Partial Differential Equations
Instructor Homework 6

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Problem 9.1.5

1. Let Γ be the fundamental solution of the heat equation in one space dimension defined by (9.2). Let $b : \mathbb{R}_+ \to \mathbb{R}$ and $a : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous. Let $B(t) = \int_0^t b(s)ds$ and $A(t) = \int_0^t a(s)ds$. Assume $A(t) > 0$ for all $t > 0$. Define

$$\tilde{\Gamma}(t,x) = \Gamma(x + B(t), A(t)), \quad x \in \mathbb{R}, t > 0.$$

Show that

$$\partial_t \tilde{\Gamma}(t,x) = b(t)\partial_x \tilde{\Gamma}(t,x) + a(t)\partial_x^2 \tilde{\Gamma}(t,x)$$

Solution: By simply applying chain rule we get

$$\partial_t \tilde{\Gamma}(t,x) = B'(t)\partial_x \tilde{\Gamma}(t,x) + A'(t)\partial_t \tilde{\Gamma}(t,x).$$

By the Fundamental Theorem of Calculus, $A'(t) = a(t)$ and $B'(t) = b(t)$. Further, by (9.3), $\partial_t \tilde{\Gamma}(t,x) = \partial_x^2 \tilde{\Gamma}(t,x)$. Hence,

$$\partial_t \tilde{\Gamma}(t,x) = b(t)\partial_x \tilde{\Gamma}(t,x) + a(t)\partial_x^2 \tilde{\Gamma}(t,x).$$

2. Show that

$$\int_{\mathbb{R}} \tilde{\Gamma}(t,x)dx = 1, \quad \int_{\mathbb{R}} x\tilde{\Gamma}(t,x)dx = -B(t),$$

and

$$\int_{\mathbb{R}} (x + B(t))^2 \tilde{\Gamma}(t,x)dx = 2A(t).$$

Solution: Let's start with the first one. Consider a change of variables similar to the tone used in the notes, $x + B(t) = y(4t)^{1/2}$. Then,

$$\begin{aligned}\int_{\mathbb{R}} \tilde{\Gamma}(t,x)dx &= \int_{\mathbb{R}} \Gamma(x+B(t), A(t))dx \\ &= \int_{\mathbb{R}} \Gamma(y(4A(t))^{1/2}, A(t)) \cdot (4A(t))^{1/2} dy \\ &= \pi^{-1/2} \int_{\mathbb{R}} e^{-y^2} dy = 1.\end{aligned}$$

For the next one, consider the change of variables $y = x + B(t)$, then

$$\begin{aligned}\int_{\mathbb{R}} x\tilde{\Gamma}(t,x)dx &= \int_{\mathbb{R}} (y - B(t))\Gamma(x+B(t), A(t))dx \\ &= \int_{\mathbb{R}} y(4\pi A(t))^{-1/2} e^{-y^2(4A(t))^{-1}} dy - \int_{\mathbb{R}} B(t)(4\pi A(t))^{-1/2} e^{-y^2(4A(t))^{-1}} dy \\ &= -A(t)^{-1/2}\pi^{-1/2} e^{-y^2(4A(t))^{-1}} \Big|_{-\infty}^{\infty} - B(t) \int_{\mathbb{R}} (4\pi A(t))^{-1/2} e^{-y^2(4A(t))^{-1}} dy \\ &= 0 - B(t) = -B(t),\end{aligned}$$

where we have used the first part of the proof for the second integral. For the last one, we use integration by parts,

$$\begin{aligned}\int_{\mathbb{R}} (x+B(t))^2 \tilde{\Gamma}(t,x)dx &= \int_{\mathbb{R}} -\sqrt{\frac{A(t)}{\pi}}(x+B(t)) \left(-(x+B(t))(2A(t))^{-1} e^{-(x+B(t))^2(4A(t))^{-1}} \right) dx \\ &= -\sqrt{\frac{A(t)}{\pi}}(x+B(t)) e^{-(x+B(t))^2(4A(t))^{-1}} \Big|_{-\infty}^{\infty} + \sqrt{\frac{A(t)}{\pi}} \int_{\mathbb{R}} e^{-(x+B(t))^2(4A(t))^{-1}} dx \\ &= 0 + \sqrt{\frac{A(t)}{\pi}} (4\pi A(t))^{1/2} \int_{\mathbb{R}} (4\pi A(t))^{-1/2} e^{-(x+B(t))^2(4A(t))^{-1}} dx \\ &= \sqrt{\frac{A(t)}{\pi}} (4\pi A(t))^{1/2} \\ &= \sqrt{\frac{A(t)}{\pi}} (4\pi A(t))^{1/2} \\ &= 2A(t),\end{aligned}$$

where we have used the first result again.