Computational Fluid Dynamics

Francisco Castillo
Homework 9

April 8, 2018

1 Introduction

In this assignment we will study a one dimensional hyperbolic PDE

$$\frac{\partial u}{\partial t} + \frac{\partial \left(\frac{u^2}{2}\right)}{\partial x} = 0, \qquad x \in [3, 7],$$

with initial condition

$$u(x, t=0) = \begin{cases} \frac{1}{4} + \frac{1}{2}\sin(\frac{\pi}{4}(x-3)) & x < 4.5 \\ \frac{1}{4} + \frac{1}{2}\sin(\frac{\pi}{4}(x-3)) + [1 + \cos(2\pi x)]\cos(8\pi x) & 4.5 \leq x \leq 5.5 \\ \frac{1}{4} + \frac{1}{2}\sin(\frac{\pi}{4}(x-3)) & x > 5.5 \end{cases}$$

and periodic boundary conditions. We will use a second order TVD scheme to solve the given hyperbolic PDE. The equations in index form of this method are detailed at the end of this document.

2 Results

In the first figure we can see the plot of the initial condition given above. In the figure 2 we can see the solution $u(x,t)$ at different values of time for $M = 256$ and $CFL = 0.1$. We can see that in fact the periodic boundary conditions are satisfied, although the initial condition was not periodic. In the figure 3 we can see the same profiles of $u(x,t)$ at the same values of time but for $M = 1024$. This value of M was the one that the GCI study determined to satisfy the accuracy requirement given in the problem (see below).

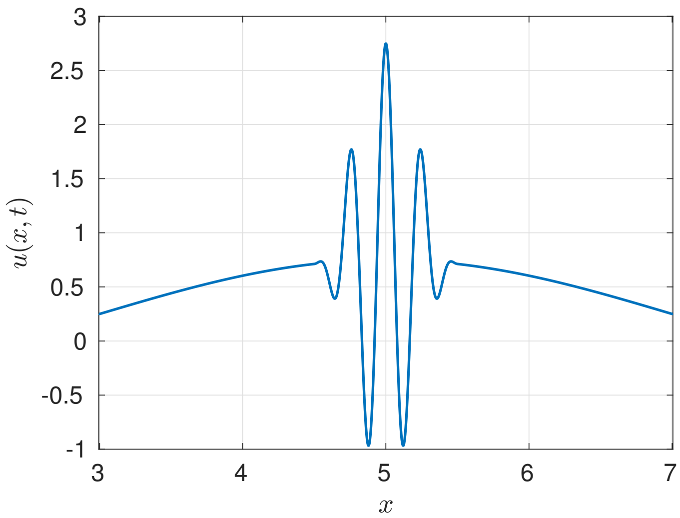
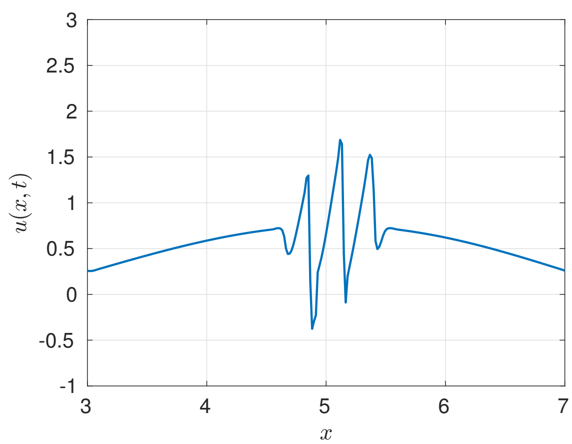
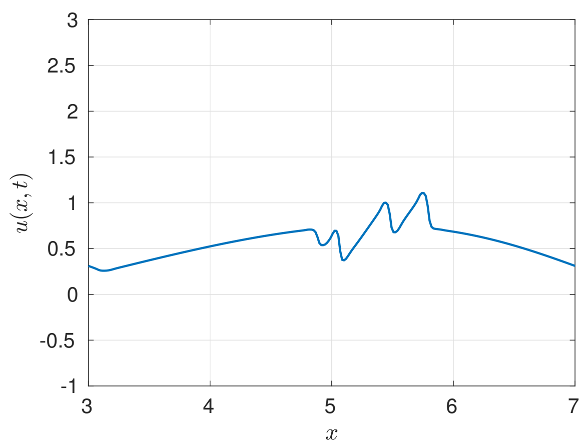


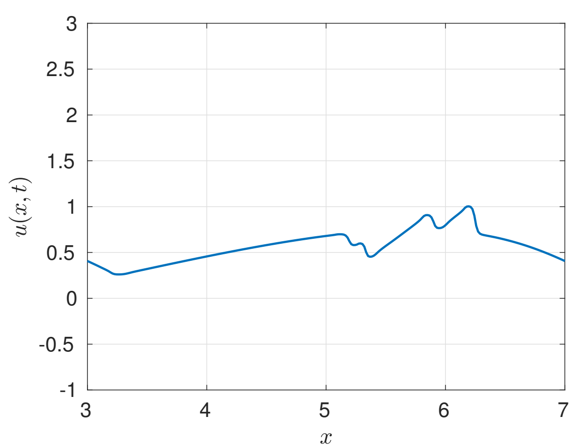
Figure 1: Initial condition for u.



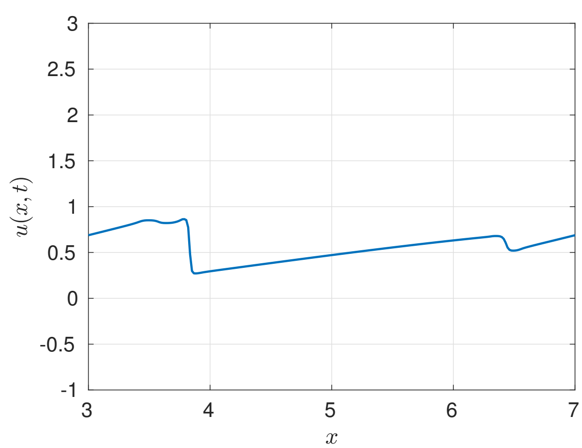
(a) $t = 0.1$ s.



(b) $t = 0.5$ s.

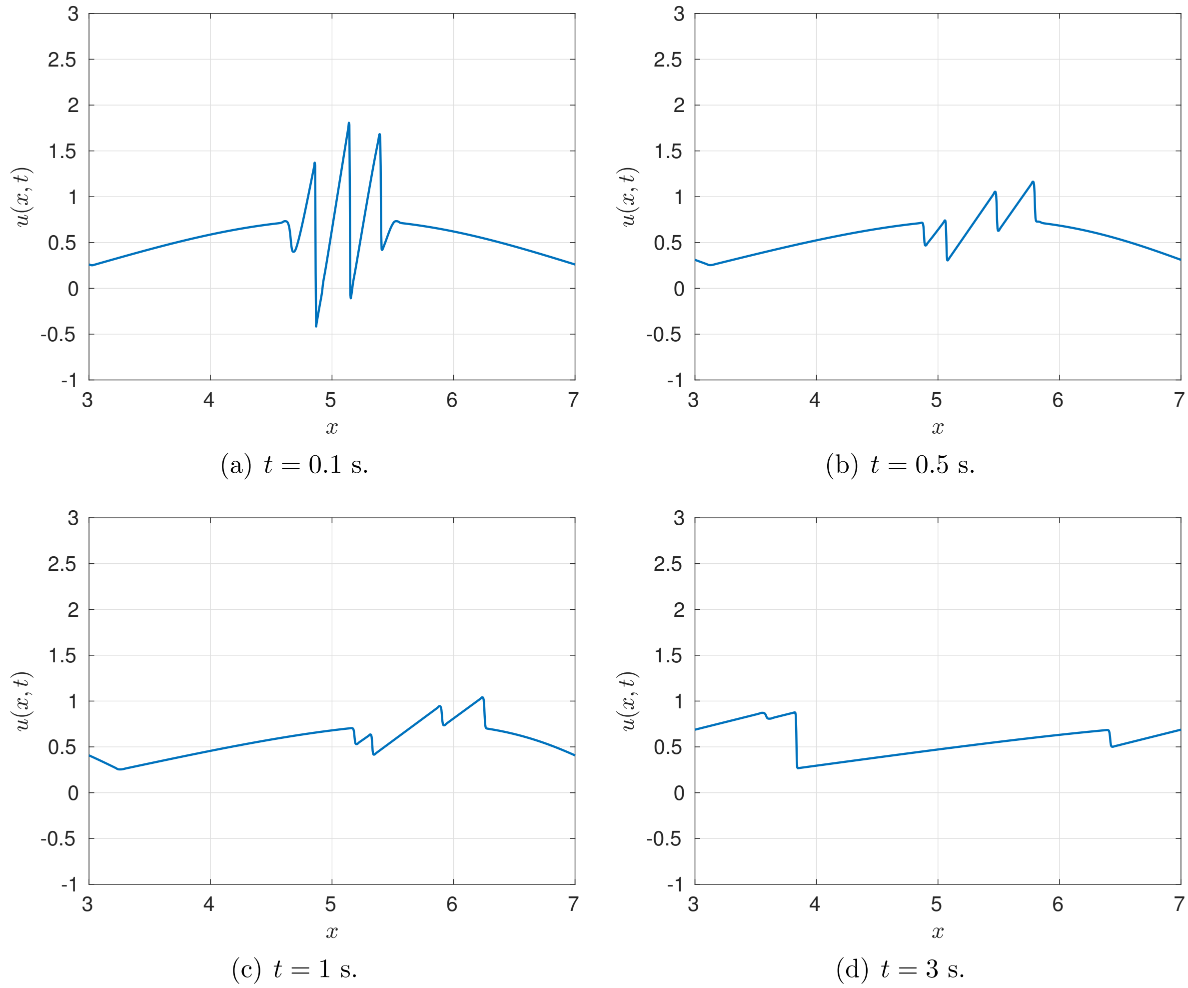


(c) $t = 1$ s.



(d) $t = 3$ s.

Figure 2: Profiles of u for M=256 and $CFL = 0.1$.



(a) $t = 0.1$ s.

(b) $t = 0.5$ s.

(c) $t = 1$ s.

(d) $t = 3$ s.

Figure 3: M=4096, CFL=0.8

The GCI analysis details are shown in the tables below. Note that

$$\beta = \frac{GCI_{12}}{GCI_{23}} r^p,$$

and $u_{h=0}$ is obtained by Richardson extrapolation. We can see that $\beta \in [0.95, 1.05]$ which implies that we are in the asymptotic range of convergence, and for the last mesh we have a GCI_{12} value less than 0.1%, the requested accuracy.

M	$u(6,1)$
64	0.863483397473951
128	0.827088879661215
256	0.797959995827032
512	0.810290219692641
1024	0.810473786888916

Table 1: GCI analysis data.

M	$u_{h=0}$	p	GCI_{12} (%)	GCI_{23} (%)	β
64	-	-	-	-	-
128	-	-	-	-	-
256	0.681178462938105	0.321270729334096	18.293763706	0.22051804272	1.03650419066
512	0.819340609477331	1.240251427765663	1.396164850	0.03349261517	0.98478295360
1024	0.810476561060975	6.069746913021977	0.000427862	0.00028746084	0.99977350631

Table 2: GCI analysis results.

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Contents

Defined functions

```
function phi=EntropyFix(y)
eps=0.1;
phi=abs(y).*(abs(y)>=eps)+((y.^2+eps^2)./(2*eps)).*(abs(y)<eps);
end


function alpha = HY_alpha(u,E,M)
eps=0.1;
alpha = zeros(M+3,1);
for i=1:M+3
    absDelta=abs(u(i+1)-u(i));
    if absDelta>=eps
        alpha(i) = (E(i+1)-E(i))/(u(i+1)-u(i));
    elseif absDelta<eps
        alpha(i) =(u(i)+u(i+1))/2;
    end
end


function beta = HY_beta(u,G,M)
eps=1e-12;
beta=zeros(M+1,1);
for i=2:M+2
    absDelta=abs(u(i+1)-u(i));
    if absDelta>=eps
        beta(i-1) = (G(i+1)-G(i))/(u(i+1)-u(i));
    elseif absDelta<eps
        beta(i-1)=0;
    end
end
end


function G=HY_G(u,alpha,dt,dx,M)
sigma=sigmaG(alpha,dt,dx);
G=zeros(M+4,1);
for i=3:M+2
    S=sign(u(i+1)-u(i));
    G(i) = S*max(0,min(sigma(i)*abs(u(i+1)-u(i)),...
            S*sigma(i-1)*(u(i)-u(i-1))));
end
% Periodic Boundary Conditions
G(1)=G(M+1);
G(2)=G(M+2);
G(M+3)=G(3);
G(M+4)=G(4);
end


function Phi = HY_Phi(u,G,alpha,beta,M)
psi=EntropyFix(alpha(2:end-1)+beta);
Phi=zeros(M+1,1);
for i=2:M+2
    Phi(i-1)=G(i+1)+G(i)-psi(i-1)*(u(i+1)-u(i));
end
end


function u = initialCondition(x,M)
u=zeros(M+4,1);
for i=3:M+2
    if (x(i)<4.5 || x(i)>5.5)
        u(i)=0.25+0.5*sin(pi/4*(x(i)-3));
    elseif (x(i)>=4.5 && x(i)<=5.5)
        u(i)=0.25+0.5*sin(pi/4*(x(i)-3))+(1+cos(2*pi*x(i)))*cos(8*pi*x(i));
    end
end
```

```
% Periodic Boundary Conditions
u(1)=u(M+1);
u(2)=u(M+2);
u(M+3)=u(3);
u(M+4)=u(4);
end


function h = numericalFlux(E,Phi,M)
h=zeros(M+1,1);
for i=2:M+2
    h(i-1)=0.5*(E(i+1)+E(i)+Phi(i-1));
end


function sigma = sigmaG(alpha,dt,dx)
sigma = 0.5*(EntropyFix(alpha)-(dt/dx)*alpha.^2);
end


function u = TVD2order(u,h,dt,dx,M)
for i=3:M+2
   u(i)=u(i)- dt/dx*(h(i-1)-h(i-2));
end
% Periodic Boundary Conditions
u(1)=u(M+1);
u(2)=u(M+2);
u(M+3)=u(3);
u(M+4)=u(4);
end
```

ProblemHomework8.m

```
clear variables
close all
clc
format long

axisSize=14;
linewidth=1.5;
L=4;
CFL=0.1;
M=32;
i=0;
T=nan(3,7);
check=1;
stp=1;
while check>0.01
```

```
    i=i+1;
    M=2*M
    dx=L/M;
    x=linspace(3-1.5*dx,7+1.5*dx,M+4)'; % Cell centered mesh with two
                                        % ghost cells at each side
```

Initialization

```
    time=0;
    u=initialCondition(x,M);
    % Plot initial condition
    if (M==256 || M==1024)
            figure(1)
            plot(x,u,'linewidth',linewidth)
            grid on
            axis([min(x) max(x) -1 3])
            xlabel('$x$','Interpreter','latex')
            ylabel('$u(x,t)$','Interpreter','latex')
            set(gca,'fontsize',axisSize)
            if M==256
                txt='Latex/FIGURES/uinitial_M256';
            elseif M==1024
                txt='Latex/FIGURES/uinitial_M1024';
            end
            saveas(gcf,txt,'epsc')
    end
    E=0.5*u.^2;
    alpha=HY_alpha(u,E,M);
    dt=CFL*dx/(max(abs(alpha)));

        outputTime=[0.1 0.5 1 3.0];
```

```
    endtime=outputTime(end);

    n=1;
    while time < endtime
```

```
        E=0.5*u.^2;
        alpha=HY_alpha(u,E,M);
        if (time < outputTime(n) && time+dt >= outputTime(n))
            dt=outputTime(n)-time;
            n=n+1;
        else
            dt=CFL*dx/(max(abs(alpha)));
        end
        G=HY_G(u,alpha,dt,dx,M);
        beta=HY_beta(u,G,M);
        psi=EntropyFix(alpha(2:end-1)+beta);
        Phi=HY_Phi(u,G,alpha,beta,M);
        h = numericalFlux(E,Phi,M);

        u=TVD2order(u,h,dt,dx,M);
        time=time+dt;
```

Plot required at part 5

```
        if (M==256 && ismember(time,outputTime))
            figure(n)
            plot(x,u,'linewidth',linewidth)
            grid on
            axis([3 7 -1 3])
            xlabel('$x$','Interpreter','latex')
            ylabel('$u(x,t)$','Interpreter','latex')
            set(gca,'fontsize',axisSize)
            txt=['Latex/FIGURES/u_M256_' num2str(n-1)];
            saveas(gcf,txt,'epsc')
        end
        if (M==1024 && ismember(time,outputTime))
            figure(n)
            plot(x,u,'linewidth',linewidth)
            grid on
            axis([3 7 -1 3])
            xlabel('$x$','Interpreter','latex')
            ylabel('$u(x,t)$','Interpreter','latex')
            set(gca,'fontsize',axisSize)
            txt=['Latex/FIGURES/u_M1024_' num2str(n-1)];
            saveas(gcf,txt,'epsc')
        end
```

u at x=6 and t=1

```
        if time==1
            uGCI(i)=(u(find(x<=6,1,'last'))+u(find(x>=6,1)))/2;
        end
```

```
    end
```

GCI analysis

```
    if i>=3
        r=2;
        Fsec=1.25;
        p(i)=log(abs(uGCI(i-2)-uGCI(i-1))/abs(uGCI(i-1)-uGCI(i)))/log(r);
        u_h0(i)=uGCI(i)+(uGCI(i)-uGCI(i-1))/(r^p(i)-1);
        GCI12(i)=Fsec*abs(1-uGCI(i-1)/uGCI(i))/(r^p(i)-1);
        GCI23(i)=Fsec*abs(1-uGCI(i-2)/uGCI(i-1))/(r^p(i)-1);
        coeff(i)=GCI12(i)*r^p(i)/GCI23(i);
        percent(i)=GCI12(i)*100;
        check=abs(percent(i));
        % Include results in a table
        T(i,2)=p(i);
        T(i,3)=u_h0(i);
        T(i,4)=GCI12(i);
        T(i,5)=GCI23(i);
        T(i,6)=coeff(i);
        T(i,7)=percent(i);
    end
    % Include results in a table
    T(i,1)=M;
```

```
end
T=array2table(T,'VariableNames',{'M','p','phi0','GCI12','GCI23','coeff','Check'})
if CFL==0.1
    save('Case1_CFL01');
elseif CFL==0.5
    save('Case2_CFL05');
end
```

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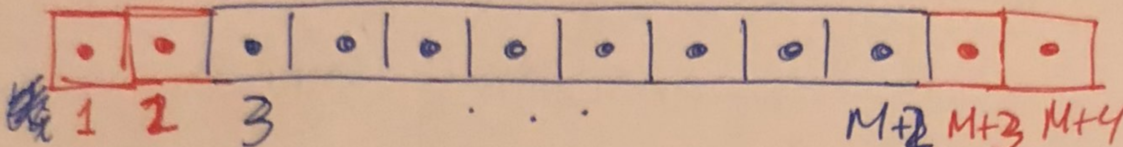

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(h^n_{i+\frac{1}{2}} - h^n_{i-\frac{1}{2}}\right),$$

$$\boxed{E_i^n = \frac{1}{2}(u_i^n)^2}$$

where
$$\begin{cases} h^n_{i+\frac{1}{2}} = \frac{1}{2}\left[\left(E^n_{i+1} + E^n_i\right) + \Phi^n_{i+\frac{1}{2}}\right] \\ h^n_{i-\frac{1}{2}} = \frac{1}{2}\left[\left(E^n_i + E^n_{i-1}\right) + \Phi^n_{i-\frac{1}{2}}\right] \end{cases}$$

and
$$\Phi^n_{i+\frac{1}{2}} = G^n_{i+1} + G^n_i - \psi\left(\alpha^n_{i+\frac{1}{2}} + \beta^n_{i+\frac{1}{2}}\right)\Delta u^n_{i+\frac{1}{2}}$$

with
$$\psi(y) = \begin{cases} |y| & |y| > \varepsilon \\ \frac{y^2+\varepsilon^2}{2\varepsilon} & |y| < \varepsilon \end{cases} \quad \text{for } 0 \le \varepsilon \le \frac{1}{8}$$

↳ choos $\varepsilon = 0.1$

$$\alpha^n_{i+\frac{1}{2}} = \begin{cases} \frac{E^n_{i+1} - E^n_i}{u^n_{i+1} - u^n_i} & |\Delta u^n_{i+1/2}| \ge \varepsilon' \\ (u^n_i + u^n_{i+1})/2 & |\Delta u^n_{i+1/2}| < \varepsilon' \end{cases}$$

$$\varepsilon' = 10^{-12}$$

$$\beta^n_{i+\frac{1}{2}} = \begin{cases} \frac{G^n_{i+1} - G^n_i}{u^n_{i+1} - u^n_i} & |\Delta u^n_{i+1/2}| \ge \varepsilon' \\ 0 & |\Delta u^n_{i+\frac{1}{2}}| < \varepsilon' \end{cases}$$

$$\alpha^n_{i-\frac{1}{2}} = \begin{cases} \frac{E^n_i - E^n_{i-1}}{u^n_i - u^n_{i-1}} & |\Delta u^n_{i-1/2}| \ge \varepsilon' \\ (u^n_{i-1} + u^n_i)/2 & |\Delta u^n_{i-\frac{1}{2}}| < \varepsilon' \end{cases}$$

$$\beta^n_{i-1/2} \begin{cases} \dfrac{G_i^n - G_{i-1}^n}{u_i^n - u_{i-1}^n} & |\Delta u^n_{i-1/2}| \geq \varepsilon' \\ 0 & |\Delta u^n_{i-1/2}| < \varepsilon' \end{cases}$$

Recall that $\Delta u^n_{i+1/2} = u^n_{i+1} - u^n_i$ and $\Delta u^n_{i-1/2} = u^n_i - u^n_{i-1}$

Then $G_i^n = S \cdot \max\left[0, \min\left(\sigma^n_{i+1/2} |\Delta u^n_{i+1/2}|, S \cdot \sigma^n_{i-1/2} \Delta u^n_{i-1/2}\right)\right]$

with $S = \text{sign}(\Delta u^n_{i+1/2})$ and

$$\sigma^n_{i+1/2} = \frac{1}{2}\left[\psi(\alpha^n_{i+1/2}) - \frac{\Delta t}{\Delta x}(\alpha^n_{i+1/2})^2\right].$$

I used a CFL $= 0.1$. Hence,

$$\Delta t = \text{CFL} \cdot \frac{\Delta x}{\max|\alpha|}$$ with $\Delta x = \frac{L}{M}$ and $L = 7 - 3 = 4$

Ghost cells BCs: periodic

left $\begin{cases} u_2 = u_{M+2} \\ u_1 = u_{M+1} \end{cases}$ right $\begin{cases} u_{M+3} = u_3 \\ u_{M+4} = u_4 \end{cases}$