A Better $\mathcal{G}$-Semi-Norm for Algorithms Based on Linear Splines

Fred J. Hickernell

Room E1-208, Department of Applied Mathematics, Illinois Institute of Technology, 10 W. 32nd St., Chicago, IL 60616

Abstract

Keywords:

1. Why Do We Need a Better $\mathcal{G}$-Semi-Norm

In [1] we presented guaranteed automatic algorithms for integration and function recovery. One weakness of these algorithms was that if $\|f'\|_1$ ($\|f'\|_\infty$) is large, but $\|f''\|_1$ ($\|f''\|_\infty$) is small, then the error bounds in the automatic algorithms for integration (function recovery) can be very conservative. The input function $f : x \mapsto 100x$ is a prime example.

Here we present a family of substitutes for $\|f'\|_p$, $p = 1, \infty$ that overcome this weakness. If the function is piecewise linear with a small number of discontinuities in f', then the algorithms in [1] converge quite quickly.

2. A Proposed New $\mathcal{G}$-Semi-Norm

Define the integral functional on the interval $[\alpha, \beta]$ as

$$I(f; \alpha, \beta) = \int_\alpha^\beta f(x)\,\mathrm{d}x,$$

and define the $\mathcal{L}_p$-norms on this as follows:

$$\|f\|_{p,[\alpha,\beta]} = \begin{cases} \left|\int_\alpha^\beta |f(x)|^p \,\mathrm{d}x\right|^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{a \le x \le b} |f(x)|, & p = \infty. \end{cases}$$

The norm over the larger interval can be written in terms of the norms of subintervals. Namely, for any α_j satisfying $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_r = \beta$, it follows that

$$\|f\|_{p,[\alpha,\beta]} = \left\| \left(\|f\|_{p,[\alpha_{k-1},\alpha_k]} \right)_{k=1}^r \right\|_{\ell_p}. \tag{1}$$

For any $m \in \mathbb{N}$, define the following equally spaced points:

$$x_{i,m} = \frac{i}{m}, \qquad i = 0, \dots m.$$

The following semi-norm compares measures the deviation of the first derivative of the function from its average value:

$$|f|_{\mathcal{G},m,p} = \left\| \left(\left\| \|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{p,[x_{i-1,m},x_{i,m}]} \right\|_{p,[x_{i-1,m},x_{i,m}]} \right)_{i=1}^{m} \right\|_{\ell_p}. \tag{2}$$

Let us verify that this indeed a semi-norm. Clearly, it is non-negative and vanishes when f is the zero function. So, the only other thing to verify is the triangle inequality, which follows since $f \mapsto f' - mI(f'; x_{i-1,m}, x_{i,m})$ is a linear operator.

It can be shown that $|f|_{\mathcal{G},rm,2} \leq |f|_{\mathcal{G},m,2}$ for positive integer r. First, note that the one may re-write the norm over $[x_{i-1,m}, x_{i,m}]$ in terms of norms over $[x_{r(i-1)+k-1,rm}, x_{r(i-1)+k,rm}]$ by (1):

$$\begin{aligned} &\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{i-1,m},x_{i,m}]} \\ &\qquad = \left\| \left(\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{r(i-1)+k-1,rm},x_{r(i-1)+k,rm}]} \right)_{k=1}^{r} \right\|_{\ell_2}. \end{aligned} \tag{3}$$

Moreover, the norms inside the expression above are no smaller than the norms comprising $|f|_{\mathcal{G},rm,2}$:

$$\begin{aligned} &\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|^2_{2,[x_{r(i-1)+k-1,rm},x_{r(i-1)+k,rm}]} \\ &\quad = \int_{x_{r(i-1)+k-1}}^{x_{r(i-1)+k,rm}} [f'(x) - mI(f'; x_{i-1,m}, x_{i,m})]^2 \, \mathrm{d}x \\ &\quad = \int_{x_{r(i-1)+k-1}}^{x_{r(i-1)+k,rm}} \big\{ [f'(x) - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})]^2 \\ &\qquad\qquad + 2m[f'(x) - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm})] \\ &\qquad\qquad\qquad \times [rI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})] \\ &\qquad\qquad + m^2[rI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]^2 \big\} \, \mathrm{d}x \\ &\quad = \left\| f' - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) \right\|^2_{2,[x_{r(i-1)+k-1},x_{r(i-1)+k,rm}]} \\ &\qquad\qquad + \frac{1}{r^2}[rI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) - I(f'; x_{i-1,m}, x_{i,m})]^2 \\ &\quad \geq \left\| f' - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) \right\|^2_{2,[x_{r(i-1)+k-1},x_{r(i-1)+k,rm}]} \end{aligned}$$

Substituting this inequality into (3) yields

$$\begin{aligned} &\|f' - mI(f'; x_{i-1,m}, x_{i,m})\|_{2,[x_{i-1,m},x_{i,m}]} \\ &\geq \left\| \left(\left\| f' - rmI(f'; x_{r(i-1)+k-1}, x_{r(i-1)+k,rm}) \right\|_{2,[x_{r(i-1)+k-1},x_{r(i-1)+k,rm}]} \right)_{k=1}^{r} \right\|_{\ell_2}. \end{aligned}$$

According to the definition of $|f|_{\mathcal{G},m,2}$ in (2), it then follows that $|f|_{\mathcal{G},rm,2} \leq |f|_{\mathcal{G},m,2}$.

3. Estimating the New $\mathcal{G}$-Semi-Norm

Let $\tilde{f}_n$ denote the linear spline based on the function values $f(x_{i,n-1})$, $i = 0, \ldots, n-1$. We estimate $|f|_{\mathcal{G},(n-1)/2,p}$ by the algorithm $G_{(n-1)/2,p,n} : f \mapsto |\tilde{f}_n|_{\mathcal{G},(n-1)/2,p}$. Then it follows that the error of our approximation to the $\mathcal{G}$-semi-norm is

$$\begin{aligned} |f|_{\mathcal{G},(n-1)/2,p} - G_{(n-1)/2,p,n}(f) &= |f|_{\mathcal{G},(n-1)/2,p} - \left|\tilde{f}_n\right|_{\mathcal{G},(n-1)/2,p} \\ &\le \left|f - \tilde{f}_n\right|_{\mathcal{G},(n-1)/2,p}, \end{aligned}$$

For any $i = 1, \ldots, n-2$, the idea is to approximate The idea is to approximate

$$\int_{x_{i-1}}^{x_{i+1}} \left| f'(x) - \frac{n-1}{2} I(f'; x_{i-1}, x_{i+1}) \right|^p \mathrm{d}x$$

by piecewise linear approximation to f, which implies a piecewise constant approximation to f'.

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4. Bounding the Error in Terms of the New $\mathcal{G}$-Semi-Norm

4.1. Integration

For the integration problem, we use the composite trapezoidal rule,

$$\begin{aligned} A_n(f) = \frac{1}{2(n-1)} \, [f(x_{0,n-1}) + 2f(x_{1,n-1}) + 2f(x_{2,n-1}) + \cdots \\ +2f(x_{n-2,n-1}) + f(x_{n-1,n-1})] \, . \end{aligned}$$

We consider how to bound the error of this approximation for integrands with given norm $|f|_{\mathcal{G},(n-1)/2,p}$.

For simplicity of notation let $x_i = x_{i,n-1}$. For a fixed $i = 1, \ldots, n-2$, consider the error in integrating over the interval $[x_{i-1}, x_{i+1}]$ using two trapezoids. The derivation proceeds by applying applying basic calculus ideas. First we express the trapezoidal rule approximation as the integral of a piecewise linear function, and then use integration by parts to write the integral in terms of the first derivative of the integrand:

$$\begin{aligned} &\left| \int_{x_{i-1}}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)} [f(x_{i-1}) + 2f(x_i) + f(x_{i+1})] \right| \\ &\quad = \left| \int_{x_{i-1}}^{x_i} \{ f(x) - (n-1)[f(x_{i-1})(x_i - x) + f(x_i)(x - x_{i-1})] \} \,\mathrm{d}x \right. \\ &\qquad\qquad \left. + \int_{x_i}^{x_{i+1}} \{ f(x) - (n-1)[f(x_i)(x_{i+1} - x) + f(x_{i+1})(x - x_i)] \} \,\mathrm{d}x \right| \\ &\quad = (n-1) \left| \int_{x_{i-1}}^{x_i} \left\{ (x_i - x) \int_{x_{i-1}}^{x} f'(t)\,\mathrm{d}t - (x - x_{i-1}) \int_{x}^{x_i} f'(t)\,\mathrm{d}t \right\} \mathrm{d}x \right. \\ &\qquad\qquad \left. + \int_{x_i}^{x_{i+1}} \left\{ (x_{i+1} - x) \int_{x_i}^{x} f'(t)\,\mathrm{d}t - (x - x_i) \int_{x}^{x_{i+1}} f'(t)\,\mathrm{d}t \right\} \mathrm{d}x \right|. \end{aligned}$$

Next the order of integration is switched and the error is expressed as a univariate integral of f' multiplied by a weight:

$$
\begin{aligned}
&\left|\int_{x_{i-1}}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{i-1}) + 2f(x_i) + f(x_{i+1})]\right| \\
&\quad = (n-1)\left|\int_{x_{i-1}}^{x_i} f'(t)\left\{\int_t^{x_i} (x_i - x)\,\mathrm{d}x - \int_{x_{i-1}}^{t} (x - x_{i-1})\,\mathrm{d}x\right\}\mathrm{d}t \right.\\
&\qquad\qquad \left. + \int_{x_i}^{x_{i+1}} f'(t)\left\{\int_t^{x_{i+1}} (x_{i+1} - x)\,\mathrm{d}x - \int_{x_i}^{t} (x - x_i)\,\mathrm{d}x\right\}\mathrm{d}t\right| \\
&\quad = (n-1)\left|\int_{x_{i-1}}^{x_i} f'(t)\left\{\frac{(t-x_i)^2 - (t-x_{i-1})^2}{2}\right\}\mathrm{d}t \right.\\
&\qquad\qquad \left. + \int_{x_i}^{x_{i+1}} f'(t)\left\{\frac{(t-x_{i+1})^2 - (t-x_i)^2}{2}\right\}\mathrm{d}t\right| \\
&\quad = \left|\int_{x_{i-1}}^{x_i} f'(t)\left(\frac{x_{i-1} + x_i - 2t}{2}\right)\mathrm{d}t + \int_{x_i}^{x_{i+1}} f'(t)\left(\frac{x_i + x_{i+1} - 2t}{2}\right)\mathrm{d}t\right|.
\end{aligned}
$$

Since the weight is odd with respect to the center of the integration interval, one may subtract a constant multiple of this weight without altering the integral, and then apply Hölder's inequality with $p^{-1} + q^{-1} = 1$.

$$
\begin{aligned}
&\left|\int_{x_{i-1}}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{i-1}) + 2f(x_i) + f(x_{i+1})]\right| \\
&\quad = \left|\int_{x_{i-1}}^{x_i} \left[f'(t) - \frac{n-1}{2}I(f'; x_{i-1}, x_{i+1})\right]\left(\frac{x_{i-1} + x_i - 2t}{2}\right)\mathrm{d}t \right.\\
&\qquad\qquad \left. + \int_{x_i}^{x_{i+1}} \left[f'(t) - \frac{n-1}{2}I(f'; x_{i-1}, x_{i+1})\right]\left(\frac{x_i + x_{i+1} - 2t}{2}\right)\mathrm{d}t\right| \\
&\quad = \left\|f' - \frac{n-1}{2}I(f', x_{i-1}, x_{i+1})\right\|_{p,[x_{i-1},x_{i+1}]} \\
&\qquad\qquad \times \left\{\int_{x_{i-1}}^{x_i} \left|\frac{x_{i-1} + x_i - 2t}{2}\right|^q \mathrm{d}t + \int_{x_i}^{x_{i+1}} \left|\frac{x_i + x_{i+1} - 2t}{2}\right|^q \mathrm{d}t\right\}^{1/q} \\
&\quad = \left[\frac{4}{(q+1)(2n-2)^{q+1}}\right]^{1/q} \left\|f' - \frac{n-1}{2}I(f'; x_{i-1}, x_{i+1})\right\|_{p,[x_{i-1},x_{i+1}]}.
\end{aligned}
$$

Finally, these error bounds over the subintervals $[x_{i-1}, x_{i+1}]$ are combined to obtain an error bound for the trapezoidal rule, again using Hölder's inequality:

$$\begin{aligned}
&\left|\int_0^1 f(x)\,\mathrm{d}x - A_n(f)\right| \\
&\qquad \le \sum_{i=1}^{(n-1)/2} \left|\int_{x_{2i}}^{x_{2i+2}} f(x)\,\mathrm{d}x - \frac{1}{2(n-1)}[f(x_{2i}) + 2f(x_{2i+1}) + f(x_{2i+2})]\right| \\
&\qquad \le \sum_{i=1}^{(n-1)/2} \left[\frac{4}{(q+1)(2n-2)^{q+1}}\right]^{1/q} \left\|f' - \frac{n-1}{2} I(f'; x_{2i}, x_{2i+2})\right\|_{p,[x_{2i},x_{2i+2}]} \\
&\qquad \le \left(\frac{1}{q+1}\right)^{1/q} \frac{|f|_{\mathcal{G},(n-1)/2,p}}{2n-2}.
\end{aligned}$$

[1] N. Clancy, Y. Ding, C. Hamilton, F. J. Hickernell, Y. Zhang, The complexity of guaranteed automatic algorithms: Cones, not balls, submitted for publication, arXiv.org:1303.2412 [math.NA] (2013).