Fourier Analysis and Wavelets
Homework 3

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Problem 5

Let

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$(\phi * \phi)(x) = \begin{cases} 1 - |x-1| & \text{if } 0 \le x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Solution: Given the definition of the convolution,

$$(\phi * \phi)(x) = \int_{-\infty}^{\infty} \phi(x-t)\phi(t)dt,$$

note that the convolution will be zero for those values of x that make the functions $\phi(t)$ and $\phi(x-t)$ not overlap. A simple graph of the functions demonstrates that they overlap as long as $x \in [0,2)$. We have now two different situations:

- $x \in [0,1)$. Then,

$$\begin{aligned} (\phi * \phi)(x) &= \int_{-\infty}^{\infty} \phi(x-t)\phi(t)dt \\ &= \int_{0}^{x} \phi(x-t)\phi(t)dt \\ &= \int_{0}^{x} dt \\ &= x, \end{aligned}$$

  which we can rewrite as

$$(\phi * \phi)(x) = 1 - (1-x)$$

- $x \in [1,2)$. Then,

$$\begin{aligned}(\phi * \phi)(x) &= \int_{-\infty}^{\infty} \phi(x-t)\phi(t)dt \\ &= \int_{x-1}^{1} \phi(x-t)\phi(t)dt \\ &= \int_{x-1}^{1} dt \\ &= 1-(x-1)\end{aligned}$$

We can now summarize both results as

$$(\phi * \phi)(x) = 1 - |x-1| \quad \text{for } x \in [0,2)$$

Problem 7

Establish the parts of *Theorem 2.6* that deal with the inverse Fourier transform. Establish the relationship between the Fourier transform and the Laplace transform given in the last part of this theorem.

Solution:

- **1.** The inverse Fourier transform is a linear operator.

  Proof:

$$\begin{aligned}\mathcal{F}^{-1}[\alpha f + \beta g](t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\alpha f(\lambda) + \beta g(\lambda)] e^{i\lambda t} d\lambda \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\alpha f(\lambda)] e^{i\lambda t} d\lambda + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\beta g(\lambda)] e^{i\lambda t} d\lambda \\ &= \alpha \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\lambda) e^{i\lambda t} d\lambda + \beta \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\lambda) e^{i\lambda t} d\lambda \\ &= \alpha \mathcal{F}^{-1}[f](t) + \beta \mathcal{F}^{-1}[g](t)\end{aligned}$$

- **3.** The inverse Fourier transform of a product of f with λ^n is given by

$$\mathcal{F}^{-1}[\lambda^n f(\lambda)](t) = (-i)^n \frac{d^n}{dt^n} \left\{ \mathcal{F}^{-1}[f](t) \right\}$$

  Proof: We start with the definition of the inverse transform

$$\mathcal{F}^{-1}[\lambda^n f(\lambda)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lambda^n f(\lambda) e^{i\lambda t} d\lambda.$$

  Using

$$\lambda^n f(\lambda) e^{i\lambda t} = (-i)^n \frac{d^n}{dt^n} \left\{ f(\lambda) e^{i\lambda t} \right\},$$

we get

$$\begin{aligned}
\mathcal{F}^{-1}[\lambda^n f(\lambda)](t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(-i)^n\frac{d^n}{dt^n}\left\{f(\lambda)e^{i\lambda t}\right\}d\lambda \\
&= (-i)^n\frac{d^n}{dt^n}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(\lambda)e^{i\lambda t}d\lambda\right\} \\
&= (-i)^n\frac{d^n}{dt^n}\left\{\mathcal{F}^{-1}[f](t)\right\},
\end{aligned}$$

and complete the proof.

- **5.** The inverse Fourier transform of the nth derivative of f is given by

$$\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = (-it)^n\mathcal{F}^{-1}[f](t)$$

Proof: We start with the definition of the inverse transform

$$\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f^{(n)}(\lambda)e^{i\lambda t}d\lambda.$$

Integrating by parts, using $u = e^{i\lambda t}$ and $dv = f^{(n)}(\lambda)d\lambda$, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f^{(n)}(\lambda)e^{i\lambda t}d\lambda = \frac{1}{\sqrt{2\pi}}f^{(n-1)}(\lambda)e^{i\lambda t}\Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(it)f^{(n-1)}(\lambda)e^{i\lambda t}d\lambda.$$

Since f vanishes at $\pm\infty$ by hypothesis, there are no boundry terms. Hence,

$$\begin{aligned}
\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(it)f^{(n-1)}(\lambda)e^{i\lambda t}d\lambda \\
&= -(it)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f^{(n-1)}(\lambda)e^{i\lambda t}d\lambda \\
&= -(it)\mathcal{F}^{-1}[f^{(n-1)}(\lambda)](t)
\end{aligned}$$

where we see that there is a transfer derivatives to factors $-it$. Repeating this process $n-1$ times we obtain

$$\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = (-it)^n\mathcal{F}^{-1}[f](t)$$

- **8.** If $f(t) = 0$ for $t < 0$, then

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\mathcal{L}[f](i\lambda),$$

where $\mathcal{L}[f]$ is the Laplace transform of f defined by

$$\mathcal{L}[f](s) = \int_0^{\infty}f(t)e^{-ts}dt.$$

Proof: We start with the definition of the Fourier transform using the assumption that the function is zero in the left half plane,

$$\begin{aligned}
\mathcal{F}[f](\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt.
\end{aligned}$$

Using the change $s = i\lambda$ we get the desired result

$$\begin{aligned}\mathcal{F}[f](\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ts} dt \\ &= \frac{1}{\sqrt{2\pi}} \mathcal{L}[f](s) \\ &= \frac{1}{\sqrt{2\pi}} \mathcal{L}[f](i\lambda)\end{aligned}$$

Problem 12

Consider the filter given by $f \to f * h$, where

$$h(t) := \begin{cases} 1/d & \text{if } 0 \leq t \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Compute and graph $\hat{h}(\lambda)$ for $0 \leq \lambda \leq 20$, for various values of d. Suppose you wish to use this filter to remove signal noise with frequency larger than 30 and retain frequencies in the range of 0 to 5. What values of d would you use? Filter the signal

$$f(t) = e^{-t} (\sin 5t + \sin 3t + \sin t + \sin 40t)$$

Solution: The maple code for this problem can be found below. I defined a two variable function $h(d, t)$ to account the dependency on d. After that I calculate the Fourier transform $\hat{h}(d, \lambda)$ and I plot the Forier transform for $d = 1, 2, 10$. In the first figure we can see how, increasing d the graph gets compressed to the left, therefore filtering more lower frequencies. Since we want to retain the frequencies in the range of 0 to 5, it is clear that a $d < 1$ is necessary. I tried a few values and I liked the result I obtained with $d = 0.6$. In the next figure we can see the Fourier transform and in the last two figures we see the signal before and after applying the filter. We can see how the general shape is conserved although the noise has been to a some extent filtered.

$restart;$

$$h := (d, t) \rightarrow piecewise\left(t < 0, 0, t \le d, \frac{1}{d}, 0\right)$$

$$(d, t) \rightarrow piecewise\left(t < 0, 0, t \le d, \frac{1}{d}, 0\right) \tag{1}$$

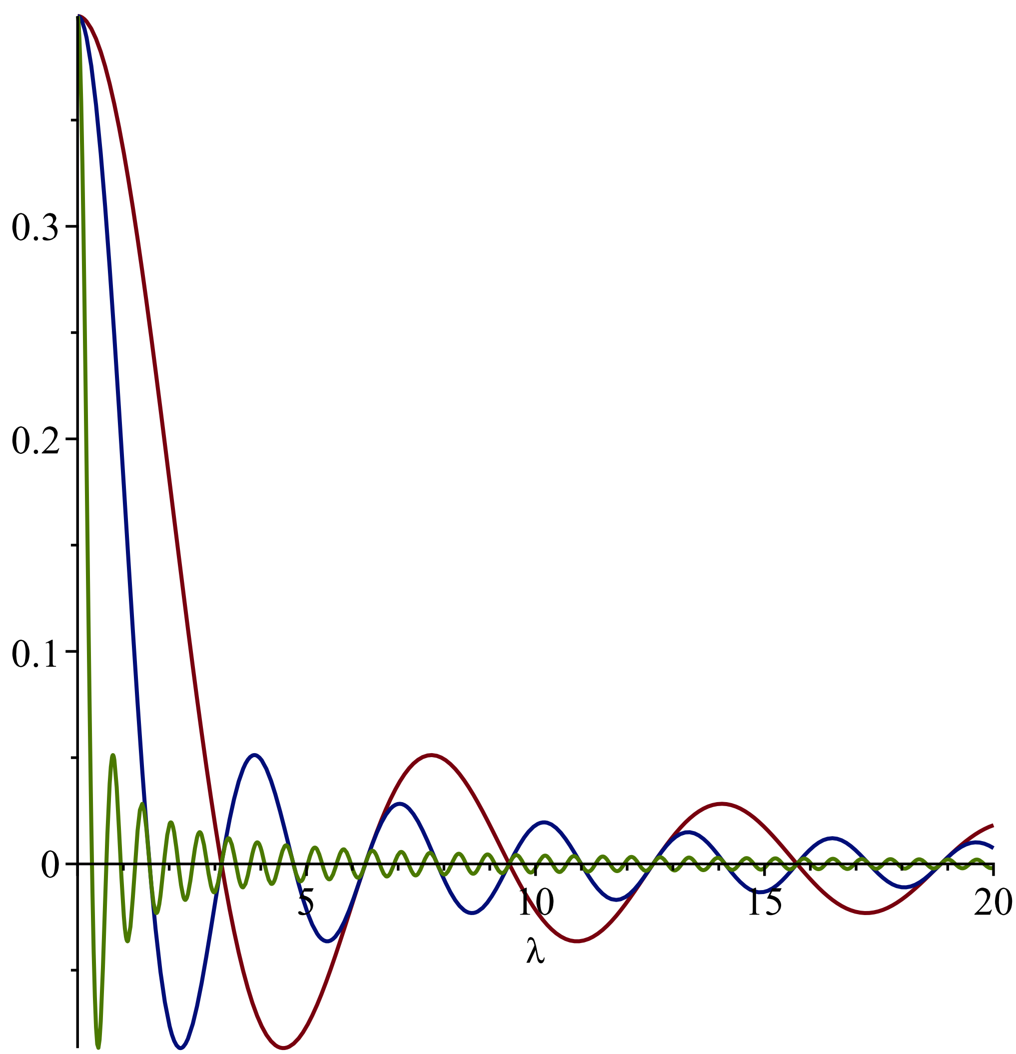
$h(d, t)$

$$\begin{cases} 0 & t < 0 \\ \frac{1}{d} & t \le d \\ 0 & otherwise \end{cases} \tag{2}$$

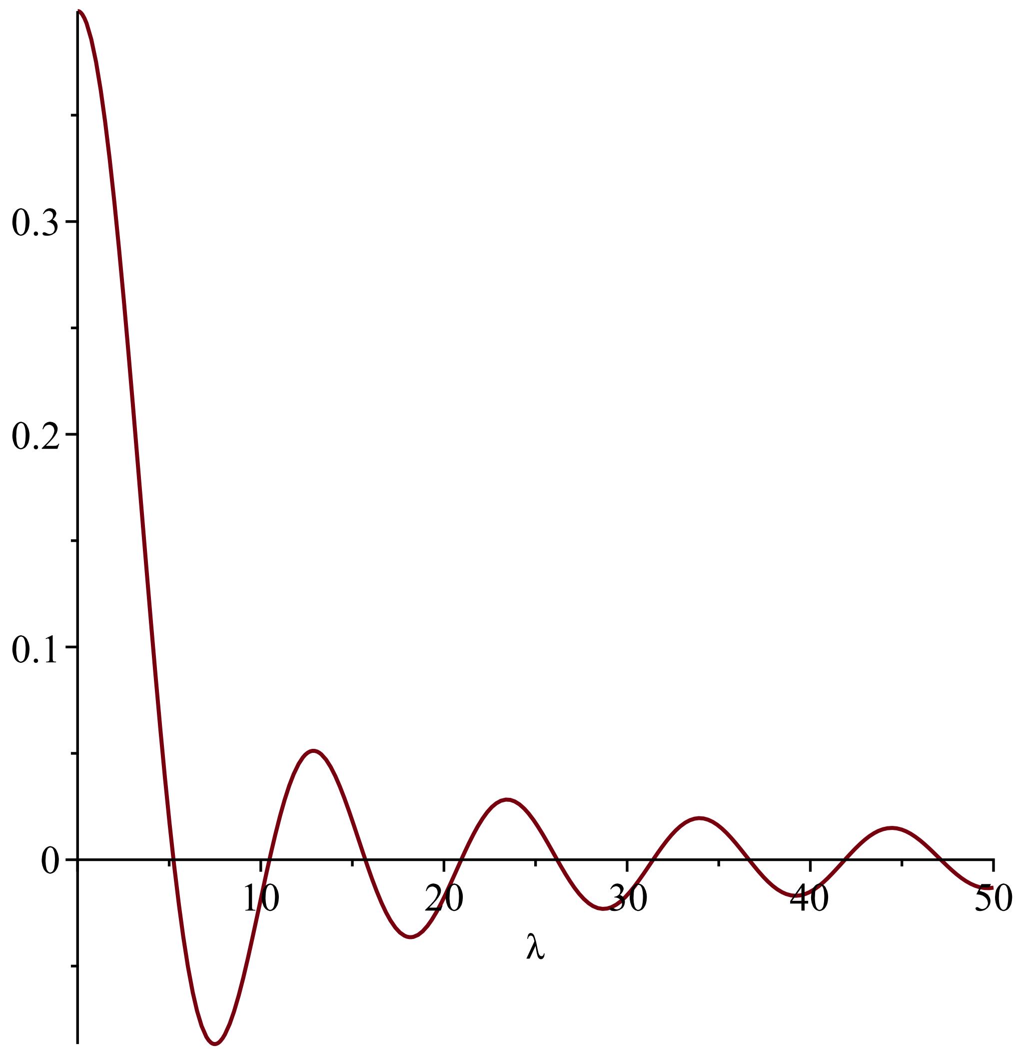
$$hhat := (d, \mathrm{lambda}) \rightarrow \frac{1}{\mathrm{sqrt}(2 \cdot \mathrm{Pi})} int(h(d, t) \cdot \exp(-I \cdot \mathrm{lambda} \cdot t), t = -\mathrm{infinity} .. \mathrm{infinity})$$

$$(d, \lambda) \rightarrow \frac{\int_{-\infty}^{\infty} h(d, t)\, \mathrm{e}^{-I \lambda t}\, \mathrm{d}t}{\sqrt{2 \pi}} \tag{3}$$

$plot(\{\mathrm{Re}(hhat(1, \mathrm{lambda})), \mathrm{Re}(hhat(2, \mathrm{lambda})), \mathrm{Re}(hhat(10, \mathrm{lambda}))\}, \mathrm{lambda} = 0 .. 20)$



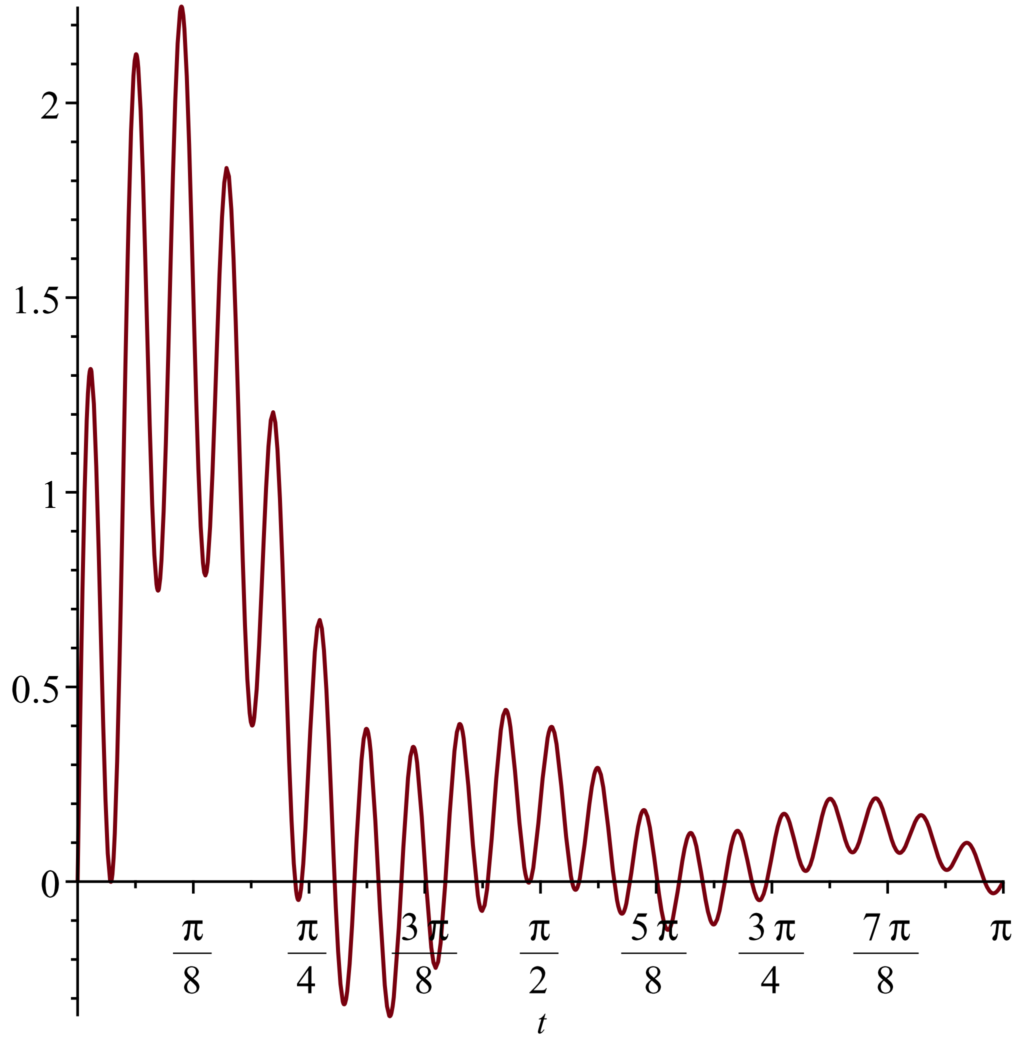
$d := 0.6$:
$plot(\mathrm{Re}(hhat(d, \mathrm{lambda})), \mathrm{lambda} = 0 ..50)$



$f := t \rightarrow \exp(-t) \cdot (\sin(5 \cdot t) + \sin(3 \cdot t) + \sin(t) + \sin(40 \cdot t))$

$$t \rightarrow e^{-t} (\sin(5\, t) + \sin(3\, t) + \sin(t) + \sin(40\, t)) \quad \textbf{(4)}$$

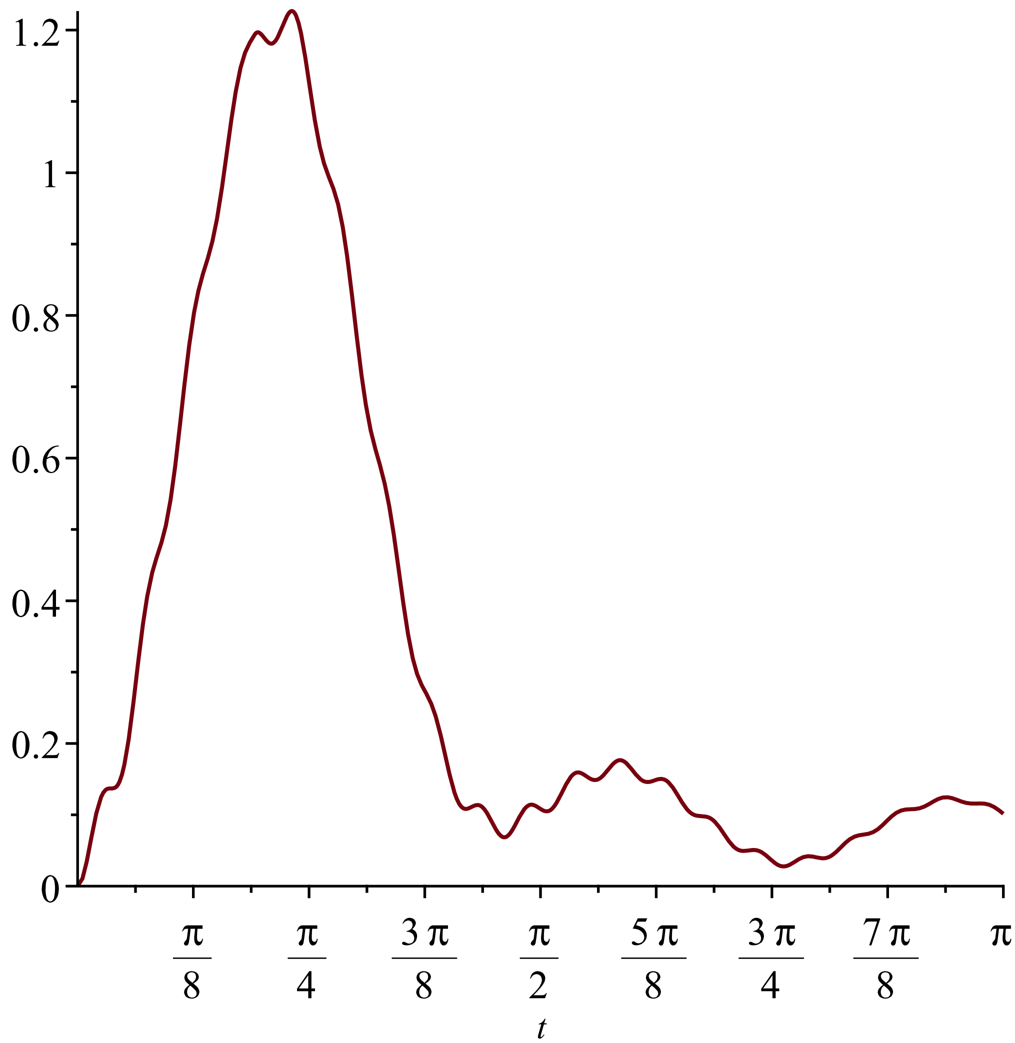
$plot(f(t), t = 0 ..\mathrm{Pi})$



$ff := t \rightarrow int(f(x) \cdot h(d, t - x), x = 0 ..\mathrm{Pi})$

$$t \rightarrow \int_0^{\pi} f(x)\, h(d, t - x)\, \mathrm{d}x \tag{5}$$

$plot(ff(t), t = 0 ..\mathrm{Pi})$



1.2
1
0.8
0.6
0.4
0.2
0
π/8
π/4
3π/8
π/2
5π/8
3π/4
7π/8
π
t