Numerical Methods for PDEs
Homework 2

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Problem 1

1. Prove that the forward Euler method for the heat equation $u_t = u_{xx}$

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

is (a) first-order accurate

Solution:
To find the local truncation error we Taylor expand

$$u(t+\Delta t) = u + \frac{\Delta t}{\Delta x^2}\left[u(x+\Delta x, t) - 2u + u(x-\Delta x, t)\right] + \tau\Delta t,$$

which gives,

$$u + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \cdots = u + \frac{\Delta t}{\Delta x^2}\left[\left(u + \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + \cdots\right) - 2u + \left(u - \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} + \cdots\right)\right] + \tau\Delta t.$$

Cancelling terms, it is easy to obtain

$$\tau = \frac{\Delta t}{2}u_{tt} \Rightarrow \tau = \mathcal{O}(\Delta t)\ .$$

Hence, the given method is first order accurate.

2. (b) conditionally stable. What is the restriction on Δt?

Solution:
Our first step will be obtain the growth function $g(k)$. Let $u_i^n = e^{ikx_i}$ and $u_i^{n+1} = g(k)e^{ikx_i}$, then

$$\begin{aligned} g(k)e^{ikx_i} &= e^{ikx_i} + \frac{\Delta t}{\Delta x^2}\left(e^{ikx_{i+1}} - 2e^{ikx_i} + e^{ikx_{i-1}}\right) \\ \Rightarrow g(k) &= 1 - 2\frac{2\Delta t}{\Delta x^2}\left(1 - \cos(k\Delta x)\right)\ , \end{aligned}$$

where we hav simply cancelled the exponential e^{ikx_i} since it is common factor in the equation, and done some algbraic operations. For the method to be stable we need to impose the condition

$\|g(k)\| \leq 1$, which lead us to

$$\left|1 - 2\frac{2\Delta t}{\Delta x^2}\left(1-\cos(k\Delta x)\right)\right| \leq 1.$$

This inequality is satisfied provided that

$$\Delta t \leq \frac{\Delta x^2}{2} \ .$$

Thus, Forward Euler is stable as long as $\Delta t \leq \frac{\Delta x^2}{2}$.

Problem 2

1. Prove that the TR method is A-stable but not L-stable for the heat equation $u_t = u_{xx}$.

Solution:

Like in the first problem, we start by obtaining $g(k)$. Using the same definitions for $u^n_{j\pm1}$ and $u^{n+1}_{j\pm1}$ in terms of complex exponentials and the growth factor, and cancelling the common factor we find

$$g(k) - g(k)\frac{\Delta t}{2\Delta x^2}\left(2\cos(k\Delta x) - 2\right) = 1 + \frac{\Delta t}{2\Delta x^2}\left(2\cos(k\Delta x)-2\right),$$

which give us

$$g(k) = \frac{1 - \frac{\Delta t}{\Delta x^2}\left(1-\cos(k\Delta x)\right)}{1 + \frac{\Delta t}{\Delta x^2}\left(1-\cos(k\Delta x)\right)} \ .$$

It is trivial that $|g(k)| \leq 1$ for all $\Delta t > 0$, proving that the TR method is A-stable. Once we have proved that the method is A-stable, we take the following limit to check for L-stability,

$$\lim_{\Delta t\to\infty} |g(k)| = \lim_{\Delta t\to\infty}\left|\frac{1 - \frac{\Delta t}{\Delta x^2}\left(1-\cos(k\Delta x)\right)}{1 + \frac{\Delta t}{\Delta x^2}\left(1-\cos(k\Delta x)\right)}\right| = 1 \neq 0 \ .$$

Hence, the TR method is not L-stable.

Problem 3

1. Show that the second-order accurate central discretization

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^2}\left(\frac{1}{4}(D_{i+1} - D_{i-1})(u_{i+1} - u_{i-1}) + D_i(u_{i+1} - 2u_i + u_{i-1})\right)$$

of $u_t = D_x u_x + D u_{xx}$ is nonconservative. (You can use $Q^n = \sum_{i=0}^{N} u_i^n \Delta x$ and $N = 4$. Then show that the terms multiplying u_2 in $(Q^{n+1} - Q^n)/\Delta t$ do not cancel unless D is constant.)

Solution:
Using the denitions given by the problem, we set $N = 4$ and calculate, paying special attention at the terms multiplying u_2^n,

$$\begin{aligned}
\frac{Q^{n+1} - Q^n}{\Delta t} &= \frac{\Delta x}{\Delta t}\left[\sum_{i=0}^{4}\left(u_i^{n+1} - u_i^n\right)\right] \\
&= \frac{1}{\Delta x}\sum_{i=0}^{4}\left[(D_{i+1} - D_{i-1})\left(u_{i+1}^n - u_{i-1}^n\right) + D_i\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right] \\
&= \frac{1}{\Delta x} = \left[\cdots + (\frac{1}{4}D_2 - \frac{1}{4}D_0 + D_1 - 2D_2 - \frac{1}{4}D_4 + \frac{1}{4}D_2 + D_3)u_2^n + \cdots\right] .
\end{aligned}$$

Hence, the terms multiplying the interior point u_2 do not cancel unless D is constant. Therefore, this discretization is not nonconservative.

Problem 4

1. (a) Using the trapezoidal rule integration method to approximate $Q^n \approx \overline{Q}(t) = \int u(x,t)dx$, show that discretizations of $u_t + f_x = 0$ of the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)$$

are conservative.

Solution:
Using the trapezoidal rule we get the following

$$Q^n = \int u^n(x,t)dx = \sum_{j=0}^{N-1}\frac{1}{2}\left(u_{j+1}^n + u_j^n\right)\Delta x ,$$

and

$$\begin{aligned}
Q^{n+1} = \int u^{n+1}(x,t)dx &= \sum_{j=0}^{N-1}\frac{1}{2}\left(u_{j+1}^{n+1} + u_j^{n+1}\right)\Delta x \\
&= \sum_{j=0}^{N-1}\frac{1}{2}\left(u_{j+1}^n + u_j^n\right)\Delta x + \frac{1}{2}\Delta t\sum_{j=0}^{N-1}\left(F_{j-1/2} - F_{j+3/2}\right) \\
&= Q^n + \frac{1}{2}\Delta t\sum_{j=0}^{N-1}\left(F_{j-1/2} - F_{j+3/2}\right).
\end{aligned}$$

From here it is easy to find,

$$\frac{dQ}{dt} \approx \frac{Q^{n+1} - Q^n}{\Delta t} = \frac{1}{2} \sum_{j=0}^{N-1} \left(F_{j-1/2} - F_{j+3/2}\right)$$
$$= \frac{1}{2} \left(F_{-1/2} + F_{1/2} - F_{N-1/2} - F_{N+1/2}\right),$$

where we have used that the series is a telescoping series. We can define the flux at the two boundaries as

$$F_0 = \frac{1}{2} \left(F_{-1/2} + F_{1/2}\right),$$

and

$$F_N = \frac{1}{2} \left(F_{N-1/2} + F_{N+1/2}\right).$$

In the previous, the nodes outside of the grid are ghost nodes. Finally we have obtained

$$\frac{dQ}{dt} \approx F_0 - F_N,$$

which proves that the discretization is conservative since the change of the conserved quantity Q is the different between the inflow and the outflow.

2. (b) For nonlinear diffusion, $f(u) = -D(u)u_x$ and $F_{i+\frac{1}{2}} = f_{i+\frac{1}{2}}$. Show that if homogeneous Neumann boundary conditions $u_x(x_L, t) = 0 = u_x(x_R, t)$ are imposed via ghost points, Q is constant in time.

Solution:

If homogeneous boundary conditions are imposed, the quantity must be conserved because there is no flux through the boundaries. Formally, we have

$$F_0 = -D(u_0)u_x|_{x=x_0} = 0,$$

and

$$F_N = -D(u_N)u_x|_{x=x_N} = 0,$$

which makes

$$\frac{dQ}{dt} \approx 0 .$$

Hence, Q is constant in time.

Problem 5

1. Derive the Fourier solution to the initial/boundary value problem for the heat equation with homogeneous Neumann boundary conditions

$$u_t = u_{xx}, \quad u_x(0,t) = 0 = u_x(\pi,t), \quad u(x,t=0) = u_0(x)$$

by making a Fourier cosine expansion (since it automatically satisfies the boundary conditions)

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(nx)$$

and solving for $a_n(t)$.

Solution:

Having the Fourier cosine series as above, we apply the PDE to find the coefficient $a_n(t)$,

$$u_t = \sum_{n=0}^{\infty} a_n'(t)\cos(nx),$$

and

$$u_{xx} = -n^2 \sum_{n=0}^{\infty} a_n(t)\cos(nx).$$

Then, we obtain the following ODE,

$$a_n'(t) = -n^2 a_n(t),$$

which has the solution

$$a_n(t) = Ke^{-n^2 t}\ .$$

To find K we impose the initial condition

$$\begin{aligned} u(x,0) &= \sum_{n=0}^{\infty} a_n(0)\cos(nx) \\ &= K\sum_{n=0}^{\infty}\cos(nx) \\ &= u_0(x). \end{aligned}$$

Thus,

$$K = \frac{u_0(x)}{\sum_{n=0}^{\infty}\cos(nx)}.$$

Therefore, the solution $u(x,t)$ is

$$u(x,t) = u_o(x)\frac{\sum_{n=0}^{\infty} e^{-n^2 t}\cos(nx)}{\sum_{n=0}^{\infty}\cos(nx)}\ .$$

It is not necessary to check the boundary conditions since the Fourier cosine series automatically satisfies homogeneous Neumann boundary conditions and the initial condition has to satisfy them as well.