Another Cone for Integration

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Abstract

Keywords: adaptive, automatic, cones, function recovery, guarantee, integration, quadrature
2010 MSC: 65D05, 65D30, 65G20

1. Introduction

In [2] we considered the problem of integration and the cone of integrands

$$\mathcal{C}_\tau := \{f \in \mathcal{V}^1 : \mathrm{Var}(f') \le \tau \left\| f' - f(1) + f(0) \right\|_1\}, \tag{1}$$

where the total variation and the $\mathcal{L}_p$ norms are defined as

$$\mathrm{Var}(f) := \sup_{\substack{n\in\mathbb{N} \\ 0=x_0<x_1<\cdots<x_n=1}} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|,$$

$$\left\| f \right\|_p := \begin{cases} \left[\displaystyle\int_0^1 \left| f(x) \right|^p \, \mathrm{d}x \right]^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{0\le x\le 1} \left| f(x) \right|, & p = \infty, \end{cases}$$

$$\mathcal{V}^k := \mathcal{V}^k[0,1] = \{f \in C[0,1] : \mathrm{Var}(f^{(k)}) < \infty\}.$$

We derived an algorithm [2, Algorithm 4] that was guaranteed for integrands in $\mathcal{C}_\tau$. In this note we consider another algorithm and other cones.

First we recall some notation and results from [2]. For all $n \in \mathcal{I} := \{0, 2, 3, \ldots\}$ we have the linear spline. By convention $A_0(f) = 0$, and for $n > 0$,

$$x_{i,n} := x_i := \frac{i-1}{n-1}, \qquad i = 1, \ldots, n, \tag{2a}$$

$$A_n(f)(x) := (n-1)\left[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i) \right] \quad \text{for } x_i \le x \le x_{i+1}. \tag{2b}$$

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The cost of each function value is one and so the cost of A_n is n. The dependence of the nodes, x_i on n is often suppressed for simplicity. Integrating the linear spline gives us the trapezoidal rule based on $n-1$ trapezoids:

$$T_n(f) := \int_0^1 A_n(f)\,\mathrm{d}x = \frac{1}{2n-2}[f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

for $n \geq 2$ and $T_0(f) = 0$.

The error of the trapezoidal rule has the following upper bound [1, (7.15)]:

$$\left| \int_0^1 f(x)\,dx - T_n(f) \right| \leq \frac{\operatorname{Var}(f')}{8(n-1)^2} \qquad n \in \mathcal{I} \setminus \{0\}. \tag{3}$$

For any $n \in \mathcal{I}$, let $\mathcal{J}_n = \{m \in \mathbb{N} : (n-1)/(m-1) \in \mathbb{N}\}$. This means that T_n integrates exactly any function that is a linear spline using m nodes for $m \in \mathcal{J}_n$. This implies that

$$\begin{aligned}\left| \int_0^1 f(x)\,dx - T_n(f) \right| &= \left| \int_0^1 [f(x) - A_m(f)(x)]\,dx - T_n(f - A_m(f)) \right| \\ &\leq \frac{\operatorname{Var}(f' - A_m(f)')}{8(n-1)^2} \qquad \forall m \in \mathcal{J}_n,\ n \in \mathcal{I} \setminus \{0\}. \end{aligned} \tag{4}$$

The variation of the first derivative of f is bounded below by the variation of the first derivative of of the linear spline of f. For all $f \in \mathcal{V}^1$ it follows that

$$\begin{aligned}\operatorname{Var}(f') \geq F_n(f) &:= \operatorname{Var}(A_n(f)') \\ &= \begin{cases} 0, & n = 0, 2, \\ (n-1)\displaystyle\sum_{i=1}^{n-2} \bigl| f(x_i) - 2f(x_{i+1}) + f(x_{i+2}) \bigr|, & n \geq 3. \end{cases}\end{aligned} \tag{5}$$

Also note that

$$F_m(f) = F_m(A_n(f)) \leq \operatorname{Var}(A_n(f)') = F_n(f) \qquad \forall m \in \mathcal{J}_n. \tag{6}$$

The bound in further implies that

$$\begin{aligned}F_n(f) \leq \operatorname{Var}(f') &\leq \operatorname{Var}(f' - A_m(f)') + \operatorname{Var}(A_m(f)) \\ &= \operatorname{Var}(f' - A_m(f)') + F_m(f). \end{aligned} \tag{7}$$

2. New Cone, New Algorithm

The new cone considered here is defined as

$$\widehat{\mathcal{C}}_{\hat{\tau}} := \{f \in \mathcal{V}^1 : \min_{m \in \mathcal{J}_n} \operatorname{Var}(f' - A_m(f)') \leq \hat{\tau}(n) F_n(f)\ \forall n \in \widehat{\mathcal{I}}\}, \tag{8}$$

Here $\hat{\tau} : \widehat{\mathcal{I}} \to [0, \infty)$ is some specified function that defines the cone, and $\widehat{\mathcal{I}} = \{N_{\min}, N_{\min} + 1, \ldots\}$, where $N_{\min}$ is some integer no smaller than 3.

Algorithm 1 (New Cone Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{T_n\}_{n\in\mathcal{I}}$ $\{F_n\}_{n\in\mathcal{I}}$, and $\widehat{\mathcal{C}}_{\hat{\tau}}$ be as described above. Set $i=1$, and let $n_1 = N_{\min}$. For any error tolerance ε and input function f, do the following:

Step 1. Bound $\mathrm{Var}(f')$ **and check for convergence.** Compute $F_{n_i}(f)$ in (5). Check whether n_i is large enough to satisfy the error tolerance, i.e.

$$\hat{\tau}(n_i)F_{n_i}(f) \le 8(n_i-1)^2\varepsilon.$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm.

Step 2. Increase the number of trapezoids. If the above condition is false, choose $n_{i+1} = 2n_i$, increment i, and go to Step 1.

3. The New Cone's Relationship to Other Cones

The cone defined in (8) makes Algorithm 1 work. In this section we show that it contains and is contained in other cones that might be more intuitive. One family of cones of interest is defined by replacing $F_n(f)$ by $\mathrm{Var}(f')$ in (8):

$$\widetilde{\mathcal{C}}_{\tilde{\tau}} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f' - A_n(f)') \le \tilde{\tau}(n)\,\mathrm{Var}(f'),\ n \in \mathcal{I}\}, \tag{9}$$

where $\tilde{\tau} : \mathcal{I} \to [0,2]$ is non-increasing. Another family of cones is related to (1) and is defined as

$$\mathcal{C}_{\overline{\tau}} := \{f \in \mathcal{V}^1 : \mathrm{Var}(f' - A_n(f)') \le \overline{\tau}(n)\,\|f' - A_n(f)'\|_1\,,\ n \in \mathcal{I}\}, \tag{10}$$

where $\overline{\tau} : \mathcal{I} \to [0,\infty]$. Under this definition $\mathcal{C}_\tau$ corresponds to defining $\overline{\tau}(2)=\tau$, $\overline{\tau}(n) = \infty$ for $n>2$.

To facilitate the comparison of $\widehat{\mathcal{C}}_{\hat{\tau}}$, $\widetilde{\mathcal{C}}_{\tilde{\tau}}$, and $\mathcal{C}_{\overline{\tau}}$ we note several inequalities. For all $f \in \mathcal{V}^1$,

$$\mathrm{Var}(f') \le \mathrm{Var}(f' - A_n(f)') + \mathrm{Var}(A_n(f)') = \mathrm{Var}(f' - A_n(f)') + F_n(f), \tag{11}$$
$$\mathrm{Var}(f' - A_n(f)') \le \mathrm{Var}(f') + \mathrm{Var}(A_n(f')) = \mathrm{Var}(f') + F_n(f). \tag{12}$$

prove the following lemma. From (5) and (12) it follows that

$$\mathrm{Var}(f' - A_n(f)') \le 2\,\mathrm{Var}(f') \qquad \forall f \in \mathcal{V}^1, \tag{13}$$

which is why $\tilde{\tau}(n) \le 2$ for all n. Moreover, if $\tilde{\tau}(n) = 2$ for all $n \in \mathcal{I}$, then $\widetilde{\mathcal{C}}_{\tilde{\tau}} = \mathcal{V}^1$.

Theorem 1. *Given the function $\hat{\tau} : \widehat{\mathcal{I}} \to [0,\infty)$, suppose that $\tilde{\tau}_j : \mathcal{I} \to [0,2]$, $j=1,2$ satisfy the inequality*

$$\tilde{\tau}_1(n) \le \frac{\hat{\tau}(n)}{1+\hat{\tau}(n)} \le \min(2, \hat{\tau}(n)) \le \tilde{\tau}_2(n) \qquad \forall n \in \widehat{\mathcal{I}}. \tag{14}$$

It follows that $\widetilde{\mathcal{C}}_{\tilde{\tau}_1} \le \widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \widetilde{\mathcal{C}}_{\tilde{\tau}_2}$.

Proof. First suppose that $f \in \widetilde{\mathcal{C}}_{\tilde{\tau}_1}$ where $\tilde{\tau}_1$ satisfies inequality (14). It then follows that

$$\begin{aligned}\operatorname{Var}(f' - A_n(f)') &= (1+\hat{\tau}(n)) \operatorname{Var}(f' - A_n(f)') - \hat{\tau}(n) \operatorname{Var}(f' - A_n(f)') \\ &\le [1+\hat{\tau}(n)]\tilde{\tau}_1(n) \operatorname{Var}(f') - \hat{\tau}(n) \operatorname{Var}(f' - A_n(f)') \quad \text{by (9)} \\ &\le \hat{\tau}(n)[\operatorname{Var}(f') - \operatorname{Var}(f' - A_n(f)')] \qquad \text{by (14)} \\ &\le \hat{\tau}(n) F_n(f) \qquad \text{by (11).}\end{aligned}$$

Thus, $\widetilde{\mathcal{C}}_{\tilde{\tau}_1} \le \widehat{\mathcal{C}}_{\hat{\tau}}$. Now suppose that $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$. It follows by (5) and (13) that $f \in \widetilde{\mathcal{C}}_{\tilde{\tau}_2}$. □

To prove the relationship between the cones defined in (8) and (10) the following bound is needed.

Lemma 1. *For all $n \in \mathcal{I}$ and all $f \in \mathcal{V}^1$ it follows that*

$$F_n(f) \le 2(n-1) \left\| f' - A_n(f)' \right\|_1 . \tag{15}$$

Proof. For all $f \in \mathcal{V}^1$ we use the triangle inequality:

$$\begin{aligned} F_n(f) &= (n-1) \sum_{i=1}^{n-2} \left| f(x_i) - 2f(x_{i+1}) + f(x_{i+2}) \right| \\ &\le (n-1) \sum_{i=1}^{n-2} \left| f(x_i) - f(x_{i+1}) + \frac{f(1)-f(0)}{n-1} \right| \\ &\qquad + (n-1) \sum_{i=1}^{n-2} \left| -f(x_{i+1}) + f(x_{i+2}) - \frac{f(1)-f(0)}{n-1} \right| \\ &\le 2(n-1) \sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(1)-f(0)}{n-1} \right| \\ &= 2(n-1) \left\| A_n(f)' - f(1) + f(0) \right\|_1 \\ &\le 2(n-1) \left\| f' - f(1) + f(0) \right\|_1 . \end{aligned}$$

□

Theorem 2. *Given the function $\hat{\tau} : \widehat{\mathcal{I}} \to [0,\infty)$, suppose that $\overline{\tau}_j : \mathcal{I} \to [0,\infty)$, $j = 1, 2$ satisfy the inequality*

$$\min\{2(n-1)\hat{\tau}(n) : n \in \widehat{\mathcal{I}}\} ?? \overline{\tau}_?(n) \tag{16}$$

It follows that $\mathcal{C}_{\overline{\tau}_1} \le \widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \mathcal{C}_{\overline{\tau}_2}$.

Proof. For all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows from (15) that

$$\operatorname{Var}(f') \le \hat{\tau}(n) F_n(f) = 2(n-1)\hat{\tau}(n) \left\| f' - f(1) + f(0) \right\|_1 \qquad \forall n \ge N_{\min}.$$

Applying the definition of τ completes the proof. □

Now we define a cone that is contained in $\widehat{\mathcal{C}}_{\hat{\tau}}$. Let

$$\widetilde{\mathcal{C}}_{\tilde{\tau}} := \{f \in \mathcal{V}^1 : \operatorname{Var}(f') \leq \tilde{\tau}(n) \left\|f' - f(1) + f(0)\right\|_1 \ \forall n \geq 3\}, \tag{17}$$

Theorem 3. *For any non-increasing $\hat{\tau} : \mathcal{I} \to (1, \infty)$, let*

$$\tau = \min\{2(n-1)\hat{\tau}(n) : n \geq N_{\min}\}.$$

It follows that $\widehat{\mathcal{C}}_{\hat{\tau}} \subseteq \mathcal{C}_{\tau}$.

Proof. For all $f \in \widehat{\mathcal{C}}_{\hat{\tau}}$ it follows from (15) that

$$\operatorname{Var}(f') \leq \hat{\tau}(n) F_n(f) = 2(n-1)\hat{\tau}(n) \left\|f' - f(1) + f(0)\right\|_1 \qquad \forall n \geq N_{\min}.$$

Applying the definition of τ completes the proof. □

References

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