Single Tailed Centelli's Inequality

Theorem 1. *with bounded variance* σ^2 *and any* $a \geq 0$*, it follows that:*

$$\mathrm{Prob}(\xi - \mathrm{E}\xi \geq a) \leq \frac{\sigma^2}{a^2 + \sigma^2}$$

Proof Let random variable $X = \xi - \mathrm{E}\xi$, S is like that:

$$S = \begin{cases} 1, & X \geq a \\ -1, & X < a \end{cases}$$

we know

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}\{X^2\} - (\mathrm{E}\{X\})^2 \\
&= \mathrm{E}\{\mathrm{E}\{X^2|S\}\} - (\mathrm{E}\{\mathrm{E}\{X|S\}\})^2 \\
&= \mathrm{E}\{\mathrm{var}(X|S) + (\mathrm{E}\{X|S\})^2\} - (\mathrm{E}\{\mathrm{E}\{X|S\}\})^2 \\
&= \mathrm{E}\{\mathrm{var}(X|S)\} + \mathrm{E}\{(\mathrm{E}\{X|S\})^2\} - (\mathrm{E}\{\mathrm{E}\{X|S\}\})^2 \\
&= \mathrm{E}\{\mathrm{var}(X|S)\} + \mathrm{var}(\mathrm{E}\{X|S\}) \qquad (1)
\end{aligned}$$

Let's look at $\mathrm{E}\{X|S\}$, obviously, $\mathrm{E}\{X|S=1\} \geq a$, and $\mathrm{E}\{\mathrm{E}\{X|S\}\} = \mathrm{E}X = 0$, e.g.

$$\mathrm{E}\{\mathrm{E}\{X|S\}\} = \mathrm{E}\{X|S=1\}P_g + \mathrm{E}\{X|S=-1\}P_l = 0 \qquad (2)$$

where $P_g = \mathrm{Prob}(X \geq a), P_l = \mathrm{Prob}(X < a)$. From (2) we have

$$(\mathrm{E}\{X|S=-1\})^2 = \left(\frac{\mathrm{E}\{X|S=1\}P_g}{P_l}\right)^2 \geq \left(\frac{aP_g}{P_l}\right)^2 \qquad (3)$$

then by (1),

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}\{\mathrm{var}(X|S)\} + \mathrm{var}(\mathrm{E}\{X|S\}) \\
&\geq \mathrm{var}(\mathrm{E}\{X|S\}) \\
&= \mathrm{E}\{\mathrm{E}\{X|S\}^2\} + (\mathrm{E}\{\mathrm{E}\{X|S\}\})^2 \\
&= \mathrm{E}\{\mathrm{E}\{X|S\}^2\} \\
&\geq a^2 P_g + P_l(\frac{aP_g}{P_l})^2 = a^2 P_g/(1-P_g)
\end{aligned}$$

Note $\mathrm{var}(X) = \mathrm{var}(\xi) = \sigma^2$, it says

$$\begin{aligned}
\sigma^2 \geq a^2 P_g/(1-P_g) &\leftrightharpoons \sigma^2(1-P_g) \geq a^2 P_g \\
&\rightleftharpoons \sigma^2 \geq (a^2+\sigma^2)P_g \\
&\rightleftharpoons P_g \leq \frac{\sigma^2}{a^2+\sigma^2}
\end{aligned}$$

also is $P\{\xi - \mathrm{E}\xi \geq a\} \leq \frac{\sigma^2}{a^2+\sigma^2}$

Theorem 2. *Proof equation(6) in [1](page 3)*

Proof From theorem 1 we know for $a > 0$

$$\mathrm{Prob}(\hat{v}_n - \sigma^2 \leq -a) = \mathrm{Prob}(\sigma^2 - \hat{v}_n \geq) a) \leq \frac{\mathrm{var}(\hat{v}_n)}{a^2 + \mathrm{var}(\hat{v}_n)}$$

Let

$$a = \sigma^2 \sqrt{\frac{1}{n}\left(\kappa + \frac{2n}{n-1}\right)\left(\frac{1-\alpha}{\alpha}\right)}$$

by theorem 5 in [1]

$$\begin{aligned}
\mathrm{Prob}(\hat{v}_n - \sigma^2 \leq -a) &\leq \frac{\frac{\sigma^4}{n}\left(\kappa + \frac{2n}{n-1}\right)}{\frac{\sigma^4}{n}\left(\kappa + \frac{2n}{n-1}\right)\left(\frac{1-\alpha}{\alpha}\right) + \frac{\sigma^4}{n}\left(\kappa + \frac{2n}{n-1}\right)} \\
&= \frac{1}{\left(\frac{1-\alpha}{\alpha}\right) + 1} = \alpha
\end{aligned} \tag{4}$$

by (4) we get equation(6) easily.

References

[1] Fred J. Hickernell, *Adaptive simple Monte Carlo*, draft, Jan. 2011.