Another Cone for Integration

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Abstract

Keywords: adaptive, automatic, cones, function recovery, guarantee, integration, quadrature
2010 MSC: 65D05, 65D30, 65G20

1. Introduction

In [2] we considered the problem of integration and the cone of integrands

$$\mathcal{C}_\tau := \{f \in \mathcal{V}^1 : \mathrm{Var}(f') \le \tau \left\| f' - f(1) + f(0) \right\|_1\}, \tag{1}$$

where the total variation and the $\mathcal{L}_p$ norms are defined as

$$\mathrm{Var}(f) := \sup_{\substack{n\in\mathbb{N} \\ 0=x_0<x_1<\cdots<x_n=1}} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|,$$

$$\left\| f \right\|_p := \begin{cases} \left[\displaystyle\int_0^1 \left| f(x) \right|^p \, \mathrm{d}x \right]^{1/p}, & 1 \le p < \infty, \\ \displaystyle\sup_{0\le x\le 1} \left| f(x) \right|, & p = \infty, \end{cases}$$

$$\mathcal{V}^k := \mathcal{V}^k[0,1] = \{f \in C[0,1] : \mathrm{Var}(f^{(k)}) < \infty\}.$$

We derived an algorithm [2, Algorithm 4] that was guaranteed for integrands in $\mathcal{C}_\tau$. In this note we consider another algorithm and other cones.

First we recall some notation and results from [2]. For all $n \in \mathcal{I} := \{0, 2, 3, \ldots\}$ we have the linear spline. By convention $A_0(f) = 0$, and for $n > 0$,

$$x_{i,n} := x_i := \frac{i-1}{n-1}, \qquad i = 1, \ldots, n, \tag{2a}$$

$$A_n(f)(x) := (n-1) \left[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i) \right] \quad \text{for } x_i \le x \le x_{i+1}. \tag{2b}$$

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The cost of each function value is one and so the cost of A_n is n. The dependence of the nodes, x_i on n is often suppressed for simplicity. Integrating the linear spline gives us the trapezoidal rule based on $n-1$ trapezoids:

$$T_n(f) := \int_0^1 A_n(f)\,\mathrm{d}x = \frac{1}{2n-2}[f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

for $n \geq 2$ and $T_0(f) = 0$.

The error of the trapezoidal rule has the following upper bound [1, (7.15)]:

$$\left| \int_0^1 f(x)\,dx - T_n(f) \right| \leq \frac{\operatorname{Var}(f')}{8(n-1)^2} \qquad n \in \mathcal{I} \setminus \{0\}. \tag{3}$$

For any $n \in \mathcal{I}$, let $\mathcal{J}_n = \{m \in \mathbb{N} : (n-1)/(m-1) \in \mathbb{N}\}$. This means that T_n integrates exactly any function that is a linear spline using m nodes for $m \in \mathcal{J}_n$. This implies that

$$\begin{aligned} \left| \int_0^1 f(x)\,dx - T_n(f) \right| &= \left| \int_0^1 [f(x) - A_m(f)(x)]\,dx - T_n(f - A_m(f)) \right| \\ &\leq \frac{\operatorname{Var}(f' - A_m(f)')}{8(n-1)^2} \qquad \forall m \in \mathcal{J}_n,\ n \in \mathcal{I} \setminus \{0\}. \end{aligned} \tag{4}$$

The variation of the first derivative of f is bounded below by the variation of the first derivative of of the linear spline of f. For all $f \in \mathcal{V}^1$ it follows that

$$\begin{aligned} \operatorname{Var}(f') \geq F_n(f) &:= \operatorname{Var}(A_n(f)') \\ &= \begin{cases} 0, & n = 0, 2, \\ (n-1) \displaystyle\sum_{i=1}^{n-2} \left| f(x_i) - 2f(x_{i+1}) + f(x_{i+2}) \right|, & n \geq 3. \end{cases} \end{aligned} \tag{5}$$

Also note that

$$F_m(f) = F_m(A_n(f)) \leq \operatorname{Var}(A_n(f)') = F_n(f) \qquad \forall m \in \mathcal{J}_n. \tag{6}$$

The bound in further implies that

$$\begin{aligned} F_n(f) \leq \operatorname{Var}(f') &\leq \operatorname{Var}(f' - A_m(f)') + \operatorname{Var}(A_m(f)) \\ &= \operatorname{Var}(f' - A_m(f)') + F_m(f). \end{aligned} \tag{7}$$

The weaker semi-norm is approximated by

$$\begin{aligned} \widetilde{F}_n(f) &:= \|A_n(f)' - f(1) + f(0)\|_1 \\ &= \begin{cases} 0, & n = 0, 2, \\ \displaystyle\sum_{i=1}^{n-1} \left| f(x_{i+1}) - f(x_i) - \frac{f(1) - f(0)}{n-1} \right|, & n \geq 3. \end{cases} \end{aligned} \tag{8}$$

Another useful fact from [2, Sect. 5.1] is that

$$\|f' - f(1) + f(0)\|_1 \leq \widetilde{F}_n(f) + \tilde{h}(n) \operatorname{Var}(f') \qquad \forall f \in \mathcal{V}^1 \tag{9}$$

where $\tilde{h}(n) := 1/(2n-2)$ for $n \geq 2$.

2. New Cone, New Algorithm

Let $\widehat{\mathcal{I}} = \{2, 4, 8, 16, \ldots\}$, and $\overline{\mathcal{I}}$ be some non-empty subset of $\mathcal{I}$ such that $i \in \overline{\mathcal{I}}$ implies that $j \in \overline{\mathcal{I}}$ for all $j \in \widehat{\mathcal{I}}$ with $j < i$. Let $\overline{\tau} : \overline{\mathcal{I}} \to (0, \infty)$ be some given function. The new cone considered here is defined as

$$\overline{\mathcal{C}}_{\overline{\tau}} := \{f \in \mathcal{V}^1 : \operatorname{Var}(f') - F_m(f) \leq \overline{\tau}(m)[\|f' - f(1) + f(0)\|_1 - \widetilde{F}_m(f)] \quad \forall m \in \overline{\mathcal{I}}\}. \tag{10}$$

Combining this cone definition (10) with the bound on the weaker norm (9), it follows that for all $m \in \overline{\mathcal{I}}$ and $n \in \mathcal{I}$

$$\begin{aligned}
\operatorname{Var}(f') &\leq F_m(f) + \overline{\tau}(m)[\|f' - f(1) + f(0)\|_1 - \widetilde{F}_m(f)] \\
&\leq F_m(f) + \overline{\tau}(m)[\widetilde{F}_n(f) + \tilde{h}(n) \operatorname{Var}(f') - \widetilde{F}_m(f)] \\
\operatorname{Var}(f') &\leq \frac{F_m(f) + \overline{\tau}(m)[\widetilde{F}_n(f) - \widetilde{F}_m(f)]}{1 - \overline{\tau}(m)\tilde{h}(n)}, \qquad \overline{\tau}(m)\tilde{h}(n) < 1, \\
&\leq \frac{F_m(f) + \overline{\tau}(m)\tilde{h}(m)F_n(f)}{1 - \overline{\tau}(m)\tilde{h}(n)}, \qquad \overline{\tau}(m)\tilde{h}(n) < 1, \\
&\leq \frac{1 + \overline{\tau}(m)\tilde{h}(m)}{1 - \overline{\tau}(m)\tilde{h}(n)} F_n(f), \qquad \overline{\tau}(m)\tilde{h}(n) < 1.
\end{aligned}$$

In light of this inequality, define

$$N_{\min} = \min\{n \in \widehat{\mathcal{I}} : \overline{\tau}(m)\tilde{h}(n) < 1 \text{ for some } m \in \overline{\mathcal{I}}\}.$$

Algorithm 1 (New Cone Adaptive Univariate Integration)**.** Let the sequence of algorithms $\{T_n\}_{n\in\mathcal{I}}$, $\{F_n\}_{n\in\mathcal{I}}$, and $\overline{\mathcal{C}}_{\overline{\tau}}$ be as described above. Set $i = 1$, and let $n_1 = N_{\min}$. For any error tolerance ε and input function f, do the following:

Step 1. Bound $\operatorname{Var}(f')$ and check for convergence. Compute $F_{n_i}(f)$ in (5). Check whether n_i is large enough to satisfy the error tolerance, i.e., whether there exists an $m \in \overline{\mathcal{I}}$ with $m \leq n_i$ such that

$$F_m(f) + \overline{\tau}(m)[\widetilde{F}_{n_i}(f) - \widetilde{F}_m(f)] \leq 8(n_i - 1)^2\varepsilon[1 - \overline{\tau}(m)\tilde{h}(n_i)].$$

If this is true, then return $T_{n_i}(f)$ and terminate the algorithm.

Step 2. Increase the number of trapezoids. If the above condition is false, choose $n_{i+1} = 2n_i$, increment i, and go to Step 1.

Note that if $\overline{\mathcal{I}} = \{2\}$, then $m = 2$, $\tilde{h}(m) = 1/2$, $F_m(f) = \widetilde{F}_m(f) = 0$, and we are back to the cone in (1). If

References

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