Problem 1. Suppose f is $2\pi-$periodic and analytic in the complex strip $|\mathrm{Im}(z)| < a$ with $|f(z)| \leq M_{f,a}$ for all z in this strip. Show that

$$\left| \int_0^{2\pi} f(x)dx - \frac{2\pi}{N} \sum_{j=0}^{N-1} f\left(2\pi j/N\right) \right| = O\left(M_{f,a} e^{-(a-\varepsilon)N} \right), \quad \text{as} \quad N \to \infty,$$

for every $\varepsilon > 0$.

Problem 2. Let $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(1+i)(n+1/2)^6} \exp(inx)$. Compute the Fourier coefficients of f using a DFT or FFT on 11 points to obtain $\hat{v_k}$ and verify the aliasing formula for the coefficients:

$$\hat{v}_k - \hat{f}_k = \sum_{m \neq 0} \hat{f}_{k+mN}.$$

Problem 3. Write a function that solves Burgers equation

$$u_t + uu_x = \varepsilon u_{xx}, \quad x \in (0,1), \quad t \in (0, t_{max}]$$

$$u(0,x) = \sin^4(\pi x)$$

and periodic boundary conditions. Use Fourier to compute derivatives in space and `ode113` to advance in time. Solve this PDE for $\varepsilon = 0.1, 0.01$, and 0.001. In each case, can you find solutions that are accurate to three digits at $t = 1$?

Problem 4. Problem 4.4 in Spectral Methods in Matlab.

Problem 5. Consider the linear advection-diffusion equation in 2–D with periodic boundary conditions:

$$w_t + uw_x + vw_y = \varepsilon(w_{xx} + w_{yy}), \quad (x,y) \in (-\pi,\pi) \times (-\pi,\pi), \quad t > 0$$

$$w(0,x,y) = \exp(-2x^2 - 5y^2),$$

where the velocity field $[u(x,y), v(x,y)]^T$ is given by

$$u(x,y) = \sin(y)\cos(x + \pi/2) \quad \text{and} \quad v = -\cos(y)\sin(x+\pi/2).$$

(a) Use `quiver` to plot the velocity field on a 64×64 grid.

(b) Solve this PDE using FFTs to compute spatial derivatives on a 256×256 grid (use $\varepsilon = 0.005$). Show the contours (`contourf`) of your solution at $t = 1, 2, 3, 8$.