POISSON EQUATION

$$\nabla \cdot (\nabla \varphi^{n+1}) = \frac{1}{\Delta t} \nabla \cdot \vec{v}^{*}$$

$$\nabla^2 \varphi^{n+1} = \frac{1}{\Delta t} \nabla \cdot \vec{v}^{*}$$

$$\frac{\partial^2 \varphi^{n+1}}{\partial x^2} + \frac{\partial^2 \varphi^{n+1}}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right)$$

Since we are trying to obtain φ^{n+1}, we will discretize the derivatives at the cell centers:

$$\left.\frac{\partial^2 \varphi^{n+1}}{\partial x^2}\right|_{i,j} = \frac{\varphi^{n+1}_{i+1,j} - 2\varphi^{n+1}_{i,j} + \varphi^{n+1}_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{\partial^2 \varphi^{n+1}}{\partial y^2}\right|_{i,j} = \frac{\varphi^{n+1}_{i,j+1} - 2\varphi^{n+1}_{i,j} + \varphi^{n+1}_{i,j-1}}{\Delta y^2} + O(\Delta y^2)$$

$$\left.\frac{\partial u^*}{\partial x}\right|_{i,j} = \frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j}}{\cancel{2}\frac{\Delta x}{\cancel{2}}} + O(\Delta x^2)$$

$$\left.\frac{\partial v^*}{\partial y}\right|_{i,j} = \frac{v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{\Delta y} + O(\Delta y^2)$$

Thus, we will solve

$$\frac{\varphi^{n+1}_{i+1,j} - 2\varphi^{n+1}_{i,j} + \varphi^{n+1}_{i-1,j}}{\Delta x^2} + \frac{\varphi^{n+1}_{i,j+1} - 2\varphi^{n+1}_{i,j} + \varphi^{n+1}_{i,j-1}}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u^*_{i+\frac{1}{2},j} - u^*_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^*_{i,j+\frac{1}{2}} - v^*_{i,j-\frac{1}{2}}}{\Delta y}\right)$$

with zero Neumann Boundary Conditions:

$$\varphi^{n+1}_{i,1} = \varphi^{n+1}_{i,2} \quad \text{(Bottom)}$$

$$\varphi^{n+1}_{i,N+2} = \varphi^{n+1}_{i,N+1} \quad \text{(Top)}$$

NOTE: Matlab indices used.

$$\varphi^{n+1}_{1,j} = \varphi^{n+1}_{2,j} \quad \text{(Left)}$$

$$\varphi^{n+1}_{M+2,j} = \varphi^{n+1}_{M+1,j} \quad \text{(Right)}$$

PROJECTION STEP

$$\frac{\partial \vec{v}}{\partial t} = -\nabla \varphi^{n+1} \Rightarrow \frac{\vec{v}^{n+1} - \vec{v}^*}{\Delta t} = -\nabla \varphi^{n+1}$$

$$\vec{v}^{n+1} = \vec{v}^* - \Delta t \nabla \varphi^{n+1} \quad \begin{cases} u^{n+1} = u^* - \Delta t \dfrac{\partial \varphi}{\partial x} \\ v^{n+1} = v^* - \Delta t \dfrac{\partial \varphi}{\partial y} \end{cases}$$

Since we want to obtain velocities, we will discretize the derivatives centering at the staggered mesh.

$$\left.\frac{\partial \varphi}{\partial x}\right|_{i+\frac{1}{2},j} = \frac{\varphi^{n+1}_{i+1,j} - \varphi^{n+1}_{i,j}}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial \varphi}{\partial y}\right|_{i,j+\frac{1}{2}} = \frac{\varphi^{n+1}_{i,j+1} - \varphi^{n+1}_{i,j}}{\Delta y} + O(\Delta y^2)$$

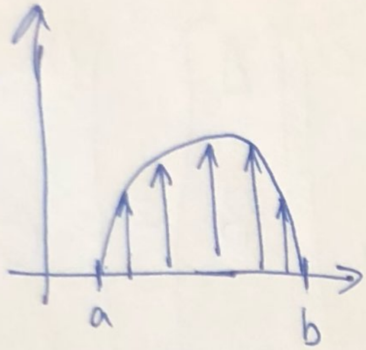
No BCs are needed since the zero Neumann BCs. for φ leaves the boundary velocities uncorrected.

Thus, we will solve:

$$u^{n+1}_{i+\frac{1}{2},j} = u^*_{i+\frac{1}{2},j} - \Delta t \frac{\varphi^{n+1}_{i+1,j} - \varphi^{n+1}_{i,j}}{\Delta x}$$

$$v^{n+1}_{i,j+\frac{1}{2}} = v^*_{i,j+\frac{1}{2}} - \Delta t \frac{\varphi^{n+1}_{i,j+1} - \varphi^{n+1}_{i,j}}{\Delta y}$$

INLETS COEFFICIENTS



$$v(x) = Ax^2 + Bx + C$$

$$v(a) = Aa^2 + Ba + C = 0 \quad (1)$$

$$v(b) = Ab^2 + Bb + C = 0 \quad (2)$$

$$v_{avg} = \frac{1}{b-a}\int_a^b v(x)\,dx = \frac{1}{b-a}\left[\frac{1}{3}A(b^3-a^3) + \frac{1}{2}B(b^2-a^2) + C(b-a)\right]$$

Hence,

$$\frac{1}{3}(b^3-a^3)A + \frac{1}{2}(b^2-a^2)B + (b-a)C = (b-a)\,v_{avg} \quad (3)$$

The system of equations (1), (2), (3) can be expressed as

$$\begin{bmatrix} \frac{1}{3}(b^3-a^3) & \frac{1}{2}(b^2-a^2) & (b-a) \\ a^2 & a & 1 \\ b^2 & b & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} (b-a)\,v_{avg} \\ 0 \\ 0 \end{bmatrix}$$

$r_3 - \frac{b^2}{a^2} r_2$:

$$\begin{bmatrix} \frac{1}{3}(b^3-a^3) & \frac{1}{2}(b^2-a^2) & (b-a) \\ a^2 & a & 1 \\ 0 & b - \frac{b^2}{a} & 1 - \frac{b^2}{a^2} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} (b-a)\,v_{avg} \\ 0 \\ 0 \end{bmatrix}$$

$r_1 - (b-a) r_2$:

$$\begin{bmatrix} \frac{1}{3}(b^3-a^3)-(b-a)a^2 & \frac{1}{2}(b^2-a^2)-(b-a)a & 0 \\ a^2 & a & 1 \\ 0 & b-\frac{b^2}{a} & 1-\frac{b^2}{a^2} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} (b-a)\,\sigma_{avg} \\ 0 \\ 0 \end{bmatrix}$$

We can obtain A, B, C using our tridiagonal solver. It also works for horizontal inlets.

I have obtained:

$$\text{Inlet 1} \begin{cases} A = -48 \\ B = 72 \\ C = -24 \end{cases}$$

$$\text{Inlet 2} \begin{cases} A = 24 \\ B = -60 \\ C = 36 \end{cases}$$

$$\text{Inlet 3} \begin{cases} A = 24 \\ B = -36 \\ C = 12 \end{cases}$$