Real Analysis Homework 8

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1 Problem 4.4.11

1. Let T, S be two closed disjoint subsets of a metric space (X, d) ($T \cap S \neq \emptyset$). Show that the function $f : X \to \mathbb{R}_+$,

$$f(x) = \frac{d(x,S)}{d(x,T) + d(x,S)} \ , \qquad x \in X,$$

separates T and S, i.e., f is well-defined and continuous, $f(x) = 1$ for all $x \in T$ and $f(x) = 0$ for all $x \in S$ and $0 < f(x) < 1$ for all $x \in X$ $(S \cup T)$.

Solution:

Proof. Since the intersection is empty, there is no point $x \in X$ that is in both S and T. Since both S and T are closed and by the contrapositive of *Problem 4.1.7* ($x \notin \overline{S} = S$ if and only if $d(x, S) \neq 0$, same for T), there is no point $x \in X$ that makes both $d(x, S)$ and $d(x, T)$ zero. Therefore, the denominator will always be positive, so the function is well defined with its domain being X. The function is also continuous for being the quotient of two real valued continuous functions. Now let $x \in T$ (therefore $x \notin S$), then $d(x, T) = 0$ and $d(x, S) > 0$:

$$f(x) = \frac{d(x,S)}{d(x,T) + d(x,S)} = \frac{d(x,S)}{0 + d(x,S)} = 1 \ , \qquad x \in X.$$

Now let $x \in S$ (therefore $x \notin T$), then $d(x, S) = 0$ and $d(x, T) > 0$:

$$f(x) = \frac{d(x,S)}{d(x,T) + d(x,S)} = 0 \ , \qquad x \in X.$$

Lastly, let $x \in X\backslash(S \cup T) = (X\backslash S) \cap (X\backslash T)$, where the equality is comes from de Morgan's laws. Then x is neither in S or in T. Therefore $d(x, T) > 0$ and $d(x, S) > 0$ and thus, $0 < f(x) < 1$. ∎

2 Problem 4.5.2

1. Let X be a complete metric space, $x \in X$, $r > 0$ and $f : \overline{U}_r(x) \to X$. Assume that there is some $k \in (0, 1)$ such that:

   i) $d(f(y), f(z)) \leq kd(y, z)$ for all $y, z \in \overline{U}_r(x)$.

   ii) $d(x, f(x)) \leq r(1 - k)$.

   Show: f has a fixed point in $\overline{U}_r(x)$.

Solution:

Proof. By *i)* it is inmediate that f is a contraction, which implies that f is a generalized contraction. On the other hand, since $\overline{U}_r(x)$ is a closed subset of the complete set X, it is also complete. Lastly, in order to use *Theorem 4.48* we need to prove that the function f maps the closed ball into itself. Let $y \in f(\overline{U}_r(x))$, then there exists a $z \in \overline{U}_r(x)$ such that $f(z) = y$. Since $z \in \overline{U}_r(x)$, $d(x, z) \leq r$. Now analyze $d(x, y)$:

$$\begin{aligned}
d(x, y) &\leq d(x, f(x)) + d(f(x), y) \\
&= d(x, f(x)) + d(f(x), f(z)) \\
&\leq r(1 - k) + kd(x, z) \\
&\leq r(1 - k) + kr \\
&= r \; .
\end{aligned}$$

Therefore, $y \in \overline{U}_r(x)$ and $f(\overline{U}_r(x)) \subseteq \overline{U}_r(x)$. So the function maps the closed ball into itself. By *Theorem 4.48*, f has an unique fixed point in $\overline{U}_r(x)$. ■

3 Problem 4.6.1

1. Let X and Y be metric spaces and $f : X \to Y$ be continuous. Let $S \subseteq X$. Show:

   a) $f(\overline{S}) \subseteq \overline{f(S)}$.

   b) If $\overline{S}$ is compact, $f(\overline{S}) = \overline{f(S)}$.

Solution:

Proof. a) Let $y \in f(\overline{S})$, then there exists an x in $\overline{S}$ such that $y = f(x)$. Since x is a limit point of S there exists a sequence (x_n) in S such that $x_n \to x$ as $n \to \infty$. Let (y_n) be the sequence in $f(S)$ such that $y_n = f(x_n)$. Since f is continuous and $x_n \to x$ as $n \to \infty$, $y_n \to y$ too as $n \to \infty$. Therefore y is a limit point of $f(S)$, $y \in \overline{f(S)}$. Thus, $f(\overline{S}) \subseteq \overline{f(S)}$. ■

Solution:

Proof. b) Since $\bar{S}$ is a compact subset of X and f is continuous, $f(\bar{S})$ is compact by Theorem 4.50. Since it is compact, it is complete by theorem 4.39. This implies $f(\bar{S})$ is closed by theorem 4.6. Then since $f(S) \subseteq f(\bar{S})$, by proposition 4.3, $\overline{f(S)} \subseteq f(\bar{S})$. Finally, since $f(\bar{S}) \subseteq \overline{f(S)}$ from the proof in part (a), $f(\bar{S}) = \overline{f(S)}$. ■

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