Computational Fluid Dynamics

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Homework 9

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1 Introduction

In this assignment we will study a one dimensional hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial \left(\frac{u^2}{2}\right)}{\partial x} = 0, \qquad x \in [3, 7],$$

with initial condition

$$u(x, t = 0) = \begin{cases} \frac{1}{4} + \frac{1}{2}\sin(\frac{\pi}{4}(x-3)) & x < 4.5 \\ \frac{1}{4} + \frac{1}{2}\sin(\frac{\pi}{4}(x-3)) + [1 + \cos(2\pi x)]\cos(8\pi x) & 4.5 \leq x \leq 5.5 \\ \frac{1}{4} + \frac{1}{2}\sin(\frac{\pi}{4}(x-3)) & x > 5.5 \end{cases}$$

and periodic boundary conditions. We will use a second order TVD scheme to solve the given hyperbolic PDE. The equations in index form of this method are detailed at the end of this document.

2 Results

In the first figure we can see the plot of the initial condition given above. In the figure 2 we can see the solution $u(x, t)$ at different values of time for $M = 256$ and $CFL = 0.1$. We can see that in fact the periodic boundary conditions are satisfied, although the initial condition was not periodic. In the figure 3 we can see the same profiles of $u(x, t)$ at the same values of time but for $M = 1024$. This value of M was the one that the GCI study determined to satisfy the accuracy requirement given in the problem (see below).

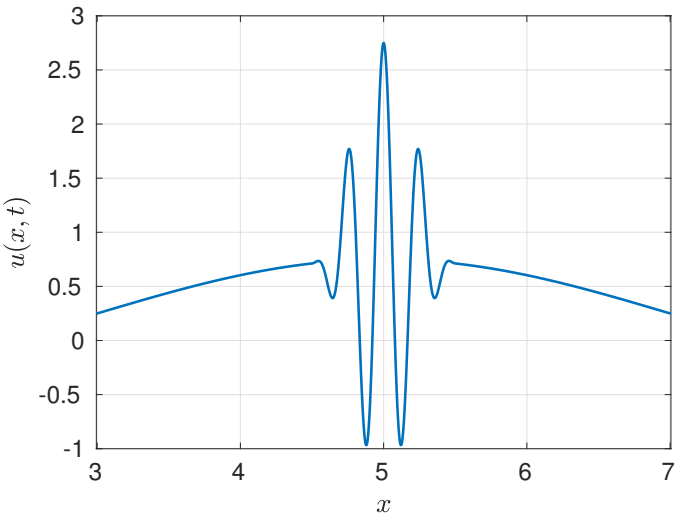
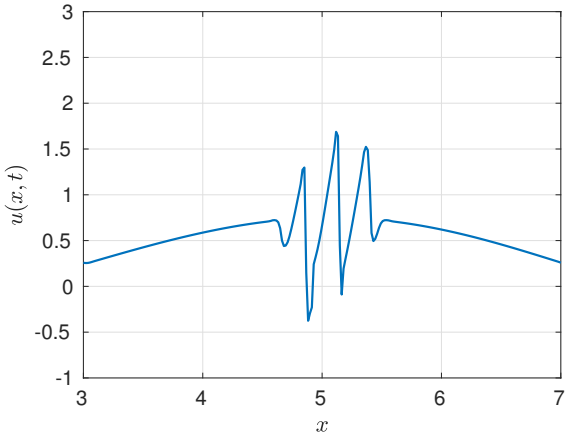
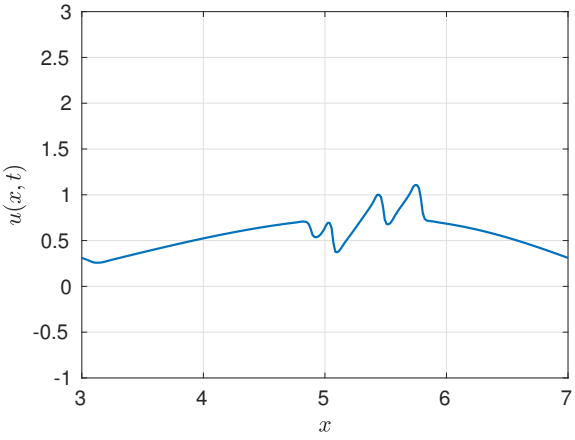


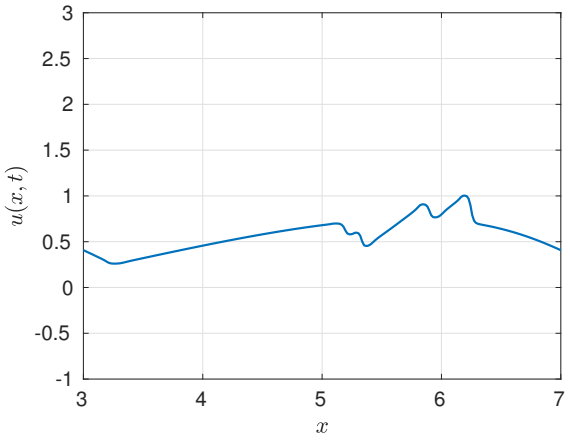
Figure 1: Initial condition for u.



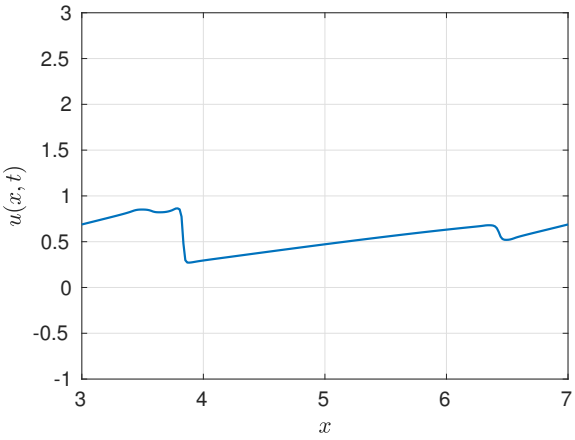
(a) $t = 0.1$ s.



(b) $t = 0.5$ s.

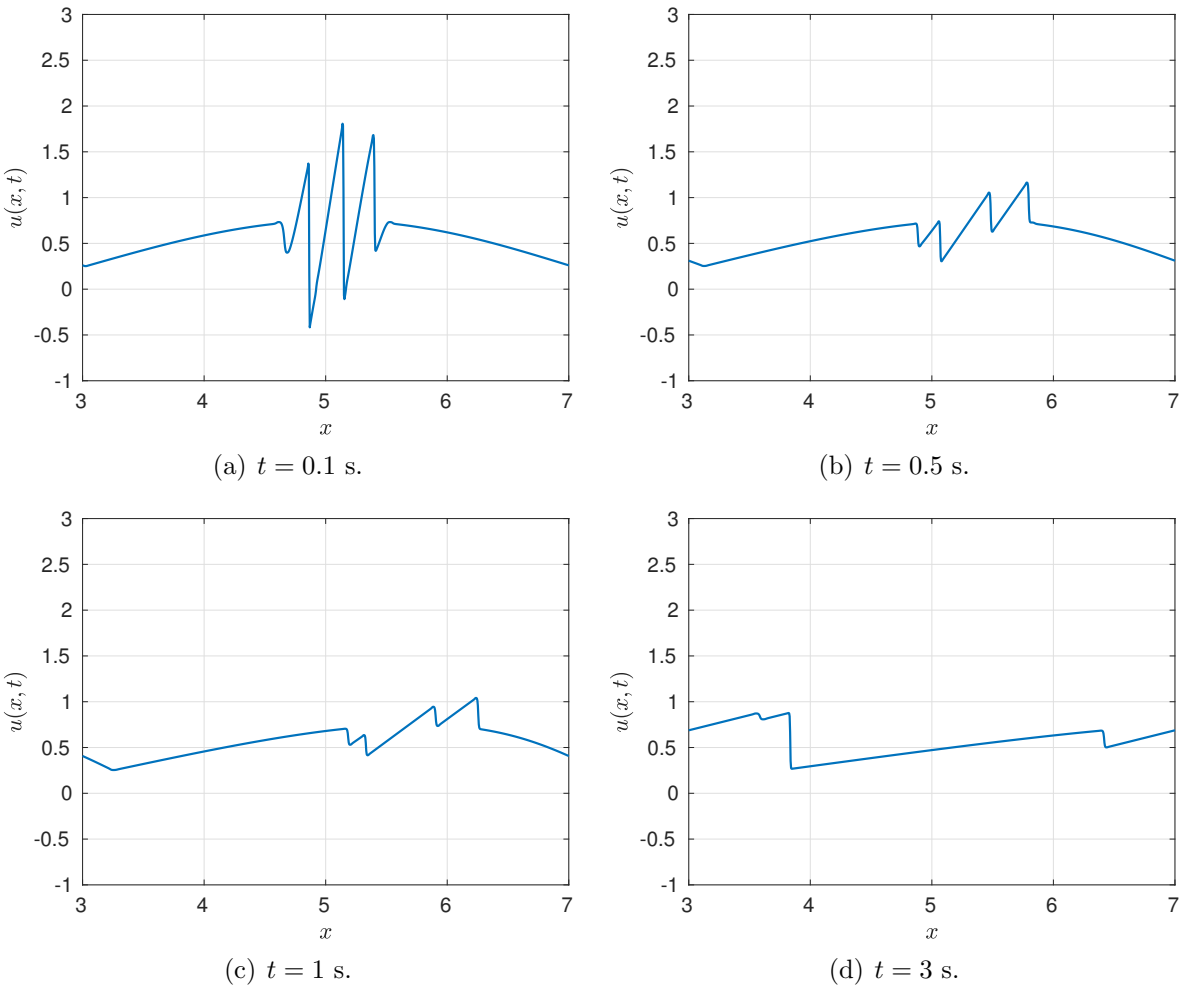


(c) $t = 1$ s.



(d) $t = 3$ s.

Figure 2: Profiles of u for M=256 and $CFL = 0.1$.



(a) $t = 0.1$ s. (b) $t = 0.5$ s.

(c) $t = 1$ s. (d) $t = 3$ s.

Figure 3: M=4096, CFL=0.8

The GCI analysis details are shown in the tables below. Note that

$$\beta = \frac{GCI_{12}}{GCI_{23}} r^p,$$

and $u_{h=0}$ is obtained by Richardson extrapolation. We can see that $\beta \in [0.95, 1.05]$ which implies that we are in the asymptotic range of convergence, and for the last mesh we have a GCI_{12} value less than 0.1%, the requested accuracy.

M	$u(6, 1)$
64	0.863483397473951
128	0.827088879661215
256	0.797959995827032
512	0.810290219692641
1024	0.810473786888916

Table 1: GCI analysis data.

M	$u_{h=0}$	p	GCI_{12} (%)	GCI_{23} (%)	β
64	-	-	-	-	-
128	-	-	-	-	-
256	0.681178462938105	0.321270729334096	18.293763706	0.22051804272	1.03650419066
512	0.819340609477331	1.240251427765663	1.396164850	0.03349261517	0.98478295360
1024	0.810476561060975	6.069746913021977	0.000427862	0.00028746084	0.99977350631

Table 2: GCI analysis results.