Computational Methods

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Homework 2

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Problem 1

(a) To prove that

$$\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 = ||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2,$$

we start by writing the left side of the equation as an inner product,

$$\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 =< f-\sum_{j=1}^{N}\lambda_j e_j|f-\sum_{j=1}^{N}\lambda_j e_j > .$$

By appllying linearity and antilinearity of the inner product we get

$$\begin{aligned}\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 &=< f-\sum_{j=1}^{N}\lambda_j e_j|f-\sum_{j=1}^{N}\lambda_j e_j >\\ &=< f|f > -\sum_{j=1}^{N}< \lambda_j e_j|f > -\sum_{j=1}^{N}< f|\lambda_j e_j > +\sum_{j=1}^{N}< \lambda_j e_j|\lambda_j e_j >\\ &= ||f||^2-\sum_{j=1}^{N}\lambda_j < e_j|f > -\sum_{j=1}^{N}\lambda_j^* < f|e_j > +\sum_{j=1}^{N}< \lambda_j e_j|\lambda_j e_j >\\ &= ||f||^2-\sum_{j=1}^{N}\lambda_j c_j^*-\sum_{j=1}^{N}\lambda_j^* c_j+\sum_{j=1}^{N}||\lambda_j||^2.\end{aligned}$$

where we have used the linearity and antilinearity properties again, and that $c_j =< f|e_j >$. By adding and substracting $\sum_{j=1}^{N} < c_j|c_j >$ and rewriting the previous equation in terms of inner product again we get

$$\begin{aligned}\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 = ||f||^2-\sum_{j=1}^{N}< \lambda_j|c_j > -\sum_{j=1}^{N}< c_j|\lambda_j > +\sum_{j=1}^{N}< \lambda_j|\lambda_j >\\ +\sum_{j=1}^{N}< c_j|c_j > -\sum_{j=1}^{N}< c_j|c_j >,\end{aligned}$$

which, again by linearity and antilinearity, we can group together as

$$\begin{aligned}\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 &= ||f||^2+\sum_{j=1}^{N}<\lambda_j-c_j|\lambda_j>-\sum_{j=1}^{N}<\lambda_j-c_j|c_j>-\sum_{j=1}^{N}<c_j|c_j>\\ &= ||f||^2+\sum_{j=1}^{N}<\lambda_j-c_j|\lambda_j-c_j>-\sum_{j=1}^{N}<c_j|c_j>\\ &= ||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2.\end{aligned}$$

Thus, we have proven that

$$\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2=||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2.$$

(b) To prove that

$$||f-f_N||\leq||f-g||,$$

we will show that their squares

$$||f-f_N||^2\leq||f-g||^2.$$

Taking into account that $f_N=\sum_{j=1}^{N}<f|e_j>e_j$ and $g=\sum_{j=1}^{N}\lambda_j e_j$ and using the previous proof we have that

$$\begin{aligned}||f-f_N||^2 &= \left|\left|f-\sum_{j=1}^{N}<f|e_j>e_j\right|\right|^2\\ &= ||f||^2+\sum_{j=1}^{N}|<f|e_j>-c_j|^2-\sum_{j=1}^{N}|c_j|^2\\ &= ||f||^2+\sum_{j=1}^{N}|<f|e_j>-<f|e_j>|^2-\sum_{j=1}^{N}|c_j|^2\\ &= ||f||^2-\sum_{j=1}^{N}|<f|e_j>|^2,\end{aligned}$$

and

$$\begin{aligned}
||f-g||^2 &= \left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 \\
&= ||f||^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2 \\
&= ||f||^2 + \sum_{j=1}^{N} |\lambda_j - < f|e_j > |^2 - \sum_{j=1}^{N} |c_j|^2 \\
&\geq ||f||^2 - \sum_{j=1}^{N} | < f|e_j > |^2 \\
&= ||f - f_N||^2.
\end{aligned}$$

Thus, we have proved that

$$||f - f_N|| \leq ||f - g||$$

Problem 2

In this problem we are going t approximate the function

$$f(x) = e^{\sin(5x)}$$

using Chebyshev's interpolation of degree ten (using eleven points) and using the orthonormal basis provided by the *Legendre polynomials*. It will be shown that the Legendre polynomials, which we show in the figure 1, give the best approximation of that degree as we proved in Problem 1.

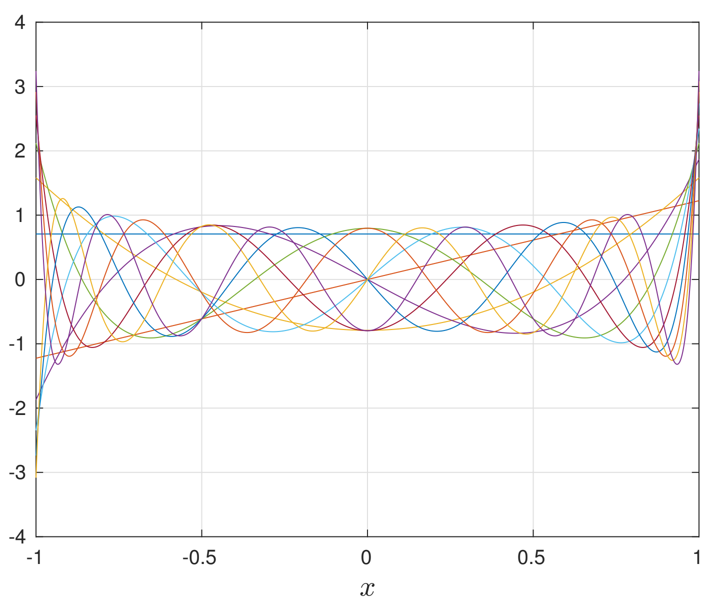

Figure 1: Legendre Polynomials.

In the following figure we show the function f and the two approximations made. To the naked eye both of them seem very accurate, however we can see how the Legendre polynomials give us a better approximation by looking at the error in figure 3.

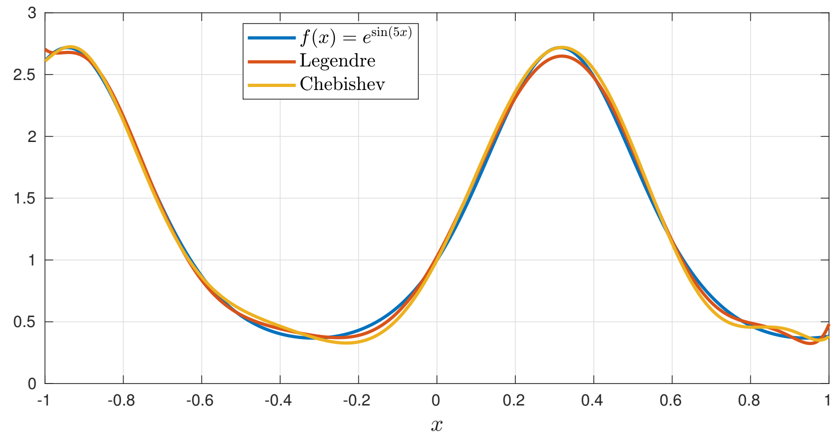

Figure 2: Approximations using Chebyshev and Legendre polynomials.

The sudden drops in the error are in fact the points used to do the approximation, so the error is indeed zero. Since we are using logarithmic scale we see thos drops. We can see how the error of Chebyshev's method is higher. We can compute the L_2 error of the two methods,

$$e_{Chebyshev} = 0.07800109011, \qquad e_{Legendre} = 0.05183155462,$$

and see how the Legendre polynomials, as proved in problem 1, give us the best approximation.

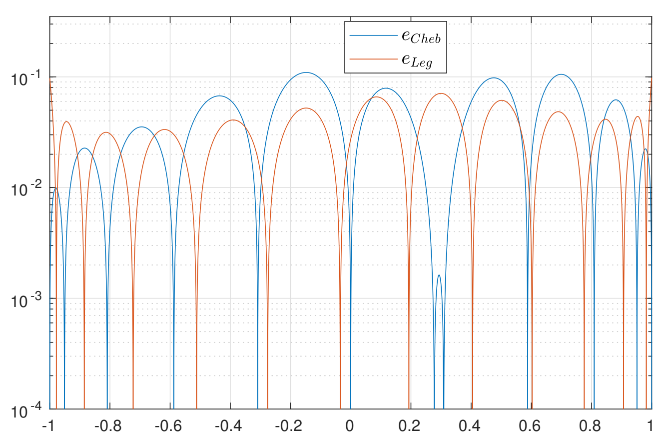

Figure 3: Error of the approximations.

Matlab code for this problem

```
%% Problem 2
clear all
close all
clc
format long
legendfontsize=12;
labelfontsize=14;
f = @(x) exp(sin(5*x));
f= chebfun(f);
f_cheb = chebfun(f,11);
xx=linspace(-1,1,1000);
P=legpoly(0:10,'norm');

figure
plot(P)
grid on
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/LegendrePols','epsc')
saveas(gcf,'Latex/FIGURES/LegendrePols','fig')

f_N=0;
for k=1:11
    f_N=f_N+(f'*P(:,k))*P(:,k);
end
%%
figure
plot(f,'linewidth',2)
hold on
plot(f_N,'linewidth',2)
plot(f_cheb,'linewidth',2)
grid on
legend({'$f(x)=e^{\sin(5x)}$','Legendre','Chebishev'}...
    ,'fontsize',legendfontsize,'interpreter','latex','location','north')
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Approximations_p2','epsc')
saveas(gcf,'Latex/FIGURES/Approximations_p2','fig')

Error_cheb=norm(f-f_cheb,2)
Error_app=norm(f-f_N,2)
%%
figure
semilogy(abs(f-f_cheb))
hold on
```

```
semilogy(abs(f-f_N))
grid on
axis([-1 1 1e-4 0.35])
legend({'$e_{Cheb}$','$e_{Leg}$'},'fontsize',legendfontsize,'interpreter','latex')
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Error_p2','epsc')
saveas(gcf,'Latex/FIGURES/Error_p2','fig')
```

Problem 3

(a) Prove that $x_1 - x_0 = \mathcal{O}(1/N^2)$.

$$\begin{aligned}
x_1 - x_0 &= -\cos\left(\frac{\pi}{N}\right) + \cos(0) \\
&= 1 - \cos\left(\frac{\pi}{N}\right) \\
&= 1 - \left(1 + \frac{1}{2}\left(\frac{\pi}{N}\right)^2 + ...\right) \\
&= -\frac{1}{2}\left(\frac{\pi}{N}\right)^2 + ... \\
&= \mathcal{O}(1/N^2),
\end{aligned}$$

where we have expanded the cosine in Taylor series around zero since $\frac{\pi}{N} \to 0$ as $N \to \infty$.

(b) Prove that $x_{N/2} - x_{N/2-1} = \mathcal{O}(1/N)$.

$$\begin{aligned}
x_{N/2} - x_{N/2-1} &= -\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\left(\frac{N}{2}-1\right)\pi}{N}\right) \\
&= \cos\left(\frac{\pi}{2} - \frac{\pi}{N}\right) \\
&= \cos\left(\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{N}\right) - \sin\left(\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{N}\right) \\
&= \sin\left(\frac{\pi}{N}\right) \\
&= \left(\frac{\pi}{N} - \frac{1}{6}\left(\frac{\pi}{N}\right)^3 + ...\right) \\
&= \mathcal{O}(1/N),
\end{aligned}$$

where we have expanded the sine in Taylor series around zero since $\frac{\pi}{N} \to 0$ as $N \to \infty$.

Problem 4

In this problem we start by plotting the lagrange polynomials

$$l(x) = \prod_{j=0}^{N}(x - x_j),$$

for equispaced nodes (figure 1), for Chebishev's nodes of the first kind (figure 2) and for Chebishev's nodes of the second kind (figure 3). The nodes are placed on the zeroes of the functions and represented by the red asterisk. We see how the Chebishev's nodes are more concentrated towards the boundaries, being very useful for boundary conditions.

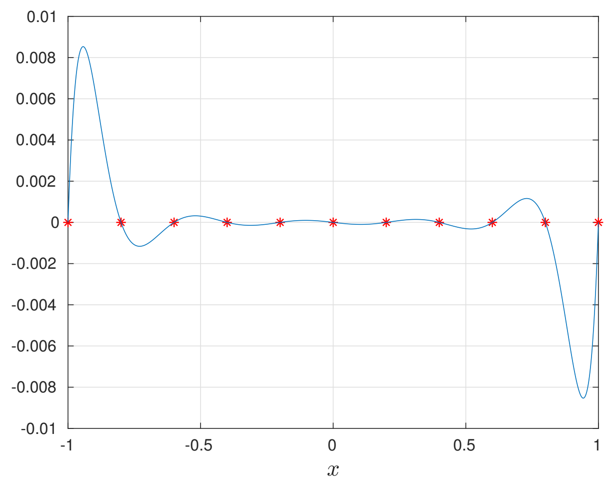

Figure 4: Equispaced nodes.

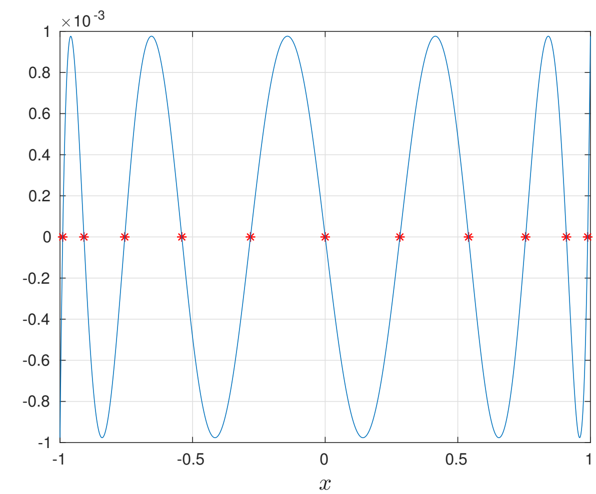

Figure 5: First kind Chebishev's nodes.

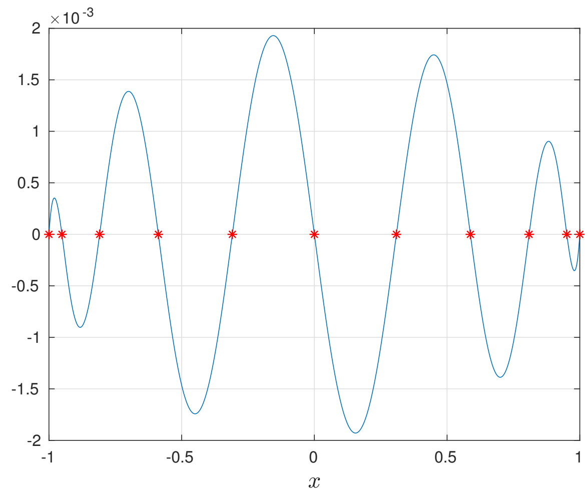

Figure 6: Second kind Chebishev's nodes.

Using those nodes we compute approximations to the function $f(x) = \cos(3x)$. In the following plots we show the error and the upper bound of the error given by the Cauchy interpolation error formula

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} l(x),$$

where $p_n(x)$ represents the approximation. Given the function to approximate, we can then find the upper bound

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} l(x) \leq \frac{3^{n+1}}{(n+1)!} l(x) := e_{Cauchy}.$$

In the following figures we can see how Cauchy error formula gives us the upper bound for equispaced nodes (figure 7), Chebishev nodes of the first kind (figure 8), Chebishev nodes of the second kind (figure 9). As in problem 2, the drops in the error are due to the fact that the error must be zero at those points used to run the interpolation and, since we are using logarithmic scale, are seen as these sudden drops in the error plot.

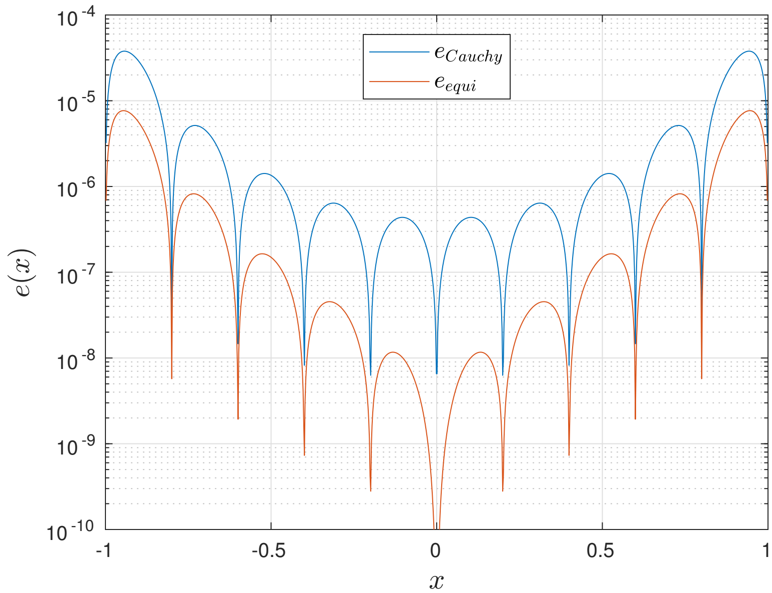

Figure 7: Error of equispaced nodes.

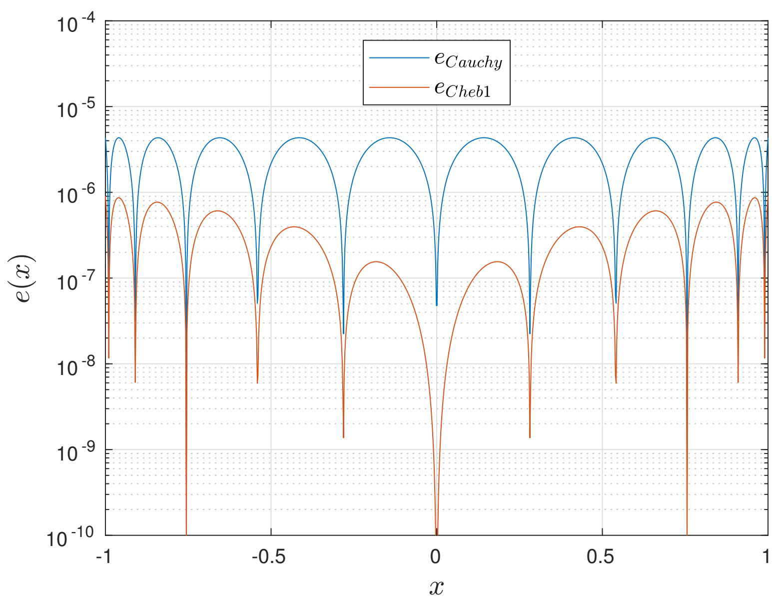

Figure 8: Error of Chebishev's nodes of first kind.

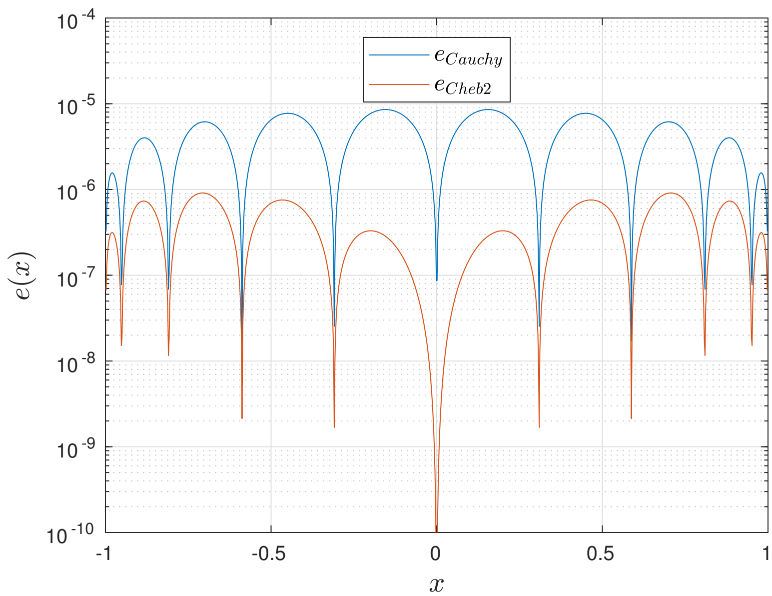

Figure 9: Error of Chebishev's nodes of second kind.

To finish, we also compute the L_2 norm of the error and check that in fact we obtain an over all upper bound for the different methdos (see table 1). As we can check, Cauchy's interpolation formula gives us a good upper bound of the error for each distribution of the nodes. Note that the error of the Chebishev's methods is smaller. As we saw in class, equispaced nodes have high errors at the boundaries.

	Equispaced	First kind Chebishev	Second kind Chebishev
e_{Cauchy}	$3.54887848499 \cdot 10^{-4}$	$9.68648228641 \cdot 10^{-5}$	$1.58167667439 \cdot 10^{-4}$
$e_{Interpolation}$	$7.05156144371 \cdot 10^{-5}$	$1.18371532261 \cdot 10^{-5}$	$1.49100947533 \cdot 10^{-5}$

Table 1: L_2 error norm of different methods.

Matlab code for this problem

```
%% Problem 4
N=10;
f = @(x) cos(3*x);
xx=linspace(-1,1,1000);
xk0=linspace(-1,1,N+1)';
xk1=chebpts(N+1,1);
xk2=chebpts(N+1,2);
l0 = @(x) prod(x-xk0);
l1 = @(x) prod(x-xk1);
l2 = @(x) prod(x-xk2);
%
figure
plot(xk0,l0(xk0),'r*')
```

```
hold on
plot(xx,l0(xx))
grid on
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Lagrange','epsc')
saveas(gcf,'Latex/FIGURES/Lagrange','fig')

%
figure
plot(xk1,l1(xk1),'r*')
hold on
plot(xx,l1(xx))
grid on
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Chebishev1','epsc')
saveas(gcf,'Latex/FIGURES/Chebishev1','fig')

%
figure
plot(xx,l2(xx))
hold on
plot(xk2,l2(xk2),'r*')
grid on
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Chebishev2','epsc')
saveas(gcf,'Latex/FIGURES/Chebishev2','fig')

w0 = baryWeights(xk0);
p0 = @(x) bary(x,f(xk0),xk0,w0);
err_bound0 = 3^(N+1)*l0(xx)/factorial(N+1);
figure
semilogy(xx,abs(err_bound0))
hold on
semilogy(xx,abs(f(xx)-p0(xx)))
grid on
axis([-1 1 1e-10 1e-4])
legend({'$e_{Cauchy}$','$e_{equi}$'},...
'fontsize',legendfontsize,'interpreter','latex','Location','north')
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
ylabel('$e(x)$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Cauchy_equi','epsc')
saveas(gcf,'Latex/FIGURES/Cauchy_equi','fig')

w1 = baryWeights(xk1);
p1 = @(x) bary(x,f(xk1),xk1,w1);
```

```
err_bound1 = 3^(N+1)*l1(xx)/factorial(N+1);
figure
semilogy(xx,abs(err_bound1))
hold on
semilogy(xx,abs(f(xx)-p1(xx)))
grid on
axis([-1 1 1e-10 1e-4])
legend({'$e_{Cauchy}$','$e_{Cheb1}$'},...
'fontsize',legendfontsize,'interpreter','latex','location','north')
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
ylabel('$e(x)$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Cauchy_cheb1','epsc')
saveas(gcf,'Latex/FIGURES/Cauchy_cheb1','fig')

w2 = baryWeights(xk2);
p2 = @(x) bary(x,f(xk2),xk2,w2);
err_bound2 = 3^(N+1)*l2(xx)/factorial(N+1);
figure
semilogy(xx,abs(err_bound2))
hold on
semilogy(xx,abs(f(xx)-p2(xx)))
grid on
axis([-1 1 1e-10 1e-4])
legend({'$e_{Cauchy}$','$e_{Cheb2}$'},...
'fontsize',legendfontsize,'interpreter','latex','location','north')
xlabel('$x$','fontsize',labelfontsize,'interpreter','latex')
ylabel('$e(x)$','fontsize',labelfontsize,'interpreter','latex')
saveas(gcf,'Latex/FIGURES/Cauchy_cheb2','epsc')
saveas(gcf,'Latex/FIGURES/Cauchy_cheb2','fig')

E_equi=norm(f(xx)-p0(xx),2)
E_Cauchy_equi=norm(err_bound0,2)
E_Chebishev1=norm(f(xx)-p1(xx),2)
E_Cauchy_cheb1=norm(err_bound1,2)
E_Chebishev2=norm(f(xx)-p2(xx),2)
E_Cauchy_cheb2=norm(err_bound2,2)
```