The algorithms used in this section on integration and the next section on function recovery are all based on quadratic splines on $[0,1]$. The node set and the quadratic spline algorithm using n function values are defined for $n \in \mathcal{I} := \{3,5,7,\ldots\}$ as follows:

$$x_i = \frac{i-1}{n-1}, \qquad i = 1,\ldots,n, \tag{1a}$$

$$\begin{aligned} A_n(f)(x) := \frac{(n-1)^2}{2} & [f(x_i)(x-x_{i+1})(x-x_{i+2}) \\ & -2f(x_{i+1})(x-x_i)(x-x_{i+2}) + f(x_{i+2})(x-x_i)(x-x_{i+1})] \\ & \text{for } x_i \le x \le x_{i+2}. \end{aligned} \tag{1b}$$

The cost of each function value is one and so the cost of A_n is n. The algorithm A_n is imbedded in the algorithm A_{3n-2}, which uses $3n-3$ subintervals. Thus, $r = 3$ is the cost multiple.

The problem to be solved is univariate integration on the unit interval, $S(f) := \mathrm{INT}(f) := \int_0^1 f(x)\,\mathrm{d}x \in \mathcal{G} := \mathbb{R}$. The fixed cost building blocks to construct the adaptive integration algorithm are the composite Simpson's rules based on $n-1$ intervals:

$$\begin{aligned} P_n(f) &:= \int_0^1 A_n(f)\,\mathrm{d}x \\ &= \frac{1}{3n-3}[f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+2f(x_5)\cdots+4f(x_{n-1})+f(x_n)]. \end{aligned} \tag{2}$$

The space of input functions is $\mathcal{F} := \mathcal{V}^3$, the space of functions whose third derivatives have finite variation. The general definitions of some relevant norms and spaces are as follows:

$$\mathrm{Var}(f) := \sup_{\substack{n\in\mathbb{N} \\ 0=x_0<x_1<\cdots<x_n=1}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \tag{3a}$$

$$\|f\|_p := \begin{cases} \left[\displaystyle\int_0^1 |f(x)|^p \,\mathrm{d}x\right]^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{0\le x\le 1} |f(x)|, & p = \infty, \end{cases} \tag{3b}$$

$$\mathcal{V}^k := \mathcal{V}^k[0,1] = \{f \in C[0,1] : \mathrm{Var}(f^{(k)}) < \infty\}, \tag{3c}$$

$$\mathcal{W}^{k,p} = \mathcal{W}^{k,p}[0,1] = \{f \in C[0,1] : \|f^{(k)}\|_p < \infty\}. \tag{3d}$$

The stronger semi-norm is $|f|_{\mathcal{F}} := \mathrm{Var}(f''')$, while the weaker semi-norm is

$|f|_{\widetilde{\mathcal{F}}} := \|f'' - A_3(f)''\|_1 = \|f'' - 4[f(1) - 2f(1/2) + f(0)]\|_1 = \mathrm{Var}(f' - A_3(f)'),$

where $A_3(f) : x \mapsto 2f(0)(x-1/2)(x-1) - 4f(1/2)x(x-1) + 2f(1)x(x-1/2)$ is the quadratic interpolant of f using the middle point and two endpoints of the integration interval. The reason for defining $|f|_{\widetilde{\mathcal{F}}}$ this way is that $|f|_{\widetilde{\mathcal{F}}}$ vanishes if f is a quadratic function, and quadratic functions are integrated exactly by the Simpson's rule. The cone of integrands is defined as

$$\mathcal{C}_\tau := \left\{ f \in \mathcal{V}^3 : \begin{cases} \operatorname{Var}(f''') \le \tau \operatorname{Var}(f'') \\ \operatorname{Var}(f'') \le \tau \|f'' - 4[f(1) - 2f(1/2) + f(0)]\|_1 \end{cases} \right\}. \tag{4}$$

The algorithm for approximating $\|f'' - 4[f(1) - 2f(1/2) + f(0)]\|_1$ is the $\widetilde{\mathcal{F}}$-semi-norm of the linear spline, $A_n(f)$:

$$\begin{aligned}
\widetilde{F}_n(f) &:= |A_n(f)|_{\widetilde{\mathcal{F}}} = \left\|A_n(f)'' - A_3(f)''\right\|_1 \\
&= \sum_{j=1}^{(n-1)/2} \int_{x_{2j-1}}^{x_{2j+1}} |A_n(f)'' - A_3(f)''| \,\mathrm{d}x, \\
&= \sum_{j=1}^{(n-1)/2} \int_{x_{2j-1}}^{x_{2j+1}} \left|(n-1)^2(f(x_{2j+1}) - 2f(x_{2j}) + f(x_{2j-1})) - 4[f(1) - 2f(1/2) + f(0)]\right| \,\mathrm{d}x, \\
&= \sum_{j=1}^{(n-1)/2} \left|(n-1)^2(f(x_{2j+1}) - 2f(x_{2j}) + f(x_{2j-1})) - 4[f(1) - 2f(1/2) + f(0)]\right| (x_{2j+1} - x_{2j-1}), \\
&= \sum_{j=1}^{(n-1)/2} \left|(n-1)^2(f(x_{2j+1}) - 2f(x_{2j}) + f(x_{2j-1})) - 4[f(1) - 2f(1/2) + f(0)]\right| \frac{2}{n-1}, \\
&= \sum_{j=1}^{(n-1)/2} \left|2(n-1)(f(x_{2j+1}) - 2f(x_{2j}) + f(x_{2j-1})) - \frac{8}{n-1}[f(1) - 2f(1/2) + f(0)]\right|.
\end{aligned} \tag{5}$$

The variation of the second derivative of the linear spline of f, i.e.,

$$F_n(f) := \operatorname{Var}(A_n(f)') = (n-1)^2 \sum_{j=1}^{(n-3)/2} \left|f(x_{2j+3}) - 2f(x_{2j+2}) + 2f(x_{2j}) - f(x_{2j-1})\right|, \tag{6}$$

provides a lower bound on $\operatorname{Var}(f')$ for $n \ge 5$.

Constructing the adaptive algorithm for integration requires an upper bound on the error of P_n and a two-sided bound on the error of $\widetilde{F}_n$. Note that $\widetilde{F}_n(f)$ never overestimates $|f|_{\widetilde{\mathcal{F}}}$ because

$$\begin{aligned}
|f|_{\widetilde{\mathcal{F}}} = \left\|f'' - A_3(f)''\right\|_1 &= \sum_{j=1}^{(n-1)/2} \int_{x_{2j-1}}^{x_{2j+1}} |f''(x) - A_3(f)''(x)| \,\mathrm{d}x \\
&\ge \sum_{j=1}^{(n-1)/2} \left| \int_{x_{2j-1}}^{x_{2j+1}} [f''(x) - A_3(f)''(x)] \,\mathrm{d}x \right| = \|A_n(f)'' - A_3(f)''\|_1 = \widetilde{F}_n(f).
\end{aligned}$$

Thus, $h_-(n) := 0$ and $\mathfrak{c}_n = \tilde{\mathfrak{c}}_n = 1$.

To find an upper bound on $|f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_n(f)$, note that

$$|f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_n(f) = |f|_{\widetilde{\mathcal{F}}} - \left|A_n(f)\right|_{\widetilde{\mathcal{F}}} \le \left|f - A_n(f)\right|_{\widetilde{\mathcal{F}}} = \left\|f'' - A_n(f)''\right\|_1,$$

since $(f - A_n(f))(x)$ vanishes for $x = 0, 1/2, 1$. Moreover,

$$\left\|f'' - A_n(f)''\right\|_1 = \sum_{j=1}^{(n-1)/2} \int_{x_{2j-1}}^{x_{2j+1}} \left|f''(x) - (n-1)^2[f(x_{2j+1}) - 2f(x_{2j}) + f(x_{2j-1})]\right| \, \mathrm{d}x. \tag{7}$$

Now we bound each integral in the summation. For $j = 1, \ldots, (n-1)/2$, let $\eta_j(x) = f''(x) - (n-1)^2[f(x_{2j+1}) - 2f(x_{2j}) + f(x_{2j-1})]$, and let p_j denote the probability that $\eta_j(x)$ is non-negative:

$$p_j = \frac{(n-1)}{2} \int_{x_{2j-1}}^{x_{2j+1}} \mathbb{1}_{[0,\infty)}(\eta_j(x)) \, \mathrm{d}x,$$

and so $1 - p_j$ is the probability that $\eta_j(x)$ is negative. Since $\int_{x_{2j-1}}^{x_{2j+1}} \eta_j(x) \, \mathrm{d}x = 0$, we know that η_j must take on both non-positive and non-negative values. Invoking the Mean Value Theorem, it follows that

$$\begin{aligned}
\frac{2p_j}{n-1} \sup_{x_{2j-1} \le x \le x_{2j+1}} \eta_j(x) &\ge \int_{x_{2j-1}}^{x_{2j+1}} \max(\eta_j(x), 0) \, \mathrm{d}x \\
&= \int_{x_{2j-1}}^{x_{2j+1}} \max(-\eta_j(x), 0) \, \mathrm{d}x \le \frac{-2(1-p_j)}{n-1} \inf_{x_{2j-1} \le x \le x_{2j+1}} \eta_j(x).
\end{aligned}$$

These bounds allow us to derive bounds on the integrals in (7):

$$\begin{aligned}
&\int_{x_{2j-1}}^{x_{2j+1}} |\eta_j(x)| \, \mathrm{d}x \\
&\quad = \int_{x_{2j-1}}^{x_{2j+1}} \max(\eta_j(x), 0) \, \mathrm{d}x + \int_{x_{2j-1}}^{x_{2j+1}} \max(-\eta_j(x), 0) \, \mathrm{d}x \\
&\quad = 4(1-p_j) \int_{x_{2j-1}}^{x_{2j+1}} \max(\eta_j(x), 0) \, \mathrm{d}x + 4p_i \int_{x_{2j-1}}^{x_{2j+1}} \max(-\eta_j(x), 0) \, \mathrm{d}x \\
&\quad \le \frac{4p_i(1-p_i)}{n-1} \left[\sup_{x_{2j-1} \le x \le x_{2j+1}} \eta_j(x) - \inf_{x_{2j-1} \le x \le x_{2j+1}} \eta_j(x) \right] \\
&\quad \le \frac{1}{(n-1)} \left[\sup_{x_{2j-1} \le x \le x_{2j+1}} f''(x) - \inf_{x_{2j-1} \le x \le x_{2j+1}} f''(x) \right],
\end{aligned}$$

since $p_i(1 - p_i) \le 1/4$.

Plugging this bound into (7) yields

$$\begin{aligned}
\left\|f'' - 4[f(1) - 2f(1/2) + f(0)]\right\|_1 - \widetilde{F}_n(f) &= |f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_n(f) \\
&\le \left\|f'' - A_n(f)''\right\|_1 \\
&\le \frac{1}{n-1} \sum_{j=1}^{(n-1)/2} \left[\sup_{x_{2j-1} \le x \le x_{2j+1}} f''(x) - \inf_{x_{2j-1} \le x \le x_{2j+1}} f''(x) \right] \\
&\le \frac{\mathrm{Var}(f'')}{n-1} = \frac{|f|_{\mathcal{F}}}{n-1},
\end{aligned}$$

and so

$$h_+(n) := \frac{1}{n-1}, \qquad \mathfrak{C}_n = \frac{1}{1 - \tau/(n-1)} \qquad \text{for } n > 1 + \tau.$$

Since $\widetilde{F}_3(f) = 0$ by definition, the above inequality for $|f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_3(f)$ implies that

$$4\left\|f' - f(1) + f(0)\right\|_1 = 4\,|f|_{\widetilde{\mathcal{F}}} \le |f|_{\mathcal{F}} = \mathrm{Var}(f'), \qquad \tau_{\min} = 4.???$$

The error of the Simpson's rule in terms of the variation of the first derivative of the integrand is:

$$\left| \int_0^1 f(x)\,dx - P_n(f) \right| \le h(n)\,\mathrm{Var}(f''')$$

$$h(n) := \frac{1}{1152(n-1)^4}, \qquad h^{-1}(\varepsilon) = \left\lceil \left(\frac{1}{1152\varepsilon} \right)^{1/4} \right\rceil + 1.$$

Given the above definitions of $h, \mathfrak{C}_n, \mathfrak{c}_n$, and $\tilde{\mathfrak{c}}_n$, it is now possible to also specify

$$h_1(n) = h_2(n) = \mathfrak{C}_n h(n) = \frac{1}{1152(n-1)^3(n-1-\tau)}, \tag{8a}$$

$$h_1^{-1}(\varepsilon) = h_2^{-1}(\varepsilon) = 1 + \left\lceil \sqrt{\frac{\tau}{8\varepsilon} + \frac{\tau^2}{16}} + \frac{\tau}{4} \right\rceil \le 2 + \frac{\tau}{2} + \sqrt{\frac{\tau}{8\varepsilon}}.??? \tag{8b}$$

Moreover, the left side of the stopping criterion inequality in the multi-stage algorithm, becomes

$$\tau h(n_i) \mathfrak{C}_{n_i} \widetilde{F}_{n_i}(f) = \frac{\tau^2 \widetilde{F}_{n_i}(f)}{1152(n_i-1)^3(2n_i-2-\tau)}. \tag{8c}$$

With these preliminaries, Algorithm **??** and Theorem **??** may be applied directly to yield the following adaptive integration algorithm and its guarantee.

Algorithm 1 (Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{P_n\}_{n\in\mathcal{I}}$, $\{\widetilde{F}_n\}_{n\in\mathcal{I}}$, and $\{F_n\}_{n\in\mathcal{I}}$ be as described above. Let $\tau \ge 4$ be the cone constant. Set $i = 1$. Let $n_1 = 2\lceil \tau/2 \rceil + 1$. For any error tolerance ε and input function f, do the following:

Stage 1. Estimate $\|f'' - 4[f(1) - 2f(1/2) + f(0)]\|_1$ and bound $\mathrm{Var}(f''')$. Compute $\widetilde{F}_{n_i}(f)$ in (5) and $F_{n_i}(f)$ in (6).

Stage 2. Check the necessary condition for $f \in \mathcal{C}_\tau$. Compute

$$\tau_{\min,n_i} = \frac{F_{n_i}(f)}{\widetilde{F}_{n_i}(f) + F_{n_i}(f)/(4n_i - 4)}.$$

If $\tau \geq \tau_{\min,n_i}$, then go to stage 3. Otherwise, set $\tau = 2\tau_{\min,n_i}$. If $n_i \geq (\tau+1)/2$, then go to stage 3. Otherwise, choose

$$n_{i+1} = 1 + (n_i - 1)\left\lceil \frac{\tau+1}{n_i - 1} \right\rceil.$$

Go to Stage 1.

Stage 3. Check for convergence. Check whether n_i is large enough to satisfy the error tolerance, i.e.

$$\widetilde{F}_{n_i}(f) \leq \frac{1152\varepsilon(n_i - 1)^3(2n_i - 2 - \tau)}{\tau}.$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm. If this is not true, choose

$$n_{i+1} = 1 + (n_i - 1)\max\left\{2, \left\lceil \frac{1}{(n_i - 1)}\sqrt{\frac{\tau \widetilde{F}_{n_i}(f)}{8\varepsilon}} \right\rceil \right\}.???$$

Go to Stage 1.