APM 506 HW 5

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Problem 1

Compute the solution to the systems for the given initial conditions using **rk4.m**. Plot your results in the phase plane (that is,$y(t)$ vs. $x(t)$).

1. Using initial conditions with $x(0)^2 + y(0)^2$ both smaller and larger than 1 (inside and outside the unit circle), solve

$$x'(t) = -4y + x(1 - x^2 - y^2), \tag{1}$$
$$y'(t) = 4x + y(1 - x^2 - y^2), \tag{2}$$

over the interval $0 < t < 10$. What is the final state of the system? Justify your answer with a plot showing the trajectories in the (x, y)-plane for a few different initial conditions. Note that you have to supply the initial conditions separately.

Solution:

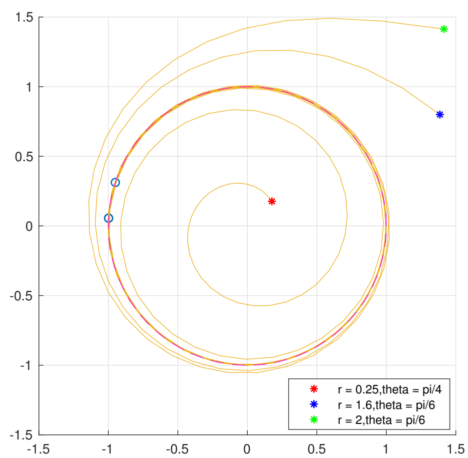

Figure 1: Maximum Error of Cubic Spline Interpolation

The solution for this problem is stable for intital conditions that fall both inside of and outside of the circle of radius 2. The solutions spiral in or out towards the circle and then remain there indefinitely.

2. Using initial conditions with $x(0)^2 + y(0)^2$ both inside and outside circles of radius 1 and 2, solve

$$x'(t) = -4y + x(1 - x^2 - y^2)(4 - x^2 - y^2), \tag{3}$$
$$y,(t) = 4x + y(1 - x^2 - y^2)(4 - x^2 - y^2), \tag{4}$$

over the interval $0 < t < 10$. What are the final states of the system? Justify your answer with a plot showing the trajectories in the (x, y)-plane for a few different initial conditions.

Solution:

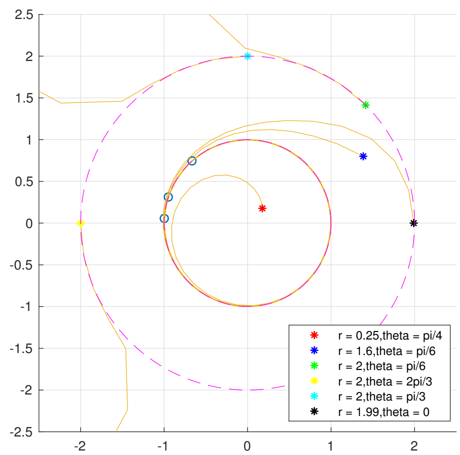

Figure 2: Maximum Error of Cubic Spline Interpolation

The solution for this problem is stable for initial conditions that fall within the circle of radius 4. If the initial condition falls on or outside of the outer circle the trajectories blow outwards. The trajectories for initial conditions inside of the outer circle spiral towards the circle of radius 2 and remain there indefinitely.

MATLAB

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- Part 2

Part 1

```
F = @(t,y) [-4*y(2) + y(1)*(1-y(1)^2-y(2)^2); ...
            4*y(1) + y(2)*(1-y(1)^2-y(2)^2)];
tspan = [0 10];
theta = pi/4;
r = 0.25;
y0 = [r*cos(theta);r*sin(theta)];
N = 200;
```

```
[t,y] = rk4(F,tspan,y0,N);

r = 1.6;
theta2 = pi/6;
y0_2 = [r*cos(theta2);r*sin(theta2)];
[t,y2] = rk4(F,tspan,y0_2,N);

r = 2;
y0_3 = [r*cos(theta);r*sin(theta)];
[t,y3] = rk4(F,tspan,y0_3,N);

figure
comet(y(:,1),y(:,2)) %phase plot
axis([-1.5 1.5 -1.5 1.5])
hold on
comet(y2(:,1),y2(:,2))
comet(y3(:,1),y3(:,2))
p1 = plot(y0(1),y0(2),'r*')
p2 = plot(y0_2(1),y0_2(2),'b*')
p3 = plot(y0_3(1),y0_3(2),'g*')
plot(cos(0:0.001:2*pi),sin(0:0.001:2*pi), '--m')
pbaspect([1 1 1])
grid on
legend([p1 p2 p3],{"r = 0.25,theta = pi/4","r = 1.6,theta = pi/6","r = 2,theta = pi/6"})
```

Part 2

```
F = @(t,y) [-4*y(2) + y(1)*(1-y(1)^2-y(2)^2)*(4-y(1)^2-y(2)^2); ...
            4*y(1) + y(2)*(1-y(1)^2-y(2)^2)*(4-y(1)^2-y(2)^2)];

tspan = [0 10];
theta = pi/4;
r = 0.25;
y0 = [r*cos(theta);r*sin(theta)];
N = 200;

[t,y] = rk4(F,tspan,y0,N);

r = 1.6;
theta2 = pi/6;
y0_2 = [r*cos(theta2);r*sin(theta2)];
[t,y2] = rk4(F,tspan,y0_2,N);

r = 2;
y0_3 = [r*cos(theta);r*sin(theta)];
[t,y3] = rk4(F,tspan,y0_3,N);
y0_4 = [r*cos(theta*4);r*sin(theta*4)];
[t,y4] = rk4(F,tspan,y0_4,N);
y0_5 = [r*cos(theta*2);r*sin(theta*2)];
[t,y5] = rk4(F,tspan,y0_5,N);
```

```
r = 1.99;
y0_6 = [r*cos(0);r*sin(0)];
[t,y6] = rk4(F,tspan,y0_6,N);

figure
comet(y(:,1),y(:,2)) %phase plot
axis([-2.5 2.5 -2.5 2.5])
hold on
comet(y2(:,1),y2(:,2))
comet(y3(:,1),y3(:,2))
comet(y4(:,1),y4(:,2))
comet(y5(:,1),y5(:,2))
comet(y6(:,1),y6(:,2))
p1 = plot(y0(1),y0(2),'r*')
p2 = plot(y0_2(1),y0_2(2),'b*')
p3 = plot(y0_3(1),y0_3(2),'g*')
p4 = plot(y0_4(1),y0_4(2),'y*')
p5 = plot(y0_5(1),y0_5(2),'c*')
p6 = plot(y0_6(1),y0_6(2),'k*')
l7 = plot(cos(0:0.001:2*pi),sin(0:0.001:2*pi), '--m')
l8 = plot(2*cos(0:0.001:2*pi),2*sin(0:0.001:2*pi), '--m')
legend([p1 p2 p3 p4 p5 p6],{"r = 0.25,theta = pi/4","r = 1.6,theta = pi/6","r = 2,theta = pi/6",...
                             "r = 2,theta = 2pi/3","r = 2,theta = pi/3","r = 1.99,theta = 0"})
pbaspect([1 1 1])
grid on
```

Problem 2

Consider the wave propogation PDE,

$$\begin{aligned} u_t &= p_x, \\ p_t &= u_x, \quad x \in (0,\pi),\ t > 0, \end{aligned}$$

with boundary conditions $u(t,0) = u(t,\pi) = 0$ and initial conditions $u(0,x) = exp(-30(x-\pi/2)^2)$ and $p(0,x) = 0$.

1. Show that the eigenvalues of the differential operator

$$\mathcal{L}\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix}$$

are integers in the imaginary axis, i.e. $0, \pm 1i, \pm 2i, \pm 3i, \ldots$

Solution:

given eigenvalues of $\mathcal{L}$, λ,

$$\mathcal{L}\begin{bmatrix} u \\ p \end{bmatrix} = \lambda \begin{bmatrix} u \\ p \end{bmatrix} \ .$$

Thus,

$$u_x = \lambda p, \quad p_x = \lambda u \quad \Rightarrow \quad u_{xx} = \lambda p_x = \lambda^2 u, \quad p_{xx} = \lambda u_x = \lambda^2 p \ .$$

Now let, $\lambda^2 = -k^2$, since $u_{xx} = -k^2 e^{ikx} = -k^2 u$ for $u = e^{ikx}$, and then

$$u = A\cos(kx) + iB\sin(kx) \ .$$

Since $u(0,t) = u(\pi,t) = 0$, $A = 0$ and $iB\sin(k\pi) = 0$. Therefor, B is either equal to 0, giving the trivial solution, or $\sin(k\pi) = 0$ which implies $k = 0, \pm 1, \pm 2, \ldots$. This gives us $\lambda = 0, \pm i, \pm 2i, \ldots$.

2. Recall the finite-difference discretization from HW3,

$$M \begin{bmatrix} u \\ p \end{bmatrix} = \frac{1}{h} \left[\begin{bmatrix} 0 & D_p \\ D_u & 0 \end{bmatrix} \right] \begin{bmatrix} u \\ p \end{bmatrix} ,$$

whose eigenvectors are

$$V_k = \begin{bmatrix} i\sin(kx_1) \\ i\sin(kx2) \\ \vdots \\ i\sin(kx_{N-1}) \\ cos(kx_{1/2}) \\ \vdots \\ cos(kx_{N-1/2}) \end{bmatrix} , \quad k = -(N-1), \ldots, -1, 0, 1, \ldots, N-1 \ .$$

Find an analytic expression for the eigenvalues of M and notice that they are also purely imaginary numbers. Use the command `eig(M)` in Matlab to double check your expression is correct.

Solution:

First solve the equation $MV = \lambda V$ by looking at a single eigenvector k. Then,

$$MV^k = \begin{cases} -V^k_{j+N-1} + V^k_{j+n} & j \le N-1 \\ V^k_1 & j = N \\ -V^k_{j-N} + V^k_{j-N+1} & N+1 \le j < 2N-1 \\ -V^k_{N-1} & j = 2N-1 \end{cases} .$$

From here we can solve for the eigenvalues for each case of j. Let $j \le N-1$. Then,

$$\begin{aligned} MV^k &= \frac{1}{h}\left[-V^k_{j+N-1} + V^k_{j+n}\right] \\ &= -\cos(kx_{j-1/2}) + \cos(kx_{j+1/2}) \\ &= -\cos\left(k\left(j - \frac{1}{2}\right)h\right) + cos\left(k\left(j + \frac{1}{2}\right)h\right) \\ &= -2\sin(kjh)\sin\left(k\frac{h}{2}\right), \quad \text{by the identity } \cos(a+b) - \cos(a-b) = -2\sin(a)(b) \ . \end{aligned}$$

Then, we implement the right hand side of the equation to get

$$\begin{aligned} & -2\sin(kjh)\sin\left(k\frac{h}{2}\right) = ih\sin(kjh)\lambda_j \\ & \Rightarrow \lambda_j = \frac{2i}{h}\sin\left(k\frac{h}{2}\right) \ . \end{aligned}$$

Next let $j = N$. Then,

$$MV^k = \frac{1}{h} V_1^k = \frac{i}{h} \sin(kh) \ ,$$

and

$$\lambda_N V^k = \lambda_N \cos(k x_{1/2}) = \lambda_N \cos\left(k\frac{h}{2}\right) \ ,$$

which implies

$$\begin{aligned}
&\frac{i}{h} \sin(kh) = \lambda_N \cos\left(k\frac{h}{2}\right) \\
&\Rightarrow i \sin(kh) = \lambda_N h \cos\left(k\frac{h}{2}\right) \\
&\Rightarrow 2i \sin\left(k\frac{h}{2}\right) \cos\left(k\frac{h}{2}\right) = \lambda_N h \cos\left(k\frac{h}{2}\right) \\
&\Rightarrow \lambda_N = \frac{2i}{h} \sin\left(k\frac{h}{2}\right) \ .
\end{aligned}$$

Now, let $j \in [N+1, 2N-1)$, then

$$\begin{aligned}
MV^k &= \frac{1}{h} \left[-V_{j-N}^k + V_{j-N+1}^k\right] \\
&= \frac{i}{h} \left[-\sin\left(k x_{j-N}\right) + \sin\left(k x_{j-N+1}\right)\right] \\
&= \frac{i}{h} \left[\sin(k(j-N+1)h) - \sin(k(j-N)h)\right] \ ,
\end{aligned}$$

and

$$\lambda_j V_j^k = \lambda_j \cos(k x_{j-N+1/2}) = \lambda_j \cos(k(j-N+1/2)h) \ .$$

This gives us

$$\begin{aligned}
&\frac{i}{h} \left[\sin(k(j-N+1)h) - \sin(k(j-N)h)\right] = \lambda_j \cos(k(j-N+1/2)h) \\
&\Rightarrow \frac{2i}{h} \cos(k(j-N+1/2)h) \sin\left(k\frac{h}{2}\right) = \lambda_j \cos(k(j-N+1/2)h) \\
&\Rightarrow \lambda_j = \frac{2i}{h} \sin\left(k\frac{h}{2}\right) \ .
\end{aligned}$$

Finally, let $j = 2N-1$, then

$$\begin{aligned}
MV^k &= -\frac{1}{h} V_{N-1}^k = -\frac{i}{h} \sin(k x_{N-1}) = -\frac{i}{h} \sin(k(N-1)h) = -\frac{i}{h} \sin(k\pi - kh) \\
&= -\frac{i}{h} \sin(k\pi)\cos(kh) + \frac{i}{h} \sin(kh)\cos(k\pi) = \frac{i}{h} \sin(kh)\cos(k\pi) \ ,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{2N-1} V^k &= \lambda_{2N-1} \cos(k(N-1/2)h) = \lambda_{2N-1} \cos\left(knh - k\frac{h}{2}\right) = \lambda_{2N-1} \cos\left(k\pi - k\frac{h}{2}\right) \\
&= \lambda_{2N-1} \left[\cos(k\pi)\cos\left(k\frac{h}{2}\right) - \sin(k\pi)\sin\left(k\frac{h}{2}\right)\right] = \lambda_{2N-1} \cos(k\pi)\cos\left(k\frac{h}{2}\right) \ .
\end{aligned}$$

This ultimately gets us

$$\lambda_{2N-1}\cos\left(k\pi - k\frac{h}{2}\right) = \frac{i}{h}\sin(kh)\cos(k\pi)$$
$$\Rightarrow \lambda_{2N-1} = \frac{2i}{h}\sin\left(k\frac{h}{2}\right) \ .$$

Thus,

$$\lambda = \frac{2i}{h}\sin\left(k\frac{h}{2}\right) \ .$$

3. Show that the eigenvalues of M are $\mathcal{O}(h^2)$ accurate if compared to the eigenvalues of part (1). More precisely, show that

$$\lambda_k = ki(1+\mathcal{O}(kh)^2), \quad k = 0, \pm 1, \ldots, \pm(N-1) \ .$$

Solution:

We begin by computing the Taylor series expansion of $\lambda_k(h)$ about 0:

$$\begin{aligned}\lambda_k(h) &= \lambda_k(0) + h\lambda_k'(0) + \frac{h^2}{2}\lambda_k''(0) + \frac{h^3}{6}\lambda_k'''(0) + \cdots \\ &= h\lambda_k'(0) + \frac{h^3}{6}\lambda_k'''(0) + \mathcal{O}(h^4) \\ &= \frac{ik}{h}h - \frac{h^3}{6}i\frac{k^3}{4h} + \mathcal{O}(h^4) \\ &= ik\left[1 + \frac{k^2h^2}{6} + \mathcal{O}(h^4)\right] \ .\end{aligned}$$

Thus,

$$\lambda_k(h) = ik\left[1 + \mathcal{O}((kh)^2)\right] \ .$$

4. Show that

$$\lambda_{\pm(N-1)} = \pm\frac{2i}{h}\cos(h/2), \quad \text{and} \quad |\lambda_{\pm(N-1)}| < \frac{2}{h} = \frac{2}{\pi}N \ .$$

Solution:

First,

$$\begin{aligned}\lambda_{\pm(N-1)} &= \frac{2i}{h}\sin\left(\pm N\frac{h}{2} \mp \frac{h}{2}\right) = \frac{2i}{h}\sin\left(\pm\frac{\pi}{2} \mp \frac{h}{2}\right) \\ &= \frac{2i}{h}\left[\sin\left(\pm\frac{\pi}{2}\right)\cos\left(\mp\frac{h}{2}\right) + \cos\left(\pm\frac{\pi}{2}\right)\sin\left(\mp\frac{h}{2}\right)\right] \\ &= \pm\frac{2i}{h}\cos\left(\frac{h}{2}\right) \ .\end{aligned}$$

Using this, we note that $0 \leq |\cos\left(\frac{\pi}{2N}\right)| < 1$ and $|i| = 1$, so,

$$|\lambda_{\pm(N-1)}| = |\pm \frac{2i}{h}\cos\left(\frac{h}{2}\right)| < \frac{2}{h} = \frac{2N}{\pi} \ .$$

5. Use `RK4` to time-step this PDE. Plot the stability region of `RK4` and find a stable Δt. Find the growth function g for this method and plot $|g|$ using contour.

Solution: We must first solve for the growth function $g(z)$. Let , $z = \Delta t\lambda$ and $f(t, y) = \lambda y$. Then,

$$\begin{aligned}
K_1 &= zy^n \\
k_2 &= \left(z + \frac{z^2}{2}\right) y^n \\
k_3 &= \left(z + \frac{1}{2}\left(z^2 + \frac{z^3}{2}\right)\right) y^n \\
k_4 &= \left(z + z\left(z + \frac{1}{2}\left(z^2 + \frac{z^3}{2}\right)\right)\right) y^n \ ,
\end{aligned}$$

and

$$y^{n+1} = y^n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)y^n \ .$$

This implies,

$$g(z) = 1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which simplifies to

$$g(z) = 1 + \frac{1}{6}\left(6z + 3z^2 + z^3 + \frac{z^4}{4}\right) \ .$$

We then plug this into Matlab and get the stability region:

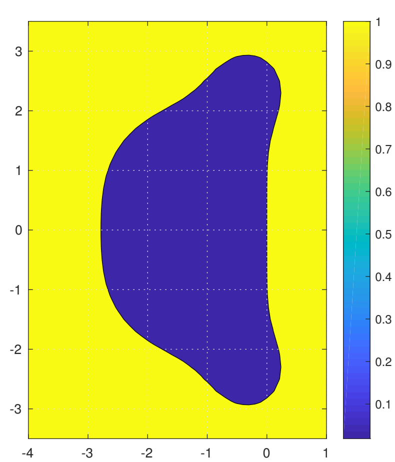


Figure 3: Stability region with RK4

6. Assuming that the boundary of the stability region crosses the imaginary axis approximately at $\pm 2.81i$, estimate the stability requirement on Δt (use part (4) to find c so that $\Delta t < ch$ for stability). For $N = 100$, plot the eigenvalues of M multiplied by Δt together with the stability region and verify that they fall inside the stability region.

Solution:

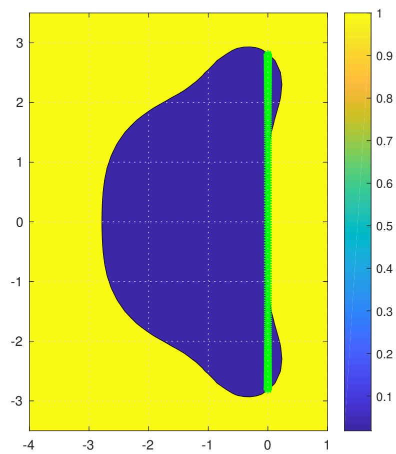
We use the conclusion from part (4), $|\lambda_{\pm(N-1)}| < \frac{2}{h}$ to see that,

$$\frac{2.81}{|\lambda_k|} > \frac{2.81}{|\lambda_{\pm(N-1)}|} > \frac{2.81}{2}h > \Delta t$$

which implies

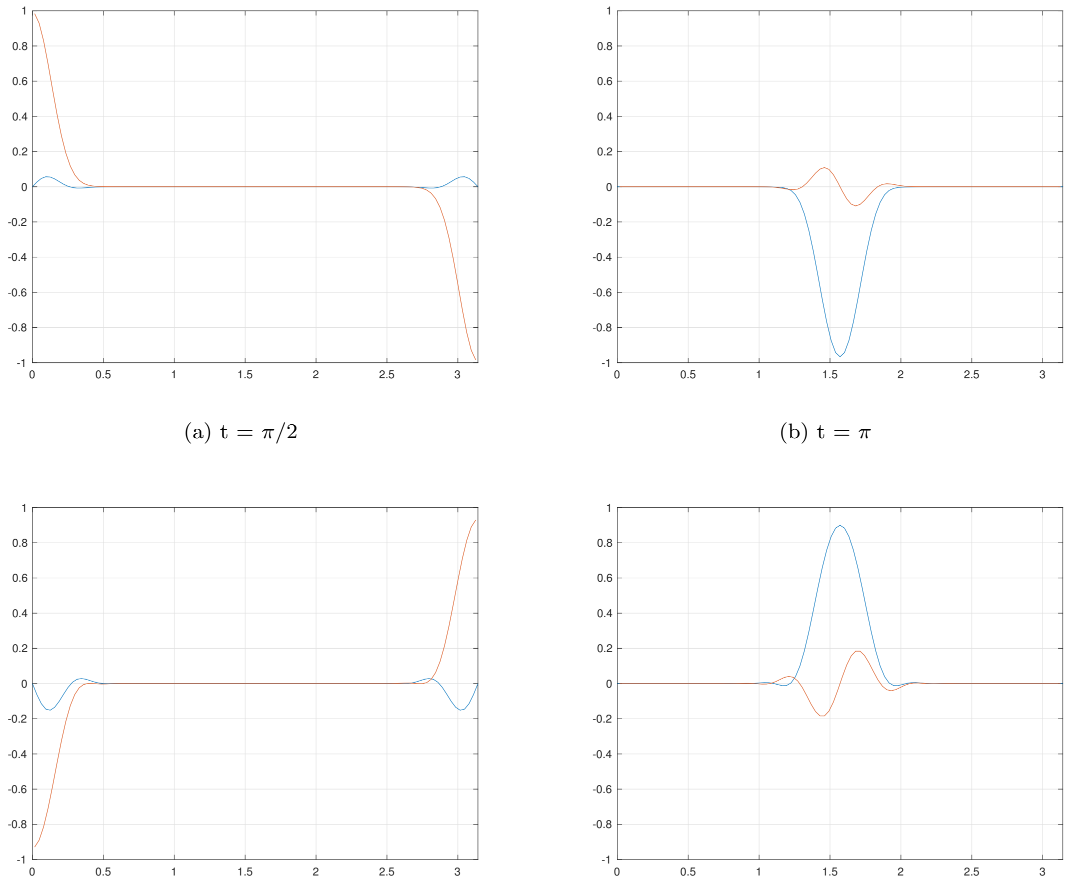
$$\Delta t = 1.405h \ .$$

The figure below shows that $(1.405h\lambda)$ falls within the stability region for all λ.



7. With $N = 100$, solve the PDE using `RK4` and plot your solution at $t = \pi/2, \pi, 3\pi/2, 2\pi$.

Solution:



(a) t = $\pi/2$ (b) t = π

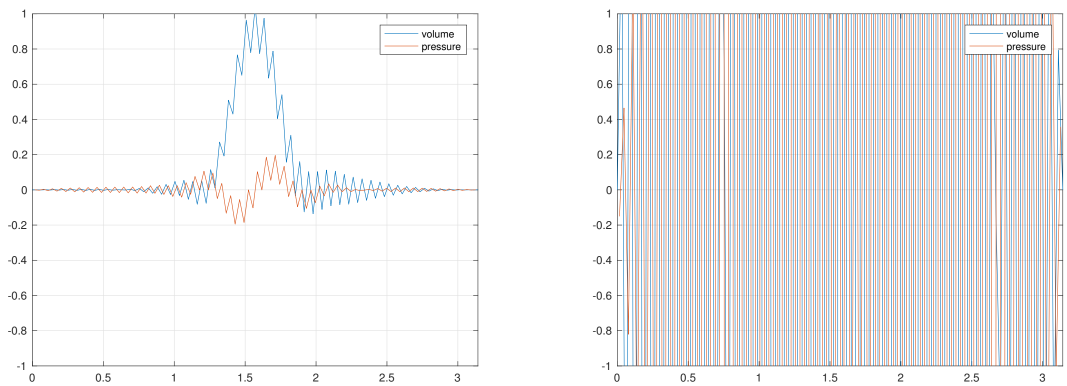
(c) t = $3\pi/2$ (d) t = 2π

Figure 4: Solutions for RK4 with stable time step

8. with $N = 100$, run you code with Δt that is slightly bigger than the optimal (stable) choice, is the computation stable?

Solution:

The solution is no longer stable once Δt becomes bigger than the optimal choice.



(a) $\Delta t = 1.48h$ (b) $\Delta t = 1.49h$

Figure 5: Solutions for RK4 with un-stable time step

As the figure above shows, the solution begins to blow up with $\Delta t = 1.48h$ and a time frame of 2π. The solution really blows up with $\Delta t = 1.49h$.

9. Using the criteria derived in part (6) for choosing Δt, plot the error at $t = 2\pi$ for several values of N and verify that the error decays as $\mathcal{O}(h^2)$. Conclude that the error is dominated by the spatial discretization in this case.

Solution:

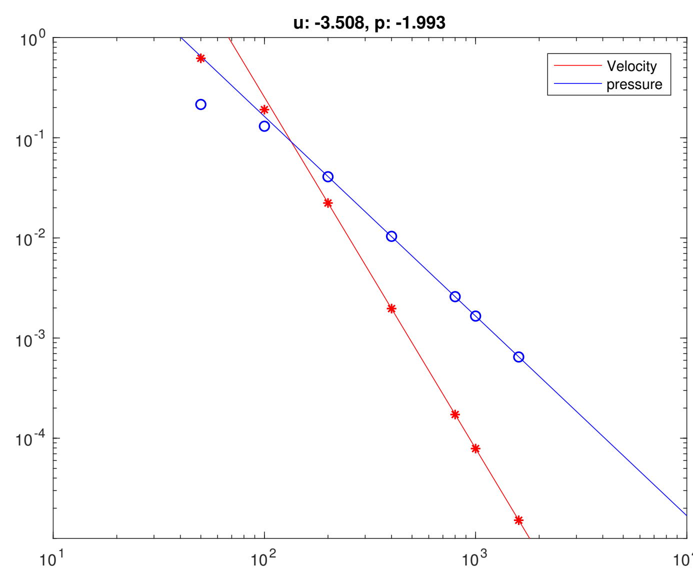


Figure 6: Error Decay as N increases

The figure above tells us that, for a stable Δt, the error decays with an approximate order of 3.508 for the velocity and 1.993 for pressure. Thus, both errors decay close to $\mathcal{O}(h^2)$ or higher.

The spatial discretization dominates the error as it decays with an order of 2 and the time stepping algorithm is of order 4, which means the spatial discretization leads the error.

10. Repeat item (7), but use 3rd order Adams-Bashforth for time-stepping. Use `RK4` to compute the first 2 time levels. Use a Δt that is close to the stability limit for `AB3`.

Solution:

Problem 3

1. Consider the predictor-corrector time-stepping scheme:

$$y^p_{n+1} = y_n + \frac{\Delta t}{12}(23f_n - 16f_{n-1} + 5f_{n-2}),$$
$$y_{n+1} = y_n + \frac{\Delta t}{12}(5f(t_{n+1}, y^p_{n+1}) + 8f)n - f_{n-1})\ .$$

Here $f_n = f(t_n, y_n)$ and $y'(t) = f(t, y)$, $y(0) = y_0$. Plot the stability region for this scheme.

Solution:
Let $y^p_{n+1} = P$ for simplicity. Then, for stability purposes, $f = \lambda y$. Finally, let $a = \frac{\Delta t \lambda}{12}$. Then,

$$\begin{cases} P = y^n + a(23y^n - 16y^{n-1} + 5y^{n-1} \\ y^{n+1} = y^n + a(5P + 8y^n - y^{n-1}) \end{cases} \Rightarrow y^{n+1} = y^n + a\left[5(y^n + a(23y^n - 16y^{n-1} + 5y^{n-1})) + 8y^n - y^{n-1}\right]\ .$$

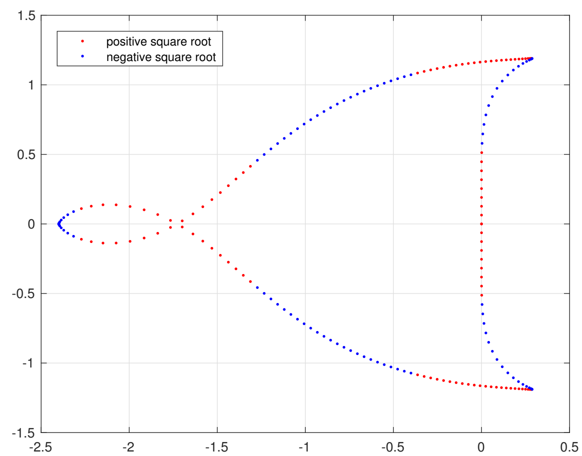
Now assume $y^{n+1} = gy^n$, and thus $y^n = g^2 y^{n-2}$. Plugging this in, we get

$$gy^n = y^n + a\left[5y^n + 115ay^n - 80\frac{a}{g}y^n + 25\frac{a}{g^2}y^n\right] + 8ay^n - \frac{a}{g}y^n\ .$$

Solving for a, we get

$$(115g^2 - 80g + 25)a^2 + (13g^2 - g)a + (g^2 - g^3) = 0$$
$$\Rightarrow a = \frac{-(13g^2 - g) \pm \sqrt{(13g^2 - g)^2 - 4(115g^2 - 80g + 25)(g^2 - g^3)}}{2(115g^2 - 80g + 25)}\ .$$

Lastly we let $z = 12a$ and plot the solution into Matlab to get the following stability region:



MATLAB

```
close all; clear variables;

theta = linspace(0,2*pi,100);
g = exp(1i*theta);
a = 115*g.^2 - 80*g+25;
b = 13*g.^2 - g;
c = g.^2-g.^3;
sq = sqrt(b.^2 - 4*a.*c);

zp = 12*((-b + sq)./(2*a));
zn = 12*((-b - sq)./(2*a));

figure
plot(real(zp),imag(zp),'.r',real(zn),imag(zn),'.b')
legend('positive square root','negative square root')
grid on
```