Computational Methods

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Homework 6

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1 Introduction

In this assignment we will study a two dimentional mixing chamber. First we will show the equations to solve for u, v and Y. We will use a staggered mesh for u and v and a cell centered mesh for Y. I apologize to the grader since I have not been able to complete the typing of this submission. I will give pdf versions of pictures of my handwritten notes which have the equations used in my code, drawing, and more details. The notation in my notes is according to *Matlab*, i.e., the staggered meshes don't have the 1/2. However, the first two sections of this submission have the development for u and v written in such staggered notation. My notes will also have details about how I treated the boundary conditions, they will be located at the end of the document. I sincerely apologize again to the grader as I know this might cause some disturbance.

2 Development for u(x,y,t)

2.1 Step 1

Let's start by rewriting our ADI equation for the Step 1 (given in class 13 slide 13), for an staggered mesh for the horizontal velocity u:

$$-d_1 u^{n+1/2}_{i-1/2,j} + (1+2d_1)u^{n+1/2}_{i+1/2,j} - d_1 u^{n+1/2}_{i+3/2,j} = d_2 u^n_{i+1/2,j+1} + (1-2d_2)u^n_{i+1/2,j} + d_2 u^n_{i+1/2,j-1},$$

which has the form

$$a u^{n+1/2}_{i-1/2,j} + b u^{n+1/2}_{i+1/2,j} + c u^{n+1/2}_{i+3/2,j} = d,$$

a tridiagonal system. The staggered mesh for u gives us a matrix u that is $(M+1)\times(N+2)$. We only need to solve the previous equation for the interior with two *for* loops in *Matlab*,

```
for j=2:N+1
for i=2:M
...
end
end
```

We will now include the boundary conditions for each case.

Case $j = 2$ and $i = 2$

In this case the general equation becomes

$$-d_1 u^{n+1/2}_{3/2,2} + (1 + 2d_1)u^{n+1/2}_{5/2,2} - d_1 u^{n+1/2}_{7/2,2} = d_2 u^n_{5/2,3} + (1 - 2d_2)u^n_{5/2,2} + d_2 u^n_{5/2,1}.$$

Note that I will be using *Maltab* indices, so the first index is 1. Since we know that $u^{n+1/2}_{3/2,2} = 0$ since it corresponds to the left wall, we can then impose that $a(1) = 0$. The value $u_{5/2,1}$ will also be determined by the boundary conditions. Since it is a cell-like boundary conditions in this case, we can impose it by updating the ghost cells. Thus, $d(1)$ is the right hand side of the previous equation.

Case $j = 2$ and $i \in [3, M - 1]$

In this case the general equation becomes

$$-d_1 u^{n+1/2}_{i-1/2,2} + (1 + 2d_1)u^{n+1/2}_{i+1/2,2} - d_1 u^{n+1/2}_{i+3/2,2} = d_2 u^n_{i+1/2,3} + (1 - 2d_2)u^n_{i+1/2,2} + d_2 u^n_{i+1/2,1}.$$

Like in the previous case, the value $u_{i+1/2,1}$ will also be determined by the boundary conditions. Since it is a cell-like boundary conditions in this case, we can impose it by updating the ghost cells. Thus, d is the right hand side of the previous equation.

Case $j = 2$ and $i = M$

In this case the general equation becomes

$$-d_1 u^{n+1/2}_{M-1/2,2} + (1 + 2d_1)u^{n+1/2}_{M+1/2,2} - d_1 u^{n+1/2}_{M+3/2,2} = d_2 u^n_{M+1/2,3} + (1 - 2d_2)u^n_{M+1/2,2} + d_2 u^n_{M+1/2,1}.$$

In this case we can see that the value $u^{n+1/2}_{M+3/2,2}$ corresponds to the horizontal velocity at the right wall, hence it is zero and so it is c in this case. Like in the previous case, the value $u_{M+1/2,1}$ will also be determined by the boundary conditions. Since it is a cell-like boundary condition in this case, we can impose it by updating the ghost cells. Thus, d is the right hand side of the previous equation.

Case $j \in [3, N]$ and $i = 2$

In this case the general equation becomes

$$-d_1 u^{n+1/2}_{3/2,j} + (1 + 2d_1)u^{n+1/2}_{5/2,j} - d_1 u^{n+1/2}_{7/2,j} = d_2 u^n_{5/2,j+1} + (1 - 2d_2)u^n_{5/2,j} + d_2 u^n_{5/2,j-1}.$$

In this case the value $u^{n+1/2}_{3/2,j}$ corresponds to the horizontal velocity at the left wall. Since its value will depend on j because of the inlet 1, it will be moved to the right hand side, getting

$$(1 + 2d_1)u^{n+1/2}_{5/2,j} - d_1 u^{n+1/2}_{7/2,j} = d_2 u^n_{5/2,j+1} + (1 - 2d_2)u^n_{5/2,j} + d_2 u^n_{5/2,j-1} + d_1 u^{n+1/2}_{3/2,j},$$

which makes $a = 0$ for this case and d being the right hand side of the previous equation.

Case $j \in [3, N]$ and $i \in [3, M-1]$

In this case the general equation does not include any boundary conditions, therefore it is unaltered.

Case $j \in [3, N]$ and $i = M$

In this case the general equation becomes

$$-d_1 u^{n+1/2}_{M-1/2,j} + (1 + 2d_1)u^{n+1/2}_{M+1/2,j} - d_1 u^{n+1/2}_{M+3/2,j} = d_2 u^n_{M+1/2,j+1} + (1 - 2d_2)u^n_{M+1/2,j} + d_2 u^n_{M+1/2,j-1}.$$

Like before the value of $u^{n+1/2}_{M+3/2,j}$ will depend on the value of j because of the inlet 2. Therefore we will move it to the right hand side and get

$$-d_1 u^{n+1/2}_{M-1/2,j} + (1 + 2d_1)u^{n+1/2}_{M+1/2,j} = d_2 u^n_{M+1/2,j+1} + (1 - 2d_2)u^n_{M+1/2,j} + d_2 u^n_{M+1/2,j-1} + d_1 u^{n+1/2}_{M+3/2,j},$$

which gives us, for this case, $c = 0$ and d being the right hand side of the previous equation.

Case $j = N + 1$ and $i = 2$

In this case the general equation becomes

$$-d_1 u^{n+1/2}_{3/2,N+1} + (1 + 2d_1)u^{n+1/2}_{5/2,N+1} - d_1 u^{n+1/2}_{7/2,N+1} = d_2 u^n_{5/2,N+2} + (1 - 2d_2)u^n_{5/2,N+1} + d_2 u^n_{5/2,N}.$$

In this case the value of $u^{n+1/2}_{3/2,N+1}$ corresponds to the horizontal velocity at the left wall, which is zero imposed by the boundary conditions. This implies that $a = 0$ for this case. The value of $u^n_{5/2,N+2}$ will be determined by the boundary conditions as well. Since it is a cell-like boundary condition in this case, we can impose it by updating the ghost cells. Thus, d is the right hand side of the previous equation.

Case $j = N + 1$ and $i \in [3, M-1]$

$$\begin{aligned}-d_1 u^{n+1/2}_{i-1/2,N+1} + (1 + 2d_1)u^{n+1/2}_{i+1/2,N+1} - d_1 u^{n+1/2}_{i+3/2,N+1} = d_2 u^n_{i+1/2,N+2} &+ (1 - 2d_2)u^n_{i+1/2,N+1} \\ &+ d_2 u^n_{i+1/2,N},\end{aligned}$$

Like in the previous case, the value of $u^n_{i+1/2,N+2}$ will be determined by the boundary conditions as well. Since it is a cell-like boundary condition in this case, we can impose it by updating the ghost cells. Thus, d is the right hand side of the previous equation. We will see how its value does not depend on the value of i since the horizontal velocity in the inlet 3 is zero.

Case $j = N + 1$ and $i = M$

$$\begin{aligned}-d_1 u^{n+1/2}_{M-1/2,N+1} + (1 + 2d_1)u^{n+1/2}_{M+1/2,N+1} - d_1 u^{n+1/2}_{M+3/2,N+1} = d_2 u^n_{M+1/2,N+2} &+ (1 - 2d_2)u^n_{M+1/2,N+1} \\ &+ d_2 u^n_{M+1/2,N},\end{aligned}$$

In this case the value of $u^{n+1/2}_{M+3/2,N+1}$ corresponds to the horizontal velocity at the right wall, imposed to be zero by the boundary conditions. Therefore $c = 0$ for this case. The value of $u^{n}_{M+1/2,N+2}$ will be determined by the boundary conditions as well. Since it is a cell-like boundary condition in this case, we can impose it by updating the ghost cells.

2.2 Step 2

We start by rewriting our ADI equation for the Step 2 (given in class 13 slide 14), for an staggered mesh for the horizontal velocity u:

$$-d_2u^{n+1}_{i+1/2,j-1} + (1+2d_2)u^{n+1}_{i+1/2,j} - d_2u^{n+1}_{i+1/2,j+1} = d_1u^{n+1/2}_{i+3/2,j} + (1-2d_1)u^{n+1/2}_{i+1/2,j} + d_1u^{n+1/2}_{i-1/2,j}$$

which has the form

$$au^{n+1}_{i+1/2,j-1} + bu^{n+1}_{i+1/2,j} + cu^{n+1}_{i+1/2,j+1} = d$$

a tridiagonal system. The staggered mesh for u gives us a matrix u that is $(M+1)\times(N+2)$. We only need to solve the previous equation for the interior with two for loops in *Matlab*,

```
for j=2:M
for i=2:N+1
...
end
end
```

We will now include the boundary conditions for each case.

Case $i = 2$ and $j = 2$

In this case the general equation becomes

$$-d_2u^{n+1}_{5/2,1} + (1+2d_2)u^{n+1}_{5/2,2} - d_2u^{n+1}_{5/2,3} = d_1u^{n+1/2}_{7/2,2} + (1-2d_1)u^{n+1/2}_{5/2,2} + d_1u^{n+1/2}_{3/2,2}.$$

The value of $u^{n+1}_{5/2,1}$ does not correspond to the interior and will be determined by the boundary conditions to be $-u^{n+1}_{5/2,2}$. Since it is a cell-like boundary condition in this case, we can impose it by updating the ghost cells. The value of $u^{n+1/2}_{3/2,2}$ corresponds to the left wall and it is then imposed to be zero. The previous equation then gives us

$$(1+3d_2)u^{n+1}_{5/2,2} - d_2u^{n+1}_{5/2,3} = d_1u^{n+1/2}_{7/2,2} + (1-2d_1)u^{n+1/2}_{5/2,2}.$$

Thus, for this case $a = 0$, $b = 1 + 3d_2$ and d is given by the right hand side.

Case $i = 2$ and $j \in [3, N]$

In this case the general equation becomes

$$-d_2u^{n+1}_{5/2,j-1} + (1+2d_2)u^{n+1}_{5/2,j} - d_2u^{n+1}_{5/2,j+1} = d_1u^{n+1/2}_{7/2,j} + (1-2d_1)u^{n+1/2}_{5/2,j} + d_1u^{n+1/2}_{3/2,j}.$$

The value of $u^{n+1/2}_{3/2,j}$ corresponds to the left wall and it is then imposed to be zero and d is given by the right hand side.

Case $i = 2$ and $j = N + 1$

In this case the general equation becomes

$$-d_2 u^{n+1}_{5/2,N} + (1 + 2d_2)u^{n+1}_{5/2,N+1} - d_2 u^{n+1}_{5/2,N+2} = d_1 u^{n+1/2}_{7/2,N+1} + (1 - 2d_1)u^{n+1/2}_{5/2,N+1} + d_1 u^{n+1/2}_{3/2,N+1}.$$

The value of $u^{n+1/2}_{3/2,N+1}$ corresponds to the left wall and it is then imposed to be zero. The value of $u^{n+1}_{5/2,N+2} = -u^{n+1}_{5/2,N+1}$ by the boundary conditions. Thus, for this case $c = 0$, $b = 1 + 3d_2$ and d is given by the right hand side.

Case $i \in [3, M - 1]$ and $j = 2$

In this case the general equation becomes

$$-d_2 u^{n+1}_{i+1/2,1} + (1 + 2d_2)u^{n+1}_{i+1/2,2} - d_2 u^{n+1}_{i+1/2,3} = d_1 u^{n+1/2}_{i+3/2,2} + (1 - 2d_1)u^{n+1/2}_{i+1/2,2} + d_1 u^{n+1/2}_{i-1/2,2}.$$

By the boundary conditions, $u^{n+1}_{i+1/2,1} = -u^{n+1}_{i+1/2,2}$ and the previous equation yields

$$(1 + 3d_2)u^{n+1}_{i+1/2,2} - d_2 u^{n+1}_{i+1/2,3} = d_1 u^{n+1/2}_{i+3/2,2} + (1 - 2d_1)u^{n+1/2}_{i+1/2,2} + d_1 u^{n+1/2}_{i-1/2,2}.$$

Thus, for this case $a = 0$, $b = 1 + 3d_2$ and d is given by the right hand side.

Case $i \in [3, M - 1]$ and $j \in [3, N]$

In this case the general equation does not include any boundary conditions, therefore it is unaltered.

Case $i \in [3, M - 1]$ and $j = N + 1$

In this case the general equation becomes

$$\begin{aligned}-d_2 u^{n+1}_{i+1/2,N} + (1 + 2d_2)u^{n+1}_{i+1/2,N+1} - d_2 u^{n+1}_{i+1/2,N+2} = d_1 u^{n+1/2}_{i+3/2,N+1} &+ (1 - 2d_1)u^{n+1/2}_{i+1/2,N+1} \\ &+ d_1 u^{n+1/2}_{i-1/2,N+1}.\end{aligned}$$

By the boundary conditions, $u^{n+1}_{i+1/2,N+2} = -u^{n+1}_{i+1/2,N+1}$ and the previous equation yields

$$-d_2 u^{n+1}_{i+1/2,N} + (1 + 3d_2)u^{n+1}_{i+1/2,N+1} = d_1 u^{n+1/2}_{i+3/2,N+1} + (1 - 2d_1)u^{n+1/2}_{i+1/2,N+1} + d_1 u^{n+1/2}_{i-1/2,N+1}.$$

Thus, for this case $c = 0$, $b = 1 + 3d_2$ and d is given by the right hand side.

Case $i = M$ and $j = 2$

In this case the general equation becomes

$$-d_2 u^{n+1}_{M+1/2,1} + (1 + 2d_2)u^{n+1}_{M+1/2,2} - d_2 u^{n+1}_{M+1/2,3} = d_1 u^{n+1/2}_{M+3/2,2} + (1 - 2d_1)u^{n+1/2}_{M+1/2,2} + d_1 u^{n+1/2}_{M-1/2,2}.$$

By the boundary conditions, $u^{n+1}_{M+1/2,1} = -u^{n+1}_{M+1/2,2}$ and $u^{n+1/2}_{M+3/2,2} = 0$ since it corresponds to the right wall. The previous equation yields

$$(1 + 3d_2)u^{n+1}_{M+1/2,2} - d_2 u^{n+1}_{M+1/2,3} = (1 - 2d_1)u^{n+1/2}_{M+1/2,2} + d_1 u^{n+1/2}_{M-1/2,2}.$$

Thus, for this case $a = 0$, $b = 1 + 3d_2$ and d is given by the right hand side.

Case $i = M$ and $j \in [3, N]$

In this case the general equation becomes

$$-d_2 u^{n+1}_{M+1/2,j-1} + (1+2d_2)u^{n+1}_{M+1/2,j} - d_2 u^{n+1}_{M+1/2,j+1} = d_1 u^{n+1/2}_{M+3/2,j} + (1-2d_1)u^{n+1/2}_{M+1/2,j} + d_1 u^{n+1/2}_{M-1/2,j}$$

The value of $u^{n+1/2}_{M+3/2,j}$ depends on j because of inlet 2. It will be imposed by the corresponding boundary condition in the code. In this case, d is given by the right hand side.

Case $i = M$ and $j = N + 1$

In this case the general equation becomes

$$\begin{aligned}-d_2 u^{n+1}_{M+1/2,N} + (1+2d_2)u^{n+1}_{M+1/2,N+1} - d_2 u^{n+1}_{M+1/2,N+2} = d_1 u^{n+1/2}_{M+3/2,N+1} &+ (1-2d_1)u^{n+1/2}_{M+1/2,N+1} \\ &+ d_1 u^{n+1/2}_{M-1/2,N+1}\end{aligned}$$

By the boundary conditions, $u^{n+1}_{M+1/2,N+2} = -u^{n+1}_{M+1/2,N+1}$ and $u^{n+1/2}_{M+3/2,N+1} = 0$ since it corresponds to the right wall. The previous equation yields

$$-d_2 u^{n+1}_{M+1/2,N} + (1+3d_2)u^{n+1}_{M+1/2,N+1} = (1-2d_1)u^{n+1/2}_{M+1/2,N+1} + d_1 u^{n+1/2}_{M-1/2,N+1}.$$

Thus, for this case $c = 0$, $b = 1 + 3d_2$ and d is given by the right hand side.

3 Development for v(x,y,t)

3.1 Step 1

Let's start by rewriting our ADI equation for the Step 1 (given in class 13 slide 13), for an staggered mesh for the vertical velocity v:

$$-d_1 v^{n+1/2}_{i-1,j+1/2} + (1+2d_1)v^{n+1/2}_{i,j+1/2} - d_1 v^{n+1/2}_{i+1,j+1/2} = d_2 v^n_{i,j+3/2} + (1-2d_2)v^n_{i,j+1/2} + d_2 v^n_{i,j-1/2},$$

which has the form

$$a v^n_{i-1,j+1/2} + b v^n_{i,j+1/2} + c v^n_{i+1,j+1/2} = d,$$

a tridiagonal system. The staggered mesh for v gives us a matrix v that is $(M+2)\times(N+1)$. We only need to solve the previous equation for the interior with two for loops in *Matlab*,

```
for j=2:N
for i=2:M+1
...
end
end
```

We will now include the boundary conditions for each case.

Case $j = 2$ **and** $i = 2$

In this case the general equation becomes

$$-d_1 v_{1,5/2}^{n+1/2} + (1+2d_1) v_{2,5/2}^{n+1/2} - d_1 v_{3,5/2}^{n+1/2} = d_2 v_{2,7/2}^{n} + (1-2d_2) v_{2,5/2}^{n} + d_2 v_{2,3/2}^{n},$$

The value $v_{1,5/2}^{n+1/2} = -v_{2,5/2}^{n+1/2}$ by the boundary conditions and $v_{2,3/2}^{n} = 0$ since it corresponds to the bottom wall. Thus the equation yields

$$(1+3d_1) v_{2,5/2}^{n+1/2} - d_1 v_{3,5/2}^{n+1/2} = d_2 v_{2,7/2}^{n} + (1-2d_2) v_{2,5/2}^{n},$$

giving $a = 0$ and $b = 1 + 3d_1$ for this case, d is given by the right hand side.

Case $j = 2$ **and** $i \in [3, M]$

In this case the general equation becomes

$$-d_1 v_{i-1,5/2}^{n+1/2} + (1+2d_1) v_{i,5/2}^{n+1/2} - d_1 v_{i+1,5/2}^{n+1/2} = d_2 v_{i,7/2}^{n} + (1-2d_2) v_{i,5/2}^{n} + d_2 v_{i,3/2}^{n},$$

Like in the previous case, the value $v_{i,3/2}^{n} = 0$ and d is the right hand side of the previous equation.

Case $j = 2$ **and** $i = M + 1$

In this case the general equation becomes

$$-d_1 v_{M,5/2}^{n+1/2} + (1+2d_1) v_{M+1,5/2}^{n+1/2} - d_1 v_{M+2,5/2}^{n+1/2} = d_2 v_{M+1,7/2}^{n} + (1-2d_2) v_{M+1,5/2}^{n} + d_2 v_{M+1,3/2}^{n},$$

Like in the previous case, the value $v_{M+1,3/2}^{n} = 0$. The value of $v_{M+2,5/2}^{n+1/2} = -v_{M+1,5/2}^{n+1/2}$ and the previous equation yields

$$-d_1 v_{M,5/2}^{n+1/2} + (1+3d_1) v_{M+1,5/2}^{n+1/2} = d_2 v_{M+1,7/2}^{n} + (1-2d_2) v_{M+1,5/2}^{n}.$$

Thus, $c = 0$, $b = 1 + 3d_1$ and d is the right hand side of the previous equation.

Case $j \in [3, N-1]$ **and** $i = 2$

In this case the general equation becomes

$$-d_1 v_{1,j+1/2}^{n+1/2} + (1+2d_1) v_{2,j+1/2}^{n+1/2} - d_1 v_{3,j+1/2}^{n+1/2} = d_2 v_{2,j+3/2}^{n} + (1-2d_2) v_{2,j+1/2}^{n} + d_2 v_{2,j-1/2}^{n}.$$

The value $v_{1,j+1/2}^{n+1/2} = -v_{2,j+1/2}^{n+1/2}$ by the boundary condition. Thus the equation yields

$$(1+3d_1) v_{2,j+1/2}^{n+1/2} - d_1 v_{3,j+1/2}^{n+1/2} = d_2 v_{2,j+3/2}^{n} + (1-2d_2) v_{2,j+1/2}^{n} + d_2 v_{2,j-1/2}^{n},$$

giving $a = 0$, $b = 1 + 3d_1$ and d given by the right hand side.

Case $j \in [3, N-1]$ and $i \in [3, M]$

In this case the general equation does not include any boundary conditions, therefore it is unaltered.

Case $j \in [3, N-1]$ and $i = M+1$

In this case the general equation becomes

$$-d_1 v^{n+1/2}_{M,j+1/2} + (1+2d_1) v^{n+1/2}_{M+1,j+1/2} - d_1 v^{n+1/2}_{M+2,j+1/2} = d_2 v^n_{M+1,j+3/2} + (1-2d_2) v^n_{M+1,j+1/2} + d_2 v^n_{M+1,j-1/2}.$$

The value $v^{n+1/2}_{M+2,j+1/2} = -v^{n+1/2}_{M+1,j+1/2}$ by the boundary condition. Thus the equation yields

$$-d_1 v^{n+1/2}_{M,j+1/2} + (1+3d_1) v^{n+1/2}_{M+1,j+1/2} = d_2 v^n_{M+1,j+3/2} + (1-2d_2) v^n_{M+1,j+1/2} + d_2 v^n_{M+1,j-1/2},$$

giving $c = 0$, $b = 1 + 3d_1$ and d given by the right hand side.

Case $j = N$ and $i = 2$

In this case the general equation becomes

$$-d_1 v^{n+1/2}_{1,N+1/2} + (1+2d_1) v^{n+1/2}_{2,N+1/2} - d_1 v^{n+1/2}_{3,N+1/2} = d_2 v^n_{2,N+3/2} + (1-2d_2) v^n_{2,N+1/2} + d_2 v^n_{2,N-1/2},$$

The value $v^{n+1/2}_{1,N+1/2} = -v^{n+1/2}_{2,N+1/2}$ by the boundary conditions and $v^n_{2,N+3/2} = 0$ since it corresponds to the top wall. Thus the equation yields

$$(1+3d_1) v^{n+1/2}_{2,N+1/2} - d_1 v^{n+1/2}_{3,N+1/2} = (1-2d_2) v^n_{2,N+1/2} + d_2 v^n_{2,N-1/2},$$

giving $a = 0$ and $b = 1 + 3d_1$ for this case, d is given by the right hand side.

Case $j = N$ and $i \in [3, M]$

In this case the general equation becomes

$$-d_1 v^{n+1/2}_{i-1,N+1/2} + (1+2d_1) v^{n+1/2}_{i,N+1/2} - d_1 v^{n+1/2}_{i+1,N+1/2} = d_2 v^n_{i,N+3/2} + (1-2d_2) v^n_{i,N+1/2} + d_2 v^n_{i,N-1/2},$$

Like in the previous case, the value $v^n_{i,N+3/2} = 0$ and d is the right hand side of the previous equation.

Case $j = N$ and $i = M+1$

In this case the general equation becomes

$$-d_1 v^{n+1/2}_{M,N+1/2} + (1+2d_1) v^{n+1/2}_{M+1,N+1/2} - d_1 v^{n+1/2}_{M+2,N+1/2} = d_2 v^n_{M+1,N+3/2} + (1-2d_2) v^n_{M+1,N+1/2} + d_2 v^n_{M+1,N-1/2}.$$

By the boundary conditions, $v^{n+1/2}_{M+2,N+1/2} = -v^{n+1/2}_{M+1,N+1/2}$ and $v^n_{M+1,N+3/2} = 0$ and the previous equation yields

$$-d_1 v^{n+1/2}_{M,N+1/2} + (1+3d_1) v^{n+1/2}_{M+1,N+1/2} = (1-2d_2) v^n_{M+1,N+1/2} + d_2 v^n_{M+1,N-1/2},$$

Thus, $c = 0$, $b = 1 + 3d_1$ and d is the right hand side of the previous equation.

3.2 Step 2

We start by rewriting our ADI equation for the Step 1 (given in class 13 slide 14), for an staggered mesh for the vertical velocity v:

$$-d_2 v^{n+1}_{i,j-1/2} + (1+2d_2) v^{n+1}_{i,j+1/2} - d_2 v^{n+1}_{i,j+3/2} = d_1 v^{n+1/2}_{i+1,j+1/2} + (1-2d_1) v^{n+1/2}_{i,j+1/2} + d_1 v^{n+1/2}_{i-1,j+1/2}$$

which has the form

$$a v^{n+1}_{i,j-1/2} + b v^{n+1}_{i,j+1/2} + c v^{n+1}_{i,j+3/2} = d,$$

a tridiagonal system. The staggered mesh for v gives vs a matrix v that is $(M+2)\times(N+1)$. We only need to solve the previous equation for the interior with two *for* loops in *Matlab*,

```
for j=2:M+1
for i=2:N
...
end
end
```

We will now include the boundary conditions for each case.

Case $i=2$ and $j=2$

In this case the general equation becomes

$$-d_2 v^{n+1}_{2,3/2} + (1+2d_2) v^{n+1}_{2,5/2} - d_2 v^{n+1}_{2,7/2} = d_1 v^{n+1/2}_{3,5/2} + (1-2d_1) v^{n+1/2}_{2,5/2} + d_1 v^{n+1/2}_{1,5/2}$$

By the boundary conditions, $v^{n+1}_{2,3/2} = 0$ and $v^{n+1/2}_{1,5/2} = -v^{n+1/2}_{2,5/2}$. Thus $a=0$ and d is given by the right hand side.

Case $i=2$ and $j\in[3,N-1]$

In this case the general equation becomes

$$-d_2 v^{n+1}_{2,j-1/2} + (1+2d_2) v^{n+1}_{2,j+1/2} - d_2 v^{n+1}_{2,j+3/2} = d_1 v^{n+1/2}_{3,j+1/2} + (1-2d_1) v^{n+1/2}_{2,j+1/2} + d_1 v^{n+1/2}_{1,j+1/2}$$

By the boundary conditions, $v^{n+1/2}_{1,j+1/2} = -v^{n+1/2}_{2,j+1/2}$ and d is given by the right hand side.

Case $i=2$ and $j=N$

In this case the general equation becomes

$$-d_2 v^{n+1}_{2,N-1/2} + (1+2d_2) v^{n+1}_{2,N+1/2} - d_2 v^{n+1}_{2,N+3/2} = d_1 v^{n+1/2}_{3,N+1/2} + (1-2d_1) v^{n+1/2}_{2,N+1/2} + d_1 v^{n+1/2}_{1,N+1/2}$$

By the boundary conditions, $v^{n+1}_{2,N+3/2} = 0$ and $v^{n+1/2}_{1,N+1/2} = -v^{n+1/2}_{2,N+1/2}$. Thus $c=0$ and d is given by the right hand side.

Case $i \in [3, M]$ and $j = 2$

In this case the general equation becomes

$$-d_2 v_{i,3/2}^{n+1} + (1 + 2d_2) v_{i,5/2}^{n+1} - d_2 v_{i,7/2}^{n+1} = d_1 v_{i+1,5/2}^{n+1/2} + (1 - 2d_1) v_{i,5/2}^{n+1/2} + d_1 v_{i-1,5/2}^{n+1/2}$$

By the boundary conditions, $v_{i,3/2}^{n+1}$ will be zero at the walls and will depend on $v_{i,5/2}^{n+1}$ and $v_{i,7/2}^{n+1}$ for the points at the outlet. Thus, $a = 0$ and the values of b and c depend on i, d is given by the right hand side.

Case $i \in [3, M]$ and $j \in [3, N - 1]$

In this case the general equation does not include any boundary conditions, therefore it is unaltered.

Case $i \in [3, M]$ and $j = N$

In this case the general equation becomes

$$-d_2 v_{i,N-1/2}^{n+1} + (1 + 2d_2) v_{i,N+1/2}^{n+1} - d_2 v_{i,N+3/2}^{n+1} = d_1 v_{i+1,N+1/2}^{n+1/2} + (1 - 2d_1) v_{i,N+1/2}^{n+1/2} + d_1 v_{i-1,N+1/2}^{n+1/2}$$

By the boundary conditions, the value of $v_{i,N+3/2}^{n+1}$ will depend on i. We will pass it to the right hand side and have

$$-d_2 v_{i,N-1/2}^{n+1} + (1 + 2d_2) v_{i,N+1/2}^{n+1} = d_1 v_{i+1,N+1/2}^{n+1/2} + (1 - 2d_1) v_{i,N+1/2}^{n+1/2} + d_1 v_{i-1,N+1/2}^{n+1/2} + d_2 v_{i,N+3/2}^{n+1}.$$

Thus, $c = 0$ and d is given by the right hand side.

Case $i = M + 1$ and $j = 2$

In this case the general equation becomes

$$-d_2 v_{M+1,3/2}^{n+1} + (1 + 2d_2) v_{M+1,5/2}^{n+1} - d_2 v_{M+1,7/2}^{n+1} = d_1 v_{M+2,5/2}^{n+1/2} + (1 - 2d_1) v_{M+1,5/2}^{n+1/2} + d_1 v_{M,5/2}^{n+1/2}$$

By the boundary conditions, $v_{M+1,3/2}^{n+1} = 0$ and $v_{M+2,5/2}^{n+1/2} = -v_{M+1,5/2}^{n+1/2}$. Thus, $a = 0$ and d is given by the right hand side.

Case $i = M + 1$ and $j \in [3, N - 1]$

In this case the general equation becomes

$$\begin{aligned}-d_2 v_{M+1,j-1/2}^{n+1} + (1 + 2d_2) v_{M+1,j+1/2}^{n+1} - d_2 v_{M+1,j+3/2}^{n+1} = d_1 v_{M+2,j+1/2}^{n+1/2} &+ (1 - 2d_1) v_{M+1,j+1/2}^{n+1/2} \\ &+ d_1 v_{M,j+1/2}^{n+1/2}\end{aligned}$$

By the boundary conditions, $v_{M+2,j+1/2}^{n+1/2} = -v_{M+1,j+1/2}^{n+1/2}$ and d is given by the right hand side.

Case $i = M+1$ and $j = N$

In this case the general equation becomes

$$-d_2 v^{n+1}_{M+1,N-1/2} + (1+2d_2) v^{n+1}_{M+1,N+1/2} - d_2 v^{n+1}_{M+1,N+3/2} = d_1 v^{n+1/2}_{M+2,N+1/2} + (1-2d_1) v^{n+1/2}_{M+1,N+1/2} + d_1 v^{n+1/2}_{M,N+1/2}$$

By the boundary conditions, $v^{n+1}_{M+1,N+3/2} = 0$ and $v^{n+1/2}_{M+2,N+1/2} = -v^{n+1/2}_{M+1,N+1/2}$. Thus, $c = 0$ and d is given by the right hand side.

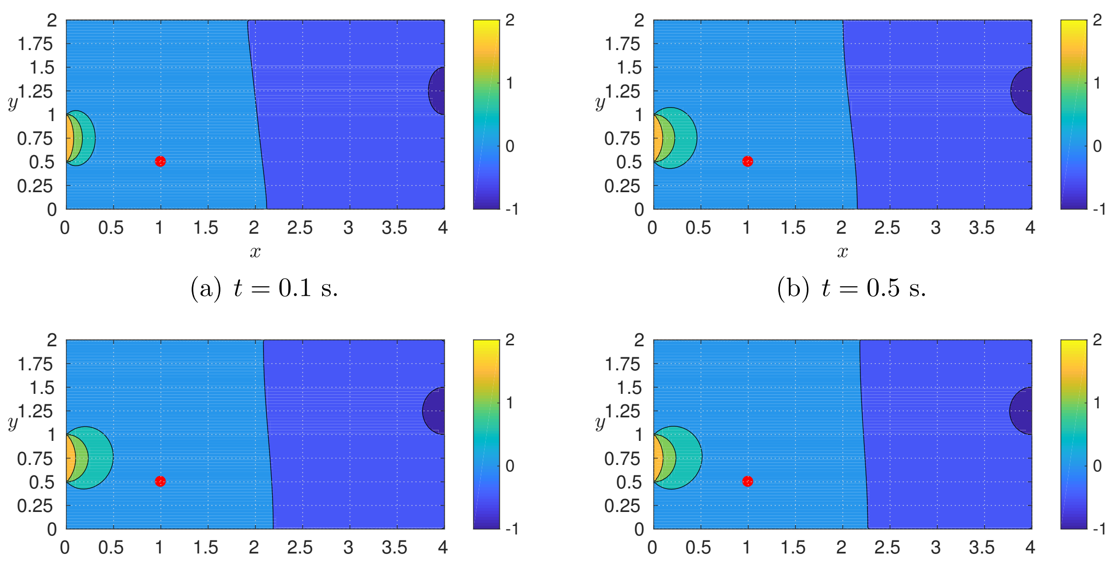
4 Development for Y(x,y,t)

4.1 Step 1

4.2 Step 2

5 Results

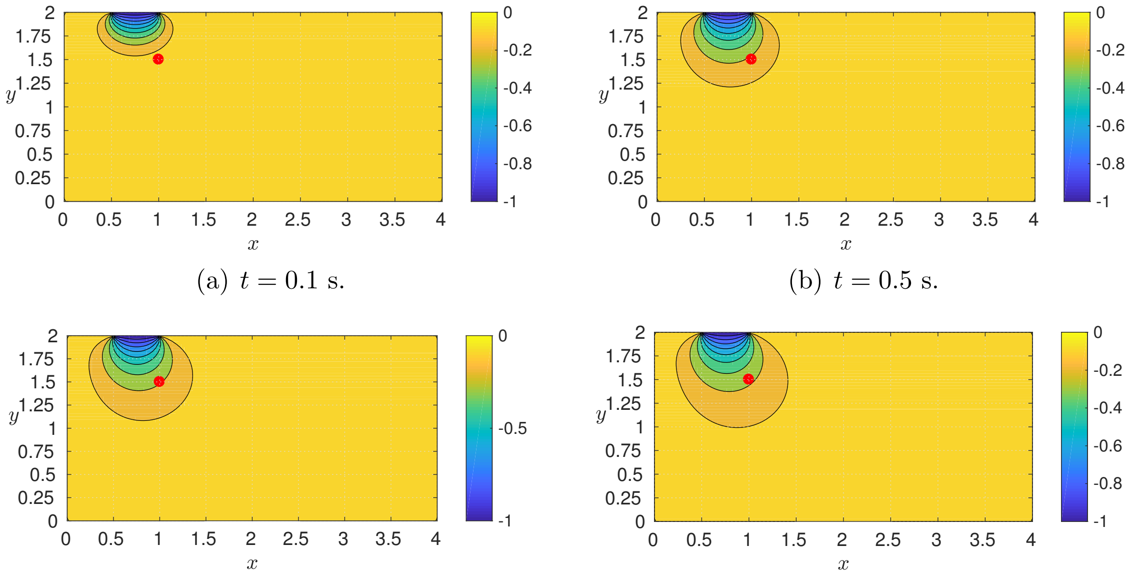
In the following figure we can see the horizontal velocity contours at different values of time. The red dot represents the probe located to meassure such quantity. As we can see we have inputs through inlets 1 (positive u) and 2 (negative u). We can see how the fluid in the domain takes seconds to start moving.



(a) $t = 0.1$ s. (b) $t = 0.5$ s. (c) $t = 1$ s. (d) $t = 10$ s.

Figure 1: Horizontal velocity contours.

In the next figure we can see the vertical velocity contours at different values of time. As we can see we only introduce fluid vertically through the inlet 3 and, like before, it takes seconds for the fluid to start moving.



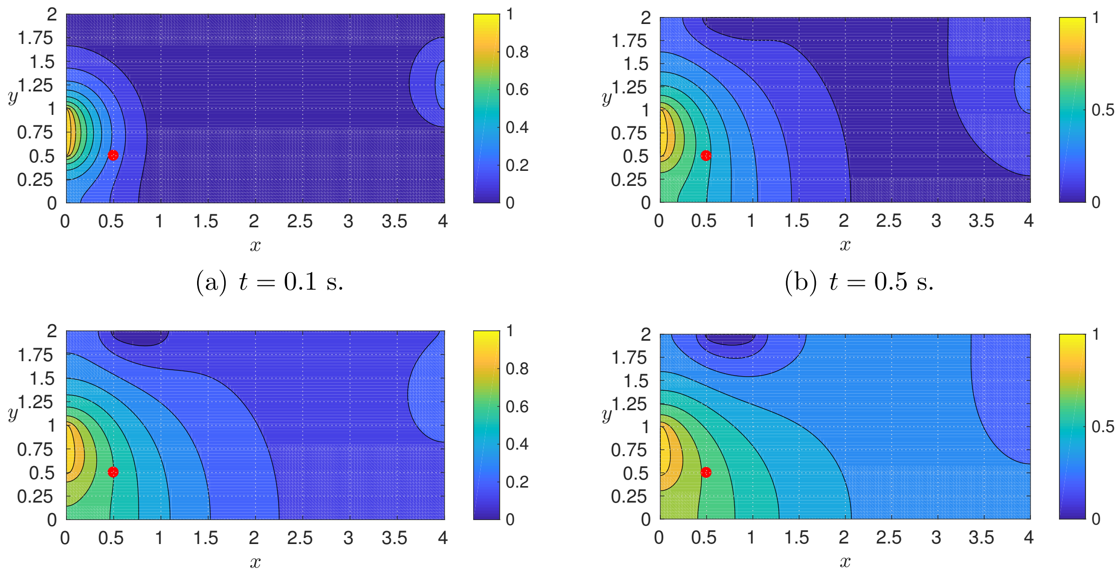
(a) $t = 0.1$ s. (b) $t = 0.5$ s. (c) $t = 1$ s. (d) $t = 10$ s.

Figure 2: Vertical velocity contours.

In the next figure we can see the mass fraction contours at different values of time. We can appreciate the Neumann boundary conditions as the lines are perpendicular to the boundary. We can also see in the inlets 1, 2 and 3 the fixed values due to the Dirichlet boundary conditions. Note how at $t = 0.1$ the majority of the domain is with $Y = 0$ and with time the fluid mixes and we have at time 10 a more clear contour indicating higher values of Y.

In the following figure we can see the probes meassurements with time during the first 2 seconds. As commented above we can see how for both u and v the fluid takes a few seconds to start moving. The further from the source the longer the time, as it is logical. We can also see how Y increases very fast as the probe is located very close to inlet 1, see contour.

To finish with the results, we proceed with the GCI analysis. In the following table we have the data needed to perform it obtained by doing simulations with different meshes.



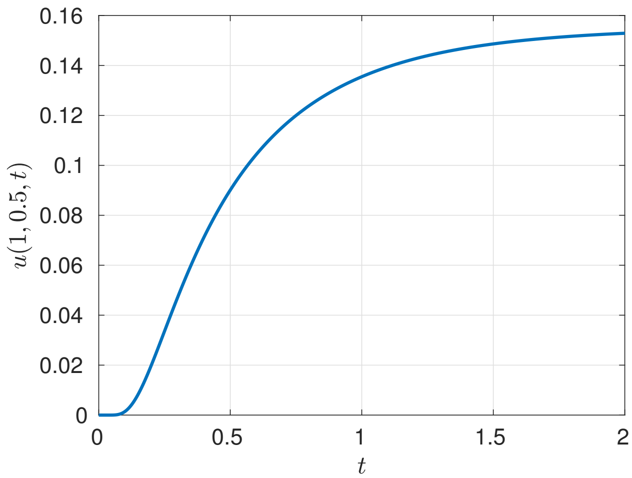
(a) $t = 0.1$ s.

(b) $t = 0.5$ s.

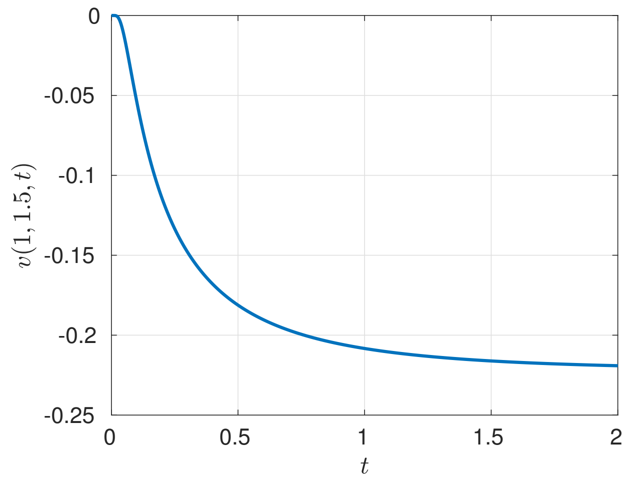
(c) $t = 1$ s.

(d) $t = 10$ s.

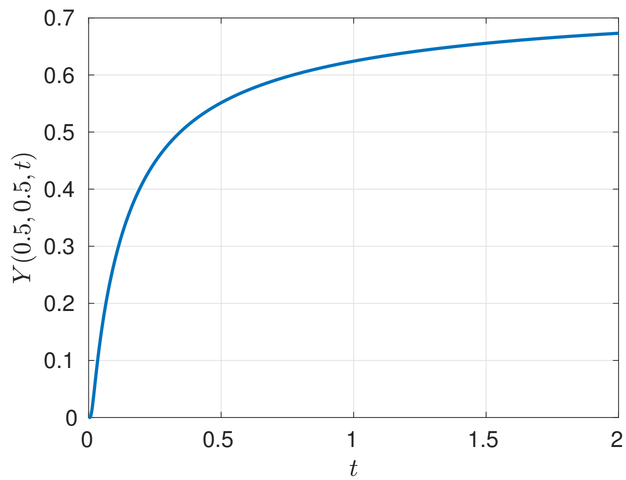
Figure 3: Mass fraction contours.



(a) Probe 1.



(b) Probe 2.



(c) Probe 3.

Figure 4: Meassures of the probes.

Grid	M	N	$u(1, 0.5, 1)$	$v(1, 1.5, 1)$	$Y(0.5, 0.5, 1)$
1	128	64	0.135485658133256	−0.208324448730563	0.624208222506483
2	64	32	0.135620186703791	−0.208302837119943	0.619301711442877
3	32	16	0.136145435462035	−0.208215577613997	0.608917592564123

Table 1: GCI analysis data.

Taking the data from u, we can calculate an order of convergence $p = 1.965088248178588$, close to the theoretical value two. Using Richardson extrapolation with the two finest grids we estimate the solution at $h = 0$,

$$u_{h=0} = 0.135439338703496.$$

We obtain the following GCI values

$$GCI_{21} = 4.273462445956238e-04, \qquad GCI_{32} = 0.001666861002506,$$

which give us the following value

$$\frac{GCI_{21}}{GCI_{32}}r^p = 1.000992935875199,$$

where $r = 2$. The previous value tells us that we are in the asymptotic range of convergence. Thus, we can say that the value meassured by the probe is

$$u(1, 0.5, 1) = 0.135439338703496 \pm 0.042734624459562\text{38}\%$$

Now doing the same for v, we can calculate an order of convergence $p = 2.013505711337656$, close to the theoretical value two. Using Richardson extrapolation with the two finest grids we estimate the solution at $h = 0$,

$$v_{h=0} = -0.208331563379270.$$

We obtain the following GCI values

$$GCI_{12} = 4.268971278965161e-05, \qquad GCI_{23} = 1.723827896664248e-04,$$

which give us the following value

$$\frac{GCI_{12}}{GCI_{23}}r^p = 0.999896259844913.$$

The previous value tells us that we are in the asymptotic range of convergence. Thus, we can say that the value meassured by the probe is

$$v(1, 1.5, 1) = -0.208331563379270 \pm 0.004268971278965\%$$

Lastly, doing the same for Y, we can calculate an order of convergence $p = 1.081609386340197$, which is not close to the theoretical value two. Using Richardson extrapolation with the two finest grids we estimate the solution at $h = 0$,

$$Y_{h=0} = 0.628603179507733.$$

We obtain the following GCI values

$$GCI_{12} = 0.008801063576995, \qquad GCI_{23} = 0.018774104553952,$$

which give us the following value

$$\frac{GCI_{12}}{GCI_{23}}r^p = 0.992139624428682.$$

The previous value tells us that we are in the asymptotic range of convergence. Thus, we can say that the value meassured by the probe is

$$Y(0.5, 0.5, 1) = 0.628603179507733 \pm 0.880106357699521\%$$