Numerical Methods for PDEs
Homework 5

Francisco Jose Castillo Carrasco

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Problem 1

Derive the modified PDE for the Lax-Friedrichs method for $u_t + cu_x = 0$. Find the coefficient $D_n \sim \{\Delta t, \Delta x\}$ of numerical diffusion in $u_t + cu_x = D_n u_{xx}$. Note that $D_n \geq 0$ iff the Courant number $r = c\Delta t/\Delta x \leq 1$.

Solution: We start Taylor expanding the following terms

$$u_i^{n+1} = u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \dots$$
$$u_{i\pm1}^n = u_i^n \pm \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + \dots$$

and substituting them into the Lax-Friedrichs method,

$$\begin{aligned}
u_i^{n+1} &= \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right),\\
u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} &= \frac{1}{2}\left(2u_i^n + \Delta x^2 u_{xx}\right) - \frac{c\Delta t}{2\Delta x} 2\Delta x u_x + \dots,\\
u_t + \frac{\Delta t}{2} u_{tt} &\approx \frac{\Delta x^2}{2\Delta t} u_{xx} - cu_x,\\
u_t + cu_x &\approx \frac{\Delta x^2}{2\Delta t} u_{xx} - \frac{\Delta t}{2} u_{tt}.
\end{aligned}$$

Using the PDE, we obtain that $u_{tt} = c^2 u_{xx}$. Thus,

$$\begin{aligned}
u_t + cu_x &\approx \frac{\Delta x^2}{2\Delta t} u_{xx} - c^2 \frac{\Delta t}{2} u_{xx}\\
u_t + cu_x &\approx \frac{\Delta x^2}{2\Delta t}\left[1 - r^2\right] u_{xx},
\end{aligned}$$

where $r = c\Delta t/\Delta x$. Hence, we have obtained that the modified PDE is

$$u_t + cu_x = D_n u_{xx},$$

with

$$D_n = \frac{\Delta x^2}{2\Delta t}\left[1 - r^2\right].$$

Note that

$$
\begin{aligned}
D_n \geq 0 &\iff \left[1 - r^2\right] \geq 0 \\
&\iff r^2 \leq 1 \\
&\iff r \leq 1
\end{aligned}
$$

Problem 2

Show that the Lax-Friedrichs method is first order (using the definition of the LTE)

Solution: We can retake the following equation from Problem 1,

$$u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} = \frac{1}{2}\left(2u_i^n + \Delta x^2 u_{xx}\right) - \frac{c\Delta t}{2\Delta x} 2\Delta x u_x + \dots,$$

and use the definition of LTE

$$
\begin{aligned}
u_i^n + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} &= \frac{1}{2}\left(2u_i^n + \Delta x^2 u_{xx}\right) - \frac{c\Delta t}{2\Delta x} 2\Delta x u_x + \Delta t \tau, \\
\Delta t u_t + \frac{\Delta t^2}{2} u_{tt} &= \frac{\Delta x^2}{2} u_{xx} - c\Delta t u_x + \Delta t \tau, \\
\Delta t \tau &= \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} - \frac{\Delta x^2}{2} u_{xx} + c\Delta t u_x, \\
\Delta t \tau &= \Delta t \cancelto{0}{(u_t + cu_x)} + \frac{\Delta t^2}{2} u_{tt} - \frac{\Delta x^2}{2} u_{xx},
\end{aligned}
$$

where we have used the PDE to cancel the term $u_t + cu_x$. Thus, the Lax-Friedrichs is indeed first order in time since

$$\tau = \frac{\Delta t}{2} u_{tt} - \frac{\Delta x^2}{2\Delta t} u_{xx}.$$

Show that the Lax-Friedrichs method is conditionally stable (using von Neumann stability analysis) for $u_t + cu_x = 0$. *Hint for stability analysis:* Show $|G(k)|^2 = G^*(k)G(k) \leq 1$ iff $r \leq 1$.

Solution: By substituting the definitions

$$u_j^n = e^{ikx_j}, \quad u_j^{n+1} = G(k)e^{ikx_j},$$

into the Lax-Friedrichs scheme we obtain

$$G(k)e^{ikx_j} = \frac{1}{2}\left(e^{ik(x_j+\Delta x)} + e^{ik(x_j-\Delta x)}\right) - c\frac{\Delta t}{2\Delta x}\left(e^{ik(x_j+\Delta x)} - e^{ik(x_j-\Delta x)}\right).$$

Dividing by e^{ikx_j},

$$G(k) = \frac{1}{2}\left(e^{ik\Delta x} + e^{-ik\Delta x}\right) - c\frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$
$$G(k) = \cos(k\Delta x) - c\frac{\Delta t}{\Delta x} i \sin(k\Delta x).$$

Finally,

$$\begin{aligned} |G(k)|^2 &= G^*(k)G(k) \\ &= \cos^2(k\Delta x) + c^2\frac{\Delta t^2}{\Delta x^2}\sin^2(k\Delta x) \\ &= \cos^2(k\Delta x) + r^2\sin^2(k\Delta x) \\ &= 1 - \left(1 - r^2\right)\sin^2(k\Delta x). \end{aligned}$$

Therefore,

$$\begin{aligned} |G(k)|^2 \le 1 &\iff 1 - \left(1 - r^2\right)\sin^2(k\Delta x) \le 1 \\ &\iff \left(1 - r^2\right)\sin^2(k\Delta x) \ge 0 \\ &\iff \left(1 - r^2\right) \ge 0 \\ &\iff r^2 \le 1 \\ &\iff r \le 1. \end{aligned}$$

Hence, Lax-Friedrichs method is conditionally stable for $u_t + cu_x = 0$.

Problem 3

Show that Lax-Friedrichs is conservative by verifying that the numerical flux function

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(f(w_i) + f(w_{i+1})\right) - \frac{\Delta x}{2\Delta t}\left(w_{i+1} - w_i\right)$$

correctly produces the Lax-Friedrichs method for $w_t + f(w)_x = 0$.

Solution: We start with the conservative form of for the given PDE,

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{\Delta x}\left(F_{i-1/2} - F_{i+1/2}\right),$$

introducing the given numerical flux function,

$$w_i^{n+1} = w_i^n + \frac{\Delta t}{\Delta x}\left[\frac{1}{2}\left(f(w_{i-1}) + f(w_i) - f(w_i) - f(w_{i+1})\right) - \frac{\Delta x}{2\Delta t}\left(w_i - w_{i-1} - w_{i+1} + w_i\right)\right].$$

Manipulating this expression we obtain the Lax-Friedrichs method,

$$\begin{aligned} w_i^{n+1} &= w_i^n + \frac{\Delta t}{2\Delta x}\left(f(w_{i-1}) - f(w_{i+1})\right) - \frac{1}{2}\left(-w_{i+1} + 2w_i - w_{i-1}\right), \\ &= \frac{\Delta t}{2\Delta x}\left(f(w_{i-1}) - f(w_{i+1})\right) - \frac{1}{2}\left(-w_{i+1} - w_{i-1}\right), \\ &= \frac{1}{2}\left(w_{i+1} + w_{i-1}\right) - \frac{\Delta t}{2\Delta x}\left(f(w_{i+1}) - f(w_{i-1})\right). \end{aligned}$$

Hence, the Lax-Friedrichs method is conservative.

Problem 4

Using von Neumann stability analysis, show downwind is unconditionally unstable for $u_t + cu_x = 0$. *Hint:* Show $|G(k)|^2 = G^*(k)G(k) > 1$ for any value of $r > 0$.

Solution: By substituting the definitions

$$u_j^n = e^{ikx_j}, \qquad u_j^{n+1} = G(k)e^{ikx_j},$$

into the Downwind scheme,

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right),$$

we obtain

$$G(k)e^{ikx_j} = e^{ikx_j} - c\frac{\Delta t}{\Delta x}\left(e^{ik(x_j+\Delta x)} - e^{ikx_j}\right).$$

Dividing by e^{ikx_j} and using the definition of Courant number $r = c\Delta t/\Delta x$, we get the growth factor

$$G(k) = 1 - r\left(e^{ik\Delta x} - 1\right).$$

To prove that the method is unconditionally unstable we study the modulus of the growth factor,

$$\begin{aligned}
|G(k)|^2 &= G^*(k)G(k), \\
&= \left[1 - r\left(e^{-ik\Delta x} - 1\right)\right]\left[1 - r\left(e^{ik\Delta x} - 1\right)\right], \\
&= 1 - r\left(2\cos(k\Delta x) - 2\right) + r^2\left(2 - 2\cos(k\Delta x)\right), \\
&= 1 + r\left(2 - 2\cos(k\Delta x)\right) + r^2\left(2 - 2\cos(k\Delta x)\right), \\
&= 1 + 2r(r+1)\left(1 - \cos(k\Delta x)\right) > 1 \quad \forall r > 0
\end{aligned}$$

Problem 5

Using von Neumann stability analysis, show that Lax-Wendroff is stable for $u_t + cu_x = 0$ as long as the CFL condition $r \leq 1$ is satisfied. *Hint:* Show $|G(k)|^2 = G^*(k)G(k) \leq 1$ iff $r \leq 1$.

Solution: By substituting the definitions

$$u_j^n = e^{ikx_j}, \quad u_j^{n+1} = G(k)e^{ikx_j},$$

into the Lax-Wendroff scheme,

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{1}{2}c^2\frac{\Delta t^2}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right),$$

we obtain

$$G(k)e^{ikx_j} = e^{ikx_j} - c\frac{\Delta t}{2\Delta x}\left(e^{ik(x_j+\Delta x)} - e^{ik(x_j-\Delta x)}\right) + \frac{1}{2}c^2\frac{\Delta t^2}{\Delta x^2}\left(e^{ik(x_j+\Delta x)} - 2e^{ikx_j} + e^{ik(x_j-\Delta x)}\right).$$

Dividing by e^{ikx_j} and using the definition of Courant number $r = c\Delta t/\Delta x$, we get the growth factor

$$\begin{aligned}
G(k) &= 1 - \frac{1}{2}r\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) + \frac{1}{2}r^2\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right), \\
&= 1 - \frac{1}{2}r2i\sin(k\Delta x) + \frac{1}{2}r^2\left(2\cos(k\Delta x) - 2\right), \\
&= 1 + r^2\left(\cos(k\Delta x) - 1\right) - ir\sin(k\Delta x),
\end{aligned}$$

To prove that the method is conditionally stable we study the modulus of the growth factor,

$$\begin{aligned}
|G(k)|^2 &= G^*(k)G(k), \\
&= \left[1 + r^2\left(\cos(k\Delta x) - 1\right) + ir\sin(k\Delta x)\right]\left[1 + r^2\left(\cos(k\Delta x) - 1\right) - ir\sin(k\Delta x)\right], \\
&= \left[1 + r^2\left(\cos(k\Delta x) - 1\right)\right]^2 + r^2\sin^2(k\Delta x), \\
&= 1 - \left[\cos(k\Delta x) - 1\right]^2 r^2 + \left[\cos(k\Delta x) - 1\right]^2 r^4.
\end{aligned}$$

Let $\beta^2 = \left[\cos(k\Delta x) - 1\right]^2$ and note that $\beta^2 \geq 0$. Then,

$$\begin{aligned}
|G(k)|^2 &= 1 - \beta^2 r^2 + \beta^2 r^4, \\
&= 1 - \beta^2 r^2\left[1 - r^2\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
|G(k)|^2 \leq 1 &\iff 1 - \beta^2 r^2\left[1 - r^2\right] \leq 1, \\
&\iff \beta^2 r^2\left[1 - r^2\right] \geq 0, \\
&\iff \left[1 - r^2\right] \geq 0, \\
&\iff r^2 \leq 1, \\
&\iff r \leq 1.
\end{aligned}$$

Hence, the Lax-Wendroff is stable for $u_t + cu_x = 0$ as long as the CFL condition $r \leq 1$ is satisfied.