Numerical Methods for PDEs
Homework 6

Francisco Jose Castillo Carrasco

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Problem 1

In the modified PDE for the Lax-Wendroff method for $u_t + cu_x = 0$, derive the coefficient $\beta = \frac{ch^2}{6}(r^2 - 1)$ of numerical dispersion in $u_t + cu_x = \beta u_{xxx}$.

Solution: We start by Taylor expanding

$$u_j^{n+1} = u_j^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \mathcal{O}(\Delta t^4),$$
$$u_{j\pm1}^n = u_j^n \pm \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} \pm \frac{\Delta x^3}{6} u_{xxx} + \mathcal{O}(\Delta x^4),$$

and substituting them (neglecting the higher order terms) into the Lax-Wendroff scheme,

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{c^2 \Delta t^2}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), \\ u_j^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} &= u_j^n - \frac{c\Delta t}{2\Delta x}\left(2\Delta x u_x + \frac{\Delta x^3}{3} u_{xxx}\right) + \frac{c^2 \Delta t^2}{2\Delta x^2} \Delta x^2 u_{xx}. \end{aligned}$$

Doing some algebraic manipulations we reach,

$$u_t + cu_x = -\frac{\Delta t}{2} u_{tt} - c\frac{\Delta x^2}{6} u_{xxx} + \frac{1}{2} c^2 \Delta t u_{xx} - \frac{\Delta t^2}{6} u_{ttt}.$$

Using the original PDE we obtain that

$$\begin{aligned} u_t = -cu_x &\Rightarrow u_{tt} = c^2 u_{xx}, \\ &\Rightarrow u_{ttt} = -c^3 u_{xxx}. \end{aligned}$$

Hence,

$$\begin{aligned} u_t + cu_x &= -\frac{\Delta t}{2} c^2 u_{xx} - c\frac{\Delta x^2}{6} u_{xxx} + \frac{1}{2} c^2 \Delta t u_{xx} + \frac{\Delta t^2}{6} c^3 u_{xxx}, \\ &= -c\frac{\Delta x^2}{6} u_{xxx} + \frac{\Delta t^2}{6} c^3 u_{xxx}, \\ &= c\frac{\Delta x^2}{6}\left(\frac{c^2 \Delta t^2}{\Delta x^2} - 1\right) u_{xxx}, \\ &= c\frac{\Delta x^2}{6}\left(r^2 - 1\right) u_{xxx}, \\ &= \beta u_{xxx}. \end{aligned}$$

Problem 2

Show that the two-step Lax-Wendroff method reduces to the original Lax-Wendroff scheme for $u_t + Au_x = 0$.

Solution: We start with the first step of the method

$$
\begin{aligned}
w_{i+1/2}^{n+1/2} &= \frac{1}{2}\left(w_i^n + w_{i+1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A \left(w_{i+1}^n - w_i^n\right), \\
w_{i-1/2}^{n+1/2} &= \frac{1}{2}\left(w_i^n + w_{i-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A \left(w_i^n - w_{i-1}^n\right),
\end{aligned}
$$

and plug the terms into the second step,

$$
\begin{aligned}
w_i^{n+1} &= w_i^n - \frac{\Delta t}{\Delta x} A \left(w_{i+1/2}^{n+1/2} - w_{i-1/2}^{n+1/2}\right), \\
&= w_i^n - \frac{\Delta t}{\Delta x} A \left[\frac{1}{2}\left(w_i^n + w_{i+1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x} A \left(w_{i+1}^n - w_i^n\right) - \frac{1}{2}\left(w_i^n + w_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x} A \left(w_i^n - w_{i-1}^n\right)\right], \\
&= w_i^n - \frac{1}{2} A \frac{\Delta t}{\Delta x}\left(w_{i+1}^n - w_{i-1}^n\right) + \frac{1}{2} A^2 \frac{\Delta t^2}{\Delta x^2}\left(w_{i+1}^n - 2w_i^n + w_{i-1}^n\right),
\end{aligned}
$$

and we obtain the desired the original Lax-Wendroff scheme for $u_t + Au_x = 0$.

Problem 3

Show that Burgers' equation $u_t + uu_x = \nu u_{xx}$ with $u(x, t = 0) = u_l,\ x < 0$ and $u(x, t = 0) = u_r < u_l,\ x > 0$ has a traveling wave solution of the form $u(x, t) = w(x - st)$ by deriving an ODE for w and showing that the ODE is solved by

$$
w(y) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh \frac{(u_l - u_r)y}{4\nu}\right]
$$

where $s = (u_l + u_r)/2$.

Solution:

Firs, let $y = x - st$. By the chain rule

$$
u_t = \frac{\partial w(y)}{\partial t} = \frac{\partial w(y)}{\partial y}\frac{\partial y}{\partial t} = -sw'.
$$

Similarly,

$$
\begin{aligned}
u_x &= w', \\
u_{xx} &= w''.
\end{aligned}
$$

With the obtained derivatives plugged into the Burgers' equation we get a second order, non linear ODE for w,

$$
-sw' + ww' = \nu w''
$$

Given w, we can compute

$$w' = -\frac{1}{2}(u_l - u_r)\text{sech}^2\left(\frac{(u_l - u_r)y}{4\nu}\right)\left(\frac{(u_l - u_r)}{4\nu}\right),$$

and,

$$w'' = \frac{(u_l - u_r)^3}{(4\nu)^2}\text{sech}^2\left(\frac{(u_l - u_r)y}{4\nu}\right)\tanh\left(\frac{(u_l - u_r)y}{4\nu}\right) \ .$$

Substituting the terms into the obtained ODE we find that the given w is indeed a solution

$$\begin{aligned}
-sw' + ww' &= -\frac{1}{2}\text{sech}^2\left(\frac{(u_l - u_r)y}{4\nu}\right)\left(\frac{(u_l - u_r)^2}{4\nu}\right)\left[-s + u_r + \frac{1}{2}(u_l - u_r)\left(1 - \tanh\left(\frac{(u_l - u_r)y}{4\nu}\right)\right)\right] \\
&= -\frac{1}{2}\text{sech}^2\left(\frac{(u_l - u_r)y}{4\nu}\right)\left(\frac{(u_l - u_r)^2}{4\nu}\right)\left[-\frac{1}{2}(u_l - u_r)\tanh\left(\frac{(u_l - u_r)y}{4\nu}\right)\right] \\
&= \nu\frac{(u_l - u_r)^3}{(4\nu)^2}\text{sech}^2\left(\frac{(u_l - u_r)y}{4\nu}\right)\tanh\left(\frac{(u_l - u_r)y}{4\nu}\right) \\
&= \nu w'' \ .
\end{aligned}$$

Problem 4

Prove that the Lax-Friedrichs (LF) method is *positivity-preserving* (i.e., if $u_j^n > 0$ for all j, then $u_j^{n+1} > 0$ for all j) for Burgers' equation $u_t + (u^2/2)_x = 0$. The LF discretization of Burgers' equation is

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right) - \frac{\Delta t}{4\Delta x}\left((u_{j+1}^n)^2 - (u_{j-1}^n)^2\right)$$

Hint: For stability, $\Delta t \leq \Delta x / \max_i\{|u_i^n|\}$.

Solution:

Let $M = \max_j |u_j|$. Then , for the method to be stable,

$$\Delta t \leq \frac{\Delta x}{M}.$$

Taking common factor in the LF discretization of Burgers' equation,

$$u_j^{n+1} = \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)\left[1 - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)\right].$$

Since we assume that $u_j^n > 0$ for all j, the factor $\frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)$ is positive for all j. Once we know that, we can procede as follows

$$\begin{aligned}
u_j^{n+1} &= \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)\left[1 - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)\right], \\
&\geq \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)\left[1 - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n + u_{j-1}^n\right)\right], \\
&\geq \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)\left[1 - \frac{\Delta t}{\Delta x}M\right],
\end{aligned}$$

where we have used that

$$\frac{u_{j+1}^n + u_{j-1}^n}{2} \leq M.$$

To finish, by the stability condition, $\frac{\Delta t}{\Delta x} M \leq 1$. Hence,

$$u_j^{n+1} \geq \frac{1}{2}\left(u_{j-1}^n + u_{j+1}^n\right)\left[1 - \frac{\Delta t}{\Delta x} M\right] \geq 0.$$

Thus LF is *positivity-preserving* for Burger's equation.

Problem 5

Show that the upwind discretization of Burgers' equation $u_t + uu_x = 0$ for $u(x,t) > 0$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n \left(u_j^n - u_{j-1}^n\right)$$

is nonconservative, by showing whether or not the scheme can be put into the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right).$$

Solution:
The upwind discretization for Burger's equation can be written as $u_t + uu_x = 0$ for $u(x,t) > 0$,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\left(u_j^n\right)^2 - u_j^n u_{j-1}^n\right) \; .$$

We let $F_{j+1/2} = \left(u_j^n\right)^2$ and note

$$F_{j+1/2} = \left(u_j^n\right)^2 \Rightarrow F_{j-1/2} = \left(u_{j-1}^n\right)^2 \neq u_j^n u_{j-1}^n \; .$$

Hence, since it cannot be written in conservative form, the upwind discretization of Burger's equation $u_t + uu_x = 0$ is not conservative.

Show that the upwind discretization of Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ for $u(x,t) > 0$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left((u_j^n)^2 - (u_{j-1}^n)^2\right)$$

is conservative, by showing whether or not the scheme can be put into the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right).$$

Solution:

The upwind discretization of Burgers' equation $u_t + (\frac{1}{2}u^2)_x = 0$ for $u(x,t) > 0$ can be written as

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left((u_j^n)^2 - (u_{j-1}^n)^2\right).$$

Define $F_{j+1/2} = \left(u_j^n\right)^2$ and $F_{j-1/2} = \left(u_{j-1}^n\right)^2$ giving,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right).$$

Thus, the upwind discretization of Burger's equation $u_t + (\frac{1}{2}u^2)_x = 0$ is conservative.