Partial Differential Equations
TA Homework 12

Francisco Jose Castillo Carrasco

April 5, 2018

Problem 6.2.1

Use the properties of the Greens function to derive Theorem 6.4 from Theorem 6.3.

Solution: Let Ω be a circular disk in $\mathbb{R}^2$ and $f : \partial\Omega \to \mathbb{R}$ be continuous and 2π-periodic. Define the sequence (f_n), $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(\theta) = \int_\theta^{\theta+\frac{1}{n}} f(\eta)d\eta = \int_0^1 f(\theta + \frac{\tau}{n})d\tau.$$

To prove that f_n is Lipschitz continuous we use the Mean Value Theorem, but first we take the derivative using Leibniz formula

$$f_n'(\theta) = n\frac{\partial}{\partial\theta}\int_\theta^{\theta+\frac{1}{n}} f(\eta)d\eta = n\left[f\left(\theta+\frac{1}{n}\right) - f(\theta)\right].$$

Thus, since f is bounded, f_n' is also bounded and

$$|f_n'(\theta)| \le M_n, \quad \forall\theta \in \mathbb{R}.$$

by the MVT, for each n, there exists an ξ_n such that,

$$f_n(x) - f_n(y) = f_n'(\xi_n)(x-y), \qquad \forall x, y \in \mathbb{R}, \quad \xi_n \in [x, y].$$

Then,

$$|f_n(x) - f_n(y)| = |f_n'(\xi_n)||x-y| \le M_n|x-y|.$$

Hence, f_n is Lipschitz continuous. Further, $f_n \to f$ as $n \to \infty$ uniformly if $|f_n - f| \to 0$ as $n \to \infty$ uniformly. Then,

$$\begin{aligned}
|f_n - f| &= \sup\left|\int_0^1 f\left(\theta+\frac{\eta}{n}\right)d\eta - f(\theta)\right| \\
&= \sup\left|\int_0^1 \left[f\left(\theta+\frac{\eta}{n}\right) - f(\theta)\right]d\eta\right| \\
&\le \sup\int_0^1 \left|f\left(\theta+\frac{\eta}{n}\right) - f(\theta)\right|d\eta \\
&= \sup\int_0^1 \left|f\left(\theta+\frac{\eta}{n}\right) - f(\theta)\right|d\eta,
\end{aligned}$$

which goes to zero as $n \to \infty$ since f is continuous. Next, define

$$v_n(r,\theta) = \int_{-\pi}^{\pi} f_n(\eta) G(r, \eta - \theta) d\eta,$$

and

$$v(r,\theta) = \int_{-\pi}^{\pi} f(\eta) G(r, \eta - \theta) d\eta$$

for $0 \leq r < a$, $\theta \in \mathbb{R}$. Then,

$$\begin{aligned}
|v_n - v| &= \sup_r \sup_\theta \left| \int_{-\pi}^{\pi} f_n(\eta) G(r, \eta - \theta) d\eta - \int_{-\pi}^{\pi} f(\eta) G(r, \eta - \theta) d\eta \right| \\
&= \sup_r \sup_\theta \left| \int_{-\pi}^{\pi} G(r, \eta - \theta) \left[f_n(\eta) - f(\eta) \right] d\eta \right| \\
&\leq \sup_r \sup_\theta \int_{-\pi}^{\pi} G(r, \eta - \theta) \left| f_n(\eta) - f(\eta) \right| d\eta.
\end{aligned}$$

Recall that the Green's function is positive and bounded. Hence, since we just proved that $f_n \to f$ as $n \to \infty$ uniformly, $v_n \to v$ as $n \to \infty$ uniformly as well. Now we would like to show that $v(r,\theta) \to f(\theta)$ as $r \to a$ uniformly. Consider

$$|v(r,\theta) - f(\theta)| \leq |v(r,\theta) - v_n(r,\theta)| + |v_n(r,\theta) - f_n(\theta)| + |f_n(\theta) - f(\theta)|,$$

where we have already proved that the first and last terms go to zero uniformly as $n \to \infty$. Therefore, we are left to prove that $|v_n(r,\theta) - f_n(\theta)| \to 0$ as $n \to a$,

$$\begin{aligned}
|v_n(r,\theta) - f_n(\theta)| &= \left| \int_{-\pi}^{\pi} G(r, \eta - \theta) f_n(\eta) d\eta - f_n(\theta) \right| \\
&= \left| \int_{-\pi}^{\pi} G(r, \eta - \theta) f_n(\eta) d\eta - \int_{-\pi}^{\pi} G(r, \eta - \theta) f_n(\theta) d\eta \right| \\
&= \left| \int_{-\pi}^{\pi} G(r, \eta - \theta) \left[f_n(\eta) - f_n(\theta) \right] d\eta \right| \\
&\leq \int_{-\pi}^{\pi} G(r, \eta - \theta) \left| f_n(\eta) - f_n(\theta) \right| d\eta \\
&\leq \int_{-\pi}^{\pi} G(r, \eta - \theta) M_n \left| \eta - \theta \right| d\eta
\end{aligned}$$

,

where we have used that $\int_{-\pi}^{\pi} G(r, \eta - \theta) d\eta = 1$. Now, since $G(r, \eta - \theta) \to 0$ as $r \to a$,

$$|v_n(r,\theta) - f_n(\theta)| \to 0 \text{ as } r \to a.$$

Thus, we also have that

$$|v(r,\theta) - f(\theta)| \to 0 \text{ as } r \to a.$$

Finally, since the Green's function is not defined at $r = a$, we extend v by making $v(a,\theta) = f(\theta)$. Since v is continuous on $[0,a) \times \mathbb{R}$ and $|v(r,\theta) - f(\theta)| \to 0$ as $r \to a$, for every $\epsilon > 0$ there exists some δ such that, if $|r - a| < \delta$, $|v(r,\theta) - f(\theta)| < \epsilon$. Hence, v is continuous on $[0.a] \times \mathbb{R}$.

Problem 6.2.2

Part a) Let $|x|$ be the Eucliden norm of $x \in \mathbb{R}^n$. Show: $\Delta|x| = \frac{n-1}{x}$ for all $x \in \mathbb{R}^n,\ x \neq 0$.

Solution: Recall that

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

Then,

$$\partial_j |x| = \frac{1}{2} \frac{2x_j}{\sqrt{\sum_{i=1}^{n} x_i^2}} = \frac{x_j}{|x|},$$

and

$$\partial_j^2 |x| = \frac{|x| - x_j \partial_j |x|}{|x|^2} = \frac{|x| - \frac{x_j^2}{|x|}}{|x|^2} = \frac{1}{|x|} - \frac{1}{|x|^3} x_j^2.$$

Finally,

$$\begin{aligned} \Delta|x| = \sum_{j=1}^{n} \partial_j^2 |x| = \sum_{j=1}^{n} \left(\frac{1}{|x|} - \frac{1}{|x|^3} x_j^2 \right) \\ &= \frac{n}{|x|} - \frac{1}{|x|^3} \sum_{j=1}^{n} x_j^2 \\ &= \frac{n}{|x|} - \frac{1}{|x|^3} |x|^2 \\ &= \frac{n-1}{|x|} \qquad \forall x \in \mathbb{R}^n,\ x \neq 0. \end{aligned}$$

Part b) Let $y \in \mathbb{R}^n$ be fixed and define $u : \mathbb{R}^n \to \mathbb{R}$ by

$$u(x) = \frac{|y|^2 - |x|^2}{|x-y|^n}.$$

Show that $\Delta u = 0$ for all $x \in \mathbb{R}^n,\ x \neq y$.

Solution: Let $z = x - y$. Then, by the previous part of the problem,

$$\partial_j |z| = \frac{z_j}{|z|},$$

and

$$\partial_j (|z|^{-n}) = -n|z|^{-n-1} \frac{z_j}{|z|} = -n|z|^{-(n+2)} z_j.$$

In addition, let's compute

$$\partial_j \left(|y|^2 - |x|^2\right) = -2|x|\partial_j|x| = -2x_j.$$

Expressing $u(x) = u(x) = \left(|y|^2 - |x|^2\right)|z|^{-n}$,

$$\begin{aligned}\partial_j u(x) &= \partial_j \left(|y|^2 - |x|^2\right)|z|^{-n} + \left(|y|^2 - |x|^2\right)\partial_j \left(|z|^{-n}\right)\\ &= -2x_j|z|^{-n} - n\left(|y|^2 - |x|^2\right)|z|^{-(n+2)}z_j.\end{aligned}$$

Further, using the product rule like before,

$$\begin{aligned}\partial_j^2 u(x) &= -2|z|^{-n} + 2nx_j|z|^{-(n+2)}z_j + 2nx_j|z|^{-(n+2)}z_j\\ &\qquad + n(n+2)\left(|y|^2 - |x|^2\right)|z|^{-(n+4)}z_j^2 - n\left(|y|^2 - |x|^2\right)|z|^{-(n+2)}\\ &= -2|z|^{-n} + 4nx_j|z|^{-(n+2)}z_j + (n^2+2n)\left(|y|^2 - |x|^2\right)|z|^{-(n+4)}z_j^2 - n\left(|y|^2 - |x|^2\right)|z|^{-(n+2)}.\end{aligned}$$

Finally,

$$\begin{aligned}\Delta u(x) &= \sum_{j=1}^{n} \partial_j^2 u(x)\\ &= -2n|z|^{-n} + 4n|z|^{-(n+2)}\sum_{j=1}^{n} x_j z_j + (n^2+2n)\left(|y|^2 - |x|^2\right)|z|^{-(n+4)}|z|^2 - n^2\left(|y|^2 - |x|^2\right)|z|^{-(n+2)}\\ &= z^{-(n+2)}\left[-2n|z|^2 + 4n\sum_{j=1}^{n} x_j z_j + (n^2+2n)\left(|y|^2 - |x|^2\right) - n^2\left(|y|^2 - |x|^2\right)\right]\\ &= z^{-(n+2)}\left[-2n|z|^2 + 4n\sum_{j=1}^{n} x_j z_j + 2n\left(|y|^2 - |x|^2\right)\right].\end{aligned}$$

We stop now to calculate two terms of the previous expression separately,

$$|z|^2 = \sum_{j=1}^{n}(x_j - y_j)^2 = (x_j^2 + y_j^2 - 2x_jy_j) = |x|^2 + |y|^2 - 2\sum_{j=1}^{n} x_jy_j,$$

and

$$\sum_{j=1}^{n} x_jz_j = \sum_{j=1}^{n} x_j(x_j - y_j) = \sum_{j=1}^{n} x_j^2 - \sum_{j=1}^{n} x_jy_j = |x|^2 - \sum_{j=1}^{n} x_jy_j.$$

Plugging the previous results in we get,

$$\Delta u(x) = z^{-(n+2)}\left[-2n|x|^2 - 2n|y|^2 + 4n\sum_{j=1}^{n} x_jy_j + 4n|x|^2 - 4n\sum_{j=1}^{n} x_jy_j + 2n|y|^2 - 2n|x|^2\right].$$

Hence, since all the terms cancel,

$$\Delta u(x) = 0 \quad \forall x \in \mathbb{R}^n,\ x \neq y.$$

Problem 6.3.1

Prove from scratch: if $u : \overline{\Omega} \to \mathbb{R}$ is continuous and is twice partially differentiable on Ω and satisfies

$$\Delta u \leq 0, \qquad x \in \Omega,$$

then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

Solution: Let us assume first that $\Delta u < 0$ for all $x \in \Omega$. Then, since u is continuous on $\overline{\Omega}$, there exists some $y \in \overline{\Omega}$ such that

$$u(y) = \min_{\overline{\Omega}} u.$$

Further, suppose that the minimum is not on the boundary, i.e., $y \in \overline{\Omega}$. Then, since it is a minimum, $\partial_j u(y) = 0$ and $\partial_j^2 u(y) \geq 0$ for all $j = 1, ..., n$. Then,

$$\Delta u(y) \geq 0,$$

which is a contradiction to our assumption. Hence, the minimum must be on the boundary,

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

It is left to prove that this is also the case if $\Delta u \leq 0$ for all $x \in \Omega$. Let $\epsilon > 0$ and

$$u_\epsilon(y) = u(y) - \epsilon|y|^2, \qquad y \in \overline{\Omega}.$$

Then,

$$\partial_j u_\epsilon(y) = \partial_j u(y) - 2\epsilon x_j,$$

and

$$\partial_j^2 u_\epsilon(y) = \partial_j^2 u(y) - 2\epsilon,$$

for $j = 1, ..., n$. Note that the derivatives were calculated with detail in the previous problem. Therefore,

$$\Delta u_\epsilon = \Delta u - 2n\epsilon < 0, \qquad x \in \Omega.$$

By the first part of this problem, since $\Delta u_\epsilon < 0, \quad x \in \Omega$,

$$\min_{\overline{\Omega}} u_\epsilon = \min_{\partial\Omega} u_\epsilon.$$

Finally, from the definition of u_ϵ we have that the minimum

$$\min_{\overline{\Omega}} u \geq \min_{\overline{\Omega}} u_\epsilon,$$

which is placed on the boundary,

$$\min_{\overline{\Omega}} u_\epsilon = \min_{\partial\Omega} u_\epsilon.$$

Since Ω is bounded, there exists some $c > 0$ such that $|x| < c$ and

$$u_\epsilon(y) \geq u(y) - \epsilon c^2.$$

Then,

$$\min_{\partial\Omega} u_\epsilon \geq \min_{\partial\Omega} u - \epsilon c^2,$$

and we have that

$$\min_{\overline{\Omega}} u \geq \min_{\overline{\Omega}} u_\epsilon = \min_{\partial\Omega} u_\epsilon \geq \min_{\partial\Omega} u - \epsilon c^2.$$

Since the inequality holds for all $\epsilon > 0$, we have proved that

$$\min_{\overline{\Omega}} u \geq \min_{\partial\Omega} u.$$

The inequality in the other direction is trivially proved since the boundary is contained in the closure of Ω, $\partial\Omega \subseteq \overline{\Omega}$. Hence,

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$