

Guaranteed Automatic Algorithms with Relative Error

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Abstract

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1. A General Error Criterion

The criterion used for the automatic algorithms in [1] is an *absolute* error criterion. Given an error tolerance, ε_a , one seeks an algorithm, A , such that

$$\|S(f) - A(f)\|_{\mathcal{H}} \leq \varepsilon_a. \quad (1)$$

This is done through a sequence of non-adaptive algorithms, A_n , with cost n . For each n one can compute from only data the quantity $\hat{\varepsilon}_n$, a reliable upper bound on $\|S(f) - A_n(f)\|_{\mathcal{H}}$, i.e.,

$$\|S(f) - A_n(f)\|_{\mathcal{H}} \leq \hat{\varepsilon}_n. \quad (2)$$

The automatic algorithms in [1] uses a sequence of A_n with n increasing until $\hat{\varepsilon}_n \leq \varepsilon_a$.

In many practical situations, one needs to approximate the answer with a certain *relative* accuracy, e.g., correct to three significant digits. In this case, given a tolerance, ε_r , one seeks an algorithm, A , such that

$$\|S(f) - A(f)\|_{\mathcal{H}} \leq \varepsilon_r \|S(f)\|_{\mathcal{H}}. \quad (3)$$

This is a global relative error criterion, rather than a point-wise relative error criterion. One may generalize the pure absolute and pure relative error criteria as follows:

$$\|S(f) - A(f)\|_{\mathcal{H}} \leq (1 - \theta)\varepsilon_a + \theta\varepsilon_r \|S(f)\|_{\mathcal{H}}, \quad 0 \leq \theta \leq 1, \quad (4)$$

where $\theta = 0$ denotes the absolute error case, and $\theta = 1$ denotes the relative error case.

Since $(1 - \theta)\varepsilon_a + \theta\varepsilon_r \|S(f)\|_{\mathcal{H}} \leq \max(\varepsilon_a, \varepsilon_r \|S(f)\|_{\mathcal{H}})$, the generalized criterion, (4), implies that either the absolute error criterion, (1) or the relative error criterion, (3), is satisfied. One can imagine a situation where one really

wants the relative error criterion to be satisfied, but since $\|S(f)\|_{\mathcal{H}}$ may be tiny in some situations, choosing $(1 - \theta)\varepsilon_a$ small but nonzero allows the algorithm to have a chance of success. In fact, one might choose

$$\theta = \frac{\varepsilon_r}{\varepsilon_a + \varepsilon_r}, \quad \varepsilon_a \geq 0, \quad \varepsilon_r \geq 0, \quad \min(\varepsilon_a, \varepsilon_r) \neq 0$$

In this case, choosing $\varepsilon_a = 0$ implies $\theta = 1$ and automatically means that one is interested purely in relative error, while choosing $\varepsilon_r = 0$ implies $\theta = 0$ and automatically means that one is interested purely in absolute error.

Using the $\hat{\varepsilon}_n$, the aforementioned reliable upper bounds on $\|S(f) - A(f)\|_{\mathcal{H}}$, the aim is to take enough samples so that the generalized error criterion can be satisfied, but not too many. The triangle inequality implies that

$$\|A_n(f)\|_{\mathcal{H}} - \|S(f) - A_n(f)\|_{\mathcal{H}} \leq \|S(f)\|_{\mathcal{H}} \leq \|A_n(f)\|_{\mathcal{H}} + \|S(f) - A_n(f)\|_{\mathcal{H}}.$$

Supposing that one can evaluate $\|A_n(f)\|_{\mathcal{H}}$ strictly from the data, this implies that any algorithm satisfying the data-dependent criterion

$$\hat{\varepsilon}_n \leq \min \left((1 - \theta)\varepsilon_a, \frac{(1 - \theta)\varepsilon_a + \theta\varepsilon_r \|A_n(f)\|_{\mathcal{H}}}{1 + \theta\varepsilon_r} \right) \quad (5)$$

must also satisfy (4). This criterion becomes the stopping criterion for the automatic Algorithm ?? below. Using the triangle inequality again implies that if

$$\hat{\varepsilon}_n \leq \min \left((1 - \theta)\varepsilon_a, \frac{(1 - \theta)\varepsilon_a + \theta\varepsilon_r \|S(f)\|_{\mathcal{H}}}{1 + 2\theta\varepsilon_r} \right), \quad (6)$$

then (5) must also be satisfied. This criterion is used to construct an upper bound on the cost of automatic Algorithm ?? in Theorem ??.

In many cases it is possible to work with a point-wise generalized error criterion. Suppose that the space of solutions, \mathcal{H} , is a vector space of real-valued functions on \mathcal{Y} , and that the \mathcal{H} -norm is a sup norm:

$$\|h\|_{\mathcal{H}} = \sup_{\mathbf{y} \in \mathcal{Y}} |h(\mathbf{y})|. \quad (7)$$

Then a point-wise generalized error criterion would take the form:

$$\left| (S - \tilde{A}_n)(f)(\mathbf{y}) \right| \leq (1 - \theta)\varepsilon_a + \theta\varepsilon_r |S(f)(\mathbf{y})| \quad \forall \mathbf{y} \in \mathcal{Y}, \quad (8)$$

where again $0 \leq \theta \leq 1$. Here \tilde{A}_n may not be the same as A_n , but as shall be seen below is defined in terms of A_n . Suppose one has a reliable pointwise upper bound on the error of a non-adaptive algorithm, A_n , with cost n :

$$\hat{\varepsilon}_n(\mathbf{y}) \geq |(S - A_n)(f)(\mathbf{y})|. \quad (9)$$

Here, $\hat{\varepsilon}_n(\mathbf{y})$ might be independent of \mathbf{y} . Furthermore, suppose that $A_n(f)(\mathbf{y})$ can be evaluated from the data.

Proposition 1. Suppose that $\theta\varepsilon_r \leq 1$. If error bound (9) holds, then the point-wise generalized error criterion (8) also holds, provided that

$$\begin{aligned}\hat{\varepsilon}_n(\mathbf{y}) &\leq (1 - \theta)\varepsilon_a + \theta\varepsilon_r \max(\hat{\varepsilon}_n(\mathbf{y}), |A_n(f)(\mathbf{y})|), \\ \tilde{A}_n(f)(\mathbf{y}) &= A_n(f)(\mathbf{y}) - \theta\varepsilon_r \operatorname{sign}(A_n(f)(\mathbf{y})) \min(\hat{\varepsilon}_n(\mathbf{y}), |A_n(f)(\mathbf{y})|).\end{aligned}$$

Proof. The cases of $\hat{\varepsilon}_n(\mathbf{y}) \leq |A_n(f)(\mathbf{y})|$ and $\hat{\varepsilon}_n(\mathbf{y}) > |A_n(f)(\mathbf{y})|$ are treated separately. In the former case the definition of $\tilde{A}_n(f)(\mathbf{y})$ and the inequality constraint on $\hat{\varepsilon}_n(\mathbf{y})$ imply that

$$\begin{aligned}& \left| \tilde{A}_n(f)(\mathbf{y}) - A_n(f)(\mathbf{y}) + \theta\varepsilon_r \operatorname{sign}(A_n(f)(\mathbf{y}))\hat{\varepsilon}_n(\mathbf{y}) \right| = 0 \\& \leq (1 - \theta)\varepsilon_a + \theta\varepsilon_r |A_n(f)(\mathbf{y})| - \hat{\varepsilon}_n(\mathbf{y}) \\& \implies -(1 - \theta)\varepsilon_a + [1 - \theta\varepsilon_r \operatorname{sign}(A_n(f)(\mathbf{y}))][A_n(f)(\mathbf{y}) + \hat{\varepsilon}_n(\mathbf{y})] \\& \leq \tilde{A}_n(f)(\mathbf{y}) \leq (1 - \theta)\varepsilon_a + [1 + \theta\varepsilon_r \operatorname{sign}(A_n(f)(\mathbf{y}))][A_n(f)(\mathbf{y}) - \hat{\varepsilon}_n(\mathbf{y})] \\& \implies -(1 - \theta)\varepsilon_a + \max\{S(f)(\mathbf{y}) - \theta\varepsilon_r |S(f)(\mathbf{y})| : |(S - A_n)(f)(\mathbf{y})| \leq \hat{\varepsilon}_n\} \\& \leq \tilde{A}_n(f)(\mathbf{y}) \\& \leq (1 - \theta)\varepsilon_a + \min\{S(f)(\mathbf{y}) + \theta\varepsilon_r |S(f)(\mathbf{y})| : |(S - A_n)(f)(\mathbf{y})| \leq \hat{\varepsilon}_n\} \\& \implies -(1 - \theta)\varepsilon_a + S(f)(\mathbf{y}) - \theta\varepsilon_r |S(f)(\mathbf{y})| \\& \leq \tilde{A}_n(f)(\mathbf{y}) \leq (1 - \theta)\varepsilon_a + S(f)(\mathbf{y}) + \theta\varepsilon_r |S(f)(\mathbf{y})| \quad \forall \mathbf{y} \in \mathcal{Y}, \\& \implies -(1 - \theta)\varepsilon_a - \theta\varepsilon_r |S(f)(\mathbf{y})| \leq \tilde{A}_n(f)(\mathbf{y}) - S(f)(\mathbf{y}) \\& \leq (1 - \theta)\varepsilon_a + \theta\varepsilon_r |S(f)(\mathbf{y})| \quad \forall \mathbf{y} \in \mathcal{Y},\end{aligned}$$

which implies (8).

In the case of $\hat{\varepsilon}_n(\mathbf{y}) > |A_n(f)(\mathbf{y})|$, the definition of $\tilde{A}_n(f)(\mathbf{y})$ and the inequality constraint on $\hat{\varepsilon}_n(\mathbf{y})$ imply that

$$\begin{aligned}& \left| \tilde{A}_n(f)(\mathbf{y}) - (1 - \theta\varepsilon_r)A_n(f)(\mathbf{y}) \right| = 0 \leq (1 - \theta)\varepsilon_a - (1 - \theta\varepsilon_r)\hat{\varepsilon}_n(\mathbf{y}) \\& \implies -(1 - \theta)\varepsilon_a + (1 - \theta\varepsilon_r)[A_n(f)(\mathbf{y}) + \hat{\varepsilon}_n(\mathbf{y})] \\& \leq \tilde{A}_n(f)(\mathbf{y}) \leq (1 - \theta)\varepsilon_a + (1 - \theta\varepsilon_r)[A_n(f)(\mathbf{y}) - \hat{\varepsilon}_n(\mathbf{y})] \\& \implies -(1 - \theta)\varepsilon_a + \max\{S(f)(\mathbf{y}) - \theta\varepsilon_r |S(f)(\mathbf{y})| : |(S - A_n)(f)(\mathbf{y})| \leq \hat{\varepsilon}_n\} \\& \leq \tilde{A}_n(f)(\mathbf{y}) \\& \leq (1 - \theta)\varepsilon_a + \min\{S(f)(\mathbf{y}) + \theta\varepsilon_r |S(f)(\mathbf{y})| : |(S - A_n)(f)(\mathbf{y})| \leq \hat{\varepsilon}_n\}\end{aligned}$$

From this point the argument showing that (8) is satisfied proceeds exactly as in the previous case. \square

Acknowledgements

- [1] Ding Y, Clancy N, Hamilton C, Hickernell FJ, Zhang Y (2012) The complexity of guaranteed automatic algorithms. In preparation