T 4.2. Let X be an IP space. If $u, v \in X$, then $|\langle u, v\rangle|^2 \le \langle u, u\rangle\langle v, v\rangle$, equality iff u, v are linearly dependent. **P 4.7**. Let M be an orthonormal set in an inner product space X over $\mathbb{K}$ and F be a finite subset of M. Then the following hold for all $u \in X$: (a) $||u - \sum_{v\in F}\langle u, v\rangle v||^2 = ||u||^2 - \sum_{v\in F}|\langle u, v\rangle|^2$. (b) If G is also a finite subset of M and $F \subseteq G, ||u - \sum_{v\in F}\langle u, v\rangle v|| \ge ||u - \sum_{v\in G}\langle u, v\rangle v||$. (c) (Bessel's inequality) $\sum_{v\in F}|\langle u, v\rangle|^2 \le ||u||^2$. (d) (Best approximation) For any choice of $\alpha_v \in \mathbb{K}, v \in F, ||u - \sum_{v\in F}\langle u, v\rangle v|| \le ||u - \sum_{v\in F}\alpha_v v||$. **T 4.8**. Let M be an orthonormal basis of the IP space X and $u \in X$. Then, for any $\epsilon > 0, \exists$ a finite set $F \subseteq M$ s.t. for all finite sets G with $F \subseteq G \subseteq M$, $||u - \sum_{v\in G}\langle u, v\rangle v|| \le \epsilon$ and $||u||^2 \le \sum_{v\in G}|\langle u, v\rangle|^2 + \epsilon^2$. **T 4.10**. Let M be a denumerable orthonormal basis of the IP space X and $f : \mathbb{N} \to M$ be bijective. Then $u = \sum_{n=1}^\infty \langle u, f(n)\rangle f(n)$ (Fourier expansion) and $||u||^2 = \sum_{n=1}^\infty |\langle u, f(n)\rangle|^2$. (Parseval's relation) **T 4.11**. Let B be a denumerable orthonormal basis of the IP space $X, I \subseteq \mathbb{R}$, and $u : I \to X$ such that, for all $v \in B, \langle u(t), v\rangle$ is a (uniformly) cont func of $t \in I$. Let $\{\alpha_v; v \in B\}$ be a family in $\mathbb{R}_+$ such that $|\langle u(t), v\rangle| \le \alpha_v$ for all $t \in I$ and $v \in B$. Assume that there is some bijective $f : \mathbb{N} \to B$ s.t. the series $\sum_{n=1}^\infty \alpha_{f(n)}^2$ converges in $\mathbb{R}$. Then u is (unif) cont. **Exercise 4.2.1** (Riesz-Fisher Theorem). Let $\{v_m; m \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space H over $\mathbb{K}$ and (α_m) a sequence in $\mathbb{K}$. Show: The series $\sum_{m=1}^\infty \alpha_m v_m$ exists in H iff $\sum_{m=1}^\infty |\alpha_m|^2 < \infty$. Further, if one and then both of these statements hold, $||\sum_{m=1}^\infty \alpha_m v_m||^2 = \sum_{m=1}^\infty |\alpha_m|^2$. *Proof.* We define partial sums in H, $x_n = \sum_{m=1}^n \alpha_m v_m$, and in $\mathbb{K}$, $\beta_n = \sum_{m=1}^n |\alpha_m|^2$. By the properties of the inner product and orthonormality, $||x_n - x_k||^2 = ||\sum_{m=k+1}^n \alpha_m v_m||^2 = \langle\sum_{m=k+1}^n \alpha_m v_m, \sum_{j=k+1}^n \alpha_j v_j\rangle = \sum_{m=k+1}^n\sum_{j=k+1}^n \alpha_m\bar{\alpha}_j\langle v_m, v_j\rangle = \sum_{m=k+1}^n |\alpha_m|^2 = |\beta_n - \beta_k|$. This shows that (x_n) is a Cauchy sequence in H iff (β_n) is a Cauchy sequence in $\mathbb{R}$. Assume that $\sum_{m=1}^\infty |\alpha_m|^2$ converges. Then (β_n) is a Cauchy sequence in $\mathbb{R}$ and (x_n) is a Cauchy sequence in H. Since H is complete, (x_n) converges, i.e., $\sum_{n=1}^\infty \alpha_n x_n$ converges. The other direction follows similarly. Finally, by continuity of the norm and orthonormality, $||\sum_{m=1}^\infty \alpha_m v_m||^2 = \lim_{n\to\infty}||x_n||^2 = \lim_{n\to\infty}\sum_{m=1}^\infty |\alpha_m|^2 = \sum_{m=1}^\infty |\alpha_m|^2$ ☐ **Exercise 4.2.2**. Let X be a Hilbert space. Let $M = \{v_m; m \in \mathbb{N}\}$ be an orthonormal subset of X. Show: $\sum_{m=1}^\infty \langle u, v_m\rangle v_m$ converges for every $u \in X$. Warning: This means that the Fourier series of u converges, but it may happen that it does not equal u (unless M is an orthonormal basis). *Proof.* Combine Bessel's inequality with Exercise 4.2.1 choosing $\alpha_v = \langle u, v\rangle$ ☐ **Exercise 4.2.3**. Let X be an inner product space and M a denumerable orthonormal subset of X. Show (a) If M is an orthonormal basis and $x \in X$, then $\langle x, v\rangle = 0$ for all $v \in M$ implies that $x = 0$. (b) If X is an Hilbert space and if, for all $x \in X, \langle x, v\rangle = 0$ for all $v \in M$ implies that $x = 0$, then M is an orthonormal basis. *Proof.* (a) Let $M = \{v_m; m \in \mathbb{N}\}$ be an orthonormal basis and $x \in X$. Then x is represented by its Fourier series, $x = \sum_{m=1}^\infty \langle x, v_m\rangle v_m$. Assume that $\langle x, v\rangle = 0$ for all $v \in M$. Then $x = 0$. (b) Let $x \in X$. By Exercise 4.2.2, the series $\sum_{m=1}^\infty \langle x, v_m\rangle v_m =: y$ converges. By orthonormality and continuity of the inner product, $\langle y, v_k\rangle = \langle x, v_k\rangle$ for all $k \in \mathbb{N}$. So $\langle y - x, v\rangle = 0$ for all $v \in M$. By assumption, $y - x = 0$ i.e., $x = y = \sum_{m=1}^\infty \langle x, v_m\rangle v_m$. This means that M is an orthonormal basis ☐ **T 4.12**. The set $B = \{v_j; j \in \mathbb{Z}\}$ with $v_j(x) = e^{ijx}$ is an orthonormal basis for the following IP spaces with the IP $\langle f, g\rangle = 1/(2\pi)\int_{-\pi}^\pi f(x)\overline{g(x)}dx$: $X = C([-\pi, \pi], \mathbb{C}), X = L^2([-\pi, \pi], \mathbb{C})$ and the space of Riemann integrable functions on $[-\pi, \pi]$ with values in $\mathbb{C}$. Further the Fourier series $\sum_{j=-\infty}^\infty \hat{f}_j e^{ijx}, \hat{f}_j = 1/(2\pi)\int_{-\pi}^\pi f(x)e^{-ijx}dx$, converges to f in the sense that for every $\epsilon > 0\exists$ some $N \in \mathbb{N}$ with $\int_{-\pi}^\pi |f(x) - \sum_{j=-k}^m \hat{f}_j e^{ijx}|^2 dx < \epsilon^2$ (4.4) for all $k, m \in \mathbb{N}$ with $k, m > N$. **P 4.14**. Let $f : \mathbb{R} \to \mathbb{C}$ be Lipschitz cont and 2π-periodic. Then $\sum_{j\in\mathbb{Z}}|\hat{f}_j|$ converges in $\mathbb{R}$. More precisely, if Λ is a Lipschitz const for f, then $\sum_{j\in M}|\hat{f}_j| \le 2\Lambda + |\hat{f}_0|$ for all finite subsets M of $\mathbb{Z}$. **R 4.15**. This proof also shows: If $g : [-\pi, \pi] \to \mathbb{C}$ is absolutely cont and 2π-periodic and $g' \in L^2[-\pi, \pi]$, then $\sum_{j\in\mathbb{Z}}|\hat{g}_j| \le 2||g'|| + |\hat{g}_0|$. Every Lipschitz cont funct f is absolutely cont with $|f'(x)| \le \Lambda$ for a.a. x. **L 4.16**. Let $f : [-L, L] \to \mathbb{C}$ be Lipschitz cont, $f(L) = f(-L)$. Extend f in an $2L$-periodic way, $f(y + 2kL) = f(y) : k \in \mathbb{Z}, -L < y \le L$. Extension of f is Lipschitz cont with same Lipschitz const. **T 4.17**. Let $f : [-L, L] \to \mathbb{C}$ be Lipschitz cont, $f(-L) = f(L)$. Then f is the uniform limit of its Fourier series, $f(x) = \sum_{j\in\mathbb{Z}}\hat{f}_j e^{i\lambda_j x}, \hat{f}_j = (1/2L)\int_{-L}^L f(y)e^{-i\lambda_j y}dy, \lambda_j = j\pi/L$. **Exercise 4.3.1**. Let $B = \{\cos(jx); j \in \mathbb{N}\} \cup \{\sin(jx); j \in \mathbb{N}\} \cup \{1/\sqrt{2}\}$. Show that B is an orthonormal basis of $L^2([-\pi, \pi], \mathbb{R})$ with inner product $\langle f, g\rangle = (1/\pi)\int_{-\pi}^\pi fg$. Hint: Use that $\{e^{ijx}; j \in \mathbb{Z}\}$ is an orthonormal basis of $L^2([-\pi, \pi], \mathbb{C})$ and express $\sin x$ and $\cos x$ in terms of e^{ix} and e^{-ix}. *Proof.* Recall that $\cos(jx) = (1/2)(e^{ijx} + e^{-ijx}), \sin(jx) = (1/2i)(e^{ijx} - e^{-ijx})$. Since e^{ijx} and e^{ikx} are orthogonal to each other for $j \ne k$, so are $\cos jx$ and $\cos kx$, and $\cos jx$ and $\sin kx$, and $\sin jx$ and $\sin kx$ for $j \ne k$. $\sin jx$ and $\cos jx$ are orthogonal to each other because their product is odd about 0 and the integral yields 0. Further $(1/\pi)\int_{-\pi}^\pi \cos jx\cos jx dx = (1/\pi)\int_{-\pi}^\pi (1/4)(e^{ijx} + e^{-ijx})(e^{ijx} + e^{-ijx})dx$. Notice that $\int_{-\pi}^\pi e^{ijx}e^{ijx} = 2\pi\langle e^{ij\cdot}, e^{-ij\cdot}\rangle_{\mathbb{C}} = 0$ and $\int_{-\pi}^\pi e^{-ijx}e^{-ijx} = 2\pi\langle e^{-ij\cdot}, e^{ij\cdot}\rangle_{\mathbb{C}} = 0$. Here $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes the inner product for $C([-\pi, \pi], \mathbb{C})$. So $(1/\pi)\int_{-\pi}^\pi \cos(jx)\cos(jx)dx = 1$. Similarly for the sines. In order to show that this is an orthonormal basis, we use Exercise 4.2.3. Let $f \in L^2([-\pi, \pi], \mathbb{R})$ and $(1/\pi)\int_{-\pi}^\pi f(x)\sin(jx)dx = 0, (1/\pi)\int_{-\pi}^\pi f(x)\cos(jx)dx = 0, j \in \mathbb{N}, (1/\pi)\int_{-\pi}^\pi f(x)(1/\sqrt{2}) = 0$. By Euler's formula, for all $j \in \mathbb{Z}, (1/2\pi)\int_{-\pi}^\pi f(x)e^{ijx}dx = (1/2\pi)\int_{-\pi}^\pi f(x)\cos(jx)dx + i(1/2\pi)\int_{-\pi}^\pi f(x)\sin(jx)dx = 0$. Since $\{e^{ijx}; j \in \mathbb{Z}\}$ is an orthonormal basis, $f = 0$ by Exercise 4.2.3(a). Exercise 4.2.3 (b) implies that B is an orthonormal basis for $L^2([-\pi, \pi], \mathbb{R})$ ☐ **Exercise 4.3.2**. Let $f : [a, b] \to \mathbb{R}$ be continuous and assume that there exists a partition $a = t_0 < \cdots < t_m = b$ such that f is differentiable with bounded derivative on each interval (t_{j-1}, t_j). Show f is Lipschitz continuous. *Proof.* Let $x, y \in [z, b], x < y$. Modifying the partition of $[a, b]$, we can find a partition $x = r_0 < \cdots < r_k = y$ such that f is continuously differentiable on each (r_{j-1}, r_j) and $L_j := \sup_{r_{j-1}<s<r_j}|f'(s)| < \infty$. More precisely $r_1, \ldots, r_{k-1} \in \{t_1, \ldots, t_{m-1}\}$. Let $j \in \{0, \ldots, k\}$. and $r_{j-1} \le s < t \le r_j$. By the mean value theorem of calculus, $f(t) - f(s) = f'(r)(t - s)$ for some $r \in (s, t)$. So $|f(t) - f(s)| \le L_j|t - s|$. Since f is continuous, we can take the limit $s \to r_{j-1}$ and $t \to r_j$ and $|f(r_j) - f(r_{j-1})| \le L_j(r_j - r_{j-1})$. We telescope, $|f(y) - f(x)| = |\sum_{j=1}^k[f(r_j) - f(r_{j-1})]| \le \sum_{j=1}^k L_j(r_j - r_{j-1})$. Set $\Lambda = \max_{j=1}^k L_j$. Then $|f(y) - f(x)| \le \Lambda\sum_{j=1}^k(r_j - r_{j-1}) = \Lambda(y - x)$. Here we have telescoped again. **Exercise 4.3.3**. Let $f : [-L, L] \to \mathbb{R}$ be Lipschitz continuous, $f(L) = f(-L)$, and A_j and B_j be the Fourier cosine and sine coeffecints respectively. Show: $\sum_{j=0}^\infty |A_j| < \infty, \sum_{j=1}^\infty |B_j| < \infty$. Hint: Use the analogous result for complex Fourier coefficients. *Proof.* After a change of variables, we can assume that $L = \pi$. Notice that, for $j \in \mathbb{N}, \hat{f}_j = 1/(2\pi)\int_{-\pi}^\pi f(x)e^{-ijx}dx = 1/(2\pi)\int_{-\pi}^\pi f(x)[\cos(jx) - i\sin(jx)]dx = (1/2)(A_j - iB_j)$. Since f has real values, A_j and B_j are real numbers and $|\hat{f}_j| = (1/2)\sqrt{A_j^2 + B_j^2} \ge (1/2)\max\{|A_j|, |B_j|\}$. Since f is Lipschitz continuous and $f(-\pi) = f(\pi)$, by Theorem 4.14, $\sum_{j=1}^\infty |A_j| \le 2\sum_{j=1}^\infty |\hat{f}_j| \le 2\sum_{j=-\infty}^\infty |\hat{f}_j| < \infty$. Similarly for $|B_j|$. ☐ **T 4.18**. Let $f : [-L, L] \to \mathbb{R}$ be Lipschitz cont and $f(L) = f(-L)$. Then $f(x) = A_0 + \sum_{j=1}^\infty(A_j\cos(\lambda_j x) + B_j\sin(\lambda_j x)), \lambda_j = (j\pi/L)$, with the convergence being uniform in $x \in [-L, L]$ and $A_j = (1/L)\int_{-L}^L f(y)\cos(\lambda_j y)dy, j \in \mathbb{N}, B_j = (1/L)\int_{-L}^L f(y)\sin(\lambda_j y)dy, j \in \mathbb{N}, A_0 = 1/(2L)\int_{-L}^L f(y)dy$. The formula of A_j, the Fourier cosine coef of f, follows by a change of variable from $A_j = (1/\pi)\int_{-\pi}^\pi f(yL/\pi)\cos(jy)dy$. Similarly for B_j, the Fourier sine coef of f. **L 4.19**. Let $f : [0, L] \to \mathbb{R}$ be Lipschitz continuous, $f(0) = 0 = f(L)$. Then $f(x) = \sum_{j=1}^\infty B_j\sin(\lambda_j x), B_j = (2/L)\int_0^L f(y)\sin(\lambda_j y)dy, \lambda_j = (j\pi/L)$, with the convergence being uniform in $[0, L]$. *Proof.* Extend f to $[-L, L]$ in an odd fashion by defining $f(-x) = -f(x)$ for $x \in (0, L]$. We first check whether this is also Lipschitz cont. Critical case is $-L \le x < 0 \le y \le L$. Since $f(0) = 0, |f(y) - f(x)| \le |f(y)| + |f(x)| = |f(y) - f(0)| + |f(0) - f(-x)| \le \Lambda(y + (-x)) \le \Lambda(y - x)$. Since $f(L) = 0, f(-L) = -f(L) = 0$ and $f(-L) = f(L)$. By T 4.18, $f(x) = A_0 + \sum_{j=1}^\infty(A_j\cos(\lambda_j x) + B_j\sin(\lambda_j x)), \lambda_j = (j\pi/L)$, with the convergence being uniform in $x \in [-L, L]$ and $A_j = (1/L)\int_{-L}^L f(y)\cos(\lambda_j y)dy, B_j = (1/L)\int_{-L}^L f(y)\sin(\lambda_j y)dy$. Since cosine is even and sine is odd, for $j \in \mathbb{N}, A_j = (1/L)\int_0^L(f(y) + f(-y))\cos(\lambda_j y)dy = 0$ and $B_j = (1/L)\int_0^L(f(y) - f(-y))\sin(\lambda_j y)dy = (2/L)\int_0^L f(y)\sin(\lambda_j y)dy, A_0 = (1/2L)\int_{-L}^L f(y)dy = 0$ ☐ **R 4.20**. Actually, $\sum_{j=1}^\infty |B_j| < \infty$ under the conds of L 4.19. See Ex 4.3.3. **Exercise 4.3.4**. Let $f : [0, L] \to \mathbb{R}$ be Lipschitz continuous. Show that $f(x) = \sum_{j=0}^\infty A_j\cos(jx\pi/L)$ with the convergence being uniform in $[0, L], \sum_{j=0}^\infty |A_j| < \infty$, and $A_j = (2/L)\int_0^L f(y)\cos(jy\pi/L)dy, j \in \mathbb{N}$, and $A_0 = (1/L)\int_0^L f(y)dy$. *Proof.* We extend f to $[-L, L]$ by defining $f(-x) = f(x)$ for $x \in (0, L]$. We first need to check whether this extension is also Lipschitz continuous. The critical case is $-L \le x < 0 \le y \le L$. By the triangle inequality, $|f(y) - f(x)| \le |f(y) - f(0)| + |f(0) - f(-x)| \le \Lambda(y + (-x)) \le \Lambda(y - x)$. By construction, $f(-L) = f(L)$. By Theorem 4.18, $f(x) = A_0 + \sum_{j=1}^\infty(A_j\cos(\lambda_j x) + B_j\sin(\lambda_j x)), \lambda_j = (j\pi/L)$, with the convergence being uniform in $x \in [-L, L]$ and $A_j = (1/L)\int_{-L}^L f(y)\cos(\lambda_j y)dy, B_j = (1/L)\int_{-L}^L f(y)\sin(\lambda_j y)dy$. By the previous exercise, $\sum_{j=0}^\infty |A_j| < \infty$. Since cosine is even and sine is odd, for $j \in \mathbb{N}, A_j = (1/L)\int_0^L(f(y) + f(-y))\cos(\lambda_j y)dy = (2/L)\int_0^L f(y)\cos(\lambda_j y)dy$ and $B_j = (1/L)\int_0^L(f(y) - f(-y))\sin(\lambda_j y)dy = 0, A_0 = 1/(2L)\int_{-L}^L f(y)dy = 1/(2L)\int_0^L(f(y) + f(-y))dy = (1/L)\int_0^L f(y)dy$ ☐ **Exercise 4.3.6**. Show that $\{v_j; j \in \mathbb{Z}_+\}$ with $v_j(x) = \cos(jx)$ and $v_0 = \sqrt{1/2}$ is an orthonormal basis of $L^2([0, \pi], \mathbb{R})$ with the inner product $\langle f, g\rangle = \int_0^\pi f(x)g(x)dx$. Conclude that, for $f \in L^2([0, \pi], \mathbb{R}), \int_0^\pi |f(x) - \sum_{j=0}^m B_j\cos(jx)dx|^2 dx \to 0, m \to \infty, B_j = (2/\pi)\int_0^\pi f(x)\cos(jx)dx, j \in \mathbb{N}, B_0 = (\sqrt{2}/\pi)\int_0^\pi f(x)dx$. *Proof.* By Exercise 4.3.1, $B = \{\cos(jx); j \in \mathbb{N}\} \cup \{\sin(jx); j \in \mathbb{N}\} \cup \{1/\sqrt{2}\}$ is an orthonormal basis of $L^2([-\pi, \pi], \mathbb{R})$, with inner product $\langle f, g\rangle = (1/\pi)\int_{-\pi}^\pi fg$. In particular $\tilde{B} = \{v_j; j \in \mathbb{N}\}$ is an orthonormal subset of $L^2([-\pi, \pi], \mathbb{R})$. So, for $j \ne k, 0 = \int_{-\pi}^\pi v_j(x)v_k(x)dx = 2\int_0^\pi v_j(x)v_k(x)dx$, because v_jv_k is an even function. By the same token, for $j \in \mathbb{N}, 1 = (1/\pi)\int_{-\pi}^\pi v_j(x)^2dx = (2/\pi)\int_0^\pi v_j(x)^2dx$. For $j = 0$ this property easily is directly verified. So $\tilde{B}$ is an orthonormal subset of $L^2([0, \pi], \mathbb{R})$. To show that it is an orthonormal basis, we use Exercise 4.2.3: We let $f = L^2([0, \pi], \mathbb{R})$ with $\int_0^\pi f(x)v_j(x)dx = 0$ for all $j \in \mathbb{Z}_+$ and show that $f = 0$. Extend f to an even function on $[-\pi, \pi]$ by setting $f(-x) = f(x)$ for $x \in (0, \pi]$. Then, for all $j \in \mathbb{Z}, \int_{-\pi}^\pi f(x)\sin(jx)dx = 0$, because $f(x)\sin(jx)$ is an odd function of x. For all $j \in \mathbb{Z}_+, \int_{-\pi}^\pi f(x)v_j(x)dx = 2\int_0^\pi f(x)v_j(x)dx = 0$, because $f(x)v_j(x)$ is an even function of x. Since B is an orthonormal basis of $L^2([-\pi, \pi], \mathbb{R}), f = 0$ by Exercise 4.2.3 (a). So $\tilde{B}$ is an orthonormal basis by Exercise 4.2.3 (b). ☐ **D 4.22**. A twice cont diff func $\phi : [0, L] \to \mathbb{R}$ with $\phi(0) = 0 = \phi(L)$ is a test func for the VSE. A func $u : [0, L] \times \mathbb{R} \to \mathbb{R}$ is called a gen sol of (4.5) if $u(t, \cdot) \in L^2[0, L]$ for all $t \in \mathbb{R}, u(x, 0) = f(x)$ for a.a. $x \in [0, L]$ and, for every test function $\phi, \int_0^L \phi(x)u(x, t)dx$ is a twice cont diff func of $t \in \mathbb{R}$ and $(d^2/dt^2)\int_0^L \phi(x)u(x, t)dx = c^2\int_0^L \phi''(x)u(x, t)dx$ and $(d/dt_{[t=0]})\int_0^L \phi(x)u(x, t)dx = \int_0^L \phi(x)g(x)dx$. **T 4.23**. Any classical sol u of (4.5) is a gen sol of (4.5). A gen sol is uniquely determined by finding its Fourier sine series, $u(x, t) = \sum_{j=1}^\infty B_j(t)\sin(\lambda_j x), \lambda_j = j\pi/L, B_j(t) = (2/L)\int_0^L u(x, t)\sin(\lambda_j x)dx, j \in \mathbb{N}$. (4.7) For each $t \in \mathbb{R}$ the convergence of $\sum_{j=1}^\infty B_j(t)\sin(\lambda_j\cdot)$ holds in $L^2[0, L]$. Let $j \in \mathbb{N}$ and choose $\phi(x) = \sin(\lambda_j x)$. ϕ is a test func and so the following derivatives exist and satisfy $(d^2/dt^2)\int_0^L u(x, t)\sin(\lambda_j x)dx = c^2\int_0^L u(x, t)(d^2/dx^2)\sin(\lambda_j x)dx = -c^2\lambda_j^2\int_0^L u(x, t)\sin(\lambda_j x)dx$ and $(d/dt_{[t=0]})\int_0^L u(x, t)\sin(\lambda_j x)dx = \int_0^L g(x)\sin(\lambda_j x)dx, \int_0^L u(x, 0)\sin(\lambda_j x)dx = \int_0^L f(x)\sin(\lambda_j x)dx$. $\implies$ Fourier sine coef of a gen sol, $B_j(t)$, satisfy the ODEs $B_j'' + c^2\lambda_j^2 B_j = 0$ and the ICs $B_j(0) = (2/L)\int_0^L f(y)\sin(\lambda_j y)dy, B_j'(0) = (2/L)\int_0^L g(y)\sin(\lambda_j y)dy$. Gen sol, $a_j\cos(c\lambda_j t) + b_j\sin(c\lambda_j t) = B_j(t)$. From IC obtain $a_j = B_j(0) = (2/L)\int_0^L f(y)\sin(\lambda_j y)dy$ (4.8) and $c\lambda_j b_j = B_j'(0) = (2/L)\int_0^L g(y)\sin(\lambda_j y)dy$. (4.9) Subst and get Fourier sine series for any gen sol, $u(x, t) = \sum_{j=1}^\infty[a_j\cos(c\lambda_j t) + b_j\sin(c\lambda_j t)]\sin(\lambda_j x)$. (4.10) The d'Alembert form provides a gen sol if f and g are cont on $[0, L]$ and $f(0) = 0 = f(L)$ and $g(0) = 0 = g(L)$ and f and g are extended in an odd and $2L$-periodic way. Set $v(x, t) = (1/2)(f(x + ct) + f(x - ct))$. (4.11) After a change of variables, $\int_0^L \phi(x)v(x, t)dx = (1/2)\int_{ct}^{L+ct}\phi(y - ct)f(y)dy + (1/2)\int_{-ct}^{L-ct}\phi(y + ct)f(y)dy$. Since ϕ is twice cont diff and f is cont, we can use the Leibnitz rule, $\implies$ expression is diff and

$(d/dt)\int_0^L \phi(x)v(x,t)dx = (c/2)[\phi(L)f(L+ct)-\phi(0)f(ct)]-(c/2)\int_{ct}^{L+ct}\phi'(y-ct)f(y)dy-(c/2)[\phi(L)f(L-ct)-\phi(0)f(-ct)]+(c/2)\int_{-ct}^{L-ct}\phi'(y+ct)f(y)dy$. Since $\phi(0) = 0 = \phi(L), (d/dt)\int_0^L \phi(x)v(x,t)dx = -(c/2)\int_{ct}^{L+ct}\phi'(y-ct)f(y)dy + (c/2)\int_{ct}^{L-ct}\phi'(y+ct)f(y)dy$ (4.12). This expression is 0 at $t = 0$. Use Leibnitz rule again, $(d^2/dt^2)\int_0^L \phi(x)v(x,t)dx = -(c^2/2)[\phi'(L)f(L+ct)-\phi'(0)f(ct)] + (c^2/2)\int_{ct}^{L+ct}\phi''(y-ct)f(y)dy - (c^2/2)[\phi'(L)f(L-ct)-\phi'(0)f(-ct)] + (c^2/2)\int_{-ct}^{L-ct}\phi''(y+ct)f(y)dy$. Since f is odd about 0 and L, the boundary terms cancel each other and, after reversing the subst $(d^2/dt^2)\int_0^L \phi(x)v(x,t)dx = \int_0^L c^2\phi''(x)v(x,t)dx$. (4.13) Set $w(x,t) = 1/(2c)\int_{x-ct}^{x+ct} g(y)dy$. (4.14) Since g is cont, w is diff wrt t and x and $\partial_t w(x,t) = (1/2)[g(x+ct)+g(x-ct)], \partial_x w(x,t) = (1/2c)[g(x+ct)-g(x-ct)]$. (4.15) At $t = 0$, the first expression is $g(x)$. Since $\partial_t w$ is cont, we can diff under the int and obtain $(d/dt)\int_0^L \phi(x)w(x,t)dx = \int_0^L \phi(x)(1/2)[g(x+ct)+g(x-ct)]dx$. The same consideration as before with g replacing f (see (4.12)) yields $(d^2/dt^2)\int_0^L \phi(x)w(x,t)dx = (-c/2)\int_{ct}^{L+ct}\phi'(y-ct)g(y)dy + (c/2)\int_{-ct}^{L-ct}\phi'(y+ct)g(y)dy$. After refersing the subst, $(d^2/dt^2)\int_0^L \phi(x)w(x,t)dx = -\int_0^L \phi'(x)(c/2)[g(x+ct)-g(x-ct)]dx$. Observe from (4.15) that $(d^2/dt^2)\int_0^L \phi(x)w(x,t)dx = -\int_0^L \phi'(x)c^2\partial_x w(x,t)dx$. We int by parts, recall $w(0,t) = 0 = w(L,t)$, and obtain $(d^2/dt^2)\int_0^L \phi(x)w(x,t)dx = \int_0^L \phi''(x)c^2 w(x,t)dx$. Since $u(x,t) = v(x,t) + w(x,t)$, so $\int_0^L \phi(x)u(x,t)dx$ is twice diff and $(d^2/dt^2)\int_0^L \phi(x)u(x,t)dx = \int_0^L c^2\phi''(x)u(x,t)dx$ and $(d/dt)\int_0^L \phi(x)u(x,t)dx = \int_0^L \phi(x)g(x)dx, t = 0$. **Heat** Let $L, a > 0$. (PDE) $(\partial_t - a\partial_x^2)u = 0, 0 \le x \le L, t > 0$, (IC) $u(x,0) = f(x), 0 \le x \le L$, (BC) $u(0,t) = 0 = u(L,t), t > 0$ (5.1). The sol, if $\exists$, can be written as a Fourier sine series (Lemma 4.19, Ex 4.3.5) $u(x,t) = \sum_{j=1}^\infty B_j(t)\sin(\lambda_j x), \lambda_j = j\pi/L = j\lambda_1$, with $B_j(t) = (2/L)\int_0^L u(y,t)\sin(\lambda_j y)dy$ (5.2), where, for fixed t, the series converges in the L^2-sense in x. If u is a sol, it is suff smooth that we can diff under the int, $B_j'(t) = (2/L)\int_0^L \partial_t u(y,t)\sin(\lambda_j y)dy = (2/L)\int_0^L a\partial_y^2 u(y,t)\sin(\lambda_j y)dy$. Since the sines and u satisfy zero boundary cond, we can int by parts twice and obtain the diff eq $B_j'(t) = -a\lambda_j^2 B_j(t)$. The IC yields $B_j(0) = (2/L)\int_0^L f(y)\sin(\lambda_j y)dy$ (5.3). Solutions $B_j(t) = B_j(0)e^{-a\lambda_j^2 t}$ (5.4). $\implies$ sol of (5.1) is uniquely determined. **Existence** **T 5.1**. Let (c_j) be a seq of non-neg numbers s.t. $\sum_{n=1}^\infty c_n < \infty$. Let $D \subseteq \mathbb{R}^N$ and (f_n) be a seq of func $f_n : D \to \mathbb{K}$ s.t. $|f_n(x)| \le c_n \forall x \in D$ and $n \in \mathbb{N}$. Then $\sum_{n=1}^\infty f_n(x)$ converges unif in $x \in D$. If each f_n is (unif) cont on D, so is $\sum_{n=1}^\infty f_n$. **T 5.3**. Let (c_j) be a seq of non-neg numbers s.t. $\sum_{n=1}^\infty c_n < \infty$. Let I_1 and I_2 be two bounded nondegenerate intervals and (f_n) be a seq of cont funcs $f_n : I_1 \times I_2 \to \mathbb{K}$. Assume that each f_n has partial derivatives wrt the first var and $|\partial_1 f_n(x,t)| \le c_n$ for all $n \in \mathbb{N}, x \in I_1, t \in I_2$. Assume $\sum_{n=1}^\infty f_n$ converges pointwise on $I_1 \times I_2$. Then $\sum_{n=1}^\infty f_n$ is partially diff wrt the first var and $\partial_1(\sum_{n=1}^\infty f_n) = (\sum_{n=1}^\infty \partial_1 f_n)$, with the second series converging uniformly; this partial derivative is bounded. If each $\partial_1 f_n$ is jointly cont on $I_1 \times I_2$, so is $\partial_1(\sum_{n=1}^\infty f_n)$. If each f_n is diff with both partial derivatives being cont and $|\partial_j f_n(x,t)| \le c_n$ for $j = 1,2$, then $\sum_{n=1}^\infty f_n$ is cont diff and we can interchange diff and sum (diff term by term). **T 5.6**. Let $f : [0,L] \to \mathbb{R}$ be Lipschitz cont, $f(L) = 0 = f(0)$. Then u is cont on $[0,L] \times [0,\infty)$ and $u(0,x) = f(x)$ for all $x \in [0,L]$. In particular, $u(x,t) \to f(x)$ as $t \to 0$ uniformly in $x \in [0,L]$. The Fourier sine series of u converges to u uniformly on $[0,L] \times [0,\infty)$. **T 5.7**. Assume that $f : [0,L] \to \mathbb{R}$ is integrable and $\int_0^L |f(x)|^2 dx < \infty$. Then the series u in (5.2) with (5.4) and (5.3) satisfies $\int_0^L |u(x,t)-f(x)|^2 dx \to 0, t \to 0$. Actually, $u(\cdot,t)$ is a uniformly cont func of $t \in \mathbb{R}_+$ with values in $L^2[0,L]$. Notice that $\int_0^L |f(x)|^2 dx < \infty$ is a stronger assumption than $\int_0^L |f(x)|dx < \infty$ in T 5.4. **T 5.8**. Let u be the sol of the heat eq with zero boundary conditions and initial data $f \in L^2([0,L],\mathbb{R})$. Then $||u(\cdot,t)|| \le ||f||e^{-a\lambda_1^2 t}, t \ge 0$. **T 5.9**. Let $f : [0,L] \to \mathbb{R}$ be integrable and $\int_0^L |f(x)|dx < \infty$ and u the sol of the heat eq from T 5.4. Then $u(x,t) \to 0$ as $t \to \infty$ uniformly in $x \in [0,L]$. *Proof.* Recall $|u(x,t)| = B_0\sum_{m=1}^\infty e^{-a\lambda_m^2 t}, B_0 = (2/L)\int_0^L |f(x)|dx$. For $t > 0$, with $\kappa = (\pi/L)^2$, since $\lambda_m^2 \le \kappa^2 m^2$, $|u(x,t)| = B_0\sum_{m=1}^\infty e^{-a\lambda_m^2 t} \le B_0\sum_{m=1}^\infty (e^{-a\kappa t})^m$. Using the geometric series formula, for $t > 0$, $|u(x,t)| = B_0(e^{-a\kappa t})/(1-e^{-a\kappa t}) \to 0, t \to \infty$ $\square$ **Exercise 5.1.3**. Let $L, a > 0$. Consider the problem (PDE) $(\partial_t - a\partial_x^2)u = 0, -\pi \le x \le \pi, t > 0$, (IC) $u(x,0) = f(x), -\pi \le x \le \pi$, (BC) $u(-\pi,t) = u(\pi,t), \partial_x u(-\pi,t) = \partial_x u(\pi,t), t > 0$ (5.15). (a) Use complex Fourier series to solve (5.15), at least as far as (PDE) and (BC) are concerned, under an appropriate condition for f. (b) Explore two assumptions for f under which (IC) is satisfied in meaningful though not necessarily literal ways. Cf. Theorem 5.6 and 5.7. (c) Show that u is real-valued if f is real-valued. (d) Show that $\int_{-\pi}^\pi u(x,t)dx = \int_{-\pi}^\pi u(x,0)dx$ for all $t \ge 0$. Hint: These integrals are related to the Fourier cosine coefficient of index zero. (e) Show that $u(x,t) \to (1/2\pi)\int_{-\pi}^\pi f(x)dx$ as $t \to \infty$, uniformly in $x \in [-\pi,\pi]$ provided that $\int_{-\pi}^\pi |f(x)|dx < \infty$. *Proof.* (a) We try to find u as a complex Fourier series $u(x,t) = \sum_{m\in\mathbb{Z}} C_m(t)e^{imx}, C_m(t) = 1/(2\pi)\int_{-\pi}^\pi u(x,t)e^{-imx}dx$ (5.16). If $\partial_t u(x,t)$ exists and is continuous on $[-\pi,\pi] \times (0,\infty)$, we can interchange time differentiation and integration, $C_m'(t) = 1/(2\pi)\int_{-\pi}^\pi \partial_t u(x,t)e^{-imx}dx = 1/(2\pi)\int_{-\pi}^\pi a\partial_x^2 u(x,t)e^{-imx}dx$. We integrate by parts twice; the boundary terms cancel because of the periodic boundary conditions, $C_m'(t) = -am^2 C_m$. Further $C_m(0) = 1/(2\pi)\int_{-\pi}^\pi f(x)e^{-imx}dx = \hat{f}_m$. We solve the ordinary differential equation, $C_m(t) = \hat{f}_m e^{imx}e^{-am^2 t}$. That the series in (5.16) converges uniformly on $[-\pi,\pi] \times [\epsilon,\infty)$ for any $\epsilon > 0$ and solves (PDE) in (5.15) is shown analogously to the proof of Theorem 5.4. Define $u_m(x,t) = \hat{f}_m e^{imx}e^{-am^2 t}$. Then u_m is infinitely often differentiable and satisfies the heat equation, $\partial_t u_m(x,t) = -am^2 u_m(x,t) = a\partial_x^2 u_m(x,t)$. For all $k, \ell \in \mathbb{Z}_+$ and $m \in \mathbb{Z}, \partial_x^k\partial_t^\ell u_m(x,t) = \hat{f}_m(im)^k(-am^2)^\ell e^{imx}e^{-am^2 t}$ and $|\partial_x^k\partial_t^\ell u_m(x,t)| = |\hat{f}_m||i|^k|m|^k a^\ell m^{2\ell}|e^{imx}|e^{-am^2 t} \le |\hat{f}_m||m|^{k+2\ell}a^\ell e^{-am^2 t}$. For $m \in \mathbb{Z}, |\hat{f}_m| \le 1/(2\pi)\int_{-\pi}^\pi |f(x)||e^{-imx}|dx \le 1/(2\pi)\int_{-\pi}^\pi |f(x)|dx =: A$. Let $\epsilon > 0$. For $m \in \mathbb{Z}$ and $k, \ell \in \mathbb{Z}_+$ and $t \in [\epsilon,\infty), |\partial_x^k\partial_t^\ell u_m(x,t)| \le A|m|^{k+2\ell}a^\ell e^{-am^2\epsilon}$. By the ration test, $\sum_{m=1}^\infty Am^{k+2\ell}a^\ell e^{-am^2\epsilon}$ converges in $\mathbb{R}$. By the Weierstraß test, for each $\ell, k \in \mathbb{Z}_+$, the following series converge uniformly for $x \in [-\pi,\pi], t \in [\epsilon,\infty), \sum_{m=1}^\infty \partial_x^k\partial_t^\ell u_m(x,t), \sum_{m=1}^\infty \partial_x^k\partial_t^\ell u_{-m}(x,t), \sum_{m\in\mathbb{Z}} \partial_x^k\partial_t^\ell u_m(x,t)$ with the third being the sum of the first and second. By Theorem 5.3, u is infinitely often partially differentiable on $[-\pi,\pi] \times (0,\infty)$ and $\partial_x^k\partial_t^\ell u(x,t) = \sum_{m\in\mathbb{Z}} \partial_x^k\partial_t^\ell u_m(x,t)$. In particular, u satisfies the heat equation on $[-\pi,\pi] \times (0,\infty)$. Since e^{imx} is 2π-periodic for all $m \in \mathbb{Z}, u(\pi,t) = u(-\pi,t)$ for all $t > 0$ follows from the uniform convergence of the series in (5.16) converges uniformly on $[-\pi,\pi] \times [\epsilon,\infty)$ for any $\epsilon > 0$. Further, $\partial_x u(x,t) = \sum_{m\in\mathbb{Z}} C_m(t)mie^{imx}$ with convergence being uniform for $x \in [-\pi,\pi], t \in [\epsilon,\infty)$ for any $\epsilon > 0$. By the same token as before $\partial_x u(\pi,t) = \partial_x u(-\pi,t), t > 0$. (b) This is analogous to Theorem 5.6 and 5.7. We first assume that f is Lipschitz continuous on $[-\pi,\pi]$ and $f(\pi) = f(-\pi)$. By Lemma 4.16, f can be extended to a Lipschitz continuous 2π-periodic function on $\mathbb{R}$. by Proposition 4.14, the following series converges, $\sum_{m\in\mathbb{Z}} |\hat{f}_m| = |\hat{f}_0| + \sum_{m=1}^\infty(|\hat{f}_m| + |\hat{f}_{-m}|)$. For all $m \in \mathbb{Z}, |C_m(t)| \le |\hat{f}_m||e^{-am^2 t}| \le |\hat{f}_m|$. Thus $|C_m(t)e^{imx}| = |C_m(t)||e^{imx}| = |C_m(t)| \le |\hat{f}_m|, m \in \mathbb{Z}$. For all $m \in \mathbb{N}, |C_m(t)e^{imx} + C_{-m}(t)e^{-imx}| \le |C_m(t)e^{imx}| + |C_{-m}(t)e^{-imx}| \le |\hat{f}_m| + |\hat{f}_{-m}|$. By the Weierstraß test. $\sum_{m\in\mathbb{Z}} C_m(t)e^{imx} = C_0(t) + \sum_{m=1}^\infty(C_m(t)e^{imx} + C_{-m}(t)e^{-imx})$ converges uniformly on $[-\pi,\pi] \times [0,\infty)$. This implies that u is continuous on $[-\pi,\pi] \times [0,\infty)$. We now assume that f is integrable and $\int_{-\pi}^\pi |f(x)|^2 dx < \infty$. Let $\langle\phi,\psi\rangle = 1/(2\pi)\int_{-\pi}^\pi \phi(x)\overline{\psi(x)}dx$ be the inner product of choice on $L^2([-\pi,\pi],\mathbb{C})$, the space of square integrable functions. Then $\{v_j; j \in \mathbb{Z}\}$ with $v_j(x) = e^{ijx}$ is an orthonormal basis. We intend to apply Theorem 4.11. Notice that $g : \mathbb{N} \to \mathbb{Z}$ with $g(2n) = n$ and $g(2n-1) = -n, n \in \mathbb{N}$, is a bijection. By the considerations at the beginning of this section, $\langle u(\cdot,t), v_m\rangle = \langle f, v_m\rangle e^{-a\lambda_m^2 t}$ (5.17), which are uniformly continous functions of $t \in \mathbb{R}_+$. Further $|\langle u(\cdot,t), v_m\rangle| \le |\langle f, v_m\rangle|$ and, by Parseval's relation (Theorem 4.10), $\sum_{m\in\mathbb{Z}} |\langle f, v_m\rangle|^2 = ||f||^2$. Theorem 4.11 implies that $U : \mathbb{R}_+ \to L^2([-\pi,\pi],\mathbb{C})$ with $U(t) = u(\cdot,t)$ is continuous. (c) This is analogous to Theorem 4.18. For fixed $t > 0, u(x,t)$ is the limit (uniformly in $x \in [-\pi,\pi]$) of Fourier sums $\sum_{m=-n}^n \hat{f}_m e^{-am^2 t}e^{imx} = \hat{f}_0 + \sum_{m=1}^n(\hat{f}_m e^{imx} + \hat{f}_{-m}e^{-imx})e^{-am^2 t}$. The same proof as for Theorem 4.18 shows that $\hat{f}_m e^{imx} + \hat{f}_{-m}e^{-imx}$ is real if f is real-valued. (d) Let $\langle\phi,\psi\rangle = 1/(2\pi)\int_{-\pi}^\pi \phi(x)\overline{\psi(x)}dx$ be the inner product on the Hilbert space $L^2([-\pi,\pi],\mathbb{C})$. Let $\{v_m; m \in \mathbb{Z}\}$ be the orthonormal basis with $v_m(x) = e^{imx}$. Notice that $v_0(x) = 1$. By orthonormality and part (a), $\langle u(\cdot,t), v_0\rangle = C_0(t) = \langle f, v_0\rangle = \langle u(\cdot,0), v_0\rangle$. This implies the assertion. (e) For all $t \ge 0, |u(x,t) - 1/(2\pi)\int_{-\pi}^\pi f(x)dx| = |\sum_{0\ne m\in\mathbb{Z}} \hat{f}_m e^{-am^2 t}e^{imx}| \le \sum_{0\ne m\in\mathbb{Z}} |\hat{f}_m|e^{-am^2 t} \le A\sum_{m=1}^\infty e^{-am^2 t}$ with $A = 1/\pi\int_{-\pi}^\pi |f(x)|dx$. The estimate can be continued by $\le A\sum_{m=1}^\infty e^{-amt} = \sum_{m=1}^\infty A(e^{-at})^m = A(e^{-at})/(1-e^{-at}) \to^{t\to\infty} 0$ $\square$ **T 5.10**. Let $T \in (0,\infty)$ and $u : \Omega \times (0,T] \to \mathbb{R}$. Assume that u is once partially diff wrt $t \in (0,T]$ and twice partially diff wrt x_k at each $x \in \Omega, t \in (0,T], k = 1,\dots,n$, Let $c : \Omega \times (0,T] \to \mathbb{R}$ be strictly neg. Assume the differential inequality $(\partial_t - L)u \le c(x,t)u, x \in \Omega, t \in (0,T]$. Then u has no positive max in $\Omega \times (0,T]$: $\exists$ no $t \in (0,T], x \in \Omega$ such that $u(x,t) \ge u(y,s)$ for all $s \in (0,T], y \in \Omega$, and $u(x,t) > 0$. **T 5.11**. Let $T \in (0,\infty)$ and $u : \bar{\Omega} \times [0,T] \to \mathbb{R}$ be cont. Let $c : \Omega \times (0,T) \to \mathbb{R}$ be bounded above. Assume that u is once partially diff wrt $t \in (0,T)$ and twice partially diff wrt x_k at each $x \in \Omega, t \in (0,T), k = 1,\dots,n$. Assume $(\partial_t - L)u \le cu, x \in \Omega, t \in (0,T), u(x,0) \le 0, x \in \bar{\Omega}, u(x,t) \le 0, x \in \partial\Omega, t \in (0,T)$. Then $u(x,t) \le 0$ for all $x \in \bar{\Omega}, t \in [0,T]$.

T 5.13. Let $T \in (0,\infty)$ and $u : \bar{\Omega} \times [0,T] \to \mathbb{R}$ be cont. Let $c, F : \Omega \times (0,T) \to \mathbb{R}$, F bounded and c non-pos. Assume that u is once partially diff wrt $t \in (0,T)$ and twice partially diff wrt x_k at each $x \in \Omega, t \in (0,T), k = 1,\dots,n$. Assume $(\partial_t - L - c)u = F(x,t), x \in \Omega, t \in (0,T)$. Let $M, N \ge 0$ s.t. $|u(x,0)| \le M, x \in \bar{\Omega}, |u(x,t)| \le M + tN, x \in \partial\Omega, t \in [0,T)$, and $|F(x,t)| \le N$ for all $x \in \Omega, t \in (0,T)$. Then $|u(x,t)| \le M + tN$ for all $x \in \bar{\Omega}, t \in [0,T]$. **T 5.14**. Let $T \in (0,\infty)$ and $u_1, u_2 : \bar{\Omega} \times [0,T] \to \mathbb{R}$ be cont. Let $c, F_1, F_2 : \Omega \times (0,T) \to \mathbb{R}, F_j$ bounded and c non-pos and $g_1, g_2 : \partial\Omega \times [0,T] \to \mathbb{R}, f_1, f_2 : \bar{\Omega} \to \mathbb{R}$. Assume u_1 and u_2 are once partially diff wrt $t \in (0,T)$ and twice partially diff wrt x_k at each $x \in \Omega, t \in (0,T), k = 1,\dots,n$. Assume, for $j = 1,2, (\partial_t - L - c)u_j = F_j(x,t), x \in \Omega, t \in (0,T), u_j(x,0) = f_j(x), x \in \Omega, u_j(x,t) = g_j(x,t), x \in \partial\Omega, t \in (0,T)$. Let $\delta, \epsilon > 0$ and $|F_1(x,t) - F_2(x,t)| \le \delta, x \in \Omega, t \in (0,T), |f_1(x) - f_2(x)| \le \epsilon, x \in \Omega, |g_1(x,t) - g_2(x,t)| \le \epsilon + \delta t, x \in \partial\Omega, t \in (0,T)$. Then $|u_1(x,t) - u_2(x,t)| \le \epsilon + \delta t$ for all $x \in \bar{\Omega}, t \in [0,T]$. **Ex 5.2.3**. Let $T > 0$ and $u : \overline{\Omega} \times [0,T]$ be cont. Let $c, F : \Omega \times (0,T) \to \mathbb{R}$, F bounded and c bounded above. Assume u is once partially diff wrt $t \in (0,T)$ and twice partially diff wrt x_k at each $x \in \Omega, k = 1,\dots,n$. Assume $(\partial_t - L - c)u = F(x,t), x \in \Omega, t \in (0,T)$, Let $M, N \ge 0$ such that $|u(x,t)| \le M$ whenever $x \in \partial\Omega, t \in [0,T]$ or $x \in \overline{\Omega}, t = 0$ and $|F(x,t)| \le N$ for all $x \in \Omega, t \in (0,T)$. Show: $|u(x,t)| \le (M + tN)e^{\kappa t}$ for all $x \in \overline{\Omega}, t \in [0,T]$, where $\kappa \ge 0$ is chosen s.t. $c(x,t) \le \kappa$ for all $x \in \Omega, t \in (0,T)$. Hint: Consider $v(x,t) = u(x,t)e^{-\kappa t}$. *Proof.* Define $v(x,t) = u(x,t)e^{-\kappa t}$. Then $\partial_t v(x,t) - (Lv)(x,t) = e^{-\kappa t}(\partial_t u(x,t) - (Lu)(x,t)) - \kappa v(x,t) = (c(x,t) - \kappa)v(x,t) + F(x,t)e^{-\kappa t}$. So $\partial_t v(x,t) - (Lv)(x,t) - \tilde{c}(x,t)v(x,t) = \tilde{F}(x,t)$, where $\tilde{c}(x,t) = c(c,t) - \kappa \le 0, \tilde{F}(x,t) = F(x,t)e^{-\kappa t}$ and so $|\tilde{F}(x,t)| \le |F(x,t)| \le N, x \in \Omega, t \in (0,T)$. Further $|v(x,t)| \le |u(x,t)| \le M$ whenever $x \in \partial\Omega, t \in [0,T]$ or $x \in \bar{\Omega}, t = 0$. By T 5.13, $|v(x,t)| \le M + Nt$. So $|u(x,t)| = |v(x,t)|e^{\kappa t} \le (M + Nt)e^{\kappa t}, x \in \bar{\Omega}, t \in [0,T]$ $\square$. **Exercise 5.2.11**. Let $L, T > 0$ and $u : [0,L] \times [0,T] \to \mathbb{R}$ be continuous, and sufficiently often differentiable and satisfy $0 \le \partial_t u(x,t) - x^3(L-x)^5\partial_x^2 u(x,t) + a\partial_x u(x,t) + (L-x)u(x,t), 0 < x < L, \quad 0 < t < T, 0 \le u(0,t), \quad u(L,t) \ge 0, \quad t \in [0,T], 0 \le u(x,0), \quad 0 \le x \le L$. Here $a \in \mathbb{R}$. Show: $u(x,t) \ge 0$ for all $x \in [0,L], t \in [0,T]$. Do not use the maximum principle, but do the proof from scratch. *Proof.* By contradiction assume there exists a $y \in [0,L]$ and an $r \in [0,T]$ such that $u(y,r) < 0$. Consider $u : [0,L] \times [0,r]$. Since u is compact u has a minimum in $[0,L] \times [0,r]$. Denote such a minimum as $u_m = u(x_m,t_m)$ with $x_m \in [0,L]$ and $t_m \in [0,r]$. Therefore $u(x_m,t_m) \le u(z,s)$ for any $z \in [0,L]$ and $s \in [0,r]$. Particularly, $u(x_m,t_m) \le u(y,r) < 0$. Since u_m is a minimum, $\partial_x u(x,t)|_{(x_m,t_m)} = 0$, and $\partial_x^2 u(x,t)|_{(x_m,t_m)} \ge 0$. Now note that $\partial_t u(x,t)|_{(x_m,t_m)} = \lim_{s\to t_m^-}(u(x_m,s) - u(x_m,t_m))/(s-t_m) \le 0$ because the numerator is positive and the denominator is negative. Having these results, we can evaluate the PDE at the minimum point (x_m,t_m). $\partial_t u(x_m,t_m) - x^3(L-x_m)^5\partial_x^2 u(x_m,t_m) + a\partial_x u(x_m,t_m) + (L-x_m)u(x_m,t_m)$. We have shown that the third term is zero so we are left with $\partial_t u(x_m,t_m) - x^3(L-x_m)^5\partial_x^2 u(x_m,t_m) + (L-x_m)u(x_m,t_m)$, We have that $(L = x_m) > 0$ and we have shown that the 1st and 3rd terms are negative and so $\partial_t u(x_m,t_m) - x^3(L-x_m)^5\partial_x^2 u(x_m,t_m) + a\partial_x u(x_m,t_m) + (L-x_m)u(x_m,t_m) < 0$ which contradicts our PDE $\square$. **Identities Sum and Difference Formula** $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. $\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$. **Double Angle Formula** $\sin(2A) = 2 \sin A \cos A$. $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$. $\tan(2A) = (2 \tan A)/(1 - \tan^2 A)$. **Half Angle Formula** $\sin(A/2) = \pm\sqrt{(1 - \cos A)/2}$. $\cos(A/2) = \pm\sqrt{(1 + \cos A)/2}$. $\tan(A/2) = (1 - \cos A)/(\sin A) = (\sin A)/(1 + \cos A)$. **Geometric Sum** $\sum_{k=1}^{\infty} q^k = q/(1 - q)$. $\sum_{k=1}^{n} q^k = (q - q^{n-1})/(1 - q)$. **General ODE Solutions** $y'' = y(t) \implies y = c_1 e^{-t} + c_2 e^t$ $\square$ $dy/dt + p(t)y = g(t) \implies y = (\int u(t)g(t))/u(t) + c$ where $u(t) = \exp(\int p(t)dt)$ $\square$ $y' = x; x' = y \implies x = c_1 \cosh t + c_2 \sinh t, y = c_1 \sinh t + c_2 \cosh t$ or $x = c_1 e^t + c_2 e^{-t}, y = c_1 e^t - c_2 e^{-t}$ $\square$ $y' = -x; x' = y \implies y = c_1 \cos t + c_2 \sin t, x = c_1 \sin t - c_2 \cos t$ $\square$ $x' = x + y; y' = -x + y \implies x = e^t(c_1 \cos t + c_2 \sin t); y = e^t(-c_1 \sin t + c_2 \cos t)$ $\square$ $v' = \gamma v, v(z, 0) = u_0 \implies v = u_0 e^{\gamma t}$ $\square$