Numerical Methods for PDEs
Homework 7

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Problem 1

Derive the entropy advection equation $s_t + us_x = 0$ from the Euler equations and the expression for the entropy $s = c_V \ln(P/\rho^\gamma)$ of a polytropic gas. *Hint:* Start with the entropy advection equation and derive $E_t + (u(E+P))_x = 0$ with $P = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$, making use of $\rho_t + (\rho u)_x = 0$ and $(\rho u)_t + (\rho u^2 + P)_x = \rho u_t + \rho u u_x + P_x = 0$ if needed.

Solution: From the definition of the entropy we obtain the partial derivatives

$$s_t = c_V \left(\frac{P_t}{P} - \gamma \frac{\rho_t}{\rho} \right),$$
$$s_x = c_V \left(\frac{P_x}{P} - \gamma \frac{\rho_x}{\rho} \right).$$

Then, the entropy advection equation gives

$$s_t + us_x = 0 \Rightarrow \rho P_t - \gamma P \rho_t + u\rho P_x - \gamma u P \rho_x = 0,$$

where we have multiplied the whole equation by ρP. We now include the easy computed partial derivatives of P,

$$P_t = (\gamma - 1) \left(E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t \right),$$
$$P_x = (\gamma - 1) \left(E_x - \frac{1}{2} u^2 \rho_x - \rho u u_x \right),$$

into the equation to get

$$\rho (\gamma - 1) \left(E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t \right) - \gamma P \rho_t + u\rho (\gamma - 1) \left(E_x - \frac{1}{2} u^2 \rho_x - \rho u u_x \right) - \gamma u P \rho_x = 0.$$

Dividing by $\rho (\gamma - 1)$ and computing the brackets we obtain

$$E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + uE_x - \frac{1}{2} u^3 \rho_x - \rho u^2 u_x - \frac{\gamma}{\gamma - 1} uP \frac{\rho_x}{\rho} = 0.$$

Using the continuity equation $\rho_t + u\rho_x + \rho u_x = 0$, we simplify the previous equation,

$$E_t - \rho u u_t - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + uE_x - \frac{1}{2} \rho u^2 u_x - \frac{\gamma}{\gamma - 1} uP \frac{\rho_x}{\rho} = 0.$$

Further, we use the Euler momentum equation in the form $-\rho u u_t = \rho u^2 u_x + u P_x$ to obtain

$$\begin{aligned}
E_t + \rho u^2 u_x + u P_x - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + u E_x - \frac{1}{2}\rho u^2 u_x - \frac{\gamma}{\gamma - 1} u P \frac{\rho_x}{\rho} &= 0, \\
E_t + u(E_x + P_x) + \frac{1}{2}\rho u^2 u_x - \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\left(\rho_t + u\rho_x\right) &= 0, \\
E_t + u(E_x + P_x) + \frac{1}{2}\rho u^2 u_x + \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\rho u_x &= 0,
\end{aligned}$$

where we have used once more the continuity equation in the last step. Note that

$$\frac{\gamma}{\gamma - 1} P = \gamma E - \frac{1}{2}\gamma \rho u^2.$$

Then,

$$\begin{aligned}
E_t + u(E_x + P_x) + u_x\left(\frac{1}{2}\rho u^2 + \gamma E - \frac{1}{2}\gamma\rho u^2\right) &= 0, \\
E_t + u(E_x + P_x) + u_x\left(\gamma E - (\gamma - 1)\frac{1}{2}\rho u^2\right) &= 0.
\end{aligned}$$

To finish, note that

$$E + P = \gamma E - (\gamma - 1)\frac{1}{2}\rho u^2.$$

Hence,

$$\begin{aligned}
E_t + u(E_x + P_x) + u_x\left(\gamma E - (\gamma - 1)\frac{1}{2}\rho u^2\right) &= 0, \\
E_t + u(E_x + P_x) + u_x\left(E + P\right) &= 0.
\end{aligned}$$

Finally, we obtain the equation that completes the proof,

$$E_t + \left(u\left(E + P\right)\right)_x = 0.$$

Problem 2

Show that the solution of the Riemann problem

$$\rho_L = 2,\ u_L = 1,\ P_L = 3; \quad \rho_R = 1,\ u_R = 0,\ P_R = 1$$

with $\gamma = 1.5$ is a single shock wave propagating to the right. Calculate the shock speed s. For $\gamma = 1.5$, $E = \frac{1}{2}\rho u^2 + 2P$. *Hint:* Use the jump conditions.

Solution: We can easily compute s as

$$s = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{2 \cdot 1 - 1 \cdot 0}{2 - 1} = 2,$$

and check that this result is right by the following:

$$s\,(\rho_L u_L - \rho_R u_R) = \rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R, \qquad s\,(E_L - E_R) = u_L(E_L + P_L) - u_R(E_R + P_R),$$
$$2\,(2\cdot 1 - 1\cdot 0) = 2\cdot 1^2 + 3 - 1\cdot 0^2 - 1, \qquad 2\cdot(7-2) = 1\cdot(7+3) - 0\cdot(2+1),$$
$$4 = 4, \qquad 10 = 10,$$

where we have calcultated $E_L = \frac{1}{2}\rho_L u_L^2 + 2P_L = 7$ and $E_R = \frac{1}{2}\rho_R u_R^2 + 2P_R = 2$

With a wall BC at $x = 1$, analytically calculate the solution after reflection of the shock wave. *Hint:* Show that the exact reflected shock solution is

$$\rho = 3.6,\ \ u = 0,\ \ P = 7.5,\ \ s_r = -1.25$$

where s_r is the velocity of the reflected shock.

Solution: In this case we repeat the process assuming the same conditions on the left. We can check that given the right values of ρ, u, and P, the value s_r obtains matches the solution:

$$s = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{2\cdot 1 - 3.6\cdot 0}{2 - 3.6} = -1.25,$$
$$s = \frac{\rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R}{\rho_L u_L - \rho_R u_R} = \frac{2\cdot 1^2 + 3 - 3.6\cdot 0^2 - 7.5}{2\cdot 1 - 3.6\cdot 0} = -1.25,$$
$$s = \frac{u_L(E_L + P_L) - u_R(E_R + P_R)}{E_L - E_R} = \frac{1\cdot(7+3) - 0\cdot(15+7.5)}{7 - 15} = -1.25,$$

where we have calcultated $E_R = \frac{1}{2}\rho_R u_R^2 + 2P_R = 15$

Problem 3

Show that the Lax-Wendroff method is second-order accurate for $u_t + Au_x = 0$ using the definition of the LTE.

Solution: To prove that the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{1}{2}A\frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}A^2\frac{\Delta t^2}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right),$$

is second order accurate for $u_t + Au_x = 0$, we start by Taylor expanding

$$u_j^{n+1} = u_j^n + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} + \mathcal{O}(\Delta t^4),$$
$$u_{j\pm1}^n = u_j^n \pm \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} \pm \frac{\Delta x^3}{6}u_{xxx} + \mathcal{O}(\Delta x^4),$$

and substituting them into the Lax-Wendroff scheme,

$$u_j^n + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} = u_j^n - \frac{A\Delta t}{2\Delta x}\left(2\Delta x u_x + \frac{\Delta x^3}{3}u_{xxx}\right) + \frac{A^2\Delta t^2}{2\Delta x^2}\Delta x^2 u_{xx} + \Delta t\tau.$$

Doing some algebraic manipulations we reach,

$$\begin{aligned} u_t + Au_x &= -\frac{\Delta t}{2}u_{tt} - A\frac{\Delta x^2}{6}u_{xxx} + \frac{1}{2}A^2\Delta t u_{xx} - \frac{\Delta t^2}{6}u_{ttt} + \tau, \\ \tau &= \frac{\Delta t}{2}u_{tt} + A\frac{\Delta x^2}{6}u_{xxx} - \frac{1}{2}A^2\Delta t u_{xx} + \frac{\Delta t^2}{6}u_{ttt}, \end{aligned}$$

where we have used that $u_t + Au_x = 0$. Using the original PDE we obtain that

$$\begin{aligned} u_t = -Au_x &\Rightarrow u_{tt} = A^2 u_{xx}, \\ &\Rightarrow u_{ttt} = -A^3 u_{xxx}. \end{aligned}$$

Hence,

$$\begin{aligned} \tau &= \frac{\Delta t}{2}A^2 u_{xx} + A\frac{\Delta x^2}{6}u_{xxx} - \frac{1}{2}A^2\Delta t u_{xx} + \frac{\Delta t^2}{6}A^3 u_{ttt}, \\ &= A\frac{\Delta x^2}{6}u_{xxx} + \frac{\Delta t^2}{6}A^3 u_{ttt}. \end{aligned}$$

Thus, the Lax-Wendroff scheme is second order accurate for $u_t + Au_x = 0$.

Problem 4

Using **weno3.m**, investigate the effects of the CFL factor r on the solution of the Riemann problem

$$\rho_L = 1,\ u_L = 0,\ p_L = 1; \quad \rho_R = 0.125,\ u_R = 0,\ p_R = 0.1$$

with $\gamma = 1.4$. Take 200 Δx and CFL factor $r = 0.1$, 0.5, 0.9. Turn in the Density plots (computed vs. exact solution) at time $t = 0.2$ for each of three cases. Briefly discuss your results.

Solution: In the following figures we can see the solution to the problem compared to the exact solution. Since in all cases the method is stable, the solutions are mostly undistinguishible. However, the smaller the CFL factor, the longer it takes the simulation to finish.

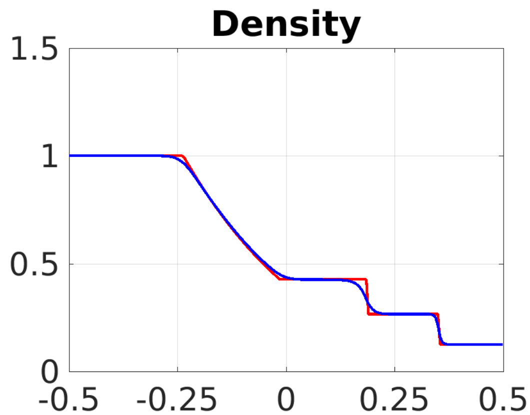

Figure 1: $CFL = 0.1$.

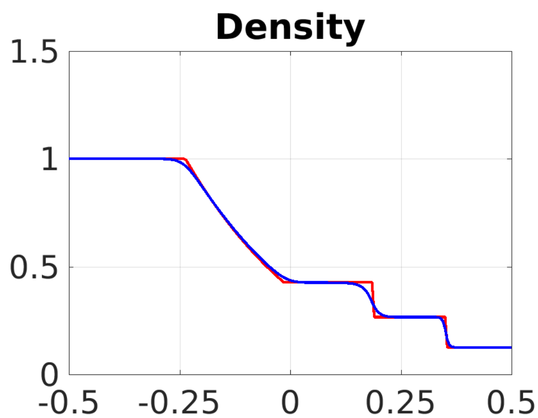

Figure 2: $CFL = 0.5$.

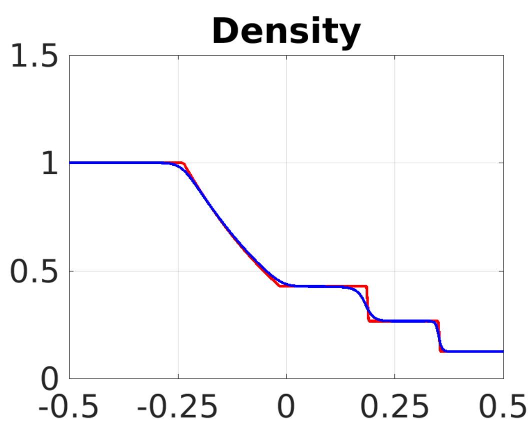

Figure 3: $CFL = 0.9$.