Partial Differential Equations
Instructor Homework 3

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Problem 4.2.1

1. Let $\{v_m; m \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space H over $\mathbb{K}$ and (α_m) a sequence in $\mathbb{K}$. Show: The series $\sum_{m=1}^{\infty} \alpha_m v_m$ exists in H if and only if $\sum_{m=1}^{\infty} |\alpha_m|^2 < \infty$. Further, if one and then both of these statements hold,

$$\left\| \sum_{m=1}^{\infty} \alpha_m v_m \right\|^2 = \sum_{m=1}^{\infty} |\alpha_m|^2$$

Solution: Let $(s_k) \in H$ be the sequence of partial sums

$$\sum_{m=1}^{k} \alpha_m v_m,$$

and $(\sigma_k) \in \mathbb{K}$ the sequence of partial sums

$$\sum_{m=1}^{k} |\alpha_m|^2.$$

Then, for $k > l$,

$$\begin{aligned}
||s_k - s_l||^2 &= \left\langle \sum_{m=1}^{k} \alpha_m v_m - \sum_{m=1}^{l} \alpha_m v_m \mid \sum_{j=1}^{k} \alpha_j v_j - \sum_{j=1}^{l} \alpha_j v_j \right\rangle \\
&= \left\langle \sum_{m=l+1}^{k} \alpha_m v_m \mid \sum_{j=l+1}^{k} \alpha_j v_j \right\rangle \\
&= \sum_{j=l+1}^{k} \sum_{m=l+1}^{k} \langle \alpha_m v_m \mid \alpha_j v_j \rangle \\
&= \sum_{j=l+1}^{k} \sum_{m=l+1}^{k} \alpha_m \alpha_j^* \langle v_m \mid v_j \rangle \\
&= \sum_{j=l+1}^{k} \sum_{m=l+1}^{k} \alpha_m \alpha_j^* \delta_{jm} \\
&= \sum_{m=l+1}^{k} \alpha_m \alpha_m^* \\
&= \sum_{m=l+1}^{k} |\alpha_m|^2 \\
&= \sum_{m=1}^{k} |\alpha_m|^2 - \sum_{m=1}^{l} |\alpha_m|^2 \\
&= \sigma_k - \sigma_l,
\end{aligned}$$

where δ_{jm} denotes the Kronecker delta and α_j^* the complex conjugate of α_j. Thus, (s_k) is Cauchy if and only if (σ_k) is Cauchy. Since H is complete by the definition of a Hilbert space, a Cauchy sequence in this space is also convergent. Therefore, (s_k) is convergent if and only if (σ_k) is convergent. Hence, $\sum_{m=1}^{\infty} \alpha_m v_m$ exists in H if and only if $\sum_{m=1}^{\infty} |\alpha_m|^2 < \infty$. Further, if one and then both of these statements hold,

$$\begin{aligned}
\left\| \sum_{m=1}^{\infty} \alpha_m v_m \right\|^2 &= \left\langle \sum_{m=1}^{\infty} \alpha_m v_m \mid \sum_{j=1}^{\infty} \alpha_j v_j \right\rangle \\
&= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \langle \alpha_m v_m \mid \alpha_j v_j \rangle \\
&= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_m \alpha_j^* \langle v_m \mid v_j \rangle \\
&= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_m \alpha_j^* \delta_{jm} \\
&= \sum_{m=1}^{\infty} \alpha_m \alpha_m^* \\
&= \sum_{m=1}^{\infty} |\alpha_m|^2.
\end{aligned}$$