 \in I$ for all $a > 0$) such that

$$\sum_{n=1}^{\infty} f_n(t_0) = \sum_{n=1}^{\infty} 0 = 0 \ .$$

Therefore, $\sum_{n=1}^{\infty} f_n(t_0)$ converges in $\mathbb{R}$. Next, define the sequence of positive numbers $M_n = \left(\frac{1}{2}\right)^n$ such that

$$||f_n'(t)|| = ||2^{-n}\cos(2^{-n}t)|| \leq ||2^{-n}|| = 2^{-n} = M_n \quad \forall n \in \mathbb{N}.$$

Observe that $\sum_{n=1}^{\infty} M_n$ converges in $\mathbb{R}$ since it is a geometric series $\sum_{n=1}^{\infty} q^n$ with $|q| < 1$. Thus, by the *Problem 6.4.1* solved above, $\sum_{n=1}^{\infty} \sin(2^{-n}t)$ converges uniformly in $t \in I$ and provides a differentiable function $f : I \to \mathbb{R}$ such that $f'(t) = \sum_{n=1}^{\infty} 2^{-n}\cos(2^{-n}t)$. Note that $f'(t)$ is continuous since it a sum of continuous functions (cosines), thus f is continuously differentiable. ∎

3 Problem 6.5.1

1. Let $f : U \to Z$ be differentiable at $x \in U$ in direction $v \in X$. Further assume there exist some $\epsilon > 0$ and $\Lambda > 0$ such that $U_\epsilon(x) \subseteq U$ and $||f(y) - f(x)|| \leq \Lambda||y - x||$ for all $y \in U_\epsilon(x)$.

   Show: $||\partial f(x, v)|| \leq \Lambda||v||$.

Solution:

Proof. Let X and Z be normed vector spaces and U an open subset of X. Let $x \in U$ and $v \in X$. Since U is open, there exists some $\epsilon > 0$ such that $U_\epsilon(x) \subseteq U$. Set $\delta \in (0, \frac{\epsilon}{1+||v||})$. Then $x + tv \in U$ for all $t \in (-\delta, \delta)$. Starting with *Definition 6.17* of the directional derivative $\partial f(x, v)$ we get

$$\begin{aligned} ||\partial f(x,v)|| &= \left|\left| \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} \right|\right| \\ &= \lim_{t\to 0} \left|\left| \frac{f(x+tv) - f(x)}{t} \right|\right| \\ &= \lim_{t\to 0} \frac{||f(x+tv) - f(x)||}{|t|} \ . \end{aligned}$$

According to the problem, there exists some $\epsilon > 0$ (same as the one defined above) and $\Lambda > 0$ such that $U_\epsilon(x) \subseteq U$ and

$$\begin{aligned} \lim_{t\to 0} \frac{||f(x+tv) - f(x)||}{|t|} &\leq \lim_{t\to 0} \frac{\Lambda||x + tv - x||}{|t|} \\ &= \lim_{t\to 0} \frac{\Lambda||tv||}{|t|} \\ &= \lim_{t\to 0} \frac{\Lambda|t|||v||}{|t|} \\ &= \Lambda||v|| \ , \end{aligned}$$

for all $y = x + tv \in U_\epsilon(x)$, which implies that $||x + tv - x|| = ||tv|| < \epsilon$. Thus,

$$||\partial f(x,v)|| \leq \Lambda||v|| \ \ \forall v \in X \text{ with } ||v|| < \frac{\epsilon}{|t|} \ .$$

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4 Problem 6.5.2

1. Let $f : U \to Z$ be Frechet differentiable at $x \in U$. Further assume there exist some $\epsilon > 0$ and $\Lambda > 0$ such that $U_\epsilon(x) \subseteq U$ and $||f(y) - f(x)|| \leq \Lambda ||y - x||$ for all $y \in U_\epsilon(x)$.

   Show: The operator norm of $Df(x)$ satisfies $||Df(x)|| \leq \Lambda$.

Solution:

Proof. Let X and Z be normed vector spaces and U an open subset of X. Let $x \in U$ and $v \in X$. Since U is open, there exists some $\epsilon > 0$ such that $U_\epsilon(x) \subseteq U$. Set $\delta \in (0, \frac{\epsilon}{1+||v||})$. Then $x + tv \in U$ for all $t \in (-\delta, \delta)$. Lastly, let $\Lambda > 0$.

Since f is Frechet differentiable at $x \in U$, by *Theorem 6.23* $f$is Gateaux differentiable at x and $\partial f(x, v) = Df(x)v$ for all $v \in X$. By the definition of Gateaux differentiable, f is differentiable at x in the direction of every v. By *Problem 6.5.1* solved above,

$$||Df(x)v|| = ||\partial f(x, v)|| \leq \Lambda ||v|| \ ,$$

which gives us

$$\frac{||Df(x)v||}{||v||} \leq \Lambda \ \forall v \neq 0.$$

Therefore Λ constitutes an upper bound of the quotient above. Since $Df(x)$ is a bounded linear operator, by *Lemma 5.3*

$$||Df(x)|| = \sup \left\{ \frac{||Df(x)v||}{||v||}; \forall v \in X, v \neq 0 \right\} \ .$$

Then, by the definition of a supremum,

$$||Df(x)|| \leq \Lambda \ .$$

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Acknowledgements

The proofs in this homework assignment have been worked and written in close collaboration with Camille Moyer.