Numerical Methods for PDEs
Homework 4

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October 15, 2018

Problem 1

a. Show that the Jacobi spectral radius $\mu = \cos(\pi h)$ for Laplace's equation on the unit square with second-order accurate central differences. Write-up only the 1D version. *Hint:* In 1D, set $A = \text{tridiag}[-1\ 2\ -1]$. Then the iteration matrix $B = \frac{1}{2}\text{tridiag}[1\ 0\ 1]$. Then show that $Bv = \cos(\pi h)\, v$ where the 1D eigenvector

$$v = [\sin(\pi h),\ \ \sin(2\pi h),\ \ \cdots,\ \ \sin(n\pi h)].$$

Note that here $h = 1/(n+1)$.

Solution: Since $A = L + D + U = \text{diag}([-1\ \ 2\ \ -1])$ and we are using the Jacobi iteration, $M = D = \text{diag}(2)$. Hence,

$$B = M^{-1}(M - A) = \text{diag}(1/2) * \text{tridiag}([1\ 0\ 1]) = \frac{1}{2}\text{tridiag}([1\ 0\ 1])$$

We start with the first point of our domain,i.e. the first row of the product Bv,

$$(Bv)_1 = \frac{1}{2}v_2 = \frac{1}{2}\sin(2\pi h) = \cos(\pi h)\sin(\pi h).$$

We continue with the interior points, where $j \in [2, n-1]$,

$$\begin{aligned}(Bv)_j = \frac{1}{2}\left(v_{j-1} + v_{j+1}\right) &= \frac{1}{2}\left[\sin(j\pi h)\cos(\pi h) - \sin(\pi h)\cos(j\pi h) + \sin(j\pi h)\cos(\pi h) + \sin(\pi h)\cos(j\pi h)\right] \\ &= \cos(\pi h)\sin(j\pi h).\end{aligned}$$

Finally, for the last point $j = n$,

$$\begin{aligned}(Bv)_n = \frac{1}{2}v_{n-1} = \frac{1}{2}\sin((n-1)\pi h) &= \frac{1}{2}\sin(n\pi h - \pi h) \\ &= \frac{1}{2}\left[\sin(n\pi h)\cos(\pi h) - \sin(\pi h)\cos(n\pi h)\right] \\ &= \frac{1}{2}\left[\sin(n\pi h)\cos(\pi h) + \sin(n\pi h)\cos(\pi h)\right] \\ &= \cos(\pi h)\sin(n\pi h),\end{aligned}$$

where we have used the following relations:

$$\begin{aligned}\sin(n\pi h) &= \sin((n+1)\pi h - \pi h) = \sin(\pi - \pi h) = \sin(\pi h) \\ \cos(n\pi h) &= \cos((n+1)\pi h - \pi h) = \cos(\pi - \pi h) = -\cos(\pi h).\end{aligned}$$

Thus, we have obtained that

$$(Bv)_j = \cos(\pi h)\sin(j\pi h),$$

which means that the eigenvalue is $\lambda = \cos(\pi h)$ and the eigenvector v has components $v_j = \sin(j\pi h)$.

Problem 2

a. For SOR/SUR, show that $\det\{B\} = (1-\omega)^n$, $0 < \omega < 2$. *Hint:* B is the product of triangular matrices ($A = L + D + U$):

$$B = (D + \omega L)^{-1}\left((1-\omega)D - \omega U\right).$$

Solution: It is simply computed that

$$\det\{(D+\omega L)^{-1}\} = \prod_{j=1}^{n} \frac{1}{d_j} = \frac{1}{\prod_{j=1}^{n} d_j},$$

since the elements of L are always multiplied by 0. For the same reason,

$$\det\{(1-\omega)D - \omega U)^{-1}\} = \prod_{j=1}^{n}(1-\omega)d_j = (1-\omega)^n \prod_{j=1}^{n} d_j.$$

Hence,

$$\det B = \frac{(1-\omega)^n \prod_{j=1}^{n} d_j}{\prod_{j=1}^{n} d_j} = (1-\omega)^n.$$

b. Derive the equation for the SOR ω_{opt}, assuming Young's formula applied to the spectral radii:

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2.$$

Here μ is the spectral radius for the Jacobi iteration method and λ is the spectral radius for the SOR iteration method. *Hint:* Set $\lambda = \omega - 1$ and minimize λ (using the quadratic formula).

Solution: We start from Young's formula and we make $\lambda = \omega - 1$ and $\mu = \rho_J$,

$$\begin{aligned}(\lambda + \omega - 1)^2 &= \lambda \omega^2 \mu^2 \\ 4\lambda^2 &= \lambda(\lambda+1)^2 \rho_J^2 \Rightarrow \\ \Rightarrow -\lambda\left(\rho_J^2\lambda^2 + (2\rho_J^2 - 4)\lambda + \rho_J^2\right) &= 0.\end{aligned}$$

The solutions to this equation are $\lambda = 0$ or

$$\lambda = \frac{4 - 2\rho_J^2 \pm \sqrt{(4-2\rho_J^2)^2 - 4\rho_J^4}}{2\rho_J^2} = \frac{4 - 2\rho_J^2 \pm 4\sqrt{1-\rho_J^2}}{2\rho_J^2} = \frac{2 - \rho_J^2 \pm 2\sqrt{1-\rho_J^2}}{\rho_J^2}.$$

Minimizing $\rho_{SOR} = \max\{\lambda\} = \omega_{opt} - 1$:

$$\rho_{SOR} = \frac{2 - \rho_J^2 - 2\sqrt{1-\rho_J^2}}{\rho_J^2},$$

and,

$$\begin{aligned}\omega_{opt} &= \frac{2 - \rho_J^2 - 2\sqrt{1-\rho_J^2}}{\rho_J^2} + 1 \\ &= 2\frac{1 - \sqrt{1-\rho_J^2}}{\rho_J^2}\end{aligned}$$

c. Show that for Laplace's equation on the unit square, the SOR $\lambda = (1 - \sin \pi h)/(1 + \sin \pi h)$.

Solution: Using that $\rho_J = \cos(\pi h)$, and the result just proved,

$$\begin{aligned}
\rho_{SOR} &= \frac{2 - \rho_J^2 - 2\sqrt{1 - \rho_J^2}}{\rho_J^2} \\
&= \frac{2 - \cos^2(\pi h) - 2\sqrt{1 - \cos^2(\pi h)}}{\cos^2(\pi h)} \\
&= \frac{2 - \cos^2(\pi h) - 2\sin(\pi h)}{\cos^2(\pi h)} \\
&= \frac{1 + \sin^2(\pi h) - 2\sin(\pi h)}{1 - \sin^2(\pi h)} \\
&= \frac{1 - \sin(\pi h)}{1 + \sin(\pi h)}
\end{aligned}$$

Problem 3

a. 2×2 *SOR example.* Calculate the first two iterates x_1 and x_2 for Jacobi, Gauss-Seidel, and SOR with $x_0 = (0, 0)$ for $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$M_{SOR} = \begin{bmatrix} \frac{2}{\omega} & 0 \\ -1 & \frac{2}{\omega} \end{bmatrix}, \quad B_{SOR} = \begin{bmatrix} 1 - \omega & \frac{\omega}{2} \\ \frac{\omega}{2}(1 - \omega) & \left(1 - \frac{\omega}{2}\right)^2 \end{bmatrix}$$

$$\omega_{opt} = 4(2 - \sqrt{3}) \approx 1.0718, \;\; \lambda_1 = \lambda_2 = \omega_{opt} - 1 = \rho_{SOR} \approx 0.0718.$$

The exact solution is $x = (1, -1)$, $x_2^J = (3/4, -3/4)$, and $x_2^{GS} = (9/8, -15/16)$. Calculate $||e_2^J||_1$, $||e_2^{GS}||_1$, and $||e_2^{SOR}||_1$. Note that the SOR x_2 is much closer to the exact solution.

Solution:

- **Jacobi:** We start with the Jacobi iteration, $x^{(k+1)} = x^{(k)} - D^{-1}r^{(k)}$, where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

  Since $x_0 = 0$, $r_0 = b$. Then,

$$x^{(1)} = D^{-1}b = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}, \quad r(1) = Ax(1) - b = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}.$$

  The second guess is

$$x^{(2)} = x^{(1)} - D^{-1}r^{(1)} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -3/4 \end{bmatrix}.$$

  Lastly, we compute the 1 error norm, $||e_J^{(2)}||_1 = 1/2$.

- **Gauss Seidel:** Now we proceed with the Gauss-Seidel iteration, $x^{(k+1)} = M^{-1}(M-A)x^{(k)} - M^{-1}b$, where

$$M = D + L = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \Rightarrow M^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},$$

and

$$M - A = -U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow M^{-1}(M-A) = \frac{1}{4}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

Since $x_0 = 0$,

$$x^{(1)} = M^{-1}b = \begin{bmatrix} -3/4 \\ -3/4 \end{bmatrix}.$$

The second guess is

$$x^{(2)} = M^{-1}(M-A)x^{(1)} - M^{-1}b = \frac{1}{4}\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3/4 \\ -3/4 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 9/8 \\ -15/16 \end{bmatrix}.$$

Finally, we compute the 1 error norm, $||e_{GS}^{(2)}||_1 = 3/16$.

- **SOR:** Next, the SOR iteration, $Mx^{(k+1)} = (M-A)x^{(k)} + b$, with $M = \frac{D}{\omega} + L$ and $M - A = (\frac{1}{\omega} - 1)D - U$. Therefore,

$$\begin{aligned} \left[\frac{D}{\omega} + L\right] x^{(k+1)} &= \left[\left(\frac{1}{\omega} - 1\right) D - U\right] x^{(k)} + b, \\ [D + \omega L]\, x^{(k+1)} &= [(1-\omega) D - \omega U]\, x^{(k)} + \omega b, \\ x^{(k+1)} &= [D + \omega L]^{-1} [(1-\omega) D - \omega U]\, x^{(k)} + \omega [D + \omega L]^{-1} b. \end{aligned}$$

Note that

$$[D + \omega L] = \begin{bmatrix} 2 & 0 \\ -\omega & 2 \end{bmatrix} \Rightarrow [D + \omega L]^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ \omega & 2 \end{bmatrix},$$

and

$$[(1-\omega)D - \omega U] = \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix} \Rightarrow [D + \omega L]^{-1}[(1-\omega)D - \omega U] = \begin{bmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega}{2}(1-\omega) & (1-\frac{\omega}{2})^2 \end{bmatrix}.$$

Since $x_0 = 0$,

$$x^{(1)} = \omega [D + \omega L]^{-1} b = \begin{bmatrix} \frac{3\omega}{2} \\ \frac{3\omega^2 - 6\omega}{4} \end{bmatrix}.$$

The second guess is

$$\begin{aligned} x^{(2)} &= [D + \omega L]^{-1} [(1-\omega) D - U]\, x^{(1)} + \omega [D + \omega L]^{-1} b \\ &= \begin{bmatrix} 1-\omega & \frac{\omega}{2} \\ \frac{\omega}{2}(1-\omega) & (1-\frac{\omega}{2})^2 \end{bmatrix}\begin{bmatrix} \frac{3\omega}{2} \\ \frac{3\omega^2 - 6\omega}{4} \end{bmatrix} + \frac{\omega}{4}\begin{bmatrix} 2 & 0 \\ \omega & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3\omega(\omega^2 - 6\omega + 8)}{8} \\ \frac{3\omega(\omega^3 - 10\omega^2 + 20\omega - 16)}{16} \end{bmatrix} = \begin{bmatrix} 1.0924 \\ -0.9687 \end{bmatrix}. \end{aligned}$$

Finally, we compute the 1 error norm, $||e_{SOR}^{(2)}||_1 = 0.1237$. As expected,

$$e_{SOR}^{(2)}||_1 < ||e_{GS}^{(2)}||_1 < ||e_J^{(2)}||_1.$$

Problem 4

a. Use conjugate gradient on the steepest descent problem we did in class:

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution: Since $x_0 = 0$, $r_0 = b$ and $\beta_1 = 0$. Then,

$$d_1 = r_0 + \beta_1 d_0 = r_0 = b,$$

and,

$$\alpha_1 = \frac{r_0^T r_0}{d_1^T A d_1} = 1/5.$$

Then new guess is

$$x_1 = x_0 + \alpha_1 d_1 = \begin{bmatrix} -2/5 \\ 2/5 \end{bmatrix},$$

and the new residual is

$$r_1 = r_0 + \alpha_1 A d_1 = \begin{bmatrix} 2/5 \\ 2/5 \end{bmatrix},$$

We repeat the process again,

$$\begin{aligned}
\beta_2 &= \frac{r_1^T r_1}{r_0^T r_0} = 1/25, \\
d_2 &= r_1 + \beta_2 d_1 = \begin{bmatrix} 8/25 \\ 12/25 \end{bmatrix}, \\
\alpha_2 &= \frac{r_1^T r_1}{d_2^T A d_2} = 5/4, \\
x_2 &= x_1 + \alpha_2 d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Problem 5

a. Compute the first two conjugate gradient iterates x_1 and x_2 with $x_0 = (0, 0)$ with and without preconditioning to the solution $x = (0, 1)$ of $Ax = b$:

$$A = \begin{bmatrix} 9 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}.$$

Calculate $||e_1^{CG}||_1$ and $||e_1^{PCG}||_1$. Note that while both CG and PCG give the exact x in two steps $x_2 = (0, 1)$, PCG gives a much better x_1.

Solution:

- **CG:** We start without using preconditioning. Since $x_0 = 0$, $r_0 = b$ and $\beta_1 = 0$. Then,

$$d_1 = r_0 + \beta_1 d_0 = r_0 = b,$$

and,

$$\alpha_1 = \frac{r_0^T r_0}{d_1^T A d_1} = 1/6.$$

Then new guess is

$$x_1 = x_0 + \alpha_1 d_1 = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix},$$

and the new residual is

$$r_1 = r_0 + \alpha_1 A d_1 = \begin{bmatrix} -2/3 \\ 2/3 \end{bmatrix},$$

We repeat the process again,

$$\beta_2 = \frac{r_1^T r_1}{r_0^T r_0} = 4/9,$$
$$d_2 = r_1 + \beta_2 d_1 = \begin{bmatrix} -2/9 \\ 10/9 \end{bmatrix},$$
$$\alpha_2 = \frac{r_1^T r_1}{d_2^T A d_2} = 3/4,$$
$$x_2 = x_1 + \alpha_2 d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- **PCG:** Now we use preconditioning. Since $x_0 = 0$, $r_0 = b$ and $\beta_1 = 0$. We solve

$$M z_0 = r_0 \Rightarrow z_0 = \begin{bmatrix} 1/9 \\ 1 \end{bmatrix}.$$

Then,

$$d_1 = z_0 + \beta_1 d_0 = r_0 = z_0,$$

and,

$$\alpha_1 = \frac{z_0^T r_0}{d_1^T A d_1} = 5/6.$$

Then new guess is

$$x_1 = x_0 + \alpha_1 d_1 = \begin{bmatrix} 5/54 \\ 5/6 \end{bmatrix},$$

and the new residual is

$$r_1 = r_0 + \alpha_1 A d_1 = \begin{bmatrix} -2/3 \\ 2/27 \end{bmatrix},$$

We repeat the process again,

$$\begin{aligned}
&Mz_1 = r_1 \Rightarrow z_1 = \begin{bmatrix} -2/27 \\ 2/27 \end{bmatrix}, \\
&\beta_2 = \frac{z_1^T r_1}{z_0^T r_0} = 4/81, \\
&d_2 = z_1 + \beta_2 d_1 = \begin{bmatrix} -50/729 \\ 10/81 \end{bmatrix}, \\
&\alpha_2 = \frac{z_1^T r_1}{d_2^T A d_2} = 27/20, \\
&x_2 = x_1 + \alpha_2 d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

To finish we compute

$$||e_{CG}^{(1)}||_1 = 1$$
$$||e_{PCG}^{(1)}||_1 = 0.2593.$$

The PCG method gives indeed a much better x_1.