

Spectral Methods

Francisco Castillo

Homework 1

February 5, 2019

Problem 1

Show that the trapezoidal rule

$$\int_0^{2\pi} f(x) dx = \frac{2\pi}{N} \sum_{j=0}^{N-1} f(x_j),$$

where $x_j = 2\pi j/N$, is exact for $f(x) = \exp(inx)$ for $|n| < N$ (but not for $|n| = N$). Conclude that the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n| < N}$.

First, we compute the integral,

$$\int_0^{2\pi} e^{inx} dx = 2\pi\delta_{n0},$$

where δ is the Kronecker-delta. For the right hand side we start considering the case where $n < |N|$ and $n \neq 0$,

$$\begin{aligned} \frac{2\pi}{N} \sum_{j=0}^{N-1} f(x_j) &= \frac{2\pi}{N} \sum_{j=0}^{N-1} e^{inx_j} \\ &= \frac{2\pi}{N} \sum_{j=0}^{N-1} e^{\left(\frac{i2\pi nj}{N}\right)} \\ &= \frac{2\pi}{N} \sum_{j=0}^{N-1} \left[e^{\left(\frac{i2\pi n}{N}\right)} \right]^j \\ &= \left(\frac{2\pi}{N} \right) \left(\frac{1 - e^{i2\pi n}}{1 - e^{\left(\frac{i2\pi n}{N}\right)}} \right) \\ &= 0, \quad \forall n < |N|, n \neq 0, \end{aligned}$$

where we have applied the geometric sum formula and that $e^{i2\pi n} = 1$. We note that, if $n = N$ the solution would not be defined. Lastly, for $n = 0$,

$$\frac{2\pi}{N} \sum_{j=0}^{N-1} f(x_j) = \frac{2\pi}{N} \sum_{j=0}^{N-1} 1 = \frac{2\pi}{N} N = 2\pi .$$

Finally, let g be a function in the span $\{e^{inx}\}_{|n|<N}$. Hence, we can write

$$g(x) = \sum_{|n|<N} c_n e^{inx},$$

with c_n being constant coefficients. Then,

$$\begin{aligned} \int_0^{2\pi} g(x) dx &= \int_0^{2\pi} \sum_{|n|<N} c_n e^{inx} dx \\ &= \sum_{|n|<N} c_n \left(\int_0^{2\pi} e^{inx} dx \right) \\ &= \sum_{|n|<N} c_n \left(\frac{2\pi}{N} \sum_{j=0}^{N-1} e^{inx_j} \right) \\ &= \frac{2\pi}{N} \sum_{j=0}^{N-1} \sum_{|n|<N} c_n e^{inx_j} \\ &= \frac{2\pi}{N} \sum_{j=0}^{N-1} g(x_j) . \end{aligned}$$

Thus, the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

Problem 2

(*Best Approximation*) Prove the following statements.

- (a) Let e_1, \dots, e_N be an orthonormal system in an inner product space H , let $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ and let $f \in H$. Then

$$\left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 = \|f\|^2 + \sum_{j=1}^N |\lambda_j - c_j|^2 - \sum_{j=1}^N |c_j|^2,$$

where $c_j = \langle f, e_j \rangle$ and $\|\cdot\|$ is the norm induced by the inner product.

To prove that the statement we start by writing the left side of the equation as an inner product,

$$\left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 = \left\langle f - \sum_{j=1}^N \lambda_j e_j, f - \sum_{j=1}^N \lambda_j e_j \right\rangle.$$

By applying linearity and antilinearity of the inner product we get

$$\begin{aligned} \left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 &= \left\langle f - \sum_{j=1}^N \lambda_j e_j, f - \sum_{j=1}^N \lambda_j e_j \right\rangle \\ &= \langle f | f \rangle - \sum_{j=1}^N \langle \lambda_j e_j | f \rangle - \sum_{j=1}^N \langle f | \lambda_j e_j \rangle + \sum_{j=1}^N \langle \lambda_j e_j | \lambda_j e_j \rangle \\ &= \|f\|^2 - \sum_{j=1}^N \lambda_j \langle e_j | f \rangle - \sum_{j=1}^N \lambda_j^* \langle f | e_j \rangle + \sum_{j=1}^N \langle \lambda_j e_j | \lambda_j e_j \rangle \\ &= \|f\|^2 - \sum_{j=1}^N \lambda_j c_j^* - \sum_{j=1}^N \lambda_j^* c_j + \sum_{j=1}^N \|\lambda_j\|^2. \end{aligned}$$

where we have used the linearity and antilinearity properties again, and that $c_j = \langle f | e_j \rangle$. By adding and subtracting $\sum_{j=1}^N \langle c_j | c_j \rangle$ and rewriting the previous equation in terms of inner product again we get

$$\begin{aligned} \left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 &= \|f\|^2 - \sum_{j=1}^N \langle \lambda_j | c_j \rangle - \sum_{j=1}^N \langle c_j | \lambda_j \rangle + \sum_{j=1}^N \langle \lambda_j | \lambda_j \rangle \\ &\quad + \sum_{j=1}^N \langle c_j | c_j \rangle - \sum_{j=1}^N \langle c_j | c_j \rangle, \end{aligned}$$

which, again by linearity and antilinearity, we can group together as

$$\begin{aligned}
\left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 &= \|f\|^2 + \sum_{j=1}^N \langle \lambda_j - c_j | \lambda_j \rangle - \sum_{j=1}^N \langle \lambda_j - c_j | c_j \rangle - \sum_{j=1}^N \langle c_j | c_j \rangle \\
&= \|f\|^2 + \sum_{j=1}^N \langle \lambda_j - c_j | \lambda_j - c_j \rangle - \sum_{j=1}^N \langle c_j | c_j \rangle \\
&= \|f\|^2 + \sum_{j=1}^N |\lambda_j - c_j|^2 - \sum_{j=1}^N |c_j|^2.
\end{aligned}$$

Thus, we have proven that

$$\left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 = \|f\|^2 + \sum_{j=1}^N |\lambda_j - c_j|^2 - \sum_{j=1}^N |c_j|^2.$$

(b) **Let** $f_N = \sum_{j=1}^N \langle f, e_j \rangle e_j$. **Then** $\|f - f_N\| \leq \|f - g\|$ **for all** g **in the span of** e_1, \dots, e_N .

To prove that

$$\|f - f_N\| \leq \|f - g\|,$$

we will show that their squares

$$\|f - f_N\|^2 \leq \|f - g\|^2.$$

Taking into account that $f_N = \sum_{j=1}^N \langle f | e_j \rangle e_j$ and $g = \sum_{j=1}^N \lambda_j e_j$ and using the previous proof we have that

$$\begin{aligned}
\|f - f_N\|^2 &= \left\| f - \sum_{j=1}^N \langle f | e_j \rangle e_j \right\|^2 \\
&= \|f\|^2 + \sum_{j=1}^N |\langle f | e_j \rangle - c_j|^2 - \sum_{j=1}^N |c_j|^2 \\
&= \|f\|^2 + \sum_{j=1}^N |\langle f | e_j \rangle - \langle f | e_j \rangle|^2 - \sum_{j=1}^N |c_j|^2 \\
&= \|f\|^2 - \sum_{j=1}^N |\langle f | e_j \rangle|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|f - g\|^2 &= \left\| f - \sum_{j=1}^N \lambda_j e_j \right\|^2 \\
&= \|f\|^2 + \sum_{j=1}^N |\lambda_j - c_j|^2 - \sum_{j=1}^N |c_j|^2 \\
&= \|f\|^2 + \sum_{j=1}^N |\lambda_j - \langle f | e_j \rangle|^2 - \sum_{j=1}^N |c_j|^2 \\
&\geq \|f\|^2 - \sum_{j=1}^N |\langle f | e_j \rangle|^2 \\
&= \|f - f_N\|^2.
\end{aligned}$$

Thus, we have proved that

$$\|f - f_N\| \leq \|f - g\|$$

Problem 3

(*Fourier-Galerkin method*) Let \mathcal{L} be a differential operator of the form $\mathcal{L} = \sum_{k=0}^m \alpha_k \frac{d^k}{dx^k}$, with $\alpha_k \in \mathbb{C}$. Suppose we are interested in solving the differential equation $\mathcal{L}u = g$. Let

$$u_N = \sum_{n=-N/2}^{N/2} c_n e^{inx},$$

where the coefficients c_n are chosen so that $\langle \mathcal{L}u_N - g, e^{ikx} \rangle = 0$ for $k = -N/2, \dots, N/2$. Show that $\|\mathcal{L}u_N - g\| \leq \|\mathcal{L}w - g\|$ for all w in the span of $\{e^{inx}\}_{|n| \leq N/2}$.

Let's first compute the result of

$$\begin{aligned} \mathcal{L}u_N &= \sum_{k=0}^m \alpha_k \frac{d^k}{dx^k} \left[\sum_{n=-N/2}^{N/2} c_n e^{inx} \right] \\ &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^m \alpha_k (in)^k e^{inx}. \end{aligned}$$

Before we continue, we work on the other piece of information, $\langle \mathcal{L}u_N - g, e^{ikx} \rangle = 0$, which implies

$$\langle \mathcal{L}u_N, e^{ilx} \rangle = \langle g, e^{ilx} \rangle,$$

where we have used the index l for future convenience. Now,

$$\begin{aligned} \langle \mathcal{L}u_N, e^{ilx} \rangle &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^m \alpha_k (in)^k \langle e^{inx}, e^{ilx} \rangle \\ &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^m \alpha_k (in)^k 2\pi \delta_{nl} \\ &= 2\pi c_l \sum_{k=0}^m \alpha_k (il)^k \\ &= \langle g, e^{ilx} \rangle. \end{aligned}$$

Hence, retaking the index n ,

$$\langle \mathcal{L}u_N, e^{inx} \rangle = 2\pi c_n \sum_{k=0}^m \alpha_k (in)^k = \langle g, e^{inx} \rangle.$$

Further, we can rewrite $\mathcal{L}u_N$ as

$$\begin{aligned}
\mathcal{L}u_N &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^m \alpha_k(in)^k e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N/2}^{N/2} 2\pi c_n \sum_{k=0}^m \alpha_k(in)^k e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N/2}^{N/2} \langle g, e^{inx} \rangle e^{inx} \\
&= \sum_{n=-N/2}^{N/2} \langle g, \frac{e^{inx}}{\sqrt{2\pi}} \rangle \frac{e^{inx}}{\sqrt{2\pi}} \\
&= \sum_{n=-N/2}^{N/2} \langle g, e_n \rangle e_n,
\end{aligned}$$

where we are denoting $e_n = \frac{e^{inx}}{\sqrt{2\pi}}$ as the orthonormal vectors. Note that the set of $\{e_n\}_{|n| \leq N/2}$ form an orthonormal base. Since $\mathcal{L}u_N$ can be written as

$$\mathcal{L}u_N = \sum_{n=-N/2}^{N/2} \langle g, e_n \rangle e_n,$$

by the previous result in problem 2, $\|\mathcal{L}u_N - g\| \leq \|\mathcal{L}w - g\|$ for all g in the span of e_1, \dots, e_N .

Problem 4

```
%% Problem 4
N = 20:10:500;
x = chebfun('x',[0 2*pi]);
f = sin(x/2).^3;

figure
plot(f,'linewidth',2)
grid on
xlabel('$x$', 'interpreter','latex')
ylabel('$f(x)=\sin^3(x/2)$', 'interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P4_1';
saveas(gcf,txt,figformat)

for j = 1:length(N)
    A = exp(1i*x*(-N(j):N(j)));
    lambda = 1/(2*pi)*A'*f;
    fn = A*lambda;
    err(j) = norm(f-fn,Inf);
end

figure
loglog(err,'*', 'MarkerSize',12)
grid on
xlabel('$N$', 'interpreter','latex')
ylabel('Error', 'interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P4_2';
saveas(gcf,txt,figformat)
```


Problem 5

```
%% Problem 5
N = 20:10:500;
x = chebfun('x',[0 2*pi]);
f = 3./(5-4*cos(x));

figure
plot(f,'linewidth',2)
grid on
xlabel('$x$', 'interpreter','latex')
ylabel('$f(x)=\frac{3}{5-4\cos(x)}$', 'interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P5_1';
saveas(gcf,txt,figformat)

for j = 1:length(N)
    A = exp(1i*x*(-N(j):N(j)));
    lambda = 1/(2*pi)*A'*f;
    fn = A*lambda;
    err(j) = norm(f-fn,Inf);
end

figure
loglog(err,'*', 'MarkerSize',12)
grid on
xlabel('$N$', 'interpreter','latex')
ylabel('Error', 'interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P5_2';
saveas(gcf,txt,figformat)
```