Numerical Methods for PDEs
Homework 5

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Problem 1

Derive the modified PDE for the Lax-Friedrichs method for $u_t + cu_x = 0$. Find the coefficient $D_n \sim \{\Delta t, \Delta x\}$ of numerical diffusion in $u_t + cu_x = D_n u_{xx}$. Note that $D_n \geq 0$ iff the Courant number $r = c\Delta t/\Delta x \leq 1$.

Solution: We start Taylor expanding the following terms

$$
\begin{aligned}
u_i^{n+1} &= u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \dots \\
u_{i\pm 1}^n &= u_i^n \pm \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + \dots
\end{aligned}
$$

and substituting them into the Lax-Friedrichs method,

$$
\begin{aligned}
u_i^{n+1} &= \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right), \\
u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} &= \frac{1}{2}\left(2u_i^n + \Delta x^2 u_{xx}\right) - \frac{c\Delta t}{2\Delta x} 2\Delta x u_x + \dots, \\
u_t + \frac{\Delta t}{2} u_{tt} &\approx \frac{\Delta x^2}{2\Delta t} u_{xx} - cu_x, \\
u_t + cu_x &\approx \frac{\Delta x^2}{2\Delta t} u_{xx} - \frac{\Delta t}{2} u_{tt}.
\end{aligned}
$$

Using the PDE, we obtain that $u_{tt} = c^2 u_{xx}$. Thus,

$$
\begin{aligned}
u_t + cu_x &\approx \frac{\Delta x^2}{2\Delta t} u_{xx} - c^2 \frac{\Delta t}{2} u_{xx} \\
u_t + cu_x &\approx \frac{\Delta x^2}{2\Delta t}\left[1 - r^2\right] u_{xx},
\end{aligned}
$$

where $r = c\Delta t/\Delta x$. Hence, we have obtained that the modified PDE is

$$u_t + cu_x = D_n u_{xx},$$

with

$$D_n = \frac{\Delta x^2}{2\Delta t}\left[1 - r^2\right].$$

Note that

$$
\begin{aligned}
D_n \geq 0 &\iff \left[1 - r^2\right] \geq 0 \\
&\iff r^2 \leq 1 \\
&\iff r \leq 1
\end{aligned}
$$

Problem 2

Show that the Lax-Friedrichs method is first order (using the definition of the LTE)

Solution: We can retake the following equation from Problem 1,

$$
u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} = \frac{1}{2}\left(2u_i^n + \Delta x^2 u_{xx}\right) - \frac{c\Delta t}{2\Delta x} 2\Delta x u_x + \ldots,
$$

and use the definition of LTE

$$
\begin{aligned}
u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} &= \frac{1}{2}\left(2u_i^n + \Delta x^2 u_{xx}\right) - \frac{c\Delta t}{2\Delta x} 2\Delta x u_x + \Delta t \tau, \\
\Delta t u_t + \frac{\Delta t^2}{2} u_{tt} &= \frac{\Delta x^2}{2} u_{xx} - c\Delta t u_x + \Delta t \tau, \\
\Delta t \tau &= \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} - \frac{\Delta x^2}{2} u_{xx} + c\Delta t u_x, \\
\Delta t \tau &= \Delta t \cancelto{0}{(u_t + cu_x)} + \frac{\Delta t^2}{2} u_{tt} - \frac{\Delta x^2}{2} u_{xx},
\end{aligned}
$$

where we have used the PDE to cancel the term $u_t + cu_x$. Thus, the Lax-Friedrichs is indeed first order in time since

$$
\tau = \frac{\Delta t}{2} u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx}.
$$

Show that the Lax-Friedrichs method is conditionally stable (using von Neumann stability analysis) for $u_t + cu_x = 0$. *Hint for stability analysis:* Show $|G(k)|^2 = G^*(k)G(k) \leq 1$ iff $r \leq 1$.

Solution: By substituting the definitions

$$
u_j^n = e^{ikx_j}, \qquad u_j^{n+1} = G(k)e^{ikx_j},
$$

into the Lax-Friedrichs scheme we obtain

$$
G(k)e^{ikx_j} = \frac{1}{2}\left(e^{ik(x_j+\Delta x)} + e^{ik(x_j-\Delta x)}\right) - c\frac{\Delta t}{2\Delta x}\left(e^{ik(x_j+\Delta x)} - e^{ik(x_j-\Delta x)}\right).
$$

Dividing by e^{ikx_j},

$$G(k) = \frac{1}{2}\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - c\frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$
$$G(k) = \cos(k\Delta x) - c\frac{\Delta t}{\Delta x} i \sin(k\Delta x).$$

Finally,

$$\begin{aligned}|G(k)|^2 &= G^*(k)G(k)\\ &= \cos^2(k\Delta x) + c^2\frac{\Delta t^2}{\Delta x^2}\sin^2(k\Delta x)\\ &= \cos^2(k\Delta x) + r^2\sin^2(k\Delta x)\\ &= 1 - \left(1 - r^2\right)\sin^2(k\Delta x).\end{aligned}$$

Therefore,

$$\begin{aligned}|G(k)|^2 \le 1 &\iff 1 - \left(1 - r^2\right)\sin^2(k\Delta x) \le 1\\ &\iff \left(1 - r^2\right)\sin^2(k\Delta x) \ge 0\\ &\iff \left(1 - r^2\right) \ge 0\\ &\iff r^2 \le 1\\ &\iff r \le 1.\end{aligned}$$

Hence, Lax-Friedrichs method is conditionally stable for $u_t + cu_x = 0$.

Problem 3

Show that Lax-Friedrichs is conservative by verifying that the numerical flux function

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(f(w_i) + f(w_{i+1})\right) - \frac{\Delta x}{2\Delta t}\left(w_{i+1} - w_i\right)$$

correctly produces the Lax-Friedrichs method for $w_t + f(w)_x = 0$.

Solution:

Problem 4

Using von Neumann stability analysis, show downwind is unconditionally unstable for $u_t + cu_x = 0$. *Hint:* Show $|G(k)|^2 = G^*(k)G(k) > 1$ for any value of $r > 0$.

Solution:

Problem 5

Using von Neumann stability analysis, show that Lax-Wendroff is stable for $u_t + cu_x = 0$ as long as the CFL condition $r \leq 1$ is satisfied. *Hint:* Show $|G(k)|^2 = G^*(k)G(k) \leq 1$ iff $r \leq 1$.

Solution: