WHEN DOES THE RECTANGLE RULE WORK

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Abstract.

1. General Setting

Let $\mathcal{H}$ be a Hilbert space of functions of interest. Suppose one wishes to bound the value of some bounded, linear functional which is very difficult to compute:

$$L : \mathcal{H} \to \mathbb{R} \qquad \text{what we want to bound}$$

in terms of some bounded, linear functional which can be computed fairly easily

$$M : \mathcal{H} \to \mathbb{R} \qquad \text{what we can compute}$$

Furthermore, let these two bounded, linear functionals have representers, ξ and ζ, respectively, i.e.,

$$L(f) = \langle \xi, f \rangle_{\mathcal{H}}, \quad M(f) = \langle \zeta, f \rangle_{\mathcal{H}}, \qquad \forall f \in \mathcal{H}.$$

We want to bound $|L(f)|$ in terms of $|M(f)|$

Decompose ξ into a term parallel to ζ and a term perpendicular to ζ:

$$\xi = a\zeta + \zeta_\perp, \qquad \text{where } a := \frac{\langle \xi, \zeta \rangle_{\mathcal{H}}}{\|\zeta\|_{\mathcal{H}}^2}, \quad \zeta_\perp := \xi - a\zeta.$$

Note that $\langle \zeta, \zeta_\perp \rangle_{\mathcal{H}} = 0$. Now any $f \in \mathcal{H}$ may be decomposed as sum of three terms:

$$f = b\zeta + b_\perp \zeta_\perp + f_\perp, \qquad \text{where } b := \frac{\langle f, \zeta \rangle_{\mathcal{H}}}{\|\zeta\|_{\mathcal{H}}^2}, \quad b_\perp := \frac{\langle f, \zeta_\perp \rangle_{\mathcal{H}}}{\|\zeta_\perp\|_{\mathcal{H}}^2}, \quad f_\perp := f - b\zeta - b_\perp \zeta_\perp.$$

Analogously, $\langle \zeta, f_\perp \rangle_{\mathcal{H}} = \langle \zeta_\perp, f_\perp \rangle_{\mathcal{H}} = 0$. The bounded linear functionals of f may be epxressed in terms of the coefficients defined above:

$$\begin{aligned}
L(f) &= \langle \xi, f \rangle_{\mathcal{H}} = \langle a\zeta + \zeta_\perp, b\zeta + b_\perp \zeta_\perp + f_\perp \rangle_{\mathcal{H}} = ab \|\zeta\|_{\mathcal{H}}^2 + b_\perp \|\zeta_\perp\|_{\mathcal{H}}^2 \\
M(f) &= \langle \zeta, f \rangle_{\mathcal{H}} = \langle \zeta, b\zeta + b_\perp \zeta_\perp + f_\perp \rangle_{\mathcal{H}} = b \|\zeta\|_{\mathcal{H}}^2 \\
\|f\|_{\mathcal{H}}^2 &= \langle b\zeta + b_\perp \zeta_\perp + f_\perp, b\zeta + b_\perp \zeta_\perp + f_\perp \rangle_{\mathcal{H}} = b^2 \|\zeta\|_{\mathcal{H}}^2 + b_\perp^2 \|\zeta_\perp\|_{\mathcal{H}}^2 \|\zeta\|_{\mathcal{H}}^2 + \|f_\perp\|_{\mathcal{H}}^2
\end{aligned}$$

Now we derive an upper bound on $|L(f)| / |M(f)|$. Suppose $|L(f)| / |M(f)| \le C$. Then $|L(f)|$, what is not known, can be bounded above by $C\,|M(f)|$, which can be computed.

$$\left| \frac{L(f)}{M(f)} \right| = \frac{\left| ab \|\zeta\|_{\mathcal{H}}^2 + b_\perp \|\zeta_\perp\|_{\mathcal{H}}^2 \right|}{|b| \|\zeta\|_{\mathcal{H}}^2} \le \frac{|ab| \|\zeta\|_{\mathcal{H}}^2 + |b_\perp| \|\zeta_\perp\|_{\mathcal{H}}^2}{|b| \|\zeta\|_{\mathcal{H}}^2} = |a| + \left| \frac{b_\perp}{b} \right| \frac{\|\zeta_\perp\|_{\mathcal{H}}^2}{\|\zeta\|_{\mathcal{H}}^2}$$

Assuming $\zeta_\perp \neq 0$, one may easily construct a *nasty* function f, i.e., choose b and $b_\perp$ such that $|b_\perp/b|$ is arbitrarily large. What we aim to do is find a nastiness criterion so that if the nastiness is bounded, then $|b_\perp/b|$ is bounded above.

We consider nastiness criteria of the form

$$\text{nasty}(f) = \frac{\langle f, f\rangle_P}{\langle f, f\rangle_{\mathcal{H}_2}},$$

where $\langle\cdot,\cdot\rangle_P$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ are semi-inner products defined on $\mathcal{H}$. Furthermore,

$$\langle f, g\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}_1} + \langle f, g\rangle_{\mathcal{H}_2}$$

where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and $\langle\cdot,\cdot\rangle_{\mathcal{H}_1}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ are the inner products defined on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Thus

$$0 = \left\langle f_{\|}, f_{\perp}\right\rangle_{\mathcal{H}} = \left\langle f_{\|,1}, f_{\perp,1}\right\rangle_{\mathcal{H}_1} + \left\langle f_{\|,2}, f_{\perp,2}\right\rangle_{\mathcal{H}_2}$$

Let $f_{\|} = b\zeta + b_{\perp}\zeta_{\perp}$, so that $f = f_{\|} + f_{\perp}$. Furthermore, let $f_{\|,j}$ and $f_{\perp,j}$, $j = 1, 2$ be the projections of $f_{\|}$ and $f_{\perp}$ into $\mathcal{H}_j$

bilinear functions. Note that using the above notation

$$\begin{aligned} P(f,f) &= P(b\zeta + b_{\perp}\zeta_{\perp} + f_{\perp}, b\zeta + b_{\perp}\zeta_{\perp} + f_{\perp}) \\ &= b^2 P(\zeta,\zeta) + 2bb_{\perp}P(\zeta,\zeta_{\perp}) + b_{\perp}^2 P(f_{\perp}, f_{\perp}) + 2P(b\zeta + b_{\perp}\zeta_{\perp}, f_{\perp}) + P(f_{\perp}, f_{\perp}) \end{aligned}$$

References

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