CHAPTER 0

L2 Inner Product: The L^2 inner product on $L^2([a,b])$ is defined as
$\langle f,g\rangle_{L^2} = \int_a^b f(t)\overline{g(t)}dt$. **l2 Inner Product:** The space l^2 is the set of all sequences $x_i \in \mathbb{C}$ with $\sum_{-\infty}^{\infty} |x_n|^2 < \infty$. The inner product on l^2 i defined as $\langle X,Y\rangle_{l^2} = \sum_{n=-\infty}^{\infty} x_n\overline{y_n}$. **Schwartz Inequality:**$|\langle X,Y\rangle| \leq ||X||||Y||$ **Triangle Inequality:**$||X+Y|| \leq ||X|| + ||Y||$ **Orthogonal Projection:** Suppose V is an inner product space and V_0 is an N-dimensional subspace with orthonormal basis $\{e_1, e_2, \ldots, e_N\}$. The orthogonal projection of a vector $v \in V$ onto V_0 is given by $v_0 = \sum_{j=1}^{N} \langle v, e_j\rangle e_j$. In addition, $||v - v_0|| = \min_{w\in V_0} ||v - w||$ **Adjoints:** If $T : V \to W$ is a linear operator between two inner product spaces, the adjoint of T is the linear operator $T^* : W \to V$, such that $\langle T(v), w\rangle_W = \langle v, T*(w)\rangle_V$.

CHAPTER 1: FOURIER SERIES

Real Fourier Series

Orthonormal Basis: The set of functions $\{\frac{\sin(k\pi x/a)}{\sqrt{\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{\cos(k\pi x/a)}{\sqrt{\pi}}\}$ with $k = 1, 2, \ldots$, is an orthonormal set of functions in $L^2([-a,a])$. **Fourier Coefficients:** If $f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi t/a) + \sum_{k=1}^{\infty} b_k \sin(k\pi t/a)$ on the interval $-a \leq t \leq a$, then $a_0 = \frac{1}{2a}\int_{-a}^{a} f(t)dt$, $a_k = \frac{1}{a}\int_{-a}^{a} f(t)\cos(k\pi t/a)dt$ and $b_k = \frac{1}{a}\int_{-a}^{a} f(t)\sin(k\pi t/a)dt$.

Complex Fourier Series

Orthonormal Basis: The set of functions $\{\frac{1}{\sqrt{2a}}e^{i\frac{n\pi}{a}t}, n = 0, \pm1, \pm2, \ldots\}$ is an orthonormal basis for $L^2([-a,a])$. **Fourier Coefficients:** If
$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i\frac{n\pi}{a}t}$, then $\alpha_n = \frac{1}{2a}\int_{-a}^{a} f(t)e^{-i\frac{n\pi}{a}t}dt$

Convergence Theorems

Riemann-Lebesgue Lemma: Suppose f is a piecewise continuous function on the interval $[a,b]$. Then $\lim_{k\to\infty}\int_a^b f(x)\cos(kx)dx = \lim_{k\to\infty}\int_a^b f(x)\sin(kx)dx = 0$. **Convergence at a Point of Continuity:** Suppose f is a continuous and 2π-periodic function. Then for each point x, where the derivative of f is defined, the Fourier series of f converges to $f(x)$. **Convergence at a Point of Discontinuity:** Suppose f is periodic function and piecewise continuous. Suppose x is a point where f is left and right differentiable (but not necessarily continuous). Then the Fourier series of f at x converges to $\frac{f(x-0)+f(x+0)}{2}$, i.e., converges to the average of the left and right limits of f. **Uniform Convergence:** The Fourier series of a continuous, piecewise smooth 2π-periodic function $f(x)$ converges uniformly to $f(x)$ on $[-\pi,\pi]$. **Lemma 1.33:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$ with $\sum_{k=1}^{\infty}|a_k| + |b : k| < \infty$. Then the Fourier series converges uniformly and absolutely to the function $f(x)$. **Convergence in the Mean:** Suppose f is an element of $L^2([-\pi,\pi])$. Let $f_N(x) = a_0 + \sum_{k=1}^{N} a_k\cos(kx) + \sum_{k=1}^{N} b_k\sin(kx)$, where a_k and b_k are the Fourier coefficients of f. Then f_N converges to f in $L^2([-\pi,\pi])$, that is, $||f_N - f||_{L^2} \to 0$ as $N \to \infty$. **Parseval's Equation - Real Version:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx) \in L^2[-\pi,\pi]$. Then $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = 2|a_0|^2 + \sum_{k=1}^{\infty}(|a_k|^2 + |b_k|^2)$. **Parseval's Equation - Complex Version:** Suppose $f(x) = \sum_{k=-\infty}^{\infty}\alpha_k e^{ikx} \in L^2[-\pi,\pi]$. Then $\frac{1}{2\pi}||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{k=-\infty}^{\infty}|\alpha_k|^2$.

CHAPTER 2:FOURIER TRANSFORM

Definition: If f is a continuously differentiable function with $\int_{-\infty}^{\infty}|f(t)|dt < \infty$, then $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\lambda t}d\lambda$, where $\hat{f}(\lambda)$ is the Fourier transform of $f(t)$ given by $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt$.
Properties:

- $\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g]$ // $\mathcal{F}^{-1}[\alpha f + \beta g] = \alpha\mathcal{F}^{-1}[f] + \beta\mathcal{F}^{-1}[g]$
- $\mathcal{F}[t^n f(t)](\lambda) = i^n\frac{d^n}{d\lambda^n}\{\mathcal{F}[f](\lambda)\}$
- $\mathcal{F}^{-1}[\lambda^n f(\lambda)]........(t) = (-i)^n\frac{d^n}{dt^n}\{\mathcal{F}^{-1}[f](t)\}$
- $\mathcal{F}[f^{(n)}(t)](\lambda) = (i\lambda)^n\mathcal{F}[f](\lambda)$
- $\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = (-it)^n\mathcal{F}^{-1}[f](t)$
- $\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda)$
- $\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f](\frac{\lambda}{b})$
- If $f(t<0) = 0$, then $\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\mathcal{L}[f](i\lambda)$, where $\mathcal{L}[f](s) = \int_0^{\infty} f(t)e^{-ts}dt$.

Convolution: Suppose f and g are two square integrable functions. The convolution of f and g is defined by $(f*g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx = \int_{-\infty}^{\infty} f(x)g(t-x)dx$. **Fourier Transform of the Convolution:**
$\mathcal{F}[f*g] = \sqrt{2\pi}\mathcal{F}[f]\cdot\mathcal{F}[g]$, $\quad \mathcal{F}^{-1}[\hat{f}\cdot\hat{g}] = \frac{1}{\sqrt{2\pi}}(f*g)$. **Pancherel Theorem:** The Fourier transform, and its inverse, preserves the L^2 inner product. $\langle\mathcal{F}[f],\mathcal{F}[g]\rangle_{L^2} = \langle f,g\rangle_{L^2}$ and $\langle\mathcal{F}^{-1}[f],\mathcal{F}^{-1}[g]\rangle_{L^2} = \langle f,g\rangle_{L^2}$.

Linear Filters

Time Invariance: A transformation L (mapping signals to signals) is said to be time-invariant if for any signal f and any real number a, $L[f_a](t) = (Lf)(t-a)$ for all t. In other words, L is time-invariant if the time shifted input signal $f(t-a)$ is transformed by L into the time shifted output signal $(Lf)(t-a)$. **Lemma 2.16:** Let L be a linear, time-invariant transformation and let λ be any fixed real number. Then, there is a function h with $L(e^{i\lambda t}) = \sqrt{2\pi}\hat{h}(\lambda)e^{i\lambda t}$. In other words, the output signal from a time-invariant filter of a sinusoidal input is also sinusoidal with the same frequency. **Convolution in Filters:** Let L be a linear, time-invariant transformation on the space of signals that are piecewise continuous functions. Then there exists an integrable function, h, such that $L(f) = f*h$ for all signals f. **Causal Filters:** A causal filter is one for which the output signal begins after the input signal has started to arrive. Let L be a time-invariant filter with response function h (i.e., $Lf = f*h$). L is a causal filter if and only if $h(t) = 0$ for all $t < 0$. **Theorem 2.20:** Suppose L is a causal filter with response function h. Then the system function associated with L is $\hat{h}(\lambda) = \frac{\mathcal{L}[h](i\lambda)}{\sqrt{2\pi}}$.

The Sampling Theorem

Definition 2.22: A function f is said to be frequency band limited if there exists a constant $\Omega > 0$ such that $f(\lambda) = 0$ for $|\lambda| > \Omega$. Note: Ω is the smallesq frequency for which the preceding equation is true. **Shannon-Whittaker Sampling Theorem:** Suppose that $\hat{f}(\lambda)$ is piecewise smooth and continuous and that $\hat{f}(\lambda) = 0$ for $|\lambda| > \Omega$, where Ω is some fixed, positive frequency. Then $f = \mathcal{F}^{-1}[f]$ is completely determined by its values at the poiints $t_j = \frac{j\pi}{\Omega}, j = 0, \pm1, \pm2, \ldots$. More precisely, f has the following series expansion: $f(t) = \sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}$, where the series converges uniformly.

CHAPTER 3: DISCRETE FOURIER TRANSFORM

Set of n-periodic sequences: Let $\mathcal{S}_n$ be the set of n-periodic sequences of complex numbers. Each element $y = y_j{}_{j=-\infty}^{\infty}$ in $\mathcal{S}_n$, can be thought of as a periodic discrete signal where y_j is the value of the signal at a time node $t = t_j$. The sequence y_j is n-periodic if $y_{k+n} = y_k$ for any integer k. **Definition:** Suppose $y = y_k$ is an element of $\mathcal{S}_n$. Let $\mathcal{F}_n(y) = \hat{y}$. That is, $\hat{y}_k = \sum_{j=0}^{n-1} y_j \overline{w}^{jk}$, where $w = e^{\frac{2\pi}{n}i}$. Then $y = \mathcal{F}^{-1}(\hat{y})$ is given by $y_j = \frac{1}{n}\sum_{k=0}^{n-1} \hat{y}_k w^{jk}$.

Properties:

- Shifts or translations. If $y \in \mathcal{S}_n$ and $z_k = y_{k+1}$, then $\mathcal{F}[z]_j = w^j \mathcal{F}[y]_j$
- Convolutions. If $y \in \mathcal{S}_n$ and $z \in \mathcal{S}_n$, then the sequence $[y * z]_k := \sum_{j=0}^{n-1} y_j z_{k-j}$ is also in $\mathcal{S}_n$. The sequence $y * z$ is called the convolution of the sequences y and z.
- The Convolution Theorem. $\mathcal{F}[y * z]_k = \mathcal{F}[y]_k \mathcal{F}[z]_k$
- If $y \in \mathcal{S}_n$ is a sequence of real numbers, then $\mathcal{F}[y]_{n-k} = \overline{\mathcal{F}[y]}_k$, for $k \in [0, n-1]$, or $\hat{y}_{n-k} = \overline{\hat{y}}_k$

APPENDIX

Identities: $\sin^2 x = (1 - \cos 2x)/2, \cos^2 x = (1 + \cos 2x)/2, e^{ix} = \cos x + i \sin x, e^{-ix} = \cos x - i \sin x,\ 2\cos x = (e^{ix} + e^{-ix}), 2i \sin x = (e^{ix} - e^{-ix}), 2\cosh x = (e^x + e^{-x}), 2\sinh x = (e^x - e^{-x}).\ \cosh^2 x - \sinh^2 x = 1.$
Sum and Difference Formula: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. $\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$. $\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$.
Double Angle Formula: $\sin(2A) = 2 \sin A \cos A$. $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$. $\tan(2A) = (2 \tan A)/(1 - \tan^2 A)$.
Sum to Product: $\sin A \pm \sin B = 2 \sin((A \pm B)/2) \cos((A \mp B)/2)$. $\cos A - \cos B = -2 \sin((A + B)/2) \sin((A - B)/2)$. $\cos A + \cos B = 2 \cos((A + B)/2) \cos((A - B)/2)$.
Geometric Sum: $\sum_{k=0}^{N} z^k = \frac{1-z^{N+1}}{1-z}$. $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$.
General ODE Solutions: $y'' = y(t) \implies y = c_1 e^{-t} + c_2 e^t$ $\square$ $dy/dt + p(t)y = g(t) \implies y = (\int u(t)g(t))/u(t) + c$ where $u(t)$ =exp($\int p(t)dt$) $\square$ $y' = x; x' = y \implies x = c_1 \cosh t + c_2 \sinh t, y = c_1 \sinh t + c_2 \cosh t$ or $x = c_1 e^t + c_2 e^{-t}, y = c_1 e^t - c_2 e^{-t}$ $\square$ $y' = -x; x' = y \implies y = c_1 \cos t + c_2 \sin t, x = c_1 \sin t - c_2 \cos t$ $\square$ $x' = x + y; y' = -x + y \implies x = e^t(c_1 \cos t + c_2 \sin t); y = e^t(-c_1 \sin t + c_2 \cos t)$ $\square$ $v' = \gamma v, v(z, 0) = u_0 \implies v = u_0 e^{\gamma t}$ $\square$