Green's function (GF) HE w/ 0 BCs, $(\partial_t - a\partial_x^2)u = 0$, on $[0,L]\times(0,\infty), u(x,0)=f(x), 0\le x\le L, u(0,t)=0=u(L,t), t>0$ (5.25), is given by the Fourier sine series $u(x,t)=\sum_{m=1}^\infty B_m e^{-a\lambda_m^2 t}\sin(\lambda_m x), \lambda_m = m\pi/L = m\lambda_1, B_m = 2/L\int_0^L f(y)\sin(\lambda_m y)dy$ (5.26). Sub 3rd into 1st eq, $u(x,t)=\sum_{m=1}^\infty(2/L\int_0^L f(y)\sin(\lambda_m y)dy)e^{-a\lambda_m^2 t}\sin(\lambda_m x)$. Interchange series and int, $u(x,t)=\int_0^L G(x,y,t)f(y)dy, G(x,y,t)=\sum_{m=1}^\infty G_m(x,y,t), t\in(0,\infty), x,y\in[0,L]$, $G_m(x,y,t) = (2/L)\sin(\lambda_m x)\sin(\lambda_m y)e^{-a\lambda_m^2 t}$ (5.27). Notice $|\partial_x^j\partial_y^k\partial_t^\ell G_m(x,y,t)| \le (2/L)a^\ell\lambda_m^{j+k+2\ell}e^{-a\lambda_m^2 t}$ (5.28). From ratio test, for $t>0, \sum_{m=1}^\infty (2/L)a^\ell\lambda_m^{j+k+2\ell}e^{-a\lambda_m^2 t}<\infty$. By T 5.2, G is inf often diff on $[0,L]^2\times(0,\infty)$ and $\partial_x^j\partial_y^k\partial_t^\ell G(x,y,t)=\sum_{m=1}^\infty \partial_x^j\partial_y^k\partial_t^\ell G_m(x,y,t)$ (5.29). This justifies (5.27). Also $|\partial_x^j\partial_y^k\partial_t^\ell G(x,y,t)|\le\sum_{m=1}^\infty(2/L)a^\ell\lambda_m^{j+k+2\ell}e^{-a\lambda_m^2 t}<\infty$ (5.30). G satisfies HE because each G_m does, $\partial_t G(x,y,t) = a\partial_x^2 G(x,y,t) = a\partial_y^2 G(x,y,t), 0 = G(0,y,t)=G(L,y,t)=G(x,0,t)=G(x,L,t), 0\le x,y\le L, t>0$ (5.31). From (5.27), $G(x,y,t)=G(y,x,t), 0\le x,y\le L, t>0$ (5.32). **P5.17**. $G(x,y,t)\ge 0\ \forall\ x,y\in[0,L], t>0$. *Prf.* Suppose $G(x,\tilde y,t)<0$ for some $x,\tilde y\in[0,L],t>0$. Then $x,\tilde y\in(0,L)$. Since G is cont, $\exists\ \delta>0$ s.t. $(\tilde y-\delta,\tilde y+\delta)\subseteq(0,L), G(x,y,t)\le-\delta, |y-\tilde y|<\delta$. Set $f(y)=[1-|\tilde y-y|/\delta]_+, y\in[0,L]$, where $[r]_+=\max\{r,0\}$ is the pos part of r. Then f is Lip cont and non-neg, $f(y)=0$ whenever $y\notin(\tilde y-\delta,\tilde y+\delta)$. Let u be sol of HE with ID f. By T 5.16, u is cont on $[0,L]\times[0,\infty)$. By T 5.11, applied to $-u, 0\le u(x,t)=\int_0^L G(x,y,t)f(y)dy=\int_{\tilde y-\delta}^{\tilde y+\delta}G(x,y,t)f(y)dy\le -\delta\int_{\tilde y-\delta}^{\tilde y+\delta}f(y)dy\le-\delta\int_{\tilde y-\delta}^{\tilde y+\delta}[1-|y-\tilde y|/\delta]dy = -\delta^2\int_{-1}^1(1-|y|)dy=-\delta^2$, a contradiction ☐ **P5.18**. (a) If $f:[0,L]\to\mathbb R$ is cont and $f(0)=0=f(L)$, then $\int_0^L G(x,y,t)f(y)dy\to f(x), t\to 0$, unif in $x\in[0,L]$. (b) $\forall\ x\in[0,L], t>0, \int_0^L G(x,y,t)dy\le 1$, and, if $0<a<b<L, \int_0^L G(x,y,t)dy\to 1, t\to 0$, unif in $x\in[a,b]$. *Prf.* (a) This follows from T 5.16 and (5.27). (b) For $n\ge 3L$, define $f_n(x)=nx, 0\le x\le 1/n, f_n(x)=1, 1/n\le x\le L-1/n, f_n(x)=n(L-x), L-1/n\le x\le L$. Then f_n is Lip cont by E 4.3.2, $0\le f_n(x)\le 1\ \forall x\in[0,L]$, and $f_n(0)=0=f_n(L)$. Let u_n be sol of HE with ID f_n and 0 BCs. By T 5.11, applied to $-u_n$ and u_n-1, we have $0\le u_n\le 1$. By (5.27), $1\ge u_n(x,t)=\int_0^L G(x,y,t)f_n(y)dy\ge\int_{1/n}^{L-1/n}G(x,y,t)dy$. For fixed $t>0, G(x,y,t)\le c_t\ \forall\ x,y\in[0,L]$ with some const $c_t>0$, we have $1\ge\int_0^L G(x,y,t)dy - c_t(2/n)$. Take the limit as $n\to\infty$ and obtain the 1st inequality. For the 2nd statement, choose $n\in\mathbb N$ so large that $(1/n)<a$ and $L-(1/n)>b$. Then $f_n(x)=1\ \forall\ x\in[a,b]$. Since u_n is unif cont on $[0,L]\times[0,1], 1\ge\int_0^L G(x,y,t)dy\ge\int_0^L G(x,y,t)f_n(y)dy\to_{t\to 0} f_n(x)=1$, unif in $x\in[a,b]$ ☐ **T5.19**. Let $f:(0,L)\to\mathbb R$ be cont and bdd, $0<a<b<L$. Then $\int_0^L G(x,y,t)f(y)dy\to f(x), t\to 0$, unif in $x\in[a,b]$. *Prf.* Consider the same fctns f_n as previous proof and choose n so large that $(1/n)<a$ and $L-(1/n)>b$. Then $f_n(x)=1\ \forall\ x\in[a,b]$. Then ff_n has a cont extension g_n on $[0,L], g_n(0)=0=g_n(L)$. Let $x\in(0,L)$. By the TI, $|\int_0^L G(x,y,t)f(y)dy-f(x)| = |\int_0^L G(x,y,t)f(y)(1-f_n(y))dy + \int_0^L G(x,y,t)f(y)f_n(y)dy - f(x)f_n(x) - f(x)(1-f_n(x))| \le |\int_0^L G(x,y,t)f(y)(1-f_n(y)dy| + |\int_0^L G(x,y,t)g_n(y)dy - g_n(x)| + |f(x)(1-f_n(x))|$. We show that each of the last 3 exps tends to 0 as $t\to 0$ unif in $x\in[a,b]$. For the 2nd exp this follows from P 5.18. The last equals 0 for $x\in[a,b]$ by our choice of n. As for the 1st, since f is bdd on $(0,L)$, choose $M>0$ s.t. $|f(x)|\le M\ \forall\ x\in(0,L)$. Then, for $x\in[a,b]$, since $G\ge 0, |\int_0^L G(x,y,t)f(y)(1-f_n(y)dy|\le\int_0^L G(x,y,t)|f(y)(1-f_n(y))|dy\le M\int_0^L G(x,y,t)(1-f_n(y))dy = M(\int_0^L G(x,y,t)dy - \int_0^L G(x,y,t)f_n(y)dy)\to_{t\to 0} M(1-f_n(x))=0$, unif in $x\in[a,b]$, by P 5.18 ☐ **C5.20**. If $f:(0,L)\to\mathbb R$ is cont and bdd, then $u(x,t)=\int_0^L G(x,y,t)f(y)dy, t>0, 0\le x\le L, u(x,t)=f(x), t=0, 0<x<L$, defines a sol of the HE in the following sense: u is defined and cont on $[0,L]\times[0,\infty)$ except at $(0,0)$ and $(L,0)$, $(\partial_t - a\partial_x^2)u=0$ on $[0,L]\times(0,\infty), u(x,0)=f(x), x\in(0,L), u(0,t)=0=u(L,t), t>0$. *Prf.* Everything has already been proved except that u is cont at points $(x,0)$ with $0<x<L$. Let $\epsilon>0$ and $0<x<L$. Choose $\delta_1>0$ s.t. $|x-\delta_1,x+\delta_1|\subseteq(0,L)$ and $|f(x)-f(y)|<\epsilon/2$ whenever $|y-x|<\delta_1$. By T 5.19, $u(y,t)\to f(y)$ as $t\to 0$, unif in $y\in|x-\delta_1,x+\delta_1|$. So $\exists\ \delta_2>0$ s.t. $|u(y,t)-f(y)|<\epsilon/2, t\in[0,\delta_2], |y-x|<\delta_1$. Set $\delta=\min\{\delta_1,\delta_2\}$. Then, if $|y-x|+|t|<\delta, |u(y,t)-u(x,0)|\le|u(y,t)-f(y)|+|f(y)-f(x)|<\epsilon/2+\epsilon/2=\epsilon$ ☐ **E 5.3.1**. Show that there is at most one GF. More precisely: $\exists$ at most one cont fctn $G:[0,L]^2\times[0,\infty)\to\mathbb R$ s.t. $u(x,t)=\int_0^L G(x,y,t)f(y)dy$ solves the HE with initial values f and zero BCs $\forall$ Lipschitz cont $f:[0,L]\to\mathbb R$ with $f(0)=0=f(L)$. Hint: Assume that there are 2 and show that they are equal. This proof is similar to the one that shows that the GF is non-neg. *Proof.* Assume that there are 2 GFs G_1 and G_2. Then $u_j(x,t)=\int_0^L G_j(x,y,t)f(y)dy$ solve the HE with initial values f and zero BCs. By C 5.12, $u_1=u_2$. So $\int_0^L G_1(x,y,t)f(y)dy=\int_0^L G_2(x,y,t)f(y)dy\ \forall$ Lipschitz cont fctns $f:[0,L]\to\mathbb R$ with $f(0)=0=f(L), t\ge 0, x\in[0,L]$. Suppose that $G_1(x,y,t)\ne G_2(x,y,t)$ for some $x,y\in[0,L], t>0$. Without LOG we assume that $G_1(x,y,t)>G_2(x,y,t)$. Since G_1 and G_2 are cont, $\exists\ \delta>0$ s.t. $G_1(x,z,t)\ge G_2(x,z,t)+\delta, z\in(y-2\delta,y+2\delta)\cap[0,L]$. We can choose $\delta>0$ so small that $y+2\delta<L$ or $y-2\delta>0$. We assume that $y+2\delta<L$. The other case is similar. We define $f(z)=0; 0\le z\le y, f(z)=z-y; y\le z\le y+\delta, f(z)=2\delta+y-z; y+\delta\le z\le y+2\delta, f(z)=0; y+2\delta\le z\le L$. Then f is Lipschitz cont, $f(0)=0=f(L)$, and $0=\int_0^L(G_1(x,z,t)-G_2(x,z,t))f(z)dz=\int_y^{y+2\delta}(G_1(x,z,t)-G_2(x,z,t))f(z)dz\ge\delta\int_y^{y+2\delta}f(z)dz=\delta(\int_y^{y+\delta}(z-y)dz+\int_{y+\delta}^{y+2\delta}(2\delta+y-z)dz=2\delta\int_0^\delta zdz=\delta^3>0$, a contradiction ☐. **E 5.3.2**. Let G be the GF. Show: $G(x,y,t+r)=\int_0^L G(x,z,t)G(z,y,r)dz, 0\le x,y\le L, t,r\ge 0$. There are several ways to prove this. One way is to use the Fourier sine representation formula and orthonormality. Another is to fix r and, for fixed by arbitrary cont $f:[0,L]\to\mathbb R$ with $f(0)=0=f(L)$ consider $u(x,t)=\int_0^L G(x,y,t+r)f(y)dy$ and $v(x,t)=\int_0^L G(x,z,t)g(z)dz$ with $g(x)=\int_0^L G(x,y,r)f(y)dy$. *Proof.* Recall that $G(x,y,t)=\sum_{m=1}^\infty(2/L)\sin(\lambda_m x)\sin(\lambda_m y)e^{-a\lambda_m^2 t}$. Then $\int_0^L G(x,z,t)G(z,y,r)dz = (2/L)\sum_{m=1}^\infty\sin(\lambda_m x)e^{-a\lambda_m^2 t}\sum_{m=1}^\infty[(2/L)\int_0^L\sin(\lambda_m z)\sin(\lambda_k z)dz]\sin(\lambda_k y)e^{-a\lambda_k^2 r}$. Because of orthonormality, the integrals in the brackets are 0 if $k\ne m$ and 1 if $k=m$. Then $\int_0^L G(x,z,t)G(z,y,r)dz=(2/L)\sum_{m=1}^\infty\sin(\lambda_m x)e^{-a\lambda_m^2 t}\sin(\lambda_m y)e^{-a\lambda_m^2 r}=(2/L)\sum_{m=1}^\infty\sin(\lambda_m x)\sin(\lambda_m y)e^{-a\lambda_m^2(t+r)}=G(x,y,t+r)$ ☐ **E 5.3.3** Let f be int and $\int_0^L|f(y)|dy<\infty$. Let u be the soln of HE with ID f and 0 BCs. Show: $\int_0^L|u(t,x)-f(x)|dx\to 0, t\to 0$. Hint: Use the GF and (without prf) that for any $\epsilon>0\ \exists$ a cont fctn $g:[0,L]\to\mathbb R$ s.t. $\int_0^L|f(y)-g(y)|dy<\epsilon$. *Prf.* Let $\epsilon>0$. Then $\exists$ a cont fctn $g:[0,L]\to\mathbb R$ s.t. $\int_0^L|f(y)-g(y)|dy<\epsilon/8$. For $n\ge 3L$, define $f_n(x)=nx, 0\le x\le 1/n; f_n(x)=1, 1/n\le x\le L-1/n; f_n(x)=n(L-x), L-1/n\le x\le L$. Then f_n is Lip cont by E4.3.2, $0\le f_n(x)\le 1\ \forall\ x\in[0,L]$, and $f_n(0)=0=f_n(L)$. Set $g_n=gf_n$. Then $\int_0^L|g-g_n|=\int_0^L|g|(1-f_n)=\int_0^{1/n}|g|+\int_{L-(1/n)}^L|g|\le(2/n)\sup_{[0,L]}|g|$. By choosing n large enough, we can achieve that $\int_0^L|g-g_n|<\epsilon/8$. By the TI, $\int_0^L|f-g_n|<\epsilon/4$. The fctn g_n is cont and $g_n(0)=0=g_n(L)$. Recall $u(x,t)=\int_0^L G(x,y,t)f(y)dy$. Set $v_n(x,t)=\int_0^L G(x,y,t)g_n(y)dy$. By T5.19, $\exists\ \delta>0$ s.t. $|v_n(x,t)-g_n(x)|\epsilon/(8L), 0\le t<\delta, 0\le x\le L$. By TI $\int_0^L|u(x,t)-f(x)|dx\le\int_0^L|u(x,t)-v_n(x,t)|dx+\int_0^L|v_n(x,t)-g_n(x)|dx+\int_0^L|g_n(x)-f(x)|dx$. We change the order of int and use that G is symmetric in x and y and so $\int_0^L G(x,y,t)dx\le 1\ \forall\ t>0, y\in[0,L]$ (P5.18(b)), $\int_0^L|u(x,t)-v_n(x,t)|dx = \int_0^L|\int_0^L G(x,y,t)(f(y)-g_n(y))dy|dx\le\int_0^L(\int_0^L G(x,y,t)|f(y)-g_n(y)|dy)dx=\int_0^L(\int_0^L G(x,y,t)dx)|f(y)-g_n(y)|dy\le\int_0^L|f(y)-g_n(y)|dy$. We subst these inequalities into each other: $\forall\ t\in(0,\delta), \int_0^L|u(x,t)-f(x)|dx\le 2\int_0^L|g_n-f|+\int_0^L\epsilon/(8L)<\epsilon/2+\epsilon/8<\epsilon$ ☐ **E 5.3.4**. Let $f:[0,L]\to\mathbb R$ be twice cont diff, $f(0)=0=f(L)$. Let G be the GF for the HE. Set $u(x,t)=\int_0^L G(x,y,t)f(y)dy, x\in[0,L], t\in(0,\infty), u(x,0)=f(x), x\in[0,L]$. (a) Show that u has cont PDs $\partial_t u(x,t)$ on $[0,L]\times(0,\infty)$ and on $(0,L)\times[0,\infty)$ and $\partial_t u(x,t)=\int_0^L G(x,y,t)af''(y)dy, x\in[0,L], t\in(0,\infty), \partial_t u(x,0)=af''(x), x\in(0,L)$ (5.33). Hint: First prove the 1st statement in (5.33) interchanging diff and int. (b) Show that $\partial_t u$ satisfies the HE with ID af'', $(\partial_t - a\partial_x^2)\partial_t u=0$ on $[0,L]\times(0,\infty), \partial_t u(x,0)=af''(x), x\in(0,L), \partial_t u(0,t)=0=\partial_t u(L,t), t\in(0,\infty)$. *Prf.* (a) Let $t\in(0,\infty)$ and $x\in[0,L]$. By (5.28), one can interchange diff and int, $\partial_t u(x,t)=\int_0^L\partial_t G(x,y,t)f(y)dy$. By (5.31) $\partial_t u(x,t)=\int_0^L a\partial_x^2 G(x,y,t)f(y)dy$. We IBP twice and use that $f(0)=0=f(L)$ and $G(x,0,t)=0=G(x,L,t), \partial_t u(x,t)=\int_0^L G(x,y,t)af''(y)dy, t\in(0,\infty), x\in[0,L]$. Define $v:[0,L]\times[0,\infty)$ by $v(x,t)=\int_0^L G(x,y,t)af''(y)dy, x\in[0,L], t\in(0,\infty), v(x,0)=af''(x), x\in[0,L]$ (5.34). By C5.20, v is cont on $[0,L]\times[0,\infty)$ except at $(0,0)$ and $(L,0)$. Let $x\in(0,L)$ and $t>0$. Notice that f is Lip cont and u is cont on $[0,L]\times[0,\infty)$ by T5.6. By the mean-value thm, $(u(x,t)-u(x,0))/(t-0)=\partial_2 u(x,s)=v(x,s)$ for some $x\in(0,t)$. Let $\epsilon>0$. Since $v(x,\cdot)$ is cont on $[0,\infty)$ and $v(x,0)=af''(x)\ \exists\ \delta>0$ s.t. $|v(x,r)-af''(x)|<\epsilon\ \forall r\in[0,\delta)$. Let $t\in[0,\delta)$. Then $s\in[0,\delta]$ and $|(u(x,t)-u(x,0))/(t-0)-af''(x)|=|v(x,s)-af''(x)|<\epsilon$. Hence $af''(x)=\lim_{t\to 0}(u(x,t)-u(x,0))/(t-0)=\partial_t u(x,0)$. (b) This follows from C5.20 ☐ **E 5.3.5**. Find a fctn like the GF for the equation $(\partial_t - a\partial_x^2)u=0, 0\le x\le L, t>0, \partial_x u(0,t)=0=\partial_x u(L,t), t>0, u(x,0)=f(x), 0\le x\le L$. *Prf.* By E 5.1.1, the solution is given by $u(x,t)=\sum_{m=0}^\infty A_m\cos(\lambda_m x)e^{-a\lambda_m^2 t}$, with $\lambda_m=m\pi/L, A_m=(2/L)\int_0^L f(y)\cos(\lambda_m y)dy, m\in\mathbb N, A_0=(1/L)\int_0^L f(y)dy$. Then $u(x,t)=\int_0^L N(x,y,t)f(y)dy$ with $N(x,y,t)=(1/L)+\sum_{m=1}^\infty(2/L)\cos(\lambda_m x)\cos(\lambda_m y)e^{-a\lambda_m^2 t}$ ☐ **Inhomog HE** with 0 BCs (Dirichlet BCs) has the form (PDE) $(\partial_t - a\partial_x^2)u=F(x,t), 0\le x\le L, t\in(0,T)$, (IC) $u(x,0)=f(x), 0<x<L$, (BC) $u(0,t)=0=u(L,t), 0<t\le T$ (5.35). Again $L,a>0$. Here $F:[0,L]\times(0,T)\to\mathbb R$ is cont and bdd and $f:(0,L)\to\mathbb R$ is cont and bdd. Assume u and $\partial_t u$ are cont on $[0,L]\times(0,T)$. Let G be the GF from the previous Section and $0<s<t<T$. Because of (5.30), we can interchange int and diff in the following, $\partial_s\int_0^L G(x,y,s)u(y,t-s)dy=\int_0^L\partial_s[G(x,y,s)u(y,t-s)]dy = \int_0^L[\partial_s G(x,y,s)u(y,t-s)+G(x,y,s)\partial_s u(y,t-s)]dy \overset{(5.31)}{=} \int_0^L a\partial_y^2 G(x,y,s)u(y,t-s)dy - \int_0^L G(x,y,s)[a\partial_y^2 u(y,t-s)+F(y,t-s)]dy = -\int_0^L G(x,y,s)F(y,t-s)dy$, where the last equation follows from IBP. We int this equation over s from ϵ to $t, 0<\epsilon<t, \int_0^L G(x,y,t)u(y,0)dy - \int_0^L G(x,y,\epsilon)u(y,t-\epsilon)dy=-\int_\epsilon^t(\int_0^L G(x,y,s)F(y,t-s)dy)ds$. Take the lim of the rhs as $\epsilon\to 0$. As for the 2nd expn on the lhs, $|\int_0^L G(x,y,\epsilon)u(y,t-\epsilon)dy - u(x,t)|\le|\int_0^L G(x,y,\epsilon)(u(y,t-\epsilon)-u(y,t))dy|+|\int_0^L G(x,y,\epsilon)u(y,t)dy - u(x,t)|$. The last expn tends to 0 by P 5.18 (a). By part (b) of this very prop, the last but one expn can be est by $\sup_{0\le y\le L}|u(y,t-\epsilon)-u(y,t)|$ which tends to 0 as $\epsilon\to 0$, because u is unif cont on $[0,L]\times[t/2,t]$. **T5.21**. If u is a soln of (5.35) and u and F are cont and bdd on $[0,L]\times(0,T)$, then $u(x,t)=\int_0^L G(x,y,t)f(y)dy+\int_0^t\int_0^L G(x,y,s)F(y,t-s)dsdy, 0<t<T, 0\le x\le L$. Under which assumptions for F is this a soln? Since we already studied the homog IVP in detail, we can set $f=0$ and, after a subst, set $\tilde u(x,t)=\int_0^t\int_0^L G(x,y,t-s)F(y,s)dsdy$ (5.36). Interchanging int and the series rep of G, (5.27), we obtain the Fourier sin rep, $\tilde u(x,t)=\sum_{m=1}^\infty\tilde u_m(x,t), \tilde u_m(x,t)=\sin(\lambda_m x)(2/L)\int_0^t\int_0^L\sin(\lambda_m y)e^{-a\lambda_m^2(t-s)}F(y,s)dsdy=\sin(\lambda_m x)e^{-a\lambda_m^2 t}(2/L)$ (5.37). We have the est $|\tilde u_m(x,t)|\le 2\sup|F|\int_0^t e^{-a\lambda_m^2(t-s)}ds\le(2\sup|F|)/(a\lambda_m^2)$. Since the series over the rhs converges (recall that λ_m is proportional to m), the series for $\tilde u$ converges unif on $[0,L]\times[0,T]$ and $\tilde u$ is cont on $[0,L]\times[0,T], \tilde u(x,0)=0\ \forall\ x\in[0,L]$. By the product rule and the fund thm of calc, $\partial_t\tilde u_m(x,t)=-a\lambda_m^2\tilde u_m(x,t)+\sin(\lambda_m x)(2/L)\int_0^L\sin(\lambda_m y)F(y,t)dy$ (5.38) $= a\partial_x^2\tilde u_m(x,t)+\sin(\lambda_m x)(2/L)\int_0^L\sin(\lambda_m y)F(y,t)dy$ (5.39). We make a change of variables in the int for $\tilde u_m$ in (5.37), $\partial_t\tilde u_m(x,t) = -a\lambda_m^2\sin(\lambda_m x)(2/L)\int_0^t\int_0^L\sin(\lambda_m y)e^{-a\lambda_m^2 s}F(y,t-s)dyds+\sin(\lambda_m x)(2/L)\int_0^L\sin(\lambda_m y)F(y,t)dy$ (5.40). We change the order of int and combine the spatial ints, $\partial_t\tilde u_m(x,t)=\sin(\lambda_m x)(2/L)\int_0^L\sin(\lambda_m y)H(y,t)dy, H(y,t)=-a\lambda_m^2\int_0^t e^{-a\lambda_m^2 s}F(y,t-s)ds+F(y,t)$. Since $a\lambda_m^2\int_0^t e^{-a\lambda_m^2 s}ds=1-e^{-a\lambda_m^2 t}, H(y,t)=a\lambda_m^2\int_0^t e^{-a\lambda_m^2 s}[F(y,t)-F(y,t-s)]ds - a\lambda_m^2\int_0^t e^{-a\lambda_m^2 s}dsF(y,t)+F(y,t) = a\lambda_m^2\int_0^t e^{-a\lambda_m^2 s}[F(y,t)-F(y,t-s)]ds+e^{-a\lambda_m^2 t}F(y,t)$. Assume F is Lip cont in the time var: $\exists\ \Lambda>0$ s.t. $|F(y,t)-F(y,s)|\le\Lambda|t-s|, 0\le y\le L, 0\le s,t\le T$. Then $|H(y,t)|\le a\lambda_m^2\int_0^t e^{-a\lambda_m^2 s}\Lambda sds+e^{-a\lambda_m^2 t}|F(y,t)|$. Subst $r=a\lambda_m^2 s, |H(y,t)|\le(L/(a\lambda_m^2)\int_0^{ta\lambda_m^2}e^{-r}rdr+e^{-a\lambda_m^2 t}|F(y,t)|\le(L/(a\lambda_m^2)+e^{-a\lambda_m^2 t}|F(y,t)|$. Now $|\partial_t\tilde u_m(x,t)|\le|\sin(\lambda_m x)|(2/L)\int_0^L|\sin(\lambda_m y)||H(y,t)|dy\le(2/L)\int_0^L|H(y,t)|dy$. Sub est $|H(y,t)|, |\partial_t\tilde u_m(x,t)|\le(2\Lambda)/(a\lambda_m^2)+e^{-a\lambda_m^2 t}(2/L)\int_0^L|F(y,t)|dy$ (5.41). By T5.3, u is PD wrt t on $[0,L]\times(0,T]$ and $\partial_t u=\sum_{m=1}^\infty\partial_t\tilde u_m$ with the series converging unif on $[0,L]\times[\epsilon,T]$ for every $\epsilon\in(0,T)$. Similarly one shows that u is twice PD wrt x and $\partial_x^2 u=\sum_{m=1}^\infty\partial_x^2\tilde u_m$ with the series converging unif on $[0,L]\times[\epsilon,T]$ for every $\epsilon\in(0,T)$.By (5.39), $\partial_t\tilde u - a\partial_x^2\tilde u = \sum_{m=1}^\infty\sin(\lambda_m x)(2/L)\int_0^L\sin(\lambda_m y)F(y,t)dy$. where the convergence on the rhs is unif on $[0,L]\times[\epsilon,T]$ for every $\epsilon\in(0,T)$. Since the sines form an orthonormal basis, the rhs equals the Fourier sine series of $F(\cdot,t)$ and so equals $F(x,t)$ for a.a. x. Further, for fixed t, the convergence holds in $L^2[0,L]$ wrt the space var. Since both $\partial_t\tilde u - a\partial_x^2\tilde u$ and F are cont, $\partial_t\tilde u - a\partial_x^2\tilde u = F$. Further $\tilde u(x,0)=0$ and $\tilde u(0,t)=0=\tilde u(L,t)$. **T 5.22**. Let $f:(0,L)\to\mathbb R$

be cont and bdd, $F : [0, L] \times [0, T] \to \mathbb{R}$ cont and Lip cont in the time var. Then $\exists$ a fctn $u : [0, L] \times [0, T] \to \mathbb{R}$ that is cont except possibly at $(0, 0)$ and $(L, 0)$ and solves (5.35). The soln u is given as in T5.21. **5.4.1 Inhomog BCs** HE: (PDE) $(\partial_t - a\partial_x^2)u = 0, 0 \le x \le L, t \in (0, T)$, (IC) $u(x, 0) = 0, 0 \le x \le L$, (BC) $u(0, t) = g(t), u(L, t) = h(t), 0 < t \le T$ (5.42). Ansatz $u(x, t) = v(x, t) + U(x, t), U(x, t) = (1 - (x/L))g(t) + (x/L)h(t)$ (5.43). Then u solves (5.42) iff v satisfies (PDE) $(\partial_t - a\partial_x^2)v = -(1 - (x/L))g'(t) - (x/L)h'(t), 0 \le x \le L, 0 < t < T$, (IC) $v(x, 0) = -U(x, 0), 0 \le x \le L$, (BC) $v(0, t) = 0, v(L, t) = 0, 0 < t \le T$ (5.44). If g' and h' are Lip cont, a soln can be found. It is given by $v(x, t) = -\int_0^L U(y, 0)G(x, y, t)dy - \int_0^t \int_0^L G(x, y, t-s)[(1-(y/L))g'(s)+(y/L)h'(s)]dyds$. **E5.4.2**. Let $u : [0, L] \times (0, T) \to \mathbb{R}$ be cont. Assume that the partial derivatives $\partial_t u(x, t)$ exist $\forall\, x \in [0, L]$ and $t \in (0, T)$ and that $\partial_t u$ is cont on $[0, L] \times (0, T)$. Show: $\forall$ cont $\phi : [0, L] \to \mathbb{R}, \int_0^L \phi(x)u(x, t)dx$ is differentiable in $t \in (0, T)$ and $d/dt \int_0^L \phi(x)u(x, t)dx = \int_0^L \phi(x)\partial_t u(x, t)dx$. Hint: Show (why?) $\int_0^L \phi(x)((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))dx \to 0, s \to t$. You may like to use the mean value thm. *Proof.* Set $v(t) = \int_0^L \phi(x)u(x, t)dx, w(t) = \int_0^L \phi(x)\partial_t u(x, t)dx$. The task is to show that v is differentiable on $(0, T)$ and $v' = w$. To this end, we work with the definition of the derivative. Let $s., t \in (0, T)$. Since integration is a linear operation, $(v(s) - v(t))/(s - t) - w(t) = \int_0^L \phi(x)((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))dx$. Then $|(v(s) - v(t))/(s - t) - w(t)| \le \int_0^L |\phi(x)||((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))|dx$. Since $\phi : [0, L] \to \mathbb{R}$ is cont, $\exists M > 0$ s.t. $|\phi(x)| \le M\ \forall x \in [0, L]$. So $|(v(s) - v(t))/(s - t) - w(t)| \le M \int_0^L |((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))|dx \le ML \sup_{0\le x\le L} |((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))|$ (5.29). Let $x \in [0, L]$. By the mean value thm, $\exists r_x$ between s and t s.t. $|((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))| = |\partial_2 u(x, r_x) - \partial_2 u(x, t)|$. Here $\partial_2 u$ denotes the partial derivative of u wrt the 2nd variable, time. Choose some $\delta_0 > 0$ s.t. $[t - \delta_0, t + \delta_0] \subseteq (0, T)$. Since $\partial_2 u$ is cont on $[0, L] \times (0, T)$, it is uniformly cont on $[0, L] \times [t - \delta_0, t + \delta_0]$. Let $\epsilon > 0$. Then $\exists\delta \in (0, \delta_0)$ s.t. $|\partial_2 u(x, r) - \partial_2 u(x, t)| < \epsilon/(2LM)$ if $|r - t| < \delta$. Let $|s - t| < \delta$. Since r_x is between s and $t, |r_x - t| \le |s - t| < \delta$ and so $|((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))| = |\partial_2 u(x, r_x) - \partial_2 u(x, t)| < \epsilon/(2LM)$. Since this holds $\forall\, x \in [0, L], \sup_{0\le x\le L} |((u(x, s) - x(x, t))/(s - t) - \partial_t u(x, t))| \le \epsilon/(2LM)$. By (5.29), $|((v(s) - v(t))/(s - t) - w(t))| \le \epsilon/2 < \epsilon$ □ **E5.4.3**. Let u be as in E5.4.2. Assume u is twice partially diff wrt x on $[0, L] \times (0, T)$ and $\partial_x u$ and $\partial_x^2 u$ are cont on $[0, L] \times (0, T)$ and $(\partial_t - a\partial_x^2)u = F(x, t), x \in [0, L], t \in (0, T), u(0, t) = 0 = u(L, t), t \in (0, T)$, where $F : [0, L] \times (0, T) \to \mathbb{R}$ is cont. Show: For every twice contly diff $\phi : [0, L] \to \mathbb{R}$ with $\phi(0) = 0 = \phi(L), \int_0^L \phi(x)u(x, t)dx$ is diff in $t \in (0, T)$ and $d/dt \int_0^L \phi(x)u(x, t)dx = \int_0^L a\phi''(x)u(x, t)dx + \int_0^L \phi(x)F(x, t)dx, 0 < t < T$. *Prf.* By the previous exercise, $\int_0^L \phi(x)u(x, t)dx$ is diff in $t \in (0, T)$ and $d/dt \int_0^L \phi(x)u(x, t)dx = \int_0^L \phi(x)\partial_t u(x, t)dx = \int_0^L \phi(x)(a\partial_x^2 u(x, t) + F(x, t))dx = \int_0^L \phi(x)a\partial_x^2 u(x, t)dx + \int_0^L \phi(x)F(x, t)dx$. Since ϕ is twice contly diff, we can IBP twice. Since $\phi(0) = 0 = \phi(L)$ and $u(0, t) = 0 = u(L, t)$, we do not obtain any terms at the int limits and $d/dt \int_0^L \phi(x)u(x, t)dx = \int_0^L a\phi''(x)u(x, t)dx + \int_0^L \phi(x)F(x, t)dx$ □ **E5.4.4**. Let $F : [0, L] \times [0, T) \to \mathbb{R}$ be cont. Define $u(x, t) = \int_0^t \int_0^L G(x, y, t - s)F(y, s)dyds, 0 < t < T, 0 \le x \le L$. Show: For every twice contly diff $\phi : [0, L] \to \mathbb{R}$ with $\phi(0) = 0 = \phi(L), \int_0^L \phi(x)u(x, t)dx$ is diff in $t \in (0, T)$ and $d/dt \int_0^L \phi(x)u(x, t)dx = \int_0^L a\phi''(x)u(x, t)dx + \int_0^L \phi(x)F(x, t)dx, \quad 0 < t < T$. Hint: Notice and prove that $\int_0^L \phi(x)u(x, t)dx = \int_0^L (\int_0^t v(y, t - s)F(y, s)ds)dy$ with $v(y, t) = \int_0^L \phi(x)G(x, y, t)dy = \int_0^L G(y, z, t)\phi(z)dz, t > 0, y \in [0, L]$ (5.45), and use E 5.3.4. You may interchange int and diff and use Leibnitz rule without prf. *Prf.* Changing the order of int, we obtain $\int_0^L \phi(x)u(x, t)dx = \int_0^L \phi(x)(\int_0^t \int_0^L G(x, y, t - s)F(y, s)dsdy)dx = \int_0^L \int_0^t (\int_0^L G(x, y, t - s)\phi(x)dx)F(y, s)dsdy = \int_0^L (\int_0^t v(y, t - s)F(y, s)ds)dy$ (5.46) with v in (5.45). Recall that $G(x, y, t) = G(y, x, t)$. By E 5.3.4, $\partial_t v(y, t) = \int_0^L G(y, z, t)a\phi''(z)dz$ $\exists$ and is cont for $t > 0, y \in [0, L]$ and is bdd on $[0, L] \times [0, \infty)$ because ϕ'' is bdd on $(0, L)$ and $\int_0^L G(y, z, t)dz \le 1$. Further v can be cont extended to $[0, L] \times [0, \infty)$ with $v(y, 0) = \phi(y)$. So we can apply Leibnitz rule and obtain $\partial_t \int_0^t v(y, t-s)F(y, s)ds = v(y, 0)F(y, t) + \int_0^t \partial_t v(y, t-s)F(y, s))ds = \phi(y)F(y, t) + \int_0^t (\int_0^L G(y, z, t)a\phi''(z))F(y, s)ds$. We interchange diff and int in (5.46), $\partial_t \int_0^L \phi(x)u(x, t)dx = \int_0^L \partial_t (\int_0^t v(y, t - s)F(y, s)ds)dy = \int_0^L (\phi(y)F(y, t) + \int_0^t (\int_0^L G(y, z, t - s)a\phi''(z)dz)F(y, x)ds)dy$. We change the order of int back and use $G(y, z, t - s) = G(z, y, t - s), \partial_t \int_0^L \phi(x)u(x, t)dx = \int_0^L \phi(x)F(y, t)dy + \int_0^L a\phi''(z)u(z, t)dz$ □ **E5.4.14**. Let $u : [0, L] \times [0, T] \to \mathbb{R}$ be cont and twice contly diff. Let $F : [0, L] \times [0, T] \to \mathbb{R}$ be cont. Assume $\partial_t u(x, t) - a\partial_x^2 u(x, t) = F(x, t), 0 \le t \le T, 0 \le x \le L, \partial_x u(0, t) = 0 = \partial_x u(L, t), t \in [0, T], u(x, 0) = 0, 0 \le x \le L$. Derive a Fourier series rep of u. You may interchange int, diff and series without prf. You do not need to discuss convergence of the series. *Prf.* Because of the BCs, we choose a Fourier cosine rep, $u(x, t) = \sum_{m=0}^\infty A_m(t) \cos(\lambda_m x), \lambda_m = (m\pi)/L, m \in \mathbb{Z}, A_m(t) = 2/L \int_0^L u(x, t) \cos(\lambda_m x)dx, m \in \mathbb{N}, A_0(t) = 1/L \int_0^L u(x, t)dx$ (5.47). Let $m \in \mathbb{N}$. Interchange diff and int, $A'_m(t) = 2/L \int_0^L \partial_t u(x, t) \cos(\lambda_m x)dx$. Use the PDE for $u, A'_m(t) = 2/L \int_0^L (a\partial_x^2 u(x, t) + F(x, t)) \cos(\lambda_m x)dx$. IBP twice and use that the boundary terms are 0 because of the BCs for u and $\sin(\lambda_m x) = 0$ for $x = 0$ and $x = L, \quad A'_m(t) = 2/L \int_0^L u(x, t)a\partial_x^2 \cos(\lambda_m x)dx + \hat{F}_m(t)$ with $\hat{F}_m(t) = 2/L \int_0^L F(x, t) \cos(\lambda_m x)dx$. By the diff properties of cosine, $A'_m(t) = -a\lambda_m^2 A_m(t) + \hat{F}(t)$. Further $A_m(0) = 0$ by the initial conditions for u. We use the variation of consts formula, $A_m(t) = \int_0^t e^{-a\lambda_m^2(t-s)} \hat{F}_m(s)ds$. Further, since u satisfies the PDE and the no-flux BCs, $A'_0(t) = 1/L \int_0^L \partial_t u(x, t)dx = 1/L \int_0^L (a\partial_x^2 u(x, t) + F(x, t))dx = (1/L)a[\partial_x u(L, t) - \partial_x u(0, t)] + 1/L \int_0^L F(x, t)dx = 1/L \int_0^L F(x, t)dx$, and $A_0 = 0$ because of the zero initial values for u. This implies $A_0(t) = \int_0^t 1/L \int_0^L F(x, s)dxds$. Together with $A_m(t) = \int_0^t e^{-a\lambda_m^2(t-s)} 2/L(\int_0^L F(x, s) \cos(\lambda_m x)dx)ds, \ m \in \mathbb{N}$, and $u(x, t) = \sum_{m=0}^\infty A_m(t) \cos(\lambda_m x), t \ge 0, 0 \le x \le L$, This provides the Fourier series representation of u □ **The LE** Let $\Omega \subseteq \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be twice diff. Then the Laplace operator is $\Delta u(x) = \sum_{j=1}^n \partial_j^2 u(x), x \in \Omega$ (6.1). The Laplace equation is for a cont fctn $u : \bar{\Omega} \to \mathbb{R}$ that is twice diff on Ω, (PDE) $\Delta u = 0$ on Ω, (BC) $u(x) = f(x), x \in \partial\Omega$ (6.2), where $\partial\Omega = \bar{\Omega} \setminus \Omega$ is the boundary of Ω and $f : \partial\Omega \to \mathbb{R}$ is given. A twice contly diff fctn u on Ω that satisfies $\Delta u = 0$ on Ω is called harmonic on Ω. **The LE on a rectangle** Let $n = 2$ and $\Omega = (0, L) \times (0, H)$ with $L, H > 0$. Then $\bar{\Omega} = [0, L] \times [0, H]$ and $\partial\Omega = \cup_{k=1}^4 B_k$ consists of four line sections. LE takes the form (PDE) $(\partial_x^2 + \partial_y^2)u(x, y) = 0, 0 < x < L, 0 < y < H$, (BC) $u(0, y) = g_1(y), u(L, y) = g_2(y), 0 < y < H, u(x, 0) = h_1(x), u(x, H) = h_2(x), 0 < x < L$ (6.3). By symmetry, it is sufficient to study the problem (PDE) $(\partial_x^2 + \partial_y^2)u(x, y) = 0, 0 < x < L, 0 < y < H$, (BC) $u(0, y) = g_1(y), u(L, y) = 0, 0 < y < H, u(x, 0) = 0, u(x, H) = 0, 0 < x < L$ (6.4). The form of the problem suggests to look for the soln in the form of a Fourier sine series in $y, u(x, y) = \sum_{m=1}^\infty B_m(x) \sin(\lambda_m y), \lambda_m = (m\pi)/H, B_m(x) = (2/H) \int_0^H u(x, y) \sin(\lambda_m y)dy$ (6.5). To determine B_m, we derive a diff eqn, $B''_m(x) = (2/H) \int_0^H \partial_x^2 u(x, y) \sin(\lambda_m y)dy = -(2/H) \int_0^H \partial_y^2 u(x, y) \sin(\lambda_m y)dy$. We IBP twice using the zero BCs for both u and the sine fctns, $B''_m(x) = -(2/H) \int_0^H u(x, y)\partial_y^2 \sin(\lambda_m y)dy = (2/H) \int_0^H u(x, y)\lambda_m^2 \sin(\lambda_m y)dy = \lambda_m^2 B_m$. Further $B_m(L) = 0, B_m(0) = (2/H) \int_0^H g_1(y) \sin(\lambda_m y)dy$ (6.6). A poss fund set of solns for this ODE is $e^{\lambda_m x}, e^{-\lambda_m x}$, but in view of the condition $B_m(L) = 0$ the fund set $\cosh(\lambda_m(L - x)), \sinh(\lambda_m(L - x))$ is more practical. Then $B_m(x) = A_m \sinh(\lambda_m(L - x))$ and, by (6.6), $A_m = 2/(H \sinh(\lambda_m L)) \int_0^H g_1(z) \sinh(\lambda_m z)dz$ (6.7). We combine (6.7) and (6.5), $u(x, y) = \sum_{m=1}^\infty u_m(x, y), u_m(x, y) = A_m \sin(\lambda_m y) \sinh(\lambda_m(L - x))$ (6.8). Then, if $0 \le x \le L$ and $0 \le y \le H, \partial_y^k u_m(x, y) = \pm\lambda_m^k A_m \sinh(\lambda_m(L - x))\{\sin(\lambda_m y)$ or $\cos(\lambda_m y)\}$ and $\partial_x^k u_m(x, y) = \pm\lambda_m^k A_m \sin(\lambda_m y)\{\sinh(\lambda_m(L - x)), k \in \mathbb{N}, k$ even, or $\cosh(\lambda_m(L - x)), k \in \mathbb{N}, k$ odd$\}$. So $|\partial_x^k \partial_y^\ell u_m(x, y)| \le \lambda_m^{k+\ell} |A_m| \{\sinh(\lambda_m(L - x))$ or $\cosh(\lambda_m(L - x))\}\ 0 \le x \le L$. Since sinh and cosh are increasing on $\mathbb{R}_+, |\partial_x^k \partial_y^\ell u_m(x, y)| \le \lambda_m^{k+\ell} |A_m| \{\sinh(\lambda_m L)$ or $\cosh(\lambda_m L)\}\ 0 \le x \le L$. Recall $\cosh z - \sinh z = (e^z + e^{-z})/2 - (e^z - e^{-z})/2 = e^{-z} \le 1$. Again, since sinh is increasing, $\cosh(\lambda_m L) \le \sinh(\lambda_m L) + 1 \le (1+1/\sinh(\lambda_1 L)) \sinh(\lambda_m L), m \in \mathbb{N}$. We combine these considerations and find a const $c > 0$ s.t. $|\partial_x^k \partial_y^\ell u_m(x, y)| \le c\lambda_m^{k+\ell} |A_m| \sinh(\lambda_m L), k, \ell \in \mathbb{N}, 0 \le x \le L$. By T 5.3 and (6.7), u is twice PD (and satisfies LE by construction) if $\infty > \sum_{m=1}^\infty \lambda_m^2 | \int_0^H g_1(z) \sin(\lambda_m z)dz| = \sum_{m=1}^\infty | \int_0^H g_1(z)(d^2/dz^2) \sin(\lambda_m z)dz|$ (6.9). If g_1 is twice contly diff and $g_1(0) = 0 = g_1(H)$, by partial int the last expression equals $\sum_{m=1}^\infty | \int_0^H g''_1(z) \sin(\lambda_m z)dz|$, which is finite if g''_1 is Lip cont and $g_1(0) = 0 = g''_1(H)$ (T 4.14 and E 4.3.3). We summarize. **T 6.1** Let $g_1 : [0, H] \to \mathbb{R}$ be twice diff, g''_1 Lip cont and $g_1(0) = 0 = g_1(H), g''_1(0) = 0 = g''_1(H)$. Then $\exists$ a twice contly diff fctn $u : [0, L] \times [0, H] \to \mathbb{R}$ that satisfies (PDE) $(\partial_x^2 + \partial_y^2)u(x, y) = 0$, (BC) $u(0, y) = g_1(y), u(L, y) = 0, u(x, 0) = 0, u(x, H) = 0, 0 \le x \le L, 0 \le y \le H$. **The LE on a disk** Now $\Omega \subseteq \mathbb{R}^2$ is the open disk with center 0 and radius $a > 0$ and $\partial\Omega$ the circle with center 0 and radius a. We represent the solution in polar coords, $u(x, y) = v(r, \theta), x = r \cos\theta, y = r \sin\theta, 0 \le r \le a, \theta \in \mathbb{R}$, where $v(r, \theta)$ is 2π-periodic in θ. The BC is easily expressed as $v(a, \theta) = f(\theta), \theta \in \mathbb{R}$ (6.15). Here $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and cont. We translate the Laplace operator into polar coords. By the chain rule, $\partial_r v(r, \theta) = \partial_x u(x, y) \cos\theta + \partial_y u(x, y) \sin\theta, \ \partial_r^2 v(r, \theta) = \partial_x^2 u(\cos\theta)^2 + 2\partial_x\partial_y u \cos\theta \sin\theta + \partial_y^2 u(\sin\theta)^2, \ \partial_\theta v(r, \theta) = \partial_x u(x, y)(-r \sin\theta) + \partial_y u(x, y)r \cos\theta, \ \partial_\theta^2 v(r, \theta) = \partial_x^2 ur^2(\sin\theta)^2 - \partial_x\partial_y u[r^2 \sin\theta \cos\theta] - \partial_x ur \cos\theta + \partial_y^2 ur^2(\cos\theta)^2 - \partial_x\partial_y u[r^2 \sin\theta \cos\theta] - \partial_y ur \sin\theta$. Thus $\partial_r^2 v(r, \theta) + (1/r^2)\partial_\theta^2 v(r, \theta) = \partial_x^2 u + \partial_y^2 u - (1/r)\partial_x u \cos\theta - (1/r)\partial_y u \sin\theta = \Delta u - (1/r)\partial_r v(r, \theta)$. The Laplacian of u takes the polar coord form $\Delta u = (\partial_r^2 + (1/r)\partial_r + (1/r^2)\partial_\theta^2)v(r, \theta)$. The LE takes the form $(r^2\partial_r^2 + r\partial_r + \partial_\theta^2)v(r, \theta) = 0, 0 \le r < a, v(a, \theta) = f(\theta), \theta \in \mathbb{R}$ (6.16), with the understanding that $v(r, \theta)$ and $f(\theta)$ are 2π-periodic in θ We write v is a complex Fourier series in $\theta, \ v(r, \theta) = \sum_{j=-\infty}^\infty \hat{v}_j(r)e^{ij\theta}, \hat{v}_j(r) = (1/(2\pi)) \int_{-\pi}^\pi v(r, \theta)e^{-ij\theta}d\theta$. If v is smooth enough, $(r^2\partial_r^2 + r\partial_r)\hat{v}_j(r) = (1/(2\pi)) \int_{-\pi}^\pi (r^2\partial_r^2 + r\partial_r)v(r, \theta)e^{-ij\theta}d\theta = (1/(2\pi)) \int_{-\pi}^\pi (-1)\partial_\theta^2 v(r, \theta)e^{-ij\theta}d\theta$. Since $v(r, \theta)$ is 2π-periodic in $\theta, \partial_\theta^k v(r, -\pi) = \partial_\theta^k v(r, \pi)$ for $r \ge 0, k = 0, 1, \ldots.$ Since the analogous properties hold for $e^{-ij\theta}$, we IBP twice and obtain $(r^2\partial_r^2 + r\partial_r - j^2)\hat{v}_j(r) = 0, j \in \mathbb{Z}$. If $j = 0, 0 = (r\partial_r^2 + \partial_r)\hat{v}_0(r) = (d/dr)(r\hat{v}'_0(r))$. So $r\hat{v}'_0(r) = \alpha_0$ and $\hat{v}_0(r) = \alpha_0 \ln r + \beta_0$. The continuity of $\hat{v}_0$ at 0 enforces $\alpha_0 = 0$ and $\hat{v}_0$ is const, $\hat{v}_0(r) = \hat{v}_0(a) = (1/(2\pi)) \int_{-\pi}^\pi f(\eta)d\eta = \hat{f}_0$. For $j \ne 0, \hat{v}_j$ satisfies Euler's equation which is solved by the ansatz $\hat{v}_j(r) = r^n$. This yields $0 = (n - 1)n + n - j^2 = n^2 - j^2$. So $n = \pm j$ and a general solution is given by $\hat{v}_j(r) = \alpha_j r^{-j} + \beta_j r^j$. Since $\hat{v}_j$ exists at $r = 0, \hat{v}_j(r) = \gamma_j r^{|j|}, j \in \mathbb{Z}, j \ne 0$. From the BC, (6.15), $\hat{v}_j(a) = (1/(2\pi)) \int_{-\pi}^\pi f(\eta)e^{-ij\eta}d\eta = \hat{f}_j$ (6.17) and $\gamma_j = a^{-|j|}\hat{f}_j$. We sub this into the formula for $\hat{v}_j, \hat{v}_j(r) = \hat{f}_j(r/a)^{|j|}, j \ne 0$. We sub this result into the Fourier series of $v, v(r, \theta) = \sum_{j\in\mathbb{Z}} \hat{f}_j(r/a)^{|j|}e^{ij\theta}$ (6.18) $= \hat{f}_0 + \sum_{j=1}^\infty \hat{f}_{-j}(r/a)^j e^{-ij\theta} + \sum_{j=1}^\infty \hat{f}_j(r/a)^j e^{ij\theta}$ (6.19). The procedure is now alalogous to the one for the HE, $v(r, \theta) = \sum_{j=0}^\infty v_j(r, \theta), 0 \le r < a, \theta \in \mathbb{R}$, with $v_0(r, \theta) = \hat{f}_0$ and $v_j(r, \theta) = (r/a)^j(\hat{f}_{-j}e^{-ij\theta} + \hat{f}_j e^{ij\theta}), 0 \le r \le a, \theta \in \mathbb{R}, j \in \mathbb{N}$. Notice that $|\hat{f}_j| \le (1/(2\pi)) \int_{-\pi}^\pi |f(\theta)|d\theta = c_f$ and $|e^{ij\theta}| = 1$. So $|\partial_r^k\partial_\theta^\ell v_j(r, \theta)| \le ((j!j^\ell)/(j - k)!)a^{-k}(r/a)^{j-k}2c_f, k \le j, 0 \le r \le a, \theta \in \mathbb{R}$, and $\partial_r^k\partial_\theta^\ell v_j(r, \theta) = 0$ if $k > j$. By the ratio test $\sum_{j=k}^\infty ((j!j^\ell)/(j - k)!)a^{-k}(r/a)^{j-k}2c_f < \infty, 0 \le r < a$. By T 5.3, $v(r, \theta)$ is infinitely often diff at $0 \le r < a, \theta \in \mathbb{R}$ and can be diff term by term. We check that each v_j satisfies the LE in polar coord. Since v_0 is const, this holds for v_0. For $j = 1, \partial_r^2 v_1 = 0, r\partial_r v_1 = v_1$ and $\partial_\theta^2 v_1 = -v_1$. So v_1 satisfies the LE. For $j \ge 2, (r^2\partial_r^2 + r\partial_r + \partial_\theta^2)v_j = (j(j - 1) + j - j^2)v_j = 0$. Since v can be diff term by term for $r < a$, v also satisfies the LE in the interior of the disk. Notice that $|v_j(r, \theta)| \le |\hat{f}_{-j}| + |\hat{f}_j|, 0 \le r \le a$. Assume that f is Lip cont. By T4.14, $\sum_{j=1}^\infty (|\hat{f}_{-j}| + |\hat{f}_j|) < \infty$. By T5.1, $v(r, \theta)$ is cont at $0 \le r \le a, \theta \in \mathbb{R}$. We summarize. **T 6.3** Let Ω be a disk in $\mathbb{R}^2$ with the origin as center and $f : \partial\Omega \to \mathbb{R}$ be Lipschitz cont. Then $\exists$ a fctn $u : \bar{\Omega} \to \mathbb{R}$ s.t. u is cont on $\bar{\Omega} \setminus \{(0, 0)\}$, u is infinitely often diff in $\Omega \setminus \{(0, 0)\}$ and $\Delta u = 0$ on $\Omega \setminus \{(0, 0)\}$ and $u = f$ on $\partial\Omega$. We do not obtain diff at the origin right away because the transformation from polar to rectangular coords is not invertible at the origin. Similarly as for the HE, we have a representation of v via a Green's type fctn. It holds, if f is just cont. We use the definition of $\hat{f}_j$ in (6.17) and interchange series and int in (6.19), $v(r, \theta) = \int_{-\pi}^\pi f(\eta)G(r, \eta - \theta)d\eta, 2\pi G(r, \theta) = \sum_{j=-\infty}^\infty (r/a)^{|j|}e^{-ij\theta}, 0 \le r < a, \theta \in \mathbb{R}$ (6.20). Similarly as before, it can be shown that G is infinitely often diff for $0 \le r < a$ and $(r^2\partial_r^2 + r\partial_r + \partial_\theta^2)G(r, \theta) = 0, 0 \le r < a, \theta \in \mathbb{R}$. Notice that, if $f \equiv 1, v \equiv 1$ is a soln of (6.16). Certainly v is sufficiently smooth to allow all the operations we have done before. We set $v \equiv f \equiv 1$ in (6.20), $\int_{-\pi}^\pi G(r, \eta - \theta)d\eta = 1, 0 \le r < a, \theta \in \mathbb{R}$ (6.21). Differently from the HE, we can obtain an explicit expression for the GF. G can be rewritten as $2\pi G(r, \theta) = \sum_{j=0}^\infty [(r/a)e^{i\theta}]^j + \sum_{j=0}^\infty [(r/a)e^{-i\theta}]^j - 1$. The geometric series converge if $r < a, 2\pi G(r, \theta) = 1/(1 - (r/a)e^{i\theta}) + 1/(1 - (r/a)e^{-i\theta}) - 1$. We bring the expression in parentheses into a common denominator $2\pi G(r, \theta) = (1 - (r/a)^2)/(1 - (r/a)(e^{i\theta} + e^{-i\theta}) + (r/a)^2)$. This simplifies to $2\pi G(r, \theta) = (a^2 - r^2)/(a^2 - 2ra \cos\theta + r^2) > 0, 0 \le r < a, \theta \in \mathbb{R}$ (6.22). We obtain Poisson's formula for the solution of the LE in polar coords, $v(r, \theta) = (1/(2\pi)) \int_{-\pi}^\pi f(\eta)(a^2 - r^2)/(a^2 - 2ra \cos(\eta - \theta) + r^2)d\eta, 0 \le r < a$

(6.23). Since $r\cos(\eta-\theta) = r(\cos\eta\cos\theta+\sin\eta\sin\theta) = x\cos\eta + y\sin\eta$, we can express the soln in rect coords, $u(x,y) = (1/(2\pi))\int_{-\pi}^{\pi} f(\eta)(a^2-x^2-y^2)/(a^2-2ax\cos\eta-2ay\sin\eta+x^2+y^2)d\eta, x^2+y^2<a^2$ (6.24). This shows that u is inf often diff at the origin as well. By continuity, it also satisfies the LE at the origin. Using the properties of the GF, we can extend T6.3 to cont boundary data. **T 6.4** Let Ω be a disk in $\mathbb{R}^2$ and $f:\partial\Omega\to\mathbb{R}$ be cont. Then there exists a cont fctn $u:\bar\Omega\to\mathbb{R}$ such that u is infinitely often diff in Ω and $\Delta u=0$ on Ω and $u=f$ on $\partial\Omega$. See E 6.2.1. Poisson's formula in rectangular coords can be rewritten in a form that can be generalized to the ball Ω in $\mathbb{R}^2$ with radius a and arbitrary center, $u(x) = (1/(A(\partial\Omega)))\int_{\partial\Omega} f(y)(a^2-|x|^2)/(a^2-2\langle x,y\rangle+|x|^2)d\sigma(y) = (1/(A(\partial\Omega)))\int_{\partial\Omega} f(y)(|y|^2-|x|^2)/(|x-y|^2)d\sigma(y), x\in\Omega$ (6.25). Notice that x and y are now vectors in $\mathbb{R}^2$, $|x|$ is the Euclidean norm of x, and $|y|=a$ for $y\in\partial\Omega$. The symbol $d\sigma$ signalized that we take the survace integral over the sphere $\partial\Omega$ in $\mathbb{R}^2$ with radius a. $A(\partial\Omega)$ is the surface area of this sphere. This formula generalizes to $\mathbb{R}^n, u(x) = (1/(A(\partial\Omega)))\int_{\partial\Omega} f(y)(|y|^2-|x|^2)/(|x-y|^n)d\sigma(y), x\in\Omega$ (6.26). See E 6.2.2. **E 6.2.1**. Use the properties of the GF to derive T6.4 from T6.3. Hint: Approximate cont boundary data by Lip cont boundary data. Then use ideas from T5.19 and C5.20. Here is a possible seq of steps. Step 1: For a cont 2π-periodic $f:\mathbb{R}\to\mathbb{R}$ construct a seq (f_n) of Lip cont 2π-periodic functions $f_n:\mathbb{R}\to\mathbb{R}$ such that $f_n\to f$ as $n\to\infty$ uniformly on $\mathbb{R}$. Step 2: Define $v_n(r,\theta)=\int_{-\pi}^{\pi} f_n(\eta)G(r,\eta-\theta)d\eta, v(r,\theta)=\int_{-\pi}^{\pi} f(\eta)G(r,\eta-\theta)d\eta$. Apply the cosiderations leading to T6.3 to v_n and v and show that $v_n(r,\theta)\to v(r,\theta)$ as $n\to\infty$ unif for $0\le r<a$ and $\theta\in\mathbb{R}$. Step 3: Show that $v(r,\theta)\to f(\theta), r\nearrow a$ unif in $\theta\in\mathbb{R}$. Step 4: Show that, if we extend v by $v(a,\theta)=f(\theta)$, v becomes cont on $[0,a]\times\mathbb{R}$. *Prf.* Step 1: We construct a seq (f_n) of Lip cont 2π-periodic functions $f_n:\mathbb{R}\to\mathbb{R}$ such that $f_n\to f$ as $n\to\infty$ unif on $\mathbb{R}$. Define $f_n(\theta)=n\int_\theta^{\theta+(1/n)} f(\eta)d\eta, n\in\mathbb{N}$. A simple change of var gives us $f_n(\theta)=n\int_0^1 f(\theta+\eta/n)d\eta, n\in\mathbb{N}$. Since f is unif cont, $f_n\to f$ as $n\to\infty$ unif on $\mathbb{R}$. Now we diff to get $f_n'(\theta)=n(f(\theta+1/n)-f(\theta))$. Since f_n' is a function of $f(\theta)$ and f is bdd, then f_n' is also bdd and thus f_n is Lip cont and since f is 2π-periodic, so is f_n. Step 2: We now define $v_n(r,\theta)=\int_{-\pi}^{\pi} f_n(\eta)G(r,\eta-\theta)d\eta, v(r,\theta)=\int_{-\pi}^{\pi} f(\eta)G(r,\eta-\theta)d\eta$. It follows from the considerations leading to T6.4 (since G and f_n are cont) that $v_n(r,\theta)$ and $v(r,\theta)$ are cont at $0\le r<a, \theta\in\mathbb{R}$. Since v_n and v are 2π- periodic in θ, v_n and v are unif cont on $[0,a)\times\mathbb{R}$. Since G is non-neg $|v(r,\theta)-v_n(r,\theta)| = |\int_{-\pi}^{\pi}(f(\eta)-f_n(\eta))G(r,\eta-\theta)d\eta| \le \int_{-\pi}^{\pi}|f(\eta)-f_n(\eta)|G(r,\eta-\theta)d\eta \le \int_{-\pi}^{\pi}G(r,\eta-\theta)d\eta(\sup_{\eta\in\mathbb{R}}|f(\eta)-f_n(\eta)|) \le (\sup_{\eta\in\mathbb{R}}|f(\eta)-f_n(\eta)|)\to 0$, as $n\to\infty$. So $v_n(r,\theta)\to v(r,\theta)$ as $n\to\infty$ unif for $0\le r<a$ and $\theta\in\mathbb{R}$. Step 3: $\forall\ n\in\mathbb{N}, 0\le r<a$ we apply the TI, $|v(r,\theta)-f(\theta)| \le |v(r,\theta)-v_n(r,\theta)|+|v_n(r,\theta)-f_n(\theta)|+|f_n(\theta)-f(\theta)|$. Let $\epsilon>0$. $\exists\ n\in\mathbb{N}$ s.t. $|f_n(\theta)-f(\theta)|\le\epsilon/4$ and $|v(r,\theta)-v_n(r,\theta)|\le\epsilon/4\ \forall\ r\in[0,a)$ and $\theta\in\mathbb{R}$. Since v_n is unif cont on $[0,a]\times\mathbb{R}, \exists\ \delta\in(0,a)$ s.t. $|v_n(r,\theta)-f_n(\theta)|\le\epsilon/4$ if $a-\delta<r<a$. Now we let $a-\delta<r<a$ and $\theta\in\mathbb{R}$. Then we put this together to get $|v(r,\theta)-f(\theta)|<\epsilon$. So $v(r,\theta)\to f(\theta), r\nearrow a$ unif in $\theta\in\mathbb{R}$. Step 4: We know that v is cont on $[0,a)\times\mathbb{R}$ and we know that f is cont. Let $\epsilon>0$. $\exists\ \delta>0$ such that $|f(\eta)-f(\theta)|<\epsilon/2$ if $|\eta-\theta|<\delta$. We can choose δ s.t. $\delta\in(0,a)$ and $|v(r,\xi)-f(\xi)|<\epsilon/2$ whenever $a-\delta<r<a, \xi\in\mathbb{R}$. We use the TI to get $|v(r,\eta)-f(\theta)|\le|v(r,\eta)-f(\eta)|+|f(\eta)-f(\theta)|<\epsilon$. The transition between polar and rect coords is cont in both directions on $\mathbb{R}^2\setminus\{(0,0)\}$ and so u is cont on $\bar\Omega\setminus\{(0,0)\}$. Equation (6.19), Poisson's formula in rectangluar coords, now shows the cont of u at $(0,0)$ ☐ **Weak max principle** Let $\Omega\subseteq\mathbb{R}^n$ be open and bdd. For a fctn $u:\Omega\to\mathbb{R}$ that is twice PD wrt x_k at each $x_k\in\Omega, k=1,\dots,n$, let $(Lu)(x)=\sum_{k=1}^n[a_k\partial_k^2u(x)+b_k\partial_ku(x)], x\in\Omega$ (6.27), with $a_k\ge0$ for $k=1,\dots,n$ and $\sum_{k=1}^n(a_k+|b_k|)>0$. **T6.7**. Assume that $u:\bar\Omega\to\mathbb{R}$ is cont and twice PD on Ω as above and satisfies $(Lu)(x)\ge0, x\in\Omega$. Then $\max_{\bar\Omega}u=\max_{\partial\Omega}u$. *Prf.* Case 1: $(Lu)(x)>0\ \forall\ x\in\Omega$. Since u is cont on $\bar\Omega, \exists\ x\in\bar\Omega$ s.t. $u(x)=\max_{\bar\Omega}u$. If $x\in\Omega$, then $\partial_ju(x)=0$ and $\partial_j^2u(x)\le0, j=1,\dots,n$. So $(Lu)(x)\le0$, a contradiction. So $x\in\partial\Omega$ and the assertion follows. Case 2: $(Lu)(x)\ge0\ \forall\ x\in\Omega$. For $\epsilon>0$, set $u_\epsilon(x)=u(x)+\epsilon c\sum_{j=1}^n\xi_jx_j+\epsilon|x|^2, x\in\bar\Omega$, where $\xi_j\in\{0,1,-1\}$ have the sign of b_j, $|x|$ denotes the Euclidean norm and $c>0$ will be determined. Then $\partial_ju_\epsilon(x)=\partial_ju(x)+\epsilon c\xi_j+2cx_j$ and $\partial_j^2u_\epsilon(x)=\partial_j^2u(x)+2\epsilon$. By (6.27) $(Lu_\epsilon)(x)=(Lu)(x)+\epsilon\sum_{j=1}^nc|b_j|+2\epsilon\sum_{j=1}^nb_jx_j+2\epsilon\sum_{j=1}^na_j$. By the Cauchy-Schwarz inequality in $\mathbb{R}^n$ and $(Lu)(x)\ge0$, with $b=(b_1,\dots,b_n), (Lu_\epsilon)(x)\ge\epsilon(c\sum_{j=1}^n|b_j|-2|b||x|+2\sum_{j=1}^na_j)$. Recall that $|b|\le\sum_{j=1}^n|b_j|$. Since Ω is bdd, $\exists\ c>0$ s.t. $|x|\le c/3\ \forall\ x\in\Omega$. So $(Lu_\epsilon)(x)\ge\epsilon\sum_{j=1}^n[|b_j|(c-2|x|)+2a_j]\ge\epsilon\sum_{j=1}^n[(c/3)|b_j|+2a_j]>0$. By case 1, $\max_{\bar\Omega}u_\epsilon=\max_{\partial\Omega}u_\epsilon$. Now, for $x\in\bar\Omega$, by the Cauchy-Schwartz inequality in $\mathbb{R}^n, u(x)\le u_\epsilon(x)+\epsilon c\sqrt{n}|x|\le u_\epsilon(x)+\epsilon c^2\sqrt{n}$ and $u_\epsilon(x)\le u(x)+\epsilon c^2(\sqrt{n}+1)$. So $\max_{\bar\Omega}u\le\max_{\bar\Omega}u_\epsilon+\epsilon c^2\sqrt n\le\max_{\partial\Omega}u_\epsilon+\epsilon c^2\sqrt n\le\max_{\partial\Omega}u+\epsilon c^2(\sqrt n+1)$. Since this holds for any $\epsilon>0, \max_{\bar\Omega}u\le\max_{\partial\Omega}u$. The opposite inequality is trivial ☐ **C6.8** Assume that $u:\bar\Omega\to\mathbb{R}$ is cont and twice PD on Ω as above and satisfies $(Lu)(x)\le0, x\in\Omega$. Then $\min_{\bar\Omega}u=\min_{\partial\Omega}u$. *Prf.* Apply T6.7 to $-u$ and use $\min u=-\max(-u)$ ☐ **C6.9** Assume that $u:\bar\Omega\to\mathbb{R}$ is cont and twice PD on Ω as above and satisfies $(Lu)(x)=0, x\in\Omega$. Then $\min_{\partial\Omega}u\le u(x)\le\max_{\partial\Omega}u, x\in\Omega$, and $\max_{\bar\Omega}|u|=\max_{\partial\Omega}|u|$. *Prf.* The 1st inequality is immediate from T6.7 and C6.8. Then, $\forall\ x\in\Omega, -\max_{\partial\Omega}|u|\le u(x)\le\max_{\partial\Omega}|u|$ and $|u(x)|\le\max_{\partial\Omega}|u|$. Since this holds $\forall\ x\in\bar\Omega, \max_{\bar\Omega}|u|\le\max_{\partial\Omega}|u|$. The opposite inequality is trivial. **C6.10**. For given $f:\bar\Omega\to\mathbb{R}$ and $g:\partial\Omega\to\mathbb{R}\ \exists$ at most one cont fctn $u:\bar\Omega\to\mathbb{R}$ that is twice diff on Ω and satisfies $Lu=f$ on $\Omega, u=g$ on $\partial\Omega$. *Prf.* Assume $\exists$ two such fctns u_1 and u_2 s.t. $Lu_j=f$ on $\Omega, u_j=g$ on $\partial\Omega, j=1,2$. Set $v=u_1-u_2$. The $Lv=0$ on $\Omega, v=0$ on $\partial\Omega$. This implies $\max_{\bar\Omega}|v|=\max_{\partial\Omega}|v|=0$. So $u_1(x)=u_2(x)\ \forall\ x\in\bar\Omega$ ☐ Assume in all these exercises that Ω is a bdd open subset of $\mathbb{R}^n$, Δ the Laplacian and L the PD operator defined in (6.27). **E6.3.1**. Prove from scratch: if $u:\bar\Omega\to\mathbb{R}$ is cont and is twice PD on Ω and satisfies $\Delta u\le0, x\in\Omega$, then $\min_{\bar\Omega}u=\min_{\partial\Omega}u$. *Prf.* Case 1: $\Delta u<0\ \forall\ x\in\Omega$. Since u is cont on the closed bdd set $\bar\Omega\subseteq\mathbb{R}^n, \exists\ x\in\bar\Omega$ s.t. $u(x)\le u(y)\ \forall\ y\in\bar\Omega$. Suppose $x\in\Omega$. Then, for $j=1,\dots,n, \partial_ju(x)=0$ and $\partial_j^2u(x)\ge0$. We sum over j and obtain $\Delta u(x)\ge0$, a contradiction. This proves that $x\in\partial\Omega$ and $\min_{\bar\Omega}u=u(x)\ge\min_{\partial\Omega}u$. The opposite inequality holds because $\partial\Omega\subseteq\bar\Omega$. Case 2: $\Delta u(x)\le0\ \forall\ x\in\Omega$. For $\epsilon>0$ set $u_\epsilon(x)=u(x)-\epsilon|x|^2, x\in\bar\Omega$. Then $\partial_ju_\epsilon(x)=\partial_ju(x)-2\epsilon x_j, \partial_j^2u_\epsilon(x)=\partial_j^2u(x)-2\epsilon$. We sum over j from 1 to $n, \Delta u_\epsilon(x)=\Delta u(x)-2n\epsilon\le-2n\epsilon<0, x\in\Omega$. By case 1: $\min_{\bar\Omega}u_\epsilon=\min_{\partial\Omega}u_\epsilon$. Now, $\forall\ x\in\bar\Omega, u(x)\ge u_\epsilon(x)$ and $u_\epsilon(x)\ge u(x)-\epsilon|x|^2\ge u(x)-\epsilon c^2$ where the const $c>0$ has been chosen s.t. $|x|\le c\ \forall\ x\in\Omega$ (recall that Ω is bdd). Then $\min_{\bar\Omega}u\ge\min_{\bar\Omega}u_\epsilon=\min_{\partial\Omega}u_\epsilon\ge\min_{\partial\Omega}u-\epsilon c^2$. Since this holds for each $\epsilon>0, \min_{\bar\Omega}u\ge\min_{\partial\Omega}u$. The opposite inequality holds because $\partial\Omega\subseteq\bar\Omega$ ☐ **E6.3.5**. Let Ω be an open bdd subset of $\mathbb{R}^2$. Let $u:\bar\Omega\to\mathbb{R}$ be cont and twice contly diff on Ω and satisfy $(\partial_x^2+\partial_y^2)u-\partial_xu+\partial_yu\ge0, (x,y)\in\Omega$. Prove from scratch that $\max_{\bar\Omega}u=\max_{\partial\Omega}u$. *Prf.* Case 1: Assume $(\partial_x^2+\partial_y^2)u-\partial_xu+\partial_yu>0, (x,y)\in\Omega$. Since u is cont, $\exists$ a point $(x,y)\in\bar\Omega$ s.t. $u(x)=\max_{\bar\Omega}u$. If $(x,y)\in\Omega, \partial_xu(x,y)=0=\partial_yu(x,y)$ and $\partial_x^2u(x,y)\le0$ and $\partial_y^2u(x,y)\le0$. So $(\partial_x^2+\partial_y^2)u-\partial_xu+\partial_yu\le0$, a contradiction. So $(x,y)\in\partial\Omega$ and the assertion follows. Case 2: Assume $(\partial_x^2+\partial_y^2)u-\partial_xu+\partial_yu\ge0, (x,y)\in\Omega$. $\forall\ \epsilon>0$ set $u_\epsilon(x,y)=u(x,y)+\epsilon(x-y)$. Then $(\partial_x^2+\partial_y^2)u_\epsilon-\partial_xu_\epsilon+\partial_yu_\epsilon=(\partial_x^2+\partial_y^2)u-\partial_xu+\partial_yu+2\epsilon>0$. By case 1, $\max_{\bar\Omega}u_\epsilon=\max_{\partial\Omega}u_\epsilon$. Since $\bar\Omega$ is bdd, $\exists\ c>0$ s.t. $|x|+|y|\le c\ \forall\ (x,y)\in\Omega$. So $\max_{\bar\Omega}u\le\max_{\bar\Omega}u_\epsilon+\epsilon c\le\max_{\partial\Omega}u_\epsilon+\epsilon c\le\max_{\partial\Omega}u+2\epsilon c$. Since this holds for any $\epsilon>0, \max_{\bar\Omega}u\le\max_{\partial\Omega}u$. Since the opposite inequality is trivially true, equality holds ☐ Consider (PDE) $(\partial_x^2+\partial_y^2)u(x,y)=0, 0<x<L, 0<y<H$, (BC) $u(0,y)=g_1(y), u(L,y)=0, 0<y<H, u(x,0)=0, u(x,H)=0, 0<x<L$ (6.28). This time we only assume that g_1 is cont and $g_1(0)=0=g_1(H)$. As in the proof of T5.16, we find a sequence of Lip cont fctns which are zero at 0 and H that converges to g_1 unif on $[0,H]$. Every Lip cont fctn that is zero at 0 and H can be unif approximated by its Fourier sine series. This implies that $\exists$ a seq of inf often diff fctns $\tilde g_n:[0,H]\to\mathbb{R}$ such that $\tilde g_n\to g_1$ as $n\to\infty$ unif on $[0,H], \tilde g_n(0)=0=\tilde g_n(H), \tilde g_n''(0)=0=\tilde g_n''(H)$. Let u_n be the solution of the BVP (6.28) with $\tilde g_n$ replacing $g_1, \Omega=(0,L)\times(0,H)$. These solutions exist by T6.1. By E 6.3.2, $\max_{\bar\Omega}|u_n-u_m|=\max_{\partial\Omega}|u_n-u_m|=\max_{[0,L]}|\tilde g_n-\tilde g_m|$. This implies that, for each $x\in[0,L], y\in[0,H], (u_n(x,y))$ is a (unif) Cauchy sequence. Let $u(x,y)=\lim_{n\to\infty}u_n(x,y)$. Then $u_n\to u$ as $n\to\infty$ unif on $\bar\Omega=[0,L]\times[0,H]$ and u is cont, and satisfies the BC in (6.28). In order to show that $\Delta u=0$ on Ω, let $z_0\in\Omega$. Since Ω is open, there is an open disk D with center z_0 and radius a such that $\bar D$ is contained in Ω. Let D_0 be the open disk with center $(0,0)$ and radius a. Set $\tilde u_n(z)=u_n(z+z_0), \tilde u(z)=u(z+z_0), z\in\bar D_0, f_n(\theta)=u_n(z_0+a(\cos\theta,\sin\theta)), f(\theta)=u(z_0+a(\cos\theta,\sin\theta)), \theta\in\mathbb{R}$. Then $\Delta\tilde u_n=0$ on D_0 and $\tilde u_n(a\cos\theta,a\sin\theta)=f_n(\theta)$ for $\theta\in\mathbb{R}$. By (6.24), $\tilde u_n(x,y)=1/(2\pi)\int_{-\pi}^{\pi}f_n(\eta)(a^2-x^2-y^2)/(a^2-2ax\cos\eta-2ay\sin\eta+x^2+y^2)d\eta, x^2+y^2<a^2$. Since $\tilde u_n\to\tilde u$ unif on $\bar D_0$ and $f_n\to f$ unif on $\mathbb{R}$, we can take the limit as $n\to\infty, \tilde u(x,y)=1/(2\pi)\int_{-\pi}^{\pi}f(\eta)(a^2-x^2-y^2)/(a^2-2ax\cos\eta-2ay\sin\eta+x^2+y^2)d\eta, x^2+y^2<a^2$. Then $\tilde u$ is infinitely often diff on D_0 and satisfies LE there. So u is infinitely often diff on D and satisfies LE on D. Since we can find such a disk around any $z_0\in\Omega$, u is infinitely often diff on Ω and $\Delta u=0$ on Ω. **T6.11**. Let $\Omega=(0,L)\times(0,H)$ and $g:\partial\Omega\to\mathbb{R}$ cont. Then $\exists$ a cont fctn $u:\bar\Omega\to\mathbb{R}$ that is infinitely often diff on Ω and satisfies $\Delta u=0$ on Ω and $u=g$ on $\partial\Omega$. *Prf.* Splitting the BVP into four parts and adding the four solutions yields a solution of the original problem provided that g is zero at the four corners of the rectangle. Suppose that this is not the case. Let $\phi(x,y)=a_0+a_1x+a_2y+a_3xy$. Then $\Delta\phi=0$. We determine the coeff in such a way that ϕ equals g at the corners of the rectangle. For the origin, we get $a_0=g(0,0)$. For $(0,L), g(L,0)=a_0+a_1L$. and so $a_1=(g(L,0)-a_0)/L$. Similarly $a_2=(g(0,H)-a_0)/H$. Finally $g(L,H)=a_0+a_1L+a_2H+a_3LH$, from which we determine a_3. By our previous consideration, $\exists$ a cont fctn $\tilde u$ which is infinitely often diff on Ω and satisfies $\Delta\tilde u=0$ and $\tilde u=g-\phi$ on $\partial\Omega$. We set $u=\tilde u+\phi$. Then u has all the required properties ☐ Let Ω be an open bdd subset of $\mathbb{R}^n$ and $f:\Omega\to\mathbb{R}^n$ be differentiable, $f(x)=(f_1(x),\dots,f_n(x))$. Then $\operatorname{div}f(x):=\sum_{j=1}^n\partial_jf_j(x), x\in\Omega$ (7.1). Ω is called normal if the divergence thm (Gauß' integral thm) holds for every cont fctn $f:\bar\Omega\to\mathbb{R}^n$ with cont bdd derivative on Ω: $\int_\Omega\operatorname{div}f(x)dx=\int_{\partial\Omega}f(x)\cdot\nu(x)d\sigma(x)$(7.2). Here $\nu(x)$ is the outer unit normal vector at $x\in\partial\Omega:\exists\ \epsilon>0$ s.t. $x+\xi\nu(x)\not\ni\bar\Omega, x-\xi\nu(x)\in\Omega, 0<\xi<\epsilon$. The notation $d\sigma(x)$ signalized that we take a surface integral. $f(x)\cdot\nu(x)=\sum_{j=1}^nf_j(x)\nu_j(x)$ is the Euclidean inner product in $\mathbb{R}^n$. Balls with respect to the three standard norms and Cartesian products of intervals are normal. For Ω to be normal, $\partial\Omega$ must allow surface integration and have a cont outer normal, but additional assumptions must be satisfied. This is an equivalent componentwise formulation of Gauß' thm. **T7.1**. Ω is normal iff $\int_\Omega\partial_jg(x)dx=\int_{\partial\Omega}g(x)\nu_j(x)d\sigma(x), j=1,\dots,n$. for every cont $g:\bar\Omega\to\mathbb{R}$ with cont and bdd derivative on Ω. *Prf.* $\Rightarrow$ Let $j\in\{1,\dots,n\}$ and set $f=(0,\dots,0,g,0,\dots,0)$ such that $f_j=g$. $\Leftarrow$ Apply this formula to $g=f_j, j=1,\dots,n$ and add over j ☐ The following result generalizes IBP. **T7.2** (Green's thm). Let Ω be normal and consider 2 cont fctns $u,v:\bar\Omega\to\mathbb{R}$ with cont bdd derivatives on Ω. Then $\int_\Omega(u\partial_jv+v\partial_ju)dx=\int_{\partial\Omega}uv\nu_jd\sigma, j=1,\dots,n$. *Prf.* Set $g=uv$ in T7.1 ☐ **T7.3** (GFs). Let Ω, u, and v be as in the previous thm. Assume that the derivative of v can and has been contly extended to $\bar\Omega$. If v has a cont bdd second derivative on $\Omega, \int_\Omega u\Delta vdx+\int_\Omega\nabla u\cdot\nabla vdx=\int_{\partial\Omega}u\partial_\nu vd\sigma$. If both u and v have bdd cont second derivatives on Ω and if the first derivatives of u and v can and have been contly extended to $\bar\Omega, \int_\Omega(u\Delta v-v\Delta u)dx=\int_{\partial\Omega}(u\partial_\nu v-v\partial_\nu u)d\sigma$. Here $\nabla u\cdot\nabla v=\sum_{j=1}^n\partial_ju\partial_jv$ and $\partial_\nu v=\nu\cdot\nabla v=\sum_{j=1}^n\nu_j\partial_jv$. We mention that $\partial_\nu v$ is called the normal derivative of v. *Prf.* In Green's thm, replace v by $\partial_jv, \int_\Omega(u\partial_j^2v+\partial_jv\partial_ju)dx=\int_{\partial\Omega}u\partial_jv\nu_jd\sigma$, and add over j. For the 2nd formula, use the symmetry in u and v and subtract ☐ **P7.4**. Let $\Omega\subseteq\mathbb{R}^n$ be open and $u:\Omega\to\mathbb{R}$ be diff and $\nabla u\equiv0$ on Ω. Then u is locally const on Ω: for each $x\in\Omega\ \exists$ some open nbhood U of x st. u is cons on U. If Ω is path-connected, then u is const on Ω. **Recall** that Ω is path-connected if, for any $x,y\in\Omega$, there is an interval $[a,b]$ and a cont "path" $\gamma:[a,b]\to\Omega$ s.t. $\gamma(a)=x$ and $\gamma(b)=y$. *Prf.* Let $x\in\Omega$. Since Ω is open, $\exists\ r>0$ s.t. $U_r(x)=\{z\in\mathbb{R}^n;|z-x|<r\}\subseteq\Omega$. Let $z\in U_r(x)$. Then, $\forall\ \xi\in[0,1], |\xi z+(1-\xi)x-x|=|\xi(z-x)|\le|z-x|<r$ and $\xi z+(1-\xi)x\in U_r(x)\subseteq\Omega$. Thus $\phi(\xi)=u(\xi z+(1-\xi)x)$ is defined $\forall\ \xi\in[0,1]$ and diff, $\phi'(\xi)=\nabla u(\xi z+(1-\xi)x)\cdot(z-x)=0$. So $u(z)=\phi(1)=\phi(0)=u(x), z\in U_r(x)$. Now assume that Ω is path-connected. Let $x,y\in\Omega$. Assume that $u(x)\ne u(y)$. There is an interval $[a,b]$ and a cont path $\gamma:[a,b]\to\Omega$ such that $\gamma(a)=x$ and $\gamma(b)=y$ and so $\gamma(a)\ne\gamma(b)$. Let $t=\sup\{s\in[a,b];u(\gamma(s))=u(\gamma(a))\}$. Then $a\le t<b$ and $u(\gamma(t))=u(\gamma(a))$ because u is cont. Since u is locally const, $\exists$ an open ball U with center $u(\gamma(t))$ such that $u(z)=u(\gamma(t))\ \forall\ z\in U$. Since $t<b$ and γ is continous, $\exists\ s>t$ s.t. $\gamma(s)\in U$ and so $u(\gamma(s))=u(\gamma(t))=u(\gamma(a))$, a contradiction to the definition of t. So $u(x)=u(y)$ for any two points x and y in Ω, i.e. u is const on Ω ☐ Consider $\Omega\subseteq\mathbb{R}^n$. $\Delta u=f$ on $\Omega, \beta(x)\partial_\nu u(x)+\alpha(x)u(x)=g(x), x\in\partial\Omega$ (7.3). Here α and β are nonneg cont real-valued fctns on $\partial\Omega, \alpha(x)+\beta(x)>0\ \forall\ x\in\partial\Omega$. Special cases: $u=g$ on $\partial\Omega$: Dirichlet BCs, $\partial_\nu u=g$ on $\partial\Omega$: Neumann BCs, $\partial_\nu u+\alpha(x)u=g$ on $\partial\Omega$: Robin BCs. **T7.5**. Any 2 solns of (7.3) that have cont bdd 2nd partial derivatives on Ω and whose 1st derivatives and themselves can be contly extended to $\bar\Omega$ have identical gradients. If Ω is path-connected, they are equal up to a const. If, in addition, $\alpha(x)\ne0$ for some $x\in\partial\Omega$, there is at most one soln. *Prf.* Let u_1 and u_2 be two solns. Then $u=u_1-u_2$ is cont on $\bar\Omega$, its 1st derivative can be contly extended to $\bar\Omega$ and u is twice contly diff on Ω. Further $\Delta u=0$ on $\Omega, \beta(x)\partial_\nu u(x)+\alpha(x)u(x)=0, x\in\partial\Omega$. By GF1, $0\le\int_\Omega|\nabla u|^2dx=\int_\Omega\nabla u\cdot\nabla udx+\int_\Omega u\Delta udx=\int_{\partial\Omega}u\partial_\nu ud\sigma$. If $x\in\partial\Omega$, there are two cases: either $\beta(x)=0$, then $\alpha(x)u(x)=0$ and $\alpha(x)>0$ and so $u(x)=0$, or $\beta(x)>0$, then $\partial_\nu u(x)=-\alpha(x)/\beta(x)u(x)$. So $0\le\int_\Omega|\nabla u|^2dx=-\int_{\partial\Omega\cap\{\beta>0\}}\alpha(x)/\beta(x)u^2(x)d\sigma(x)\le0$. This implies that $0=\int_\Omega|\nabla u|^2dx$. Since ∇u is cont, $\nabla u\equiv0$ on Ω. If Ω is path-connected, u is const by P 7.4. Then $0 \equiv \alpha u$ on $\partial\Omega$. So, if $\alpha(x) > 0$ for some $x \in \partial\Omega, u \equiv 0$ on Ω $\square$ Neumann BVP $\Delta u = f$ on $\Omega, \partial_\nu u(x) = g(x), x \in \partial\Omega$ (7.4). From GF1, for sufficiently smooth $v, \int_\Omega \nabla v \cdot \nabla u dx + \int_\Omega v \Delta u dx = \int_{\partial\Omega} v \partial_\nu u d\sigma$. So, for $v \equiv 1, \int_\Omega \Delta u dx = \int_{\partial\Omega} \partial_\nu u d\sigma$ and so $\int_\Omega f dx = \int_{\partial\Omega} g d\sigma$. **T7.6**. The Neumann BVP $\Delta u = f$ on $\Omega, \partial_\nu u = g$ on $\partial\Omega$ only has a soln if $\int_\Omega f dx = \int_{\partial\Omega} g d\sigma$. If u is a soln, then $\tilde{u}(x) = u(x) + c$ with a const c is also a soln. HE in several space dims, $(\partial_t - a\Delta_x - c(x,t))u = f(x,t), x \in \Omega, t \in (0,T), \beta(x,t)\partial_\nu u(x,t) + \alpha(x,t)u(x,t) = 0, x \in \partial\Omega, t \in (0,T)$ (7.5). Ω is a normal subset of $\mathbb{R}^n$, a is a pos const, $\Delta_x = \sum_{j=1}^n \partial^2/\partial x_j^2, \alpha$ and β are cont non-neg fctns on $\partial\Omega \times (0,T)$ and $\alpha + \beta$ is strictly pos. Further c and f are cont bdd fctns on $\Omega \times (0,T)$. We derive an energy estimate for $\int_\Omega u^2(x,t)dx$. Assume that u is cont on $\bar{\Omega} \times [0,T]$ and that the PDs $\partial_t u$ exist and are cont on $\bar{\Omega} \times (0,T)$. Assume that $u(x,t)$ is twice contly diff in x and that $\nabla_x u$ can be contly extended to $\bar{\Omega} \times (0,T)$. Then the derivative $\partial_t u^2 = 2u\partial_t u$ exists and is cont on $\bar{\Omega} \times (0,T)$. So $\int_\Omega u^2(x,t)dx$ is diff in $t \in (0,T)$ and $d/dt \int_\Omega u^2(x,t)dx = \int_\Omega \partial_t u^2(x,t)dx = \int_\Omega 2u(x,t)\partial_t u(x,t)dx = \int_\Omega 2u(x,t)((a\Delta_x + c(x,t))u(x,t) + f(x,t))dx = 2a \int_\Omega u(x,t)\Delta_x u(x,t)dx + 2\int_\Omega c(x,t)u^2(x,t)dx + 2\int_\Omega u(x,t)f(x,t)dx$. By GF1, $\int_\Omega u(x,t)\Delta_x u(x,t)dx = -\int_\Omega \nabla_x u(x,t) \cdot \nabla_x u(x,t)dx + \int_{\partial\Omega} u(x,t)\partial_\nu u(x,t)d\sigma(x) = -\int_\Omega |\nabla u(x,t)|^2 dx - \int_{\partial\Omega \cap \{\beta > 0\}} \alpha(x,t)/\beta(x,t)u^2(x,t)d\sigma(x) \leq 0$. So $\partial_t \int_\Omega u^2(x,t) \; dx \leq 2\int_\Omega c(x,t)u^2(x,t)dx + 2\int_\Omega u(x,t)f(x,t)dx$. Let $\bar{c} = \sup_{\Omega \times (0,T)} c$. Choose some $\epsilon > 0$. By the Cauchy-Schwarz ineq, $\partial_t \int_\Omega u^2(x,t)dx \leq 2\bar{c}\int_\Omega u^2(x,t)dx + 2(\int_\Omega u^2(x,t)dx)^{1/2}(\int_\Omega f^2(x,t)dx)^{1/2} \leq (2\bar{c} + \epsilon)\int_\Omega u^2(x,t)dx + 1/\epsilon \int_\Omega f^2(x,t)dx$. Here we have used the ineq $2rs \leq \epsilon s^2 + (1/\epsilon)r^2$. Set $\kappa = 2\bar{c} + \epsilon$. Using an integrating factor, we obtain $\int_\Omega u^2(x,t)dx \leq e^{\kappa t}\int_\Omega u^2(x,0)dx + 1/\epsilon \int_0^t e^{k(t-s)}(\int_\Omega f^2(x,s)dx)ds$. Assume $\bar{c} < 0$. Then choose $\epsilon > 0$ s.t. $\kappa < 0 \implies e^{\kappa t}\int_\Omega u^2(x,0)dx \to 0, t \to \infty$ and $1/\epsilon \int_0^t e^{k(t-s)}(\int_\Omega f^2(x,s)dx)ds \leq \sup_{s \in (0,t)} \int_\Omega f^2(s,x)dx 1/(-\kappa\epsilon)$. Notice that $-\kappa\epsilon = -(2\bar{c} + \epsilon)\epsilon = (2|\bar{c}| - \epsilon)\epsilon$ takes its maximum at $\epsilon = |\bar{c}|$ where it is $\bar{c}^2$. So we pick the estimate, $\int_\Omega u^2(x,t)dx \leq e^{\bar{c}t}\int_\Omega u^2(x,0)dx + 1/\bar{c}^2 \sup_{0<s<t}\int_\Omega f^2 \; (x,s)dx$. Consider the WE on a normal subset Ω of $\mathbb{R}^n$. $(\partial_t^2 - c^2\Delta_x)u = 0, x \in \Omega, t \in (0,T), u(x,t) = g(x), x \in \partial\Omega, t \in (0,T), u(x,0) = \phi(x), x \in \Omega, \partial_t u(x,0) = \psi(x), x \in \Omega$ (7.6). Define $E(t) = 1/2\int_\Omega((\partial_t u(x,t))^2 + c^2|\nabla_x u(x,t)|^2)dx$ (7.7). Assume that u is twice contly diff on $\Omega \times (0,T)$ and that the 1st and 2nd PDs are bdd. Assume that $\nabla_x u$ and $\partial_t u$ can be contly extended to $\Omega \times (0,T)$. Then $\partial_t(\partial_t u)^2 = 2\partial_t u \partial_t^2 u$ exists and is cont and bdd on $\Omega \times (0,T)$. Moreover $\partial_t|\nabla_x u|^2 = 2\nabla_x u \cdot \partial_t \nabla_x u = 2\nabla_x u \cdot \nabla_x \partial_t u$ exists and is cont and bdd on $\Omega \times (0,T)$. This implies that E is diff and diff and int can be interchanged and $E'(t) = \int_\Omega(\partial_t u(x,t)\partial_t^2 u(x,t) + c^2\nabla_x u(x,t) \cdot \partial_t \nabla_x u(x,t))dx = c^2(\int_\Omega \partial_t u(x,t)\Delta_x u(x,t)dx + \int_\Omega \nabla_x u(x,t) \cdot \nabla_x \partial_t u(x,t))dx)$. By GF1, $E'(t) = c^2\int_{\partial\Omega} \partial_t u(x,t)\partial_\nu u(x,t)d\sigma(x)$. Since $u(x,t) = g(x)$ for $x \in \partial\Omega, t \in (0,T), \partial_t u(x,t) = 0$ for $x \in \partial\Omega, t \in (0,T)$. So $E'(t) = 0 \; \forall \; t \in (0,T)$. Thus $E(t) = E(0) = 1/2\int_\Omega(\psi^2(x) + c^2|\nabla\phi(x)|^2)dx$. In the following, Ω is always a normal set contained in $\mathbb{R}^n$. **E7.3.4**. Consider the Newmann boundary problem for the LE. $\Delta u(x) + c(x)u(x) = f(x), \quad x \in \Omega, \partial_\nu u(x) = g(x), \quad x \in \partial\Omega$. Assume that Ω is path-connected, $c : \Omega \to \mathbb{R}$ is nonpos and cont and $c(x) < 0$ for some $x \in \Omega$. Show that there is at most one soln u. *Prf.* Let u_1 and u_2 be two solns and $u = u_1 - u_2$. Then $\Delta u(x) + c(x)u(x) = 0, \quad x \in \Omega, \partial_\nu u(x) = 0, \quad x \in \partial\Omega$. By GF1, $0 \leq \int_\Omega |\nabla u(x)|^2 = \int_\Omega \nabla u(x) \cdot \nabla u(x)dx + \int_\Omega(\Delta u(x) + c(x)u(x))u(x)dx = \int_\Omega \nabla u(x) \cdot \nabla u(x)dx + \int_\Omega \Delta u(x)u(x) + \int_\Omega c(x)u(x)^2 dx = \int_{\partial_\Omega} u(x)\partial_\nu u(x)d\sigma + \int_\Omega c(x)u(x)^2 dx = \int_\Omega c(x)u(x)^2 dx$. Because $c(x) < 0$ we have that $0 \leq \int_\Omega |\nabla u(x)|^2 = \int_\Omega c(x)u(x)^2 dx \leq 0$. And so by P 7.4, $\nabla u(x) = 0$ on Ω and so $u(x)$ is const. Also $0 = \int_\Omega c(x)u(x)^2 dx = u(x)^2\int_\Omega c(x)dx$. Because $c(x) \neq 0, \int_\Omega c(x)dx \neq 0$ and therefore $u(x)^2 = 0$ and so $u(x) = 0$ which shows that $u_1 = u_2$ and so there is at most one soln u $\square$ **E7.3.5**. Consider the HE with mixed BC $(\partial_t - a\Delta_x - c(t,x))u = f(t,x), x \in \Omega, \; t \in (0,T), \beta(x,t)\partial_\nu u(x,t) + \alpha(x,t)u(x,t) = g(x,t), x \in \partial\Omega, \; t \in (0,T), u(x,0) = \phi(x), x \in \Omega$, with a bdd cont fctn $c : \Omega \times (0,T) \to \mathbb{R}$ and α and β as before. Show: Given f, g, ϕ, there is at most one soln u that satisfies smoothness assumptions of Section 7.2. *Prf.* Let u_1 and u_2 be 2 solutions. Set $v = u_1 - u_2$. Then $(\partial_t - a\Delta_x - c(t,x))v = 0, x \in \Omega, \; t \in (0,T), \beta(x,t)\partial_\nu v(x,t) + \alpha(x,t)v(x,t) = 0, x \in \partial\Omega, \; t \in (0,T), v(x,0) = 0, x \in \Omega$, By the considerations in Section 7.2, $\int_\Omega v^2(x,t)dx \leq 0$. Since this integral is non-neg, it is 0. Since v^2 is cont and non-neg, $v^2(x,t) = 0$ and so we have that $u_1(x,t) = u_2(x,t) \; \forall \; t \in (0,T), x \in \Omega$ $\square$ Consider $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ solving $\partial_t u(x,t) = \partial_x^2 u(x,t), x \in \mathbb{R}, t > 0, u(x,0) = u_0(x), x \in \mathbb{R}$ (9.1), for $u_0 : \mathbb{R} \to \mathbb{R}$. Let $\Gamma(x,t) := (4\pi t)^{-1/2}e^{-x^2(4t)^{-1}}, x \in \mathbb{R}, t > 0$ (9.2). Γ is strictly pos and inf often diff, $\partial_x\Gamma(x,t) = -x/(2t)^{-1}\Gamma(x,t), \partial_x^2\Gamma(x,t) = x^2(2t)^{-2}\Gamma(x,t) - (2t)^{-1}\Gamma(x,t), \partial_t\Gamma(x,t) = -2\pi(4\pi t)^{-3/2}e^{-x^2(4t)^{-1}} + x^2 4/(4t)^{-2}\Gamma(x,t) = \partial_x^2\Gamma(x,t)$ (9.3). Notice Γ and these PDs are bdd and integrable on $\mathbb{R} \times [\epsilon, c]$ for any $c > \epsilon > 0$ as are all higher derivatives. We combine the 2nd and 3rd eq and reorg, $\partial_t\Gamma(x,t) = [x^2 - (2t)](2t)^{-2}\Gamma(x,t)$. **L9.1**. (a) for each $x \in \mathbb{R}, \Gamma(x,\cdot)$ is strictly increasing on $(0, x^2/2)$ and strictly decreasing on $(x^2/2, \infty)$. (b) For each $t > 0, \Gamma(\cdot,t)$ is strictly increasing on $(-\infty, 0)$, and strictly decreasing on $(0,\infty)$. Also, by the change of variables $x = y(4t)^{1/2}, \int_\mathbb{R} \Gamma(x,t)dx = (\pi)^{-1/2}\int_\mathbb{R} e^{-y^2}dx = 1$ (9.4). Notice that, for any $t > 0, \Gamma(\cdot,t)$ is the probability density of a normal (or Gauß) distribution with mean 0 and variance $2t$ (E 9.1.2). By the same change of variables, $\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]}\Gamma(x,t)dx = 2\int_\epsilon^\infty (4\pi t)^{-1/2}e^{-x^2(4t)^{-1}}dx = 2\int_{\epsilon/(4t)^{-1/2}}^\infty (\pi)^{-1/2}e^{-y^2}dy \to 0, t \to 0$ (9.5). Define $u(x,t) = \int_\mathbb{R} \Gamma(x-y,t)u_0(y)dy, x \in \mathbb{R}, t > 0$ (9.6). We substitute $y = x - z, u(x,t) = \int_\mathbb{R} \Gamma(z,t)u_0(x-z)dz$ (9.7). Assume that u_0 is measurable and either integrable or bdd. Because of the props of Γ, we can interchange diff and int in (9.6) and $\partial_t u(x,t) = \int_\mathbb{R} \partial_t\Gamma(x-y,t)u_0(y)dy = \int_\mathbb{R} \partial_x^2\Gamma(x-y,t)u_0(y)dy = \partial_x^2 u(x,t)$. **P9.2**. Let $u_0 : \mathbb{R} \to \mathbb{R}$ be bdd and cont. Then the fctn $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ defined by $u(x,t) = \int_\mathbb{R} \Gamma(t, x-y)u_0(y)dy, t > 0, x \in \mathbb{R}, u(x,0) = u_0(x), x \in \mathbb{R}$, is cont and bdd. Hint: Show $u(x,t) \to u_0(x)$ as $t \to 0$ unif on each bdd subset of $\mathbb{R}$. *Prf.* Since u_0 is bdd, its abs value has a sup in $\mathbb{R}$ and $|u(x,t)| \leq \int_\mathbb{R} \Gamma(t, x-y)\sup_{z\in\mathbb{R}}|u_0(z)|dy = \int_\mathbb{R} \Gamma(t,z)dz \sup_{z\in\mathbb{R}}|u_0(z)| = \sup_{z\in\mathbb{R}}|u_0(z)|$. By (9.4). By (9,7) and (9.4), $u(x,t) - u_0(x) = \int_\mathbb{R} \Gamma(z,t)(u_0(x-z) - u_0(x))dz$. Hence $|u(x,t) - u_0(x)| \leq \int_\mathbb{R} \Gamma(z,t)|u_0(x-z) - u_0(x)|dz$. Recall that u_0 is bdd and cont. Let $\delta \in (0,1)$. Split up the int; $\forall x \in \mathbb{R}, |u(x,t) - u_0(x)| \leq \int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)|u_0(x-z) - u_0(x)|dz + \int_{-\delta}^\delta \Gamma(z,t)|u_0(x-z) - u_0(x)|dz \leq 2\sup|u_0|\int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)dz + \int_{-\delta}^\delta \Gamma(z,t)dz \sup_{|z|<\delta}|u_0(x-z) - u_0(x)| \leq 2\sup|u_0|\int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)dz + \sup_{|z|<\delta}|u_0(x-z) - u_0(x)|$. In the last inequality we used $\int_{-\delta}^\delta \Gamma(z,t)dz \leq \int_\mathbb{R} \Gamma(z,t)dz = 1$. Let B be a bdd subset of $\mathbb{R}$. Then $\exists \; n \in \mathbb{N}$ s.t. $B \subseteq [-n,n]$. Let $\epsilon > 0$. Since u_0 is unif cont on $[-(n+1), n+1], \exists \; \delta \in (0,1)$ s.t. $|u_0(y) - u_0(x)| < \epsilon/3 \; \forall y, x \in [-(n+1), n+1]$ with $|y - x| < \delta$. Let $x \in [-n,n]$. If $|z| < \delta \leq 1$, then $x - z \in [-(n+1), n+1]$ and $|(x-z) - x| = |z| < \delta$, hence $|u_0(x-z) - u_0(x)| < \epsilon/3$. This implies $\sup_{|z|<\delta}|u_0(x-z) - u_0(x)| \leq \epsilon/3 < \epsilon/2, x \in B \subseteq [-n,n]$. By (9.5), $\exists \; \eta > 0$ s.t. $2\sup|u_0|\int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)dz < \epsilon/2, 0 < t < \eta$. So $|u(x,t) - u_0(x)| < \epsilon \; \forall \; x \in B$ if $0 < t < \eta$. Continuity of u at any point (x,t) with $t > 0$ follows from the properties of Γ. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. Since $u(t,y) \to u_0(y)$ as $t \to 0$ unif for $y \in [x-1, x+1], \exists \; \eta > 0$ s.t. $|u(t,y) - u_0(y)| < \epsilon/2$ whenever $t \in [0,\eta)$ and $y \in [x-1, x+1]$. Since u_0 is cont, $\exists \; \delta_1 > 0$ s.t. $|u_0(x) - u_0(y)| < \epsilon/2 \; \forall y \in \mathbb{R}$ with $|y - x| < \delta_1$. Set $\delta = \min\{\delta_1, \eta, 1\}$. Let $y \in \mathbb{R}$ and $|y - x| + t < \delta$. Then $|y - x| < \delta$ and $|y - x| < 1$ and so $y \in [x-1, x+1]$. Further $t < \eta$. By the TI, $|u(y,t) - u(x,0)| \leq |u(y,t) - u_0(y)| + |u_0(y) - u_0(x)| < \epsilon/2 + \epsilon/2 = \epsilon$ $\square$ **T9.3**. If u_0 is measurable and either integrable or bdd, formula (9.6) provides a solution of the HE on $\mathbb{R} \times (0,\infty)$. If u_0 is bdd and cont on $\mathbb{R}$ and $u(x,0) := u_0(x)$ for $x \in \mathbb{R}$, then u is bdd and cont on $\mathbb{R} \times [0,\infty)$ $\square$. We investigate under which conditions on u, a soln u of (9.1) would necessarily be given by (9.6). To start, we assume that $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is cont and bdd on $\mathbb{R} \times [0,T]$ for every $T \in (0,\infty)$. Fix $t > 0$ and $x \in \mathbb{R}$. Define $v(s) = \int_\mathbb{R} \Gamma(t-s, x-y)u(y,s)dy, 0 \leq s < t$, and, for each $n \in \mathbb{N}, v_n(s) = \int_{-n}^n \Gamma(t-s, x-y)u(y,s)dy, 0 \leq s < t$. Choose $\epsilon \in (0, t/2)$ and assume that u and $\partial_t u, \partial_x u$ and $\partial_x^2 u$ are bdd on $[\epsilon, t]$. Then by the properties of Γ, v_n is diff on (ϵ, t) and $v_n'(s) = \int_{-n}^n \partial_s[\Gamma(t-s, x-y)u(y,s)]dy$. By the product rule, $v_n'(s) = \int_{-n}^n[\partial_s\Gamma(t-s, x-y)]u(y,s)dy + \int_{-n}^n \Gamma(t-s, x-y)\partial_s u(y,s)dy$. Using the respective PDEs, $v_n'(s) = \int_{-n}^n[-\partial_y^2\Gamma(t-s, x-y)]u(y,s)dy + \int_{-n}^n \Gamma(t-s, x-y)\partial_y^2 u(y,s)dy$. We IBP, $v_n'(s) = \partial_y\Gamma(t-s, x+n)u(-n,s) - \partial_y\Gamma(t-s, x-n)u(n,s) + \Gamma(t-s, x-n)\partial_y u(n,s) - \Gamma(t-s, x+n)\partial_y u(-n,s)$. By the props of Γ and u, the rhs converges to 0 as $n \to \infty$ unif for $s \in (\epsilon, t-\epsilon)$. Further $v_n(s) \to v(s)$ as $n \to \infty, s \in (\epsilon, t-\epsilon)$. This implies that v is diff on $(\epsilon, t-\epsilon)$ and $v'(s) = 0$. So v is const on $[\epsilon, t-\epsilon]$ and $\int_\mathbb{R} \Gamma(\epsilon, x-y)u(y, t-\epsilon)dy = \int_\mathbb{R} \Gamma(t-\epsilon, x-y)u(y,\epsilon)dy$ (9.8). Let us assume that $u, \partial_t u, \partial_x u, \partial_x^2 u$ are bdd on $\mathbb{R} \times [\delta, 1/\delta]$ for every $\delta \in (0,1)$. Then $u(y, t-\epsilon) \to u(y,t)$ as $\epsilon \to 0$ unif in $x \in \mathbb{R}$ and $u(\cdot,t)$ is unif cont. So $|\int_\mathbb{R} \Gamma(\epsilon, x-y)u(y, t-\epsilon)dy - u(x,t)| \leq |\int_\mathbb{R} \Gamma(\epsilon, x-y)u(y, t-\epsilon)dy - \int_\mathbb{R} \Gamma(\epsilon, x-y)u(y,t)dy| + |\int_\mathbb{R} \Gamma(\epsilon, x-y)u(y,t)dy - u(x,t)|$. Since the last term tends to 0 as $\epsilon \to 0$, we only need to deal with the last but one term estimated by $\int_\mathbb{R} \Gamma(\epsilon, x-y)|u(y, t-\epsilon) - u(y,t)|dy \leq \int_\mathbb{R} \Gamma(\epsilon, x-y)dy \sup_y |u(y, t-\epsilon) - u(y,t)| \to 0, \epsilon \to 0$. Further, by Lebesque' thm of dominated convergence, $\int_\mathbb{R} \Gamma(t-\epsilon, x-y)u(y,\epsilon)dy \to \int_\mathbb{R} \Gamma(t, x-y)u(y,0)dy$. Notice that, for $0 < \epsilon < t/2, \Gamma(t-\epsilon, x) \leq (2\pi t)^{-1/2}e^{-x^2(4t)^{-1}} \leq \sqrt{2}\Gamma(x,t)$ (9,9). Take the lim of (9.8) as $\epsilon \to 0$ and obtain $u(x,t) = \int_\mathbb{R} \Gamma(t, x-y)u(y,0)dy$. **T9.4**. Let $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be cont and bdd on $\mathbb{R} \times [0,T]$ for every $T \in (0,\infty)$. Further, let $\partial_t u, \partial_x u, \partial_x^2 u$ be bdd on $\mathbb{R} \times [\epsilon, 1/\epsilon]$ for every $\epsilon \in (0,1)$. Then, if u solves (9.1), $u(x,t) = \int_\mathbb{R} \Gamma(t, x-y)u_0(y)dy, t > 0$. Further, if u_0 is unif cont and bdd, this formula gives a sol of (9.1) with the props above. Fix $r > 0$ and define $u(x,t) = \Gamma(t+r, x), x \in \mathbb{R}, t \geq 0$. Then $\partial_t u = \partial_x^2 u$ and u satisfies all the assumptions we made before. This yields $\Gamma(t+r, x) = \int_\mathbb{R} \Gamma(t, x-y)u(y,0)dy = \int_\mathbb{R} \Gamma(t, x-y)\Gamma(r,y)dy$ (9.10). This means that Γ satisfies the Chapman-Kolmogorov equation. Consider the HE in the space of integrable fctns $L^1(\mathbb{R})$. Similarly as before, u given by (9,6) solves the PDE on $\mathbb{R} \times (0,\infty)$. We assume that u_0 is non-neg and measurable with finite integral $\int_\mathbb{R} u_0(y)dy$ and u given by (9.6). Then u is non-neg and, by Tonelli's thrm, $\int_\mathbb{R} u(x,t)dx = \int_\mathbb{R}(\int_\mathbb{R} \Gamma(t, x-y)dx)u_0(y)dy = \int_\mathbb{R} u_0(y)dy, t > 0$. But, by (9.2), $u(t,x) \leq (4\pi t)^{-1/2}\int_\mathbb{R} u_0(y)dy \to 0, t \to \infty$, unif for $x \in \mathbb{R}$. In what sense does u given by (9.6) satisfy the initial condition? **T9.5**. Let u_0 be measurable with finite integral $\int_\mathbb{R} |u_0(y)|dy$ and u be given by (9.6). Then $\int_\mathbb{R} |u(x,t) - u_0(x)|dx \to 0$ as $t \to 0$. We use a result from integration theory, namely, that $C_0(\mathbb{R})$, the space of cont fctns with compact support, is dense in $L^1(\Omega)$. Let $\epsilon > 0$. Then $\exists \; f \in C_0(\mathbb{R})$ s.t. $\int_\mathbb{R} |u_0(x) - f(x)|dx < \epsilon/6$. By the TI, $\int_\mathbb{R} |u(x,t) - u_0(x)|dx \leq \int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)u_0(y)dy - f(x)|dx + \int_\mathbb{R} |f(x) - u_0(x)|dx \leq \int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)[u_0(y) - f(x)]dy|dx + \int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx + \int_\mathbb{R} |f(x) - u_0(x)|dx \leq \int_\mathbb{R}(\int_\mathbb{R} \Gamma(t, x-y)|u_0(y) - f(y)|dy)dx + \int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx + \int_\mathbb{R} |f(x) - u_0(x)|dx$. By Fubini's thrm, $\int_\mathbb{R}(\int_\mathbb{R} \Gamma(t, x-y)|u_0(y) - f(y)|dy)dx = \int_\mathbb{R}(\int_\mathbb{R} \Gamma(t, x-y)dx)|u_0(y) - f(y)|dy = \int_\mathbb{R} |u_0(y) - f(y)|dy$. So $\int_\mathbb{R} |u(x,t) - u_0(x)|dx \leq \int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx + 2\int_\mathbb{R} |f(x) - u_0(x)|dx$ (9.11). Since $f \in C_0(\mathbb{R})$, by E 9.1.1, $|\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)| \to 0, t \to 0$, unif for $x \in \mathbb{R}$. Further $\exists \; b > 0$ s.t. $f(x) = 0$ for $x \in \mathbb{R} \setminus (-b, b)$. We split up the 1st integral on the rhs of inequality (9.11), $\int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx \leq \int_{-2b}^{2b}|\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx + \int_{2b}^\infty(\int_\mathbb{R} \Gamma(t, x-y)|f(y)|dy)dx + \int_{-\infty}^{-2b}(\int_\mathbb{R} \Gamma(t, x-y)|f(y)|dy)dx$ (9.12). $\exists \; \delta_1 > 0$ s.t. $|\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)| < \epsilon/(36b), t \in (0, \delta_1)$, and so $\int_{-2b}^{2b}|\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx < \epsilon/(9b), t \in (0, \delta_1)$ (9.13). Now, with $c = \sup|f|$, by Funbini's thm, $\int_{2b}^\infty(\int_\mathbb{R} \Gamma(t, x-y)|f(y)|dy)dx \leq \int_{2b}^\infty(\int_{-b}^b \Gamma(t, x-y)cdy)dx = c\int_{-b}^b(\int_{2b}^\infty \Gamma(t, x-y)dx)dy$. By a change of variables, $\int_{2b}^\infty \int_\mathbb{R} \Gamma(t, x-y)f(y)dydx \leq c\int_{-b}^b(\int_{2b-y}^\infty \Gamma(t,x)dx)dy \leq c\int_{-b}^b(\int_b^\infty \Gamma(t,x)dx)dy \leq 2cb\int_b^\infty (4\pi t)^{-1/2}e^{-x^2(4t)^{-1}}dx$. By a change of variables, $\int_{2b}^\infty \int_\mathbb{R} \Gamma(t, x-y)f(y)dydx \leq 2cb\pi^{-1/2}\int_{b(4\pi t)^{-1/2}}^\infty e^{-x^2}dx \to 0$ as $t \to 0$. So $\exists \; \delta_2 > 0$ s.t. $\int_{2b}^\infty \int_\mathbb{R} \Gamma(t, x-y)f(y)dydx < \epsilon/9, t \in (0, \delta_2)$ (9.14). Similarly, $\exists \; \delta_3 > 0$ s.t. $\int_{-\infty}^{-2b}\int_\mathbb{R} \Gamma(t, x-y)|f(y)|dydx < \epsilon/9, t \in (0, \delta_3)$ (9.15). We set $\delta = \min_{j=1}^3 \delta_j$ and combine the inequalities (9.12) - (9.15), $\int_\mathbb{R} |\int_\mathbb{R} \Gamma(t, x-y)f(y)dy - f(x)|dx < \epsilon/3, t \in (0,\delta)$. This, combined with (9.11) yields $\int_\mathbb{R} |u(x,t) - u_0(x)|dx < \epsilon, t \in (0,\delta)$. **E9.1.1**. Let $u_0 : \mathbb{R} \to \mathbb{R}$ be bdd and unif cont. Define $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by $u(x,t) = \int_\mathbb{R} \Gamma(x-y,t)u_0(y)dy, t > 0, x \in \mathbb{R}, u(x,0) = u_0(x), x \in \mathbb{R}$. Show (a) $u(x,t) \to u(x,0)$ as $t \to 0$, unif for $x \in \mathbb{R}$. (b) For any $t > 0, u(x,t)$ is a unif cont fctn of $x \in \mathbb{R}$. *Prf.* (a) By (9.7) and (9.4) $u(x,t) - u_0(x) = \int_\mathbb{R} \Gamma(z,t)(u_0(x-z) - u_0(x))dx$. Hence $|u(x,t) - u_0(x)| \leq \int_\mathbb{R} \Gamma(z,t)|(u_0(x-z) - u_0(x))|dx$. Assume that u_0 is bdd and unif cont. Let $\epsilon > 0$. $\exists\delta > 0$ s.t. $|u_0(x-z) - u_0(x)| < \epsilon/2$ if $z, x \in \mathbb{R}$ and $|z| < \delta$. Hence $|u(x,t) - u_0(x)| \leq \int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)|u_0(x-z) - u_0(x)|dz + \int_{-\delta}^\delta \Gamma(z,t)|u_0(x-z) - u_0(x)|dz \leq 2\sup|u_0|\int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)dz + \epsilon/2\int_{-\delta}^\delta \Gamma(z,t)dz \leq 2\sup|u_0|\int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)dz + \epsilon/2$. Now $\exists \; \eta > 0$ s.t. $2\sup|u_0|\int_{\mathbb{R}\setminus[-\delta,\delta]}\Gamma(z,t)dz < \epsilon/2, 0 < t < \eta$. So $|u(x,t) - u_0(x)| < \epsilon \; \forall \; x \in \mathbb{R}$ if $0 < t < \eta$. (b) Choose $\delta > 0$ s.t. $|u_0(x) - u_0(\tilde{x})| < \epsilon \; \forall \; x, \tilde{x} \in \mathbb{R}$ with $|x - \tilde{x}| < \delta$. Let $x, \tilde{x} \in \mathbb{R}$ with $|x - \tilde{x}| < \delta$. Then $|(x-z) - (\tilde{x}-z)| < \delta \; \forall \; z \in \mathbb{R}$ and, by (9.7) and (9.4), $|u(x,t) - u(\tilde{x},t))| \leq \int_{\mathbb{R}} \Gamma(t,z)|u_0(x-z) - u_0(\tilde{x}-z)|dz \leq \int_{\mathbb{R}} \Gamma(t,z)\epsilon dz = \epsilon$ $\square$ **Higher space dimension**. We look for a function $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ solving $\partial_t u(x,t) = \Delta_x u(x,t), x \in \mathbb{R}^n, t > 0, u(x,0) = u_0(x), x \in \mathbb{R}^n$ (9.19), for a given function $u_0 : \mathbb{R}^n \to \mathbb{R}$. **P9.6**. Define $G(x,t) = \Pi_{i=1}^n \Gamma(x_i,t), t > 0, x = (x_1,\ldots,x_n) \in \mathbb{R}^n$. Then: (a) $\partial_t G(x,t) = \Delta_x G(x,t), t > 0, x \in \mathbb{R}^n$. (b) $\int_{\mathbb{R}^n} G(x,t)dx = 1 \ \forall \ t > 0$. (c) $G(x,t) = (4\pi t)^{-n/2} e^{-||x||^2(4t)^{-1}}$, where $||\cdot||$ is the Euclidean norm on $\mathbb{R}^n$. (d) For every $\epsilon > 0, \int_{||x||\geq\epsilon} G(x,t)dx \to 0$ as $t \to 0$. (e) (Chapman-Kolmogorov eq) $\forall\, t,r \in [0,\infty), x \in \mathbb{R}^n, G(x,t+r) = \int_{\mathbb{R}^n} G(x-y,t)G(y,r)dy$. *Prf.* (a) By the product rule, $\partial_t G(x,t) = \sum_{j=1}^n (\Pi_{i\neq j}\Gamma(x_i,t))\partial_t \Gamma(x_j,t) = \sum_{j=1}^n (\Pi_{i\neq j}\Gamma(x_i,t))\partial_{x_j}^2 \Gamma(x_j,t) = \sum_{j=1}^n \partial_{x_j}^2 (\Pi_{i\neq j}\Gamma(x_i,t))\Gamma(x_j,t) = \Delta_x G(x,t)$. (b) We use induction over n. Write G_n for G. We know the statement to be true for $n = 1$. Let $n \in \mathbb{N}$ be arbitrary and assume that the statement is true for n. Notice that, for $y \in \mathbb{R}^n$ and $z \in \mathbb{R}, G_{n+1}((y,z),t) = G_n(y,t)\Gamma(z,t)$. Now, by the thms of Fubini or Tonelli, $\int_{\mathbb{R}^{n+1}} G_{n+1}(x,t)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}} G_n(y,t)\Gamma(z,t)dydz = (\int_{\mathbb{R}} G_n(y,t)dy)(\int_{\mathbb{R}}\Gamma(z,t)dz) = 1 \cdot 1 = 1$. (c) $G(x,t) = \Pi_{i=1}^n((4\pi t)^{-1/2} e^{-x_i^2(4t)^{-1}}) = (4\pi t)^{-n/2}\Pi_{i=1}^n e^{-x_i^2(4t)^{-1}} = (4\pi t)^{-n/2}\exp(-\sum_{i=1}^n x_i^2(4t)^{-1}) = (4\pi t)^{-n/2} e^{-||x||^2(4t)^{-1}}$. (d) Let $\epsilon > 0$. We substitute $x = t^{1/2}y, \int_{||x||\geq\epsilon} G(x,t)dx = (4\pi)^{-n/2}\int_{||y||\geq\epsilon t^{-1/2}} e^{-||y||^2/4} = \int_{||y||\geq\epsilon t^{-1/2}} G(y,1)dy \to 0$, as $t \to 0$. (e) Let $t,r \in [0,\infty), x \in \mathbb{R}^n. \int_{\mathbb{R}^n} G(x-y,t)G(y,r)dy = \int_{\mathbb{R}^n}\Pi_{j=1}^n\Gamma(x_j-y_j,t)\Pi_{j=1}^n\Gamma(y_j,r)dy = \int_{\mathbb{R}^n}\Pi_{j=1}^n[\Gamma(x_j-y_j,t)\Gamma(y_j,r)]dy = \Pi_{j=1}^n\int_{\mathbb{R}}[\Gamma(x_j-y_j,t)\Gamma(y_j,r)]dy_j = \Pi_{j=1}^n\Gamma(x_j.t+r) = G(x,t+r)$ $\square$ **R9.7** $\int_{||x||\geq\epsilon} G(x,t)dx = \pi^{-2/n}\int_{||z||\geq\epsilon(4t)^{-1/2}} e^{-||z||^2}dz = \pi^{-2/n}\mathrm{vol}(U_1(0))\int_{\epsilon(4t)^{-1/2}}^\infty nr^{n-1}e^{-r^2}dr$. For $n = 2$, using polar coords, $\int_{||x||\geq\epsilon} G(x,t)dx = e^{-\epsilon^2(4t)^{-1}}$. For $n = 4, \int_{||x||\geq\epsilon} G(x,t)dx = \pi^{-1/2}\mathrm{vol}(U_1^4(0))2(\int_{\epsilon(4t)^{-1/2}}^\infty r^2(d/dr)e^{-r^2}dr) = \pi^{-1/2}\mathrm{vol}(U_1^4(0))2(\epsilon^2(4t)^{-1}e^{-\epsilon^2(4t)^{-1}} + \int_{\epsilon(4t)^{-1/2}}^\infty 2re^{-r^2}dr) = \pi^{-1/2}\mathrm{vol}(U_1^4(0))2(\epsilon^2(4t)^{-1}e^{-\epsilon^2(4t)^{-1}} + e^{-\epsilon^2(4t)^{-1}} = \pi^{-1/2}\mathrm{vol}(U_1^4(0))2(\epsilon^2(4t)^{-1}+1)e^{-\epsilon^2(4t)^{-1}}$. **T9.8**. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be bdd and unif cont. Define the fctn $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ by $u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_0(y)dy, t > 0, x \in \mathbb{R}^n$. Then (a) $u(x,t) \to u(x,0)$ as $t \to 0$, unif for $x \in \mathbb{R}^n$. (b) For any $t > 0, u(x,t)$ is a bdd unif cont fctn of $x \in \mathbb{R}$. *Prf.* By a change of variables $y \mapsto x-z, u(x,t) = \int_{\mathbb{R}^n} G(z,t)u_0(x-z)dz, x \in \mathbb{R}^n, t \in (0,\infty)$ (9.20). (a) By (9.20) and P 9.6 (b) and the nonneg of $G, |u(x,t)| \leq \int_{\mathbb{R}^n} G(z,t)|u_0(x-z)|dz \leq \sup_{\mathbb{R}^n}|u_0|$ (9.21). So u is bdd and $\sup_{x\in\mathbb{R}^n,t>0}|u(x,t)| \leq \sup_{\mathbb{R}^n}|u_0|$. Again by (9.20), P 9.6 (b), $u(x,t) - u_0(x) = \int_{\mathbb{R}^n} G(z,t)(u_0(x-z) - u_0(x))dx$. Since G is non-neg, $|u(x,t) - u_0(x)| = \int_{\mathbb{R}^n} G(z,t)|(u_0(x-z) - u_0(x))|dx$. Assume that u_0 is bdd and unif cont. Let $\epsilon > 0$. Then $\exists\ \delta > 0$ s.t. $|u_0(x-z) - u_0(x)| < \epsilon/2$ if $z,x \in \mathbb{R}^n$ and $||z|| < \delta$. We split up the integral accordingly, $|u(x,t) - u_0(x)| \leq \int_{||z||\geq\delta} G(z,t)|u_0(x-z) - u_0(x)|dz + \int_{||z||<\delta} G(z,t)|u_0(x-z) - u_0(x)|dz \leq 2\sup|u_0|\int_{||z||\geq\delta} G(z,t)dz + \epsilon/2\int_{||z||<\delta} G(z,t)dz \leq 2\sup|u_0|\int_{||z||\geq\delta} G(z,t)dz + \epsilon/2$. By P 9.6 (d), $\exists\ \eta > 0$ s.t. $2\sup|u_0|\int_{||z||\geq\delta} G(z,t)dz < \epsilon/2, 0 < t < \eta$. So $|u(x,t) - u_0(x)| < \epsilon\ \forall\ x \in \mathbb{R}^n$ if $0 < t < \eta$. (b) Choose $\delta > 0$ s.t. $|u_0(x) - u_0(\tilde{x})| < \epsilon\ \forall\ x,\tilde{x}, \in \mathbb{R}^n$ with $||x-\tilde{x}|| < \delta$. Let $x,\tilde{x} \in \mathbb{R}^n$ with $||x-\tilde{x}|| < \delta$. Then $||(x-z) - (\tilde{x}-z)|| < \delta\ \forall\ z \in \mathbb{R}$ and, by (9.20) and P 9.6 (b), $|u(x,t) - u(\tilde{x},t)| \leq \int_{\mathbb{R}^n} G(t,z)|u_0(x-z) - u_0(\tilde{x}-z)|dz \leq \int_{\mathbb{R}^n} G(t,z)\epsilon dz = \epsilon$ $\square$ Let $X = BUC(\mathbb{R}^n)$ be the Banach space of unif cont bdd fctns from $\mathbb{R}^n$ to $\mathbb{R}$ with the sup norm. For $t > 0$, define $(S(t)f)(x) = \int_{R^n} G(x-y,t)f(y)dy, f \in X$. Then $S(t)$ is a bdd linear operator on X with $||S(t)|| = 1, S(t)S(r) = S(t+r), r,t > 0$ (9.22) and $S(t)f \to f, t \to 0, f \in X$ (9.23). Families of operators with these properties are called C_0-(operator-)semi-groups. That $S(t)$ maps X into X follows from part (b) of the last thm while (9.23) follows from part (a). (9.22) follows from the Chapman-Kolmogorov equations by switching the order of integration. Let $f \in BUC(\mathbb{R}^n), t,r > 0, x \in \mathbb{R}^n$, and $g = S(r)f, [S(t)S(r)f](x) = [S(t)g](x) = \int_{\mathbb{R}^n} G(x-y,t)g(y)dy = \int_{\mathbb{R}^n} G(x-y,t)(\int_{\mathbb{R}^n} G(y-z,t)f(z)dz)dy = \int_{\mathbb{R}^n}(\int_{\mathbb{R}^n} G(x-y,t)G(y-z,t)dy)f(z)dz$. We substitute $y = \tilde{y}+z$ and use the Chapman-Kolmogorov equations, $[S(t)S(r)f](x) = \int_{\mathbb{R}^n}(\int_{\mathbb{R}^n} G(x-\tilde{y}-z,t)G(\tilde{y},t)d\tilde{y})f(z)dz = \int_{\mathbb{R}^n} G(x-z,t+r)f(z)dz = [S(t+r)f]x$. Since this holds $\forall\ x \in \mathbb{R}^n$, we have $S(t)S(r)f = S(t+r)f$. The linearity of $S(t)$ follows from the linearity of the integral. The boundedness of $S(t)$ and $||S(t)|| = 1$ follows from $\int_{\mathbb{R}^n} G(xt)dx = 1, ||S(t)f||_\infty \leq ||f||_\infty$, with $||f||_\infty = \sup_{x\in\mathbb{R}^n}|f(x)|$. See (9.21). Notice that $BUC(\mathbb{R}^n)$ contains the const functions and $S(t)f = f$ for any const function $f : \mathbb{R}^n \to \mathbb{R}$.