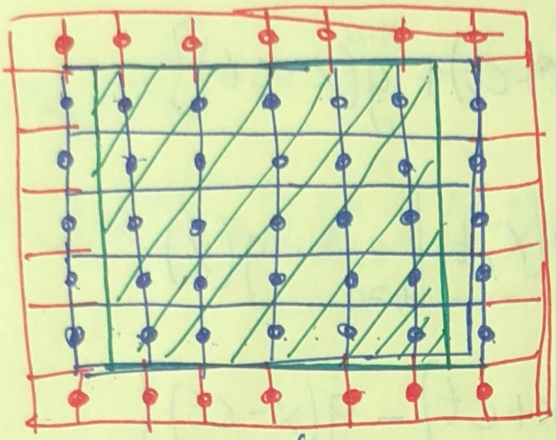
Development for $u(x,y)$:

$\boxed{U_{(M+1)\times(N+2)}}$

$\boxed{\tilde{U}_{(M-1)\times N}}$



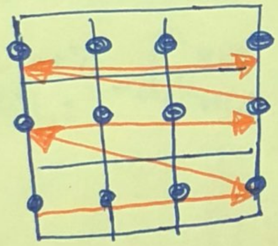
$N=5$

$N+2$

$M=6$

ONLY SOLVE FOR THE INTERIOR

STEP 1



(olumns togeth

$V((j-1)M+i) = U(i,j)$

$\boxed{\cancel{V((j-1)M+i-1) = u(i,j)}}$

$V((j-2)(M-1)+i-1) = u(i,j)$

$i-1,\ i,\ i+1$

To keep ~~columns~~ together and account only the interior

$$\underbrace{-d_1}_{a} u^{n+1/2}_{i-1,j} + \underbrace{(1+2d_1)}_{b} u^{n+1/2}_{i,j} \underbrace{-d_1}_{c} u^{n+1/2}_{i+1,j} =$$

$$= \underbrace{d_2 u^{n}_{i,j+1} + (1-2d_2) u^{n}_{i,j} + d_2 u^{n}_{i,j-1}}_{d}$$

```
for j = 2:N+1
    for i = 2:M
        ...
    end
end
```

For $i = 2:M$ and $j = 2:N+1$

Include BCs when $i = 2, M$ and $j = 2, N+1$

Horizontally node based, vertically cell based.

If $j = 2$ and $i = 2$

$$-d_1 u_{1,2}^{n+1/2} + (1+2d_1) u_{2,2}^{n+1/2} - d_1 u_{3,2}^{n+1/2} = d_2 u_{2,3}^{n} + (1-2d_2) u_{2,2}^{n} + d_2 u_{2,1}^{n}$$

$$(1+2d_1) u_{2,2}^{n+1/2} - d_1 u_{3,2}^{n+1/2} = \underbrace{d_2 u_{2,3}^{n} + (1-2d_2) u_{2,2}^{n} + d_2 BC_{2,1} + d_1 BC_{1,2}}_{d(1)}$$

$a = 0$

$\cancel{d(M)}$

$a\left(\cancel{(2-2)(M-1)+2-1} = a(M)\right) = 0$

$a(1) = 0$ ✓

If $j = 2$ and $i \in [3, M-1]$

$$-d_1 u_{i-1,2}^{n+1/2} + (1+2d_1) u_{i,2}^{n+1/2} - d_1 u_{i+1,2}^{n+1/2} = \underbrace{d_2 u_{i,3}^{n} + (1-2d_2) u_{i,2}^{n} + \underbrace{d_2 u_{i,1}^{n}}_{+d_2 BC_{i,1}}}_{\cancel{d(M+i-2)}}$$

$d(i-1)$

If $j = 2$ and $i = M$:

$$-d_1 u_{M-1,2}^{n+1/2} + (1+2d_1) u_{M,2}^{n+1/2} - d_1 u_{M+1,2}^{n+1/2} = d_2 u_{M,3}^{n} + (1-2d_2) u_{M,2}^{n} + d_2 u_{M,1}^{n}$$

$$-d_1 u_{M-1,2}^{n+1/2} + (1+2d_1) u_{M,2}^{n+1/2} = \underbrace{d_2 u_{M,3}^{n} + (1-2d_2) u_{M,2}^{n} + d_1 BC_{M+1,2} + d_2 BC_{M,1}}_{\cancel{d(2(M-1))}}$$

$d(M-1)$

$c\left(\cancel{(2-1)(M-1)+M-1} = c(2(M-1)) = 0\right)$

$c(M-1) = 0$ ✓

If $j \in [3, N]$ and $i = 2$:

$$(1+2d_1)u_{2,j}^{n+1/2} - d_1 u_{3,j}^{n+1/2} = \underbrace{d_2 u^n_{2,j+1} + (1-2d_2)u^n_{2,j} + d_2 u^n_{2,j-1} + d_1 BC_{1,j}}_{d((j-2)(M-1)+1)}$$

$a((j-2)(M-1)+1) = 0$ ✓

If $j \in [3, N]$ and $i \in [3, M-1]$

general eq.

If $j \in [3, N]$ and $i = M$:

$$-d_1 u^{n+1/2}_{M-1,j} + (1+2d_1)u^{n+1/2}_{M,j} = \underbrace{d_2 u^n_{M,j+1} + (1-2d_2)u^n_{M,j} + d_2 u^n_{M,j-1} + \underset{d_1 BC_{M+1,j}}{\underline{d_1 u^n_{M+1,j}}}}_{d((j-2)(M-1)+M-1)}$$

$c((j-2)(M-1)+M-1) = 0$ ✓

If $j = N+1$ and $i = 2$:

$$-d_1 u^{n+1/2}_{1,N+1} + (1+2d_1) u^{n+1/2}_{2,N+1} - d_1 u^{n+1/2}_{3,N+1} = d_2 u^n_{2,N+2} + (1-2d_2) u^n_{2,N+1} + d_2 u^n_{2,N}$$

$$(1+2d_1) u^{n+1/2}_{2,N+1} - d_1 u^{n+1/2}_{3,N+1} = \underbrace{(1-2d_2) u^n_{2,N+1} + d_2 u^n_{2,N} + d_1 BC_{1,N+1} + d_2 BC_{2,N+2}}_{d((N-1)(M-1)+1)\ \cancel{d(N(M-1)+1)}}$$

$$a((N-1)(M-1)+1) = 0 \checkmark \quad \cancel{a(N(M-1)+1) = 0}$$

If $j = N+1$ and $i \in [3, M-1]$:

$$-d_1 u^{n+1/2}_{i-1,N+1} + (1+2d_1) u^{n+1/2}_{i,N+1} - d_1 u^{n+1/2}_{i+1,N+1} = d_2 u^n_{i,N+2} + (1-2d_2) u^n_{i,N+1} + d_2 u^n_{i,N}$$

$$-d_1 u^{n+1/2}_{i-1,N+1} + (1+2d_1) u^{n+1/2}_{i,N+1} - d_1 u^{n+1/2}_{i+1,N+1} = \underbrace{(1-2d_2) u^n_{i,N+1} + d_2 u^n_{i,N} + d_2 BC_{i,N+2}}_{d((N-1)(M-1)+i-1)}$$

If $j = N+1$ and $i = M$:

$$-d_1 u^{n+1/2}_{M-1,N+1} + (1+2d_1) u^{n+1/2}_{M,N+1} - d_1 u^{n+1/2}_{M+1,N+1} = d_2 u^n_{M,N+2} + (1-2d_2) u^n_{M,N+1} + d_2 u^n_{M,N}$$

$$-d_1 u^{n+1/2}_{M-1,N+1} + (1+2d_1) u^{n+1/2}_{M,N+1} = \underbrace{(1-2d_2) u^n_{M,N+1} + d_2 u^n_{M,N} + d_1 BC_{M+1,N+1} + d_2 BC_{M,N+2}}_{d((N-1)(M-1)+M-1) = d((M-1)N)}$$

$$c((M-1)N) = 0 \checkmark$$

STEP 2

$V\left((i-2)N+j-1\right) = u(i,j)$

To keep $j-1, j, j+1$ together and account only for the interior

$$\underbrace{-d_2}_{a} u^{n+1}_{i,j-1} + \underbrace{(1+2d_2)}_{b} u^{n+1}_{i,j} \underbrace{-d_2}_{c} u^{n+1}_{i,j+1} =$$

$$= \underbrace{d_1 u^{n+1/2}_{i+1,j} + (1-2d_1) u^{n+1/2}_{i,j} + d_1 u^{n+1/2}_{i-1,j}}_{d}$$

for $i = 2:M$
for $j = 2:N+1$
...
end
end

Include BCs when $i = 2, M$ and $j = 2, N+1$.

If $i=2$ and $j=2$

$$\underline{-d_2 u^{n+1}_{2,1}} + (1+2d_2) u^{n+1}_{2,2} - d_2 u^{n+1}_{2,3} = d_1 u^{n+1/2}_{3,2} + (1-2d_1) u^{n+1/2}_{2,2} + \underline{d_1 u^{n+1/2}_{1,2}}$$

$= +d_2 u^{n+1}_{2,2}$

$$(1+3d_2) u^{n+1}_{2,2} - d_2 u^{n+1}_{2,3} = \underbrace{d_1 u_{3,2} + (1-2d_1) u^{n+1/2}_{2,2} + \underline{d_1 u_{1,2} + \cancel{d_2 u_{2,1}}}}_{d(1)}$$ BCs

$b(1) = 1+3d_2$ ✓

$a(1) = 0$ ✓

If $i=2$ and $j \in [3, N]$

$$-d_2 u^{n+1}_{2,j-1} + (1+2d_2) u^{n+1}_{2,j} - d_2 u^{n+1}_{2,j+1} = \underbrace{d_1 u^{n+1/2}_{3,j} + (1-2d_1) u^{n+1/2}_{2,j} + d_1 u^{n+1/2}_{1,j}}_{d(j-1)}$$

If $i=2$ and $j=N+1$

$$-d_2 u_{2,N}^{n+1} + (1+2d_2) u_{2,N+1}^{n+1} \underbrace{- d_2 u_{2,N+2}^{n+1}}_{+d_2 u_{2,N+1}^{n+1}} = d_1 u_{3,N+1}^{n+1/2} + (1-2d_1) u_{2,N+1}^{n+1/2} + d_1 u_{1,N+1}^{n+1/2}$$

$$-d_2 u_{2,N}^{n+1} + (1+3d_2) u_{2,N+1}^{n+1} = \underbrace{d_1 u_{3,N+1}^{n+1/2} + (1-2d_1) u_{2,N+1}^{n+1/2} + d_1 u_{1,N+1}^{n+1/2} + \cancel{d_2 u_{2,N+2}^{n+1/2}}}_{d(N)}$$

$b(N) = 1+3d_2$ ✓

$c(N) = 0$ ✓

If $i\in[3,M-1]$ and $j=2$

$$\underbrace{-d_2 u_{i,1}^{n+1}}_{=d_2 u_{i,2}^{n+1}} + (1+2d_2) u_{i,2}^{n+1} - d_2 u_{i,3}^{n+1} = d_1 u_{i+1,2}^{n+1/2} + (1-2d_1) u_{i,2}^{n+1/2} + d_1 u_{i-1,2}^{n+1/2}$$

$$(1+3d_2) u_{i,2}^{n+1} - d_2 u_{i,3}^{n+1} = \underbrace{d_1 u_{i+1,2}^{n+1/2} + (1-2d_1) u_{i,2}^{n+1/2} + d_1 u_{i-1,2}^{n+1/2} + \cancel{d_2 u_{i,1}}}_{d((i-2)N+1}$$

$b((i-2)N+1) = 1+3d_2$

$a((i-2)N+1) = 0$ ✓

If $i\in[3,M-1]$ and $j\in[3,N]$

general eq.

If $i\in[3,M-1]$ and $j=N+1$

$$-d_2 u_{i,N}^{n+1} + (1+2d_2) u_{i,N+1}^{n+1} - d_2 u_{i,N+2}^{n+1 = d_2 u_{i,N+1}^{n+1}} = d_1 u_{i+1,N+1}^{n+1/2} + (1-2d_1) u_{i,N+1}^{n+1/2} + d_1 u_{i-1,N+1}^{n+1/2}$$

$$-d_2 u_{i,N}^{n+1} + (1+3d_2) u_{i,N+1}^{n+1} = \underbrace{d_1 u_{i+1,N+1}^{n+1/2} + (1-2d_1) u_{i,N+1}^{n+1/2} + d_1 u_{i-1,N+1}^{n+1/2} + \cancel{d_2 u_{i,N+2}^{n+1}}}_{d((i-2)N+N) = d((i-1)N)}$$

$b((i-1)N) = 1+3d_2$

$c((i-1)N) = 0$ ✓

If $i=M$ and $j=2$

$$\underline{-d_2 u^{n+1}_{M,1}} + (1+2d_2) u^{n+1}_{M,2} - d_2 u^{n+1}_{M,3} = \underline{d_1 u^{n+1/2}_{M+1,2}} + (1-2d_1) u^{n+1/2}_{M,2} + d_1 u^{n+1/2}_{M-1,2}$$

$$= d_2 u^{n+1}_{M,2}$$

$$(1+3d_2) u^{n+1}_{M,2} - d_2 u^{n+1}_{M,3} = \underbrace{(1-2d_1) u^{n+1/2}_{M,2} + d_1 u^{n+1/2}_{M-1,2} + \cancel{d_1 u^{n+1/2}_{M+1,2}} \; d_1 u^{n+1/2}_{M+1,2} + \cancel{d_2 u^{n+1/2}_{M,1}}}_{d((M-2)N+1)}$$

$b((M-2)N+1) = 1+3d_2$

$a((M-2)N+1) = 0$ ✓

If $i=M$ and $j\in[3,N]$

$$-d_2 u^{n+1}_{M,j-1} + (1+2d_2) u^{n+1}_{M,j} - d_2 u^{n+1}_{M,j+1} = \underbrace{(1-2d_1) u^{n+1/2}_{M,j} + d_1 u^{n+1/2}_{M-1,j} + d_1 u^{n+1/2}_{M+1,j}}_{d((M-2)N+j-1)}$$

If $i=M$ and $j=N+1$

$$-d_2 u^{n+1}_{M,N} + (1+2d_2) u^{n+1}_{M,N+1} - \underline{d_2 u^{n+1}_{M,N+2}} = \underline{d_1 u^{n+1/2}_{M+1,N+1}} + (1-2d_1) u^{n+1/2}_{M,N+1} + d_1 u^{n+1/2}_{M-1,N+1}$$

$$= d_2 u^{n+1}_{M,N+1}$$

$$-d_2 u^{n+1}_{M,N} + (1+3d_2) u^{n+1}_{M,N+1} = \underbrace{(1-2d_1) u^{n+1/2}_{M,N+1} + d_1 u^{n+1/2}_{M-1,N+1} + d_1 u^{n+1/2}_{M+1,N+1} + \cancel{d_2 u^{n+1}_{M,N+2}}}_{d((M-2)N+N) = d((M-1)N)}$$

$b((M-1)N) = 1+3d_2$ ✓

$c((M-1)N) = 0$ ✓

Boundary conditions for u :

* Node-like left and right boundaries :

~~left : $u_{1,j} = BC_{i,1} = \{ 0$ if y~~

$$\text{left} : u_{1,j} = BC_{1,j} = \begin{cases} 0 & \text{if } y_j \notin [0.5, 1] \\ 2 & \text{if } y_j \in [0.5, 1] \end{cases}$$

$$\text{right} : u_{M+1,j} = BC_{M+1,j} = \begin{cases} 0 & \text{if } y_j \notin [1, 1.5] \\ -1 & \text{if } y_j \in [1, 1.5] \end{cases}$$

---

* Cell-like top and bottom boundaries

$$\text{Top}: \quad u_{i,N+1+\frac{1}{2}} = \frac{u_{i,N+1} + u_{i,N+2}}{2} \Rightarrow$$

$$\Rightarrow u_{i,N+2} = 2\, u_{i,N+1+\frac{1}{2}} - u_{i,N+1} = \{$$

$$u_{i,N+2} = BC_{i,N+2} = \{ \; - u_{i,N+1} \quad \forall i$$

$$u_{i,N+2} = BC_{i,N+2} = - u_{i,N+1} .$$

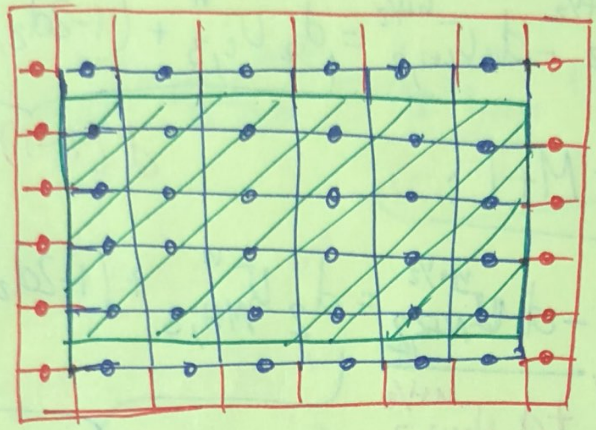
$$\text{Bottom}: \quad u_{i,1} = 2\, u_{i,1+\frac{1}{2}} - u_{i,2} \; ; \quad u_{i,1+\frac{1}{2}} = \{ 0$$

$$u_{i,1} = BC_{i,1} = - u_{i,2} \checkmark$$

Development for: $v(x,y)$:

$v_{(M+2)\times(N+1)}$

$\tilde{v}_{M\times(N-1)}$



N

M

$V\big((j-2)M+i-1\big)=u(i,j)$

To keep $i-1, i, i+1$ together.
+ only interior

STEP 1

$$\underbrace{-d_1}_{a} u_{i-1,j}^{n+1/2} + \underbrace{(1+2d_1)}_{b} v_{i,j}^{n+1/2} \underbrace{-d_1}_{c} u_{i+1,j}^{n+1/2} = \underbrace{d_2 u_{i,j+1}^{n} + (1-2d_2) u_{i,j}^{n} + d_2 u_{i,j-1}^{n}}_{d}$$

For $i=2:M+1$ and $j=2:N$

for $j=2:N$
for $i=2:M+1$
...
end end
end

If $j=2$ and $i=2$:

$$-d_1 v_{1,2}^{n+1/2} + (1+2d_1) v_{2,2}^{n+1/2} - d_1 v_{3,2}^{n+1/2} = d_2 u_{2,3}^{n} + (1-2d_2) u_{2,2}^{n} + d_2 v_{2,1}^{n}$$

$+d_1 v_{2,2}^{n+1/2}$

$$(1+3d_1) v_{2,2}^{n+1/2} - d_1 v_{3,2}^{n+1/2} = \underbrace{d_2 v_{2,3}^{n} + (1-2d_2) v_{2,2}^{n} + d_2 \cancel{v_{2,1}^{n}}}_{d(1)}$$

BC

Made this 0 Since it is not likely to be outlet

$a(1)=0$ ✓

$b(1)=1+3d_1$ ✓

If $j=2$, and $i \in [3, M]$:

$$-d_1 U_{i-1,2}^{n+1/2} + (1+2d_1) U_{i,2}^{n+1/2} - d_1 U_{i+1,2}^{n+1/2} = \underbrace{d_2 U_{i,3}^{n} + (1-2d_2) U_{i,2}^{n} + \underline{d_2 U_{i,1}^{n}}_{\text{B.C}}}_{d(i-1)}$$

If $j=2$ and $i=M+1$:

$$-d_1 U_{M,2}^{n+1/2} + (1+2d_1) U_{M+1,2}^{n+1/2} \underline{- d_1 U_{M+2,2}^{n+1/2}}_{+d_1 U_{M+1,2}^{n+1/2}} = \underbrace{d_2 U_{M+1,3}^{n} + (1-2d_2) U_{M+1,2}^{n} + \underline{d_2 U_{M+1,1}^{n}}_{\text{B.C.}}}_{d(M)}$$

$b(M) = 1+3d_1$ ✓

$c(M) = 0$ ✓

If $j \in [3, N-1]$ and $i=2$

$$\underline{-d_1 U_{1,j}^{n+1/2}}_{+d_1 U_{2,j}} + (1+2d_1) U_{2,j}^{n+1/2} - d_1 U_{3,j}^{n+1/2} = \underbrace{d_2 U_{2,j+1}^{n} + (1-2d_2) U_{2,j}^{n} + d_2 U_{2,j-1}^{n}}_{d((j-2)M+1)}$$

$a((j-2)M+1) = 0$ ✓ ✓

$b((j-2)M+1) = 1+3d_1$

If $j \in [3, N-1]$ and $i \in [3, M]$

general eqn.

If $j \in [3, N-1]$ and $i = M+1$

$$-d_1 U_{M,j}^{n+1/2} + (1+2d_1) U_{M+1,j}^{n+1/2} \underline{- d_1 U_{M+2,j}^{n+1/2}}_{+d_1 U_{M+1,j}} = \underbrace{d_2 U_{M+1,j+1}^{n} + (1-2d_2) U_{M+1,j}^{n} + d_2 U_{M+1,j-1}^{n}}_{d((j-1)M)}$$

$b((j-2)M+M) = b((j-1)M) = 1+3d_1$ ✓

$c((j-1)M) = 0$ ✓

If $j=N$ and $i=2$ <u></u> Ignore change $d_1 \leftrightarrow d_2$

$\cancel{-d_1 \delta t_{...}}$

$$\underline{-d_1 v_{1,N}^{n+1/2}} + (1+2d_1) v_{2,N}^{n+1/2} - d_1 v_{3,N}^{n+1/2} = \underbrace{\underline{d_2 v_{2,N+1}^{n}}_{B.C} + (1-2d_2) v_{2,N}^{n} + d_2 v_{2,N-1}^{n}}_{d((N-2)M+1)}$$

$+ d_1 v_{2,N}$

$a((N-2)M+1) = 0$ ✓

$b((N-2)M+1) = 1+3d_1$ ✓

If $j=N$ and $i \in [3, M]$

$$-d_1 v_{i-1,N}^{n+1/2} + (1+2d_1) v_{i,N}^{n+1/2} - d_1 v_{i+1,N}^{n+1/2} = \underbrace{\underline{d_2 v_{i,N+1}^{n}}_{B.C.} + (1-2d_2) v_{i,N}^{n} + d_2 v_{i,N-1}^{n}}_{d((N-2)M+i-1)}$$

If $j=N$ and $i = M+1$

$$-d_1 v_{M,N}^{n+1/2} + (1+2d_1) v_{M+1,N}^{n+1/2} \underline{- d_1 v_{M+2,N}^{n+1/2}} = \underbrace{\underline{d_2 v_{M+1,N+1}^{n}}_{B.C.} + (1-2d_2) v_{M+1,N}^{n} + d_2 v_{M+1,N-1}^{n}}_{d((N-1)M)}$$

$+ d_1 v_{M+1,N}$

$b((N-2)M+M) = b((N-1)M) = 1+3d_1$ ✓

$c((N-1)M) = 0$ ✓

STEP 2

~~$V((i-2)(N-1)+j-1)$~~

$$V((i-2)(N-1)+j-1) = v(i,j)$$

$$\underbrace{-d_2}_{a} u_{i,j-1}^{n+1} + \underbrace{(1+2d_2)}_{b} u_{i,j}^{n+1} \underbrace{-d_2}_{c} u_{i,j+1}^{n+1} = \underbrace{d_1 u_{i+1,j}^{n+1/2} + (1-2d_1) u_{i,j}^{n+1/2} + d_1 u_{i-}^{n+}}_{d}$$

```
for i = 2:M+1
    for j = 2:N
    ...
end end
```

If $i=2$ and $j=2$

$$\underbrace{-d_2 \cancel{u_{2,1}^{n+1}}}_{B.C.} \to 0 + (1+2d_2) u_{2,2}^{n+1} - d_2 u_{2,3}^{n+1} = d_1 u_{3,2}^{n+1/2} + (1-2d_1) u_{2,2}^{n+1/2} + \underbrace{d_1 u_{1,2}^{n+1/2}}_{-d_1 v_{2,2}}$$

$$(1+2d_2) v_{2,2}^{n+1} - d_2 v_{2,3}^{n+1} = \underbrace{d_1 v_{3,2}^{n+1/2} + (1-3d_1) v_{2,2}^{n+1/2} + \underbrace{\cancel{d_2 v_{2,1}^{n+1}}}_{B.C} \to 0}_{d(1)}$$

made it 0 since not outlet

~~Careful, b could change~~

$a(1) = 0$ ✓

If $i=2$ and $j \in [3, N-1]$

$$-d_2 v_{2,j-1}^{n+1} + (1+2d_2) v_{2,j}^{n+1} - d_2 v_{2,j+1}^{n+1} = \underbrace{d_1 v_{3,j}^{n+1/2} + (1-2d_1) v_{2,j}^{n+1/2} + \underbrace{d_1 v_{1,j}^{n+1/2}}_{-d_1 v_{2,j}^{n+1/2}}}$$

$$d(j-1) = d_1 v_{3,j}^{n+1/2} + (1-3d_1) v_{2,j}^{n+1/2}$$

If $i=2$ and $j=N$

$$-d_2 v_{2,N-1}^{n+1}+(1+2d_2)v_{2,N}^{n+1}-\underbrace{d_2 v_{2,N+1}^{n+1}}_{B.C.}=d_1 v_{3,N}^{n+1/2}+(1-2d_1)v_{2,N}^{n+1/2}+\frac{d_1 v_{1,N}^{n+1/2}}{-d_1 v_{2,N}}$$

$$-d_2 v_{2,N-1}^{n+1}+(1+2d_2)v_{2,N}^{n+1}=\underbrace{d_1 v_{3,N}^{n+1/2}+(1-3d_1)v_{2,N}^{n+1/2}+\underbrace{d_2 v_{2,N+1}^{n+1}}_{B.C}}_{d(N-1)}$$

$c(N-1)=0$ ✓

If $i\in[3,M]$ and $j=2$

$$\underbrace{-d_2 v_{i,1}^{n+1}}_{B.C.}+(1+2d_2)v_{i,2}^{n+1}-d_2 v_{i,3}^{n+1}=\underbrace{d_1 v_{i+1,2}^{n+1/2}+(1-2d_1)v_{i,2}^{n+1/2}+d_1 v_{i-1,2}^{n+1/2}\ \cancel{+B.C.}}_{d((i-2)(N-1)+1)}$$

$a((i-2)(N-1)+1)=0$ ✓

If $i\in[3,M]$ and $j\in[3,N-1]$

General equ.

If $i\in[3,M]$ and $j=N$

$$-d_2 v_{i,N-1}+(1+2d_2)v_{i,N}-\underbrace{d_2 v_{i,N+1}}_{B.C}=\underbrace{d_1 v_{i+1,N}^{n+1/2}+(1-2d_1)v_{i,N}^{n+1/2}+d_1 v_{i-1,N}^{n+1/2}}\ +$$

$\cancel{c((i-2)(N-1)+N-1)}=$

$c((i-1)(N-1))=0$ ✓

$d((i-2)(N-1)+N-1)=$

$=d((i-1)(N-1))$

If $i = M+1$ and $j = 2$

$$\underbrace{-d_2 v_{M+1,1}^{n+1}}_{\substack{B.C.=0 \\ \text{Not outlet}}} + (1+2d_2) v_{M+1,2}^{n+1} - d_2 v_{M+1,3}^{n+1} = \underbrace{\underbrace{d_1 v_{M+2,2}^{n+1/2}}_{-d_1 v_{M+1,2}^{n+1/2}} + (1-\cancel{3}d_1) v_{M+1,2}^{n+1/2} + d_1 v}_{d((M-1)(N-1)+1)}$$

$a((M-1)(N-1)+1) = 0$ ✓

If $i = M+1$ and $j \in [3, N-1]$

$$-d_2 v_{M+1,j-1}^{n+1} + (1+2d_2) v_{M+1,j}^{n+1} - d_2 v_{M+1,j+1}^{n+1} = \underbrace{\underbrace{d_1 v_{M+2,j}^{n+1/2}}_{-d_1 v_{M+1,j}^{n+1/2}} + (1-\cancel{3}d_1) v_{M+1,j}^{n+1/2} + d_1 v_{M,j}^{n+1/2}}_{d((M-1)(N-1)+j-1)}$$

If $i = M+1$ and $j = N$

$$-d_2 v_{M+1,N-1}^{n+1} + (1+2d_2) v_{M+1,N}^{n+1} \underbrace{- d_2 v_{M+1,N+1}^{n+1}}_{B.C} = \underbrace{\underbrace{d_1 v_{M+2,N}^{n+1/2}}_{-d_1 v_{M+1,N}^{n+1/2}} + (1-\cancel{3}d_1) v_{M+1,N}^{n+1/2} + d_1 v_{M,N}^{n+1/2}}_{d((N-1)M)}$$

$+$

$c((M-1)(N-1)+N-1) =$

$= c((N-1)M) = 0$

Boundary Conditions for v

* ~~Node~~ Cell-like ~~top~~ left and ~~bottom~~ right boundaries:

left: $$v_{1+\frac{1}{2},j} = \frac{v_{1,j} + v_{2,j}}{2} = 0 \quad \forall j \in [1, N+1]$$

$$v_{1,j} = -v_{2,j}$$

right: $$v_{M+1+\frac{1}{2},j} = \frac{v_{M+1,j} + v_{M+2,j}}{2} = 0 \quad \forall j \in [1, N+1]$$

$$v_{M+2,j} = -v_{M+1,j}$$

* Node-like top and bottom boundaries:

top: $$v_{i,N+1} = \begin{cases} 0 & x_i \notin [0.5, 1] \\ -1 & x_i \in [0.5, 1] \end{cases}$$

Bottom $$v_{i,1} = \begin{cases} 0 & x_i \notin [1.5, 2] \\ \cancel{v_{i,2} \quad x_i \in [1.5, 2]} \end{cases}$$

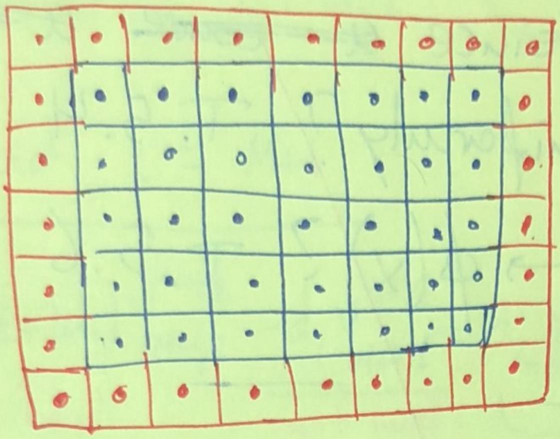
$$\cancel{\left.\frac{\partial v}{\partial y}\right|_{out} = v_{i,2} - v_{i,1}} = 0 \Rightarrow$$

$x_i \in [1.5, 2]$

2^{nd} order FFT: $$\left.\frac{\partial v}{\partial y}\right|_{out} = -\frac{v_{i,3} - 4v_{i,2} + 3v_{i,1}}{2h} = 0 \Rightarrow v_{i,1} = \frac{4}{3}v_{i,2} - \frac{1}{3}v_{i,3}$$

Development for $Y(x,y)$:

$Y_{(M+2)\times(N+2)}$



$V((j-2)M+i-1)$ ..

STEP 1

$$-d_1 Y^{n+1/2}_{i-1,j} + (1+2d_1) Y^{n+1/2}_{i,j} - d_1 Y^{n+1/2}_{i+1,j} = d_2 Y^{n}_{i,j+1} + (1-2d_2) Y^{n}_{i,j} + d_2 Y^{n}_{i,j-1}$$

For $j = 2:N+1$ and $i = 2:M+1$ $\left\{ \begin{array}{l} \text{for } j=2:N+1 \\ \quad \text{for } i=2:M+1 \\ \quad\quad \cdots \\ \quad \text{end} \\ \text{end} \end{array} \right.$

~~If j=2 and i=2~~

If $j=2$ and $i=2$

$$-d_1 Y^{n+1/2}_{1,2} + (1+2d_1) Y^{n+1/2}_{2,2} - d_1 Y^{n+1/2}_{3,2} = d_2 Y^{n}_{2,3} + (1-2d_2) Y^{n}_{2,2} + \underline{d_2 Y^{n}_{2,1}}$$

$d_2 Y^n_{3,2}$ (no need)

It is not inlet 1. Thus, $-d_1 Y^{n+1/2}_{2,2}$

$$(1+1d_1) Y^{n+1/2}_{2,2} - d_1 Y^{n+1/2}_{3,2} = \underbrace{d_2 Y^{n}_{2,3} + (1-1d_2) Y^{n}_{2,2} + \cancel{d_2 Y^{n}_{2,1}}}_{d(1)}$$

$a(1) = 0$ ✓

$b(1) = 1 + d_1$ ✓

If $j=2$ and $i\in[3,M]$

$$-d_1 Y^{n+1/2}_{i-1,2} + (1+2d_1) Y^{n+1/2}_{i,2} - d_1 Y^{n+1/2}_{i+1,2} = \underbrace{d_2 Y^n_{i,3} + (1-2d_2) Y^n_{i,2} + d_2 \underbrace{Y^n_{i,1}}_{\substack{\text{B.C.}\\ \text{(no need)}}}}_{d(i-1)}$$

If $j=2$ and $i=M+1$:

$$-d_1 Y^{n+1/2}_{M,2} + \underbrace{(1+2d_1)}_{(1+d_1)} Y^{n+1/2}_{M+1,2} \underbrace{-d_1 Y^{n+1/2}_{M+2,2}}_{\substack{\text{not inlet 2}\\ -d_1 Y^{n+1/2}_{M+1,2}}} = \underbrace{d_2 Y^n_{M+1,3} + (1-2d_2) Y^n_{M+1,2} + d_2 \underbrace{Y^n_{M+1,1}}_{\substack{\text{B.C.}\\ \text{(no need)}}}}_{d}$$

$b(M) = 1+d_1$ ✓

$c(M) = 0$ ✓

If $j\in[3,N]$ and $i=2$

$$\underbrace{-d_1 Y^{n+1/2}_{1,j}}_{\text{B.C}} + (1+2d_1) Y^{n+1/2}_{2,j} - d_1 Y^{n+1/2}_{3,j} = \underbrace{d_2 Y^n_{2,j+1} + (1-2d_2) Y^n_{2,j} + d_2 Y^n_{2,j-1}}_{d}$$

$a((j-2)M+1) = 0$ ✓

- $Y_j \in [0.5, 1]$: $Y_{1,j} = 2 - Y_{2,j}$

$$(1+3d_1) Y^{n+1/2}_{2,j} - d_1 Y^{n+1/2}_{3,j} = \underbrace{d_2 Y^n_{2,j+1} + (1-2d_2) Y^n_{2,j} + d_2 Y^n_{2,j-1} + 2d_1}_{d((j-2)M+1)}$$

$b((j-2)M+1) = 1+3d_1$ ✓

- $Y_j \notin [0.5, 1]$: $Y_{1,j} = Y_{2,j}$

$$(1+d_1) Y^{n+1/2}_{2,j} - d_1 Y^{n+1/2}_{3,j} = \underbrace{d_2 Y^n_{2,j+1} + (1-2d_2) Y^n_{2,j} + d_2 Y^n_{2,j-1}}_{d((j-2)M+1)}$$

$b((j-2)M+1) = 1+d_1$ ✓

If $j \in [3, N]$ and $i \in [3, M]$

general eqn

If $j \in [3, N]$ and $i = M+1$

$$-d_1 Y_{M,j}^{n+1/2} + (1+2d_1) Y_{M+1,j}^{n+1/2} - d_1 \underbrace{Y_{M+2,j}^{n+1/2}}_{B.C.} = d_2 Y_{M+1,j+1}^{n} + (1-2d_2) Y_{M+1,j}^{n} + d_2 Y_{M+1,j-1}^{n}$$

- $Y_j \in [1, 1.5]$: $Y_{M+2,j} = 0.5 - Y_{M+1,j}$

$$-d_1 Y_{M,j}^{n+1/2} + (1+3d_1) Y_{M+1,j}^{n+1/2} = \underbrace{d_2 Y_{M+1,j+1} + (1-2d_2) Y_{M+1,j} + d_2 Y_{M+1,j-1} + 0.5 d_1}_{d((j-1)M)}$$

$c((j-2)M+M) = c((j-1)M) = 0$ ✓

$b((j-1)M) = 1+3d_1$ ✓

- $Y_j \notin [1, 1.5]$: $Y_{M+2,j} = Y_{M+1,j}$

$$-d_1 Y_{M,j}^{n+1/2} + (1+1d_1) Y_{M+1,j}^{n+1/2} = d_2 Y_{M+1,j+1}^{n} + (1-2d_2) Y_{M+1,j}^{n} + d_2 Y_{M+1,j-1}^{n}$$

$b((j-1)M) = 1+d_1$ ✓

If $j = N+1$ and $i = 2$

$$\underbrace{-d_1 Y_{1,N+1}^{n+1/2}}_{\text{Assume it is not inlet 1}} + (1+2d_1) Y_{2,N+1}^{n+1/2} - d_1 Y_{3,N+1}^{n+1/2} = \underbrace{d_2 Y_{2,N+2}^{n}}_{\text{Assume it is not inlet 3}} + (1-2d_2) Y_{2,N+1}^{n} + d_2 Y_{2,N}^{n}$$

Assume it is not inlet 1

$-d_1 Y_{2,N+1}^{n+1/2}$

$b((N-1)M+1) = 1+d_1$ ✓

$a((N-1)M+1) = 0$ ✓

Assume it is not inlet 3

$d_2 Y_{2,N+1}$

(no need)

$d((N-1)M+1)$

If $j = N+1$ and $i \in [3, M]$

$$-d_1 Y_{i-1,N+1}^{n+1/2} + (1+2d_1) Y_{i,N+1}^{n+1/2} - d_1 Y_{i,N+1}^{n+1/2} = \underbrace{\underline{d_2 Y_{i,N+2}^{n}}_{\text{(no need)}} + (1-2d_2) Y_{i,N+1}^{n} + d_2 Y_{i,N}^{n}}_{d}$$

If $j = N+1$ and $i = M+1$:

$$-d_1 Y_{M,N+1}^{n+1/2} + (1+2d_1) Y_{M+1,N+1}^{n+1/2} \underline{- d_1 Y_{M+2,N+1}^{n+1/2}} = \underbrace{\underline{d_2 Y_{M+1,N+2}^{n}}_{\text{(no need)}} + (1-2d_2) Y_{M+1,N+1}^{n} + d_2 Y_{M+1,N}^{n}}_{d}$$

Assume not inlet 2.

$-d_1 Y_{M+1,N+1}$

$b(M \cdot N) = 1 + d_1$ ✓

$c((N-1)M + M) = c(M \cdot N) = 0$ ✓

$\boxed{\text{STEP 2}}$ $V((i-2)N + j - 1) = Y(i,j)$

$$-d_2 Y_{i,j-1}^{n+1} + (1+2d_2) Y_{i,j}^{n+1} - d_2 Y_{i,j+1}^{n+1} = d_1 Y_{i+1,j}^{n+1/2} + (1-2d_1) Y_{i,j}^{n+1/2} + d_1 Y_{i-1,j}^{n+1/2}$$

```
for i = 2 : M+1
    for j = 2 : N+1
        ...
    end
end
```

If $i = 2$ and $j = 2$

$$\underline{-d_2 Y_{2,1}^{n+1}} + (1+2d_2) Y_{2,2}^{n+1} - d_2 Y_{2,3}^{n+1} = d_1 Y_{3,2}^{n+1/2} + (1-2d_1) Y_{2,2}^{n+1/2} + \underline{d_1 Y_{1,2}^{n+1/2}}_{\text{(no need)}}$$

$-d_2 Y_{2,2}^{n+1}$

$a(1) = 0$ ✓

$b(1) = 1 + d_2$ ✓

If $i=2$ and $j\in[3,N]$

$$-d_2 Y_{2,j-1}^{n+1}+(1+2d_2)Y_{2,j}^{n+1}-d_2 Y_{2,j+1}^{n+1}=d_1 Y_{3,j}^{n+1/2}+(1-2d_1)Y_{2,j}^{n+1/2}+\underbrace{d_1 Y_{1,j}^{n+1/2}}_{\text{(no need)}}$$

If $i=2$ and $j=N+1$

$$-d_2 Y_{2,N}^{n+1}+(1+2d_2)Y_{2,N+1}^{n+1}-\underbrace{d_2 Y_{2,N+2}^{n+1}}_{\substack{\text{Assume not}\\ \text{inlet 3}\\ -d_2 Y_{2,N+1}}}=\underbrace{d_1 Y_{3,N+1}^{n+1/2}+(1-2d_1)Y_{2,N+1}^{n+1/2}+\underbrace{d_1 Y_{1,N+1}^{n+1/2}}_{\text{(no need)}}}_{d(N)}$$

$c(N)=0$ ✓

$b(N)=1+d_2$ ✓

If $i\in[3,M]$ and $j=2$

$$-\underbrace{d_2 Y_{i,1}^{n+1}}_{-d_2 Y_{i,2}}+(1+2d_2)Y_{i,2}^{n+1}-d_2 Y_{i,3}^{n+1}=\underbrace{d_1 Y_{i+1,2}^{n+1/2}+(1-2d_1)Y_{i,2}^{n+1/2}+d_1 Y_{i-1,2}^{n+1/2}}$$

$a((i-2)N+1)=0$ ✓

$b((i-2)N+1)=1+d_2$ ✓

If $i\in[3,M]$ and $j\in[3,N]$

General equ.

If $i \in [3, M]$ and $j = N+1$

$$-d_2 Y_{i,N}^{n+1} + (1+2d_2) Y_{i,N+1}^{n+1} - \underbrace{d_2 Y_{i,N+2}^{n+1}}_{\text{B.C.}} = \underbrace{d_1 Y_{i+1,N+1}^{n+1/2} + (1-2d_1) Y_{i,N+1}^{n+1/2} + d_1 Y_{i-1,N+1}^{n+1/2}}_{d((i-1)N).}$$

$c((i-2)N+N) = c((i-1)N) = 0$ ✓

- $x_i \in [0,5,1]$: $Y_{i,N+2} = -Y_{i,N+1}$

  $b((i-1)N) = 1+3d_2$ ✓

- $x_i \notin [0.5,1]$: $Y_{i,N+2} = Y_{i,N+1}$

  $b((i-1)N) = 1+d_2$ ✓

If $i = M+1$ and $j = 2$

$$\underbrace{-d_2 Y_{M+1,1}^{n+1}}_{-d_2 Y_{M+1,2}} + (1+2d_2) Y_{M+1,2}^{n+1} - d_2 Y_{M+1,3}^{n+1} = \underbrace{d_1 Y_{M+2,2}^{n+1/2}}_{\text{(no need)}} + (1-2d_1) Y_{M+1,2}^{n+1/2} + d_1 Y_{M,2}^{n+1/2}$$

$a((M-1)N+1) = 0$ ✓

$b((M-1)N+1) = 1+d_2$ ✓

If $i = M+1$ and $j \in [3, N]$

$-d_2 Y_{M+1,j-1}$ - - - - like gen. eqn.

$i = M+1$ and $j = N+1$

$$-d_2 Y^{n+1}_{M+1,N} + (1+2d_2) Y^{n+1}_{M+1,N+1} - \underbrace{d_2 Y^{n+1}_{M+1,N+2}}_{\text{assume not inlet 3}} = \underbrace{\underbrace{d_1 Y^{n+1/2}_{M+2,N+1}}_{\text{no need}} + (1-2d_1) Y^{n+1/2}_{M+1,N+1} + d_1 Y^{n+1/2}_{M,N+1}}$$

$-d_2 Y^{n+1}_{M+1,N+1}$

$c(MN) = 0$ ✓

$b(MN) = 1+d_2$ ✓

$\cancel{f(L+x) = f(y)+2}$

Top Boundary

$$Y_{i,N+1+1/2} = \frac{Y_{i,N+1} + Y_{i,N+2}}{2} = 0 \quad \text{for inlet 3}$$

$$Y_{i,N+2} = -Y_{i,N+1} \quad \text{for } x_i \in [0.5, 1]$$

$$\left.\frac{\partial Y}{\partial y}\right|_{i,N+1+1/2} = \frac{Y_{i,N+2} - Y_{i,N+1}}{2\frac{h_y}{2}} = 0 \Rightarrow Y_{i,N+2} = Y_{i,N+1} \quad \text{for } x_i \notin [0.5, 1]$$

Bottom Boundary

$$\left.\frac{\partial Y}{\partial y}\right|_{i,1+1/2} = \frac{Y_{i,2} - Y_{i,1}}{2\frac{h_y}{2}} = 0 \quad \text{for all the boundary}$$

$$Y_{i,1} = Y_{i,2} \quad \forall\, x_i.$$

Boundary Conditions for $Y(x,y)$:

Left Boundary

$$Y_{1+1/2;j} = \frac{Y_{1,j} + Y_{2,j}}{2} = 1 \quad \text{for inlet 1}$$

$$Y_{1,j} = 2 - Y_{2,j} \quad \text{for } y_j \in [0.5, 1]$$

For wall:

$$\left.\frac{\partial Y}{\partial x}\right|_{1+1/2,j} = \frac{Y_{2,j} - Y_{1,j}}{2\frac{h_x}{2}} = 0 \Rightarrow Y_{1,j} = Y_{2,j} \quad \text{for } y_j \notin [0.5, 1]$$

Right Boundary

$$Y_{M+1+1/2,j} = \frac{Y_{M+1,j} + Y_{M+2,j}}{2} = 0.25 \quad \text{for inlet 2}$$

$$Y_{M+2,j} = 0.5 - Y_{M+1,j} \quad \text{for } y_j \in [1, 1.5]$$

For wall:

$$\left.\frac{\partial Y}{\partial x}\right|_{M+1+1/2,j} = \frac{Y_{M+2,j} - Y_{M+1,j}}{2\frac{h_x}{2}} = 0 \Rightarrow Y_{M+2,j} = Y_{M+1,j} \quad \text{for } y_j \notin [1, 1.5]$$