Riemann Zeta Function - The Functional Equation

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Abstract

We first introduce the Riemann zeta function on an open half-plane and use Weierstrass's uniform convergence theorem to address the holomorphy of the function. By deriving its Euler product representation we deduce the absence of roots on the function's domain. Once we have defined the gamma function and examined some of its identities including Euler's reflection formula, we create a link to the Riemann zeta function. We manage to meromorphically continue said function and extend it to the whole complex plane by utilising the analytic character of contour integrals. This leads us directly to a meromorphic integral expression with only one simple pole finally paving the way to the functional equation. One fundamental result of this new expression is that the Riemann zeta function has no nontrivial zeros outside the critical strip.

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Chapter 1

Introduction

The Riemann zeta function is one of the most important functions in analytic number theory, particularly since it encodes a lot of information about prime numbers. Thus it is well worth taking a closer look at its properties and behaviour in the complex plane. Special attention should be paid to the zeros of the function as they might give some indication of the mysterious distribution of primes.
This work is mainly based on Apostol (1976) and uses some definitions and theorems from Ahlfors (1966) as well.
Henceforth let $s \in \mathbb{C}$ be a complex number where $\sigma := \Re(s)$ and $t := \Im(s)$. The space of all holomorphic functions on a domain $S \subseteq \mathbb{C}$ shall be denoted by $\mathcal{H}(S) := \left\{ f \in \mathbb{C}^S \colon f \textit{ holomorphic} \right\}$.

1.1 Elementary Properties of ζ

First of all we want to draw some conclusions about the convergence of the Riemann zeta function and its holomorphy on its domain.

Definition 1.1. *We refer to*

$$\zeta \colon \{s \in \mathbb{C} \colon \sigma > 1\} \to \mathbb{C},$$
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

as the Riemann zeta function.

1.1.1 Absolute Convergence and Holomorphy

Theorem 1.1.1. $\zeta(s)$ *converges absolutely for* $\sigma > 1$.

Proof. Let $x \in \mathbb{R}_{>0}$. We quickly note that

$$|x^s| = \left|e^{s\log(x)}\right| = \left|e^{(\sigma+it)\log(x)}\right| = \left|x^\sigma e^{it\log(x)}\right| = x^\sigma$$

as

$$\forall \theta \in \mathbb{R}\colon \left|e^{i\theta}\right| = 1.$$

Hence we obtain

$$\sum_{n=1}^{\infty} \left|n^{-s}\right| = \sum_{n=1}^{\infty} n^{-\sigma}.$$

We know that the right sum converges for $\sigma > 1$. Consequently $\zeta(s)$ converges absolutely. $\square$

Lemma 1.1.2 (Weierstrass's uniform convergence theorem)**.** *Let $S \subseteq \mathbb{C}$ be open, $f \in \mathbb{C}^S$ and $(f_n)_{n\in\mathbb{N}} \in \mathcal{H}(S)^{\mathbb{N}}$ with $\lim_{n\to\infty} f_n = f$. If*

$$\forall D \subset S \text{ compact}: f_n\big|_D \to f\big|_D \text{ uniformly,}$$

then

$$f \in \mathcal{H}(S) \tag{1.1}$$

and

$$f_n^{(k)}\big|_D \to f^{(k)}\big|_D \text{ uniformly for } k \in \mathbb{N}. \tag{1.2}$$

Proof. Let $\triangle \subset S$ denote a closed solid triangle. As $\triangle$ is compact and due to the uniform convergence of $f_n\big|_\triangle \to f\big|_\triangle$ and Goursat's lemma we know that

$$\forall \triangle \subset S\colon \int_{\partial\triangle} f(z)dz = \int_{\partial\triangle} \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_{\partial\triangle} f_n(z)dz = 0.$$

Morera's theorem finally gives us

$$f \in \mathcal{H}(S).$$

To show the second claim, let $r \in \mathbb{R}_{>0}$ be the radius of an arbitrary closed disk $D_r \subset S$ and let the positively orientated boundary of D_r be denoted by ∂D_r. Since $f_n^{(k)} \in \mathcal{H}(S)$ we can apply Cauchy's integral formula

$$\begin{aligned} f^{(k)}(z) &= \frac{k!}{2\pi i} \int_{\partial D_r} \frac{f(\xi)}{(\xi - z)^{k+1}} d\xi \\ &= \frac{k!}{2\pi i} \int_{\partial D_r} \lim_{n\to\infty} \frac{f_n(\xi)}{(\xi - z)^{k+1}} d\xi \\ &= \lim_{n\to\infty} \frac{k!}{2\pi i} \int_{\partial D_r} \frac{f_n(\xi)}{(\xi - z)^{k+1}} d\xi = \lim_{n\to\infty} f_n^{(k)}(z) \end{aligned}$$

for $z \in D_r^\circ$. For $l < r$ we obtain

$$\begin{aligned}\left\| f_n^{(k)} - f^{(k)} \right\|_{D_l} &\leq \sup_{z \in D_l} \left\{ \frac{k!}{2\pi} \int_{\partial D_r} \left| \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{k+1}} \right| |d\xi| \right\} \\ &\leq \sup_{z \in D_l} \left\{ \frac{k! r}{\operatorname{dist}(z, \partial D_r)^{k+1}} \left\| f_n - f \right\|_{\partial D_r} \right\} \\ &\leq \frac{k! r}{(r-l)^{k+1}} \left\| f_n - f \right\|_{\partial D_r}.\end{aligned}$$

Hence $f_n^{(k)}\big|_{D_l} \to f^{(k)}\big|_{D_l}$ uniformly. Since we can always find a finite cover of open disks D^i for any compact set $D \subset S$ such that $\bigcup \overline{D^i} \subset S$, we have provided conclusive evidence for our claim. $\square$

Theorem 1.1.3. *ζ is holomorphic for $\sigma > 1$.*

Proof. Let $R = [a, b] \times i\,[c, d] \subset \mathbb{C}$ with $1 < \delta \leq a$. By the definition of uniform convergence we write

$$\begin{aligned}&\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall m \geq N\ \forall s \in R: \\ &\left| \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{m-1} n^{-s} \right| = \left| \sum_{n=m}^{\infty} n^{-s} \right| \leq \sum_{n=m}^{\infty} \left| n^{-s} \right| = \sum_{n=m}^{\infty} n^{-\sigma} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \leq \sum_{n=m}^{\infty} n^{-\delta} < \varepsilon.\end{aligned}$$

Therefore $\zeta_m\big|_R \to \zeta\big|_R$ uniformly and according to Weierstrass ζ is holomorphic for $\sigma > 1$. $\square$

1.2 The Euler Product

In this section we introduce infinite products and talk about their convergence. After deriving ζ's Euler product representation we can make some assertions about the occurrence of zeros for $\sigma > 1$.

1.2.1 Infinite Convergent Products

Definition 1.2. *Let $(a_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. The infinite product*

$$\prod_{n \in \mathbb{N}} (1 + a_n)$$

is said to be convergent if

$$\lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n) \in \mathbb{C} \setminus \{0\}\,.$$

We speak of absolute convergence if the same holds true for $|a_n|$*.*

Remark. *We particularly don't allow convergent products to be zero here, since any infinite product being zero would converge if only one factor was zero, hence the convergence wouldn't depend on the whole sequence of factors. Furthermore this convention would be too radical for certain applications.*

Lemma 1.2.1. *Let* $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$*. The following statements are equivalent*

$$(i) \quad \prod_{n\in\mathbb{N}} (1+a_n) \;\; \textit{converges}$$
$$(ii) \quad \sum_{n\in\mathbb{N}} \log(1+a_n) \;\; \textit{converges.}$$

Proof. Let the infinite product be denoted by P, its partial products by P_n and respectively the series by S and its partial sums by S_n. If P_n converges then

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \lim_{n\to\infty} (1+a_n) = 1.$$

Hence $(a_n)_{n\in\mathbb{N}}$ has to be a null sequence. To prove $(i) \implies (ii)$ we assume that $P_n \to P$ and we choose the principal value of $\log(P)$, namely

$$\log(P) = \log(|P|) + i\arg(P).$$

Now we determine $\arg(P_n)$ by the condition

$$\arg(P) - \pi < \arg(P_n) \le \arg(P) + \pi$$

and set $\log(P_n) = \log(|P_n|) + i\arg(P_n)$. Since log is a multivalued function we get

$$S_n = \log(P_n) + h_n 2\pi i$$

for $h_n \in \mathbb{Z}$. For two consecutive terms we can write

$$(h_{n+1} - h_n)2\pi i = \log(1+a_{n+1}) + \log(P_n) - \log(P_{n+1}).$$

If we choose n large enough we get

$$\begin{aligned}\left|\arg(1+a_{n+1})\right| &< \frac{2\pi}{3} \\ \left|\arg(P)-\arg(P_n)\right| &< \frac{2\pi}{3} \\ \left|\arg(P)-\arg(P_{n+1})\right| &< \frac{2\pi}{3}.\end{aligned}$$

These inequalities imply

$$|h_{n+1}-h_n| < 1,$$

hence for a sufficiently large n we have $h_n = h_{n+1} = h$ and have

$$\lim_{n\to\infty} S_n = S = \log(P) + h2\pi i.$$

To show $(ii) \implies (i)$ we assume that $S_n \to S$ and by writing $P_n = e^{S_n}$ for the principal branch of log we conclude that

$$\lim_{n\to\infty} P_n = P = e^S \in \mathbb{C}\setminus 0.$$

□

Lemma 1.2.2. *Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. The following statements are equivalent*

$$\begin{aligned}&(i) \quad \prod_{n\in\mathbb{N}} \big(1+|a_n|\big) \ \textit{converges} \\ &(ii) \quad \sum_{n\in\mathbb{N}} |a_n| \ \textit{converges.}\end{aligned}$$

Proof. We can write

$$0 < 1 + \sum_{n\in\mathbb{N}} |a_n| \le \prod_{n\in\mathbb{N}} \big(1+|a_n|\big) = e^{\sum_{n\in\mathbb{N}} \log(1+|a_n|)} \le e^{\sum_{n\in\mathbb{N}}|a_n|}$$

as

$$\forall x \in \mathbb{R}_{\ge 0}\colon \ \log(1+x) \le x.$$

□

1.2.2 Euler Product Representation of ζ

Definition 1.3. *An arithmetic function $f \in \mathbb{C}^{\mathbb{N}}$ satisfying*

$$\forall n, m \in \mathbb{N}\colon\ \gcd(n,m) = 1 \implies f(nm) = f(n)f(m)$$

is called multiplicative and we particularly have $f(1) = 1$. If even

$$\forall n, m \in \mathbb{N}\colon f(nm) = f(n)f(m)$$

holds true we say f is completely multiplicative.

Theorem 1.2.3. *Let $\mathcal{P} = \{p \in \mathbb{N}\colon\ p \text{ prime number}\}$ and let $f \in \mathbb{C}^{\mathbb{N}}$ be an arithmetic function. If $\sum_{n\in\mathbb{N}} f(n)$ converges absolutely and if f is multiplicative, the series can be expressed by an absolutely convergent infinite product*

$$\sum_{n\in\mathbb{N}} f(n) = \prod_{p\in\mathcal{P}} \left(1 + \sum_{i=1}^{\infty} f(p^i)\right).$$

If f even is completely multiplicative and fulfils $\left|f(p)\right| < 1\ \forall p \in \mathcal{P}$ we have

$$\sum_{n\in\mathbb{N}} f(n) = \prod_{p\in\mathcal{P}} \frac{1}{1 - f(p)}.$$

Proof. Let $\mathcal{P}_{\leq x} \coloneqq \{p \in \mathcal{P}\colon\ p \leq x\}$. Take the product

$$Q(x) = \prod_{p\in\mathcal{P}_{\leq x}} \left(1 + \sum_{i=1}^{\infty} f(p^i)\right).$$

Since the product is finite and our series converge absolutely we can expand the product and rearrange its terms without altering the sum. For an arbitrary multiplication of two factors involving primes p_k and p_l we get

$$\begin{aligned} &\left(1 + \sum_{i=1}^{\infty} f(p_k^i)\right)\left(1 + \sum_{j=1}^{\infty} f(p_l^j)\right) \\ &= 1 + \sum_{i=1}^{\infty} f(p_k^i) + \sum_{j=1}^{\infty}\left(f(p_l^j) + \sum_{i=1}^{\infty} f(p_k^i) f(p_l^j)\right) \end{aligned}$$

and because f is multiplicative, we observe that after calculating the whole product we end up with summands of the form

$$f(p_{i_1}^{\alpha_1}) f(p_{i_2}^{\alpha_2}) \ldots f(p_{i_r}^{\alpha_r}) = f(p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \ldots p_{i_r}^{\alpha_r}),$$

where $p_{i_1}^{\alpha_1} p_{i_2}^{\alpha_2} \dots p_{i_r}^{\alpha_r}$ occurs only once within the whole sum as the above representation is unique due to the unique factorisation theorem. This allows us to write

$$Q(x) = \sum_{n \in A} f(n),$$

where A consists of those n having all their prime factors $\leq x$, hence

$$\sum_{n \in \mathbb{N}} f(n) - Q(x) = \sum_{n \in B} f(n),$$

where B is the set of n having at least one prime factor $> x$. Thus

$$\left| \sum_{n \in \mathbb{N}} f(n) - Q(x) \right| \leq \sum_{n \in B} \left| f(n) \right| \leq \sum_{n > x} \left| f(n) \right|$$

converges to 0 as $x \to \infty$ implying

$$\lim_{x \to \infty} Q(x) = \sum_{n \in \mathbb{N}} f(n).$$

To show the absolute convergence of $Q(\infty)$ we recall that the infinite product $\prod_{n \in \mathbb{N}} (1 + a_n)$ converges absolutely if and only if $\sum_{n \in \mathbb{N}} |a_n| < \infty$. We write

$$\forall x \in \mathbb{N}: \quad \sum_{p \in \mathcal{P}_{\leq x}} \left| \sum_{i=1}^{\infty} f(p^i) \right| \leq \sum_{p \in \mathcal{P}_{\leq x}} \sum_{i=1}^{\infty} \left| f(p^i) \right| \leq \sum_{n=2}^{\infty} \left| f(n) \right| < \infty$$

which implies

$$\sum_{p \in \mathcal{P}} \left| \sum_{i=1}^{\infty} f(p^i) \right| < \infty.$$

Therefore $Q(\infty)$ is absolutely convergent. If f is completely multiplicative and fulfils $\left| f(p) \right| < 1 \; \forall p \in \mathcal{P}$ the infinite product simplifies to

$$\sum_{n \in \mathbb{N}} f(n) = \prod_{p \in \mathcal{P}} \left(1 + \sum_{i=1}^{\infty} f(p^i) \right) = \prod_{p \in \mathcal{P}} \left(\sum_{i=0}^{\infty} f(p)^i \right) = \prod_{p \in \mathcal{P}} \frac{1}{1 - f(p)}.$$

□

Corollary 1.2.3.1. *We have ζ's Euler product representation*

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} \quad \text{for } \sigma > 1.$$

Proof. Since $f(n) = n^{-s}$ is a completely multiplicative arithmetic function, $\left|f(n)\right| = n^{-\sigma} < 1$ for $\sigma > 1$ and the series converges absolutely the above statement follows directly from the previous theorem. □

Corollary 1.2.3.2. *$\zeta(s)$ has no zeros for $\sigma > 1$.*

Proof. According to the last theorem we know that the infinite product representation of $\zeta(s)$ for $\sigma > 1$ is absolutely convergent, which implies that $\zeta(s)$ can't vanish for $\sigma > 1$. □

Chapter 2

Meromorphic Continuation

Analytic continuation provides a way to extend the domain of a given analytic complex function. We can expand this function into a power series which is only valid within its radius of convergence. Under certain circumstances these expansions have a larger-than-expected radius of convergence and its power series can be used to define the function outside of its domain. If said function has at most countable many poles we speak of meromorphic continuation. We won't explicitly use power series for our meromorphic continuation of the Riemann zeta function but a ruse where contour integrals are involved.

2.1 The Gamma Function

The gamma function is closely related to the Riemann zeta function and plays an important role in ζ's meromorphic continuation beyond the $\sigma = 1$ line. Thus we want to acquire some of its properties including Euler's reflection formula, which helps us to make assertions about zero-free regions of the Riemann zeta function.

Definition 2.1. *The function*

$$\Gamma\colon \{s \in \mathbb{C}\colon \sigma > 0\} \to \mathbb{C},$$
$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

is called the gamma function.

2.1.1 Convergence and Holomorphy

Theorem 2.1.1. $\Gamma(s)$ *converges absolutely for* $\sigma > 0$

Proof. We can write

$$\begin{aligned}\big|\Gamma(s)\big| &= \left|\int_0^\infty x^{s-1}e^{-x}dx\right| \leq \int_0^\infty \left|x^{s-1}e^{-x}\right| dx \\ &= \int_0^1 x^{\sigma-1}e^{-x}dx + \int_1^k x^{\sigma-1}e^{-x}dx + \int_k^\infty x^{\sigma-1}e^{-x}dx\end{aligned}$$

for $k \in (1, \infty)$. For the first integral we find

$$\int_0^1 x^{\sigma-1}e^{-x}dx \leq \int_0^1 x^{\sigma-1}dx = \frac{1}{\sigma},$$

since $e^{-x} \leq 1$ for $x > 0$. As the second integral clearly exists for every k, we proceed to the third integral. Note that

$$\forall r \in \mathbb{R}\colon\ \lim_{x\to\infty} x^r e^{-\frac{x}{2}} = 0,$$

hence

$$\exists k \in (0,\infty)\ \forall x \geq k\colon\ x^r e^{-\frac{x}{2}} \leq 1.$$

Thus we have

$$\int_k^\infty x^{\sigma-1}e^{-x}dx = \int_k^\infty \left(x^{\sigma-1}e^{-\frac{x}{2}}\right) e^{-\frac{x}{2}}dx \leq \int_k^\infty e^{-\frac{x}{2}}dx = 2e^{-\frac{k}{2}}.$$

□

Theorem 2.1.2. $\Gamma(s)$ *is holomorphic for* $\sigma > 0$*.*

Proof. Let $\partial\triangle \subset \{s \in \mathbb{C}\colon\ \sigma > 0\}$ denote the boundary of a solid triangle. Since Γ converges absolutely for $\sigma > 0$ we can use Fubini's theorem for integrable functions in order to interchange the double integral and we get

$$\int_{\partial\triangle} \Gamma(s)ds = \int_{\partial\triangle}\int_0^\infty x^{s-1}e^{-x}dxds = \int_0^\infty \int_{\partial\triangle} x^{s-1}e^{-x}dsdx = 0.$$

As the integrand of Γ is a holomorphic function of s Goursat's lemma justifies that the whole expression has to be 0. Therefore Morera's theorem finally gives us that $\Gamma(s)$ is holomorphic for $\sigma > 0$. □

Theorem 2.1.3. *We have the identity*

$$\Gamma(s+1) = s\Gamma(s).$$

Proof. By applying integration by parts we get

$$\Gamma(s+1) = \int_0^\infty x^s e^{-x} dx = \Big[-x^s e^{-x}\Big]_0^\infty + s\int_0^\infty x^{s-1} e^{-x} dx = s\Gamma(s).$$

□

2.1.2 Meromorphic Continuation

Theorem 2.1.4. $\Gamma(s)$ *is meromorphically continuable for all* $s \in \mathbb{C}$ *with simple poles at* $s = -n \in \mathbb{Z}_{\leq 0}$ *and we have*

$$Res\,\big(\Gamma(s), -n\big) = \frac{(-1)^n}{n!}.$$

Proof. For fixed $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ let $n \in \mathbb{N}_0$ be such that $\sigma + n > 0$. Then we formally have

$$\Gamma(s) = \frac{1}{s}\Gamma(s+1) = \cdots = \frac{1}{s\,(s+1)\ldots(s+n-1)}\Gamma(s+n) = \frac{\Gamma(s+n)}{P_n(s)}.$$

Thus Γ's holomorphy in the negative plane is reducible to the holomorphy of the function in the positive plane which we've already shown. Hence $\Gamma(s)$ is well defined and its poles are just the simple roots of $P_n(s)$ for $n \in \mathbb{N}$, namely $\mathbb{Z}_{\leq 0}$.
Now we want to calculate the residues of $\Gamma(s)$ and find

$$Res\,\big(\Gamma(s), -n\big) = \lim_{s \to -n}(s+n)\,\frac{\Gamma(s+n+1)}{s\,(s+1)\ldots(s+n)} = \frac{(-1)^n}{n!}.$$

□

Lemma 2.1.5. *In particular we have*

$$\Gamma(n+1) = n! \textit{ for } n \in \mathbb{N}_0.$$

Proof. We easily check that

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n!\Gamma(1) = n!.$$

□

2.1.3 Euler's Reflection Formula

Lemma 2.1.6. *It holds that*

$$\forall \sigma \in \mathbb{R}\colon \lim_{|t|\to\infty} \Gamma(\sigma + it) = 0,$$

so $\Gamma(\sigma+it)$ is bounded for a fixed $\sigma \in \mathbb{R}$ and $|t|$ large enough.

Proof. Let $\sigma > 0$ be fixed. For the substitution $y = \frac{1}{2\pi}\log(x)$ we get

$$\begin{aligned}\Gamma(\sigma + it) &= \int_0^\infty x^{\sigma+it-1} e^{-x} dx \\ &= \int_{-\infty}^\infty e^{2\pi iyt}\left(2\pi e^{2\pi y\sigma} e^{-e^{2\pi y}}\right) dy = \int_{-\infty}^\infty e^{2\pi iyt}\hat{g}(y)dy.\end{aligned}$$

By definition of the Fourier transform, the above formula says that the function

$$\begin{aligned}&g\colon \mathbb{R} \to \mathbb{C},\\ &g(t) = \Gamma(\sigma + it)\end{aligned}$$

is the inverse Fourier transform of $\hat{g}$. Since the (inverse) Fourier transform describes an isomorphism on the Schwartz space $\mathcal{S}$, we deduce that g has to decrease more rapidly for an increasing $|t|$ than any polynomial. In other words it in fact follows that

$$\lim_{|t|\to\infty} g(t) = \lim_{|t|\to\infty} \Gamma(\sigma + it) = 0.$$

By meromorphic continuation of Γ this holds for all $\sigma \in \mathbb{R}$. $\square$

Theorem 2.1.7 (Euler's reflection formula)**.** *We have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \textit{for } s \in \mathbb{C}\setminus\mathbb{Z}.$$

Hence $\Gamma(s)$ has no zeros and its reciprocal function $\frac{1}{\Gamma(s)}$ is well defined with zeros at the simple poles of $\Gamma(s)$.

Proof. Let us consider

$$f(s) = \sin(\pi s)\Gamma(s)\Gamma(1-s).$$

We want to show that $f(s)$ is constant. It is easy to check that $f(s)$ is entire as

$$\lim_{s\to 0} \frac{\sin(s)}{s} = 1,$$

hence $\sin(s)$ eradicates the simple poles of $\Gamma(s)$. Furthermore $f(s)$ satisfies

$$\begin{aligned} f(s+1) &= \sin(\pi\,(s+1))\Gamma(s+1)\Gamma(-s) \\ &= -\sin(\pi s)s\Gamma(s)\left(-\frac{1}{s}\right)\Gamma(1-s) = f(s). \end{aligned}$$

So for fixed t $f(\sigma+it)$ is 1-periodic and $f \in C^\infty(\mathbb{R}/\mathbb{Z})$. Therefore $f(s)$ has a rapidly decreasing converging Fourier series

$$f(s) = \sum_{n\in\mathbb{Z}} c_n(t)e^{i2\pi n\sigma}.$$

Let us consider the m-th term of the Fourier series for fixed s. By definition we have

$$\begin{aligned} f_m(s) &= c_m(t)e^{i2\pi m\sigma} \\ &= \left(\int_0^1 f(x+it)e^{-i2\pi mx}dx\right)e^{i2\pi n\sigma} = \int_0^1 f(x+it)e^{-i2\pi m(x-\sigma)}dx \\ &= \int_{-\sigma}^{1-\sigma} f(x+\sigma+it)e^{-i2\pi mx}dx = \int_0^1 f(x+s)e^{-i2\pi mx}dx \\ &= \int_0^1 h(s,x)dx. \end{aligned}$$

Since $h(s,x)$ is defined for $(s,x) \in \mathbb{C}\times[0,1]$ and $h(s,x)$ is holomorphic in s for each x and since $h(s,x)$ is continuous on $\mathbb{C}\times[0,1]$ theorem 5.4 from Stein and Shakarchi (2003) gives us that f_m is holomorphic, hence analytic. Therefore we can apply the Cauchy-Riemann equations which imply that

$$\frac{d}{dt}\big(c_m(t)\cos(2\pi m\sigma)\big) = -\frac{d}{d\sigma}\big(c_m(t)\sin(2\pi m\sigma)\big)$$

or equivalently

$$\frac{d}{dt}c_m(t) = -2\pi m c_m(t).$$

This represents a linear homogenous differential equation with the general solution

$$c_m(t) = c_m e^{-2\pi mt}.$$

Finally we get

$$f(s) = \sum_{m\in\mathbb{Z}} c_m e^{i2\pi ms}.$$

Let us now suppose $m > 0$ and $t \to -\infty$. Then we have

$$f_m(\sigma + it) = c_m e^{i2\pi m\sigma} e^{2\pi m|t|} = \widetilde{c}_m e^{2\pi m|t|}.$$

On the other hand we know that

$$\left|f_m(s)\right| = \left|\int_0^1 f(x+s)e^{-i2\pi mx}dx\right| \leq \int_0^1 \left|f(x+s)\right| dx.$$

According to the previous lemma $\left|\Gamma(\sigma + it)\right|$ is bounded for $|t|$ large enough and we get

$$\begin{aligned}
|\widetilde{c}_m| \, e^{2\pi m|t|} &\leq \int_0^1 \left|\sin(x+s)\right| \left|\Gamma(x+s)\Gamma(x+1-s)\right| dx \\
&\leq \int_0^1 ce^{\pi|t|}dx \leq de^{\pi|t|},
\end{aligned}$$

where $c, d \in \mathbb{R}_{>0}$. Thus by comparing the growth rates we conclude that $\widetilde{c}_m = 0$ for $m \neq 0$ leaving $f(s)$ constant. The value is easily determined by

$$\begin{aligned}
\lim_{s\to 0} f(s) &= \lim_{s\to 0} \sin(s)\Gamma(s)\Gamma(1-s) \\
&= \lim_{s\to 0} \frac{\sin(\pi s)}{s}\Gamma(s+1)\Gamma(1-s) = \pi.
\end{aligned}$$

□

2.2 Contour Integral Representation

After deriving an integral representation of the product of the gamma and the Riemann zeta function, we manage to meromorphically continue ζ beyond the $\sigma = 1$ line.

Theorem 2.2.1. *We have the identity*

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}}dx \text{ for } \sigma > 1. \tag{2.1}$$

Proof. First we assume that $s \in \mathbb{R}_{>1}$. Let $x = (n+1)\,t$ where $n \in \mathbb{N}_{\geq 0}$. By substitution we get

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx = (n+1)^s \int_0^\infty e^{-(n+1)t}t^{s-1}dt,$$

equivalently

$$(n+1)^{-s}\,\Gamma(s) = \int_0^\infty e^{-(n+1)t} t^{s-1} dt.$$

After summing up both sides over n ranging from 0 to ∞ we obtain

$$\zeta(s)\Gamma(s) = \sum_{n=0}^{\infty} (n+1)^{-s}\,\Gamma(s) = \sum_{n=0}^{\infty} \int_0^\infty e^{-(n+1)t} t^{s-1} dt.$$

Since the integral can be treated as a Lebesgue integral and since the integrand is nonnegative, Levi's convergence theorem tells us that the series

$$\sum_{n=0}^{\infty} e^{-(n+1)t} t^{s-1}$$

converges almost everywhere to a Lebesgue-integrable sum function on $[0, \infty)$ and that we can interchange sum and integral such that

$$\zeta(s)\Gamma(s) = \int_0^\infty \sum_{n=0}^{\infty} e^{-(n+1)t} t^{s-1} dt.$$

If $t > 0$ then $0 < e^{-t} < 1$. Thus

$$\sum_{n=0}^{\infty} e^{-nt} = \frac{1}{1-e^{-t}}$$

is a geometric series. Bringing it together we have

$$\sum_{n=0}^{\infty} e^{-(n+1)t} t^{s-1} = \frac{e^{-t} t^{s-1}}{1-e^{-t}}$$

everywhere on $[0, \infty)$, except for $t = 0$, so we get

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{e^{-t} t^{s-1}}{1-e^{-t}} dt$$

for real $s > 1$. To extend it to all complex s with $\sigma > 1$ we note that the left side is analytic for $\sigma > 1$. To show that the right side is analytic we assume $1 + \delta \le \sigma \le c$, where $c > 1$ and $\delta > 0$ and write

$$\int_0^\infty \left| \frac{e^{-t} t^{s-1}}{1-e^{-t}} \right| dt \le \int_0^\infty \frac{e^{-t} t^{\sigma-1}}{1-e^{-t}} dt = \left(\int_0^1 + \int_1^\infty \right) \frac{e^{-t} t^{\sigma-1}}{1-e^{-t}} dt.$$

If $0 \le t \le 1$ we have $t^{\sigma-1} \le t^{\delta}$ and if $t \ge 1$ we have $t^{\sigma-1} \le t^{c-1}$. Also, since $e^t - 1 \ge t$ for $t > 0$ we have

$$\int_0^1 \frac{e^{-t}t^{\sigma-1}}{1-e^{-t}}dt \le \int_0^1 \frac{t^{\delta}}{e^t-1} \le \int_0^1 t^{\delta-1} = \frac{1}{\delta}$$

and

$$\int_1^{\infty} \frac{e^{-t}t^{\sigma-1}}{1-e^{-t}}dt \le \int_1^{\infty} \frac{e^{-t}t^{c-1}}{1-e^{-t}}dt \le \int_0^{\infty} \frac{e^{-t}t^{c-1}}{1-e^{-t}}dt = \Gamma(c)\zeta(c).$$

This shows that the integral converges uniformly in every strip $1+\delta \le \sigma \le c$, where $\delta > 0$, and therefore represents an analytic function in every such strip, hence also in the half-plane $\sigma > 1$. Therefore, by analytic continuation, our identity holds for all s with $\sigma > 1$. $\square$

Theorem 2.2.2. *Let $C = C_1 + C_2 + C_3$ denote a positively oriented contour as shown in Figure 2.1 looping around the negative real axis and enclosing*

$$\begin{aligned} C_1 &\colon [c,\infty) \to \mathbb{C},\ C_1(r) = re^{i\pi} \\ C_2 &\colon (-\pi,\pi) \to \mathbb{C},\ C_2(\theta) = ce^{i\theta} \\ C_3 &\colon [c,\infty) \to \mathbb{C},\ C_3(r) = re^{-i\pi} \end{aligned}$$

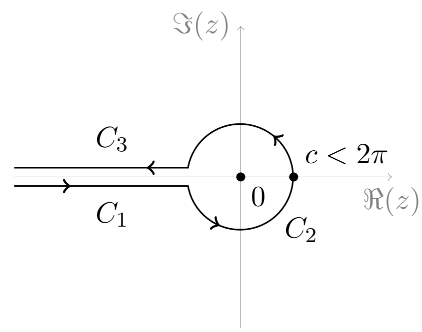

Figure 2.1: Contour C

the point $z = 0$. The function

$$\begin{aligned} &I \colon \mathbb{C} \to \mathbb{C}, \\ &I(s) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^z}{1-e^z}dz \end{aligned} \tag{2.2}$$

is entire and

$$\zeta(s) = \Gamma(1-s)I(s) \text{ for } \sigma > 1. \tag{2.3}$$

Proof. As the integrand

$$\begin{aligned} &f_z \colon \mathbb{C} \to \mathbb{C}, \\ &f_z(s) = \frac{z^{s-1}e^z}{1-e^z} \end{aligned}$$

is an entire function of s for $z \in \{z \in \mathbb{C} \colon \left|\Im(z)\right| < 2\pi \wedge z \neq 0\}$ we only have to show that the integrals along C_1 and C_3 converge uniformly on every compact disk $D_R = \{s \in \mathbb{C} \colon |s| \leq R\}$ for $R \in \mathbb{R}_{>0}$. Along C_1 for $r \geq 1$ we have

$$\left|z^{s-1}\right| = r^{\sigma-1}\left|e^{-\pi i(\sigma-1+it)}\right| = r^{\sigma-1}e^{\pi t} \leq r^{R-1}e^{\pi R}$$

and analogously along C_3 for $r \geq 1$ we get

$$\left|z^{s-1}\right| = r^{\sigma-1}\left|e^{\pi i(\sigma-1+it)}\right| = r^{\sigma-1}e^{-\pi t} \leq r^{R-1}e^{\pi R}.$$

Hence since $e^r - 1 > \frac{e^r}{2}$ for $r \geq \log(2)$ either on C_1 or C_3 we have

$$\left|\frac{z^{s-1}e^z}{1-e^z}\right| \leq \frac{r^{R-1}e^{\pi R}e^{-r}}{1-e^{-r}} = \frac{r^{R-1}e^{\pi R}}{e^r - 1} \leq A(R)r^{R-1}e^{-r}$$

for $r \geq 1$ and a constant $A(R) \in \mathbb{R}_{>0}$ only depending on R. As

$$\forall c, R \in \mathbb{R}_{>0} \colon \int_c^\infty r^{R-1}e^{-r}dr < \infty$$

we have shown that C_1 and C_3 converge uniformly on every disk D_R.
For the second claim along C_1 and C_3 we have

$$g(z) := \frac{e^z}{1-e^z} = \frac{e^{re^{\pm\pi i}}}{1-e^{re^{\pm\pi i}}} = \frac{e^{-r}}{1-e^{-r}} = g(-r).$$

We write

$$\begin{aligned}
2\pi i I_c(s) &= \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) z^{s-1}g(z)dz \\
&= \int_\infty^c r^{s-1}e^{-\pi is}g(-r)dr + i\int_{-\pi}^{\pi} c^{s-1}e^{(s-1)i\theta}ce^{i\theta}g(ce^{\theta})d\theta \\
&\qquad\qquad\qquad + \int_c^\infty r^{s-1}e^{\pi is}g(-r)dr \\
&= 2i\sin(\pi s)\int_c^\infty r^{s-1}g(-r)dr + ic^s\int_{-\pi}^{\pi} e^{is\theta}g(ce^{i\theta})d\theta.
\end{aligned}$$

Once we have divided by $2i$ on both sides we get

$$\pi I_c(s) = \sin(\pi s)I_{1,c}(s) + I_{2,c}(s).$$

Now we let $c \to 0$. For $I_{1,c}$ the previous theorem gives us

$$\lim_{c\to 0} I_{1,c}(s) = \int_0^\infty \frac{r^{s-1}e^{-r}}{1-e^{-r}}dr = \Gamma(s)\zeta(s).$$

For $I_{2,c}$ we have to keep in mind that $g(z)$ is holomorphic for $|z| < 2\pi$ except a simple pole at $z=0$. Thus $zg(z)$ is holomorphic anywhere inside $|z| = c < 2\pi$ and is bounded there. So we get $\big|g(z)\big| < \frac{B}{|z|}$ for some constant $B \in \mathbb{R}_{>0}$ and obtain the approximation

$$\big|I_{2,c}(s)\big| \leq \frac{c^\sigma}{2}\int_{-\pi}^{\pi} e^{-t\sigma}\frac{B}{c}d\theta \leq Be^{\pi|t|}c^{\sigma-1}.$$

If $\sigma > 1$ then $\lim\limits_{c\to 0} I_{2,c}(s) = 0$. Hence we get

$$\pi I(s) = \sin(\pi s)\Gamma(s)\zeta(s).$$

By multiplying both sides with $\Gamma(1-s)$ and plugging in Euler's reflection formula we finally obtain

$$\zeta(s) = \Gamma(1-s)I(s).$$

□

Definition 2.2. *As $\Gamma(1-s)$ is analytic for $\sigma \leq 1, s \neq 1$ and $I(s)$ is entire we can meromorphically continue $\zeta(s)$ and define*

$$\zeta(s) = \Gamma(1-s)I(s) \text{ for } \sigma \leq 1, s \neq 1.$$

Theorem 2.2.3. *We have $\zeta \in \mathcal{H}(\mathbb{C}\setminus\{1\})$ with a simple pole in $s=1$ and its residue $Res\big(\zeta(s), 1\big) = 1$.*

Proof. The only singularities that come into question are the poles of $\Gamma(1-s)$ as $I(s)$ is entire, but we already know that $\zeta(s)$ is analytic for $\sigma > 1$ which leaves $s = 1$ the single possible pole.
To prove the above statement, assume $s = 1$. The integrals along C_1 and C_3 take the same values and cancel each other out leaving

$$\begin{aligned} I(1) &= \frac{1}{2\pi i}\int_{C_2}\frac{z^0 e^z}{1-e^z}dz = Res\left(\frac{e^z}{1-e^z}, 1\right) \\ &= \lim_{z\to 0} z\frac{e^z}{1-e^z} = \lim_{z\to 0}\frac{z}{e^{-z}-1} = -\lim_{z\to 0}\frac{1}{e^{-z}} = -1. \end{aligned}$$

Now we can compute the residue

$$\begin{aligned} Res\,\big(\zeta(s),1\big) &= \lim_{s\to 1}\,(s-1)\,\zeta(s) \\ &= -\lim_{s\to 1}\,(1-s)\,\Gamma(1-s)I(s) \\ &= -I(1)\lim_{s\to 1}\Gamma(2-s) = \Gamma(1) = 1. \end{aligned}$$

□

2.3 The Functional Equation

The functional equation gives a new powerful closed expression of ζ and allows us to make further assertions about the occurrence of zeros in the whole complex plane.

Lemma 2.3.1. *Let $r \in (0,\pi]$ and $U = \mathbb{C} \setminus \{2n\pi i \colon n \in \mathbb{Z}\}$. If*

$$S(r) = \big\{z \in \mathbb{C}\colon \forall n \in \mathbb{Z}\colon\ |z - 2n\pi i| \geq r\big\} \subset U$$

and

$$g\colon U \to \mathbb{C},\ g(z) = \frac{e^z}{1-e^z}$$

then $g(z)$ is bounded in $S(r)$ such that

$$\exists A(r) \in \mathbb{R}^+\ \forall z \in S(r)\colon\ \big|g(z)\big| \leq A(r).$$

Proof. Let $z = x+iy \in \mathbb{C}$. First we assume $|x| \leq 1$. Consider the punctured rectangle

$$Q(r) = \big\{z \in \mathbb{C}\colon\ |x| \leq 1, |y| \leq \pi, |z| \geq r\big\} \subset S(r)$$

As g is continuous and $Q(r)$ is compact in U we know that $g\big|_{Q(r)}$ has to be bounded by a $B(r) \in \mathbb{R}^+$. The same holds for g in the punctured infinite strip $S(r) \cap \big\{z \in \mathbb{C}\colon\ |x| \leq 1\big\}$ because of its $2\pi i$-periodicity.
Now suppose $|x| \geq 1$. We have

$$\big|g(z)\big| = \left|\frac{e^z}{1-e^z}\right| = \frac{e^x}{|1-e^z|} \leq \frac{e^x}{|1-e^x|}.$$

For $x \geq 1$ we get

$$\big|g(z)\big| \leq \frac{e^x}{e^x - 1} = \frac{1}{1-e^{-x}} \leq \frac{1}{1-e^{-1}} = \frac{e}{e-1}$$

and analogously for $x \leq -1$

$$\left|g(z)\right| \leq \frac{e^x}{1-e^x} \leq \frac{1}{1-e^x} \leq \frac{1}{1-e^{-1}} = \frac{e}{e-1}$$

and therefore

$$\forall z \in S(r)\colon \ \left|g(z)\right| \leq \max\left\{\frac{e}{e-1}, B(r)\right\} =: A(r).$$

□

Theorem 2.3.2. *We find that*

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\Gamma(s)\cos(\frac{\pi s}{2})\zeta(s) \text{ for } \sigma > 1 \tag{2.4}$$

or equivalently

$$\zeta(s) = 2\,(2\pi)^{s-1}\,\Gamma(1-s)\sin(\frac{\pi s}{2})\zeta(1-s) \text{ for } \sigma < 0. \tag{2.5}$$

This is the so called functional equation of ζ. By meromorphic continuation the domain is extendable to $\mathbb{C}\setminus\{1\}$.

Proof. Let $C_N = C_{1,N} + C_{2,N} + C_{3,N} + C_{4,N}$ denote a negatively orientated contour for $N \in \mathbb{N}$ as shown in Figure 2.2, where $0 < c < \pi$ describes the

$$\begin{aligned} C_{1,N}\colon \ \left[c, R(N)\right] &\to \mathbb{C},\ C_{1,N}(r) = re^{i\pi} \\ C_{2,N}\colon \ (-\pi,\pi) &\to \mathbb{C},\ C_{2,N}(\theta) = ce^{i\theta} \\ C_{3,N}\colon \ \left[c, R(N)\right] &\to \mathbb{C},\ C_{3,N}(r) = re^{-i\pi} \\ C_{4,N}\colon \ (-\pi,\pi) &\to \mathbb{C},\ C_{4,N}(\theta) = R(N)e^{i\theta} \end{aligned}$$

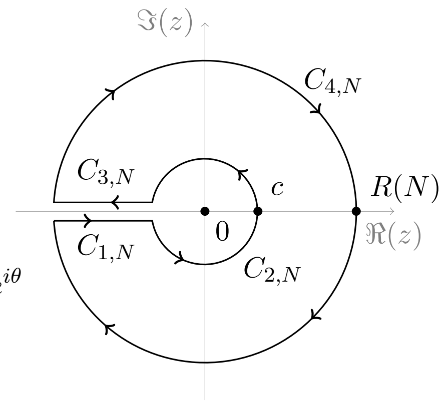

Figure 2.2: Contour C_N

inner and $R(N) = 2(N+1)\pi$ the outer radius of our two circles $C_{2,N}$ and $C_{4,N}$. Consider

$$I_N(s) = \frac{1}{2\pi i}\int_{C_N} \frac{z^{s-1}e^z}{1-e^z}dz.$$

We first want to prove that $\lim\limits_{N\to\infty} I_N(s) = I(s)$ for $\sigma < 0$. As

$$\begin{aligned}
\lim_{N\to\infty} I_N(s) &= \lim_{N\to\infty} \frac{1}{2\pi i}\left(\int_{C_{1,N}} + \int_{C_{2,N}} + \int_{C_{3,N}} + \int_{C_{4,N}}\right)\frac{z^{s-1}e^z}{1-e^z}dz \\
&= \frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \lim_{N\to\infty}\int_{C_{4,N}}\right)\frac{z^{s-1}e^z}{1-e^z}dz
\end{aligned}$$

where C_1, C_2 and C_3 are as defined in Figure 2.1, it suffices to show that $\lim\limits_{N\to\infty} \int_{C_{4,N}} \cdots = 0$. For $z \in \left|C_{4,N}\right|$ we know that

$$\left|z^{s-1}\right| = \left|z^{\sigma-1}\right| = R(N)^{\sigma-1}e^{-t\sigma} \leq R(N)^{\sigma-1}e^{\pi|t|}$$

and from the previous lemma we deduce

$$\left|\frac{e^z}{1-e^z}\right| \leq A(\pi),$$

which leads to

$$\begin{aligned}
\left|\int_{C_{4,N}} \frac{z^{s-1}e^z}{1-e^z}dz\right| &\leq \int_{-\pi}^{\pi}\left|\frac{z^{s-1}e^z}{1-e^z}z'\right|d\theta \\
&\leq \int_{-\pi}^{\pi} R(N)^{\sigma}e^{\pi|t|}A(\pi)d\theta \\
&= 2\pi R(N)^{\sigma}e^{\pi|t|}A(\pi),
\end{aligned}$$

which converges to 0 for $N \to \infty$ if $\sigma < 0$. Accordingly we can write

$$\lim_{N\to\infty} I_N(1-s) = I(1-s) \ \textit{for}\ \sigma > 1.$$

Now we want to determine $I_N(1-s)$ explicitly by using Cauchy's residue theorem. Let

$$R(n) = Res\left(\frac{z^{-s}e^z}{1-e^z}, 2n\pi i\right),$$

where

$$\begin{aligned}
R(n) &= \lim_{z\to 2n\pi i}(z-2n\pi i)\frac{z^{-s}e^z}{1-e^z} \\
&= (2n\pi i)^{-s}e^{2n\pi i}\lim_{z\to 2n\pi i}\frac{z-2n\pi i}{1-e^z} \\
&= -(2n\pi i)^{-s}.
\end{aligned}$$

We get

$$\begin{aligned}
I_N(1-s) &= -\sum_{\substack{n=-N \\ n\neq 0}}^{N} R(n) = -\sum_{n=1}^{N} R(n) + R(-n) \\
&= \sum_{n=1}^{N} (2n\pi i)^{-s} + \sum_{n=1}^{N} (-2n\pi i)^{-s} \\
&= \left(\frac{e^{-\frac{\pi i s}{2}}}{(2\pi)^s} + \frac{e^{\frac{\pi i s}{2}}}{(2\pi)^s}\right) \sum_{n=1}^{N} n^{-s} \\
&= \frac{2}{(2\pi)^s} \cos(\frac{\pi s}{2}) \sum_{n=1}^{N} n^{-s}.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\zeta(1-s) &= \Gamma(s) I(1-s) = \Gamma(s) \lim_{N\to\infty} I_N(1-s) \\
&= \frac{2}{(2\pi)^s} \Gamma(s) \cos(\frac{\pi s}{2}) \zeta(s)
\end{aligned}$$

and by substituting s with $1-s$ we have

$$\zeta(s) = 2\,(2\pi)^{s-1}\,\Gamma(1-s) \sin(\frac{\pi s}{2}) \zeta(1-s).$$

□

2.3.1 Trivial Zeros and Critical Strip

Corollary 2.3.2.1. *$\zeta(s)$ has zeros at $s = -2n$ for $n \in \mathbb{N}_0$. These are called the trivial zeros.*

Proof. The functional equation of $\zeta(s)$ becomes zero whenever $\sin(\frac{\pi s}{2})$ vanishes, which in fact happens when $s = -2n$. □

Corollary 2.3.2.2. *$\zeta(s)$ can only have nontrivial zeros within the strip $0 \leq \sigma \leq 1$. This region is also known as the critical strip.*

Proof. Consider the second functional equation from above

$$\zeta(s) = 2\,(2\pi)^{s-1}\,\Gamma(1-s) \sin(\frac{\pi s}{2}) \zeta(1-s) \text{ for } \sigma < 0.$$

We already know that $\zeta(s)$ has no zeros for $\sigma > 1$. We also have shown that $\Gamma(s)$ doesn't vanish on its domain and the only remaining factor is $\sin(\frac{\pi s}{2})$, which in fact generates the trivial zeros. Thus $\zeta(s)$ has no nontrivial zeros for $\sigma < 0$. □

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