Spectrall Methods

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Homework 2

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Problem 1

Suppose f is 2πperiodic and analytic in the complex strip $|Im(z)| < a$ with $|f(z)| \leq M_{f,a}$ for all z in this strip. Show that

$$\left| \int_0^{2\pi} f(x)dx - \frac{2\pi}{N} \sum_{j=0}^{N-1} f(2\pi j/N) \right| = \mathcal{O}\left(M_{f,a} e^{-(a-\varepsilon)N} \right), \quad \textbf{as } N \to \infty$$

for every $\varepsilon > 0$

Let us recall that

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx,$$

and

$$\hat{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) e^{-ikx_j},$$

with $x_j = 2\pi j/N$, $j = 0, \dots, N-1$. Therefore, using the aliasing formula as well,

$$\begin{aligned}
\left| \int_0^{2\pi} f(x)dx - \frac{2\pi}{N} \sum_{j=0}^{N-1} f(2\pi j/N) \right| &= 2\pi \left| \hat{f}_0 - \hat{v}_0 \right| \\
&= 2\pi \left| \hat{f}_0 - \hat{f}_0 - \sum_{m \neq 0} \hat{f}_m N \right| \\
&= 2\pi \left| \sum_{m \neq 0} \hat{f}_m N \right|.
\end{aligned}$$

Since the function is analytic in the complex strip $|Im(z)| < a$, we can say that it is analytic in the complex strip $|Im(z)| \leq a - \varepsilon$, for every $\varepsilon > 0$. Hence, we can use the bound studied in class,

$$\begin{aligned}\left|\int_0^{2\pi} f(x)dx - \frac{2\pi}{N}\sum_{j=0}^{N-1} f(2\pi j/N)\right| &= 2\pi\left|\sum_{m\neq 0}\hat{f}_m N\right| \\ &= 4\pi M_{f,a}\sum_{m\neq 0} e^{-(a-\varepsilon)|mN|} \\ &= 8\pi M_{f,a}\sum_{m=1}^{\infty} e^{-(a-\varepsilon)mN} \\ &\leq 8\pi M_{f,a}\sum_{k=N}^{\infty} e^{-(a-\varepsilon)k} \\ &\leq \frac{8\pi M_{f,a}}{a-\varepsilon} e^{-(a-\varepsilon)N}.\end{aligned}$$

Thus,

$$\left|\int_0^{2\pi} f(x)dx - \frac{2\pi}{N}\sum_{j=0}^{N-1} f(2\pi j/N)\right| = \mathcal{O}\left(M_{f,a}e^{-(a-\varepsilon)N}\right), \quad \text{as } N\to\infty$$

for every $\varepsilon > 0$

Problem 2

Let $f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(1+i)(n+1/2)^6} e^{inx}$. Compute the Fourier coefficients of f using a DFT or FFT on 11 points to obtain $\hat{v}_k$ and verify the aliasing formula for the coefficients:

$$\hat{v}_k - \hat{f}_k = \sum_{m \neq 0} \hat{f}_{k+mN}$$

For this problem we inmediately have

$$\hat{f}_k = \frac{1}{(1+i)(k+1/2)^6}.$$

To compute $\hat{v}_k = \frac{1}{N} \sum_0^{N-1} f(x_j) e^{-ikx_j}$ we need first $f(x_j)$. We can compute the values of the function using the information given by the problem, truncating the sum at $n = 1000$ (because for higher values of n we reach machine precision when calculating the remainder terms of the sum):

$$f(x_j) = \sum_{n=-1000}^{1000} \frac{1}{(1+i)(n+1/2)^6} e^{inx_j}.$$

Once we have these values it is easy to compute $\hat{v}_k$. The right hand side of the equation is calculated for every value of k in a loop in MATLAB, see code below. We have obtained an accuracy of 10^{-14} between the two sides of the equation. So the aliasing formula is verified.

Matlab code for this problem

```
%% Problem 2
x0=0;
xf=2*pi;
N=11;
K=5;
L=1e3;
x=linspace(x0,xf,N+1);
x(end)=[];
k=-K:K;
l=-L:L;

fhat = 1./((1+1i)*(k+(1/2)).^6);
fxj = 1./((1+1i)*(l+(1/2)).^6)*exp(1i*l'*x);
vhat = (1/N)*fxj*exp(-1i*x'*k);

lhs = vhat-fhat;
m = 1;
```

```
m(1001) = [];

rhs = zeros(1,11);
for l = 1:length(k)
   rhs(l) = sum(1./((1+1i)*((k(l)+m*N)+(1/2)).^6));
end
diff = lhs-rhs
```

Problem 3

Write a function that solves Burgers equation

$$u_t + uu_x = \varepsilon u_{xx}, \qquad x \in (0,1), \quad t \in (0, t_{max}]$$
$$u(0,x) = sin^4(\pi x)$$

and periodic boundary conditions. Use Fourier to compute derivatives in space and ode113 to advance in time. Solve this PDE for $\varepsilon = 0.1, 0.01$, and 0.001. In each case, can you find solutions that areaccurate to three digits at $t = 1$?

Since the domain is 1-periodic, in order to use Fourier DFT comfortably we will change the space variable x to work with a PDE and boundary conditions 2π-periodic. The convenient change of variable is $y = 2\pi x$. Now y is 2π-periodic. Then, the PDE is changed as follows,

$$u_t + 2\pi uu_y = 4\pi^2 \varepsilon u_{yy}, \qquad y \in (0, 2\pi), \quad t \in (0, t_{max}]$$
$$u(0,y) = sin^4(y/2).$$

We can easily solve this PDE using Fourier Derivatives, see code below. The errors I have obtained, using the infinity norm, are:

ε	N	Error
0.1	64	$0.4192 \cdot 10^{-3}$
0.01	64	$0.4839 \cdot 10^{-3}$
0.001	512	$0.9414 \cdot 10^{-3}$

We can see how, the smaller the ε (and therefore more insignificant the diffusion), the more difficult it is to obtain an accurate solution.

Matlab code for this problem

```
%% Problem 3
tmax = 1;
eps = [1e-1 1e-2 1e-3]';
tol = 1e-3;
mesh = zeros(length(eps),1);
E = zeros(length(eps),1);
for m=1:length(eps)
    j = 5;
    N = 2^j;
    error = 1;
    u = PDE_solve(N,tmax,eps(m),false);
    while (error>tol && N<2048)
        j = j+1;
        N = 2^j;
        ufine = PDE_solve(N,tmax,eps(m),false);
```

```
        error = norm(u(end,:)-ufine(end,2:2:end),inf);
        u = ufine;
        j
    end
    m
    mesh(m) = N;
    E(m) = error;
end
mesh
E

function [u,x]=PDE_solve(N,tmax,eps,movie)
x0 = 0;
xf = 2*pi;
h = (xf-x0)/N;
x = x0 + h*(1:N)';
u0 = sin(x/2).^4;

tic;
[t,u] = ode113(@(t,u) rhs(u,N,eps), [0 tmax], u0);
toc
if movie
    figure
    for k = 1:2:length(t)
        plot(x,u(k,:));
        axis([x0 xf -.2 1.2]);
        drawnow
    end
end

end

function y = rhs(u,N,eps)
    vhat = fft(u);
    what = 1i*[0:N/2-1 0 -N/2+1:-1]' .* vhat;
    what2 = -([0:N/2-1 N/2 -N/2+1:-1].^2)' .* vhat;

    ux = real(ifft(what));
    uxx = real(ifft(what2));

    y = 4*pi^2*eps*uxx-2*pi*u.*ux;
end
```

Problem 4

Modify Program 7 so that you can verify that the data in the first curve of Output 7 match the prediction of Theorem 4(a). Verify also that the third and fourth curves match the predictions of parts (c) and (d).

In this problem we verify the convergence rates of Theorem 4. For all cases let w be the vth spectral derivative on the grid $h\mathbb{Z}$. Since we are going to be working with the first derivative, $v = 1$.

First, we focus on the function $u = |sin(x)|^3$, which has $2 = p - 1$ continuous derivatives in $L^2(\mathbb{R})$ and a $p = 3$rd derivative of bounded variation. Hence,

$$|w_j - u^{(v)}(x_j)| = \mathcal{O}(h^{p-v}) = \mathcal{O}(h^{3-1}) = \mathcal{O}(h^2) \text{ as } h \to 0,$$

which we can verify in the next figure.

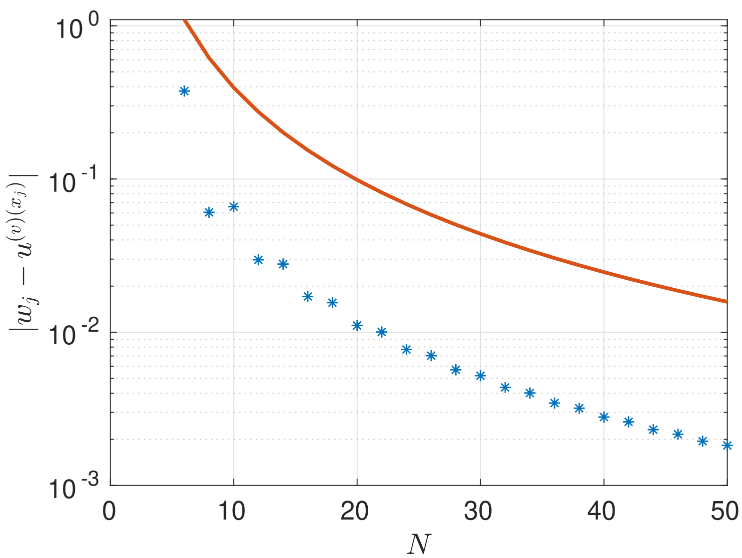

Figure 1: Error convergence for $u = |sin(x)|^3$.

Second, the function $u = \frac{1}{1+\sin^2(x/2)}$ is analytic in a strip in the complex plane. We find this strip as follows

$$1 + \sin^2(z/2) = 0 \ \Rightarrow\ a = \text{Imag}(2\arcsin(i)) \ \Rightarrow\ a \approx 1.76\ .$$

Then, according to the theorem,

$$|w_j - u^{(v)}(x_j)| = \mathcal{O}\left(e^{-(a-\varepsilon)/h)}\right) \text{ as } h \to 0\ ,$$

for every $\varepsilon > 0$ (we have chosen 10^{-4}). We can verify this result in the following figure.

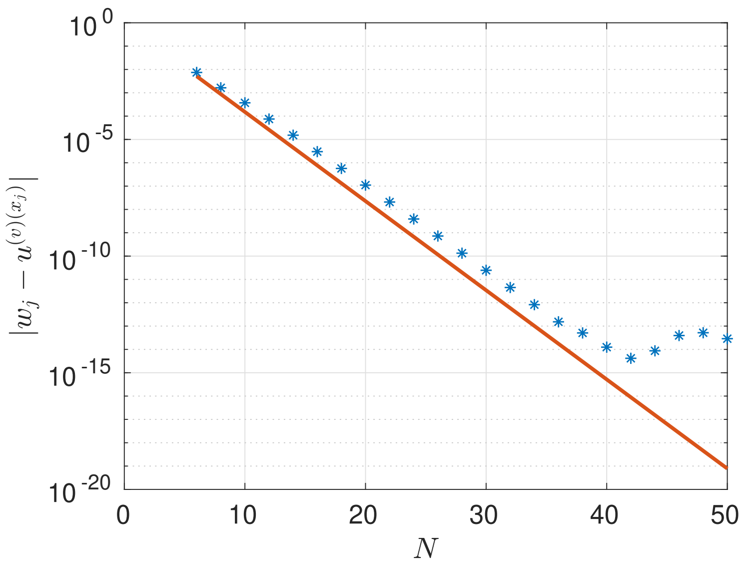

Figure 2: Error convergence for $u = \frac{1}{1+\sin^2(x/2)}$.

Lastly, $u = \sin(10x)$ is band-limited. It is possible to find $a > 0$ such that u can be extended to an entire function and for $z \in \mathbb{C}$, $|u(z)| = o(e^{a|z|})$ as $z \to \infty$. Note that

$$|\sin(10z)| = \left|\frac{e^{i10z} - e^{-i10z}}{2i}\right| = \left|\frac{e^{i10x}e^{-10y} - e^{-i10x}e^{10y}}{2i}\right| \leq \left|\frac{e^{-10y} - e^{10y}}{2}\right| \leq e^{10|z|} \ ,$$

where we have made $z = x + iy$. Further,

$$\lim_{z\to\infty} \frac{|\sin(10z)|}{|e^{a|z|}|} \leq \lim_{z\to\infty} \frac{e^{10|z|}}{e^{a|z|}} = 0 \ \Rightarrow \ a > 10.$$

According to the theorem,

$$w_j = u^{(v)}(x_j),$$

as long as $h = 2\pi/N \leq \pi/a$. Hence,

$$w_j = u^{(v)}(x_j),$$

for $N > 2a = 20$, which is clearly seen in the next figure, where for $N > 20$ the error is machine precision.

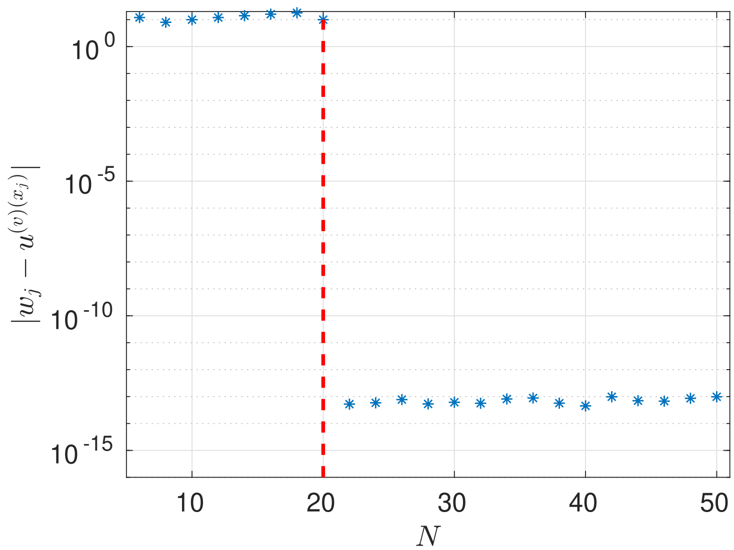

Figure 3: Error convergence for $u = \sin(10x)$.

Matlab code for this problem

```
%% Problem 4
close all
figformat='epsc';
linewidth=2;
% Compute derivatives for various values of N:
Nmax = 50; E = zeros(3,Nmax/2-2);
for N = 6:2:Nmax
    h = 2*pi/N; x = h*(1:N)';
    column = [0 .5*(-1).^(1:N-1).*cot((1:N-1)*h/2)]';
    D = toeplitz(column,column([1 N:-1:2]));
    v = abs(sin(x)).^3;                      % 3rd deriv in BV
    vprime = 3*sin(x).*cos(x).*abs(sin(x));
    E(1,N/2-2) = norm(D*v-vprime,inf);
    v = exp(-sin(x/2).^(-2));                % C-infinity
    vprime = .5*v.*sin(x)./sin(x/2).^4;
    E(2,N/2-2) = norm(D*v-vprime,inf);
    v = 1./(1+sin(x/2).^2);                  % analytic in a strip
    vprime = -sin(x/2).*cos(x/2).*v.^2;
    E(3,N/2-2) = norm(D*v-vprime,inf);
    v = sin(10*x); vprime = 10*cos(10*x);    % band-limited
    E(4,N/2-2) = norm(D*v-vprime,inf);
end
Nvector = 6:2:Nmax;
hvector = 2*pi./Nvector;
```

```
p = 3;
v = 1;

figure
semilogy(Nvector,E(1,:),'*')
hold on
semilogy(Nvector,hvector.^(p-v),'linewidth',linewidth)
grid on
xlabel('$N$','interpreter','latex')
ylabel('$|w_j-u^{(v)(x_j)}|$','interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P4_1';
saveas(gcf,txt,figformat)

a = 1.76;
e = 1e-4;

figure
semilogy(Nvector,E(3,:),'*')
hold on
semilogy(Nvector,exp(-pi*(a-e)./hvector),'linewidth',linewidth)
grid on
xlabel('$N$','interpreter','latex')
ylabel('$|w_j-u^{(v)(x_j)}|$','interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P4_2';
saveas(gcf,txt,figformat)

figure
semilogy(Nvector,E(4,:),'*')
hold on
plot([20 20],[1e-16 10],'r--','linewidth',linewidth)
axis([5 51 1e-16 2e1])
grid on
xlabel('$N$','interpreter','latex')
ylabel('$|w_j-u^{(v)(x_j)}|$','interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P4_3';
saveas(gcf,txt,figformat)
```