Real Analysis Homework 12

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1 Problem 5.2.4.

1. Let X be a normed vector space and u, $x \in X$.

   Show: If $x^*x = x^*u$ for all $x^* \in X^*$, then $x = u$.

Solution:

Proof. Let us assume $x^*x = x^*u$ for all $x^* \in X^*$. Then, by Lemma 5.9, since $x - u \in X$, there exists an $x^* \in X^*$ such that

$$\|x - u\| = x^*(x - u) = x^*x - x^*u = 0 \ .$$

Thus, by the properties of norms, $\|x - u\| = 0$ implies $x = u$.

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2 Problem 6.1.2.

1. Let X and Y be metric spaces and let d denote the metric on both. Let $S \subseteq X$ and $f : S \to Y$. Let $x \in X$ be an accumulation point of S.

   Then L is the limit of f at x if and only if for every sequence (s_n) in $S\backslash\{x\}$ with $s_n \to x$ also $f(s_n) \to L$ as $n \to \infty$.

Solution:

Proof. $(\Rightarrow)$

Let $\varepsilon > 0$. Since L is a limit point of f at x, there exists $\delta > 0$ such that for $s \in S\backslash\{x\}$,

$$0 < d(s, x) < \delta \Rightarrow d(f(s), L) < \varepsilon \ .$$

Let (s_n) be a sequence in $S\backslash\{x\}$ such that $s_n \to x$ as $n \to \infty$. Then there exists an $N \in \mathbb{N}$ with $N > 0$ such that

$$d(s_n, x) < \delta \ \text{ when } n > N \ ,$$

which implies

$$d(f(s_n), L) < \varepsilon \ \text{ for } n > N \ .$$

($\Leftarrow$) By contrapositive.

Let L not be a limit of F at X. Then there exists a $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $s \in S\backslash\{x\}$ such that

$$d(s,x) < \delta \quad \text{but} \quad d(f(s),L) \geq \varepsilon \ .$$

In particular, for every $n \in \mathbb{N}$ with $\delta = \frac{1}{n}$, there exists some $s_n \in S\backslash\{x\}$ such that

$$d(s_n,x) < \frac{1}{n} \quad \text{but} \quad d(f(s_n),L) \geq \varepsilon \ .$$

Thus we have a sequence $(s_n) \in S\backslash\{x\}$ with $s_n \to x$ but $f(s_n)$ does not converge to L as $n \to \infty$.

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3 Problem 6.2.2.

1. (Product rule of differentiation). Let X be a normed vector space over $\mathbb{K}$ and $f : [a,b] \to X$, $\phi : [a,b] \to \mathbb{K}$. If $t \in [a,b]$ and f and ϕ are differentiable at t, then ϕf, defined by $(\phi f)(t) = \phi(t)f(t)$, is differentiable at t and $(\phi f)'(t) = \phi'(t)f(t) + \phi(t)f'(t)$.

Solution:

Proof. Let $s \in (a,b)$ and $t \in [a,b]$ with f and ϕ differentiable at t. As $s \to t$,

$$\begin{aligned}
&\|(\phi f)'(t) - (\phi'(t)f(t) + \phi(t)f'(t))\| = \|\frac{(\phi f)(s) - (\phi f)(t)}{s-t} - (\phi'(t)f(t) + \phi(t)f'(t))\| \\
&= \|\frac{\phi(s)f(s) - \phi(t)f(t) + \phi(s)f(t) - \phi(s)f(t)}{s-t} - (\phi'(t)f(t) + \phi(t)f'(t))\| \\
&= \|\phi(s)\frac{f(s)-f(t)}{s-t} + f(t)\frac{\phi(s)-\phi(t)}{s-t} - \phi'(t)f(t) - \phi(t)f'(t)\| \\
&\leq \|\phi(s)\frac{f(s)-f(t)}{s-t} - \phi(t)f'(t)\| + \|f(t)\frac{\phi(s)-\phi(t)}{s-t} - f(t)\phi'(t)\| \\
&= \|\phi(s)\frac{f(s)-f(t)}{s-t} - \phi(t)f'(t) + \phi(s)f'(t) - \phi(s)f'(t)\| + \|f(t)\frac{\phi(s)-\phi(t)}{s-t} - f(t)\phi'(t)\| \\
&\leq \|\phi(s)\frac{f(s)-f(t)}{s-t} - \phi(s)f'(t)\| + \|\phi(s)f'(t) - \phi(t)f'(t)\| + |f(t)\frac{\phi(s)-\phi(t)}{s-t} - f(t)\phi'(t)\| \\
&= |\phi(s)|\|\frac{f(s)-f(t)}{s-t} - f'(t)\| + |f'(t)|\|\phi(s)-\phi(t)\| + |f(t)|\|\frac{\phi(s)-\phi(t)}{s-t} - \phi'(t)\| \to 0 \ ,
\end{aligned}$$

by the differentiability of f and ϕ at t and because $\|\phi(s) - \phi(t)\| \to 0$ as $s \to t$ since ϕ must be continuous in order to be differentiable. Therefore, $(\phi f)'(t) = \phi'(t)f(t) + \phi(t)f'(t)$.

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4 Problem 6.3.4.

1. Let $f : [a, b] \to X$, f continuous on $[a, b]$ and differentiable on (a, b).

   Show: if f' is bounded on (a, b), then f is Lipschitz continuous and $\sup_{t\in(a,b)} \|f'(t)\|$ is a Lipschitz constant for f.

Solution:

Proof. Let f' be bounded on (a, b). Then, there exists $\sup_{t\in(a,b)} f'(t)$ and also $\sup_{t\in(a,b)} \|f'(t)\|$. Since f is a continuous function that is differentiable on (a, b), then by theorem 6.13, there exists some $t \in (a, b)$ such that

$$\|f(b) - f(a)\| \leq \|f'(t)\|(b - a) \ .$$

Then, by definition of a supremum,

$$\|f(b) - f(a)\| \leq \sup_{t\in(a,b)} \|f'(t)\|(b - a) \ .$$

Therefore, f is Lipschitz continuous on $[a, b]$ with Lipschitz constant $\sup_{t\in(a,b)} \|f'(t)\|$.

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