Partial Differential Equations
TA Homework 13

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Problem 6.3.2

Let $g_1, g_2 : \partial\Omega \to \mathbb{R}$ be continuous and $u_1, u_2 : \overline{\Omega} \to \mathbb{R}$ be continuous on $\overline{\Omega}$ and twice differentiable on Ω and

$$Lu_j = 0 \quad \text{on } \Omega, \quad u_j = g_j \quad \text{on } \partial\Omega, \quad j = 1, 2.$$

Show:

$$\max_{\overline{\Omega}} |u_1 - u_2| = \max_{\partial\Omega} |g_1 - g_2|$$

Solution: Let $v = u_1 - u_2$. Therefore $v : \overline{\Omega} \to \mathbb{R}$ is continuous and twice partially differentiable on Ω. Then,

$$Lv = Lu_1 - Lu_2 = 0, \quad \text{on } \Omega,$$

and

$$v = g_1 - g_2 \quad \text{on } \partial\Omega.$$

Then, by *Corollary 6.9*,

$$\max_{\Omega} |v| = \max_{\partial\Omega} |v| = \max_{\partial\Omega} |u_1 - u_2| = \max_{\partial\Omega} |g_1 - g_2|.$$

Problem 6.3.5

Let Ω be an open bounded subset of $\mathbb{R}^2$. Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous and twice continuously differentiable on Ω and satisfy

$$(\partial_x^2 + \partial_y^2)u - \partial_x u + \partial_y u \geq 0, \qquad (x, y) \in \Omega.$$

Prove from scratch that $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

Solution: Case 1: Let

$$(\partial_x^2 + \partial_y^2)u - \partial_x u + \partial_y u > 0, \qquad (x, y) \in \Omega.$$

Since u is continuous on $\overline{\Omega}$, there exists some $(x, y) \in \overline{\Omega}$ such that

$$u(x, y) = \max_{\overline{\Omega}} u.$$

If $(x, y) \in \Omega$, then $\partial_x u = \partial_y u = 0$ and $\partial_x^2 u \leq 0$, $\partial_y^2 u \leq 0$, so

$$(\partial_x^2 + \partial_y^2)u - \partial_x u + \partial_y u = (\partial_x^2 + \partial_y^2)u \leq 0, \qquad (x, y) \in \Omega,$$

a contradiction. Hence, $(x, y) \in \partial\Omega$ and the assertion follows.
Case 2: Let

$$(\partial_x^2 + \partial_y^2)u - \partial_x u + \partial_y u \geq 0, \qquad (x, y) \in \Omega.$$

For $\epsilon > 0$, set

$$u_\epsilon(x, y) = u(x, y) + \epsilon c(-x + y) + \epsilon|(x, y)|^2, \quad (x, y) \in \overline{\Omega},$$

where $|(x, y)|$ denotes the Euclidean norm and $c > 0$ will be determined. Then,

$$\partial_x u_\epsilon = \partial_x u - \epsilon c + 2\epsilon x,$$

and

$$\partial_y u_\epsilon = \partial_x u + \epsilon c + 2\epsilon y.$$

Now the second derivatives,

$$\partial_x^2 u_\epsilon = \partial_x^2 u + 2\epsilon,$$

and

$$\partial_y^2 u_\epsilon = \partial_x^2 u + 2\epsilon.$$

Thus,

$$(\partial_x^2 + \partial_y^2 - \partial_x + \partial_y)u_\epsilon = (\partial_x^2 + \partial_y^2 - \partial_x + \partial_y)u + 4\epsilon + 2\epsilon c + 2\epsilon(y - x), \qquad (x, y) \in \Omega.$$

Using the fact that $(\partial_x^2 + \partial_y^2 - \partial_x + \partial_y)u \geq 0$ and $\epsilon > 0$,

$$(\partial_x^2 + \partial_y^2 - \partial_x + \partial_y)u_\epsilon > 2\epsilon c + 2\epsilon(y - x).$$

We can express the last term of the previous equation as

$$2\epsilon(y-x) = 2\epsilon \sum_{j=1}^{2} b_j x_j,$$

where $x_1 = x$, $b_1 = -1$, $x_2 = y$, $b_2 = 1$. Then,

$$\begin{aligned}
2\epsilon(y-x) &= 2\epsilon \sum_{j=1}^{2} b_j x_j \\
&> -2\epsilon \left| \sum_{j=1}^{2} b_j x_j \right| \\
&\geq -2\epsilon \left| b \right| \sqrt{x^2+y^2} \\
&> -2\epsilon\sqrt{x^2+y^2}.
\end{aligned}$$

Since Ω is bounded, we can find a c such that $c > \sqrt{x^2+y^2}$ and

$$\begin{aligned}
(\partial_x^2 + \partial_y^2 - \partial_x + \partial_y)u_\epsilon &> 2\epsilon c + 2\epsilon(y-x) \\
&> 2\epsilon c - 2\epsilon\sqrt{x^2+y^2} \\
&= 2\epsilon\left(c - \sqrt{x^2+y^2}\right) \\
&> 0.
\end{aligned}$$

By Case 1,

$$\max_{\overline{\Omega}} u_\epsilon = \max_{\partial\Omega} u_\epsilon.$$

Problem 7.3.3

Consider the following version of the Neumann boundary problem for the Laplace equation.

$$\begin{aligned} \Delta u(x) + c(x)u(x) &= f(x), \quad x \in \Omega, \\ \partial_\nu u(x) &= g(x), \quad x \in \partial\Omega. \end{aligned}$$

Find a sign condition for the function c that guarantees that there is at most one solution u even if Ω is not path-connected.

Solution: Let c be continuous and negative for all $x \in \Omega$. Let u, v be solutions to he previous PDE and boundary conditions. Define $w : \overline{\Omega} \to \mathbb{R}$ by $w(x) = u(x) - v(x)$. Then,

$$\begin{aligned} \Delta w(x) + c(x)w(x) &= 0, \quad x \in \Omega, \\ \partial \nu w(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Making use of THEOREM 7.3 for w we get

$$\int_\Omega w\Delta w dx + \int_\Omega \nabla w \cdot \nabla w dx = \int_{\partial\Omega} w\partial_\nu w d\sigma.$$

Using the PDE into the previous equation,

$$-\int_\Omega c(x)w^2(x)dx + \int_\Omega \nabla w \cdot \nabla w dx = 0.$$

Multiplying the equation by -1 and noting that the inner product is always non-negative,

$$0 = \int_\Omega c(x)w^2(x)dx - \int_\Omega \nabla w \cdot \nabla w dx \leq \int_\Omega c(x)w^2(x)dx \leq 0,$$

since $c(x) < 0$ for all $x \in \Omega$. Therefore,

$$\int_\Omega c(x)w^2(x)dx = 0,$$

which implies that $w(x) = 0$ for all $x \in \Omega$ since $c(x)w^2(x)$ is continuous on the domain. Finally, since $c(x) \neq 0$, $w(x) = 0$ for all $x \in \Omega$. Hence, we have that $u = v$ for all $x \in \Omega$, proving that there exists at most one solution for the PDE and boundary conditions given provided that $c(x) < 0$ for all $x \in \Omega$.

Problem 7.3.4

Consider the following version of the Neumann boundary problem for the Laplace equation.

$$\begin{aligned} \Delta u(x) + c(x)u(x) &= f(x), \quad x \in \Omega, \\ \partial_\nu u(x) &= g(x), \quad x \in \partial\Omega. \end{aligned}$$

Assume that Ω is path-connected, $c : \Omega \to \mathbb{R}$ is nonpositive and continuous and $c(x) < 0$ for some $x \in \Omega$. Show that there is at most one solution u.

Solution: Let c be continuous and non-positive for all $x \in \Omega$. Assume there exists an $x^* \in \Omega$ such that $c(x^*) < 0$. Let u, v be solutions to he previous PDE and boundary conditions. Define $w : \overline{\Omega} \to \mathbb{R}$ by $w(x) = u(x) - v(x)$. Then,

$$\begin{aligned} \Delta w(x) + c(x)w(x) &= 0, \quad x \in \Omega, \\ \partial\nu w(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Making use of THEOREM 7.3 for w we get

$$\int_\Omega w\Delta w dx + \int_\Omega \nabla w \cdot \nabla w dx = \int_{\partial\Omega} w\partial_\nu w d\sigma.$$

Using the PDE into the previous equation,

$$-\int_\Omega c(x)w^2(x)dx + \int_\Omega \nabla w \cdot \nabla w dx = 0,$$

and,

$$\int_\Omega \nabla w \cdot \nabla w dx = \int_\Omega c(x)w^2(x)dx.$$

Since the inner product is non-negative,

$$0 \leq \int_\Omega \nabla w \cdot \nabla w dx = \int_\Omega c(x)w^2(x)dx \leq 0,$$

where we have used that $c(x)w^2(x) \leq 0$ for all $x \in \Omega$. Hence,

$$\int_\Omega c(x)w^2(x)dx = 0, \tag{1}$$

and,

$$\int_\Omega \nabla w \cdot \nabla w dx = 0.$$

The latter implies that $\nabla w = 0$ on Ω. Since Ω is path-connected, by PROPOSITION 7.4, w is constant on Ω. Retaking equation (1), since $c(x)w^2(x) \leq 0$ and continuous on the domain, we have that $c(x)w^2(x) = 0$ for all $x \in \Omega$. Since there we had that $c(x^*) < 0$, $w(x^*) = 0$. Since w is constant on Ω, $w = 0$ for all $x \in \Omega$. Hence, we have that $u = v$ for all $x \in \Omega$, proving that there exists at most one solution for the PDE and boundary conditions given provided that $c(x) \leq 0$ for all $x \in \Omega$ and Ω is path-connected.