Fourier Analysis and Wavelets
Homework 2

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Problem 10

Consider the function $f(x) = \pi - x$, $0 \leq x \leq \pi$.

(a) Sketch the even, 2π-periodic extension of f. Find the Fourier cosine series for f.

Solution: In the following figure we can see the even extension of f.

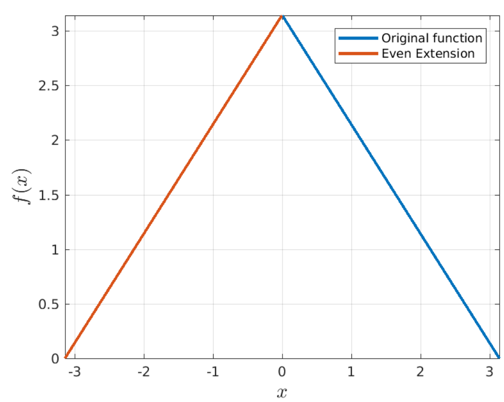


Figure 1: Even extension of the function f.

Since the extended function is even, the Fourier series is going to be only a series of cosines. We now calculate the coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)dt = \frac{1}{\pi}\left(\pi^2 - \frac{1}{2}\pi^2\right) = \frac{\pi}{2},$$

and

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt &= \frac{2}{\pi}\int_0^{\pi} (\pi - t)\cos(nt)dt \\
&= 2\cancelto{0}{\int_0^{\pi} \cos(nt)dt} - \frac{2}{\pi}\int_0^{\pi} t\cos(nt)dt \\
&= \frac{2}{n\pi}\int_0^{\pi} \sin(nt)dt \\
&= -\frac{2}{n^2\pi}\cos(nt)\Big]_0^{\pi},
\end{aligned}
$$

where we have used intergration by parts and that $\sin(n\pi) = \sin(0) = 0$. Thus,

$$
a_n = \frac{2}{n^2\pi}\left[1 - \cos(n\pi)\right] = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n^2\pi} & n \text{ odd} \end{cases}
$$

Finally, the Fourier cosine series for f is

$$
FS(f) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k+1)^2}\cos(kx)
$$

(b) Sketch the odd, 2π-periodic extension of f. Find the Fourier sine series for f.

Solution: In the following figure we can see the odd extension of f.

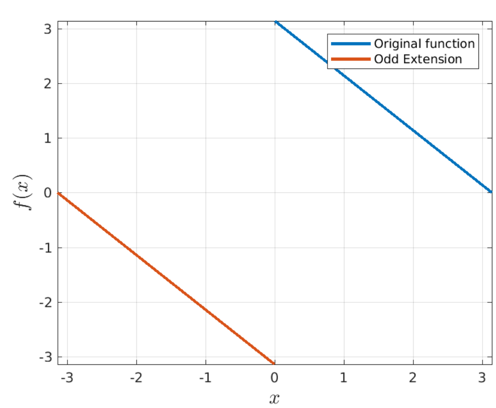


Figure 2: Odd extension of the function f.

Since the extended function is odd, the Fourier series is going to be only a series of sines. We now calculate the coefficients:

$$
\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt &= \frac{2}{\pi}\int_0^{\pi}(\pi - t)\sin(nt)dt \\
&= 2\int_0^{\pi}\sin(nt)dt - \frac{2}{\pi}\int_0^{\pi} t\sin(nt)dt \\
&= -\frac{2}{n}\cos(nt)]_0^{\pi} + \frac{2}{n\pi}\, t\cos(nt)]_0^{\pi} - \cancelto{0}{\frac{2}{n\pi}\int_0^{\pi}\cos(nt)dt} \\
&= \frac{2}{n}\left[1 - \cos(n\pi)\right] + \frac{2}{n\pi}\left[\pi\cos(n\pi)\right] \\
&= \frac{2}{n}
\end{aligned}
$$

Finally, the Fourier sine series for f is

$$
FS(f) = \sum_{n=1}^{\infty}\frac{2}{n}\sin(nx)
$$

(c) Sketch the π-periodic extension of f. Find the Fourier series for f.

Solution: In the following figure we can see the odd extension of f.

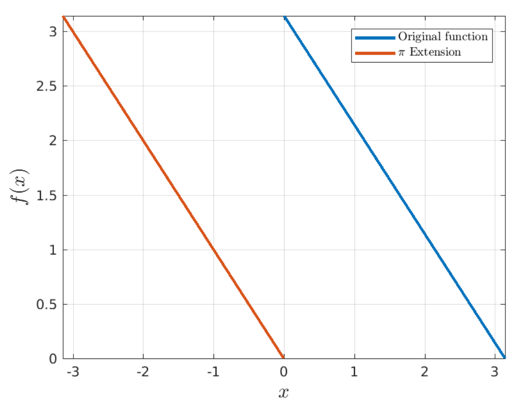


Figure 3: Extension of the function f.

We first calculate the cosine coefficients starting with

$$
a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi}\int_{-\pi}^{0} -t dt + \frac{1}{2\pi}\int_0^{\pi}(\pi - t)dt = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.
$$

For the rest of the cosine coefficients we note that we can apply the change of variables $y = -t$ the following way,

$$
\begin{aligned}
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt &= \frac{1}{\pi}\int_{-\pi}^{0} -t\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\cos(nt)dt \\
&= \frac{1}{\pi}\int_{\pi}^{0} -(-y)\cos(-ny)(-dy) + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\cos(nt)dt \\
&= -\frac{1}{\pi}\int_{\pi}^{0} y\cos(ny)dy + \int_{0}^{\pi}\cos(nt)dt - \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} y\cos(ny)dy + \int_{0}^{\pi}\cos(nt)dt - \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt \\
&= \int_{0}^{\pi}\cos(nt)dt \\
&= 0
\end{aligned}
$$

where we have saved doing two of the integrals. Further, using the same change of variables,

$$
\begin{aligned}
b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt &= \frac{1}{\pi}\int_{-\pi}^{0} -t\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\sin(nt)dt \\
&= \frac{1}{\pi}\int_{\pi}^{0} -(-y)\sin(-ny)(-dy) + \frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\sin(nt)dt \\
&= \frac{1}{\pi}\int_{\pi}^{0} y\sin(ny)dy + \int_{0}^{\pi}\sin(nt)dt - \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt \\
&= -\frac{1}{\pi}\int_{0}^{\pi} y\sin(ny)dy + \int_{0}^{\pi}\sin(nt)dt - \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt \\
&= -\frac{2}{\pi}\int_{0}^{\pi} y\sin(ny)dy + \int_{0}^{\pi}\cos(nt)dt \\
&= \frac{2}{n\pi}\left[y\cos(ny)\right]_0^{\pi} - \cancelto{0}{\frac{2}{n\pi}\int_{0}^{\pi}\cos(ny)dy} + \int_{0}^{\pi}\sin(nt)dt \\
&= \frac{2}{n}\cos(n\pi) - \frac{1}{n}\left(\cos(n\pi) - 1\right) \\
&= \frac{1 + \cos(n\pi)}{n}\begin{cases} \frac{2}{n} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}
$$

Finally, the Fourier series for f is

$$
FS(f) = \frac{\pi}{2} + \sum_{k=1}^{\infty}\frac{1}{k}\sin(kx)
$$

Problem 18

Let f and g be 2π-periodic, piecewise smooth functions having Fourier series $f(x) = \sum_n \alpha_n e^{inx}$ and $g(x) = \sum_n \beta_n e^{inx}$, and define the convolution of f and g to be $f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)dt$. Show that the complex form of the Fourier series for $f * g$ is

$$f * g(x) = \sum_{n=-\infty}^{\infty} \alpha_n \beta_n e^{inx} \ .$$

Solution: To prove the result is rather simple, it suffices with plugging in the Fourier series of the different functions in the convolution definition and operate:

$$\begin{aligned} f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(x-t)dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_n \alpha_n e^{int} \sum_m \beta_m e^{im(x-t)} dt \\ &= \frac{1}{2\pi}\sum_n \sum_m \alpha_n \beta_m e^{imx} \int_{-\pi}^{\pi} e^{int} e^{-imt} dt \\ &= \frac{1}{2\pi}\sum_n \sum_m \alpha_n \beta_m e^{imx} \int_{-\pi}^{\pi} e^{i(n-m)t} dt. \end{aligned}$$

We know that the set of complex exponentials is orthogonal. Hence, the integral is zero if $n \neq m$ and 2π if $n = m$. Therefore,

$$\begin{aligned} f * g(x) &= \frac{1}{2\pi}\sum_n \sum_m \alpha_n \beta_m e^{imx} \int_{-\pi}^{\pi} e^{i(n-m)t} dt \\ &= \sum_n \alpha_n \beta_n e^{inx}, \end{aligned}$$

and the result is proved.

Problem 23

Sketch two periods of the pointwise limit of the Fourier series for each of the following functions. State wether or not each function's Fourier series converges uniformly.

(e) $f(x) = \cos(x) + |\cos(x)|$, $-\pi \leq x \leq \pi$.

Solution: As we can see in the next figure, the function f is continuous, piecewise smooth and 2π-periodic. Hence, by *Theorem .30*, the Fourier Series for this function converges uniformly to it.

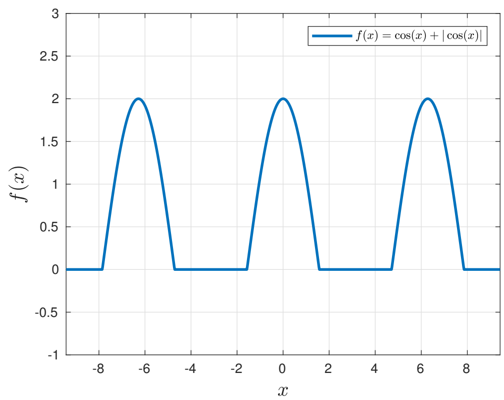


Figure 4: Graph of the given functon.

Problem 34

Consider the function

$$f(x) = e^{-x^2/10} \left(\cos 2x + 2 \sin 4x + 0.4 \cos 2x \cos 40x \right) .$$

For what values of n would you expect the Fourier coefficients $a(n)$ and $b(n)$ to be significant. Why? Compute the $a(n)$ and $b(n)$ through $n = 50$ and see if you are right. Plot the partial Fourier series through $n = 6$ and compare with the plot of the original $f(x0$.

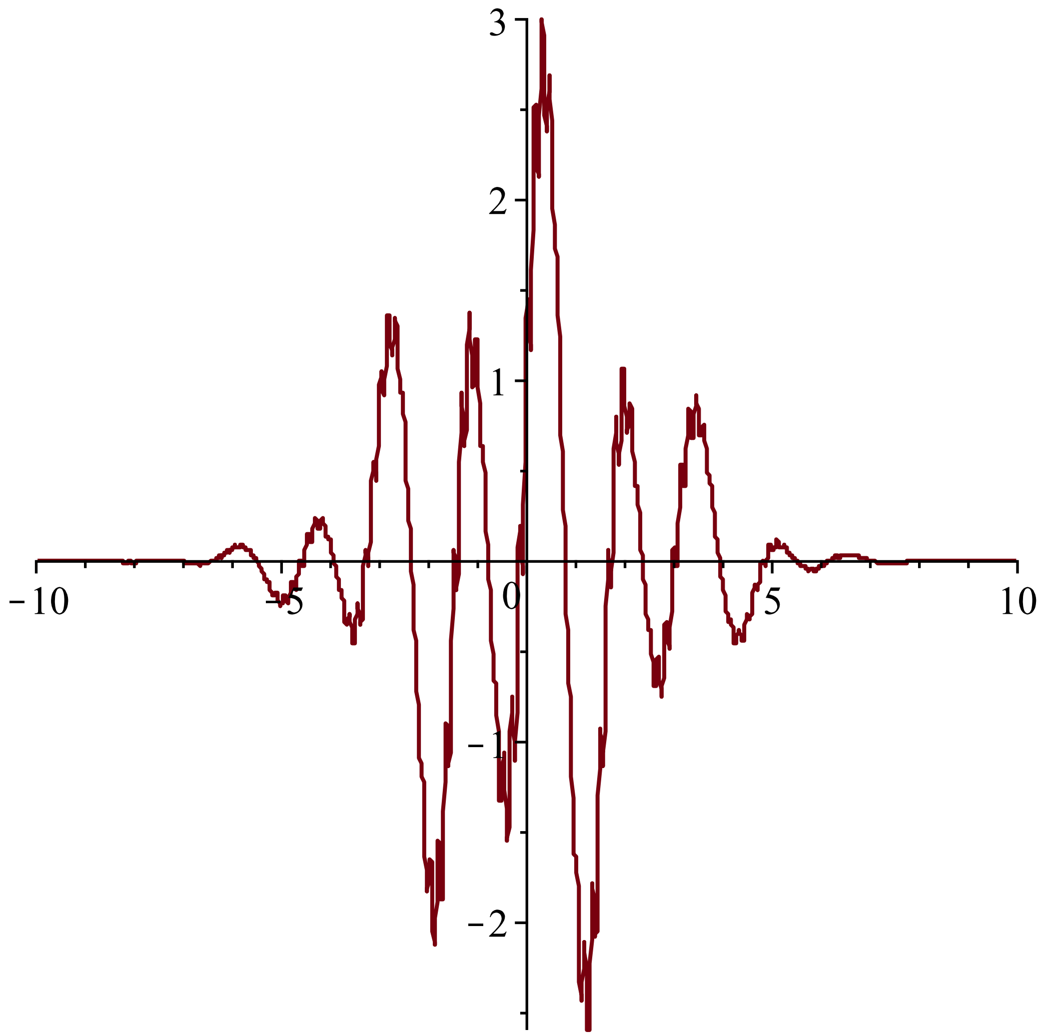
Solution: Given the function f I expect the coefficients to be significant for values of n closer to 2 and 4. Maybe a bit significant closer to 40 but not as much since I believe that is a noise term. I expect this because those are the frequencies of the cosines that appear in the definition of f. In the following page the maple code can be found with the 50 first coefficients a_n and b_n. I think I was right, since the major values of the coefficients appear to be when $n = 2, 4$. In addition they do have another peak close to 40, not as high and accurate (since it is a multiplication of two cosines). In the plot we can appreciate how six terms in the series apporximate the function f well enough as long as $x \in [-4, 4]$.

$restart;$

$f := x \to (\cos(2 \cdot x) + 2 \cdot \sin(4 \cdot x) + 0.4 \cdot \cos(2 \cdot x) \cdot \cos(40 \cdot x)) \cdot \exp\left(-\frac{x^2}{10}\right)$

$$x \to (\cos(2\,x) + 2\,\sin(4\,x) + 0.4\,\cos(2\,x)\,\cos(40\,x))\,\mathrm{e}^{-\frac{1}{10}\,x^2} \quad \textbf{(1)}$$

$plot(f, -10..10, numpoints = 2000)$



$a0 := value(\ (1/(2\cdot \mathrm{Pi}))*Int(f(x), x=-\mathrm{Pi}..\mathrm{Pi})\ );$ # *the value of a_0*

$$-0.01939716410 - 7.321127380\ 10^{-15}\ \mathrm{I} \quad \textbf{(2)}$$

$a := n \to value(\ (1/(\mathrm{Pi}))*Int(f(x)*\cos(n*x), x=-\mathrm{Pi}..\mathrm{Pi})\ );$ # *the value of a_n*

$$n \to value\left(\frac{\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,\mathrm{d}x}{\pi}\right) \quad \textbf{(3)}$$

$b := n \to value(\ (1/(\mathrm{Pi}))*Int(f(x)*\sin(n*x), x=-\mathrm{Pi}..\mathrm{Pi})\ );$ # *the value of b_n*

$$n \rightarrow value\left(\frac{\int_{-\pi}^{\pi} f(x)\,\sin(n\,x)\,\mathrm{d}x}{\pi}\right) \tag{4}$$

abs($a0$)

$$0.01939716410 \tag{5}$$

for j **from** 1 **by** 1 **to** 50 **do** *print*([j], abs($a(j)$)) **end do**

[1], 0.1476872214

[2], 0.7445477843

[3], 0.1422293659

[4], 0.02149944693

[5], 0.01003136555

[6], 0.005926364799

[7], 0.003969783887

[8], 0.002871757925

[9], 0.002188597981

[10], 0.001732390166

[11], 0.001411692307

[12], 0.001177360422

[13], 0.001000940386

[14], 0.0008649783387

[15], 0.0007582674793

[16], 0.0006733448947

[17], 0.0006050957368

[18], 0.0005499380238

[19], 0.0005053288305

[20], 0.0004694573254

[21], 0.0004410519426

[22], 0.0004192615147

[23], 0.0004035885961

[24], 0.0003938648562

[25], 0.0003902675324

[26], 0.0003933856890

[27], 0.0004043593937

[28], 0.0004251411185

[29], 0.0004589829907

[30], 0.0005113772273

[31], 0.0005919861954

[32], 0.0007189579587

[33], 0.0009297561837

[34], 0.001313016999

[35], 0.002126626146

[36], 0.004413440527

[37], 0.02855314628

[38], 0.1488080948

[39], 0.02963351236

[40], 0.007849913064

[41], 0.02962380552

[42], 0.1488275819

[43], 0.02852373085

[44], 0.004373870827

[45], 0.002076594105

[46], 0.001252126110

[47], 0.0008575132591

[48], 0.0006347625537

[49], 0.0004951172555

[50], 0.0004009763730 **(6)**

for j **from** 1 **by** 1 **to** 50 **do** $print([j], \mathrm{abs}(b(j)))$ **end do**

[1], 0.01091687016

[2], 0.03459209603

[3], 0.2753152502

[4], 1.500943508

[5], 0.2765252713

[6], 0.03726288640

[7], 0.01569455126

[8], 0.008398605247

[9], 0.005128640916

[10], 0.003403251820

[11], 0.002392010828

[12], 0.001754049513

[13], 0.001328960626

[14], 0.001033460984

[15], 0.0008209713323

[16], 0.0006638742324

[17], 0.0005450045873

[18], 0.0004532703805

$[19], 0.0003812656244$

$[20], 0.0003239054464$

$[21], 0.0002776133758$

$[22], 0.0002398203080$

$[23], 0.0002086461556$

$[24], 0.0001826921482$

$[25], 0.0001609021128$

$[26], 0.0001424678719$

$[27], 0.0001267635071$

$[28], 0.0001132989032$

$[29], 0.0001016864089$

$[30], 0.00009161657464$

$[31], 0.00008284026687$

$[32], 0.00007515532835$

$[33], 0.00006839651869$

$[34], 0.00006242785255$

$[35], 0.00005713670894$

$[36], 0.00005242926239$

$[37], 0.00004822691080$

$[38], 0.00004446346069$

$[39], 0.00004108289374$

$[40], 0.00003803758254$

$[41], 0.00003528685600$

$[42], 0.00003279583906$

$[43], 0.00003053450916$

$[44], 0.00002847692439$

$[45], 0.00002660058933$

$[46], 0.00002488593129$

$[47], 0.00002331586568$

$[48], 0.00002187543397$

$[49], 0.00002055150067$

$[50], 0.00001933249879$ **(7)**

$N := 6:$

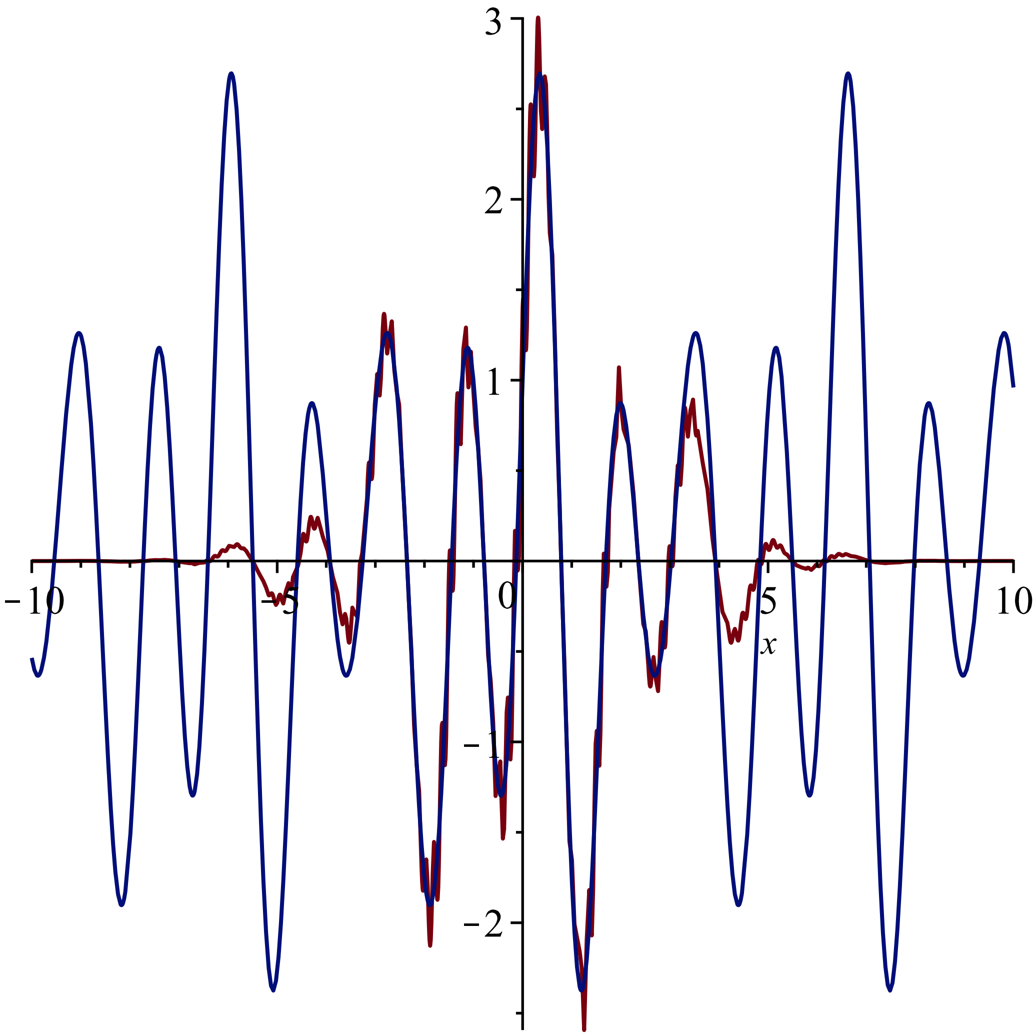
$a0 + Sum(a(n)\cdot\cos(n\cdot x), n = 1..N) + Sum(b(n)\cdot\sin(n\cdot x), n = 1..N);$

$$-0.01939716410 - 7.321127380\ 10^{-15}\,\mathrm{I} + \sum_{n=1}^{6} 0.08920620578\,\mathrm{e}^{-2.500000000\,n^2 - 210.\,n}\,\big(\mathrm{e}^{420.\,n - 4410.}\,\mathrm{erf}(0.9934588266 + 66.40783086\,\mathrm{I} - 1.581138830\,\mathrm{I}\,n) + \mathrm{e}^{-3610. + 20.\,n}\,\mathrm{erf}(0.9934588266 + 60.08327554\,\mathrm{I} + 1.581138830\,\mathrm{I}\,n) \tag{8}$$

$$+ e^{400.n - 3610.} \operatorname{erf}(0.9934588266 + 60.08327554\,I - 1.581138830\,I\,n)$$
$$+ 5.\,e^{-10. + 200.n} \operatorname{erf}(0.9934588266 + 3.162277660\,I + 1.581138830\,I\,n)$$
$$+ 5.\,e^{220.n - 10.} \operatorname{erf}(0.9934588266 + 3.162277660\,I - 1.581138830\,I\,n)$$
$$+ 5.\,e^{220.n - 10.} \operatorname{erf}(0.9934588266 - 3.162277660\,I + 1.581138830\,I\,n)$$
$$+ 5.\,e^{-10. + 200.n} \operatorname{erf}(0.9934588266 - 3.162277660\,I - 1.581138830\,I\,n)$$
$$+ e^{400.n - 3610.} \operatorname{erf}(0.9934588266 - 60.08327554\,I + 1.581138830\,I\,n)$$
$$+ e^{-3610. + 20.n} \operatorname{erf}(0.9934588266 - 60.08327554\,I - 1.581138830\,I\,n)$$
$$+ e^{420.n - 4410.} \operatorname{erf}(0.9934588266 - 66.40783086\,I + 1.581138830\,I\,n)$$
$$+ 5.772112761\;10^{-1916} \operatorname{erf}(0.9934588266 + 66.40783086\,I + 1.581138830\,I\,n)$$
$$+ 5.772112761\;10^{-1916} \operatorname{erf}(0.9934588266 - 66.40783086\,I - 1.581138830\,I\,n)\Big) \cos(n\,x)$$
$$+ \sum_{n=1}^{6} \Big(-0.8920620578\, e^{-2.500000000\, n^2 - 20.\,n - 40.} \big(-1.\, e^{40.\,n} \operatorname{erf}(0.9934588266$$
$$+ 6.324555320\,I - 1.581138830\,I\,n) - 1.\, e^{40.\,n} \operatorname{erf}(0.9934588266 - 6.324555320\,I$$
$$+ 1.581138830\,I\,n) + \operatorname{erf}(0.9934588266 + 6.324555320\,I + 1.581138830\,I\,n)$$
$$+ \operatorname{erf}(0.9934588266 - 6.324555320\,I - 1.581138830\,I\,n)\big) \sin(n\,x)\Big)$$

$S := value(\%)$:

$plot(\{f(x), S\}, x = -10\,..10)$



3
2
1
−1
−2
−10
−5
0
5
10
x

Problem 35

Consider the function

$$g(x) = e^{-x^2/8}\left(\cos 2x + 2\sin 4x + 0.4\cos 2x\cos 10x\right) \ .$$

Compute the partial Fourier series through $N = 25$. Throw away any coefficients that are smaller than 0.01 in absolute value. Plot the resulting series and compare with the original function $g(x)$. Try experimenting with different tolerances.

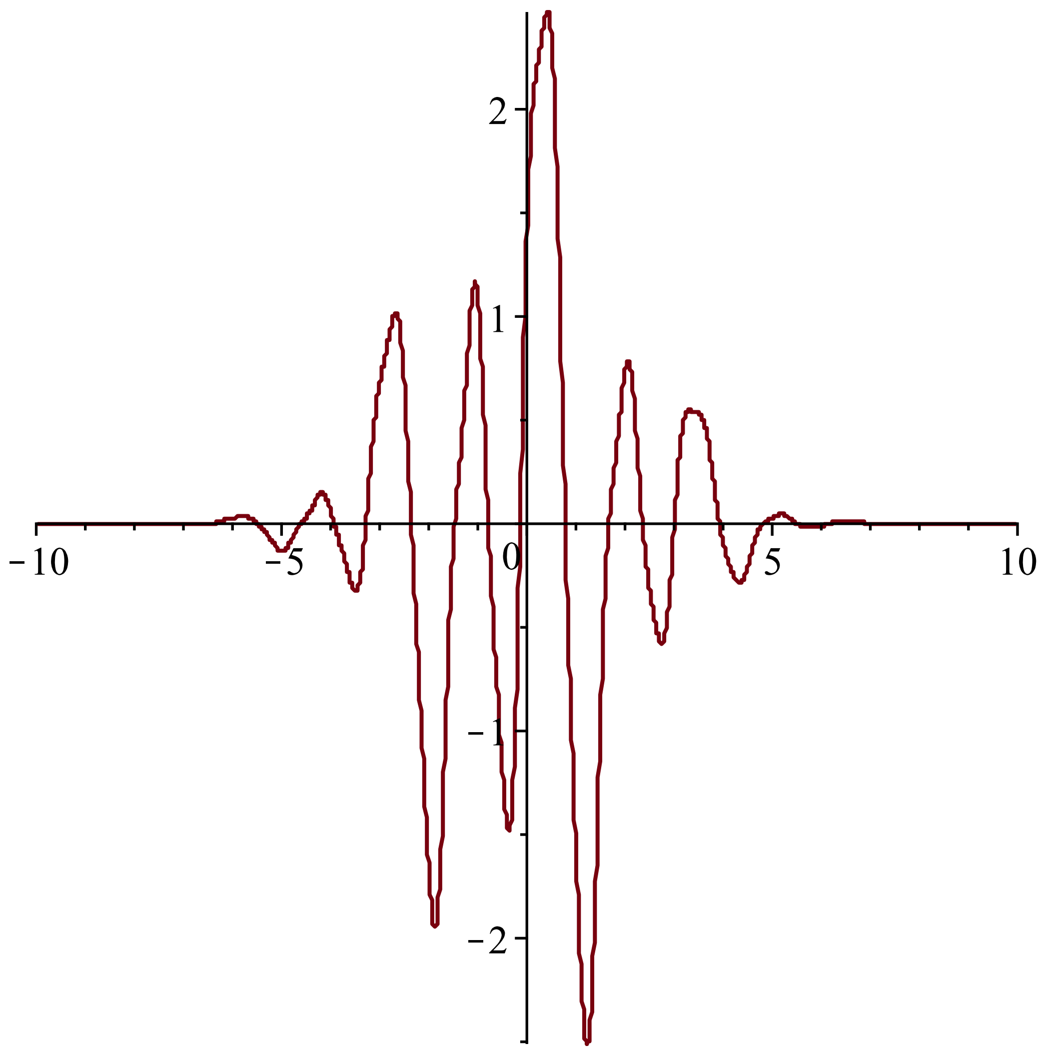
Solution: In the maple code I have computed the first 25 coefficients of the series for this function. First, I have taken out those that were not larger than 0.01 obtaining the very good approximation shown in the figure using 19 terms (above the threshold). I have also done another try using a threshold of 0.3 which led me to use only 4 terms obtaining the approximation shown in the last figure. It is obviously a worse approximation but it is surprisingly good considering that only 4 terms were used compared to the 19 in the previous one.

restart;

$f := x \rightarrow (\cos(2 \cdot x) + 2 \cdot \sin(4 \cdot x) + 0.4 \cdot \cos(2 \cdot x) \cdot \cos(10 \cdot x)) \cdot \exp\left(-\frac{x^2}{8}\right)$

$$x \rightarrow (\cos(2\,x) + 2\,\sin(4\,x) + 0.4\,\cos(2\,x)\,\cos(10\,x))\,e^{-\frac{1}{8}x^2} \quad \textbf{(1)}$$

plot(*f*, −10 ..10, *numpoints* = 2000)



$a0 := value(\ (1/(2\cdot \mathrm{Pi})) * Int(f(x), x = -\mathrm{Pi}..\mathrm{Pi})\ )$; # *the value of a_0*

$$-0.01831626531 - 3.819718633\ 10^{-15}\ \mathrm{I} \quad \textbf{(2)}$$

$a := n \rightarrow value(\ (1/(\mathrm{Pi})) * Int(f(x) * \cos(n * x), x = -\mathrm{Pi}..\mathrm{Pi})\ )$; # *the value of a_n*

$$n \rightarrow value\left(\frac{\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,\mathrm{d}x}{\pi}\right) \quad \textbf{(3)}$$

$b := n \rightarrow value(\ (1/(\mathrm{Pi})) * Int(f(x) * \sin(n * x), x = -\mathrm{Pi}..\mathrm{Pi})\ )$; # *the value of b_n*

$$n \rightarrow value\left(\frac{\int_{-\pi}^{\pi} f(x)\,\sin(n\,x)\,\mathrm{d}x}{\pi}\right) \tag{4}$$

abs($a0$)

$$0.01831626531 \tag{5}$$

for j **from** 1 **by** 1 **to** 25 **do** *print*([j], abs($a(j)$)) **end do**

[1], 0.1689352965

[2], 0.6997913956

[3], 0.1639504845

[4], 0.02131923542

[5], 0.01173682840

[6], 0.009845704196

[7], 0.03653179993

[8], 0.1372609244

[9], 0.03582556008

[10], 0.008915777563

[11], 0.03504938027

[12], 0.1389502637

[13], 0.03358240613

[14], 0.004845510193

[15], 0.002664029127

[16], 0.001784969863

[17], 0.001331466062

[18], 0.001056006945

[19], 0.0008706973172

[20], 0.0007372293492

[21], 0.0006363674129

[22], 0.0005574061177

[23], 0.0004939059027

[24], 0.0004417517322

[25], 0.0003981843462 **(6)**

for j **from** 1 **by** 1 **to** 25 **do** *print*([j], abs($b(j)$)) **end do**

[1], 0.01050139527

[2], 0.03191534929

[3], 0.3171675791

[4], 1.412572635

[5], 0.3183464179

$$[6], 0.03451620402$$
$$[7], 0.01514965726$$
$$[8], 0.008154429497$$
$$[9], 0.004990677877$$
$$[10], 0.003315434140$$
$$[11], 0.002331820945$$
$$[12], 0.001710634903$$
$$[13], 0.001296442641$$
$$[14], 0.001008383427$$
$$[15], 0.0008011744704$$
$$[16], 0.0006479430091$$
$$[17], 0.0005319759580$$
$$[18], 0.0004424681294$$
$$[19], 0.0003722023342$$
$$[20], 0.0003162218538$$
$$[21], 0.0002710395211$$
$$[22], 0.0002341498905$$
$$[23], 0.0002037191620$$
$$[24], 0.0001783828166$$
$$[25], 0.0001571104239 \tag{7}$$

 # With tolerance of 0.01 we obtain

$a0 + Sum(a(n)\cdot\cos(n\cdot x), n=1..5) + Sum(a(n)\cdot\cos(n\cdot x), n=7..9) + Sum(a(n)\cdot\cos(n\cdot x), n=11..13) + Sum(b(n)\cdot\sin(n\cdot x), n=1..7);$

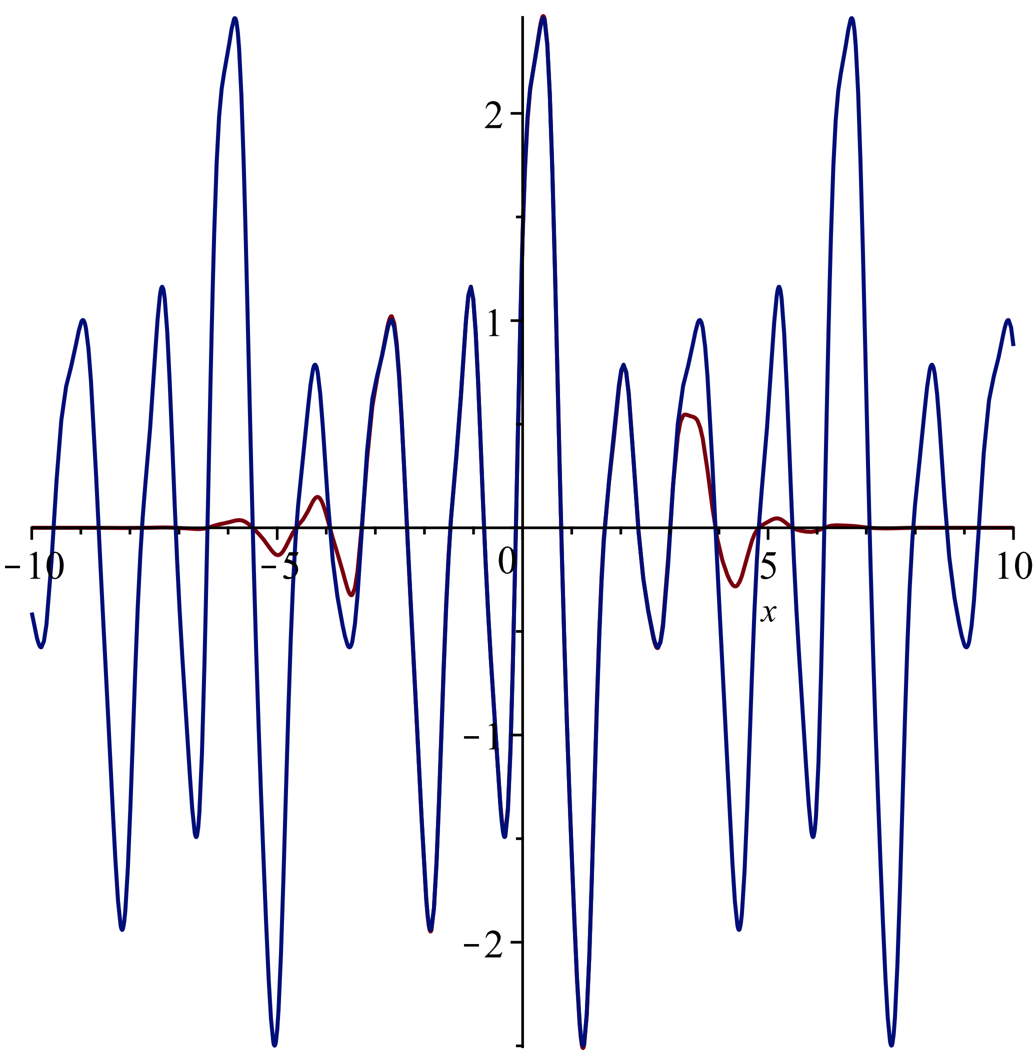
$$-0.01831626531 - 3.819718633\ 10^{-15}\,\mathrm{I} + \sum_{n=1}^{5} \tag{8}$$

$$0.07978845607 \big(8.378942534\ 10^{-126}\,\mathrm{erf}(1.110720735 + 16.97056275\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)$$
$$+ 8.378942534\ 10^{-126}\,\mathrm{erf}(1.110720735 - 16.97056275\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n)$$
$$+ \mathrm{e}^{96.\,n - 288.}\,\mathrm{erf}(1.110720735 + 16.97056275\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n)$$
$$+ \mathrm{e}^{-128. + 16.\,n}\,\mathrm{erf}(1.110720735 + 11.31370850\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)$$
$$+ \mathrm{e}^{80.\,n - 128.}\,\mathrm{erf}(1.110720735 + 11.31370850\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n)$$
$$+ 5.\,\mathrm{e}^{-8. + 40.\,n}\,\mathrm{erf}(1.110720735 + 2.828427125\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)$$
$$+ 5.\,\mathrm{e}^{56.\,n - 8.}\,\mathrm{erf}(1.110720735 + 2.828427125\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n)$$
$$+ 5.\,\mathrm{e}^{56.\,n - 8.}\,\mathrm{erf}(1.110720735 - 2.828427125\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)$$
$$+ 5.\,\mathrm{e}^{-8. + 40.\,n}\,\mathrm{erf}(1.110720735 - 2.828427125\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n)$$
$$+ \mathrm{e}^{80.\,n - 128.}\,\mathrm{erf}(1.110720735 - 11.31370850\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)$$
$$+ \mathrm{e}^{-128. + 16.\,n}\,\mathrm{erf}(1.110720735 - 11.31370850\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n)$$
$$+ \mathrm{e}^{96.\,n - 288.}\,\mathrm{erf}(1.110720735 - 16.97056275\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)\big)\,\mathrm{e}^{-2.\,n^2 - 48.\,n}\cos(n\,x)$$

$$+\sum_{n=7}^{9}0.07978845607\,\big(8.378942534\,10^{-126}\,\mathrm{erf}(1.110720735+16.97056275\,\mathrm{I}$$
$$+1.414213562\,\mathrm{I}\,n)+8.378942534\,10^{-126}\,\mathrm{erf}(1.110720735-16.97056275\,\mathrm{I}$$
$$-1.414213562\,\mathrm{I}\,n)+\mathrm{e}^{96.\,n-288.}\,\mathrm{erf}(1.110720735+16.97056275\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{-128.+16.\,n}\,\mathrm{erf}(1.110720735+11.31370850\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{80.\,n-128.}\,\mathrm{erf}(1.110720735+11.31370850\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{-8.+40.\,n}\,\mathrm{erf}(1.110720735+2.828427125\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{56.\,n-8.}\,\mathrm{erf}(1.110720735+2.828427125\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{56.\,n-8.}\,\mathrm{erf}(1.110720735-2.828427125\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{-8.+40.\,n}\,\mathrm{erf}(1.110720735-2.828427125\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{80.\,n-128.}\,\mathrm{erf}(1.110720735-11.31370850\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{-128.+16.\,n}\,\mathrm{erf}(1.110720735-11.31370850\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{96.\,n-288.}\,\mathrm{erf}(1.110720735-16.97056275\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)\big)\,\mathrm{e}^{-2.\,n^2-48.\,n}\cos(n\,x)$$
$$+\sum_{n=11}^{13}0.07978845607\,\big(8.378942534\,10^{-126}\,\mathrm{erf}(1.110720735+16.97056275\,\mathrm{I}$$
$$+1.414213562\,\mathrm{I}\,n)+8.378942534\,10^{-126}\,\mathrm{erf}(1.110720735-16.97056275\,\mathrm{I}$$
$$-1.414213562\,\mathrm{I}\,n)+\mathrm{e}^{96.\,n-288.}\,\mathrm{erf}(1.110720735+16.97056275\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{-128.+16.\,n}\,\mathrm{erf}(1.110720735+11.31370850\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{80.\,n-128.}\,\mathrm{erf}(1.110720735+11.31370850\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{-8.+40.\,n}\,\mathrm{erf}(1.110720735+2.828427125\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{56.\,n-8.}\,\mathrm{erf}(1.110720735+2.828427125\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{56.\,n-8.}\,\mathrm{erf}(1.110720735-2.828427125\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+5.\,\mathrm{e}^{-8.+40.\,n}\,\mathrm{erf}(1.110720735-2.828427125\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{80.\,n-128.}\,\mathrm{erf}(1.110720735-11.31370850\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{-128.+16.\,n}\,\mathrm{erf}(1.110720735-11.31370850\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{e}^{96.\,n-288.}\,\mathrm{erf}(1.110720735-16.97056275\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)\big)\,\mathrm{e}^{-2.\,n^2-48.\,n}\cos(n\,x)$$
$$+\sum_{n=1}^{7}\Big(-0.7978845607\,\mathrm{e}^{-2.\,n^2-16.\,n-32.}\,\big(-1.\,\mathrm{e}^{32.\,n}\,\mathrm{erf}(1.110720735-5.656854249\,\mathrm{I}$$
$$+1.414213562\,\mathrm{I}\,n)-1.\,\mathrm{e}^{32.\,n}\,\mathrm{erf}(1.110720735+5.656854249\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)$$
$$+\mathrm{erf}(1.110720735-5.656854249\,\mathrm{I}-1.414213562\,\mathrm{I}\,n)+\mathrm{erf}(1.110720735$$
$$+5.656854249\,\mathrm{I}+1.414213562\,\mathrm{I}\,n)\big)\sin(n\,x)\Big)$$

$S := value(\%)$:

$plot(\{f(x), S\}, x=-10..10)$



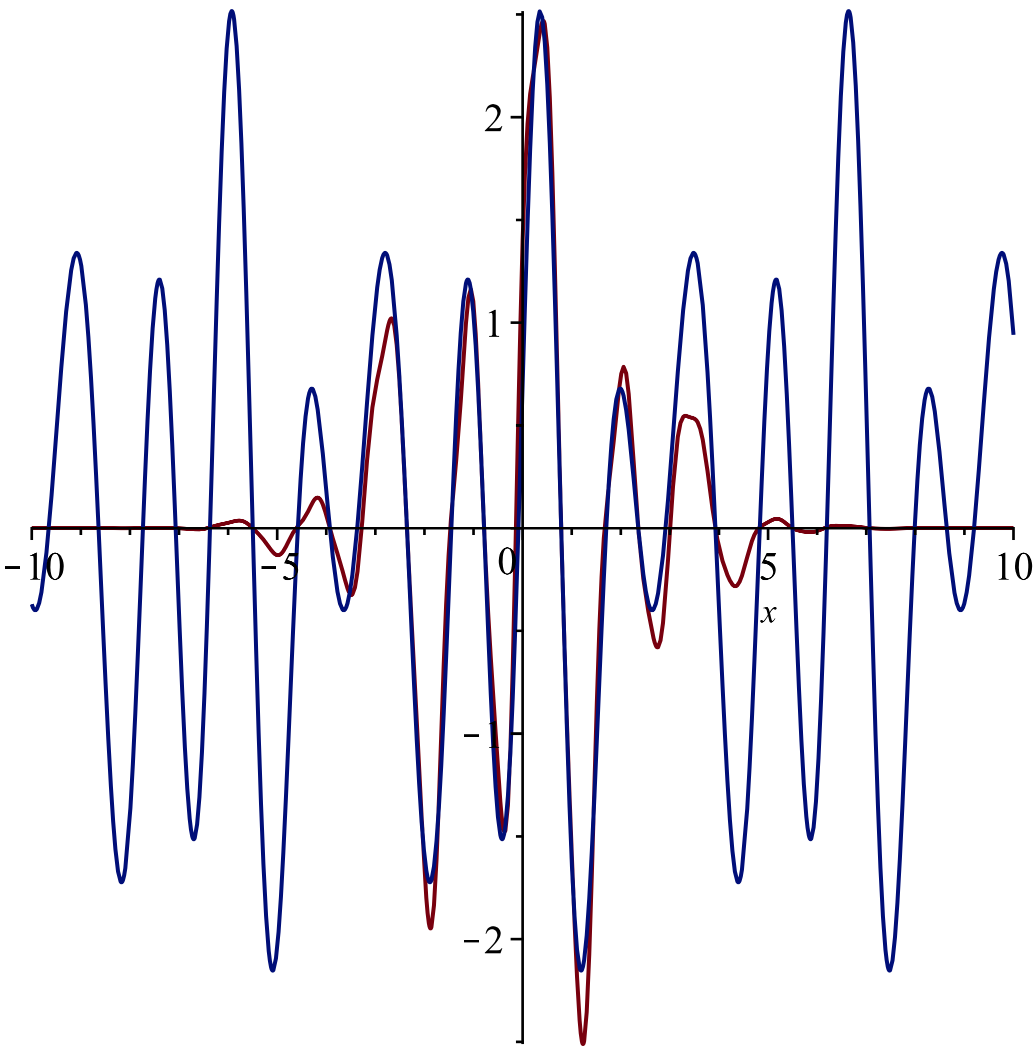
 # With tolerance of 0.3 we obtain

$a(2)\cdot\cos(2\cdot x) + Sum(b(n)\cdot\sin(n\cdot x), n = 3..5);$

$$(0.6997913956 + 0.\,\mathrm{I})\cos(2x) + \sum_{n=3}^{5}\Big(-0.7978845607\,\mathrm{e}^{-2.\,n^2 - 16.\,n - 32.}\big(-1.\,\mathrm{e}^{32.\,n}\,\mathrm{erf}(1.110720735 - 5.656854249\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n) - 1.\,\mathrm{e}^{32.\,n}\,\mathrm{erf}(1.110720735 + 5.656854249\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n) + \mathrm{erf}(1.110720735 - 5.656854249\,\mathrm{I} - 1.414213562\,\mathrm{I}\,n) + \mathrm{erf}(1.110720735 + 5.656854249\,\mathrm{I} + 1.414213562\,\mathrm{I}\,n)\big)\sin(n\,x)\Big) \tag{9}$$

$S := value(\%)$:

$plot(\{f(x), S\}, x = -10..10)$



2
1
−1
−2
−10
−5
0
5
10
x