Guaranteed Automatic Algorithms with a Generalized Error Criterion

Yuhan Ding, Fred J. Hickernell, Yizhi Zhang

Room E1-208, Department of Applied Mathematics, Illinois Institute of Technology, 10 W. 32nd St., Chicago, IL 60616

Abstract

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1. A General Global Error Criterion

The criterion used for the automatica algorithms in [**?**] is an *absolute* error criterion. Given an error tolerance, ε_a, one seeks an algorithm, A, such that

$$\|S(f) - A(f)\|_{\mathcal{H}} \le \varepsilon_a. \tag{1}$$

This is done through a sequence of non-adaptive algorithms, A_n, with cost n. For each n one can compute from only data the quantity $\hat{\varepsilon}_n$, a reliable upper bound on $\|S(f) - A_n(f)\|_{\mathcal{H}}$, i.e.,

$$\|S(f) - A_n(f)\|_{\mathcal{H}} \le \hat{\varepsilon}_n. \tag{2}$$

The automatic algorithms in [**?**] uses a sequence of A_n with n increasing until $\hat{\varepsilon}_n \le \varepsilon_a$.

In many practical situations, one needs to approximate the answer with a certain *relative* accuracy, e.g., correct to three significant digits. In this case, given a tolerance, ε_r, one seeks an algorithm, A, such that

$$\|S(f) - A(f)\|_{\mathcal{H}} \le \varepsilon_r \|S(f)\|_{\mathcal{H}}. \tag{3}$$

This is a global relative error criterion, rather than a point-wise relative error criterion. One may generalize the pure absolute and pure relative error criteria as follows:

$$\|S(f) - A(f)\|_{\mathcal{H}} \le \operatorname{tol}(\varepsilon_a, \varepsilon_r \|S(f)\|_{\mathcal{H}}). \tag{4}$$

Here $\operatorname{tol} : [0,\infty) \times [0,\infty) \to [0,\infty)$ is non-decreasing in each of its arguments and satisfies a Lipschitz condition in terms of its second argument:

$$|\operatorname{tol}(a, b) - \operatorname{tol}(a, b')| \le |b - b'| \quad \forall a, b, b' \ge 0. \tag{5}$$

Two examples that one might choose are

$$\operatorname{tol}(a,b) = \max(a,b), \tag{6}$$

$$\operatorname{tol}(a,b) = (1-\theta)a + \theta b, \quad 0 \le \theta \le 1. \tag{7}$$

Both of these examples include absolute error and relative error as special cases.

Using the $\hat{\varepsilon}_n$, the aforementioned reliable upper bounds on $\|S(f) - A(f)\|_{\mathcal{H}}$, the aim is to take enough samples so that the generalized error criterion can be satisfied, but not too many. The triangle inequality implies that

$$\|A_n(f)\|_{\mathcal{H}} - \|S(f) - A_n(f)\|_{\mathcal{H}} \le \|S(f)\|_{\mathcal{H}} \le \|A_n(f)\|_{\mathcal{H}} + \|S(f) - A_n(f)\|_{\mathcal{H}}.$$

Supposing that one can evaluate $\|A_n(f)\|_{\mathcal{H}}$ strictly from the data, this implies that any algorithm satisfying the data-dependent criterion

$$\hat{\varepsilon}_n \le \operatorname{tol}\big(\varepsilon_a, \varepsilon_r \max(\|A_n(f)\|_{\mathcal{H}} - \hat{\varepsilon}_n, 0)\big) \tag{8}$$

must also satisfy (4). This criterion becomes the stopping criterion for the automatic Algorithm **??** below. Using the triangle inequality again implies that if

$$\hat{\varepsilon}_n \le \operatorname{tol}\big(\varepsilon_a, \varepsilon_r \max(\|S(f)\|_{\mathcal{H}} - 2\hat{\varepsilon}_n, 0)\big), \tag{9}$$

then (8) must also be satisfied. This criterion is used to construct an upper bound on the cost of automatic Algorithm **??** in Theorem **??**.

2. A General Pointwise Error Criterion

In many cases it is possible to work with a point-wise generalized error criterion. Suppose that the space of solutions, $\mathcal{H}$, is a vector space of real-valued functions on $\mathcal{Y}$, and that the $\mathcal{H}$-norm is a sup norm:

$$\|h\|_{\mathcal{H}} = \sup_{\boldsymbol{y} \in \mathcal{Y}} |h(\boldsymbol{y})|. \tag{10}$$

Then a point-wise generalized error criterion would take the form:

$$\left|(S - \tilde{A}_n)(f)(\boldsymbol{y})\right| \le \operatorname{tol}(\varepsilon_a, \varepsilon_r |S(f)(\boldsymbol{y})|) \quad \forall \boldsymbol{y} \in \mathcal{Y}, \tag{11}$$

where again $0 \le \theta \le 1$. Here $\tilde{A}_n$ may not be the same as A_n, but as shall be seen below is defined in terms of A_n. Suppose one has a reliable pointwise upper bound on the error of a non-adaptive algorithm, A_n, with cost n:

$$\hat{\varepsilon}_n(\boldsymbol{y}) \ge |(S - A_n)(f)(\boldsymbol{y})| \quad \forall \boldsymbol{y} \in \mathcal{Y}. \tag{12}$$

Here, $\hat{\varepsilon}_n(\boldsymbol{y})$ might be independent of $\boldsymbol{y}$. Furthermore, suppose that $A_n(f)(\boldsymbol{y})$ can be evaluated from the data.

Proposition 1. *Suppose that $\varepsilon_r \le 1$, and define*

$$\begin{aligned}
&\Delta_{\pm}(\boldsymbol{y}) \\
&\quad = \frac{1}{2}\left[\operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr) \pm \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr)\right],
\end{aligned} \tag{13}$$

and the approximation

$$\tilde{A}_n(f)(\boldsymbol{y}) = A_n(f)(\boldsymbol{y}) + \Delta_{-}(\boldsymbol{y}). \tag{14}$$

If point-wise error bound (12) *holds, where $\hat{\varepsilon}_n(\boldsymbol{y})$ satisfies the following condition,*

$$\hat{\varepsilon}_n(\boldsymbol{y}) \le \Delta_{+}(\boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \mathcal{Y}, \tag{15}$$

then the point-wise generalized error criterion for $\tilde{A}_n(f)$, (11)*, is satisfied. If $\hat{\varepsilon}_n(\boldsymbol{y})$ becomes small enough to satisfy*

$$\begin{aligned}
&\hat{\varepsilon}_n(\boldsymbol{y}) \\
&\qquad \le \frac{1}{2}\left[\operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|S(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr) + \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|S(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr)\right],
\end{aligned} \tag{16}$$

for all $\boldsymbol{y} \in \mathcal{Y}$, then (15) *must be satisfied.*

Proof. The proof follows by applying the condition (12) that bounds the absolute error in terms of $\hat{\varepsilon}_n(\boldsymbol{y})$, upper bound condition (15) on $\hat{\varepsilon}_n(\boldsymbol{y})$, the definition of the approximate solution in (14), and definition (13):

$$\begin{aligned}
&\left|\tilde{A}_n(f)(\boldsymbol{y}) - A_n(f)(\boldsymbol{y}) - \Delta_{-}(\boldsymbol{y})\right| = 0 \le \Delta_{+}(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y}) \quad \text{by (14) and (15)} \\
&\quad \Longleftrightarrow\ A_n(f)(\boldsymbol{y}) + \Delta_{-}(\boldsymbol{y}) - \Delta_{+}(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y}) \le \tilde{A}_n(f)(\boldsymbol{y}) \\
&\qquad\qquad \le A_n(f)(\boldsymbol{y}) + \Delta_{-}(\boldsymbol{y}) + \Delta_{+}(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y}) \\
&\quad \Longleftrightarrow\ A_n(f)(\boldsymbol{y}) - \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr) + \hat{\varepsilon}_n(\boldsymbol{y}) \le \tilde{A}_n(f)(\boldsymbol{y}) \\
&\qquad\qquad \le A_n(f)(\boldsymbol{y}) + \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr) - \hat{\varepsilon}_n(\boldsymbol{y}) \quad \text{by (13)} \\
&\quad \Longrightarrow\ S(f)(\boldsymbol{y}) - \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|S(f)(\boldsymbol{y})\right|\bigr) \le \tilde{A}_n(f)(\boldsymbol{y}) \\
&\qquad\qquad \le S(f)(\boldsymbol{y}) + \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|S(f)(\boldsymbol{y})\right|\bigr) \\
&\qquad\qquad \text{by (12) and since } b \mapsto b \pm \operatorname{tol}(a, \varepsilon_r |b|) \text{ is non-decreasing by (5)} \\
&\quad \Longleftrightarrow\ \left|\tilde{A}_n(f)(\boldsymbol{y}) - S(f)(\boldsymbol{y})\right| \le \operatorname{tol}\bigl(\varepsilon_a, \varepsilon_r \left|S(f)(\boldsymbol{y})\right|\bigr).
\end{aligned}$$

This completes the proof. $\square$

For the special case of (6)

$$\begin{aligned}
\Delta_+ &= \frac{1}{2}\left[\max\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr) + \max\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr)\right] \\
&= \begin{cases}
\max\bigl(\left|A_n(f)(\boldsymbol{y})\right|, \hat{\varepsilon}_n(\boldsymbol{y})\bigr), & \varepsilon_a \le \varepsilon_r \left|\left|A_n(f)(\boldsymbol{y})\right| - \hat{\varepsilon}_n(\boldsymbol{y})\right|, \\
\frac{1}{2}\left[\varepsilon_a + \varepsilon_r\bigl(\left|A_n(f)(\boldsymbol{y})\right| + \hat{\varepsilon}_n(\boldsymbol{y})\bigr)\right], & \\
\qquad \varepsilon_r \left|\left|A_n(f)(\boldsymbol{y})\right| - \hat{\varepsilon}_n(\boldsymbol{y})\right| \le \varepsilon_a \le \varepsilon_r\bigl(\left|A_n(f)(\boldsymbol{y})\right| + \hat{\varepsilon}_n(\boldsymbol{y})\bigr), & \\
\varepsilon_a, & \varepsilon_r\bigl(\left|A_n(f)(\boldsymbol{y})\right| + \hat{\varepsilon}_n(\boldsymbol{y})\bigr) \le \varepsilon_a.
\end{cases} \\
\Delta_- &= \frac{1}{2}\left[\max\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr) - \max\bigl(\varepsilon_a, \varepsilon_r \left|A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\bigr)\right] \\
&= \begin{cases}
-\operatorname{sign}(A_n(f)(\boldsymbol{y}))\min\bigl(\left|A_n(f)(\boldsymbol{y})\right|, \hat{\varepsilon}_n(\boldsymbol{y})\bigr), & \\
\qquad \varepsilon_a \le \varepsilon_r \left|\left|A_n(f)(\boldsymbol{y})\right| - \hat{\varepsilon}_n(\boldsymbol{y})\right|, & \\
\frac{-\operatorname{sign}(A_n(f)(\boldsymbol{y}))}{2}\left[\varepsilon_r\bigl(\left|A_n(f)(\boldsymbol{y})\right| + \hat{\varepsilon}_n(\boldsymbol{y})\bigr) - \varepsilon_a\right], & \\
\qquad \varepsilon_r \left|\left|A_n(f)(\boldsymbol{y})\right| - \hat{\varepsilon}_n(\boldsymbol{y})\right| \le \varepsilon_a \le \varepsilon_r\bigl(\left|A_n(f)(\boldsymbol{y})\right| + \hat{\varepsilon}_n(\boldsymbol{y})\bigr), & \\
0, & \varepsilon_r\bigl(\left|A_n(f)(\boldsymbol{y})\right| + \hat{\varepsilon}_n(\boldsymbol{y})\bigr) \le \varepsilon_a.
\end{cases}
\end{aligned}$$

For the special case of (7)

$$\begin{aligned}
\Delta_+ &= (1-\theta)\varepsilon_a + \theta\varepsilon_r \frac{1}{2}\left[\left|A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right| + \left|A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\right] \\
&= (1-\theta)\varepsilon_a + \theta\varepsilon_r \max(\left|A_n(f)(\boldsymbol{y})\right|, \hat{\varepsilon}_n(\boldsymbol{y})) \\
\Delta_- &= \theta\varepsilon_r \frac{1}{2}\left[\left|A_n(f)(\boldsymbol{y}) - \hat{\varepsilon}_n(\boldsymbol{y})\right| - \left|A_n(f)(\boldsymbol{y}) + \hat{\varepsilon}_n(\boldsymbol{y})\right|\right] \\
&= -\theta\varepsilon_r \operatorname{sign}(A_n(f)(\boldsymbol{y}))\min(\left|A_n(f)(\boldsymbol{y})\right|, \hat{\varepsilon}_n(\boldsymbol{y}))
\end{aligned}$$

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