Solve PDE: $\frac{\partial \phi}{\partial t} + a(x,t)\frac{\partial \phi}{\partial x} = 0$

~~Discretization~~

Index form: $\left.\frac{\partial \phi^n}{\partial t}\right|_i = -a_i^n \left.\frac{\partial \phi^n}{\partial x}\right|_i$

For the time derivative: TVD-RK3 method.

$$\phi_i^{(1)} = \phi_i^n - \alpha_{1,0}\left(a_i^n \Delta t \left.\frac{\partial \phi^n}{\partial x}\right|_i\right)$$

$$\phi_i^{(2)} = \phi_i^{(1)} - \alpha_{2,0}\left(a_i^n \Delta t \left.\frac{\partial \phi^n}{\partial x}\right|_i\right) - \alpha_{2,1}\left(a_i^n \Delta t \left.\frac{\partial \phi^{(1)}}{\partial x}\right|_i\right)$$

$$\phi_i^{n+1} = \phi_i^{(2)} - \alpha_{3,0}\left(a_i^n \Delta t \left.\frac{\partial \phi^n}{\partial x}\right|_i\right) - \alpha_{3,1}\left(a_i^n \Delta t \left.\frac{\partial \phi^{(1)}}{\partial x}\right|_i\right) - \alpha_{3,2}\left(a_i^n \Delta t \left.\frac{\partial \phi^{(2)}}{\partial x}\right|_i\right)$$

with $\alpha_{1,0} = 1$

$\alpha_{2,0} = \frac{3}{4}$; $\alpha_{2,1} = \frac{1}{4}$

$\alpha_{3,0} = -\frac{1}{12}$; $\alpha_{3,1} = -\frac{1}{12}$; $\alpha_{3,2} = \frac{3}{2}$

To do this we need the space derivative, obtained with the WENO-5 method.

For the space derivative: WENO-5 method.

- For $a_i^n > 0$

$$\left.\frac{\partial \phi^n}{\partial x}\right|_i^- = \frac{1}{12\Delta x}\left(-\Delta^+\phi^n_{i-2} + 7\Delta^+\phi^n_{i-1} + 7\Delta^+\phi^n_i - \Delta^+\phi^n_{i+1}\right)$$

$$- \Psi_{WENO}\left(\frac{\Delta^-\Delta^+\phi^n_{i-2}}{\Delta x}, \frac{\Delta^-\Delta^+\phi^n_{i-1}}{\Delta x}, \frac{\Delta^-\Delta^+\phi^n_i}{\Delta x}, \frac{\Delta^-\Delta^+\phi^n_{i+1}}{\Delta x}\right)$$

Nothe that:

$$-\Delta^+\phi_{i-2} + 7\Delta^+\phi_{i-1} + 7\Delta^+\phi_i - \Delta^+\phi_{i+1} =$$

$$= -\phi_{i-1} + \phi_{i-2} + \cancel{7\phi_i} - 7\phi_{i-1} + 7\phi_{i+1} - \cancel{7\phi_i} - \phi_{i+2} + \phi_{i+1}$$

$$= \phi_{i-2} - 8\phi_{i-1} + 8\phi_{i+1} - \phi_{i+2}.$$

Further,

$$\frac{\Delta^-\Delta^+\phi_i}{\Delta x} = \frac{\Delta^-(\phi_{i+1} - \phi_i)}{\Delta x} = \frac{\Delta^-\phi_{i+1} - \Delta^-\phi_i}{\Delta x} = \frac{\phi_{i+1} - \phi_i - \phi_i + \phi_{i-1}}{\Delta x}$$

$$= \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x}$$

Hence,

$$\frac{\Delta^-\Delta^+\phi_{i-2}}{\Delta x} = \frac{\phi_{i-1} - 2\phi_{i-2} + \phi_{i-3}}{\Delta x},$$

$$\frac{\Delta^-\Delta^+\phi_{i-1}}{\Delta x} = \frac{\phi_i - 2\phi_{i-1} + \phi_{i-2}}{\Delta x},$$

$$\frac{\Delta^-\Delta^+\phi_{i+1}}{\Delta x} = \frac{\phi_{i+2} - 2\phi_{i+1} + \phi_i}{\Delta x}.$$

Thus,

$$\left.\frac{\partial \phi^n}{\partial x}\right|_i^- = \frac{1}{12\Delta x}\left(\phi^n_{i-2} - 8\phi^n_{i-1} + 8\phi^n_{i+1} - \phi^n_{i+2}\right)$$
$$- \Psi_{WENO}\left(\frac{\phi^n_{i-1} - 2\phi^n_{i-2} + \phi^n_{i-3}}{\Delta x}, \frac{\phi^n_i - 2\phi^n_{i-1} + \phi^n_{i-2}}{\Delta x}, \frac{\phi^n_{i+1} - 2\phi^n_i + \phi^n_{i-1}}{\Delta x}, \frac{\phi^n_{i+2} - 2\phi^n_{i+1} + \phi^n_i}{\Delta x}\right)$$

- <u>For $a_i^n < 0$</u>

$$\left.\frac{\partial \phi^n}{\partial x}\right|_i^+ = \frac{1}{12\Delta x}\left(-\Delta^+\phi_{i-2} + 7\Delta^+\phi_{i-1} + 7\Delta^+\phi_i - \Delta^+\phi_{i+1}\right)$$
$$+ \Psi_{WENO}\left(\frac{\Delta^-\Delta^+\phi_{i+2}}{\Delta x}, \frac{\Delta^-\Delta^+\phi_{i+1}}{\Delta x}, \frac{\Delta^-\Delta^+\phi_i}{\Delta x}, \frac{\Delta^-\Delta^+\phi_{i-1}}{\Delta x}\right)$$

Mirroing the results from $a_i^n \geq 0$,

$$\left.\frac{\partial \phi^n}{\partial x}\right|_i^+ = \frac{1}{12\Delta x}\left(\phi_{i-2} - 8\phi^n_{i-1} + 8\phi^n_{i+1} - \phi^n_{i+2}\right)$$
$$+ \Psi_{WENO}\left(\frac{\phi^n_{i+3} - 2\phi^n_{i+2} + \phi^n_{i+1}}{\Delta x}, \frac{\phi^n_{i+2} - 2\phi^n_{i+1} + \phi^n_i}{\Delta x}, \frac{\phi^n_{i+1} - 2\phi^n_i + \phi^n_{i-1}}{\Delta x}, \frac{\phi^n_i - 2\phi^n_{i-1} + \phi^n_{i-2}}{\Delta x}\right)$$

After every step of the RK method, we need to update the ghost cells using the BCs.

- Left ghost cells: Dirichlet

MATLAB Indices

$$\cancel{\phi(3) = 2\phi^n_{BC}} \quad \phi^n_3 = 2\phi^n_{BC} - \phi^n_4$$

$$\phi^n_2 = 2\phi^n_{BC} - \phi^n_5$$

$$\phi^n_1 = 2\phi^n_{BC} - \phi^n_6$$

where $\phi^n_{BC} = \phi_{BC}(t_n) = \phi(x=-1, t_n)$ given in the problem.

- Right ghost cells: zero Neumann

$$\phi^n_{M+4} = \phi^n_{M+3}$$

$$\phi^n_{M+5} = \phi^n_{M+2}$$

$$\phi^n_{M+6} = \phi^n_{M+1}.$$

Time step.

Stable time step for 1st order upwind:

$$\Delta t \le \frac{\Delta x}{a}$$

For our method we include the CFL factor and, since a depends on x ~~and~~ for each value of time,

$$\Delta t \le CFL \frac{\Delta x}{\max_{1\le i \le M+6} |a^n_i|}$$