CHAPTER 0

L2 Inner Product: The L^2 inner product on $L^2([a,b])$ is defined as $\langle f,g\rangle_{L^2} = \int_a^b f(t)\overline{g(t)}dt$. **l2 Inner Product:** The space l^2 is the set of all sequences $x_i \in \mathbb{C}$ with $\sum_{-\infty}^{\infty} |x_n|^2 < \infty$. The inner product on l^2 i defined as $\langle X,Y\rangle_{l^2} = \sum_{n=-\infty}^{\infty} x_n\overline{y_n}$. **Schwartz Inequality:**$|\langle X,Y\rangle| \leq ||X||||Y||$ **Triangle Inequality:**$||X+Y|| \leq ||X|| + ||Y||$ **Orthogonal Projection:** Suppose V is an inner product space and V_0 is an N-dimensional subspace with orthonormal basis $\{e_1, e_2, \ldots, e_N\}$. The orthogonal projection of a vector $v \in V$ onto V_0 is given by $v_0 = \sum_{j=1}^{N} \langle v, e_j\rangle e_j$. In addition, $||v - v_0|| = \min_{w \in V_0} ||v - w||$ **Adjoints:** If $T: V \to W$ is a linear operator between two inner product spaces, the adjoint of T is the linear operator $T^*: W \to V$, such that $\langle T(v), w\rangle_W = \langle v, T*(w)\rangle_V$.

CHAPTER 1: FOURIER SERIES

Real Fourier Series

Orthonormal Basis: The set of functions $\{\frac{\sin(k\pi x/a)}{\sqrt{\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{\cos(k\pi x/a)}{\sqrt{\pi}}\}$ with $k = 1, 2, \ldots,$ is an orthonormal set of functions in $L^2([-a,a])$. **Fourier Coefficients:** If $f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi t/a) + \sum_{k=1}^{\infty} b_k \sin(k\pi t/a)$ on the interval $-a \leq t \leq a$, then $a_0 = \frac{1}{2a}\int_{-a}^{a} f(t)dt$, $a_k = \frac{1}{a}\int_{-a}^{a} f(t)\cos(k\pi t/a)dt$ and $b_k = \frac{1}{a}\int_{-a}^{a} f(t)\sin(k\pi t/a)dt$.

Complex Fourier Series

Orthonormal Basis: The set of functions $\{\frac{1}{\sqrt{2a}}e^{i\frac{n\pi}{a}t}, n = 0, \pm1, \pm2, \ldots\}$ is an orthonormal basis for $L^2([-a,a])$. **Fourier Coefficients:** If $f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i\frac{n\pi}{a}t}$, then $\alpha_n = \frac{1}{2a}\int_{-a}^{a} f(t)e^{-i\frac{n\pi}{a}t}dt$

Convergence Theorems

Riemann-Lebesgue Lemma: Suppose f is a piecewise continuous function on the interval $[a,b]$. Then $\lim_{k\to\infty}\int_a^b f(x)\cos(kx)dx = \lim_{k\to\infty}\int_a^b f(x)\sin(kx)dx = 0$. **Convergence at a Point of Continuity:** Suppose f is a continuous and 2π-periodic function. Then for each point x, where the derivative of f is defined, the Fourier series of f converges to $f(x)$. **Convergence at a Point of Discontinuity:** Suppose f is periodic function and piecewise continuous. Suppose x is a point where f is left and right differentiable (but not necessarily continuous). Then the Fourier series of f at x converges to $\frac{f(x-0)+f(x+0)}{2}$, i.e., converges to the average of the left and right limits of f. **Uniform Convergence:** The Fourier series of a continuous, piecewise smooth 2π-periodic function $f(x)$ converges uniformly to $f(x)$ on $[-\pi,\pi]$. **Convergence in the Mean:** Suppose f is an element of $L^2([-\pi,\pi])$. Let $f_N(x) = a_0 + \sum_{k=1}^{N} a_k\cos(kx) + \sum_{k=1}^{N} b_k\sin(kx)$, where a_k and b_k are the Fourier coefficients of f. Then f_N converges to f in $L^2([-\pi,\pi])$, that is, $||f_N - f||_{L^2} \to 0$ as $N \to \infty$. **Parseval's Equation - Real Version:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx) \in L^2[-\pi,\pi]$. Then $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = 2|a_0|^2 + \sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right)$. **Parseval's Equation - Complex Version:** Suppose $f(x) = \sum_{k=-\infty}^{\infty}\alpha_k e^{ikx} \in L^2[-\pi,\pi]$. Then $\frac{1}{2\pi}||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{k=-\infty}^{\infty}|\alpha_k|^2$.

CHAPTER 2: FOURIER TRANSFORM

Definition: If f is a continuously differentiable function with $\int_{-\infty}^{\infty}|f(t)|dt < \infty$, then $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\lambda t}d\lambda$, where $\hat{f}(\lambda)$ is the Fourier transform of $f(t)$ given by $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt$ **Properties:**

- $\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g]$ // $\mathcal{F}^{-1}[\alpha f + \beta g] = \alpha\mathcal{F}^{-1}[f] + \beta\mathcal{F}^{-1}[g]$

CHAPTER 3:

Identities Sum and Difference Formula $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$. $\cos(A \mp B) = \cos A\cos B \pm \sin A\sin B$. **Double Angle Formula** $\sin(2A) = 2\sin A\cos A$. $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$. $\tan(2A) = (2\tan A)/(1 - \tan^2 A)$. **Half Angle Formula** $\sin(A/2) = \pm\sqrt{(1-\cos A)/2}$. $\cos(A/2) = \pm\sqrt{(1+\cos A)/2}$. $\tan(A/2) = (1-\cos A)/(\sin A) = (\sin A)/(1+\cos A)$. **Geometric Sum** $\sum_{k=1}^{\infty} q^k = q/(1-q)$. $\sum_{k=1}^{n} q^k = (q - q^{n-1})/(1-q)$. **General ODE Solutions** $y'' = y(t) \implies y = c_1e^{-t} + c_2e^t$ $\square$ $dy/dt + p(t)y = g(t) \implies y = (\int u(t)g(t))/u(t) + c$ where $u(t)$ =exp$(\int p(t)dt)$ $\square$ $y' = x; x' = y \implies x = c_1\cosh t + c_2\sinh t, y = c_1\sinh t + c_2\cosh t$ or $x = c_1e^t + c_2e^{-t}, y = c_1e^t - c_2e^{-t}$ $\square$ $y' = -x; x' = y \implies y = c_1\cos t + c_2\sin t, x = c_1\sin t - c_2\cos t$ $\square$ $x' = x + y; y' = -x + y \implies x = e^t(c_1\cos t + c_2\sin t); y = e^t(-c_1\sin t + c_2\cos t)$ $\square$ $v' = \gamma v, v(z,0) = u_0 \implies v = u_0e^{\gamma t}$ $\square$