# Partial Differential Equations
# Instructor Homework 2

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## Problem 3.3.8

1. Let $\phi : [0, L] \times [0, \infty) \to \mathbb{R}$ have continuous partial derivatives with respect to $x$, $\phi(0, t) = 0 = \phi(L, t)$ for all $t \geq 0$. Solve the vibrating string equation with external force,

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)u(x,t) &= \phi(x,t), \quad t \geq 0, 0 \leq x \leq L,\\ u(x,0) &= 0, \quad x \in [0, L],\\ \partial_t u(x,0) &= 0, \quad x \in [0, L],\\ u(0,t) = u(L,t) &= 0, \quad t \geq 0.\end{aligned}$$

Show that the formula you provide is a solution indeed.

**Solution:** Given the zero boundary condition at $x = 0$, we extend $\phi$ to $[-L, L]$ in an odd fashion:

$$\phi(-x, t) = -\phi(x, t), \qquad x \in [-L, L],$$

To take care of the zero boundary condition at $x = L$ we perform a $2L-$periodic extension of $\phi$ which gives us a function defined on all $\mathbb{R}$,

$$\phi(x + 2kL) := \phi(x), \quad k \in \mathbb{Z},\ x \in [-L, L],$$

Now we can rewrite the PDE for the extended $\phi$,

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)u(x,t) = \phi(x,t), &\quad x, t \in \mathbb{R},\\ u(x,0) = 0, &\quad x, t \in \mathbb{R},\\ \partial_t u(x,0) = 0, &\quad x, t \in \mathbb{R},\\ u(0,t) = u(L,t) = 0, &\quad t \in \mathbb{R}.\end{aligned}$$

The PDE above has the following solution

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} \phi(\rho, s)d\rho\right) ds,$$

as it is solved in the notes. The above solution satisfies the PDE and the initial condition, we will check now that it also satisfies the boundary conditions.

$$\begin{aligned}
u(0,t) &= \frac{1}{2c}\int_0^t \left(\int_{-c(t-s)}^{c(t-s)} \phi(\rho,s)d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \phi(\rho,s)d\rho - \int_0^{-c(t-s)} \phi(\rho,s)d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \phi(\rho,s)d\rho + \int_0^{c(t-s)} \phi(-\rho,s)d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \left[\phi(\rho,s) + \phi(-\rho,s)\right] d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \left[\phi(\rho,s) - \phi(\rho,s)\right] d\rho\right) ds = 0
\end{aligned}$$

since $\phi$ is an odd function around zero. Similarly, we prove the boundary condition for $x = L$,

$$\begin{aligned}
u(L,t) &= \frac{1}{2c}\int_0^t \left(\int_{L-c(t-s)}^{L+c(t-s)} \phi(\rho,s)d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{L+c(t-s)} \phi(\rho,s)d\rho - \int_0^{L-c(t-s)} \phi(\rho,s)d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \phi(L+\rho,s)d\rho + \int_0^{c(t-s)} \phi(L-\rho,s)d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \left[\phi(L+\rho,s) + \phi(L-\rho,s)\right] d\rho\right) ds \\
&= \frac{1}{2c}\int_0^t \left(\int_0^{c(t-s)} \left[\phi(L+\rho,s) - \phi(L+\rho,s)\right] d\rho\right) ds = 0,
\end{aligned}$$

since $\phi$ is an odd function around $L$. The function $u$ is twice differentiable with respect to $t$ and to $x$ if the extended $\phi$ is continuous and has partial derivatives with respect to $x$ which are continuous. Since the original $\phi$ has continuous partial derivatives with respect to $x$, and $\phi(0,t) = 0 = \phi(L,t)$, the extended $\phi$ has continuous partial derivatives with respect to $x$ as well. Thus, the function $u$ is twice differentiable and the solution to our problem.