CHAPTER 1: FOURIER SERIES

REAL FOURIER SERIES **Orthonormal Basis:** The set of functions $\{\frac{\sin(k\pi x/a)}{\sqrt{\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{\cos(k\pi x/a)}{\sqrt{\pi}}\}$ with $k = 1, 2, \dots,$ is an orthonormal set of functions in $L^2([-a,a])$. **Fourier Coefficients:** If $f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi t/a) + \sum_{k=1}^{\infty} b_k \sin(k\pi t/a)$ on the interval $-a \le t \le a$, then $a_0 = \frac{1}{2a}\int_{-a}^{a} f(t)dt$, $a_k = \frac{1}{a}\int_{-a}^{a} f(t)\cos(k\pi t/a)dt$ and $b_k = \frac{1}{a}\int_{-a}^{a} f(t)\sin(k\pi t/a)dt$.

COMPLEX FOURIER SERIES **Orthonormal Basis:** The set of functions $\{\frac{1}{\sqrt{2a}}e^{i\frac{n\pi}{a}t}, n = 0, \pm1, \pm2, \dots\}$ is an orthonormal basis for $L^2([-a,a])$. **Fourier Coefficients:** If $f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i\frac{n\pi}{a}t}$, then $\alpha_n = \frac{1}{2a}\int_{-a}^{a} f(t)e^{-i\frac{n\pi}{a}t}dt$

CONVERGENCE THEOREMS **Riemann-Lebesgue Lemma:** Suppose f is a piecewise continuous function on the interval $[a,b]$. Then $\lim_{k\to\infty}\int_a^b f(x)\cos(kx)dx = \lim_{k\to\infty}\int_a^b f(x)\sin(kx)dx = 0$. **Convergence at a Point of Continuity:** Suppose f is a continuous and 2π-periodic function. Then for each point x, where the derivative of f is defined, the Fourier series of f converges to $f(x)$. **Convergence at a Point of Discontinuity:** Suppose f is periodic function and piecewise continuous. Suppose x is a point where f is left and right differentiable (but not necessarily continuous). Then the Fourier series of f at x converges to $\frac{f(x-0)+f(x+0)}{2}$, i.e., converges to the average of the left and right limits of f. **Uniform Convergence:** The Fourier series of a continuous, piecewise smooth 2π-periodic function $f(x)$ converges uniformly to $f(x)$ on $[-\pi,\pi]$. **Lemma 1.33:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$ with $\sum_{k=1}^{\infty}|a_k| + |b : k| < \infty$. Then the Fourier series converges uniformly and absolutely to the function $f(x)$. **Convergence in the Mean:** Suppose f is an element of $L^2([-\pi,\pi])$. Let $f_N(x) = a_0 + \sum_{k=1}^{N} a_k\cos(kx) + \sum_{k=1}^{N} b_k\sin(kx)$, where a_k and b_k are the Fourier coefficients of f. Then f_N converges to f in $L^2([-\pi,\pi])$, that is, $||f_N - f||_{L^2} \to 0$ as $N \to \infty$. **Parseval's Equation - Real Version:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx) \in L^2[-\pi,\pi]$. Then $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = 2|a_0|^2 + \sum_{k=1}^{\infty}(|a_k|^2 + |b_k|^2)$. **Parseval's Equation - Complex Version:** Suppose $f(x) = \sum_{k=-\infty}^{\infty}\alpha_k e^{ikx} \in L^2[-\pi,\pi]$. Then
$\frac{1}{2\pi}||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{k=-\infty}^{\infty}|\alpha_k|^2$.

CHAPTER 2:FOURIER TRANSFORM

Definition: If f is a continuously differentiable function with $\int_{-\infty}^{\infty}|f(t)|dt < \infty$, then $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\lambda)e^{i\lambda t}d\lambda$, where $\hat{f}(\lambda)$ is the Fourier transform of $f(t)$ given by $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\lambda t}dt$. **Properties:**

- $\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g]$ // $\mathcal{F}^{-1}[\alpha f + \beta g] = \alpha\mathcal{F}^{-1}[f] + \beta\mathcal{F}^{-1}[g]$
- $\mathcal{F}[t^n f(t)](\lambda) = i^n \frac{d^n}{d\lambda^n}\{\mathcal{F}[f](\lambda)\}$
- $\mathcal{F}^{-1}[\lambda^n f(\lambda)](t) = (-i)^n \frac{d^n}{dt^n}\{\mathcal{F}^{-1}[f](t)\}$
- $\mathcal{F}[f^{(n)}(t)](\lambda) = (i\lambda)^n\mathcal{F}[f](\lambda)$
- $\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = (-it)^n\mathcal{F}^{-1}[f](t)$
- $\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a}\mathcal{F}[f](\lambda)$
- $\mathcal{F}[f(bt)](\lambda) = \frac{1}{b}\mathcal{F}[f](\frac{\lambda}{b})$
- If $f(t<0) = 0$, then $\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\mathcal{L}[f](i\lambda)$, where $\mathcal{L}[f](s) = \int_0^{\infty} f(t)e^{-ts}dt$.

Convolution: Suppose f and g are two square integrable functions. The convolution of f and g is defined by $(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx = \int_{-\infty}^{\infty} f(x)g(t-x)dx$. **Fourier Transform of the Convolution:** $\mathcal{F}[f * g] = \sqrt{2\pi}\mathcal{F}[f]\cdot\mathcal{F}[g]$, $\quad \mathcal{F}^{-1}[\hat{f}\cdot\hat{g}] = \frac{1}{\sqrt{2\pi}}(f * g)$. **Pancherel Theorem:** The Fourier transform, and its inverse, preserves the L^2 inner product.
$\langle\mathcal{F}[f],\mathcal{F}[g]\rangle_{L^2} = \langle f, g\rangle_{L^2}$ and $\langle\mathcal{F}^{-1}[f],\mathcal{F}^{-1}[g]\rangle_{L^2} = \langle f, g\rangle_{L^2}$.

LINEAR FILTERS **Time Invariance:** A transformation L (mapping signals to signals) is said to be time-invariant if for any signal f and any real number a, $L[f_a](t) = (Lf)(t-a)$ for all t. In other words, L is time-invariant if the time shifted input signal $f(t-a)$ is transformed by L into the time shifted output signal $(Lf)(t-a)$. **Lemma 2.16:** Let L be a linear, time-invariant transformation and let λ be any fixed real number. Then, there is a function h with $L(e^{i\lambda t}) = \sqrt{2\pi}\hat{h}(\lambda)e^{i\lambda t}$. In other words, the output signal from a time-invariant filter of a sinusoidal input is also sinusoidal with the same frequency. **Convolution in Filters:** Let L be a linear, time-invariant transformation on the space of signals that are piecewise continuous functions. Then there exists an integrable function, h, such that $L(f) = f * h$ for all signals f. **Causal Filters:** A causal filter is one for which the output signal begins after the input signal has started to arrive. Let L be a time-invariant filter with response function h (i.e., $Lf = f * h$). L is a causal filter if and only if $h(t) = 0$ for all $t < 0$. **Theorem 2.20:** Suppose L is a causal filter with response function h. Then the system function associated with L is $\hat{h}(\lambda) = \frac{\mathcal{L}[h](i\lambda)}{\sqrt{2\pi}}$.

THE SAMPLING THEOREM **Definition 2.22:** A function f is said to be frequency band limited if there exists a constant $\Omega > 0$ such that $f(\lambda) = 0$ for $|\lambda| > \Omega$. Note: Ω is the smallesq frequency for which the preceding equation is true. **Shannon-Whittaker Sampling Theorem:** Suppose that $\hat{f}(\lambda)$ is piecewise smooth and continuous and that $\hat{f}(\lambda) = 0$ for $|\lambda| > \Omega$, where Ω is some fixed, positive frequency. Then $f = \mathcal{F}^{-1}[f]$ is completely determined by its values at the poiints $t_j = \frac{j\pi}{\Omega}, j = 0, \pm1, \pm2, \dots$. More precisely, f has the following series expansion: $f(t) = \sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right)\frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}$, where the series converges uniformly.

CHAPTER 3: DISCRETE FOURIER TRANSFORM

Set of n-periodic sequences: Let $\mathcal{S}_n$ be the set of n-periodic sequences of complex numbers. Each element $y = {y_j}_{j=-\infty}^{\infty}$ in $\mathcal{S}_n$, can be thought of as a periodic discrete signal where y_j is the value of the signal at a time node $t = t_j$. The sequence y_j is n-periodic if $y_{k+n} = y_k$ for any integer k. **Definition:** Suppose $y = y_k$ is an element of $\mathcal{S}_n$. Let $\mathcal{F}_n(y) = \hat{y}$. That is, $\hat{y}_k = \sum_{j=0}^{n-1} y_j\overline{w}^{jk}$, where $w = e^{\frac{2\pi}{n}i}$. Then $y = \mathcal{F}^{-1}(\hat{y})$ is given by $y_j = \frac{1}{n}\sum_{k=0}^{n-1}\hat{y}_k w^{jk}$. **Properties:**

- Shifts or translations. If $y \in \mathcal{S}_n$ and $z_k = y_{k+1}$, then $\mathcal{F}[z]_j = w^j\mathcal{F}[y]_j$
- Convolutions. If $y \in \mathcal{S}_n$ and $z \in \mathcal{S}_n$, then the sequence $[y * z]_k := \sum_{j=0}^{n-1} y_j z_{k-j}$ is also in $\mathcal{S}_n$. The sequence $y * z$ is called the convolution of the sequences y and z.
- The Convolution Theorem. $\mathcal{F}[y * z]_k = \mathcal{F}[y]_k\mathcal{F}[z]_k$
- If $y \in \mathcal{S}_n$ is a sequence of real numbers, then $\mathcal{F}[y]_{n-k} = \overline{\mathcal{F}[y]}_k$, for $k \in [0, n-1]$, or $\hat{y}_{n-k} = \overline{\hat{y}}_k$

CHAPTER 4: HAAR WAVELET ANALYSIS

Haar Scaling function: The Haar scaling function is defined as $\phi(x) = 1$ if $x \in [0, 1]$. **Definition:** Suppose j is any nonnegative integer. The space of step functions at level j, denoted by V_j, is defined to be the space spanned by the set $\{\dots, \phi(2^j + 1), \phi(2^j), \phi(2^j - 1), \phi(2^j - 2), \dots\}$. **Theorem 4.5:** A function $f(x)$ belongs to $V_0//V_j$ if and only if $f(2^j x)//f(2^{-j}x))$ belongs to $V_j//V_0$. **Theorem 4.6:** The set of functions $\{2^{j/2}\phi(2^j x - k); k \in \mathbb{Z}\}$ is an orthonormal basis of V_j. **Haar Wavelet:** The Haar wavelet is function $\psi(x) = \phi(2x) - \phi(2x - 1)$. **Theorem 4.8:** Let W_j be the space of functions of the form $\sum_{k\in\mathbb{Z}} a_k \psi(2^j x - k)$, $a_k \in \mathbb{R}$ (only a finite number of a_k are nonzero). W_j is the orthogonal complement of V_j in V_{j+1} and $V_{j+1} = V_j \bigoplus W_j$. **Theorem 4.9:** The space $L^2(\mathbb{R})$ can be decomposed as an infinite orthogonal direct sum $L^2(\mathbb{R}) = V_0 \bigoplus W_0 \bigoplus W_1 \dots$. In particular, each $f \in L^2(\mathbb{R})$ can be written as $f = f_0 + \sum_{j=0}^{\infty} w_j$, where $f_0 \in V_0$ and $w_j \in W_j$.

SAMPLE If the signal is continuous, $y = f(t)$, where t represents time, choose the top level $j = J$ so that 2^j is larger than the Nyquist rate for the signal. Let $a_k^J = f(k/2^J)$. The top level a_k^J is set equal to the kth term in the sampled signal, and 2^J is taken to be the sampling rate. In any case, we have the highest-level approximation to f given by $f_J = \sum_{k\in\mathbb{Z}} a_k^j \phi(2^J x - k)$.

DECOMPOSITION **Lemma 4.10:** The following relations hold for all $x \in \mathbb{R}$. $\phi(2^j x) = \left(\phi(2^{j-1}x) + \psi(2^{j-1}x)\right)/2$. $\phi(2^j x - 1) = \left(\phi(2^{j-1}x) - \psi(2^{j-1}x)\right)/2$. **Theorem 4.12:** Suppose $f_j(x) = \sum_{k\in\mathbb{Z}} a_k^j \phi(2^j x - k) \in V_j$. Then f_j can be decomposed as $f_j = w_{j-1} + f_{j-1}$, where $w_{j-1} = \sum_{k\in\mathbb{Z}} b_k^{j-1} \psi(2^{j-1}x - k) \in W_{j-1}$ and $f_{j-1} = \sum_{k\in\mathbb{Z}} a_k^{j-1} \phi(2^{j-1}x - k) \in V_{j-1}$, with $b_k^{j-1} = \frac{a_{2k}^j - a_{2k+1}^j}{2}$ and $a_k^{j-1} = \frac{a_{2k}^j + a_{2k+1}^j}{2}$.

RECONSTRUCTION **Theorem 4.12:** If $f = f_0 + w_0 + w_1 + \cdots + w_{j-1}$ with $f_0(x) = \sum_{k\in\mathbb{Z}} a_k^0 \phi(x - k) \in V_0$ and $w_j = \sum_{k\in\mathbb{Z}} b_k^j \psi(2^j x - k) \in W_j$ for $0 \le j \le J$, then $f(x) = f_J(x) = \sum_{k\in\mathbb{Z}} a_k^J \phi(2^J x - k) \in V_J$. The a_k^J are determined recursively by $a_k^j = a_l^{j-1} + b_l^{j-1}$ if $k = 2l$ is even and $a_k^j = a_l^{j-1} - b_l^{j-1}$ if $k = 2l + 1$ is odd.

CHAPTER 5: MULTIRESOLUTION ANALYSIS

Definition: Let $V_j, j = \cdots - 1, 0, 1 \dots$, be a sequence of subspaces of cuntions in $L^2(\mathbb{R})$. The collection of spaces $\{V_j, j \in \mathbb{Z}\}$ is called a *multiresolution analysis with scaling funciton* ϕ if the following conditions hold. **1.** (Nested) $V_j \subset V_{j+1}$. **2.** (Density) $\overline{\cup V_j} = L^2(\mathbb{R})$. **3.** (Separation) $\cap V_j = \{0\}$ **4.** (Scaling) See Theorem 4.5(b) **5.** (Orthonormal basis) The function ϕ belongs to V_0 and the set $\{\phi(x - k); k \in \mathbb{Z}\}$ is an orthonormal basis for V_0. **Theorem 5.5:** Suppose $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis with scaling function ϕ. Then for any $\in \mathbb{Z}$, the set of functions $\{\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k); k \in \mathbb{Z}\}$ is an orthonormal basis for V_j.

THE SCALING RELATION **Theorem 5.6:** Suppose $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis with scaling function ϕ. Then the following scaling relation holds: $\phi(x) = \sum_{k\in\mathbb{Z}} p_k \phi(2x - k)$, where $p_k = 2\int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x - k)}dx$. Moreover, we also have $\phi_{j-1,l} = 2^{-1/2} \sum_{k\in\mathbb{Z}} p_{k-2l} \phi_{jk}$.
Remark: When the support of ϕ is compact, only a finite number of p_k are nonzero, because when $|k|$ is large enough, the support of $\phi(2x - k)$ will be outside of the support of $\phi(x)$. **Theorem 5.9:** Suppose $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis with scaling function ϕ. Then, provided the scaling relation can be integrated termwise, the following equalities hold: **1.** $\sum_{k\in\mathbb{Z}} p_{k-2l}\overline{p_k} = 2\delta_{l0}$. **2.** $\sum_{k\in\mathbb{Z}} |p_k|^2 = 2$. **3.** $\sum_{k\in\mathbb{Z}} p_k = 2$. **4.** $\sum_{k\in\mathbb{Z}} p_{2k} = 1$ and $\sum_{k\in\mathbb{Z}} p_{2k+1} = 1$. **Theorem 5.10:** Suppose $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis with scaling function $\phi = \sum_k p_k \phi(2x - k)$. Let W_j be the span of $\{\psi(2^j x - k); k \in \mathbb{Z}\}$ where $\psi(x) = \sum_{k\in\mathbb{Z}} (-1)^k \overline{p_{1-k}} \phi(2x - k)$. Then, $W_j \subset V_{j+1}$ is the orthogonal complement of V_j in V_{j+1}. Furthermore, $\{\psi_{jk}(x) := 2^{j/2}\psi(2^j x - k); k \in \mathbb{Z}\}$ is an orthonormal basis for the W_j. **Theorem 5.11:** Let $\{V_j, j \in \mathbb{Z}\}$ be a multiresolution analysis with scaling function ϕ. Let W_j be the orthogonal complement of V_j in V_{j+1}. Then $L^2(\mathbb{R}) = \cdots \bigoplus W_{-1} \bigoplus W_0 \bigoplus W_1 \bigoplus \dots$. In particular, each $f \in L^2(\mathbb{R})$ can be uniquely expressed as a sum $\sum_{k\in\mathbb{Z}} w_k$ with $w_k \in W_k$ and where the w_k's are mutually orthogonal. Equivalently, the set of all wavelets, $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$, is an orthonormal basis for $L^2(\mathbb{R})$.

DECOMPOSITION **Orthogonal Form:** $a_l^{j-1} = 2^{-1}\sum_{k\in\mathbb{Z}} \overline{p_{k-2l}} a_k^j$ and $b_l^{j-1} = 2^{-1}\sum_{k\in\mathbb{Z}} (-1)^k p_{1-k+2l} a_k^j$ **ON Form:** $\langle f, \phi_{j-1,l}\rangle = 2^{-1/2}\sum_{k\in\mathbb{Z}} \overline{p_{k-2l}} \langle f, \phi_{jk}\rangle$ and $\langle f, \psi_{j-1,l}\rangle = 2^{-1/2}\sum_{k\in\mathbb{Z}} (-1)^k p_{1-k+2l} \langle f, \phi_{jk}\rangle$

RECONSTRUCTION **Orthogonal Form:** $a_k^j = \sum_{l\in\mathbb{Z}} p_{k-2l} a_l^{j-1} + \sum_{l\in\mathbb{Z}} (-1)^k \overline{p_{1-k+2l}} b_l^{j-1}$ **ON Form:** $\langle f, \phi_{jk}\rangle = 2^{-1/2}\sum_{l\in\mathbb{Z}} p_{k-2l} \langle f, \phi_{j-1,l}\rangle + 2^{-1/2}\sum_{l\in\mathbb{Z}} (-1)^k \overline{p_{1-k+2l}}$

APPENDIX

Identities: $\sin^2 x = (1 - \cos 2x)/2, \cos^2 x = (1 + \cos 2x)/2$, $e^{ix} = \cos x + i \sin x, e^{-ix} = \cos x - i \sin x$,
$e^{ix} + e^{-ix} = 2\cos x$,
$e^{ix} - e^{-ix} = 2i \sin x$.
Sum and Difference Formula: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. $\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$. $\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$.
Double Angle Formula: $\sin(2A) = 2 \sin A \cos A$. $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$. $\tan(2A) = (2 \tan A)/(1 - \tan^2 A)$.
Sum to Product: $\sin A \pm \sin B = 2 \sin((A \pm B)/2) \cos((A \mp B)/2)$. $\cos A - \cos B = -2 \sin((A + B)/2) \sin((A - B)/2)$. $\cos A + \cos B = 2 \cos((A + B)/2) \cos((A - B)/2)$.
Geometric Sum: $\sum_{k=0}^{N} z^k = \frac{1 - z^{N+1}}{1 - z}$. $\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}$.
Integrals:

$$\int (a + bx)\cos(kx)dx = \frac{(a + bx)\sin(kx)}{k} + \frac{bcos(kx)}{k^2} + C$$

$$\int (a + bx)\sin(kx)dx = \frac{b\sin(kx)}{k^2} - \frac{(a + bx)cos(kx)}{k} + C$$

$$\int (a + bx)e^{ikx}dx = \frac{e^{ikx}(b - ik(a + bx))}{k^2} + C$$