Locally Adaptive Automatic Algorithms

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Abstract

Keywords:

1. The Basic Problem

Let $\mathcal{F}$ be a separable Banach space of real-valued input functions with domain $\mathcal{X}$, and let $\mathcal{F}$ have a semi-norm $|\cdot|_{\mathcal{F}}$. Let $\mathcal{H}$ be a separable Banach space of output functions with norm $\|\cdot\|_{\mathcal{H}}$, and let S be a solution operator, $S : \mathcal{F} \to \mathcal{H}$. We have three examples in mind for this setting. Let a, b be two fixed real numbers with $a < b$, and $\mathcal{X} = [a, b]$. Here are three problems:

$$\begin{aligned}
\text{Integration (INT)} \qquad & S : f \mapsto \int_a^b f(x)\,\mathrm{d}x, \qquad S : \mathcal{W}^{4,1}[a,b] \to \mathbb{R}, \\
\text{Approximation (APP)} \qquad & S : f \mapsto f, \qquad S : \mathcal{W}^{3,\infty}[a,b] \to \mathcal{L}_\infty[a,b], \\
\text{Optimization (OPT)} \qquad & S : f \mapsto \min_{a \le x \le b} f(x) \qquad S : \mathcal{W}^{3,\infty}[a,b] \to \mathbb{R}.
\end{aligned}$$

The Sobolev and Lebesgue spaces and their (semi-)norms are defined as follows. For all real numbers α, β with $\alpha < \beta$,

$$\mathcal{W}^{k,p} := \mathcal{W}^{k,p}[a,b], \qquad \mathcal{W}^{k,p}[\alpha,\beta] := \{f \in C[\alpha,\beta] : \left\|f^{(k)}\right\|_{p,[\alpha,\beta]} < \infty\},$$

$$\mathcal{L}_p := \mathcal{L}_p[a,b], \qquad \mathcal{L}_p[\alpha,\beta] := \mathcal{W}^{0,p}[\alpha,\beta],$$

$$|f|_{\mathcal{W}^{k,p}[\alpha,\beta]} := \left\|f^{(k)}\right\|_{p,[\alpha,\beta]}, \qquad \|f\|_{\mathcal{L}_p[\alpha,\beta]} := \|f\|_{p,[\alpha,\beta]},$$

$$\|f\|_{p,[\alpha,\beta]} := \begin{cases} \left[\int_\alpha^\beta |f(x)|^p \,\mathrm{d}x\right]^{1/p}, & 1 \le p < \infty, \\ \max_{\alpha \le x \le \beta} |f(x)|, & p = \infty. \end{cases}$$

Although these are problems involving univariate functions, one could extend these to multivariate problems, and the general analysis would still apply.

2. Solving the Problems on Partitions

For any $\mathcal{Y} \subset \mathcal{X}$, let let $f_{\mathcal{Y}}$ denote f restricted to the set $\mathcal{Y}$, i.e.,

$$f_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{R}, \qquad f_{\mathcal{Y}} : \boldsymbol{x} \mapsto f(\boldsymbol{x}).$$

Moreover, let $\mathcal{F}_{\mathcal{Y}}$ denote the space of functions in $\mathcal{F}$ restricted to the set $\mathcal{Y}$, i.e., $\mathcal{F}_{\mathcal{Y}} = \{f_{\mathcal{Y}} : f \in \mathcal{F}\}$.

It is also assumed that their exists, $\mathcal{T}$, some sets of measurable subsets of $\mathcal{X}$ for which one can define norms, solution operators, and approximation operators. It is assumed that

- For each $\mathcal{Y} \in \mathcal{T}$, the subspace $\mathcal{F}_{\mathcal{Y}}$ has a semi-norm, $|\cdot|_{\mathcal{F}_{\mathcal{Y}}}$ satisfying $|f_{\mathcal{Y}}|_{\mathcal{F}_{\mathcal{Y}}} \le |f|_{\mathcal{F}}$. For simplicity of notation, we let $|f|_{\mathcal{F}_{\mathcal{Y}}} = |f_{\mathcal{Y}}|_{\mathcal{F}_{\mathcal{Y}}}$
- For each $\mathcal{Y} \in \mathcal{T}$ there exists a solution operator $S(\cdot;\mathcal{Y}) : \mathcal{F} \to \mathcal{H}$, for which $S(f,\mathcal{Y})$ is actually only a function of $f_{\mathcal{Y}}$.

Partitions of $\mathcal{X}$ are finite subsets of $\mathcal{T}$ such that the following conditions hold:

- There exists a function $\Phi(\cdot;\mathcal{P}) : \mathcal{H}^{|\mathcal{P}|} \to \mathcal{H}$ that combines the solutions defined on the subsets to reconstruct the true solution:
$$S(f) = \Phi(\boldsymbol{S}_{\mathcal{P}}(f)), \quad \boldsymbol{S}_{\mathcal{P}}(f) := \{S(f;\mathcal{Y})\}_{\mathcal{Y}\in\mathcal{P}}, \qquad \forall f \in \mathcal{F}.$$
Here $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$.
- There exists a pair of functions $(\widetilde{\Phi}, \mathrm{err})(\cdot,\cdot;\mathcal{P}) : \mathcal{H}^{|\mathcal{P}|} \times [0,\infty)^{|\mathcal{P}|} \to \mathcal{H} \times [0,\infty)$ that combine approximate solutions defined on the subsets with error bounds to reconstruct an approximation to the true solution. If $\left\|S(f;\mathcal{Y}) - \widetilde{S}_{\mathcal{Y}}\right\|_{\mathcal{H}} \le \varepsilon_{\mathcal{Y}}$ for all $\mathcal{Y} \in \mathcal{P}$, $\widetilde{\boldsymbol{S}}_{\mathcal{P}} := \{\widetilde{S}_{\mathcal{Y}}\}_{\mathcal{Y}\in\mathcal{P}}$, and $\tilde{\boldsymbol{\varepsilon}}_{\mathcal{P}} := \{\tilde{\varepsilon}_{\mathcal{Y}}\}_{\mathcal{Y}\in\mathcal{P}}$, then
$$\left\|S(f) - \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \tilde{\boldsymbol{\varepsilon}}_{\mathcal{P}};\mathcal{P})\right\|_{\mathcal{H}} \le \mathrm{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \tilde{\boldsymbol{\varepsilon}}_{\mathcal{P}};\mathcal{P}) \qquad \forall f \in \mathcal{F}.$$
Here $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$. Moreover,
$$\mathrm{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{0};\mathcal{P}) = 0, \qquad \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{0};\mathcal{P}) = \Phi(\boldsymbol{S}_{\mathcal{P}}(f)) = S(f).$$

Note that the subsets of $\mathcal{X}$ comprising the partition $\mathcal{P}$ need not have nonempty intersection.

For our three problems, these partitions take the form of subintervals of $[a,b]$:
$$\mathcal{T} = \{[\alpha,\beta] : a \le \alpha < \beta \le b\}$$
$$\mathcal{P} = \{[t_0,t_1],[t_1,t_2],\dots,[t_{L-1},t_L]\}, \qquad a = t_0 < t_1 < \cdots < t_L = b.$$

The semi-norms and solution operators defined on the elements of $\mathcal{T}$, and the functions Φ, $\widetilde{\Phi}$, and err that combine the solutions on the sets in the partition into the full solution, the approximate, and the upper error bound are the following:

$$\text{INT} \qquad \left\|f_{[\alpha,\beta]}\right\|_{\mathcal{F}_{[\alpha,\beta]}} = \left\|f^{(4)}\right\|_{1,[\alpha,\beta]}, \qquad S_{[\alpha,\beta]} : f \mapsto \int_\alpha^\beta f(x)\,\mathrm{d}x,$$
$$\Phi(\widetilde{\boldsymbol{S}}_{\mathcal{P}};\mathcal{P}) = \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \sum_{l=1}^{L} \widetilde{S}_{[t_{l-1},t_l]},$$
$$\mathrm{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \left\|\boldsymbol{\varepsilon}_{\mathcal{P}}\right\|_1,$$

$$
\begin{aligned}
&\text{APP} \qquad \left\|f_{[\alpha,\beta]}\right\|_{\mathcal{F}_{[\alpha,\beta]}} = \left\|f^{(3)}\right\|_{\infty,[\alpha,\beta]}, \qquad S_{[\alpha,\beta]} : f \mapsto f\mathbb{1}_{[\alpha,\beta]}, \\
&\Phi(\widetilde{\boldsymbol{S}}_{\mathcal{P}};\mathcal{P}) = \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \sum_{l=1}^{L} \widetilde{S}_{[t_{l-1},t_l]}, \\
&\mathrm{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \left\|\boldsymbol{\varepsilon}_{\mathcal{P}}\right\|_{\infty}, \\
&\text{OPT} \qquad \left\|f_{[\alpha,\beta]}\right\|_{\mathcal{F}_{[\alpha,\beta]}} = \left\|f^{(3)}\right\|_{\infty,[\alpha,\beta]}, \qquad S_{[\alpha,\beta]} : f \mapsto \min_{\alpha\le x\le\beta} f(x), \\
&\Phi(\widetilde{\boldsymbol{S}}_{\mathcal{P}};\mathcal{P}) = \min \widetilde{\boldsymbol{S}}_{\mathcal{P}}, \\
&\widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \frac{1}{2}\left\{\min(\widetilde{\boldsymbol{S}}_{\mathcal{P}} + \boldsymbol{\varepsilon}_{\mathcal{P}}) + \min(\widetilde{\boldsymbol{S}}_{\mathcal{P}} - \boldsymbol{\varepsilon}_{\mathcal{P}})\right\}, \\
&\mathrm{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \frac{1}{2}\left\{\min(\widetilde{\boldsymbol{S}}_{\mathcal{P}} + \boldsymbol{\varepsilon}_{\mathcal{P}}) - \min(\widetilde{\boldsymbol{S}}_{\mathcal{P}} - \boldsymbol{\varepsilon}_{\mathcal{P}})\right\}.
\end{aligned}
$$

Here the operator "min" applied to a vector takes the minimum of the elements.

3. Algorithms

Now we consider numerical algorithms for solving the problems on a subset of the whole domain. Suppose that

- For each $\mathcal{Y} \in \mathcal{T}$ and each $n \in \mathcal{I}$ there exists a non-adaptive approximation operator $A_n(\cdot;\mathcal{Y}) : \mathcal{F} \to \mathcal{H}$ that uses n function values sampled only in $\mathcal{Y}$.
- There exists an error bound function $h : \mathcal{I} \times \mathcal{T} \to [0,\infty)$ such that $h(\cdot,\mathcal{Y})$ is non-increasing, and

$$\left\|S(f;\mathcal{Y}) - A_n(f;\mathcal{Y})\right\|_{\mathcal{H}} \le h(n,\mathcal{Y})\, |f|_{\mathcal{F}_{\mathcal{Y}}}, \qquad \forall f \in \mathcal{F}.$$

For our three problems, the non-adaptive algorithms A_n are based on piecewise quadratic polynomial approximation of the function.

For the integration problem this corresponds to Simpson's rule. The number of possible function values is the odd integers $\mathcal{I} = \{2i-1 : i \in \mathbb{N}\}$

$$x_i = \alpha + (\beta - \alpha)\frac{i-1}{n-1}, \qquad i = 1,\ldots,n, \tag{1a}$$

$$
\begin{aligned}
A_n(f,[\alpha,\beta]) = \frac{\beta-\alpha}{3(n-1)}\,[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots \\
+4f(x_{n-1}) + f(x_n)]\,.
\end{aligned} \tag{1b}
$$

From ??? it is known that discretization error of Simpson's rule is

$$\left|\int_{\alpha}^{\beta} f(x)\,\mathrm{d}x - A_n(f,[\alpha,\beta])\right| \le \left\|f^{(4)}\right\|_{1,[\alpha,\beta]} h(n,[\alpha,\beta]), \tag{1c}$$

$$h(n,[\alpha,\beta]) = ???. \tag{1d}$$

For approximation

For optimization

4. Functions in Cones

The challenge in applying the error bounds for the non-adaptive integration, function recovery, and optimization algorithms is that $|f|_{\mathcal{F}_\mathcal{Y}}$ is not known a priori. Our approach is to assume that the input functions lie inside cones. For partitions $\mathcal{P}$ suppose that

- For each $\mathcal{Y} \in \mathcal{T}$ there exists a semi-norm $|\cdot|_{\mathcal{G}_\mathcal{Y}}$ defined on the space $\mathcal{F}_\mathcal{Y}$ that is weaker than $|\cdot|_{\mathcal{F}_\mathcal{Y}}$.
- For a fixed function non-increasing $\tau : (0,1) \to (0,\infty)$, define the cone

$$\mathcal{C}_\tau = \{ f \in \mathcal{F} : |f|_{\mathcal{F}_\mathcal{Y}} \le \tau(\operatorname{vol}(\mathcal{Y}))\, |f|_{\mathcal{G}_\mathcal{Y}} \}, \tag{2}$$

  where $\operatorname{vol}(\mathcal{Y})$ denotes relative volume of $\mathcal{Y}$, i.e., the Lebesgue measure $\mathcal{Y}$ divided by the Lebesgue measure of $\mathcal{X}$.

- For each $\mathcal{Y} \in \mathcal{T}$ and each $n \in \mathcal{I}$ there exists a non-adaptive approximation operator $G_n(\cdot;\mathcal{Y}) : \mathcal{F} \to \mathcal{H}$ that uses n function values sampled only in $\mathcal{Y}$.
- There exists error bound functions $g_\pm : \mathcal{I} \times \mathcal{T} \to [0,\infty)$ such that $g(\cdot,\mathcal{Y})$ is non-increasing, and

$$-g_-(n,\mathcal{Y})\, |f|_{\mathcal{F}_\mathcal{Y}} \le |f|_{\mathcal{G}_\mathcal{Y}} - G_n(f;\mathcal{Y}) \le g_+(n,\mathcal{Y})\, |f|_{\mathcal{F}_\mathcal{Y}}, \qquad \forall f \in \mathcal{F}.$$

  Invoking the definition of the cone implies a two sided bound for $|f|_{\mathcal{G}_\mathcal{Y}}$ and $|f|_{\mathcal{G}_\mathcal{Y}}$ in terms of $G_n(f;\mathcal{Y})$:

$$-\tau(\operatorname{vol}(\mathcal{Y}))g_-(n,\mathcal{Y})\, |f|_{\mathcal{G}_\mathcal{Y}} \le |f|_{\mathcal{G}_\mathcal{Y}} - G_n(f;\mathcal{Y}) \le \tau(\operatorname{vol}(\mathcal{Y}))g_+(n,\mathcal{Y})\, |f|_{\mathcal{G}_\mathcal{Y}}, \qquad \forall f \in \mathcal{C}_\tau.$$

$$\begin{aligned}
\frac{G_n(f;\mathcal{Y})}{1+\tau(\operatorname{vol}(\mathcal{Y}))g_-(n,\mathcal{Y})} &\le |f|_{\mathcal{G}_\mathcal{Y}} \le \frac{G_n(f;\mathcal{Y})}{1-\tau(\operatorname{vol}(\mathcal{Y}))g_+(n,\mathcal{Y})}, \quad \forall f \in \mathcal{C}_\tau, \\
\frac{\tau(\operatorname{vol}(\mathcal{Y}))G_n(f;\mathcal{Y})}{1+\tau(\operatorname{vol}(\mathcal{Y}))g_-(n,\mathcal{Y})} &\le |f|_{\mathcal{F}_\mathcal{Y}} \le \frac{\tau(\operatorname{vol}(\mathcal{Y}))G_n(f;\mathcal{Y})}{1-\tau(\operatorname{vol}(\mathcal{Y}))g_+(n,\mathcal{Y})}, \quad \forall f \in \mathcal{C}_\tau.
\end{aligned}$$

Algorithm 1. Given . . . .