Monte Carlo Algorithms Where the Integrand Size is Unknown

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Joint work with Lan Jiang, Yuewei Liu, and Art Owen

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Feb. 16, 2012

Hypothetical Conversation

Practitioner	You, the Expert
I need to evaluate integrals $$\mu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \rho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$ for many different f, where ρ is a given probability density function.	

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How large should I make n?	As large as your computational budget allows.

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How large should I make n to obtain $\lvert\mu - \hat{\mu}\rvert \le \varepsilon$?	The Central Limit Theorem says $$n = \left\lceil \left(\frac{1.96\sigma}{\varepsilon}\right)^2 \right\rceil$$ where σ^2 is the variance of the integrand.

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How do I find σ^2?	

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How large should I make n to obtain $\lvert \mu - \hat{\mu} \rvert \le \varepsilon$?	The Central Limit Theorem says $$n = \left\lceil \left(\frac{1.96\hat{\sigma}}{\varepsilon} \right)^2 \right\rceil$$
How do I find σ^2?	Try the sample variance times a variance inflation factor: $$\hat{\sigma}^2 = \frac{\mathfrak{C}^2}{n_\sigma - 1} \sum_{i=1}^{n_\sigma} [f(\boldsymbol{X}_i) - \hat{\mu}_\sigma]^2.$$

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How do I find σ^2?	Try the sample variance times a variance inflation factor: $$\hat{\sigma}^2 = \frac{\mathfrak{C}^2}{n_\sigma - 1}\sum_{i=1}^{n_\sigma}[f(\boldsymbol{X}_i) - \hat{\mu}_\sigma]^2.$$
Does theory guarantee that this algorithm works (at least 95% of the time)?	

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How large should I make n to obtain $\lvert \mu - \hat{\mu} \rvert \le \varepsilon$?	The Central Limit Theorem says $$n = \left\lceil \left(\frac{1.96\hat{\sigma}}{\varepsilon} \right)^2 \right\rceil$$
How do I find σ^2?	Try the sample variance times a variance inflation factor: $$\hat{\sigma}^2 = \frac{\mathfrak{C}^2}{n_\sigma - 1} \sum_{i=1}^{n_\sigma} [f(\boldsymbol{X}_i) - \hat{\mu}_\sigma]^2.$$
Does theory guarantee that this algorithm works (at least 95% of the time)?	Yes! This algorithm, with minor modifications, carries a limited warranty.

Three Perspectives

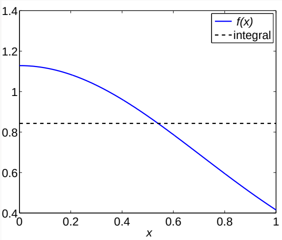
$$\mu = E[f(\boldsymbol{X})] = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \rho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = ?$$

Algorithm Design Construct an automatic multivariate integrator analogous to MATLAB's quad for univariate integrals.

Information-Based Complexity Construct an algorithm, A, satisfying $|\mu - A(f)| \le \epsilon$ (definitely, with high probability, or on average) with $\mathrm{cost}(\varepsilon, A, f)$ depending reasonably on ε and the unknown $\mathrm{size}(f)$.

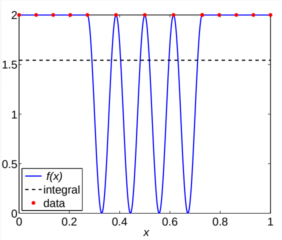
Statistics Find a nonparametric confidence interval of prescribed half-width ε for μ from a reasonable number of samples $Y_i = f(\boldsymbol{X}_i)$.

MATLAB's Quadrature Routine quad Works Well, *but It Can Be Fooled*



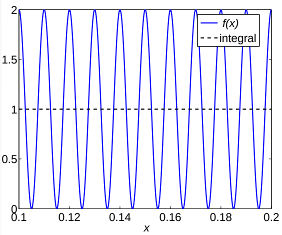
$$\frac{2}{\sqrt{\pi}} \int_0^1 \mathrm{e}^{-x^2} \, \mathrm{d}x$$
$$= 0.8427007929497149$$

quad $\to$
0.8427007929497149 in
0.160521 seconds.



$$\int_0^1 f(x) \, \mathrm{d}x = 1.5436$$

but quad $\to$
2
in 0.007092 seconds.



$$\int_0^1 [1 + \cos(200\pi x)] \, \mathrm{d}x = 1$$

but quad $\to$
0.7636784919876782
in 0.205272 seconds.

Can We Have a Guarantee Like This?

For nice integrands, f, quad will provide $\int_a^b f(x)\,\mathrm{d}x$ with an error $\le \varepsilon$ in a reasonable amount of time, or your money back.

A nice integrand, f, satisfies the following conditions:

- ..., i.e., quad won't be fooled,
- ..., i.e., the number of function values required is moderate.

If f is not nice (nasty), then this guarantee is void, and quad may return an incorrect answer.

An Impractical Guarantee

For integrands, f, satisfying $\|f''\|_\infty \le M$, a trapezoidal rule with $n = \sqrt{(b-a)^3 M/(12\varepsilon)}$ trapezoids will provide $\int_a^b f(x)\,\mathrm{d}x$ with an absolute error $\le \varepsilon$.

To apply this guarantee, one must know M in advance, which is impractical. This is why quad (adaptive recursive Simpson's rule) estimates the error and adaptively determines the number of function evaluations, n.

If the algorithm works for f, it should normally work for cf.

Recall Our Hypothetical Conversation

Practitioner	You, the Expert
I need to evaluate integrals $$\mu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$ for many different f, where ρ is a given probability density function.	Try a sample average, $$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{X}_{i+n_\sigma}),$$ where the $\boldsymbol{X}_i$ are i.i.d. $\sim \rho$.
	The Central Limit Theorem says
How large should I make n to obtain $\lvert\mu - \hat{\mu}\rvert \le \varepsilon$?	$$n = \left\lceil \left(\frac{1.96\hat{\sigma}}{\varepsilon}\right)^2 \right\rceil$$
	Try the sample variance times a variance inflation factor:
How do I find σ^2?	$$\hat{\sigma}^2 = \frac{\mathfrak{C}^2}{n_\sigma - 1}\sum_{i=1}^{n_\sigma}[f(\boldsymbol{X}_i) - \hat{\mu}_\sigma]^2.$$
Does theory guarantee that this algorithm works (at least 95% of the time)?	Yes! This algorithm, with minor modifications, carries a limited warranty.

Guarantee the Variance

The sample variance, v is an unbiased estimate of $\sigma^2 = \int_{\mathbb{R}^d} [f(\boldsymbol{x}) - \mu]^2 \, \rho(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$.

$$v = \frac{1}{n_\sigma - 1} \sum_{i=1}^{n_\sigma} [f(\boldsymbol{X}_i) - \hat{\mu}_{n_\sigma}]^2, \qquad \hat{\mu}_{n_\sigma} = \frac{1}{n_\sigma} \sum_{i=1}^{n_\sigma} f(\boldsymbol{X}_i), \qquad \boldsymbol{X}_1, \boldsymbol{X}_2, \ldots \text{ i.i.d.} \sim \rho$$

$$E[v] = \sigma^2, \qquad \operatorname{var}(v) = \frac{\sigma^4}{n_\sigma}\left(\kappa - \frac{n_\sigma - 3}{n_\sigma - 1}\right), \qquad \kappa := \frac{\int_{\mathbb{R}^d} [f(\boldsymbol{x}) - \mu]^4 \, \rho(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}}{\sigma^4}$$

Cantelli's Inequality (Lin and Bai, 2010, 6.1e) guarantees that an inflated sample variance bounds the variance from above with uncertainty $\tilde{\alpha}$,

$$\hat{\sigma}^2 := \mathfrak{C}^2 v, \qquad \operatorname{Prob}(\hat{\sigma}^2 \geq \sigma^2) \geq 1 - \tilde{\alpha}, \qquad \mathfrak{C} > 1$$

provided that the kurtosis of the integrand, κ, is not too large, i.e.,

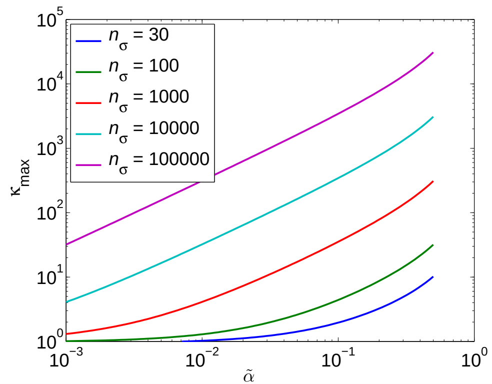
$$\kappa \leq \frac{n_\sigma - 3}{n_\sigma - 1} + \left(\frac{\tilde{\alpha} n_\sigma}{1 - \tilde{\alpha}}\right)\left(1 - \frac{1}{\mathfrak{C}^2}\right)^2 =: \kappa_{\max}(\tilde{\alpha}, n_\sigma, \mathfrak{C}).$$

Guarantee the Variance

$$\hat{\sigma}^2 = \frac{\mathfrak{C}^2}{n_\sigma - 1} \sum_{i=1}^{n_\sigma} [f(\boldsymbol{X}_i) - \hat{\mu}_{n_\sigma}]^2,$$

$$\text{Prob}(\hat{\sigma}^2 \geq \sigma^2) \geq 1 - \tilde{\alpha}$$

if $\kappa \leq \kappa_{\max}(\tilde{\alpha}, n_\sigma, \mathfrak{C})$



$\mathfrak{C} = 1.5$

Guarantee the Integral (Mean)

The Central Limit Theorem gives an asymptotic result for fixed $z \ge 0$:

$$\mathrm{Prob}\left[\left|\underbrace{\int_{\mathbb{R}^d} f(\boldsymbol{x})\,\rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{\mu} - \underbrace{\frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{X}_{i+n_\sigma})}_{\hat{\mu}}\right| \le \frac{z\sigma}{\sqrt{n}}\right] \to 1 - 2\Phi(-z) \quad \text{as } n \to \infty$$

A non-uniform Berry-Esseen Inequality (Petrov, 1995, Theorem 5.16, p. 168) gives a hard upper bound:

$$\mathrm{Prob}\left[|\mu - \hat{\mu}| \le \frac{z\sigma}{\sqrt{n}}\right] \ge 1 - 2\left(\Phi(-z) + \frac{0.56\kappa^{3/4}}{\sqrt{n}}(1+|z|)^{-3}\right)$$

This guarantees that $\mathrm{Prob}\left[|\mu - \hat{\mu}| \le \varepsilon\right] \ge 1 - \tilde{\alpha}$ if the sample size is large enough:

$$n \ge N_B(\varepsilon/\sigma, \tilde{\alpha}, \kappa) := \min\left\{m \in \mathbb{N} : \Phi\left(-\varepsilon\sqrt{m}/\sigma\right) + \frac{0.56\kappa^{3/4}}{\sqrt{m}\left(1+\varepsilon\sqrt{m}/\sigma\right)^3} \le \frac{\tilde{\alpha}}{2}\right\}$$

$$\asymp \frac{\sigma^2}{\varepsilon^2} \qquad \text{as } \frac{\varepsilon}{\sigma} \to 0$$

cubMC

To evaluate $\mu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\,\rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$

given input f, ρ, ε, α, n_σ, $\mathfrak{C}$, and $N_{\max}$:

- Compute $\tilde{\alpha} = 1 - \sqrt{1-\alpha}$, and the maximum kurtosis allowed, $\kappa_{\max}(\tilde{\alpha}, n_\sigma, \mathfrak{C})$.
- Overestimate the variance: $\hat{\sigma}^2 = \dfrac{\mathfrak{C}^2}{n_\sigma - 1} \displaystyle\sum_{i=1}^{n_\sigma} [f(\boldsymbol{X}_i) - \hat{\mu}_{n_\sigma}]^2$.
- Choose the new sample size, $n = \min\left(\max\left(n_\sigma, N_B(\varepsilon/\hat{\sigma}, \tilde{\alpha}, \kappa_{\max})\right), N_{\max}\right)$, for the sample mean.
- Finally, compute the sample mean: $\hat{\mu} = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} f(\boldsymbol{X}_{i+n_\sigma})$.

Then $\mathrm{Prob}\left[|\mu - \hat{\mu}| \le \varepsilon\right] \ge 1 - \alpha$ provided $\kappa \le \kappa_{\max}$ and $n < N_{\max}$.

Guarantee the Time (Sample Size)

Cantelli's inequality also tells us that the estimated variance, $\hat{\sigma}^2$, will not overestimate the true variance, σ^2, by much, and so the number of function values needed is not unnecessarily large:

$$\begin{aligned}\mathrm{cost}(\varepsilon, \mathrm{cubMC}, \sigma) &= \sup_{f:\kappa\le\kappa_{\max}} \min_N \{\mathrm{Prob}[n_\sigma + n \le N] \ge 1-\beta\} \\ &\le n_\sigma + \max(n_\sigma, N_B(\varepsilon/(\sigma\gamma), \tilde{\alpha}, \kappa_{\max}^{3/4})) \asymp \frac{\sigma^2}{\varepsilon^2}, \\ \gamma &:= \mathfrak{C}\left\{1 + \sqrt{\left(\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)\left(\frac{1-\beta}{\beta}\right)\left(1-\frac{1}{\mathfrak{C}^2}\right)^2}\right\}^{1/2}.\end{aligned}$$

Cost depends on $\sigma^2 = \mathrm{var}(f)$, but the algorithm does not need to know σ^2.

Guarantee for cubMC

For nice integrands cubMC will provide the value of $\mu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \rho(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$ with an absolute error of $\le \varepsilon$, with probability $1-\alpha$, in time $\asymp (\sigma/\varepsilon)^2$ with probability $1-\beta$, or your money back.

A nice integrand, f, satisfies the following conditions:

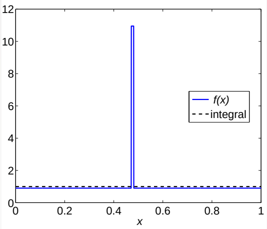
- the kurtosis is not too large, i.e., $\kappa \le \kappa_{\max}(\tilde{\alpha}, n_\sigma, \mathfrak{C})$, and
- the variance is not overwhelming, i.e., $\sigma^2 \le c\varepsilon^2 N_{\max}/d$, where $N_{\max}$ is the maximum number of scalar samples.

If f is not nice (nasty), cubMC may return the wrong answer.

Peak Function — cubMC

$$f(x) = \begin{cases} 1 + \sigma\sqrt{\frac{1-p}{p}}, & 0 \le x - z \pmod 1 \le p, \\ 1 - \sigma\sqrt{\frac{p}{1-p}}, & p < x - z \pmod 1 \le 1, \end{cases}$$

$$\mu = 1, \qquad \kappa = \frac{1}{p(1-p)} - 3$$



$$z \sim U(0,1)$$

$$p \in \left[10^{-5}, 1/2\right], \qquad \sigma \in [0.1, 10]$$

$\log(p), \log(\sigma) \sim$ Uniform

$$\alpha = 5\%, \qquad \mathfrak{C} = 1.5,$$

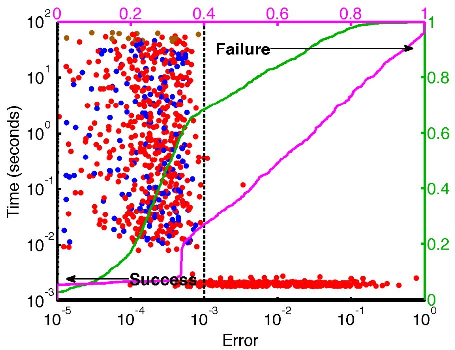
$$n_\sigma = 1024, \qquad \varepsilon = 0.001$$

$$\kappa_{\max} = 9.2, \qquad N_{\max} = 10^9$$

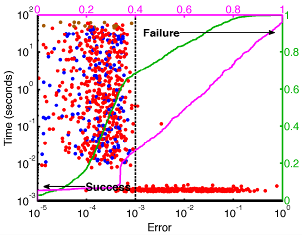
covered by guarantee

kurtosis too large

truncated sample



Peak Function — cubMC vs. quad & quadgk



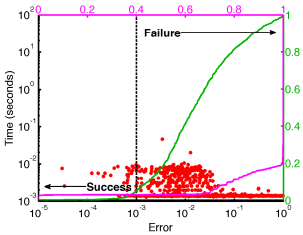
my cubMC

$\varepsilon = 0.001$

covered by guarantee

kurtosis too large

truncated sample



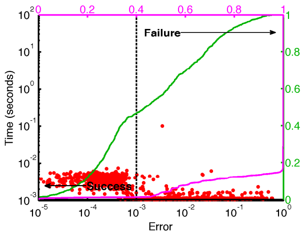
MATLAB's quad

$\varepsilon = 0.001$

fast

tolerance rarely met

no guarantee



MATLAB's quadgk

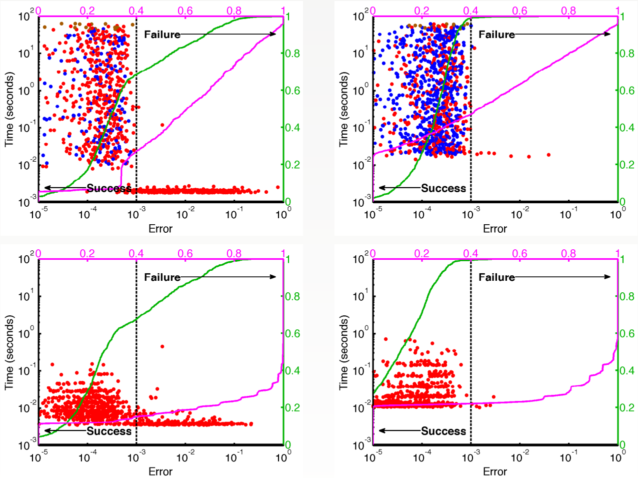
$\varepsilon = 0.001$

fast

tolerance rarely met

no guarantee

Peak Function — cubMC i.i.d. vs. Sobol', Plus More Robustness

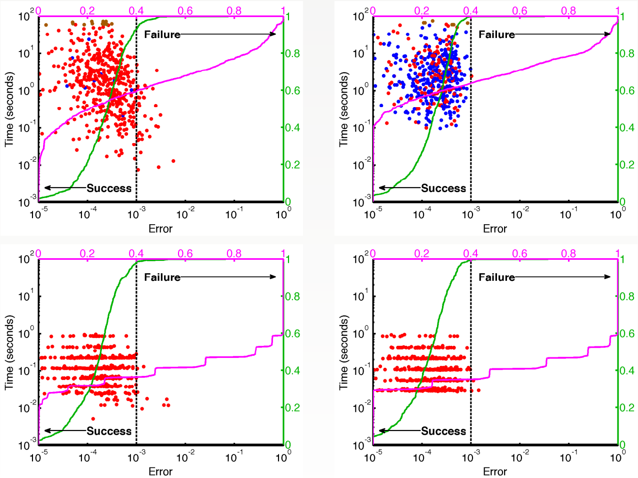


i.i.d. sampling
covered by guarantee
kurtosis too large
truncated sample

Sobol' sampling
no guarantee yet
faster

$n_\sigma = 1024,\ \kappa_{\max} = 9.2$ $\quad n_\sigma = 131072,\ \kappa_{\max} = 1050$ $\quad \varepsilon = 0.001$

Peak Function for $d = 3$

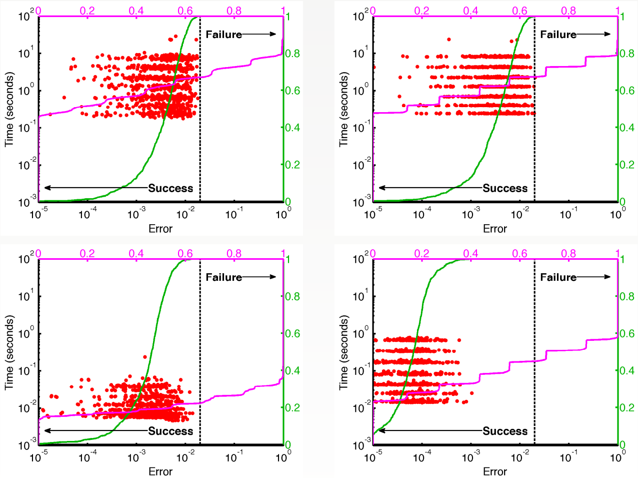


i.i.d. sampling
covered by guarantee
kurtosis too large
truncated sample

Sobol' sampling
quasi-standard error
no guarantee yet
faster

$n_\sigma = 1024, \ \kappa_{\max} = 9.2$ $\quad n_\sigma = 131072, \ \kappa_{\max} = 1050$ $\quad \varepsilon = 0.001$

Asian Geometric Mean Call, $d = 1, 2, 4, \ldots, 64$



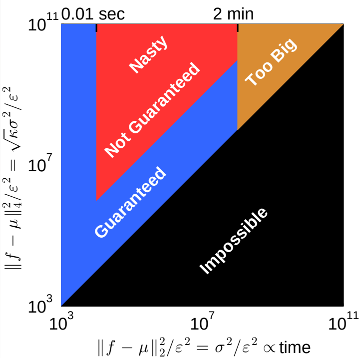
{ i.i.d. sampling

{ Sobol' sampling

$n_\sigma = 1024, \ \kappa_{\max} = 9.2$ $\quad n_\sigma = 131072, \ \kappa_{\max} = 1050$ $\quad \varepsilon = 0.02$

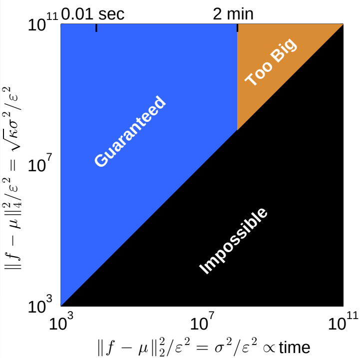
Why an Adaptive Algorithm?

cubMC



- Cost depends on size of integrand
- Algorithm parameters determine robustness to nasty integrands
- Tiny integrands handled regardless
- Huge integrands cannot be handled

non-adaptive algorithm



- Cost is fixed and high if you want it reach tolerance for lots of integrands
- Huge integrands cannot be handled

Why Make cubMC Dependent on Kurtosis?

$$\boxed{\sigma = \text{difficulty} \qquad \kappa = \text{nastiness}}$$

The kurtosis of a random variable or function,

$$\kappa = \frac{E[(Y-\mu)^4]}{\underbrace{\{E[(Y-\mu)^2]\}^2}_{\sigma^4}} \quad \text{or} \quad \kappa := \frac{\displaystyle\int_{\mathbb{R}^d} [f(\boldsymbol{x}) - \mu]^4 \, \rho(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}}{\underbrace{\left\{\displaystyle\int_{\mathbb{R}^d} [f(\boldsymbol{x}) - \mu]^2 \, \rho(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}\right\}^2}_{\sigma^4}}$$

is difficult to estimate. Why should cubMC's guarantee depend on bounded κ?

- Practically, we need κ bounded to justify the estimates of σ^2.
- Bounded κ yields sets of probability distributions or functions that are non-convex.
  - Nonparametric confidence intervals are impossible for convex sets of distributions (Bahadur and Savage, 1956, Corollary 2). How we break convexity
  - Adaptive information does not help for convex sets of integrands in the worst case and probabilistic settings (Traub et al, 1988, Chapter 4, Theorem 5.2.1; Chapter 8, Corollary 5.3.1). How we break convexity

Quasi-Standard Error (Internal Replications) for Quasi-Random Sequences

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be a (random or deterministic) sequence, let r be fixed, and let

$$\hat{\mu}_m = \frac{1}{2^m} \sum_{i=1}^{2^m} f(\boldsymbol{X}_i) = \frac{1}{2^r} \sum_{j=1}^{2^r} \hat{\mu}_{m,j}, \qquad \hat{\mu}_{m,j} = \frac{1}{2^{m-r}} \sum_{i=1}^{2^{m-r}} f(\boldsymbol{X}_{(j-1)2^{m-r}+i})$$

The quasi-standard error (Owen, 1997) measures the variation of among the means of parts of the whole sample

$$\mathrm{qse}_m = \sqrt{\frac{1}{2^r(2^r-1)} \sum_{j=1}^{2^r} (\hat{\mu}_{m,j} - \hat{\mu}_m)^2}$$

Given error tolerance, ε, and parameters $r \in \mathbb{N}$, $m_1 \geq r$, and $\mathfrak{C} > 1$, for $m = m_1, m_1 + 1, \ldots,$

- Compute $f(\boldsymbol{X}_{2^{m-1}+1}), \ldots, f(\boldsymbol{X}_{2^m})$, and $\hat{\mu}_{m,1}, \ldots, \hat{\mu}_{m,2^r}, \hat{\mu}_m$.
- If $\mathfrak{C}\,\mathrm{qse}_m \leq \varepsilon$, then stop. Else continue.

When this works

Further Work

Quasi-Monte Carlo Sampling — What is a good measure of an integrand being nasty or nice?

Variance Reduction Techniques — Can we preserve the guarantee?

Different Error Criteria — Worst case? Randomized?

Lower Bounds on Cost — The typical fooling functions are nasty (high kurtosis). Does assuming moderate kurtosis make the problem easier?

Relative Error Tolerances — Both the variance and the mean are needed to determine the eventual sample size.

Unbounded or Infinite d — Can automatic integrators for finite d be used in multilevel methods to improve efficiency?

Look here

Other Problems — Are there any guarantees for MATLAB's `quad`, or any other univariate adaptive quadrature routine that estimates error? What about guarantees for function approximation?

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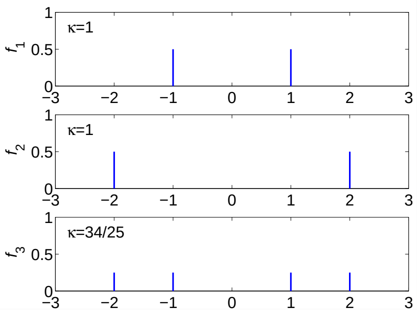
A Set of Distributions with Bounded Kurtosis is Non-Convex

$$\mathrm{Prob}(Y_1 = y) = 0.5, \ \ y = \pm 1$$
$$\mathrm{Prob}(Y_2 = y) = 0.5, \ \ y = \pm 2$$
$$f_3 = \frac{1}{2} f_1 + \frac{1}{2} f_2$$
$$\mathrm{Prob}(Y_3 = y) = 0.25, \ \ y = \pm 1, \pm 2$$
$$\kappa_3 = \frac{34}{25} > 1 = \kappa_1 = \kappa_2$$



Return

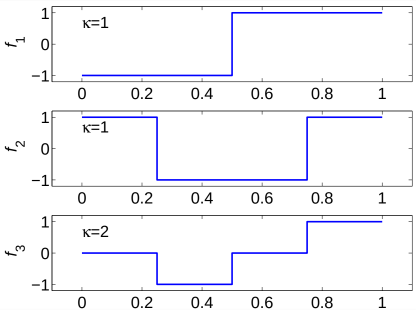
A Set of Integrands with Bounded Kurtosis is Non-Convex

$$f_1(x) = \begin{cases} -1, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1 \end{cases}$$

$$f_2(x) = \begin{cases} 1, & 0 \le x < 1/4 \\ -1, & 1/4 \le x \le 3/4 \\ 1, & 3/4 \le x \le 1 \end{cases}$$

$$\begin{aligned} f_3(x) &= \tfrac{1}{2} f_1(x) + \tfrac{1}{2} f_2(x) \\ &= \begin{cases} 0, & 0 \le x < 1/4, \\ -1, & 1/4 \le x \le 1/2, \\ 0, & 1/2 \le x \le 3/4, \\ 1, & 3/4 \le x \le 1, \end{cases} \end{aligned}$$

$$\kappa_3 = 2 > 1 = \kappa_1 = \kappa_2$$



Return

When Does Quasi-Standard Error Work?

Quasi-standard error has been proposed by Warnock and studied by Owen (1997); Snyder (2000); Halton (2005); Owen (2006). Suppose that $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ is a scrambled digital (t,d)-sequence in base 2, $P_m = \{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{2^m}\}$, and the integrand can be expanded in a Walsh series:

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\pi\sqrt{-1}\boldsymbol{k}\otimes\boldsymbol{x}}, \qquad (S_B f)(\boldsymbol{x}) := \sum_{\boldsymbol{k} \in B} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\pi\sqrt{-1}\boldsymbol{k}\otimes\boldsymbol{x}} \quad \text{(filtered } f\text{)}$$

$$\mu = \int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = S_{\{\boldsymbol{0}\}} f, \qquad \hat{\mu}_m = \frac{1}{2^m} \sum_{i=1}^{2^m} f(\boldsymbol{X}_i) = (S_{P_m^\perp} f)(\boldsymbol{X}_1),$$

where $\otimes$ is a bitwise dot product modulo 2, and $P_m^\perp = \{\boldsymbol{k} \in \mathbb{N}_0^d : \boldsymbol{k} \otimes \boldsymbol{x} = 0 \ \forall \boldsymbol{x} \in P_m\}$ is the dual net (wavenumbers aliased with $\boldsymbol{0}$). The quasi-standard error may be expressed as

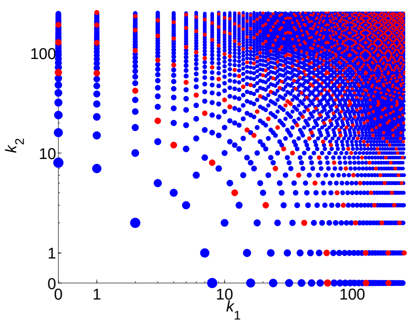
$$\mathrm{qse}_m = \sqrt{\frac{1}{2^r - 1} \sum_{\boldsymbol{l} \in (P_{m-r}^\perp / P_m^\perp) \setminus \{\boldsymbol{0}\}} (S_{P_m^\perp \oplus \boldsymbol{l}} f)^2 (\boldsymbol{X}_1)}$$

which is a surrogate for $|\mu - \hat{\mu}_m| = \left| (S_{P_m^\perp \setminus \{\boldsymbol{0}\}} f)(\boldsymbol{X}_1) \right|$.

When Does Quasi-Standard Error Work?

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\pi\sqrt{-1}\boldsymbol{k}\otimes\boldsymbol{x}}$$

$$|\mu - \hat{\mu}_m| = \left|(S_{P_m^\perp \setminus \{\boldsymbol{0}\}} f)(\boldsymbol{X}_1)\right| \qquad \bullet P_6^\perp \setminus \{\boldsymbol{0}\}$$



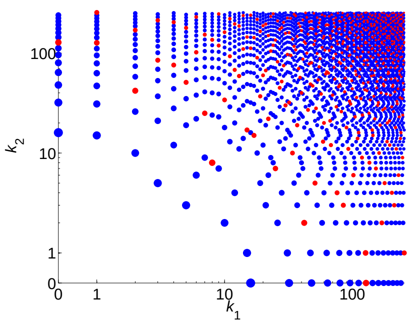
$$\mathrm{qse}_m = \sqrt{\frac{1}{7} \sum_{\boldsymbol{l}\in(P_{m-3}^\perp / P_m^\perp)\setminus\{\boldsymbol{0}\}} (S_{P_m^\perp \oplus \boldsymbol{l}} f)^2(\boldsymbol{X}_1)} \qquad \bullet P_3^\perp \setminus P_6^\perp$$

Return

When Does Quasi-Standard Error Work?

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}(\boldsymbol{k})\mathrm{e}^{\pi\sqrt{-1}\boldsymbol{k}\otimes\boldsymbol{x}}$$

$$|\mu - \hat{\mu}_m| = \left|(S_{P_m^\perp\setminus\{\boldsymbol{0}\}}f)(\boldsymbol{X}_1)\right| \qquad \bullet P_7^\perp \setminus \{\boldsymbol{0}\}$$



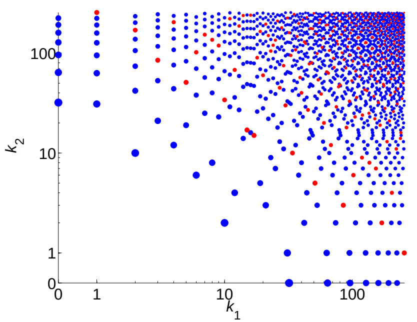
$$\mathrm{qse}_m = \sqrt{\frac{1}{7}\sum_{\boldsymbol{l}\in(P_{m-3}^\perp/P_m^\perp)\setminus\{\boldsymbol{0}\}} (S_{P_m^\perp\oplus\boldsymbol{l}}f)^2(\boldsymbol{X}_1)} \qquad \bullet P_4^\perp \setminus P_7^\perp$$

Return

When Does Quasi-Standard Error Work?

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\pi \sqrt{-1} \boldsymbol{k} \otimes \boldsymbol{x}}$$

$$|\mu - \hat{\mu}_m| = \left| (S_{P_m^\perp \setminus \{\boldsymbol{0}\}} f)(\boldsymbol{X}_1) \right| \qquad \bullet P_8^\perp \setminus \{\boldsymbol{0}\}$$



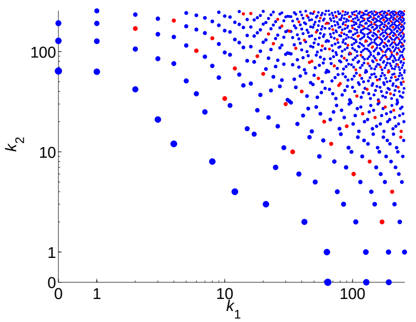
$$\mathrm{qse}_m = \sqrt{\frac{1}{7} \sum_{\boldsymbol{l} \in (P_{m-3}^\perp / P_m^\perp) \setminus \{\boldsymbol{0}\}} (S_{P_m^\perp \oplus \boldsymbol{l}} f)^2 (\boldsymbol{X}_1)} \qquad \bullet P_5^\perp \setminus P_8^\perp$$

Return

When Does Quasi-Standard Error Work?

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\pi\sqrt{-1}\boldsymbol{k}\otimes\boldsymbol{x}}$$

$$|\mu - \hat{\mu}_m| = \left|(S_{P_m^\perp\setminus\{\mathbf{0}\}} f)(\boldsymbol{X}_1)\right| \qquad \bullet P_9^\perp \setminus \{\mathbf{0}\}$$



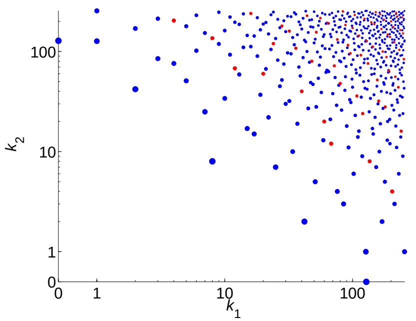
$$\mathrm{qse}_m = \sqrt{\frac{1}{7} \sum_{\boldsymbol{l}\in(P_{m-3}^\perp/P_m^\perp)\setminus\{\mathbf{0}\}} (S_{P_m^\perp\oplus\boldsymbol{l}} f)^2(\boldsymbol{X}_1)} \qquad \bullet P_6^\perp \setminus P_9^\perp$$

Return

When Does Quasi-Standard Error Work?

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^d} \hat{f}(\boldsymbol{k}) \mathrm{e}^{\pi\sqrt{-1}\boldsymbol{k}\otimes\boldsymbol{x}}$$

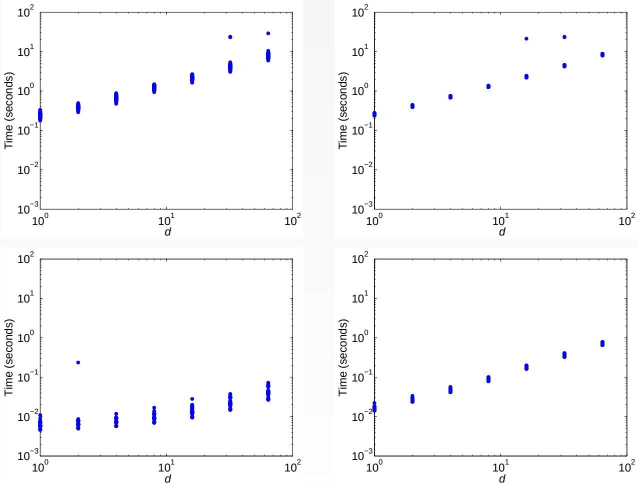
$$|\mu - \hat{\mu}_m| = \left|(S_{P_m^\perp\setminus\{\mathbf{0}\}} f)(\boldsymbol{X}_1)\right| \qquad \bullet P_{10}^\perp \setminus \{\mathbf{0}\}$$



$$\mathrm{qse}_m = \sqrt{\frac{1}{7} \sum_{\boldsymbol{l}\in(P_{m-3}^\perp/P_m^\perp)\setminus\{\mathbf{0}\}} (S_{P_m^\perp\oplus\boldsymbol{l}} f)^2(\boldsymbol{X}_1)} \qquad \bullet P_7^\perp \setminus P_{10}^\perp$$

Return

Asian Geometric Mean Call Execution Times



{ i.i.d. sampling

{ Sobol' sampling

$n_\sigma = 1024,\ \kappa_{\max} = 9.2$

$n_\sigma = 131072,\ \kappa_{\max} = 1050$

Return