Another Cone for Integration

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Abstract

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1. Introduction

In [2] we considered the problem of integration and the cone of integrands

$$\mathcal{C}_\tau := \{f \in \mathcal{V}^1 : \mathrm{Var}(f') \le \tau \left\| f' - f(1) + f(0) \right\|_1\}, \tag{1}$$

where the total variation and the $\mathcal{L}_p$ norms are defined as

$$\mathrm{Var}(f) := \sup_{\substack{n\in\mathbb{N} \\ 0=x_0<x_1<\cdots<x_n=1}} \sum_{i=1}^n \left| f(x_i) - f(x_{i-1}) \right|,$$

$$\left\| f \right\|_p := \begin{cases} \left[\displaystyle\int_0^1 \left| f(x) \right|^p \, \mathrm{d}x \right]^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{0\le x\le 1} \left| f(x) \right|, & p = \infty, \end{cases}$$

$$\mathcal{V}^k := \mathcal{V}^k[0,1] = \{f \in C[0,1] : \mathrm{Var}(f^{(k)}) < \infty\}.$$

We derived an algorithm [2, Algorithm 4] that was guaranteed for integrands in $\mathcal{C}_\tau$. In this note we consider another algorithm and other cones.

First we recall some notation and results from [2]. For all $n \in \mathcal{I} := \{0, 2, 3, \ldots\}$ we have the linear spline. By convention $A_0(f) = 0$, and for $n > 0$,

$$x_{i,n} := x_i := \frac{i-1}{n-1}, \qquad i = 1, \ldots, n, \tag{2a}$$

$$A_n(f)(x) := (n-1) \left[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i) \right] \quad \text{for } x_i \le x \le x_{i+1}. \tag{2b}$$

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The cost of each function value is one and so the cost of A_n is n. The dependence of the nodes, x_i on n is often suppressed for simplicity. Integrating the linear spline gives us the trapezoidal rule based on $n-1$ trapezoids:

$$T_n(f) := \int_0^1 A_n(f)\,\mathrm{d}x = \frac{1}{2n-2}[f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

for $n \geq 2$ and $T_0(f) = 0$.

The error of the trapezoidal rule has the following upper bound [1, (7.15)]:

$$\left|\int_0^1 f(x)\,dx - T_n(f)\right| \leq \frac{\operatorname{Var}(f')}{8(n-1)^2} \qquad n \in \mathcal{I} \setminus \{0\}. \tag{3}$$

For any $n \in \mathcal{I}$, let $\mathcal{J}_n = \{m \in \mathbb{N} : (n-1)/(m-1) \in \mathbb{N}\}$. This means that T_n integrates exactly any function that is a linear spline using m nodes for $m \in \mathcal{J}_n$. This implies that

$$\begin{aligned}\left|\int_0^1 f(x)\,dx - T_n(f)\right| &= \left|\int_0^1 [f(x) - A_m(f)(x)]\,dx - T_n(f - A_m(f))\right| \\ &\leq \frac{\operatorname{Var}(f' - A_m(f)')}{8(n-1)^2} \qquad \forall m \in \mathcal{J}_n,\ n \in \mathcal{I} \setminus \{0\}.\end{aligned} \tag{4}$$

The variation of the first derivative of f is bounded below by the variation of the first derivative of of the linear spline of f. For all $f \in \mathcal{V}^1$ it follows that

$$\begin{aligned}\operatorname{Var}(f') \geq F_n(f) &:= \operatorname{Var}(A_n(f)') \\ &= \begin{cases} 0, & n = 0, 2, \\ (n-1)\displaystyle\sum_{i=1}^{n-2}\bigl|f(x_i) - 2f(x_{i+1}) + f(x_{i+2})\bigr|, & n \geq 3. \end{cases}\end{aligned} \tag{5}$$

Also note that

$$F_m(f) = F_m(A_n(f)) \leq \operatorname{Var}(A_n(f)') = F_n(f) \qquad \forall m \in \mathcal{J}_n. \tag{6}$$

The bound in further implies that

$$\begin{aligned}F_n(f) \leq \operatorname{Var}(f') &\leq \operatorname{Var}(f' - A_m(f)') + \operatorname{Var}(A_m(f)) \\ &= \operatorname{Var}(f' - A_m(f)') + F_m(f).\end{aligned} \tag{7}$$

Another useful fact from ??? is that

$$\|f' - f(1) + f(0)\|_1 \leq \widetilde{F}_n(f) + \tilde{h}(n)\operatorname{Var}(f') \qquad \forall f \in \mathcal{V}^1 \tag{8}$$

2. New Cone, New Algorithm

Let $\overline{\mathcal{I}}$ be some non-empty subset of $\{(\ell, m, n) \in \{2, 3, \ldots\}^3 : \ell \in \mathcal{J}_m,\ m \in \mathcal{J}_n\}$, and let $\overline{\tau} : \overline{\mathcal{I}} \to (0, \infty)$ be some given function. The new cone considered here is defined as

$$\overline{\mathcal{C}}_{\overline{\tau}} := \{f \in \mathcal{V}^1 : \operatorname{Var}(f'-A_m(f)') \le F_n(f-A_m(f)) + \overline{\tau}(\ell,m,n) \left\|f' - A_\ell(f)'\right\|_1) \\ \forall (\ell,m,n) \in \overline{\mathcal{I}}\}. \tag{9}$$

Note that for all $(\ell,m,n) \in \overline{\mathcal{I}}$

$$\begin{aligned}
\left\|f' - A_\ell(f)'\right\|_1) &= \left\|f' - A_m(f)' + A_m(f)' - A_\ell(f)'\right\|_1 \\
&\le \left\|f' - A_m(f)'\right\|_1 + \widetilde{F}_m(f - A_\ell(f)) \\
&\le \widetilde{F}_m(f - A_m(f)) + \tilde{h}(m)\operatorname{Var}(f' - A_m(f)') + \widetilde{F}_m(f - A_\ell(f)) \\
&= \tilde{h}(m)\operatorname{Var}(f' - A_m(f)') + \widetilde{F}_m(f - A_\ell(f)) \qquad \text{by (8)}
\end{aligned}$$

For all $f \in \overline{\mathcal{C}}_{\overline{\tau}}$ it then follows that

$$\begin{aligned}
\operatorname{Var}(f' - A_m(f)') &\le F_n(f - A_m(f)) + \overline{\tau}(\ell,m,n)\widetilde{F}_\ell(f) \\
&\le F_n(f - A_m(f)) + \overline{\tau}(\ell,m,n)[\tilde{h}(m)\operatorname{Var}(f' - A_m(f)') \\
&\qquad + \widetilde{F}_m(f - A_\ell(f))] \\
\operatorname{Var}(f' - A_m(f)') &\le \frac{F_n(f - A_m(f)) + \overline{\tau}(\ell,m,n)\widetilde{F}_m(f - A_\ell(f))}{1 - \overline{\tau}(\ell,m,n)\tilde{h}(m)}, \\
&\qquad \text{provided } \overline{\tau}(\ell,m,n)\tilde{h}(m) < 1.
\end{aligned}$$

Here $\hat{\tau} : \widehat{\mathcal{I}} \to [0,\infty)$ is some specified function that defines the cone, and $\widehat{\mathcal{I}} = \{N_{\min}, N_{\min}+1, \ldots\}$, where $N_{\min}$ is some integer no smaller than 3.

Algorithm 1 (New Cone Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{T_n\}_{n\in\mathcal{I}}$ $\{F_n\}_{n\in\mathcal{I}}$, and $\widehat{\mathcal{C}}_{\hat{\tau}}$ be as described above. Set $i = 1$, and let $n_1 = N_{\min}$. For any error tolerance ε and input function f, do the following:

Step 1. Bound $\operatorname{Var}(f')$ **and check for convergence.** Compute $F_{n_i}(f)$ in (5). Check whether n_i is large enough to satisfy the error tolerance, i.e.

$$\hat{\tau}(n_i)F_{n_i}(f) \le 8(n_i - 1)^2\varepsilon.$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm.

Step 2. Increase the number of trapezoids. If the above condition is false, choose $n_{i+1} = 2n_i$, increment i, and go to Step 1.

3. New Cone, New Algorithm

The new cone considered here is defined as

$$\widehat{\mathcal{C}}_{\hat{\tau}} := \{f \in \mathcal{V}^1 : \min_{m\in\mathcal{J}_n} \operatorname{Var}(f' - A_m(f)') \le \hat{\tau}(n)F_n(f) \ \forall n \in \widehat{\mathcal{I}}\}, \tag{10}$$

Here $\hat{\tau} : \widehat{\mathcal{I}} \to [0,\infty)$ is some specified function that defines the cone, and $\widehat{\mathcal{I}} = \{N_{\min}, N_{\min}+1, \ldots\}$, where $N_{\min}$ is some integer no smaller than 3.

Algorithm 2 (New Cone Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{T_n\}_{n\in\mathcal{I}}$ $\{F_n\}_{n\in\mathcal{I}}$, and $\widehat{\mathcal{C}}_{\hat\tau}$ be as described above. Set $i=1$, and let $n_1 = N_{\min}$. For any error tolerance ε and input function f, do the following:

Step 1. Bound $\operatorname{Var}(f')$ **and check for convergence.** Compute $F_{n_i}(f)$ in (5). Check whether n_i is large enough to satisfy the error tolerance, i.e.

$$\hat\tau(n_i)F_{n_i}(f) \le 8(n_i-1)^2\varepsilon.$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm.

Step 2. Increase the number of trapezoids. If the above condition is false, choose $n_{i+1} = 2n_i$, increment i, and go to Step 1.

4. The New Cone's Relationship to Other Cones

The cone defined in (10) makes Algorithm 2 work. In this section we show that it contains and is contained in other cones that might be more intuitive. One family of cones of interest is defined by replacing $F_n(f)$ by $\operatorname{Var}(f')$ in (10):

$$\widetilde{\mathcal{C}}_{\tilde\tau} := \{f\in\mathcal{V}^1 : \operatorname{Var}(f'-A_n(f)') \le \tilde\tau(n)\operatorname{Var}(f'),\ n\in\mathcal{I}\}, \tag{11}$$

where $\tilde\tau:\mathcal{I}\to[0,2]$ is non-increasing. Another family of cones is related to (1) and is defined as

$$\mathcal{C}_{\overline{\tau}} := \{f\in\mathcal{V}^1 : \operatorname{Var}(f'-A_n(f)') \le \overline{\tau}(n)\left\|f'-A_n(f)'\right\|_1,\ n\in\mathcal{I}\}, \tag{12}$$

where $\overline{\tau}:\mathcal{I}\to[0,\infty]$. Under this definition $\mathcal{C}_\tau$ corresponds to defining $\overline{\tau}(2)=\tau$, $\overline{\tau}(n)=\infty$ for $n>2$.

To facilitate the comparison of $\widehat{\mathcal{C}}_{\hat\tau}$, $\widetilde{\mathcal{C}}_{\tilde\tau}$, and $\mathcal{C}_{\overline{\tau}}$ we note several inequalities. For all $f\in\mathcal{V}^1$,

$$\operatorname{Var}(f') \le \operatorname{Var}(f'-A_n(f)') + \operatorname{Var}(A_n(f)') = \operatorname{Var}(f'-A_n(f)') + F_n(f), \tag{13}$$

$$\operatorname{Var}(f'-A_n(f)') \le \operatorname{Var}(f') + \operatorname{Var}(A_n(f')) = \operatorname{Var}(f') + F_n(f). \tag{14}$$

prove the following lemma. From (5) and (14) it follows that

$$\operatorname{Var}(f'-A_n(f)') \le 2\operatorname{Var}(f') \qquad \forall f\in\mathcal{V}^1, \tag{15}$$

which is why $\tilde\tau(n)\le 2$ for all n. Moreover, if $\tilde\tau(n)=2$ for all $n\in\mathcal{I}$, then $\widetilde{\mathcal{C}}_{\tilde\tau}=\mathcal{V}^1$.

Theorem 1. *Given the function* $\hat\tau:\widehat{\mathcal{I}}\to[0,\infty)$*, suppose that* $\tilde\tau_j:\mathcal{I}\to[0,2]$*,* $j=1,2$ *satisfy the inequality*

$$\tilde\tau_1(n) \le \frac{\hat\tau(n)}{1+\hat\tau(n)} \le \min(2,\hat\tau(n)) \le \tilde\tau_2(n) \qquad \forall n\in\widehat{\mathcal{I}}. \tag{16}$$

It follows that $\widetilde{\mathcal{C}}_{\tilde\tau_1} \le \widehat{\mathcal{C}}_{\hat\tau} \subseteq \widetilde{\mathcal{C}}_{\tilde\tau_2}$.

Proof. First suppose that $f \in \widetilde{\mathcal{C}}_{\tilde{\tau}_1}$ where $\tilde{\tau}_1$ satisfies inequality (16). It then follows that

$$\begin{aligned}
\operatorname{Var}(f' - A_n(f)') &= (1+\hat{\tau}(n)) \operatorname{Var}(f' - A_n(f)') - \hat{\tau}(n) \operatorname{Var}(f' - A_n(f)') \\
&\le [1+\hat{\tau}(n)]\tilde{\tau}_1(n) \operatorname{Var}(f') - \hat{\tau}(n) \operatorname{Var}(f' - A_n(f)') \quad \text{by (11)} \\
&\le \hat{\tau}(n)[\operatorname{Var}(f') - \operatorname{Var}(f' - A_n(f)')] \qquad \text{by (16)} \\
&\le \hat{\tau}(n) F_n(f) \qquad \text{by (13).}
\end{aligned}$$

Thus, $\widetilde{\mathcal{C}}_{\tilde{\tau}_1} \le \widehat{\mathcal{C}}_{\hat{\tau}}$. Now suppose that $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$. It follows by (5) and (15) that $f \in \widetilde{\mathcal{C}}_{\tilde{\tau}_2}$. □

To prove the relationship between the cones defined in (10) and (12) the following bound is needed.

Lemma 1. *For all $n \in \mathcal{I}$ and all $f \in \mathcal{V}^1$ it follows that*

$$F_n(f) \le 2(n-1) \left\| f' - A_n(f)' \right\|_1 . \tag{17}$$

Proof. For all $f \in \mathcal{V}^1$ we use the triangle inequality:

$$\begin{aligned}
F_n(f) &= (n-1) \sum_{i=1}^{n-2} \left| f(x_i) - 2f(x_{i+1}) + f(x_{i+2}) \right| \\
&\le (n-1) \sum_{i=1}^{n-2} \left| f(x_i) - f(x_{i+1}) + \frac{f(1)-f(0)}{n-1} \right| \\
&\qquad + (n-1) \sum_{i=1}^{n-2} \left| -f(x_{i+1}) + f(x_{i+2}) - \frac{f(1)-f(0)}{n-1} \right| \\
&\le 2(n-1) \sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(1)-f(0)}{n-1} \right| \\
&= 2(n-1) \left\| A_n(f)' - f(1) + f(0) \right\|_1 \\
&\le 2(n-1) \left\| f' - f(1) + f(0) \right\|_1 .
\end{aligned}$$

□

Theorem 2. *Given the function $\hat{\tau} : \widehat{\mathcal{I}} \to [0,\infty)$, suppose that $\overline{\tau}_j : \mathcal{I} \to [0,\infty)$, $j = 1,2$ satisfy the inequality*

$$\min\{2(n-1)\hat{\tau}(n) : n \in \widehat{\mathcal{I}}\}??\overline{\tau}_?(n) \tag{18}$$

It follows that $\mathcal{C}_{\overline{\tau}_1} \le \widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \mathcal{C}_{\overline{\tau}_2}$.

Proof. For all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows from (17) that

$$\operatorname{Var}(f') \le \hat{\tau}(n) F_n(f) = 2(n-1)\hat{\tau}(n) \left\| f' - f(1) + f(0) \right\|_1 \qquad \forall n \ge N_{\min}.$$

Applying the definition of τ completes the proof. □

Now we define a cone that is contained in $\widehat{\mathcal{C}}_{\hat{\tau}}$. Let

$$\widetilde{\mathcal{C}}_{\tilde{\tau}} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f') \leq \tilde{\tau}(n) \left\| f' - f(1) + f(0) \right\|_1 \ \forall n \geq 3\}, \tag{19}$$

Theorem 3. *For any non-increasing $\hat{\tau} : \mathcal{I} \to (1, \infty)$, let*

$$\tau = \min\{2(n-1)\hat{\tau}(n) : n \geq N_{\min}\}.$$

It follows that $\widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \mathcal{C}_\tau$.

Proof. For all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows from (17) that

$$\mathrm{Var}(f') \leq \hat{\tau}(n) F_n(f) = 2(n-1)\hat{\tau}(n) \left\| f' - f(1) + f(0) \right\|_1 \qquad \forall n \geq N_{\min}.$$

Applying the definition of τ completes the proof. □

References

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