Fourier Analysis and Wavelets
Homework 5

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Problem 2

Let ϕ and ψ be the Haar scaling and wavelet functions, respectively. Let V_j and W_j be the spaces generatd by $\phi(2^k x - k), k \in \mathbb{Z}$, and $\psi(2^j x - k), k \in \mathbb{Z}$, respectively. Consider the function defined on $0 \leq x \leq 1$ given by

$$f(x) = \begin{cases} 2 & 0 \leq x < 1/4, \\ -3 & 1/4 \leq x < 1/2, \\ 1 & 1/2 \leq x < 3/4, \\ 3 & 3/4 \leq x < 1. \end{cases}$$

Express f first in terms of the basis for V_2 and then decompose f into its component parts in W_1, W_0 and V_0. In other words, find the Haar wavelet decomposition for f. Sketch each of these components.

Solution: The function f is easily expressed as

$$f(x) = 2\phi(4x) - 3\phi(4x-1) + \phi(4x-2) + 3\phi(4x-3) \in V_2$$

For the decomposition we use the following relations

$$\phi\left(2^j x\right) = \frac{\phi\left(2^{j-1}x\right) + \psi\left(2^{j-1}x\right)}{2},$$
$$\phi\left(2^j x - 1\right) = \frac{\phi\left(2^{j-1}x\right) - \psi\left(2^{j-1}x\right)}{2}.$$

For this case we use

$$\phi\left(4x\right) = \frac{\phi\left(2x\right) + \psi\left(2x\right)}{2},$$
$$\phi\left(4x-1\right) = \frac{\phi\left(2x\right) - \psi\left(2x\right)}{2},$$
$$\phi\left(4x-2\right) = \phi\left(4(x-1/2)\right) = \frac{\phi\left(2x-1\right) + \psi\left(2x-1\right)}{2},$$
$$\phi\left(4x-3\right) = \phi\left(4(x-1/2)-1\right) = \frac{\phi\left(2x-1\right) - \psi\left(2x-1\right)}{2}.$$

With these relations we decompose f into its components in W_1 and V_1:

$$f(x) = \frac{5}{2}\psi\left(2x\right) - \psi\left(2x-1\right) - \frac{1}{2}\phi\left(2x\right) + 2\phi\left(2x-1\right) \in W_1 \bigoplus V_1.$$

Next, we decompose the V_1 component, $f_{V_1} = -\frac{1}{2}\phi\left(2x\right) + 2\phi\left(2x-1\right)$, into its components in W_0 and V_0 using

$$\phi\left(2x\right) = \frac{\phi\left(x\right) + \psi\left(x\right)}{2},$$
$$\phi\left(2x-1\right) = \frac{\phi\left(x\right) - \psi\left(x\right)}{2}.$$

Then,

$$f_{V_1} = -\frac{5}{4}\psi\left(x\right) + \frac{3}{4}\phi\left(x\right) \in W_0 \bigoplus V_0.$$

Hence,

$$f\left(x\right) = \frac{5}{2}\psi\left(2x\right) - \psi\left(2x-1\right) - \frac{5}{4}\psi\left(x\right) + \frac{3}{4}\phi\left(x\right) \in W_1 \bigoplus W_0 \bigoplus V_0.$$

Every component of the function and the function itself are graphed in the following figures.

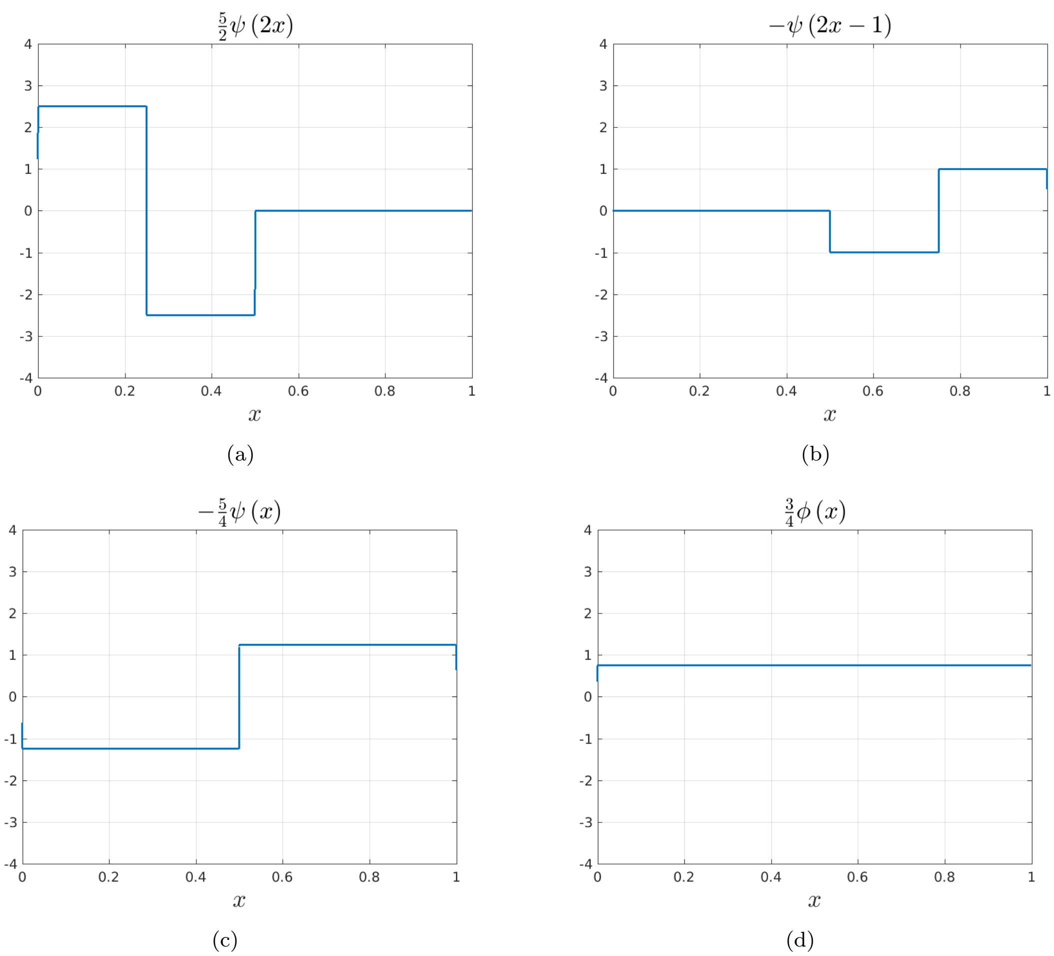


Figure 1: Haar components of the function g.

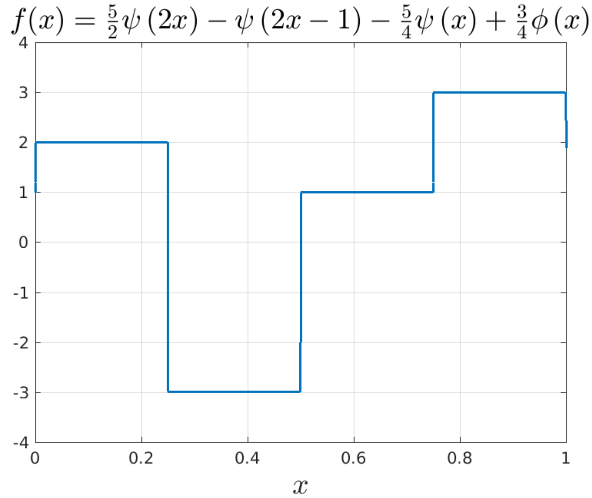


Figure 2: Function $g = \sum_{k=0}^{7} a_k^3 \phi(8x - k)$.

Problem 4

Let V_n be the spaces generated by $\phi(2^n x - k), k \in \mathbb{Z}$, where ϕ is the Haar scaling function.

- On the interval $0 \leq x \leq 1$, what are the dimensions of the spaces V_n and W_n for $n \geq 0$?

Solution: The functions $\phi(2^n x - k), k \in \mathbb{Z}$ represent step functions of width 2^{-n}. Given the interval $0 \leq x \leq 1$, it is inmediate that we need 2^n functions to cover the whole interval. More formally, by *Theorem 4.6*, we know that $\{2^{n/2}\phi(2^n x - k), k \in \mathbb{Z}\}$ is a basis for V_n and $\{2^{n/2}\psi(2^n x - k), k \in \mathbb{Z}\}$ is a basis for W_n. In this case, because of the given interval, $k = 0, \ldots, 2^n - 1$, which makes the dimension of both spaces V_n and W_n equal to 2^n.

- Using the result

$$\dim(A \bigoplus B) = \dim(A) + \dim(B),$$

count the dimension of the space on the right side of the equality

$$V_n = W_{n-1} \bigoplus W_{n-2} \bigoplus \cdots \bigoplus W_0 \bigoplus V_0$$

Solution: We proceed using the previous relation

$$\begin{aligned}
\dim(V_n) &= \dim(W_{n-1}) + \dim(W_{n-2}) + \cdots + \dim(W_1) + \dim(W_0) + \dim(V_0), \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 + 2^0, \\
&= 2^0 + \sum_{k=0}^{n-1} 2^k = 1 + 2^n - 1 = 2^n,
\end{aligned}$$

we get the same result as in the first part.

Problem 6

Reconstruct $g \in V_3$, given these coefficients in its Haar wavelet decomposition:

$$a^2 = [1/2, 2, 5/2, -3/2],$$
$$b^2 = [-3/2, -1, 1/2, -1/2].$$

The first entry in each list corresponds to $k = 0$. Sketch g.

Solution: It is easy to reconstruct $g \in V_3$ using the relations from *Theorem 4.14*,

$$g(x) = \sum_{k=0}^{2^3-1} a_k^3 \phi(2^3 x - k) \in V_3,$$

where

$$a_k^3 = \begin{cases} a_l^2 + b_l^2 & \text{if } k = 2l \text{ is even,} \\ a_l^2 - b_l^2 & \text{if } k = 2l+1 \text{ is odd.} \end{cases}$$

Therefore,

$$a^3 = [a_0^2 + b_0^2, a_0^2 - b_0^2, a_1^2 + b_1^2, a_1^2 - b_1^2, a_2^2 + b_2^2, a_2^2 - b_2^2, a_3^2 + b_3^2, a_3^2 - b_3^2]$$
$$a^3 = [-1, 2, 1, 3, 3, 2, -2].$$

Hence,

$$g(x) = \sum_{k=0}^{7} a_k^3 \phi(8x - k),$$

which we can expand as

$$g(x) = -\phi\left(8x\right) + 2\phi\left(8x - 1\right) + \phi\left(8x - 2\right) + 3\phi\left(8x - 3\right) + 3\phi\left(8x - 4\right) + 2\phi\left(8x - 5\right) - 2\phi\left(8x - 6\right) - \phi\left(8x - 7\right),$$

and it is graphed in the next figure.

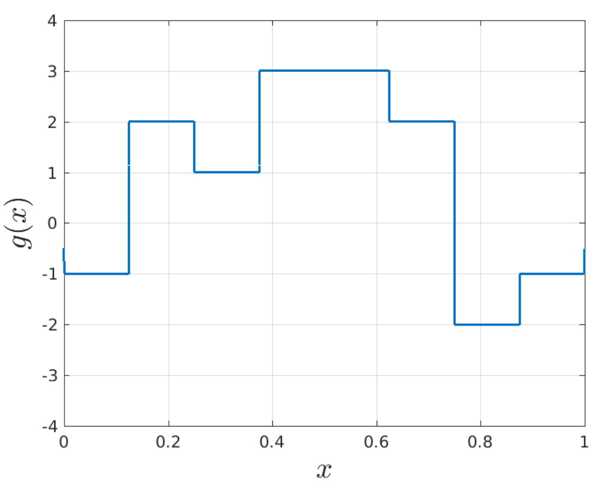


Figure 3: Function $g = \sum_{k=0}^{7} a_k^3 \phi(8x - k)$.

Problem 9

Let

$$f(t) = e^{-t^2/10}\left(\sin(2t) + 2\cos(4t) + 0.4\sin(t)\sin(50t)\right).$$

Discretize the function f over the interval $0 \le t \le 1$ as described in Step 1of Section 4.4. Use $n = 8$ as the top level (so there are 2^8 nodes in the discretization). Implement the decomposition algorithm described in Step 2 of Section 4.4 using the Haar wavelets. Plot the resulting levels, $f_{j-1} \in V_{j-1}$ for $j = 8 \ldots 1$ and compare with the original signal.

Solution: *The solution to this problem below, with the Matlab code.*

Problem 10

Filter the wavelet coefficients computed in exercise 9 by setting to zero any wavelet coefficient whose absolute value is less than $tol = 0.1$. Then reconstruct the signal as described in Section 4.4. Plot the reconstructed f_8 and compare with the original signal. Compute the relative l^2 difference between the original signal and the compressed signal. Experiment with various tolerances. Keep track of the percentage of wavelet coefficients that have been filetered out.

Solution: *The solution to this problem below, with the Matlab code.*

MATLAB CODE - FRANCISCO CASTILLO

Table of Contents

Preliminary Commands

```
clear all
close all
clc
linewidth=1.6;
labelfontsize=18;
legendfontsize=12;
```

Introduction

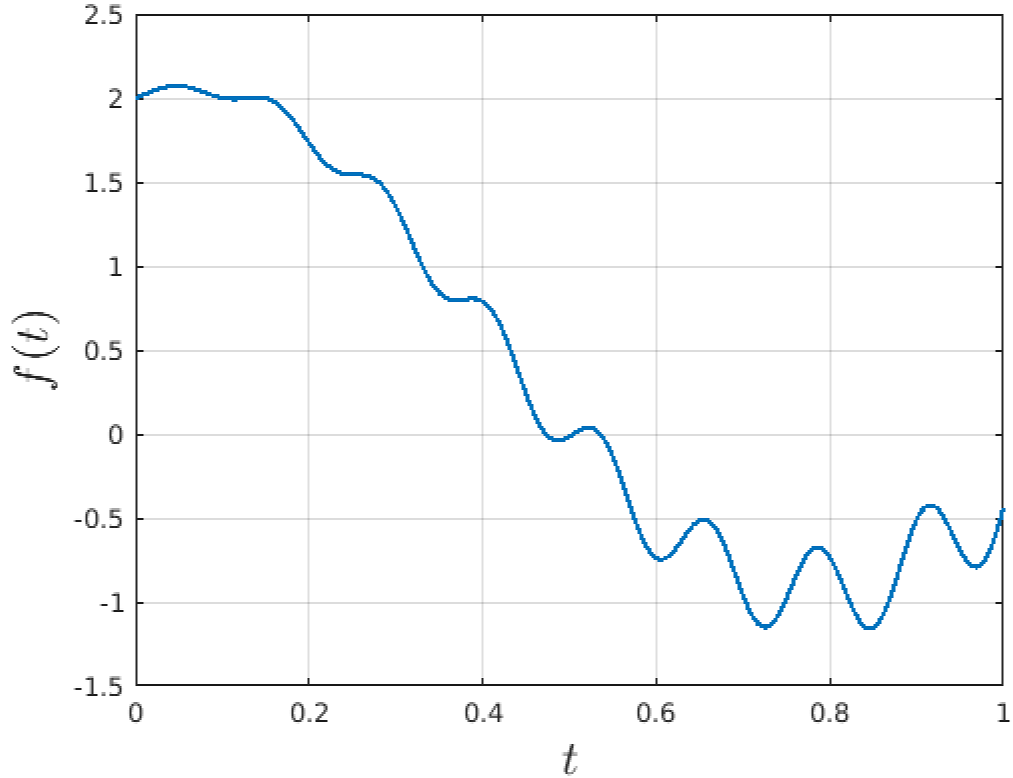
This code solves both problems 9 and 10 of the present homework assignment.

Define the function

```
t=linspace(0,1,256); % interval [0,1] partitioned into 2^8=256 pieces
y = exp(-t.^2/10).*(sin(2*t) + 2*cos(4*t) + .4*sin(t).*sin(50*t));
```

In the following figure we can see the function that we are going to work with in this two problems.

```
figure
plot(t,y,'linewidth',linewidth);
xlabel('$t$','interpreter','latex','fontsize',labelfontsize)
ylabel('$f(t)$','interpreter','latex','fontsize',labelfontsize)
grid on
axis([0 1 -1.5 2.5])
```

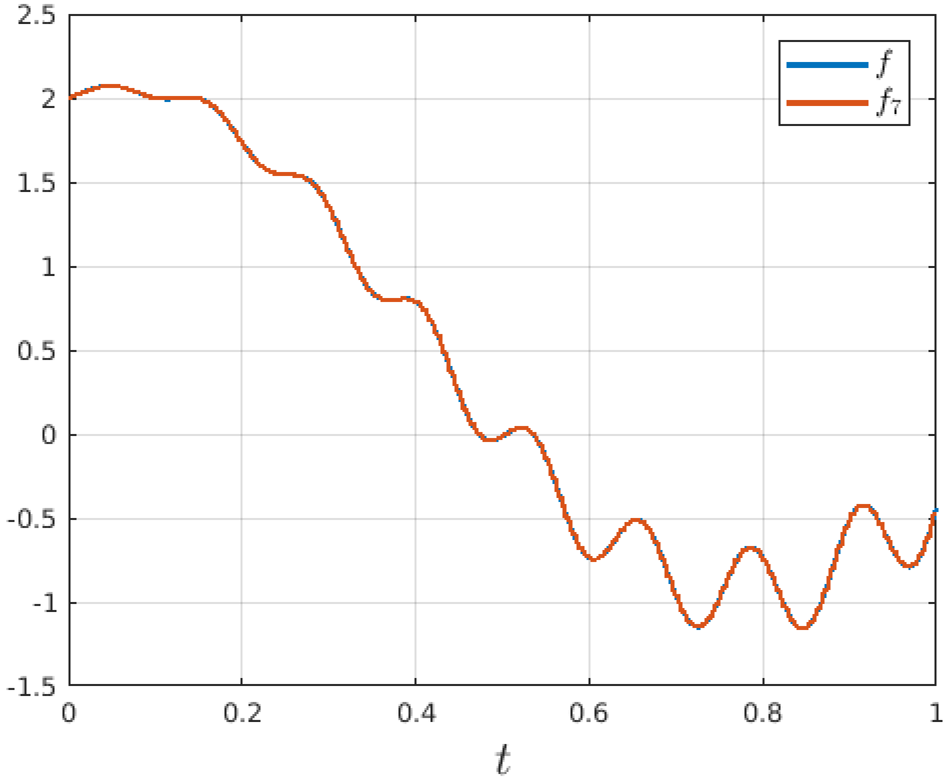


The goal in problem 9 will be decompose the signal using Haar wavelets and plot the resulting levels. In problem 10 the goal will be filter the high frequency noise and obtain a filtered signal, which we will compare to the initial.

Decomposition (problem 9)

In this section we implement the decomposition algorithm described in Step 2 of Section 4.4.

```
for i=1:8
    [C,L] = wavedec(y,i,'db1'); % i-th level decomposition
    Ai = upcoef('a',C(1:L(1)),'db1',i,length(y));
    figure
    plot(t,y,t,Ai,'linewidth',linewidth); % Plot original function and
 the i-th level decomposition
    grid on
    xlabel('$t$','interpreter','latex','fontsize',labelfontsize)
    axis([0 1 -1.5 2.5])
    h=legend('$f$',['$f_' num2str(8-i) '$']);
    set(h,'interpreter','latex','fontsize',legendfontsize);
    saveas(gcf,['f_' num2str(8-i) '_p9'],'png')
end
```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0.2
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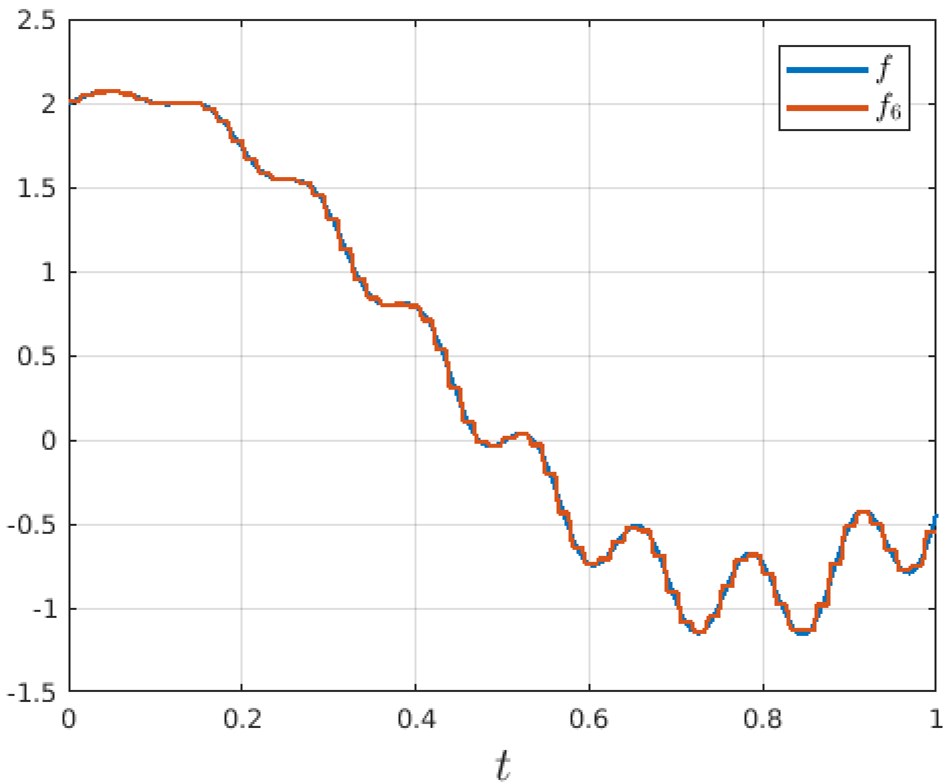
t
f
f_7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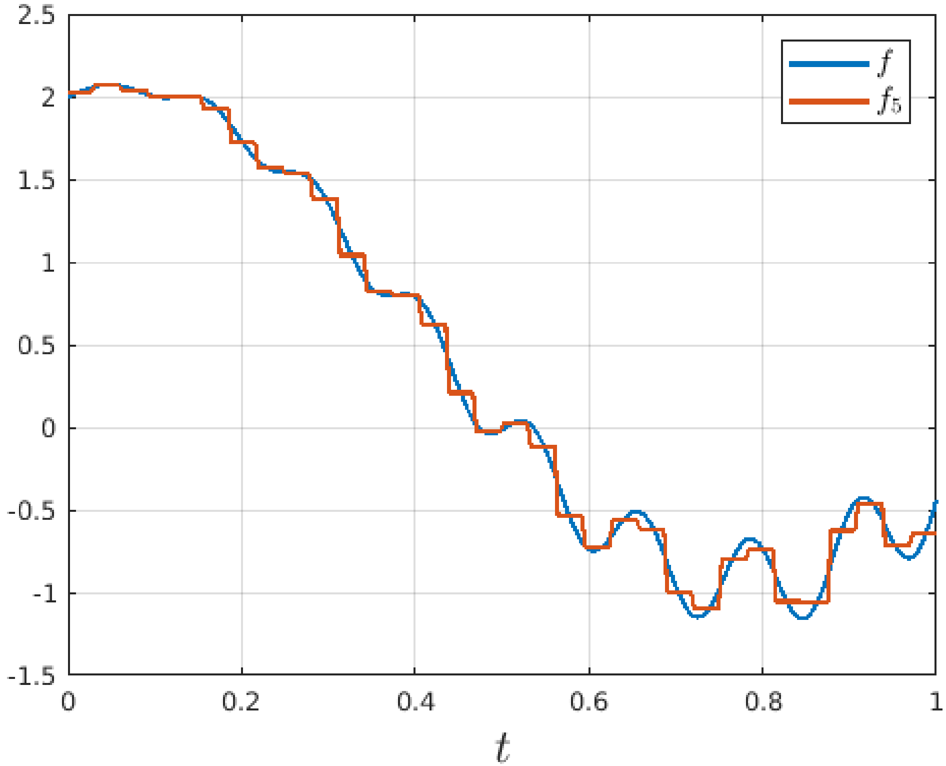
f_6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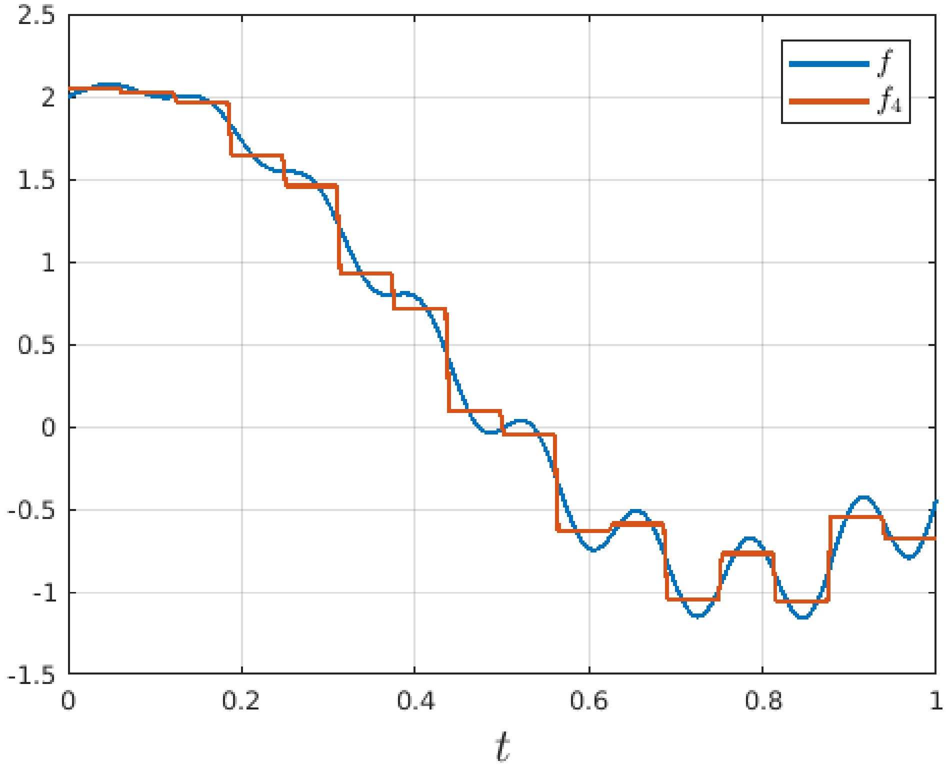
f_5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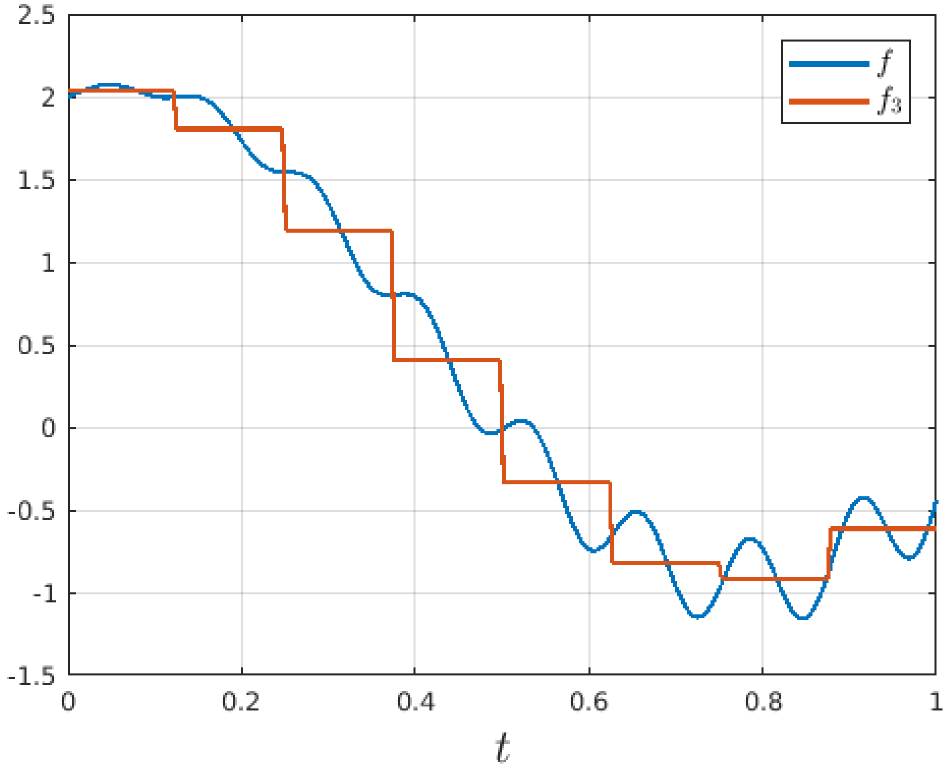
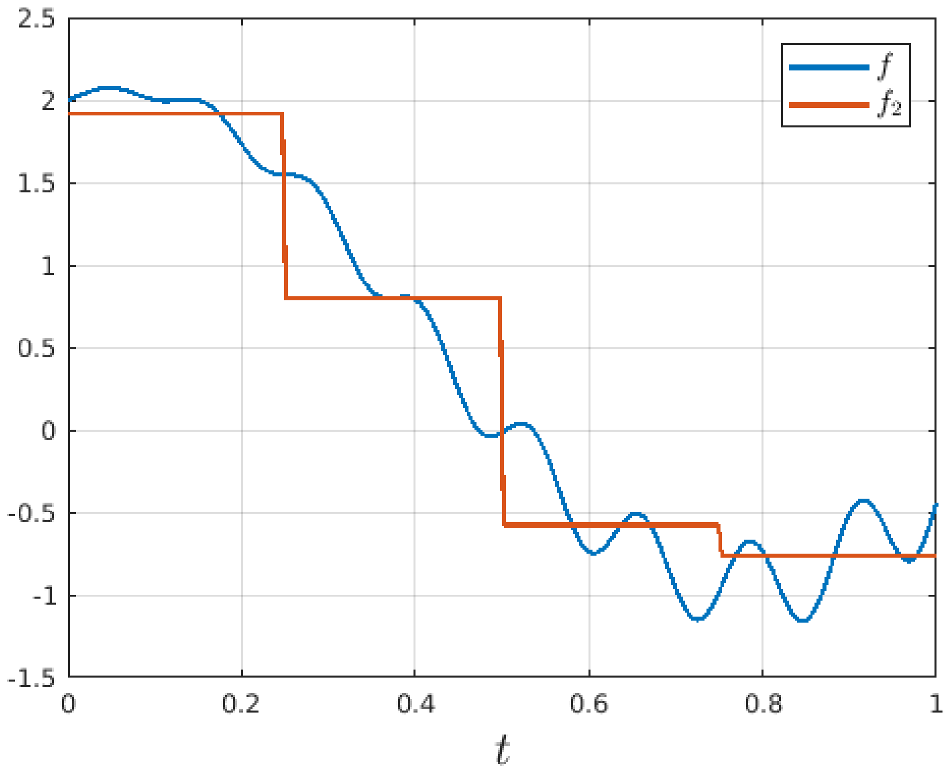
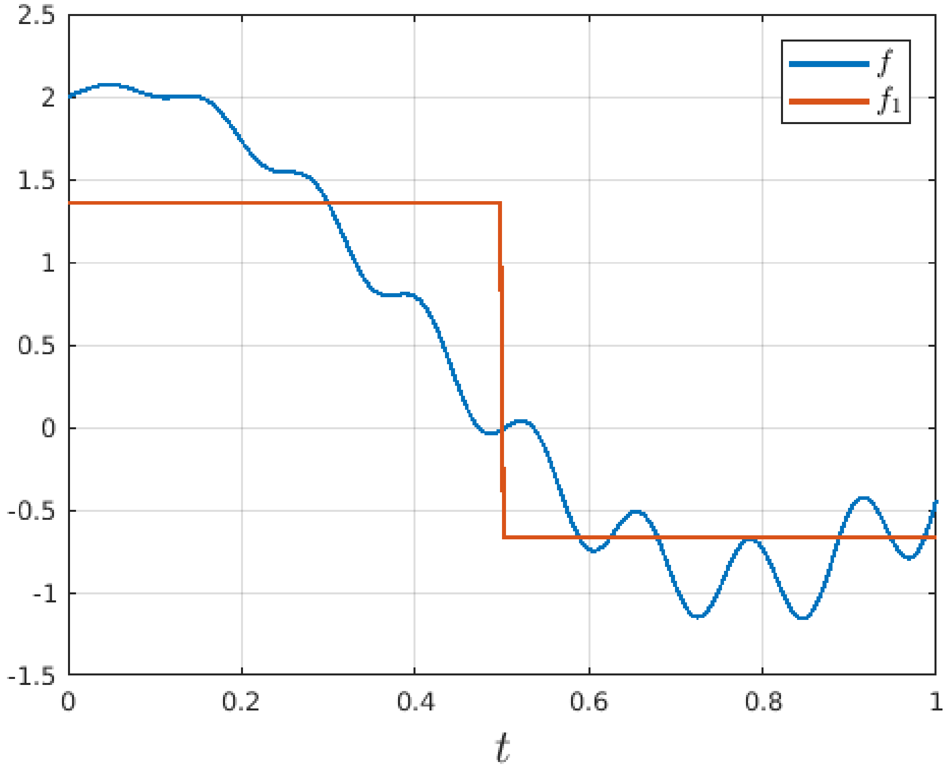

f
f_3
t


f
f_2
t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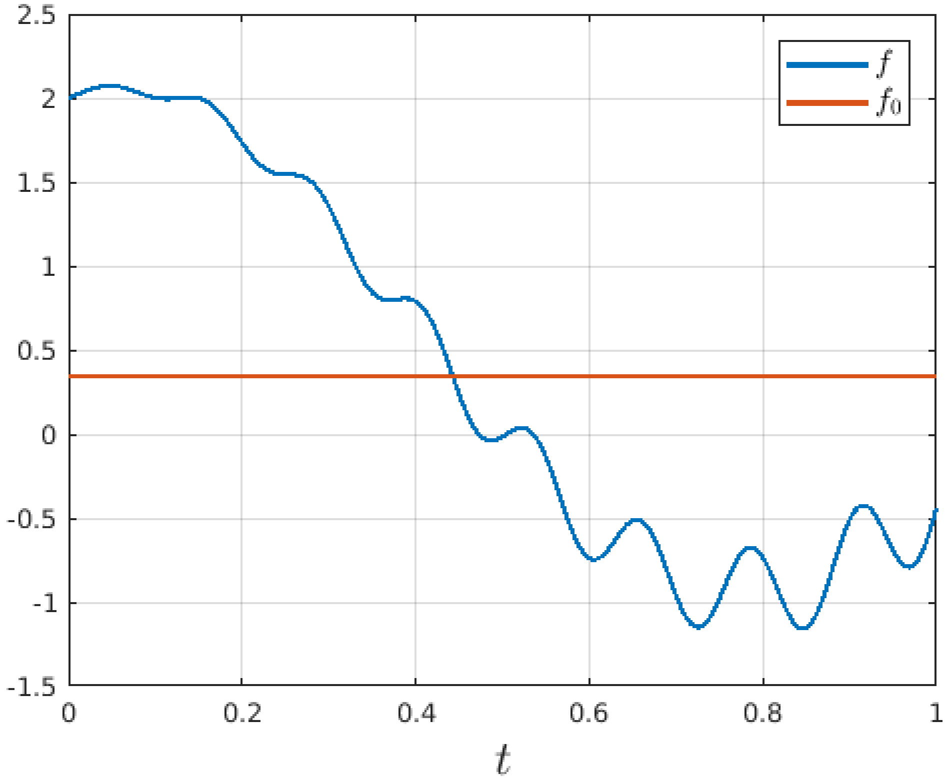
t
f
f_1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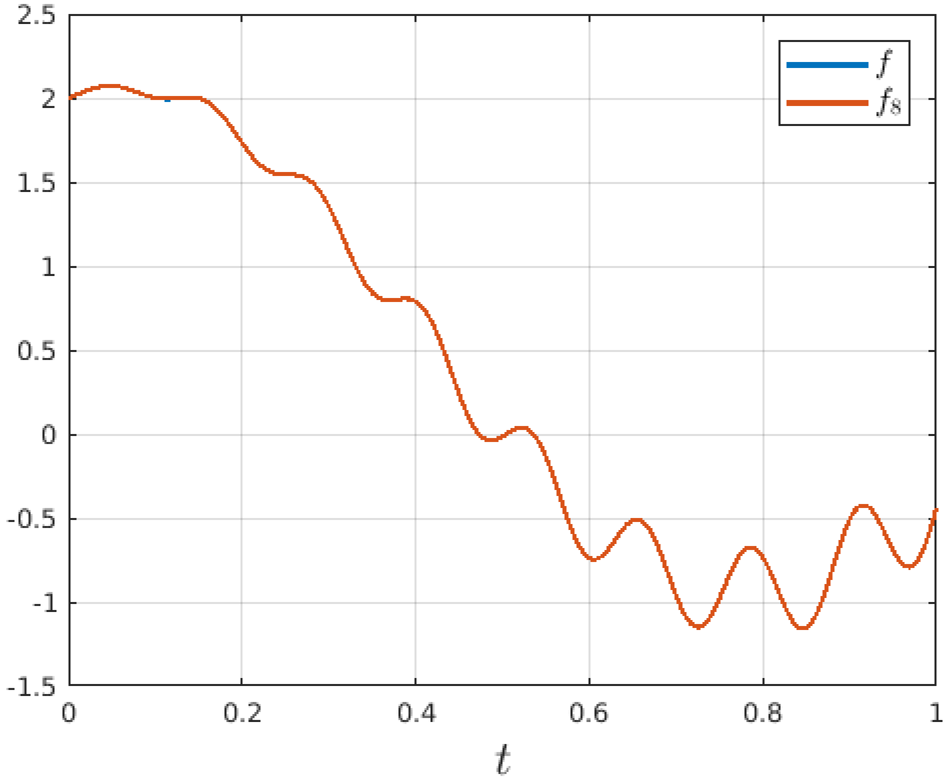
t
f
f_0

As we can see in the figures, the further we take the decomposition, the more inacurate the approximation. This is expected because for lower js the decomposed signal does not catch the higher frequencies.

Filter out high frequencies (Problem 10)

In this section we will filter out those coefficients smaller than a preset tolerance by setting them to zero. Once that is done, we will reconstruct the signal and compare it to the original one.

```
N=2^8;
tol=[1e-3 5e-3 1e-2 5e-2 1e-1 5e-1 1 5];
for i=1:length(tol)
    count(i)=0;
    [C,L]=wavedec(y,8,'db1');   %level 8 decomposition; analogous to
 full fft
    for j=1:N
        if (abs(C(j))<tol(i))
            C(j)=0;
            count(i)=count(i)+1;
        end
    end
    yc=waverec(C,L,'db1');   %reconstruct - analogous to ifft
    figure
    plot(t,y,t,yc,'linewidth',linewidth); % Plot original function and
 the i-th level decomposition
    grid on
    xlabel('$t$','interpreter','latex','fontsize',labelfontsize)
    axis([0 1 -1.5 2.5])
    h=legend('$f$','$f_8$');
    set(h,'interpreter','latex','fontsize',legendfontsize);
    saveas(gcf,['f_tol' num2str(i) '_p10'],'png')
    e(i)=norm(y-yc)/norm(y);
end
```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f_8
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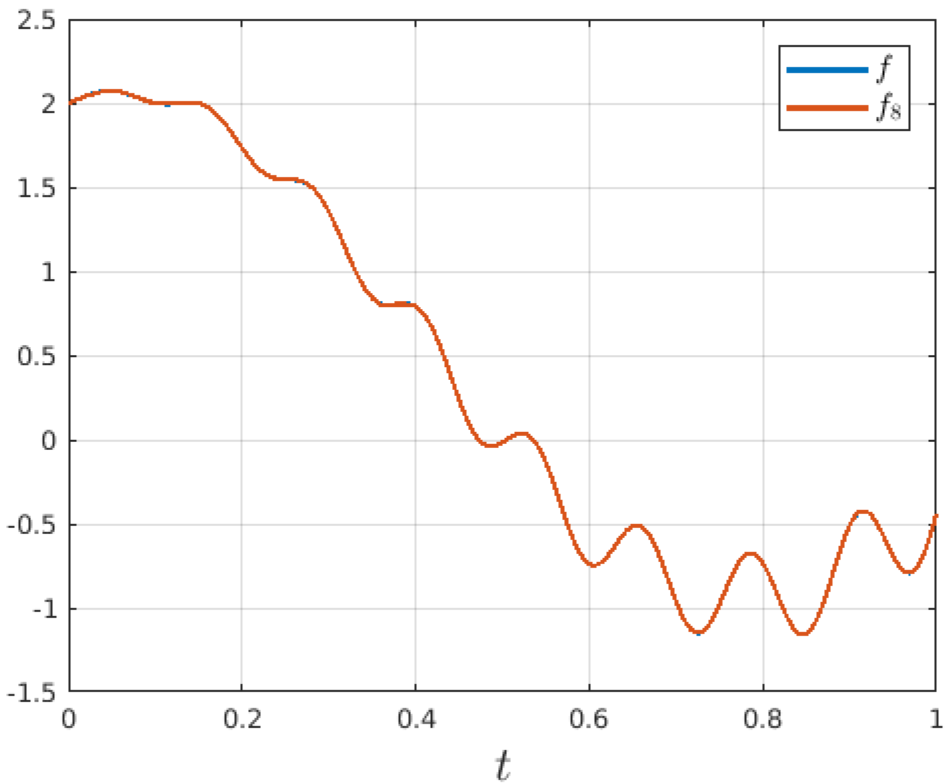
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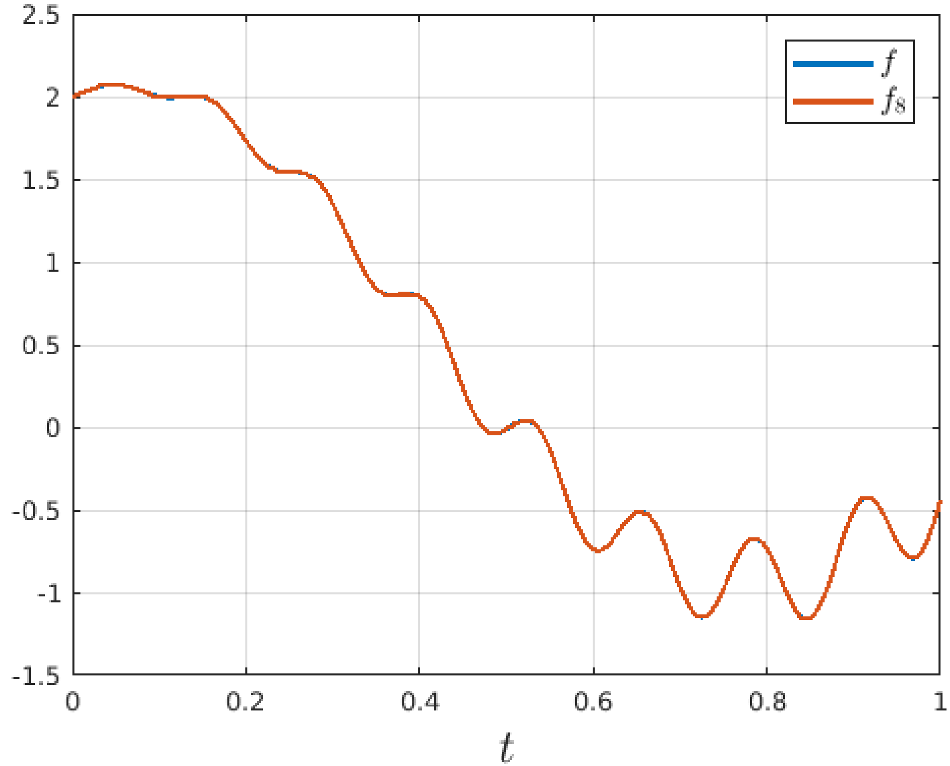
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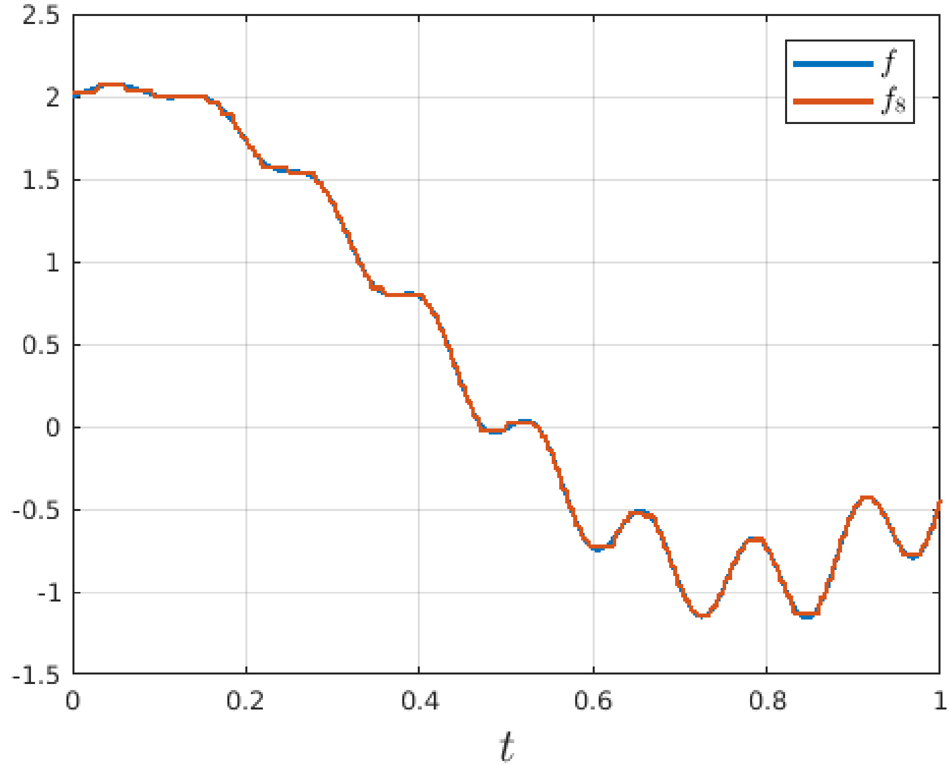
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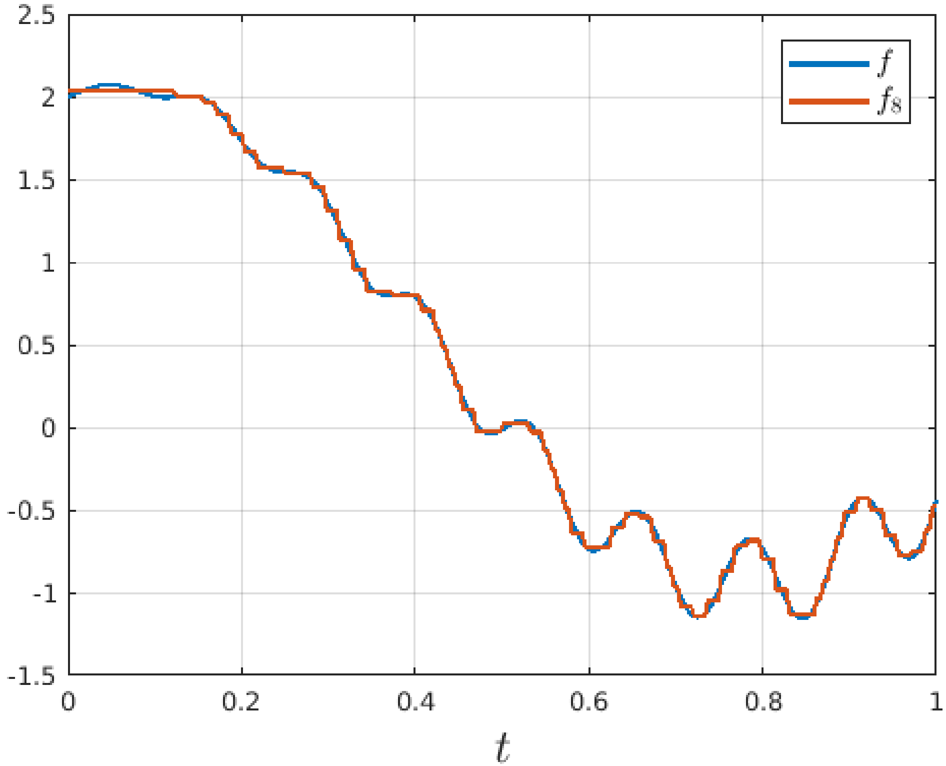
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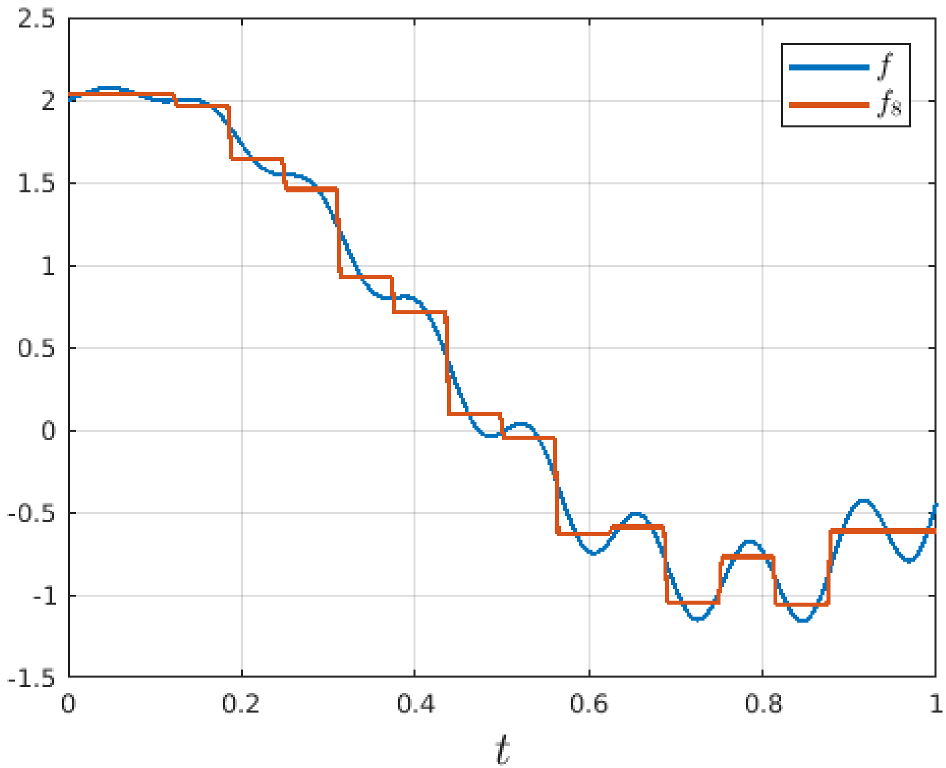
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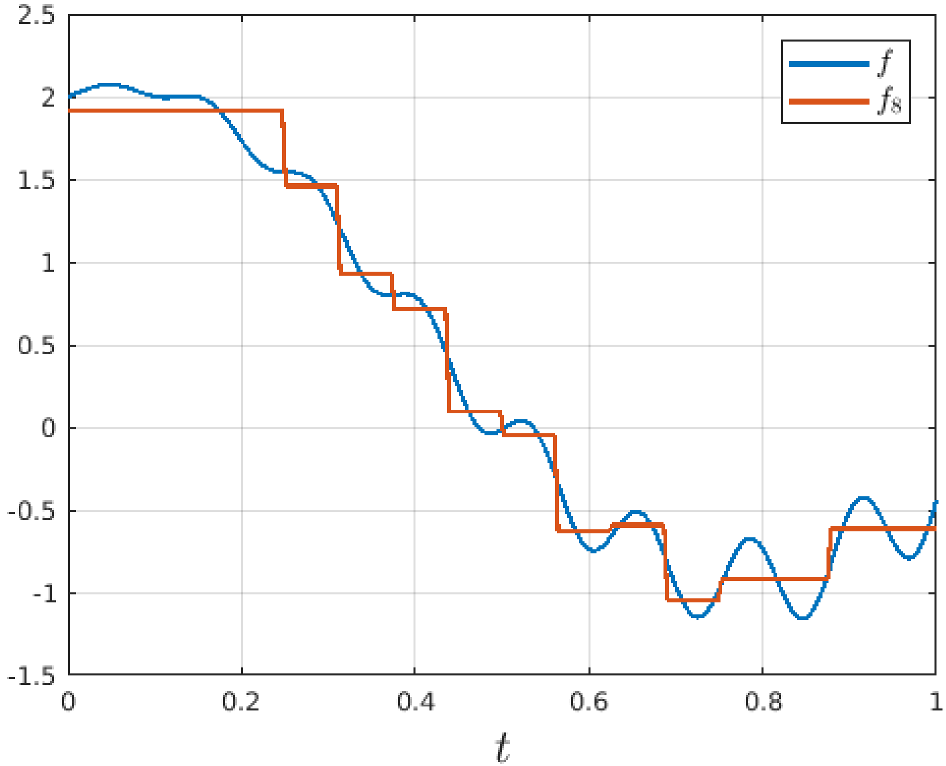
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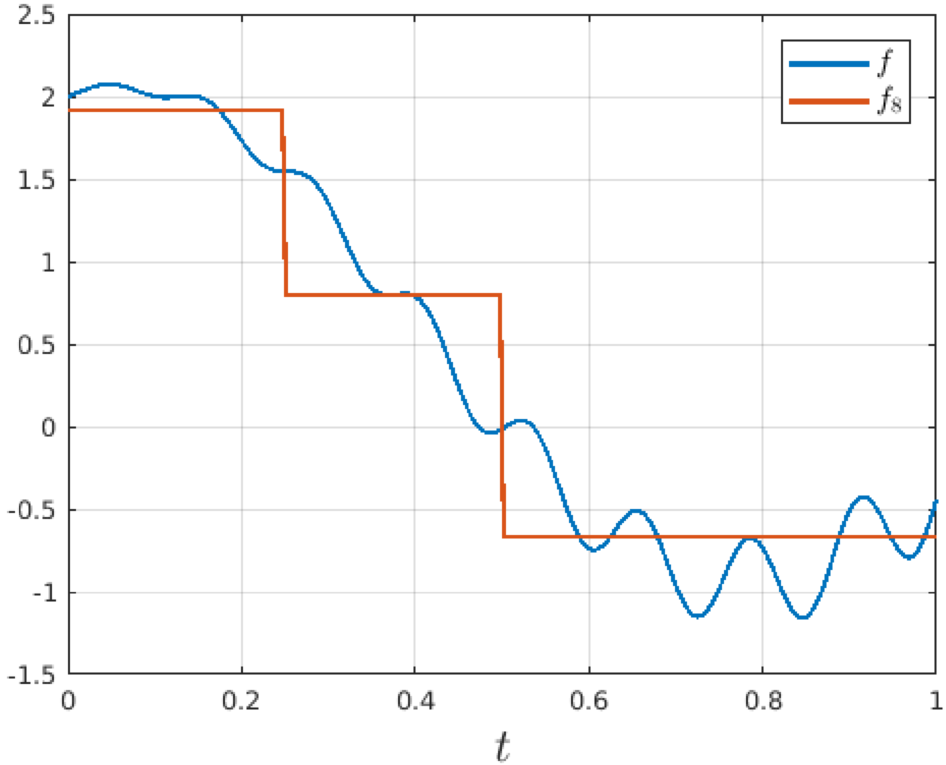
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t
f
f_8

In the figures above it can be seen how we increase the tolerance and the reconstructed signal gets more and more inaccurate compared to the original signal. I have used the following tolerances:

```
tol


tol =

  Columns 1 through 7

    0.0010    0.0050    0.0100    0.0500    0.1000    0.5000    1.0000

  Column 8

    5.0000
```

Which have produced the removal of the following number of coefficients, respectively:

```
count


count =

     9    40    65   173   200   242   245   253
```

This represent, respectively, the following percentages of the total number of data points

```
percentage=count*100/N


percentage =

  Columns 1 through 7

    3.5156   15.6250   25.3906   67.5781   78.1250   94.5312   95.7031

  Column 8

   98.8281
```

To finish, the relative error obtained is detail below:

```
e


e =

  Columns 1 through 7

    0.0001    0.0009    0.0021    0.0158    0.0258    0.0843    0.1161

  Column 8
```

```
    0.2908
```

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