Advance Numerical Methods for PDEs

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Outline

1. Discretization

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Discretization

PDE

$$\partial_t u(x,t) + a\partial_x u(x,t) = 0$$

We can discretize using finite differences, forward in time and central in space:

$$u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

NOT STABLE FOR THIS PDE!

Instead, we substitute u_j^n by the average of its two neighboring grid points.

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - a\frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

Lax-Friedrichs Discretization

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - ac\left(u_{j+1}^n - u_{j-1}^n\right)$$

This change introduces an artificial viscosity.

Artificial Viscosity

$$2b = \frac{\Delta x^2}{\Delta t}$$

Courant Number

$$c = \frac{\Delta t}{\Delta x}$$

Changing to Matrix Form

Rearranging terms,

$$\begin{aligned} u_j^{n+1} &= \frac{1}{2}\left(1+ac\right)u_{j-1}^n + \frac{1}{2}\left(1-ac\right)u_{j+1}^n, \\ &= Au_{j-1}^n + Bu_j^n + Cu_{j+1}^n, \end{aligned}$$

where $A = \frac{1}{2}\left(1+ac\right)$, $B = 0$ and $C = \frac{1}{2}\left(1-ac\right)$.

Lax-Friedrichs Matrix Form

$$\vec{u}^{n+1} = M\vec{u}^n,$$

Tridiagonal Matrix

$$M = \begin{pmatrix} B & C & & & \\ A & B & C & & \\ & \ddots & \ddots & \ddots & \\ & & A & B & C \\ & & & A & B \end{pmatrix}$$

Boundary Conditions

We will solve the matrix system **only for the interior**.

Solve for the interior

$$\vec{u}^{n+1} = \tilde{M}\vec{u}^{n},$$

$\tilde{M}$ is M where we have removed the first and last rows.

The algorithm updates the solution for the interior, then updates the end points using the periodic boundary conditions.

Periodic Boundary Conditions

$$u_j = u_{j\pm N}$$

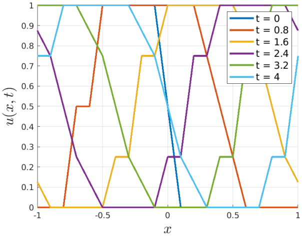
More specifically,

$$\begin{aligned} u_0^{n+1} &= Au_{-1}^n + Bu_0^n + Cu_1^n, \\ &= Au_{N-1}^n + Bu_0^n + Cu_1^n, \\ u_N^{n+1} &= Au_{N-1}^n + Bu_N^n + Cu_{N+1}^n \\ &= Au_{N-1}^n + Bu_N^n + Cu_1^n \end{aligned}$$

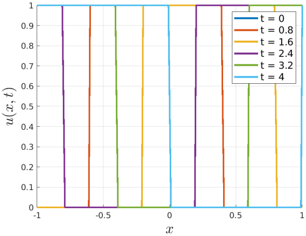
Applied Boundary Conditions

$$\begin{aligned} u_0^{n+1} &= Au_{-1}^n + Bu_0^n + Cu_1^n \\ u_N^{n+1} &= Au_{N-1}^n + Bu_N^n + Cu_{N+1}^n \end{aligned}$$

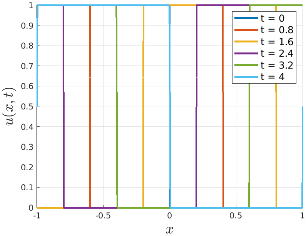
*NOTE: in MATLAB the subindeces are shifted +1 since the arrays start at 1, not at 0.



$\Delta x = 0.1$



$\Delta x = 0.01$



$\Delta x = 0.001$

Thank you