A Better $\mathcal{G}$-Semi-Norm for Algorithms Based on Linear Splines

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Abstract

Keywords:

1. Why Do We Need a Better $\mathcal{G}$-Semi-Norm

In [1] we presented guaranteed automatic algorithms for integration and function recovery. One weakness of these algorithms was that if $\|f'\|_1$ ($\|f'\|_\infty$) is large, but $\|f''\|_1$ ($\|f''\|_\infty$) is small, then the error bounds in the automatic algorithms for integration (function recovery) can be very conservative. The input function $f : x \mapsto 100x$ is a prime example.

Here we present a substitute for $\|f'\|_p$, $p = 1, \infty$ that overcomes this weakness. If the function is nearly linear, then the algorithms in [1] converges quite quickly.

The following semi-norm compares measures the deviation of the first derivative of the function from its average value:

$$|f|_{\mathcal{G},p} = \|f' - f(1) + f(0)\|_p . \tag{1}$$

Let us verify that this indeed a semi-norm. Clearly, it is non-negative and vanishes when f is the zero function. So, the only other thing to verify is the triangle inequality, which follows since $f \mapsto f' - f(1) - f(0)$ is a linear operator.

2. Estimating the New $\mathcal{G}$-Semi-Norm

Given $n = 2, 3, \ldots$, let $x_i = (i-1)/(n-1)$ for $i = 1, \ldots, n$. Let $\tilde{f}_n$ denote the linear spline based on the function values $f(x_{i,n-1})$, $i = 0, \ldots, n-1$. We estimate $|f|_{\mathcal{G},p}$ by the algorithm $G_{p,n} : f \mapsto \left|\tilde{f}_n\right|_{\mathcal{G},p}$.

This algorithm never overestimates the norm. To see why, first note that $|f|_{\mathcal{G},p} = |f + f_{\text{lin}}|_{\mathcal{G},p}$ and $G_{p,n}(f) = G_{p,n}(f + f_{\text{lin}})$ for any linear function f_{lin}. For any given f, choose f_{lin} to interpolate f at the two endpoints of the interval, i.e., $f_{\text{lin}} : x \mapsto f(1)x + f(0)(1-x)$. Then it follows that

$$|f|_{\mathcal{G},p} - G_{p,n}(f) = \left|f - f_{\text{lin}}\right|_{\mathcal{G},p} - G_{p,n}(f - f_{\text{lin}}) = \|f' - f'_{\text{lin}}\|_p - \left\|\tilde{f}' - f'_{\text{lin}}\right\|_p,$$

since $f - f_{\mathrm{lin}}$ vanishes at 0 and 1. For the case $1 \le p < \infty$ it follows that

$$\begin{aligned}
&\|f' - f'_{\mathrm{lin}}\|_p^p - \left\|\tilde{f}' - f'_{\mathrm{lin}}\right\|_p^p \\
&\quad = \int_0^1 \left[\left|(f' - f'_{\mathrm{lin}})(x)\right|^p - \left|(\tilde{f}' - f'_{\mathrm{lin}})(x)\right|^p\right] \mathrm{d}x \\
&\quad = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left[\left|(f' - f'_{\mathrm{lin}})(x)\right|^p - \left|(\tilde{f}' - f'_{\mathrm{lin}})(x)\right|^p\right] \mathrm{d}x \\
&\quad = \sum_{i=1}^{n-1} \left[\int_{x_i}^{x_{i+1}} \left|(f' - f'_{\mathrm{lin}})(x)\right|^p - \left|\int_{x_i}^{x_{i+1}} (\tilde{f}' - f'_{\mathrm{lin}})(x)\,\mathrm{d}x\right|^p\right] \\
&\quad \ge 0,
\end{aligned}$$

since $\tilde{f}' - f'_{\mathrm{lin}}$ is constant over each interval $[x_i, x_{i+1}]$. The proof is similar for $p = \infty$.

An upper bound on the the error of our approximation to the $\mathcal{G}$-semi-norm is

$$|f|_{\mathcal{G},p} - G_{p,n}(f) = |f|_{\mathcal{G},p} - \left|\tilde{f}_n\right|_{\mathcal{G},p} \le \left|f - \tilde{f}_n\right|_{\mathcal{G},p} = \left\|f' - \tilde{f}'_n\right\|_p,$$

since $(f - \tilde{f}_n)(x)$ vanishes for $x = 0, 1$. We are most interested in the cases $p = 1, \infty$.

For any $i = 1, \ldots, n-1$, the idea is to estimate how large $(f - \tilde{f}_n)(x)$ can become for $x \in [x_i, x_{i+1}]$. Note that

$$\begin{aligned}
f'(x) - \tilde{f}'_n(x) &= f'(x) - (n-1)[f(x_{i+1}) - f(x_i)] \\
&= -(n-1) \int_{x_i}^{x_{i+1}} v(t,x) f''(t)\,\mathrm{d}t
\end{aligned} \tag{2}$$

where

$$v(t,x) = \begin{cases} x_i - t, & x_i \le t \le x, \\ t - x_{i+1}, & x < t \le x_{i+1}. \end{cases}$$

This implies the following upper bound on a piece of $\|f'\|_1$:

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} |f'(x) - \tilde{f}'_n(x)|\,\mathrm{d}x \\
&\quad \le (n-1) \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} |v(t,x)||f''(t)|\mathrm{d}t\,\mathrm{d}x \\
&\quad \le (n-1) \int_{x_i}^{x_{i+1}} 2(t - x_i)(x_{i+1} - t)|f''(t)|\,\mathrm{d}t \\
&\quad \le (n-1) \max_{x_i \le t \le x_{i+1}} |2(t - x_i)(x_{i+1} - t)| \int_{x_i}^{x_{i+1}} |f''(t)|\,\mathrm{d}t \\
&\quad \le \frac{1}{2(n-1)} \int_{x_i}^{x_{i+1}} |f''(t)|\,\mathrm{d}t.
\end{aligned}$$

Applying this inequality for $i = 1, \ldots, n-1$ leads to

$$\begin{aligned}
\left| |f|_{\mathcal{G},1} - G_{1,n}(f) \right| &\leq \left\| f' - \tilde{f}_n \right\|_1 \\
&= \sum_{i=1}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} |f'(x)| \, \mathrm{d}x - |f(x_{i+1}) - f(x_i)| \right\} \\
&\leq \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |f''(t)| \, \mathrm{d}t = \frac{\|f''\|_1}{2(n-1)}.
\end{aligned}$$

The bound for the case $p = \infty$ follows by a similar argument. Starting from (2), it follows that for $x \in [x_i, x_{i+1}]$,

$$\begin{aligned}
|f'(x) - \tilde{f}_n(x)| &\leq (n-1) \left| \int_{x_i}^{x_{i+1}} v(t,x) f''(t) \, \mathrm{d}t \right| \\
&\leq (n-1) \|f''\|_\infty \int_{x_i}^{x_{i+1}} |v(t,x)| \, \mathrm{d}t \\
&= (n-1) \|f''\|_\infty \left\{ \frac{1}{2(n-1)^2} - (x - x_i)(x_{i+1} - x) \right\} \\
&\leq \frac{\|f''\|_\infty}{2(n-1)}.
\end{aligned}$$

Applying the above argument for $i = 1, \ldots, n-1$ leads to

$$\left| |f|_{\mathcal{G},\infty} - G_{\infty,n}(f) \right| \leq \left\| f' - \tilde{f}_n \right\|_\infty \leq \frac{\|f''\|_\infty}{2(n-1)}.$$

3. Bounding the Error in Terms of the New $\mathcal{G}$-Semi-Norm

3.1. Integration

For the integration problem, we use the composite trapezoidal rule,

$$A_n(f) = \frac{1}{2(n-1)} \left[f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n) \right].$$

We consider how to bound the error of this approximation for integrands with given norm $|f|_{\mathcal{G},1}$. Using the trick of subtracting a the function f_{lin} defined above, note that

$$\begin{aligned}
\left| \int_0^1 f(x) \, \mathrm{d}x - A_n(f) \right| &= \left| \int_0^1 (f - f_{\mathrm{lin}})(x) \, \mathrm{d}x - A_n(f - f_{\mathrm{lin}}) \right| \\
&\leq \frac{\|f' - f'_{\mathrm{lin}}\|_1}{8(n-1)^2} = \frac{|f|_{\mathcal{G},1}}{8(n-1)^2}
\end{aligned}$$

3.2. Approximation

For the approximation problem, we use linear splines. Using again the trick of subtracting a the function f_{lin} defined above, note that

$$\begin{aligned}|f(x) - A_n(f)(x)| &= |(f - f_{\text{lin}})(x)\,\mathrm{d}x - A_n(f - f_{\text{lin}})(x)| \\ &\le \frac{\|f' - f'_{\text{lin}}\|_\infty}{8(n-1)^2} = \frac{|f|_{\mathcal{G},\infty}}{8(n-1)^2}.\end{aligned}$$

4. Lower Bound Using the New $\mathcal{G}$-Semi-Norm

With the new $\mathcal{G}$-semi-norm, the function $f_0 : x \mapsto x$ is no longer works because for this choice one would have $|f_0|_{\mathcal{G},p} = 0$. Instead we choose:

$$f_0(x) = x(1-x).$$

For $p = 1$ one has

$$|f_0|_{\mathcal{G},1} = \|f_0'\|_1 = \frac{1}{2}, \qquad \|f_0''\|_1 = 2, \qquad \tau_0 = 4.$$

For $p = \infty$ one has

$$|f_0|_{\mathcal{G},1} = \|f_0'\|_1 = 1, \qquad \|f_0''\|_1 = 2, \qquad \tau_0 = 2.$$

Using these, the complexity lower bound for integration is

$$\min\left(\frac{(\tau-4)\,|f'|_{\mathcal{G},1}}{15(\tau-2)\varepsilon}, \sqrt{\frac{\sqrt{3}\tau(\tau-4)\,|f'|_{\mathcal{G},1}}{160(\tau-2)\varepsilon}}\right) - 1,$$

provided $\tau > 4$. The complexity lower bound for approximation is

$$\min\left(\frac{3\sqrt{3}(\tau-2)\,|f'|_{\mathcal{G},1}}{64(\tau-1)\varepsilon}, \sqrt{\frac{\tau(\tau-2)\,|f'|_{\mathcal{G},1}}{128(\tau-1)\varepsilon}}\right) - 1,$$

provided $\tau > 2$.

[1] N. Clancy, Y. Ding, C. Hamilton, F. J. Hickernell, Y. Zhang, The complexity of guaranteed automatic algorithms: Cones, not balls, submitted for publication, arXiv.org:1303.2412 [math.NA] (2013).