Real Analysis Homework 8

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1 Problem 4.2.1

1. Consider $\mathbb{R}$ with the absolute value. Let $a, b \in \mathbb{R}$ and $a < b$. Show that $I = (a, b)$ is open.

Solution:

Proof. Let $y \in I$, then $a < y < b$ and $d(y, x) < (b - a)\ \forall\ x \in I$. Set $\varepsilon = (b - a) - d(y, x) > 0$.
Now, let $z \in U_\varepsilon(y)$ such that $d(z, y) < \varepsilon$. Then, $\forall x \in I$, by the triangle inequality,

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \varepsilon + d(y,x) \\ &= (b-a) - d(y,x) + d(y,x) \\ &= (b-a)\ . \end{aligned}$$

Thus $z \in I$. Since z was chosen arbitrarily, $U_\varepsilon(y) \subseteq I$ and y is an interior point of I. Thus, every point in I is an interior point of I which implies, I is open. ■

2 Problem 4.2.3

1. Consider $\mathbb{R}$ with the metric induced by the absolute value. Let $S \subseteq \mathbb{R}$. S is called *order-dense* if for any $a, b \in \mathbb{R}$ with $a < b$ there exists some $s \in S$ such that $a < s < b$. Show that S is order-dense if and only if S is dense (i.e. $\mathbb{R} \subseteq \bar{S}$).

Solution:

Proof. ($\Rightarrow$) Let $x \in \mathbb{R}$. Since S is *order-dense*, for each $n \in \mathbb{N}$ there exists some $s_n \in S$ and $a, b \in \mathbb{R}$ with $a = x - \frac{1}{n}$ and $b = x + \frac{1}{n}$ such that:

$$a < s_n < b\ ,$$

$$x - \frac{1}{n} < s_n < x + \frac{1}{n}\ .$$

The last equation means that $|s_n - x| < \frac{1}{n}$, which implies that $s_n \to x$ as $n \to \infty$. Therefore $x \in \bar{S}$. Thus, $\mathbb{R} \subseteq \bar{S}$ and S is dense.

($\Leftarrow$) Assume S is dense and let $a, b \in \mathbb{R}$ with $a < b$. Consider $I = (a, b)$. Let $x \in I$. In the previous problem we showed that, for some ϵ, we can define an open ball $U_\epsilon(x) \subseteq I$. Since S is dense in $\mathbb{R}$, x is a limit point of S and, by Lemma 4.11, $U_\epsilon(x) \cap S \neq \emptyset$. Therefore, there exists some $s \in S$ that is also in $U_\epsilon(x)$ and then $a < s < b$. Thus, S is *order-dense* in $\mathbb{R}$. ■

3 Problem 4.2.14

1. Let X be a metric space with metric d and S and T be subsets of X.
   Show: $int(S \cap T) = \breve{S} \cup \breve{T}$

Solution:

Proof. Observe that,

$$X \setminus (\breve{S} \cap \breve{T}) = X \setminus \breve{S} \ \cup \ X \setminus \breve{T}$$

By DeMorgan's law.
Then, by proposition 4.17,

$$X \setminus \breve{S} \ \cup \ X \setminus \breve{T} = \overline{X \setminus S} \ \cup \ \overline{X \setminus T} \ = \overline{X \setminus S \ \cup \ X \setminus T} \ .$$

Then, by DeMorgan's Law and proposition 4.17,

$$\overline{X \setminus S \ \cup \ X \setminus T} = \overline{X \setminus S \cap T} = X \setminus Int(S \cap T) \ .$$

Thus,

$$X \setminus (\breve{S} \cap \breve{T}) = X \setminus Int(S \cap T)$$
$$\Rightarrow \breve{S} \cap \breve{T} = Int(S \cap T) \ .$$

∎

2. $\breve{S} \cup \breve{T} \subseteq Int(S \cup T)$ but equality fails in general.

Solution:

Proof. Let $x \in \breve{S} \cap \breve{T}$. Then,
Case 1: Let $x \in \breve{S}$, $\exists \ \varepsilon_1 > 0$ such that,

$$U_{\varepsilon_1}(x) \subseteq S \subseteq S \cup T \ .$$

Thus $x \in Int(S \cup T)$.

Case 2: Let $x \in \breve{T}$, $\exists \ \varepsilon_2 > 0$ such that,

$$U_{\varepsilon_2}(x) \subseteq T \subseteq S \cup T \ .$$

Thus, $x \in Int(S \cup T)$.

Case 3: Let $x \in \breve{S} \cup \breve{T}$, $\exists \varepsilon_3 > 0$ such that,

$$U_{\varepsilon_3}(x) \subseteq S \cap T \subseteq S \cup T \ .$$

Thus, $x \in Int(S \cup T)$.

Then $x \in \breve{S} \cap \breve{T} \Rightarrow x \in Int(S \cup T)$. Thus,

$$\breve{S} \cup \breve{T} \subseteq Int(S \cup T)$$

Now, lets show that equality fails in general. Let $(\mathbb{R}, d)$ be the real number line under Euclidean topology. Let $S = [a, \cdots, b]$ and $T = [b, \cdots, c]$ $\forall$ $a, b, c \in \mathbb{R}$. Then,

$$\begin{aligned} Int(S \cup T) &= Int([a, \cdots, b] \cup [b, \cdots, c]) \\ &= Int([a, \cdots, c]) \\ &= (a, \cdots, c) \ , \end{aligned}$$

and

$$\begin{aligned} \breve{S} \cup \breve{T} &= Int([a, \cdots, b]) \cup Int([b, \cdots, c]) \\ &= (a, \cdots, b) \cup (b, \cdots, c) \\ &\neq (a, \cdots, c) \ . \end{aligned}$$

Thus, the equality fails.

■

4 Problem 4.3.1

1. Consider the sequence space l^∞ with the supremum norm and the subset $D = \{x = (x_n) \in \mathbb{R}^{\mathbb{N}}; |x_n| \leq 1/n \ \ \forall n \in \mathbb{N}\}$. Show that D is totally bounded.

Solution:

Proof. D is totally bounded if and only if every sequence in D has a subsequence that is a Cauchy sequence. Let $(x^m)_{m=1}^\infty$ be a sequence in D. Since we are in the space of sequences, a sequence in this space is in fact a sequence of sequences $x^m = (x_1^m, x_2^m, ..., x_n^m, x_{n+1}^m, ...)$. Since $(x^m)_{m=1}^\infty \in D$, $|x_n^m| < \frac{1}{n}$ for all $n \in \mathbb{N}$. For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$, $N > \frac{2}{\varepsilon}$ such that:

$$|x_n^m| < \frac{1}{n} < \frac{1}{N} < \frac{\varepsilon}{2} \ .$$

Now let (x^{m_j}) be subsequence of (x^m). Since every bounded set in $\mathbb{R}^N$ is totally bounded, for each $k = \{1, ..., N\}$, $\exists N_k > 0$ such that

$$|x_k^{m_i} - x_k^{m_j}| < \varepsilon, \quad \text{for } i, j > N_k \ .$$

Let $i, j > N_k$ and $k > N$:

$$|x_k^{m_i} - x_k^{m_j}| < |x_k^{m_i}| + |x_k^{m_j}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

Choose now $N_2 = max_{k=1}^N N_k$. Then:

$$||x_k^{m_i} - x_k^{m_j}|| < \varepsilon \ \ \forall i, j > N_2 \text{ and } \forall k > N \ .$$

Therefore,

$$||x^{m_i} - x^{m_j}|| < \varepsilon \quad \forall i, j > N_2 \ .$$

Thus, the subsequence is Cauchy and D is totally bounded. $\blacksquare$