Partial Differential Equations
TA Homework 4

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Problem 3.3.4

1. Solve the wave equation

$$\begin{aligned} &\partial_t^2 u - \partial_x^2 u = 0, && x, t \in \mathbb{R} \\ &u(x,x) = f(x), \quad \partial_t u(x,0) = g(x), && x \in \mathbb{R}, \end{aligned}$$

where $g, f : \mathbb{R} \to \mathbb{R}$.

State appropriate assumptions for f and g such that you really have a solution.

Notice that $c = 1$.

Solution: Given the previous PDE, the wave equation, we can express the solution $u(x,t)$ as

$$u(x,t) = F(x+t) + G(x-t),$$

where we have already taken into account that $c = 1$. Imposing the first initial condition,

$$u(x,x) = F(2x) + G(0) = f(x). \tag{1}$$

For the second initial condition, we calculate first $\partial_t u$,

$$\partial_t u(x,t) = F'(x+t) - G'(x-t),$$

and impose the initial condition,

$$\partial u(x,0) = F'(x) - G'(x) = g(x).$$

Integrating the equation we get

$$F(x) - G(x) = F(0) - G(0) + \int_0^x g(y) dy,$$

or

$$F(2x) - G(2x) = F(0) - G(0) + \int_0^{2x} g(y) dy.$$

Substract now equation (1) from the previous equation and get

$$-G(2x) - G(0) = F(0) - G(0) - f(x) + \int_0^{2x} g(y) dy,$$

$$-G(2x) = F(0) - f(x) + \int_0^{2x} g(y) dy, G(2x) = f(x) - F(0) - \int_0^{2x} g(y) dy,$$

or

$$G(x) = f(\frac{x}{2}) - F(0) - \int_0^x g(y)dy.$$

From the equation above we obtain $G(x-t)$ needed for our solution,

$$G(x-t) = f(\frac{x-t}{2}) - F(0) - \int_0^{x-t} g(y)dy.$$

Now, we retake equation (1) and rewrite it as

$$F(x+t) = f(\frac{x+t}{2}) - G(0).$$

Thus, the solution is

$$u(x,t) = f(\frac{x+t}{2}) + f(\frac{x-t}{2}) - (F(0) + G(0)) - \int_0^{x-t} g(y)dy,$$

and it is left to calculate $F(0) + G(0)$, which we do by evaluating equation (1) at $x = 0$,

$$F(0) + G(0) = f(0).$$

Hence,

$$u(x,t) = f(\frac{x+t}{2}) + f(\frac{x-t}{2}) - f(0) - \int_0^{x-t} g(y)dy,$$

which satisfies the initial conditions:

- $u(x,x) = f(x)$.

  $$u(x,x) = f(\frac{2x}{2}) + f(0) - f(0) - \int_0^0 g(y)dy = f(x).$$

- $\partial_t u(x,0) = g(x)$.

  $$\partial_t u(x,0) = \frac{1}{2}f(\frac{x}{2}) - \frac{1}{2}f(\frac{x}{2}) + g(x) = g(x),$$

  where to evaluate the derivative of the integral we have used the Fundamental Theorem of Calculus.

Thus, since the $u(x,t)$ given satisfies the PDE and the initial conditions, it is the solution provided that f is twice differentiable and g is once differentiable.

Problem 3.3.7

1. Let $L > 0$. Solve the wave equation

$$\begin{aligned} \partial_t^2 u - c^2 \partial_x^2 u &= 0, && 0 \le x \le L, t \ge 0, \\ u(x,0) &= f(x), && 0 \le x \le L, \\ \partial_t u(x,0) &= g(x), && 0 \le x \le L, \\ \partial_x u(0,t) = 0 &= \partial_x u(L,t), && t \ge 0. \end{aligned}$$

Hint: Extend f, g in an even and $2L$-periodic fashion.

Which assumptions do f and g have to satisfy to make u a solution?

Solution: Given the zero Neumann boundary condition at $x = 0$, we extend f and g to $[-L, L]$ in an even fashion:

$$f(-x) = f(x), \quad x \in [0, L],$$
$$g(-x) = g(x), \quad x \in [0, L],$$

and noticing that the derivative of f is odd

$$f'(-x) = -f'(x), \quad x \in [0, L].$$

To take care of the zero Neumann boundary condition at $x = L$ we perform a $2L-$periodic extension of f and g which gives us functions defined on all $\mathbb{R}$,

$$f(x + 2kL) := f(x), \quad k \in \mathbb{Z}, \ x \in [-L, L],$$
$$g(x + 2kL) := g(x), \quad k \in \mathbb{Z}, \ x \in [-L, L].$$

We can prove that the extended f is $2L$ periodic and even in a similar way as the *Lemma 3.13* is proved in the notes. Indeed, let $x \in \mathbb{R}$. Then $x = y + 2kL$ with $-L \le y \le L$ and $k \in \mathbb{Z}$. By extension,

$$f(x + 2L) = f(y + 2(k+1)L) = f(y) = f(y + 2kL) = f(x).$$

Further

$$f(-x) = f(-y - 2kL) = f(-y) = f(y) = f(x),$$

so f is even around zero. Since f is even about zero and $2L$-periodic,

$$f(L + x) = f(L + x - 2L) = f(-L + x) = f(-(L - x)) = f(L - x),$$

it is also even about L. We can prove the same for g. The conditions for f and g imply that their extensions to $\mathbb{R}$ are twice and once differentiable, respectively. The D'Alembert's formula provides a solution to the PDE and the initial conditions,

$$u(x,t) = \frac{1}{2}\left[f(x + ct) + f(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

We check now that the boundary conditions are satisfied. First we calculate

$$\partial_x u(x,t) = \frac{1}{2}\left[f'(x + ct) + f'(x - ct)\right] + \frac{1}{2c}\left[g(x + ct) - g(x - ct)\right],$$

where we have used the Fundamental Theorem of Calculus to differentiate the integral, and we make $x = 0$,

$$\begin{aligned}
\partial_x u(0,t) &= \frac{1}{2}\left[f'(ct) + f'(-ct)\right] + \frac{1}{2c}\left[g(ct) - g(-ct)\right] \\
&= \frac{1}{2}\left[f'(ct) - f'(ct)\right] + \frac{1}{2c}\left[g(ct) - g(ct)\right] \\
&= 0.
\end{aligned}$$

For the boundary condition at $x = L$,

$$\begin{aligned}
\partial_x u(0,t) &= \frac{1}{2}\left[f'(L+ct) + f'(L-ct)\right] + \frac{1}{2c}\left[g(L+ct) - g(L-ct)\right] \\
&= \frac{1}{2}\left[f'(L+ct) - f'(L+ct)\right] + \frac{1}{2c}\left[g(L+ct) - g(L+ct)\right] \\
&= 0.
\end{aligned}$$

since f' and g are odd and even around L, respectively. Thus, $u(x,t)$ is the solution provided that extended f and g are twice and once differentiable, respectively. The extended f is twice differentiable if and only if the original f is twice differentiable and

$$f(0) = f(L), \qquad f'(0) = 0 = f'(L).$$

The extended g is once differentiable if and only if the original g is once differentiable and

$$g(0) = g(L).$$

Problem 3.3.9

1. Let u solve

$$\begin{aligned}
(\partial_t^2 - c^2\partial_x^2)u(x,t) &= \phi(x,t), \quad x,t \in \mathbb{R}. \\
u(x,0) &= 0, \quad x \in \mathbb{R}, \\
\partial_t u(x,0) &= 0, \quad x \in \mathbb{R}
\end{aligned}$$

and $\tilde{u}$ solve

$$\begin{aligned}
(\partial_t^2 - c^2\partial_x^2)\tilde{u}(x,t) &= 0, \quad x,t \in \mathbb{R}. \\
\tilde{u}(x,0) &= f(x), \quad x \in \mathbb{R}, \\
\partial_t \tilde{u}(x,0) &= g(x), \quad x \in \mathbb{R}
\end{aligned}$$

Prove that $U = u + \tilde{u}$ solves

$$\begin{aligned}
(\partial_t^2 - c^2\partial_x^2)U(x,t) &= \phi(x,t), \quad x,t \in \mathbb{R}. \\
U(x,0) &= f(x), \quad x \in \mathbb{R}, \\
\partial_t U(x,0) &= g(x), \quad x \in \mathbb{R}
\end{aligned}$$

This is a special case of the so-called principle of superposition. It works here because the problem is linear.

Solution: We start by proving that U satisfies the PDE,

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)U &= (\partial_t^2 - c^2\partial_x^2)(u+\tilde{u})\\ &= \partial_t^2(u+\tilde{u}) - c^2\partial_x^2(u+\tilde{u})\\ &= \partial_t^2 u + \partial_t^2\tilde{u} - c^2\partial_x^2 u - c^2\partial_x^2\tilde{u} & &= (\partial_t^2 - c^2\partial_x^2)u + (\partial_t^2 - c^2\partial_x^2)\tilde{u}\\ &= \phi(x,t) + 0 & &= \phi(x,t)\end{aligned}$$

Now we check that it satisfies the first initial condition,

$$\begin{aligned}U(x,0) &= u(x,0) + \tilde{u}(x,0)\\ &= 0 + f(x)\\ &= f(x).\end{aligned}$$

To check the second initial condition we first calculate

$$\partial_t U(x,t) = \partial_t(u+\tilde{u}) = \partial_t u(x,t) + \partial_t\tilde{u}(x,t),$$

and make $t=0$,

$$\begin{aligned}\partial_t U(x,0) &= \partial_t u(x,0) + \partial_t\tilde{u}(x,0)\\ &= 0 + g(x)\\ &= g(x).\end{aligned}$$

Hence, since $U = u + \tilde{u}$ satisfies the PDE and the initial conditions, it is the solution.

Problem 3.3.10

1. Solve the vibrating string equation with external force,

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)u(x,t) &= t\sin(x), \quad t\geq 0, 0\leq x\leq\pi,\\ u(x,0) &= \sin(x), \quad x\in[0,\pi],\\ \partial_t u(x,0) &= \sin(x), \quad x\in[0,\pi],\\ u(0,t) &= 0 = u(\pi,t), \quad t\geq 0.\end{aligned}$$

Show that the solution is of the form $u(x,t) = \psi(t)\sin(x)$. Determine $\psi(t)$ using dAlembert. **Do not assume that the solution is of this form.**

Solution: We start performing a 2π-periodic extension of $f(x) = \sin x$ in an odd and 2π-periodic fashion. We express the PDE as

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)u(x,t) =& t\sin(x), & x\in\mathbb{R}, t\in\mathbb{R}\\ u(x,0) =& \sin(x), & x\in\mathbb{R}\\ \partial_t u(x,0) =& \sin(x), & x\in\mathbb{R}\\ u(0,t) = 0 =& u(\pi,t), & t\in\mathbb{R}\end{aligned}$$

Like in the previous problem, we can separate the PDE in two and, by the superposition principle, add the solutions of the two following PDEs:

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)u_1(x,t) =& t\sin(x), \qquad x \in \mathbb{R}, t \in \mathbb{R}\\ u_1(x,0) =& 0, \qquad x \in \mathbb{R}\\ \partial_t u_1(x,0) =& 0, \qquad x \in \mathbb{R}\\ u_1(0,t) = 0 =& u_1(\pi,t), \qquad t \in \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}(\partial_t^2 - c^2\partial_x^2)u_2(x,t) =& 0, \qquad x \in \mathbb{R}, t \in \mathbb{R}\\ u_2(x,0) =& \sin(x), \qquad x \in \mathbb{R}\\ \partial_t u_2(x,0) =& \sin(x), \qquad x \in \mathbb{R}\\ u_2(0,t) = 0 =& u_2(\pi,t), \qquad t \in \mathbb{R}\end{aligned}$$

We start with the first PDE. As it is proved in the notes, we can solve this inhomogeneous wave equation and the solution is

$$\begin{aligned}u_1(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-r)}^{x+c(t-r)} r\sin\rho \; d\rho dr\\ &= \frac{1}{2c}\int_0^t r\left[-\cos\rho\right]_{x-c(t-r)}^{x+c(t-r)} dr\\ &= \frac{1}{2c}\int_0^t r\left[\cos\left(x-c(t-r)\right) - \cos\left(x+c(t-r)\right)\right] dr.\end{aligned}$$

By the trigonometric identities, $\cos\left(x-c(t-r)\right) - \cos\left(x+c(t-r)\right) = 2\sin(x)\cos(ct-cr)$. Therefore,

$$\begin{aligned}u_1(x,t) &= \frac{1}{2c}\int_0^t 2r\sin(x)\cos(ct-cr)dr\\ &= \frac{\sin(x)}{c}\int_0^t r\cos(ct-cr)dr\\ &= \frac{\sin(x)}{c}\left(\left[\frac{r}{c}\cos(ct-cr)\right]_0^t - \frac{1}{c}\int_0^t \cos(ct-cr)dr\right)\\ &= \sin(x)\left(\frac{t}{c^2} - \frac{\sin(ct)}{c^3}\right),\end{aligned}$$

where we have used integration by parts in the last steps. For the second PDE, d'Alembert formula gives us the solution

$$\begin{aligned}u_2(x,t) &= \frac{1}{2}\left(f(x+ct)+f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds\\ &= \frac{1}{2}\left(\sin(x+ct)+\sin(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)ds\\ &= \frac{1}{2}\left(\sin(x)\cos(ct)+\sin(ct)\cos(x)+\sin(x)\cos(ct)-\sin(ct)\cos(x)\right) + \frac{1}{2c}\left[-\cos(s)\right]_{x-ct}^{x+ct}\\ &= \sin(x)\cos(ct) + \frac{1}{2c}\left[\cos(x-ct)-\cos(x+ct)\right]\\ &= \sin(x)\cos(ct) + \frac{1}{2c}2\sin(x)\sin(ct)\\ &= \sin(x)\left[\cos(ct) + \frac{1}{c}\sin(ct)\right].\end{aligned}$$

Finally, by superposition principle, we obtain

$$u(x,t) = \sin(x)\left[\frac{t}{c^2} + \cos(ct) + \frac{1}{c}\sin(ct) - \frac{\sin(ct)}{c^3}\right].$$

We check now if the solution satisfies the initial condtions

$$u(x,0) = \sin(x),$$

and

$$\partial_t u(x,t) = \sin(x)\left[\frac{1}{c^2} - c\sin(ct) + \cos(ct) - \frac{\cos(ct)}{c^2}\right],$$
$$\partial_t u(x,0) = \sin(x)\left[\frac{1}{c^2} + 1 - \frac{1}{c^2}\right] = \sin(x).$$

To finish, we check that satisfies the boundry conditions

$$u(0,t) = 0 = u(\pi,t),$$

since the sine mutlplying the whole expression is zero at those points of x. Thus, $u(x,t)$ given is the solution to our problem.