Partial Differential Equations
TA Homework 4

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Problem 2

Let X be an inner product space and M a denumerable orthonormal subset of X. Show

(a) If M is an orthonormal basis and $x \in X$, then $\langle x|v\rangle = 0$ for all $v \in M$ implies that $x = 0$.

Solution: By *Theorem 4.10*, since M is a denumerable orthonormal basis of the inner product space X, we can express x as *Fourier* expansion,

$$x = \sum_{m=1}^{\infty} \langle u|v_m\rangle\, v_m, \quad v_m \in M.$$

Then, if $\langle u|v_m\rangle$ for every m, it is inmediate that $x = 0$.

Show:

(b) If X is a Hilbert space and if, for all $x \in X$, $\langle x|v\rangle = 0$ for all $v \in M$ implies that $x = 0$, then M is an orthonormal basis.

Solution: Let $x \in X$ and define $y = \sum_{m=1}^{\infty} \langle x|v_m\rangle\, v_m$. Since X is a Hilbert space, $y \in X$ by Problem 4.2.1. Let $v \in M$ be arbitrary but fixed. Then,

$$\begin{aligned}
\langle y - x|v\rangle &= \langle y|v\rangle - \langle x|v\rangle \\
&= \sum_{m=1}^{\infty} \langle \langle x|v_m\rangle\, v_m|v\rangle - \langle x|v\rangle \\
&= \sum_{m=1}^{\infty} \langle x|v_m\rangle \langle v_m|v\rangle - \langle x|v\rangle \\
&= \langle x|v\rangle \langle v|v\rangle - \langle x|v\rangle \\
&= \langle x|v\rangle - \langle x|v\rangle \\
&= 0,
\end{aligned}$$

where we have used orthonormality of the set M. By assumption, $\langle y - x | v \rangle = 0$ implies that $y - x = 0$. Thus,

$$x = y = \sum_{m=1}^{\infty} \langle x | v_m \rangle \, v_m.$$

Hence, since any $x \in X$ can be expressed as a *Fourier* expansion of the elements of the orthonormal set M, M is an orthonormal basis.

Problem 7

Let X be a Hilbert space. Let $M = \{v_m; m \in \mathbb{N}\}$ be an orthonormal subset of X.

Show: $\sum_{m=1}^{\infty} \langle u|v_m \rangle v_m$ converges for every $u \in X$.

Warning: This means that the Fourier series of u converges, but it may happen that it does not equal u (unless M is an orthonormal basis).

Solution: As we showed in Problem 4.2.1, $\sum_{m=1}^{\infty} \langle u|v_m \rangle v_m$ converges in X if and only if $\sum_{m=1}^{\infty} |\langle u|v_m \rangle|^2 < \infty$. To prove the latter, we define the increasing sequence of partial sums $(s_n) \in M$,

$$s_n = \sum_{m=1}^{n} |\langle u|v_m \rangle|^2 ,$$

with $s = \lim_{n\to\infty} s_n$. Now, by Bessel's Inequality,

$$s_n = \sum_{m=1}^{n} |\langle u|v_m \rangle|^2 \leq ||u||^2.$$

Therefore,

$$s = \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} ||u||^2 = ||u||^2.$$

Thus,

$$s = \sum_{m=1}^{\infty} |\langle u|v_m \rangle|^2 \leq ||u||^2 < \infty,$$

and by Problem 4.2.1 the series $\sum_{m=1}^{\infty} \langle u|v_m \rangle v_m$ converges in X.

Problem 11

Let $B = \{\cos(jx); j \in \mathbb{N}\} \cup \{\sin(jx); j \in \mathbb{N}\} \cup \{\frac{1}{\sqrt{2}}\}$.

Show that B is an orthonormal basis of $L^2([-\pi,\pi],\mathbb{R})$ with inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg$.

Hint: Use that $\{e^{ijx}; j \in Z\}$ is an orthonormal basis of $L^2([-\pi,\pi],\mathbb{C})$ and express $\sin(x)$ and $\cos(x)$ in terms of e^{ix} and e^{-ix}.

Solution: We first show that B is an orthonormal set. Let $j \neq k$,

$$\begin{aligned}
\langle \cos(jx)|\cos(kx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(jx)\cos(kx)dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{ijx}+e^{-ijx}\right)\left(e^{ikx}+e^{-ikx}\right)dx \\
&= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}e^{ijx}e^{ikx}dx + \int_{-\pi}^{\pi}e^{ijx}e^{-ikx}dx + \int_{-\pi}^{\pi}e^{-ijx}e^{ikx}dx + \int_{-\pi}^{\pi}e^{-ijx}e^{-ikx}dx\right) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle \cos(jx)|\cos(jx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(jx)\cos(jx)dx \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{ijx}+e^{-ijx}\right)\left(e^{ijx}+e^{-ijx}\right)dx \\
&= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}e^{ijx}e^{ijx}dx + \int_{-\pi}^{\pi}e^{ijx}e^{-ijx}dx + \int_{-\pi}^{\pi}e^{-ijx}e^{ijx}dx + \int_{-\pi}^{\pi}e^{-ijx}e^{-ijx}dx\right) \\
&= \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}e^{2ijx}dx + 2\int_{-\pi}^{\pi}dx + \int_{-\pi}^{\pi}e^{-2ijx}dx\right) \\
&= \frac{1}{4\pi}(0+4\pi+0) \\
&= 1,
\end{aligned}$$

where we have used that $\{e^{ijx}; j \in Z\}$ is an orthonormal basis of $L^2([-\pi,\pi],\mathbb{C})$. Now for the sine, let $j \neq k$,

$$\begin{aligned}
\langle \sin(jx)|\sin(kx)\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(jx)\sin(kx)dx \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{ijx}-e^{-ijx}\right)\left(e^{ikx}-e^{-ikx}\right)dx \\
&= -\frac{1}{4\pi}\left(\int_{-\pi}^{\pi}e^{ijx}e^{ikx}dx - \int_{-\pi}^{\pi}e^{ijx}e^{-ikx}dx - \int_{-\pi}^{\pi}e^{-ijx}e^{ikx}dx + \int_{-\pi}^{\pi}e^{-ijx}e^{-ikx}dx\right) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle \sin(jx) | \sin(jx) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(jx)\sin(jx)dx \\
&= -\frac{1}{4\pi} \int_{-\pi}^{\pi} \left(e^{ijx} - e^{-ijx}\right)\left(e^{ijx} - e^{-ijx}\right) dx \\
&= -\frac{1}{4\pi} \left(\int_{-\pi}^{\pi} e^{ijx}e^{ijx}dx - \int_{-\pi}^{\pi} e^{ijx}e^{-ijx}dx - \int_{-\pi}^{\pi} e^{-ijx}e^{ijx}dx + \int_{-\pi}^{\pi} e^{-ijx}e^{-ijx}dx \right) \\
&= -\frac{1}{4\pi} \left(\int_{-\pi}^{\pi} e^{2ijx}dx - 2\int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} e^{-2ijx}dx \right) \\
&= -\frac{1}{4\pi}\left(0 - 4\pi + 0\right) \\
&= 1,
\end{aligned}$$

where we have used again that $\{e^{ijx}; j \in Z\}$ is an orthonormal basis of $L^2([-\pi,\pi],\mathbb{C})$. Now,

$$\langle \cos(jx) | \sin(kx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(jx)\sin(kx)dx = 0,$$

for any values of j, k since the product of cosine and sine is an odd function that is zero when integrated from $-\pi$ to pi. We continue with

$$\left\langle \frac{1}{\sqrt{2}} \middle| \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = \frac{1}{2\pi}2\pi = 1.$$

To conclude,

$$\begin{aligned}
\left\langle \cos(jx) \middle| \frac{1}{\sqrt{2}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \cos(jx)dx \\
&= \frac{2}{\sqrt{2}\pi} \int_{0}^{\pi} \cos(jx)dx \\
&= \frac{2}{\sqrt{2}\pi} \left[\sin(jx)\right]_0^{\pi} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \sin(jx) \middle| \frac{1}{\sqrt{2}} \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} \sin(jx)dx \\
&= \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} \cos(jx)dx \\
&= 0,
\end{aligned}$$

where we have used that $\cos(x)$ is an even function $\sin(x)$ is an odd function. Hence, we have proved that B is an orthonormal subset of $X = L^2([-\pi,\pi],\mathbb{R})$. Let $f \in X$ and assume that $\langle f, g \rangle = 0$ for all $g \in B$. By problem 4.2.3, if we show that then $f = 0$, then B is an orthonormal basis of X.

$$\begin{aligned}
\langle f(x), \cos(x) \rangle &= \left\langle f(x), \frac{e^{ijx} + e^{-ijx}}{2} \right\rangle \\
&= \frac{1}{2}\left(\left\langle f(x), e^{ijx} \right\rangle + \left\langle f(x), e^{-ijx} \right\rangle\right) = 0,
\end{aligned}$$

$$\langle f(x), \sin(x) \rangle = \left\langle f(x), \frac{e^{ijx} - e^{-ijx}}{2} \right\rangle$$
$$= \frac{1}{2} \left(\left\langle f(x), e^{ijx} \right\rangle - \left\langle f(x), e^{-ijx} \right\rangle \right) = 0.$$

From the previous system of two equations we obtain that

$$\left\langle f(x), e^{ijx} \right\rangle = 0,$$

and

$$\left\langle f(x), e^{-ijx} \right\rangle = 0.$$

Since $\{e^{ijx}; j \in Z\}$ is an orthonormal basis of $L^2([-\pi, \pi], \mathbb{C})$, the previous implies that $f = 0$. Thus, by Problem 4.2.3, B is an orthonormal basis of $L^2([-\pi, \pi], \mathbb{R})$.

Problem 14

Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that there exists a partition $a = t_0 < \ldots < t_m = b$ such that f is differentiable with bounded derivative on each interval (t_{j-1}, t_j). Show: f is Lipschitz continuous.

Solution: Let $x, y \in [a, b]$ and without loss of generality let $y > x$. We can construct another partition $x = q_0 < q_1 < \ldots < q_n = y$ where the elements of the constructed partition q_j are also from the initial one. By the Mean Value Theorem,

$$|f'(c_i)| = \left| \frac{f(q_i) - f(q_{i-1})}{q_i - q_{i-1}} \right| \leq M_i, \quad i = 0, \ldots, n$$

since f is continuous in the close interval, differentiable in the open and the derivative is bounded on each interval. In addition, define
$M = \max\{M_i; i = 0, \ldots n\}$. Therefore,

$$|f'(c_i)| < M, \quad i = 0, \ldots, n,$$

and

$$|f(q_i) - f(q_{i-1})| \leq M|q_i - q_{i-1}|, \quad i = 0, \ldots, n$$

Now

$$\begin{aligned}
|f(y) - f(x)| &= |f(y) - f(q_{n-1}) + f(q_{n-1}) - f(q_{n-2}) + f(q_{n-2}) \ldots + f(q_1) - f(x)| \\
&\leq |f(y) - f(q_{n-1})| + |f(q_{n-1}) - f(q_{n-2})| + \ldots + |f(q_1) - f(x)| \\
&\leq M \left(|y - q_{n-1}| + |q_{n-1} - q_{n-2}| + \ldots + |q_1 - x| \right) \\
&= M|y - x|.
\end{aligned}$$

Thus, f is Lipschitz continuous in $[a, b]$.

Problem 23

Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that there exists a partition $a = t_0 < \ldots < t_m = b$ such that f is differentiable with bounded derivative on each interval (t_{j-1}, t_j). Show: f is Lipschitz continuous.

Solution: Let $x, y \in [a, b]$ and without loss of generality let $y > x$. We can construct another partition $x = q_0 < q_1 < ... < q_n = y$ where the elements of the constructed partition q_j are also from the initial one. By the Mean Value Theorem,

$$|f'(c_i)| = \left| \frac{f(q_i) - f(q_{i-1})}{q_i - q_{i-1}} \right| \leq M_i, \quad i = 0, ..., n$$

since f is continuous in the close interval, differentiable in the open and the derivative is bounded on each interval. In addition, define
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$$|f'(c_i)| < M, \quad i = 0, ..., n,$$

and

$$|f(q_i) - f(q_{i-1})| \leq M|q_i - q_{i-1}|, \quad i = 0, ..., n$$

Now

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f(q_{n-1}) + f(q_{n-1}) - f(q_{n-2}) + f(q_{n-2}) ... + f(q_1) - f(x)| \\ &\leq |f(y) - f(q_{n-1})| + |f(q_{n-1}) - f(q_{n-2})| + ... + |f(q_1) - f(x)| \\ &\leq M \left(|y - q_{n-1}| + |q_{n-1} - q_{n-2}| + ... + |q_1 - x|\right) \\ &= M|y - x|. \end{aligned}$$

Thus, f is Lipschitz continuous in $[a, b]$.