

# Numerical Methods for PDEs

## Homework 7

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### Problem 1

Derive the entropy advection equation  $s_t + us_x = 0$  from the Euler equations and the expression for the entropy  $s = c_V \ln(P/\rho^\gamma)$  of a polytropic gas. *Hint:* Start with the entropy advection equation and derive  $E_t + (u(E + P))_x = 0$  with  $P = (\gamma - 1)(E - \frac{1}{2}\rho u^2)$ , making use of  $\rho_t + (\rho u)_x = 0$  and  $(\rho u)_t + (\rho u^2 + P)_x = \rho u_t + \rho u u_x + P_x = 0$  if needed.

**Solution:** From the definition of the entropy we obtain the partial derivatives

$$s_t = c_V \left( \frac{P_t}{P} - \gamma \frac{\rho_t}{\rho} \right),$$
$$s_x = c_V \left( \frac{P_x}{P} - \gamma \frac{\rho_x}{\rho} \right).$$

Then, the entropy advection equation gives

$$s_t + us_x = 0 \Rightarrow \rho P_t - \gamma P \rho_t + u \rho P_x - \gamma u P \rho_x = 0,$$

where we have multiplied the whole equation by  $\rho P$ . We now include the easy computed partial derivatives of  $P$ ,

$$P_t = (\gamma - 1) \left( E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t \right),$$
$$P_x = (\gamma - 1) \left( E_x - \frac{1}{2} u^2 \rho_x - \rho u u_x \right),$$

into the equation to get

$$\rho(\gamma - 1) \left( E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t \right) - \gamma P \rho_t + u \rho(\gamma - 1) \left( E_x - \frac{1}{2} u^2 \rho_x - \rho u u_x \right) - \gamma u P \rho_x = 0.$$

Dividing by  $\rho(\gamma - 1)$  and computing the brackets we obtain

$$E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + u E_x - \frac{1}{2} u^3 \rho_x - \rho u^2 u_x - \frac{\gamma}{\gamma - 1} u P \frac{\rho_x}{\rho} = 0.$$

Using the continuity equation  $\rho_t + u \rho_x + \rho u_x = 0$ , we simplify the previous equation,

$$E_t - \rho u u_t - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + u E_x - \frac{1}{2} \rho u^2 u_x - \frac{\gamma}{\gamma - 1} u P \frac{\rho_x}{\rho} = 0.$$

Further, we use the Euler momentum equation in the form  $-\rho uu_t = \rho u^2 u_x + P_x$  to obtain

$$\begin{aligned} E_t + \rho u^2 u_x + u P_x - \frac{\gamma}{\gamma-1} P \frac{\rho_t}{\rho} + u E_x - \frac{1}{2} \rho u^2 u_x - \frac{\gamma}{\gamma-1} u P \frac{\rho_x}{\rho} &= 0, \\ E_t + u(E_x + P_x) + \frac{1}{2} \rho u^2 u_x - \frac{\gamma}{\gamma-1} \frac{P}{\rho} (\rho_t + u \rho_x) &= 0, \\ E_t + u(E_x + P_x) + \frac{1}{2} \rho u^2 u_x + \frac{\gamma}{\gamma-1} \frac{P}{\rho} \rho u_x &= 0, \end{aligned}$$

where we have used once more the continuity equation in the last step. Note that

$$\frac{\gamma}{\gamma-1} P = \gamma E - \frac{1}{2} \gamma \rho u^2.$$

Then,

$$\begin{aligned} E_t + u(E_x + P_x) + u_x \left( \frac{1}{2} \rho u^2 + \gamma E - \frac{1}{2} \gamma \rho u^2 \right) &= 0, \\ E_t + u(E_x + P_x) + u_x \left( \gamma E - (\gamma-1) \frac{1}{2} \rho u^2 \right) &= 0. \end{aligned}$$

To finish, note that

$$E + P = \gamma E - (\gamma-1) \frac{1}{2} \rho u^2.$$

Hence,

$$\begin{aligned} E_t + u(E_x + P_x) + u_x \left( \gamma E - (\gamma-1) \frac{1}{2} \rho u^2 \right) &= 0, \\ E_t + u(E_x + P_x) + u_x (E + P) &= 0. \end{aligned}$$

Finally, we obtain the equation that completes the proof,

$$E_t + (u(E + P))_x = 0.$$

## Problem 2

Show that the solution of the Riemann problem

$$\rho_L = 2, \quad u_L = 1, \quad P_L = 3; \quad \rho_R = 1, \quad u_R = 0, \quad P_R = 1$$

with  $\gamma = 1.5$  is a single shock wave propagating to the right. Calculate the shock speed  $s$ . For  $\gamma = 1.5$ ,  $E = \frac{1}{2} \rho u^2 + 2P$ . *Hint:* Use the jump conditions.

**Solution:** We can easily compute  $s$  as

$$s = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{2 \cdot 1 - 1 \cdot 0}{2 - 1} = 2,$$

and check that this result is right by the following:

$$\begin{aligned} s(\rho_L u_L - \rho_R u_R) &= \rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R, & s(E_L - E_R) &= u_L(E_L + P_L) - u_R(E_R + P_R), \\ 2(2 \cdot 1 - 1 \cdot 0) &= 2 \cdot 1^2 + 3 - 1 \cdot 0^2 - 1, & 2 \cdot (7 - 2) &= 1 \cdot (7 + 3) - 0 \cdot (2 + 1), \\ 4 &= 4, & 10 &= 10, \end{aligned}$$

where we have calculated  $E_L = \frac{1}{2}\rho_L u_L^2 + 2P_L = 7$  and  $E_R = \frac{1}{2}\rho_R u_R^2 + 2P_R = 2$

With a wall BC at  $x = 1$ , analytically calculate the solution after reflection of the shock wave. *Hint:* Show that the exact reflected shock solution is

$$\rho = 3.6, \quad u = 0, \quad P = 7.5, \quad s_r = -1.25$$

where  $s_r$  is the velocity of the reflected shock.

**Solution:** In this case we repeat the process assuming the same conditions on the left. We can check that given the right values of  $\rho$ ,  $u$ , and  $P$ , the value  $s_r$  obtains matches the solution:

$$\begin{aligned} s &= \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{2 \cdot 1 - 3.6 \cdot 0}{2 - 3.6} = -1.25, \\ s &= \frac{\rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R}{\rho_L u_L - \rho_R u_R} = \frac{2 \cdot 1^2 + 3 - 3.6 \cdot 0^2 - 7.5}{2 \cdot 1 - 3.6 \cdot 0} = -1.25, \\ s &= \frac{u_L(E_L + P_L) - u_R(E_R + P_R)}{E_L - E_R} = \frac{1 \cdot (7 + 3) - 0 \cdot (15 + 7.5)}{7 - 15} = -1.25, \end{aligned}$$

where we have calculated  $E_R = \frac{1}{2}\rho_R u_R^2 + 2P_R = 15$

### Problem 3

Show that the Lax-Wendroff method is second-order accurate for  $u_t + Au_x = 0$  using the definition of the LTE.

**Solution:** To prove that the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{1}{2}A \frac{\Delta t}{\Delta x} (u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}A^2 \frac{\Delta t^2}{\Delta x^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

is second order accurate for  $u_t + Au_x = 0$ , we start by Taylor expanding

$$\begin{aligned} u_j^{n+1} &= u_j^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \mathcal{O}(\Delta t^4), \\ u_{j\pm 1}^n &= u_j^n \pm \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} \pm \frac{\Delta x^3}{6} u_{xxx} + \mathcal{O}(\Delta x^4), \end{aligned}$$

and substituting them into the Lax-Wendroff scheme,

$$u_j^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} = u_j^n - \frac{A \Delta t}{2 \Delta x} \left( 2 \Delta x u_x + \frac{\Delta x^3}{3} u_{xxx} \right) + \frac{A^2 \Delta t^2}{2 \Delta x^2} \Delta x^2 u_{xx} + \Delta t \tau.$$

Doing some algebraic manipulations we reach,

$$u_t + Au_x = -\frac{\Delta t}{2}u_{tt} - A\frac{\Delta x^2}{6}u_{xxx} + \frac{1}{2}A^2\Delta t u_{xx} - \frac{\Delta t^2}{6}u_{ttt} + \tau,$$

$$\tau = \frac{\Delta t}{2}u_{tt} + A\frac{\Delta x^2}{6}u_{xxx} - \frac{1}{2}A^2\Delta t u_{xx} + \frac{\Delta t^2}{6}u_{ttt},$$

where we have used that  $u_t + Au_x = 0$ . Using the original PDE we obtain that

$$u_t = -Au_x \Rightarrow u_{tt} = A^2u_{xx},$$

$$\Rightarrow u_{ttt} = -A^3u_{xxx}.$$

Hence,

$$\tau = \frac{\Delta t}{2}A^2u_{xx} + A\frac{\Delta x^2}{6}u_{xxx} - \frac{1}{2}A^2\Delta t u_{xx} + \frac{\Delta t^2}{6}A^3u_{ttt},$$

$$= A\frac{\Delta x^2}{6}u_{xxx} + \frac{\Delta t^2}{6}A^3u_{ttt}.$$

Thus, the Lax-Wendroff scheme is second order accurate for  $u_t + Au_x = 0$ .

## Problem 4

Using **weno3.m**, investigate the effects of the CFL factor  $r$  on the solution of the Riemann problem

$$\rho_L = 1, u_L = 0, p_L = 1; \quad \rho_R = 0.125, u_R = 0, p_R = 0.1$$

with  $\gamma = 1.4$ . Take 200  $\Delta x$  and CFL factor  $r = 0.1, 0.5, 0.9$ . Turn in the Density plots (computed vs. exact solution) at time  $t = 0.2$  for each of three cases. Briefly discuss your results.

**Solution:**