Guaranteed Automatic Algorithms with Relative Error

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Abstract

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1. A General Error Criterion

The criterion used for the automatica algorithms in [1] is an *absolute* error criterion. Given an error tolerance, ε_a, one seeks an algorithm, A, such that

$$\|S(f) - A(f)\|_{\mathcal{H}} \le \varepsilon_a. \tag{1}$$

This is done through a sequence of non-adaptive algorithms, A_n, with cost n. For each n one can compute from only data the quantity $\hat{\varepsilon}_n$, a reliable upper bound on $\|S(f) - A(f)\|_{\mathcal{H}}$, i.e.,

$$\|S(f) - A(f)\|_{\mathcal{H}} \le \hat{\varepsilon}_n. \tag{2}$$

The automatic algorithms in [1] uses a sequence of A_n with n increasing until $\hat{\varepsilon}_n \le \varepsilon_a$.

In many practical situations, one needs to approximate the answer with a certain *relative* accuracy, e.g., correct to three significant digits. In this case, given a tolerance, ε_r, one seeks an algorithm, A, such that

$$\|S(f) - A(f)\|_{\mathcal{H}} \le \varepsilon_r \|S(f)\|_{\mathcal{H}}. \tag{3}$$

This is a global relative error criterion, rather than a point-wise relative error criterion. One may generalize the pure absolute and pure relative error criteria as follows:

$$\|S(f) - A(f)\|_{\mathcal{H}} \le (1-\theta)\varepsilon_a + \theta\varepsilon_r \|S(f)\|_{\mathcal{H}}, \qquad 0 \le \theta \le 1, \tag{4}$$

where $\theta = 0$ denotes the absolute error case, and $\theta = 1$ denotes the relative error case.

Since $(1-\theta)\varepsilon_a + \theta\varepsilon_r \|S(f)\|_{\mathcal{H}} \le \max(\varepsilon_a, \varepsilon_r \|S(f)\|_{\mathcal{H}})$, the generalized criterion, (4), implies that either the absolute error criterion, (1) or the relative error criterion, (3), is satisfied. One can imagine a situation where one really wants the relative error criterion to be satisfied, but since $\|S(f)\|_{\mathcal{H}}$ may be tiny in some situations, choosing $(1-\theta)\varepsilon_a$ small but nonzero allows the algorithm to have a chance of success. In fact, one might choose

$$\theta = \frac{\varepsilon_r}{\varepsilon_a + \varepsilon_r}, \quad \varepsilon_a \geq 0,\ \varepsilon_r \geq 0,\ \min(\varepsilon_a, \varepsilon_r) \neq 0$$

In this case, choosing $\varepsilon_a = 0$ implies $\theta = 1$ and automatically means that one is interested purely in relative error, while choosing $\varepsilon_r = 0$ implies $\theta = 0$ and automatically means that one is interested purely in absolute error.

Using the $\hat{\varepsilon}_n$, the aforementioned reliable upper bounds on $\|S(f) - A(f)\|_{\mathcal{H}}$, the aim is to take enough samples so that the generalized error criterion can be satisfied, but not too many. The triangle inequality implies that

$$\|A_n(f)\|_{\mathcal{H}} - \|S(f) - A_n(f)\|_{\mathcal{H}} \leq \|S(f)\|_{\mathcal{H}} \leq \|A_n(f)\|_{\mathcal{H}} + \|S(f) - A_n(f)\|_{\mathcal{H}}.$$

Supposing that one can evaluate $\|A_n(f)\|_{\mathcal{H}}$ strictly from the data, this implies that any algorithm satisfying the data-dependent criterion

$$\hat{\varepsilon}_n \leq \min\left((1-\theta)\varepsilon_a, \frac{(1-\theta)\varepsilon_a + \theta\varepsilon_r \|A_n(f)\|_{\mathcal{H}}}{1+\theta\varepsilon_r}\right) \tag{5}$$

must also satisfy (4). This criterion is becomes the stopping criterion for the automatic Algorithm **??** below. Using the triangle inequality again implies that if

$$\hat{\varepsilon}_n \leq \min\left((1-\theta)\varepsilon_a, \frac{(1-\theta)\varepsilon_a + \theta\varepsilon_r \|S(f)\|_{\mathcal{H}}}{1+2\theta\varepsilon_r}\right), \tag{6}$$

then (5) must also be satisfied. This criterion is used to construct an upper bound on the cost of automatic Algorithm **??** in Theorem **??**.

In many cases it is possible to work with a point-wise generalized error criterion. Suppose that the space of solutions, $\mathcal{H}$, is a vector space of real-valued functions on $\mathcal{Y}$, and that the $\mathcal{H}$-norm is a sup norm:

$$\|h\|_{\mathcal{H}} = \sup_{\boldsymbol{y}\in\mathcal{Y}} |h(\boldsymbol{y})|. \tag{7}$$

Then a point-wise generalized error criterion would take the form:

$$\left|(S - \tilde{A}_n)(f)(\boldsymbol{y})\right| \leq (1-\theta)\varepsilon_a + \theta\varepsilon_r |S(f)(\boldsymbol{y})| \quad \forall \boldsymbol{y} \in \mathcal{Y}, \tag{8}$$

where again $0 \leq \theta \leq 1$. Here $\tilde{A}_n$ may not be the same as A_n, but as shall be seen below is defined in terms of A_n. Suppose one has a reliable pointwise upper bound on the error of a non-adaptive algorithm, A_n, with cost n:

$$\hat{\varepsilon}_n(\boldsymbol{y}) \geq |(S - A_n)(f)(\boldsymbol{y})|. \tag{9}$$

Here, $\hat{\varepsilon}_n(\boldsymbol{y})$ might be independent of $\boldsymbol{y}$. Furthermore, suppose that $A_n(f)(\boldsymbol{y})$ can be evaluated from the data.

Proposition 1. *Suppose that $\theta_r \varepsilon_r \le 1$. If error bound* (9) *holds, then the point-wise generalized error criterion* (8) *also holds, provided that*

$$\hat{\varepsilon}_n(\boldsymbol{y}) \le (1-\theta)\varepsilon_a + \theta\varepsilon_r \max(\hat{\varepsilon}_n(\boldsymbol{y}), |A_n(f)(\boldsymbol{y})|),$$
$$\tilde{A}_n(f)(\boldsymbol{y}) = A_n(f)(\boldsymbol{y}) - \theta\varepsilon_r \operatorname{sign}(A_n(f)(\boldsymbol{y})) \min(\hat{\varepsilon}_n(\boldsymbol{y}), |A_n(f)(\boldsymbol{y})|).$$

Proof. The cases of $\hat{\varepsilon}_n(\boldsymbol{y}) \le |A_n(f)(\boldsymbol{y})|$ and $\hat{\varepsilon}_n(\boldsymbol{y}) > |A_n(f)(\boldsymbol{y})|$ are treated separately. In the former case the definition of $\tilde{A}_n(f)(\boldsymbol{y})$ and the inequality constraint on $\hat{\varepsilon}_n(\boldsymbol{y})$ imply that

$$\begin{aligned}
&\left|\tilde{A}_n(f)(\boldsymbol{y}) - A_n(f)(\boldsymbol{y}) + \theta\varepsilon_r \operatorname{sign}(A_n(f)(\boldsymbol{y}))\hat{\varepsilon}_n(\boldsymbol{y})\right| = 0 \\
&\qquad\qquad \le (1-\theta)\varepsilon_a + \theta\varepsilon_r\,|A_n(f)(\boldsymbol{y})| - \hat{\varepsilon}_n(\boldsymbol{y}) \\[1em]
&\implies -(1-\theta)\varepsilon_a + [1-\theta\varepsilon_r \operatorname{sign}(A_n(f)(\boldsymbol{y}))][A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})] \\
&\qquad \le \tilde{A}_n(f)(\boldsymbol{y}) \le (1-\theta)\varepsilon_a + [1+\theta\varepsilon_r \operatorname{sign}(A_n(f)(\boldsymbol{y}))][A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})] \\[1em]
&\implies -(1-\theta)\varepsilon_a + \max\{S(f)(\boldsymbol{y}) - \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| : |(S-A_n)(f)(\boldsymbol{y})| \le \hat{\varepsilon}_n\} \\
&\qquad \le \tilde{A}_n(f)(\boldsymbol{y}) \\
&\qquad \le (1-\theta)\varepsilon_a + \min\{S(f)(\boldsymbol{y}) + \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| : |(S-A_n)(f)(\boldsymbol{y})| \le \hat{\varepsilon}_n\} \\[1em]
&\implies -(1-\theta)\varepsilon_a + S(f)(\boldsymbol{y}) - \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| \\
&\qquad \le \tilde{A}_n(f)(\boldsymbol{y}) \le (1-\theta)\varepsilon_a + S(f)(\boldsymbol{y}) + \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| \qquad \forall \boldsymbol{y} \in \mathcal{Y}, \\[1em]
&\implies -(1-\theta)\varepsilon_a - \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| \le \tilde{A}_n(f)(\boldsymbol{y}) - S(f)(\boldsymbol{y}) \\
&\qquad \le (1-\theta)\varepsilon_a + \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| \qquad \forall \boldsymbol{y} \in \mathcal{Y},
\end{aligned}$$

which implies (8).

In the case of $\hat{\varepsilon}_n(\boldsymbol{y}) > |A_n(f)(\boldsymbol{y})|$, the definition of $\tilde{A}_n(f)(\boldsymbol{y})$ and the inequality constraint on $\hat{\varepsilon}_n(\boldsymbol{y})$ imply that

$$\begin{aligned}
&\left|\tilde{A}_n(f)(\boldsymbol{y}) - (1-\theta\varepsilon_r)A_n(f)(\boldsymbol{y})\right| = 0 \le (1-\theta)\varepsilon_a - (1-\theta\varepsilon_r)\hat{\varepsilon}_n(\boldsymbol{y}) \\[1em]
&\implies -(1-\theta)\varepsilon_a + (1-\theta\varepsilon_r)[A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})] \\
&\qquad \le \tilde{A}_n(f)(\boldsymbol{y}) \le (1-\theta)\varepsilon_a + (1-\theta\varepsilon_r)[A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})] \\[1em]
&\implies -(1-\theta)\varepsilon_a + \max\{S(f)(\boldsymbol{y}) - \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| : |(S-A_n)(f)(\boldsymbol{y})| \le \hat{\varepsilon}_n\} \\
&\qquad \le \tilde{A}_n(f)(\boldsymbol{y}) \\
&\qquad \le (1-\theta)\varepsilon_a + \min\{S(f)(\boldsymbol{y}) + \theta\varepsilon_r\,|S(f)(\boldsymbol{y})| : |(S-A_n)(f)(\boldsymbol{y})| \le \hat{\varepsilon}_n\}
\end{aligned}$$

From this point the argument showing that (8) is satisfied proceeds exactly as in the previous case. □

Acknowledgements

[1] Ding Y, Clancy N, Hamilton C, Hickernell FJ, Zhang Y (2012) The complexity of guaranteed automatic algorithms. In preparation