Computational Fluid Dynamics

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Homework 8

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1 Introduction

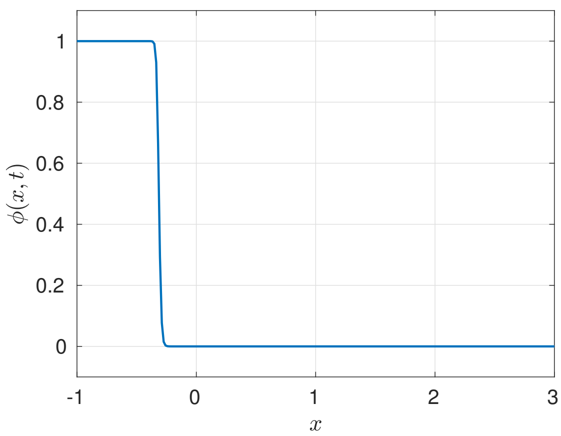
In this assignment we will study a one dimensional hyperbolic PDE

$$\frac{\partial \phi}{\partial t} + a(x,t)\frac{\partial \phi}{\partial x} = 0,$$

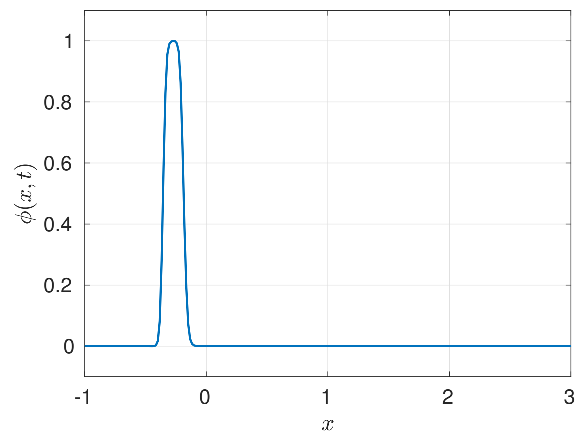
with a discontinuous boundary condition and a sinusoidal velocity $a(x,t)$. We will use a the total variation diminishing (TVD) third order Runge-Kutta method, abbreviated $TVD\ RK3$, to solve the PDE in time and a fith order Weighted Essentially Non-Oscillatory Scheme, abreviated $WENO-5$, to solve the spatial derivative.

2 Results

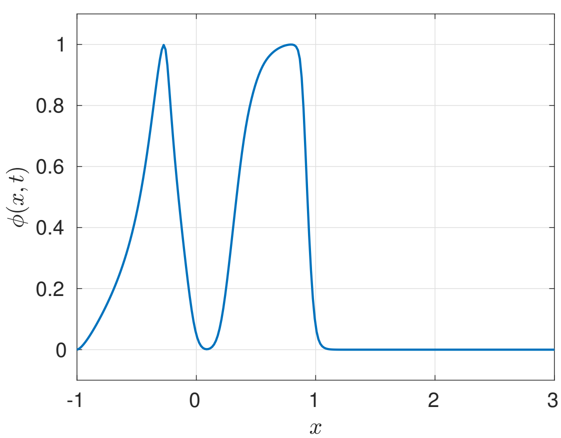
In the first figure we can see the solution $\phi(x,t)$ at different values of time for $M = 256$ and $CFL = 0.8$. We can see the strange solution profiles that are due to the form of the left boundary condition.



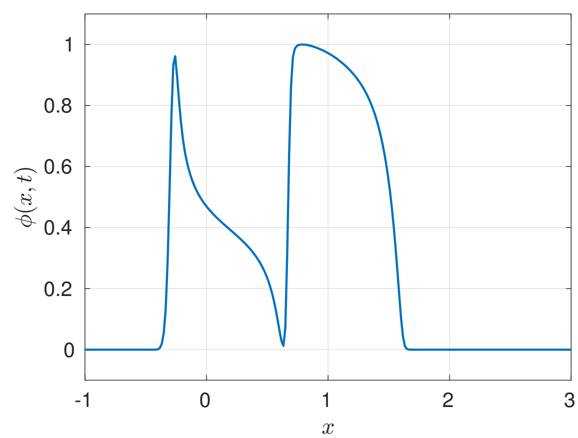
(a) $t = 0.25$ s.



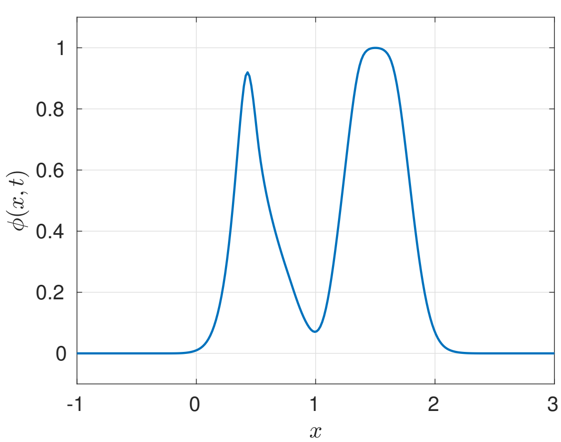
(b) $t = 0.5$ s.



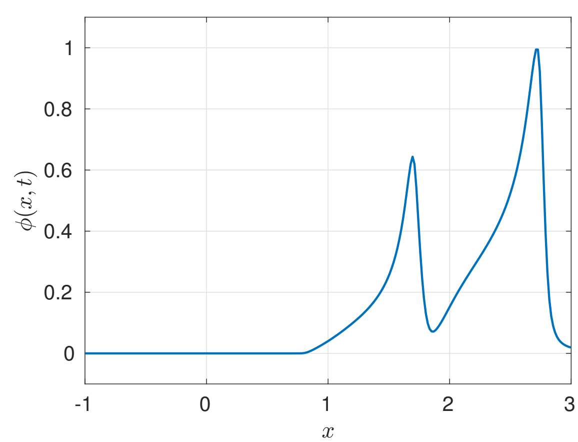
(c) $t = 1$ s.



(d) $t = 1.25$ s.



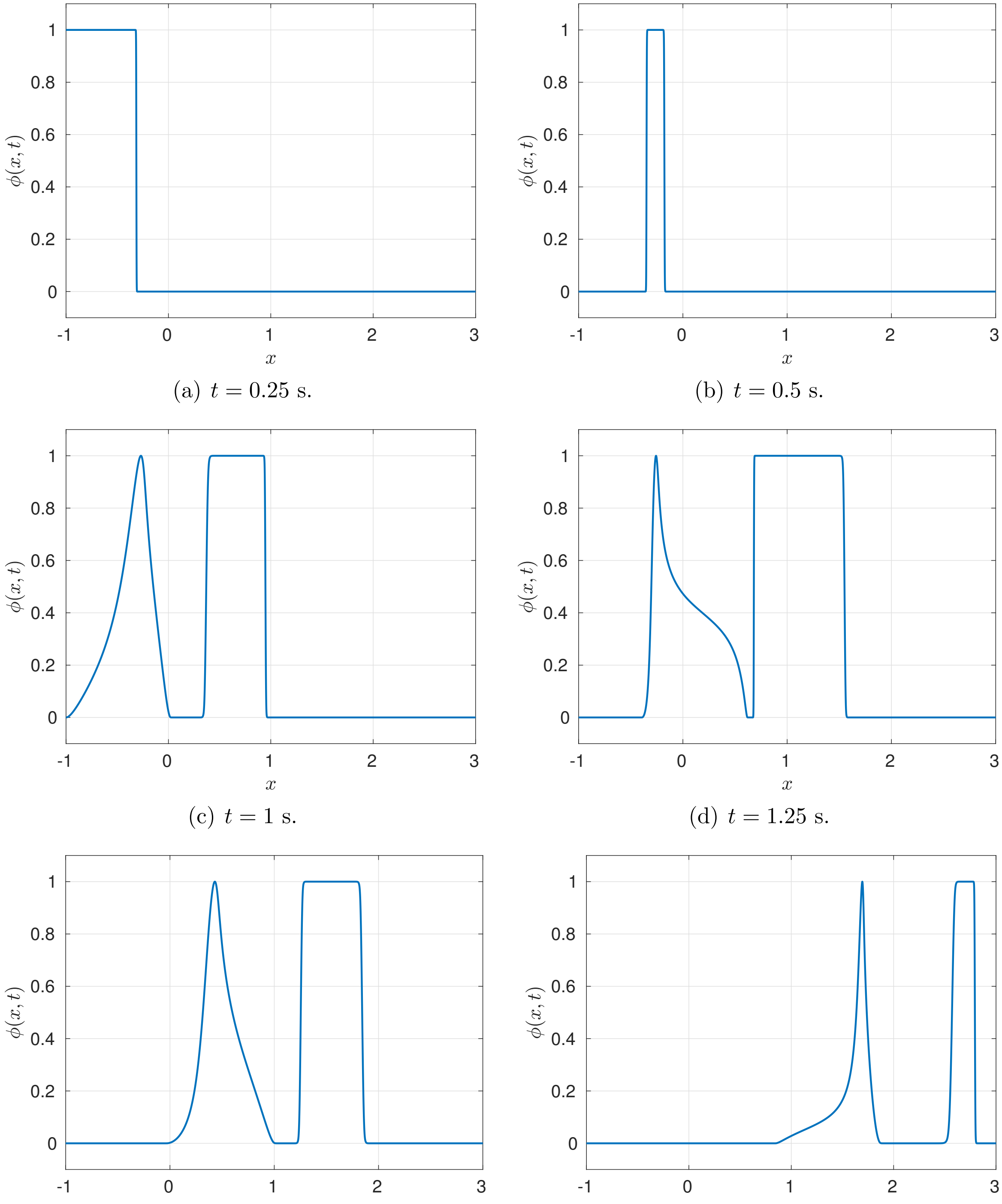
(e) $t = 1.5$ s.



(f) $t = 2.1$ s.

Figure 1: M=256.

In the next two figures we see the solution profiles for $M = 4096$, $CFL = 0.8$ (Figure 2) and $M = 1024$, $CFL = 0.5$ (Figure 3). Both configurations satisfy the accuracy requirement as we show in the tables below. In this case I have found more beneficial when it comes to computational cost, to reduce the time step instead of going to such fine meshes. However, we can see in figure 4 that if we want an outstanding precision in space to catch the discontinuities, we should increase the number of elements. However, reducing the time step, and using 4 times less elements (Figure 4(c)), gives us a very similar result much faster.



(a) $t = 0.25$ s.

(b) $t = 0.5$ s.

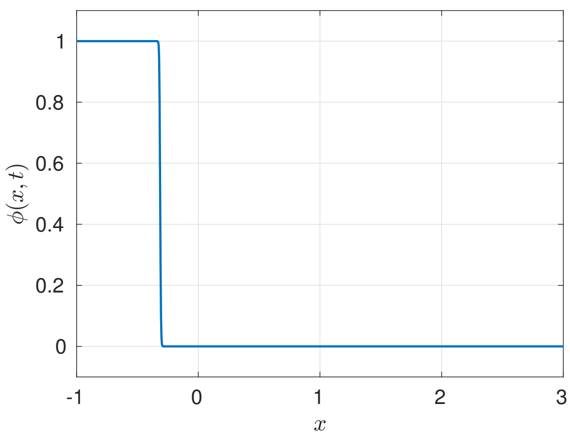
(c) $t = 1$ s.

(d) $t = 1.25$ s.

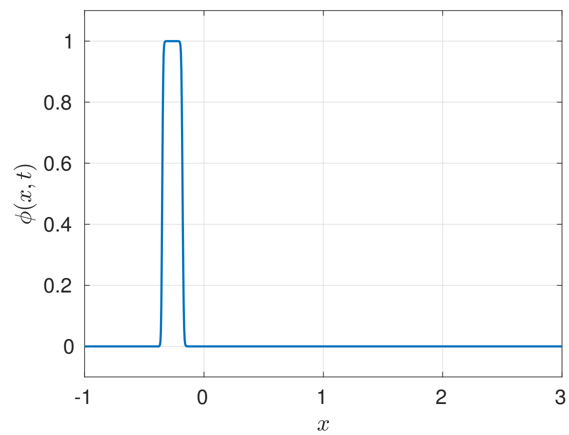
(e) $t = 1.5$ s.

(f) $t = 2.1$ s.

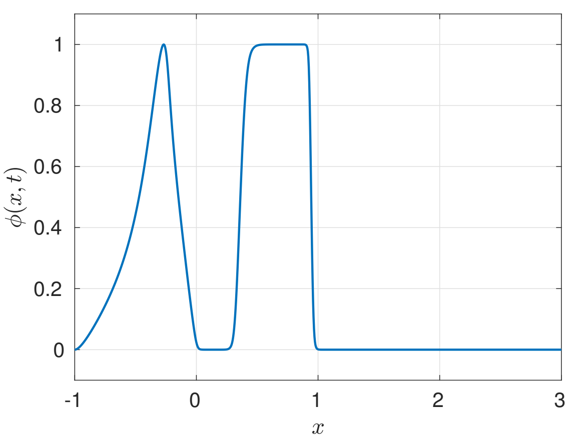
Figure 2: M=4096, CFL=0.8



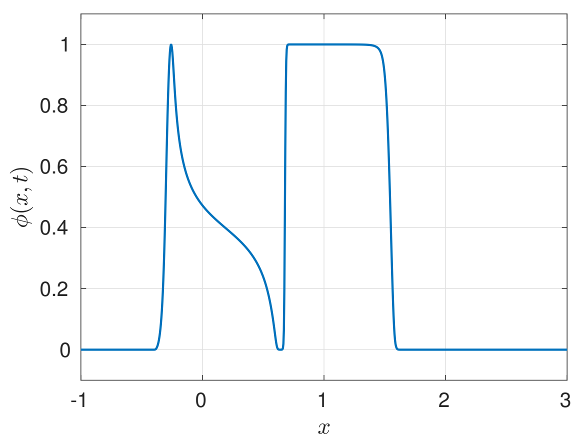
(a) $t = 0.25$ s.



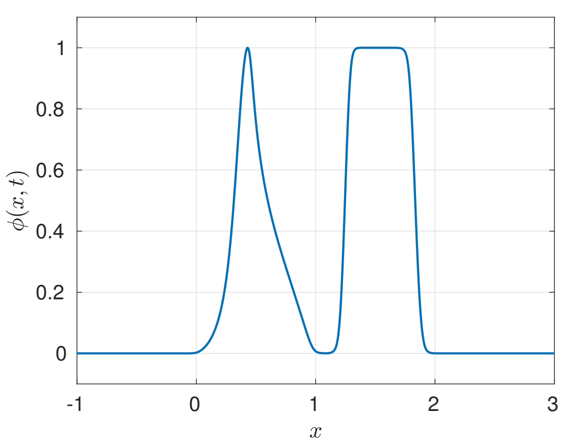
(b) $t = 0.5$ s.



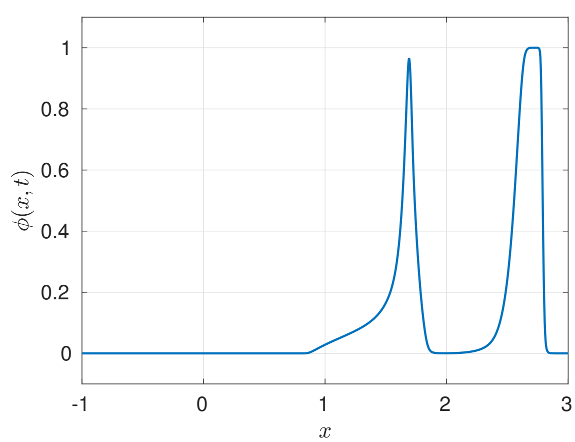
(c) $t = 1$ s.



(d) $t = 1.25$ s.

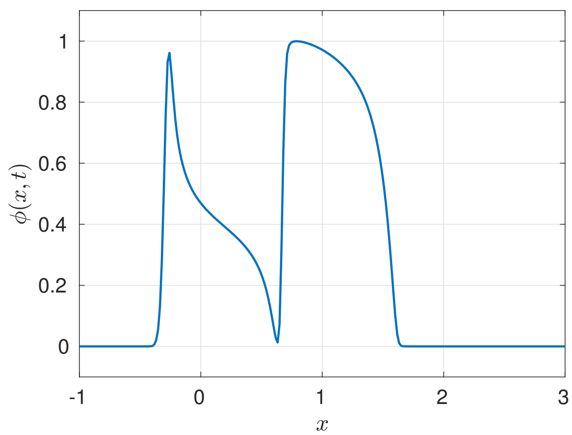


(e) $t = 1.5$ s.

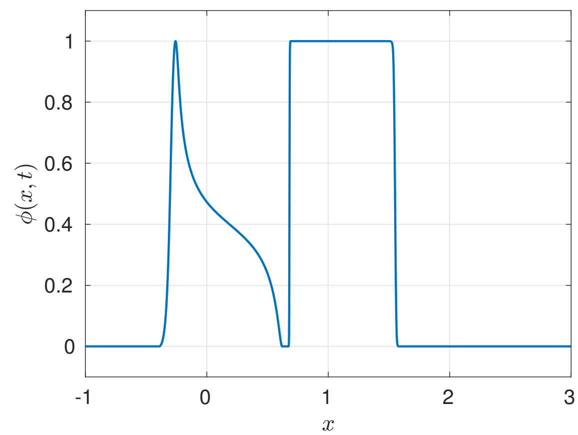


(f) $t = 2.1$ s.

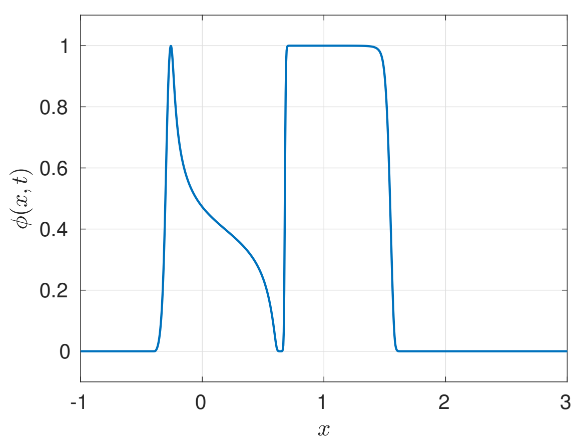
Figure 3: M=4096, CFL=0.5



(a) $M = 256$, $CFL = 0.8$.



(b) $M = 4096$, $CFL = 0.8$.



(c) $M = 1024$, $CFL = 0.5$.

Figure 4: Comparison of the different solutions.

The GCI analysis details for both CFL values are shown in the tables below. Note that

$$\beta = \frac{GCI_{12}}{GCI_{23}} r^p,$$

and $\phi_{h=0}$ is obtained by Richardson extrapolation. We can see that $\beta \in [0.95, 1.05]$ which implies that we are in the asymptotic range of convergence, and for the last mesh we have a GCI_{12} value less than 0.1%, the requested accuracy.

M	$\phi(0, 1.25)$
128	0.464517670619715
256	0.469387263974194
512	0.471586585046651
1024	0.472388657559251
2048	0.472790327155522
4096	0.472989985610756

Table 1: GCI analysis data for $CFL = 0.8$.

M	$\phi_{h=0}$	p	GCI_{12} (%)	GCI_{23} (%)	β
128	-	-	-	-	-
256	-	-	-	-	-
512	0.473398015743182	1.146743067686559	0.4801426593676	1.0680817516835	0.99533
1024	0.472849076939131	1.455253657646967	0.1218327780824	0.3346394735644	0.99830
2048	0.473193267438895	0.997723408586615	0.1065324997757	0.2129099277600	0.99915
4096	0.473187318780311	1.008475093835296	0.0521504618381	0.1049597469500	0.99957

Table 2: GCI analysis results for $CFL = 0.8$.

M	$\phi(0, 1.25)$
128	0.470667415947558
256	0.471166273852559
512	0.472263797941595
1024	0.472683229511829

Table 3: GCI analysis data for $CFL = 0.5$.

M	$\phi_{h=0}$	p	GCI_{12} (%)	GCI_{23} (%)	β
128	-	-	-	-	-
256	-	-	-	-	-
512	0.470251726509905	-1.137551764106874	-0.5325602556401	-0.2426286094229	0.99767
1024	0.472942667327806	1.387745244157744	0.0686077376399	0.1796848999356	0.99911

Table 4: GCI analysis results for $CFL = 0.5$.