

# Fourier Analysis and Wavelets

## Homework 1

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### Problem 2

Verify that the function  $\langle, \rangle$  defined in Example 0.3 is an inner product.

**Solution:** Given the inner product on  $C^2$  defined by

$$\langle v, w \rangle = (\bar{w}_1 \quad \bar{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

it is easy to check the properties.

- Positivity:
- Conjugate symmetry:

$$\begin{aligned} \langle v, w \rangle &= (\bar{w}_1 \quad \bar{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (2\bar{w}_1 + i\bar{w}_2 \quad 3\bar{w}_2 - i\bar{w}_1) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= 2v_1\bar{w}_1 + iv_1\bar{w}_2 + 3v_2\bar{w}_2 - iv_2\bar{w}_1 \\ &= \overline{2\bar{v}_1w_1 - i\bar{v}_1w_2 + 3\bar{v}_2w_2 + i\bar{v}_2w_1} \\ &= \overline{(2\bar{v}_1 + i\bar{v}_2 \quad 3\bar{v}_2 - i\bar{v}_1) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}} \\ &= (\bar{v}_1 \quad \bar{v}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \overline{\langle w, v \rangle} \end{aligned}$$

- Homogeneity:

$$\begin{aligned} \langle cv, w \rangle &= (\bar{w}_1 \quad \bar{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix} \\ &= (\bar{w}_1 \quad \bar{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} c \\ &= \langle v, w \rangle c \\ &= c \langle v, w \rangle, \end{aligned}$$

where we have taken the complex scalar  $c$  out of the vector  $v$  since it is common in all its components.

- Linearity:

$$\begin{aligned}
\langle u + v, w \rangle &= (\overline{w}_1 \quad \overline{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \\
&= (\overline{w}_1 \quad \overline{w}_2) (2(u_1 + v_1) - i(u_2 + v_2) \quad i(u_1 + v_1) + 3(u_2 + v_2)) \\
&= (\overline{w}_1 \quad \overline{w}_2) (2u_1 - iu_2 + 2v_1 - iv_2 \quad iu_1 + 3u_2 + iv_1 + 3v_2) \\
&= (\overline{w}_1 \quad \overline{w}_2) [(2u_1 - iu_2 \quad iu_1 + 3u_2) + (2v_1 - iv_2 \quad iv_1 + 3v_2)] \\
&= (\overline{w}_1 \quad \overline{w}_2) \left[ \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right] \\
&= (\overline{w}_1 \quad \overline{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (\overline{w}_1 \quad \overline{w}_2) \begin{pmatrix} 2 & -i \\ i & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \langle u, w \rangle + \langle v, w \rangle
\end{aligned}$$

## Problem 7

For  $n > 0$ , let

$$f_n(t) = \begin{cases} \sqrt{n}, & 0 \leq t \leq 1/n^2 \\ 0, & \text{otherwise} \end{cases}$$

Show that  $f_n \rightarrow 0$  in  $L^2[0, 1]$  but that  $f_n(0)$  does not converge to zero.

**Solution:** For the first proof we need to prove that

$$\|f_n(t) - 0\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then,

$$\begin{aligned} \|f_n(t) - 0\| &= \sqrt{\langle f_n - 0, f_n - 0 \rangle_{L^2}} \\ &= \sqrt{\int_0^1 [f_n(t) - 0]^2 dt} \\ &= \sqrt{\int_0^1 [f_n(t)]^2 dt} \\ &= \sqrt{\int_0^{1/n^2} n dt + \int_{1/n^2}^1 0 dt} \\ &= \sqrt{\frac{1}{2} n \frac{1}{n^2}} \\ &= \sqrt{\frac{1}{2n}}. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|f_n(t) - 0\| = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{2n}} = 0.$$

However,

$$\lim_{n \rightarrow \infty} f_n(0) = \lim_{n \rightarrow \infty} \sqrt{n} = \infty \neq 0.$$

## Problem 11

Show that if a differentiable function ,  $f$ , is orthogonal to  $\cos(t)$  on  $L^2[0, \pi]$  then  $f'$  is orthogonal to  $\sin(t)$  on  $L^2[0, \pi]$ .

**Solution:** Since  $f$  is orthogonal to  $\cos(t)$  we know that

$$\langle f, \cos(t) \rangle = \int_0^\pi f(t) \cos(t) dt = 0.$$

Then,

$$\begin{aligned} \langle f', \sin(t) \rangle &= \int_0^\pi f'(t) \sin(t) dt \\ &= \int_0^\pi \left( \frac{d}{dt} [f(t) \sin(t)] - f(t) \cos(t) \right) dt \\ &= \int_0^\pi d[f(t) \sin(t)] - \int_0^\pi f(t) \cos(t) dt \\ &= \underbrace{f(t) \sin(t)}_{\substack{\nearrow \\ 0}} \Big|_0^\pi - \int_0^\pi f(t) \cos(t) dt \\ &= - \int_0^\pi f(t) \cos(t) dt = 0, \end{aligned}$$

where we have used that  $\sin(\pi) = \sin(0) = 0$  and the fact that  $f$  is orthogonal to  $\cos(t)$ . Hence, it has been proved that  $f'$  is orthogonal to  $\sin(t)$  provided that  $f$  is orthogonal to  $\cos(t)$ .

## Problem 14

Find the  $L^2[-\pi, \pi]$  projection of the function  $f(x) = x^2$  onto the space  $V_n \in L^2[-\pi, \pi]$  spanned by

$$\left\{ \frac{\sin(jx)}{\sqrt{2\pi}}, \frac{\cos(jx)}{\sqrt{\pi}}; j = 1, 2, \dots, n \right\}$$

for  $n=2$ . Plot these projections along with  $f$  using a computer algebra system. Repeat for  $g(x) = x^3$ .

**Solution:** Let  $a_j = \frac{\sin(jx)}{\sqrt{2\pi}}$  and  $b_j = \frac{\cos(jx)}{\sqrt{\pi}}$ . Then, the projection,  $f_0$ , of  $f$  onto the space  $V_n$  is

$$f_0 = \sum_{j=1}^n \langle f, a_j \rangle a_j + \sum_{j=1}^n \langle f, b_j \rangle b_j,$$

where  $n = 2$ . First, we calculate the inner products

$$\langle f, a_j \rangle = \int_{-\pi}^\pi x^2 \frac{\sin(jx)}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^\pi x^2 \sin(jx) dx = 0,$$

since  $x^2 \sin(jx)$  is odd. Thus,

$$\langle f, a_j \rangle = 0 \quad \forall j \in \mathbb{Z}.$$

Now,

$$\begin{aligned}
\langle f, b_j \rangle &= \int_{-\pi}^{\pi} x^2 \frac{\cos(jx)}{\sqrt{\pi}} dx = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x^2 \cos(jx) dx \\
&= \frac{1}{\sqrt{\pi}} \left. \frac{x^2 \sin(jx)}{j} \right|_{-\pi}^{\pi} - \frac{2}{\sqrt{\pi}} \int_{-\pi}^{\pi} \frac{1}{j} x \sin(jx) dx \\
&= -\frac{2}{j\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin(jx) dx \\
&= +\frac{2}{j^2\sqrt{\pi}} x \cos(jx) \Big|_{-\pi}^{\pi} - \frac{2}{j^2\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(jx) dx \\
&= +\frac{2}{j^2\sqrt{\pi}} x \cos(jx) \Big|_{-\pi}^{\pi} - \frac{2}{j^3\sqrt{\pi}} \left. \sin(jx) \right|_{-\pi}^{\pi} \\
&= +\frac{2}{j^2\sqrt{\pi}} (\pi \cos(j\pi) + \pi \cos(-j\pi)) \\
&= +\frac{4\sqrt{\pi}}{j^2} \cos(j\pi),
\end{aligned}$$

where we have integrated by parts twice. Therefore,

$$\begin{aligned}
f_0 &= \sum_{j=1}^n \langle f, a_j \rangle a_j + \sum_{j=1}^n \langle f, b_j \rangle b_j \\
&= \langle f, b_1 \rangle b_1 + \langle f, b_2 \rangle b_2 \\
&= -4\sqrt{\pi} \frac{\cos(x)}{\sqrt{\pi}} + \sqrt{\pi} \frac{\cos(2x)}{\sqrt{\pi}}.
\end{aligned}$$

Hence, the projection

$$f_0 = \cos(2x) - 4\cos(x).$$

Further, we repeat the same process for the function  $g(x) = x^3$ . In this case, the parity of the function has changed from even to odd. Hence, in this case the inner products  $\langle g, b_j \rangle = 0$  because  $x^3 \cos(jx)$  is an odd function. We calculate then the inner products

$$\begin{aligned}
\langle g, a_j \rangle &= \int_{-\pi}^{\pi} x^3 \frac{\sin(jx)}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^3 \sin(jx) dx \\
&= -\frac{1}{\sqrt{2\pi}} \left. \frac{x^3 \cos(jx)}{j} \right|_{-\pi}^{\pi} + \frac{1}{\sqrt{2\pi}} \frac{3}{j} \int_{-\pi}^{\pi} x^2 \cos(jx) dx \\
&= -\frac{1}{j\sqrt{2\pi}} (\pi^3 \cos(j\pi) + \pi^3 \cos(-j\pi)) + \frac{3}{j\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{x^2 \cos(jx)}{\sqrt{\pi}} dx \\
&= -\frac{2\pi^2}{j\sqrt{2}} \cos(j\pi) + \frac{3}{j\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{x^2 \cos(jx)}{\sqrt{\pi}} dx \\
&= -\frac{\pi^2\sqrt{2}}{j} \cos(j\pi) + \frac{3}{j\sqrt{2\pi}} \frac{4\sqrt{\pi}}{j^2} \cos(j\pi) \\
&= \frac{6\sqrt{2}}{j^3} \cos(j\pi) - \frac{\pi^2\sqrt{2}}{j} \cos(j\pi).
\end{aligned}$$

Similarly than with  $f$ , we have

$$\begin{aligned}
 g_0 &= \sum_{j=1}^n \langle g, a_j \rangle a_j + \sum_{j=1}^n \langle g, b_j \rangle b_j \xrightarrow{0} \\
 &= \langle g, a_1 \rangle a_1 + \langle g, a_2 \rangle a_2 \\
 &= \sqrt{2} (\pi^2 - 6) \frac{\sin(x)}{\pi\sqrt{2}} + \frac{3 - 2\pi^2}{2\sqrt{2}} \frac{\sin(2x)}{\pi\sqrt{2}} \\
 &= \frac{(\pi^2 - 6)}{\pi} \sin(x) + \frac{3 - 2\pi^2}{4\pi} \sin(2x)
 \end{aligned}$$

## Problem 23

Show that a set of orthonormal vectors is linearly independent.

**Solution:** Consider the set of orthonormal vectors

$$S = \{e_1, e_2, \dots, e_N; \langle e_j, e_k \rangle = \delta_{jk}\},$$

where  $\delta_{jk}$  is the Kronecker delta. Assume that

$$\sum_{j=1}^N \alpha_j e_j = 0.$$

Taking the inner product

$$\begin{aligned}
 0 &= \left\langle \sum_{j=1}^N \alpha_j e_j, e_k \right\rangle = \sum_{j=1}^N \langle \alpha_j e_j, e_k \rangle \\
 &= \sum_{j=1}^N \alpha_j \langle e_j, e_k \rangle \\
 &= \sum_{j=1}^N \alpha_j \delta_{jk} \\
 &= \alpha_k.
 \end{aligned}$$

Hence, we have obtained that all the  $\alpha_k = 0$  and therefore it is proved that a set of orthonormal vectors is linearly independent.