Partial Differential Equations
TA Homework 10

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Problem 5.3.3

Let f be integrable and $\int_0^L |f(y)|dy < \infty$. Let u be the solution of the heat equation with initial data f and 0 boundary conditions. Show:

$$\int_0^L |u(t,x) - f(x)|dx \to 0, \qquad t \to 0.$$

Solution: Let $\epsilon > 0$. For $n \geq 3L$, define $f_n(x)$ as in the proof of *Proposition 5.18 (b)* and set $g_n(x) = g(x) \cdot f_n(x)$. Recall that $u(x,t)$ is given by

$$u(x,t) = \int_0^L G(x,y,t)f(y)dy,$$

and define

$$v_n(x,t) = \int_0^L G(x,y,t)g_n(y)dy.$$

Then, by triangle inequality,

$$\begin{aligned}\int_0^L |u(x,t) - f(x)|dx \leq & \int_0^L |u(x,t) - v_n(x,t)|dx + \int_0^L |v_n(x,t) - g_n(x)|dx \\ & + \int_0^L |g_n(x) - g(x)|dx + \int_0^L |g(x) - f(x)|dx.\end{aligned}$$

We will study each integral individually. First,

$$\begin{aligned}\int_0^L |g_n(x) - g(x)|dx &= \int_0^L |g(x)f_n(x) - g(x)|dx \\ &= \int_0^L |g(x)[f_n(x) - 1]|dx \\ &\leq \frac{1}{n} \sup_{[0,L]} |g| = \frac{\epsilon}{6}.\end{aligned}$$

Furhter,

$$\begin{aligned}
\int_0^L |u(x,t) - v_n(x,t)|dx &= \int_0^L \left| \int_0^L G(x,y,t)f(y)dy - \int_0^L G(x,y,t)g_n(y)dy \right| dx \\
&= \int_0^L \left| \int_0^L G(x,y,t)\left[f(y) - g_n(y)\right] dy \right| dx \\
&\le \int_0^L \int_0^L G(x,y,t)\left|f(y) - g_n(y)\right| dydx,
\end{aligned}$$

where we haven't written the absolute value of the Green's function since we know it is nonnegative. Reorganizing the last expression we get

$$\begin{aligned}
\int_0^L |u(x,t) - v_n(x,t)|dx &\le \int_0^L \left[\int_0^L G(x,y,t)dx\right] \left|f(y) - g_n(y)\right| dy \\
&\le \int_0^L \left|f(y) - g_n(y)\right| dy \qquad \text{by Proposition 5.18 (b)} \\
&\le \int_0^L \left|f(y) - g(y)\right| dy + \int_0^L \left|g(y) - g_n(y)\right| dy \qquad \text{by Triangle Inequality} \\
&< \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.
\end{aligned}$$

Next,

$$\int_0^L |v_n(x,t) - g_n(x)|dx = \int_0^L \left| \int_0^L G(x,y,t)g_n(y)dy - g_n(x) \right| dx.$$

Since

$$\int_0^L G(x,y,t)g_n(y)dy - g_n(x) \to 0$$

by *Theorem 5.19*,

$$\int_0^L |v_n(x,t) - g_n(x)|dx = \int_0^L \left| \int_0^L G(x,y,t)g_n(y)dy - g_n(x) \right| dx < \frac{\epsilon}{3}.$$

To finish,

$$\begin{aligned}
\int_0^L |u(x,t) - f(x)|dx &\le \int_0^L |u(x,t) - v_n(x,t)|dx + \int_0^L |v_n(x,t) - g_n(x)|dx \\
&\quad + \int_0^L |g_n(x) - g(x)|dx + \int_0^L |g(x) - f(x)|dx \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \epsilon,
\end{aligned}$$

where we have used the hint for the last integral. Thus,

$$\int_0^L |u(x,t) - f(x)|dx \to 0, \qquad t \to 0$$

Problem 5.3.4

Let $f : [0, L] \to \mathbb{R}$ be twice continuously differentiable, $f(0) = 0 = f(L)$. Let G be the Greens function for the heat equation. Set

$$
\begin{aligned}
&u(x,t) = \int_0^L G(x,y,t)f(y)dy, \quad x \in [0,L], t \in (0,\infty),\\
&u(x,0) = f(x), \quad x \in [0,L].
\end{aligned}
$$

(a) Show that u has continuous partial derivatives $\partial_t u(x,t)$ on $[0,L] \times (0,\infty)$ and on $(0,L) \times [0,\infty)$ and

$$
\begin{aligned}
&\partial_t u(x,t) = \int_0^L G(x,y,t)af''(y)dy, \quad x \in [0,L], t \in (0,\infty),\\
&\partial_t u(x,0) = af''(x), \quad x \in (0,L).
\end{aligned}
$$

(b) Show that $\partial_t u$ satisfies the heat equation with initial data af'',

$$
\begin{aligned}
&(\partial_t - a\partial_x^2)\partial_t u = 0 \quad \text{on} \quad [0,L] \times (0,\infty)\\
&\partial_t u(x,0) = af''(x), \quad x \in (0,L)\\
&\partial_t u(0,t) = 0 = \partial_t u(L,t), \quad t \in (0,\infty).
\end{aligned}
$$

Solution: Let

$$u(x,t) = \int_0^L G(x,y,t)f(y)dy.$$

Then,

$$
\begin{aligned}
\partial_t u(x,t) &= \int_0^L \partial_t G(x,y,t)f(y)dy\\
&= \int_0^L \partial_t \left[\frac{2}{L}\sum_{m=1}^{\infty} \sin(\lambda_m x)\sin(\lambda_m y)e^{-a\lambda_m^2 t}\right] f(y)dy\\
&= \int_0^L a\left[-\frac{2}{L}\sum_{m=1}^{\infty} \lambda_m^2 \sin(\lambda_m x)\sin(\lambda_m y)e^{-a\lambda_m^2 t}\right] f(y)dy\\
&= \int_0^L a\partial_y^2 G(x,y,t)f(y)dy.
\end{aligned}
$$

Integrating by parts twice,

$$
\begin{aligned}
\partial_t u(x,t) &= \int_0^L a\partial_y^2 G(x,y,t)f(y)dy\\
&= a\left[\partial_y G(x,y,t)f(y)\right]_0^L - a\int_0^L \partial_y G(x,y,t)f'(y)dy\\
&= -a\int_0^L \partial_y G(x,y,t)f'(y)dy\\
&= -a\left[G(x,y,t)f'(y)\right]_0^L + a\int_0^L G(x,y,t)f''(y)dy\\
&= \int_0^L G(x,y,t)af''(y)dy.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\partial_t u(x,0) &= \int_0^L G(x,y,0)af''(y)dy \\
&= \int_0^L \left[\sum_{m=1}^{\infty} \frac{2}{L}\sin(\lambda_m x)\sin(\lambda_m y)\right] af''(y)dy \\
&= \sum_{m=1}^{\infty} \left[\frac{2}{L}\int_0^L \sin(\lambda_m y)af''(y)dy\right] \sin(\lambda_m x) \\
&= af''(x),
\end{aligned}
$$

where the last equality comes from the Fourier sine expansion of the function $af''(x)$. For the second part of the problem let $g(x) = af''(x)$ and $v(x,t) = \partial_t u(x,t)$. We can now rewrite the previous equations as

$$
T(n) = \begin{cases} \int_0^L G(x,y,t)g(y)dy & t > 0, \quad x \in [0,L] \\ g(x) & t = 0, \quad x \in [0,L] \end{cases}
$$

Note that $g(x)$ is continuous since $u(x,t)$ is twice differentiable. Therefore $g(x)$ is also bounded since $[0,L]$ is a compact set. By *Corollary 5.20*, $v(x,t)$ defines a solution to the heat equation with $v(x,0) = g(x)$. This proves the statement for $\partial_t u(x,t)$.