Riemann Zeta Function - The Critical Strip

University of Vienna

Florian Jesacher

September 21, 2016

Abstract

By taking the complex logarithm of the Euler product representation of the Riemann zeta function we first show that said function doesn't vanish on the lines with the real part constantly being zero and one. Furthermore we manage to establish growth estimations for the zeta function, its reciprocal function and its derivative under certain conditions. After introducing the prime-counting function pi and Tchebychev's psi function we then deduce their asymptotics by defining two auxiliary functions in order to complete our proof of the prime number theorem. Moreover, once we defined the Mellin transformation and stated the Mellin inversion theorem we use the xi function to derive a new formula, which finally provides us the necessary properties for the conclusive evidence that the Riemann zeta function has infinitely many roots on the critical line.

Contents

Chapter 1

Introduction

The Riemann zeta function has many interesting properties in the critical strip that have a huge impact on the analytic number theory, especially concerning the distribution of prime numbers. We want to have a closer look at its zero-free regions and find good estimates of the function's growth under certain circumstances. These will be needed later on in order to prove the prime number theorem and to show that the Riemann zeta function has infinitely many zeros on the critical line.
This essay is a continuation of Jesacher (2016) and is mainly based on Stein and Shakarchi (2003) and Titchmarsh (1986).
Henceforth let $s \in \mathbb{C}$ be a complex number where $\sigma := \Re(s)$ and $t := \Im(s)$ and if nothing else is stated let $p \in \mathbb{N}$ be a prime number.

1.1 Properties of ζ in the critical strip

First of all we want to draw some conclusions about the function's behaviour at the borders of the critical strip and show the absence of zeros on the lines $\sigma = 0$ and $\sigma = 1$.

Lemma 1.1.1. *If* $\sigma > 1$ *then*

$$\log(\zeta(s)) = \sum_{p,m} \frac{p^{-ms}}{m}.$$

Proof. Let $0 < x < 1$. Taking the logarithm of ζ's Euler product representation and using the power series expansion for the logarithm yields

$$\log\left(\frac{1}{1-x}\right) = -\log(1 + (-x)) = \sum_{m=1}^{\infty}\left(\frac{x^m}{m}\right).$$

Let $s > 1$ be real. We find that

$$\begin{aligned}
\log(\zeta(s)) &= \log\left(\prod_p \frac{1}{1-p^{-s}}\right) \\
&= \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s},
\end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{m}, & \text{for } n = p^m, \\ 0, & \text{otherwise.} \end{cases}$$

By analytic continuation this also holds for $s \in \mathbb{C}$ with $\sigma > 1$. In particular $\log(\zeta(s))$ is well defined as $\zeta(s) \neq 0$ for $\sigma > 1$. □

Lemma 1.1.2. *If $\theta \in \mathbb{R}$ then*

$$3 + 4\cos(\theta) + \cos(2\theta) \geq 0.$$

Proof. The application of the well known trigonometric identity

$$\cos(\theta)^2 = \frac{1}{2}(1 + \cos(2\theta))$$

immediately gives

$$3 + 4\cos(\theta) + \cos(2\theta) = 2 + 4\cos(\theta) + 2\cos(\theta)^2 = 2(1 + \cos(\theta))^2 \geq 0.$$

□

Corollary 1.1.2.1. *If $\sigma > 1$ then*

$$\log(\left|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)\right|) \geq 0.$$

Proof. We write

$$\begin{aligned}
\Re(n^{-s}) &= \Re(e^{-(\sigma+it)\log(n)}) \\
&= e^{-\sigma \log(n)} \cos(t \log(n)) = n^{-\sigma} \cos(t \log(n))
\end{aligned}$$

and for $r > 0, \theta \in \mathbb{R}$ we have

$$\Re(\log(re^{i\theta}) = \Re(\log(r) + i\theta + k2i\pi) = \log(r) = \log(\left|re^{i\theta}\right|).$$

Therefore Lemma 1.1.1 and Lemma 1.1.2 directly give us our final result

$$\log(\left|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)\right|) =$$
$$3\log(\left|\zeta(\sigma)\right|) + 4\log(\left|\zeta(\sigma+it)\right|) + \log(\left|\zeta(\sigma+2it)\right|) =$$
$$3\Re(\log(\zeta(\sigma))) + 4\Re(\log(\zeta(\sigma+it))) + \Re(\log(\zeta(\sigma+2it))) =$$
$$\sum_{n=1}^{\infty} c_n n^{-\sigma}(3 + 4\cos(t\log(n)) + \cos(2t\log(n))) \geq 0.$$

□

Theorem 1.1.3. *It holds that $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$.*

Proof. Let us suppose there exists a constant $t_0 \neq 0$ such that $\zeta(1+t_0) = 0$. We know that ζ is holomorphic on $\mathbb{C}\setminus\{1\}$, particularly in $s = 1+it_0$. Hence

$$\lim_{h\to 0} \frac{\zeta(1+it_0+h) - \zeta(1+it_0)}{h} = \lim_{\sigma\to 1} \frac{\zeta(\sigma+it_0)}{\sigma-1} \in \mathbb{C}.$$

Thus

$$\left|\zeta^4(\sigma+it_0)\right| = \mathcal{O}\left(|\sigma-1|^4\right) \text{ as } \sigma \to 1.$$

Since ζ has a simple pole in $s = 1$ we get

$$\lim_{\sigma\to 1} \zeta(\sigma)(\sigma-1) \in \mathbb{C}.$$

Thus we find that

$$\left|\zeta^3(\sigma)\right| = \mathcal{O}\left(|\sigma-1|^{-3}\right) \text{ as } \sigma \to 1.$$

Since ζ is also holomorphic in $s = 1+2it_0$ the function is locally bounded there, therefore we have

$$\left|\zeta(\sigma+2it)\right| = \mathcal{O}(1) \text{ as } \sigma \to 1.$$

Hence we get

$$\lim_{\sigma\to 1} \left|\zeta(\sigma)^3\zeta(\sigma+it_0)^4\zeta(\sigma+2it_0)\right| = 0.$$

But this contradicts Corollary 1.1.2.1 since $\log(x) < 0$ for all $x \in (0,1)$. □

Corollary 1.1.3.1. *It holds that $\zeta(it) \neq 0$ for all $t \in \mathbb{R}$.*

Proof. This proof relies on the fact that the xi function (see Definition 3.2) satisfies the symmetry $\xi(s) = \xi(1-s)$ (see Proposition 3.2.1). This, and the fact that only ζ can produce zeros in ξ, immediately yields

$$\zeta(1+it) \neq 0 \implies \xi(1+it) = \xi(-it) \neq 0 \implies \zeta(-it) \neq 0,$$

for all $t \in \mathbb{R}$. □

1.2 Estimates involving ζ

We want to study the growth of ζ, ζ' and ζ^{-1} in the half-plane $\sigma > 0$ in more detail in order to find good estimates for later applications.

Proposition 1.2.1. *There exists a sequence of entire functions* $\{\delta_n(s)\}_{n=1}^{\infty}$ *that satisfy the estimate* $|\delta_n(s)| \le \frac{|s|}{n^{\sigma+1}}$ *and such that*

$$\sum_{1\le n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\le n<N} \delta_n(s), \tag{1.1}$$

whenever N *is an integer* > 1 *and* $\sigma > 0$.

Proof. In order to prove our claim we compare $\sum_{1\le n<N} n^{-s}$ with $\sum_{1\le n<N} \int_n^{n+1} x^{-s}dx$, and set

$$\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx. \tag{1.2}$$

Let $f(x) = x^{-s}$. We have

$$\begin{aligned}\left|\frac{1}{n^s} - \frac{1}{x^s}\right| &= \left|\int_n^x f'(y)dy\right| \le \int_n^x |f'(y)|\, dy \\ &\le \sup_{y\in[n,x]} |f'(y)| = |f'(n)| = \frac{|s|}{n^{\sigma+1}},\end{aligned}$$

whenever $x \in [n, n+1]$. Therefore $|\delta_n(s)| \le \frac{|s|}{n^{\sigma+1}}$, and since

$$\int_1^N \frac{dx}{x^s} = \sum_{1\le n<N} \int_n^{n+1} \frac{dx}{x^s},$$

our proof is complete. □

Corollary 1.2.0.1. *For* $\sigma > 0$ *we have*

$$\zeta(s) - \frac{1}{s-1} = H(s),$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ *is holomorphic in the half-plane* $\sigma > 0$.

Proof. We first assume that $\sigma > 1$. When $N \to \infty$ in formula 1.1 we observe that by the estimate $|\delta_n(s)| \le \frac{|s|}{n^{\sigma+1}}$ we have uniform convergence of the series $\sum_{n=1}^{\infty} \delta_n(s)$ in any half plane $\sigma \ge \delta$ when $\delta > 0$. Since $\sigma > 1$ the series $\sum_{n=1}^{\infty} n^{-s}$ converges to $\zeta(s)$. This proves our assertion when $\sigma > 1$. The uniform convergence shows that $\sum_{n=1}^{\infty} \delta_n(s)$ is holomorphic when $\sigma > 0$ and thus shows that ζ is extendable to that half-plane and that the identity continues to hold there. □

Proposition 1.2.2. *For each $\sigma_0 \in [0,1]$ and every $\varepsilon > 0$, there exists a constant c_ε, so that*

$$\left|\zeta(s)\right| \leq c_\varepsilon \left|t\right|^\varepsilon, \text{ if } \sigma_0 \leq \sigma \text{ and } |t| \geq 1 \tag{1.3}$$

and

$$\left|\zeta'(s)\right| \leq c_\varepsilon \left|t\right|^\varepsilon, \text{ if } 1 \leq \sigma \text{ and } |t| \geq 1. \tag{1.4}$$

Proof. For the proof we recall the estimate $\left|\delta_n(s)\right| \leq \frac{|s|}{n^{\sigma+1}}$. We also have the estimate $\left|\delta_n(s)\right| \leq \frac{2}{n^\sigma}$, which follows from the expression for δ_n given by (1.2) and the fact that $\left|n^{-s}\right| = n^{-\sigma}$ and $\left|x^{-s}\right| \leq n^{-\sigma}$ if $x \geq n$. We then combine these two estimates for $\left|\delta_n(s)\right|$ via the observation that $A = A^\delta A^{1-\delta}$, to obtain the bound

$$\left|\delta_n(s)\right| \leq \left(\frac{|s|}{n^{\sigma_0+1}}\right)^\delta \left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta} \leq \frac{2\left|s\right|^\delta}{n^{\sigma_0+\delta}},$$

as long as $\delta \geq 0$. Now choose $\delta = 1 - \sigma_0 + \varepsilon$ and apply the identity to Corollary 1.2.0.1. Then, with $\sigma \geq \sigma_0$, we find

$$\left|\zeta(s)\right| \leq \left|\frac{1}{s-1}\right| + 2\left|s\right|^{1-\sigma_0+\varepsilon} \sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}},$$

and conclusion (1.3) is proved. By the Cauchy integral formula,

$$\zeta'(s) = \frac{1}{2\pi r} \int_0^{2\pi} \zeta(s + re^{i\theta}) e^{i\theta} d\theta,$$

where the integration is taken over a circle of radius r centred at s. Now choose $r = \varepsilon$ and observe that this circle lies in the half-plane $\sigma \geq 1 - \varepsilon$, and so the estimate (1.4) follows as a consequence of the estimate (1.3) on replacing 2ε by ε. $\square$

Proposition 1.2.3. *For every $\varepsilon > 0$ we have $\frac{1}{|\zeta(s)|} \leq c_\varepsilon \left|t\right|^\varepsilon$ for $c_\varepsilon \in \mathbb{R}_{>0}$ if $\sigma > 1$ and $|t| \geq 1$.*

Proof. From the previous observation we have that

$$\left|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)\right| \geq 1,$$

whenever $\sigma > 1$. Using our previous estimate for ζ we find that

$$\left|\zeta^4(\sigma+it)\right| \geq c\left|\zeta^{-3}(\sigma)\right| \left|t\right|^{-\varepsilon} \geq c'(\sigma-1)^3 \left|t\right|^{-\varepsilon}$$

for all $\sigma \geq 1$ and $|t| \geq 1$. Thus

$$\left|\zeta(\sigma + it)\right| \geq c'(\sigma - 1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} .$$

We now consider two cases, depending on whether $\sigma - 1 \geq A|t|^{-5\varepsilon}$ holds for an appropriate constant $A \in \mathbb{R}_{>0}$ or not. If this inequality does hold it follows

$$\left|\zeta(\sigma + it)\right| \geq A' |t|^{-4\varepsilon} .$$

By replacing 4ε by ε we can conclude the proof in this case. If, however, $\sigma - 1 < A|t|^{-5\varepsilon}$ then we first choose $\sigma' > \sigma$ with $\sigma' - 1 = A|t|^{-5\varepsilon}$. The triangle inequality then implies

$$\left|\zeta(\sigma + it)\right| \geq \left|\zeta(\sigma' + it)\right| - \left|\zeta(\sigma' + it) - \zeta(\sigma + it)\right| ,$$

and combined with the previous estimate (1.4) for ζ' we obtain

$$\begin{aligned}\left|\zeta(\sigma' + it) - \zeta(\sigma + it)\right| = \left|\int_\sigma^{\sigma'} \zeta'(x + it)dx\right| &\leq \int_\sigma^{\sigma'} \left|\zeta'(x + it)\right| dx \\ &\leq c''(\sigma' - \sigma) |t|^{\varepsilon} \\ &\leq c''(\sigma' - 1) |t|^{\varepsilon} .\end{aligned}$$

These observations, together with our result above where we set $\sigma = \sigma'$, show that

$$\left|\zeta(\sigma + it)\right| \geq c'(\sigma' - 1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} - c''(\sigma' - 1) |t|^{\varepsilon} .$$

We can now choose $A = \left|\frac{c'}{2c''}\right|^4$, and recall that $\sigma' - 1 = A|t|^{-5\varepsilon}$. This gives precisely

$$c'(\sigma' - 1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} = 2c''(\sigma' - 1) |t|^{\varepsilon}$$

and therefore

$$\left|\zeta(\sigma + it)\right| \geq A'' |t|^{-4\varepsilon} .$$

By replacing 4ε by ε the desired inequality is established and therefore the proof is complete. $\square$

Corollary 1.2.0.2. *For every $\varepsilon > 0$ we have $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq c_\varepsilon |t|^{\varepsilon}$ for $c_\varepsilon \in \mathbb{R}_{>0}$ if $\sigma \geq 1$ and $|t| \geq 1$.*

Proof. Our desired result immediately follows from combining the previous estimates for ζ' and $\frac{1}{\zeta}$. $\square$

1.3 Mellin Transform

Definition 1.1. *The function defined by*

$$\{\mathcal{M}f\}(s) = \int_0^\infty x^{s-1} f(x) dx.$$

is called the Mellin transform of the function f. Conversely

$$\left\{\mathcal{M}^{-1}\varphi\right\}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s) ds.$$

defines the inverse Mellin transform.

Theorem 1.3.1 (Mellin's Inversion Formula)**.** *Let $\varphi(s)$ be analytic in $a < \sigma, c < b$ and $\varphi(s) \to 0$ uniformly as $t \to \pm\infty$ with its integral along the line $\sigma = c$ converging absolutely. Then if*

$$f(x) = \left\{\mathcal{M}^{-1}\varphi\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} x^{-s}\varphi(s) ds$$

we have that

$$\varphi(s) = \{\mathcal{M}f\} = \int_0^\infty x^{s-1} f(x) dx.$$

Conversely suppose f is piecewise continuous on the positive real numbers and suppose the integral

$$\varphi(s) = \int_0^\infty x^{s-1} f(x) dx$$

is absolutely convergent in $a < \sigma < b$. Then f is recoverable via the inverse Mellin transform from its Mellin transform.

Proof. This theorem follows from the inversion formula for the Laplace transform and the proof can, for example, be found in *McLachlan* (1953). □

1.4 The Poisson Summation Formula

Definition 1.2. *For each $a > 0$ we define $\mathcal{F}_a$ by the space of all functions f that satisfy the following two conditions:*

(i) The function f is holomorphic on the horizontal strip

$$S_a = \left\{s \in \mathbb{C}:\ |t| < a\right\}.$$

(ii) There exists a constant $A > 0$ such that

$$\left|f(s)\right| \leq \frac{A}{1+\sigma^2} \quad \textit{for all } \sigma \in \mathbb{R}, |y| < a.$$

Theorem 1.4.1 (Poisson Summation Formula)**.** *For $f \in \mathcal{F}_a$, the Poisson summation may be stated as*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k),$$

where $\hat{f}$ is the Fourier transform of f.

Proof. The entire proof can be found in Stein and Shakarchi (2003). $\square$

Chapter 2

The Prime Number Theorem

The prime number theorem is a very fundamental statement about the asymptotic behaviour of the distribution of prime numbers. Based on the fact which we established in the previous chapter, that the Riemann zeta function doesn't have zeros on the line $\sigma = 1$ and by introducing Tchebychev's psi-function together with two auxiliary functions, we will have enough tools to manage to prove said theorem.

2.1 Tchebychev's ψ function

Definition 2.1. *We define the function*

$$\pi \colon \mathbb{R} \to \mathbb{N},$$
$$\pi(x) = \sum_{p \leq x} 1,$$

as the prime number counting function summing over all primes $p \leq x$.

Definition 2.2. *The function*

$$\psi \colon \mathbb{R} \to \mathbb{R},$$
$$\psi(x) = \sum_{p^m \leq x} \log(p),$$

summing over all primes p and $m \in \mathbb{N}_{>0}$ such that $p^m \leq x$ is often referred to as Tchebychev's psi function.

Lemma 2.1.1. *We have*

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log(x)}{\log(p)} \right] \log(p).$$

where $[u]$ denotes the greatest integer $\leq u$.

Proof. First we define

$$\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^m, \\ 0, & \text{otherwise,} \end{cases}$$

then it's clear that

$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n).$$

With this observation and the fact that if $p^m \leq x$ then $m \leq \frac{\log(x)}{\log(p)}$ our formula follows immediately. $\square$

2.2 Asymptotics of ψ and ψ_1

Lemma 2.2.1. *If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim \frac{x}{\log(x)}$ as $x \to \infty$.*

Proof. By definition we have to prove the inequalities

$$1 \leq \liminf_{x \to \infty} \pi(x) \frac{\log(x)}{x}$$

and

$$\limsup_{x \to \infty} \pi(x) \frac{\log(x)}{x} \leq 1.$$

At first we can make the crude estimate

$$\psi(x) = \sum_{p \leq x} \left[\frac{\log(x)}{\log(p)} \right] \log(p) \leq \sum_{p \leq x} \frac{\log(x)}{\log(p)} \log(p) = \pi(x) \log(x)$$

and dividing by x yields

$$\frac{\psi(x)}{x} \leq \frac{\pi(x) \log(x)}{x}.$$

This asymptotic condition $\psi(x) \sim x$ implies the first inequality. For the second inequality we fix $0 < \alpha < 1$ and note that

$$\psi(x) \geq \sum_{p \leq x} \log(p) \geq \sum_{x^\alpha < p \leq x} \log(p) \geq \big(\pi(x) - \pi(x^\alpha)\big) \log(x^\alpha),$$

and therefore

$$\psi(x) + \alpha \pi(x^\alpha) \log(x) \geq \alpha \pi(x) \log(x).$$

Dividing by x, noting that $\pi(x^\alpha) \leq x^\alpha$ and $\psi(x) \sim x$, gives

$$1 \geq \alpha \limsup_{x\to\infty} \pi(x)\frac{\log(x)}{x}.$$

Since $\alpha < 1$ is arbitrary the proof is complete. $\square$

Definition 2.3. *The function*

$$\begin{aligned}
&\psi_1 \colon \mathbb{R} \to \mathbb{R},\\
&\psi_1(x) = \int_1^x \psi(u)du,
\end{aligned}$$

is closely related to the ψ-function.

Lemma 2.2.2. *If $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$.*

Proof. It suffices to show that $\psi(x) \sim x$. This can easily be seen from the fact that if $\alpha < 1 < \beta$, then

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} \psi(u)du \leq \psi(x) \leq \frac{1}{(\beta-1)x}\int_{x}^{\beta x} \psi(u)du.$$

The proof of this double inequality relies simply on the fact that ψ is increasing. Consequently we find that

$$\psi(x) \leq \frac{1}{(\beta-1)x}\left(\psi_1(\beta x) - \psi_1(x)\right),$$

and therefore

$$\frac{\psi(x)}{x} \leq \frac{1}{\beta-1}\left(\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\right).$$

In turn this implies

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \frac{1}{\beta-1}\left(\frac{1}{2}\beta^2 - \frac{1}{2}\right) = \frac{1}{2}(\beta+1).$$

Since this result holds for all $\beta > 1$, we have proved that

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq 1.$$

For showing

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1$$

we can use a similar argument with $\alpha < 1$. $\square$

Lemma 2.2.3. *For $\sigma > 1$ we have*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof. We know that for $\sigma > 1$ we have

$$\log(\zeta(s)) = \sum_{m,p} \frac{p^{-ms}}{m}.$$

Differentiating both expressions leads to

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m,p} p^{-ms}\log(p) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

□

Lemma 2.2.4. *If $c > 0$ then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}ds = \begin{cases} 0, & \text{if } 0 < a < 1, \\ 1 - \frac{1}{a}, & \text{if } 1 \le a. \end{cases}$$

$$\begin{aligned} C(T)\colon\ [0,\pi] &\to \mathbb{C}, \\ \phi &\mapsto c + Te^{i(\phi+\frac{\pi}{2})}, \\ S(T)\colon\ [-T,T] &\to \mathbb{C}, \\ t &\mapsto c + it \end{aligned}$$

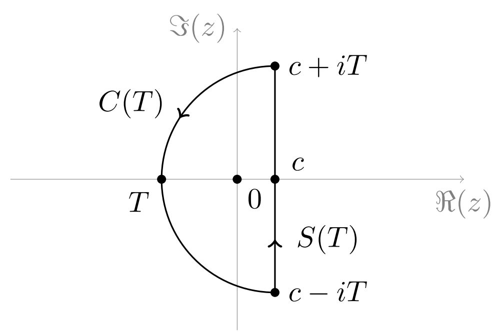

Figure 2.1: Contour $\Gamma(T)$

Proof. First note that since $|a^s| = a^c$, the integral converges. Suppose that $1 \le a$, and write $a = e^{\beta}$ with $\beta = \log(a) \ge 0$. Let

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}.$$

Then $Res(f(s), 0) = 1$ and $Res(f(s), 1) = -\frac{1}{a}$. For $T > 0$, consider the path $\Gamma(T)$ shown in Figure 2.1. Now, we choose T large enough such that

0 and -1 are contained in the interior of the contour $\Gamma(T)$. By the residue formula we get

$$\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)ds = 1 - \frac{1}{a}.$$

Since

$$\int_{\Gamma(T)} f(s)ds = \int_{S(T)} f(s)ds + \int_{C(T)} f(s)ds,$$

it suffices to show that the integral over $C(T)$ goes to 0 as $T \to \infty$. Note that if $s \in C(T)$, then for all large T we have

$$\left|s(s+1)\right| \geq \frac{1}{2}T^2.$$

and since $\sigma \leq c$ we also have the estimate $\left|e^{\beta s}\right| \leq e^{\beta c}$. Therefore

$$\lim_{T\to\infty}\left|\int_{C(T)} f(s)ds\right| \leq \frac{C}{T^2}2\pi T = 0$$

and the case where $a \geq 1$ is proved. If $0 < a < 1$ we consider the half circle lying to the right of $S(T)$. Noting there are no poles in the interior of the contour we can give a similar argument that the integral over the half circle vanishes for $T \to \infty$. $\square$

Proposition 2.2.1. *For all $c > 1$ we have*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds.$$

Proof. We first observe that

$$\psi(u) = \sum_{n=1}^{\infty} \Lambda(n) f_n(u),$$

where

$$f_n(u) = \begin{cases} 1, & \text{if } n \leq u, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \psi_1(x) = \int_0^x \psi(u)du &= \sum_{n=1}^{\infty}\int_0^x \Lambda(n) f_n(u) \\ &= \sum_{n\leq x}\Lambda(n)\int_n^x du = \sum_{n\leq x}\Lambda(n)(x-n). \end{aligned}$$

This fact, together with Lemma 2.2.4 where $a = \frac{x}{n}$, gives

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds &= x\sum_{n=1}^{\infty}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left(\frac{x}{n}\right)^s}{s(s+1)}ds\\ &= x\sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right)\\ &= \psi_1(x),\end{aligned}$$

as was to be shown. □

Theorem 2.2.5. *We have that* $\psi_1(x) \sim \frac{x^2}{2}$ *as* $x \to \infty$.

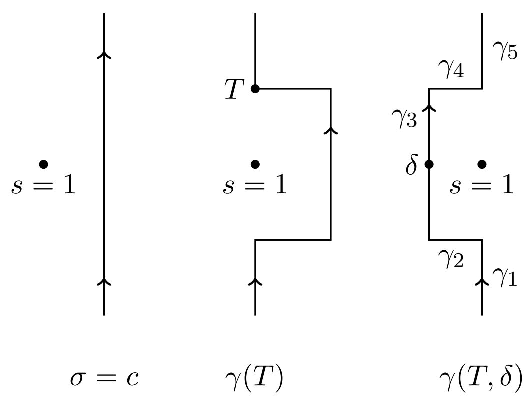

Figure 2.2: Three stages: the line $\sigma = c$, the contours $\gamma(T)$ and $\gamma(T, \delta)$

Proof. We first fix $c > 1$, say $c = 2$ and assume that x is also fixed with $x \geq 2$. Let $F(s)$ denote the integrand

$$F(s) = \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right).$$

Now, we deform the vertical line from $\sigma = c$ to the path $\gamma(T)$ shown in Figure 2.2 where $T \geq 3$. Cauchy's integral theorem allows us to see that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds. \tag{2.1}$$

Indeed, we know on the basis of Proposition 2.2.1 that $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq A\,|t|^{\eta}$ for any fixed $\eta > 0$ whenever $\sigma \geq 1$ and $|t| \geq 1$. Thus we have $\left|F(s)\right| \leq A'\,|t|^{-2+\eta}$ in the two rectangles bounded by the line $\sigma = c$ and $\gamma(T)$. Furthermore F is regular and its decrease at infinity is rapid enough, hence the assertion (2.1) is established. Next we pass from the contour $\gamma(T)$ to the contour $\gamma(T, \delta)$. For fixed T, we chose $\delta > 0$ small enough so that ζ has no zeros in the box

$$\left\{ s \in \mathbb{C} \colon 1 - \delta \leq \sigma \leq 1, |t| \leq T \right\}.$$

Such a choice can be made since ζ does not vanish on the line $\sigma = 1$. Now $F(s)$ has a simple pole at $s = 1$. In fact, by Corollary 1.2.0.1, we know that $\zeta(s) = \frac{1}{s-1} + H(s)$, where $H(s)$ is regular near $s = 1$. Hence $-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + h(s)$, where $h(s)$ is holomorphic near $s = 1$, and so we have $Res(F(s), 1) = \frac{x^2}{2}$. As a result we receive

$$\frac{1}{2\pi i} \int_{\gamma(T)} F(s) ds = \frac{x^2}{2} + \frac{1}{2\pi i} \int_{\gamma(T,\delta)} \frac{x^{s+1}}{s(s+1)} F(s) ds.$$

We now decompose the contour $\gamma(T, \delta)$ as $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ and estimate each of the integrals over $\gamma_j, j = 1, 2, 3, 4, 5$. First we contend that there exists T so large that

$$\left| \int_{\gamma_1} F(s) ds \right| \leq \frac{\varepsilon}{2} x^2$$

and

$$\left| \int_{\gamma_5} F(s) ds \right| \leq \frac{\varepsilon}{2} x^2,$$

for $\varepsilon > 0$. To see this, we first note that for $s \in \gamma_1$ one has

$$\left| x^{1+s} \right| = x^{1+\sigma} = x^2.$$

Then by Proposition 2.2.1 we have, for example, that $\left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq A \left| t \right|^{\frac{1}{2}}$, so

$$\left| \int_{\gamma_1} F(s) ds \right| \leq C x^2 \int_T^\infty \frac{|t|^{\frac{1}{2}}}{t^2} dt.$$

Since the integral converges, we can make the right-hand side $\leq \varepsilon \frac{x^2}{2}$ upon taking T sufficiently large. The argument for the integral over γ_5 is the same. Having now fixed T, we choose δ appropriately small. On γ_3, note that

$$\left| x^{1+s} \right| = x^{1+1-\delta} = x^{2-\delta},$$

from which we conclude that there exists a constant C_T (dependent on such T) such that

$$\left|\int_{\gamma_3} F(s)ds\right| \le C_T x^{2-\delta}.$$

Finally, on the small horizontal segment γ_2 (and similarly on γ_4), we can estimate the integral as follows:

$$\left|\int_{\gamma_2} F(s)ds\right| \le C'_T \int_{1-\delta}^{1} x^{1+\sigma} d\sigma \le C'_T \frac{x^2}{\log(x)}.$$

We conclude that there exist constants C_T and C'_T (possibly different from the others above) such that

$$\left|\psi_1(x) - \frac{x^2}{2}\right| \le \varepsilon x^2 + C_T x^{2-\delta} + C'_T \frac{x^2}{\log(x)}.$$

Dividing through by $\frac{x^2}{2}$, we see that

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \le 2\varepsilon + 2C_T x^{-\delta} + 2C'_T \frac{1}{\log(x)},$$

and therefore, for all large x we have

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \le 4\varepsilon.$$

This concludes the proof that $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$. Therefore we have finally proved the prime number theorem. $\square$

Chapter 3

The Critical Line

In this chapter we will introduce the theta function, define the xi function and prove some important identities in order to come up with an integral formula that allows us to draw a far-reaching conclusion about the occurrence of zeros of the Riemann zeta function on the critical line.

3.1 The Theta Function

Definition 3.1. *The function*

$$\vartheta \colon \mathbb{R}_{>0} \to \mathbb{C},$$
$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$$

is known as a version of the renowned "theta" function. For further references we also introduce two auxiliary functions

$$\omega(x) = \frac{1}{2}\big(\vartheta(x) - 1\big) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$$

and

$$\theta(x) = \omega(x) - \frac{1}{2\sqrt{t}} = \sum_{n=1}^{\infty} e^{-\pi n^2 x} - \frac{1}{2\sqrt{t}}.$$

Remark. *It is known that the well-behaved monotonically increasing function θ is defined for $0 \le x < \infty$.*

Proposition 3.1.1. *If* $f(x) = e^{-\pi x^2}$ *then* $f(x) = \hat{f}(\xi)$*, where* $\hat{f}$ *is the Fourier transform of* f*, or more formally*

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx = e^{-\pi\xi^2}. \tag{3.1}$$

$$\begin{aligned}
\gamma_{1_+} \colon\ & [-R, R] \to \mathbb{C}, \\
& \quad r \mapsto r, \\
\gamma_{1_-} \colon\ & [-R, R] \to \mathbb{C}, \\
& \quad r \mapsto -r + i\xi, \\
\gamma_{2_+} \colon\ & [0, \xi] \to \mathbb{C}, \\
& \quad t \mapsto R + it, \\
\gamma_{2_-} \colon\ & [0, \xi] \to \mathbb{C}, \\
& \quad t \mapsto -R + i(\xi - t)
\end{aligned}$$

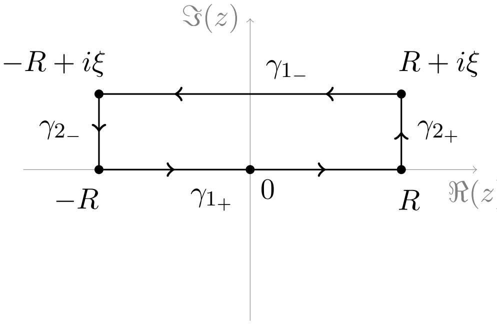

Figure 3.1: Contour γ

Proof. If $\xi = 0$ we already know very well that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Now, let us suppose $\xi > 0$ and consider the function $f(z) = e^{-\pi z^2}$ for $z \in \mathbb{C}$. f is clearly entire. According to Cauchy's theorem we have

$$\int_{\gamma} f(z) dz = 0.$$

For the integral over γ_{1_+} we simply have

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi x^2} dx = 1.$$

For the integral over γ_{2_+} for a fixed ξ we have

$$\begin{aligned}
\lim_{R\to\infty} I(R) &= \lim_{R\to\infty} \int_0^{\xi} f(R + iy) i dy \\
&= \lim_{R\to\infty} \int_0^{\xi} e^{-\pi(R^2 + 2iRy - y^2)} i dy = 0,
\end{aligned}$$

since we may estimate it by

$$\big|I(R)\big| \leq Ce^{-\pi R^2}.$$

The integral for $\gamma_{2\_}$ also goes to zero for the same reasons. Finally, the remaining integral over $\gamma_{1\_}$ is

$$\int_R^{-R} e^{-\pi(x+i\xi)^2}dx = -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi ix\xi}dx.$$

Thus for $R \to \infty$ the limit gives

$$1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ix\xi}dx = 0$$

and our desired formula is established. In the case $\xi < 0$, we then consider the rectangle in the lower half plane. $\square$

Lemma 3.1.1. *For $x > 0$ we have the identity*

$$\vartheta(x) = \frac{1}{\sqrt{x}}\vartheta(\frac{1}{x}).$$

Proof. The change of variables $x \to \sqrt{y}x$ in the integral formula (3.1) shows that the Fourier transform of the function $f(x) = e^{-\pi yx^2}$ is in fact $\hat{f}(\xi) = \frac{1}{\sqrt{y}}e^{-\frac{\pi\xi^2}{y}}$. Apparently $f(x)$ belongs to $\mathcal{F}_a$. Thus we may apply the Poisson summation formula to the pair $\left(f, \hat{f}\right)$ and we get

$$\vartheta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi xn^2} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{x}}e^{-\frac{\pi\xi^2}{x}} = \frac{1}{\sqrt{x}}\vartheta(\frac{1}{x}).$$

$\square$

Corollary 3.1.1.1. *We have the identities*

$$\omega(x) = \frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}.$$

and

$$\theta(x) = \frac{1}{\sqrt{x}}\theta(\frac{1}{x}).$$

Proof. For the first identity we have

$$2\omega(x) + 1 = \vartheta(x) = \frac{1}{\sqrt{x}}\vartheta(\frac{1}{x}) = \frac{1}{\sqrt{x}}\left(2\omega(\frac{1}{x}) + 1\right)$$

and just solve the equation for $\omega(x)$. The second identity follows directly from the first by writing

$$\begin{aligned}\theta(x) = \omega(x) - \frac{1}{2\sqrt{x}} &= \frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right) - \frac{1}{2}\\ &= \frac{1}{\sqrt{x}}\left(\omega\left(\frac{1}{x}\right) - \frac{\sqrt{x}}{2}\right) = \frac{1}{\sqrt{x}}\theta(x).\end{aligned}$$

□

3.2 The Xi Function

Definition 3.2. *Let*

$$\begin{aligned}&\xi\colon \mathbb{C} \to \mathbb{C},\\ &\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{s}{2})\zeta(s)\end{aligned}$$

be the xi function and for further reference we define

$$\xi^*(s) = \frac{2}{s(s-1)}\xi(s) = \pi^{-\frac{1}{2}s}\Gamma(\frac{s}{2})\zeta(s)$$

Remark. *ξ is entire since s and $(s-1)$ cancel out the simple poles of Γ in $s = 0$ and ζ in $s = 1$ and since the trivial zeros of ζ eradicate the simple poles of Γ in $s = -2n$ for $n \in \mathbb{N}_{>0}$.*

Proposition 3.2.1. *We have the identity*

$$\xi(s) = \xi(1-s).$$

Proof. Let us recall the formula

$$\Gamma(\frac{1}{2}s) = \int_0^\infty e^{-y}y^{\frac{1}{2}s-1}dy,$$

whenever $\sigma > 0$. We observe that if $n \geq 1$ and by substituting $y = x\pi n^2$ we may write

$$\int_0^\infty e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx = n^{-s}\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s).$$

Hence if $\sigma > 1$ we get

$$\xi^*(s) = \pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{1}{2}s-1}e^{-n^2\pi x}dx$$
$$= \int_0^{\infty} x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty} e^{-n^2\pi x}dx = \int_0^{\infty} x^{\frac{1}{2}s-1}\omega(x)dx,$$

where we can interchange sum and integral because of absolute convergence. We now use Corollary 3.1.1.1 and may now write

$$\xi^*(s) =$$
$$\int_0^1 x^{\frac{1}{2}s-1}\omega(x)dx + \int_1^{\infty} x^{\frac{1}{2}s-1}\omega(x)dx =$$
$$\int_0^1 x^{\frac{1}{2}s-1}\left(\frac{1}{\sqrt{x}}\omega\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right)dx + \int_1^{\infty} x^{\frac{1}{2}s-1}\omega(x)dx =$$
$$\frac{1}{s-1} - \frac{1}{s} + \int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\omega(x)dx + \int_1^{\infty} x^{\frac{1}{2}s-1}\omega(x)dx =$$
$$\frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty}\left(x^{-\frac{1}{2}s-\frac{1}{2}} + x^{\frac{1}{2}s-1}\right)\omega(x)dx.$$

Since the function ω has exponential decay at infinity the last integral defines an entire function. Thus the formula holds by analytic continuation for all $s \in \mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. The right-hand side remains unchanged when s is replaced by $1 - s$. Hence we finally get

$$\xi(s) = \frac{1}{2}s(1-s)\xi^*(s) = \frac{1}{2}(1-s)s\xi^*(1-s) = \xi(1-s).$$

□

Lemma 3.2.1. *Let $f(x) = 2\theta(x^2)$. For $0 < \sigma < 1$ we have*

$$\xi^*(s) = \int_0^{\infty} x^{s-1}f(x)dx,$$

hence $\xi^*(s) = \{\mathcal{M}f\}$ and therefore by Mellin's inversion formula $f = \{\mathcal{M}^{-1}\xi^*\}$ whenever $0 < \sigma < 1$.

Proof. This formula is essentially the same as stated in Edwards (1974) [p. 213] after the substitution $u \to \frac{1}{x}$. □

Definition 3.3. *Let*

$$\Xi\colon \mathbb{R} \to \mathbb{C},$$
$$\Xi(t) = \xi(\frac{1}{2} + it) = -\frac{1}{2}(t^2 + \frac{1}{4})\pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma(\frac{1}{4} + \frac{1}{2}it)\zeta(\frac{1}{2} + it)$$

be the restriction of the xi function to $\sigma = \frac{1}{2}$.

Lemma 3.2.2. Ξ *is even.*

Proof. Let $t \in \mathbb{R}$. We have

$$\Xi(-t) = \xi(\frac{1}{2} - it) = \xi(1 - (\frac{1}{2} + it)) = \xi(\frac{1}{2} + it) = \Xi(t).$$

□

Lemma 3.2.3. Ξ *is a real function.*

Proof. We first recall Corollary 3.1.1.1 which immediately yields the symmetric form

$$x^{-\frac{1}{4}}\theta(\frac{1}{x}) = x^{\frac{1}{4}}\theta(x).$$

Plugging this into the integral form of the functional equation of ξ^* from Lemma 3.2.1 for $0 < \sigma < 1$ gives

$$\xi^*(s) = \int_0^\infty x^{\frac{s-\frac{1}{2}}{2}} x^{\frac{1}{4}-1}\theta(x)dx = \int_0^\infty x^{-\frac{s-\frac{1}{2}}{2}} x^{\frac{1}{4}-1}\theta(x)dx.$$

By addition of the two sides we obtain

$$\xi^*(s) = \int_0^\infty \cosh\left(\frac{1}{2}\left(s - \frac{1}{2}\right)\log(x)\right) x^{\frac{1}{4}-1}\theta(x)dx.$$

Further, by applying the addition theorem

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

for all $x, y \in \mathbb{R}$ we may deduce that

$$\begin{aligned}\xi^*(s) = &\int_0^\infty \cosh\left(\frac{1}{2}\left(\sigma - \frac{1}{2}\right)\log(x)\right)\cos\left(\frac{t}{2}\log(x)\right) x^{\frac{1}{4}-1}\theta(x)dx\\ &+ \int_0^\infty \sinh\left(\frac{1}{2}\left(\sigma - \frac{1}{2}\right)\log(x)\right)\sin\left(\frac{t}{2}\log(x)\right) x^{\frac{1}{4}-1}\theta(x)dx.\end{aligned}$$

We now in fact observe that

$$\Im(\Xi(t)) = \lim_{\sigma\to\frac{1}{2}} \Im(\xi(\sigma + it)) = \lim_{\sigma\to\frac{1}{2}} \Im\left(\frac{(\sigma + it)(\sigma + it - 1)}{2}\xi^*(\sigma + it)\right) = 0,$$

hence Ξ is entirely real. □

3.3 Roots of ζ on the Critical Line

Proposition 3.3.1. *We have the identity*

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cos(xt)dt = \frac{1}{2}\pi(e^{\frac{1}{2}x} - 2e^{-\frac{1}{2}x}\omega(e^{-2x})).$$

Proof. Let $f(t) = \left|\gamma(t)\right|^2 = \lambda(it)\lambda(-it)$ where λ is analytic and let

$$\Phi(x) = \int_0^\infty f(t)\Xi(t)\cos(xt)dt.$$

By substituting $y = e^x$ we get

$$\begin{aligned}
\Phi(x) &= \int_0^\infty \lambda(it)\lambda(-it)E(t)\frac{1}{2}(y^{it} + y^{-it})dt \\
&= \frac{1}{2}\int_{-\infty}^\infty \lambda(it)\lambda(-it)E(t)y^{it}dt \\
&= \frac{1}{2}\int_{-\infty}^\infty \lambda(it)\lambda(-it)\xi(\frac{1}{2}+it)y^{it}dt
\end{aligned}$$

Again the substitution $s = \frac{1}{2} + it$ gives

$$\begin{aligned}
\Phi(x) &= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \lambda(s-\frac{1}{2})\lambda(\frac{1}{2}-s)\xi(s)y^s dt \\
&= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \lambda(s-\frac{1}{2})\lambda(\frac{1}{2}-s)(s-1)\Gamma(1+\frac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)y^s ds.
\end{aligned}$$

Finally setting $\lambda(s) = \frac{1}{s+\frac{1}{2}}$ together with Lemma 3.2.1 yields

$$\begin{aligned}
\Phi(x) &= -\frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \frac{1}{s}\Gamma(1+\frac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)y^s ds \\
&= -\frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \Gamma(\frac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)y^s ds \\
&= -\frac{\pi}{2\sqrt{y}}\left(\frac{1}{2i\pi}\int_{\frac{1}{2}-\infty}^{\frac{1}{2}+\infty} \xi^*(s)\left(\frac{1}{y}\right)^{-s} ds\right) \\
&= -\frac{\pi}{\sqrt{y}}\omega\left(\frac{1}{y^2}\right) + \frac{1}{2}\pi\sqrt{y}.
\end{aligned}$$

After re-substituting we receive the identity we wanted to show. $\square$

Theorem 3.3.1. *ζ has infinitely many roots in $s = \frac{1}{2} + t$ for $t \in \mathbb{R}$.*

Proof. We substitute $x = -i\alpha$ and plug it into the previously shown identity

$$\begin{aligned}\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cosh(\alpha t)dt &= e^{-\frac{1}{2}\alpha} - 2e^{\frac{1}{2}i\alpha}\omega(e^{2i\alpha}) \\ &= 2\cos(\frac{1}{2}\alpha) - 2e^{\frac{1}{2}i\alpha}\left(\frac{1}{2}+\omega(e^{2i\alpha})\right).\end{aligned}$$

Since $\zeta(\frac{1}{2}+it) = \mathcal{O}(t^A)$ and $\Xi(t) = \mathcal{O}(t^A e^{-\frac{1}{4}\pi t})$ and since $\cosh(\alpha t) = \mathcal{O}(e^{\alpha t})$, for $\alpha < \frac{1}{4}\pi$ we can interchange integral and differentiation according to the dominated convergence theorem and differentiate both sides with respect to α on both sides $2n$-times and receive

$$\begin{aligned}&\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\alpha t)dt \\ &\qquad = \frac{(-1)^n\cos(\frac{1}{2}\alpha)}{2^{2n-1}} - 2\left(\frac{d}{d\alpha}\right)^{2n} e^{\frac{1}{2}i\alpha}\left(\frac{1}{2}+\omega(e^{2i\alpha})\right).\end{aligned}$$

In order to show that the last term vanishes as $\alpha \to 0$ for every fixed n we use Corollary 3.1.1.1 and write

$$\begin{aligned}\omega(i+\delta) = \sum_{n=1}^\infty e^{-n^2\pi(i+\delta)} &= \sum_{n=1}^\infty (-1)^n e^{-n^2\pi\delta} \\ &= 2\omega(4\delta) - \omega(\delta) \\ &= \frac{1}{\sqrt{\delta}}\omega\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\omega\left(\frac{1}{\delta}\right) - \frac{1}{2}.\end{aligned}$$

Obviously along every route in an angle $\arg(x-i) < \frac{1}{2}\pi$ we have

$$\lim_{x\to i}\omega(x) + \frac{1}{2} = 0.$$

Therefore we have shown that

$$\lim_{\alpha\to\frac{1}{4}\pi}\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\alpha t)dt = \frac{(-1)^n\cos(\frac{1}{8}\pi)}{2^{2n-1}}.$$

Let's assume that Ξ were ultimately of one sign, say positive for $t \geq T$ where $T \in \mathbb{R}$. Then for all $L > 0$ and $T' \geq T$ we have

$$\lim_{\alpha\to\frac{1}{4}\pi}\frac{2}{\pi}\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\alpha t)dt = L.$$

Hence

$$\lim_{\alpha\to\frac{1}{4}\pi}\frac{2}{\pi}\int_T^{T'}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\alpha t)dt \le L.$$

Hence for $\alpha = \frac{1}{4}\pi$ we have

$$\frac{2}{\pi}\int_T^{T'}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\frac{1}{4}\pi t)dt \le L.$$

Thus the integral

$$\frac{2}{\pi}\int_0^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\frac{1}{4}\pi t)dt$$

is convergent and therefore the integral is uniformly convergent with respect to $0 \le \alpha \le \frac{1}{4}\pi$ and it follows that

$$\int_0^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\frac{1}{4}\pi t)dt = \frac{(-1)^n\pi\cos(\frac{1}{8}\pi)}{2^{2n}}.$$

Now we choose n to be odd. The right-hand side is negative and therefore

$$\begin{aligned}\int_T^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\frac{1}{4}\pi t)dt &< -\int_0^{T}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\frac{1}{4}\pi t)dt \\ &< KT^{2n},\end{aligned}$$

where K is independent of n. But by hypothesis there is a $m = m(T)$ such that

$$\frac{\Xi(t)}{t^2+\frac{1}{4}} \ge m$$

for $2T \le t \le 2T+1$. Hence

$$\int_T^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh(\frac{1}{4}\pi t)dt \ge \int_{2T}^{2T+1} mt^{2n}dt \ge m(2T)^{2n}.$$

Hence

$$m2^{2n} < K,$$

which is false for sufficiently large n. That proves our theorem by contradiction. □

Bibliography

Ahlfors, Lars V. (1966), *Complex Analysis: An introduction to the theory of analytic functions of one complex variable*, second edition, volume 313. McGraw-Hill Book Company.

Apostol, Tom M. (1976), *Introduction to Analytic Number Theory*, volume 329. Springer-Verlag, New York.

Edwards, H. M. (1974), *Riemann's Zeta Function*. Academic Press, New York.

Franca, Guilherme and LeClair Andre (2014), *A Theory for the Zeros of Riemann Zeta and other L-functions*. URL `http://arxiv.org/pdf/1407.4358v1.pdf`.

Jesacher, F. (2016), *Riemann Zeta Function - The Functional Equation*. URL `https://github.com/fje/Riemann-Zeta-Function-The-Functional-Equation/blob/master/Thesis.pdf`.

McLachlan, N. W. (1953), *Complex Variable Theory and Transform Calculus*. Cambridge University Press.

Stein, E. M. and R. Shakarchi (2003), *Complex Analysis*, volume 379. Princeton University Press.

Titchmarsh, E. C. (1986), *The Theory of the Riemann Zeta-Function*, second edition, volume 406. Clarendon Press, Oxford.