

# A Better $\mathcal{G}$ -Semi-Norm for Algorithms Based on Linear Splines

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## Abstract

*Keywords:*

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## 1. Why Do We Need a Better $\mathcal{G}$ -Semi-Norm

In [1] we presented guaranteed automatic algorithms for integration and function recovery. One weakness of these algorithms was that if  $\|f'\|_1$  ( $\|f'\|_\infty$ ) is large, but  $\|f''\|_1$  ( $\|f''\|_\infty$ ) is small, then the error bounds in the automatic algorithms for integration (function recovery) can be very conservative. The input function  $f : x \mapsto 100x$  is a prime example.

Here we present a family of substitutes for  $\|f'\|_p$ ,  $p = 1, \infty$  that overcome this weakness. If the function is piecewise linear with a small number of discontinuities in  $f'$ , then the algorithms in [1] converge quite quickly.

## 2. A Proposed New $\mathcal{G}$ -Semi-Norm

Define the integral functional on the interval  $[\alpha, \beta]$  as

$$I(f; \alpha, \beta) = \int_{\alpha}^{\beta} f(x) \, dx,$$

and define the  $\mathcal{L}_p$ -norms on this as follows:

$$\|f\|_{p, [\alpha, \beta]} = \begin{cases} \left| \int_{\alpha}^{\beta} |f(x)|^p \, dx \right|^{1/p}, & 1 \leq p < \infty, \\ \sup_{a \leq x \leq b} |f(x)|, & p = \infty. \end{cases}$$

The norm over the larger interval can be written in terms of the norms of subintervals. Namely, for any  $\alpha_j$  satisfying  $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_r = \beta$ , it follows that

$$\|f\|_{p, [\alpha, \beta]} \leq \left\| \left( \|f\|_{p, [\alpha_{k-1}, \alpha_k]} \right)_{k=1}^r \right\|_{\ell_p}. \quad (1)$$

The reason for possible lack of equality is that there might be jumps at the endpoints of the subintervals.

For any  $m \in \mathbb{N}$ , define the following equally spaced points:

$$x_{i,m} = \frac{i}{m}, \quad i = 0, \dots, m.$$

The following semi-norm compares measures the deviation of the first derivative of the function from its average value:

$$|f|_{\mathcal{G},m,p} = \left\| \left( \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{p, [x_{i-1,m}, x_{i,m}]} \right)_{i=1}^m \right\|_{\ell_p}. \quad (2)$$

Let us verify that this indeed a semi-norm. Clearly, it is non-negative and vanishes when  $f$  is the zero function. So, the only other thing to verify is the triangle inequality, which follows since  $f \mapsto f' - mI(f'; x_{i-1,m}, x_{i,m})$  is a linear operator.

It can be shown that  $|f|_{\mathcal{G},rm,2} \leq |f|_{\mathcal{G},m,2}$  for positive integer  $r$ . First, note that the one may re-write the norm over  $[x_{i-1,m}, x_{i,m}]$  in terms of norms over  $[x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]$  by (1):

$$\begin{aligned} & \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2, [x_{i-1,m}, x_{i,m}]} \\ &= \left\| \left( \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2, [x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]} \right)_{k=1}^r \right\|_{\ell_2}. \quad (3) \end{aligned}$$

Moreover, the norms inside the expression above are no smaller than the norms comprising  $|f|_{\mathcal{G},rm,2}$  :

$$\begin{aligned} & \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2, [x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]}^2 \\ &= \int_{x_{r(i-1)+k-1,rm}}^{x_{r(i-1)+k,rm}} [f'(x) - mI(f'; x_{i-1,m}, x_{i,m})]^2 dx \\ &= \int_{x_{r(i-1)+k-1,rm}}^{x_{r(i-1)+k,rm}} \{ [f'(x) - rmI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm})] \\ &\quad + 2m[f'(x) - rmI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm})] \\ &\quad \times [rI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})] \\ &\quad + m^2[rI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]^2 \} dx \\ &= \|f' - rmI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm})\|_{2, [x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]}^2 \\ &\quad + \frac{1}{r^2} [rI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]^2 \\ &\geq \|f' - rmI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm})\|_{2, [x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]}^2 \end{aligned}$$

Substituting this inequality into (3) yields

$$\begin{aligned} & \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2, [x_{i-1,m}, x_{i,m}]} \\ &\geq \left\| \left( \|f' - rmI(f'; x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm})\|_{2, [x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]} \right)_{k=1}^r \right\|_{\ell_2}. \end{aligned}$$

According to the definition of  $|f|_{\mathcal{G},m,2}$  in (2), it then follows that  $|f|_{\mathcal{G},rm,2} \leq |f|_{\mathcal{G},m,2}$ .

### 3. Estimating the New $\mathcal{G}$ -Semi-Norm

Let  $\tilde{f}_n$  denote the linear spline based on the function values  $f(x_{i,n-1})$ ,  $i = 0, \dots, n-1$ . We estimate  $|f|_{\mathcal{G},(n-1)/2,p}$  by the algorithm  $G_{(n-1)/2,p,n} : f \mapsto |\tilde{f}_n|_{\mathcal{G},(n-1)/2,p}$ . Then it follows that the error of our approximation to the  $\mathcal{G}$ -semi-norm is

$$\begin{aligned} |f|_{\mathcal{G},(n-1)/2,p} - G_{(n-1)/2,p,n}(f) &= |f|_{\mathcal{G},(n-1)/2,p} - |\tilde{f}_n|_{\mathcal{G},(n-1)/2,p} \\ &\leq |f - \tilde{f}_n|_{\mathcal{G},(n-1)/2,p}, \end{aligned}$$

For any  $i = 1, \dots, n-2$ , the idea is to approximate The idea is to approximate

$$\int_{x_{i-1}}^{x_{i+1}} \left| f'(x) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right|^p dx$$

by piecewise linear approximation to  $f$ , which implies a piecewise constant approximation to  $f'$ .

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### 4. Bounding the Error in Terms of the New $\mathcal{G}$ -Semi-Norm

#### 4.1. Integration

For the integration problem, we use the composite trapezoidal rule,

$$\begin{aligned} A_n(f) &= \frac{1}{2(n-1)} [f(x_{0,n-1}) + 2f(x_{1,n-1}) + 2f(x_{2,n-1}) + \dots \\ &\quad + 2f(x_{n-2,n-1}) + f(x_{n-1,n-1})]. \end{aligned}$$

We consider how to bound the error of this approximation for integrands with given norm  $|f|_{\mathcal{G},(n-1)/2,p}$ .

For simplicity of notation let  $x_i = x_{i,n-1}$ . For a fixed  $i = 1, \dots, n-2$ , consider the error in integrating over the interval  $[x_{i-1}, x_{i+1}]$  using two trapezoids. The derivation proceeds by applying applying basic calculus ideas. First we express the trapezoidal rule approximation as the integral of a piecewise linear function, and then use integration by parts to write the integral in terms of the

first derivative of the integrand:

$$\begin{aligned}
& \left| \int_{x_{i-1}}^{x_{i+1}} f(x) \, dx - \frac{1}{2(n-1)} [f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right| \\
&= \left| \int_{x_{i-1}}^{x_i} \{f(x) - (n-1)[f(x_{i-1})(x_i - x) + f(x_i)(x - x_{i-1})]\} \, dx \right. \\
&\quad \left. + \int_{x_i}^{x_{i+1}} \{f(x) - (n-1)[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i)]\} \, dx \right| \\
&= (n-1) \left| \int_{x_{i-1}}^{x_i} \left\{ (x_i - x) \int_{x_{i-1}}^x f'(t) \, dt - (x - x_{i-1}) \int_x^{x_i} f'(t) \, dt \right\} \, dx \right. \\
&\quad \left. + \int_{x_i}^{x_{i+1}} \left\{ (x_{i+1} - x) \int_{x_i}^x f'(t) \, dt - (x - x_i) \int_x^{x_{i+1}} f'(t) \, dt \right\} \, dx \right|.
\end{aligned}$$

Next the order of integration is switched and the error is expressed as a univariate integral of  $f'$  multiplied by a weight:

$$\begin{aligned}
& \left| \int_{x_{i-1}}^{x_{i+1}} f(x) \, dx - \frac{1}{2(n-1)} [f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right| \\
&= (n-1) \left| \int_{x_{i-1}}^{x_i} f'(t) \left\{ \int_t^{x_i} (x_i - x) \, dx - \int_{x_{i-1}}^t (x - x_{i-1}) \, dx \right\} \, dt \right. \\
&\quad \left. + \int_{x_i}^{x_{i+1}} f'(t) \left\{ \int_t^{x_{i+1}} (x_{i+1} - x) \, dx - \int_{x_i}^t (x - x_i) \, dx \right\} \, dt \right| \\
&= (n-1) \left| \int_{x_{i-1}}^{x_i} f'(t) \left\{ \frac{(t - x_i)^2 - (t - x_{i-1})^2}{2} \right\} \, dt \right. \\
&\quad \left. + \int_{x_i}^{x_{i+1}} f'(t) \left\{ \frac{(t - x_{i+1})^2 - (t - x_i)^2}{2} \right\} \, dt \right| \\
&= \left| \int_{x_{i-1}}^{x_i} f'(t) \left( \frac{x_{i-1} + x_i - 2t}{2} \right) \, dt + \int_{x_i}^{x_{i+1}} f'(t) \left( \frac{x_i + x_{i+1} - 2t}{2} \right) \, dt \right|.
\end{aligned}$$

Since the weight is odd with respect to the center of the integration interval, one may subtract a constant multiple of this weight without altering the integral,

and then apply Hölder's inequality with  $p^{-1} + q^{-1} = 1$ .

$$\begin{aligned}
& \left| \int_{x_{i-1}}^{x_{i+1}} f(x) \, dx - \frac{1}{2(n-1)} [f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right| \\
&= \left| \int_{x_{i-1}}^{x_i} \left[ f'(t) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right] \left( \frac{x_{i-1} + x_i - 2t}{2} \right) dt \right. \\
&\quad \left. + \int_{x_i}^{x_{i+1}} \left[ f'(t) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right] \left( \frac{x_i + x_{i+1} - 2t}{2} \right) dt \right| \\
&= \left\| f' - \frac{n-1}{2} I(f', x_{i-1}, x_{i+1}) \right\|_{p, [x_{i-1}, x_{i+1}]} \\
&\quad \times \left\{ \int_{x_{i-1}}^{x_i} \left| \frac{x_{i-1} + x_i - 2t}{2} \right|^q dt + \int_{x_i}^{x_{i+1}} \left| \frac{x_i + x_{i+1} - 2t}{2} \right|^q dt \right\}^{1/q} \\
&= \left[ \frac{4}{(q+1)(2n-2)^{q+1}} \right]^{1/q} \left\| f' - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right\|_{p, [x_{i-1}, x_{i+1}]}.
\end{aligned}$$

Finally, these error bounds over the subintervals  $[x_{i-1}, x_{i+1}]$  are combined to obtain an error bound for the trapezoidal rule, again using Hölder's inequality:

$$\begin{aligned}
& \left| \int_0^1 f(x) \, dx - A_n(f) \right| \\
&\leq \sum_{i=1}^{(n-1)/2} \left| \int_{x_{2i}}^{x_{2i+2}} f(x) \, dx - \frac{1}{2(n-1)} [f(x_{2i}) + 2f(x_{2i+1}) + f(x_{2i+2})] \right| \\
&\leq \sum_{i=1}^{(n-1)/2} \left[ \frac{4}{(q+1)(2n-2)^{q+1}} \right]^{1/q} \left\| f' - \frac{n-1}{2} I(f'; x_{2i}, x_{2i+2}) \right\|_{p, [x_{2i}, x_{2i+2}]} \\
&\leq \left( \frac{1}{q+1} \right)^{1/q} \frac{|f|_{\mathcal{G}, (n-1)/2, p}}{2n-2}.
\end{aligned}$$

- [1] N. Clancy, Y. Ding, C. Hamilton, F. J. Hickernell, Y. Zhang, The complexity of guaranteed automatic algorithms: Cones, not balls, submitted for publication, arXiv.org:1303.2412 [math.NA] (2013).