Advanced Numerical Methods for PDEs

Homework 1

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Consider the initial - boundary value problem for the scalar advection diffusion equation

$$\partial_t u(x,t) + a\partial_x u(x,t) - b\partial_x^2 u(x,t) = 0, \qquad u(x,t=0) = u^I(x), \tag{1}$$

on the interval $x \in [-1,1]$ with periodic boundary conditions $u(x+2,t) = u(x,t), \forall x,t$. Consider the explicit difference method

$$U(x,t+\Delta t) = U(x,t) - \frac{a\Delta t}{2\Delta x}(T - T^{-1})U(x,t) + \frac{b}{\Delta x^2}(T - 2 + T^{-1})U(x,t) \tag{2}$$

for the problem (1).

Problem 1

1. Derive an analytic expression for the solution $u(x,t)$ of problem (1) for a general initial function $u^I(x)$ and general stepsizes Δx, Δt, using Fourier transforms.

Solution: We will use the Fourier transform

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} u(x,t) e^{-i\omega x} dx, \tag{3}$$

to turn our $1-D$ PDE into an ODE. Taking the Fourier transform of the PDE (1) we obtain

$$\partial_t \hat{u}(w,t) + iaw\hat{u}(w,t) + bw^2\hat{u}(w,t) = 0,$$

which can be manipulated into

$$\partial_t \hat{u}(w,t) = -(bw^2 + iaw)\hat{u}(w,t).$$

The previous equation has a simple analytical solution,

$$\hat{u}(w,t) = e^{-(bw^2+iaw)t}\hat{u}(w,0),$$

which we can conveniently rewrite as

$$\hat{u}(w,t) = e^{-b\omega^2 t}\hat{u}^I(w)e^{-iawt} \tag{4}$$

To obtain the solution, we will use the following property of Fourier transforms,

$$\mathcal{F}\left[f * g\right] = \mathcal{F}\left[f\right]\mathcal{F}\left[g\right], \tag{5}$$

where $*$ represents convolution. In our case, we have

$$\begin{aligned}\mathcal{F}\left[f\right](w,t) &= e^{-b\omega^2 t},\\ \mathcal{F}\left[g\right](w,t) &= \hat{u}^I(w)e^{-iawt}.\end{aligned}$$

By using the inverse Fourier transform on the previous equations we obtain

$$\begin{aligned}f(x,t) = \mathcal{F}^{-1}\left[\mathcal{F}\left[f\right](w,t)\right](x,t) &= \frac{e^{-x^2/4bt}}{\sqrt{2bt}}, \qquad (6)\\ g(x,t) = \mathcal{F}^{-1}\left[\mathcal{F}\left[g\right](w,t)\right](x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}^I(w)e^{-iw(x-at)}dw\\ &= u^I(x-at).\end{aligned}$$

Hence, we have proved so far that

$$\hat{u}(w,t) = e^{-b\omega^2 t}\hat{u}^I(w)e^{-iawt} = \mathcal{F}\left[f\right](w,t)\mathcal{F}\left[g\right](w,t) = \mathcal{F}\left[f * g\right](w,t), \tag{7}$$

Then, we can finally obtain the solution to the problem,

$$\begin{aligned}u(x,t) &= \mathcal{F}^{-1}\left[\hat{u}(w,t)\right](x,t)\\ &= \mathcal{F}^{-1}\left[\mathcal{F}\left[f * g\right](w,t)\right](x,t)\\ &= \left[f * g\right](x,t).\end{aligned}$$

To conclude,

$$u(x,t) = \left[f * u^I(x-at)\right](x,t), \tag{8}$$

with f given by (6).

2. Derive an analytic expression for the solution $U(x,t)$ of problem (2) for a general initial function $u^I(x)$ and general stepsizes Δx, Δt, using Discrete Fourier transforms.

Solution: We will use the Discrete Fourier transform

$$\hat{u}(w_\nu, t_n) = \frac{\Delta x}{\sqrt{2\pi}} \sum_{j=-N}^{N} u(x_j, t_n) e^{-i\omega_\nu x_j}, \tag{9}$$

and its inverse,

$$u(x_j, t_n) = \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} \hat{u}(w_\nu, t_n) e^{i x_j w_\nu}, \tag{10}$$

where $x_j = j\Delta_x$, $t_n = n\Delta t$, $w_\nu = \nu\Delta w$ and $N\Delta x\Delta w = \pi$. It is simple to prove that

$$\begin{aligned}
TU(x_j, t_n) = U(x_{j+1}, t_n) &= \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} \hat{u}(w_\nu, t_n) e^{-i(x_{j+1})w_\nu} \\
&= \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} e^{-i\Delta x w_\nu} \hat{u}(w_\nu, t_n) e^{-i x_j w_\nu}, \\
T^{-1}U(x_j, t_n) = U(x_{j-1}, t_n) &= \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} \hat{u}(w_\nu, t_n) e^{-i(x_{j-1})w_\nu} \\
&= \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} e^{i\Delta x w_\nu} \hat{u}(w_\nu, t_n) e^{-i x_j w_\nu},
\end{aligned}$$

We can substitute into equation(11) and simplify to obtain,

$$\begin{aligned}
\hat{u}(w_\nu, t_{n+1}) &= \left[1 - \frac{a\Delta t}{2\Delta x}(e^{-i\Delta x w_\nu} - e^{i\Delta x w_\nu}) + \frac{b}{\Delta x^2}(e^{-i\Delta x w_\nu} - 2 + e^{i\Delta x w_\nu})\right] \hat{u}(x_j, t_n) \\
&= \left[1 + i\frac{a\Delta t}{\Delta x}\sin(\Delta x w_\nu) - \frac{4b}{\Delta x^2}\sin^2\left(\frac{\Delta x w_\nu}{2}\right)\right] \hat{u}(w_\nu, t_n).
\end{aligned} \tag{11}$$

Define $g(\omega_\nu)$ as

$$g(\omega_\nu) = \left[1 + i\frac{a\Delta t}{\Delta x}\sin(\Delta x w_\nu) - \frac{4b}{\Delta x^2}\sin^2\left(\frac{\Delta x w_\nu}{2}\right)\right]. \tag{12}$$

Then, $\hat{u}(w_\nu, t_{n+1}) = g(\omega_\nu)\hat{u}(w_\nu, t_n)$. We can now find the solution, in frequency domain, as a function of the initial condition.

$$\begin{aligned}
\hat{u}(w_\nu, t_n) &= g(\omega_\nu)\hat{u}(w_\nu, t_{n-1}) = g^2(\omega_\nu)\hat{u}(w_\nu, t_{n-2}) = g^3(\omega_\nu)\hat{u}(w_\nu, t_{n-3}), \\
&= \cdots = g^n(\omega_\nu)\hat{u}(w_\nu, 0), \\
&= g^n(\omega_\nu)\hat{u}^I(w_\nu).
\end{aligned}$$

We now use equation (10) to obtain the solution in space domain,

$$\begin{aligned} u(x_j, t_n) &= \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} \hat{u}(w_\nu, t_n) e^{i x_j w_\nu} \\ &= \frac{\Delta w}{\sqrt{2\pi}} \sum_{\nu=-N}^{N} g^n(\omega_\nu) \hat{u}^I(w_\nu) e^{i x_j w_\nu}. \end{aligned} \tag{13}$$

To finish, it must be mentioned that the above solution is stable if and only if $|g(w)| \leq 1, \forall w \in [-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x}]$. This condition will produce the following conditions on the parameters:

$$\Delta t a^2 \leq 2b, \text{ and } 2\Delta t b \leq \Delta x^2.$$

Problem 2

Problem 3