Real Analysis TA Homework 11

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November 3, 2017

1 Problem 4.6.5

1. Let (X, d) be a metric space, $f : X \to \mathbb{R}$ be continuous, and D a compact subset of X. Then there exist x^o and x_0 in D such that $f(x^o) = \sup f(D)$ and $f(x_0) = \inf f(D)$.

Solution:

Proof. Since D is a compact subset of X and f is continuous, then by *Theorem 4.50*, $f(D)$ is compact and f is uniformly continuous on D. Then, by *Theorem 4.39*, $f(D)$ is also complete and totally bounded. Next, by *Lemma 4.23*, $f(D)$ is bounded. Therefore there exists a $c \in \mathbb{R}$ such that

$$-c \leq f(x) \leq c \quad \forall x \in D \ .$$

Then, by the definitions of infimum and supremum,

$$-c \leq \inf f(D) \leq f(x) \leq \sup f(D) \leq c \quad \forall x \in D \ .$$

We know from advanced caluclus that there exists a sequence such that the limit is the supremum. Let (y_n) in $f(D)$ be such a sequence and let (x_n) be a sequence in D. Then there exists an x' in $\mathbb{R}$ such that

$$y_n = f(x_n) \to \sup f(D) = f(x') = y' \in \mathbb{R} \quad .$$

Since $x_n \in D$ and D is compact, there exists a subsequence (x_{n_j}) of (x_n) that converges in D. Let x^o in D be such that $x_{n_j} \to x^o$. Since f is uniformly continuous (and therefore continuous),

$$y_{n_j} = f(x_{n_j}) \to f(x^o) = y^o \in f(D) \ .$$

By *Theorem 2.17*, since (y_{n_j}) is a subsequence of the convergent sequence (y_n), (y_{n_j}) converges itself to the same limit, the supremum of $f(D)$. Thus, there exists a point $x^o \in D$ such that $f(x^o) = \sup f(D)$. Same procedure to prove that there exists a point $x_0 \in D$ such that $f(x_0) = \inf f(D)$.

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2 Problem 4.6.7

1. Let X be a compact metric space and Z a normed vector space. Let $\mathcal{F}$ be an equicontinuous subset of $C(X,Z)$, the space of continuous functions from X to Z. Show that $\mathcal{F}$ is uniformly equicontinuous.

Solution:

Proof. Let's prove the statement by contradiction. Assume $\mathcal{F}$ is not uniformly continuous. Then, there exists an $\varepsilon > 0$ such that, for all $\delta > 0$ there is an x and a y in X with $d(x,y) < \delta$ but $||f(x) - f(y)|| \geq \varepsilon$ for some $f \in \mathcal{F}$. Now, for each $n \in \mathbb{N}$ and $\delta = \frac{1}{n}$, there exists a sequence (x_n) and (y_n) in X with $d(x_n, y_n) < \delta$ but $||f_n(x_n) - f_n(y_n)|| \geq \varepsilon$. Since X is compact, there exists a subsequence (x_{n_j}) and (y_{n_j}) such that $x_{n_j} \to x \in X$ and $y_{n_j} \to y \in X$ with $d(x_{n_j}, y_{n_j}) < \frac{1}{n_j} \to 0$ as $j \to \infty$. By triangle inequality:

$$0 \leq d(y,x) \leq d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x) \to 0 \quad \text{as } j \to \infty \ ,$$

which implies that $y_{n_j} \to x$. Therefore $x = y$ by uniqueness of limits. Then, let $\varepsilon > 0$ and δ be defined as above, then, since $\mathcal{F}$ is equicontinuous there exists $N_1,\ N_2 \in \mathbb{N}$ such that,

$$d(x_{n_j}, x) < \delta \quad \Rightarrow \quad ||f_{n_j}(x_{n_j}) - f_{n_j}(x)|| < \frac{\varepsilon}{2} \ \text{ for } f \in \mathcal{F} \text{ with } n_j > N_1 \ ,$$

and

$$d(y_{n_j}, x) < \delta \quad \Rightarrow \quad ||f_{n_j}(y_{n_j}) - f_{n_j}(x)|| < \frac{\varepsilon}{2} \ \text{ for } f \in \mathcal{F} \text{ with } n_j > N_2 \ .$$

Lastly, let $N = \max\{N_1, N_2\}$, then

$$\begin{aligned} ||f_{n_j}(x_{n_j}) - f_{n_j}(y_{n_j})|| &\leq \|f_{n_j}(x_{n_j}) - f_{n_j}(x)\| + \|f_{n_j}(x) - f_{n_j}(y_{n_j})\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \text{ for } \ n_j > N \ . \end{aligned}$$

finding a contradiction. Thus, $\mathcal{F}$ is uniformly equicontinuous. ■

3 Problem 4.6.11

1. Let X be a metric space, K a compact subset of X and B a bounded subset of X. Show: For any sequence (x_n) in B there exists a subsequence (x_{n_j}) of (x_n) and a continuous function $f : K \to \mathbb{R}$ such that $d(x_{n_j}, x) \to f(x)$ as $j \to \infty$ uniformly for $x \in K$.

Solution:

Proof. Define the set $\mathcal{F} = f_y(x) := d(y,x); y \in B, x \in K$. Let$y \in B$ and $x, z \in K$, from *Proposition 1.22*,

$$|d(y,x) - d(y,z)| \leq d(x,z) \ ,$$

which in terms of f yields

$$|f_y(x) - f_y(z)| \leq d(x,z) \ .$$

Therefore, if $d(x,z) \to 0$, $|f_y(x) - f_y(z)| \to 0$ as well, proving that $\mathcal{F}$ is equicontinuous. Now let $z_1 \in B$ and $z_2 \in K$ be fixed but arbitrary, also let $x \in K$ and $y \in B$. Then, for all $f \in \mathcal{F}$:

$$\begin{aligned} f_y(x) = d(y,x) &\leq d(y,z_1) + d(z_1,z_2) + d(z_2,x) \\ &\leq \Delta B + d(z_1,z_2) + \Delta K \ . \end{aligned}$$

Therefore the set $S = \{f(x), f \in \mathcal{F}\} \subseteq \mathbb{R}$ is bounded and, by *Theorem 4.54*, it is also totally bounded. Lastly, since K is compact, by *Theorem 4.39*, it is totally bounded and by *Theorem 4.34*, it is separable.

Finally, let $(x_n) \in B$ such that (f_{x_n}) is a sequence in $\mathcal{F}$. Then, by *Theorem 4.59*, there exists a subsequence (x_{n_j}) and a continuous function $f : K \to \mathbb{R}$ such that $f_{x_{n_j}}(x) = d(x_{n_j}, x) \to f(x)$ as $j \to \infty$. ■

4 Problem 4.6.13

1. Let X be a metric space (f_n) be sequence of continuous functions from X to $\mathbb{R}$ such that $\{f_n; n \in \mathbb{N}\}$ is equicontinuous. Let f be a continuous function from X to $\mathbb{R}$ and assume that there exists a dense subset A of X such that $f_n(x) \to f(x)$ pointwise for $x \in A$. Show that $f_n \to f$ pointwise on X.

Solution:

Proof. Let $x \in X$ and $\varepsilon > 0$. Since $X \subseteq \overline{A}$ (because A is dense in X), there exists a sequence (x_k) in A, and therefore also in X, sucht that $x_k \to x$. By triangle inequality, for all $n \in \mathbb{N}$,

$$d(f_n(x), f(x)) \leq d(f_n(x), f_n(x_k)) + d(f_n(x_k), f(x_k)) + d(f(x_k), f(x)) \ .$$

Since $\{f_n; n \in \mathbb{N}\}$ is equicontinuous there exists some $\delta_1 > 0$ such that , for every $x_k \in X$ and every $f_n \in \{f_n; n \in \mathbb{N}\}$:

$$d(x, x_k) < \delta_1 \Rightarrow d(f_n(x), f_n(x_k)) < \varepsilon/3 \ .$$

Similarly, since f is continuous on X there exists some $\delta_2 > 0$ such that for every $x_k \in X$:

$$d(x, x_k) < \delta_2 \Rightarrow d(f(x_k), f(x)) < \varepsilon/3 \ .$$

Pick k such that, for $x_k = y$, $d(x,y) < \delta = \min\{\delta_1, \delta_2\}$. Lastly, since $f_n \to f$ pointwise on A as $n \to \infty$, there exists an $N \in \mathbb{N}$ such that

$$d(f_n(y), f(y)) < \varepsilon/3 \ , \quad \forall n \in \mathbb{N}, \text{ with } n > N \ .$$

Therefore, for $d(x,y) < \delta$:

$$d(f_n(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ \ \forall n > N.$$

Thus, since x is a point defined in X, $f_n \to f$ pointwise on X. ■

Acknowledgements

The proofs in this homework assignment have been worked and written in close collaboration with Camille Moyer.