Partial Differential Equations
TA Homework 4

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August 24, 2018

Problem 2

Verify that the function $<,>$ defined in Example 0.3 is an inner product.

Solution: Given the inner product on C^2 defined by

$$\langle v, w\rangle = \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix},$$

it is easy to check the properties.

- Positivity:
- Conjugate symmetry:

$$\begin{aligned}\langle v, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\ &= \begin{pmatrix}2\overline{w}_1 + i\overline{w}_2 & 3\overline{w}_2 - i\overline{w}_1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\ &= 2v_1\overline{w}_1 + iv_1\overline{w}_2 + 3v_2\overline{w}_2 - iv_2\overline{w}_1\\ &= \overline{2\overline{v}_1 w_1 - i\overline{v}_1 w_2 + 3\overline{v}_2 w_2 + i\overline{v}_2 w_1}\\ &= \overline{\begin{pmatrix}2\overline{v}_1 + i\overline{v}_2 & 3\overline{v}_2 - i\overline{v}_1\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}}\\ &= \overline{\begin{pmatrix}\overline{v}_1 & \overline{v}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}}\\ &= \overline{\langle w, v\rangle}\end{aligned}$$

- Homogeneity:

$$\begin{aligned}\langle cv, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}cv_1\\ cv_2\end{pmatrix}\\ &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}c\\ &= \langle v, w\rangle c\\ &= c\langle v, w\rangle,\end{aligned}$$

  where we have taken the complex scalar c out of the vector v since it is common in all its components.

- Linearity:

$$
\begin{aligned}
\langle u+v, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1+v_1\\ u_2+v_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2(u_1+v_1)-i(u_2+v_2) & i(u_1+v_1)+3(u_2+v_2)\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2u_1-iu_2+2v_1-iv_2 & iu_1+3u_2+iv_1+3v_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\left[\begin{pmatrix}2u_1-iu_2 & iu_1+3u_2\end{pmatrix}+\begin{pmatrix}2v_1-iv_2 & iv_1+3v_2\end{pmatrix}\right]\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\left[\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right]\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}
$$

Problem 7

For $n > 0$, let

$$f_n(t) = 1$$

Show that $f_n \to 0$ in $L^2[0,1]$ but that $f_n(0)$ does not converge to zero.

Solution:

$$f_n(t) = \begin{cases} \sqrt{n}, & 0 \le t \le 1/n^2 \\ 0, & \text{otherwise} \end{cases}$$

Problem 11

Show that if a differentiable function , f, is orthogonal to $cos(t)$ on $L^2[0, \pi]$ then f' is orthogonal to $sin(t)$ on $L^2[0, \pi]$.

Solution:

Problem 14

Find the $L^2[-\pi, \pi]$ projection of the function $f(x) = x^2$ onto the space $V_n \in L^2[-\pi, \pi]$ spanned by for n=2. Plot these projections along with f using a computer algebra system. Repeat for $g(x) = x^3$.

Solution:

Problem 23

Show that a set of orthonormal vectors is linearly independent.

Solution: