Spectrall Methods

Francisco Castillo
Homework 1

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Problem 1

Show that the trapezoidal rule

$$\int_0^{2\pi} f(x)dx = \frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j),$$

where $x_j = 2\pi j/N$, is exact for $f(x) = \exp(inx)$ for $|n| < N$ (but not for $|n| = N$). Conclude that the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

First, we compute the integral,

$$\int_0^{2\pi} e^{inx}\ dx = 2\pi\delta_{n0},$$

where δ is the Kronecker-delta. For the right hand side we start considering the case where $n < |N|$ and $n \neq 0$,

$$\begin{aligned}
\frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j) &= \frac{2\pi}{N}\sum_{j=0}^{N-1} e^{inx_j} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} e^{\left(\frac{i2\pi nj}{N}\right)} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} \left[e^{\left(\frac{i2\pi n}{N}\right)}\right]^j \\
&= \left(\frac{2\pi}{N}\right)\left(\frac{1-e^{i2\pi n}}{1-e^{\left(\frac{i2\pi n}{N}\right)}}\right) \\
&= 0, \qquad \forall n < |N|, n \neq 0,
\end{aligned}$$

where we have applied the geometric sum formula and that $e^{i2\pi n} = 1$. We note that, if $n = N$ the solution would not be defined. Lastly, for $n = 0$,

$$\frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j) = \frac{2\pi}{N}\sum_{j=0}^{N-1} 1 = \frac{2\pi}{N}N = 2\pi\ .$$

Finally, let g be a function in the span$\{e^{inx}\}_{|n|<N}$. Hence, we can write

$$g(x) = \sum_{|n|<N} c_n e^{inx},$$

with c_n being constant coefficients. Then,

$$\begin{aligned}
\int_0^{2\pi} g(x)\ dx &= \int_0^{2\pi} \sum_{|n|<N} c_n e^{inx}\ dx \\
&= \sum_{|n|<N} c_n \left(\int_0^{2\pi} e^{inx}\ dx\right) \\
&= \sum_{|n|<N} c_n \left(\frac{2\pi}{N}\sum_{j=0}^{N-1} e^{inx_j}\right) \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1}\sum_{|n|<N} c_n e^{inx_j} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} g(x_j)\ .
\end{aligned}$$

Thus, the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

Problem 2

(*Best Approximation*) Prove the following statements.

(a) **Let $e_1, \cdots, e_N$ be an orthonormal system in an inner product space H, let $\lambda_1, \cdots, \lambda_N \in \mathbb{C}$ and let $f \in H$. Then**

$$\left|\left| f - \sum_{j=1}^{N} \lambda_j e_j \right|\right|^2 = ||f||^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2,$$

where $c_j = \langle f, e_j \rangle$ and $\| \cdot \|$ is the norm induced by the inner product.

To prove that the statement we start by writing the left side of the equation as an inner product,

$$\left|\left| f - \sum_{j=1}^{N} \lambda_j e_j \right|\right|^2 =< f - \sum_{j=1}^{N} \lambda_j e_j | f - \sum_{j=1}^{N} \lambda_j e_j > .$$

By appllying linearity and antilinearity of the inner product we get

$$\begin{aligned}
\left|\left| f - \sum_{j=1}^{N} \lambda_j e_j \right|\right|^2 &=< f - \sum_{j=1}^{N} \lambda_j e_j | f - \sum_{j=1}^{N} \lambda_j e_j > \\
&=< f|f > - \sum_{j=1}^{N} < \lambda_j e_j | f > - \sum_{j=1}^{N} < f | \lambda_j e_j > + \sum_{j=1}^{N} < \lambda_j e_j | \lambda_j e_j > \\
&= ||f||^2 - \sum_{j=1}^{N} \lambda_j < e_j | f > - \sum_{j=1}^{N} \lambda_j^* < f | e_j > + \sum_{j=1}^{N} < \lambda_j e_j | \lambda_j e_j > \\
&= ||f||^2 - \sum_{j=1}^{N} \lambda_j c_j^* - \sum_{j=1}^{N} \lambda_j^* c_j + \sum_{j=1}^{N} ||\lambda_j||^2.
\end{aligned}$$

where we have used the linearity and antilinearity properties again, and that $c_j =< f|e_j >$. By adding and substracting $\sum_{j=1}^{N} < c_j | c_j >$ and rewriting the previous equation in terms of inner product again we get

$$\begin{aligned}
\left|\left| f - \sum_{j=1}^{N} \lambda_j e_j \right|\right|^2 = ||f||^2 &- \sum_{j=1}^{N} < \lambda_j | c_j > - \sum_{j=1}^{N} < c_j | \lambda_j > + \sum_{j=1}^{N} < \lambda_j | \lambda_j > \\
&+ \sum_{j=1}^{N} < c_j | c_j > - \sum_{j=1}^{N} < c_j | c_j >,
\end{aligned}$$

which, again by linearity and antilinearity, we can group together as

$$\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 = ||f||^2+\sum_{j=1}^{N}<\lambda_j-c_j|\lambda_j>-\sum_{j=1}^{N}<\lambda_j-c_j|c_j>-\sum_{j=1}^{N}<c_j|c_j>$$
$$= ||f||^2+\sum_{j=1}^{N}<\lambda_j-c_j|\lambda_j-c_j>-\sum_{j=1}^{N}<c_j|c_j>$$
$$= ||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2.$$

Thus, we have proven that

$$\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 = ||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2.$$

(b) **Let $f_N = \sum_{j=1}^{N}\langle f, e_j\rangle e_j$. Then $\|f-f_N\| \leq \|f-g\|$ for all g in the span of $e_1,\cdots,e_N$.**

To prove that

$$||f-f_N|| \leq ||f-g||,$$

we will show that their squares

$$||f-f_N||^2 \leq ||f-g||^2.$$

Taking into account that $f_N = \sum_{j=1}^{N} < f|e_j > e_j$ and $g = \sum_{j=1}^{N}\lambda_j e_j$ and using the previous proof we have that

$$||f-f_N||^2 = \left|\left|f-\sum_{j=1}^{N}<f|e_j>e_j\right|\right|^2$$
$$= ||f||^2+\sum_{j=1}^{N}|<f|e_j>-c_j|^2-\sum_{j=1}^{N}|c_j|^2$$
$$= ||f||^2+\sum_{j=1}^{N}|<f|e_j>-<f|e_j>|^2-\sum_{j=1}^{N}|c_j|^2$$
$$= ||f||^2-\sum_{j=1}^{N}|<f|e_j>|^2,$$

and

$$\begin{aligned}
||f-g||^2 &= \left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 \\
&= ||f||^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2 \\
&= ||f||^2 + \sum_{j=1}^{N} |\lambda_j - < f|e_j > |^2 - \sum_{j=1}^{N} |c_j|^2 \\
&\geq ||f||^2 - \sum_{j=1}^{N} | < f|e_j > |^2 \\
&= ||f-f_N||^2.
\end{aligned}$$

Thus, we have proved that

$$||f-f_N|| \leq ||f-g||$$

Problem 3

(*Fourier-Galerkin method*) Let $\mathcal{L}$ be a differential operator of the form $\mathcal{L} = \sum_{k=0}^{m} \alpha_k \frac{d^k}{dx^k}$, with $\alpha_k \in \mathbb{C}$. Suppose we are interested in solving the differential equation $\mathcal{L}u = g$. Let

$$u_N = \sum_{n=-N/2}^{N/2} c_n e^{inx},$$

where the coefficients c_n are chosen so that $\langle \mathcal{L}u_N - g, e^{ikx} \rangle = 0$ for $k = -N/2, \ldots, N/2$. Show that $||\mathcal{L}u_N - g|| \leq ||\mathcal{L}w - g||$ for all w in the span of $\{e^{inx}\}_{|n| \leq N/2}$.

Let's first compute the result of

$$\begin{aligned}
\mathcal{L}u_N &= \sum_{k=0}^{m} \alpha_k \frac{d^k}{dx^k} \left[\sum_{n=-N/2}^{N/2} c_n e^{inx} \right] \\
&= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k e^{inx}.
\end{aligned}$$

Before we continue, we work on the other piece of information, $\langle \mathcal{L}u_N - g, e^{ikx} \rangle = 0$, which implies

$$\langle \mathcal{L}u_N, e^{ilx} \rangle = \langle g, e^{ilx} \rangle,$$

where we have used the index l for future convenience. Now,

$$\begin{aligned}
\langle \mathcal{L}u_N, e^{ilx} \rangle &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k \langle e^{inx}, e^{ilx} \rangle \\
&= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k 2\pi \delta_{nl} \\
&= 2\pi c_l \sum_{k=0}^{m} \alpha_k (il)^k \\
&= \langle g, e^{ilx} \rangle.
\end{aligned}$$

Hence, retaking the index n,

$$\langle \mathcal{L}u_N, e^{inx} \rangle = 2\pi c_n \sum_{k=0}^{m} \alpha_k (in)^k = \langle g, e^{inx} \rangle.$$

Further, we can rewrite $\mathcal{L}u_N$ as

$$\begin{aligned}
\mathcal{L}u_N &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N/2}^{N/2} 2\pi c_n \sum_{k=0}^{m} \alpha_k (in)^k e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N/2}^{N/2} \langle g, e^{inx} \rangle e^{inx} \\
&= \sum_{n=-N/2}^{N/2} \langle g, \frac{e^{inx}}{\sqrt{2\pi}} \rangle \frac{e^{inx}}{\sqrt{2\pi}} \\
&= \sum_{n=-N/2}^{N/2} \langle g, e_n \rangle e_n,
\end{aligned}$$

where we are denoting $e_n = \frac{e^{inx}}{\sqrt{2\pi}}$ as the orthonormal vectors. Note that the set of $\{e_n\}_{|n| \leq N/2}$ form an orthonormal base. Since $\mathcal{L}u_N$ can be written as

$$\mathcal{L}u_N = \sum_{n=-N/2}^{N/2} \langle g, e_n \rangle e_n,$$

by the previous result in problem 2, $||\mathcal{L}u_N - g|| \leq ||\mathcal{L}w - g||$ for all g in the span of $e_1, \cdots, e_N$.