Real Analysis Homework 8

Francisco Jose Castillo Carrasco

October 20, 2017

1 Problem 4.3.4

1. Consider the sequence space l^∞ with supremum norm and the subset $D = \{x = (x_n); x_n \to 0, n \to \infty\}$. Show: D is not totally bounded.

Solution:

Proof. Let $n, j, k \in \mathbb{N}$. Consider the sequence $(e^n) \in l^\infty$ where e^n is the sequence where all terms are 0 except the $n-th$ term which is 1. It is inmediate that $\lim_{n\to\infty} e^n = 0$, meaning that $(e^n) \in D$. Consider the subsequence (e^{n_j}) of (e^n), $(e^{n_j}) \in D$. Let $\varepsilon = \frac{1}{2}$. Without loss of generality let $n_j > n_k$. Then:

$$||e^{n_j} - e^{n_k}||_\infty = 1 > \varepsilon \;\; \forall n_j, n_k \;,$$

where $||\cdot||_\infty$ represents the supremum norm. Therefore $\nexists N \in \mathbb{N}$ such that, $\forall \varepsilon > 0$

$$||e^{n_j} - e^{n_k}||_\infty < \varepsilon \;\; \forall n_j, n_k > N \;.$$

Thus, the subsequence (e^{n_j}) is not Cauchy. Since e^{n_j} was chose arbitrarily, D is not totally bounded according to *Theorem 4.24*. ■

2 Problem 4.3.9

1. Show: For each $n \in \mathbb{N}$, $\mathbb{R}^n$ is separable. Hint: Show $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Solution:

Proof. Let $x = (x^1, ..., x^n) \in \mathbb{R}^n$ where the upper indices are not powers. Since $\mathbb{R} \subseteq \overline{\mathbb{Q}}$ (by *example 4.9*), we can define $(x_k) \in \mathbb{Q}^n$ such that each $x_k = (x_k^1, ..., x_k^n) \in \mathbb{Q}^n$ and $\lim x_k = x$. Thus, (x_k) is a sequence of rational vectors converging to the vector x which is in $\mathbb{R}^n$. In addition, each $x \in \mathbb{R}^n$ is a limit point of $\mathbb{Q}^n$ and therefore $x \in \overline{Q^n}$. Thus, $\mathbb{R}^n \subseteq \overline{Q^n}$ and $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$. By *definition 4.31*, $\mathbb{R}^n$ is separable. ■

3 Problem 4.3.11

1. Let X be a metric space. Show that the countable union of separable sets is separable: If $\{S_n; n \in \mathbb{N}\}$ is a countable family of separable set S_n in X, then $\cup_{n\in\mathbb{N}} S_n$ is separable.

Solution:

Proof. Let $x \in S_n$. Therefore x is also in $\cup_{n\in\mathbb{N}} S_n$. Since S_n is separable, $\exists M_n$ such that

$$M_n \subseteq S_n \subseteq \overline{M_n} \ ,$$

and, since M_n is dense in S_n, $\exists (x_n) \in M_n$ such that $x_n \to x$. Therefore, $x \in S_n$ is a limit point of M_n and $x \in \overline{M_n}$. Similarly, $\forall x \in \cup_{n\in\mathbb{N}} S_n$, $\exists (x_n) \in \cup_{n\in\mathbb{N}} M_n$ such that $x_n \to x$, so x is a limit point of $\cup_{n\in\mathbb{N}} M_n$, then $x \in \cup_{n\in\mathbb{N}} \overline{M_n}$. Thus, $\cup_{n\in\mathbb{N}} S_n \subseteq \cup_{n\in\mathbb{N}} \overline{M_n}$ meaning that $\cup_{n\in\mathbb{N}} S_n$ is separable. ■

4 Problem 4.4.5

1. Let (X, d) be a metric space and A and B be subsets of X. Show: If A and B are compact sets, so is $A \cup B$.

Solution:

Proof. Let $n, j, k \in \mathbb{N}$. Assume A and B are compact. Let (x_n) be a sequence in $A \cup B$. Let (x_{n_j}) be a subsequence of (x_n) that is entirely in either A or in B. Without loss of generality assume that (x_{n_j}) is entirely in A. Since A is compact, the subsequence has itself a subsequence $(x_{n_{j_k}})$ which has a limit in x in A, and therefore $x \in A \cup B$. Similarly in the case of the subsequence being entirley in B. Therefore, it has been proven that the sequence $(x_n) \in A \cup B$ has a subsequence $(x_{n_{j_k}})$ which has a limit in $A \cup B$. Thus, $A \cup B$ is compact. ■

2. If $A \subseteq B$ and A is closed and B is compact, then A is compact.

Solution:

Proof. Let (x_n) be a sequence in A. Since $A \subseteq B$, (x_n) is also in B. Then, since B is compact, there exists a convergent subsequence (x_{n_j}) of (x_n) and its limit x is in B. Since A is closed, the limit $x \in A$. Therefore A is compact. ■

3. If A is closed and B is compact, then $A \cap B$a is compact.

Solution:

Proof. Let (x_n) be an arbitrary sequence in $A \cap B$, then $(x_n) \in B$. Since B is compact, there exists a subsequence $(x_{n_j}) \in B$ which has a limit x in B. Since (x_n) is in the $A \cap B$, it is also in A, and so is the subsequence (x_{n_j}). Then, since A is closed, the limit x of (x_{n_j}) is also in A. Thus, since (x_n) was chosen arbitrarily, A is compact. ■

Acknowledgements

The proofs in this homework assignment have been worked and written in close collaboration with Camille Moyer.