Real Analysis Homework 14

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1 Problem 7.3.1

1. Let μ be a measure on a ring $\mathcal{B}$. Show that μ is finitely subadditive and σ-subadditive. (See Lemma 7.11).

Solution:

Proof. Let μ be a measure on a ring $\mathcal{B}$. By *Definition 7.10*, μ is additive. Then, by *Lemma 7.8*, μ is subbadditive, i.e. $\mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2)$. We prove that μ is finitely subadditive,

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu\left(A_j\right) \ , \tag{1}$$

by induction. It is trivial for $n = 1$ is given by the definition of subadditivity for $n = 2$. Assume that (1) is true for n. Now

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{n+1} A_j\right) &= \mu\left(\bigcup_{j=1}^{n} A_j \cup A_{n+1}\right) \\
&\leq \mu\left(\bigcup_{j=1}^{n} A_j\right) + \mu\left(A_{n+1}\right), \quad \text{by subadditivity for two sets,} \\
&\leq \sum_{j=1}^{n} \mu\left(A_j\right) + \mu\left(A_{n+1}\right), \quad \text{by induction hypothesis (1),} \\
&= \sum_{j=1}^{n+1} \mu\left(A_j\right).
\end{aligned}$$

Thus, μ is finitely subadditive.

Now we prove that μ is σ-subadditive. By *Definition 7.10*, μ is continuous from below and we just proved that it is finitely subadditive. Let (A_n) be a sequence of sets in $\mathcal{B}$ such that $\cup_{n\in\mathbb{N}} A_n \in \mathcal{B}$. Set $B_n = \bigcup_{j=1}^{n} A_j$. Then (B_n) is an increasing sequence in $\mathcal{B}$ with $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Since μ is continuous from below,

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n).$$

Since μ is finitely subadditive,

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right)$$
$$\leq \lim_{n\to\infty} \sum_{j=1}^{n} \mu(A_j)$$
$$= \sum_{j=1}^{\infty} \mu(A_j).$$

Thus,

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j),$$

and μ is σ-subadditive. ■

2 Problem 7.3.7

1. Let μ_1, $\mu_2 : \mathcal{B} \to [0, \infty]$ be two measures on the σ-ring $\mathcal{B}$ on Ω. Define $\mu(B) = \mu_1(B) + \mu_2(B)$, $B \in \mathcal{B}$. Show that μ is a measure on $\mathcal{B}$.

Solution:

Proof. a) To prove non-negativity we take into account that since μ_1 and μ_2 are measures themselves, by *Definition 7.10* they are non-negative. Then, since μ is the sum of two non-negative quantities, it must be itsel non-negative.

b) To prove that μ is additive let A,B be disjoint sets in $\mathcal{B}$. Since μ_1 and μ_2 are both measures, by *Definition 7.10*, they are both additive,

$$\begin{aligned}\mu\left(A \uplus B\right) &= \mu_1\left(A \uplus B\right) + \mu_2\left(A \uplus B\right)\\ &= \mu_1(A) + \mu_1(B) + \mu_2(A) + \mu_2(B)\\ &= \mu_1(A) + \mu_2(A) + \mu_1(B) + \mu_2(B)\\ &= \mu(A) + \mu(B).\end{aligned}$$

Thus μ is additive.

c) To prove that μ is continuous from below let (A_n) be an increasing sequence in $\mathcal{B}$ with $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}$. By *Definition 7.10*, μ_1 and μ_2 are continuous from below,

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) + \mu_2\left(\bigcup_{n=1}^{\infty} A_n\right)\\ &= \lim_{n\to\infty} \mu_1(A_n) + \lim_{n\to\infty} \mu_2(A_n)\\ &= \lim_{n\to\infty} \left(\mu_1(A_n) + \mu_2(A_n)\right)\\ &= \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

Thus, μ is continuous from below.

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3 Problem 7.4.2

1. Prove *Lemma 7.23.* The product, maximum, and minimum of two simple functions are simple. The maximum of two functions f and g is defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$. The minimum of two functions f and g is defined by $(f \wedge g)(x) = \min\{f(x), g(x)\}$.

Solution:

Proof. Let $f, g : \Omega \to \mathbb{R}_+$ be simple (not necessarily in canonical representation),

$$f = \sum_{j=1}^{m} \alpha_j \chi_{A_j}, \quad g = \sum_{k=1}^{n} \beta_k \chi_{B_k}.$$

We set

$$A = \bigcup_{j=1}^{m} A_j, \quad B = \bigcup_{k=1}^{n} B_k,$$

$$A_{m+1} = (A \cup B) \setminus A, \quad B_{n+1} = (A \cup B) \setminus B,$$

$$\alpha_{m+1} = 0 = \beta_{n+1}.$$

Since $\mathcal{B}$ is a ring, A, B, $A \cup B$, and A_{m+1} and B_{n+1} are elements in $\mathcal{B}$. Define

$$C_{jk} = A_j \cap B_k, \quad j = 1, ..., m+1, \quad k = 1, ..., n+1.$$

Let $C_{jk} \neq \emptyset$ and note that for each j, k, $C_{jk} \in \mathcal{B}$.

a) The product of two simple functions is a simple function.

The functions f and g can be expressed as

$$f = \sum_{j=1}^{m} \alpha_j \chi_{A_j} = \sum_{j=1}^{m+1} \alpha_j \sum_{k=1}^{n+1} \chi_{C_{jk}} = \sum_{j=1}^{m+1} \sum_{k=1}^{n+1} \alpha_j \chi_{C_{jk}},$$

and

$$g = \sum_{k=1}^{n} \beta_k \chi_{B_k} = \sum_{k=1}^{n+1} \beta_k \sum_{j=1}^{m+1} \chi_{C_{jk}} = \sum_{j=1}^{m+1} \sum_{k=1}^{n+1} \beta_k \chi_{C_{jk}}.$$

Therefore, the product fg can be written as

$$fg = \sum_{j=1}^{m+1} \sum_{k=1}^{n+1} \alpha_j \beta_k \chi_{C_{jk}}.$$

Thus, fg can be written as a linear combination of indicator functions and therefore it is a simple function.

b) The maximum of two simple functions is a simple function.

The maximum of f and g can be expressed as follows:

$$\begin{aligned}
(f \vee g)(x) &= \max\{f(x), g(x)\} \\
&= \max\left\{\sum_{j=1}^{m} \alpha_j \chi_{A_j}(x), \sum_{k=1}^{n} \beta_k \chi_{B_k}(x)\right\} \\
&= \max\left\{\sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \alpha_j \chi_{C_{jk}}(x), \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \beta_k \chi_{C_{jk}}(x)\right\} \\
&= \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \max\{\alpha_j, \beta_k\}\, \chi_{C_{jk}}(x) \\
&= \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \gamma_{jk} \chi_{C_{jk}}(x).
\end{aligned}$$

Where we have made $\gamma_{jk} = \max\{\alpha_j, \beta_k\}$. Therefore,

$$(f \vee g) = \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \gamma_{jk} \chi_{C_{jk}},$$

and as it happened before, $(f \vee g)$ can be written as a linear combination of indicator functions and therefore it is a simple function.

c) The minimum of two simple functions is a simple function.

The minimum of f and g can be expressed as follows:

$$\begin{aligned}
(f \wedge g)(x) &= \min\{f(x), g(x)\} \\
&= \min\left\{\sum_{j=1}^{m} \alpha_j \chi_{A_j}(x), \sum_{k=1}^{n} \beta_k \chi_{B_k}(x)\right\} \\
&= \min\left\{\sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \alpha_j \chi_{C_{jk}}(x), \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \beta_k \chi_{C_{jk}}(x)\right\} \\
&= \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \min\{\alpha_j, \beta_k\}\, \chi_{C_{jk}}(x) \\
&= \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \delta_{jk} \chi_{C_{jk}}(x).
\end{aligned}$$

Where we have made $\delta_{jk} = \min\{\alpha_j, \beta_k\}$. Therefore,

$$(f \wedge g) = \sum_{j=1}^{m+1}\sum_{k=1}^{n+1} \delta_{jk} \chi_{C_{jk}},$$

and as it happened before, $(f \wedge g)$ can be written as a linear combination of indicator functions and therefore it is a simple function.

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