Asymptotic Error Estimation for Trapezoidal Rule

Fred J. Hickernell

Room E1-208, Department of Applied Mathematics, Illinois Institute of Technology, 10 W. 32nd St., Chicago, IL 60616

Abstract

Keywords: adaptive, cones, function recovery, integration, quadrature
2010 MSC: 65D05, 65D30, 65G20

---

1. Problem

Consider the integral

$$I(f) = \int_0^1 f(x)\,\mathrm{d}x$$

that is approximated by the trapezoidal rule

$$\hat{I}_n(f) = \frac{1}{n}\left[\frac{1}{2}f(0) + f(1/n) + f(2/n) + \cdots + f(1-1/n) + \frac{1}{2}f(1)\right].$$

The error is often estimated as

$$E_n(f) := I(n) - \hat{I}_n(f) \approx \frac{\hat{I}_n(f) - \hat{I}_{n/2}(f)}{3} =: \widehat{E}_n(f).$$

We want to find a bound on $E_n(f) - \widehat{E}_n(f)$.

2. Error for One or Two Trapezoids

Looking at a piece of the trapezoidal rule, given a and h we have

$$\begin{aligned}
&\int_a^{a+h} f(x)\,\mathrm{d}x - \frac{h}{2}[f(a)+f(a+h)] \\
&\qquad = f(x)(x-a-h/2)\Big|_a^{a+h} - \int_a^{a+h} f'(x)(x-a-h/2)\,\mathrm{d}x \\
&\qquad\qquad\qquad - \frac{h}{2}[f(a)+f(a+h)] \\
&\qquad = -\int_a^{a+h} f'(x)(x-a-h/2)\,\mathrm{d}x \\
&\qquad = f'(x)\frac{(x-a)(a+h-x)}{2}\Big|_a^{a+h} - \int_a^{a+h} f''(x)\frac{(x-a)(a+h-x)}{2}\,\mathrm{d}x \\
&\qquad = -\int_a^{a+h} f''(x)\frac{(x-a)(a+h-x)}{2}\,\mathrm{d}x
\end{aligned}$$

Thus, for the integral over $[a, a+2h]$ using one trapezoid, the error is

$$\begin{aligned}
&\int_a^{a+2h} f(x)\,\mathrm{d}x - h[f(a)+f(a+2h)] \\
&\qquad\qquad\qquad = -\int_a^{a+2h} f''(x)\frac{(x-a)(a+2h-x)}{2}\,\mathrm{d}x, \qquad (1)
\end{aligned}$$

while the error of the same integral using two trapezoids is

$$\begin{aligned}
E_2(f) &= \int_a^{a+2h} f(x)\,\mathrm{d}x - \frac{h}{2}[f(a)+2f(a+h)+f(a+2h)] \\
&= -\int_a^{a+2h} f''(x)\left\{\left[\frac{(x-a)(a+h-x)}{2}\right]1_{[a,a+h]}(x)\right. \\
&\qquad\qquad \left. + \left[\frac{(x-a-h)(a+2h-x)}{2}\right]1_{[a+h,a+2h]}(x)\right\}\mathrm{d}x \\
&= -\int_a^{a+2h} f''(x)\left[\frac{|x-a-h|\,(h-|x-a-h|)}{2}\right]\mathrm{d}x \qquad (2a) \\
&= \int_a^{a+2h} f''(x)u(x)\,\mathrm{d}x, \qquad (2b)
\end{aligned}$$

where

$$u(x) = -\left[\frac{|x-a-h|\,(h-|x-a-h|)}{2}\right] = \Theta(h^2). \qquad (2c)$$

Since $u(x) = \Theta(h^2)$, it follows that $E_2(f) = \Theta(h^3)$, as expected.

An expression for

$$\begin{aligned}\widehat{E}_2(f) &= \frac{1}{3}\left\{\frac{h}{2}[f(a)+2f(a+h)+f(a+2h)] - h[f(a)+f(a+2h)]\right\}\\ &= \frac{h}{6}[-f(a)+2f(a+h)-f(a+2h)]\end{aligned}$$

may be obtained from (1) and (2) as follows:

$$\begin{aligned}\widehat{E}_2(f) &= \int_a^{a+2h} f''(x)v(x)\,\mathrm{d}x,\\ v(x) &= \frac{1}{3}\left\{\left[\frac{|x-a-h|\,(h-|x-a-h|)}{2}\right] - \frac{(x-a)(a+2h-x)}{2}\right\}\\ &= \frac{1}{6}\left\{|x-a-h|\,(h-|x-a-h|) - (x-a)(a+2h-x)\right\}\\ &= \frac{1}{6}\left\{|x-a-h|\,(h-|x-a-h|) - [h^2-|x-a-h|^2]\right\}\\ &= \frac{h}{6}\left\{|x-a-h|-h\right\} = \Theta(h^2).\end{aligned}$$

Since $v(x) = \Theta(h^2)$, it follows that $\widehat{E}_2(f) = \Theta(h^3)$, as expected.

It is hoped that $E_2(f) - \widehat{E}_2(f) = o(h^3)$. The difference between the true and approximate errors is

$$\begin{aligned}E_2(f) - \widehat{E}_2(f) &= \int_a^{a+2h} f''(x)w(x)\,\mathrm{d}x,\\ w(x) &= u(x) - v(x)\\ &= -\left[\frac{|x-a-h|\,(h-|x-a-h|)}{2}\right] - \frac{h}{6}\left\{|x-a-h|-h\right\}\\ &= \frac{1}{6}\left[3(x-a-h)^2 - 4h\,|x-a-h| + h^2\right]\\ &= \frac{1}{6}(3\,|x-a-h|-h)(|x-a-h|-h) = \Theta(h^2).\end{aligned}$$

Now let

$$\begin{aligned}W(x) &= \int_a^x w(t)\,\mathrm{d}t\\ &= \frac{1}{6}\int_a^x \left[3(t-a-h)^2 - 4h\,|t-a-h| + h^2\right]\,\mathrm{d}t\\ &= \frac{1}{6}\left[(t-a-h)^3 - 2h\,|t-a-h|\,(t-a-h) + h^2(t-a-h)\right]\Big|_a^x\\ &= \frac{1}{6}\left[(x-a-h)^3 - 2h\,|x-a-h|\,(x-a-h) + h^2(x-a-h)\right]\\ &= \frac{1}{6}(x-a-h)\left[(x-a-h)^2 - 2h\,|x-a-h| + h^2\right]\\ &= \frac{1}{6}(x-a-h)(|x-a-h|-h)^2 = \Theta(h^3).\end{aligned}$$

Note that $W(a) = W(a+2h) = 0$. Thus,

$$
\begin{aligned}
E_2(f) - \widehat{E}_2(f) &= \int_a^{a+2h} f''(x) w(x)\,\mathrm{d}x, \\
&= f''(x) W(x)\Big|_a^{a+h} - \int_a^{a+2h} f'''(x) W(x)\,\mathrm{d}x \\
&= -\int_a^{a+2h} f'''(x) W(x)\,\mathrm{d}x, \\
\left|E_2(f) - \widehat{E}_2(f)\right| &= \left|\int_a^{a+2h} f'''(x) W(x)\,\mathrm{d}x\right| \\
&\le \sup_{a \le x \le a+2h} |W(x)| \times \int_a^{a+2h} |f'''(x)|\;\mathrm{d}x.
\end{aligned}
$$

Noting that $|W(x)|$ attains its maximum where $w(x)$ vanishes, i.e., at $x = a + h \pm h/3$, and that

$$
|W(a+h \pm h/3)| = \frac{1}{6} \times \frac{h}{3} \times \left(\frac{2h}{3}\right)^2 = \frac{2h^3}{81}
$$

it follows that

$$
\left|E_2(f) - \widehat{E}_2(f)\right| \le \frac{2h^3}{81} \int_a^{a+2h} |f'''(x)|\;\mathrm{d}x = \Theta(h^4).
$$

If $f^{(4)}$ is integrable, then one may bound the difference of the true and approximate error better as follows. Let

$$
\begin{aligned}
\widetilde{W}(x) &= \int_a^x W(t)\,\mathrm{d}t \\
&= \frac{1}{6} \int_a^x (x-a-h)(|x-a-h| - h)^2\,\mathrm{d}t \\
&= \frac{1}{6}\left[\frac{1}{4}(t-a-h)^4 - \frac{2h}{3}\,|t-a-h|^3 + \frac{h^2}{2}(t-a-h)^2\right]\Bigg|_a^x \\
&= \frac{1}{72}\left[3(t-a-h)^4 - 8h\,|t-a-h|^3 + 6h^2(t-a-h)^2 - h^4\right],
\end{aligned}
$$

and note that $\widetilde{W}(a) = \widetilde{W}(a+2h) = 0$. Then it follows that

$$\begin{aligned}
E_2(f) - \widehat{E}_2(f) &= -\int_a^{a+2h} f'''(x) W(x)\,\mathrm{d}x, \\
&= -f'''(x)\widetilde{W}(x)\Big|_a^{a+h} + \int_a^{a+2h} f^{(4)}(x)\widetilde{W}(x)\,\mathrm{d}x \\
&= \int_a^{a+2h} f^{(4)}(x)\widetilde{W}(x)\,\mathrm{d}x, \\
\left|E_2(f) - \widehat{E}_2(f)\right| &= \left|\int_a^{a+2h} f^{(4)}(x)\widetilde{W}(x)\,\mathrm{d}x\right| \\
&\le \sup_{a \le x \le a+2h} \left|\widetilde{W}(x)\right| \times \int_a^{a+2h} \left|f^{(4)}(x)\right|\,\mathrm{d}x.
\end{aligned}$$

Noting that $\left|\widetilde{W}(x)\right|$ attains its maximum where $W(x)$ vanishes, i.e., at $x = a+h$, and that

$$\left|\widetilde{W}(a+h)\right| = \frac{h^4}{72},$$

it follows that

$$\left|E_2(f) - \widehat{E}_2(f)\right| \le \frac{h^4}{72} \int_a^{a+2h} \left|f^{(4)}(x)\right|\,\mathrm{d}x = \Theta(h^5).$$

3. Error for the Whole Integral

Applying the work from the previous section with $h = 1/n$, and $a = x_i$ for $i = 0, 2, 4 \ldots, n-2$ it follows that

$$\begin{aligned}
\left|E_n(f) - \widehat{E}_n/2(f)\right| &\le \frac{2h^3}{81} \sum_{j=0}^{n/2-1} \int_{x_{2j}}^{x_{2j+2}} |f'''(x)|\,\mathrm{d}x \\
&= \frac{2}{81n^3} \int_0^1 |f'''(x)|\,\mathrm{d}x = \frac{2}{81n^3} \|f'''\|_1, \\
\left|E_n(f) - \widehat{E}_n/2(f)\right| &\le \frac{h^4}{72} \sum_{j=0}^{n/2-1} \int_{x_{2j}}^{x_{2j+2}} \left|f^{(4)}(x)\right|\,\mathrm{d}x \\
&= \frac{1}{72n^4} \int_0^1 \left|f^{(4)}(x)\right|\,\mathrm{d}x = \frac{1}{72n^4} \left\|f^{(4)}\right\|_1.
\end{aligned}$$