Partial Differential Equations
TA Homework 4

Francisco Jose Castillo Carrasco

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Problem 2

Verify that the function $<,>$ defined in Example 0.3 is an inner product.

Solution: Given the inner product on C^2 defined by

$$\langle v, w \rangle = \begin{pmatrix} \overline{w}_1 & \overline{w}_2 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

it is easy to check the properties.

- Positivity:
- Conjugate symmetry:

$$\begin{aligned}
\langle v, w \rangle &= \begin{pmatrix} \overline{w}_1 & \overline{w}_2 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \begin{pmatrix} 2\overline{w}_1 + i\overline{w}_2 & 3\overline{w}_2 - i\overline{w}_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= 2v_1\overline{w}_1 + iv_1\overline{w}_2 + 3v_2\overline{w}_2 - iv_2\overline{w}_1 \\
&= \overline{2\overline{v}_1 w_1 - i\overline{v}_1 w_2 + 3\overline{v}_2 w_2 + i\overline{v}_2 w_1} \\
&= \overline{\begin{pmatrix} 2\overline{v}_1 + i\overline{v}_2 & 3\overline{v}_2 - i\overline{v}_1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}} \\
&= \overline{\begin{pmatrix} \overline{v}_1 & \overline{v}_2 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}} \\
&= \overline{\langle w, v \rangle}
\end{aligned}$$

- Homogeneity:

$$\begin{aligned}
\langle cv, w \rangle &= \begin{pmatrix} \overline{w}_1 & \overline{w}_2 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} \\
&= \begin{pmatrix} \overline{w}_1 & \overline{w}_2 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} c \\
&= \langle v, w \rangle c \\
&= c \langle v, w \rangle,
\end{aligned}$$

where we have taken the complex scalar c out of the vector v since it is common in all its components.

- Linearity:

$$
\begin{aligned}
\langle u+v, w\rangle &= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1+v_1\\ u_2+v_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2(u_1+v_1)-i(u_2+v_2) & i(u_1+v_1)+3(u_2+v_2)\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2u_1-iu_2+2v_1-iv_2 & iu_1+3u_2+iv_1+3v_2\end{pmatrix}\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\left[\begin{pmatrix}2u_1-iu_2 & iu_1+3u_2\end{pmatrix}+\begin{pmatrix}2v_1-iv_2 & iv_1+3v_2\end{pmatrix}\right]\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\left[\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right]\\
&= \begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}\overline{w}_1 & \overline{w}_2\end{pmatrix}\begin{pmatrix}2 & -i\\ i & 3\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}\\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}
$$

Problem 7

For $n > 0$, let

$$f_n(t) = \begin{cases} \sqrt{n}, & 0 \le t \le 1/n^2 \\ 0, & \text{otherwise} \end{cases}$$

Show that $f_n \to 0$ in $L^2[0,1]$ but that $f_n(0)$ does not converge to zero.

Solution: For the first proof we need to prove that

$$||f_n(t) - 0|| \to 0 \text{ as } n \to \infty.$$

Then,

$$\begin{aligned} ||f_n(t) - 0|| &= \sqrt{\langle f_n - 0, f_n - 0\rangle_{L^2}} \\ &= \sqrt{\int_0^1 [f_n(t) - 0]^2\, dt} \\ &= \sqrt{\int_0^1 [f_n(t)]^2\, dt} \\ &= \sqrt{\int_0^{1/n^2} n dt + \int_{1/n^2}^1 0 dt} \\ &= \sqrt{\frac{1}{2} n \frac{1}{n^2}} \\ &= \sqrt{\frac{1}{2n}}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} ||f_n(t) - 0|| = \lim_{n\to\infty} \sqrt{\frac{1}{2n}} = 0.$$

However,

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \sqrt{n} = \infty \neq 0.$$

Problem 11

Show that if a differentiable function , f, is orthogonal to $cos(t)$ on $L^2[0, \pi]$ then f' is orthogonal to $sin(t)$ on $L^2[0, \pi]$.

Solution: Since f is orthogonal to $cos(t)$ we know that

$$\langle f, cos(t) \rangle = \int_0^\pi f(t)cos(t)dt = 0.$$

Then,

$$\begin{aligned}
\langle f', sin(t) \rangle &= \int_0^\pi f'(t)sin(t)dt \\
&= \int_0^\pi \left(\frac{d}{dt} [f(t)sin(t)] - f(t)cos(t) \right) dt \\
&= \int_0^\pi d[f(t)sin(t)] - \int_0^\pi f(t)cos(t)dt \\
&= \cancelto{0}{f(t)sin(t)|_0^\pi} - \int_0^\pi f(t)cos(t)dt \\
&= -\int_0^\pi f(t)cos(t)dt = 0,
\end{aligned}$$

where we have used that $sin(\pi) = sin(0) = 0$ and the fact that f is orthogonal to $cos(t)$. Hence, it has been proved that f' is orthogonal to $sin(t)$ provided that f is orthogonal to $cos(t)$.

Problem 14

Find the $L^2[-\pi, \pi]$ projection of the function $f(x) = x^2$ onto the space $V_n \in L^2[-\pi, \pi]$ spanned by for n=2. Plot these projections along with f using a computer algebra system. Repeat for $g(x) = x^3$.

Solution:

Problem 23

Show that a set of orthonormal vectors is linearly independent.

Solution: