Partial Differential Equations
TA Homework 7

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Problem 4.4.1

Consider the wave equation

$$\begin{aligned}
(PDE) \quad & (\partial_t^2 - \partial_x^2)u = 0, && 0 \le x \le \pi, t \in \mathbb{R}, \\
(IC) \quad & u(x,0) = f(x), && 0 \le x \le \pi, \\
& \partial_t u(x,0) = g(x), && 0 \le x \le \pi, \\
(BC) \quad & \partial_x u(0,t) = 0 = \partial_x u(L,t), && t \in \mathbb{R}.
\end{aligned} \tag{1}$$

with f and g in $C[0,\pi]$.

(c) Check whether the d'Alembert formula provides a generalized solution.

Solution: Let $\phi : [0,\pi] \to \mathbb{R}$ be a twice differentiable function with $\phi'(0) = 0 = \phi'(\pi)$. Define $v(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$ and $w(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$. Recall that f and g are continuous on $[0,\pi]$ and extended in an even and 2π-periodic way. Then, using Leibniz rule,

$$\begin{aligned}
\frac{d}{dt}\int_0^\pi \phi(x)v(x,t)dx =& \frac{c}{2}\left[\phi(\pi)f(\pi+ct) - \phi(0)f(ct)\right] - \frac{c}{2}\int_{ct}^{\pi+ct} \phi'(y-ct)f(y)dy \\
& - \frac{c}{2}\left[\phi(\pi)f(\pi-ct) - \phi(0)f(-ct)\right] + \frac{c}{2}\int_{-ct}^{\pi-ct} \phi'(y+ct)f(y)dy \\
=& \frac{c}{2}\int_{-ct}^{\pi-ct} \phi'(y+ct)f(y)dy - \frac{c}{2}\int_{ct}^{\pi+ct} \phi'(y-ct)f(y)dy,
\end{aligned}$$

where we have used that f is even around 0 and π. Notice that

$$\frac{d}{dt}\int_0^\pi \phi(x)v(x,t)dx\bigg|_{t=0} = 0.$$

Using Leibniz rule again,

$$\begin{aligned}\frac{d^2}{dt^2}\int_0^\pi \phi(x)v(x,t)dx =& -\frac{c^2}{2}\left[\phi'(\pi)f(\pi+ct)-\phi'(0)f(ct)\right]+\frac{c^2}{2}\int_{ct}^{\pi+ct}\phi''(y-ct)f(y)dy\\ &-\frac{c^2}{2}\left[\phi'(\pi)f(\pi-ct)-\phi'(0)f(-ct)\right]+\frac{c^2}{2}\int_{-ct}^{\pi-ct}\phi''(y+ct)f(y)dy\\ =&\frac{c^2}{2}\int_{-ct}^{\pi-ct}\phi''(y+ct)f(y)dy+\frac{c^2}{2}\int_{ct}^{\pi+ct}\phi''(y-ct)f(y)dy,\end{aligned}$$

and reversing the change of variables we get

$$\begin{aligned}\frac{d^2}{dt^2}\int_0^\pi \phi(x)v(x,t)dx &= \frac{c^2}{2}\int_0^\pi \phi''(x)f(x-ct)dx+\frac{c^2}{2}\int_0^\pi \phi''(x)f(x+ct)dx\\ &=\int_0^\pi c^2\phi''(x)\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]dx\\ &=\int_0^\pi c^2\phi''(x)v(x,t)dx.\end{aligned}$$

Now we start working on w. Since g is continuous, w is differentiable with respect to t and x and

$$\partial_t w(x,t)=\frac{1}{2}\left[g(x+ct)+g(x-ct)\right],$$

$$\partial_x w(x,t)=\frac{1}{2c}\left[g(x+ct)-g(x-ct)\right].$$

Note that

$$\partial_t w(x,t)\big|_{t=0}=g(x).$$

Since $\partial_t w(x,t)$ is continuous, we can differentiate under the integral and obtain

$$\frac{d}{dt}\int_0^\pi \phi(x)w(x,t)dx=\int_0^\pi \phi(x)\frac{1}{2}\left[g(x+ct)+g(x-ct)\right]dx.$$

Same consideration as before with g replacin f gives us

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x)w(x,t)dx=-\frac{c}{2}\int_{ct}^{\pi+ct}\phi'(y-ct)g(y)dy+\frac{c}{2}\int_{-ct}^{\pi-ct}\phi'(y+ct)g(y)dy,$$

which, after reversing the substitution yields

$$\begin{aligned}\frac{d^2}{dt^2}\int_0^\pi \phi(x)w(x,t)dx &= -\int_0^\pi \phi'(x)\frac{c}{2}\left[g(x+ct)-g(x-ct)\right]\\ &=-\int_0^\pi \phi'(x)c^2\partial_x w(x,t)dx.\end{aligned}$$

Integrating by parts, recalling that $\phi'(0)=0=\phi'(\phi)$, we obtain

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x)w(x,t)dx=\int_0^\pi \phi''(x)c^2 w(x,t)dx.$$

Since, by d'Alembert formula, $u(x,t) = v(x,t) + w(x,t)$, we have shown that $\int_0^\pi \phi(x)u(x,t)dx$ is twice differentiable and

$$\frac{d^2}{dt^2}\int_0^\pi \phi(x)u(x,t)dx = \int_0^\pi c^2\phi''(x)u(x,t)dx$$

and

$$\frac{d}{dt}\int_0^\pi \phi(x)u(x,t)dx = \int_0^\pi \phi(x)g(x)dx, \qquad t=0.$$

Problem 5.1.1

Let L, $a > 0$. Consider the problem

$$\begin{aligned} \text{(PDE)} \quad & (\partial_t - a\partial_x^2)u = 0, \quad 0 \le x \le L.t > 0, \\ \text{(IC)} \quad & u(x,0) = f(x), \quad 0 \le x \le L, \\ \text{(BC)} \quad & \partial_x u(0,t) = 0 = \partial_x u(L,t), \quad t > 0. \end{aligned}$$

This equation is a model for heat diffusion in a (finite) rod of length L. The no flux boundary condition means that both ends of the rod are insulated.

(a) Use Fourier cosine series to solve (5.14), at least as far as (PDE) and (BC) are concerned, under an appropriate condition for f.

(b) Explore two assumptions for f under which (IC) is satisfied in meaningful though not necessarily literal ways.

(c) Show that $\int_0^L u(x,t)dx = \int_0^L u(x,0)dx$ for all $t \ge 0$.
Hint: These integrals are related to the Fourier cosine coefficient of index zero.

(d) Show that $u(x,t) \to \frac{1}{L}\int_0^L f(x)dx$ as $t \to \infty$, uniformly in $x \in [0,L]$.

Solution: The solution, if exists, can be expressed as a Fourier cosine series,

$$u(x,t) = \sum_{j=0}^{\infty} A_j(t)\cos(\lambda_j x), \qquad \lambda_j = j\frac{\pi}{L},$$

with

$$A_j(t) = \frac{2}{L}\int_0^L u(y,t)\cos(\lambda_j y)dy.$$

If u is a solution, it is sufficiently smooth that we can differentiate under the integral

$$\begin{aligned} A_j'(t) &= \frac{2}{L}\int_0^L \partial_t u(y,t)\cos(\lambda_j y)dy \\ &= \frac{2}{L}\int_0^L a\partial_y^2 u(y,t)\cos(\lambda_j y)dy. \end{aligned}$$

Since the cosines and u satisfy zero Neumann boundary conditions, we can integrate by parts twice and get

$$A_j'(t) = -a\lambda_j^2\frac{2}{L}\int_0^L u(y,t)\cos(\lambda_j y)dy = -a\lambda_j^2 A_j(t).$$

We have obtain the following ODE for $A_j(t)$,

$$A_j'(t) + a\lambda_j^2 A_j(t) = 0,$$

which has the following solution,

$$A_j(t) = A_j(0)e^{-a\lambda_j^2 t}, \quad j > 0.$$

For the case $j = 0$, since $\lambda_0 = 0$, the ODE is

$$A_0'(t) = 0,$$

and

$$A_0(t) = A_0(0).$$

Recall that

$$A_j(0) = \frac{2}{L}\int_0^L f(y)\cos(\lambda_j y)dy, \qquad j > 0,$$

and

$$A_0(0) = \frac{1}{L}\int_0^L f(y)dy, \qquad j > 0,$$

Thus, the solution $u(x,t)$ is uniquely determined. We now concentrate in provine uniqueness.

Claim *Let $f \in L^1([0,L],\mathbb{R})$. Then the series u with $A_j(t)$ and $A_j(0)$ given above converges uniformly on $[0,L]x[\epsilon,\infty)$ for all $\epsilon > 0$. Moreover u is inifitely often differentiable on $[0,L]x(0,\infty)$ and satisfies the PDE and BC.*

Proof. For $m \in \mathbb{N}$, set

$$u_m(x,t) = A_m(0)cos(\lambda_m x)e^{-a\lambda_m^2 t}.$$

Then u is infinitely differentiable and

$$\partial_x^k \partial_t^l u_m(x,t) = A_m(0)\frac{d^k}{dx^k}cos(\lambda_m x)\frac{d^l}{dt^l}e^{-a\lambda_m^2 t}.$$

Notice that

$$|A_m(0)| \leq \frac{2}{L}\int_0^L |f(x)|dx =: M_0 < \infty.$$

Let $t \geq \epsilon > 0$. Then, by the form of λ_m,

$$\begin{aligned}\left|\partial_x^k \partial_t^l u_m(x,t)\right| &\leq |A_m(0)|\lambda_m^{k+2l}a^l e^{-a\lambda_m^2 t}\\ &\leq M_0 \lambda_m^{k+2l}a^l e^{-a\lambda_m^2 \epsilon}\\ &\leq M_0\ c\ m^{k+2l}\eta^{(m^2)},\end{aligned}$$

where $\eta = e^{-a\lambda_1^2\epsilon}$. The ratio test implies that $\sum_{m=1}^{\infty} M_0\ c\ m^{k+2l}\eta^{(m^2)}$ converges. Then, by *Theorem 5.1*, each series $\sum_{m=1}^{\infty}\partial_x^k\partial_t^l u_m(x,t)$ (in particular the series for u) converges uniformly on $[0,L]x(0,\infty)$. Let $x \in [0,L] = I_1$ and $t > 0$. Choose $I_2 = (t_1,t_2)$ with $0 < t_1 < t_2 < \infty$. Applying *Theorem 5.3* repeatedly implies that $u = \sum_{m=1}^{\infty} u_m$ has partial derivatives of all order and can be differentiated term by term on $I_x I_2$. Since each u_m satisfies the PDE and BC, so does u on $[0,L]x(0,\infty)$. Hence, we have proved existence. For part b) we will prove two claims.

Claim *Let $f : [0,L] \to \mathbb{R}$ be Lipschitz continuous, $f(L) = 0 = f(0)$. Then u is continuous on $[0,L]x[0,\infty)$ and $u(x,0) = f(x)$ for all $x \in [0,L]$. In particular, $u(x,t) \to f(x)$ as $t \to 0$, uniformly in $x \in [0,L]$.*

Proof. Notice that the functions u_m satisfy the estimate

$$|u_m(x,t)| \leq |A_m(0)|, \qquad x \in [0,L],\ t \geq 0,\ m \in \mathbb{N},$$

with $A_m(0)$ given above. Since f is Lipschitz continuous, and $f(0) = 0 = f(L)$, the series

$$\sum_{m=0}^{\infty} |A_m(0)| < \infty,$$

by *Exercise 4.3.3* and even extension. By *Theorem 5.1*, $u = \sum m = 1^{\infty} u_m$ converges uniformly and is continuous on $[0, L]x[0, \infty)$. In Particular, $u(x, 0) = f(x)$ by *Exercise 4.3.4*. Let $\epsilon > 0$. Then there exists some $\delta > 0$ such that

$$|u(x,t) - u(y,0)| < \epsilon \text{ whenever } |x - y| + |t - 0| < \delta.$$

In particular

$$\begin{aligned} |u(x,t) - u(x,0)| &< \epsilon \text{ whenever } 0 \leq t < \delta, \\ |u(x,t) - f(x)| &< \epsilon \text{ whenever } 0 \leq t < \delta. \end{aligned}$$

Claim 2 *Assume that $f : [0, L] \to \mathbb{R}$ is intergable and $\int_0^L |f(x)|^2 dx < \infty$. Then the series u defined satisfies*

$$\int_0^L |u(x,t) - f(x)|^2 dx \to 0, \qquad t \to 0.$$

Proof. Let $\langle \phi, \psi \rangle = \frac{2}{L} \int_0^L \phi(x)\psi(x) dx$ be the inner product of choice on $L^2([0, L], \mathbb{R})$, the space of square integrable functions. Then $v_j; j \in \mathbb{N}$ with $v_j = cos(\lambda_j x)$ and $v_0 = \frac{1}{\sqrt{2}}$ is an orthonormal basis. By the considerations at the beginning of section 5.1,

$$\langle u(\cdot, t), v_m \rangle = \langle f, v_m \rangle \, e^{-a\lambda_m^2 t},$$

which are uniformly continuous functions on $\mathbb{R}_+$. Further,

$$|\langle u(\cdot, t), v_m \rangle| \leq |\langle f, v_m \rangle| \, ,$$

and, by Parseval's relation,

$$\sum_{m \in \mathbb{N}} |\langle f, v_m \rangle|^2 = ||f||^2.$$

The assertion now follows from *Theorem 4.11*.

For part c) we just compute both integrals seperately and we find the same result. We start with

$$\begin{aligned} \int_0^L u(x,t) dx &= \int_0^L \sum_{j=0}^{\infty} A_j(t) \cos(\lambda_j x) dx \\ &= \int_0^L A_0(t) dx + \sum_{j=1}^{\infty} A_j(t) \int_0^L \cos(\lambda_j x) dx \\ &= \int_0^L A_0(t) dx \\ &= A_0(t) L = A_0(0) L, \end{aligned}$$

where we have used that $\lambda_0 = 0$, the integrals of the cosines are zero and recall that $A_0(t) = A_0(0)$. Now we calculate

$$\begin{aligned}\int_0^L u(x,0)dx &= \int_0^L \sum_{j=0}^{\infty} A_j(0)\cos(\lambda_j x)dx \\ &= \int_0^L A_0(0)dx + \sum_{j=1}^{\infty} A_j(0)\int_0^L \cos(\lambda_j x)dx \\ &= A_0(0)L.\end{aligned}$$

Thus,

$$\int_0^L u(x,t)dx = \int_0^L u(x,0)dx, \quad \text{for all } t \geq 0.$$

For part d) notice that the limit to prove is no other than A_0, then

$$\begin{aligned}|u(x,t) - A_0| &= \left|\sum_{j=0}^{\infty} A_j \cos(\lambda_j x)e^{-a\lambda_j^2 t} - A_0\right| \\ &= \left|\sum_{j=1}^{\infty} A_j \cos(\lambda_j x)e^{-a\lambda_j^2 t}\right| \\ &\leq \sum_{j=1}^{\infty} |A_j|\, e^{-a\lambda_j^2 t} \\ &\leq A_0 \sum_{j=1}^{\infty} e^{-a\lambda_j^2 t} \\ &\leq A_0 \sum_{j=1}^{\infty} \left(e^{-akt}\right)^j, \quad \text{for } t > 0,\end{aligned}$$

where we have made $k = (\pi/L)^2$ and using that $\lambda_j^2 \leq k^2 j^2$. Now, using the geometric series formula,

$$|u(x,t) - A_0| \leq A_0 \frac{e^{-akt}}{1 - e^{-akt}} \to 0 \text{ as } t \to \infty.$$

Hence,

$$u(x,t) \to \frac{1}{L}\int_0^L f(x)dx$$

as $t \to \infty$ uniformly in $x \in [0, L]$.

Problem 5.1.2

Let $f : [0, L] \to \mathbb{R}$ be integrable and $\int_0^L |f(x)|dx < \infty$, i.e. $f \in L^1([0, L], \mathbb{R})$. Consider the heat equation with zero boundary condition and initial data f. Show: there exists a function $u : [0, L] \times (0, \infty) \to \mathbb{R}$ that solves (PDE) and (BC) and satisfies the initial condition in the following weak sense: if $\phi : [0, L] \to \mathbb{R}$ is Lipschitz continuous and $\phi(0) = 0 = \phi(L)$, then

$$\int_0^L \phi(x)u(x,t)dx \to \int_0^L \phi(x)f(x)dx, \quad t \to 0.$$

Hint: Notice (and prove) that

$$\int_0^L \phi(x)u(x,t)dx = \int_0^L f(x)v(x,t)dx$$

where v is the solution of the heat equation with initial data ϕ.

Solution: Let us express the solutions of the PDE $u(x,t)$ and $v(x,t)$ as

$$u(x,t) = \sum_{j=0}^{\infty} B_j(t)\sin(\lambda_j x),$$

with

$$B_j(t) = B_j(0)e^{-a\lambda_j t} = \frac{2}{L}e^{-a\lambda_j t}\int_0^L f(y)\sin(\lambda_j y)dy$$

and

$$v(x,t) = \sum_{j=0}^{\infty} \tilde{B}_j(t)\sin(\lambda_j x),$$

with

$$\tilde{B}_j(t) = \tilde{B}_j(0)e^{-a\lambda_j t} = \frac{2}{L}e^{-a\lambda_j t}\int_0^L \phi(y)\sin(\lambda_j y)dy$$

Then,

$$\begin{aligned}
\int_0^L \phi(x)u(x,t)dx &= \int_0^L \phi(x)\sum_{j=0}^{\infty} B_j(t)\sin(\lambda_j x)dx \\
&= \int_0^L \phi(x)\sum_{j=0}^{\infty} \frac{2}{L}e^{-a\lambda_j t}\int_0^L f(y)\sin(\lambda_j y)dy\sin(\lambda_j x)dx \\
&= \sum_{j=0}^{\infty} \frac{2}{L}e^{-a\lambda_j t}\int_0^L \phi(x)\sin(\lambda_j x)dx\int_0^L f(y)\sin(\lambda_j y)dy \\
&= \sum_{j=0}^{\infty} \tilde{B}_j(t)\int_0^L f(y)\sin(\lambda_j y)dy \\
&= \int_0^L f(y)\sum_{j=0}^{\infty} \tilde{B}_j(t)\sin(\lambda_j y)dy \\
&= \int_0^L f(y)v(y,t)dy.
\end{aligned}$$

Hence,

$$\int_0^L \phi(x)u(x,t)dx = \int_0^L f(x)v(x,t)dx.$$

Then,

$$\begin{aligned}
\lim_{t\to 0}\int_0^L \phi(x)u(x,t)dx &= \lim_{t\to 0}\int_0^L f(x)v(x,t)dx \\
&= \int_0^L \lim_{t\to 0} f(x)v(x,t)dx \\
&= \int_0^L f(x)\lim_{t\to 0} v(x,t)dx \\
&= \int_0^L f(x)\phi(x)dx,
\end{aligned}$$

where we have introduced the limit inside the integral since $v(x,t) \to \phi(x)$ as $t \to 0$ uniformly in $x \in [0,L]$ according to *Theorem 5.6.* Thus,

$$\int_0^L \phi(x)u(x,t)dx \to \int_0^L \phi(x)f(x)dx, \qquad t \to 0.$$