Spectrall Methods

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Homework 1

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Problem 1

Show that the trapezoidal rule

$$\int_0^{2\pi} f(x)dx = \frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j),$$

where $x_j = 2\pi j/N$, is exact for $f(x) = \exp(inx)$ for $|n| < N$ (but not for $|n| = N$). Conclude that the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

First, we compute the integral,

$$\int_0^{2\pi} e^{inx}\, dx = 2\pi\delta_{n0},$$

where δ is the Kronecker-delta. For the right hand side we start considering the case where $n < |N|$ and $n \neq 0$,

$$\begin{aligned}
\frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j) &= \frac{2\pi}{N}\sum_{j=0}^{N-1} e^{inx_j} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} e^{\left(\frac{i2\pi nj}{N}\right)} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} \left[e^{\left(\frac{i2\pi n}{N}\right)}\right]^j \\
&= \left(\frac{2\pi}{N}\right)\left(\frac{1-e^{i2\pi n}}{1-e^{\left(\frac{i2\pi n}{N}\right)}}\right) \\
&= 0, \qquad \forall n < |N|, n \neq 0,
\end{aligned}$$

where we have applied the geometric sum formula and that $e^{i2\pi n} = 1$. We note that, if $n = N$ the solution would not be defined. Lastly, for $n = 0$,

$$\frac{2\pi}{N}\sum_{j=0}^{N-1} f(x_j) = \frac{2\pi}{N}\sum_{j=0}^{N-1} 1 = \frac{2\pi}{N}N = 2\pi \ .$$

Finally, let g be a function in the span$\{e^{inx}\}_{|n|<N}$. Hence, we can write

$$g(x) = \sum_{|n|<N} c_n e^{inx},$$

with c_n being constant coefficients. Then,

$$\begin{aligned}
\int_0^{2\pi} g(x)\ dx &= \int_0^{2\pi} \sum_{|n|<N} c_n e^{inx}\ dx \\
&= \sum_{|n|<N} c_n \left(\int_0^{2\pi} e^{inx}\ dx\right) \\
&= \sum_{|n|<N} c_n \left(\frac{2\pi}{N}\sum_{j=0}^{N-1} e^{inx_j}\right) \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1}\sum_{|n|<N} c_n e^{inx_j} \\
&= \frac{2\pi}{N}\sum_{j=0}^{N-1} g(x_j) \ .
\end{aligned}$$

Thus, the trapezoidal rule is exact for all functions in the span of $\{\exp(inx)\}_{|n|<N}$.

Problem 2

(*Best Approximation*) Prove the following statements.

(a) **Let $e_1, \cdots, e_N$ be an orthonormal system in an inner product space H, let $\lambda_1, \cdots, \lambda_N \in \mathbb{C}$ and let $f \in H$. Then**

$$\left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 = ||f||^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2,$$

where $c_j = \langle f, e_j \rangle$ and $\| \cdot \|$ is the norm induced by the inner product.

To prove that the statement we start by writing the left side of the equation as an inner product,

$$\left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 =< f - \sum_{j=1}^{N} \lambda_j e_j | f - \sum_{j=1}^{N} \lambda_j e_j > .$$

By appllying linearity and antilinearity of the inner product we get

$$\begin{aligned}
\left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 &=< f - \sum_{j=1}^{N} \lambda_j e_j | f - \sum_{j=1}^{N} \lambda_j e_j > \\
&=< f|f > - \sum_{j=1}^{N} < \lambda_j e_j | f > - \sum_{j=1}^{N} < f | \lambda_j e_j > + \sum_{j=1}^{N} < \lambda_j e_j | \lambda_j e_j > \\
&= ||f||^2 - \sum_{j=1}^{N} \lambda_j < e_j | f > - \sum_{j=1}^{N} \lambda_j^* < f | e_j > + \sum_{j=1}^{N} < \lambda_j e_j | \lambda_j e_j > \\
&= ||f||^2 - \sum_{j=1}^{N} \lambda_j c_j^* - \sum_{j=1}^{N} \lambda_j^* c_j + \sum_{j=1}^{N} ||\lambda_j||^2.
\end{aligned}$$

where we have used the linearity and antilinearity properties again, and that $c_j =< f|e_j >$. By adding and substracting $\sum_{j=1}^{N} < c_j | c_j >$ and rewriting the previous equation in terms of inner product again we get

$$\begin{aligned}
\left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 = ||f||^2 - \sum_{j=1}^{N} < \lambda_j | c_j > - \sum_{j=1}^{N} < c_j | \lambda_j > + \sum_{j=1}^{N} < \lambda_j | \lambda_j > \\
+ \sum_{j=1}^{N} < c_j | c_j > - \sum_{j=1}^{N} < c_j | c_j >,
\end{aligned}$$

which, again by linearity and antilinearity, we can group together as

$$\begin{aligned}\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 &= ||f||^2+\sum_{j=1}^{N}<\lambda_j-c_j|\lambda_j>-\sum_{j=1}^{N}<\lambda_j-c_j|c_j>-\sum_{j=1}^{N}<c_j|c_j> \\ &= ||f||^2+\sum_{j=1}^{N}<\lambda_j-c_j|\lambda_j-c_j>-\sum_{j=1}^{N}<c_j|c_j> \\ &= ||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2.\end{aligned}$$

Thus, we have proven that

$$\left|\left|f-\sum_{j=1}^{N}\lambda_j e_j\right|\right|^2 = ||f||^2+\sum_{j=1}^{N}|\lambda_j-c_j|^2-\sum_{j=1}^{N}|c_j|^2.$$

(b) **Let $f_N=\sum_{j=1}^{N}\langle f,e_j\rangle e_j$. Then $\|f-f_N\|\leq\|f-g\|$ for all g in the span of $e_1,\cdots,e_N$.**

To prove that

$$||f-f_N||\leq||f-g||,$$

we will show that their squares

$$||f-f_N||^2\leq||f-g||^2.$$

Taking into account that $f_N=\sum_{j=1}^{N}<f|e_j>e_j$ and $g=\sum_{j=1}^{N}\lambda_j e_j$ and using the previous proof we have that

$$\begin{aligned}||f-f_N||^2 &= \left|\left|f-\sum_{j=1}^{N}<f|e_j>e_j\right|\right|^2 \\ &= ||f||^2+\sum_{j=1}^{N}|<f|e_j>-c_j|^2-\sum_{j=1}^{N}|c_j|^2 \\ &= ||f||^2+\sum_{j=1}^{N}|<f|e_j>-<f|e_j>|^2-\sum_{j=1}^{N}|c_j|^2 \\ &= ||f||^2-\sum_{j=1}^{N}|<f|e_j>|^2,\end{aligned}$$

and

$$\begin{aligned}
||f-g||^2 &= \left|\left|f - \sum_{j=1}^{N} \lambda_j e_j\right|\right|^2 \\
&= ||f||^2 + \sum_{j=1}^{N} |\lambda_j - c_j|^2 - \sum_{j=1}^{N} |c_j|^2 \\
&= ||f||^2 + \sum_{j=1}^{N} |\lambda_j - < f|e_j > |^2 - \sum_{j=1}^{N} |c_j|^2 \\
&\geq ||f||^2 - \sum_{j=1}^{N} | < f|e_j > |^2 \\
&= ||f - f_N||^2.
\end{aligned}$$

Thus, we have proved that

$$||f - f_N|| \leq ||f - g||$$

Problem 3

(*Fourier-Galerkin method*) Let $\mathcal{L}$ be a differential operator of the form $\mathcal{L} = \sum_{k=0}^{m} \alpha_k \frac{d^k}{dx^k}$, with $\alpha_k \in \mathbb{C}$. Suppose we are interested in solving the differential equation $\mathcal{L}u = g$. Let

$$u_N = \sum_{n=-N/2}^{N/2} c_n e^{inx},$$

where the coefficients c_n are chosen so that $\langle \mathcal{L}u_N - g, e^{ikx} \rangle = 0$ for $k = -N/2, \ldots, N/2$. Show that $||\mathcal{L}u_N - g|| \leq ||\mathcal{L}w - g||$ for all w in the span of $\{e^{inx}\}_{|n| \leq N/2}$.

Let's first compute the result of

$$\begin{aligned}
\mathcal{L}u_N &= \sum_{k=0}^{m} \alpha_k \frac{d^k}{dx^k} \left[\sum_{n=-N/2}^{N/2} c_n e^{inx} \right] \\
&= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k e^{inx}.
\end{aligned}$$

Before we continue, we work on the other piece of information, $\langle \mathcal{L}u_N - g, e^{ikx} \rangle = 0$, which implies

$$\langle \mathcal{L}u_N, e^{ilx} \rangle = \langle g, e^{ilx} \rangle,$$

where we have used the index l for future convenience. Now,

$$\begin{aligned}
\langle \mathcal{L}u_N, e^{ilx} \rangle &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k \langle e^{inx}, e^{ilx} \rangle \\
&= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k 2\pi \delta_{nl} \\
&= 2\pi c_l \sum_{k=0}^{m} \alpha_k (il)^k \\
&= \langle g, e^{ilx} \rangle.
\end{aligned}$$

Hence, retaking the index n,

$$\langle \mathcal{L}u_N, e^{inx} \rangle = 2\pi c_n \sum_{k=0}^{m} \alpha_k (in)^k = \langle g, e^{inx} \rangle.$$

Further, we can rewrite $\mathcal{L}u_N$ as

$$\begin{aligned}
\mathcal{L}u_N &= \sum_{n=-N/2}^{N/2} c_n \sum_{k=0}^{m} \alpha_k (in)^k e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N/2}^{N/2} 2\pi c_n \sum_{k=0}^{m} \alpha_k (in)^k e^{inx} \\
&= \frac{1}{2\pi} \sum_{n=-N/2}^{N/2} \langle g, e^{inx} \rangle e^{inx} \\
&= \sum_{n=-N/2}^{N/2} \langle g, \frac{e^{inx}}{\sqrt{2\pi}} \rangle \frac{e^{inx}}{\sqrt{2\pi}} \\
&= \sum_{n=-N/2}^{N/2} \langle g, e_n \rangle e_n,
\end{aligned}$$

where we are denoting $e_n = \frac{e^{inx}}{\sqrt{2\pi}}$ as the orthonormal vectors. Note that the set of $\{e_n\}_{|n| \leq N/2}$ form an orthonormal base. Since $\mathcal{L}u_N$ can be written as

$$\mathcal{L}u_N = \sum_{n=-N/2}^{N/2} \langle g, e_n \rangle e_n,$$

by the previous result in problem 2, $||\mathcal{L}u_N - g|| \leq ||\mathcal{L}w - g||$ for all g in the span of $e_1, \cdots, e_N$.

Problem 4

Let $f(x) = \sin^3(x/2)$. Compute its Fourier approximation for $N = 20 : 10 : 500$ and plot the error using *loglog*. Does this convergence plot agree with the error bound derived in class? Explain.

The first thing to notice is that the function is not 2π-periodic, as we can see in the figure below.

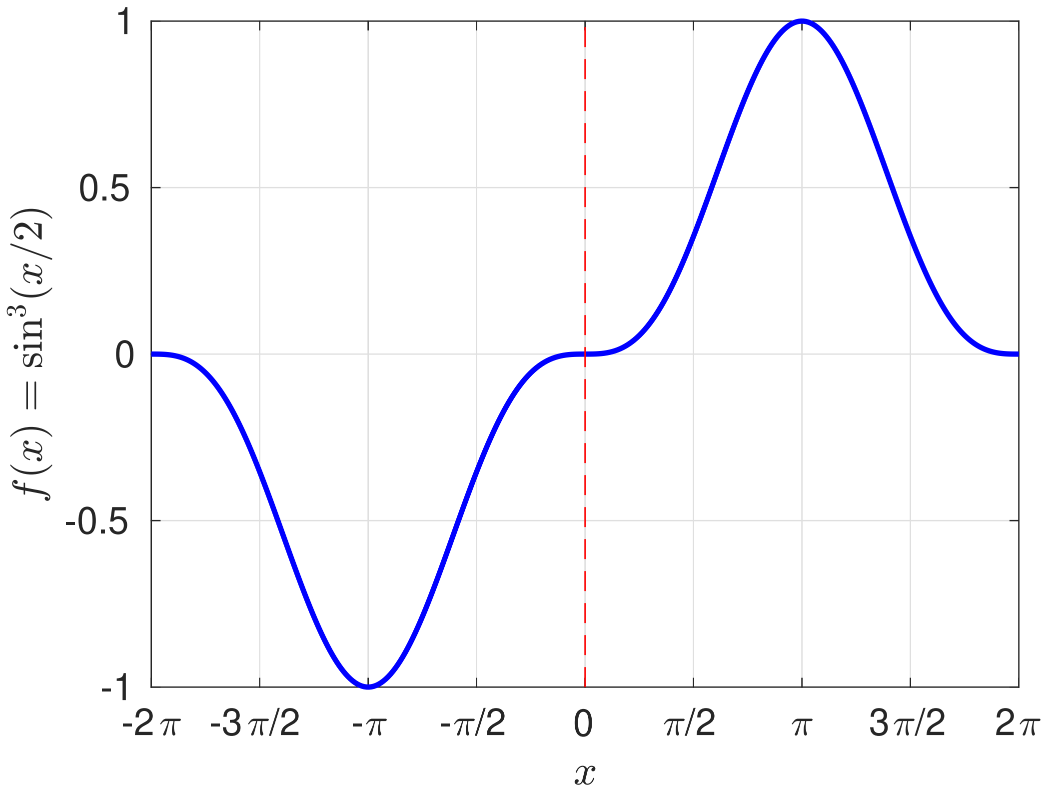

Figure 1: Original function $f(x)$.

Hence, we will approximate the error using the following upper bound,

$$|\mathcal{F}[u](x) - \mathcal{F}_N[u](x)| \leq \frac{2}{m-1}||u^{(m)}||_2 \frac{1}{N^{m-1}}.$$

The theory states that the function u, its $m-1$ derivatives and their periodic extensions must be continuous. Therefore we construct the periodic extension of f, $f_p(x) = |\sin^3(x/2)|$, shown below.

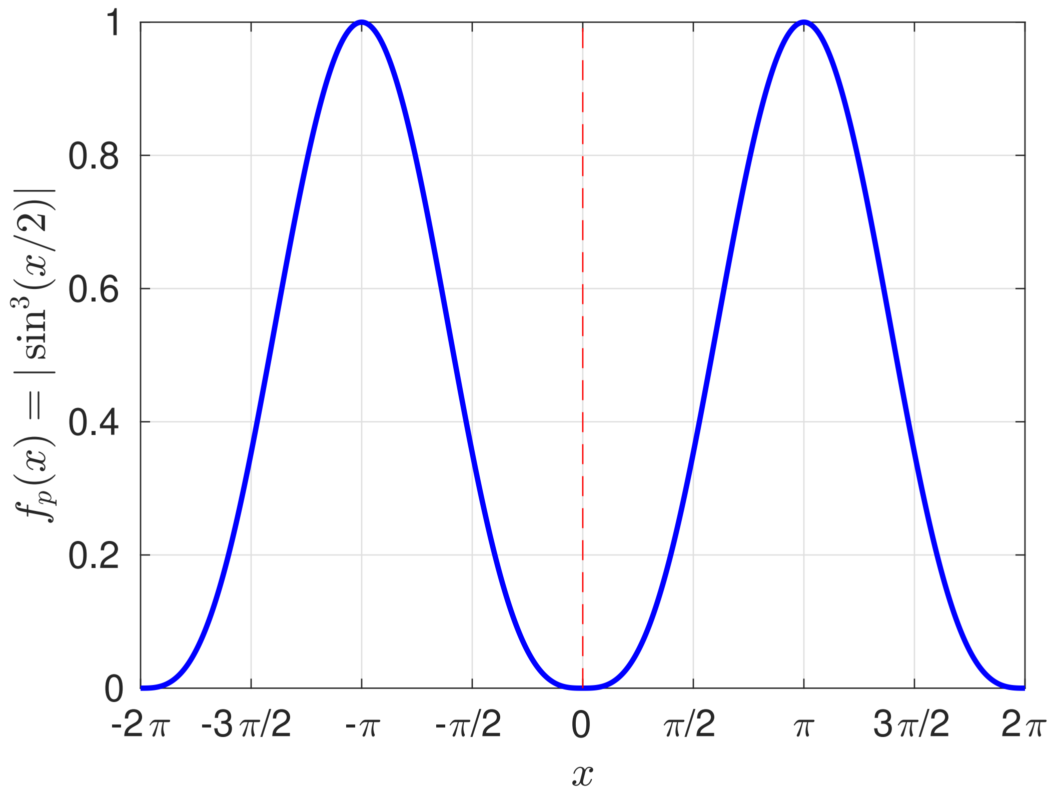

Figure 2: Periodic extension of f.

Now we can check how many continuous derivatives does f_p have:

- First Derivative:

$$f_p'(x) =$$

- Second Derivative:

$$f_p''(x) =$$

- Third Derivative:

$$f_p'''(x) =$$

Since the third derivative is discontinuous at $x = 0$, $m = 3$. Thus, our error is

$$|\mathcal{F}[u](x) - \mathcal{F}_N[u](x)| \leq ||u^{(m)}||_2 \frac{1}{N^2} \propto N^{-2}.$$

However, this is a conservative bound. It is no taking into consideration the particular function we are working with. In this case, it is possible to perform one more integreation by parts and the error in fact decays as N^{-3}, as show in the figure below.

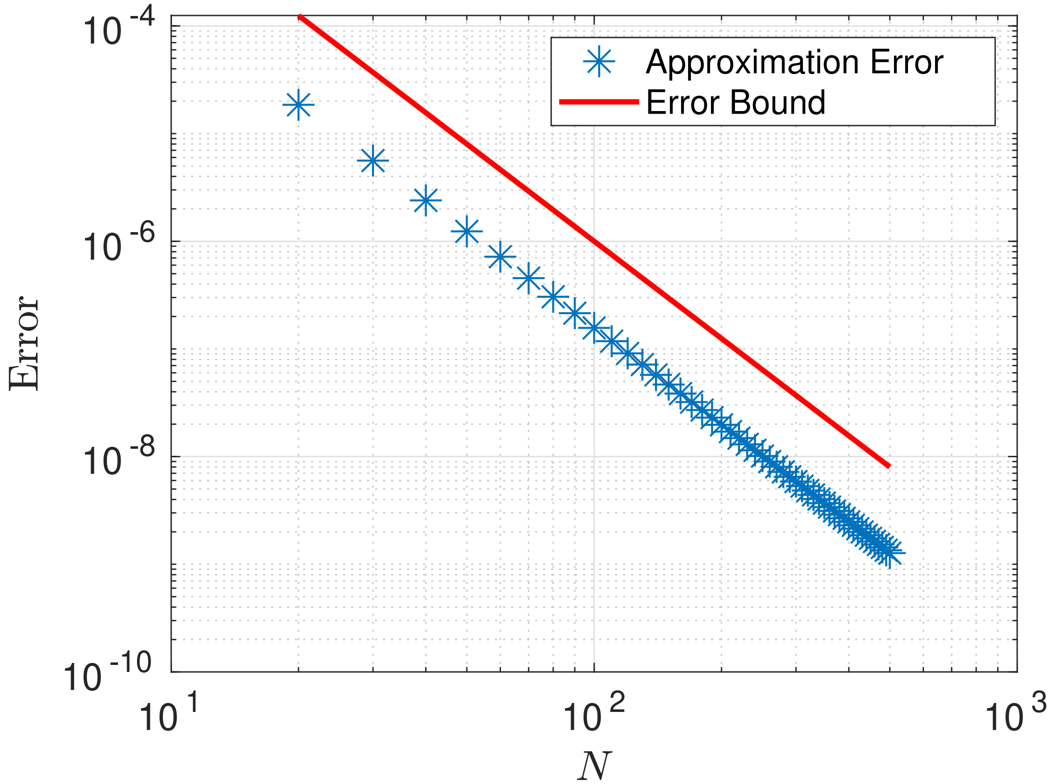

Figure 3: Approximation error and its bound.

Matlab code for this problem

```
%% Problem 4
clear variables; close all; clc
figformat='png';

% Plot function
x0 = -2*pi;
xf = 2*pi;
x = chebfun('x',[x0 xf]);
f = sin(x/2).^3;

figure
plot(f,'b-','linewidth',2)
hold on
plot([0 0],[-1 1],'r--')
grid on
axis([x0 xf -1 1])
xlabel('$x$','interpreter','latex')
ylabel('$f(x)=\sin^3(x/2)$','interpreter','latex')
set(gca,'fontsize',14)
```

```
set(gca,'XTick',x0:pi/2:xf)
xticklabels({'-2\pi','-3\pi/2','-\pi','-\pi/2','0','\pi/2','\pi','3\pi/2','2\pi'})
txt='Latex/FIGURES/P4_1';
saveas(gcf,txt,figformat)

x0 = -2*pi;
xf = 0;
x = chebfun('x',[x0 xf]);
fp1 = -sin(x/2).^3;
x0 = 0;
xf = 2*pi;
x = chebfun('x',[x0 xf]);
fp2 = sin(x/2).^3;

x0 = -2*pi;
xf = 2*pi;
figure
plot(fp1,'b-','linewidth',2)
hold on
plot(fp2,'b-','linewidth',2)
plot([0 0],[0 1],'r-')
grid on
axis([x0 xf 0 1])
xlabel('$x$','interpreter','latex')
ylabel('$f_p(x)=|\sin^3(x/2)|$','interpreter','latex')
set(gca,'fontsize',14)
set(gca,'XTick',x0:pi/2:xf)
xticklabels({'-2\pi','-3\pi/2','-\pi','-\pi/2','0','\pi/2','\pi','3\pi/2','2\pi'})
txt='Latex/FIGURES/P4_2';
saveas(gcf,txt,figformat)

% Approximation and Error
N = 20:10:500;
x0 = 0;
xf = 2*pi;
x = chebfun('x',[x0 xf]);
f = sin(x/2).^3;

parfor j = 1:length(N)
    A = exp(1i*x*(-N(j):N(j)));
    lambda = 1/(2*pi)*A'*f;
    fn = A*lambda;
    err(j) = norm(f-fn,Inf);
end
```

```
% Plot Error
figure
loglog(N,err,'*','MarkerSize',12)
hold on
loglog(N,abs(N).^-3,'r-')
grid on
% axis([20 510 1e-10 1e-1])
xlabel('$N$','interpreter','latex')
ylabel('Error','interpreter','latex')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P4_3';
saveas(gcf,txt,figformat)
```

Problem 5

Let $f(x) = 3/(5-4\cos(x))$. Compute its Fourier approximation for $N = 20:10:500$ and plot the error using *loglog*. What is the largest strip around the real axis for which f can be extended to an analytic function? How does this relate to the error bound for analytic functions derived in class?

In this case the function f is 2π-periodic, as we can see in figure 1, and analytic inside the strip $|\Im(z)| \leq a$. To determine the value of a, we solve

$$5 - 4\cos(z) = 0 \Rightarrow z = \cos^{-1}(5/4) \approx \ln(2)i.$$

Hence, $a \approx \ln(2)$.

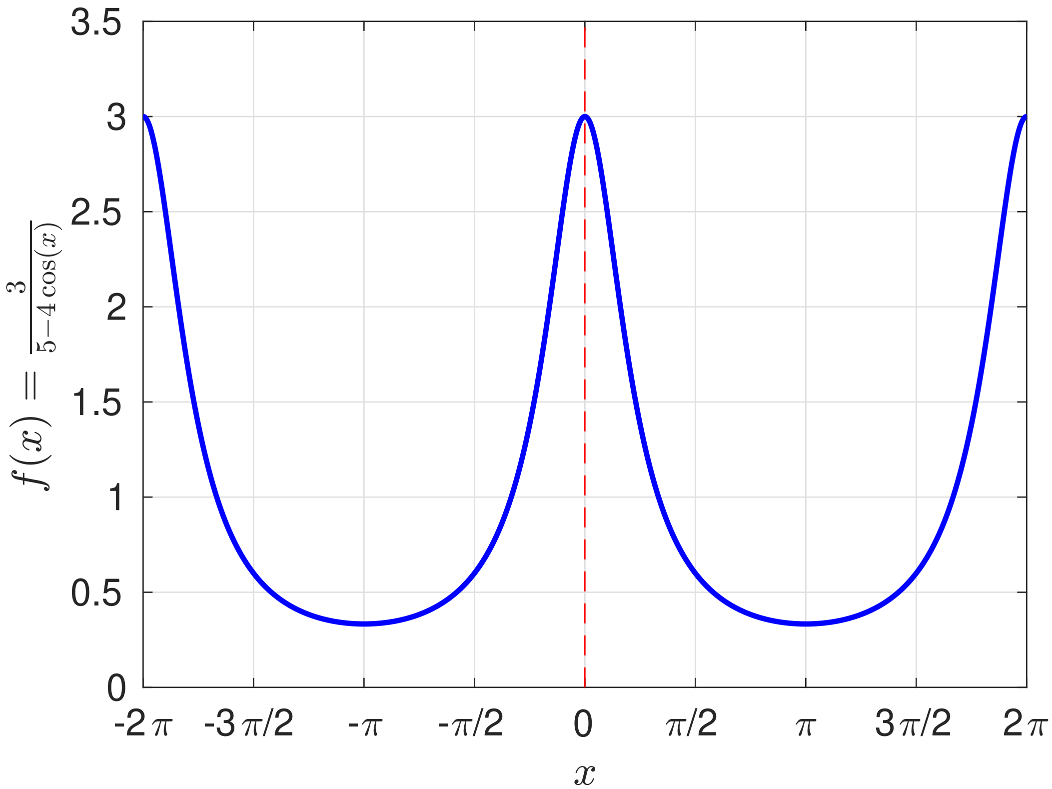

Figure 4: Graph of the function to approximate.

Given that the function is 2π-periodic and analytic inside the strip $|\Im(z)| \leq a$, the error is bounded as follows

$$|\mathcal{F}[u](x) - \mathcal{F}_N[u](x)| \leq \frac{4\pi M}{a} e^{-aN} = \frac{4\pi M}{\ln(2)} e^{-\ln(2)N} = \frac{4\pi M}{\ln(2)} 2^{-N} \propto 2^{-N},$$

where $M = \max_{x\in[0,2\pi]} |f(x - ai)|$. We can check in the last figure how this convergence is much faster. The points that separate from the error bound are due to round-off errors.

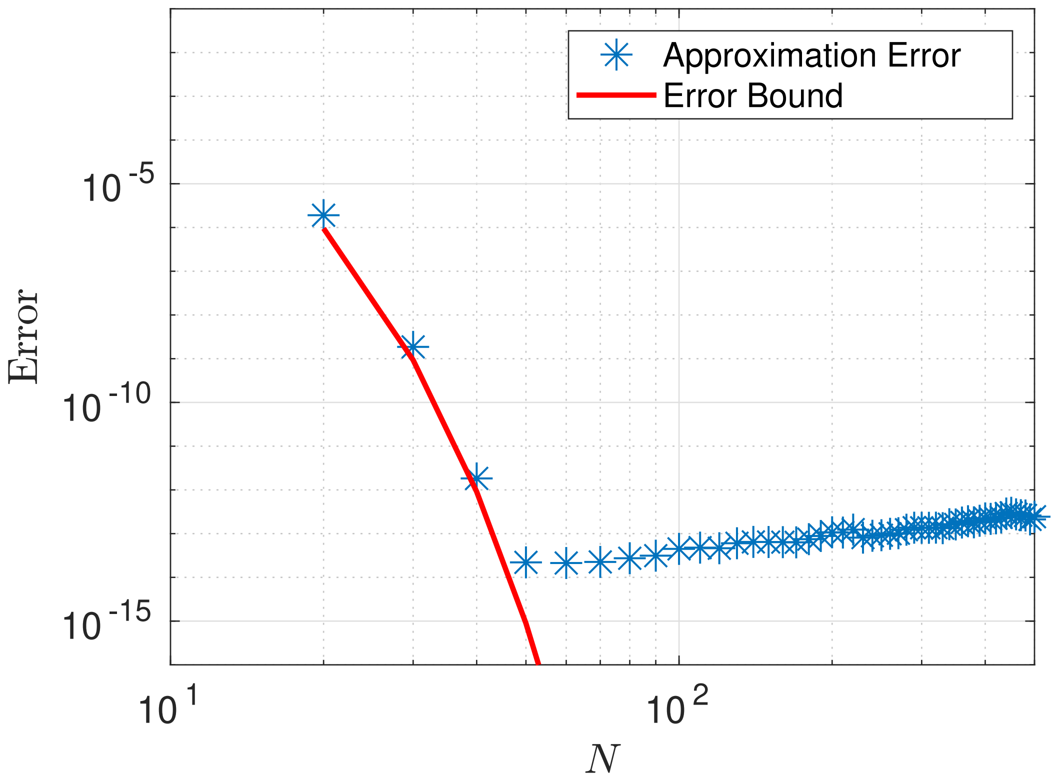

Figure 5: Approximation error and its bound.

Matlab code for this problem

```
%% Problem 5
clear variables; close all; clc
figformat='epsc';

% Plot function

x0 = -2*pi;
xf = 2*pi;
x = chebfun('x',[x0 xf]);
f = 3./(5-4*cos(x));

figure
plot(f,'b-','linewidth',2)
hold on
plot([0 0],[0 3.5],'r--')
grid on
xlabel('$x$','interpreter','latex')
ylabel('$f(x)=\frac{3}{5-4\cos(x)}$','interpreter','latex')
set(gca,'fontsize',14)
set(gca,'XTick',x0:pi/2:xf)
```

```
xticklabels({'-2\pi','-3\pi/2','-\pi','-\pi/2','0','\pi/2','\pi','3\pi/2','2\pi'})
txt='Latex/FIGURES/P5_1';
saveas(gcf,txt,figformat)

%%
% Approximation and error
N = 20:10:500;
x0 = 0;
xf = 2*pi;
x = chebfun('x',[x0 xf]);
f = 3./(5-4*cos(x));

parfor j = 1:length(N)
    A = exp(1i*x*(-N(j):N(j)));
    lambda = 1/(2*pi)*A'*f;
    fn = A*lambda;
    err(j) = norm(f-fn,Inf);
end
%%
figure
loglog(N,err,'*','MarkerSize',12)
hold on
loglog(N,2.^(-abs(N)),'r','linewidth',2)
grid on
axis([10 500 1e-16 1e-1])
xlabel('$N$','interpreter','latex')
ylabel('Error','interpreter','latex')
legend('Approximation Error','Error Bound')
set(gca,'fontsize',14)
txt='Latex/FIGURES/P5_2';
saveas(gcf,txt,figformat)
```