

The algorithms used in this section on integration and the next section on function recovery are all based on linear splines on  $[a, b]$ . The node set and the linear spline algorithm using  $n$  function values are defined for  $n \in \mathcal{I} := \{2, 3, \dots\}$  as follows:

$$x_i = a + \frac{i-1}{n-1}(b-a), \quad i = 1, \dots, n, \quad (20a)$$

$$A_n(f)(x) := \frac{n-1}{b-a} [f(x_i)(x_{i+1}-x) + f(x_{i+1})(x-x_i)]$$

for  $x_i \leq x \leq x_{i+1}$ . (20b)

The cost of each function value is one and so the cost of  $A_n$  is  $n$ . The algorithm  $A_n$  is imbedded in the algorithm  $A_{2n-1}$ , which uses  $2n-2$  subintervals. Thus,  $r=2$  is the cost multiple as described in Section 1.2.

The problem to be solved is univariate integration on the unit interval,  $S(f) := \text{INT}(f) := \int_a^b f(x) dx \in \mathcal{G} := \mathbb{R}$ . The fixed cost building blocks to construct the adaptive integration algorithm are the composite trapezoidal rules based on  $n-1$  trapezoids:

$$T_n(f) := \int_a^b A_n(f) dx = \frac{b-a}{2n-2} [f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)].$$

The space of input functions is  $\mathcal{F} := \mathcal{V}^1$ , the space of functions whose first derivatives have finite variation. The general definitions of some relevant norms and spaces are as follows:

$$\text{Var}(f) := \sup_{\substack{n \in \mathbb{N} \\ a=x_0 < x_1 < \dots < x_n=b}} \sum_{i=1}^n |f(x_i) - f(x_{i-1})|, \quad (21a)$$

$$\|f\|_p := \begin{cases} \left[ \int_a^b |f(x)|^p dx \right]^{1/p}, & 1 \leq p < \infty, \\ \sup_{a \leq x \leq b} |f(x)|, & p = \infty, \end{cases} \quad (21b)$$

$$\mathcal{V}^k := \mathcal{V}^k[a, b] = \{f \in C[a, b] : \text{Var}(f^{(k)}) < \infty\}, \quad (21c)$$

$$\mathcal{W}^{k,p} = \mathcal{W}^{k,p}[a, b] = \{f \in C[a, b] : \|f^{(k)}\|_p < \infty\}. \quad (21d)$$

The stronger semi-norm is  $|f|_{\mathcal{F}} := \text{Var}(f')$ , while the weaker semi-norm is

$$|f|_{\tilde{\mathcal{F}}} := \|f' - A_2(f)'\|_1 = \left\| f' - \frac{f(b) - f(a)}{b-a} \right\|_1 = \text{Var}(f - A_2(f)),$$

where  $A_2(f) : x \mapsto [f(a)(b-x) + f(b)(x-a)]/(b-a)$  is the linear interpolant of  $f$  using the two endpoints of the integration interval. The reason for defining  $|f|_{\tilde{\mathcal{F}}}$  this way is that  $|f|_{\tilde{\mathcal{F}}}$  vanishes if  $f$  is a linear function, and linear functions

are integrated exactly by the trapezoidal rule. The cone of integrands is defined as

$$\mathcal{C}_{\tau_{a,b}} := \left\{ f \in \mathcal{V}^1 : \text{Var}(f') \leq \frac{\tau_{a,b}}{b-a} \left\| f' - \frac{f(b) - f(a)}{b-a} \right\|_1 \right\}. \quad (22)$$

The algorithm for approximating  $\left\| f' - \frac{f(b) - f(a)}{b-a} \right\|_1$  is the  $\tilde{\mathcal{F}}$ -semi-norm of the linear spline,  $A_n(f)$ :

$$\begin{aligned} \tilde{F}_n(f) &:= |A_n(f)|_{\tilde{\mathcal{F}}} = \|A_n(f)' - A_2(f)'\|_1 \\ &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |A_n(f)' - A_2(f)'| dx \\ &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left| \frac{n-1}{b-a} (f(x_{i+1}) - f(x_i)) - \frac{f(b) - f(a)}{b-a} \right| dx \\ &= \frac{n-1}{b-a} \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left| (f(x_{i+1}) - f(x_i)) - \frac{f(b) - f(a)}{n-1} \right| dx \\ &= \frac{n-1}{b-a} \frac{b-a}{n-1} \sum_{i=1}^{n-1} \left| (f(x_{i+1}) - f(x_i)) - \frac{f(b) - f(a)}{n-1} \right| \\ &= \sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(b) - f(a)}{n-1} \right|. \end{aligned} \quad (23)$$

The variation of the first derivative of the linear spline of  $f$ , i.e.,

$$\begin{aligned} F_n(f) &:= \text{Var}(A_n(f)') = \text{Var} \left( \frac{n-1}{b-a} [f(x_{i+1}) - f(x_i)] \right) \\ &= \sup_{\substack{n \in \mathbb{N} \\ a=x_0 < x_1 < \dots < x_n=b}} \sum_{i=1}^{n-1} |A_{i+1}(f)' - A_i(f)'|, \\ &= \sum_{i=1}^{n-2} \frac{n-1}{b-a} |f(x_{i+2}) - f(x_{i+1}) - (f(x_{i+1}) - f(x_i))| \\ &= \frac{n-1}{b-a} \sum_{i=1}^{n-2} |f(x_i) - 2f(x_{i+1}) + f(x_{i+2})|, \end{aligned} \quad (24)$$

provides a lower bound on  $\text{Var}(f')$  for  $n \geq 3$ , and can be used in the necessary condition that  $f$  lies in  $\mathcal{C}_{\tau_{a,b}}$  as described in Remark 4. The Mean Value Theorem implies that

$$\begin{aligned} F_n(f) &= (n-1) \sum_{i=1}^{n-1} |[f(x_{i+2}) - f(x_{i+1})] - [f(x_{i+1}) - f(x_i)]| \\ &= \sum_{i=1}^{n-1} |f'(\xi_{i+1}) - f'(\xi_i)| \leq \text{Var}(f'), \end{aligned}$$

where  $\xi_i$  is some point in  $[x_i, x_{i+1}]$ .

### 5.1. Adaptive Algorithm and Upper Bound on the Cost

Constructing the adaptive algorithm for integration requires an upper bound on the error of  $T_n$  and a two-sided bound on the error of  $\tilde{F}_n$ . Note that  $\tilde{F}_n(f)$  never overestimates  $|f|_{\tilde{\mathcal{F}}}$  because

$$\begin{aligned} |f|_{\tilde{\mathcal{F}}} &= \|f' - A_2(f)'\|_1 = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} |f'(x) - A_2(f)'(x)| \, dx \\ &\geq \sum_{i=1}^{n-1} \left| \int_{x_i}^{x_{i+1}} [f'(x) - A_2(f)'(x)] \, dx \right| = \|A_n(f)' - A_2(f)'\|_1 = \tilde{F}_n(f). \end{aligned}$$

Thus,  $h_-(n) := 0$  and  $\mathbf{c}_n = \tilde{\mathbf{c}}_n = 1$ .

To find an upper bound on  $|f|_{\tilde{\mathcal{F}}} - \tilde{F}_n(f)$ , note that

$$|f|_{\tilde{\mathcal{F}}} - \tilde{F}_n(f) = |f|_{\tilde{\mathcal{F}}} - |A_n(f)|_{\tilde{\mathcal{F}}} \leq |f - A_n(f)|_{\tilde{\mathcal{F}}} = \|f' - A_n(f)'\|_1,$$

since  $(f - A_n(f))(x)$  vanishes for  $x = a, b$ . Moreover,

$$\|f' - A_n(f)'\|_1 = \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left| f'(x) - \frac{n-1}{b-a} [f(x_{i+1}) - f(x_i)] \right| \, dx. \quad (25)$$

Now we bound each integral in the summation. For  $i = 1, \dots, n-1$ , let  $\eta_i(x) = f'(x) - \frac{n-1}{b-a} [f(x_{i+1}) - f(x_i)]$ , and let  $p_i$  denote the probability that  $\eta_i(x)$  is non-negative:

$$p_i = \frac{n-1}{b-a} \int_{x_i}^{x_{i+1}} \mathbb{1}_{[0, \infty)}(\eta_i(x)) \, dx,$$

and so  $1-p_i$  is the probability that  $\eta_i(x)$  is negative. Since  $\int_{x_i}^{x_{i+1}} \eta_i(x) \, dx = 0$ , we know that  $\eta_i$  must take on both non-positive and non-negative values. Invoking the Mean Value Theorem, it follows that

$$\begin{aligned} \frac{p_i(b-a)}{n-1} \sup_{x_i \leq x \leq x_{i+1}} \eta_i(x) &\geq \int_{x_i}^{x_{i+1}} \max(\eta_i(x), 0) \, dx \\ &= \int_{x_i}^{x_{i+1}} \max(-\eta_i(x), 0) \, dx \leq \frac{-(1-p_i)(b-a)}{n-1} \inf_{x_i \leq x \leq x_{i+1}} \eta_i(x). \end{aligned}$$

These bounds allow us to derive bounds on the integrals in (25):

$$\begin{aligned}
& \int_{x_i}^{x_{i+1}} |\eta_i(x)| \, dx \\
&= \int_{x_i}^{x_{i+1}} \max(\eta_i(x), 0) \, dx + \int_{x_i}^{x_{i+1}} \max(-\eta_i(x), 0) \, dx \\
&= 2(1-p_i) \int_{x_i}^{x_{i+1}} \max(\eta_i(x), 0) \, dx + 2p_i \int_{x_i}^{x_{i+1}} \max(-\eta_i(x), 0) \, dx \\
&\leq \frac{2p_i(1-p_i)(b-a)}{n-1} \left[ \sup_{x_i \leq x \leq x_{i+1}} \eta_i(x) - \inf_{x_i \leq x \leq x_{i+1}} \eta_i(x) \right] \\
&\leq \frac{b-a}{2(n-1)} \left[ \sup_{x_i \leq x \leq x_{i+1}} f'(x) - \inf_{x_i \leq x \leq x_{i+1}} f'(x) \right],
\end{aligned}$$

since  $p_i(1-p_i) \leq 1/4$ .

Plugging this bound into (25) yields

$$\begin{aligned}
\left\| f' - \frac{f(b) - f(a)}{b-a} \right\|_1 - \tilde{F}_n(f) &= |f|_{\tilde{\mathcal{F}}} - \tilde{F}_n(f) \\
&\leq \|f' - A_n(f)'\|_1 \\
&\leq \frac{b-a}{2n-2} \sum_{i=1}^{n-1} \left[ \sup_{x_i \leq x \leq x_{i+1}} f'(x) - \inf_{x_i \leq x \leq x_{i+1}} f'(x) \right] \\
&\leq \frac{b-a}{2n-2} \text{Var}(f') = \frac{b-a}{2n-2} |f|_{\mathcal{F}},
\end{aligned}$$

and so

$$h_+(n) := \frac{b-a}{2n-2}, \quad \mathfrak{C}_n = \frac{1}{1 - \tau_{a,b}/(2n-2)} \quad \text{for } n > 1 + \tau_{a,b}/2.$$

Since  $\tilde{F}_2(f) = 0$  by definition, the above inequality for  $|f|_{\tilde{\mathcal{F}}} - \tilde{F}_2(f)$  implies that

$$2 \left\| f' - \frac{f(b) - f(a)}{b-a} \right\|_1 = 2 |f|_{\tilde{\mathcal{F}}} \leq (b-a) |f|_{\mathcal{F}} = (b-a) \text{Var}(f'), \quad \tau_{\min} = 2/(b-a).$$

The error of the trapezoidal rule in terms of the variation of the first derivative of the integrand is given in [? , (7.15)]:

$$\begin{aligned}
& \left| \int_a^b f(x) \, dx - T_n(f) \right| \leq h(n) \text{Var}(f') \\
& h(n) := \frac{(b-a)^2}{8(n-1)^2}, \quad h^{-1}(\varepsilon) = \left\lceil (b-a) \sqrt{\frac{1}{8\varepsilon}} \right\rceil + 1.
\end{aligned}$$

Given the above definitions of  $h$ ,  $\mathfrak{C}_n$ ,  $\mathfrak{c}_n$ , and  $\tilde{\mathfrak{c}}_n$ , it is now possible to also specify

$$h_1(n) = h_2(n) = \mathfrak{C}_n h(n) = \frac{(b-a)^2}{4(n-1)(2n-2-\tau_{a,b})}, \quad (26a)$$

$$h_1^{-1}(\varepsilon) = h_2^{-1}(\varepsilon) = 1 + \left\lceil \sqrt{\frac{(b-a)^2 \tau_{a,b}}{8\varepsilon} + \frac{\tau_{a,b}^2}{16} + \frac{\tau_{a,b}}{4}} \right\rceil \leq 2 + \frac{\tau_{a,b}}{2} + (b-a) \sqrt{\frac{\tau_{a,b}}{8\varepsilon}}. \quad (26b)$$

Moreover, the left side of (13), the stopping criterion inequality in the multi-stage algorithm, becomes

$$\frac{\tau_{a,b}}{b-a} h(n_i) \mathfrak{C}_{n_i} \tilde{F}_{n_i}(f) = \frac{\tau_{a,b} \tilde{F}_{n_i}(f)(b-a)}{4(n_i-1)(2n_i-2-\tau_{a,b})}. \quad (26c)$$

With these preliminaries, Algorithm 3 and Theorem 3 may be applied directly to yield the following adaptive integration algorithm and its guarantee.

**Algorithm 4** (Adaptive Univariate Integration). Let the sequence of algorithms  $\{T_n\}_{n \in \mathcal{I}}$ ,  $\{\tilde{F}_n\}_{n \in \mathcal{I}}$ , and  $\{F_n\}_{n \in \mathcal{I}}$  be as described above. Choose integer  $n_{\text{lo}}, n_{\text{hi}}$ , such that  $n_{\text{lo}} \leq n_{\text{hi}}$ . Set  $i = 1$ . Let  $n_1 = \max \left\{ \left\lceil n_{\text{hi}} \left( \frac{n_{\text{lo}}}{n_{\text{hi}}} \right)^{\frac{1}{1+b-a}} \right\rceil, 3 \right\}$ . Let  $\tau_{a,b} = 2n_1 - 3$ . For any error tolerance  $\varepsilon$  and input function  $f$ , do the following:

**Stage 1. Estimate**  $\left\| f' - \frac{f(b)-f(a)}{b-a} \right\|_1$  **and bound**  $\text{Var}(f')$ . Compute  $\tilde{F}_{n_i}(f)$  in (23) and  $F_{n_i}(f)$  in (24).

**Stage 2. Check the necessary condition for**  $f \in \mathcal{C}_{\tau_{a,b}}$ . Compute

$$\tau_{\min, n_i} = \frac{F_{n_i}(f)}{\tilde{F}_{n_i}(f) + (b-a)F_{n_i}(f)/(2n_i-2)}.$$

If  $\tau_{a,b} \geq \tau_{\min, n_i}$ , then go to stage 3. Otherwise, set  $\tau_{a,b} = 2\tau_{\min, n_i}$ . If  $n_i \geq (\tau_{a,b} + 1)/2$ , then go to stage 3. Otherwise, choose

$$n_{i+1} = 1 + (n_i - 1) \left\lceil \frac{\tau_{a,b} + 1}{2n_i - 2} \right\rceil.$$

Go to Stage 1.

**Stage 3. Check for convergence.** Check whether  $n_i$  is large enough to satisfy the error tolerance, i.e.

$$\tilde{F}_{n_i}(f) \leq \frac{4\varepsilon(n_i-1)(2n_i-2-\tau_{a,b})}{\tau_{a,b}(b-a)}.$$

If this is true, then return  $T_{n_i}(f)$  and terminate the algorithm. If this is not true, choose

$$n_{i+1} = 1 + (n_i - 1) \max \left\{ 2, \left\lceil \frac{1}{(n_i - 1)} \sqrt{\frac{\tau_{a,b}(b-a)\tilde{F}_{n_i}(f)}{8\varepsilon}} \right\rceil \right\}.$$

Go to Stage 1.

**Theorem 7.** *Let  $\sigma > 0$  be some fixed parameter, and let  $\mathcal{B}_\sigma = \{f \in \mathcal{V}^1 : \text{Var}(f') \leq \sigma\}$ . Let  $T \in \mathcal{A}(\mathcal{B}_\sigma, \mathbb{R}, \text{INT}, \Lambda^{\text{std}})$  be the non-adaptive trapezoidal rule defined by Algorithm 1, and let  $\varepsilon > 0$  be the error tolerance. Then this algorithm succeeds for  $f \in \mathcal{B}_\sigma$ , i.e.,  $|\text{INT}(f) - T(f, \varepsilon)| \leq \varepsilon$ , and the cost of this algorithm is  $\lceil (b-a)\sqrt{\sigma/(8\varepsilon)} \rceil + 1$ , regardless of the size of  $\text{Var}(f')$ .*

Now let  $T \in \mathcal{A}(\mathcal{C}_{\tau_{a,b}}, \mathbb{R}, \text{INT}, \Lambda^{\text{std}})$  be the adaptive trapezoidal rule defined by Algorithm 4, and let  $\tau_{a,b}$ ,  $n_1$ , and  $\varepsilon$  be as described there. Let  $\mathcal{C}_{\tau_{a,b}}$  be the cone of functions defined in (22). Then it follows that Algorithm 4 is successful for all functions in  $\mathcal{C}_{\tau_{a,b}}$ , i.e.,  $|\text{INT}(f) - T(f, \varepsilon)| \leq \varepsilon$ . Moreover, the cost of this algorithm is bounded below and above as follows:

$$\begin{aligned} & \max \left( \left\lceil \frac{\tau_{a,b} + 1}{2} \right\rceil, \left\lceil (b-a) \sqrt{\frac{\text{Var}(f')}{8\varepsilon}} \right\rceil \right) + 1 \\ & \leq \max \left( \left\lceil \frac{\tau_{a,b} + 1}{2} \right\rceil, \left\lceil (b-a) \sqrt{\frac{\tau_{a,b} \left\| f' - \frac{f(b)-f(a)}{b-a} \right\|_1}{8\varepsilon}} \right\rceil \right) + 1 \\ & \leq \text{cost}(T, f; \varepsilon) \\ & \leq (b-a) \sqrt{\frac{\tau_{a,b} \|f' - f(1) + f(0)\|_1}{2\varepsilon}} + \tau_{a,b} + 4 \leq (b-a) \sqrt{\frac{\tau_{a,b}(b-a) \text{Var}(f')}{4\varepsilon}} + \tau_{a,b} + 4. \end{aligned} \tag{27}$$

The algorithm is computationally stable, meaning that the minimum and maximum costs for all integrands,  $f$ , with fixed  $\|f' - f(1) + f(0)\|_1$  or  $\text{Var}(f')$  are an  $\varepsilon$ -independent constant of each other.