

Fourier Analysis and Wavelets

Homework 4

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Problem 2

Prove THEOREM 3.4.

- Shifts or translations. If $y \in S_n$ and $z_k = y_{k+1}$, then $\mathcal{F}[z]_j = w^j \mathcal{F}[y]_j$, where $w = e^{i\frac{2\pi}{n}}$.

Solution: This is simply proved by calculating the Discrete Fourier Transform of z_k .

$$\begin{aligned}
 \mathcal{F}[z]_k &= \sum_{j=0}^{n-1} z_j \bar{w}^{jk} \\
 &= \sum_{j=0}^{n-1} y_{j+1} \bar{w}^{jk} \quad \text{making } l = j + 1 \\
 &= \sum_{l=1}^n y_l \bar{w}^{(l-1)k} \\
 &= \sum_{l=1}^n y_l \bar{w}^{lk} \bar{w}^{-k} \\
 &= \bar{w}^{-k} \sum_{l=1}^n y_l \bar{w}^{lk} \\
 &= \bar{w}^{-k} \sum_{l=0}^{n-1} y_l \bar{w}^{lk} \\
 &= w^k \mathcal{F}[y]_k,
 \end{aligned}$$

where in the last step we have applied the definition of the *DFT* and not that $\bar{w}^{-k} = w^k$.

- Convolutions. If $y \in S_n$ and $z \in S_n$, then the sequence defined by

$$[y * z]_k := \sum_{j=0}^{n-1} y_j z_{k-j}$$

is also in S_n .

Solution: To prove periodicity of the convolution we compute

$$[y * z]_{k+n} = \sum_{j=0}^{n-1} y_j z_{k-j+n} = \sum_{j=0}^{n-1} y_j z_{k-j} = [y * z]_k,$$

where we have used that $z \in S_n$. Hence, $[y * z]_k \in S_n$.

- The Convolution Theorem. $\mathcal{F}[y * z]_k = \mathcal{F}[y]_k \mathcal{F}[z]_k$

Solution: This is easily proved starting from the *DFT* of the convolution,

$$\begin{aligned} \mathcal{F}[y * z]_k &= \sum_{j=0}^{n-1} [y * z]_j \bar{w}^{jk} \\ &= \sum_{j=0}^{n-1} \sum_{m=0}^{n-1} y_m z_{j-m} \bar{w}^{jk} \\ &= \sum_{j=0}^{n-1} \sum_{m=0}^{n-1} y_m z_{j-m} \bar{w}^{(j-m)k} \bar{w}^{mk} \\ &= \sum_{m=0}^{n-1} y_m \bar{w}^{mk} \sum_{j=0}^{n-1} z_{j-m} \bar{w}^{(j-m)k}. \end{aligned}$$

Applying periodicity to the second sum, $\sum_{j=0}^{n-1} z_{j-m} \bar{w}^{(j-m)k} = \sum_{j=0}^{n-1} z_j \bar{w}^{jk}$, we obtain the desired result:

$$\mathcal{F}[y * z]_k = \sum_{m=0}^{n-1} y_m \bar{w}^{mk} \sum_{j=0}^{n-1} z_j \bar{w}^{jk} = \mathcal{F}[y]_k \mathcal{F}[z]_k$$

- If $y \in S_n$ is a sequence of real numbers, then

$$\mathcal{F}[y]_{n-k} = \overline{\mathcal{F}[y]_k}, \quad \text{for } 0 \leq k - n \leq n - 1.$$

Solution: Like in the previous cases, computations suffice to prove the statement. We start with

$$\mathcal{F}[y]_{n-k} = \sum_{j=0}^{n-1} y_j \bar{w}^{j(n-k)} = \sum_{j=0}^{n-1} y_j \bar{w}^{jn} \bar{w}^{-jk}.$$

Note that $\bar{w}^{jn} = 1$. Then,

$$\mathcal{F}[y]_{n-k} = \sum_{j=0}^{n-1} y_j \bar{w}^{-jk} = \sum_{j=0}^{n-1} \overline{y_j \bar{w}^{jk}} = \overline{\sum_{j=0}^{n-1} y_j \bar{w}^{jk}} = \overline{\mathcal{F}[y]_k}.$$

Problem 7

Filtering. Let

$$f(t) = e^{-t^2/10} (\sin 2t + 2 \cos 4t + 0.4 \sin t \sin 50t).$$

Discretize f by setting $y_k = f(2k\pi/256)$, $k = 1 \dots 256$. Use the fast Fourier transform to compute \hat{y}_k for $0 \leq k \leq 256$. Filter out the high-frequency terms by setting $\hat{y}_k = 0$ for $m \leq k \leq 255 - m$ with $m = 6$; then apply the inverse fast Fourier transform to this new set of \hat{y}_k to compute the y_k (now filtered); plot the new values of y_k and compare with the original function. Experiment with other values of m .

Solution: *The solution to this problem below, with the Matlab code.*

Problem 8

Compression. Let $tol = 1.0$. In exercise 7, if $|\hat{y}_k| < tol$ then set \hat{y}_k equal to zero. Apply the inverse fast Fourier transform to this new set of \hat{y}_k to compute the y_k ; plot the new values of y_k and compare with the original function. Experiment with other values of tol . Keep track of the percentage of Fourier coefficients which have been filtered out. Compute the relative l^2 error of the compressed signal as compared with the original signal.

Solution: *The solution to this problem below, with the Matlab code.*

MATLAB CODE - FRANCISCO CASTILLO

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Preliminary Commands

```
clear all
close all
clc
linewidth=1.6;
labelfontsize=18;
legendfontsize=12;
```

Introduction

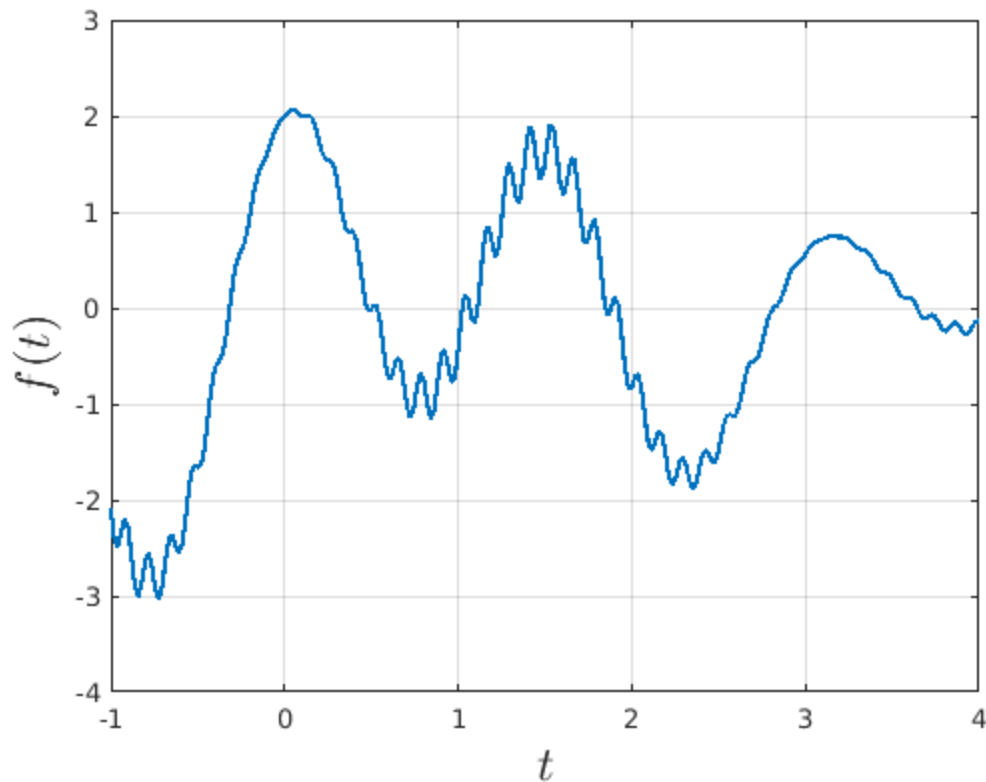
This code solves both problems 7 and 8 of the present homework assignment. As we can see, in the next section we define the function to filter (problem 7) and compress (problem 8).

Define the function

```
t=linspace(-1,4,256); % interval [-1,4] partitioned into 256 pieces
y = exp(-t.^2/10).*(sin(2*t) + 2*cos(4*t) + .4*sin(t).*sin(50*t));
```

In the following figure we can see the function that we are going to work with in this two problems.

```
figure
plot(t,y,'linewidth',linewidth);
xlabel('$t$', 'interpreter', 'latex', 'fontsize', labelfontsize)
ylabel('$f(t)$', 'interpreter', 'latex', 'fontsize', labelfontsize)
grid on
```

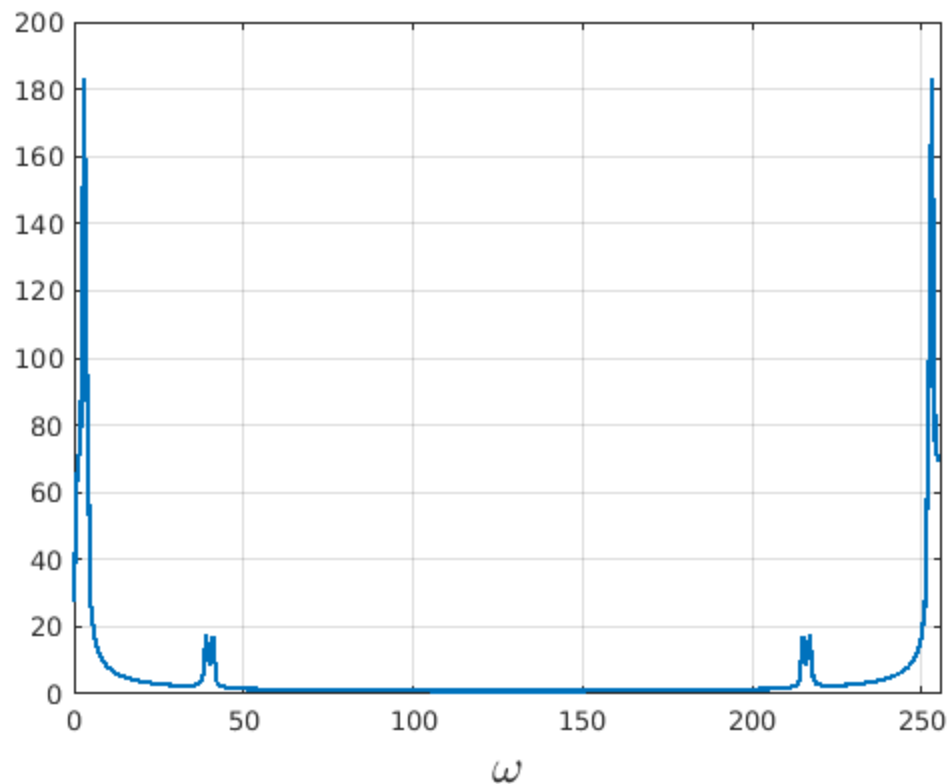


The goal in problem 7 will be filter the high frequency noise and obtain a filtered signal, which we will compare to the initial. In problem 8 we will take the filtered signal and compress by picking only the main frequencies. We will also compare it to the initial signal.

FFT Commands

In this section we simply compute the FFT of the signal f (shown in the next figure).

```
yhat=fft(y);  
freq=0:255;  
figure  
plot(freq,abs(yhat),'linewidth',linewidth); % plot of the absolute  
value of frequencies  
grid on  
xlabel('$\omega$', 'interpreter', 'latex', 'fontsize', labelfontsize)  
axis([0 256 0 200])
```



As we can see in the figure, the FFT does not give us reliable information of the high frequencies, they are simply a "copy" of the low frequencies. If we wanted to know how the FFT looks at frequency 256, we would need to sample up to $N=528$ (or more to be safe).

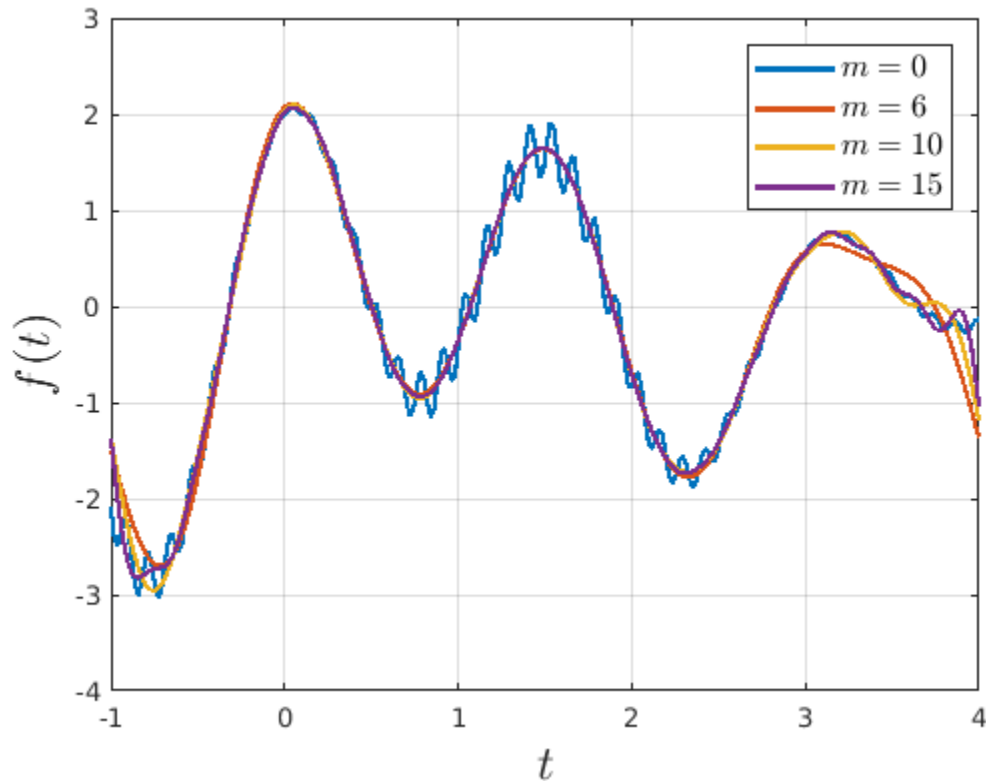
Filter out high frequencies (Problem 7)

In this section we will filter the m higher frequencies out. Once we have the filter Fourier transform of the signal, we use the inverse Fourier transform to obtain the filtered signal. In the figure, we can see the different filtered signals, depending on m , compared to the original signal $f(t)$ (corresponding to $m = 0$).

```
M=[6 10 20];
figure
plot(t,y,'linewidth',linewidth); % plot of original signal and
    filtered signal
grid on
hold on
for i=1:length(M)
    m=M(i);
    yhatf=[yhat(1:m), zeros(1,256-2*m), yhat(256-m+1:256)]; %keep
        lowes m frequencies
    yf=ifft(yhatf); % reconstruct filtered signal with ifft
    plot(t, yf,'linewidth',linewidth); % plot of original signal and
        filtered signal
end
xlabel('$t$', 'interpreter', 'latex', 'fontsize', labelfontsize)
```

```
ylabel('$f(t)$','interpreter','latex','fontsize',labelfontsize)
h1=legend('$m=0$', '$m=6$', '$m=10$', '$m=15$');
set(h1,'interpreter','latex','fontsize',legendfontsize);
```

Warning: Imaginary parts of complex X and/or Y arguments ignored
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Warning: Imaginary parts of complex X and/or Y arguments ignored



As we can see, m only makes a difference on the extremes. This is due to the function not being periodic. Hence, the filtered signals match very well for average t values.

Compression (Problem 8)

In this part we will compress the signal f . This means we will take only those frequencies whose power is above the threshold tol . If they are not, they will be set to zero.

```
TOL=[1 2 4 8];
figure
plot(t,y,'linewidth',linewidth);
xlabel('$t$', 'interpreter','latex','fontsize',labelfontsize)
ylabel('$f(t)$','interpreter','latex','fontsize',labelfontsize)
grid on
hold on
for i=1:length(TOL)
    tol=TOL(i);
```

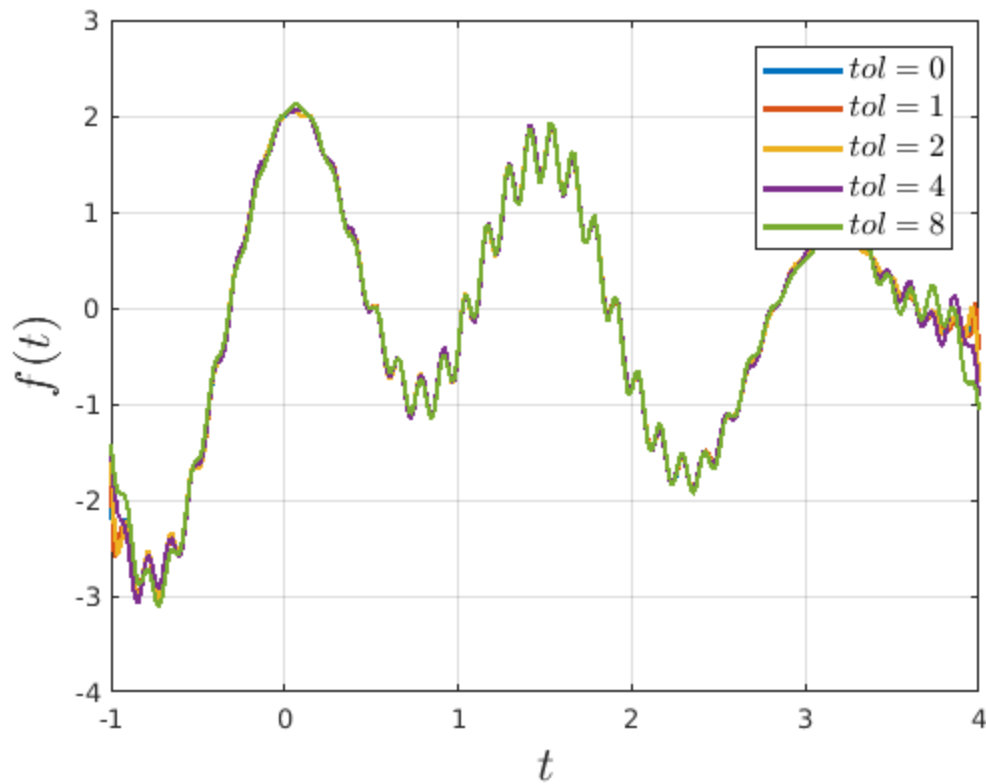
```
count(i)=0; % for compression - keep frequencies which are larger
than tol
for j=1:256
    if (abs(yhat(j))<tol)
        yhatc(j)=0;
        count(i)=count(i)+1;
    else
        yhatc(j)=yhat(j);
    end
end
yc=ifft(yhatc); %reconstruct compressed signal with ifft
plot(t,yc,'linewidth',linewidth); % plot of original and
compressed signals
end
h1=legend('$tol=0$', '$tol=1$', '$tol=2$', '$tol=4$', '$tol=8$');
set(h1,'interpreter','latex','fontsize',legendfontsize);
TOL
count          %count = number of zeroed-out terms
```

TOL =

1 2 4 8

count =

87 167 211 231



We see how, except on the extremes, all compressions match the original signal very well, even having $tol = 8$ (which results in cancelling 236 out of 256 frequencies). However, If we want to be more precise on the extremes we would have to take one of the first two values. The difference between the values of $tol 1$ and 2 is minimum. However, we cancel double as many terms using $tol = 2$. For this reason, having rather the same behaviour while having less data to store, I would pick a $tol = 2$.

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