Partial Differential Equations
Instructor Homework 4

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Problem 6.1.3

1. Solve the Laplace equation with mixed boundary conditions,

$$\begin{aligned} &(\partial_x^2 + \partial_y^2)u(x,y) = 0, \qquad && x \in [0,L],\ y \in [0,H], \\ &u(0,y) = g(y), \quad u(L,y) = 0, && y \in [0,H], \\ &\partial_y u(x,0) = 0 = \partial_y u(x,H), && x \in [0,L]. \end{aligned}$$

   Make an educated guess which conditions g must satisfy for u to be a solution. Explain why you chose these conditions. Is u unique?

Solution: Since we have zero Neumann boundary conditions, we express the solution u as a Fourier cosine series,

$$u(x,y) = A_0(x) + \sum_{m=1}^{\infty} A_m(x)\cos(\lambda_m y), \qquad \lambda_m = m\frac{\pi}{H},$$

where

$$A_m(y) = \frac{2}{H}\int_0^H u(x,y)\cos(\lambda_m y)dy, \qquad m > 1,$$

and

$$A_0(y) = \frac{1}{H}\int_0^H u(x,y)dy.$$

Differentiating twice,

$$\begin{aligned} A_m''(x) &= \frac{2}{H}\int_0^H \partial_x^2 u(x,y)\cos(\lambda_m y)dy \\ &= -\frac{2}{H}\int_0^H \partial_y^2 u(x,y)\cos(\lambda_m y)dy \\ &= \frac{2}{H}\lambda_m^2 \int_0^H u(x,y)\cos(\lambda_m y)dy \\ &= \lambda_m^2 A_m(x), \end{aligned}$$

where we have used the PDE for the first step, integrated by parts using the Neumann boundary conditions and that $\sin(\lambda_m H) = 0$ in the second step and the definition of A_m in the last. Doing the same for $A_0(x)$,

$$\begin{aligned}
A_0''(x) &= \frac{1}{H}\int_0^H \partial_x^2 u(x,y)dy \\
&= -\frac{1}{H}\int_0^H \partial_y^2 u(x,y)dy \\
&= -\frac{1}{H}\int_0^H \partial(\partial_y u(x,y)) \\
&= -\frac{1}{H}\left[\partial_y u(x,y)\right]_0^H \\
&= 0.
\end{aligned}$$

Given the ODEs, we have the following solutions

$$\begin{aligned}
A_m(x) &= k_1\cosh(\lambda_m(L-x)) + k_2\sinh(\lambda_m(L-x)), \qquad m > 1, \\
A_0(x) &= k_3(L-x) + k_4.
\end{aligned}$$

To obtain the values of the constants we use the boundary conditions,

$$A_m(L) = \frac{2}{H}\int_0^H u(L,y)\cos(\lambda_m y)dy = 0 = k_1\cosh(0) + k_2\sinh(0) = k_1 \Rightarrow \quad \boxed{k_1 = 0}.$$

Then,

$$u(L,y) = 0 = A_0(L) + \sum_{m=1}^{\infty} A_m(L)\cos(\lambda_m y) = A_0(L) = k_4 \Rightarrow \quad \boxed{k_4 = 0}.$$

Further,

$$\begin{aligned}
A_m(0) = \frac{2}{H}\int_0^H g(y)\cos(\lambda_m y)dy = k_2\sinh(\lambda_m L) &\Rightarrow \quad \boxed{k_2 = \frac{2}{H\sinh(\lambda_m L)}\int_0^H g(y)\cos(\lambda_m y)dy}, \\
A_0(0) = \frac{1}{H}\int_0^H g(y)dy = k_3 L &\Rightarrow \quad \boxed{k_3 = \frac{1}{HL}\int_0^H g(y)dy}.
\end{aligned}$$

Let $C_m = \frac{2}{H\sinh(\lambda_m L)}\int_0^H g(z)\cos(\lambda_m z)dz$. Now we can express u as

$$\begin{aligned}
u(x,y) &= u_0(x,y) + \sum_{m=1}^{\infty} u_m(x,y), \\
u_0(x,y) &= \frac{(L-x)}{H}\int_0^H g(z)dz, \\
u_m(x,y) &= C_m\sinh(\lambda_m(L-x))\cos(\lambda_m y)
\end{aligned}$$

Then, if $0 \le x \le L$ and $0 \le y \le H$,

$$\partial_y^l u_m(x,y) = \pm\lambda_m^l C_m\sinh(\lambda_m(L-x))\begin{cases}\sin(\lambda_m y) & l\in\mathbb{N},\ l \text{ odd}, \\ \cos(\lambda_m y) & l\in\mathbb{N},\ l \text{ even},\end{cases}$$

and

$$\partial_x^k u_m(x,y) = \pm\lambda_m^k C_m \cos(\lambda_m y) \left\{ \begin{array}{ll} \cosh(\lambda_m(L-x)) & l \in \mathbb{N},\ l \text{ odd}, \\ \sinh(\lambda_m(L-x)) & l \in \mathbb{N},\ l \text{ even}. \end{array} \right.$$

Therefore,

$$\left|\partial_x^k \partial_y^l u_m(x,y)\right| \leq \lambda_m^{k+l} |C_m| \left\{ \begin{array}{c} \cosh(\lambda_m(L-x)) \\ \sinh(\lambda_m(L-x)) \end{array} \right\} \qquad x \in [0,L]$$

Since the hyperbolic functions are increasing on $\mathbb{R}_+$,

$$\left|\partial_x^k \partial_y^l u_m(x,y)\right| \leq \lambda_m^{k+l} |C_m| \left\{ \begin{array}{c} \cosh(\lambda_m L) \\ \sinh(\lambda_m L) \end{array} \right\} \qquad x \in [0,L]$$

As it is done in the notes we know that the previous result gives us the following bound,

$$\left|\partial_x^k \partial_y^l u_m(x,y)\right| \leq c\lambda_m^{k+l} |C_m| \sinh(\lambda_m L), \qquad k,l \in \mathbb{N}, \quad x \in [0,L],$$

for some constant $c > 0$. Thus, by *Theorem 5.3* and the definition of C_m, $\sum_{m=1}^{\infty} u_m$ is twice partially differentiable and satisfies the Laplace equation by construction if

$$\sum_{m=1}^{\infty} \lambda_m^2 \left| \int_0^H g(z) \cos(\lambda_m z) dz \right| = -\sum_{m=1}^{\infty} \left| \int_0^H g(z) \frac{d}{dz^2} \cos(\lambda_m z) dz \right| < \infty.$$

If g is twice continuously differentiable and $g'(0) = 0 = g'(L)$, integrating by parts,

$$-\sum_{m=1}^{\infty} \left| \int_0^H g(z) \frac{d}{dz^2} \cos(\lambda_m z) dz \right| = -\sum_{m=1}^{\infty} \left| \int_0^H g''(z) \cos(\lambda_m z) dz \right|$$

which is finite if g'' is Lipschitz continuous and $g''(0) = 0 = g''(H)$. Since $u_0''(x,y) = 0$, it gives us no extra conditions on g. Hence, u is solution provided that g satisfies the conditions mentioned. The solution is unique since it is expressed as an unique Fourier cosine series.