弹性力学

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1　平面问题的极坐标解答

1.1　极坐标中的平衡微分方程

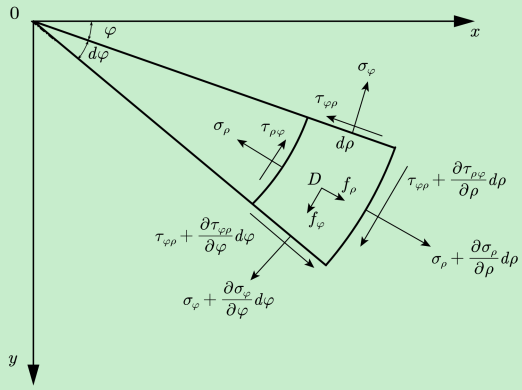


图 1

1.1.1　极坐标中的平衡微分方程

$$\begin{cases} \frac{\partial\sigma_\rho}{\partial\rho}+\frac{1}{\rho}\frac{\partial\tau_{\rho\varphi}}{\partial\varphi}+\frac{\sigma_\rho-\sigma_\varphi}{\rho}+f_\rho=0 \\ \frac{1}{\rho}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{\partial\tau_{\rho\varphi}}{\partial\rho}+\frac{2\tau_{\rho\varphi}}{\rho}+f_\varphi=0 \end{cases} \tag{1}$$

1.2　极坐标中的几何方程和物理方程

1.2.1　极坐标中的几何方程

$$\begin{cases} \varepsilon_\rho=\frac{\partial u_\rho}{\partial\rho} \\ \varepsilon_\varphi=\frac{u_\rho}{\rho}+\frac{1}{\rho}\frac{\partial u_\rho}{\partial\varphi} \\ \gamma_{\rho\varphi}=\frac{1}{\rho}\frac{\partial u_\rho}{\partial\varphi}+\frac{\partial u_\rho}{\partial\rho}-\frac{u_\varphi}{\rho} \end{cases} \tag{2}$$

1.2.2　极坐标中的物理方程

$$\begin{cases} \varepsilon_\rho=\frac{1}{E}\left(\sigma_\rho-\mu\sigma_\varphi\right) \\ \varepsilon_\varphi=\frac{1}{E}\left(\sigma_\varphi-\mu\sigma_\rho\right) \\ \gamma_{\rho\varphi}=\frac{1}{G}\tau_{\rho\varphi}=\frac{2(1+\mu)}{E}\tau_{\rho\varphi} \end{cases} \tag{3}$$

平面应变的情况下，需将上式中的E换为$\frac{E}{1-\mu^2}$，μ换为$\frac{\mu}{1-\mu}$。

1.3 坐标变换

1.3.1 位移分量的坐标变换

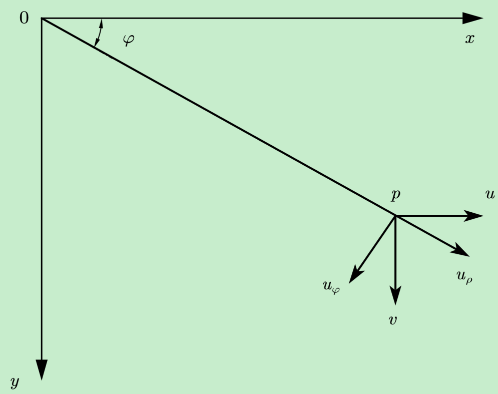


图 2

$$\begin{cases} u_\rho = u\cos\varphi + v\sin\varphi \\ u_\varphi = -u\sin\varphi + v\cos\varphi \end{cases} \Longrightarrow \begin{pmatrix} u_\rho \\ u_\varphi \end{pmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{pmatrix} u_\rho \\ u_\varphi \end{pmatrix} \tag{4}$$

1.3.2 应力分量的坐标变换

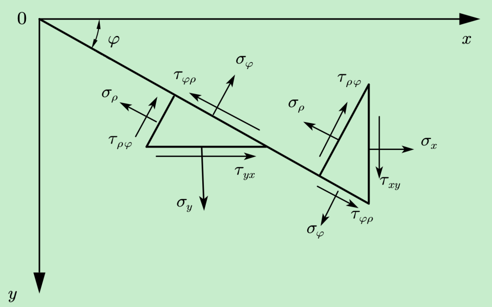


图 3

$$\begin{cases} \sigma_x = \sigma_\rho\cos^2\varphi + \sigma_\varphi\sin^2\varphi - 2\tau_{\rho\varphi}\sin\varphi\cos\varphi \\ \sigma_y = \sigma_\rho\sin^2\varphi + \sigma_\varphi\cos^2\varphi + 2\tau_{\varphi\rho}\sin\varphi\cos\varphi \\ \tau_{xy} = \frac{\sigma_\rho-\sigma_\varphi}{2}\sin 2\varphi + \tau_{\rho\varphi}\cos 2\varphi \end{cases} \tag{5}$$

$$\begin{cases} \sigma_x = \frac{\sigma_\rho+\sigma_\varphi}{2} + \frac{\sigma_\rho-\sigma_\varphi}{2}\cos 2\varphi - \tau_{\rho\varphi}\sin 2\varphi \\ \sigma_y = \frac{\sigma_\rho+\sigma_\varphi}{2} - \frac{\sigma_\rho-\sigma_\varphi}{2}\cos 2\varphi - \tau_{\rho\varphi}\sin 2\varphi \\ \tau_{xy} = \frac{\sigma_\rho-\sigma_\varphi}{2}\sin 2\varphi + \tau_{\rho\varphi}\cos 2\varphi \end{cases} \tag{6}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \sigma_\rho & \tau_{\varphi\rho} \\ \tau_{\rho\varphi} & \sigma_\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}^T \tag{7}$$

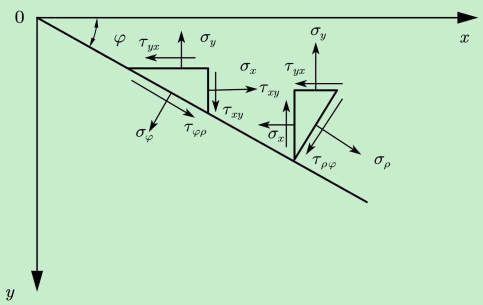


图 4

$$\begin{cases} \sigma_\rho = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\tau_{xy}\sin\varphi\cos\varphi \\ \sigma_\varphi = \sigma_x\sin^2\varphi + \sigma_y\cos^2\varphi - 2\tau_{xy}\sin\varphi\cos\varphi \\ \tau_{\rho\varphi} = \left(\sigma_y - \sigma_x\right)\sin\varphi\cos\varphi + \tau_{xy}\left(\cos^2\varphi - \sin^2\varphi\right) \end{cases} \tag{8}$$

$$\begin{cases} \sigma_\rho = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\varphi + \tau_{xy}\sin 2\varphi \\ \sigma_\varphi = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\varphi - \tau_{xy}\sin 2\varphi \\ \tau_{\varphi\rho} = \frac{\sigma_x-\sigma_y}{2}\sin 2\varphi + \tau_{xy}\cos 2\varphi \end{cases} \tag{9}$$

$$\begin{bmatrix} \sigma_\rho & \tau_{\varphi\rho} \\ \tau_{\rho\varphi} & \sigma_\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}^T \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \tag{10}$$

1.4 极坐标中的应力函数与相容方程

当$f_\rho = f_\varphi = 0$时：

$$\begin{cases} \sigma_\rho = \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} \\ \sigma_\varphi = \frac{\partial^2\Phi}{\partial\rho^2} \\ \tau_{\rho\varphi} = -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\right) \end{cases} \tag{11}$$

1.4.1 相容方程

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)^2\Phi=0 \tag{12}$$

1.5 轴对称应力和相容的位移

所谓轴对称问题，是指物体几何形状或某物理量是绕某一轴对称的，凡通过此轴的任何面均为对称面。如果该物体所受外部荷载也对称于该轴，那么相应所产生的应力也必对称该轴。

令$\Phi=\Phi(\rho)$

$$\sigma_\rho=\frac{1}{\rho}\frac{d\Phi}{d\rho},\sigma_\varphi=\frac{d^2\Phi}{d\rho^2},\tau_{\rho\varphi}=0 \tag{13}$$

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\Phi=0 \tag{14}$$

即

$$\Phi=A\ln\rho+B\rho^2\ln\rho+C\rho^2+D \tag{15}$$

应力分量

$$\begin{cases}\sigma_\rho=\frac{A}{\rho^2}+B(1+2\ln\rho)+2C\\ \sigma_\varphi=-\frac{A}{\rho^2}+B(3+2+\ln\rho)+2C\\ \tau_{\rho\varphi}=0\end{cases} \tag{16}$$

代入物理方程和几何方程可得位移分量

$$\begin{cases}u_\rho=\frac{1}{E}\left[-(1+\mu)\frac{A}{\rho}+2(1-\mu)B\rho(\ln\rho-1)+(1-3\mu)B\rho+2(1-\mu)C\rho\right]+I\cos\varphi+K\sin\varphi\\ u_\varphi=\frac{4B\rho\varphi}{E}+H\rho-I\sin\varphi+K\cos\varphi\end{cases} \tag{17}$$

(1). 在轴对称应力条件下，应力、应变和位移的通解，适用于任何轴对称应力问题。

(2). 在轴对称应力条件下，应变也是轴对称的，但位移不一定是轴对称的。

(3). 实现轴对称应力的条件是：物体形状、体力和面力应是轴对称的。

(4). 轴对称应力及对应的位移的通解已满足相容方程，它们还需满足边界条件及多连体中的位移单值条件，并由此求出系数A、B、C。

注 欧拉方程：

$$x^ny^{(n)}+P_1x^{n-1}y^{(n-1)}+\cdots+P_{n-1}xy'+P_ny=0$$

特征方程为：

$$[k(k-1)\cdots(k-n+1)+P_1[k(k-1)\cdots(k-n+2)]+\cdots+P_{n-1}k+P_n]=0$$

关于K的n次代数方程，可解得n个特征根k_1、k_2、$\cdots k_n$

(1). 当它们是互不相等的实根时，通解具有幂函数的形式

$$y = C_1 x^{k_1} + C_2 x^{k_2} + \cdots + C_n x^{k_n}$$

(2). 每当出现重根时，每多一重根，就多乘一个$\ln x$

如：当k_1为$m\,(m < n)$重根时，通解为

$$y = C_1 x^{k_1} + C_2 x^{k_2} \ln x + \cdots + C_m x^{k_1} \ln^{m-1} x + C_{m+1} x^{k_{m+1}} + \cdots + C_n x^{k_n}$$

(3). 出现共轭复根时，则虚部是三角函数因子

如：当$k_{1,2} = \alpha \pm i\beta$，通解为

$$y = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x) + C_3 x^{k_3} + \cdots + C_n x^{k_n}$$

(4). 当出现复重根，则实部要多乘因子$\ln x$

如：$k_{1,2} = \alpha \pm i\beta$为$m\left(m < \frac{n}{2}\right)$重共轭复根时，通解为

$$\begin{aligned} y = &\left[C_1 x^{\alpha} + C_2 x^{\alpha} \ln x + \cdots + C_m x^{\alpha} \ln^{m-1} x\right] \cos(\beta \ln x) + \\ &\left[C_{m+1} x^{\alpha} + C_{m+2} x^{\alpha} \ln x + \cdots + C_{2m} x^{\alpha} \ln^{m-1} x\right] \sin(\beta \ln x) + \\ &C_{2m+1} x^{k_{2m+1}} + \cdots + C_n x^{k_n} \end{aligned}$$

例 1.1 试求解平面轴对称应力问题的相容方程

$$\rho^4 \Phi^{(4)} + 2\rho^3 \Phi^{(3)} - \rho^2 \Phi'' + \rho \Phi' = 0$$

解 特征方程：

$$k(k-1)(k-2)(k-3) + 2k(k-1)(k-2) - k(k-1) + k = 0$$

解得：

$$k_{1,2} = 0, k_{3,4} = 2$$

得：

$$\Phi = A \ln \rho + B\rho^2 \ln \rho + C\rho^2 + D$$

例 1.2 试求解如下的常微分方程

$$\rho^4 f^{(4)}(\rho) + 2\rho^3 f'''(\rho) - 9\rho^2 f''(\rho) + 9\rho f'(\rho) = 0$$

解 特征方程：

$$k(k-1)(k-2)(k-3) + 2k(k-1)(k-2) - 9k(k-1) + 9k = 0$$

解得：

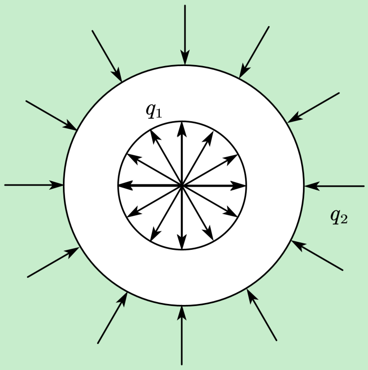
$$k_1 = 4, k_2 = 2, k_3 = 0, k_4 = -2$$

得：

$$f(\rho) = A\rho^4 + B\rho^2 + C + D\rho^{-2}$$

1.6 圆环或圆筒受均布压力

例 1.3 设有圆环（平面应力问题）或圆筒（平面应变问题）受均匀内压力q_1，和外压力q_2作用；内半径为r;外半径为R。试求应力分量；位移分量。



解 轴对称应力问题的应力通解：

$$
\begin{cases}
\sigma_\rho = \frac{A}{\rho^2} + B\left(1 + 2\ln\rho\right) + 2C \\
\sigma_\varphi = -\frac{A}{\rho^2} + B\left(3 + 2 + \ln\rho\right) + 2C \\
\tau_{\rho\varphi} = 0
\end{cases}
$$

应力边界条件：

$$
\begin{cases}
\left(\sigma_\rho\right)_{\rho=r} = -q_1 \\
\left(\sigma_\rho\right)_{\rho=R} = -q_2 \\
\left(\tau_{\rho\varphi}\right)_{\rho=r} = \left(\tau_{\rho\varphi}\right)_{\rho=R} = 0
\end{cases}
\implies
\begin{cases}
\frac{A}{r^2} + B\left(1 + 2\ln r\right) + 2C = -q_1 \\
\frac{A}{R^2} + B\left(1 + 2\ln r\right) + 2C = -q_2
\end{cases}
$$

圆环或圆筒具有贯穿孔洞，为多连体，故需进一步考虑位移单值条件：由

$$
u_\varphi = \frac{4B\rho\varphi}{E} + H\rho - I\sin\varphi + K\cos\varphi
$$

考虑(ρ,φ)和$(\rho,\varphi+2\pi)$同一点只能有一确定的位移，故$B=0$。则:

$$
\begin{cases}
A = \frac{r^2R^2(q_2-q_1)}{R^2-r^2} \\
C = \frac{q_1r^2-q_2R^2}{R^2-r^2}
\end{cases}
$$

得：

$$
\begin{cases}
\sigma_\rho = \dfrac{\frac{R^2}{\rho^2}-1}{\frac{R^2}{r^2}-1}q_1 - \dfrac{1-\frac{r^2}{\rho^2}}{1-\frac{r^2}{R^2}}q_2 \\
\sigma_\varphi = \dfrac{\frac{R^2}{\rho^2}+1}{\frac{R^2}{r^2}-1}q_1 - \dfrac{1+\frac{r^2}{\rho^2}}{1-\frac{r^2}{R^2}}q_2
\end{cases}
$$

例 1.4 轴对称应力条件下的应力和位移的通解，可以应用于各种应力边界条件和位移边界条件的情形，试考虑下列圆环或圆筒的问题应如何求解

(1). 内边界受均布压力q_1作用，而外边界为固定边.

(2). 外边界受均布压力q_2作用，而内边界为固定边.

(3). 外边界受到强迫的均匀位移$u_\rho = -\varDelta$，而内边界为自由边（如车辆的轮毂的作用.

(4). 内边界受到强迫的均匀位移$u_\rho = \varDelta$，而外边界为自由边.

解

(1). 应力边界条件:

$$\begin{cases} \left(\sigma_\rho\right)_{\rho=r} = -q_1 \\ \left(\tau_{\rho\varphi}\right)_{\rho=r} = 0 \end{cases} \implies \frac{A}{r^2} + 2C = -q_1$$

位移边界条件:

$$\begin{cases} \left(u_\rho\right)_{\rho=R} = 0 \\ \left(u_\varphi\right)_{\rho=R} = 0 \end{cases} \implies \begin{cases} H = I = K = 0 \\ -(1+\mu)\frac{A}{R} + 2(1-\mu)CR = 0 \end{cases}$$

(2). 应力边界条件:

$$\begin{cases} \left(\sigma_\rho\right)_{\rho=R} = -q_2 \\ \left(\tau_{\rho\varphi}\right)_{\rho=R} = 0 \end{cases} \implies \frac{A}{R^2} + 2C = -q_2$$

位移边界条件:

$$\begin{cases} \left(u_\rho\right)_{\rho=r} = 0 \\ \left(u_\varphi\right)_{\rho=r} = 0 \end{cases} \implies \begin{cases} H = I = K = 0 \\ -(1+\mu)\frac{A}{r} + 2(1-\mu)Cr = 0 \end{cases}$$

(3). 应力边界条件:

$$\begin{cases} \left(\sigma_\rho\right)_{\rho=r} = 0 \\ \left(\tau_{\rho\varphi}\right)_{\rho=r} = 0 \end{cases} \implies \frac{A}{r^2} + 2C = 0$$

位移边界条件:

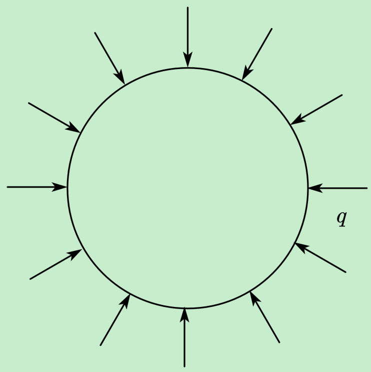
$$\begin{cases} \left(u_\rho\right)_{\rho=R} = -\varDelta \\ \left(u_\varphi\right)_{\rho=R} = 0 \end{cases} \implies \begin{cases} H = I = K = 0 \\ -(1+\mu)\frac{A}{R} + 2(1-\mu)CR = -E\varDelta \end{cases}$$

(4). 应力边界条件:

$$\begin{cases} \left(\sigma_\rho\right)_{\rho=R} = 0 \\ \left(\tau_{\rho\varphi}\right)_{\rho=R} = 0 \end{cases} \implies \frac{A}{R^2} + 2C = 0$$

位移边界条件:

$$\begin{cases} \left(u_\rho\right)_{\rho=r} = -\varDelta \\ \left(u_\varphi\right)_{\rho=r} = 0 \end{cases} \implies \begin{cases} H = I = K = 0 \\ -(1+\mu)\frac{A}{r} + 2(1-\mu)Cr = -E\varDelta \end{cases}$$



例 1.5 实心圆盘在$\rho = r$的圆周上受有均布压力q的作用，试求其应力分量。

解 平面轴对称应用问题，应力通解为：

$$\begin{cases} \sigma_\rho = \frac{A}{\rho^2} + B\left(1 + 2\ln\rho\right) + 2C \\ \sigma_\varphi = -\frac{A}{\rho^2} + B\left(3 + 2 + \ln\rho\right) + 2C \\ \tau_{\rho\varphi} = 0 \end{cases}$$

应力的有界性，圆盘中心处（即$\rho = 0$）处的应力值应当有界，不能是无限大的故$A = B = 0$
应力边界条件:

$$\begin{cases} \left(\sigma_\rho\right)_{\rho=r} = -q \\ \left(\tau_{\rho\varphi}\right)_{\rho=r} = 0 \end{cases} \Longrightarrow C = -\frac{q}{2}$$

故实心圆盘的应力解答：

$$\begin{cases} \sigma_\rho = \sigma_\varphi = -q \\ \tau_{\rho\varphi} = 0 \end{cases}$$

1.7 压力隧洞