

CHAPTER 0

L2 Inner Product: The L^2 inner product on $L^2([a, b])$ is defined as $\langle f, g \rangle_{L^2} = \int_a^b f(t)g(t)dt$. **12 Inner Product:** The space l^2 is the set of all sequences $x_i \in \mathbb{C}$ with $\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty$. The inner product on l^2 is defined as $\langle X, Y \rangle_{l^2} = \sum_{n=-\infty}^{\infty} x_n \bar{y}_n$. **Schwartz Inequality:** $|\langle X, Y \rangle| \leq \|X\| \|Y\|$. **Triangle Inequality:** $\|X + Y\| \leq \|X\| + \|Y\|$. **Orthogonal Projection:** Suppose V is an inner product space and V_0 is an N -dimensional subspace with orthonormal basis $\{e_1, e_2, \dots, e_N\}$. The orthogonal projection of a vector $v \in V$ onto V_0 is given by $v_0 = \sum_{j=1}^N \langle v, e_j \rangle e_j$. In addition, $\|v - v_0\| = \min_{w \in V_0} \|v - w\|$. **Adjoint:** If $T : V \rightarrow W$ is a linear operator between two inner product spaces, the adjoint of T is the linear operator $T^* : W \rightarrow V$, such that $\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V$.

CHAPTER 1: FOURIER SERIES

Real Fourier Series

Orthonormal Basis: The set of functions $\{\frac{\sin(k\pi x/a)}{\sqrt{\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{\cos(k\pi x/a)}{\sqrt{\pi}}\}$ with $k = 1, 2, \dots$, is an orthonormal set of functions in $L^2([-a, a])$. **Fourier Coefficients:** If $f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi t/a) + \sum_{k=1}^{\infty} b_k \sin(k\pi t/a)$ on the interval $-a \leq t \leq a$, then $a_0 = \frac{1}{2a} \int_{-a}^a f(t)dt$, $a_k = \frac{1}{a} \int_{-a}^a f(t) \cos(k\pi t/a)dt$ and $b_k = \frac{1}{a} \int_{-a}^a f(t) \sin(k\pi t/a)dt$.

Complex Fourier Series

Orthonormal Basis: The set of functions $\{\frac{1}{\sqrt{2a}} e^{i \frac{n\pi}{a} t}, n = 0, \pm 1, \pm 2, \dots\}$ is an orthonormal basis for $L^2([-a, a])$. **Fourier Coefficients:** If $f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i \frac{n\pi}{a} t}$, then $\alpha_n = \frac{1}{2a} \int_{-a}^a f(t) e^{-i \frac{n\pi}{a} t} dt$.

Convergence Theorems

Riemann-Lebesgue Lemma: Suppose f is a piecewise continuous function on the interval $[a, b]$. Then $\lim_{k \rightarrow \infty} \int_a^b f(x) \cos(kx)dx = \lim_{k \rightarrow \infty} \int_a^b f(x) \sin(kx)dx = 0$. **Convergence at a Point of Continuity:** Suppose f is a continuous and 2π -periodic function. Then for each point x , where the derivative of f is defined, the Fourier series of f converges to $f(x)$. **Convergence at a Point of Discontinuity:** Suppose f is periodic function and piecewise continuous. Suppose x is a point where f is left and right differentiable (but not necessarily continuous). Then the Fourier series of f at x converges to $\frac{f(x-0) + f(x+0)}{2}$, i.e., converges to the average of the left and right limits of f . **Uniform Convergence:** The Fourier series of a continuous, piecewise smooth 2π -periodic function $f(x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$. **Lemma 1.33:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$ with $\sum_{k=1}^{\infty} |a_k| + |b_k| < \infty$. Then the Fourier series converges uniformly and absolutely to the function $f(x)$. **Convergence in the Mean:** Suppose f is an element of $L^2([-\pi, \pi])$. Let $f_N(x) = a_0 + \sum_{k=1}^N a_k \cos(kx) + \sum_{k=1}^N b_k \sin(kx)$, where a_k and b_k are the Fourier coefficients of f . Then f_N converges to f in $L^2([-\pi, \pi])$, that is, $\|f_N - f\|_{L^2} \rightarrow 0$ as $N \rightarrow \infty$. **Parseval's Equation - Real Version:** Suppose $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx) \in L^2[-\pi, \pi]$. Then $\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = 2|a_0|^2 + \sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2)$. **Parseval's Equation - Complex Version:** Suppose $f(x) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikx} \in L^2[-\pi, \pi]$. Then $\frac{1}{2\pi} \|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |\alpha_k|^2$.

CHAPTER 2:FOURIER TRANSFORM

Definition: If f is a continuously differentiable function with $\int_{-\infty}^{\infty} |f(t)|dt < \infty$, then $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda$, where $\hat{f}(\lambda)$ is the Fourier transform of $f(t)$ given by $\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt$. **Properties:**

- $\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g]$ // $\mathcal{F}^{-1}[\alpha f + \beta g] = \alpha \mathcal{F}^{-1}[f] + \beta \mathcal{F}^{-1}[g]$
- $\mathcal{F}[t^n f(t)](\lambda) = i^n \frac{d^n}{d\lambda^n} \{\mathcal{F}[f](\lambda)\}$
- $\mathcal{F}^{-1}[\lambda^n f(\lambda)](\dots)(t) = (-i)^n \frac{d^n}{dt^n} \{\mathcal{F}^{-1}[f](t)\}$
- $\mathcal{F}[f^{(n)}(t)](\lambda) = (i\lambda)^n \mathcal{F}[f](\lambda)$
- $\mathcal{F}^{-1}[f^{(n)}(\lambda)](t) = (-it)^n \mathcal{F}^{-1}[f](t)$
- $\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a} \mathcal{F}[f](\lambda)$
- $\mathcal{F}[f(bt)](\lambda) = \frac{1}{b} \mathcal{F}[f](\frac{\lambda}{b})$
- If $f(t < 0) = 0$, then $\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \mathcal{L}[f](i\lambda)$, where $\mathcal{L}[f](s) = \int_0^{\infty} f(t) e^{-ts} dt$.

Convolution: Suppose f and g are two square integrable functions. The convolution of f and g is defined by $(f * g)(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx = \int_{-\infty}^{\infty} f(x)g(t-x)dx$. **Fourier Transform of the Convolution:** $\mathcal{F}[f * g] = \sqrt{2\pi} \mathcal{F}[f] \cdot \mathcal{F}[g]$, $\mathcal{F}^{-1}[\hat{f} \cdot \hat{g}] = \frac{1}{\sqrt{2\pi}} (f * g)$. **Pancherel Theorem:** The Fourier transform, and its inverse, preserves the L^2 inner product. $\langle \mathcal{F}[f], \mathcal{F}[g] \rangle_{L^2} = \langle f, g \rangle_{L^2}$ and $\langle \mathcal{F}^{-1}[f], \mathcal{F}^{-1}[g] \rangle_{L^2} = \langle f, g \rangle_{L^2}$.

Linear Filters

Time Invariance: A transformation L (mapping signals to signals) is said to be time-invariant if for any signal f and any real number a , $L[f_a](t) = (Lf)(t-a)$ for all t . In other words, L is time-invariant if the time shifted input signal $f(t-a)$ is transformed by L into the time shifted output signal $(Lf)(t-a)$. **Lemma 2.16:** Let L be a linear, time-invariant transformation and let λ be any fixed real number. Then, there is a function h with $L(e^{i\lambda t}) = \sqrt{2\pi} \hat{h}(\lambda) e^{i\lambda t}$. In other words, the output signal from a time-invariant filter of a sinusoidal input is also sinusoidal with the same frequency. **Convolution in Filters:** Let L be a linear, time-invariant transformation on the space of signals that are piecewise continuous functions. Then there exists an integrable function, h , such that $L(f) = f * h$ for all signals f . **Causal Filters:** A causal filter is one for which the output signal begins after the input signal has started to arrive. Let L be a time-invariant filter with response function h (i.e., $Lf = f * h$). L is a causal filter if and only if $h(t) = 0$ for all $t < 0$. **Theorem 2.20:** Suppose L is a causal filter with response function h . Then the system function associated with L is $\hat{h}(\lambda) = \frac{\mathcal{L}[h](i\lambda)}{\sqrt{2\pi}}$.

The Sampling Theorem

Definition 2.22: A function f is said to be frequency band limited if there exists a constant $\Omega > 0$ such that $f(\lambda) = 0$ for $|\lambda| > \Omega$. Note: Ω is the smallest frequency for which the preceding equation is true. **Shannon-Whittaker Sampling Theorem:** Suppose that $\hat{f}(\lambda)$ is piecewise smooth and continuous and that $\hat{f}(\lambda) = 0$ for $|\lambda| > \Omega$, where Ω is some fixed, positive frequency. Then $f = \mathcal{F}^{-1}[f]$ is completely determined by its values at the points $t_j = \frac{j\pi}{\Omega}, j = 0, \pm 1, \pm 2, \dots$. More precisely, f has the following series expansion: $f(t) = \sum_{j=-\infty}^{\infty} f\left(\frac{j\pi}{\Omega}\right) \frac{\sin(\Omega t - j\pi)}{\Omega t - j\pi}$, where the series converges uniformly.

CHAPTER 3: DISCRETE FOURIER TRANSFORM

Set of n-periodic sequences: Let \mathcal{S}_n be the set of n -periodic sequences of complex numbers. Each element $y = y_j \in \mathcal{S}_n$, can be thought of as a periodic discrete signal where y_j is the value of the signal at a time node $t = t_j$. The sequence y_j is n -periodic if $y_{k+n} = y_k$ for any integer k . **Definition:** Suppose $y = y_k$ is an element of \mathcal{S}_n . Let $\hat{\mathcal{F}}_n(y) = \hat{y}$. That is, $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$, where $w = e^{\frac{2\pi i}{n}}$. Then $y = \mathcal{F}^{-1}(\hat{y})$ is given by $y_j = \frac{1}{n} \sum_{k=0}^{n-1} \hat{y}_k w^{jk}$. **Properties:**

- Shifts or translations. If $y \in \mathcal{S}_n$ and $z_k = y_{k+1}$, then $\mathcal{F}[z]_j = w^j \mathcal{F}[y]_j$
- Convolutions. If $y \in \mathcal{S}_n$ and $z \in \mathcal{S}_n$, then the sequence $[y * z]_k := \sum_{j=0}^{n-1} y_j z_{k-j}$ is also in \mathcal{S}_n . The sequence $y * z$ is called the convolution of the sequences y and z .
- The Convolution Theorem. $\mathcal{F}[y * z]_k = \mathcal{F}[y]_k \mathcal{F}[z]_k$
- If $y \in \mathcal{S}_n$ is a sequence of real numbers, then $\mathcal{F}[y]_{n-k} = \overline{\mathcal{F}[y]_k}$, for $k \in [0, n-1]$, or $\hat{y}_{n-k} = \bar{\hat{y}}_k$

APPENDIX

Identities: $\sin^2 x = (1 - \cos 2x)/2, \cos^2 x = (1 + \cos 2x)/2, e^{ix} = \cos x + i \sin x, e^{-ix} = \cos x - i \sin x, 2 \cos x = (e^{ix} + e^{-ix}), 2i \sin x = (e^{ix} - e^{-ix}), 2 \cosh x = (e^x + e^{-x}), 2 \sinh x = (e^x - e^{-x}), \cosh^2 x - \sinh^2 x = 1$. **Sum and Difference Formula:** $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B, \cos(A \mp B) = \cos A \cos B \pm \sin A \sin B, \tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$. **Double Angle Formula:** $\sin(2A) = 2 \sin A \cos A, \cos(2A) = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A, \tan(2A) = (2 \tan A)/(1 - \tan^2 A)$. **Sum to Product:** $\sin A \pm \sin B = 2 \sin((A \pm B)/2) \cos((A \mp B)/2), \cos A - \cos B = -2 \sin((A+B)/2) \sin((A-B)/2), \cos A + \cos B = 2 \cos((A+B)/2) \cos((A-B)/2)$. **Geometric Sum:** $\sum_{k=0}^N z^k = \frac{1-z^{N+1}}{1-z}, \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. **General ODE Solutions:** $y'' = y(t) \implies y = c_1 e^{-t} + c_2 e^t \quad \square \quad dy/dt + p(t)y = g(t) \implies y = (\int u(t)g(t))/u(t) + c$ where $u(t) = \exp(\int p(t)dt) \quad \square \quad y' = x; x' = y \implies x = c_1 \cosh t + c_2 \sinh t, y = c_1 \sinh t + c_2 \cosh t$ or $x = c_1 e^t + c_2 e^{-t}, y = c_1 e^t - c_2 e^{-t} \quad \square \quad y' = -x; x' = y \implies y = c_1 \cos t + c_2 \sin t, x = c_1 \sin t - c_2 \cos t \quad \square \quad x' = x+y; y' = -x+y \implies x = e^t (c_1 \cos t + c_2 \sin t); y = e^t (-c_1 \sin t + c_2 \cos t) \quad \square \quad v' = \gamma v, v(z, 0) = u_0 \implies v = u_0 e^{\gamma t} \quad \square$