Partial Differential Equations
Homework 1

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Problem 3.1.3

1. Solve the Cauchy problem

$$x_1^2 \partial_1 u + x_2^2 \partial_2 u = u^2 \ ,$$
$$u(x_1, x_2) = 1 \text{ on the line } x_2 = 2x_1 \ .$$

Where is the solution defined?

Solution: The PDE above has the form of the equation (3.2) from the notes. Since c and a_j are partially differentiable in all variables and these partial derivatives are continuous, the solution u must be unique acording to *Theorem 3.5*. Looking at the initial condition we identify the hyper-surface $S = \{(z, 2z); z \in \mathbb{R}\}$. Therefore $S = g(\mathbb{R})$ with $g(z) = (z, 2z)$. Now, we can write our *Characteristic System* with the *Initial Conditions*:

$$\begin{aligned} \partial_t \xi_1(z,t) &= \xi_1^2(z,t), \quad && \xi_1(z,0) = z, \\ \partial_t \xi_2(z,t) &= \xi_2^2(z,t), \quad && \xi_2(z,0) = 2z, \\ \partial_t v(z,t) &= v^2(z,t), \quad && v(z,0) = 1. \end{aligned}$$

We integrate the equation for ξ_1,

$$\xi_1(z,t) = -\frac{1}{t + f_1(z)}.$$

Imposing the initial condition for ξ_1 we find the function $f_1(z)$,

$$\xi_1(z,0) = -\frac{1}{f_1(z)} = z,$$

obtaining $f_1(z) = -1/z$. Therefore we have found, doing some modifications,

$$\xi_1(z,t) = \frac{z}{1 - zt}.$$

We integrate now the equation for ξ_2,

$$\xi_2(z,t) = -\frac{1}{t + f_2(z)}.$$

Imposing the initial condition for ξ_2 we find the function $f_2(z)$,

$$\xi_2(z,0) = -\frac{1}{f_2(z)} = 2z,$$

obtaining $f_2(z) = -1/2z$. Therefore we have found, doing some modifications,

$$\xi_2(z,t) = \frac{2z}{1-2zt}.$$

Lastly we integrate the equation for v

$$v(z,t) = -\frac{1}{t+f_3(z)}.$$

Imposing the initial condition for v we find the function $f_3(z)$,

$$v(z,0) = -\frac{1}{f_3(z)} = 1,$$

obtaining $f_3(z) = -1$. Therefore we have found

$$v(z,t) = \frac{1}{1-t} = v(t).$$

We want to find the solution to the PDE

$$u(x_1,x_2) = v(z(x_1,x_2), t(x_1,x_2)) = v(t(x_1,x_2)).$$

To do so we need to find first $t(x_1,x_2)$ using

$$x_1 = \xi_1 = \frac{z}{1-zt},$$
$$x_2 = \xi_2 = \frac{2z}{1-2zt}.$$

From the first we can isolate z,

$$z = \frac{x_1}{1+x_1 t},$$

and insert it in the second,

$$x_2 = \frac{\frac{2x_1}{1+x_1 t}}{1 - \frac{2x_1}{1+x_1 t}t} = \frac{2x_1}{1-x_1 t}.$$

From the previous equation we find t and, consequently u,

$$t = \frac{x_2 - 2x_1}{x_1 x_2},$$

$$u(x_1,x_2) = v(t(x_1,x_2)) = \frac{1}{1 - \frac{x_2-2x_1}{x_1x_2}},$$

$$u(x_1,x_2) = \frac{x_1x_2}{x_1x_2 - x_2 + 2x_1}$$

This solution is defined in $\mathbb{R}^2 \setminus \left\{(x_1,0),(0,x_2),\left(x_1,\frac{2x_1}{1-x_1}\right)\right\}$. The solution doesn't exist if either of x_1 or x_2 is zero since it would make t infinite. Also the soltion is not defined in those points that make the denominator of the solution zero.

Problem 3.1.4

1. Solve the Cauchy problem

$$u\partial_1 u + \partial_2 u = 2\ ,$$
$$u(x_1, x_2) = 0 \text{ on the line } x_1 = x_2.$$

Where is the solution defined and where is it differentiable?

Solution: The PDE above has the form of the equation (3.2) from the notes. Since c and a_j are partially differentiable in all variables and these partial derivatives are continuous, the solution u must be unique acording to *Theorem 3.5.* Looking at the initial condition we identify the hypersurface $S = (z, z); z \in \mathbb{R}$. So $S = g(\mathbb{R})$ with $g(z) = (z, z)$. Now we can write our *Characteristic System* with the *Initial Conditions*:

$$\begin{aligned} \partial_t \xi_1(z,t) &= v(z,t), \quad & \xi_1(z,0) &= z, \\ \partial_t \xi_2(z,t) &= 1, \quad & \xi_2(z,0) &= z, \\ \partial_t v(z,t) &= 2, \quad & v(z,0) &= 0. \end{aligned}$$

Let's first integrate for ξ_2,

$$\partial_t \xi_2(z,t) = 1 \quad \Rightarrow \quad \xi_2(z,t) = t + f_2(z).$$

Imposing the initial condition,

$$\xi_2(z,0) = f_2(z) = z,$$

we obtain

$$\xi_2(z,t) = z + t.$$

Now we solve for v,

$$\partial_t v(z,t) = 2 \quad \Rightarrow \quad v(z,t) = 2t + f_3(z).$$

Imposing the initial condition,

$$v(z,0) = f_3(z) = 0,$$

we obtain

$$v(z,t) = 2t = v(t).$$

Finally we can integrate ξ_1,

$$\partial_t \xi_1(z,t) = v(t) = 2t \quad \Rightarrow \quad \xi_1(z,t) = t^2 + f_1(z).$$

Imposing the initial condition,

$$\xi_1(z,0) = f_1(z) = z,$$

we obtain

$$\xi_1(z,t) = z + t^2.$$

Again, to find the solution we need to find $t(x_1, x_2)$ using that

$$x_1 = \xi_1 = z + t^2,$$
$$x_2 = \xi_2 = z + t.$$

Substracting the two previous equations we find

$$x_1 - x_2 = t^2 - t \quad \Rightarrow \quad t^2 - t + x_2 - x_1 = 0,$$

which we can solve for t,

$$t = \frac{1 \pm \sqrt{1 - 4(x_2 - x_1)}}{2}.$$

Now we can obtain the solution

$$u(x_1, x_2) = v(t(x_1, x_2)) = 2t(x_1, x_2) = 1 \pm \sqrt{1 - 4(x_2 - x_1)}.$$

The solution must be unique so, to get rid of one sign, we impose the initial condition for u,

$$u(x_1, x_1) = 1 \pm 1 = 0,$$

which implies that

$$u(x_1, x_2) = 1 - \sqrt{1 - 4(x_2 - x_1)}.$$

The solution is defined for all $\mathbb{R}^2$ except for those points that don't satisfy $x_2 - x_1 \leq 1/4$, i.e.,$\mathbb{R}^2 \setminus \{(x_1, x_2); x_2 - x_1 > 1/4\}$. To see when the solution is differentiable we calculate

$$\partial_1 u = \frac{-2}{\sqrt{1 - 4(x_2 - x_1)}}, \qquad \partial_2 u = \frac{2}{\sqrt{1 - 4(x_2 - x_1)}}.$$

Clearly, u is differentiable in $\mathbb{R}^2 \setminus \{(x_1, x_2); x_2 - x_1 \geq 1/4\}$, since $x_2 - x_1 = 1/4$ would make the derivatives to go to infinity.

Problem 3.1.6

1. Solve the Cauchy problem

$$u\partial_1 u + x_2 \partial_2 u = u \ ,$$
$$u(z, z) = z^2, \qquad z \in \mathbb{R}.$$

Where is the solution defined and where is it partially differentiable?

Solution: The PDE above has the form of the equation (3.2) from the notes. Since c and a_j are partially differentiable in all variables and these partial derivatives are continuous, the solution u must be unique acording to *Theorem 3.5*. Looking at the initial condition we identify the hypersurface $S = (z, z); z \in \mathbb{R}$. So $S = g(\mathbb{R})$ with $g(z) = (z, z)$. Now we can write our *Characteristic System* with the *Initial Conditions*:

$$\begin{aligned} \partial_t \xi_1(z,t) &= v(z,t), \qquad \xi_1(z,0) = z, \\ \partial_t \xi_2(z,t) &= \xi_2(z,t), \qquad \xi_2(z,0) = z, \\ \partial_t v(z,t) &= v(z,t), \qquad v(z,0) = z^2. \end{aligned}$$

We integrate first the differential equation for v,

$$\partial_t v(z,t) = v(z,t) \quad \Rightarrow \quad v(z,t) = f_3(z)e^t.$$

Imposing the initial condition,

$$v(z,0) = f_3(z) = z^2,$$

we obtain $f_3(z)$ and

$$v(z,t) = z^2 e^t.$$

Using this result we integrate the differential equation for ξ_1,

$$\partial_t \xi_1(z,t) = v(z,t) \quad \Rightarrow \quad \xi_1(z,t) = z^2 e^t + f_1(z).$$

Imposing the initial condition,

$$\xi_1(z,0) = z^2 + f_1(z) = z,$$

obtaining $f_1(z) = z - z^2$. Thus,

$$\xi_1(z,t) = z - z^2(1 - e^t).$$

Now we integrate for ξ_2,

$$\partial_t \xi_2(z,t) = \xi_2(z,t) \quad \Rightarrow \quad \xi_2(z,t) = f_2(z)e^t.$$

Imposing the initial condition,

$$\xi_2(z,0) = f_2(z) = z,$$

obtaining $f_2(z) = z$. Thus,

$$\xi_2(z,t) = ze^t.$$

To find the solution we need to find $z(x_1,x_2)$ and $t(x_1,x_2)$ using that

$$\begin{aligned} x_1 &= \xi_1 = z - z^2(1 - e^t), \\ x_2 &= \xi_2 = ze^t. \end{aligned}$$

From the last equation we obtain

$$e^t = \frac{x_2}{z},$$

and

$$x_1 = z - z^2 + z^2 \frac{x_2}{z} = z - z^2 + x_2 z = (1 + x_2)z - z^2,$$

$$z^2 - (1+x_2)z + x_1 = 0.$$

We can solve now for z,

$$z = \frac{1 + x_2 \pm \sqrt{(1+x_2)^2 - 4x_1}}{2}.$$

At this point, using $e^t = x_2/z$, we can obtain the solution

$$u(x_1, x_2) = z^2 \frac{x_2}{z} = zx_2 = x_2 \frac{1 + x_2 \pm \sqrt{(1+x_2)^2 - 4x_1}}{2}.$$

The solution must be unique, imposing the initial condition for u we can discard one of the signs:

$$\begin{aligned} u(z,z) &= z\frac{1 + z \pm \sqrt{(1+z)^2 - 4z}}{2} \\ &= z\left(1 + z \pm (1-z)\right) = z. \end{aligned}$$

Therefore, the minus sign gives us the wrong solution. Thus,

$$u(x_1, x_2) = \frac{x_2}{2}\left(1 + x_2 + \sqrt{(1+x_2)^2 - 4x_1}\right).$$

The solution is defined for all $\mathbb{R}^2$ except for those points that don't satisfy $(1+x_2)^2 \geq 4x_1$, i.e.,$\mathbb{R}^2 \setminus \left\{(x_1, x_2); (1+x_2)^2 - 4x_1 < 0\right\}$. To see when the solution is differentiable we calculate

$$\partial_1 u = \frac{-x_2}{\sqrt{(1+x_2)^2 - 4x_1}}, \qquad \partial_2 u = \frac{1}{2} + x_2 + \frac{1}{2}\sqrt{(1+x_2)^2 - 4x_1} + \frac{x_2(1+x_2)}{\sqrt{(1+x_2)^2 - 4x_1}}.$$

Clearly, u is differentiable in $\mathbb{R}^2 \setminus \left\{(x_1, x_2); (1+x_2)^2 - 4x_1 \leq 0\right\}$. Both its partial derivatives are not defined for points that don't satisfy $(1+x_2)^2 - 4x_1 > 0$.

Problem 3.1.8

1. Which of the PDEs in the exercises you have been assigned are linear, semilinear, quasilinear, nonlinear?

Solution: The PDE of Problem 3.1.3 is semilinear since the solution u is not in the coefficients a_j, but appears in c in a nonlinear way, $c = u^2$. The PDEs of Problem 3.1.4 and Problem 3.1.6 is quasilinear because the solution u appears in the coefficients a_j which multiply the partial derivatives of u.