Another Cone for Integration

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Abstract

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1. Introduction

In [1] we considered the problem of integration and the cone of integrands

$$\mathcal{C}_\tau := \{f \in \mathcal{V}^1 : \operatorname{Var}(f') \le \tau \left\| f' - f(1) + f(0) \right\|_1\}, \tag{1}$$

where the total variation and the $\mathcal{L}_p$ norms are defined as

$$\operatorname{Var}(f) := \sup_{\substack{n\in\mathbb{N} \\ 0=x_0<x_1<\cdots<x_n=1}} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|,$$

$$\left\| f \right\|_p := \begin{cases} \left[\displaystyle\int_0^1 \left| f(x) \right|^p \, \mathrm{d}x \right]^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{0\le x\le 1} \left| f(x) \right|, & p = \infty, \end{cases}$$

$$\mathcal{V}^k := \mathcal{V}^k[0,1] = \{f \in C[0,1] : \operatorname{Var}(f^{(k)}) < \infty\}.$$

We derived an algorithm [1, Algorithm 4] that was guaranteed for integrands in $\mathcal{C}_\tau$. In this note we consider another algorithm and other cones.

First we recall some notation and results from [1]. For all $n \in \mathcal{I} := \{2, 3, \ldots\}$ we have the linear spline:

$$x_{i,n} := x_i := \frac{i-1}{n-1}, \qquad i = 1, \ldots, n, \tag{2a}$$

$$A_n(f)(x) := (n-1)\left[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i) \right] \quad \text{for } x_i \le x \le x_{i+1}. \tag{2b}$$

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The cost of each function value is one and so the cost of A_n is n. The dependence of the nodes, x_i on n is often suppressed for simplicity. Integrating the linear spline gives us the trapezoidal rule based on $n-1$ trapezoids:

$$T_n(f) := \int_0^1 A_n(f)\,\mathrm{d}x = \frac{1}{2n-2}[f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

The error of the trapezoidal rule has the following upper bound:

$$\left|\int_0^1 f(x)\,dx - T_n(f)\right| \le \frac{\mathrm{Var}(f' - A_n(f)')}{8(n-1)^2} \le \frac{\mathrm{Var}(f')}{8(n-1)^2}. \tag{3}$$

The variation of the first derivative of f is bounded below by the variation of the first derivative of of the linear spline of f:

$$\begin{aligned} \mathrm{Var}(f') \ge F_n(f) &:= \mathrm{Var}(A_n(f)') \\ &= \begin{cases} 0, & n = 2, \\ (n-1)\displaystyle\sum_{i=1}^{n-2} \left|f(x_i) - 2f(x_{i+1}) + f(x_{i+2})\right|, & n \ge 3. \end{cases} \end{aligned} \tag{4}$$

2. New Cone, New Algorithm

The new cone considered here is defined as

$$\widehat{\mathcal{C}}_{\hat\tau} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f' - A_n(f)') \le \hat\tau(n) F_n(f)\ \forall n \in \mathcal{I}\}, \tag{5}$$

Here $\hat\tau : \mathcal{I} \to [0, \infty]$ is some specified non-increasing function that defines the cone.

Algorithm 1 (New Cone Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{T_n\}_{n\in\mathcal{I}}$ $\{F_n\}_{n\in\mathcal{I}}$, and $\widehat{\mathcal{C}}_{\hat\tau}$ be as described above. Set $i = 1$, and let $n_1 = N_{\min}$. For any error tolerance ε and input function f, do the following:

Step 1. Bound $\mathrm{Var}(f')$ and check for convergence. Compute $F_{n_i}(f)$ in (4). Check whether n_i is large enough to satisfy the error tolerance, i.e.

$$\hat\tau(n_i) F_{n_i}(f) \le 8(n_i - 1)^2 \varepsilon.$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm.

Step 2. Increase the number of trapezoids. If the above condition is false, choose

$$n_{i+1} = \min\{n \in \mathbb{N} : (n-1)/(n_i - 1) \in \{2, 3, \ldots\},\ \hat\tau(n) F_{n_i}(f) \le 8(n-1)^2\}.$$

Go to Step 1.

3. The New Cone's Relationship to Other Cones

The cone defined in (5) makes Algorithm 1 work. In this section we show that it contains and is contained in other cones that might be more intuitive. One family of cones of interest is defined by replacing $F_n(f)$ by $\mathrm{Var}(f')$ in (5):

$$\widetilde{\mathcal{C}}_{\tilde{\tau}} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f' - A_n(f)') \le \tilde{\tau}(n)\,\mathrm{Var}(f'),\ n \in \mathcal{I}\}, \tag{6}$$

where $\tilde{\tau} : \mathcal{I} \to [0, 2]$ is non-increasing. Another family of cones is related to (1) and is defined as

$$\mathcal{C}_{\overline{\tau}} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f' - A_n(f)') \le \overline{\tau}(n)\,\|f' - A_n(f)'\|_1\,,\ n \in \mathcal{I}\}, \tag{7}$$

where $\overline{\tau} : \mathcal{I} \to [0, \infty]$. Under this definition $\mathcal{C}_\tau$ corresponds to defining $\overline{\tau}(2) = \tau$, $\overline{\tau}(n) = \infty$ for $n > 2$.

To facilitate the comparison of $\widehat{\mathcal{C}}_{\hat{\tau}}$, $\widetilde{\mathcal{C}}_{\tilde{\tau}}$, and $\mathcal{C}_{\overline{\tau}}$ we note several inequalities. For all $f \in \mathcal{V}^1$,

$$\mathrm{Var}(f') \le \mathrm{Var}(f' - A_n(f)') + \mathrm{Var}(A_n(f)') = \mathrm{Var}(f' - A_n(f)') + F_n(f), \tag{8}$$
$$\mathrm{Var}(f' - A_n(f)') \le \mathrm{Var}(f') + \mathrm{Var}(A_n(f')) = \mathrm{Var}(f') + F_n(f). \tag{9}$$

prove the following lemma. From (4) and (9) it follows that

$$\mathrm{Var}(f' - A_n(f)') \le 2\,\mathrm{Var}(f') \qquad \forall f \in \mathcal{V}^1,$$

which is why $\tilde{\tau}(n) \le 2$ for all n.

Theorem 1. *Given the function $\hat{\tau} : \mathcal{I} \to [0, \infty]$, suppose that $\tilde{\tau}_j : \mathcal{I} \to [0, 2]$, $j = 1, 2$ satisfy the inequality*

$$\tilde{\tau}_1(n) \le \frac{\hat{\tau}(n)}{1 + \hat{\tau}(n)} \le \min(2, \hat{\tau}(n)) \le \tilde{\tau}_2(n) \qquad \forall n \in \mathcal{I}.$$

It follows that $\widetilde{\mathcal{C}}_{\tilde{\tau}_1} \le \widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \widetilde{\mathcal{C}}_{\tilde{\tau}_2}$.

Proof. □

Integrands in this cone satisfy the following useful properties.

Lemma 1. *Let $N_{\min} = \min\{n \in \mathcal{I} : \hat{\tau}(n) < 1\}$, and $\widehat{\mathcal{I}} = \{N_{\min}, N_{\min} + 1, \ldots\}$. For $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows that*

$$\mathrm{Var}(f') \le \frac{F_n(f)}{1 - \hat{\tau}(n)}, \quad \mathrm{Var}(f' - A_n(f)') \le \frac{\hat{\tau}(n) F_n(f)}{1 - \hat{\tau}(n)}, \qquad \forall n \ge \widehat{\mathcal{I}}.$$

Proof. Because $f \mapsto \mathrm{Var}(f')$ is a semi-norm, it follows that for all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ and all $n \in \mathcal{I}$,

$$\mathrm{Var}(f') \le \mathrm{Var}(f' - A_n(f)') + \mathrm{Var}(A_n(f)') \le \hat{\tau}(n)\,\mathrm{Var}(f') + F_n(f).$$

Rearranging the terms in this inequality leads to the desired results provided that $\hat{\tau}(n) < 1$. □

The cone defined in (5) makes Algorithm 1 work. In this section we show that it contains and is contained in other cones that might be more intuitive. The family of cones of interest are defined as

$$\mathcal{C}_{\tilde{\tau}} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f' - A_n(f)') \le \tilde{\tau}(n) \left\| f' - A_n(f)' \right\|_1, \ n \in \mathcal{I}\}, \tag{10}$$

where $\tilde{\tau} : \mathcal{I} \to (1, \infty)$. Under this definition $\mathcal{C}_\tau$ corresponds to defining $\tilde{\tau}(2) = \tau$, $\tilde{\tau}(n) = \infty$ for $n > 2$. To facilitate the comparison of $\mathcal{C}_{\tilde{\tau}}$ and $\widehat{\mathcal{C}}_{\hat{\tau}}$ we prove the following lemma.

Lemma 2. *Let $n = p(m-1)+1$ for some positive integer p. Then*

$$F_n(f) \le 2???? \left\| f' - A_n(f)' \right\|_1, \qquad \forall f \in \mathcal{V}^1$$

Proof. For all f

$$\begin{aligned}
F_n(f) &= (n-1) \sum_{i=1}^{n-2} \left| f\left(\frac{i-1}{n-1}\right) - 2f\left(\frac{i}{n-1}\right) + f\left(\frac{i+1}{n-1}\right) \right| \\
&= (n-1) \sum_{k=1}^{m} \Bigg[\sum_{j=1}^{p-1} \Bigg| f\left(\frac{k-1}{m-1} + \frac{j-1}{p(m-1)}\right) - 2f\left(\frac{k-1}{m-1} + \frac{j}{p(m-1)}\right) \\
&\qquad + f\left(\frac{k-1}{m-1} + \frac{j+1}{p(m-1)}\right) \Bigg| \\
&= (n-1) \sum_{k=1}^{m} \sum_{j=1}^{p-2} \left| f(x_{p(k-1)+j}) - 2f(x_{p(k-1)+j+1}) + f(x_{p(k-1)+j+2}) \right| \\
&\le (n-1) \sum_{i=1}^{n-2} \left| f(x_i) - f(x_{i+1}) + \frac{f(1) - f(0)}{n-1} \right| \\
&\qquad + (n-1) \sum_{i=1}^{n-2} \left| -f(x_{i+1}) + f(x_{i+2}) - \frac{f(1) - f(0)}{n-1} \right| \\
&\le 2 \sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(1) - f(0)}{n-1} \right| \\
&= 2(n-1) \left\| A_n(f)' - f(1) + f(0) \right\|_1 \\
&\le 2(n-1) \left\| f' - f(1) + f(0) \right\|_1.
\end{aligned} \tag{11}$$

□

Recall from [1] that

$$\begin{aligned}
\left\| f' - f(1) + f(0) \right\|_1 \ge \widetilde{F}_n(f) &:= \left\| A_n(f)' - A_2(f)' \right\|_1 \\
&= \sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(1) - f(0)}{n-1} \right|.
\end{aligned}$$

Morever, for all $f \in \mathcal{V}^1$

$$
\begin{aligned}
F_n(f) &= (n-1)\sum_{i=1}^{n-2}\bigl|f(x_i) - 2f(x_{i+1}) + f(x_{i+2})\bigr| \\
&\le (n-1)\sum_{i=1}^{n-2}\left|f(x_i) - f(x_{i+1}) + \frac{f(1)-f(0)}{n-1}\right| \\
&\qquad + (n-1)\sum_{i=1}^{n-2}\left|-f(x_{i+1}) + f(x_{i+2}) - \frac{f(1)-f(0)}{n-1}\right| \\
&\le 2\sum_{i=1}^{n-1}\left|f(x_{i+1}) - f(x_i) - \frac{f(1)-f(0)}{n-1}\right| \\
&= 2(n-1)\bigl\|A_n(f)' - f(1) + f(0)\bigr\|_1 \\
&\le 2(n-1)\bigl\|f' - f(1) + f(0)\bigr\|_1. \qquad (12)
\end{aligned}
$$

Theorem 2. *For any non-increasing $\hat{\tau} : \mathcal{I} \to (1, \infty)$, let*

$$\tau = \min\{2(n-1)\hat{\tau}(n) : n \ge N_{\min}\}.$$

It follows that $\widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \mathcal{C}_\tau$.

Proof. For all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows from (12) that

$$\operatorname{Var}(f') \le \hat{\tau}(n)F_n(f) = 2(n-1)\hat{\tau}(n)\bigl\|f' - f(1) + f(0)\bigr\|_1 \qquad \forall n \ge N_{\min}.$$

Applying the definition of τ completes the proof. $\square$

Now we define a cone that is contained in $\widehat{\mathcal{C}}_{\hat{\tau}}$. Let

$$\widetilde{\mathcal{C}}_{\tilde{\tau}} := \{f \in \mathcal{V}^1 : \operatorname{Var}(f') \le \tilde{\tau}(n)\,\|f' - f(1) + f(0)\|_1 \ \ \forall n \ge 3\}, \qquad (13)$$

Theorem 3. *For any non-increasing $\hat{\tau} : \mathcal{I} \to (1, \infty)$, let*

$$\tau = \min\{2(n-1)\hat{\tau}(n) : n \ge N_{\min}\}.$$

It follows that $\widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \mathcal{C}_\tau$.

Proof. For all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows from (12) that

$$\operatorname{Var}(f') \le \hat{\tau}(n)F_n(f) = 2(n-1)\hat{\tau}(n)\bigl\|f' - f(1) + f(0)\bigr\|_1 \qquad \forall n \ge N_{\min}.$$

Applying the definition of τ completes the proof. $\square$

References

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