The algorithms used in this section on integration and the next section on function recovery are all based on linear splines on $[a, b]$. The node set and the linear spline algorithm using n function values are defined for $n \in \mathcal{I} := \{2, 3, \dots\}$ as follows:

$$x_i = a + \frac{i-1}{n-1}(b-a), \qquad i = 1, \dots, n, \tag{20a}$$

$$A_n(f)(x) := \frac{n-1}{b-a}\left[f(x_i)(x_{i+1}-x) + f(x_{i+1})(x-x_i)\right] \qquad \text{for } x_i \le x \le x_{i+1}. \tag{20b}$$

The cost of each function value is one and so the cost of A_n is n. The algorithm A_n is imbedded in the algorithm A_{2n-1}, which uses $2n-2$ subintervals. Thus, $r = 2$ is the cost multiple as described in Section 1.2.

The problem to be solved is univariate integration on the unit interval, $S(f) := \mathrm{INT}(f) := \int_a^b f(x)\,\mathrm{d}x \in \mathcal{G} := \mathbb{R}$. The fixed cost building blocks to construct the adaptive integration algorithm are the composite trapezoidal rules based on $n-1$ trapezoids:

$$T_n(f) := \int_a^b A_n(f)\,\mathrm{d}x = \frac{b-a}{2n-2}[f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

The space of input functions is $\mathcal{F} := \mathcal{V}^1$, the space of functions whose first derivatives have finite variation. The general definitions of some relevant norms and spaces are as follows:

$$\mathrm{Var}(f) := \sup_{\substack{n \in \mathbb{N} \\ a = x_0 < x_1 < \cdots < x_n = b}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \tag{21a}$$

$$\|f\|_p := \begin{cases} \left[\displaystyle\int_a^b |f(x)|^p \,\mathrm{d}x\right]^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{a \le x \le b} |f(x)|, & p = \infty, \end{cases} \tag{21b}$$

$$\mathcal{V}^k := \mathcal{V}^k[a,b] = \{f \in C[a,b] : \mathrm{Var}(f^{(k)}) < \infty\}, \tag{21c}$$

$$\mathcal{W}^{k,p} = \mathcal{W}^{k,p}[a,b] = \{f \in C[a,b] : \|f^{(k)}\|_p < \infty\}. \tag{21d}$$

The stronger semi-norm is $|f|_{\mathcal{F}} := \mathrm{Var}(f')$, while the weaker semi-norm is

$$|f|_{\widetilde{\mathcal{F}}} := \|f' - A_2(f)'\|_1 = \left\| f' - \frac{f(b) - f(a)}{b-a} \right\|_1 = \mathrm{Var}(f - A_2(f)),$$

where $A_2(f) : x \mapsto [f(a)(b-x) + f(b)(x-a)]/(b-a)$ is the linear interpolant of f using the two endpoints of the integration interval. The reason for defining $|f|_{\widetilde{\mathcal{F}}}$ this way is that $|f|_{\widetilde{\mathcal{F}}}$ vanishes if f is a linear function, and linear functions are integrated exactly by the trapezoidal rule. The cone of integrands is defined as

$$\mathcal{C}_{\tau_{a,b}} := \left\{ f \in \mathcal{V}^1 : \mathrm{Var}(f') \le \frac{\tau_{a,b}}{b-a} \left\| f' - \frac{f(b)-f(a)}{b-a} \right\|_1 \right\}. \tag{22}$$

The algorithm for approximating $\left\| f' - \frac{f(b)-f(a)}{b-a} \right\|_1$ is the $\widetilde{\mathcal{F}}$-semi-norm of the linear spline, $A_n(f)$:

$$\begin{aligned}
\widetilde{F}_n(f) &:= |A_n(f)|_{\widetilde{\mathcal{F}}} = \left\| A_n(f)' - A_2(f)' \right\|_1 \\
&= \sum_{i=1}^{n-1} \int_{x_i}^{x_{x+1}} |A_n(f)' - A_2(f)'| dx \\
&= \sum_{i=1}^{n-1} \int_{x_i}^{x_{x+1}} \left| \frac{n-1}{b-a}(f(x_{i+1}) - f(x_i)) - \frac{f(b)-f(a)}{b-a} \right| dx \\
&= \frac{n-1}{b-a} \sum_{i=1}^{n-1} \int_{x_i}^{x_{x+1}} \left| (f(x_{i+1}) - f(x_i)) - \frac{f(b)-f(a)}{n-1} \right| dx \\
&= \frac{n-1}{b-a} \frac{b-a}{n-1} \sum_{i=1}^{n-1} \left| (f(x_{i+1}) - f(x_i)) - \frac{f(b)-f(a)}{n-1} \right| \\
&= \sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(b)-f(a)}{n-1} \right|.
\end{aligned} \tag{23}$$

The variation of the first derivative of the linear spline of f, i.e.,

$$\begin{aligned}
F_n(f) &:= \mathrm{Var}(A_n(f)') = \mathrm{Var}\left(\frac{n-1}{b-a}[f(x_{i+1}) - f(x_i)] \right) \\
&= \sup_{\substack{n \in \mathbb{N} \\ a = x_0 < x_1 < \cdots < x_n = b}} \sum_{i=1}^{n-1} |A_{i+1}(f)' - A_i(f)'|, \\
&= \sum_{i=1}^{n-2} \frac{n-1}{b-a} |f(x_{i+2}) - f(x_{i+1}) - (f(x_{i+1}) - f(x_i))| \\
&= \frac{n-1}{b-a} \sum_{i=1}^{n-2} \left| f(x_i) - 2f(x_{i+1}) + f(x_{i+2}) \right|,
\end{aligned} \tag{24}$$

provides a lower bound on $\mathrm{Var}(f')$ for $n \ge 3$, and can be used in the necessary condition that f lies in $\mathcal{C}_{\tau_{a,b}}$ as described in Remark 4. The Mean Value Theorem implies that

$$\begin{aligned}
F_n(f) &= (n-1) \sum_{i=1}^{n-1} \left| [f(x_{i+2}) - f(x_{i+1})] - [f(x_{i+1}) - f(x_i)] \right| \\
&= \sum_{i=1}^{n-1} |f'(\xi_{i+1}) - f'(\xi_i)| \le \mathrm{Var}(f'),
\end{aligned}$$

where ξ_i is some point in $[x_i, x_{i+1}]$.

5.1. Adaptive Algorithm and Upper Bound on the Cost

Constructing the adaptive algorithm for integration requires an upper bound on the error of T_n and a two-sided bound on the error of $\widetilde{F}_n$. Note that $\widetilde{F}_n(f)$ never overestimates $|f|_{\widetilde{\mathcal{F}}}$ because

$$\begin{aligned}|f|_{\widetilde{\mathcal{F}}} &= \left\|f' - A_2(f)'\right\|_1 = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |f'(x) - A_2(f)'(x)| \, \mathrm{d}x \\ &\geq \sum_{i=1}^{n-1} \left| \int_{x_i}^{x_{i+1}} [f'(x) - A_2(f)'(x)] \, \mathrm{d}x \right| = \|A_n(f)' - A_2(f)'\|_1 = \widetilde{F}_n(f).\end{aligned}$$

Thus, $h_-(n) := 0$ and $\mathfrak{c}_n = \tilde{\mathfrak{c}}_n = 1$.

To find an upper bound on $|f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_n(f)$, note that

$$|f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_n(f) = |f|_{\widetilde{\mathcal{F}}} - \left|A_n(f)\right|_{\widetilde{\mathcal{F}}} \leq \left|f - A_n(f)\right|_{\widetilde{\mathcal{F}}} = \left\|f' - A_n(f)'\right\|_1,$$

since $(f - A_n(f))(x)$ vanishes for $x = a, b$. Moreover,

$$\left\|f' - A_n(f)'\right\|_1 = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left| f'(x) - \frac{n-1}{b-a}[f(x_{i+1}) - f(x_i)] \right| \mathrm{d}x. \tag{25}$$

Now we bound each integral in the summation. For $i = 1, \ldots, n-1$, let $\eta_i(x) = f'(x) - \frac{n-1}{b-a}[f(x_{i+1}) - f(x_i)]$, and let p_i denote the probability that $\eta_i(x)$ is non-negative:

$$p_i = \frac{n-1}{b-a} \int_{x_i}^{x_{i+1}} \mathbb{1}_{[0,\infty)}(\eta_i(x)) \, \mathrm{d}x,$$

and so $1 - p_i$ is the probability that $\eta_i(x)$ is negative. Since $\int_{x_i}^{x_{i+1}} \eta_i(x) \, \mathrm{d}x = 0$, we know that η_i must take on both non-positive and non-negative values. Invoking the Mean Value Theorem, it follows that

$$\begin{aligned}\frac{p_i(b-a)}{n-1} \sup_{x_i \leq x \leq x_{i+1}} \eta_i(x) &\geq \int_{x_i}^{x_{i+1}} \max(\eta_i(x), 0) \, \mathrm{d}x \\ &= \int_{x_i}^{x_{i+1}} \max(-\eta_i(x), 0) \, \mathrm{d}x \leq \frac{-(1-p_i)(b-a)}{n-1} \inf_{x_i \leq x \leq x_{i+1}} \eta_i(x).\end{aligned}$$

These bounds allow us to derive bounds on the integrals in (25):

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} |\eta_i(x)| \,\mathrm{d}x \\
&\quad = \int_{x_i}^{x_{i+1}} \max(\eta_i(x), 0) \,\mathrm{d}x + \int_{x_i}^{x_{i+1}} \max(-\eta_i(x), 0) \,\mathrm{d}x \\
&\quad = 2(1-p_i) \int_{x_i}^{x_{i+1}} \max(\eta_i(x), 0) \,\mathrm{d}x + 2p_i \int_{x_i}^{x_{i+1}} \max(-\eta_i(x), 0) \,\mathrm{d}x \\
&\quad \le \frac{2p_i(1-p_i)(b-a)}{n-1} \left[\sup_{x_i \le x \le x_{i+1}} \eta_i(x) - \inf_{x_i \le x \le x_{i+1}} \eta_i(x) \right] \\
&\quad \le \frac{b-a}{2(n-1)} \left[\sup_{x_i \le x \le x_{i+1}} f'(x) - \inf_{x_i \le x \le x_{i+1}} f'(x) \right],
\end{aligned}$$

since $p_i(1-p_i) \le 1/4$.

Plugging this bound into (25) yields

$$\begin{aligned}
\Big\| f' - \frac{f(b)-f(a)}{b-a} \Big\|_1 - \widetilde{F}_n(f) = |f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_n(f) &\le \big\| f' - A_n(f)' \big\|_1 \\
&\le \frac{b-a}{2n-2} \sum_{i=1}^{n-1} \left[\sup_{x_i \le x \le x_{i+1}} f'(x) - \inf_{x_i \le x \le x_{i+1}} f'(x) \right] \\
&\le \frac{b-a}{2n-2} \operatorname{Var}(f') = \frac{b-a}{2n-2} \, |f|_{\mathcal{F}} \,,
\end{aligned}$$

and so

$$h_+(n) := \frac{b-a}{2n-2}, \qquad \mathfrak{C}_n = \frac{1}{1 - \tau_{a,b}/(2n-2)} \qquad \text{for } n > 1 + \tau_{a,b}/2.$$

Since $\widetilde{F}_2(f) = 0$ by definition, the above inequality for $|f|_{\widetilde{\mathcal{F}}} - \widetilde{F}_2(f)$ implies that

$$2 \Big\| f' - \frac{f(b)-f(a)}{b-a} \Big\|_1 = 2\, |f|_{\widetilde{\mathcal{F}}} \le (b-a)\, |f|_{\mathcal{F}} = (b-a) \operatorname{Var}(f'), \qquad \tau_{\min} = 2/(b-a).$$

The error of the trapezoidal rule in terms of the variation of the first derivative of the integrand is given in [**?** , (7.15)]:

$$\left| \int_a^b f(x)\, dx - T_n(f) \right| \le h(n) \operatorname{Var}(f')$$

$$h(n) := \frac{(b-a)^2}{8(n-1)^2}, \qquad h^{-1}(\varepsilon) = \left\lceil (b-a) \sqrt{\frac{1}{8\varepsilon}} \right\rceil + 1.$$

Given the above definitions of $h, \mathfrak{C}_n, \mathfrak{c}_n$, and $\tilde{\mathfrak{c}}_n$, it is now possible to also specify

$$h_1(n) = h_2(n) = \mathfrak{C}_n h(n) = \frac{(b-a)^2}{4(n-1)(2n-2-\tau_{a,b})}, \tag{26a}$$

$$h_1^{-1}(\varepsilon) = h_2^{-1}(\varepsilon) = 1 + \left\lceil \sqrt{\frac{(b-a)^2\tau_{a,b}}{8\varepsilon} + \frac{\tau_{a,b}^2}{16}} + \frac{\tau_{a,b}}{4} \right\rceil \le 2 + \frac{\tau_{a,b}}{2} + (b-a)\sqrt{\frac{\tau_{a,b}}{8\varepsilon}}. \tag{26b}$$

Moreover, the left side of (13), the stopping criterion inequality in the multistage algorithm, becomes

$$\frac{\tau_{a,b}}{b-a} h(n_i)\mathfrak{C}_{n_i}\widetilde{F}_{n_i}(f) = \frac{\tau_{a,b}\widetilde{F}_{n_i}(f)(b-a)}{4(n_i-1)(2n_i-2-\tau_{a,b})}. \tag{26c}$$

With these preliminaries, Algorithm 3 and Theorem 3 may be applied directly to yield the following adaptive integration algorithm and its guarantee.

Algorithm 4 (Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{T_n\}_{n\in\mathcal{I}}$, $\{\widetilde{F}_n\}_{n\in\mathcal{I}}$, and $\{F_n\}_{n\in\mathcal{I}}$ be as described above. Choose integer $n_{\text{lo}}, n_{\text{hi}}$, such that $n_{\text{lo}} \le n_{\text{hi}}$. Set $i=1$. Let $n_1 = \max\left\{\lceil n_{\text{hi}} \left(\frac{n_{\text{lo}}}{n_{\text{hi}}}\right)^{\frac{1}{1+b-a}}\rceil, 3\right\}$. Let $\tau_{a,b} = 2n_1 - 3$. For any error tolerance ε and input function f, do the following:

Stage 1. Estimate $\left\|f' - \frac{f(b)-f(a)}{b-a}\right\|_1$ and bound $\mathrm{Var}(f')$. Compute $\widetilde{F}_{n_i}(f)$ in (23) and $F_{n_i}(f)$ in (24).

Stage 2. Check the necessary condition for $f \in \mathcal{C}_{\tau_{a,b}}$. Compute

$$\tau_{\min,n_i} = \frac{F_{n_i}(f)}{\widetilde{F}_{n_i}(f) + (b-a)F_{n_i}(f)/(2n_i-2)}.$$

If $\tau_{a,b} \ge \tau_{\min,n_i}$, then go to stage 3. Otherwise, set $\tau_{a,b} = 2\tau_{\min,n_i}$. If $n_i \ge (\tau_{a,b}+1)/2$, then go to stage 3. Otherwise, choose

$$n_{i+1} = 1 + (n_i - 1)\left\lceil \frac{\tau_{a,b}+1}{2n_i-2} \right\rceil.$$

Go to Stage 1.

Stage 3. Check for convergence. Check whether n_i is large enough to satisfy the error tolerance, i.e.

$$\widetilde{F}_{n_i}(f) \le \frac{4\varepsilon(n_i-1)(2n_i-2-\tau_{a,b})}{\tau_{a,b}(b-a)}.$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm. If this is not true, choose

$$n_{i+1} = 1 + (n_i - 1)\max\left\{2, \left\lceil \frac{1}{(n_i-1)}\sqrt{\frac{\tau_{a,b}(b-a)\widetilde{F}_{n_i}(f)}{8\varepsilon}} \right\rceil \right\}.$$

Go to Stage 1.

Theorem 7. *Let $\sigma > 0$ be some fixed parameter, and let $\mathcal{B}_\sigma = \{f \in \mathcal{V}^1 : \operatorname{Var}(f') \le \sigma\}$. Let $T \in \mathcal{A}(\mathcal{B}_\sigma, \mathbb{R}, \mathrm{INT}, \Lambda^{\mathrm{std}})$ be the non-adaptive trapezoidal rule defined by Algorithm 1, and let $\varepsilon > 0$ be the error tolerance. Then this algorithm succeeds for $f \in \mathcal{B}_\sigma$, i.e., $|\mathrm{INT}(f) - T(f,\varepsilon)| \le \varepsilon$, and the cost of this algorithm is $\left\lceil (b-a)\sqrt{\sigma/(8\varepsilon)} \right\rceil + 1$, regardless of the size of $\operatorname{Var}(f')$.*

Now let $T \in \mathcal{A}(\mathcal{C}_{\tau_{a,b}}, \mathbb{R}, \mathrm{INT}, \Lambda^{\mathrm{std}})$ be the adaptive trapezoidal rule defined by Algorithm 4, and let $\tau_{a,b}$, n_1, and ε be as described there. Let $\mathcal{C}_{\tau_{a,b}}$ be the cone of functions defined in (22). Then it follows that Algorithm 4 is successful for all functions in $\mathcal{C}_{\tau_{a,b}}$, i.e., $|\mathrm{INT}(f) - T(f,\varepsilon)| \le \varepsilon$. Moreover, the cost of this algorithm is bounded below and above as follows:

$$\begin{aligned}
&\max\left(\left\lceil \frac{\tau_{a,b}+1}{2} \right\rceil, \left\lceil (b-a)\sqrt{\frac{\operatorname{Var}(f')}{8\varepsilon}} \right\rceil\right) + 1 \\
&\qquad \le \max\left(\left\lceil \frac{\tau_{a,b}+1}{2} \right\rceil, \left\lceil (b-a)\sqrt{\frac{\tau_{a,b}\left\| f' - \frac{f(b)-f(a)}{b-a} \right\|_1}{8\varepsilon}} \right\rceil\right) + 1 \\
&\qquad\qquad \le \operatorname{cost}(T, f; \varepsilon) \\
&\le (b-a)\sqrt{\frac{\tau_{a,b}\left\| f' - f(1) + f(0) \right\|_1}{2\varepsilon}} + \tau_{a,b} + 4 \le (b-a)\sqrt{\frac{\tau_{a,b}(b-a)\operatorname{Var}(f')}{4\varepsilon}} + \tau_{a,b} + 4.
\end{aligned} \tag{27}$$

The algorithm is computationally stable, meaning that the minimum and maximum costs for all integrands, f, with fixed $\left\| f' - f(1) + f(0) \right\|_1$ or $\operatorname{Var}(f')$ are an ε-independent constant of each other.

5.2. Lower Bound on the Computational Cost

Next, we derive a lower bound on the cost of approximating functions in the ball $\mathcal{B}_\sigma$ and in the cone $\mathcal{C}_\tau$ by constructing fooling functions. Following the arguments of Section 4, we choose the triangle shaped function with zero end point value $f_0 : x \mapsto 1/2 - |a+b-2x|/(2b-2a)$. Then

$$|f_0|_{\widetilde{\mathcal{F}}} = \left\| f_0' - \frac{f_0(b) - f_0(a)}{b-a} \right\|_1 = \frac{1}{b-a}\int_a^b |\operatorname{sign}(a+b-2x)| \, \mathrm{d}x = 1,$$
$$|f_0|_{\mathcal{F}} = \operatorname{Var}(f_0') = 2 = \tau_{\min}.$$

(Deduction:)

$$
\begin{aligned}
f_0' &= -\frac{1}{2(b-a)}|a+b-2x|', \\
&= \frac{1}{2(b-a)} 2\,\mathrm{sign}(a+b-2x), \\
&= \frac{1}{b-a}\,\mathrm{sign}(a+b-2x).
\end{aligned}
$$

So

$$
\begin{aligned}
|f_0|_{\widetilde{\mathcal{F}}} &= \frac{1}{b-a}\int_a^b |\mathrm{sign}(a+b-2x)|\,dx, \\
&= \frac{1}{b-a}\int_a^b 1dx, \\
&= \frac{1}{b-a}(b-a) = 1.
\end{aligned}
$$

For any $n \in \mathcal{J} := \mathbb{N}_0$, suppose that the one has the data $L_i(f) = f(\xi_i)$, $i = 1, \ldots, n$ for arbitrary ξ_i, where $a = \xi_0 \le \xi_1 < \cdots < \xi_n \le \xi_{n+1} = b$. There must be some $j = 0, \ldots, n$ such that $\xi_{j+1} - \xi_j \ge (b-a)/(n+1)$. The function f_1 is defined as a triangle function on the interval $[\xi_j, \xi_{j+1}]$:

$$
f_1(x) := \begin{cases} \dfrac{\xi_{j+1} - \xi_j - |\xi_{j+1} + \xi_j - 2x|}{8} & \xi_j \le x \le \xi_{j+1}, \\ 0 & \text{otherwise.} \end{cases}
$$

This is a piecewise linear function whose derivative changes from 0 to 1/4 to $-1/4$ to 0 provided $0 < \xi_j < \xi_{j+1} < 1$, and so $|f_1|_{\mathcal{F}} = \mathrm{Var}(f_1') \le 1$. Moreover,

$$
\begin{gathered}
\mathrm{INT}(f) = \int_0^1 f_1(x)\,\mathrm{d}x = \frac{(\xi_{j+1} - \xi_j)^2}{16} \ge \frac{(b-a)^2}{16(n+1)^2} =: g(n), \\
g^{-1}(\varepsilon) = \left\lceil (b-a)\sqrt{\frac{1}{16\varepsilon}} \right\rceil - 1.
\end{gathered}
$$

Using these choices of f_0 and f_1, along with the corresponding g above, one may invoke Theorems 4–6, and Corollary 1 to obtain the following theorem.

Theorem 8. *For $\sigma > 0$ let $\mathcal{B}_\sigma = \{f \in \mathcal{V}^1 : \mathrm{Var}(f') \le \sigma\}$. The complexity of integration on this ball is bounded below as*

$$
\mathrm{comp}(\varepsilon, \mathcal{A}(\mathcal{B}_\sigma, \mathbb{R}, \mathrm{INT}, \Lambda^{\mathrm{std}}), \mathcal{B}_s) \ge \left\lceil (b-a)\sqrt{\frac{\min(s,\sigma)}{16\varepsilon}} \right\rceil - 1.
$$

Algorithm 1 using the trapezoidal rule has optimal order in the sense of Theorem 5.

For $\tau > 2$, the complexity of the integration problem over the cone of functions $\mathcal{C}_\tau$ defined in (22) is bounded below as

$$\mathrm{comp}(\varepsilon, \mathcal{A}(\mathcal{C}_\tau, \mathbb{R}, \mathrm{INT}, \Lambda^{\mathrm{std}}), \mathcal{B}_s) \geq \left\lceil (b-a)\sqrt{\frac{(\tau-2)s}{32\tau\varepsilon}} \right\rceil - 1.$$

The adaptive trapezoidal Algorithm 4 has optimal order for integration of functions in $\mathcal{C}_\tau$ in the sense of Corollary 1.