Partial Differential Equations
TA Homework 4

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Problem 2

Verify that the function <, > defined in Example 0.3 is an inner product.

Solution: Given the inner product on C^2 defined by

$$\langle v, w\rangle = \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix},$$

it is easy to check the properties.

- Positivity:
- Conjugate symmetry:

$$\begin{aligned}
\langle v, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\
&= \begin{pmatrix}2\overline{w}_1 + i\overline{w}_2 & 3\overline{w}_2 - i\overline{w}_1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\
&= 2v_1\overline{w}_1 + iv_1\overline{w}_2 + 3v_2\overline{w}_2 - iv_2\overline{w}_1\\
&= \overline{2\overline{v}_1 w_1 - i\overline{v}_1 w_2 + 3\overline{v}_2 w_2 + i\overline{v}_2 w_1}\\
&= \overline{\begin{pmatrix}2\overline{v}_1 + i\overline{v}_2 & 3\overline{v}_2 - i\overline{v}_1\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}}\\
&= \overline{\begin{pmatrix}\overline{v}_1 & \overline{v}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix}}\\
&= \overline{\langle w, v\rangle}
\end{aligned}$$

- Homogeneity:

$$\begin{aligned}
\langle cv, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}cv_1\\ cv_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} c\\
&= \langle v, w\rangle\, c\\
&= c\,\langle v, w\rangle\,,
\end{aligned}$$

  where we have taken the complex scalar c out of the vector v since it is common in all its components.

- Linearity:

$$
\begin{aligned}
\langle u+v, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1+v_1\\ u_2+v_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2(u_1+v_1)-i(u_2+v_2) & i(u_1+v_1)+3(u_2+v_2)\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2u_1-iu_2+2v_1-iv_2 & iu_1+3u_2+iv_1+3v_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\left[\begin{pmatrix}2u_1-iu_2 & iu_1+3u_2\end{pmatrix}+\begin{pmatrix}2v_1-iv_2 & iv_1+3v_2\end{pmatrix}\right]\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\left[\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right]\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}
$$

Problem 7

For $n > 0$, let

$$f_n(t) = \begin{cases} \sqrt{n}, & 0 \le t \le 1/n^2 \\ 0, & \text{otherwise} \end{cases}$$

Show that $f_n \to 0$ in $L^2[0,1]$ but that $f_n(0)$ does not converge to zero.

Solution: For the first proof we need to prove that

$$||f_n(t) - 0|| \to 0 \text{ as } n \to \infty.$$

Then,

$$\begin{aligned} ||f_n(t) - 0|| &= \sqrt{\langle f_n - 0, f_n - 0 \rangle_{L^2}} \\ &= \sqrt{\int_0^1 [f_n(t) - 0]^2 \, dt} \\ &= \sqrt{\int_0^1 [f_n(t)]^2 \, dt} \\ &= \sqrt{\int_0^{1/n^2} n dt + \int_{1/n^2}^1 0 dt} \\ &= \sqrt{\frac{1}{2} n \frac{1}{n^2}} \\ &= \sqrt{\frac{1}{2n}}. \end{aligned}$$

Hence,

$$\lim_{n \to \infty} ||f_n(t) - 0|| = \lim_{n \to \infty} \sqrt{\frac{1}{2n}} = 0.$$

However,

$$\lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} \sqrt{n} = \infty \neq 0.$$

Problem 11

Show that if a differentiable function , f, is orthogonal to $cos(t)$ on $L^2[0,\pi]$ then f' is orthogonal to $sin(t)$ on $L^2[0,\pi]$.

Solution: Since f is orthogonal to $cos(t)$ we know that

$$\langle f, cos(t)\rangle = \int_0^\pi f(t)cos(t)dt = 0.$$

Then,

$$\begin{aligned}
\langle f', sin(t)\rangle &= \int_0^\pi f'(t)sin(t)dt \\
&= \int_0^\pi \left(\frac{d}{dt}\left[f(t)sin(t)\right] - f(t)cos(t)\right)dt \\
&= \int_0^\pi d\left[f(t)sin(t)\right] - \int_0^\pi f(t)cos(t)dt \\
&= \cancelto{0}{f(t)sin(t)|_0^\pi} - \int_0^\pi f(t)cos(t)dt \\
&= -\int_0^\pi f(t)cos(t)dt = 0,
\end{aligned}$$

where we have used that $sin(\pi) = sin(0) = 0$ and the fact that f is orthogonal to $cos(t)$. Hence, it has been proved that f' is orthogonal to $sin(t)$ provided that f is orthogonal to $cos(t)$.

Problem 14

Find the $L^2[-\pi,\pi]$ projection of the function $f(x) = x^2$ onto the space $V_n \in L^2[-\pi,\pi]$ spanned by

$$\left\{\frac{\sin(jx)}{\sqrt{2}\pi}, \frac{\cos(jx)}{\sqrt{\pi}}; j = 1, 2, ..., n\right\}$$

for n=2. Plot these projections along with f using a computer algebra system. Repeat for $g(x) = x^3$.

Solution: Let $a_j = \frac{\sin(jx)}{\sqrt{2}\pi}$ and $b_j = \frac{\cos(jx)}{\sqrt{\pi}}$. Then, the projection, f_0, of f onto the space V_n is

$$f_0 = \sum_{j=1}^{n} \langle f, a_j\rangle a_j + \sum_{j=1}^{n} \langle f, b_j\rangle b_j,$$

where $n = 2$. First, we calculate the inner products

$$\langle f, a_j\rangle = \int_{-\pi}^{\pi} x^2 \frac{\sin(jx)}{\sqrt{2}\pi} dx = \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} x^2 \sin(jx)dx = 0,$$

since $x^2\sin(jx)$ is odd. Thus,

$$\langle f, a_j\rangle = 0 \quad \forall j \in \mathbb{Z}.$$

Now,

$$\begin{aligned}
\langle f, b_j \rangle = \int_{-\pi}^{\pi} x^2 \frac{\cos(jx)}{\sqrt{\pi}} dx &= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x^2 \cos(jx) dx \\
&= \frac{1}{\sqrt{\pi}} \cancelto{0}{\frac{x^2 \sin(jx)}{j} \Bigg|_{-\pi}^{\pi}} - \frac{2}{\sqrt{\pi}} \int_{-\pi}^{\pi} \frac{1}{j} x \sin(jx) dx \\
&= -\frac{2}{j\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin(jx) dx \\
&= +\frac{2}{j^2\sqrt{\pi}} \, x\cos(jx)|_{-\pi}^{\pi} - \frac{2}{j^2\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(jx) dx \\
&= +\frac{2}{j^2\sqrt{\pi}} \, x\cos(jx)|_{-\pi}^{\pi} - \cancelto{0}{\frac{2}{j^3\sqrt{\pi}} \sin(jx)|_{-\pi}^{\pi}} \\
&= +\frac{2}{j^2\sqrt{\pi}} \left(\pi\cos(j\pi) + \pi\cos(-j\pi)\right) \\
&= +\frac{4\sqrt{\pi}}{j^2} \cos(j\pi),
\end{aligned}$$

where we have integrated by parts twice. Therefore,

$$\begin{aligned}
f_0 &= \cancelto{0}{\sum_{j=1}^{n} \langle f, a_j \rangle} a_j + \sum_{j=1}^{n} \langle f, b_j \rangle b_j \\
&= \langle f, b_1 \rangle b_1 + \langle f, b_2 \rangle b_2 \\
&= -4\sqrt{\pi} \frac{\cos(x)}{\sqrt{\pi}} + \sqrt{\pi} \frac{\cos(2x)}{\sqrt{\pi}}.
\end{aligned}$$

Hence, the projection

$$f_0 = \cos(2x) - 4\cos(x).$$

Further, we repeat the same process for the function $g(x) = x^3$. In this case, the parity of the function has changed from even to odd. Hence, in this case the inner products $\langle g, b_j \rangle = 0$ because $x^3 \cos(jx)$ is an odd function. We calculate then the inner products

$$\begin{aligned}
\langle g, a_j \rangle = \int_{-\pi}^{\pi} x^3 \frac{\sin(jx)}{\sqrt{2}\pi} dx &= \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} x^3 \sin(jx) dx \\
&= -\frac{1}{\sqrt{2}\pi} \frac{x^3 \cos(jx)}{j} \Bigg|_{-\pi}^{\pi} + \frac{1}{\sqrt{2}\pi} \frac{3}{j} \int_{-\pi}^{\pi} x^2 \cos(jx) dx \\
&= -\frac{1}{j\sqrt{2}\pi} \left(\pi^3 \cos(j\pi) + \pi^3 \cos(-j\pi)\right) + \frac{3}{j\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{x^2 \cos(jx)}{\sqrt{\pi}} dx \\
&= -\frac{2\pi^2}{j\sqrt{2}} \cos(j\pi) + \frac{3}{j\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{x^2 \cos(jx)}{\sqrt{\pi}} dx \\
&= -\frac{\pi^2\sqrt{2}}{j} \cos(j\pi) + \frac{3}{j\sqrt{2\pi}} \frac{4\sqrt{\pi}}{j^2} \cos(j\pi) \\
&= \frac{6\sqrt{2}}{j^3} \cos(j\pi) - \frac{\pi^2\sqrt{2}}{j} \cos(j\pi).
\end{aligned}$$

Similarly than with f, we have

$$\begin{aligned}
g_0 &= \sum_{j=1}^{n} \langle g, a_j \rangle a_j + \cancelto{0}{\sum_{j=1}^{n} \langle g, b_j \rangle} b_j \\
&= \langle g, a_1 \rangle a_1 + \langle g, a_2 \rangle a_2 \\
&= \sqrt{2}\left(\pi^2 - 6\right) \frac{\sin(x)}{\pi\sqrt{2}} + \frac{3 - 2\pi^2}{2\sqrt{2}} \frac{\sin(2x)}{\pi\sqrt{2}} \\
&= \frac{\left(\pi^2 - 6\right)}{\pi} \sin(x) + \frac{3 - 2\pi^2}{4\pi} \sin(2x)
\end{aligned}$$

Problem 23

Show that a set of orthonormal vectors is linearly independent.

Solution: Consider the set of orthonormal vectors

$$S = \{e_1, e_2, ..., e_N; \langle e_j, e_k \rangle = \delta_{jk}\},$$

where δ_{jk} is the Kronecker delta. Assume that

$$\sum_{j=1}^{N} \alpha_j e_j = 0.$$

Taking the inner product

$$\begin{aligned}
0 = \left\langle \sum_{j=1}^{N} \alpha_j e_j, e_k \right\rangle &= \sum_{j=1}^{N} \langle \alpha_j e_j, e_k \rangle \\
&= \sum_{j=1}^{N} \alpha_j \langle e_j, e_k \rangle \\
&= \sum_{j=1}^{N} \alpha_j \delta_{jk} \\
&= \alpha_k.
\end{aligned}$$

Hence, we have obtained that all the $\alpha_k = 0$ and therefore it is proved that a set of orthonormal vectors is linearly independent.