APM 524

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Problem 1

1. Show that the trapezoidal rule

$$\int_0^{2\pi} f(x)dx = \frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j),$$

where $x_j = 2\pi j/N$, is exact for $f(x) = \exp(inx)$ for $|n| < N$ (but not for $|n| = N$). Conclude that the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

Solution:

We begin by assuming $n < |N|$ and $n \neq 0$. Then, for the right hand side,

$$\begin{aligned}
\frac{2\pi}{N}\sum_{j=0}^{N-1} e^{inx_j} &= \frac{2\pi}{N}\sum_{j=0}^{N-1} e^{in\left(\frac{2\pi j}{N}\right)} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} \left(e^{in\left(\frac{2\pi}{N}\right)}\right)^j \\
&= \left(\frac{2\pi}{N}\right)\left(\frac{1-e^{in2\pi}}{1-e^{in\left(\frac{2\pi}{N}\right)}}\right) \\
&= 0\ .
\end{aligned}$$

We note, that if $n = N$ the solution would not be defined. Next, for the left hand side, we find

$$\int_0^{2\pi} e^{inx}\ dx = \frac{1}{in}\left[e^{inx}\right]_0^{2\pi} = \frac{1}{in}\left[\cos(2\pi n) + i\sin(2\pi n) - 1\right] = 0\ .$$

Thus the equality holds.

Now assume $n = 0$. We compute,

$$\frac{2\pi}{N}\sum_{j=0}^{N-1} e^{i(0)x_j} = \frac{2\pi}{N}\sum_{j=0}^{N-1} 1 = \left(\frac{2\pi}{N}\right)N = 2\pi\ ,$$

and

$$\int_0^{2\pi} e^{i(0)x}\ dx = \int_0^{2\pi}\ dx = 2\pi\ .$$

Finally, let g be a function such that $g \in \operatorname{span}\{e^{inx}\}_{|n|<N}$. Thus it is of the form $g(x) = \sum_{|n|<N} c_n e^{inx}$ with c_n being constant coefficients. Then,

$$\begin{aligned}
\int_0^{2\pi} g(x)\,dx &= \sum_{|n|<N} c_n \left(\int_0^{2\pi} e^{inx}\,dx \right) \\
&= \sum_{|n|<N} c_n \left(\frac{2\pi}{N} \sum_{j=0}^{N-1} e^{inx_j} \right) \\
&= \frac{2\pi}{N} \sum_{j=0}^{N-1} \sum_{|n|<N} c_n e^{inx_j} \\
&= \frac{2\pi}{N} \sum_{j=0}^{N-1} g(x_j) \ .
\end{aligned}$$

Thus, the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

Problem 2

(*Best Approximation*) Prove the following statements.

1. Let $e_1, \cdots, e_N$ be an orthonormal system in an inner product space H, let $\lambda_1, \cdots, \lambda_N \in \mathbb{C}$ and let $f \in H$. Then

$$\left\| f - \sum_{j=1}^{N} \lambda_j e_j \right\|^2 = \|f\|^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2$$

, where $c_j = \langle f, e_j \rangle$ and $\| \cdot \|$ is the norm induced by the inner product.

Solution:

Using the rules for normed vector spaces with functions and vectors, as well as the fact that $\lambda_i \bar{\lambda_i} = |\lambda_i|^2$ we get

$$
\begin{aligned}
\left\| f - \sum_{j=1}^{N} \lambda_j e_j \right\|^2 &= \langle f - \sum_{j=1}^{N} \lambda_j e_j, f - \sum_{j=1}^{N} \lambda_j e_j \rangle \\
&= \|f\|^2 - \langle f, \sum_{j=1}^{N} \lambda_j e_j \rangle - \langle \sum_{j=1}^{N} \lambda_j e_j, f \rangle + \langle \sum_{j=1}^{N} \lambda_j e_j, \sum_{j=1}^{N} \lambda_j e_j \rangle \\
&= \|f\|^2 - \sum_{j=1}^{N} \overline{\lambda_j} \langle f, e_j \rangle + \sum_{j=1}^{N} \lambda_j \langle \overline{f, e_j} \rangle + \sum_{j,i=1}^{N} \langle \lambda_j e_j, \lambda_i e_i \rangle \\
&= \|f\|^2 - \sum_{j=1}^{N} \overline{\lambda_j} c_j - \sum_{j=1}^{N} \lambda_j \overline{c_j} + \sum_{j,i=1}^{N} \lambda_j \overline{\lambda_i} \langle e_j, e_i \rangle \\
&\text{(note that all terms for } \langle e_j, e_i \rangle = 0 \text{ for } j \neq i \text{ and 1 for } j = i) \\
&= \|f\|^2 - \sum_{j=1}^{N} \overline{\lambda_j} c_j - \sum_{j=1}^{N} \lambda_j \overline{c_j} + \sum_{j=1}^{N} \lambda_j \overline{\lambda_j} \\
&= \|f\|^2 - \sum_{j=1}^{N} \overline{\lambda_j} c_j - \sum_{j=1}^{N} \lambda_j \overline{c_j} + \sum_{j,i=1}^{N} \lambda_j \overline{\lambda_j} + \sum_{j=1}^{N} c_j \overline{c_j} - \sum_{j=1}^{N} c_j \overline{c_j} \\
&= \|f\|^2 + \sum_{j=1}^{N} (-\overline{\lambda_j} c_j - \lambda_j \overline{c_j} + \lambda_j \overline{\lambda_j} + c_j \overline{c_j}) - \sum_{j=1}^{N} c_j \overline{c_j} \\
&= \|f\|^2 + \sum_{j=1}^{N} [\lambda_j (\overline{\lambda_j} - \overline{c_j}) - c_j (\overline{\lambda_j} - \overline{c_j})] - \sum_{j=1}^{N} c_j \overline{c_j} \\
&= \|f\|^2 + \sum_{j=1}^{N} [(\lambda_j - c_j)(\overline{\lambda_j - c_j})] - \sum_{j=1}^{N} c_j \overline{c_j} \\
&= \|f\|^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2 \ .
\end{aligned}
$$

2. Let $f_N = \sum_{j=1}^{N} \langle f, e_j \rangle e_j$. Then $\|f - f_N\| \leq \|f - g\|$ for all g in the span of $e_1, \cdots, e_N$.

Solution:

Note that if $\|f - f_N\| \leq \|f - g\|$ then $\|f - f_N\|^2 \leq \|f - g\|^2$ and that $g = \sum_{j=1}^{N} \lambda_j e_j$. Calculating $\|f - f_N\|^2$ we get

$$\begin{aligned}
\|f - f_N\|^2 &= \left\| f - \sum_{j=1}^{N} \langle f, e_j \rangle e_j \right\|^2 \\
&= \langle f - \sum_{j=1}^{N} \langle f, e_j \rangle e_j, f - \sum_{j=1}^{N} \langle f, e_j \rangle e_j \rangle \\
&= \|f\|^2 - \langle f, \sum_{j=1}^{N} c_j e_j \rangle - \langle \sum_{j=1}^{N} c_j e_j, f \rangle + \langle \sum_{j=1}^{N} c_j e_j, \sum_{i=1}^{N} c_i e_i \rangle \\
&\text{(Using the same reasoning for the last term as above)} \\
&= \|f\|^2 - \sum_{j=1}^{N} \overline{c_j} c_j - \sum_{j=1}^{N} c_j \overline{c_j} + \sum_{j=1}^{N} c_j \overline{c_j} \\
&= \|f\|^2 - \sum_{j=1}^{N} |c_j|^2 \ .
\end{aligned}$$

Then, using the equation from part 1 and the fact that $\sum_{j=1}^{N} |\lambda_j - c_j|^2$ is a nonnegative term,

$$\|f\|^2 - \sum_{j=1}^{N} |c_j|^2 \ \leq \ \|f\|^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2 \ ,$$

which implies $\|f - f_N\|^2 \leq \|f - g\|^2$ and in turn $\|f - f_N\| \leq \|f - g\|$.

Problem 3

1. **Solution:**

Problem 4

1. **Solution:**

Problem 5

1. **Solution:**