This PDE, $\sum_{j=1}^n a_j(x,u(x))\partial u/\partial x_j(x) = c(x,u(x))$, is called **quasilinear** if the solution u appears in the coefficients a_j which multiply the partial derivatives of u. If u is not in a_j, but appears in c in a nonlinear way like $c(x,u)=\gamma(x)u^2$ or $c(x,u)=\gamma(x)e^u$, the PDE is called **semilinear**. If u appears linearly in c like $c(x,u)=\gamma(x)u$, the PDE is called **homogeneous linear**. If u appears in c like $c(x,u)=\gamma(x)u+\tilde{\gamma}(x)$, then the PDE is called **inhomogeneous linear**. The heat equation $u_t = cu_{xx}+f(x,t)$ is a linear **parabolic** PDE. The wave equation $u_{tt}=c^2u_{xx}+f(x,t)$ is a linear **hyperbolic** PDE. The Poisson equation, $\Delta u + f(x) = 0$, is an **elliptic** PDE. $\Delta u = u_{xx}$. **3.1P** Let u be a solution of $\sum_{j=1}^n a_j(x,u(x))\partial_j u(x) = c(x,u(x))$ and let u, a_j and c be continuously differentiable. Let $\xi_1,\dots\xi_n, v : I \to \mathbb{R}$ be a solution of the ODE system $\xi_j' = a_j(\xi,v), j=1,\dots n, v' = c(\xi,v)$, where $\xi(t) = (\xi_1(t),\dots,\xi_n(t))$ and I some interval. Then $v(t)=u(\xi(t))$ for all $t\in I$, if it holds for at least one $t_0 \in I$. **Example3.4**. Find a solution u of two real variables x,y to $xu_x+(x+y)u_y = u+1, u(x,0)=x^2$. Solution. In the language used before, this is $x_1\partial_1 u + (x_1+x_2)\partial_2 u = u+1, u(x_1,0) = x_1^2$, a nonhomogeneous linear PDE. We identify the hypersurface $S=\{(x,0); x\in\mathbb{R}\}$. So $S=g(\mathbb{R})$ with $g(z)=(z,0)$. We further identify $u_0(z,0)=z^2, z\in\mathbb{R}$. The characteristic system is $\partial_t\xi_1 = \xi_1, \partial_t\xi_2 = \xi_1+\xi_2, \partial_t v = v+1$, with the initial conditions $\xi_1(z,0)=z, \xi_2(z,0)=0, v(z,0)=z^2$. We integrate the equation for $\xi_1, \xi_1(z,t)=ze^t$. We substitute this into the differential equation for $\xi_2, \partial_t = ze^t+\xi_2$. Recall the variation of constants formula or use an integrating factor. Since $\xi_2(z,0)=0, \xi_2(z,t)=\int_0^t ze^se^{t-s}ds = tze^t$. By the same token, $v(z,t)=v(z,0)e^t+\int_0^t e^sds = z^2e^t+e^t-1$, or set $w=v+1, \partial_t w = w, w(z,0)=z^2+1, w(z,t)=(z^2+1)e^t, v(z,t)=(z^2+1)e^t-1$. In order to find u with $v(z,t)=u(\xi(z,t))$ (3.10), we solve $x=\xi_1=ze^t, y=\xi_2=tze^t$ for z and t, with x,y being given, Notice that $y/x=t$ and so $z=xe^{-t}=xe^{-y/x}$. By (3.10), $u(x,y)=v(z,t)=(xe^{-y/x})^2e^{y/x}+e^{y/x}-1 = x^2e^{-y/x}+e^{y/x}-1$. It is readily checked that u solves our Cauchy problem for $x\neq 0$. $\square$ **3.5T** Let $S=g(\Omega)$ be a hypersurface in $\mathbb{R}^n$. Let $\xi_1,\dots,\xi_n,v$ be a solution of the characteristic system (ODE) $\{\partial_t\xi_j = a_j(\xi,v), j=1,\dots,n, \partial_t v = c(\xi,v)\}$ on $V\times I$, (IC) $\{\xi(z,0)=g(z), v(z,0)=u_0(g(z))\} z\in V$, where V is an open subset of $\Omega\subset\mathbb{R}^{n-1}$ and I an open interval containing 0. Suppose that v is differentiable on $V\times I$ and that there exists some open set U in $\mathbb{R}^n$ such that $\xi=(\xi_1,\dots,\xi_n)$ is one-to-one and onto from $V\times I$ to $U, U\cap S = g(V)$, and that the inverse ξ^{-1} is differentiable on U. Then the function $u: U\to\mathbb{R}$ defined by $u(x)=v(\xi^{-1}(x)), x\in U$, is a solution of $\sum_{j=1}^n a_j(x,u)\partial_j u = c(x,u), x\in U, u(x)=u_0(x), x\in U\cap S$. If $c,a,\dots,a_n$ are partially differentiable in all variables and these partial derivatives are continuous, then u is the unique solution. $\square$ **3.7T** Let $S=g(\Omega)$ be a hypersurface in $\mathbb{R}^n$ with $g:\Omega\to\mathbb{R}^n$, and $u_0: S\to\mathbb{R}$. Let $\check{z}\in\Omega$ and $\check{x}=g(\check{z})$. Assume that g and $u_0\circ g$ are continuously differentiable in an open neighborhood of $\check{z}$ contained in Ω. Further let $a_1,\dots,a_n,c$ be defined and continuously differentiable in an open neighborhood of $(\check{x},u_0(\check{x}))$ in $\mathbb{R}^{n-1}$. Finally let $\det(g'(\check{z}), a(\check{x},u_0(\check{x})))\neq 0$ with $g'(z) = \begin{pmatrix}\partial_1 g_1(z) & \cdots & \partial_{n-1}g_1(z)\\ \vdots & & \vdots \\ \partial_1 g_n(z) & \cdots & \partial_{n-1}g_n(z)\end{pmatrix}$ and $a(x,v)=\begin{pmatrix}a_1(x,v)\\ \vdots \\ a_n(x,v)\end{pmatrix}$. Then there exists an open neighborhood U of $\check{x}$ and a uniquely determined function $u: U\to\mathbb{R}$ such that $\sum_{j=1}^n a_j(x,u)\partial_j u = c(x,u), x\in U, u(x)=u_0(x), x\in U\cap S$. We call the determinant in this theorem the characteristic determinant. **Eq (3.17)** (PDE) $\partial u/\partial t + \sum_{j=1}^{n-1} b_j(t,u)\partial u/\partial y_j = \gamma u, y\in\mathbb{R}^{n-1}, t>0$ (IC) $u(y,0)=u_0(y), y\in\mathbb{R}^{n-1}$. **3.11T** Let $b:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ be continuous and have continuous partial derivatives b_u. Further let u_0 be continuously differentiable. Assume that $T>0$ and $\zeta(z,t)\to\pm\infty, z\to\pm\infty, t\in[0,T)$, and $\zeta_z(z,t)>0$ for all $z\in\mathbb{R}$ and $t\in[0,T)$. Then the Cauchy problem (3.17) has a unique solution on $\mathbb{R}\times[0,T)$. The solution u satisfies $u(\zeta(z,t),t)=u_0(z)e^{\gamma t}, z\in\mathbb{R}, t\in[0,T)$. *Proof.* It follows from the preceeding considerations that, for fixed $t\in[0,T)$, the function $\zeta(\cdot,t)$ is bijective from $\mathbb{R}$ to $\mathbb{R}$. So there exists a function $\phi:\mathbb{R}\times[0,T)\to\mathbb{R}$ such that $\zeta(\phi(z,t),t)=z, \phi(\zeta(z,t),t)=z, z\in\mathbb{R}, t\in[0,T)$. It follows from our assumptions that ζ is continuously differentiable. By the implicit function theorem, ϕ is continuously differentiable. Define $u(y,t)=v(\phi(y,t),t)$. By Theorem 3.5, with $\xi(z,t)=(\zeta(z,t),t)$ and $\xi^{-1}(y,t)=(\phi(y,t),t)$, u is differentiable and satisfies (3.17) on $\mathbb{R}\times[0,T)$. $\square$ **3.12C** Let $b:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ be continuous and have continuous partial derivatives b_u and let $u_0:\mathbb{R}\to\mathbb{R}$ be continuously differentiable. Assime that $b_u\geq 0$ on $\mathbb{R}\times[0,T)$ and $u_0'\geq 0$ on $\mathbb{R}$ (or $b_u\leq 0$ and $u_0'\leq 0$). Then there exists a unique solution to (3.17) on $\mathbb{R}\times[0,\infty)$. **d'Alembert's solution** $u(x,t)=(1/2)(f(x+ct)+f(x-ct)+(1/c)\int_{x-ct}^{x+ct} g(y)dy)$. *Proof.* Let $z\geq 0$. Then $b(s,u)$ is an increasing function of u and u_0 is an increasing function and, by (3.20), $\zeta(z,t)\geq z+\int_0^t b(s,u_0(0)e^{\gamma s})ds\to\infty, z\to\infty$. Let $z\leq 0$. Then by the same toekn, $\zeta(z,t)\leq z+\int_0^t b(s,u_0(0)e^{\gamma s})ds\to-\infty, z\to-\infty$. Further, by (3.23), $\zeta_z(z,t)\geq 1$. The statemet now follows frim Theorem 3.11. $\square$ **3.13L** The extended f is $2L$-periodic and odd around 0 and L. *Proof.* By construction, f is $2L$-periodic. Indeed, let $x\in\mathbb{R}$. Then $x=y+2kL$ with $-L\leq y\leq L$ and $k\in\mathbb{Z}$. By the extension, $f(x+2L)=f(y+2(k+1)L)=f(y)=f(y+2kL)=f(x)$. Further $f(-x)=f(-y-2kL)=f(-y)=-f(y)=-f(x)$. So f is odd around 0. f is also odd around L, i.e. $f(L+x)=-f(L-x), x\in\mathbb{R}$. Indeed, since f is odd about 0 and $2L$-periodic, $f(L+x)=-f(-L-x)=-f(L-x)$. $\square$ **3.14T** Let $f,g:[0,L]\to\mathbb{R}$. Extend f and g in an odd and $2L$-periodic fashion. Then the d'Alembert formula provides a solution of the vibrating string equations provided that f is twice differentiable, g is once differentiable, and $f(0)=0=f(L), f''(0)=0=f''(L), g(0)=0=g(L)$. **Proof**. As we mentioned before,the conditions for f and g imply that their extensions to $\mathbb{R}$ are twice and once differentiable, respectively. So the d'Alembert formula provides a solution to the PDE and the initial conditions. We check the boundary condition at $L, u(L,t)=(1/2)(f(L+ct)+f(L-ct))+(1/2c)\int_{L-ct}^{L+ct} g(s)ds$. Since f is odd around L, let $s=r+L, u(L,t)=(1/2c)\int_{-ct}^{ct} g(L+r)dr = (1/2c)\int_0^{ct} g(L+s)+g(L-s))ds$ after splitting the integral at 0 and changing of variables. Since g is odd around $L, u(x,t)=0$. The boundary condition at $x=0$ is checked similarly $\square$ **Inhomogeneous wave equation** $u(x,t)=(1/2c)\int_0^t(\int_{x-c(t-s)}^{x+c(t-s)}\phi(\rho,s)d\rho)ds$. **Leibniz rule** $d/dx\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x)-f(g(x))g'(x)$ **Int by parts** $\int udv = uv-\int vdu$ **E3.1.1** (a) Let $f:\mathbb{R}_+\to\mathbb{R}$ be differentiable. Show: The function $u:\mathbb{R}^2\to\mathbb{R}$ defined by $u(x,y)=f(x^2+y^2)$ satisfies the PDE $yu_x-xu_y=0$. (b) Assume that a differentiable function $u:\mathbb{R}^2\to\mathbb{R}$ satisfies the PDE $yu_x-xu_y=0$. Show: $u(x,y)=f(x^2+y^2)$ for all $x,y\in\mathbb{R}$ with some function $f:\mathbb{R}_+\to\mathbb{R}$. Hint: Consider $w(z,t)=u(z\cos t, z\sin t)$. Proof: (a) Define $u(x,y)=f(x^2+y^2)$ for $x,y\in\mathbb{R}$. Then $yu_x-xu_y = y2xf'-x2yf'=0$. So u is a solution of the PDE. (b) Let u be a solution of the PDE. Set $w(z,t)=u(z\cos t,z\sin t)$. Then $\partial_t w(z,t) = u_x(z\cos t,z\sin t)(-z\sin t)+u_y(z\cos t,z\sin t)(z\cos t)=0$. So $w(z,t)=\tilde{f}(z)$ with an appropriate function f and $u(z\cos t,z\sin t)=f(z)$. If $x=z\cos t$ and $y=z\sin t$, then $z^2=x^2+y^2$. So $u(x,y)=\tilde{f}(\sqrt{x^2+y^2})=f(x^2+y^2)$ with $f(r)=\tilde{f}(\sqrt{r})$. $\square$ **E3.1.2** Solve the Cauchy problem $-yu_x+xu_y=0, u(x,x^2)=x^3$. **Solution**. The curve $y=x^2$ is parameterized by $g(z)=(z,z^2)'$. The characteristic equations are $\partial_r\xi_1=-\xi_2, \xi_1(z,0)=z, \partial_r\xi_2=\xi_1, \xi_2(z,0)=z^2, \partial_r v=0, v(z,0)=z^3$. We solve the last equation to get $v=c$ and initial conditions gives $v=z^3$. We look at the first two equations, $\partial_r^2\xi_1=-\xi_1$. The general solution of this linear second order PDE is $\xi_1(z,r)=c_1\cos r+c_2\sin r$. Using initial conditions we get $\xi_1(z,0)=z=c_1\cos(0)+c_2\sin(0)=c_1 \implies c_1=z$. $\partial_r\xi_1(z,r)=-z\sin r+c_2\cos r=-\xi_2$ and $\xi_2(z,0)=z^2$ so $-z\sin(0)+c_2\cos(0)=-z^2 \implies c_2=-z^2$. So we solve $x=z\cos r-z^2\sin r, y=zsinr+z^2\cos r$. We try $x^2+y^2=z^2\cos^2 r-2z^3\cos r\sin r+z^4\sin^2 r+z^2\sin^2 r+2z^3\cos r\sin r+z^4\cos^2 r=z^2+z^4$. So $z^4+z^2-(x^2+y^2)=0$. Notice that this is a quadratic equation in z^2 so $z^2=(-1\pm\sqrt{1+4(x^2+y^2)})/2 \implies z=\sqrt{(-1\pm\sqrt{1+4(x^2+y^2)})/2}$ which gives $v=((-1\pm\sqrt{1+4(x^2+y^2)})/2)^{3/2}$. We check initial conditions for the sign. $((-1\pm\sqrt{1+4x^2+4x^4})/2)^{3/2}=((-1\pm\sqrt{(2x^2+1)(2x^2+1)})/2)^{3/2}=((-1\pm(2x^2+1))/2)^{3/2}$. We take the positive to get $(x^2)^{3/2}=x^3$ so $u(x,y)=((-1\pm\sqrt{1+4(x^2+y^2)})/2)^{3/2}$. **E3.1.5** Solve $-x_2\partial_1 u+x_1\partial_2 u=u$ on $\mathbb{R}^2, u(x_1,0)=u_0(x_1^2), x_1\in\mathbb{R}$. Answer: $u=u_0(x_1^2+x_2^2)\exp(\arctan(x_2/x_1))$. Solution. The hypersurface (in this case a curve) is parameterized by $g(z)=(z;0)$. The equations for the characteristic curves are $\partial_r\xi_1=-\xi_2, \xi_1(z,0)=z, \partial_r\xi_2=\xi_1, \xi_2(z,0)=0, \partial_r v=v, v(z,0)=u_0(z^2)$. Hence $\partial_r^2\xi_1=-\xi_1, \xi_1(z,0)=z, \partial_r\xi_1(z,0)=0$. The general solution of this linear second order ODE is $\xi_1(z,r)=c_1(z)\cos(r)+c_2(z)\sin(r)$. Using the initial conditions we find, $\xi_1(z,r)=z\cos r$, and $\xi_2(z,r)=-\partial_r\xi_1(z,r)=z\sin r$. Finally $v(z,r)=u_0(z^2)e^r$. From $x_1=z\cos r, x_2=z\sin r$ we have $z^2=x_1^2+x_2^2, x_2/x_1=\tan r$. This implies the above answer. $\square$ **E3.1.7** Determine the solution of $u=u(y,t)$ of $yu_y+uu_t=t, y,t\in\mathbb{R}, u(y,0)=1, y\in\mathbb{R}$. **Solution**. Characteristic curves are $\partial_r\xi_1=\xi_1, \xi_1(z,0)=z, \partial_r\xi_2=v, \xi_2(z,0)=0, \partial_r v=\xi_2, v(z,0)=1$. We solve the first equation, $\xi_1=c_1e^r$ with initial conditions we have $\xi_1=ze^r$. The general solution of the second two equations is $\xi_2=c_1\cosh r+c_2\sinh r, v=-c_1\sinh r+c_2\cosh r$. When we apply the initial conditions we get $\xi_2=\sinh r, v=\cosh r$. Now we solve $y=ze^r$ and $t=\sinh r$. We apply identities to get $2x_2=e^r-e^{-r}$. We multiply through by e^r and rearrange to get $(e^r)^2-2x_2(e^r)-1=0$. This is a quadradic equation in e^r so we get $e^r=(1/2)2t\pm\sqrt{4t^2-4(1)(-1)}=t\pm\sqrt{t^2+1}$. $u(y,t)=v(z,r)=\cosh r=(1/2)e^r+e^-r)=(1/2)(t\pm\sqrt{t^2+1}+(t\pm\sqrt{t^2+1})^{-1})$. We check the initial condition and see that we have $u(y,t)=(1/2)(t+\sqrt{t^2+1}+(t+\sqrt{t^2+1})^{-1})$ which simplifies to $\sqrt{t^2+1}$. **E3.1.X** Solve $-x_2\partial_1 u+x_1\partial_2 u=u$ on $\mathbb{R}^2, u(z,z)=v_0(z^2), z\in\mathbb{R}$, with $v_0:\mathbb{R}\to\mathbb{R}$. Solution. The hypersurface (in this case a curve) is parameterized by $g(z)=(z;z)$. The equations for the characteristic curves are $\partial_r\xi_1=-\xi_2, \xi_1(z,0)=z, \partial_r\xi_2=\xi_1, \xi_2(z,0)=z, \partial_r v=v, v(z,0)=v_0(z^2)$. Hence $\partial_r^2\xi_1=-\xi_1, \xi_1(z,0)=z, \partial_r\xi_1(z,0)=0$. The general solution of this linear second order ODE is $\xi_1(z,r)=c_1(z)\cos(r)+c_2(z)\sin(r)$ and $\xi_2(z,r)=-\partial_r\xi_1(z,r)=c_1\sin(r)-c_2\cos(r)$. Finally $v(z,r)=v_0(z^2)e^r$. From the initial conditions, $z=c_1(z), c_2(z)=-z$. So $\xi_1(z,r)=z(\cos r-\sin r), \xi_2(z,r)=z(\cos r+\sin r)$. From the differential equations and initial conditions for ξ, we find $\xi_1^2+\xi_2^2=2z^2$. To find the inverse of ξ, we solve $x_1=z(\cos r-\sin r), x_2=z(\cos r+\sin r)$. We already know $x_1^2+x_2^2=2z^2$. Further $x_1+x_2=2z\cos r, x_2-x_1=2z\sin r$, and so $\tan r=(x_2-x_1)/(x_2+x_1)$. We obtain $u(x_1,x_2)=v_0((1/2)(x_1^2+x_2^2))exp(arctan((x_2-x_1)/(x_2+x_1))$. $\square$ **E3.1.10** Solve $\sum_{j=1}^n x_j^2\partial_j u = \alpha u, u(x_1,\dots,x_{n-1},b)=v_0(x_1,\dots,x_{n-1}), x_1,\dots,x_{n-1}\in$

$\mathbb{R}$, where $v_0 : \mathbb{R}^{n-1} \to \mathbb{R}$ is a given function and $b > 0$ and α are given real numbers. Where is the solution defined? Determine the characteristic determinant and ponder whether there is a connection between your result and where the solution is defined. Solution. The equations for the characteristic curves take the form $\partial_t \xi_j(z,t) = \xi_j^2(z,t), \xi_j(z,0) = z_j, j = 1,\dots,n-1, \xi_n(z,0) = b. \partial_t v(z,t) = \alpha v(z,t), v(z,0) = u_0(z)$. The equations are solved by $\xi_j(z,t) = 1/((1/z_j) - t), z_j \neq 0, \xi_j(z,t) = 0, z_j = 0, j = 1,\dots,n-1, \xi_n(z,t) = 1/((1/b) - t), v(z,t) = u_0(z)e^{\alpha t}$. In order to find the inverse function of ξ, we solve the system $x_j = 1/((1/z_j) - t), j = 1,\dots,n-1, x_n = 1/((1/b) - t)$. Hence $t = (1/b) - (1/x_n)$ and $z_j = 1/((1/x_j) + t) = 1/((1/x_j) - (1/x_n) + (1/b)) = x_j/(1 - (x_j/x_n) + (x_j/b))$. Notice that the last expression gives us $z_j = 0$ iff $x_j = 0$. As $u(x) = v(z,t)$ we obtain $u(x) = u_0((x_j/(1 - (x_j/x_n) + (x_j/b)))_{1 \le j \le n-1}) \exp((\alpha/b) - (\alpha/x_n))$. Since the initial condition is posed at $x_n = b > 0$ and $x_n \neq 0$ to make the solution defined, we impose $x_n > 0$ on the domain of definition. Further, if $x_n \neq b$, we require $x_j \neq (x_n b)/(b - x_n), j = 1,\dots,n-1$. ☐ **E3.1.12** Solve $(y + x)u_x + (y - x)u_y = u, u = 1$ on the circle $x^2 + y^2 = 1$. Proof. The characteristic system is (for the time being we ignore the initial conditions) $\partial_t \xi_1 = \xi_1 + \xi_2, \partial_t \xi_2 = -\xi_1 + \xi_2, \partial_t v = v$. The first two equations form a linear subsystem with matrix $A = [1\ 1; -1\ 1]$. We solve the characteristic equation $0 = [(1-\lambda)\ 1; -1\ (1-\lambda)] = (\lambda - 1)^2 + 1$. The solutions of the characteristic equation are $\lambda = 1 \pm i$. So ξ_1 is of the form $\xi_1(z,t) = e^t(c_1(z)\cos t + c_2(z)\sin t)$. Since $\xi_2 = \partial_t \xi_1 - \xi_1, \xi_2(z,t) = e^t(-c_1(z)\sin t + c_2(z)\cos t)$. We parameterize the initial surface by $[\cos z; \sin z] = g(z) = \xi(z,0)$. This yields $c_1(z) = \cos z, c_2(z) = \sin z$ and $\xi_1(z,t) = e^t(\cos z \cos t + \sin z \sin t) = e^t \cos(z - t), \xi_2(z,t) = e^t(-\cos z \sin t + \sin z \cos t) = e^t \sin(z - t)$. For v we obtain, $v(z,t) = e^t$. To find u, we solve $x = e^t \cos(z-t), y = e^t \sin(z-t)$. This yields $x^2 + y^2 = (e^t)^2$. So $u(x,y) = \pm\sqrt{x^2+y^2}$. Because $u = 1$ on the circle with radius $1, u(x,y) = \sqrt{x^2+y^2}$. **E3.1.14** Solve $\partial_1 u + u\partial_2 u = 0, u(x_1,x_2) = \gamma$ on the line $x_1 = x_2$. For which γ can you solve the problem? Determine the characteristic determinant and ponder whether there is a connection. Proof. By inspection, $u(x_1,x_2) = \gamma$ for all $x_1,x_2 \in \mathbb{R}$ is a solution. To check whether this is the only solution, we solve the characteristic system $\partial_t \xi_1(z,t) = 1, \xi_1(z,t) = z, \partial_t \xi_2(z,t) = v, \xi_2(z,t) = z, \partial_t v(z,t) = 0, v(z,0) = \gamma$. So $\xi_1(z,t) = t + z, v(z,t) = \gamma, \xi_2(z,t) = \gamma t + z$. The same proof as for Proposition 3.1 shows that $u(\xi(z,t)) = v(z,t) = \gamma$. So any solution only takes the value γ. The characteristic determinant is given by $[1\ 1; 1\ \gamma] = \gamma - 1$. While we cannot invert ξ if $\gamma = 1$, in this case a zero characteristic determinant does not indicate that there is a problem with existence or uniqueness. ☐. **E3.1.15** Determine all solutions $u = u(x_1,x_2)$ of $(1-u)\partial_{x_1} u + (1+u)\partial_{x_2} u = 1, x_1, x_2 \in \mathbb{R}, u(x_1,x_2) = 0, x_1 = x_2$. Where are the solutions defined? Interpret your results in the light of the general local existence theorem. *Proof.* We identify $g(z) = (z,z), z \in \mathbb{R}$ and $u_0(z,z) = 0$.The characteristic system is $\partial_t \xi_1 = 1 - v, \xi_1(z,0) = z, \partial_t \xi_2 = 1 + v, \xi_2(z,0) = z, \partial_t v = 1, v(z,0) = 0$. We solve the equation for v and get $v = t + c_v$. With initial conditions this gives us $v = t$. We put this into the equations for ξ_1 and ξ_2 to get $\partial_t \xi_1 = 1 - t \implies \xi_1 = c_1 + t - t^2/2, \partial_t \xi_2 = 1 + t \implies \xi_1 = c_1 + t + t^2/2$. When we add the initial conditions we get $\xi_1 = z + t - t^2/2$, and $\xi_2 = z + t + t^2/2$. To find u we solve $x_1 = z + t - t^2/2$ and $x_2 = z + t + t^2/2. x_2 - x_1 = t^2 \implies t = \pm\sqrt{x_2 - x_1}$.From our equation above we have that $v = t$ and so $u(x) = \pm\sqrt{x_2 - x_1}$. These solutions are only defined where $x_2 \ge x_1$. In the light of the general local existence theorem we find the determinant of the characteristic matrix. $det[1\ 1-u; 1\ 1+u] = 1 + u - (1-u) = 2u$. Since $u(x_1,x_2) = 0$ when $x_1 = x_2$, this equals zero when $x_1 = x_2$. So the assumptions of the general local existence theorem are not met and therefore we can have two solutions. **E3.2.1**. Consider the Cauchy problem $\partial_t u + b(t,u)\partial_y u = -\alpha u, \quad t > 0, y \in \mathbb{R}, u(y,0) = u_0(y)$ with $\alpha > 0$. Assume the following properties for the given functions $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, u_0 : \mathbb{R} \to \mathbb{R} : b, u_0$ are continuously differentiable, $|b_u(t,u)| \le c_1, |u_0'(y)| \le c_2$ for all $y,t,u \in \mathbb{R}$ where c_1, c_2 are positive constants satisfying $c_1 c_2 \le \alpha$. Show: There exists a solution $u = u(y,t)$ which is defined for all $t \ge 0, y \in \mathbb{R}$. **Solution**. This is a great candidate for Theorem 3.11. We have $\zeta(z,t) = z + \int_0^t b(s, u_0(z)e^{-\alpha s})ds$, and $\zeta_z(z,t) = 1 + u_0'(z)\int_0^t b_u(s, u_0(z)e^{-\alpha s})e^{-\alpha s}ds$. Now we apply the properties of absolute value to get $\zeta_z(z,t) \ge 1 - |u_0'(z)| \int_0^t |b_u(s, u_0(z)e^{-\alpha s})| e^{-\alpha s}ds$. Now we apply the given assumptions and do some manipulation $\zeta_z(z,t) \ge 1 - c_1 c_2 \int_0^t e^{-\alpha s}ds = 1 - (c_1c_2)/\alpha(1 - e^{-\alpha t}) = 1 - (c_1c_2)/\alpha + (c_1c_2)/\alpha e^{-\alpha t}$. And so by our assuption that $c_1c_2 \le \alpha$ we have that $\zeta_z(z,t) > (c_1c_2)/\alpha e^{-\alpha t} \ge e^{-\alpha t} > 0$, for $z \in \mathbb{R}$ and $t \ge 0$. Now we use the mean value theorem. For for $z \in \mathbb{R}$, with some $\tilde{z}$ betwen 0 and $z, \zeta(z,t) = \zeta(0,t) + z\zeta_z(\tilde{z},t)$. And so for $z > 0$, we have that $\zeta(z,t) \ge \zeta(0,t) + ze^{-\alpha t}$. Therefore $\zeta(z,t) \to \infty$ as $z \to \infty$. Moreover, for $z < 0, \zeta(z,t) \le \zeta(0,t) + ze^{-\alpha t}$. Therefore $\zeta(z,t) \to -\infty$ as $z \to -\infty$. And so by Theorem 3.11, there exists a solution $u = u(y,t)$ which is defined for all $t \ge 0, y \in \mathbb{R}$. ☐. **E3.2.2** Consider the Cauchy problem $\partial_t u + \cos(wt)u\partial_x u = 0, u(x,0) = u_0(x)$. Assume that u_0 is continuously differentiable on $\mathbb{R}$ and $\sup_x |u_0'(x)| \le M$ for some $M > 0$. (a) Show that the solution exists for all $t \ge 0$ provided that w is large enough. (b) What can be done if w is not sufficiently large? Solution. In order to apply Theorem 3.11, we identify $b(t,u) = \cos(wt)u, u_0(y) = f(y)$. By (3.20) $\zeta(z,t) = z + \int_0^t \cos(ws)u_0(z)ds = z + (1/w)\sin(wt)u_0(z)$. Further $\zeta_z(z,t) = 1 + u_0'(z)\int_0^t \cos(ws)ds = 1 + (u_0'(z)/w)\sin(wt)$. (a) $\zeta_z(z,t) \ge 1 - (|u_0'(z)|/w)|\sin(wt)| \ge 1 - (M/w)$. Choose $w > M$. Then $\zeta_z(z,t) > 0$ for all $z \in \mathbb{R}, t \ge 0$. Let $t \ge 0, z \in \mathbb{R}$. By the mean value theorem $\zeta(z,t) = \zeta(0,t) + z\zeta_z(\tilde{z},t)$ with some $\tilde{z}$ between 0 and z (which depends on z and t). If $z \ge 0, \zeta(z,0) \ge \zeta(z,0) + z(1 - (M/w)) \to \infty, z \to \infty$. If $z \le 0, \zeta(z,0) \le \zeta(z,0) + z(1 - (M/w)) \to -\infty, z \to -\infty$. So, by Theorem 3.11, there exists a solution u on $\mathbb{R} \times [0,\infty)$. (b) Alternatively, by (3.20), $\zeta_z(z,t) = 1 + u_0'(z)\int_0^t \cos(ws)ds \ge 1 - Mt$. Choose $T = 1/M$. Then $\zeta_z(z,t) > 0$ for all $t \in [0,T)$. Similarly as in (a), $\zeta(z,t) \to \pm\infty$ as $z \to \pm\infty$. By Theorem 3.11, there exists a unique solution u on $\mathbb{R} \times [0, 1/M)$. ☐ **E3.3.2** Solve the wave equation $\partial_t^2 u - c^2\partial_x^2 u = 0, x,t \in \mathbb{R}, u(x,0) = f(x), u(cx,x) = g(x), x \in \mathbb{R}$ where $f, g : \mathbb{R} \to \mathbb{R}$. State appropriate assumptions for f and g such that you really have a solution. Proof. The general solution for this wave equation is $u(x,t) = F(x+ct) + G(x-ct)$. F and G are to be determined from the inital and diagonal data, $f(x) = u(x,0) = F(x) + G(x), g(x) = u(cx,x) = F(2cx) + G(0)$. Replacing $2cx$ by x in the second equation, $f(x) = F(x) + G(x), g(x/(2c)) = F(x) + G(0)$. We subtract the equations, $f(x) - g(x/(2c)) = G(x) - G(0)$. We substitute this result into the first equation and rearrange, $F(x) = f(x) - G(x) = g(x/(2c)) - G(0)$. We substitute this into the general solution, $u(x,t) = g((x+ct)/(2c)) - g((x-ct)/(2c)) + f(x-ct)$. For u to be twice differentiable, we need f and g to be twice differentiable. u satisfies the diagonal condition iff $f(0) = g(0)$. ☐ **E3.3.6** Let u be a solution of the wave equation $(\partial_t^2 - c^2\partial_x^2)u(x,t) = 0$. Show the "parallelogram rule" $u(A) + u(C) = u(B) + u(D)$ where A, B, C, and D are arbitrary points of the form $C = (x,t), D = (x+cr, t+r), B = (x-cs, t+s), A = (x+cr-cs, t+r+s)$. Why is this formula called this way? **Proof**. Substitute in for the points $u(x+cr-cs, t+r+s) + u(x,t) = u(x-cs, t+s) + u(x+cr, t+r)$ Set $u(x,t) = F(x+ct) + G(x-ct)$. Then $F(x+cr-cs+ct+cr+cs) + G(x+cr-cs-ct-cr-cs) + F(x+ct) + G(x-ct) = F(x-cs+ct+cs) + G(x-cs-ct-cs) + F(x+cr+ct+cr) + G(x+cr-ct-cr) \implies F(x+2cr+ct) + G(x-2cs-ct) + F(x+ct) + G(x-ct) = F(x+ct) + G(x-2cs-ct) + F(x+2cr+ct) + G(x-ct)$ which are indeed equal. The slopes of the sides are $BC = (x-cs-x)/(t+s-t) = (-cs)/s = -c$, and $AD = (x+cr-cs-x-cr)/(t+r+s-t-r) = (-cs)/s = -c$ and so BC and AD are parallel lines. $CD = (x-x-cr)/(t-t-r) = (-cr)/(-r) = c$ and $BA = (x-cs-x-cr+cs)/(t+s-t-r-s) = (-cr)/(-r) = c$ and so CD and BA are parallel lines. The slopes of adjacent lines are the additive inverse of each other. ☐ **E3.3.9** Let u solve $(\partial_t^2 - c^2\partial_x^2)u(x,t) = \phi(x,t), x,t \in \mathbb{R}, u(x,0) = 0, x \in \mathbb{R}, \partial_t(x,0) = 0, x \in \mathbb{R}$. And $\tilde{u}$ solve $(\partial_t^2 - c^2\partial_x^2)\tilde{u}(x,t) = \phi(x,t), x,t \in \mathbb{R}, \tilde{u}(x,0) = f(x), x \in \mathbb{R}, \partial_t\tilde{u}(x,0) = g(x), x \in \mathbb{R}$. Prove that $U = u + \tilde{u}$ solves $(\partial_t^2 - c^2\partial_x^2)U(x,t) = \phi(x,t), x,t \in \mathbb{R}, U(x,0) = f(x), x \in \mathbb{R}, \partial_t U(x,0) = g(x), x \in \mathbb{R}$. This is a special case of the so-called principle of superposition. It works here because the problem is linear. *Solution.* By assuption $U(x,t) = u(x,t) + \tilde{u}(x,t)$. We differentiate with respect to t to get $\partial_t U(x,t) = \partial_t u(x,t) + \partial_t \tilde{u}(x,t)$. It follows that $\partial_t U(x,0) = \partial_t u(x,0) + \partial_t \tilde{u}(x,0)$. We rearrange the equations for u and $\tilde{u}$ to get $\partial_t^2 u(x,t) - c^2\partial_x^2 u(x,t) = \phi(x,t), u(x,0) = 0, \partial_t u(x,0) = 0, \partial_t^2 \tilde{u}(x,t) - c^2\partial_x^2 \tilde{u}(x,t) = \phi(x,t), \tilde{u}(x,0) = f(x), \partial_t \tilde{u}(x,0) = g(x)$. We add these two sets of equations to get $\partial_t^2(u(x,t) + \tilde{u}(x,t)) - c^2\partial_x^2(u(x,t) + \tilde{u}(x,t)) = \phi(x,t), u(x,0) + \tilde{u}(x,0) = f(x), \partial_t u(x,0) + \partial_t \tilde{u}(x,0) = g(x)$. And we get $\partial_t^2(U(x,t)) - c^2\partial_x^2(U(x,t)) = \phi(x,t), U(x,0) = f(x), \partial_t U(x,0) = g(x)$. ☐ **Extensions** Extend odd if boundary condition in u. Extend even if boundary condition in u_x or u_t. **Identities** $2\cos x = (e^{ix} + e^{-ix}), 2i\sin x = (e^{ix} - e^{-ix}), 2\cosh x = (e^x + e^{-x}), 2\sinh x = (e^x - e^{-x}). \cosh^2 x - \sinh^2 x = 1$. **Sum and Difference Formula** $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$. $\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$. $\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$. **Double Angle Formula** $\sin(2A) = 2\sin A\cos A$. $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$. $\tan(2A) = (2\tan A)/(1 - \tan^2 A)$. **Half Angle Formula** $\sin(A/2) = \pm\sqrt{(1-\cos A)/2}$. $\cos(A/2) = \pm\sqrt{(1+\cos A)/2}$. $\tan(A/2) = (1-\cos A)/(\sin A) = (\sin A)/(1+\cos A)$. **Product to Sum** $\cos A\cos B = (1/2)(\cos(A+B) + \cos(A-B))$. $\sin A \sin B = (1/2)(\cos(A-B) - \cos(A+B)$. $\sin A\cos B = (1/2)(\sin(A+B) + \sin(A-B)$. **Sum to Product** $\sin A \pm \sin B = 2\sin((A \pm B)/2)\cos((A \mp B)/2)$. $\cos A - \cos B = -2\sin((A+B)/2)\sin((A-B)/2)$. $\cos A + \cos B = 2\cos((A+B)/2)\cos((A-B)/2)$. **Geometric Sum** $\sum_{k=1}^{\infty} q^k = q/(1-q)$. $\sum_{k=1}^{n} q^k = (q - q^{n-1})/(1-q)$.

General ODE Solutions $y'' = y(t) \implies y = c_1e^{-t} + c_2e^t$ ☐ $dy/dt + p(t)y = g(t) \implies y = (\int u(t)g(t))/u(t) + c$ where $u(t)$ =exp$(\int p(t)dt)$ ☐ $y' = x; x' = y \implies x = c_1\cosh t + c_2\sinh t, y = c_1\sinh t + c_2\cosh t$ or $x = c_1e^t + c_2e^{-t}, y = c_1e^t - c_2e^{-t}$ ☐ $y' = -x; x' = y \implies y = c_1\cos t + c_2\sin t, x = c_1\sin t - c_2\cos t$ ☐ $x' = x + y; y' = -x + y \implies x = e^t(c_1\cos t + c_2\sin t); y = e^t(-c_1\sin t + c_2\cos t)$ ☐ $v' = \gamma v, v(z,0) = u_0 \implies v = u_0e^{\gamma t}$ ☐

3.11.T $\frac{\partial u}{\partial t} + \sum_{j=1}^{n-1} b_j(t,u)\frac{\partial u}{\partial y_j} = \gamma u, \ u(y,0) = u_0(y)$. $\zeta(z,t) = z + \int_0^t b(s, u_0(z)e^{\gamma s})ds, \zeta_z(z,t) = 1 + u_0'(z)\int_0^t b_u(s, u_0(z)e^{\gamma s})e^{\gamma s}ds$. Prove $\zeta \to \infty$ and $\zeta_z > 0$. **Wave equation:** $\partial_t^2 u(x,t) - c^2\partial_x^2 = 0 \to u(x,t) = F(x+ct) + G(x-ct)$. **Vibrating String, 3.14.T:** $\partial_t^2 u(x,t) - c^2\partial_x^2 = 0, \ u(x,0) = f(x), \partial_t u(x,0) = g(x), \ u(0,t) = 0 = u(L,t), \ x \in [0,L], t \in \mathbb{R} \to$ **d'Alembert formula** $u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$. **Inhomoge-**

neous wave equation: $\partial_t^2 u(x,t) - c^2\partial_x^2 = 0,\ u(x,0) = 0,\ \partial_t u(x,0) = 0,\quad x,t \in \mathbb{R} \to u = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \phi(\rho,s)d\rho ds.$ **Remember:** only the odd functions give $f(0) = 0 = f(L)$.