The Knife Edge Viscometer

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Outline

1. Introduction
   - The Knife Edge Viscometer
   - Quick Motivation
   - Governing Equations
   - Boundary Conditions

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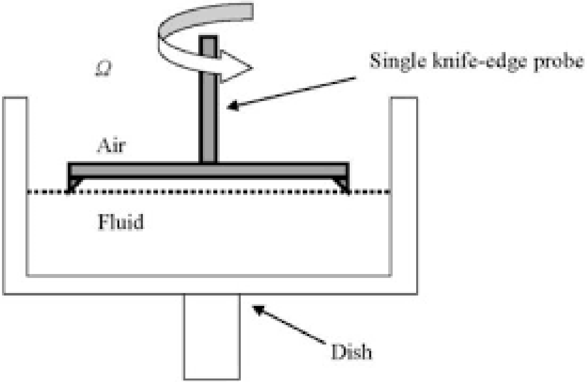
   - The Knife Edge Viscometer
   - Quick Motivation
   - Governing Equations
   - Boundary Conditions
2. Results
   - Observables
   - Parameter Sweep
   - Time-series at the bifurcation
   - Movies

The Knife Edge Viscometer

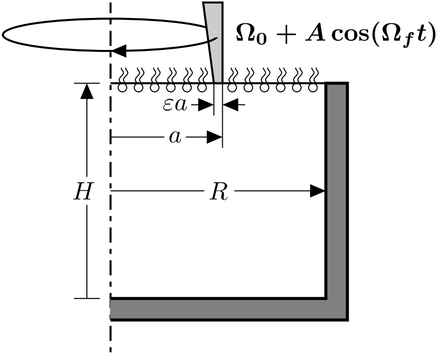


Quick Motivation

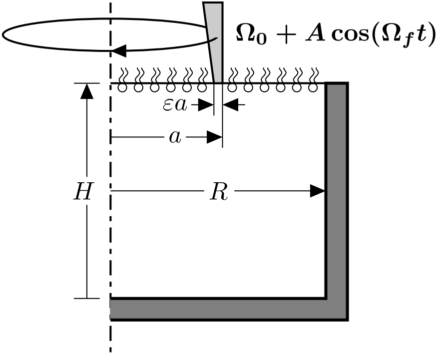
Why do we study this problem?

- How to measure surface shear viscosity remains a controversial issue.
- Monomolecular layers are key in broad areas
  - Pharmaceuticals: interfacial processing.
  - Food processing: surfactants.
  - Natural: Gas absorption into a fluid (lungs, oceans).
- Applications where a high degree of mixing is desired (microbioreactors), although at a low level of shear stress.

Governing Equations



Governing Equations



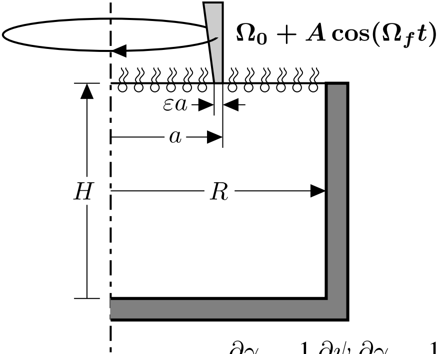
Cylindrical Coordinates

$\psi \equiv$ Stream function
$\gamma \equiv$ Angular momentum
$\eta \equiv$ Azimuthal vorticity

$$\mathbf{u} = (u, v, w) = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z}, \frac{\gamma}{r}, \frac{1}{r}\frac{\partial \psi}{\partial r}\right)$$

$$\nabla \times \mathbf{u} = \left(-\frac{1}{r}\frac{\partial \gamma}{\partial z}, \eta, \frac{1}{r}\frac{\partial \gamma}{\partial r}\right)$$

Governing Equations



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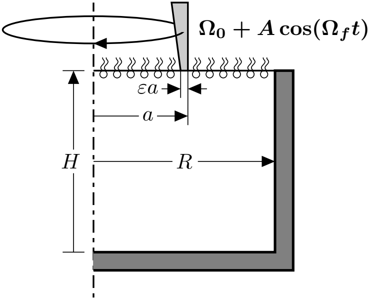
$$\frac{\partial \gamma}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \gamma}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \gamma}{\partial z} = \frac{1}{Re}\left(\frac{\partial^2 \gamma}{\partial z^2} + \frac{\partial^2 \gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \gamma}{\partial r}\right)$$

$$\frac{\partial \eta}{\partial t} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \eta}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \eta}{\partial z} + \frac{\eta}{r^2}\frac{\partial \psi}{\partial z} - \frac{2\gamma}{r^3}\frac{\partial \gamma}{\partial z} = \frac{1}{Re}\left(\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} - \frac{\eta}{r^2}\right)$$

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = -r\eta$$

Boundary Conditions



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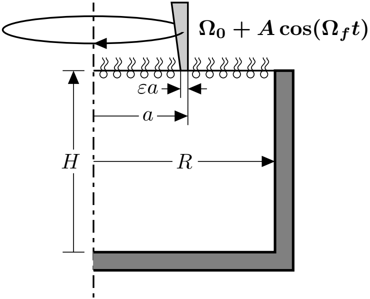

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$$\psi = \gamma = 0 \text{ at } z = 0$$

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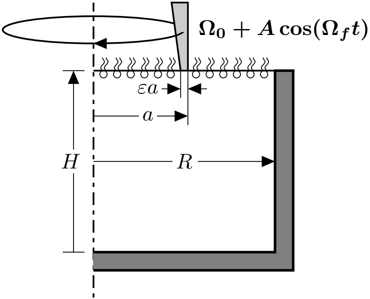

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End Wall (no slip)

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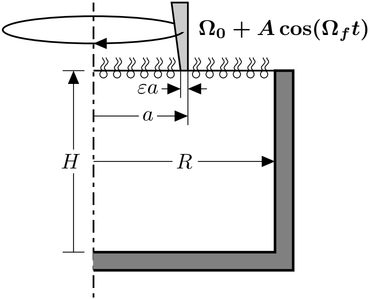

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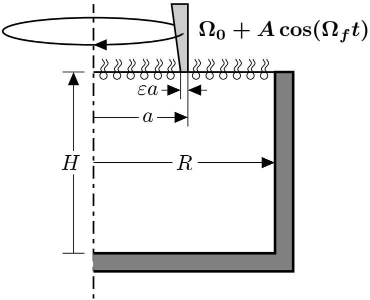
Axis (symmetry)

$$\psi = \gamma = \eta = 0 \ \text{ at } r = 0$$

Are we sure? $\left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\bigg|_{r=0} = 0\right)$

Boundary Conditions (II)



$$v = \frac{\gamma}{r}$$

Top (stress balance)

$$\psi = 0 \quad \text{at } z = A_H$$

$$\eta = -\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2}\bigg|_{z=A_H}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = \frac{1}{Bo}\frac{\partial v}{\partial z} \quad \forall r \neq A_R/2, z = A_H$$

$$v = 1 \quad r = A_R/2, z = A_H$$

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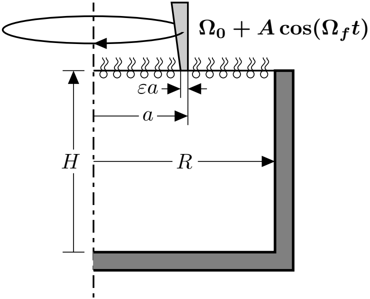

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The limit $Bo \to \infty$

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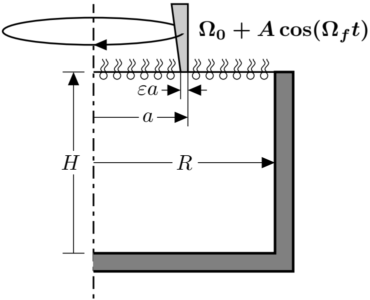

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$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} = 0$$

The bulk flow and the monolayer are decoupled!
Analytic solution possible

Observables

Global

- Kinetic Energy

$$E_k = \frac{1}{2}\int \|\mathbf{u}\|^2 dV$$

- Enstrophy

$$E_w = \int \|\nabla \times \mathbf{u}\|^2 dV$$

- Angular Momentum

$$E_\gamma = \int \gamma^2 dV$$

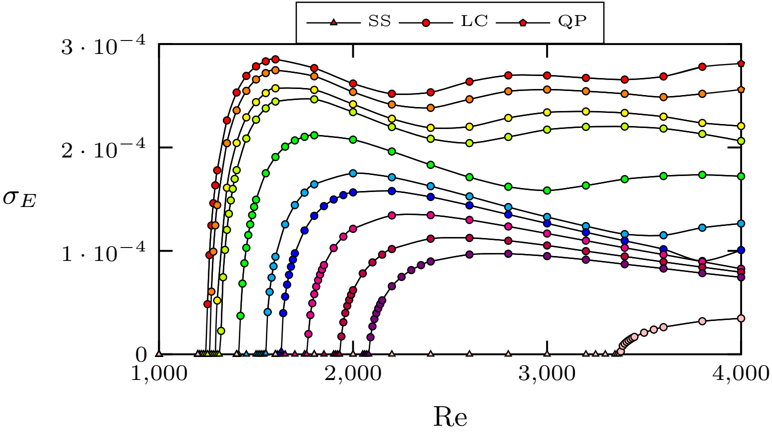
Local

We probe the value of the three velocity components:

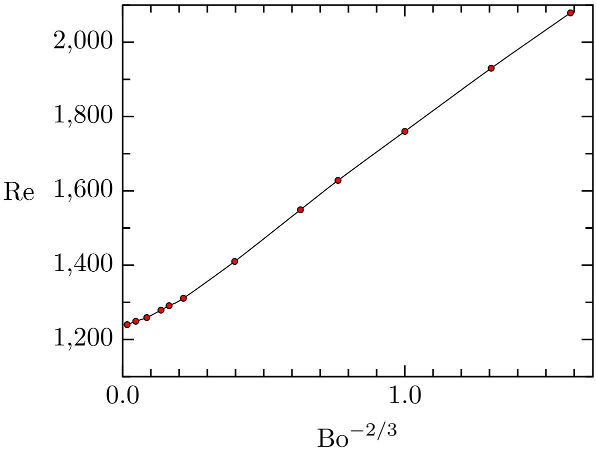
$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z}$$
$$v = \frac{\gamma}{r}$$
$$w = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

at the point $\left(\frac{3}{4}A_H, \frac{3}{4}A_R\right)$.

Parameter Sweep



Scaling



At the Hopf Bifurcation

We now focus on the results for $Bo = 20$ and $Re = 1270, 1300$ (right after the bifurcation).

▸ Zero Forcing Frequency Webpage

Thanks to Jason Yalim.

Flow Field Animations

MOVIES IMPOSSIBLE TO EMBED IN THE BEAMER PRESENTATION

Thank you.