Real Analysis Instructor Homework

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1 Problem 5.1.1.

1. Let B be an $m \times n$ matrix $(\alpha_{jk}; j = 1, ..., m, k = 1, ..., n)$. As we know from Example 5.4,

$$[Bx]_j = \sum_{k=1}^{n} \alpha_{jk} x_k, \quad j = 1, \cdots, m, \; x = (x_1, \cdots, x_n)$$

defines a bounded linear operator B from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Show: if both spaces are equipped with the sum-norm, then

$$\|B\| = \max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}|$$

Solution:

Proof. First let us show $\|B\| \leq \max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}|$.
Let $[Bx]_j = \sum_{k=1}^{n} \alpha_{jk} x_k, \quad j = 1, \cdots, m, \; x = (x_1, \cdots, x_n)$ and $x \neq 0$. Then,

$$Bx = \begin{bmatrix} \sum_{k=1}^{n} \alpha_{1k} \\ \sum_{k=1}^{n} \alpha_{2k} \\ \vdots \\ \sum_{k=1}^{n} \alpha_{mk} \end{bmatrix} .$$

By triangle inequality,

$$\begin{aligned} \|Bx\|_1 &= \sum_{j=1}^{m} |[Bx]_j| = \sum_{j=1}^{m} |\sum_{k=1}^{n} \alpha_{jk} x_k| \leq \sum_{j=1}^{m} \sum_{k=1}^{n} |\alpha_{jk} x_k| = \sum_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}||x_k| \\ &\leq \sum_{k=1}^{n} \left(\max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}| \right) |x_k| = \left(\max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}| \right) \|x\|_1 \; . \end{aligned}$$

Therefore,

$$\frac{\|Bx\|_1}{\|x\|_1} \leq \max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}| \; , \; x \neq 0 \; .$$

Thus, by *Lemma 5.3* and the definition of a supremum,

$$\|B\| \leq \max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}| \ .$$

To show the inequality in the other sense, choose k such that

$$\max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}| = \sum_{j=1}^{m} |\alpha_{jk}| \ .$$

Now consider e^k to be the k^{th} canonical basis vector with all components being zero except for the $k-th$ component which is 1. It is inmediate that $\|e^k\|_1 = 1$. Then,

$$\frac{\|Be^k\|_1}{\|e^k\|_1} = \|Be^k\|_1 = \sum_{j=1}^{m} |\alpha_{jk}| \ .$$

Again, by *Lemma 5.3* and the definition of a supremum,

$$\max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}| = \sum_{j=1}^{m} |\alpha_{jk}| = \frac{\|Be^k\|_1}{\|e^k\|_1} \leq \|B\| \ .$$

We combine this inequality with the one obtained before to get the desired result:

$$\|B\| = \max_{k=1}^{n} \sum_{j=1}^{m} |\alpha_{jk}|$$

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