Partial Differential Equations
TA Homework 10

Francisco Jose Castillo Carrasco

March 29, 2018

Problem 5.3.5

Find a function like the Green's function for the equation

$$(\partial_t - \partial_x^2)u = 0, \qquad x \in [0, L],\ t \in (0, \infty),$$
$$\partial_x u(0,t) = 0 = \partial_x u(L,t), \qquad t \in (0,\infty),$$
$$u(x,0) = f(x), \qquad x \in [0, L].$$

Solution: The solution to the PDE with zero Neumann boundary conditions is given by the Fourier cosine series

$$u(x,t) = \sum_{m=1}^{\infty} A_m e^{-a\lambda_m^2 t}\cos(\lambda_m x), \qquad \lambda_m = m\frac{\pi}{L},$$

with

$$A_m = \frac{2}{L}\int_0^L f(y)\cos(\lambda_m y)dy, \qquad m \geq 1,$$

and

$$A_0 = \frac{1}{L}\int_0^L f(y)dy.$$

Note that $\lambda_0 = 0$. Then,

$$u(x,t) = \frac{1}{L}\int_0^L f(y)dy + \sum_{m=1}^{\infty}\frac{2}{L}\int_0^L f(y)\cos(\lambda_m y)dy e^{-a\lambda_m^2 t}\cos(\lambda_m x).$$

We can reorganize the last expression and get

$$u(x,t) = \int_0^L \left[\frac{1}{L} + \sum_{m=1}^{\infty}\frac{2}{L}\cos(\lambda_m x)\cos(\lambda_m y)e^{-a\lambda_m^2 t}\right] f(y)dy.$$

Let $\tilde{G}_0 = \frac{1}{L}$ and

$$\tilde{G}_m(x,y,t) = \frac{2}{L}\cos(\lambda_m x)\cos(\lambda_m y)e^{-a\lambda_m^2 t}, \quad m \geq 1.$$

We can then substitute in the last equation

$$u(x,t) = \int_0^L \sum_{m=0}^{\infty} \tilde{G}_m(x,y,t) f(y) dy = \int_0^L \tilde{G}(x,y,t) f(y) dy.$$

Thus, the "modified Green's function" for the heat equation with Neumann boundary conditions is

$$\tilde{G}(x,y,t) = \sum_{m=0}^{\infty} \tilde{G}_m(x,y,t),$$

where $\tilde{G}_m(x,y,t)$ is specified above for all $m \geq 0$.

Problem 5.4.3

Let u be as in *Problem 5.4.2*. Also assume that u is twice partially differentiable with respect to x on $[0,L] \times (0,T)$ and $\partial_x u$ and $\partial_x^2 u$ are continuous on $[0,L] \times (0,T)$ and

$$\begin{aligned} (\partial_t - a\partial_x^2)u = F(x,t), &\qquad x \in [0,L], t \in (0,T), \\ u(0,t) = 0 = u(L,t), &\qquad t \in (0,T), \end{aligned}$$

where $F : [0,L] \times (0,T) \to \mathbb{R}$ is continuous.

Show: For every twice continuously differentiable $\phi : [0,L] \to \mathbb{R}$ with $\phi(0) = 0 = \phi(L)$, $\int_0^L \phi(x) u(x,t) dx$ is differentiable in $t \in (0,T)$ and

$$\frac{d}{dt} \int_0^L \phi(x) u(x,t) dx = \int_0^L a\phi''(x) u(x,t) dx + \int_0^L \phi(x) F(x,t) dx, \qquad t \in (0,T).$$

Solution: First, since u and ϕ satisfy the assumptions of the *Problem 5.4.2*, the integral $\int_0^L \phi(x) u(x,t) dx$ is differentiable by the result of said problem. Then,

$$\begin{aligned} \frac{d}{dt} \int_0^L \phi(x) u(x,t) dx &= \int_0^L \phi(x) \frac{\partial}{\partial_t} u(x,t) dx \\ &= \int_0^L \phi(x) \left[a\partial_x^2 u(x,t) + F(x,T) \right] dx \\ &= a \int_0^L \partial_x^2 u(x,t) \phi(x) dx + \int_0^L \phi(x) F(x,T) dx. \end{aligned}$$

Integrating the first integral of the right hand side by parts twice and using that $\phi(0) = 0 = \phi(L)$,

$$\frac{d}{dt} \int_0^L \phi(x) u(x,t) dx = \int_0^L a\phi''(x) u(x,t) dx + \int_0^L \phi(x) F(x,T) dx.$$

Problem 5.4.4

Let $F : [0, L] \times [0, T) \to \mathbb{R}$ be continuous. Define

$$u(x,t) = \int_0^L \int_0^t G(x,y,t-s)F(y,s)dsdy, \qquad x \in [0,L],\ t \in (0,T).$$

Show: For every twice continuously differentiable function $\phi : [0, L] \to \mathbb{R}$ with $\phi(0) = 0\phi(L)$, $\int_0^L \phi(x)u(x,t)dx$ is differentiable in $t \in (0,T)$ and

$$\frac{d}{dt}\int_0^L \phi(x)u(x,t)dx = \int_0^L a\phi''(x)u(x,t)dx + \int_0^L \phi(x)F(x,t)dx, \qquad t \in (0,T).$$

Solution: First, since given its definition, u is continuous on $[0, L] \times (0, T)$ and ϕ is continuous, they satisfy the assumptions of the *Problem 5.4.2*. Therefore, the integral $\int_0^L \phi(x)u(x,t)dx$ is differentiable by the result of said problem. Define

$$v(y,t) = \int_0^L \phi(x)G(x,y,t)dx = \int_0^L G(y,z,t)\phi(z)dz, \qquad t > 0,\ y \in [0,L].$$

Then,

$$\begin{aligned}\int_0^L \phi(x)u(x,t)dx &= \int_0^L \phi(x)\left(\int_0^L \int_0^t G(x,y,t-s)F(y,s)dsdy\right)dx \\ &= \int_0^L \int_0^t \left(\int_0^L G(y,x,t-s)\phi(x)dx\right)F(y,s)dsdy \\ &= \int_0^L \int_0^t v(y,t-s)F(y,s)dsdy,\end{aligned}$$

where we have used that $G(x,y,t) = G(y,x,t)$. Then,

$$\begin{aligned}\frac{d}{dt}\int_0^L \phi(x)u(x,t)dx &= \frac{d}{dt}\int_0^L \int_0^t v(y,t-s)F(y,s)dsdy \\ &= \int_0^L \frac{\partial}{\partial t}\int_0^t v(y,t-s)F(y,s)dsdy.\end{aligned}$$

Using Leibniz rule,

$$\frac{\partial}{\partial t}\int_0^t v(y,t-s)F(y,s)ds = v(y,0)F(y,t) + \int_0^t F(y,s)\partial_t v(y,t-s)ds,$$

where

$$\partial_t v(y,t-s) = \partial_t \int_0^L G(y,z,t-s)\phi(z)dz = \int_0^L a\phi''(z)G(y,z,t-s)dz$$

by *Problem 5.3.4*. Putting all these pieces together we get

$$\begin{aligned}\frac{d}{dt}\int_0^L \phi(x)u(x,t)dx &= \int_0^L \frac{\partial}{\partial t}\int_0^t v(y,t-s)F(y,s)dsdy \\ &= \int_0^L \left[v(y,0)F(y,t) + \int_0^t F(y,s)\left(\int_0^L a\phi''(z)G(y,z,t-s)dz\right)ds\right]dy.\end{aligned}$$

Note that $v(y,0) = f(y)$ by *Proposition 5.18* and reorganize the terms

$$\begin{aligned}\frac{d}{dt}\int_0^L \phi(x)u(x,t)dx &= \int_0^L \phi(y)F(y,t)dy + \int_0^L a\phi''(z)\left(\int_0^t\int_0^L G(z,y,t-s)F(y,s)dsdy\right)dz \\ &= \int_0^L \phi(y)F(y,t)dy + \int_0^L a\phi''(z)u(z,t)dz.\end{aligned}$$

Rewriting the integrals in terms of x and reorganizing we obtain the desired result,

$$\frac{d}{dt}\int_0^L \phi(x)u(x,t)dx = \int_0^L a\phi''(x)u(x,t)dx + \int_0^L \phi(x)F(x,t)dx.$$

Problem 5.4.14

Let $u : [0,L]\times[0,T] \to \mathbb{R}$ be continuous a twice continuously differentiable. Let $F : [0,L]\times[0,T] \to \mathbb{R}$ be continuous. Assume that

$$\begin{aligned}(\partial_t - \partial_x^2)u = F(x,t), &\qquad x \in [0,L],\ t \in [0,T], \\ \partial_x u(0,t) = 0 = \partial_x u(L,t), &\qquad t \in [0,T], \\ u(x,0) = 0, &\qquad x \in [0,L].\end{aligned}$$

Derive a Fourier series representation of u.

Solution: Let $\tilde{G}$ be the "modified Green's function" from *Problem 3.5.3* and set

$$u(x,t) = \int_0^L\int_0^t \tilde{G}(x,y,t-s)F(y,s)dsdy.$$

Interchanging integration and the series representation of $\tilde{G}$, we obtain the Fourier cosine representation of u,

$$\begin{aligned}u(x,t) &= \int_0^L\int_0^t \sum_{m=0}^{\infty} \tilde{G}_m(x,y,t-s)F(y,s)dsdy \\ &= \sum_{m=0}^{\infty}\int_0^L\int_0^t \tilde{G}_m(x,y,t-s)F(y,s)dsdy \\ &= \sum_{m=0}^{\infty} u_m(x,t), \qquad \lambda_m = m\frac{\pi}{L},\end{aligned}$$

where

$$\begin{aligned}u_m(x,t) &= \int_0^L\int_0^t \tilde{G}_m(x,y,t-s)F(y,s)dsdy \\ &= \frac{2}{L}\cos(\lambda_m x)\int_0^L\int_0^t \cos(\lambda_m y)e^{-a\lambda_m^2(t-s)}F(y,s)dsdy, \qquad m \geq 1,\end{aligned}$$

and

$$u_0(x,t) = \frac{1}{L}\int_0^L \int_0^t F(y,s)dsdy.$$

Let us define $\alpha(s) = \int_0^L \cos(\lambda_m y)F(y,s)dy$ to make the process shorter, then we have that

$$u_m(x,t) = \frac{2}{L}\cos(\lambda_m x)\int_0^t \alpha(s)e^{-a\lambda_m^2(t-s)}ds, \qquad m \geq 1,$$

We now prove that u satisfy the PDE by first inspecting the terms separately. First we calculate $\partial_t u_m$ using Leibniz rule,

$$\begin{aligned}
\partial_t u_m(x,t) &= \frac{2}{L}\cos(\lambda_m x)\left[\alpha(t) + \int_0^t \alpha(s)\left(-a\lambda_m^2 e^{-a\lambda_m^2(t-s)}\right)\right]ds \\
&= \frac{2}{L}\cos(\lambda_m x)\alpha(t) - a\lambda_m^2\frac{2}{L}\cos(\lambda_m x)\int_0^t \alpha(s)e^{-a\lambda_m^2(t-s)}ds \\
&= \frac{2}{L}\cos(\lambda_m x)\alpha(t) - a\lambda_m^2 u_m(x,t).
\end{aligned}$$

On the other hand,

$$\partial_x^2 u_m(x,t) = -\lambda_m^2 u_m(x,t).$$

Then,

$$\begin{aligned}
(\partial_t - a\partial_x^2)u_m &= \frac{2}{L}\cos(\lambda_m x)\alpha(t) - a\lambda_m^2 u_m(x,t) + a\lambda_m^2 u_m(x,t) \\
&= \frac{2}{L}\cos(\lambda_m x)\int_0^L \cos(\lambda_m y)F(y,t)dy \\
&= \frac{2}{L}\int_0^L \cos(\lambda_m y)F(y,t)dy\cos(\lambda_m x) =: F_m(x,t).
\end{aligned}$$

For the case of $m = 0$,

$$\partial_t u_0(x,t) = \partial_t \frac{1}{L}\int_0^L \int_0^t F(y,s)dsdy = \frac{1}{L}\int_0^L F(y,t)dy,$$

$$\partial_x^2 u_0(x,t) = 0.$$

Then,

$$(\partial_t - a\partial_x^2)u_0 = \frac{1}{L}\int_0^L F(y,t)dy =: F_0(x,t)$$

Hence, we see that u satisfies the PDE,

$$\begin{aligned}(\partial_t - a\partial_x^2)u(x,t) &= (\partial_t - a\partial_x^2)\sum_{m=0}^{\infty} u_m(x,t) \\ &= \sum_{m=0}^{\infty} (\partial_t - a\partial_x^2)u_m(x,t) \\ &= (\partial_t - a\partial_x^2)u_0(x,t) + \sum_{m=1}^{\infty} (\partial_t - a\partial_x^2)u_m(x,t) \\ &= F_0(x,t) + \sum_{m=1}^{\infty} F_m(x,t) = F(x,t).\end{aligned}$$

To check that satisfies the boundary conditions it sufices to check that each term of the series satisfies them,

$$\partial_x u_m(x,t) = \lambda_m \frac{2}{L}\sin(\lambda_m x)\int_0^t \alpha(s)e^{-a\lambda_m^2(t-s)}ds$$

with

$$\partial_x u_m(0,t) = 0\partial_x u_m(L,t),$$

since the sine cancels at the boundaries. Since $u_0(x,t)$ does not depend on x, the boundary conditions are trivially satisfied. Note that u satisfies the initial conditions as well.