Partial Differential Equations
TA Homework 3

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Problem 3.2.1

1. Consider the Cauchy problem

$$\partial_t u + b(t,u)\partial_y u = -\alpha u, \qquad t>0,\; y\in\mathbb{R}, y(y,0) = u_0(y),$$

with $\alpha > 0$. Assume the following properties for the given functions $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $u_0 : \mathbb{R} \to \mathbb{R}$; b, u_0 are continuosuly differentiable, $|b_u(t,u)| \leq c_1$, $|u_0'(y)| \leq c_2$ for all $y, t, u \in \mathbb{R}$ where c_1, c_2 are positive constants satisfying $c_1 c_2 \leq \alpha$.

Show: There exists a solution $u = u(y,t)$ which is defined for all $t \geq 0, y \in \mathbb{R}$.

Solution: We can prove the desired result using *Theorem 3.11*. We need to prove that $\zeta_z > 0$, for all $z \in \mathbb{R}$ and $t \in [0,T)$ with $T > 0$, and that $\zeta \to \pm\infty$ as $z \to \pm\infty$. Let's start by proving that $\zeta_z(z,t) > 0$. Given the differential equation above we know that

$$\zeta(z,t) = z + \int_0^t b\left(s, u_0(z)e^{-\alpha s}\right) ds.$$

Thus, differentiating with respect to z we obtain

$$\zeta_z(z,t) = 1 + u_0'(z)\int_0^t b_u\left(s, u_0(z)e^{-\alpha s}\right) e^{-\alpha s} ds.$$

Given the previous form of ζ we can find the lower bound

$$\begin{aligned}
\zeta_z(z,t) &\geq 1 - \left| u_0'(z)\int_0^t b_u\left(s, u_0(z)e^{-\alpha s}\right) e^{-\alpha s} ds \right| \\
&= 1 - |u_0'(z)| \left| \int_0^t b_u\left(s, u_0(z)e^{-\alpha s}\right) e^{-\alpha s} ds \right| \\
&\geq 1 - |u_0'(z)| \int_0^t \left| b_u\left(s, u_0(z)e^{-\alpha s}\right) \right| e^{-\alpha s} ds \\
&\geq 1 - c_2 \int_0^t c_1 e^{-\alpha s} ds \\
&\geq 1 - \alpha \int_0^t e^{-\alpha s} ds \\
&= 1 - \left(1 - e^{-\alpha t}\right) \\
&= e^{-\alpha t} > 0,
\end{aligned}$$

where we have used the inequalities given by the problem as well as the triangle inequality when introducing the absolutve value inside the integral. Thus, we have

$$\zeta_z(z,t) > 0.$$

Now it is left to prove that $\zeta \to \pm\infty$ as $z \to \pm\infty$. By the Mean Value Theorem,

$$\zeta_z(\hat{z},t) = \frac{\zeta(z,t)-\zeta(0,t)}{z-0},$$

where $\zeta(0,t)$ is independent of z, $\hat{z}$ is some value between 0 and z, and $\zeta_z(\hat{z},t) > 0$ as we just proved. Rearranging terms we get

$$\zeta(z,t) = z\zeta_z(\hat{z},t) + \zeta(0,t).$$

Now we study two cases:

- If $z > 0$,

$$\begin{aligned}\zeta(z,t) &= z\zeta_z(\hat{z},t) - \zeta(0,t)\\ &> ze^{-\alpha t} - \zeta(0,t) \to \infty \text{ as } z \to \infty.\end{aligned}$$

- If $z < 0$,

$$\begin{aligned}\zeta(z,t) &= z\zeta_z(\hat{z},t) - \zeta(0,t)\\ &< ze^{-\alpha t} - \zeta(0,t) \to -\infty \text{ as } z \to -\infty.\end{aligned}$$

Thus, by *Theorem 3.11* there exists a solution $u = u(y,t)$ which is defined for all $t \geq 0, y \in \mathbb{R}$.

Problem 3.2.3

1. Consider the Cauchy problem

$$\begin{aligned}&\frac{\partial u}{\partial t} + \sin(\omega t)u\frac{\partial u}{\partial x} = 0,\\ &u(x,0) = u_0(x).\end{aligned}$$

Assume that u_0 is continuously differentiable on $\mathbb{R}$ and $\sup_x |u_0'(x)| \leq M$ for some $M > 0$.

(a) Show that the solution exists for all $t \geq 0$ provided that ω is large enough.

(b) What can be done if ω is not sufficiently large?

Solution: We start the like in the previous problem, by proving that $\zeta_z > 0$. Generally

$$\zeta(z,t) = z + \int_0^t b\left(s, u_0(z)e^{\gamma s}\right) ds,$$

and

$$\zeta_z(z,t) = 1 + u_0'(z)\int_0^t b_u\left(s, u_0(z)e^{\gamma s}\right) e^{\gamma s} ds,$$

In our problem $b(t,u) = \sin(\omega t) u$ and $\gamma = 0$. Therefore,

$$\zeta(z,t) = z + u_0(z) \int_0^t sin(\omega s) ds$$

which we can integrate and obtain

$$\zeta(z,t) = z + \frac{u_0(z)}{\omega} \left[1 - \cos(\omega t)\right].$$

Differentiating with respect to z we easily obtain

$$\zeta_z(z,t) = 1 + \frac{u_0'(z)}{\omega} \left[1 - \cos(\omega t)\right].$$

Like in the previous problem we can find a lower bound

$$\begin{aligned} \zeta_z(z,t) &\geq 1 - \left| \frac{u_0'(z)}{\omega} \left[1 - \cos(\omega t)\right] \right| \\ &= 1 - \frac{|u_0'(z)|}{|\omega|} \left[1 - \cos(\omega t)\right] \\ &\geq 1 - \frac{M}{|\omega|} \left[1 - \cos(\omega t)\right] \\ &\geq 1 - 2\frac{M}{|\omega|}. \end{aligned}$$

Thus, if $|\omega| > 2M$, $\zeta_z(z,t) > 0$ for all $z \in \mathbb{R}$ and $t \in [0,T) = [0,\infty)$. To prove that $\zeta \to \pm\infty$ as $z \to \pm\infty$ we use the Mean Value Theorem,

$$\zeta_z(\hat{z},t) = \frac{\zeta(z,t) - \zeta(0,t)}{z - 0},$$

where $\zeta(0,t)$ is independent of z, $\hat{z}$ is some value between 0 and z, and $\zeta_z(\hat{z},t) > 0$ for $\omega > 2M$ as we just proved. Rearranging terms we get

$$\zeta(z,t) = z\zeta_z(\hat{z},t) + \zeta(0,t).$$

Now we study two cases:

- If $z > 0$,

$$\begin{aligned} \zeta(z,t) &= z\zeta_z(\hat{z},t) - \zeta(0,t) \\ &> z(1 - 2M) - \zeta(0,t) \to \infty \text{ as } z \to \infty, \end{aligned}$$

  since $1 - 2M > 0$ and constant.

- If $z < 0$,

$$\begin{aligned} \zeta(z,t) &= z\zeta_z(\hat{z},t) - \zeta(0,t) \\ &< z(1 - 2M) - \zeta(0,t) \to -\infty \text{ as } z \to -\infty, \end{aligned}$$

since $1 - 2M > 0$ and constant.

Thus, if $|\omega| > 2M$, $\zeta_z(z,t) > 0$ and $\zeta \to \pm\infty$ as $z \to \pm\infty$ for all $z \in \mathbb{R}$ and $t \in [0,T) = [0,\infty)$. Therefore, by *Theorem 3.11*, the Cauchy problem has a unique solution u on $\mathbb{R} \times [0,\infty)$.

For part b), if we want our solution to exist, but ω is not sufficiently large, i.e., $\omega < 2M$, we have restrict T. We retake

$$\begin{aligned}\zeta_z(z,t) &= 1 + u_0'(z)\int_0^t b_u\left(s, u_0(z)e^{\gamma s}\right)e^{\gamma s}ds \\ &= 1 + u_0'(z)\int_0^t \sin(\omega s)ds \\ &\geq 1 - \left|u_0'(z)\int_0^t \sin(\omega s)ds\right| \\ &\geq 1 - |u_0'(z)|\int_0^t |\sin(\omega s)|\,ds \\ &\geq 1 - M\int_0^t ds \\ &\geq 1 - Mt,\end{aligned}$$

which, since we need $\zeta_z(z,t)$ to be strictly greater than zero, gives us

$$1 - Mt > 0,$$

and we can obtain a condition for t,

$$t < \frac{1}{M} = T^*.$$

Thus, if ω is not large enough, $\zeta_z(z,t) > 0$ for all $z \in \mathbb{R}$ and $t \in [0, T^*)$. To prove that $\zeta \to \pm\infty$ as $z \to \pm\infty$ we use again the Mean Value Theorem. Like above,

$$\zeta(z,t) = z\zeta_z(\hat{z},t) + \zeta(0,t).$$

Now we study two cases:

- If $z > 0$,

$$\begin{aligned}\zeta(z,t) &= z\zeta_z(\hat{z},t) - \zeta(0,t) \\ &> z(1 - Mt) - \zeta(0,t) \to \infty \text{ as } z \to \infty,\end{aligned}$$

  since $1 - Mt > 0$ and constant.

- If $z < 0$,

$$\begin{aligned}\zeta(z,t) &= z\zeta_z(\hat{z},t) - \zeta(0,t) \\ &< z(1 - Mt) - \zeta(0,t) \to -\infty \text{ as } z \to -\infty,\end{aligned}$$

since $1 - Mt > 0$ and constant.

Thus, if $|\omega| < 2M$, $\zeta_z(z,t) > 0$ and $\zeta \to \pm\infty$ as $z \to \pm\infty$ for all $z \in \mathbb{R}$ and $t \in [0, T^*)$. Therefore, by *Theorem 3.11*, the Cauchy problem has a unique solution u on $\mathbb{R} \times [0, T^*)$.

Problem 3.2.4

1. Solve the linear size-structured population problem

$$\begin{aligned} \partial_t u + \gamma(t)\partial_x u + \mu u &= f(x,t), \qquad x,t \in \mathbb{R}, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned}$$

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ and $u_0 : \mathbb{R} \to \mathbb{R}$ are continuously differentiable. Further $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $\mu \geq 0$ is a constant. Show also: If $u_0(x) = 0$ for $x \leq 0$ and $f(x,t) = 0$ for all $x \leq 0$, $t \geq 0$, then $u(x,t) = 0$ for all $x \leq 0$, $t \geq 0$.

Solution: We start by writing the *Characteristic System* corresponding to this problem,

$$\begin{aligned} \partial_t \xi_1(z,t) &= \gamma(t), \qquad \xi_1(z,0) = z, \\ \partial_t \xi_2(z,t) &= 1, \qquad \xi_2(z,0) = 0, \\ \partial_t v(z,t) &= f(x,t) - \mu v, \qquad v(z,0) = u_0(z). \end{aligned}$$

We start by solving for ξ_2,

$$\xi_2(z,t) = t + h_2(z).$$

Imposing the initial condition

$$\xi_2(z,0) = h_2(z) = 0,$$

we get

$$\xi_2(z,t) = t.$$

Now we solve for ξ_1,

$$\xi_1(z,t) = \int_0^t \gamma(s)ds + h_1(z).$$

Imposing the initial condition

$$\xi_1(z,0) = h_1(z) = z,$$

we get

$$\xi_1(z,t) = z + g(t),$$

where we have defined

$$g(t) = \int_0^t \gamma(s)ds.$$

Observe that $g(0) = 0$ and, since $\gamma(s) > 0$ for all s, $g(t_2) > g(t_1)$ if $t_2 > t_1$. Finally, we solve for v retaking its ordinary differential equation

$$\partial_t v(z,t) = -\mu v(z,t) + f(z + g(t), t).$$

By Duhamel's formula we have

$$v(z,t) = u_0(z)e^{-\mu t} + \int_0^t e^{-\mu(t-s)} f(z + g(s), s)ds.$$

Now we can obtain the solution $u(x,t)$ by plugging in $z = x - g(t)$,

$$u(x,t) = u_0(x - g(t))e^{-\mu t} + \int_0^t e^{-\mu(t-s)} f(x - (g(t) - g(s)), s)ds.$$

Now we analyze the argument of the function u_0 and the first argument of f. Since $g(t)$ is nonnegative for all t, if x is negative, then $x - g(t)$ is also negative and $u(x - g(t) = 0$. On the other hand,

$$g(t) - g(s) = \int_0^t \gamma(r)dr - \int_0^s \gamma(r)dr,$$

and, since $t \geq s$, it is nonnegative. Therefore, if x is negative and t nonnegative, then $x-(g(t)-g(s))$ is also negative and $f(x - (g(t) - g(s)), s) = 0$. Thus, for all $x \leq 0$ and $t \geq 0$, $u(x,t) = 0$.

Problem 3.3.3

1. Solve the wave equation

$$\begin{aligned} &\partial_t^2 u - c^2 \partial_x^2 u = 0, && x, t \in \mathbb{R}, \\ &u(0,t) = f(t), && t \in \mathbb{R}, \\ &u(cx, x) = g(x), && x \in \mathbb{R}. \end{aligned}$$

where $f, g : \mathbb{R} \to \mathbb{R}$. State appropriate assumptions for f and g such that you really have a solution.

Solution: Before starting to solve the PDE, we can obtain one condition that f and g must satisfy. We have that

$$u(0,0) = f(0),$$

and also

$$u(0,0) = g(0).$$

Thus, $f(0) = g(0)$. We continue with the general solution of the wave equation

$$u(x,t) = F(x + ct) + G(x - ct),$$

and imposing the boundary conditions

$$\begin{aligned} u(0,t) &= F(ct) + G(-ct) = f(t), \\ u(cx,x) &= F(2xc) + G(0) = g(x). \end{aligned}$$

From the previous equations we obtain the following system

$$\begin{aligned} F(x) + G(-x) &= f\left(\frac{x}{c}\right), \\ F(x) + G(0) &= g\left(\frac{x}{2c}\right), \end{aligned}$$

where we have substituted ct for x in the first equation and $2cx$ for x in the second one. Substracting both equations we get

$$G(-x) = G(0) + f\left(\frac{x}{c}\right) - g\left(\frac{x}{2c}\right).$$

Thus,

$$G(x-ct) = G(0) + f\left(\frac{ct-x}{c}\right) - g\left(\frac{ct-x}{2c}\right).$$

Coming back to $F(x) = -G(0) + g\left(\frac{x}{2c}\right)$ we can obtain

$$F(x+ct) = -G(0) + g\left(\frac{x+ct}{2c}\right).$$

We can now write the solution of the PDE

$$u(x,t) = F(x+ct) + G(x-ct) = f\left(\frac{ct-x}{c}\right) - g\left(\frac{ct-x}{2c}\right) + g\left(\frac{x+ct}{2c}\right),$$

and we can check that satisfies the boundary conditions

$$u(cx,x) = f(0) - g\left(0\right) + g\left(x\right) = g(x),$$

and

$$u(0,t) = f(t) - g\left(\frac{t}{2}\right) + g\left(\frac{t}{2}\right) = f(t).$$

In order for the previous solution to be really a soluton we need f and g to be twice differentiable and $f(0) = g(0)$ to satisfy the boundary conditions.