Partial Differential Equations
TA Homework 9

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Problem 5.2.1

Let $T > 0$ and $u : \overline{\Omega} \times [0, T]$ be continuous. Further let $c, F : \Omega \times (0, T) \to \mathbb{R}$, F non-negative and c bounded above and $f : \Omega \to \mathbb{R}$ be nonnegative. Let the partial differential operator L be as in the text before. Assume that u is once partially differentiable with respect to $t \in (0, T)$ and twice partially differentiable with respect to x_k at each $x \in \Omega$, $k = 1, \dots, n$. Assume that

$$\begin{aligned} (\partial_t - L - c)u = F(x,t), &\quad x \in \Omega, t \in (0,T), \\ u(x,t) = 0, &\quad x \in \partial\Omega, t \in (0,T), \\ u(x,0) = f(x), &\quad x \in \Omega. \end{aligned}$$

Show that $u(x,t) \geq 0$ for all $x \in \overline{\Omega}$, $t \in [0,T]$.

Solution: Define $v : \overline{\Omega} \times [0, T]$, $v(x,t) = -u(x,t)$. Then the PDE yields

$$\begin{aligned} -(\partial_t - L)v + cv = F(x,t), &\quad x \in \Omega, t \in (0,T), \\ v(x,t) = 0, &\quad x \in \partial\Omega, t \in (0,T), \\ v(x,0) = -f(x), &\quad x \in \Omega. \end{aligned}$$

Since $F(x,t)$ and $f(x)$ are nonnegative,

$$\begin{aligned} (\partial_t - L)v \leq cv, &\quad x \in \Omega, t \in (0,T), \\ v(x,t) = 0, &\quad x \in \partial\Omega, t \in (0,T), \\ v(x,0) \leq 0, &\quad x \in \Omega. \end{aligned}$$

Then, by *Theorem 5.11*, $v(x,t) \leq 0$ for all $x \in \overline{\Omega}$, $t \in [0,T]$. Thus, $u(x,t) \geq 0$ for all $x \in \overline{\Omega}$, $t \in [0,T]$.

Problem 5.2.3

Let $T > 0$ and $u : \overline{\Omega} \times [0, T]$ be continuous. Further let $c, F : \Omega \times (0, T) \to \mathbb{R}$, F bounded and c bounded above. Assume that u is once partially differentiable with respect to $t \in (0, T)$ and twice partially differentiable with respect to x_k at each $x \in \Omega$, $k = 1, \ldots, n$. Assume that

$$(\partial_t - L - c)u = F(x, t), \quad x \in \Omega, t \in (0, T).$$

Let $M, N \geq 0$ such that

$$|u(x, t)| \leq M \text{ whenever } x \in \partial\Omega, t \in [0, T] \text{ or } x \in \overline{\Omega}, t = 0$$

and

$$|F(x, t)| \leq N \text{ for all } x \in \Omega, t \in (0, T).$$

Show: $|u(x, t)| \leq (M + tN)e^{kt}$ for all $x \in \overline{\Omega}$, $t \in [0, T]$, where $k \geq 0$ is chosen such that $c(x, t) \leq k$ for all $x \in \Omega$, $t \in (0, T)$.

Solution: Since c is bounded, there exists some $k \geq 0$ such that $c(x, t) \leq k$ for all $x \in \Omega$, $t \in (0, T)$. Define $v(x, t) = u(x, t)e^{-kt}$. Then,

$$\begin{aligned} (\partial_t - L - c)\, v(x, t) &= e^{-kt} (\partial_t - L - c)\, u(x, t) - kv(x, t) \\ &= e^{-kt} F(x, t) - kv(x, t). \end{aligned}$$

Reorganizing the terms we get

$$(\partial_t - L - (c - k))\, v(x, t) = F(x, t)e^{-kt}.$$

Define $\beta = c - k \leq 0$. Then the previous equation yields

$$(\partial_t - L - \beta)\, v(x, t) = F(x, t)e^{-kt}.$$

We continue with

$$|v(x, 0)| = |u(x, 0)| \leq M, \qquad x \in \overline{\Omega},$$

$$|v(x, t)| = \left|u(x, t)e^{-kt}\right| = |u(x, t)|\, e^{-kt} \leq |u(x, t)| \leq M \leq M + tN, \qquad x \in \partial\Omega, t \in [0, T),$$

and

$$\left|F(x, t)e^{-kt}\right| = |F(x, t)|\, e^{-kt} \leq |F(x, t)| \leq N, \qquad x \in \Omega, t \in (0, T).$$

Then, by *Theorem 5.13*, $|v(x, t)| \leq M + tN$ for all $x \in \overline{\Omega}, t \in [0, T]$. Therefore,

$$\begin{aligned} |v(x, t)| &= \left|u(x, t)e^{-kt}\right| \\ &= |u(x, t)|\, e^{-kt} \\ &\leq M + tN. \end{aligned}$$

Thus,

$$|u(x, t)| \leq (M + tN)e^{kt}, \qquad x \in \overline{\Omega}, t \in [0, T].$$

Problem 5.2.6

a) Find the solution w of

$$\begin{aligned} -\partial_x^2 w(x) &= 2, \qquad 0 < x < 1, \\ w(0) &= 0 = w(1). \end{aligned}$$

Solution: We can rewrite the PDE as

$$\begin{aligned} \partial_x^2 w(x) &= -2, \qquad 0 < x < 1, \\ w(0) &= 0 = w(1), \end{aligned}$$

which clearly has a polynomial solution

$$w(x) = -x^2 + C_1 x + C_2.$$

Imposing the boundary conditions

$$\begin{aligned} w(0) &= C_2 = 0, \\ w(1) = -1 + C_1 + C_2 &= -1 + C_1 = 0, \end{aligned}$$

we get $C_1 = 1$ and $C_2 = 0$. Hence, the solution is

$$w(x) = x(1-x), \qquad x \in [0,1].$$

b) Suppose that u is the solution to

$$\begin{aligned} (\partial_t - \partial_x^2)u &= 2 && \text{on } (0,1) \times (0,\infty), \\ u(0,t) = 0 &= u(1,t) && t \geq 0 \\ u(x,0) &= 0 && 0 \leq x \leq 1. \end{aligned}$$

Show that

$$x(1-x)\left(1 - e^{-8t}\right) \leq u(x,t) \leq x(1-x), \quad x \in [0,1].$$

Solution: Set $v(x,t) = x(1-x)$. Then

$$(\partial_t - \partial_x^2)v(x,t) = 0 - (-2) = 2.$$

Define $w(x,t) = u(x,t) - v(x,t)$. Then

$$(\partial_t - \partial_x^2)w(x,t) = (\partial_t - \partial_x^2)u(x,t) - (\partial_t - \partial_x^2)v(x,t) = 2 - 2 = 0 \leq 0, \qquad x \in [0,1], t \in (0,\infty).$$

Further,

$$w(x,0) = u(x,0) - v(x,0) = 0 - x(1-x) = x(x-1) \leq 0, \qquad x \in [0,1],$$

and

$$w(0,t) = u(0,t) - v(0,t) = 0 = u(1,t) - v(1,t) = w(1,t), \qquad t \in [0,\infty).$$

Therefore, by *Theorem 5.11*, $w(x,t) \le 0$ for all $x \in [0,1]$ and $t \in [0,\infty)$. Thus,

$$u(x,t) \le v(x,t) = x(1-x), \qquad x \in [0,1].$$

To prove the other side of the inequality we proceed similarly. Now set $v(x,t) = x(1-x)\left(1 - e^{-8t}\right)$. Then

$$\begin{aligned}(\partial_t - \partial_x^2)v(x,t) &= 8x(1-x)e^{-8t} + 2\left(1 - e^{-8t}\right) \\ &= 8x(1-x)e^{-8t} + 2 - 2e^{-8t}.\end{aligned}$$

The function $f(x) = x(1-x)$ has a maximum at $x = \frac{1}{2}$ of value $\frac{1}{4}$. Therefore

$$\begin{aligned}(\partial_t - \partial_x^2)v(x,t) &= 8x(1-x)e^{-8t} + 2 - 2e^{-8t} \\ &\le 2e^{-8t} + 2 - 2e^{-8t} \\ &= 2.\end{aligned}$$

Define $w(x,t) = v(x,t) - u(x,t)$. Then

$$(\partial_t - \partial_x^2)w(x,t) = (\partial_t - \partial_x^2)v(x,t) - (\partial_t - \partial_x^2)u(x,t) \le 0, \qquad x \in [0,1], t \in (0,\infty).$$

Further,

$$w(x,0) = v(x,0) - u(x,0) = x(1-x)(1-1) - 0 = 0 \le 0, \qquad x \in [0,1],$$

and

$$w(0,t) = v(0,t) - u(0,t) = 0 = v(1,t) - u(1,t) = w(1,t), \qquad t \in [0,\infty).$$

Therefore, by *Theorem 5.11*, $w(x,t) \le 0$ for all $x \in [0,1]$ and $t \in [0,\infty)$. Thus,

$$v(x,t) \le u(x,t), x(1-x)\left(1 - e^{-8t}\right) \le u(x,t), \qquad x \in [0,1], t \in [0,\infty).$$

Hence, we have proved that

$$x(1-x)\left(1 - e^{-8t}\right) \le u(x,t) \le x(1-x), \qquad x \in [0,1], t \in [0,\infty).$$

Problem 5.2.11

Let $L, T > 0$ and $u : [0,L]x[0,T] \to \mathbb{R}$ be continuous, and suffieciently often differentiable and satisfy

$$\begin{aligned}&0 \le \partial_t u(x,t) - x^3(L-x)^5\partial_x^2 u(x,t) + a\partial_x u(x,t) + (L-x)u(x,t), \quad x \in (0,L), t \in (0,T), \\ &0 \le u(0,t), \quad u(L,t) \ge 0 \quad t \in [0,T], \\ &0 \le u(x,0) \quad x \in [0,L]\end{aligned}$$

Show: $u(x,t) \ge 0$ for all $x \in [0,L], t \in [0,T]$.

Solution: We will prove the statement by contradiction. Assume that there exists a $y \in [0, L]$ and an $r \in [0, T]$ such that $u(y, r) < 0$. Consider $u : [0, L] \times [0, r]$. Since u is continuous and $[0, L] \times [0, r]$ is compact, u has a minimum in $[0, L] \times [0, r]$. Denote such minimum as $u_m = u(x_m, t_m)$ with $x_m \in [0, L]$ and $t_m \in [0, r]$. Therefore, $u(x_m, t_m) \leq u(z, s)$ for any $z \in [0, L]$ and $s \in [0, r]$. Particularly,

$$u(x_m, t_m) \leq u(y, r) < 0.$$

Since u_m is a minimum,

$$\partial_x u(x, t)|_{(x_m, t_m)} = 0,$$

and

$$\partial_x^2 u(x, t)\big|_{(x_m, t_m)} \geq 0.$$

Lastly, note that

$$\partial_t u(x, t)|_{(x_m, t_m)} = \lim_{s \to t_m^-} \frac{u(x_m, s) - u(x_m, t_m)}{s - t_m} \leq 0.$$

Having the previous results, we can evaluate the PDE at the minimum point (x_m, t_m),

$$\begin{aligned} &\partial_t u(x_m, t_m) - x_m^3 (L - x_m)^5 \partial_x^2 u(x_m, t_m) + a \partial_x u(x_m, t_m) + (L - x_m) u(x_m, t_m) = \\ &= \partial_t u(x_m, t_m) - x_m^3 (L - x_m)^5 \partial_x^2 u(x_m, t_m) + (L - x_m) u(x_m, t_m) < 0, \end{aligned}$$

which contradicts the PDE of the problem and we found our contradiction. Thus, $u(x, t) \geq 0$ for all $x \in [0, L], t \in [0, T]$.