Real Analysis Homework 6

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1 Problem 4.1.3

1. Let (X, d) be a metric space. Then $BUC(X) = BUC(X, \mathbb{K})$ denotes the set of bounded uniformly continuous functions on X with values in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Show: $BUC(X)$ is a closed linear subspace of $B(X)$ with the supremum norm.

Solution:

Proof. Since the zero function, $\mathbb{O}(x) = 0 \ \forall \ x \in X$, is uniformly continuous and obviously bounded, it is an element of $BUC(X)$. Therefore $BUC(X)$ is nonempty.

Now let $f, g \in BUC(X)$ and $\alpha, \beta \in \mathbb{K}$. From a previous result we know that $B(X)$ is a linear subspace of $\mathbb{R}^x$, therefore $\alpha f + \beta g$ is a bounded function.

Let $(x_n), (y_n)$ be sequences in X with $d(x_n, y_n) \to 0$ as $n \to \infty$. Since f and g are uniformly continuous,

$$d(f(x_n), f(y_n)) \to 0 \ ,$$

and

$$d(g(x_n), g(y_n)) \to 0$$

as $n \to \infty$.

Consider now

$$\begin{aligned}
d(\alpha f(x_n) + \beta g(x_n), \alpha f(y_n) + \beta g(y_n)) &= sup\{|\alpha f(x_n) + \beta g(x_n) - \alpha f(y_n) - \beta g(y_n)|\} \\
&\leq sup\{|\alpha(f(x_n) - f(y_n))| + |\beta(g(x_n) - g(y_n))|\} \\
&\leq sup\{|\alpha(f(x_n) - f(y_n))|\} + sup\{|\beta(g(x_n) - g(y_n))|\} \\
&= \|\alpha(f(x_n) - f(y_n))\|_\infty + \|\beta(g(x_n) - g(y_n))\|_\infty \\
&= |\alpha| d(f(x_n), f(y_n)) + |\beta| d(g(x_n), g(y_n)) \to 0 \text{ as } n \to \infty \ ,
\end{aligned}$$

by the uniform continuity of both f and g. Therefore, $\alpha f + \beta g$ is uniformly continuous by theorem 3.12. Thus, $\alpha f + \beta g \in BUC(X)$.

To show that $BUC(X)$ is closed let (f_n) be a sequence of functions in $BUC(X)$ and $f \in B(X)$ such that $f_n \to f$ uniformly,

$$\|f_n - f\|_\infty \to 0 \text{ as } n \to \infty \ .$$

Let $\varepsilon > 0$, then $\exists\ k \in \mathbb{N}$ such that,

$$|f_k(x) - f(x)| \leq \frac{\varepsilon}{3}\ \forall x \in X\ .$$

Since f_k is uniformly continuous, $\exists\ \delta > 0$ such that

$$|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}\ \forall x, y \in X \text{ with } d(x,y) < \delta\ .$$

Consider $x, y \in X$ with $d(x,y) < \delta$, then

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\ .$$

Concluding that f is uniformly continuous as well, so $f \in BUC(X)$. Thus, $BUC(X)$ is a closed set.

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2 Problem 4.1.8

1. Let (X, d) be a metric space and Y a nonempty subset of X, $x \in X$. Show: $d(x, \bar{Y}) = d(x, Y)$.

Solution:

Proof. Let $h \in Y$ and $y \in \bar{Y}$. Since Y is a subset of $\bar{Y}$, $h \in \bar{Y}$ as well. Then:

$$d(x, \bar{Y}) = inf\{d(x,h); h \in \bar{Y}\}\ ,$$
$$d(x, \bar{Y}) \leq d(x,h)\ .$$

Since h was chosen arbitrarily, $d(x, \bar{Y})$ is a lower bound of $\{d(x,h); h \in Y\}$. By the definition of an infimum,

$$d(x, Y) = inf\{d(x,h); h \in Y\} \geq d(x, \bar{Y})\ .$$

Next, since y is a limit point of Y, $\exists (y_n) \in Y$ such that $(y_n) \to y$ as $n \to \infty$. By lemma 2.3,

$$d(x,y) = \lim_{n\to\infty} d(x, y_n)\ .$$

Similarly as done before,

$$d(x, Y) = inf\{d(x, y_n); y_n \in Y\} \leq d(x, y_n) \quad \forall n \in \mathbb{N}$$
$$\Rightarrow d(x, Y) \leq d(x, y)\ .$$

So $d(x, Y)$ is a lower bound, and by the definition of the infimum,

$$d(x, Y) \leq inf\{d(x,y); y \in \bar{Y}\} = d(x, \bar{Y})\ .$$

We prove the statement by the two inequalities put together,

$$d(x, Y) = d(x, \bar{Y})\ .$$

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3 Problem 4.1.15

1. Let X and Y be metric spaces, S a nonempty subset of X, and (f_n) be a sequence of continuous functions from X to Y and f a continuous function from X to Y . Assume that $f_n \to f$ uniformly on S. Show that $f_n \to f$ uniformly on $\bar{S}$.

Solution:

Proof. Assume $f_n \to f$ as $n \to \infty$ uniformly on S and let the same d represent different metrics on X and Y. Then, $\forall\ \varepsilon > 0$, $\exists\ N_1 \in \mathbb{N}$ such that,

$$d(f_n(t), f(t)) < \frac{\varepsilon}{3} \quad \forall\ n \in \mathbb{N},\ \text{with } n > N_1 \text{ and } \forall\ t \in S\ .$$

Let $x_1 \in X$. Then, since (f_n) is a sequence of continuous functions on X, $\exists\ \delta_n > 0$ such that, $\forall\ x \in X$ with $d(x, x_1) < \delta_n$,

$$d(f_n(x), f_n(x_1)) < \frac{\varepsilon}{3}\ .$$

Similarly, since f is a continuous function, $\exists\ \delta_f > 0$, such that, $\forall\ x \in X$ with $d(x, x_1) < \delta_f$

$$d(f(x), f(x_1)) < \frac{\varepsilon}{3}\ .$$

Now let $s \in \bar{S}$ and (s_n) be a sequence in S such that $s_n \to s$ as $n \to \infty$. Define $\delta = inf\{(\delta_n), \delta_f\}$ and $N \in \mathbb{N}$ with $N \geq N_1$ such that,

$$d(s_n, s) < \delta \quad \forall\ n \in \mathbb{N},\ \text{with } n > N\ .$$

Then,

$$\begin{aligned} d(f_n(s), f(s)) &\leq d(f_n(s), f_n(s_n)) + d(f_n(s_n), f(s_n)) + d(f(s_n), f(s)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \forall\ n > N\ . \end{aligned}$$

Thus, since s was chosen arbitrarily, $f_n \to f$ uniformly on $\bar{S}$.

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