Fourier Analysis and Wavelets
Homework 5

Francisco Jose Castillo Carrasco

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Problem 2

Let ϕ and ψ be the Haar scaling and wavelet functions, respectively. Let V_j and W_j be the spaces generatd by $\phi(2^k x - k), k \in \mathbb{Z}$, and $\psi(2^j x - k), k \in \mathbb{Z}$, respectively. Consider the function defined on $0 \leq x \leq 1$ given by

$$f(x) = \begin{cases} 2 & 0 \leq x < 1/4, \\ -3 & 1/4 \leq x < 1/2, \\ 1 & 1/2 \leq x < 3/4, \\ 3 & 3/4 \leq x < 1. \end{cases}$$

Express f first in terms of the basis for V_2 and then decompose f into its component parts in W_1, W_0 and V_0. In other words, find the Haar wavelet decomposition for f. Sketch each of these components.

Solution: The function f is easily expressed as

$$f(x) = 2\phi(4x) - 3\phi(4x-1) + \phi(4x-2) + 3\phi(4x-3) \in V_2$$

For the decomposition we use the following relations

$$\phi\left(2^j x\right) = \frac{\phi\left(2^{j-1}x\right) + \psi\left(2^{j-1}x\right)}{2},$$
$$\phi\left(2^j x - 1\right) = \frac{\phi\left(2^{j-1}x\right) - \psi\left(2^{j-1}x\right)}{2}.$$

For this case we use

$$\phi\left(4x\right) = \frac{\phi\left(2x\right) + \psi\left(2x\right)}{2},$$
$$\phi\left(4x-1\right) = \frac{\phi\left(2x\right) - \psi\left(2x\right)}{2},$$
$$\phi\left(4x-2\right) = \phi\left(4(x-1/2)\right) = \frac{\phi\left(2x-1\right) + \psi\left(2x-1\right)}{2},$$
$$\phi\left(4x-3\right) = \phi\left(4(x-1/2)-1\right) = \frac{\phi\left(2x-1\right) - \psi\left(2x-1\right)}{2}.$$

With these relations we decompose f into its components in W_1 and V_1:

$$f(x) = \frac{5}{2}\psi\left(2x\right) - \psi\left(2x-1\right) - \frac{1}{2}\phi\left(2x\right) + 2\phi\left(2x-1\right) \in W_1 \bigoplus V_1.$$

Next, we decompose the V_1 component, $f_{V_1} = -\frac{1}{2}\phi(2x) + 2\phi(2x-1)$, into its components in W_0 and V_0 using

$$\phi(2x) = \frac{\phi(x) + \psi(x)}{2},$$
$$\phi(2x-1) = \frac{\phi(x) - \psi(x)}{2}.$$

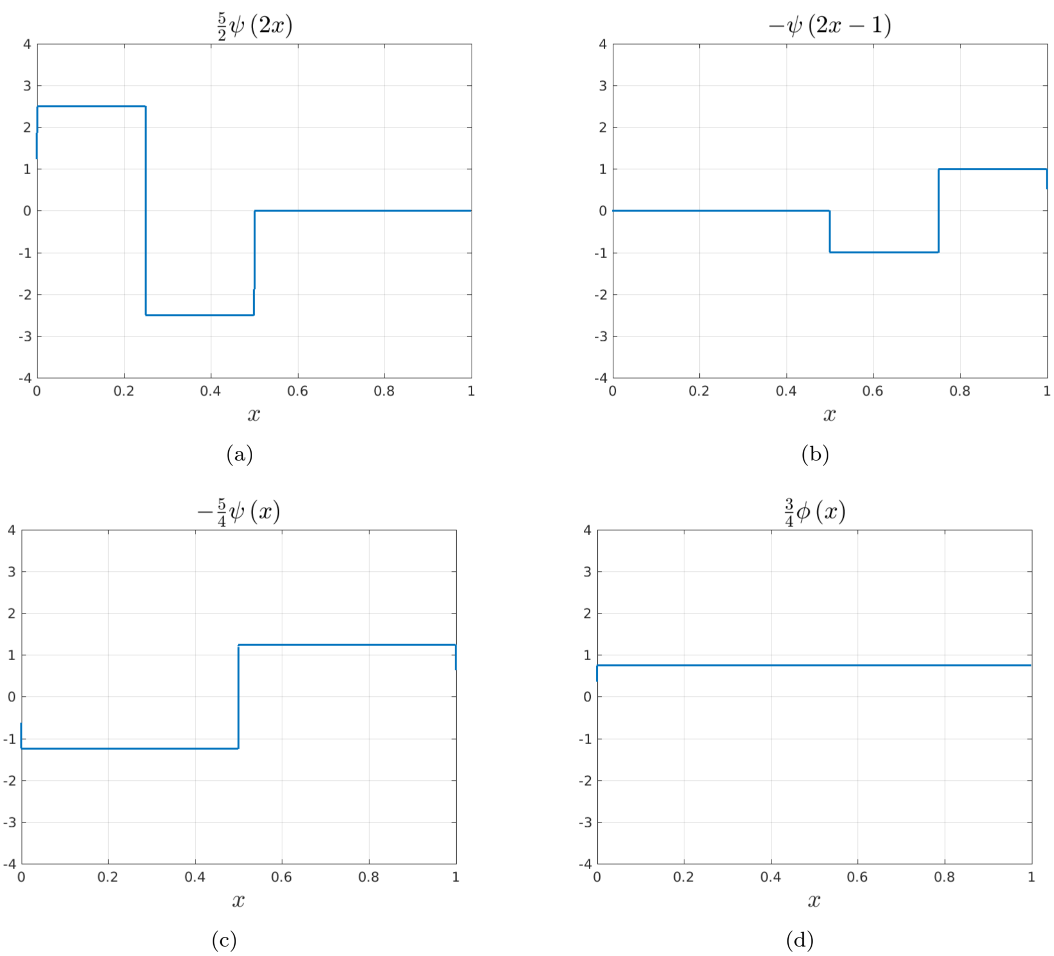
Then,

$$f_{V_1} = -\frac{5}{4}\psi(x) + \frac{3}{4}\phi(x) \in W_0 \bigoplus V_0.$$

Hence,

$$f(x) = \frac{5}{2}\psi(2x) - \psi(2x-1) - \frac{5}{4}\psi(x) + \frac{3}{4}\phi(x) \in W_1 \bigoplus W_0 \bigoplus V_0.$$

Every component of the function and the function itself are graphed in the following figures.



(a) (b) (c) (d)

Figure 1: Haar components of the function g.

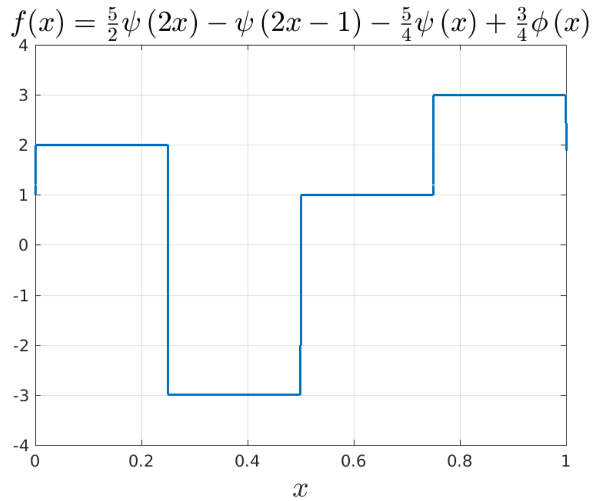


Figure 2: Function $g = \sum_{k=0}^{7} a_k^3 \phi(8x-k)$.

Problem 4

Let V_n be the spaces generated by $\phi(2^n x - k), k \in \mathbb{Z}$, where ϕ is the Haar scaling function.

- On the interval $0 \leq x \leq 1$, what are the dimensions of the spaces V_n and W_n for $n \geq 0$?

Solution: The functions $\phi(2^n x - k), k \in \mathbb{Z}$ represent step functions of width 2^{-n}. Given the interval $0 \leq x \leq 1$, it is inmediate that we need 2^n functions to cover the whole interval. More formally, by *Theorem 4.6*, we know that $\{2^{n/2}\phi(2^n x - k), k \in \mathbb{Z}\}$ is a basis for V_n and $\{2^{n/2}\psi(2^n x - k), k \in \mathbb{Z}\}$ is a basis for W_n. In this case, because of the given interval, $k = 0, \ldots, 2^n - 1$, which makes the dimension of both spaces V_n and W_n equal to 2^n.

- Using the result

$$\dim(A \bigoplus B) = \dim(A) + \dim(B),$$

count the dimension of the space on the right side of the equality

$$V_n = W_{n-1} \bigoplus W_{n-2} \bigoplus \cdots \bigoplus W_0 \bigoplus V_0$$

Solution: We proceed using the previous relation

$$\begin{aligned}
\dim(V_n) &= \dim(W_{n-1}) + \dim(W_{n-2}) + \cdots + \dim(W_1) + \dim(W_0) + \dim(V_0), \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 + 2^0, \\
&= 2^0 + \sum_{k=0}^{n-1} 2^k = 1 + 2^n - 1 = 2^n,
\end{aligned}$$

we get the same result as in the first part.

Problem 6

Reconstruct $g \in V_3$, given these coefficients in its Haar wavelet decomposition:

$$a^2 = [1/2, 2, 5/2, -3/2],$$
$$b^2 = [-3/2, -1, 1/2, -1/2].$$

The first entry in each list corresponds to $k = 0$. Sketch g.

Solution: It is easy to reconstruct $g \in V_3$ using the relations from *Theorem 4.14*,

$$g(x) = \sum_{k=0}^{2^3-1} a_k^3 \phi(2^3 x - k) \in V_3,$$

where

$$a_k^3 = \begin{cases} a_l^2 + b_l^2 & \text{if } k = 2l \text{ is even,} \\ a_l^2 - b_l^2 & \text{if } k = 2l+1 \text{ is odd.} \end{cases}$$

Therefore,

$$a^3 = [a_0^2 + b_0^2, a_0^2 - b_0^2, a_1^2 + b_1^2, a_1^2 - b_1^2, a_2^2 + b_2^2, a_2^2 - b_2^2, a_3^2 + b_3^2, a_3^2 - b_3^2]$$
$$a^3 = [-1, 2, 1, 3, 3, 2, -2].$$

Hence,

$$g(x) = \sum_{k=0}^{7} a_k^3 \phi(8x - k),$$

which we can expand as

$$g(x) = -\phi\left(8x\right) + 2\phi\left(8x - 1\right) + \phi\left(8x - 2\right) + 3\phi\left(8x - 3\right) + 3\phi\left(8x - 4\right) + 2\phi\left(8x - 5\right) - 2\phi\left(8x - 6\right) - \phi\left(8x - 7\right),$$

and it is graphed in the next figure.

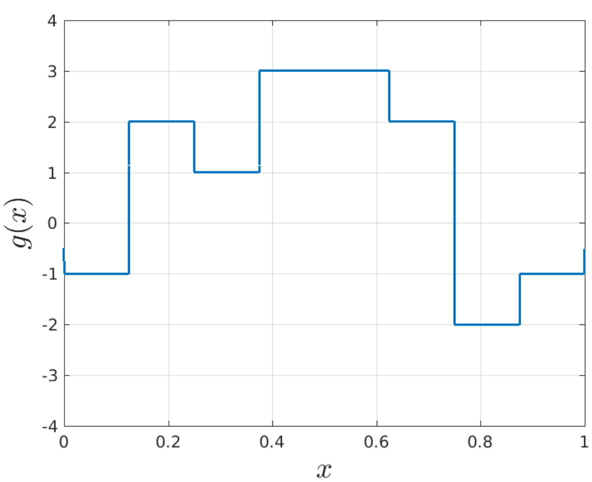


Figure 3: Function $g = \sum_{k=0}^{7} a_k^3 \phi(8x - k)$.

Problem 9

Let

$$f(t) = e^{-t^2/10} \left(\sin(2t) + 2\cos(4t) + 0.4 \sin(t) \sin(50t) \right).$$

Discretize the function f over the interval $0 \le t \le 1$ as described in Step 1of Section 4.4. Use $n = 8$ as the top level (so there are 2^8 nodes in the discretization). Implement the decomposition algorithm described in Step 2 of Section 4.4 using the Haar wavelets. Plot the resulting levels, $f_{j-1} \in V_{j-1}$ for $j = 8 \ldots 1$ and compare with the original signal.

Solution: *The solution to this problem below, with the Matlab code.*

Problem 10

Filter the wavelet coefficients computed in exercise 9 by setting to zero any wavelet coefficient whose absolute value is less than $tol = 0.1$. Then reconstruct the signal as described in Section 4.4. Plot the reconstructed f_8 and compare with the original signal. Compute the relative l^2 difference between the original signal and the compressed signal. Experiment with various tolerances. Keep track of the percentage of wavelet coefficients that have been filetered out.

Solution: *The solution to this problem below, with the Matlab code.*