REVIEWER REPORT

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Title: Empirical Bernstein and betting confidence intervals for randomized quasi-Monte Carlo
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General assessment

Randomized quasi-Monte Carlo (RQMC) provides unbiased estimators of an integral (an expectation) whose variance converges at a faster rate than classical Monte Carlo (MC), under appropriate smoothness conditions on the integrand. The variance would typically converge nearly as $\mathcal{O}(n^{-\theta})$ for n sample points, where θ is often 2 or 3, whereas $\theta = 1$ for MC. One issue with RQMC is the computation of a confidence interval (CI) on the mean. Since the n samples are not independent, in general one cannot rely on the central limit theorem (CLT) even if n is large. The usual approach is to make R independent replicates of the RQMC estimator by randomizing the points R times, independently, and estimate the RQMC variance by their sample variance. Various ways of computing a CI from that are compared in [23] and the authors found in their empirical experiments that Student CIs were quite robust. However, these CIs do not provide a guaranteed coverage.

The aim of this paper is to construct CIs with guaranteed coverage for any values of R and n, for RQMC, and find at what rates R and n should increase when the total computing budget $N = nR$ increases. In their section 2.3, the authors recall that without appropriate conditions on the distribution of the RQMC estimator, there are no useful CIs with such guaranteed coverage. If we assume that the RQMC estimator X takes its values only in a known bounded interval, which can be taken as the interval $[0, 1]$ without loss of generality, then a CI with guaranteed coverage becomes possible. In particular, CIs based on Hoeffding's inequality do that. However, as shown in Equation (4), these Hoeffding intervals are independent of the variance of X, so a lower RQMC variance has no impact on the width of the CI. This implies that the optimal budget allocation would be to take $n = 1$ and $R = N$. These CIs also tend to be extremely conservative.

The authors then consider CIs whose width depend on the variance to a certain extent. The Bennett CI in (5) depend on $\sigma^2 = \mathrm{Var}[X]$, which is generally unknown, while the (slightly larger) empirical Bernstein CI in (6) depends on the empirical variance s^2, which is easily computable. In Section 2.4, the authors then go on to study valid CI sequences, for which one can stop after any number of RQMC replications (e.g., when the current CI is narrow enough) and return this current CI.

In Section 3, the authors examine how best to allocate a total budget of N function evaluations into R RQMC replicates of size n, so $nR = N$, asymptotically when $N \to \infty$. When looking at (5) or (6), we see that when the variance gets very small, the middle term in these equations (the first term of the width) becomes very small and the width of the CI then depends essentially on the last term, which does not depend on the variance. To reduce the width in that case, we must increase R as much as possible, because increasing n has little effect. This also explains why using RQMC instead of MC with those intervals does not decrease the width by much. All of this is reflected in Theorem 1, which states that the optimal n increases very slowly with N, as $\mathcal{O}(N^{-1/(\theta+1)})$. That is, when the RQMC variance converges at a faster rate (θ is larger), increasing n faster brings negligible gain and it is more effective to increase R instead.

My takeaway: these types of CIs provide a guaranteed coverage, but there is a price to pay for that. They must be quite conservative in general, and using RQMC does not reduce their width by much, as shown in Theorem 2. With RQMC they are also likely to become much more conservative than with MC.

I think that understanding all of this is very useful and therefore I recommend acceptance of this paper for publication, after minor corrections.

Specific comments and details

1. The confidence interval at the bottom of page 6 makes sense if $\hat{\mu}_i$ is approximately normally distributed. It is exact for any fixed R if $\hat{\mu}_i$ is exactly normally distributed. I think this should be pointed out.
2. Page 8, line 20: I find the $F \sim \mathbb{R}$ confusing.
3. Page 10, line 27: λ_i should be $\lambda_i(m)$.
4. Page 15, line 51: $[0, 1]$ should be $[0, 1]^d$.