Fourier Analysis and Wavelets
Homework 6

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Problem 4

Use Parseval's equation

$$\langle f, g \rangle = \sum_{k=1}^{\infty} a_k \bar{b}_k,$$

to the show that

$$\langle \psi_{0m}, \psi_{0l} \rangle = \frac{1}{2} \sum_{k \in \mathbb{Z}} \overline{p_{1-k+2m}} p_{1-k+2l},$$

where ψ is defined as

$$\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{p_{1-k}} \phi(2x - k).$$

Solution: To start, let $\psi_{jk} = 2^{j/2} \psi(2^j x - k)$, $k \in \mathbb{Z}$. Then,

$$\begin{aligned} \psi_{0m} = \psi(x - m) &= \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} \overline{p_{1-k_1}} \phi(2x - (k_1 + 2m)), \\ \psi_{0l} = \psi(x - l) &= \sum_{k_2 \in \mathbb{Z}} (-1)^{k_2} \overline{p_{1-k_2}} \phi(2x - (k_2 + 2l)). \end{aligned}$$

Next,

$$\begin{aligned} \langle \psi_{0m}, \psi_{0l} \rangle &= \left\langle \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1} \overline{p_{1-k_1}} \phi(2x - (k_1 + 2m)), \sum_{k_2 \in \mathbb{Z}} (-1)^{k_2} \overline{p_{1-k_2}} \phi(2x - (k_2 + 2l)) \right\rangle \\ &= \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} (-1)^{k_1 + k_2} \overline{p_{1-k_1}} p_{1-k_2} \left\langle \phi(2x - (k_1 + 2m)), \phi(2x - (k_2 + 2l)) \right\rangle. \end{aligned}$$

Since the set $\left\{ \phi_{jk} = 2^{j/2} \phi(2^j x - k); k \in \mathbb{Z} \right\}$ is an orthonormal base,

$$\langle \phi(2x - (k_1 + 2m)), \phi(2x - (k_2 + 2l)) \rangle = \delta_{k_1 + 2m, k_2 + 2l}.$$

Further, making $k_2 = k_1 + 2m - 2l$,

$$\begin{aligned}\langle \psi_{0m}, \psi_{0l} \rangle &= \sum_{k_1 \in \mathbb{Z}} (-1)^{k_1+k_1+2m-2l} \overline{p_{1-k_1}} p_{1-k_1-2m+2l} \frac{1}{2} \\ &= \frac{1}{2} \sum_{k_1 \in \mathbb{Z}} (-1)^{2(k_1+m-l)} \overline{p_{1-k_1}} p_{1-k_1-2m+2l} \\ &= \frac{1}{2} \sum_{k_1 \in \mathbb{Z}} \overline{p_{1-k_1}} p_{1-k_1-2m+2l}.\end{aligned}$$

Finally, we make $k = k_1 + 2m$ and obtain the desired result,

$$\langle \psi_{0m}, \psi_{0l} \rangle = \frac{1}{2} \sum_{k \in \mathbb{Z}} \overline{p_{1-k+2m}} p_{1-k+2l}.$$

Problem 9

For $j \in \mathbb{Z}$, let V_j be the space of all finite energy signals f that are continuous and piecewise linear, with possible corners occurring only at the dyadic points $k/2^j$, $k \in \mathbb{Z}$.

(a) Show that $\{V_j\}_{j \in \mathbb{Z}}$ satisfies properties 1, 3, and 4 in the definition of a multiresolution analysis.

Solution:

- (Nested) $V_j \subset V_{j+1}$.
  As it happens with Haar, the nested property holds since the set of multiples of 2^{-j} consists in every other element of the set of multiples of $2^{-(j+1)}$. Therefore, the set of multiples of 2^{-j} is contained in the set of multiples of $2^{-(j+1)}$. This translates in that the functions in V_{j+1} can always represent the functions in V_j but the inverse is not always true.
- (Separation) $\bigcap V_j = \{0\}$.
  Let $\varphi(2^j x - k)$ represent the "tent function" of spread 2^j and its translates. Let $f \in V_{-j}$ with $j > 0$. Then f must be a linear combination of $\varphi(x/2^j - k)$ whose elements are positive on the base of the tent of length 2^j. As j gets larger, this base gets larger as well. Since the support of f must remain finite, if it belongs to all the V_{-j} as $j \to \infty$, then the positive values of f must be zero.

(b) Let $\varphi(x)$ be the "tent function",

$$\varphi(x) = \begin{cases} x+1, & -1 \le x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & |x| > 1. \end{cases}$$

Show that $\{\varphi(x-k)\}_{k \in \mathbb{Z}}$ is a (nonorthonormal) basis for V_0. Find the scaling relation for φ.

Solution: We will show that $\varphi(x)$ is a linear combination of $\varphi(2x+1)$, $\varphi(2x)$ and $\varphi(2x-1)$:

$$\varphi(x) = \frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1).$$

Given the scaling by the factor of two and the translation of the function φ, we can encounter the following cases:

- Suppose $x < -1$, then $2x+1 < -1$, $2x < -2$ and $2x-1 < -3$. Therefore,

$$\frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1) = \frac{1}{2}0 + 0 + \frac{1}{2}0 = 0 = \varphi(x)$$

- Suppose $x \in [-1, -\frac{1}{2}]$, then $2x+1 \in [-1, 0]$, $2x \in [-2, -1]$ and $2x-1 \in [-3, -2]$. Therefore,

$$\frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1) = \frac{1}{2}(2x+2) + 0 + \frac{1}{2}0 = x + 1 = \varphi(x)$$

- Suppose $x \in [-\frac{1}{2}, 0]$, then $2x+1 \in [0, 1]$, $2x \in [-1, 0]$ and $2x-1 \in [-2, -1]$. Therefore,

$$\frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1) = \frac{1}{2}(1 - 2x - 1) + (2x+1) + \frac{1}{2}0 = x + 1 = \varphi(x)$$

- Suppose $x \in [0, \frac{1}{2}]$, then $2x+1 \in [1, 2]$, $2x \in [0, 1]$ and $2x-1 \in [-1, 0]$. Therefore,

$$\frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1) = \frac{1}{2}0 + (1 - 2x) + \frac{1}{2}(2x - 1 + 1) = 1 - x = \varphi(x)$$

- Suppose $x \in [\frac{1}{2}, 1]$, then $2x+1 \in [2, 3]$, $2x \in [1, 2]$ and $2x-1 \in [0, 1]$. Therefore,

$$\frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1) = \frac{1}{2}0 + 0 + \frac{1}{2}(1 - 2x + 1) = 1 - x = \varphi(x)$$

- Suppose $x > 1$, then $2x+1 > 3$, $2x > 2$ and $2x-1 > 1$. Therefore,

$$\frac{1}{2}\varphi(2x+1) + \varphi(2x) + \frac{1}{2}\varphi(2x-1) = \frac{1}{2}0 + 0 + \frac{1}{2}0 = 0 = \varphi(x)$$

Thus, the relation has been proven.

Problem 16

Prove the second part of *Theorem 5.18* using the proof of the first part as a guide.

Solution: We start the proof with the orthonormality condition stated as

$$\int \psi(x)\overline{\phi(x-l)}dx = 0,$$

where δ is the Kronecker-delta and $l \in \mathbb{Z}$. By the Plancherel's identity for the Fourier Transform,

$$\begin{aligned}\int_{-\infty}^{\infty} \psi(x)\overline{\phi(x-l)}dx &= \int_{-\infty}^{\infty} \hat{\psi}(\eta)\overline{\hat{\phi}(\eta)e^{-il\eta}}d\eta, \\ &= \int_{-\infty}^{\infty} \hat{\psi}(\eta)\overline{\hat{\phi}(\eta)}e^{il\eta}d\eta.\end{aligned}$$

Like in the first part of the proof, we divide the real line into the intervals $I_j = [2\pi j, 2\pi(j+1)]$ for $j \in \mathbb{Z}$ and the equation can be written as

$$\sum_{j\in\mathbb{Z}} \int_{2\pi j}^{2\pi(j+1)} \hat{\psi}(\eta)\overline{\hat{\phi}(\eta)}e^{il\eta}d\eta = 0.$$

Now replace η by $\eta + 2\pi j$,

$$\int_0^{2\pi} \sum_{j\in\mathbb{Z}} \hat{\psi}(\eta+2\pi j)\overline{\hat{\phi}(\eta+2\pi j)}e^{il\eta}d\eta = 0,$$

where we have used that $e^{2\pi ilj} = 1$ for $j, l \in \mathbb{Z}$. Let

$$G(\eta) = 2\pi \sum_{j\in\mathbb{Z}} \hat{\psi}(\eta+2\pi j)\overline{\hat{\phi}(\eta+2\pi j)}.$$

The orthonormality condition can be then expressed as

$$\frac{1}{2\pi}\int_0^{2\pi} G(\eta)e^{il\eta}d\eta = 0.$$

Note that $G(\eta)$ is periodic,

$$\begin{aligned}G(\eta+2\pi) &= 2\pi \sum_{j\in\mathbb{Z}} \hat{\psi}(\eta+2\pi+2\pi j)\overline{\hat{\phi}(\eta+2\pi+2\pi j)}, \\ &= 2\pi \sum_{j\in\mathbb{Z}} \hat{\psi}(\eta+2\pi(j+1))\overline{\hat{\phi}(\eta+2\pi(j+1))}, \\ &= 2\pi \sum_{j'\in\mathbb{Z}} \hat{\psi}(\eta+2\pi j'))\overline{\hat{\phi}(\eta+2\pi j'))}, \\ &= G(\eta).\end{aligned}$$

Hence, $G(\eta)$ accepts a Fourier representation,

$$G(\eta) = \sum_{l\in\mathbb{Z}} \alpha_l e^{ilx},$$

where the coefficients are given by

$$\alpha_l = \frac{1}{2\pi}\int_0^{2\pi} G(\eta)e^{-il\eta}d\eta.$$

This implies that the orthogonality condition can be written as $\alpha_{-l} = 0$ and this implies that $G(\eta) = 0$. Therefore,

$$2\pi \sum_{j\in\mathbb{Z}} \hat{\psi}(\eta+2\pi j)\overline{\hat{\phi}(\eta+2\pi j)} = 0.$$

The proof is finished by simply dividing by 2π and taking the conjugate of the whole equation,

$$\sum_{j\in\mathbb{Z}} \hat{\phi}(\eta+2\pi j)\overline{\hat{\psi}(\eta+2\pi j)} = 0.$$