Proof of Error of G_n

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Given the data sites $x_i = (i-1)/(n-1)$, $i = 0, \ldots, n$, let the $\mathcal{L}_\infty$ norm of the f' be approximated by

$$G_n(f) = (n-1) \sup_{i=1,\ldots,n-1} |f(x_{i+1}) - f(x_i)|$$

For any $x \in [x_i, x_{i+1}]$, note that

$$\begin{aligned}
&|f'(x)| - (n-1)\,|f(x_{i+1}) - f(x_i)| \\
&\qquad \le |f'(x) - (n-1)[f(x_{i+1}) - f(x_i)]| \\
&\qquad = \left| \int_{x_i}^{x_{i+1}} f''(t)[(n-1)(t-x_i) - 1_{[x,x_{i+1}]}(t)]\,\mathrm{d}t \right| \\
&\qquad \le \sup_{x_i \le t \le x_{i+1}} |f''(t)| \int_{x_i}^{x_{i+1}} \left|(n-1)(t-x_i) - 1_{[x,x_{i+1}]}(t)\right|\,\mathrm{d}t \\
&\qquad = \sup_{x_i \le t \le x_{i+1}} |f''(t)| \left\{ \frac{1}{2(n-1)} - (x-x_i)[1-(n-1)(x-x_i)] \right\} \\
&\qquad \le \frac{1}{2(n-1)} \sup_{x_i \le t \le x_{i+1}} |f''(t)|
\end{aligned}$$

Furthermore, this inequality is tight if f'' is constant second derivative and f' does not change sign over $[x_i, x_{i+1}]$, and $x = x_i$ or x_{i+1}. Applying the above argument for $i = 0, \ldots, n-1$ implies that

$$\|f'\|_\infty - G_n(f) \le \frac{\|f''\|_\infty}{2(n-1)}$$

with equality holding when has a constant second derivative and its first derivative does not change sign over $[0, 1]$.

Preprint submitted to Journal of Complexity *December 17, 2012*