A Better $\mathcal{G}$-Semi-Norm for Algorithms Based on Linear Splines

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Abstract

Keywords:

1. Why Do We Need a Better $\mathcal{G}$-Semi-Norm

In [1] we presented guaranteed automatic algorithms for integration and function recovery. One weakness of these algorithms was that if $\|f'\|_1$ ($\|f'\|_\infty$) is large, but $\|f''\|_1$ ($\|f''\|_\infty$) is small, then the error bounds in the automatic algorithms for integration (function recovery) can be very conservative. The input function $f : x \mapsto 100x$ is a prime example.

Here we present a family of substitutes for $\|f'\|_p$, $p = 1, \infty$ that overcome this weakness. If the function is piecewise linear with a small number of discontinuities in f', then the algorithms in [1] converge quite quickly.

2. A Proposed New $\mathcal{G}$-Semi-Norm

Define the integral functional on the interval $[\alpha, \beta]$ as

$$I(f; \alpha, \beta) = \int_\alpha^\beta f(x)\,\mathrm{d}x,$$

and define the $\mathcal{L}_p$-norms on this as follows:

$$\|f\|_{p,[\alpha,\beta]} = \begin{cases} \left| \int_\alpha^\beta |f(x)|^p\,\mathrm{d}x \right|^{1/p}, & 1 \le p < \infty, \\ \sup_{a \le x \le b} |f(x)|, & p = \infty. \end{cases}$$

The norm over the larger interval can be written in terms of the norms of subintervals. Namely, for any α_j satisfying $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_r = \beta$, it follows that

$$\|f\|_{p,[\alpha,\beta]} \le \left\| \left(\|f\|_{p,[\alpha_{k-1},\alpha_k]} \right)_{k=1}^r \right\|_{\ell_p}. \tag{1}$$

The reason for possible lack of equality is that there might be jumps at the endpoints of the subintervals.

For any $m \in \mathbb{N}$, define the following equally spaced points:

$$x_{i,m} = \frac{i}{m}, \qquad i = 0, \ldots m.$$

The following semi-norm compares measures the deviation of the first derivative of the function from its average value:

$$|f|_{\mathcal{G},m,p} = \left\| \left(\left\| \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{p,[x_{i-1,m},x_{i,m}]} \right\|_{p,[x_{i-1,m},x_{i,m}]} \right)_{i=1}^{m} \right\|_{\ell_p}. \tag{2}$$

Let us verify that this indeed a semi-norm. Clearly, it is non-negative and vanishes when f is the zero function. So, the only other thing to verify is the triangle inequality, which follows since $f \mapsto f' - mI(f'; x_{i-1,m}, x_{i,m})$ is a linear operator.

It can be shown that $|f|_{\mathcal{G},rm,2} \leq |f|_{\mathcal{G},m,2}$ for positive integer r. First, note that the one may re-write the norm over $[x_{i-1,m}, x_{i,m}]$ in terms of norms over $[x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]$ by (1):

$$\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{i-1,m},x_{i,m}]}$$
$$= \left\| \left(\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{r(i-1)+k-1,rm},x_{r(i-1)+k,rm}]} \right)_{k=1}^{r} \right\|_{\ell_2}. \tag{3}$$

Moreover, the norms inside the expression above are no smaller than the norms comprising $|f|_{\mathcal{G},rm,2}$:

$$\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{r(i-1)+k-1,rm},x_{r(i-1)+k,rm}]}^{2}$$
$$= \int_{x_{r(i-1)+k-1}}^{x_{r(i-1)+k,rm}} [f'(x) - mI(f'; x_{i-1,m}, x_{i,m})]^2 \, dx$$
$$= \int_{x_{r(i-1)+k-1}}^{x_{r(i-1)+k,rm}} \Big\{ [f'(x) - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})]^2$$
$$+ 2m[f'(x) - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})]$$
$$\times [rI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]$$
$$+ m^2[rI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]^2 \Big\} \, dx$$
$$= \|f' - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})\|_{2,[x_{r(i-1)+k-1},x_{r(i-1)+k,rm}]}^{2}$$
$$+ \frac{1}{r^2}[rI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]^2$$
$$\geq \|f' - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})\|_{2,[x_{r(i-1)+k-1},x_{r(i-1)+k,rm}]}^{2}$$

Substituting this inequality into (3) yields

$$\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{i-1,m},x_{i,m}]}$$
$$\geq \left\| \left(\|f' - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})\|_{2,[x_{r(i-1)+k-1},x_{r(i-1)+k,rm}]} \right)_{k=1}^{r} \right\|_{\ell_2}.$$

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According to the definition of $|f|_{\mathcal{G},m,2}$ in (2), it then follows that $|f|_{\mathcal{G},rm,2} \leq |f|_{\mathcal{G},m,2}$.

3. Estimating the New $\mathcal{G}$-Semi-Norm

Let $\tilde{f}_n$ denote the linear spline based on the function values $f(x_{i,n-1})$, $i = 0, \ldots, n-1$. We estimate $|f|_{\mathcal{G},(n-1)/2,p}$ by the algorithm $G_{(n-1)/2,p,n} : f \mapsto |\tilde{f}_n|_{\mathcal{G},(n-1)/2,p}$. Then it follows that the error of our approximation to the $\mathcal{G}$-semi-norm is

$$|f|_{\mathcal{G},(n-1)/2,p} - G_{(n-1)/2,p,n}(f) = |f|_{\mathcal{G},(n-1)/2,p} - |\tilde{f}_n|_{\mathcal{G},(n-1)/2,p}$$
$$\leq |f - \tilde{f}_n|_{\mathcal{G},(n-1)/2,p},$$

For any $i = 1, \ldots, n-2$, the idea is to approximate The idea is to approximate

$$\int_{x_{i-1}}^{x_{i+1}} \left| f'(x) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right|^p \, \mathrm{d}x$$

by piecewise linear approximation to f, which implies a piecewise constant approximation to f'.

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4. Bounding the Error in Terms of the New $\mathcal{G}$-Semi-Norm

4.1. Integration

For the integration problem, we use the composite trapezoidal rule,

$$A_n(f) = \frac{1}{2(n-1)} \left[f(x_{0,n-1}) + 2f(x_{1,n-1}) + 2f(x_{2,n-1}) + \cdots \right.$$
$$\left. + 2f(x_{n-2,n-1}) + f(x_{n-1,n-1}) \right].$$

We consider how to bound the error of this approximation for integrands with given norm $|f|_{\mathcal{G},(n-1)/2,p}$.

For simplicity of notation let $x_i = x_{i,n-1}$. For a fixed $i = 1, \ldots, n-2$, consider the error in integrating over the interval $[x_{i-1}, x_{i+1}]$ using two trapezoids. The derivation proceeds by applying applying basic calculus ideas. First we express the trapezoidal rule approximation as the integral of a piecewise linear function, and then use integration by parts to write the integral in terms of the

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first derivative of the integrand:

$$\left| \int_{x_{i-1}}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right|$$

$$= \left| \int_{x_{i-1}}^{x_i} \{f(x) - (n-1)[f(x_{i-1})(x_i - x) + f(x_i)(x - x_{i-1})]\}\,\mathrm{d}x \right.$$

$$\left. + \int_{x_i}^{x_{i+1}} \{f(x) - (n-1)[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i)]\}\,\mathrm{d}x \right|$$

$$= (n-1)\left| \int_{x_{i-1}}^{x_i} \left\{ (x_i - x)\int_{x_{i-1}}^{x} f'(t)\,\mathrm{d}t - (x - x_{i-1})\int_{x}^{x_i} f'(t)\,\mathrm{d}t \right\}\,\mathrm{d}x \right.$$

$$\left. + \int_{x_i}^{x_{i+1}} \left\{ (x_{i+1} - x)\int_{x_i}^{x} f'(t)\,\mathrm{d}t - (x - x_i)\int_{x}^{x_{i+1}} f'(t)\,\mathrm{d}t \right\}\,\mathrm{d}x \right|.$$

Next the order of integration is switched and the error is expressed as a univariate integral of f' multiplied by a weight:

$$\left| \int_{x_{i-1}}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right|$$

$$= (n-1)\left| \int_{x_{i-1}}^{x_i} f'(t)\left\{ \int_{t}^{x_i} (x_i - x)\,\mathrm{d}x - \int_{x_{i-1}}^{t} (x - x_{i-1})\,\mathrm{d}x \right\}\,\mathrm{d}t \right.$$

$$\left. + \int_{x_i}^{x_{i+1}} f'(t)\left\{ \int_{t}^{x_{i+1}} (x_{i+1} - x)\,\mathrm{d}x - \int_{x_i}^{t} (x - x_i)\,\mathrm{d}x \right\}\,\mathrm{d}t \right|$$

$$= (n-1)\left| \int_{x_{i-1}}^{x_i} f'(t)\left\{ \frac{(t - x_i)^2 - (t - x_{i-1})^2}{2} \right\}\,\mathrm{d}t \right.$$

$$\left. + \int_{x_i}^{x_{i+1}} f'(t)\left\{ \frac{(t - x_{i+1})^2 - (t - x_i)^2}{2} \right\}\,\mathrm{d}t \right|$$

$$= \left| \int_{x_{i-1}}^{x_i} f'(t)\left(\frac{x_{i-1} + x_i - 2t}{2} \right)\,\mathrm{d}t + \int_{x_i}^{x_{i+1}} f'(t)\left(\frac{x_i + x_{i+1} - 2t}{2} \right)\,\mathrm{d}t \right|.$$

Since the weight is odd with respect to the center of the integration interval, one may subtract a constant multiple of this weight without altering the integral,

and then apply Hölder's inequality with $p^{-1} + q^{-1} = 1$.

$$\left| \int_{x_{i-1}}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right|$$

$$= \left| \int_{x_{i-1}}^{x_i} \left[f'(t) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right] \left(\frac{x_{i-1} + x_i - 2t}{2} \right) \mathrm{d}t \right.$$

$$\left. + \int_{x_i}^{x_{i+1}} \left[f'(t) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right] \left(\frac{x_i + x_{i+1} - 2t}{2} \right) \mathrm{d}t \right|$$

$$= \left\| f' - \frac{n-1}{2} I(f', x_{i-1}, x_{i+1}) \right\|_{p,[x_{i-1},x_{i+1}]}$$

$$\times \left\{ \int_{x_{i-1}}^{x_i} \left| \frac{x_{i-1} + x_i - 2t}{2} \right|^q \mathrm{d}t + \int_{x_i}^{x_{i+1}} \left| \frac{x_i + x_{i+1} - 2t}{2} \right|^q \mathrm{d}t \right\}^{1/q}$$

$$= \left[\frac{4}{(q+1)(2n-2)^{q+1}} \right]^{1/q} \left\| f' - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right\|_{p,[x_{i-1},x_{i+1}]}.$$

Finally, these error bounds over the subintervals $[x_{i-1}, x_{i+1}]$ are combined to obtain an error bound for the trapezoidal rule, again using Hölder's inequality:

$$\left| \int_0^1 f(x)\,\mathrm{d}x - A_n(f) \right|$$

$$\leq \sum_{i=1}^{(n-1)/2} \left| \int_{x_{2i}}^{x_{2i+2}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{2i}) + 2f(x_{2i+1}) + f(x_{2i+2})] \right|$$

$$\leq \sum_{i=1}^{(n-1)/2} \left[\frac{4}{(q+1)(2n-2)^{q+1}} \right]^{1/q} \left\| f' - \frac{n-1}{2} I(f'; x_{2i}, x_{2i+2}) \right\|_{p,[x_{2i},x_{2i+2}]}$$

$$\leq \left(\frac{1}{q+1} \right)^{1/q} \frac{|f|_{\mathcal{G},(n-1)/2,p}}{2n-2}.$$

[1] N. Clancy, Y. Ding, C. Hamilton, F. J. Hickernell, Y. Zhang, The complexity of guaranteed automatic algorithms: Cones, not balls, submitted for publication, arXiv.org:1303.2412 [math.NA] (2013).