Advanced Numerical Methods for PDEs

Homework 1

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Consider the initial - boundary value problem for the scalar advection diffusion equation

$$\partial_t u(x,t) + a\partial_x u(x,t) - b\partial_x^2 u(x,t) = 0, \qquad u(x,t=0) = u^I(x), \tag{1}$$

on the interval $x \in [-1,1]$ with periodic boundary conditions $u(x+2,t) = u(x,t), \forall x,t$. Consider the explicit difference method

$$U(x,t+\Delta t) = U(x,t) - \frac{a\Delta t}{2\Delta x}(T - T^{-1})U(x,t) + \frac{b}{\Delta x^2}(T - 2 + T^{-1})U(x,t) \tag{2}$$

for the problem (1).

Problem 1

1. Derive an analytic expression for the solution $u(x,t)$ of problem (1) for a general initial function $u^I(x)$ and general stepsizes Δx, Δt, using Fourier transforms.

Solution: We will use the Fourier transform

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} u(x,t)e^{-i\omega x}dx, \tag{3}$$

to turn our $1-D$ PDE into an ODE. Taking the Fourier transform of the PDE (1) we obtain

$$\partial_t \hat{u}(w,t) + iaw\hat{u}(w,t) + bw^2\hat{u}(w,t) = 0,$$

which can be manipulated into

$$\partial_t \hat{u}(w,t) = -(bw^2 + iaw)\hat{u}(w,t).$$

The previous equation has a simple analytical solution,

$$\hat{u}(w,t) = e^{-(bw^2+iaw)t}\hat{u}(w,0),$$

which we can conveniently rewrite as

$$\hat{u}(w,t) = e^{-b\omega^2 t}\hat{u}^I(w)e^{-iawt} \tag{4}$$

To obtain the solution, we will use the following property of Fourier transforms,

$$\mathcal{F}\left[f * g\right] = \mathcal{F}\left[f\right]\mathcal{F}\left[g\right], \tag{5}$$

where $*$ represents convolution. In our case, we have

$$\begin{aligned} \mathcal{F}\left[f\right](w,t) &= e^{-b\omega^2 t}, \\ \mathcal{F}\left[g\right](w,t) &= \hat{u}^I(w)e^{-iawt}. \end{aligned}$$

By using the inverse Fourier transform on the previous equations we obtain

$$\begin{aligned} f(x,t) &= \mathcal{F}^{-1}\left[\mathcal{F}\left[f\right](w,t)\right](x,t) = \frac{e^{-x^2/4bt}}{\sqrt{2bt}}, \\ g(x,t) &= \mathcal{F}^{-1}\left[\mathcal{F}\left[g\right](w,t)\right](x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}^I(w)e^{-iw(x-at)}dw \\ &= u^I(x-at). \end{aligned} \tag{6}$$

Hence, we have proved so far that

$$\hat{u}(w,t) = e^{-b\omega^2 t}\hat{u}^I(w)e^{-iawt} = \mathcal{F}\left[f\right](w,t)\mathcal{F}\left[g\right](w,t) = \mathcal{F}\left[f * g\right](w,t), \tag{7}$$

Then, we can finally obtain the solution to the problem,

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\hat{u}(w,t)\right](x,t) \\ &= \mathcal{F}^{-1}\left[\mathcal{F}\left[f * g\right](w,t)\right](x,t) \\ &= \left[f * g\right](x,t). \end{aligned}$$

To conclude,

$$u(x,t) = \left[f * u^I(x-at)\right](x,t), \tag{8}$$

with f given by (6).

2. Derive an analytic expression for the solution $U(x,t)$ of problem (2) for a general initial function $u^I(x)$ and general stepsizes Δx, Δt, using discrete Fourier transforms.

Solution: CACA

Problem 2

Problem 3