Numerical Methods for PDEs
Homework 7

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Problem 1

Derive the entropy advection equation $s_t + us_x = 0$ from the Euler equations and the expression for the entropy $s = c_V \ln(P/\rho^\gamma)$ of a polytropic gas. *Hint:* Start with the entropy advection equation and derive $E_t + (u(E+P))_x = 0$ with $P = (\gamma - 1)\left(E - \frac{1}{2}\rho u^2\right)$, making use of $\rho_t + (\rho u)_x = 0$ and $(\rho u)_t + (\rho u^2 + P)_x = \rho u_t + \rho u u_x + P_x = 0$ if needed.

Solution: From the definition of the entropy we obtain the partial derivatives

$$s_t = c_V \left(\frac{P_t}{P} - \gamma \frac{\rho_t}{\rho} \right),$$
$$s_x = c_V \left(\frac{P_x}{P} - \gamma \frac{\rho_x}{\rho} \right).$$

Then, the entropy advection equation gives

$$s_t + us_x = 0 \Rightarrow \rho P_t - \gamma P \rho_t + u\rho P_x - \gamma u P \rho_x = 0,$$

where we have multiplied the whole equation by ρP. We now include the easy computed partial derivatives of P,

$$P_t = (\gamma - 1) \left(E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t \right),$$
$$P_x = (\gamma - 1) \left(E_x - \frac{1}{2} u^2 \rho_x - \rho u u_x \right),$$

into the equation to get

$$\rho\,(\gamma - 1) \left(E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t \right) - \gamma P \rho_t + u\rho\,(\gamma - 1) \left(E_x - \frac{1}{2} u^2 \rho_x - \rho u u_x \right) - \gamma u P \rho_x = 0.$$

Dividing by $\rho\,(\gamma - 1)$ and computing the brackets we obtain

$$E_t - \frac{1}{2} u^2 \rho_t - \rho u u_t - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + uE_x - \frac{1}{2} u^3 \rho_x - \rho u^2 u_x - \frac{\gamma}{\gamma - 1} uP \frac{\rho_x}{\rho} = 0.$$

Using the continuity equation $\rho_t + u\rho_x + \rho u_x = 0$, we simplify the previous equation,

$$E_t - \rho u u_t - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + uE_x - \frac{1}{2} \rho u^2 u_x - \frac{\gamma}{\gamma - 1} uP \frac{\rho_x}{\rho} = 0.$$

Further, we use the Euler momentum equation in the form $-\rho u u_t = \rho u^2 u_x + u P_x$ to obtain

$$\begin{aligned} E_t + \rho u^2 u_x + u P_x - \frac{\gamma}{\gamma - 1} P \frac{\rho_t}{\rho} + u E_x - \frac{1}{2}\rho u^2 u_x - \frac{\gamma}{\gamma - 1} u P \frac{\rho_x}{\rho} &= 0, \\ E_t + u(E_x + P_x) + \frac{1}{2}\rho u^2 u_x - \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\left(\rho_t + u\rho_x\right) &= 0, \\ E_t + u(E_x + P_x) + \frac{1}{2}\rho u^2 u_x + \frac{\gamma}{\gamma - 1}\frac{P}{\rho}\rho u_x &= 0, \end{aligned}$$

where we have used once more the continuity equation in the last step. Note that

$$\frac{\gamma}{\gamma - 1} P = \gamma E - \frac{1}{2}\gamma \rho u^2.$$

Then,

$$\begin{aligned} E_t + u(E_x + P_x) + u_x\left(\frac{1}{2}\rho u^2 + \gamma E - \frac{1}{2}\gamma\rho u^2\right) &= 0, \\ E_t + u(E_x + P_x) + u_x\left(\gamma E - (\gamma - 1)\frac{1}{2}\rho u^2\right) &= 0. \end{aligned}$$

To finish, note that

$$E + P = \gamma E - (\gamma - 1)\frac{1}{2}\rho u^2.$$

Hence,

$$\begin{aligned} E_t + u(E_x + P_x) + u_x\left(\gamma E - (\gamma - 1)\frac{1}{2}\rho u^2\right) &= 0, \\ E_t + u(E_x + P_x) + u_x\left(E + P\right) &= 0. \end{aligned}$$

Finally, we obtain the equation that completes the proof,

$$E_t + \left(u\left(E + P\right)\right)_x = 0.$$

Problem 2

Show that the solution of the Riemann problem

$$\rho_L = 2,\ u_L = 1,\ P_L = 3; \quad \rho_R = 1,\ u_R = 0,\ P_R = 1$$

with $\gamma = 1.5$ is a single shock wave propagating to the right. Calculate the shock speed s. For $\gamma = 1.5$, $E = \frac{1}{2}\rho u^2 + 2P$. *Hint:* Use the jump conditions.

Solution: We can easily compute s as

$$s = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{2 \cdot 1 - 1 \cdot 0}{2 - 1} = 2,$$

and check that this result is right by the following:

$$s\,(\rho_L u_L - \rho_R u_R) = \rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R, \qquad s\,(E_L - E_R) = u_L(E_L + P_L) - u_R(E_R + P_R),$$
$$2\,(2\cdot 1 - 1\cdot 0) = 2\cdot 1^2 + 3 - 1\cdot 0^2 - 1, \qquad 2\cdot(7-2) = 1\cdot(7+3) - 0\cdot(2+1),$$
$$4 = 4, \qquad 10 = 10,$$

where we have calcultated $E_L = \frac{1}{2}\rho_L u_L^2 + 2P_L = 7$ and $E_R = \frac{1}{2}\rho_R u_R^2 + 2P_R = 2$

With a wall BC at $x = 1$, analytically calculate the solution after reflection of the shock wave. *Hint:* Show that the exact reflected shock solution is

$$\rho = 3.6,\ u = 0,\ P = 7.5,\ s_r = -1.25$$

where s_r is the velocity of the reflected shock.

Solution: In this case we repeat the process assuming the same conditions on the left. We can check that given the right values of ρ, u, and P, the value s_r obtains matches the solution:

$$s = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R} = \frac{2\cdot 1 - 3.6\cdot 0}{2 - 3.6} = -1.25,$$
$$s = \frac{\rho_L u_L^2 + P_L - \rho_R u_R^2 - P_R}{\rho_L u_L - \rho_R u_R} = \frac{2\cdot 1^2 + 3 - 3.6\cdot 0^2 - 7.5}{2\cdot 1 - 3.6\cdot 0} = -1.25,$$
$$s = \frac{u_L(E_L + P_L) - u_R(E_R + P_R)}{E_L - E_R} = \frac{1\cdot(7+3) - 0\cdot(15+7.5)}{7 - 15} = -1.25,$$

where we have calcultated $E_R = \frac{1}{2}\rho_R u_R^2 + 2P_R = 15$

Problem 3

Show that the Lax-Wendroff method is second-order accurate for $u_t + Au_x = 0$ using the definition of the LTE.

Solution: To prove that the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{1}{2}A\frac{\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{1}{2}A^2\frac{\Delta t^2}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right),$$

is second order accurate for $u_t + Au_x = 0$, we start by Taylor expanding

$$u_j^{n+1} = u_j^n + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} + \mathcal{O}(\Delta t^4),$$
$$u_{j\pm 1}^n = u_j^n \pm \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} \pm \frac{\Delta x^3}{6}u_{xxx} + \mathcal{O}(\Delta x^4),$$

and substituting them into the Lax-Wendroff scheme,

$$u_j^n + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} = u_j^n - \frac{A\Delta t}{2\Delta x}\left(2\Delta x u_x + \frac{\Delta x^3}{3}u_{xxx}\right) + \frac{A^2\Delta t^2}{2\Delta x^2}\Delta x^2 u_{xx} + \Delta t\tau.$$

Doing some algebraic manipulations we reach,

$$
\begin{aligned}
u_t + Au_x &= -\frac{\Delta t}{2}u_{tt} - A\frac{\Delta x^2}{6}u_{xxx} + \frac{1}{2}A^2\Delta t u_{xx} - \frac{\Delta t^2}{6}u_{ttt} + \tau, \\
\tau &= \frac{\Delta t}{2}u_{tt} + A\frac{\Delta x^2}{6}u_{xxx} - \frac{1}{2}A^2\Delta t u_{xx} + \frac{\Delta t^2}{6}u_{ttt},
\end{aligned}
$$

where we have used that $u_t + Au_x = 0$. Using the original PDE we obtain that

$$
\begin{aligned}
u_t = -Au_x &\Rightarrow u_{tt} = A^2 u_{xx}, \\
&\Rightarrow u_{ttt} = -A^3 u_{xxx}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\tau &= \frac{\Delta t}{2}A^2 u_{xx} + A\frac{\Delta x^2}{6}u_{xxx} - \frac{1}{2}A^2\Delta t u_{xx} + \frac{\Delta t^2}{6}A^3 u_{ttt}, \\
&= A\frac{\Delta x^2}{6}u_{xxx} + \frac{\Delta t^2}{6}A^3 u_{ttt}.
\end{aligned}
$$

Thus, the Lax-Wendroff scheme is second order accurate for $u_t + Au_x = 0$.

Problem 4

Using **weno3.m**, investigate the effects of the CFL factor r on the solution of the Riemann problem

$$\rho_L = 1,\ u_L = 0,\ p_L = 1; \quad \rho_R = 0.125,\ u_R = 0,\ p_R = 0.1$$

with $\gamma = 1.4$. Take 200 Δx and CFL factor $r = 0.1$, 0.5, 0.9. Turn in the Density plots (computed vs. exact solution) at time $t = 0.2$ for each of three cases. Briefly discuss your results.

Solution: