Fourier Analysis and Wavelets
Homework 4

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Problem 2

Prove THEOREM 3.4.

- Shifts or translations. If $y \in S_n$ and $z_k = y_{k+1}$, then $\mathcal{F}[z]_j = w^j \mathcal{F}[y]_j$, where $w = e^{i\frac{2\pi}{n}}$.

Solution: This is simply proved by calculating the Discrete Fourier Transform of z_k.

$$
\begin{aligned}
\mathcal{F}[z]_k &= \sum_{j=0}^{n-1} z_j \overline{w}^{jk} \\
&= \sum_{j=0}^{n-1} y_{j+1} \overline{w}^{jk} \qquad \text{making} \ \ l = j+1 \\
&= \sum_{l=1}^{n} y_l \overline{w}^{(l-1)k} \\
&= \sum_{l=1}^{n} y_l \overline{w}^{lk} \overline{w}^{-k} \\
&= \overline{w}^{-k} \sum_{l=1}^{n} y_l \overline{w}^{lk} \\
&= \overline{w}^{-k} \sum_{l=0}^{n-1} y_l \overline{w}^{lk} \\
&= w^k \mathcal{F}[y]_k,
\end{aligned}
$$

where in the last step we have applied the definition of the DFT and not that $\overline{w}^{-k} = w^k$.

- Convolutions. If $y \in S_n$ and $z \in S_n$, then the sequence defined by

$$
[y * z]_k := \sum_{j=0}^{n-1} y_j z k - j
$$

is also in S_n.

Solution: To prove periodicity of the convolution we compute

$$[y * z]_{k+n} = \sum_{j=0}^{n-1} y_j z_{k-j+n} = \sum_{j=0}^{n-1} y_j z_{k-j+n} = [y * z]_k,$$

where we have used that $z \in S_n$. Hence, $[y * z]_k \in S_n$.

- The Convolution Theorem. $\mathcal{F}[y * z]_k = \mathcal{F}[y]_k \mathcal{F}[z]_k$

Solution: This is easily proved starting from the DFT of the convolution,

$$\begin{aligned}
\mathcal{F}[y * z]_k &= \sum_{j=0}^{n-1} [y * z]_j \overline{w}^{jk} \\
&= \sum_{j=0}^{n-1} \sum_{m=0}^{n-1} y_m z_{j-m} \overline{w}^{jk} \\
&= \sum_{j=0}^{n-1} \sum_{m=0}^{n-1} y_m z_{j-m} \overline{w}^{(j-m)k} \overline{w}^{mk} \\
&= \sum_{m=0}^{n-1} y_m \overline{w}^{mk} \sum_{j=0}^{n-1} z_{j-m} \overline{w}^{(j-m)k}.
\end{aligned}$$

Applying perioidicity to the second sum, $\sum_{j=0}^{n-1} z_{j-m} \overline{w}^{(j-m)k} = \sum_{j=0}^{n-1} z_j \overline{w}^{jk}$, we obtain the desired result:

$$\mathcal{F}[y * z]_k = \sum_{m=0}^{n-1} y_m \overline{w}^{mk} \sum_{j=0}^{n-1} z_j \overline{w}^{jk} = \mathcal{F}[y]_k \mathcal{F}[z]_k$$

- If $y \in S_n$ is a sequence of real numbers, then

$$\mathcal{F}[y]_{n-k} = \overline{\mathcal{F}[y]_k}, \qquad \text{for} \;\; 0 \leq k - n \leq n - 1.$$

Solution: Like in the previous cases, computations suffice to prove the statement. We start with

$$\mathcal{F}[y]_{n-k} = \sum_{j=0}^{n-1} y_j \overline{w}^{j(n-k)} = \sum_{j=0}^{n-1} y_j \overline{w}^{jn} \overline{w}^{-jk}.$$

Note that $\overline{w}^{jn} = 1$. Then,

$$\mathcal{F}[y]_{n-k} = \sum_{j=0}^{n-1} y_j \overline{w}^{-jk} = \sum_{j=0}^{n-1} \overline{y_j \overline{w}^{jk}} = \overline{\sum_{j=0}^{n-1} y_j \overline{w}^{jk}} = \overline{\mathcal{F}[y]_k}.$$

Problem 7

Filtering. Let

$$f(t) = e^{-t^2/10} \left(\sin 2t + 2 \cos 4t + 0.4 \sin t \sin 50t \right).$$

Discretize f by setting $y_k = f(2k\pi/256)$, $k = 1 \ldots 256$. Use the fast Fourier transform to compute $\hat{y}_k$ for $0 \leq k \leq 256$. Filter out the high-frequency terms by setting $\hat{y}_k = 0$ for $m \leq k \leq 255 - m$ with $m = 6$; then apply the inverse fast Fourier transform to this new set of $\hat{y}_k$ to compute the y_k(now filtered); plot the new values of y_k and compare with the original function. Experiment with other values of m.

Solution: *The solution to this problem below, with the Matlab code.*

Problem 8

Compression. Let $tol = 1.0$. In exercise 7, if $|\hat{y}_k| < tol$ then set $\hat{y}_k$ equal to zero. Apply the inverse fast Fourier transform to this new set of $\hat{y}_k$ to compute the y_k; plot the new values of y_k and compare with the original function. Experiment with other values of tol. Keep track of the percentage of Fourier coefficients which have been filtered out. Compute hte relative l^2 error of the compressed signal as compared with the original signal.

Solution: *The solution to this problem below, with the Matlab code.*