Partial Differential Equations
TA Homework 2

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Problem 3.1.11

1. Show that the Cauchy problem

$$(u-1)\partial_1 u + \partial_2 u = u, \quad u(0,x_2) = 1,$$

has at least two solutions for $x_1 \geq 0, x_2 \in \mathbb{R}$. Why does this result not contradict the local uniqueness result we proved in class?

Solution: From the PDE above we can write its *Characteristic System*

$$\begin{aligned} \partial_t \xi_1(z,t) &= v(z,t) - 1, \quad \xi_1(z,0) = 0, \\ \partial_t \xi_2(z,t) &= 1, \quad \xi_1(z,0) = z, \\ \partial_t v(z,t) &= v, \quad v(z,0) = 1. \end{aligned}$$

We solve first for v

$$v(z,t) = f_3(z)e^t,$$

and imposing the initial condition we find $f_3(z) = 1$, so

$$v(z,t) = e^t = v(t).$$

Now we solve for ξ_1

$$\xi_1(z,t) = e^t - t + f_1(z),$$

and imposing the initial condition we find $f_1(z) = -1$, so

$$\xi_1(z,t) = e^t - t - 1.$$

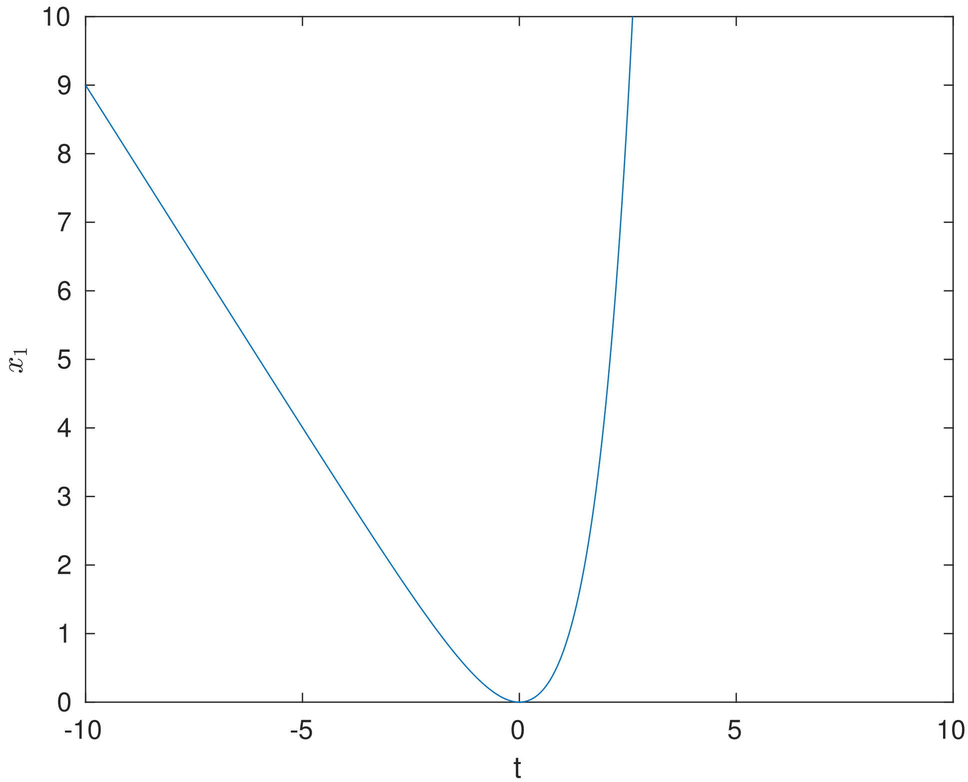
Now we solve for ξ_2

$$\xi_2(z,t) = t + f_2(z),$$

and imposing the initial condition we find $f_2(z) = z$, so

$$\xi_2(z,t) = z + t.$$

We can obtain the solution $u(x_1,x_2) = v(t(x_1,x_2))$ by finding t as a function of x_1 and x_2. However, as we see in the figure, for the same value of x_1 there exists two possible values of t and therefore two possible solutions. Thus, the solution u is not uniquely defined.



Given the initial condition $u(0, x_2) = 1$, we can find the parametrized line

$$g(z) = \begin{pmatrix} 0 \\ z \end{pmatrix} \ \Rightarrow \ g'(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and looking at the PDE we can observe that

$$a(x, u) = \begin{pmatrix} u - 1 \\ 1 \end{pmatrix} \ \Rightarrow \ a(g(z), u_0(g(z))) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can know compute the *Characteristic Determinant*

$$det(g'(z), a(g(z), u_0(g(z)))) = 0,$$

and by *Theorem 3.7* we cannot guarantee that there exists an open neighborhood U and an uniquely determined function $u : U \to \mathbb{R}$ that is solution to the Cauchy problem, as we already discussed above.

Problem 3.1.13

1. Find the solution u of two variables $x, y \in \mathbb{R}$ of

$$(y+x)u_x + (y-x)u_y = u^2,$$
$$u = 1 \text{ on the circle } x^2+y^2=1.$$

Where is the solution defined?

Solution: We can rewrite the PDE in terms of x_1, x_2 by making $x_1 = x$ and $x_2 = y$.

$$(x_2+x_1)\partial_1 u + (x_2-x_1)\partial_2 u = u^2,$$
$$u = 1 \text{ on the circle } x_1^2+x_2^2=1.$$

We can also parametrize the initial condition using sine and cosine functions,

$$u(\cos z, \sin z) = 1.$$

Therefore we get

$$g(z) = \begin{pmatrix} \cos z \\ \sin z \end{pmatrix}.$$

We can now rewrite the PDE as a system of ODE, the *Characteristic System*,

$$\begin{aligned} &\partial_t \xi_1(z,t) = \xi_2 + \xi_1, && \xi_1(z,0) = \cos z, \\ &\partial_t \xi_2(z,t) = \xi_2 - \xi_1, && \xi_1(z,0) = \sin z, \\ &\partial_t v(z,t) = v^2, && v(z,0) = 1. \end{aligned}$$

We solve first for ξ_1, ξ_2, which form a linear system of differential equations, we can solve it using the matrix exponential. We can rewrite the system in matrix form,

$$\xi' = A\xi,$$

$$\begin{pmatrix} \partial_t \xi_1 \\ \partial_t \xi_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

We can easily calculate the eigenvalues of the matrix A, $\lambda = 1 \pm i$, and its eigenvectors

$$V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Thus, the solution of the system is

$$\xi(z,t) = f_1(z)e^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + f_2(z)e^{(1-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$

which we can express in terms of sines and cosines,

$$\begin{aligned} \xi(z,t) &= f_1(z)e^t\left[\cos t + i\sin t\right]\begin{pmatrix} 1 \\ i \end{pmatrix} + f_2(z)e^t\left[\cos t - i\sin t\right]\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \left[f_1(z)+f_2(z)\right]e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\left[f_1(z)-f_2(z)\right]e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \\ &= K_1(z)e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + iK_2(z)e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

Imposing the initial conditions

$$\begin{aligned}\xi(z,0) &= K_1(z)\begin{pmatrix}1\\0\end{pmatrix} + iK_2(z)\begin{pmatrix}0\\1\end{pmatrix}\\ &= \begin{pmatrix}K_1(z)\\ iK_2(z)\end{pmatrix} = \begin{pmatrix}\cos z\\ \sin z\end{pmatrix},\end{aligned}$$

we can find K_1 and K_2

$$K_1(z) = \cos z, \qquad K_2(z) = -i\sin z.$$

Now we can write the solution for ξ

$$\xi(z,t) = \cos(z)e^t\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + \sin(z)e^t\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.$$

Now we solve for v

$$v(z,t) = \frac{-1}{t+f_3(z)},$$

with initial condition

$$v(z,0) = \frac{-1}{f_3(z)} = 1,$$

giving us $f_3(z) = -1$. Therefore

$$v(z,t) = \frac{1}{1-t} = v(t),$$

and we only have to find $t(x_1,x_2)$ to obtain the solution $u(x_1,x_2) = v(t(x_1,x_2))$. In order to find $t(x_1,x_2)$ we need to solve

$$\begin{aligned}x_1 &= e^t\left(\cos z\cos t + \sin z\sin t\right),\\ x_2 &= e^t\left(\sin z\cos t - \cos z\sin t\right).\end{aligned}$$

Let us compute $x_1^2+x_2^2$,

$$\begin{aligned}x_1^2+x_2^2 = e^{2t}(&\cos^2 z\cos^2 t + \sin^2 z\sin^2 t + 2\sin z\cos z\sin t\cos t\\ &+ \sin^2 z\cos^2 t + \cos^2 z\sin^2 t - 2\sin z\cos z\sin t\cos t),\end{aligned}$$

$$\begin{aligned}x_1^2+x_2^2 &= e^{2t}\left[\left(\sin^2 z+\cos^2 z\right)\cos^2 t + \left(\sin^2 z+\cos^2 z\right)\sin^2 t\right]\\ &= e^{2t}.\end{aligned}$$

Now we can isolate $t(x_1,x_2)$ from the previous equation

$$t(x_1,x_2) = \frac{1}{2}\ln\left(x_1^2+x_2^2\right),$$

which gives us the solution to the PDE

$$u(x_1,x_2) = \frac{1}{1-\frac{1}{2}\ln\left(x_1^2+x_2^2\right)}.$$

It is clear that the solution is not defined at the origin since $\ln(0)$ is not defined. In addition we know that the denominator cannot be zero, therefore the solution is not defined in points that satisfy $x_1^2+x_2^2 = e^2$ which describes the circle of radius $e > 1$. Since the domain of existence has to be connected, the solution exists on the annulus $0 < x_1^2+x_2^2 < e^2$.

Problem 3.1.15

1. Determine all solutions $u = u(x_1, x_2)$ of

$$(1-u)\frac{\partial u}{\partial x_1} + (1+u)\frac{\partial u}{\partial x_2} = 1, \quad x_1, x_2 \in \mathbb{R},$$
$$u(x_1, x_2) = 0, \quad x_1 = x_2.$$

Where are the solutions defined? Interpret your results in the light of the genereal local existence theorem.

Solution: We can rewrite the PDE as

$$(1-u)\partial_1 u + (1+u)\partial_2 u = 1,$$

with a parametrized initial condition

$$u(z, z) = 0.$$

Therefore we can obtain

$$g(z) = \begin{pmatrix} z \\ z \end{pmatrix},$$

and

$$a(x, u) = g(z) = \begin{pmatrix} 1-u \\ 1+u \end{pmatrix} \;\Rightarrow\; a(g(z), u_0(g(z))) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We now rewrite the PDE as a system of ODE, the *Characteristic System*,

$$\begin{aligned} \partial_t \xi_1 &= 1 - v, && \xi_1(z, 0) = z \\ \partial_t \xi_2 &= 1 + v, && \xi_2(z, 0) = z \\ \partial_t v &= 1, && v(z, 0) = 0. \end{aligned}$$

Given the previous system, we have to start by solving for v

$$v(z, t) = t + f_3(z).$$

By imposing the initial condition we find that $f_3(z) = 0$ and

$$v(z, t) = v(t) = t.$$

Now we solve for ξ_1

$$\partial_t \xi_1 = 1 - t \;\Rightarrow\; \xi_1(z, t) = t - \frac{1}{2}t^2 + f_1(z),$$

and imposing the initial condition we find that $f_1(z) = z$ and

$$\xi_1(z, t) = z + t - \frac{1}{2}t^2.$$

Now we solve for ξ_2

$$\partial_t \xi_2 = 1 + t \;\Rightarrow\; \xi_2(z, t) = t + \frac{1}{2}t^2 + f_2(z),$$

and imposing the initial condition we find that $f_2(z) = z$ and

$$\xi_2(z,t) = z + t + \frac{1}{2}t^2.$$

To obtain the solution $u(x_1, x_2) = v(t(x_1, x_2))$ we need to find $t(x_1, x_2)$ by solving

$$x_1(z,t) = z + t - \frac{1}{2}t^2, x_2(z,t) = z + t + \frac{1}{2}t^2.$$

Let us compute $x_2 - x_1$

$$x_2 - x_1 = t^2 \ \Rightarrow \ t = \pm\sqrt{x_2 - x_1},$$

and finally

$$u(x_1, x_2) = \pm\sqrt{x_2 - x_1}$$

and we see that by applying the initial condition we cannot get rid of any of the two signs,

$$u(x_1, x_1) = \pm\sqrt{x_1 - x_1} = 0.$$

Therefore we see that there is not an unique solution. In order to relate this to the general local existence theorem, we compute first

$$g(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and now the determinant

$$det(g'(z), a(g(z), u_0(g(z)))) = det\left(\begin{pmatrix} 11 \\ 11 \end{pmatrix}\right) = 0.$$

Thus, according to the *Theorem 3.7*, we cannot guarantee that there exists an open neighborhood U and an uniquely determined function $u : U \to \mathbb{R}$ that is solution to the Cauchy problem, which is in agreement with the results obtained.

Problem 3.1.16

1. Solve the age-structured population problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu(a,t)u = 0, \quad a, t > 0, a \neq t$$
$$u(a,0) = u_0(a), \quad a > 0,$$
$$u(0,t) = b(t), \quad t > 0.$$

Under which conditions is u continuous at $a = t$?

Solution: First of all, note that we can rewrite the above PDE as

$$\partial_1 u + \partial_2 u = -\mu(x_1, x_2)u$$

Let $v(z,y) = u(y, z+y)$ and $w(z,y) = u(z+y, y)$. Now compute

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}u(y, z+y) = \partial_1 u + \partial_2 u = -\mu(y, z+y)u(y, z+y) = -\mu(y, z+y)v(z,y),$$

and observe that we obtained an ODE for $v(z,t)$ that we are able to solve,

$$v(z,y) = f_1(z)e^{-\int_0^y \mu(s,z+s)ds}.$$

If we repeat the same procedure for $w(z,y)$ we obtain

$$w(z,y) = f_2(z)e^{-\int_0^y \mu(z+s,s)ds}.$$

Imposing the initial condition for $v(z,y)$ we obtain $f_1(z)$

$$v(z,0) = u(0,z),$$
$$f_1(z) = b(z),$$

and for $w(z,y)$ we obtain $f_2(z)$

$$w(z,0) = u(z,0),$$
$$f_2(z) = u_0(z).$$

Thus, the solutions for $v(z,y)$ and $w(z,y)$ are

$$v(z,y) = b(z)e^{-\int_0^y \mu(s,z+s)ds},$$
$$w(z,y) = u_0(z)e^{-\int_0^y \mu(z+s,s)ds}.$$

Since the argument z of both previous functions must be positive we obtain two situations. First, if $t > a > 0$

$$u(a,t) = u(a, t-a+a) = v(t-a, a) = b(t-a)e^{-\int_0^a \mu(s,t-a+s)ds},$$

and if $a > t > 0$

$$u(a,t) = u(a-t+t, t) = w(a-t, t) = u_0(a-t)e^{-\int_0^t \mu(a-t+s,s)ds}.$$

To find the condition under which u would be continuous we simply make $t = a$

$$u(a,a) = v(0,a) = b(0)e^{-\int_0^a \mu(s,s)ds},$$
$$u(a,a) = w(0,t) = u_0(0)e^{-\int_0^t \mu(s,s)ds}.$$

Hence, u will be continuous if and only if $b(0) = u_0(0)$.