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Estimating means of bounded random variables by betting

Ian Waudby-Smith[1] and Aaditya Ramdas[1,2]

[1]Department of Statistics & Data Science, Carnegie Mellon University, Pittsburgh, PA, USA
[2]Machine Learning Department, Carnegie Mellon University, Pittsburgh, PA, USA

Address for correspondence: Aaditya Ramdas, Machine Learning, Carnegie Mellon University, 5000 Forbes Ave, 132 Baker Hall, Pittsburgh, PA 15213, USA. Email: aramdas@cmu.edu

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Abstract

We derive confidence intervals (CIs) and confidence sequences (CSs) for the classical problem of estimating a bounded mean. Our approach generalizes and improves on the celebrated Chernoff method, yielding the best closed-form "empirical-Bernstein" CSs and CIs (converging exactly to the oracle Bernstein width) as well as non-closed-form "betting" CSs and CIs. Our method combines new composite nonnegative (super) martingales with Ville's maximal inequality, with strong connections to testing by betting and the method of mixtures. We also show how these ideas can be extended to sampling without replacement. In all cases, our bounds are adaptive to the unknown variance, and empirically vastly outperform prior approaches, establishing a new state-of-the-art for four fundamental problems: CSs and CIs for bounded means, when sampling with and without replacement.

1 Introduction

This work presents a new approach to two fundamental problems: (Q1) how do we produce a confidence interval for the mean of a distribution with (known) bounded support using n independent observations? (Q2) given a fixed list of N (nonrandom) numbers with known bounds, how do we produce a confidence interval for their mean by sampling $n \le N$ of them without replacement in a random order? We work in a nonasymptotic and nonparametric setting, meaning that we do not employ asymptotics or parametric assumptions. Both (Q1) and (Q2) are well studied questions in probability and statistics, but we bring new conceptual tools to bear, resulting in state-of-the-art solutions to both.

We also consider sequential versions of these problems where observations are made one-by-one; we derive time-uniform confidence sequences, or equivalently, confidence intervals that are valid at arbitrary stopping times. In fact, we first describe our techniques in the sequential regime, because the employed proof techniques naturally lend themselves to this setting. We then instantiate the derived bounds for the more familiar setting of a fixed sample size when a batch of data is observed all at once. Our supermartingale techniques can be thought of as generalizations of classical methods for deriving concentration inequalities, but we prefer to present them in the language of betting, since this is a more accurate reflection of the authors' intuition.

Arguably, the most famous concentration inequality for bounded random variables was derived by Hoeffding (1963). What is now referred to as 'Hoeffding's inequality' was in fact improved upon in the same paper where he derived a Bernoulli-type upper bound on the moment generating function of bounded random variables (Hoeffding, 1963, equation (3.4)). While these bounds are already reasonably tight in a worst-case sense, the resulting confidence intervals do not adapt to non-Bernoulli distributions with lower variance. Inequalities by Bennett (1962), Bernstein (1927), and Bentkus (2004) improve upon Hoeffding's, but such

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improvements require knowledge of nontrivial upper bounds on the variance. This led to the development of so-called 'empirical Bernstein inequalities' by Audibert et al. (2007) and Maurer and Pontil (2009), which outperform Hoeffding's method for low-variance distributions at large sample sizes by estimating the variance from the data. Our new, and arguably quite simple, approaches to developing bounds significantly outperform these past works (e.g., Figure 1).[1] We also show that the same conceptual (betting) framework extends to without-replacement sampling, resulting in significantly tighter bounds than classical ones by Serfling (1974), improvements by Bardenet and Maillard (2015), and previous state-of-the-art methods due to Waudby-Smith and Ramdas (2020).

For providing intuition, our approach can be described in words as follows: *If we are allowed to repeatedly bet against the mean being m, and if we make a lot of money in the process, then we can safely exclude m from the confidence set.* The rest of this paper makes the above claim more precise by showing smart, adaptive strategies for (automated) betting, quantifying the phrase 'a lot of money', and explaining why such an exclusion is mathematically justified. At the risk of briefly losing the unacquainted reader, here is a slightly more detailed high-level description:

> For each $m \in [0, 1]$, we set up a 'fair' multi-round game of statistician against nature whose payoff rules are such that if the true mean happened to equal m, then the statistician can neither gain nor lose wealth in expectation (their wealth in the mth game is a nonnegative martingale), but if the mean is not m, then it is possible to bet smartly and make money. Each round involves the statistician making a bet on the next observation, nature revealing the observation and giving the appropriate (positive or negative) payoff to the statistician. The statistician then plays all these games (one for each m) in parallel, starting each with one unit of wealth, and possibly using a different, adaptive, betting strategy in each. The $1 - \alpha$ confidence set at time t consists of all $m \in [0, 1]$ such that the statistician's money in the corresponding game has not crossed $1/\alpha$. The true mean μ will be in this set with high probability.

Our choice of language above stems from a game-theoretic approach toward probability, as developed in the books by Shafer and Vovk (2001, 2019) and a recent paper by Shafer (2021), but from a purely mathematical viewpoint, our results are extensions of a unified supermartingale approach toward nonparametric concentration and estimation described in Howard et al. (2020, 2021); related supermartingale approaches were studied by Kaufmann and Koolen (2021) and Jun and Orabona (2019). We elaborate on this viewpoint in Section 4.1. The most directly related works to our own are by Hendriks (2018), whose preprint has initial explorations of methods similar to ours for with-replacement sequential testing and estimation, and Stark (2020), who credits Kaplan for a computationally intractable variant of our approach for sequential testing in the without-replacement case. Apart from several novel results, the present paper extends these past works in *depth, breadth, and unity*: our work contains a deeper empirical and theoretical investigation from statistical and computational viewpoints, places our work in a broader context of related work in both settings, and unifies the with- and without-replacement methodology for both testing and estimation in both fixed-time and sequential settings.

We now have the appropriate context for a concrete formalization of our problem, which is slightly more general than introduced above. After that, we describe the game, why the rules of engagement result in valid statistical inference and derive computationally and statistically efficient betting strategies.

Outline. We summarize the broad approach in Section 2. As a warmup, we derive a new predictable plug-in method for deriving confidence sequences using exponential supermartingales (Section 3), which already leads to computationally efficient and visually appealing empirical Bernstein confidence intervals and sequences. We then further improve on the aforementioned methods by developing a new martingale approach to deriving time-uniform and fixed-time confidence sets for means of bounded random variables, and connect the developed ideas to betting (Section 4). Online Supplementary Material, Section B discusses some principles to derive

[1] https://github.com/WannabeSmith/betting-paper-simulations has code to reproduce figures. The `betting` module of the Python package in https://github.com/gostevehoward/confseq has the main algorithms, but the package also contains implementations from other papers.

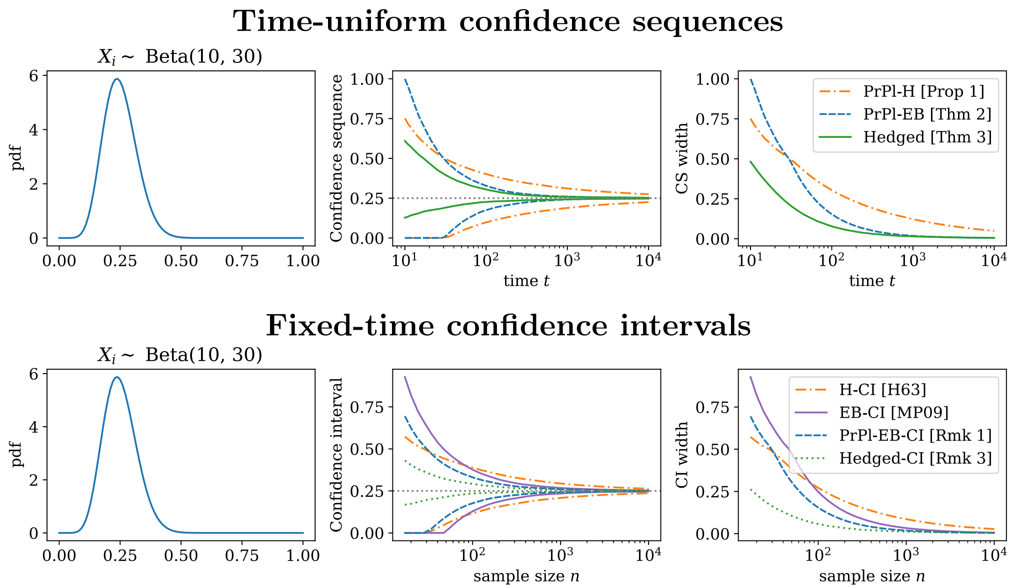

Figure 1. Time-uniform 95% confidence sequences (upper row) and fixed-time 95% confidence intervals (lower row) for the mean of independent and identically distributed (iid) draws from a Beta(10, 30) distribution (unknown to the methods). The betting approaches (Hedged and Hedged-CI) adapt to both the small variance and asymmetry of the data, outperforming the other methods. For a detailed empirical comparison under a larger variety of settings, see Online Supplementary Material, Section C; for additional comparisons under non-iid data, see Online Supplementary Material, Section E.5.

powerful betting strategies to obtain tight confidence sets. We then show how our techniques also extend to sampling without replacement (Section 5). Revealing simulations are performed along the way to demonstrate the efficacy of the new methods, with a more extensive comparison with past work in Online Supplementary Material, Section C. Section 6 summarizes how betting ideas have shaped mathematics, outside of our paper's focus on statistical inference. We postpone proofs to Online Supplementary Material, Section A and further theoretical insights to Online Supplementary Material, Section E.

2 Concentration inequalities via nonnegative supermartingales

To set the stage, let $\mathcal{Q}^m$ be the set of all distributions on $[0, 1]$, where each distribution has mean m. Note that $\mathcal{Q}^m$ is a convex set of distributions and it has no common dominating measure, since it consists of both discrete and continuous distributions.

Consider the setting where we observe a (potentially infinite) sequence of $[0, 1]$-valued random variables with conditional mean μ for some unknown $\mu \in [0, 1]$. We write this as $(X_t)_{t=1}^{\infty} \sim P$ for some $P \in \mathcal{P}^\mu$, where $\mathcal{P}^\mu$ is the set of all distributions P on $[0, 1]^\infty$ such that $\mathbb{E}_P(X_t \mid X_1, \ldots, X_{t-1}) = \mu$. This includes familiar settings such as independent observations, where $X_i \sim Q_i \in \mathcal{Q}^\mu$, or i.i.d. observations where all Q_i's are identical, but captures more general settings where the conditional distribution of X_t given the past is an element of $\mathcal{Q}^\mu$. When one only observes n outcomes, it suffices to imagine throwing away the rest, so that in what follows, we avoid a new notation for distributions P over finite length sequences.

We are interested in deriving tight confidence sets for μ, typically intervals, with no further assumptions. Specifically, for a given error tolerance $\alpha \in (0, 1)$, a $(1-\alpha)$ confidence interval (CI) is a random set $C_n \equiv C(X_1, \ldots, X_n) \subseteq [0, 1]$ such that

$$\forall n \geq 1, \quad \inf_{P \in \mathcal{P}^\mu} P(\mu \in C_n) \geq 1 - \alpha. \tag{1}$$

As mentioned earlier, the inequality by Hoeffding (1963) implies that we can choose

$$C_n := \left(\overline{X}_n \pm \sqrt{\frac{\log(2/\alpha)}{2n}} \right) \cap [0, 1]. \tag{2}$$

Above, we write $(a \pm b)$ to mean $(a - b, a + b)$ for brevity.

This inequality is derived by what is now known as the Chernoff method (Boucheron et al., 2013), involving an analytic upper bound on the moment generating function of a bounded random variable. However, we will proceed differently; we adopt a hypothesis testing perspective and couple it with a generalization of the Chernoff method. As mentioned in the introduction, we first consider the sequential regime where data are observed one after another over time, since nonnegative supermartingales—the primary mathematical tools used throughout this paper—naturally arise in this setup. As we will see, these sequential bounds can be instantiated for a fixed sample size, yielding tight confidence intervals for this more familiar setting. These will be much tighter than the Hoeffding confidence interval (2), which is itself one such fixed-sample-size instantiation (Howard et al., 2020, Figures 4 and 6).

Let us briefly review some terminology. For succinctness, we use the notation $X_1^t := (X_1, \ldots, X_t)$. Define the sigma-field $\mathcal{F}_t := \sigma(X_1^t)$ generated by X_1^t with $\mathcal{F}_0$ being the trivial sigma-field. The *canonical filtration* $\mathcal{F} := (\mathcal{F}_t)_{t=0}^{\infty}$ refers to the increasing sequence of sigma-fields $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. A stochastic process $(M_t)_{t=0}^{\infty}$ is called a *test supermartingale* for P if $(M_t)_{t=0}^{\infty}$ is a nonnegative process adapted to $\mathcal{F}$, $M_0 = 1$, and

$$\mathbb{E}_P(M_t \mid \mathcal{F}_{t-1}) \le M_{t-1} \quad \text{for each } t \ge 1. \tag{3}$$

$(M_t)_{t=0}^{\infty}$ is called a *test martingale* for P if the above '$\le$' is replaced with '='. We sometimes shorten $(M_t)_{t=0}^{\infty}$ to just (M_t) for brevity. If the above property holds simultaneously for all $P \in \mathcal{P}$, we call (M_t) a test (super)martingale for $\mathcal{P}$. We say that a sequence $(\lambda_t)_{t=1}^{\infty}$ is *predictable* if λ_t is $\mathcal{F}_{t-1}$-measurable for each $t \ge 1$, meaning that λ_t can only depend on X_1^{t-1}. (In)equalities are interpreted in an almost sure sense.

2.1 Confidence sequences and the method(s) of mixtures

Even though the concentration inequalities thus far have been described in a setting where the sample size n is fixed in advance, all of our ideas stem from a sequential approach toward uncertainty quantification. The goal there is not to produce one confidence set C_n, but to produce an infinite sequence $(C_t)_{t=1}^{\infty}$ such that

$$\sup_{P \in \mathcal{P}^{\mu}} P(\exists t \ge 1 : \mu \notin C_t) \le \alpha. \tag{4}$$

Such a $(C_t)_{t=1}^{\infty}$ is called a *confidence sequence* (CS), and preferably $\lim_{t \to \infty} C_t = \{\mu\}$. It is known (Howard et al., 2021, Lemma 3) that (4) is equivalent to requiring that $\sup_{P \in \mathcal{P}^{\mu}} P(\mu \notin C_{\tau}) \le \alpha$ for arbitrary stopping times τ with respect to $\mathcal{F}$.

As detailed in the next subsection, one general way to construct a CS is to invert a family of sequential tests based on applying Ville's maximal inequality (Ville, 1939) to a test (super)martingale. In fact, Ramdas et al. (2020) proved that this is (in some formal sense) a universal method to construct CSs, meaning that any other approach can in principle be recovered or dominated by the aforementioned one.

Designing test supermartingales is nontrivial, and the task of making it have 'power one' against composite alternatives is often accomplished via the *method of mixtures*. This can arguably be traced back (in a nonstochastic context) to Ville's 1939 thesis and (in a stochastic context) to Wald (1945). Robbins and collabarators (Darling & Robbins, 1967a; Robbins, 1970; Robbins & Siegmund, 1968) applied the method to derive CSs, and these ideas have been extended to a variety of nonparametric settings by Howard et al. (2020, 2021). The latter paper describes several variants: conjugate mixtures, discrete mixtures, stitching, and inverted stitching.

These works form our vantage point for the rest of the paper, but we extend them in several ways. First, we describe a 'predictable plug-in' technique that is implicit in the work of Ville. It can be viewed as a nonparametric extension of a passing remark in the parametric setting in the textbook by Wald (1945, equation 10:10) and later explored in the parametric case by Robbins and Siegmund (1974).

Like Ville's work in the binary setting, the predictable plug-in method connects the game-theoretic approach and the aforementioned mixture methods—succinctly, the plugged-in value determines the bet, where each bet is implicitly targeting a different alternative (much like the components of a mixture). Following this translation, prior work on using the method mixtures for confidence sequences can be viewed as using the same betting strategy (mixture distribution) for every value of m. We find that there is significant statistical benefit to betting differently for each m (but tied together in a specific way, not in an ad hoc manner). One must typically specify the mixture distribution in advance of observing data, but betting can be viewed as building up a data-dependent mixture distribution on the fly (this led us to previously name our approach as the 'predictable mixture' method). These sequential perspectives are powerful, even if only interested in fixed-sample CIs.

2.2 Nonparametric confidence sequences via sequential testing

As seen above, it is straightforward to derive a confidence interval for μ by resorting to a nonparametric concentration inequality like Hoeffding's. In contrast, it is also well known that CIs are inversions of families of hypothesis tests (as we will see below), so one could presumably derive CIs by first specifying tests. However, the literature on nonparametric concentration inequalities, such as Hoeffding's, has not commonly utilized a hypothesis testing perspective to derive concentration bounds; for example the excellent book on concentration by Boucheron et al. (2013) has no examples of such an approach. This is presumably because the underlying nonparametric, composite hypothesis tests may be quite challenging themselves, and one may not have nonasymptotically valid solutions or closed-form analytic expressions for these tests. This is in contrast to simple parametric nulls, where it is often easy to calculate a p-value based on likelihood ratios. In abandoning parametrics, and thus abandoning likelihood ratios, it may be unclear how to define a powerful test or calculate a nonasymptotically valid p-value. This is where betting and test (super)martingales come to the rescue. Ramdas et al. (2020, Proposition 4) prove that not only do likelihood ratios form test martingales, but every (nonparametric, composite) test martingale is also a (nonparametric, composite) likelihood ratio.

Theorem 1 (4-step procedure for supermartingale confidence sets). On observing $(X_t)_{t=1}^{\infty} \sim P$ from $P \in \mathcal{P}^{\mu}$ for some unknown $\mu \in [0, 1]$, do

(a) Consider the composite null hypothesis $H_0^m : P \in \mathcal{P}^m$ for each $m \in [0, 1]$.
(b) For each index $m \in [0, 1]$, construct a nonnegative process $M_t^m \equiv M^m(X_1, \ldots, X_t)$ such that the process $(M_t^{\mu})_{t=0}^{\infty}$ indexed by μ has the following property: for each $P \in \mathcal{P}^{\mu}$, $(M_t^{\mu})_{t=0}^{\infty}$ is upper-bounded by a test (super)martingale for P, possibly a different one for each P.
(c) For each $m \in [0, 1]$ consider the sequential test $(\phi_t^m)_{t=1}^{\infty}$ defined by

$$\phi_t^m := \mathbf{1}(M_t^m \geq 1/\alpha),$$

where $\phi_t^m = 1$ represents a rejection of H_0^m after t observations.
(d) Define C_t as the set of $m \in [0, 1]$ for which ϕ_t^m fails to reject H_0^m:

$$C_t := \{m \in [0, 1] : \phi_t^m = 0\}.$$

Then $(C_t)_{t=1}^{\infty}$ is a $(1-\alpha)$-confidence sequence for μ: $\sup_{P \in \mathcal{P}^{\mu}} P(\exists t \geq 1 : \mu \notin C_t) \leq \alpha$.

The above result relies centrally on Ville's inequality (Ville, 1939), which states that if $(L_t) \equiv (L_t)_{t=1}^{\infty}$ is (upper bounded by) a test martingale for P, then we have $P(\exists t \geq 1 : L_t \geq 1/\alpha) \leq \alpha$. See Howard et al. (2020, Section 6) for a short proof.

Proof of Theorem 1. By Ville's inequality, ϕ_t^m is a level-α sequential hypothesis test, in the sense that for any $P \in \mathcal{P}^\mu$, we have $P(\exists t \geq 1 : \phi_t^\mu = 1) \leq \alpha$. Now, by definition of the sets $(C_t)_{t=1}^{\infty}$, we have that $\mu \notin C_t$ at some time $t \geq 1$ if and only if there exists a time $t \geq 1$ such that $\phi_t^\mu = 1$, and hence

$$\sup_{P \in \mathcal{P}^\mu} P(\exists t \geq 1 : \mu \notin C_t) = \sup_{P \in \mathcal{P}^\mu} P(\exists t \geq 1 : \phi_t^\mu = 1) \leq \alpha, \tag{5}$$

which completes the proof. $\square$

At a high level, this approach is not new. Composite test supermartingales for $\mathcal{P}$ have been used in past works on concentration inequalities and/or confidence sequences (which are related but different), from the initial series of works by Robbins and collaborators in the 1960s and 1970s, to de la Peña et al. (2007), to recent work by Jun and Orabona (2019, Section 7.2) and Howard et al. (2020, 2021). Test martingales have also been explicitly considered in some hypothesis testing problems (Shafer et al., 2011; Vovk et al., 2005); the latter paper popularized the term 'test martingale' that we borrow, but unlike us, used it primarily for singleton $\mathcal{P} = \{P\}$. We highlight an (independently developed) unpublished preprint by Hendriks (2018) that has overlaps with the current paper in the with-replacement setting, and some complementary results. For singleton (parametric) classes $\mathcal{P}$, Wald's sequential likelihood ratio statistic is a test martingale, so all of the above methods can be viewed as inverting nonparametric or composite generalizations of Wald's tests.

Nevertheless, we make two additional comments. First, the requirement in step (b) of the algorithm that the process (M_t^m) be *upper-bounded* by a test (super)martingale for each $P \in \mathcal{P}$ was posited by Howard et al. (2020) and has recently been christened an e-process for $\mathcal{P}$ (Ramdas et al., 2021) (see also Grünwald et al., 2019). E-processes are strictly more general than test (super)martingales for $\mathcal{P}$ in the sense that there exist many interesting classes $\mathcal{P}$ for which nontrivial test (super)martingales do not exist, but one can design powerful e-processes for $\mathcal{P}$. Second, one must take care to design test (super)martingales for each m that are tied together across m in a nontrivial manner that improves statistical power while maintaining computational tractability. All the confidence sets in this paper (both in the sequential and batch settings) will be based on this 4-step procedure, but with different carefully chosen processes (M_t^m). In the language of betting, we will come up with new, powerful ways to bet for each m, and also tie together the betting strategies for different m.

2.3 Connections to the Chernoff method

By virtue of $(C_t)_{t=1}^{\infty}$ being a time-uniform confidence sequence, we also have that C_n is a $(1-\alpha)$-confidence interval for μ for any fixed sample size n. In fact, the celebrated Chernoff method results in such a confidence interval. So, how exactly are the two approaches related? The answer is simple: Theorem 1 generalizes and improves on the Chernoff method. To elaborate, recall that Hoeffding proved that

$$\sup_{P \in \mathcal{P}^\mu} \mathbb{E}_P[\exp(\lambda(X-\mu) - \lambda^2/8)] \leq 1, \quad \text{for any } \lambda \in \mathbb{R}, \tag{6}$$

and so if X_1^n are independent (say), the following process can be used in step (b):

$$M_t^m := \prod_{i=1}^{t} \exp(\lambda(X_i - m) - \lambda^2/8). \tag{7}$$

Usually, the only fact that matters for the Chernoff method is that $\mathbb{E}_P[M_t^m] \leq 1$, and Markov's inequality is applied (instead of Ville's) in step (c). To complete the story, the Chernoff method then involves a smart choice for λ. Setting $\lambda := \sqrt{8\log(1/\alpha)/n}$ recovers the familiar Hoeffding inequality for the batch sample-size setting. Taking a union bound over X_1^n and $-X_1^n$ yields the Hoeffding confidence interval (2) exactly. Using our four-step approach, the resulting confidence sequence is a time-uniform generalization of Hoeffding's inequality, recovering the latter precisely including constants at time n; see Howard et al. (2020) for this and other generalizations.

In recent parlance, a statistic like M_t^m, which has at most unit expectation under the null, has been called a betting score (Shafer, 2021) or an *e*-value (Vovk, 2021) and their relationship to sequential testing (Grünwald et al., 2019) and estimation (Ramdas et al., 2020) as an alternative to *p*-values has been recently examined. In parametric settings with singleton nulls and alternative hypotheses, the likelihood ratio is an *e*-value. For composite null testing, the split likelihood ratio statistic (Wasserman et al., 2020) (and its variants) are *e*-values. However, our setup is more complex: $\mathcal{P}^m$ is highly composite, there is no common dominating measure to define likelihood ratios, but Hoeffding's result yields an *e*-value. (In fact, it yields test supermartingale and hence an e-process, which is an *e*-value even at stopping times.)

In summary, the Chernoff method is simply one powerful, but as it turns out, rather limited way to construct an *e*-value. This paper provides better constructions of M_t^m, whose expectation is exactly equal to one, thus removing one source of looseness in the Hoeffding-type approach above, as well as better ways to pick the tuning parameter λ, which will correspond to our bet.

3 Warmup: exponential supermartingales and predictable plug-ins

A central technique for constructing confidence sequences (CSs) is Robbins' *method of mixtures* (Robbins, 1970), see also Darling and Robbins (1967a); Robbins and Siegmund (1968, 1970, 1972, 1974). Related ideas of 'pseudo-maximization' or Laplace's method were further popularized and extended by de la Peña et al. (2004, 2007, 2009) and has led to several other followup works (Abbasi-Yadkori et al., 2011; Balsubramani, 2014; Howard et al., 2020; Kaufmann & Koolen, 2021).

However, beyond the case when the data are (sub)-Gaussian, the method of mixtures rarely leads to a closed-form CS; it yields an *implicit* construction for C_t which can sometimes be computed efficiently (e.g., using conjugate mixtures Howard et al. (2021)), but is otherwise analytically opaque and computationally tedious. Below, we provide an alternative construction—called the 'predictable plug-in'—that is exact, explicit, and efficient (computationally and statistically).

In the next section, our CSs avoid exponential supermartingales and are much tighter than the recent state-of-the-art in Howard et al. (2021). The ones in this section match the latter but are simpler to compute, so we present them first.

3.1 Predictable plug-in Cramer–Chernoff supermartingales

Suppose $(X_t)_{t=1}^{\infty} \sim P$ for some $P \in \mathcal{P}^\mu$ where $\mathcal{P}^\mu$ is the set of all distributions on $\prod_{i=1}^{\infty}[0, 1]$ so that $\mathbb{E}_P(X_t \mid \mathcal{F}_{t-1}) = \mu$ for each t. The Hoeffding process $(M_t^H(m))_{t=0}^{\infty}$ for a given candidate mean $m \in [0, 1]$ is given by

$$M_t^H(m) := \prod_{i=1}^{t} \exp\left(\lambda(X_i - m) - \psi_H(\lambda)\right) \tag{8}$$

with $M_0^H(m) \equiv 1$ by convention. Here, $\psi_H(\lambda) := \lambda^2/8$ is an upper bound on the cumulant generating function (CGF) for $[0, 1]$-valued random variables with $\lambda \in \mathbb{R}$ chosen in some strategic way. For example, to maximize $M_n^H(m)$ at a fixed sample size n, one would set $\lambda := \sqrt{8\log(1/\alpha)/n}$ as in the classical fixed-time Hoeffding inequality (Hoeffding, 1963).

Following Howard et al. (2021), we have that $(M_t^H(\mu))_{t=0}^{\infty}$ is a nonnegative supermartingale with respect to the canonical filtration. Therefore, by Ville's maximal inequality for nonnegative supermartingales (Howard et al., 2020; Ville, 1939),

$$P(\exists t \geq 1 : M_t^H(\mu) \geq 1/\alpha) \leq \alpha. \tag{9}$$

Robbins' method of mixtures proceeds by noting that $\int_{\lambda \in \mathbb{R}} M_t^H(m)\, \mathrm{d}F(\lambda)$ is also a supermartingale for any 'mixing' probability distribution $F(\lambda)$ on $\mathbb{R}$ and thus

$$P\big(\exists t \geq 1 : \textstyle\int_{\lambda \in \mathbb{R}} M_t^H(\mu)\, \mathrm{d}F(\lambda) \geq 1/\alpha\big) \leq \alpha. \tag{10}$$

In this particular case, if $F(\lambda)$ is taken to be the Gaussian distribution, then the above integral can be computed in a closed-form (Howard et al., 2020). For other distributions or altogether different supermartingales (i.e., other than Hoeffding), the integral may be computationally tedious or intractable.

To combat this, instead of fixing $\lambda \in \mathbb{R}$ or integrating over it, consider constructing a sequence $\lambda_1, \lambda_2, \ldots$ which is predictable, and thus λ_t can depend on X_1^{t-1}. Then,

$$M_t^{\text{PrPl-H}}(m) := \prod_{i=1}^{t} \exp\left(\lambda_i (X_i - m) - \psi_H(\lambda_i)\right) \tag{11}$$

is also a test supermartingale for $\mathcal{P}^m$ (and hence Ville's inequality applies). We call such a sequence $(\lambda_t)_{t=1}^{\infty}$ a *predictable plug-in*. While not always explicitly referred to by this exact name, predictable plug-ins have appeared in works on parametric sequential analysis by Wald (1947, equation (10:10)), Robbins and Siegmund (1974, equation (4)), Dawid (1984), and Lorden and Pollak (2005) as well as in the information theory literature (Rissanen, 1984). As we will see, these techniques also prove useful in nonparametric testing and estimation problems both in sequential and batch settings.

Using $M_t^{\text{PrPl-H}(m)}$ as the process in step (b) of Theorem 1 results in a lower CS for μ, while constructing an analogous supermartingale using $(-X_t)_{t=1}^{\infty}$ yields an upper CS. Combining these by taking a union bound results in the predictable plug-in Hoeffding CS which we introduce now.

Proposition 1 (Predictable plug-in Hoeffding CS [**PrPl-H**]). Suppose that $(X_t)_{t=1}^{\infty} \sim P$ for some $P \in \mathcal{P}^{\mu}$. For any chosen real-valued predictable $(\lambda_t)_{t=1}^{\infty}$,

$$C_t^{\text{PrPl-H}} := \left(\frac{\sum_{i=1}^{t} \lambda_i X_i}{\sum_{i=1}^{t} \lambda_i} \pm \frac{\log(2/\alpha) + \sum_{i=1}^{t} \psi_H(\lambda_i)}{\sum_{i=1}^{t} \lambda_i} \right)$$

forms a $(1-\alpha)$-CS for μ,

as does its running intersection, $\bigcap_{i \leq t} C_i^H$.

A sensible choice of predictable plug-in is given by

$$\lambda_t^{\text{PrPl-H}} := \sqrt{\frac{8 \log(2/\alpha)}{t \log(t+1)}} \wedge 1, \tag{12}$$

for reasons which will be discussed in Section 3.3. The proof of Proposition 1 is provided in Online Supplementary Material, Section A.1. As alluded to earlier, predictable plug-ins are actually the *least* interesting when using Hoeffding's sub-Gaussian bound because of the available closed form Gaussian-mixture boundary. However, the story becomes more interesting when either (a) the method of mixtures is computationally opaque or complex, or (b) the optimal choice of λ is based on unknown but estimable quantities. Both (a) and (b) are issues that arise when computing empirical Bernstein-type CSs and CIs. In the following section, we present predictable plug-in empirical Bernstein-type CSs and CIs which are both computationally and statistically efficient.

3.2 Application: closed-form empirical Bernstein confidence sets

To prepare for the results that follow, consider the empirical Bernstein-type process,

$$M_t^{\text{PrPl-EB}}(m) := \prod_{i=1}^{t} \exp\left\{\lambda_i (X_i - m) - v_i \psi_E(\lambda_i)\right\} \tag{13}$$

where, following Howard et al. (2020, 2021), we have defined $v_i := 4(X_i - \widehat{\mu}_{i-1})^2$ and

$$\psi_E(\lambda) := (-\log(1-\lambda) - \lambda)/4 \quad \text{for } \lambda \in [0, 1). \tag{14}$$

As we revisit later, the appearance of the constant 4 is to facilitate easy comparison to ψ_H, since $\lim_{\lambda \to 0^+} \psi_E(\lambda)/\psi_H(\lambda) = 1$. In short, ψ_E is nonnegative, increasing on $[0, 1)$, and grows quadratically near 0.

Using $M_t^{\text{PrPl-EB}}(m)$ in step (b) in Theorem 1—and applying the same procedure but with $(X_t)_{t=1}^\infty$ and m replaced by $(-X_t)_{t=1}^\infty$ and $-m$ combined with a union bound over the resulting CSs—we get the following CS.

Theorem 2 (Predictable plug-in empirical Bernstein CS [**PrPl-EB**]). Suppose $(X_t)_{t=1}^\infty \sim P$ for some $P \in \mathcal{P}^\mu$. For any $(0, 1)$-valued predictable $(\lambda_t)_{t=1}^\infty$,

$$C_t^{\text{PrPl-EB}} := \left(\frac{\sum_{i=1}^t \lambda_i X_i}{\sum_{i=1}^t \lambda_i} \pm \frac{\log(2/\alpha) + \sum_{i=1}^t v_i \psi_E(\lambda_i)}{\sum_{i=1}^t \lambda_i} \right)$$

forms a $(1-\alpha)$-CS for μ,

as does its running intersection, $\bigcap_{i \le t} C_i^{\text{PrPl-EB}}$.

In particular, we recommend the predictable plug-in $(\lambda_t^{\text{PrPl-EB}})_{t=1}^\infty$ given by

$$\lambda_t^{\text{PrPl-EB}} := \sqrt{\frac{2\log(2/\alpha)}{\widehat{\sigma}_{t-1}^2 t \log(1+t)}} \wedge c, \quad \widehat{\sigma}_t^2 := \frac{1/4 + \sum_{i=1}^t (X_i - \widehat{\mu}_i)^2}{t+1}, \quad \widehat{\mu}_t := \frac{1/2 + \sum_{i=1}^t X_i}{t+1} \tag{15}$$

for some $c \in (0, 1)$ (a reasonable default being 1/2 or 3/4). This choice was inspired by the fixed-time empirical Bernstein as well as the widths of time-uniform CSs (more details are provided in Section 3.3). The sequences of estimators $(\widehat{\mu}_t)_{t=1}^\infty$ and $(\widehat{\sigma}_t^2)_{t=1}^\infty$ can be interpreted as predictable, regularized sample means and variances. This technique was employed by Kotłowski et al. (2010) for misspecified exponential families in the so-called *maximum likelihood plug-in strategy*.

The proof of Theorem 2 relies on establishing that $M_t^{\text{PrPl-EB}}(m)$ is a test supermartingale for $\mathcal{P}^m$. This latter fact is related to, but cannot be derived directly from, a powerful deterministic inequality for bounded numbers due to Fan et al. (2015). One needs an additional trick from Howard et al. (2021, Section A.8) which swaps $(X_i - m)^2$ with $(X_i - \widehat{\mu}_{i-1})^2$, for any predictable $\widehat{\mu}_{i-1}$, within the variance term v_i. It is this additional piece which yields both tighter and *closed-form* CSs; details are in Online Supplementary Material, Section A.2. We remark that before taking the running intersection, the above intervals are symmetric around the weighted sample mean, but this symmetry will not carry forward to other CSs in the paper.

Figure 2 compares the conjugate mixture empirical-Bernstein CS (CM-EB) due to Howard et al. (2021) with our predictable plug-in empirical-Bernstein CS (PrPl-EB). The two CSs perform similarly, but our closed-form PrPl-EB is over 500 times faster to compute than CM-EB (in our experience) which requires root finding at each step. However, our later bounds will be tighter than both of these.

Remark 1 Theorem 2 yields computationally and statistically efficient empirical Bernstein-type CIs for a fixed sample size n. Recalling (15), we recommend using $\bigcap_{i \le n} C_i^{\text{PrPl-EB}}$ along with the predictable sequence

$$\lambda_t^{\text{PrPl-EB}(n)} := \sqrt{\frac{2\log(2/\alpha)}{n\widehat{\sigma}_{t-1}^2}} \wedge c. \tag{16}$$

We call the resulting confidence interval the 'predictable plug-in empirical Bernstein confidence interval' or [**PrPl-EB-CI**] for short; see Figure 3.

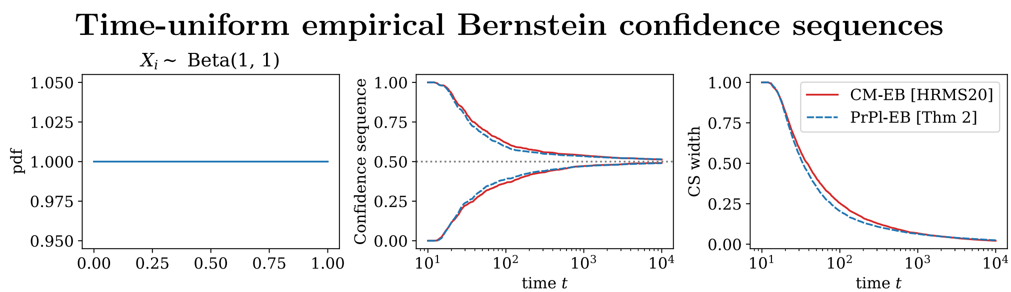

Figure 2. Empirical Bernstein CSs produced via a predictable plug-in (PrPl) with $(\lambda_t)_{t=1}^{\infty}$ from (15) match (or slightly improve) those obtained via conjugate mixtures (CM) by Howard et al. (2021); the former is closed-form, but the latter is not and requires numerical methods.

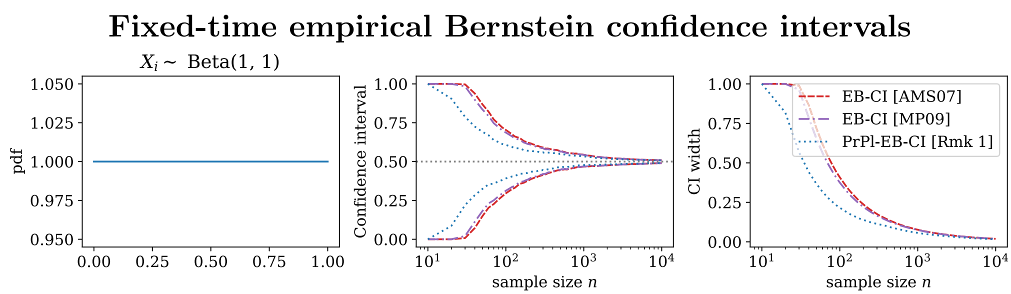

Figure 3. Our predictable plug-in (PrPl) empirical Bernstein (EB) CI is significantly tighter than those of Maurer and Pontil (2009) and Audibert et al. (2007).

If $X_1, \ldots, X_n$ are independent, then at the expense of computation, the above CI can be effectively derandomized to remove the effect of the ordering of variables. One can randomly permute the data B times to obtain $(\widetilde{X}_{1,b}, \ldots, \widetilde{X}_{n,b})$ and correspondingly compute $\widetilde{M}_{n,b}^{\text{PrPl-EB}}(m)$, one for each permutation $b \in \{1, \ldots, B\}$. Averaging over these permutations, define $\widetilde{M}_n^{\text{PrPl-EB}}(m) := 1/B \sum_{b=1}^{B} \widetilde{M}_{n,b}^{\text{PrPl-EB}}(m)$. For each b, $M_{n,b}^{\text{PrPl-EB}}(\mu)$ has expectation at most one (by linearity of expectation). Thus, $\widetilde{M}_n^{\text{PrPl-EB}}(\mu)$ is a e-value (i.e., it has expectation at most 1). By Markov's inequality, $\widetilde{C}_n^{\text{PrPl-EB}} := \{m \in [0, 1] : \widetilde{M}_n^{\text{PrPl-EB}}(m) < 1/\alpha\}$ is a $(1-\alpha)$-CI for μ. This set is not available in a closed-form and the intersection $\bigcap_{i \le n} \widetilde{C}_i^{\text{PrPl-EB}}$ no longer yields a valid CI. In our experience, this derandomization procedure neither helps nor hurts. In any case, both $\bigcap_{i \le n} C_i$ and $\widetilde{C}_n$ will be significantly improved in Section 4.4.

In Online Supplementary Material, Section E.3, we show that in iid settings the width of [PrPl-EB-CI] scales with the true (unknown) standard deviation:

$$\sqrt{n}\left(\frac{\log(2/\alpha) + \sum_{i=1}^{n} v_i \psi_E(\lambda_i)}{\sum_{i=1}^{n} \lambda_i}\right) \xrightarrow{\text{a.s.}} \sigma\sqrt{2\log(2/\alpha)}. \tag{17}$$

Notice that (17) is the same asymptotic behavior that one would observe for CIs based on Bernstein's or Bennett's inequalities, both of which require knowledge of the true variance σ^2, while [PrPl-EB-CI] does not. This is in contrast to the empirical Bernstein CIs of Maurer and Pontil (2009) whose limit would be $\sigma\sqrt{2\log(4/\alpha)}$. In the maximum variance case where $\sigma = 1/2$, (17) yields the same asymptotic behavior as Hoeffding's CI (2).

Until now, we presented various predictable plug-ins—$(\lambda_t^{\text{PrPl-H}})_{t=1}^{\infty}$, $(\lambda_t^{\text{PrPl-EB}})_{t=1}^{\infty}$, and $(\lambda_t^{\text{PrPl-EB}(n)})_{t=1}^{n}$—but have not provided intuition for why these are sensible choices. Next, we discuss guiding principles for deriving predictable plug-ins.

3.3 Guiding principles for deriving predictable plug-ins

Let us begin our discussion with the predictable plug-in Hoeffding process (11) and the resulting CS in Proposition 1, which has a half-width

$$W_t = \frac{\log(2/\alpha) + \sum_{i=1}^t \lambda_i^2/8}{\sum_{i=1}^t \lambda_i}$$

To ensure that $W_t \longrightarrow 0$ as $t \longrightarrow \infty$, it is clear that we want $\lambda_t \xrightarrow{\text{a.s.}} 0$, but at what rate? As a sensible default, we recommend setting $\lambda_t \asymp 1/\sqrt{t \log t}$ so that $W_t = \widetilde{O}(\sqrt{\log t / t})$ which matches the width of the conjugate mixture Hoeffding CS (Howard et al., 2020, Proposition 2) (here $\widetilde{O}$ treats $O(\log\log t)$ factors as constants). See Table 1 for a comparison between rates for λ_t and their resulting CS widths.

Now consider the predictable plug-in empirical Bernstein process (13) and the resulting CS of Theorem 2, which has a half-width

$$W_t = \frac{\log(2/\alpha) + \sum_{i=1}^t 4(X_i - \widehat{\mu}_{i-1})^2 \psi_E(\lambda_i)}{\sum_{i=1}^t \lambda_i}$$

By two applications of L'Hôpital's rule, we have that

$$\frac{\psi_E(\lambda)}{\psi_H(\lambda)} \overset{\lambda \to 0^+}{\longrightarrow} 1. \tag{18}$$

Performing some approximations for small λ_i to help guide our choice of $(\lambda_t)_{t=1}^\infty$ (without compromising validity of resulting confidence sets) we have that

$$W_t \approx \frac{\log(2/\alpha) + \sum_{i=1}^t 4(X_i - \mu)^2 \lambda_i^2/8}{\sum_{i=1}^t \lambda_i}. \tag{19}$$

Thus, in the special case of i.i.d. X_i with variance σ^2, for large enough t,

$$\mathbb{E}_P(W_t \mid \mathcal{F}_{t-1}) \lesssim \frac{\log(2/\alpha) + \sigma^2 \sum_{i=1}^t \lambda_i^2/2}{\sum_{i=1}^t \lambda_i}. \tag{20}$$

If we were to set $\lambda_1 = \lambda_2 = \cdots = \lambda^\star \in \mathbb{R}$ and minimize the above expression for a specific time $t^\star$, this amounts to minimizing

$$\frac{\log(2/\alpha) + \sigma^2 t^\star {\lambda^\star}^2/2}{t^\star \lambda^\star}, \tag{21}$$

which is achieved by setting

$$\lambda^\star := \sqrt{\frac{2\log(2/\alpha)}{\sigma^2 t^\star}}. \tag{22}$$

This is precisely why we suggested the predictable plug-in $(\lambda_t^{\text{PrPl}})_{t=1}^\infty$ given by (15), where the additional $\log(t+1)$ is included in an attempt to enforce $W_t = \widetilde{O}(\sqrt{\log t/t})$.

The above calculations are only used as guiding principles to sharpen the confidence sets, but *all* such schemes retain the validity guarantee. As long as $(\lambda_t)_{t=1}^\infty$ is $[0, 1)$-valued and predictable, we have that $(M_t^E(\mu))_{t=0}^\infty$ is a test supermartingale for $\mathcal{P}^\mu$ which can be used in Theorem 1 to obtain different valid CSs for μ.

Table 1. Below, we think of $\log x$ as $\log(x+1)$ to avoid trivialities

Strategy $(\lambda_i)_{i=1}^{\infty}$	$\sum_{i=1}^{t} \lambda_i$	$\sum_{i=1}^{t} \lambda_i^2$	Width W_t
$\asymp 1/i$	$\asymp \log t$	$\asymp 1$	$1/\log t$
$\asymp \sqrt{\log i / i}$	$\asymp \sqrt{t \log t}$	$\asymp \log^2 t$	$\asymp \log^{3/2} t/\sqrt{t}$
$\asymp 1/\sqrt{i}$	$\asymp \sqrt{t}$	$\asymp \log t$	$\asymp \log t/\sqrt{t}$
$\asymp 1/\sqrt{i \log i}$	$\asymp \sqrt{t/\log t}$	$\asymp \log\log t$	$\asymp \sqrt{\log t/t}$
$\asymp 1/\sqrt{i \log i \log\log i}$	$\asymp \sqrt{t/\log t}$	$\asymp \log\log\log t$	$\asymp \sqrt{\log t/t}$

Notes. The claimed rates are easily checked by approximating the sums as integrals, and taking derivatives. For example, $\mathrm{d}/\mathrm{d}x \log\log x = 1/x \log x$, so the sum of $\sum_{i \le t} 1/i \log i \asymp \log\log t$. It is worth remarking that for $t = 10^{80}$, the number of atoms in the universe, $\log\log t \approx 5.2$, which is why we treat $\log\log t$ as a constant when expressing the rate for W_t. The iterated logarithm pattern in the the last two lines can be continued indefinitely.

Foreshadowing our attempt to generalize this procedure in the next section, notice that the exponential function was used throughout to ensure nonnegativity, but that any other test supermartingale would have sufficed. In fact, if a martingale is used in place of a supermartingale, then Ville's inequality is tighter.

Next, we present a test *martingale*, removing a source of looseness in the confidence sets derived thus far. We discuss its betting interpretation, provide other guiding principles for setting λ_i (equivalently, for betting), which will involve attempting to maximize the expected log-wealth in the betting game.

4 The capital process, betting, and martingales

In Section 3, we generalized the Cramer–Chernoff method to derive predictable plug-in exponential supermartingales and used this result to obtain tight empirical Bernstein CSs and CIs. In this section, we consider an alternative process which can be interpreted as the wealth accumulated from a series of bets in a game. This process is a central object of study in the game-theoretic probability literature where it is referred to as the *capital process* (Shafer & Vovk, 2001). We discuss its connections to the purely statistical goal of constructing CSs and CIs and demonstrate how these sets improve on Cramer–Chernoff approaches, including the empirical Bernstein confidence sets of the previous section.

Consider the same setup as in Section 3: we observe an infinite sequence of conditionally mean-μ random variables, $(X_t)_{t=1}^{\infty} \sim P$ from some distribution $P \in \mathcal{P}^{\mu}$. Define the *capital process* $\mathcal{K}_t(m)$ for any $m \in [0, 1]$,

$$\mathcal{K}_t(m) := \prod_{i=1}^{t} (1 + \lambda_i(m) \cdot (X_i - m)), \tag{23}$$

with $\mathcal{K}_0(m) := 1$ and where $(\lambda_t(m))_{t=1}^{\infty}$ is a $(-1/(1-m), 1/m)$-valued predictable sequence, and thus $\lambda_t(m)$ can depend on X_1^{t-1}. Note that for each $t \ge 1$, we have $X_t \in [0, 1]$, $m \in [0, 1]$ and $\lambda_t(m) \in (-1/(1-m), 1/m)$. Here and below, $1/m$ should be interpreted as ∞ when $m = 0$ and similarly for $1/(1-m)$ and $m = 1$, respectively. Importantly, $(1 + \lambda_t(m) \cdot (X_t - m)) \in [0, \infty)$, and thus $\mathcal{K}_t(m) \ge 0$ for all $t \ge 1$. Following similar techniques to the previous section, the reader may easily check that $\mathcal{K}_t(\mu)$ is a test martingale. Moreover, we have the stronger result summarized in the following central proposition.

Proposition 2 Suppose a draw from some distribution P yields a sequence $X_1, X_2, \ldots$ of $[0, 1]$-valued random variables, and let $\mu \in [0, 1]$ be a constant. The following four statements imply each other:

(a) $\mathbb{E}_P(X_t \mid \mathcal{F}_{t-1}) = \mu$ for all $t \in \mathbb{N}$, where $\mathcal{F}_{t-1} = \sigma(X_1, \ldots, X_{t-1})$.
(b) There exists a constant $\lambda \in \mathbb{R}\backslash\{0\}$ for which $(\mathcal{K}_t(\mu))_{t=0}^{\infty}$ is a strictly positive test martingale for P.

(c) For every fixed $\lambda \in (-(1/(1-\mu)), 1/\mu)$, $(\mathcal{K}_t(\mu))_{t=0}^{\infty}$ is a test martingale for P.
(d) For every $(-(1/(1-\mu)), 1/\mu))$-valued predictable sequence $(\lambda_t)_{t=1}^{\infty}$, $(\mathcal{K}_t(\mu))_{t=0}^{\infty}$ is a test martingale for P.

Further, the intervals $(-(1/(1-\mu)), 1/\mu)$ mentioned above can be replaced by any subinterval containing at least one nonzero value, like $[-1, 1]$ or $(-0.5, 0.5)$. Finally, every test martingale for $\mathcal{P}^{\mu}$ is of the form $(\mathcal{K}_t(\mu))$ for some predictable sequence (λ_t).

The proof can be found in Online Supplementary Material, Section A.3. While the subsequent theorems will primarily make use of $(a) \Rightarrow (d)$, the above proposition establishes a core fact: the assumption of the (conditional) means being identically μ is an *equivalent restatement* of our capital process being a test martingale. Thus, test martingales are not simply 'technical tools' to deal with means of bounded random variables, they are fundamentally at the very heart of the problem definition itself.

Proposition 2 can be generalized to another remarkable, yet simple, result: for any set of distributions $\mathcal{S}$, *every* test martingale for $\mathcal{S}$ has the same form.

Proposition 3 (Universal representation). For any arbitrary set of (possibly unbounded) distributions $\mathcal{S}$, (M_t) is a test martingale for $\mathcal{S}$ if and only if $M_t = \prod_{i=1}^{t}(1 + \lambda_i Z_i)$ for some $Z_i \geq -1$ such that $\mathbb{E}_S[Z_i|\mathcal{F}_{i-1}] = 0$ for every $S \in \mathcal{S}$, and some predictable λ_i such that $\lambda_i Z_i \geq -1$. The same claim also holds for test supermartingales for $\mathcal{S}$, with the aforementioned '$= 0$' replaced by '≤ 0'.

The proof can be found in Online Supplementary Material, Section A.4. The above proposition immediately makes this paper's techniques actionable for a wide class of nonparametric testing and estimation problems. We give an example relating to quantiles later.

4.1 Connections to betting

It is worth pausing to clarify how the capital process $\mathcal{K}_t(m)$ and Proposition 2 can be viewed in terms of betting. We imagine that nature implicitly posits a hypothesis H_0^m—which we treat as a game providing us a chance to make money if the hypothesis is wrong, by repeatedly betting some of our capital against H_0^m. We start the game with a capital of 1 (i.e., $\mathcal{K}_0(m) := 1$), and design a bet of $b_t := s_t|\lambda_t^m|$ at each step, where $s_t \in \{-1, 1\}$. Setting $s_t := 1$ indicates that we believe that $\mu > m$ while $s_t := -1$ indicates the opposite. $|\lambda_t^m|$ indicates the amount of our capital that we are willing to put at stake at time t: setting $\lambda_t^m = 0$ results in neither losing nor gaining any capital regardless of the outcome, while setting $\lambda_t^m \in \{-1/(1-m), 1/m\}$ means that we are willing to risk all of our capital on the next outcome.

However, if H_0^m is true (i.e., $m = \mu$), then by Proposition 2, our capital process is a martingale. In betting terms, no matter how clever a betting strategy $(\lambda_t^m)_{t=1}^{\infty}$ we devise, we cannot expect to make (or lose) money at each step. If on the other hand, H_0^m is false, then a clever betting strategy will make us a lot of money. In statistical terms, when our capital exceeds $1/\alpha$, we can confidently reject the hypothesis H_0^m since if it were true (and the game were fair) then by Ville's inequality (Ville, 1939), the a priori probability of this *ever* occurring is at most α. We imagine simultaneously playing this game with $H_0^{m'}$ for each $m' \in [0, 1]$. At any time t, the games $m' \in [0, 1]$ for which our capital is small ($<1/\alpha$) form a CS.

Both the Cramer–Chernoff processes of Section 3 and $\mathcal{K}_t(m)$ are nonnegative and tend to increase when $\mu > m$. However, only $\mathcal{K}_t(m)$ is a *test martingale* when $m = \mu$; the others are test supermartingales. A test martingale is the wealth accumulated in a 'fair game' where our capital stays constant in expectation, while a test supermartingale is the wealth accumulated in a game where our capital is expected to decrease (not strictly). Larger values of capital correspond to rejecting H_0^m more readily. Therefore, test supermartingales tend to yield conservative tests compared to their martingale counterparts.

More generally, *every* nonnegative supermartingale can be regarded as the wealth process of a gambler playing a game with odds that are fair or stacked against them. In other words, there is a one-to-one correspondence between wealths of hypothetical gamblers and nonnegative supermartingales. Taking this perspective, every statement involving nonnegative supermartingales (and thus likelihood ratios) are statements about betting, and vice versa. Mixture methods that combine nonnegative supermartingales are simply strategies to hedge across various instruments available to the gambler. Thus, the gambling analogy can be entirely dropped, and our results would find themselves comfortably nestled in the rich literature on martingale methods for concentration inequalities, but we mention the betting analogy for intuition so that the mathematics are animated and easier to absorb.

Ville introduced martingales into modern mathematical probability theory, and centered them around their betting interpretation. Since then, ideas from betting have appeared in various fields, including probability theory, statistical testing and estimation, information theory, and online learning theory. While our paper focuses on the utility of betting in some statistical inference tasks, Online Supplementary Material, Section F provides a brief overview of the use of betting in other mathematical disciplines.

4.2 Connections to likelihood ratios

As alluded to in the previous subsection, useful intuition is provided via the connection to likelihood ratios. $\mathcal{K}_t(m)$ is a 'composite' test martingale for $\mathcal{P}^m$, meaning that it is a nonnegative martingale starting at one for every $P \in \mathcal{P}^m$ (recall that P is a distribution over infinite sequences of observations with conditional mean m).

If we were dealing with a single distribution such as Q^∞, meaning a product distribution where every observation is drawn iid from Q, then one may pick any alternative Q' that is absolutely continuous with respect to Q, to observe that the likelihood ratio $\prod_{i=1}^t Q'(X_i)/Q(X_i)$ is a test martingale for Q^∞.

However, since $\mathcal{P}^m$ is highly composite and nonparametric and is not even dominated by a single measure (as it contains atomic measures, continuous measures, and all their mixtures), it is unclear how one can even begin to write down a likelihood ratio. Nevertheless, Ramdas et al. (2020, Proposition 4) show that if (M_t) is a composite test martingale for any $\mathcal{S}$, then for every distribution $Q \in \mathcal{S}$, M_t equals the likelihood ratio of some Q' against Q (where Q' depends on Q).

Thus, not only is every likelihood ratio a test martingale, but every (composite) test martingale can also be represented as a likelihood ratio. Hence, in a formal sense, test martingales are nonparametric composite generalizations of likelihood ratios, which are at the very heart of statistical inference. When this observation is combined with Proposition 2, it should be no surprise any longer that the capital process $\mathcal{K}_t(m)$ (even devoid of any betting interpretation) is fundamental to the problem at hand. In Online Supplementary Material, Section E.6 we also observe connections to the empirical likelihood of Owen (2001) and the dual likelihood of (Mykland, 1995).

4.3 Adaptive, constrained adversaries

Despite the analogies to betting, the game described so far appears to be purely stochastic in the sense that nature simply commits to a distribution $P \in \mathcal{P}^\mu$ for some unknown $\mu \in [0, 1]$ and presents us observations from P. However, Proposition 2 can be extended to a more adversarial setup, but with a constrained adversary.

To elaborate, recall the difference between $\mathcal{Q}$ and $\mathcal{P}$ from the start of Section 2 and consider a game with three players: an adversary, nature, and the statistician. First, the adversary commits to a $\mu \in [0, 1]$. Then, the game proceeds in rounds. At the start of round t, the statistician publicly discloses the bets for every m, which could depend on $X_1, \ldots, X_{t-1}$. The adversary picks a distribution $Q_t \in \mathcal{Q}^\mu$, which could depend on $X_1, \ldots, X_{t-1}$ and the statistician's disclosed bets, and hands Q_t to nature. Nature simply acts like an arbitrator, first verifying that the adversary chose a Q_t with mean μ, and then draws $X_t \sim Q_t$ and presents X_t to the statistician.

In this fashion, the adversary does not need to pick μ and $P \in \mathcal{P}^\mu$ at the start of the interaction, which is the usual stochastic setup, but can instead build the distribution P in a data-dependent fashion over time. In other words, the adversary does not commit to a distribution P, but instead to a *rule for building P* from the data. Of course, they do not need to disclose this rule, or even be able express what this rule would do on any other hypothetical outcomes other than the one observed. The results in this paper, which build on the central Proposition 2, continue to hold in this more general interaction model.

A geometric reason why we can move from the stochastic model first described to the above (constrained) adversarial model, is that the above distribution P lies in the 'fork convex hull' of $\mathcal{P}^\mu$. Fork-convexity is a sequential analogue of convexity (Ramdas et al., 2021). Informally, the fork-convex hull of a set of distributions over sequences is the set of predictable plug-ins of these distributions, and is much larger than their convex hull (mixtures). If a process is a nonnegative martingale under every distribution in a set, then it is also a nonnegative martingale under every distribution in the fork convex hull of that set. No results about fork convexity are used anywhere in this paper, and we only mention it for the mathematically curious.

4.4 The hedged capital process

We now return to the purely statistical problem of using the capital process $\mathcal{K}_t(m)$ to construct time-uniform CSs and fixed-time CIs. We might be tempted to use $\mathcal{K}_t(\mu)$ as the nonnegative martingale in Theorem 1 to conclude that $\mathfrak{B}_t := \{m \in [0, 1] : \mathcal{K}_t(m) < 1/\alpha\}$ forms a $(1-\alpha)$-CS for μ. Unlike the empirical Bernstein CS of Section 3, $\mathfrak{B}_t$ cannot be computed in a closed-form. Instead, we theoretically need to compute the family of processes $\{\mathcal{K}_t(m)\}_{m\in[0,1]}$ and include those $m \in [0, 1]$ for which $\mathcal{K}_t(m)$ remains below $1/\alpha$. This is not practical as the parameter space [0, 1] is uncountably infinite. But if we know a priori that $\mathfrak{B}_t$ is guaranteed to produce an interval for each t, then it is straightforward to find a superset of $\mathfrak{B}_t$ by either performing a grid search on $(0, 1/g, 2/g, \ldots, (g-1)/g, 1)$ for some large $g \in \mathbb{N}$, or by employing root-finding algorithms. This motivates the *hedged capital process*, defined for any $\theta, m \in [0, 1]$ as

$$\begin{aligned} &\mathcal{K}_t^{\pm}(m) := \max\{\theta\mathcal{K}_t^{+}(m), (1-\theta)\mathcal{K}_t^{-}(m)\}, \\ \text{where}\quad &\mathcal{K}_t^{+}(m) := \prod_{i=1}^{t}(1+\lambda_i^{+}(m)\cdot(X_i-m)), \\ \text{and}\quad &\mathcal{K}_t^{-}(m) := \prod_{i=1}^{t}(1-\lambda_i^{-}(m)\cdot(X_i-m)), \end{aligned} \tag{24}$$

and $(\lambda_t^{+}(m))_{t=1}^{\infty}$ and $(\lambda_t^{-}(m))_{t=1}^{\infty}$ are predictable sequences of $[0, 1/m)$- and $[0, 1/(1-m))$-valued random variables, respectively.

$\mathcal{K}_t^{\pm}(m)$ can be viewed from the betting perspective as dividing one's capital into proportions of θ and $(1-\theta)$ and making two series of simultaneous bets, positing that $\mu \geq m$, and $\mu < m$, respectively which accumulate capital in $\mathcal{K}_t^{+}(m)$ and $\mathcal{K}_t^{-}(m)$. If $\mu \neq m$, then we expect that one of these strategies will perform poorly, while we expect the other to make money in the long term. If $\mu = m$, then we expect neither strategy to make money. The maximum of these processes is upper-bounded by their convex combination,

$$\mathcal{M}_t^{\pm} := \theta\mathcal{K}_t^{+} + (1-\theta)\mathcal{K}_t^{-}.$$

Both $\mathcal{K}_t^{\pm}$ and $\mathcal{M}_t^{\pm}$ can be used for step (b) of Theorem 1 to yield a CS. Empirically, both yield intervals, but only the former provably so.

Theorem 3 (Hedged capital CS [**Hedged**]). Suppose $(X_t)_{t=1}^{\infty} \sim P$ for some $P \in \mathcal{P}^\mu$. Let $(\tilde{\lambda}_t^{+})_{t=1}^{\infty}$ and $(\tilde{\lambda}_t^{-})_{t=1}^{\infty}$ be real-valued predictable sequences not depending on m, and for each $t \geq 1$ let

$$\lambda_t^{+}(m) := |\tilde{\lambda}_t^{+}| \wedge \frac{c}{m}, \quad \lambda_t^{-}(m) := |\tilde{\lambda}_t^{-}| \wedge \frac{c}{1-m}, \tag{25}$$

for some $c \in [0, 1)$ (some reasonable defaults being $c = 1/2$ or $3/4$). Then

$$\mathfrak{B}_t^{\pm} := \{m \in [0, 1] : \mathcal{K}_t^{\pm}(m) < 1/\alpha\} \quad \text{forms a } (1-\alpha)\text{-CS for } \mu,$$

as does its running intersection $\bigcap_{i\leq t}\mathfrak{B}_i^{\pm}$. Further, $\mathfrak{B}_t^{\pm}$ is an interval for each $t \geq 1$. Finally, replacing $\mathcal{K}_t^{\pm}(m)$ by $\mathcal{M}_t^{\pm}(m)$ yields a tighter $(1-\alpha)$-CS for μ.

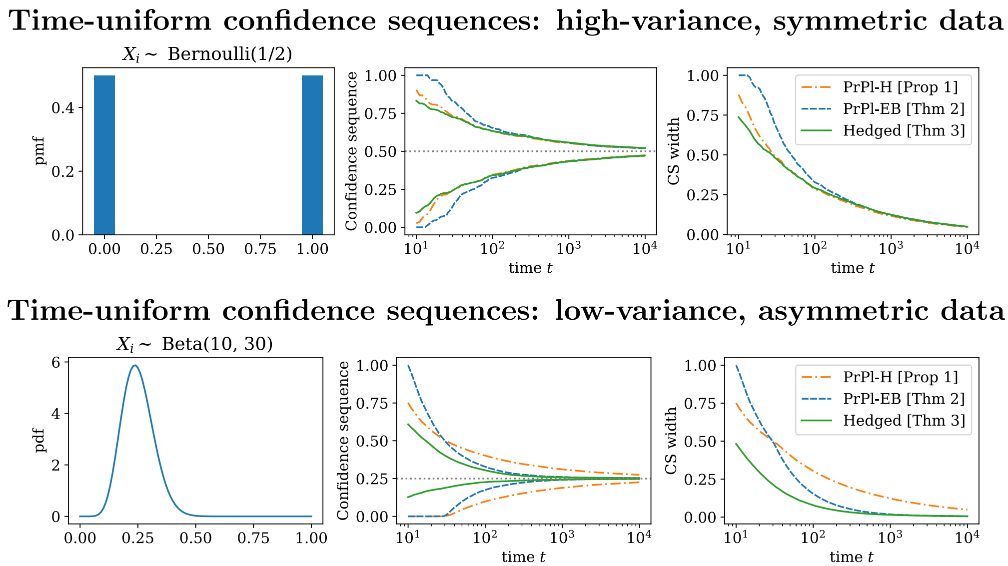

Figure 4. Predictable plug-in Hoeffding, empirical Bernstein, and hedged capital CSs under two distributional scenarios. Notice that the latter roughly matches the others in the Bernoulli(1/2) case, but shines in the low-variance, asymmetric scenario.

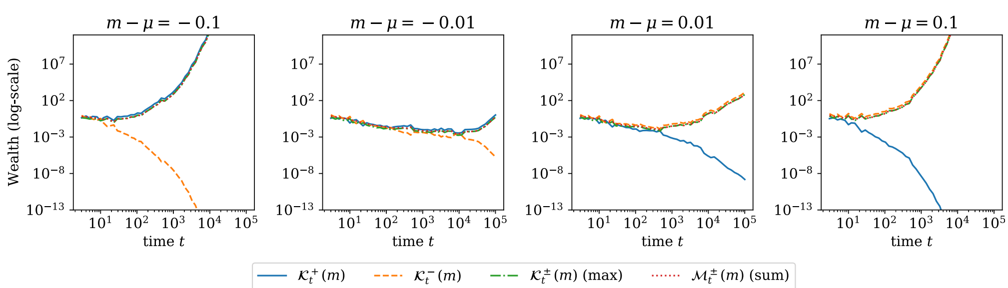

Figure 5. A comparison of capital processes $\mathcal{K}_t^+(m)$, $\mathcal{K}_t^-(m)$, the hedged capital process $\mathcal{K}_t^\pm(m)$, and its upper-bounding nonnegative martingale, $\mathcal{M}_t^\pm(m)$ under four alternatives (from left to right): $m \ll \mu$, $m < \mu$, $m > \mu$, $m \gg \mu$. When $m < \mu$, we see that $\mathcal{K}_t^+(m)$ increases, while $\mathcal{K}_t^-(m)$ approaches zero, but the opposite is true when $m > \mu$. Notice that not much is gained by taking a sum $\mathcal{M}_t^\pm(m)$ rather than a maximum $\mathcal{K}_t^\pm(m)$, since one of $\mathcal{K}_t^+(m)$ and $\mathcal{K}_t^-(m)$ vastly dominates the other, depending on whether $m > \mu$ or $m < \mu$.

For reasons given in Online Supplementary Material, Section B.1, we recommend setting $\tilde{\lambda}_t^+ = \tilde{\lambda}_t^- = \lambda_t^{\mathrm{PrPl}\pm}$ as

$$\lambda_t^{\mathrm{PrPl}\pm} := \sqrt{\frac{2\log(2/\alpha)}{\widehat{\sigma}_{t-1}^2 t \log(t+1)}}, \quad \widehat{\sigma}_t^2 := \frac{1/4 + \sum_{i=1}^t (X_i - \widehat{\mu}_i)^2}{t+1}, \quad \text{and} \quad \widehat{\mu}_t := \frac{1/2 + \sum_{i=1}^t X_i}{t+1}, \tag{26}$$

for each $t \geq 1$, and truncation level $c := 1/2$ or $3/4$; see Figure 4. A reasonable point estimator for μ is $\operatorname{argmin}_{m \in [0,1]} \mathcal{K}_t^\pm(m)$ or $\operatorname{argmin}_{m \in [0,1]} \mathcal{M}_t^\pm(m)$ (see Online Supplementary Material, Figure 18).

Remark 2 Since $\mathcal{K}_t^\pm(m) \leq \mathcal{M}_t^\pm(m)$, the latter confidence sequence is tighter. In the proof of Theorem 3, we use a property of the max function to establish quasiconvexity of $\mathcal{K}_t^\pm(m)$, implying that $\mathfrak{B}_t^\pm$ is an interval. We find the difference in empirical performance negligible (Figure 5). For the interested reader,

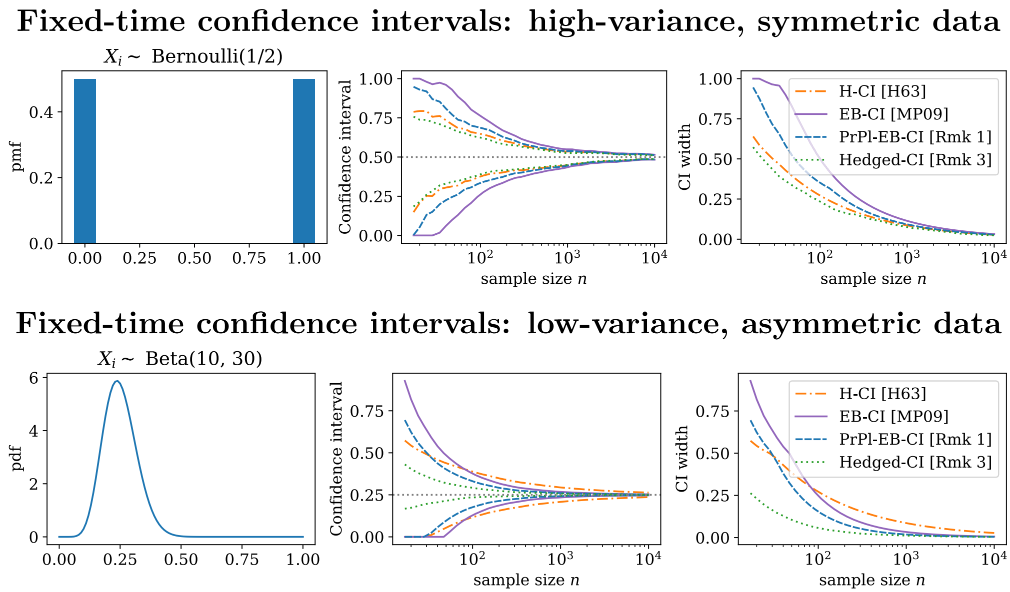

Figure 6. Hoeffding (H), empirical Bernstein (EB), and hedged capital CIs under two distributional scenarios. Similar to the time-uniform setting, the betting approach tends to outperform the other bounds, especially for low-variance, asymmetric data.

Online Supplementary Material, Section E.4 constructs a (pathological) CS that is not almost surely an interval.

Remark 3 Theorem 3 yields tight hedged CIs for a fixed sample size n. Recalling (26), we recommend using $\bigcap_{i \le n} \mathfrak{B}_i^{\pm}$, and setting $\tilde{\lambda}_t^+ = \tilde{\lambda}_t^- = \tilde{\lambda}_t^{\pm}$ given by

$$\tilde{\lambda}_t^{\pm} := \sqrt{\frac{2\log(2/\alpha)}{n\widehat{\sigma}_{t-1}^2}}. \tag{27}$$

We refer to the resulting CI as the 'hedged capital confidence interval' or **[Hedged-CI]** for short, and demonstrate its superiority to past work in Figure 6.

Similar to the discussion after Remark 1, if $X_1, \ldots, X_n$ are independent, then one can permute the data many times and average the resulting capital processes to effectively derandomize the procedure.

The proof of Theorem 3 is in Online Supplementary Material, Section A.5. Unlike the empirical Bernstein-type CSs and CIs of Section 3, those based on the hedged capital process are not necessarily symmetric. In fact, we empirically find through simulations that these CSs and CIs are able to adapt and benefit from this asymmetry (see Figures 4 and 6). While it is not obvious from the definition of $\mathfrak{B}_t^{\pm}$, bets can be chosen such that hedged capital CSs and CIs converge at the optimal rates of $O(\sqrt{\log\log t/t})$ and $O(1/\sqrt{n})$, respectively (see Online Supplementary Material, Section E.2) and such that for sufficiently large n, hedged capital CIs almost surely dominate those based on Hoeffding's inequality (see Online Supplementary Material, Section E.1). However, the implications of time-uniform convergence rates are subtle, and optimal rates are not always desirable in practical applications (see Howard et al., 2021, Section 3.5). Nevertheless, we find that hedged capital CSs and CIs significantly outperform past works even for small sample sizes (see Online Supplementary Material, Section C). Some additional tools for visualizing CSs across α and t are provided in Online Supplementary Material, Section D.5.

In Online Supplementary Material, Section B, we discuss some guiding principles for deriving powerful betting strategies, presenting the hedged capital CSs and CIs as special cases along with the following game-theoretic betting schemes:

- Growth rate adaptive to the particular alternative (GRAPA),
- Approximate GRAPA (aGRAPA),
- Lower-bound on the wealth (LBOW),
- Online Newton step-m (ONS-m),
- Diversified Kelly betting (dKelly),
- Confidence boundary bets (ConBo), and
- Sequentially rebalanced portfolio (SRP).

Each of these betting strategies have their respective benefits, whether computational, conceptual, or statistical which are discussed further in Online Supplementary Material, Section B.

5 Betting while sampling without replacement (WoR)

This section tackles a slightly different problem, that of sampling without replacement (WoR) from a finite set of real numbers in order to estimate its mean. Importantly, the N numbers in the finite population $(x_1, \ldots, x_N)$ are fixed and nonrandom. What is random is only the order of observation; the model for sampling uniformly at random without replacement (WoR) posits that at time $t \geq 1$,

$$X_t \mid (X_1, \ldots, X_{t-1}) \sim \text{Uniform}((x_1, \ldots, x_N) \backslash (X_1, \ldots, X_{t-1})). \tag{28}$$

All probabilities are thus to be understood as solely arising from observing fixed entities in a random order, with no distributional assumptions being made on the finite population. We consider the same canonical filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0}^N$ as before. For $t \geq 1$, let $\mathcal{F}_t := \sigma(X_1^t)$ be the sigma-field generated by $X_1, \ldots, X_t$ and let $\mathcal{F}_0$ be the empty sigma-field. For succinctness, we use the notation $[a] := \{1, \ldots, a\}$.

For each $m \in [0, 1]$, let $\mathcal{L}^m := \{x_1^N \in [0, 1]^N : \sum_{i=1}^N x_i/N = m\}$ be the set of all unordered lists of $N \geq 2$ real numbers in $[0, 1]$ whose average is m. For instance, $\mathcal{L}^0$ and $\mathcal{L}^1$ are both singletons, but otherwise $\mathcal{L}^m$ is uncountably infinite. Let $\mathcal{P}^m$ be the set of all measures on $\mathcal{F}_N$ that are formed as follows: pick an arbitrary element of $\mathcal{L}^m$, apply a uniformly random permutation, and reveal the elements one by one. Thus, every element of $\mathcal{P}^m$ is a uniform measure on the $N!$ permutations of some element in $\mathcal{L}^m$, so there is a one-to-one mapping between $\mathcal{L}^m$ and $\mathcal{P}^m$.

Define $\mathcal{P} := \bigcup_m \mathcal{P}^m$ and let μ represent the true unknown mean, meaning that the data is drawn from some $P \in \mathcal{P}^\mu$. For every $m \in [0, 1]$, we posit a composite null hypothesis $H_m^0 : P \in \mathcal{P}^m$, but clearly only one of these nulls is true. We will design betting strategies to test these nulls and thus find efficient confidence intervals or sequences for μ. It is easier to present the sequential case first, since that is arguably more natural for sampling WoR, and discuss the fixed-time case later.

5.1 Existing (super)martingale-based confidence sequences or tests

Several papers have considered estimating the mean of a finite set of nonrandom numbers when sampling WoR, often by constructing concentration inequalities (Bardenet & Maillard, 2015; Hoeffding, 1963; Serfling, 1974; Waudby-Smith & Ramdas, 2020). Notably, Hoeffding (1963) showed that the same bound for sampling with replacement (2) can be used when sampling WoR. Serfling (1974) improved on this bound, which was then further refined by Bardenet and Maillard (2015). While test supermartingales appeared in some of the aforementioned works, Waudby-Smith and Ramdas (2020) identified better test supermartingales which yield explicit Hoeffding- and empirical Bernstein-type concentration inequalities and CSs for sampling WoR that significantly improved on previous bounds. Consider their exponential Hoeffding-type supermartingale,

$$M_t^{\text{H-WoR}} := \exp\left\{\sum_{i=1}^{t}\left[\lambda_i\left(X_i - \mu + \frac{1}{N-(i-1)}\sum_{j=1}^{i-1}(X_j-\mu)\right) - \psi_H(\lambda_i)\right]\right\}, \tag{29}$$

and their exponential empirical Bernstein-type supermartingale,

$$M_t^{\text{EB-WoR}} := \exp\left\{\sum_{i=1}^{t}\left[\lambda_i\left(X_i - \mu + \frac{1}{N-(i-1)}\sum_{j=1}^{i-1}(X_j-\mu)\right) - v_i\psi_E(\lambda_i)\right]\right\}, \tag{30}$$

where $(\lambda_t)_{t=1}^N$ is any predictable λ-sequence (real-valued for $M_t^{\text{H-WoR}}$, but $[0, 1)$-valued for $M_t^{\text{EB-WoR}}$), $v_i = 4(X_i - \widehat{\mu}_{i-1})^2$ as before, and $\psi_H(\cdot)$ and $\psi_E(\cdot)$ are defined as in Section 3. Defining $M_0^{\text{H-WoR}} \equiv M_0^{\text{EB-WoR}} := 1$, Waudby-Smith and Ramdas (2020) prove that $(M_t^{\text{H-WoR}})_{t=0}^N$ and $(M_t^{\text{EB-WoR}})_{t=0}^N$ are test supermartingales with respect to $\mathcal{F}$, and hence can be used in step (b) of Theorem 1.

In recent work on election audits, Stark (2020) credits Harold Kaplan for proposing

$$M_t^K := \int_0^1 \prod_{i=1}^{t}\left(1 + \gamma\left[X_i\frac{1-(i-1)/N}{\mu - \sum_{j=1}^{i-1}X_j/N} - 1\right]\right)d\gamma. \tag{31}$$

The 'Kaplan martingale' $(M_t^K)_{t=0}^N$ was employed for election auditing, but it is a polynomial of degree t and is computationally expensive for large t (Stark, 2020).

Next, we mimic the approach of Section 4 to derive a capital process for sampling WoR. We then derive WoR analogues of the efficient betting strategies from Online Supplementary Material, Section B.

5.2 The capital process for sampling without replacement

Define the predictable sequence $(\mu_t^{\text{WoR}})_{t\in[N]}$ where

$$\mu_t^{\text{WoR}} := \mathbb{E}[X_t|\mathcal{F}_{t-1}] = \frac{N\mu - \sum_{i=1}^{t-1}X_i}{N-(t-1)}. \tag{32}$$

It is clear that $\mu_t^{\text{WoR}} \in [0, 1]$, since it is the mean of the unobserved elements of $\{x_i\}_{i\in[N]}$. $(\mu_t^{\text{WoR}})_{t\in[N]}$ is unobserved since μ is unknown, so it is helpful to define

$$m_t^{\text{WoR}} := \frac{Nm - \sum_{i=1}^{t-1}X_i}{N-(t-1)}. \tag{33}$$

Now, let $(\lambda_t(m))_{t=1}^N$ be a predictable sequence such that $\lambda_t(m)$ is $(-1/(1-m_t^{\text{WoR}}), 1/m_t^{\text{WoR}})$-valued. Define the *without-replacement capital process* $\mathcal{K}_t^{\text{WoR}}(m)$,

$$\mathcal{K}_t^{\text{WoR}}(m) := \prod_{i=1}^{t}(1 + \lambda_i(m)\cdot(X_i - m_i^{\text{WoR}})) \tag{34}$$

with $\mathcal{K}_0^{\text{WoR}}(m) := 1$. The following result is analogous to Proposition 2.

Proposition 4 Let X_1^N be a WoR sample from $x_1^N \in [0, 1]^N$. The following two statements imply each other:

(a) $\mathbb{E}_P(X_t \mid \mathcal{F}_{t-1}) = \mu_t^{\text{WoR}}$ for each $t \in [N]$.
(b) For every predictable sequence with $\lambda_t(m) \in (-1/(1-\mu_t^{\text{WoR}}),\, 1/\mu_t^{\text{WoR}})$, $(\mathcal{K}_t^{\text{WoR}}(\mu))_{t=0}^N$ is a test martingale.

The other claims within Proposition 2 also hold above with minor modification, but we do not mention them again for brevity. Further, Proposition 3 technically covers WoR sampling as well. We now present a 'hedged' capital process and powerful betting schemes for sampling WoR, to construct a CS for $\mu = 1/N \sum_{i=1}^N x_i$.

5.3 Powerful betting schemes

Similar to Section 4.4, define the hedged capital process for sampling WoR:

$$\mathcal{K}_t^{\text{WoR},\pm}(m) := \max\left\{\theta \prod_{i=1}^t (1 + \lambda_i^+(m) \cdot (X_i - m_i^{\text{WoR}})),\right.$$
$$\left.(1-\theta) \prod_{i=1}^t (1 - \lambda_i^-(m) \cdot (X_i - m_i^{\text{WoR}}))\right\}$$

for some predictable $(\lambda_t^+(m))_{t=1}^N$ and $(\lambda_t^-(m))_{t=1}^N$ taking values in $[0,\, 1/m_i^{\text{WoR}}]$ and $[0,\, 1/(1-m_i^{\text{WoR}})]$ at time t, respectively. Using $(\mathcal{K}_t^{\text{WoR},\pm}(m))_{t=0}^N$ as the process in Step (b) of Theorem 1, we obtain the CS summarized in the following theorem.

Theorem 4 (WoR hedged capital CS [**Hedged-WoR**]). Given a finite population $x_1^N \in [0, 1]^N$ with mean $\mu := 1/N \sum_{i=1}^N x_i = \mu$, suppose that $X_1, X_2, \ldots X_N$ are sampled WoR from x_1^N. Let $(\dot{\lambda}_t^+)_{t=1}^N$ and $(\dot{\lambda}_t^-)_{t=1}^N$ be real-valued predictable sequences not depending on m, and for each $t \geq 1$ let

$$\lambda_t^+(m) := |\dot{\lambda}_t^+| \wedge \frac{c}{m_t^{\text{WoR}}}, \quad \lambda_t^-(m) := |\dot{\lambda}_t^-| \wedge \frac{c}{1 - m_t^{\text{WoR}}},$$

for some $c \in [0, 1)$ (some reasonable defaults being $c = 1/2$ or $3/4$). Then

$$\mathfrak{B}_t^{\pm,\text{WoR}} := \{m \in [0, 1] : \mathcal{K}_t^{\pm,\text{WoR}}(m) < 1/\alpha\} \quad \text{forms a } (1-\alpha)\text{-CS for } \mu,$$

as does $\bigcap_{i \leq t} \mathfrak{B}_i^{\pm,\text{WoR}}$. Furthermore, $\mathfrak{B}_t^{\pm,\text{WoR}}$ is an interval for each $t \geq 1$.

The proof of Theorem 4 is in Online Supplementary Material, Section A.9. We recommend setting $\dot{\lambda}_t^+ = \dot{\lambda}_t^- = \lambda_t^{\text{PrPl}\pm}$ as was done earlier in (26); for each $t \geq 1$, and $c := 1/2$, let

$$\lambda_t^{\text{PrPl}\pm} := \sqrt{\frac{2\log(2/\alpha)}{\widehat{\sigma}_{t-1}^2 t \log(t+1)}}, \quad \widehat{\sigma}_t^2 := \frac{1/4 + \sum_{i=1}^t (X_i - \widehat{\mu}_i)^2}{t+1}, \quad \text{and} \quad \widehat{\mu}_t := \frac{1/2 + \sum_{i=1}^t X_i}{t+1},$$

See Figure 7 for a comparison to the best prior work.

Remark 4 As before, we can use Theorem 4 to derive powerful CIs for the mean of a non-random set of bounded numbers given a fixed sample size $n \leq N$. We recommend using $\bigcap_{i \leq n} \mathfrak{B}_i^{\pm,\text{WoR}}$, and setting $\dot{\lambda}_t^+ = \dot{\lambda}_t^- = \dot{\lambda}_t^\pm$ as in (27): $\dot{\lambda}_t^\pm := \sqrt{[2\log(2/\alpha)]/n\widehat{\sigma}_{t-1}^2}$. We refer to the resulting CI as [**Hedged-WoR-CI**]; see Figure 8.

Remark 5 For some values of m near 0 or 1, m_t^{WoR} could lie outside of $[0, 1]$, leading $\mathcal{K}_t^{\pm,\text{WoR}}(m)$ to potentially be negative. However, it is impossible for $\mathcal{K}_t^{\pm,\text{WoR}}(\mu)$ to be negative since $\mu_t \in [0, 1]$ always. In fact, a negative m_t implies

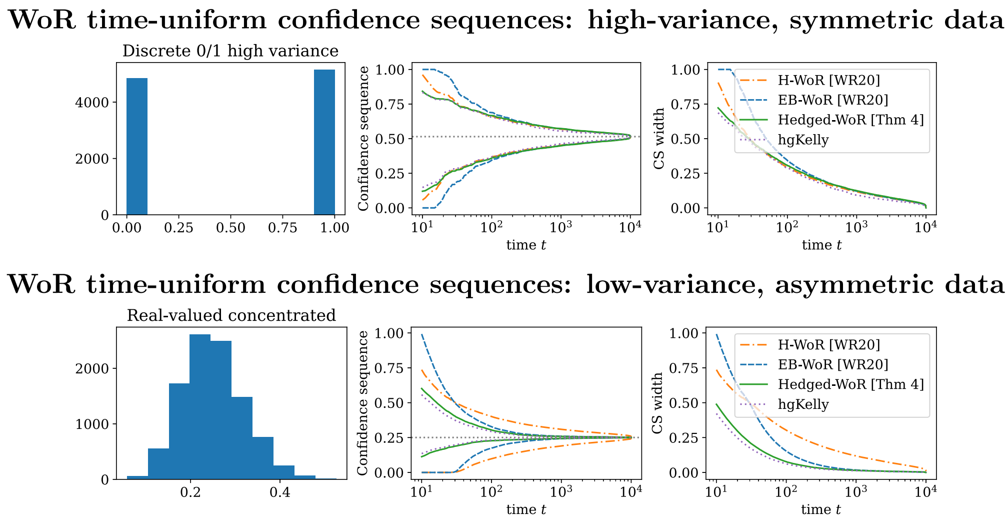

Figure 7. Without-replacement betting CSs versus the predictable plug-in supermartingale-based CSs (Waudby-Smith & Ramdas, 2020). Similar to the with-replacement case, the betting approach matches or vastly outperforms past state-of-the art methods.

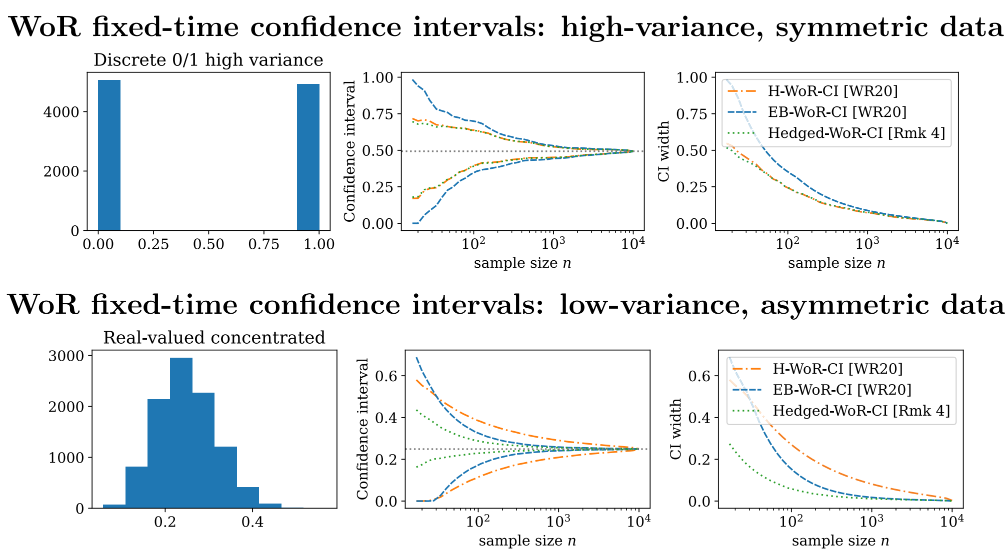

Figure 8. WoR analogue of the hedged capital CI versus the WoR Hoeffding- and empirical Bernstein-type CIs (Waudby-Smith & Ramdas, 2020). Similar to with-replacement, the betting approach has excellent performance.

that the value of m being tested is impossible, and thus one can reject that null immediately. In particular, when running our method, one can instead use the modified capital process

$$\tilde{\mathcal{K}}_t^{\pm,\text{WoR}}(m) := |\mathcal{K}_t^{\pm,\text{WoR}}(m)|/1(m_t \in [0, 1])$$

which takes on the value $+\infty$ if the denominator evaluates to zero. Note that $\tilde{\mathcal{K}}_t^{\pm,\text{WoR}}(\mu)$ still forms a nonnegative martingale since its denominator is always one when $m = \mu$.

Notice that constructing a WoR test martingale only relies on changing the fixed conditional mean μ to the time-varying conditional mean $\mu_t^{\mathrm{WoR}} := (N\mu - \sum_{i=1}^{t-1} X_i)/(N - t + 1)$ and now designing $(-1/(1-\mu_t^{\mathrm{WoR}}),\ 1/\mu_t^{\mathrm{WoR}})$-valued bets instead of $(-1/(1-\mu),\ 1/\mu)$-valued ones. In this way, it is possible to adapt any of the betting strategies in Online Supplementary Material, Section B to sampling WoR, yielding a wide array of solutions to this estimation problem.

5.4 Relationship to composite null testing

This paper focuses primarily on estimation, but we end with a note that our CSs (or CIs) yield valid, sequential (or batch) tests for composite null hypotheses $H_0 : \mu \in S$ for any $S \subset [0, 1]$. Specifically, for any of our capital processes $\mathcal{K}_t(m)$,

$$\mathfrak{p}_t := \sup_{m \in S} \frac{1}{\mathcal{K}_t(m)}$$

is an 'anytime-valid p-value' for H_0, as is $\widetilde{\mathfrak{p}}_t := \inf_{s \le t}\ \mathfrak{p}_s$, meaning that

$$\sup_{P \in \cup_{m \in S} \mathcal{P}^m} P(\ \mathfrak{p}_\tau \le \alpha) \le \alpha \ \text{ for arbitrary stopping times } \tau.$$

Alternately, $\mathfrak{p}_t$ is also the smallest α for which our $(1-\alpha)$-CS does not intersect S. Similarly, $\mathfrak{e}_t := \inf_{m \in S} \mathcal{K}_t(m)$ is an 'e-process' for H_0, meaning that

$$\sup_{P \in \cup_{m \in S} \mathcal{P}^m} \mathbb{E}_P[\mathfrak{e}_\tau] \le 1 \ \text{ for arbitrary stopping times } \tau.$$

For more details on inference at arbitrary stopping times, we refer the reader to Howard et al. (2020, 2021), Grünwald et al. (2019), Ramdas et al. (2020).

6 A brief selective history on betting and its mathematical applications

From a purely statistical perspective, this paper could be viewed as tackling the problem of deriving sharp confidence sets for means of bounded random variables. In this pursuit, we find that a technique with excellent empirical performance happens to have strong connections to the topics of betting and gambling. While we provide a more detailed discussion in Online Supplementary Material, Section F, here we briefly summarize some of the ways in which betting ideas have appeared in and shaped probability, statistical inference, information theory, and online learning, in the broad context of our paper. Figure 9 gives a chronological illustration of the discussion below, highlighting which prominent authors worked with these ideas, and in which subfield.

- *Probability*: The 1939 PhD thesis of Ville (1939) brought betting and martingales to the forefront of modern probability theory, by giving actionable interpretations to Kolmogorov's newly developed measure-theoretic probability, and dealing a near-fatal blow to the theory of collectives by von Mises. Ville showed that for *any* event A of probability measure zero (like sequences violating the law of large numbers), he could design an explicit betting strategy that never bets more than it has, whose wealth (a test martingale) grows without to infinity if the event A occurs. Ville worked with binary sequences, but his result holds more generally; see Shafer and Vovk (2001).

  One may view Ville's result as a theorem in measure-theoretic probability theory; what he effectively proved was: the event that a test (super)martingale exceeds $1/\alpha$ has probability at most α (Ville's inequality in this paper). This holds for any $\alpha \in [0, 1]$, treating $1/0 \equiv +\infty$, with the $\alpha = 0$ case being the most remarkable part. But Ville's result is also an axiomatic building block for *game-theoretic probability* (Shafer & Vovk, 2001, 2019; Vovk, 1993). Many classical results in probability can been derived in completely game-theoretic terms (Shafer & Vovk, 2001, 2019). The capital processes used for deriving CSs are of the same form as those used to derive these foundational theorems of game-theoretic probability, despite the two goals being quite different.

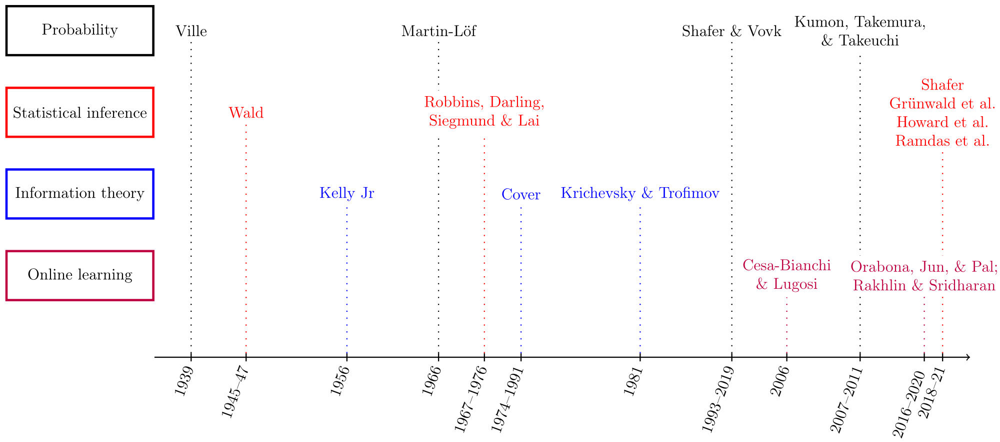

Figure 9. A brief selective history of betting ideas appearing (often implicitly) in various literatures. As discussed further in Online Supplementary Material, Section F, these subfields have rarely cited each other, but ideas are now beginning to permeate. Several authors did not explicitly use the language of betting, and their inclusion above is due to reinterpreting their work in hindsight.

- *Statistical inference*: The famous book of Wald (1945) was the first to thoroughly present and study sequential hypothesis testing. Despite not being presented in this way by Wald, we know in hindsight that the sequential probability ratio test (SPRT) is quite centrally based on the fact that the likelihood ratio is a nonnegative martingale. Two decades later, Robbins and colleagues built on Wald's sequential testing work in several ways, including to estimation via confidence sequences (Darling & Robbins, 1967a, 1967b, 1967c; Lai, 1976; Robbins, 1970; Robbins & Siegmund, 1968, 1969, 1970, 1972, 1974). The recent work of Howard et al. (2020, 2021), Ramdas et al. (2021), Wasserman et al. (2020) extends the early work of Wald, Robbins and colleagues to a broader class of problems using exponential supermartingales and 'e-processes', which can be seen as nonparametric, composite generalizations of the SPRT martingale. Connections between *betting* and the works of Wald, Robbins et al., and Howard et al. are implicit in those works, but can now be seen in hindsight, and our paper makes these connections explicit.
- *Information theory*: Working in the new field of information theory, Kelly (1956) made direct connections to betting by showing that the capacity of a channel (itself fundamentally related to entropy and the Kullback–Leibler divergence) is given by the maximal rate of growth of wealth of a gambler in a simple game with iid Bernoulli(p) observations and known p. Breiman (1961) generalized Kelly's results significantly, and Krichevsky and Trofimov (1981) extended these results beyond the case of known p using a mixture method. Thomas Cover's interest in these techniques spans several decades (R. Bell & Cover, 1988; R. M. Bell & Cover, 1980; Cover, 1974, 1984, 1987), culminating in his famous universal portfolio algorithm (Cover, 1991). The results of Krichevsky–Trofimov and Cover are essentially regret inequalities, leading directly to the final subfield below.
- *Online learning*: The techniques of Krichevsky, Trofimov and Cover found extensive applications to *sequential prediction with the logarithmic loss* (Cesa-Bianchi & Lugosi, 2006). Here, one derives *regret inequalities* for the total loss accumulated when predicting the next observation from a potentially adversarial sequence. This problem is fundamentally connected to online convex optimization, for which Orabona and colleagues use parameter-free betting algorithms to derive regret inequalities (Cutkosky & Orabona, 2018; Jun & Orabona, 2019; Jun et al., 2017; Orabona & Pal, 2016; Orabona & Tommasi, 2017). Rakhlin and Sridharan (2017) articulated a deep connection between martingale concentration and deterministic regret inequalities, and Jun and Orabona (2019, Section 7.1) derive concentration bounds for the general setting of Banach space-valued observations with sub-exponential noise.

7 Summary

Nonparametric confidence sequences are particularly useful in sequential estimation because they enable valid inference at arbitrary stopping times, but they are underappreciated as powerful tools to provide accurate inference even at fixed times. Recent work (Howard et al., 2020, 2021) has developed several time-uniform generalizations of the Cramer–Chernoff technique utilizing 'line-crossing' inequalities and using various variants of Robbins' method of mixtures (discrete mixtures, conjugate mixtures and stitching) to convert them to 'curve-crossing' inequalities.

This work adds new techniques to the toolkit: to complement the aforementioned mixture methods, we develop a 'predictable plug-in' approach. When coupled with existing nonparametric supermartingales, it yields (for example) computationally efficient empirical-Bernstein confidence sequences. One of our major contributions is to thoroughly develop the theory and methodology for a new nonnegative martingale approach to estimating means of bounded random variables in both with- and without-replacement settings. These convincingly outperform all existing published work that we are aware of, for CIs and CSs, both with and without replacement.

Our methods are particularly easy to interpret in terms of evolving capital processes and sequential testing by betting (Shafer, 2021) but we go much further by developing powerful and efficient betting strategies that lead to state-of-the-art variance-adaptive confidence sets that are significantly tighter than past work in all considered settings. In particular, Shafer espouses *complementary* benefits of such approaches, ranging from improved scientific communication, ties to historical advances in probability, and reproducibility via continued experimentation (also see Grünwald et al., 2019), but our focus here has been on developing a new state of the art for a set of classical, fundamental problems.

There appear to be nontrivial connections to online learning theory (Cutkosky & Orabona, 2018; Kotłowski et al., 2010; Kumon et al., 2011; Orabona & Tommasi, 2017), and to empirical and dual likelihoods (see Online Supplementary Material, Section E.6 and an extended historical review of betting in Online Supplementary Material, Section F). The reductions from regret inequalities to concentration bounds described in Rakhlin and Sridharan (2017) and Jun and Orabona (2019) are fascinating, but existing published bounds are loose in the constants and not competitive in practice compared to our direct approach. Exploring deeper connections may yield other confidence sequences or betting strategies.

It is clear to us, and hopefully to the reader as well, that the ideas behind this work (adaptive statistical inference by betting) form the tip of the iceberg—they lead to powerful, efficient, nonasymptotic, nonparametric inference and can be adapted to a range of other problems. As just one example, let $\mathcal{P}^{p,q}$ represent the set of all continuous distributions such that the p-quantile of X_t, conditional on the past, is equal to q. This is also a nonparametric, convex set of distributions with no common reference measure. Nevertheless, for any predictable (λ_i), it is easy to check that

$$M_t = \prod_{i=1}^{t} (1 + \lambda_i(\mathbf{1}_{X_i \le q} - p))$$

is a test martingale for $\mathcal{P}^{p,q}$. Setting $p = 1/2$ and $q = 0$, for example, we can sequentially test if the median of the underlying data distribution is the origin. The continuity assumption can be relaxed, and this test can be inverted to get a confidence sequence for any quantile. We do not pursue this idea further in the current paper because the recent (rather different) nonnegative martingale methods of Howard and Ramdas (2022) already provide a challenging benchmark for that problem. Typically, one test martingale-based method cannot uniformly dominate another, and the large gains in this paper were made possible because all previous published approaches implicitly or explicitly employed test *super*martingales, while we employ test martingales that are computationally simple to implement.

To conclude, we opine that 'game-theoretic statistical inference' is in its nascency, and we expect much theoretical and practical progress in coming years. We hope the reader shares our excitement in this regard.

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Data availability

No new data were generated or analyzed in support of this research.

Supplementary material

Supplementary material are available at *Journal of the Royal Statistical Society: Series B* online.

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Discussion Paper Contribution

Proposer of the vote of thanks to Waudy-Smith and Ramdas and contribution to the Discussion of 'Estimating means of bounded random variables by betting'

Peter Grünwald[1,2]

[1]Machine Learning Group, Centrum Wiskunde & Informatica, Amsterdam, The Netherlands
[2]Mathematical Institute, Leiden University, Leiden, The Netherlands

Address for correspondence: Peter Grünwald, Centrum Wiskunde & Informatica, Science Park 123, 1097 AG Amsterdam, The Netherlands. Email: pdg@cwi.nl

The authors derive non-asymptotic anytime-valid confidence sequences for the mean of a sequence $X_1, X_2, \ldots$ of bounded random variables. When put to practice, their new methods beat the best known bounds, sometimes by vast margin—*even for the fixed-sample size, not anytime-valid setting*. It is rare in statistics that one can get such substantial improvements on a decades-old problem, and I congratulate Waudby-Smith and Ramdas on this remarkable achievement. It illustrates once more the relevance of *e-process-based anytime-valid methods* (Grünwald et al., 2024) even when anytime-validity is not required. In particular their results have repercussions for PAC-Bayesian machine learning theory, which relies on concentration bounds for bounded i.i.d. X_t—the main (but not only) setting the authors (WSR from now on) consider and on which I will also focus. So, let $X_1, X_2, \ldots$ be i.i.d. $\sim P$ with P an arbitrary distribution on [0, 1]. The null, denoted $\mathcal{P}^{\mu}$, consists of *all* distributions on [0, 1] with some fixed mean μ. We want to test whether the mean is μ, against alternative $\bigcup_{\mu' \neq \mu} \mathcal{P}^{\mu'}$.

1 An embarrassment of Neyman–Pearson theory

Assume the X_t arrive sequentially. A company's data science team is instructed to find out whether it can rule out, with high certainty, that $\mu < \mu_0$ for some fixed given value μ_0. They plan to await 5,000 outcomes and then check if the lower end of the $1 - \alpha$ confidence interval is above μ_0, for $\alpha = 0.001$.

But now suppose their boss is impatient and, at $t = 1{,}000$, wants to know if there is already sufficiently strong evidence to rule out $\mu < \mu_0$. He thus asks the data science team to peek at the data. They find they already have a significant result, so they stop sampling. As is well-known, this invalidates confidence intervals, and may be viewed as *p-hacking*. What is less known though, is that *even if they had not found a significant result at* $t = 1{,}000$ *and therefore had decided to keep sampling until* $t = 5{,}000$ *after all*, they would already have invalidated the $(1 - \alpha)$-coverage—by the mere act of just checking, even if based on the particular data they saw they did not change course after the check. In this sense, classical methods seem almost like quantum mechanics: you may already destroy the validity of your conclusions merely by looking at the data! Anytime-valid methods like WSR's avoid this issue altogether.

2 An embarrassment of Bayes theory

Does not Bayesian statistics fare better on this problem? It has often been claimed that 'optional stopping is no problem for Bayesians'. While such claims are problematic anyway

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(Hendriksen et al., 2021), here I focus on a different issue: the simple problem addressed by WSR is incredibly difficult to solve via a full Bayesian analysis, which requires specifying a prior distribution on some set $\mathcal{P}$ containing P. How to choose $\mathcal{P}$ if, like WSR, we want to make no assumptions at all on P? Even if one adapts a standard non-parametric $\mathcal{P}$ and corresponding prior, one still rules out many possible and reasonable P…While such points have been made since the 1950s, the issue is brought to light particularly clearly in WSR's bounded support setting, since they really need to assume nothing further about P at all and require only *two* parameters to get their results.

Still, their approach does have a *pseudo-Bayesian* flavour. They employ *capital processes* $(\mathcal{K}_t(\mu))_{t=0}^{\infty}$ which are really test martingales relative to the null $\mathcal{P}^{\mu}$, of the form

$$\mathcal{K}_t(\mu) = \prod_{i=1}^{t} (1 + \lambda_i(\mu) \cdot (X_i - \mu)), \tag{1}$$

with $\mathcal{K}_0(\mu) := 1$ and $(\lambda_t(\mu))_{t=1}^{\infty}$ any $\Lambda(\mu)$-valued predictable sequence, with $\Lambda(\mu) = (-1/(1-\mu), 1/\mu)$. Thus, one can let $\lambda_t(\mu)$ depend on X_1^{t-1} and in this way one can learn 'good' values of λ based on past data. In their arguably most sophisticated approach for determining the λ_t's, *GRAPA*, WSR determine a $\hat{\lambda}_t(\mu)$ directly, via a plug-in approach, but, they point out, it can also be done via the *method of mixtures*, by putting a prior density w_{μ} on $\Lambda(\mu)$ and using in (1) the 'posterior-mean' $\tilde{\lambda}_t(\mu) := \int w_{\mu}(\lambda \mid X_1^{t-1}) d\lambda$, based on 'pseudo-posterior mixture'

$$w_{\mu}(\lambda \mid X_1^{t-1}) \propto w_{\mu}(\lambda) \cdot \prod_{i=1}^{t-1} (1 + \lambda(X_i - \mu)). \tag{2}$$

Orabona and Jun (2021) take this approach, with w_{μ} generalising Jeffreys' prior for the Bernoulli model.

3 GRAPA vs. REGROW vs. KLinf

What is a good martingale to use in the first place? (Grünwald et al., 2024) strongly argued that, if a simple alternative P is given, then the *Kelly criterion* (which they called P-GRO, standing for *growth-rate optimal* relative to P) is the natural anytime-valid replacement for the traditional goal of optimising power. The P-GRO martingale $(M_t)_{t=0}^{\infty}$ (if it exists) maximises

$$\mathbf{E}_P[\log M_t] \tag{3}$$

for all t. The natural extension of (3) in case of a large (rather than simple) alternative hypothesis is called REGROW (for *relative growth-optimality in worst-case*) by Grünwald et al. (2024). WSR's GRAPA can be viewed as approximating the REGROW martingale. This follows from WSR's Proposition 2, Part (d), which shows that *any* test martingale for testing μ must be of the form (1) for some predictable λ_t. Thus, for any P in the alternative $\bigcup_{\mu' \neq \mu} \mathcal{P}^{\mu'}$, there must be some sequence $\{\lambda_t^{(P)}\}_t$ for which the corresponding $\mathcal{K}_t(\mu)$ is GRO. The arguments of Koolen and Grünwald (2021) imply that $\lambda_t^{(P)}$ must be the same for all t; let us denote it as λ_P^*. REGROW then amounts to finding a test martingale $(M_t)_{t=0}^{\infty}$ for testing $\mathcal{H}_0$ for which

$$\max_{P \in \mathcal{H}_1} \mathbf{E}_P\left[\log \mathcal{K}_t^{(\lambda_P^*)}(\mu) - \log M_t\right] \tag{4}$$

is small for each t, where we use $\mathcal{K}_t^{(\lambda)}(\mu)$ to denote the fixed-λ-capital process with $\lambda_t = \lambda$ for all t. Alternatively, one may consider the *expected regret*, given by replacing λ_P^* in (4) by $\lambda^{hs}(X^t)$ ('optimal fixed λ with hindsight'), the λ for which $\mathcal{K}_t^{(\lambda)}(\mu)$ is maximised at time t.

GRAPA can be thought of as finding an (almost-) REGROW $\mathcal{K}_t(\mu)$ by setting each λ_t to the $\lambda^{hs}(X^{t-1})$ that would have maximised the empirical counterpart based on the data seen in the past. Orabona and Jun, in contrast, show that for their prior the regret ((4) with λ_P^* replaced by

$\lambda^{bs}(X^t)$) is within $(1/2)\log t + O(1)$. We suspect that GRAPA will deliver similar REGROWth and regret, taking as our cue the parametric setting, where both REGROW and regret of order $(1/2)\log t + O(1)$ is achieved for both the 'prequential' ML plug-in method (of which GRAPA is a non-parametric analogue) and the Bayesian mixture (for which Orabona and Jun's approach is the non-parametric analogue).

Imposing a regret-minimising prior on λ's in (1) is also central to the *KLinf method* in the bandit literature (Agrawal et al., 2021), which directly links growth-optimality of (1) to KL divergence, providing a 3parametric analogue of the duality between KL divergence and GRO established by Grünwald et al. (2024) in the parametric case. A further theoretical analysis of the precise relation between GRAPA, KLinf and regret should lead to better understanding and propel potential extensions such as *bounded regression*.

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Gergely Neu

Department of Information and Communication Technologies, Universitat Pompeu Fabra, Barcelona, Spain

Address for correspondence: Gergely Neu, Department of Information and Communication Technologies, Universitat Pompeu Fabra, C/ Roc Boronat 138, 08024 Barcelona, Spain. Email: gergely.neu@gmail.com

The paper proposes an elegant new framework for deriving time-uniform confidence intervals for the mean of a sequence of random variables. The proposed technique is based on a game-theoretic view of statistical inference, rephrasing the problem of building confidence intervals as identifying a set of coin-betting games with a certain 'plausibility' property: if one can get 'implausibly rich' in a game by betting on the mean being lower (resp. higher) than a certain threshold, then the mean must be higher (resp. lower) than said threshold. A confidence interval can then be derived as the interval between the lowest and highest values that cannot be ruled out as 'implausible'. This game-theoretic view extends the pioneering work of Ville (1939) and Shafer and Vovk (2019) who used similar betting games to lay an alternative set of foundations of probability theory that is free from measure theory.

The authors of this work develop a set of new concentration inequalities from this framework that are both aesthetically pleasing and empirically tight. In many ways, the framework appears to be so natural and profound that the reader wonders why this connection with betting has not been exploited in this depth previously. Indeed, the language of betting is so closely tied with the theory of test martingales that one wonders why almost all past work on deriving confidence intervals (and more generally, testing) have focused on using supermartingales that inevitably lead to at least some looseness when used for this purpose. In other words, I believe that the authors have contributed a valuable fundamental insight which will undoubtedly lead to many interesting future results.

Besides this appraisal, I would also like to mention some limitations of the results presented in the paper and particularly highlight connections with the concurrent work of Orabona and Jun (2021) that proposes a similar framework for deriving confidence sequences.

1. One slightly unsatisfying aspect of the paper is that, while the authors develop a new theory of mean estimation through the lens of playing a sequential game, they do not seem to make full advantage of this connection by going 'all in' on this connection. In particular, most of the proposed estimators are based on betting heuristics rather than betting strategies with already existing theoretical guarantees on their growth rate (or other properties of interest). There are several such algorithms for a range of known betting games, and I am surprised to see that these are only mentioned as an afterthought in the appendix (if at all). In particular, it remains unclear what betting algorithm one should use to, say, minimise the confidence width. The authors criticise the use of principled betting strategies, saying that those are all optimised for worst-case price relatives, and the i.i.d. case considered in the paper should be much easier to handle. On one hand, this is hardly an appropriate justification to replace principled methods with heuristics. On the other hand, this argument is not entirely well-founded, as several well-studied betting strategies are 'equalisers' in the sense that they have the same regret for all sequences (i.e. worst-case or i.i.d.). See the detailed discussion on this issue by Orabona and Jun (2021).
2. The confidence intervals depend on the order in which the data are presented to the algorithm. This is natural for non-i.i.d. data but is hard to justify when the data are i.i.d. The authors propose a 'derandomisation' strategy consisting of shuffling the data a number of times, executing the algorithm on all resulting sequences, and aggregating the results. Besides being somewhat unsatisfactory from a conceptual point of view, this approach also destroys the efficiency that was part of the magic of the proposed method. I wish to point out that this aspect has been addressed very neatly by Orabona and Jun (2021), whose reduction to the universal portfolio optimisation method of Cover and Ordentlich (1996) produces estimates that are independent of the order of the data without such derandomisation.
3. The method is limited to random variables bounded almost surely in [0, 1]. This seems like an inherent limitation, although it seems likely that it can be removed either by combining the current approach with the 'moment truncation' estimator of Catoni (2012) (as one of the authors did in a related paper, Wang & Ramdas, 2023). Still, it is unclear if such extensions can retain the appealing 'purity' of the approach proposed in the present paper, or if a more elegant and natural treatment of unbounded random variables would be possible.

I wish to emphasise that these latter comments are not meant to diminish the value of the contributions of the paper but rather to highlight some outstanding issues that future work has to address. There are of course several more open questions left behind that this note can hardly do justice to. That said, opening so many potential directions for future research is a sign of great fundamental work, and I wish to congratulate the authors (as well as the concurrent pioneers; Orabona & Jun, 2021) for bringing this set of tools to the forefront of statistical inference.

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The vote of thanks was passed by acclamation.

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Ruodu Wang's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Ruodu Wang

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON N2L3G1, Canada

Address for correspondence: Ruodu Wang, Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON N2L3G1, Canada. Email: wang@uwaterloo.ca

I congratulate Ian Waudby-Smith and Aaditya Ramdas for this excellent contribution to the theory of e-testing (or testing by betting). Over the past several years, this theory has been developed rapidly in multiple exciting directions. The contribution of Waudby-Smith and Ramdas is an important piece in this active stream of literature. I have a few comments on the approach in the current paper and its generalisations.

The main approach taken in the paper (see its Theorem 1), which is standard in the field of game-theoretical statistics (Shafer & Vovk, 2019), relies on constructing an *e-process* $(M_t^\theta)_{t\in\mathbb{T}}$ for each parameter value $\theta \in \Theta$ (which is the mean m in the paper). The e-process $(M_t^\theta)_{t\in\mathbb{T}}$ is often (but not always) constructed by combining several *sequential e-variables* $E^\theta = (E_t^\theta)_{t\in\mathbb{T}}$ from the data (i.e. satisfying $\mathbb{E}^\theta[E_t^\theta \mid \mathcal{F}_{t-1}] \le 1$ for each $t \in \mathbb{T}$ and $\theta \in \Theta$), via a method of martingale: $M_t^\theta = \prod_{s=1}^t (1 - \lambda_s^\theta(E_s^\theta - 1))$, $t \in \mathbb{T}$, where $\lambda = (\lambda_t^\theta)_{t\in\mathbb{T}}$ is a predictable process. The abstract problem of combining sequential e-variables is studied in Vovk and Wang (2022), where it is shown that the above martingale method is the *only admissible way* to combine sequential e-variables into one e-variable. Therefore, anytime validity (i.e. validity under optional sampling) is obtained automatically if the goal is to make a decision based on a combined e-value.

Although the above method of e-testing is by now standard and its validity is easy to show, the highly non-trivial tasks are to build suitable E^θ and to find powerful λ^θ. The validity is guaranteed even when the data-generating procedure varies arbitrarily over time, as long as the parameter of interest (mean m in this paper) in the null hypothesis is specified. Nevertheless, the power and optimality of λ^θ depend crucially on how data are generated. Most methods (such as GRAPA and aGRAPA introduced in the current paper) use sample mean, sample variance, or the empirical distribution to decide λ^θ. This requires some 'stationarity', 'predictability', or 'temporal structure' of the data. In some applications involving dynamic decision making (data depend on previous decisions), such as backtesting financial risk prediction, such stationarity cannot be assumed. In this context, some options of powerful betting strategies are studied by Wang et al. (2022).

The e-testing approach can be applied to many other quantities in a model-free fashion, similar to the mean with bounded supported treated in this paper. Wang et al. (2022) developed one-sided e-tests for other quantities, including mean (with one-side bounded support), variance, quantile, and the risk measure Expected Shortfall. A main advantage of such methods is that they do not assume any knowledge of the data-generating probability or its temporal structure.

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Anastasios N. Angelopoulos' contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Anastasios N. Angelopoulos

Department of Electrical Engineering and Computer Science, University of California, Berkeley, CA, USA

Address for correspondence: Anastasios N. Angelopoulos, Department of Electrical Engineering and Computer Science, University of California, 253 Cory Hall, Berkeley, CA 94720-1770, USA. Email: angelopoulos@berkeley.edu

I congratulate Waudby-Smith and Ramdas for their paper, which develops algorithms for confidence sequences motivated by betting. An impressive consequence of their work is the derivation of confidence intervals for a fixed sample size n, given by Theorem 3/Remark 3 and Theorem 4/Remark 4. These bounds, which my colleagues and I have referred to eponymously as the *WSR bounds*, prove quite useful for quantifying the uncertainty of artificial intelligence algorithms. I will give an example application from our work *en route* towards showcasing their utility.

In order to quantify and guarantee the reliability of a machine learning model on a sample point X, we may seek to form a *risk-controlling prediction set* (RCPS) $\mathcal{T}(X)$ (Angelopoulos et al., 2021; Bates et al., 2021). More formally, given a test data point (X, Y) and a calibration dataset $\mathcal{D}_{\mathrm{cal}}$, we want to calibrate the model to produce sets that control some [0, 1]-bounded loss L in high

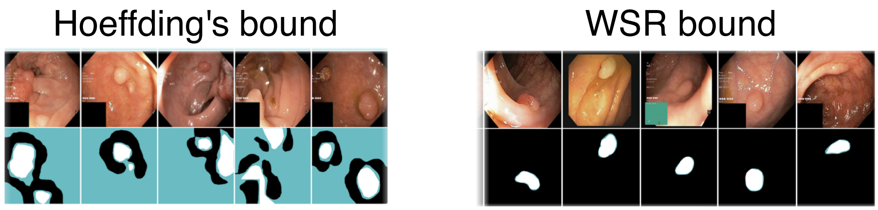

Figure 1. The application of the WSR bound to tumour segmentation. True positives are white, true negatives are black, false positives are blue, and false negatives are red.

probability:

$$\mathrm{P}(\mathbb{E}[L(\mathcal{T}(X), Y)|\mathcal{D}_{\mathrm{cal}}] > \alpha) \leq \delta. \tag{1}$$

The loss quantifies the reliability of the set; an example is the false-negative proportion $L(\mathcal{T}(x), y) = |y \cap \mathcal{T}(x)|/|y|$ in the case that y itself is set-valued. Critically, we seek to satisfy (1) in a distribution-free way, and in finite samples, by concentrating on the empirical risk.

The central point of my commentary is to highlight the significance of the WSR inequality in the formation of an RCPS. Any concentration inequality will work to construct an RCPS, but the tightness of the bound greatly affects the usability of the prediction sets. In our practical experiments, the WSR bound outperformed all other concentration inequalities, such as those of Hoeffding (1994), Bentkus (2004), and Maurer and Pontil (2009), and furthermore, was nearly tight, adapting quite strongly to the variance of the loss. Figure 1 depicts a practical application of the WSR bound, demonstrating its use in segmenting gut tumours through machine learning. Here, we focus on controlling the false-negative rate—essentially, the proportion of the tumour that is incorrectly excluded from the segmentation mask. This is of crucial importance in ensuring that *virtually all cancerous cells* are excised with a high degree of certainty.

Comparing the WSR's practical improvements over Hoeffding's inequality, the difference is clear, and the bound is tight for all practical purposes. This has been evident in our works on risk control and machine learning-assisted inference (Angelopoulos et al., 2023). I extend my gratitude to the authors for this practical contribution.

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The following contributions were received in writing after the meeting

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Anthony C Davison and Igor Rodionov's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Anthony C. Davison and Igor Rodionov

Institute of Mathematics, Ecole Polytechnique Fédérale de Lausanne (EPFL), EPFL-FSB-MATH-STAT, Station 8, 1015 Lausanne, Switzerland

Address for correspondence: Anthony C. Davison, Institute of Mathematics, Ecole Polytechnique Fédérale de Lausanne (EPFL), EPFL-FSB-MATH-STAT, Station 8, 1015 Lausanne, Switzerland. Email: Anthony.Davison@epfl.ch

We congratulate the authors on a wide-ranging contribution to a statistical hardy perennial. The restriction that the support be known and bounded appears highly restrictive, yet Proposition 3 suggests much wider potential for applications, one of which we briefly explore below.

The Pickands–Balkema–de Haan theorem (Balkema & de Haan, 1974; Pickands, 1975) establishes that the distribution of a random variable X, conditioned on its exceeding a high threshold u can in rather wide generality be approximated by the generalised Pareto distribution (GPD). This result, a cornerstone of the statistical analysis of rare events, is typically applied by choosing a threshold u empirically from a random sample of n observations, fitting the GPD to the k observations that exceed u, and using this fit to estimate quantiles or other measures of risk. Numerous procedures have been suggested to choose u, or equivalently k, often based on threshold-stability properties of the GPD; informal graphical approaches were proposed by Davison and Smith (1990), and in many settings more formal procedures are complemented by these or other graphs. The Hill plot (Hill, 1975) is used when X lies in the Fréchet max-domain of attraction with tail index $1/\gamma$ and is related to the Rényi representation for exponential order statistics. One potential use of the ideas in Section 4.4 of the paper is to aid in the choice of k, using the sample order statistics $X_{(1)} \leq \cdots \leq X_{(n)}$ and the simple martingale

$$M_k = (X_{(n-k)}/X_{(n-k+1)})^k, \quad k = 1, \ldots, n;$$

both (M_k) and its expectation $1/(1+\gamma)$ when the scaled differences $k(\log X_{(n-k+1)} - \log X_{(n-k)})$ are independent exponential variables lie in the unit interval. This leads to a conservative overall approach to choosing the number of upper order statistics k used in the Hill estimator.

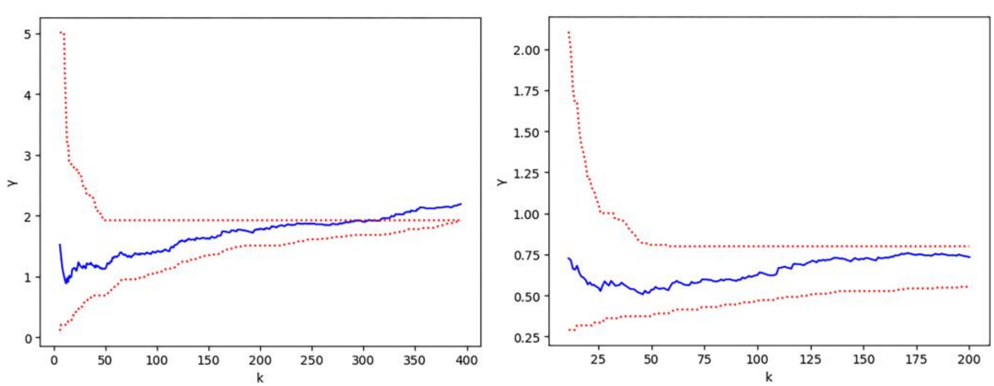

Figure 1. Hill plots showing the estimate of γ (solid) and martingale 95% confidence limits (dotted) based on a log-normal sample (left) and the Danish insurance data (right).

Figure 1 shows confidence intervals for γ for a log-normal sample and for the Danish fire insurance data (Embrechts et al., 1997). The first uses $\lambda_k^+ = (\gamma+1)/4$ and $\lambda_k^- = (\gamma+1)/(4\gamma)$, and the second uses the λs suggested in expressions (25) and (26) of the paper. The theory does not apply to the first, for which $\gamma = 0$, but extremes of finite log-normal samples are typically well-approximated by a distribution with a heavier tail; the martingale shows the expected lack of stability. The second is more stable, suggesting that $\gamma \approx 0.75$; the corresponding confidence interval is similar to that in the top left-hand panel of Figure 6.4.3 of Embrechts et al. (1997).

It should be clear that we found the paper very stimulating.

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Steven R Howard's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Steven R. Howard

The Voleon Group, Berkeley, CA, USA

Address for correspondence: Steven R. Howard, The Voleon Group, 2484 Shattuck Ave, Berkeley, CA, USA. Email: steve@stevehoward.org

I congratulate the authors on this creative and well-written contribution. It's always a delight to see new insights into such a fundamental problem as inference for the mean of bounded observations, and certainly inspires excitement about the future of game-theoretic statistical inference. Besides the impressive technical contributions, I appreciate how the authors draw previously unappreciated connections between work in information theory, finance, and online learning, hopefully increasing future collaboration between researchers in these fields.

In their conclusion, the authors allude to the possibility of estimating functionals other than the mean, using the quantile as an example. Are there fruitful applications of betting methods to estimation of general, possibly vector-valued functionals θ defined by possibly vector-valued estimating equations of the form $\mathbb{E}\left[\psi(X_t, \theta) \,\middle|\, X_1, \ldots, X_{t-1}\right] = 0$ a.s. for all t, assuming such a value of θ exists (Angelopoulos et al., 2023)? This would be interesting even for univariate functionals beyond the mean, though the real prize would be practical methods for estimating vector-valued functionals such as population regression coefficients, for which we have few tractable results (such as the multivariate normal mixture (Abbasi-Yadkori et al., 2011), Hermitian dilation (Tropp, 2012), and Banach space methods (Pinelis, 1994)).

Martingale methods have found adoption in practical applications, and martingale-based confidence sequences specifically have found use in some online A/B testing systems, especially in platforms designed for use by general audience with little statistical training (Johari et al., 2022; Maharaj et al., 2023). However, there remains resistance due to that fact that fixed-sample confidence intervals based on the CLT, and group sequential methods which are generally supported by CLT arguments, yield tighter intervals and higher power. Of course, CLT-based confidence intervals do not come with a nonasymptotic guarantee, and can seriously break in cases of practical interest, so we should not be surprised that nonasymptotic intervals are wider than those implied by the CLT. However, for example, even in the ideal case of Bernoulli(1/2) observations, no matter how large the sample size, Hoeffding's bound remains wider than the CLT interval by a factor of $\sqrt{2\log(2/\alpha)}/z_{1-\alpha/2}$, where z_p is the p-quantile of the standard normal distribution. This factor is about 1.4 at $\alpha = 0.95$, implying an experimenter must gather roughly twice as many samples to obtain the same interval width with a Hoeffding bound that she would have achieved with the CLT; this can be a hard sell. Ideally, a nonasymptotic confidence interval C_n with width $|C_n|$ might have a guarantee like the following: for i.i.d. (X_i) with variance σ^2, we have $\sqrt{n}|C_n| \xrightarrow{\text{p}} 2z_{1-\alpha/2}\sigma$ as $n \to \infty$. On the other hand, a linear sub-Gaussian uniform boundary is tight for Brownian motion, suggesting that martingale-based methods are ideal for extremely frequent 'peeking' but not for fixed-sample testing or sparse 'peeking'. We are left wondering whether nonasymptotic analysis based on martingales and betting can be made similarly competitive in the latter regime.

Lastly, the authors discuss in Supplementary Material Section E.5 a simulation with non-i.i.d. data, but the framework still excludes nonstationary settings in which the conditional mean changes with time. This is in contrast to the exponential supermartingale approach which gives a confidence sequence for a sequence of possibly random 'estimands' given by the average conditional means $\theta_t = t^{-1}\sum_{i=1}^{t}\mathbb{E}(X_i \mid X_1, \ldots, X_{i-1})$ (Howard et al., 2021). Such robustness to general nonstationary is reassuring in general and useful in specific cases such as the estimation of average treatment effects in a potential outcomes model. Do we have counterexamples showing that betting-based confidence sequences, derived under the assumption of constant conditional mean, substantially break down in this more general setting? Is it possible that the bounds in this paper are valid in the more general nonstationary setting and simply await a new analysis?

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Supplementary material

Supplementary material is available online at *Journal of the Royal Statistical Society: Series B*.

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Rong Jiang and Keming Yu's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Rong Jiang[1] and Keming Yu[2]

[1]School of Mathematics Physics and Statistics, Shanghai Polytechnic University, People's Republic of China
[2]Department of Mathematics, Brunel University, London UB83PH, UK

Address for correspondence: Keming Yu, Department of Mathematics, Brunel University, London UB83PH, UK. Email: keming.yu@brunel.ac.uk

We want to congratulate the authors on estimating means of bounded random variables in both with- and without-replacement settings. The authors constructed confidence intervals and time-uniform confidence sequences for the mean of a bounded random variable using test supermartingale technique. It deepens our understanding of the confidence sequence. Confidence sequence is one particular tool in sequential design that facilitates anytime-valid inference. In particular, confidence sequence is a sequence of confidence intervals that is valid at data-dependent stopping times. We offer three comments.

First, the $C_t^{PrPl-EB}$ in Theorem 2 involves $(\lambda_t)_{t=1}^{\infty}$. The authors recommend the predictable plug-in $(\lambda_t^{PrPl-EB})_{t=1}^{\infty}$ given by

$$\lambda_t^{PrPl-EB} = \sqrt{\frac{2\log(2/\alpha)}{\hat{\sigma}_{t-1}^2 t \log(1+t)}} \wedge c,\ \hat{\sigma}_t^2 = \frac{1/4 + \sum_{i=1}^t (X_i - \hat{\mu}_i)^2}{t+1},\ \hat{\mu}_t = \frac{1/2 + \sum_{i=1}^t X_i}{t+1}.$$

Whether so many parameter estimators $\lambda_t^{PrPl-EB}$ will lead to the superposition of errors and the failure of the method. The Hoeffding process $(M_t^H(m))_{t=0}^{\infty}$ in equation (8) only need one λ. In particular, when $t < 100$, is $C_t^{PrPl-EB}$ still correct? See Figure 2, we can see the results are bed when $t < 100$. Thus, whether Theorem 2 needs to add restrictions on t. Moreover, whether $C_t^{PrPl-EB}$ is sensitive to c in $\lambda_t^{PrPl-EB}$, and if so, how to select c, although 1/2 or 3/4 is recommended.

Second, the author mentioned their test supermartingle can also be inverted to get a confidence sequence for any quantile. How about mode (Chernoff, 1964), expectile (Newey & Powell, 1987), and extremile (Daouia et al., 2019)?

Third, the authors consider arbitrary distribution but bounded distribution which implies all moments exist and require a Chernoff-type assumption on the distribution resulting in $O(\sqrt{\log t/t})$ shrinkage rates for the confidence sequences. Recently, Wang and Ramdas (2023) show that employing Catoni's estimator improves the rate to $O(\sqrt{\log\log 2t/t})$ under weaker assumptions on the distribution ((1+δ)-th moment bound). We wonder if the methods and results can be generalised to unbounded observations, since the σ^2-bounded-variance assumption (Wang & Ramdas, 2023) is more realistic and easier to verify.

Conflict of interests: None declared.

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Martin Larsson and Johannes Ruf's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Martin Larsson[1] and Johannes Ruf[2]

[1]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
[2]Department of Mathematics, London School of Economics, London, UK

Address for correspondence: Johannes Ruf, Department of Mathematics, London School of Economics, Columbia House, Houghton St, London WC2A 2AE, UK. Email: j.ruf@lse.ac.uk

We congratulate Ian Waudby-Smith and Aaditya Ramdas on their comprehensive and insightful paper.

The authors construct time-uniform confidence sequences for the mean of a sequence of [0, 1]-valued random variables $X_1, X_2, \ldots$ that all have the same conditional mean, assumed to be deterministic. Specifically, fix $m \in (0, 1)$ and define $\mathcal{P}^m$ as the set of probability measures on the canonical sequence space under which, for each $t \in \mathbb{N}$, the conditional expectation of X_t given the history $X_1, \ldots, X_{t-1}$ is equal to the deterministic number m. The authors construct nonnegative processes (K_t) that satisfy $K_0 = 1$ and are $\mathcal{P}^m$-martingales, i.e. $\mathbb{P}$-martingales for each $\mathbb{P} \in \mathcal{P}^m$. These martingales are then used to construct anytime-valid statistical tests that in turn can be transformed into confidence sequences (see also Ramdas et al., 2022).

In keeping with the game-theoretic probability literature, the authors refer to the processes (K_t) as *capital processes*. Let us ponder this terminology, starting with the following simple but interesting observation made by the authors: every nonnegative $\mathcal{P}^m$-martingale (K_t) with $K_0 = 1$ is of the form

$$K_t = \prod_{s=1}^{t} (1 + \lambda_s(X_s - m)) \tag{1}$$

for some predictable process (λ_t) with values in $[-(1-m)^{-1}, m^{-1}]$. *Predictable* means that each λ_t only depends on $X_1, \ldots, X_{t-1}$. One may interpret λ_t as the proportion of one's capital K_{t-1} that is invested in an asset with return $X_t - m$, keeping whatever is left over 'in the pocket'. The fact that λ_t can be greater than one, or negative, poses no issue as this simply means that one may *borrow* cash to purchase more of the asset than one could otherwise afford, or *sell the asset short* to generate additional cash income. Crucially, one's capital must always remain nonnegative.

The upshot is this: not only is (K_t) the capital process produced by repeated betting; thanks to the representation (1) there is *always* an explicit trading strategy, operating on one single asset, that generates the capital process. Indeed, one has $\lambda_t = (K_t/K_{t-1} - 1)/(X_t - m)$. Given any particular (K_t) of interest, we believe insight can be gained by computing the associated trading strategy. For example, the 'diversified Kelly' capital process considered by the authors is

$$K_t^{\text{dKelly}} = \frac{1}{D} \sum_{d=1}^{D} \prod_{s=1}^{t} (1 + \lambda_s^d (X_s - m)),$$

built from D separate strategies (λ_t^d), $d = 1, \ldots, D$. This is equivalent to the single strategy

$$\lambda_t = \frac{\sum_{d=1}^{D} K_{t-1}^d \lambda_t^d}{\sum_{d=1}^{D} K_{t-1}^d},$$

where (K_t^d) is the capital process generated by the dth strategy. In other words, diversified Kelly arises from executing the capital-weighted average of the given strategies. This links it to Cover's universal portfolios (Cover, 1991).

Many of the capital processes proposed in the paper are specified in terms of a strategy (λ_t). What we find worth emphasising is that such a (λ_t) can *always* be found, and is likely to yield insights. Finally, let us point out that the representation (1) is of course specific to the particular structure of $\mathcal{P}^m$. An interesting question is to what extent analogous representations exist for other, more complex, statistical hypotheses.

Conflict of interests: None declared.

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Jiayi Li, Yuantong Li and Xiaowu Dai's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Jiayi Li[1], Yuantong Li[1] and Xiaowu Dai[1,2]

[1]Department of Statistics and Data Science, University of California, Los Angeles, CA, USA
[2]Department of Biostatistics, University of California, Los Angeles, CA, USA

Address for correspondence: Xiaowu Dai, University of California, 8125 Math Sciences Bldg #951554, Los Angeles, CA 90095-1554. Email: dai@stat.ucla.edu

We congratulate Waudby-Smith and Ramdas for their interesting paper Waudby-Smith and Ramdas (2020b) in generating confidence intervals and time-uniform confidence sequences for mean estimation with bounded observations. Their methodology utilises composite non-negative martingales and establishes a connection to game-theoretic probability. Our comments will focus on numerical comparisons with alternative methods. The corresponding code is available at https://github.com/Likelyt/Estimate-mean-with-betting.

1 Methods

Consider a sequence of random variables $(X_t)_{t=1}^{\infty}$, drawn from a distribution $P \in \mathcal{P}^{\mu}$, where $\mathcal{P}^{\mu}$ represents the set of all distributions on $[0, 1]^{\infty}$. We assume $\mathbb{E}_P[X_t | X_1, \ldots, X_{t-1}] = \mu$ for some unknown $\mu \in [0, 1]$. The objective is to construct a time-uniform confidence sequence $(C_t)_{t=1}^{\infty}$ that satisfies the condition

$$\sup_{P \in \mathcal{P}^{\mu}} P(\exists t \geq 1 : \mu \notin C_t) \leq \alpha. \tag{1}$$

The confidence sequence guarantees that, for any fixed n, C_n is a $(1 - \alpha)$-confidence interval for μ. We review three methods for generating such confidence sequences.

(a) The *predictable plug-in empirical Bernstein* (Pr-EB) method, discussed in Section 3 of Waudby-Smith and Ramdas (2020b), combines the Robbins' method of mixture (Robbins, 1970) with exponential supermartingales.
(b) The *hedged capital process* (Betting) method, introduced in Section 4 of Waudby-Smith and Ramdas (2020b), is a *novel* approach that enjoys the interpretation of wealth accumulation in a game and has connections to the game-theoretic probability (Shafer & Vovk, 2005).
(c) The *Bootstrap* resampling method is implemented using the R package BOOT (Canty & Ripley, 2022). With $B = 200$ bootstrap replicates and $L = 10$ batches, we calculate a separate confidence interval for data sequence $2^l \leq t < 2^{l+1}$ within each batch $l = 1, \ldots, L$. The confidence intervals are constructed using the $\frac{\alpha}{2L}$- and $(1 - \frac{\alpha}{2L})$-quantiles of the bootstrap means.

2 Numerical studies

Synthetic data example. We sequentially generate data from Beta(10, 30) distribution for $1 \leq t \leq 10^4$, and construct a 95% confidence sequence C_t in equation (1) using methods (a)–(c) in Section 1.

Figure 1 shows that the Betting method outperforms the Bootstrap method with a higher lower bound in the confidence sequence and consistently tighter intervals for $t \geq 1{,}500$. This aligns with the intuition that the Bootstrap method results in wider intervals due to dividing the confidence

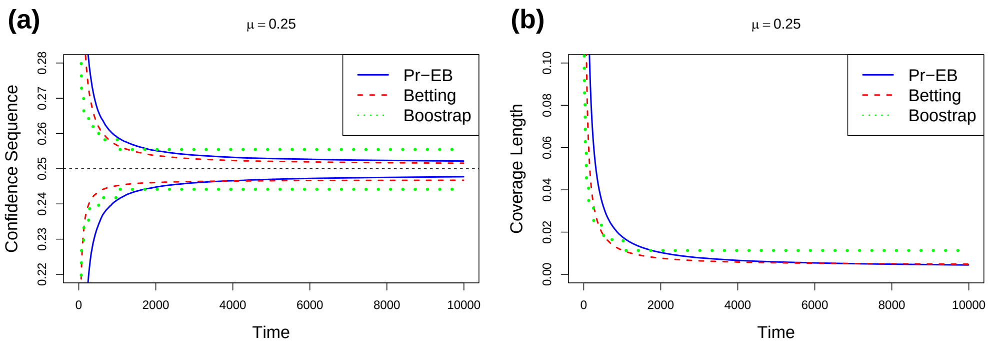

Figure 1. Comparisons based on the synthetic data of Beta distribution. (a) Confidence sequence and (b) coverage length.

budget among data subsets when dealing with a large number of sequential data points. Moreover, the Betting method outperforms the Pr-EB method with consistently tighter confidence sequences for $t \leq 3{,}000$. For $t \geq 3{,}000$, both methods yield comparable coverage lengths. The Betting method's sequence shows a slight shift towards smaller values, reflecting the left-skewed ground truth distribution, while the Pr-EB method produces a symmetric interval around the estimated mean.

Real data example. We analyse a batting dataset of 18 Major League players from the 1970 season, available in Efron and Hastie (2021) or the R package EfronMorris. The goal is to construct a 95% confidence interval for the true batting level of each player based on their first 45 at-bats. The following results are obtained from 100 data replications.

Figure 2a shows the average confidence intervals, where the black circle represents the true batting level of each player. Note that while the Bootstrap method fails to cover the true batting level of player 17, the Betting method successfully includes it. Figure 2b presents the coverage probability of both methods. The Betting method achieves higher coverage probability for the true batting levels of players 4 and 17 compared to the Bootstrap method. These results indicate that the Betting method outperforms the Bootstrap method in constructing accurate confidence intervals for this baseball data.

3 Extensions

The work in Waudby-Smith and Ramdas (2020b) could inspire several directions for future research. One of such directions is the generalisation of the Betting method to handle multidimensional observations. Currently, the Betting method relies on the assumption that the underlying capital process is a martingale. However, when estimating the mean of multidimensional data, applying Ville's inequality (Waudby-Smith & Ramdas, 2020a) for confident rejection of the null hypothesis becomes more challenging. Defining the hedged capital process in a vector space

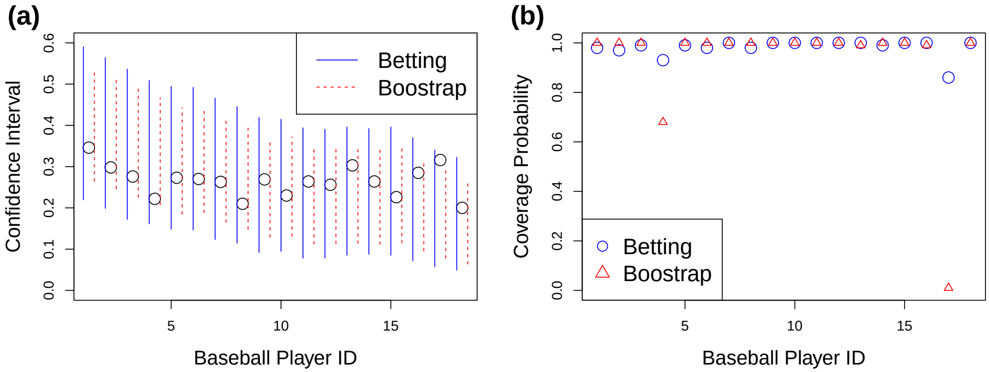

Figure 2. Comparisons based on the baseball batting data from Efron and Hastie (2021). (a) Confidence interval and (b) coverage probability.

introduces complexities when simultaneously estimating multiple attributes. Another direction is extending the Betting method to online decision-making scenarios, such as dynamic treatment, online recommendation, online matching, and dynamic pricing (Dai & Jordan, 2021a, 2021b; Kamenica & Gentzkow, 2011; Mansour et al., 2020). These tasks frequently involve constructing confidence sequences to determine optimal actions. It is of interest to study the applications of the Betting method to these contexts.

Conflict of interest: None to be declared.

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Ryan Martin's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Ryan Martin

Department of Statistics, North Carolina State University, Raleigh, NC, USA

Address for correspondence: Ryan Martin, Department of Statistics, North Carolina State University, 2311 Stinson Drive, 5109 SAS Hall, Raleigh, NC 27695, USA. Email: rgmarti3@ncsu.edu

Congratulations to Waudby-Smith and Ramdas (WSR) for their excellent contribution to the rapidly growing literature on anytime-valid inference. Since some of my work involves imprecise probability, Professor Ramdas asked me privately what, if any, connections there are between e-values, etc. and imprecise probability. The answer to his question might be of general interest, so I'll share it here.

Following WSR, let $(X_t : t \geq 1)$ be a $[0, 1]$-valued process with distribution $P \in \mathcal{P}^\mu$ having unknown mean μ. Write $X^t = (X_1, \ldots, X_t)$ and x^t for a generic realisation. Take $M(X^t, m)$ to be an e-value for testing $H_0^m : \mu = m$. Given x^t, define

$$\gamma_{x^t}(m) = M(x^t, m)^{-1} \wedge 1, \quad m \in [0, 1].$$

This function determines an imprecise probability for uncertainty quantification about μ, one whose upper probability is a sub-additive *possibility measure* given by

$$\overline{\Gamma}_{x^t}(A) = \sup_{m \in A} \gamma_{x^t}(m), \quad A \subseteq [0, 1].$$

$\overline{\Gamma}_{x^t}$ is a coherent upper probability, so it determines a nonempty, closed, and convex credal set $\mathscr{C}(\overline{\Gamma}_{x^t}) = \{Q : Q(\cdot) \leq \overline{\Gamma}_{x^t}(\cdot)\}$ of probability distributions dominated by $\overline{\Gamma}_{x^t}$. The credal set facilitates probabilistic uncertainty quantification about μ: a hypothesis H is discredited if $Q(H)$ is small for all $Q \in \mathscr{C}(\overline{\Gamma}_{x^t})$, i.e. if $\overline{\Gamma}_{x^t}(H)$ is small. These are not Bayesian-style subjective degrees of belief about μ—they're frequentistly justified. Indeed, WSR's confidence sets "C_t" are exactly the γ-level sets: $C_t \equiv \{m : \gamma_{x^t}(m) > \alpha\}$, and there's a uniform validity property (e.g. Cella & Martin, 2023),

$$\sup_{\mu} \sup_{P \in \mathcal{P}^\mu} P\left[\bigcup_{H \subseteq [0,1] : H \ni \mu} \{X^t : \overline{\Gamma}_{X^t}(H) \leq \alpha\}\right] \leq \alpha, \quad \alpha \in [0, 1],$$

that prevents systematic discreditation of *any* true hypothesis about μ.

But there's another possibility measure that's even better than $\overline{\Gamma}_{x^t}$. Define

$$\pi_{x^t}(m) = \sup_{P \in \mathcal{P}^m} P\{M(X^t, m) \geq M(x^t, m)\}, \quad m \in [0, 1]. \tag{1}$$

If M is such that $\sup_{m \in [0,1]} \pi_{x^t}(m) = 1$ for all x^t, then $\overline{\Pi}_{x^t}(A) = \sup_{m \in A} \pi_{x^t}(m)$ is a possibility measure. It can be shown (Martin, 2022a, 2022b) that $\overline{\Pi}_{X^t}$ enjoys the same properties as $\overline{\Gamma}_{X^t}$, but is more efficient, i.e. $\pi_{X^t}(\cdot) \leq \gamma_{X^t}(\cdot)$ and, hence, $\mathscr{C}(\overline{\Pi}_{X^t}) \subseteq \mathscr{C}(\overline{\Gamma}_{X^t})$.

For WSR's nonparametric case, computation of π_{x^t} in (1) is a challenge, so one might opt for the computationally simpler but statistically less efficient upper bound γ_{x^t}. In other cases, however, more can be done. For example, in the context of parametric models, I argued (Martin, 2023) that one can efficiently balance frequentist desiderata with the likelihood principle by excluding those unrealistic and impractically extreme stopping rules, e.g. sample until the test rejects. This exclusion is achieved by suitably shrinking the domain over which an optimisation like in (1) is taken.

Conflict of interest: None declared.

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Hien Nguyen's contribution to the Discussion of "Estimating means of bounded random variables by betting" by Waudby-Smith and Ramdas

Hien Nguyen

Department of Mathematics, School of Mathematics and Physics, University of Queensland, St. Lucia, 4072 Queensland, Australia

Address for correspondence: Hien Nguyen, School of Mathematics and Physics, University of Queensland, St. Lucia, 4072 Queensland, Australia. Email: h.nguyen7@uq.edu.au

I would like to congratulate and thank the authors for their comprehensive work on non-parametric anytime-valid confidence sets (CSs) for the mean of bounded random variables. When performing statistical inference, a general expectation is that parametric procedures, when available for the same task, will typically be more efficient than non-parametric equivalents. To assess this conjecture, I consider the use of the universal inference (UI) procedure from Wasserman et al. (2020) to generate anytime-valid CSs for the mean parameter. That is, suppose that $(X_t)_{t=1}^{\infty}$ is an iid sequence of random variables, where the distribution of each X_t is characterised by the density $f(x; \theta_0, \psi_0)$, with parameters $(\theta_0, \psi_0) \in \mathcal{U} \subset \mathcal{T} \times \mathcal{S}$. If $(\hat{\theta}_t)_{t=1}^{\infty}$ and $(\hat{\psi}_t)_{n=1}^{\infty}$ are predictable sequences of estimators for the parameter of interest θ_0 and the nuisance parameter ψ_0, respectively, with respect to the canonical filtration, then, the profile likelihood form of (Wasserman et al., 2020, Theorem 11) yields, for each $\delta \in (0, 1)$:

$$\inf_{\theta_0 \in \mathcal{T}} P_{\theta_0}\left(\forall t \geq 1{:}\theta_0 \in C_t^{\delta}\right) \geq 1 - \delta, \text{ where}$$

$$C_t^{\delta} = \left\{\theta{:} \frac{\prod_{i=1}^{t} f\left(X_i; \hat{\theta}_i, \hat{\psi}_i\right)}{\sup_{\psi \in \mathcal{S}(\theta)} \prod_{i=1}^{t} f\left(X_i; \theta, \psi\right)} \leq \frac{1}{\delta}\right\}, \text{ with } \mathcal{S}(\theta) = \left\{\psi \in \mathcal{S}{:}(\theta, \psi) \in \mathcal{U}\right\}.$$

When θ_0 is the mean μ of X_t, the CSs $(C_t^{\delta})_{t=1}^{\infty}$ provide an alternative to the constructions from the text.

To make comparisons to authors' approaches, I apply the UI procedure to obtain CSs for the mean of beta distributions with parameters $(\alpha, \beta) = (1, 1)$ and $(10, 40)$. Here, θ_0 and ψ_0 are the

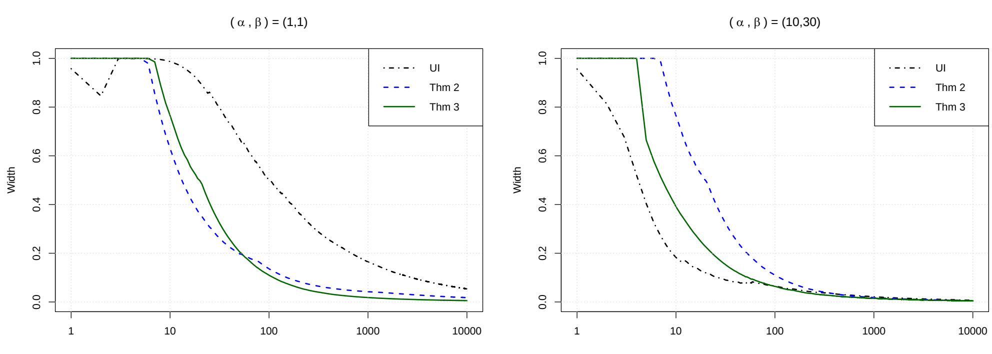

Figure 1. Widths of 95%-CSs for the mean μ.

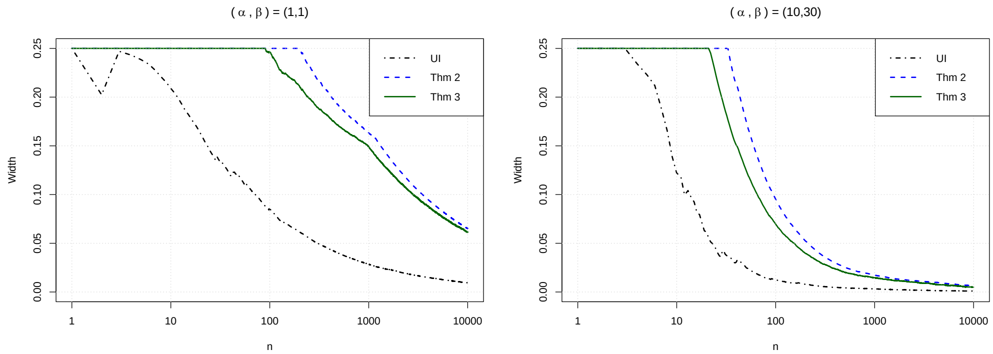

Figure 2. Widths of 95%-CSs for the variance σ^2.

mean $\mu = \alpha/(\alpha+\beta)$ and size $\nu = \alpha+\beta$, respectively. Method of moments estimators are used for $(\hat{\theta}_t)_{t=1}^{\infty}$ and $(\hat{\psi}_t)_{n=1}^{\infty}$. Comparisons with CSs constructed using Theorems 2 and 3 are presented in Figure 1.

Contrary to expectations, the authors' methods appear to provide far more efficient confidence sets (CSs) when $(\alpha, \beta) = (1, 1)$. However, there may be some justification for parametric methods in the low variance and asymmetric case. Nevertheless, the validity of the UI CSs depends on the beta distribution of the data, while the authors' techniques remain non-parametrically robust.

A problem where parametric CSs appear to outperform the authors' approaches is that of constructing CSs for the variance of iid beta random variables $(X_t)_{t=1}^{\infty}$. Here, I apply the same UI procedure, now identifying θ_0 and ψ_0 with the variance σ^2 and mean μ, respectively, where $(\alpha, \beta) = [\mu(1-\mu)/\sigma^2 - 1](\mu, 1-\mu)$. To construct CSs for the variance using the authors' techniques, I observe that if $(Y_s)_{s=1}^{\infty} \sim P$ and $(Z_t)_{t=1}^{\infty} \sim Q$, where $P \sim \mathcal{P}^{\mu_1}$ and $Q \sim \mathcal{P}^{\mu_2}$, and $(A_s^{\delta/2})_{s=1}^{\infty}$ and $(B_t^{\delta/2})_{t=1}^{\infty}$ are anytime-valid $(1-\delta/2)$%-CSs for μ_1 and μ_2, constructed using $(Y_t)_{t=1}^{\infty}$ and $(Z_s)_{s=1}^{\infty}$, respectively, then

$$\inf_{P,Q} \Pr_{P,Q}\left(\forall s, t \geq 1 : \mu_1 \in A_t^{\delta/2}, \mu_2 \in B_s^{\delta/2}\right) \geq 1-\delta,$$

where $\Pr_{P,Q}$ is a probability measure compatible with P and Q. Letting $Y_t = X_t$ and $Z_t = X_t^2$, I use Theorems 2 and 3 to construct CSs for the variance $\sigma^2 = \mu_2 - \mu_1^2$. Figure 2 shows comparisons with the UI CSs. Although less efficient, I must note again that the non-parametric CSs are robust and remain valid for data arising from any distribution having fixed conditional mean and variance.

Conflict of interests: None declared.

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Art Owen's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Art B. Owen

Department of Statistics, Stanford University, Stanford, CA 94305, USA

Address for correspondence: Art B. Owen, Department of Statistics, Stanford University, Sequoia Hall, 390 Jane Stanford Way, Stanford, CA 94305, USA. Email: owen@stanford.edu

I congratulate the authors on some very interesting work making connections between likelihood, betting and martingales.

What caught my eye was the connection to the empirical likelihood (EL) (Owen, 2001) and the dual likelihood (Mykland, 1995). The hindsight optimal λ solves $0 = \sum_{i=1}^{t} (x_i - m)/(1 + \lambda^{\mathrm{T}}(x_i - m)) = 0$ corresponding to the observation weights $w_i \propto 1/(1 + \lambda^{\mathrm{T}}(x_i - m))$ used in EL. Two common alternatives use weights $w_i \propto (1 - \lambda^{\mathrm{T}}(x_i - m))$ and $w_i \propto \exp(-\lambda^{\mathrm{T}}(x_i - m))$ arising from L_2 and entropy criteria, respectively (for different vectors λ). The entropy weights connect to exponential tilting and logistic regression; see, for instance, Hainmueller (2012).

The EL weights perform a kind of reciprocal tilting that gives them some special power properties. Kitamura (2003) shows that EL tests cannot be dominated by other regular tests for moment restrictions, in a large deviations sense. His result is a nonparametric counterpart to the likelihood test optimality result of Hoeffding (1965) for multinomial distributions. Lazar and Mykland (1998) find that for true parametric models, EL matches their power to second order and at third order either the empirical or the parametric tests could have greater power. In some overspecified moment models, EL has such high power for detecting lack of fit that, under lack of fit, there cannot exist any pseudo-true value of the parameter for which the maximum EL estimate is root-n consistent (Schennach, 2007). We can now add the present authors' hindsight optimality to this list of power properties.

A similar power optimality is achieved by the confidence bands of Berk and Jones (1979). They use the most significant binomial likelihood ratio from all n order statistics to form confidence bands with greater power than any weighted Kolmogorov–Smirnov test. It would be very interesting to see if the authors' methods could produce an always valid version of the Berk–Jones bands.

Conflict of interests: None declared.

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David Siegmund's contribution to the Discussion of "Estimating means of bounded random variables by betting" by Waudby-Smith and Ramdas

David Siegmund

Department of Statistics, Stanford University, Stanford, CA, USA

Address for correspondence: David Siegmund, Department of Statistics, Stanford University, 390 Jane Stanford Way, Stanford, CA 94305, USA. Email: siegmund@stanford.edu

This contribution by Waudby-Smith and Ramdas provides a host of new insights combined with a concise, yet comprehensive, historical review of "any time valid inference", which has blossomed in recent years after the hiatus that followed its beginnings in the mid-20th Century. Particularly, attractive is the authors' systematic use of martingales of the form (29) to facilitate the betting interpretation and give unified derivations of a large diversity of inequalities.

Given the success of this paper in studying the mean of a distribution, one might ask how to estimate the entire distribution. A special case of Kelly's gambling theory involves a histogram having N cells with probabilities $p_1, \ldots, p_N$, which pays a constant amount N when an observation falls into the nth cell for $n = 1, \ldots, N$. The gambler's expected log return is maximised by repeatedly betting for each n a fraction p_n of the gambler's current fortune on the nth cell; and the procedure can be effectively simulated by a statistician who does not know the cell probabilities, but judiciously estimates them sequentially (cf. Robbins & Siegmund, 1974). Can one schedule a sequence of refinements of the histograms to approximate a continuous probability density function and estimate the density efficiently?

A second exercise is to give a sequence of anytime valid confidence bands for a distribution function. A hint in this direction occurs in Siegmund (1988), but one might want to use an estimator that pays more attention to the tails of the distribution.

The subject of anytime valid inference disappeared from the literature for 40-some years after an initial burst of theoretical activity, presumably because applications were lacking. The huge amount of data available in some internet experiments changes that situation, but a genuine scientific application would, in my view, be an important contribution. My favourite candidate is sequential clinical trials, where experimental expense and government regulations usually mandate tests having a fixed horizon. Is there evidence that the flexibility of anytime valid stopping rules to deal with changed horizons outweighs their disadvantages? (I found disappointing what I was able to show 40 years ago.)

Conflict of interest: None declared.

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Philip B. Stark's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Philip B. Stark

Department of Statistics, University of California, Berkeley, CA, USA
Address for correspondence: Philip B. Stark, University of California, Berkeley, CA, USA. Email: pbstark@berkeley.edu

This paper is a gem, and I predict it will have lasting impact on theoretical and applied statistics.

Drawing on intellectual heritage spanning more than 80 years, illustrating and leveraging deep connections among results in probability, finance, information theory, computer science, and statistics, Waudby-Smith and Ramdas present mature, intuitive, flexible, computable, and powerful methods for two fundamental (indeed, canonical) nonparametric inference problems: conservative confidence bounds for the expected value of a bounded random variable from IID observations and conservative confidence bounds for the mean of a finite, bounded population from a simple random sample. Sequentially valid methods have been proposed for those problems and the related problem of conservative tests for nonnegative or bounded means [including some based on Ville's (1939) inequality, explicitly or through Wald's (1945) sequential probability ratio test: Howard et al., 2021; Kaplan, 1987, 2012; Stark, 2009, 2020, 2023; Stark & Teague, 2014], but their performance varies and their derivations provide little insight into how to sharpen the methods. In contrast, the betting martingale framework leads naturally to techniques that approximately optimise power, are computationally tractable, perform well 'out of the box' for a broad range of bounded populations and distributions, and can incorporate prior information without compromising coverage. Moreover, the sequential validity makes them 'safe' against common practices that invalidate other approaches (optional stopping and optional continuation) (Grünwald et al., 2023; Ramdas et al., 2021; Shafer, 2021).

It would be interesting to compare these new methods to the approach of Orabona and Jun (2021) based on regret bounds for a universal portfolio and to the conjectured bound of Gaffke (2005) and Learned-Miller and Thomas (2019) for fixed sample size, even though the latter bound has not been proved to be valid in general. Accommodating Bernoulli sampling is trivial (see, e.g. Ottoboni et al., 2020; Stark, 2023), but tied to the question of derandomisation, especially for fixed sample size. [Kaplan (1987) does this in a crude way using the union bound; Waudby-Smith and Ramdas give a number of sharper approaches, and work in progress makes further improvements; Ramdas & Manole, 2023.] Accommodating stratified sampling using union–of–intersection tests is straightforward (Spertus et al., 2023; Spertus & Stark, 2022; Stark, 2023) because independent E-values can be combined by multiplication or averaging (Vovk & Wang, 2021); alternatively, the E-values can be converted to P-values and then combined. Optimising union–of–intersection tests based on test martingales yields an interesting generalisation of the multi-arm bandit (MAB) problem, a 'gang of bandits', where N MABs are probed in parallel: at each stage, the statistician chooses an arm, which pulls the same arm of all N MABs, as if they were 'ganged' like switches. The statistician has a fortune for each MAB and bets separately on each, but seeks to maximise their minimum fortune across the bandits (Stark, 2023) by choosing the sequence of arms and bets. Improved methods for stratified inference have immediate applications ranging from election audits (Ottoboni et al., 2020; Spertus & Stark, 2022; Stark, 2008, 2009, 2020, 2023; Stark & Teague, 2014; Waudby-Smith et al., 2021) and financial audits to randomised experiments with blocking.

In closing, I highly recommend appendix F for a brief history of connections among gambling, portfolio theory, martingales, concentration inequalities, and hypothesis testing.

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Philip S Thomas, Erik Learned-Miller and My Phan's contribution to the Discussion of 'Estimating means of bounded random variables by betting' by Waudby-Smith and Ramdas

Philip S. Thomas, Erik Learned-Miller and My Phan

Manning College of Information and Computer Sciences, University of Massachusetts, MA, USA

Address for correspondence: Philip S. Thomas, Manning College of Information and Computer Sciences, University of Massachusetts, 140 Governors Dr., Amherst, MA 01003-9264, USA. Email: pthomas@cs.umass.edu

We enthusiastically agree about the importance of developing tighter confidence intervals and sequences for the mean—the topic of this paper. There are increasing concerns about the safety and fairness of machine learning systems, and we believe one powerful tool for ensuring that machine learning systems are safe, fair, and otherwise reliable is for them to provide guarantees in the form of their confidence that the models they produce will satisfy certain safety or fairness guarantees. At the core of such methods are mechanisms for producing confidence intervals on parameters related to fairness and safety. However, often these confidence intervals must be constructed from samples from extreme distributions for which normality assumptions remain unreasonable even with relatively large numbers of samples. In such cases, confidence intervals like those proposed in this paper are critical, because they do not rely on normality assumptions yet remain relatively tight.

We appreciate the comparison of the Hedged-CI method to our own recent confidence interval for the mean ('PTL21' in the paper), and find the further increase in tightness compelling, as it could result in improved data-efficiency of safe and fair machine learning methods. Furthermore, we find the extension of the proposed methods to the confidence sequence setting compelling, as it could allow for the re-testing of the safety or fairness of machine learning models as additional data becomes available.

One additional comparison that would be interesting to see would be to the confidence interval proposed by Gaffke (2005). (Among the three statistics presented by Gaffke, we are referring to statistic 'K'.) Showing that this confidence interval has (or does not have) guaranteed coverage remains an open question. Nevertheless, given the remarkable tightness of Gaffke's bound and the substantial effort that has gone into finding distributions for which it does not exhibit coverage (with no success), it seems like a promising direction for future analysis.

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Vladimir Vovk's contribution to the Discussion of "Estimating means of bounded random variables by betting" by Waudby-Smith and Ramdas

Vladimir Vovk

Department of Computer Science, Royal Holloway, University of London, Egham, Surrey, UK

Address for correspondence: Vladimir Vovk, Department of Computer Science, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, UK. Email: v.vovk@rhul.ac.uk

Let me start by congratulating the authors on an excellent paper clearly explaining the method of mixtures and proposing its novel uses. They trace this method back to Ville and Wald and state it (in Section 3.1) in the form 'a mixture of test supermartingales is a test supermartingale'. But it can be regarded as a version of an older and equally important method of mixtures: a mixture of probability measures is a probability measure. In simple finite-horizon probability spaces, there is a one-to-one correspondence between probability measures Q and test supermartingales M: a test supermartingale M is the likelihood ratio of a probability measure Q to the underlying probability measure.

In more complex cases, we no longer have a one-to-one correspondence, and the two versions are distinct albeit closely related. The mixture method for probability measures is the basis of Bayesian statistics: a statistical model $\{P_\theta \mid \theta \in \Theta\}$ together with a prior distribution μ gives us the Bayes mixture $\int P_\theta \mu(d\theta)$, a single probability measure explaining the data. In some cases, we even start from the Bayes mixture, as in Bayesian modelling (Bernardo & Smith, 2000, Chapter 4) based on de Finetti's theorem.

In game-theoretic probability (Shafer & Vovk, 2019), the mixture method for probability measures loses its importance, as we avoid using all-encompassing probability measures and replace them by more modest kinds of statistical modelling. The mixture method for test supermartingales, however, remains fundamental.

The mixture method for probability measures was also an important source for online learning as described in the last subsection of Section 6. Sequential prediction with the logarithmic loss is only part of online learning. The loss function can be, for example, quadratic, which provides us with methods for performing regression free of any statistical assumptions (Vovk, 2001, Section 3). The case of the logarithmic loss was treated in the pioneering paper by DeSantis et al. (1988), who simply applied the Bayes mixture to the problem of prediction establishing worst-case guarantees for it. Another simple case, classification with the 0-1-loss, was considered independently in Littlestone and Warmuth (1994) and Vovk (1992, Theorem 5), who both came up with the Weighted Majority Algorithm (in Littlestone and Warmuth's terminology). The first generic mixture algorithm covering the Bayes mixture, the Weighted Majority Algorithm, and Cover's universal portfolio was the Aggregating Algorithm (Vovk, 1990), which can be applied not only to classical regression, as in Vovk (2001, Section 3), but also to a wide range of other prediction problems (Adamskiy et al., 2019; Kalnishkan, 2022; Kalnishkan & Vyugin, 2008).

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The authors replied later in writing as follows.

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Authors' reply to the Discussion of 'Estimating means of bounded random variables by betting'

Ian Waudby-Smith and Aaditya Ramdas

Department of Statistics and Data Science, Carnegie Mellon University, Pittsburgh, PA, USA

Address for correspondence: Aaditya Ramdas, Department of Statistics and Data Science, Carnegie Mellon University, Baker 132H, Pittsburgh, PA 15213, USA. Email: aramdas@stat.cmu.edu

We are grateful to each of the discussants for their rich and insightful discussions of our paper. We were thrilled to see that our work has piqued their interest, and it was a delight reading through discussions written by colleagues and friends, old and new. More broadly, we are delighted to see the widespread interest in betting, concentration inequalities, and anytime-valid inference from experts in a wide variety of fields including probability, mathematical statistics, machine learning, and online learning. The eclectic range of discussants aptly represents the diverse set of fields in which modern betting tools have their roots. In the paragraphs to follow, we will reply to discussants in the order that we received their discussions, saving our more detailed replies to the votes of thanks (of Peter Grünwald and Gergely Neu) for the end.

We are in particular honoured that Anastasios Angelopoulos and colleagues have started using the name 'WSR bounds' to refer to the betting-based CIs we developed. We also thank Angelopoulos for the striking illustration of how our bounds can improve on existing methods when used within the *Risk Controlling Prediction Set* (RCPS) framework applied to tumor segmentation (Bates et al., 2021). More broadly, we view the RCPS and *Prediction Powered Inference* frameworks of Bates et al. (2021) and Angelopoulos et al. (2023) as much-needed tools in the assumption-lean statistical inference toolbox, and we are delighted that our bounds have helped sharpen parts of these tools.

Art Owen makes some thoughtful observations relating the hindsight-optimal bets (optimal values of λ) to empirical likelihood (EL) weights, building on our observations in Appendix E. In particular, Owen notes that alternative weights are also sometimes used in empirical likelihood, for example $\lambda := \exp\{-\lambda^T(x_i - m)\}$ is motivated by entropy criteria. We note that such a formula happens to correspond exactly with the first Taylor expansion in the derivation of our 'approximate GRAPA' strategy, leading to perhaps another connection between betting and empirical likelihood that we were not aware of earlier. Owen also mentions interesting power and optimality guarantees that EL inherits from these weights. While we suspect that this may translate to a certain optimal power guarantee for the GRAPA betting strategy, it is not yet obvious to us what precise statements can be made.

David Siegmund raises an interesting open question about sequentially estimating the density of a distribution using histogram binning. This task is only possible if some amount of smoothness is imposed (e.g. if the density were assumed to be Lipschitz continuous with a known constant). There may be hope to perform data-dependent binning (without sample-splitting) using the quantile-conditional histogram binning techniques (Gupta & Ramdas, 2021) that arose in the calibration literature, but it is indeed an open problem to sequentialise such a construction, for example progressively adding points to bins and then adaptively splitting bins with too many points. Siegmund also mentions applications to cumulative distribution functions (CDFs). Both Howard and Ramdas (2022) and Manole and Ramdas (2023) provide time- and quantile-uniform confidence bands for the CDF, without any restrictions on the underlying distribution. Notably, Howard et al. (2021, Theorem 5) attain widths that depend on the sample size *and* on the quantile being estimated, as Siegmund alludes to. These bounds may also be useful for Owen's open problem connected to the Berk–Jones confidence bands (Berk & Jones, 1979).

While we fully agree with Siegmund that methods often used in sequential clinical trials (e.g. group-sequential methods and repeated confidence intervals Jennison & Turnbull, 1984, 1989) can be tighter than anytime-valid CSs when applied to problems with fixed horizons, we view these two paradigms as complementary and targeting somewhat different sequential settings. Firstly, as Howard also brings up in his discussion, group-sequential methods are typically justified via asymptotic normality while much of the anytime-valid literature is focused on non-asymptotic guarantees. This is sometimes a matter of philosophical preference, but there are cases where non-asymptotics are quite important to the problem at hand. For example, in a post-election risk-limiting audit (RLA) (Stark, 2008), paper ballots are examined sequentially (often without replacement) from a stack to ensure that an erroneously announced winning candidate will be corrected with probability at least $1-\alpha$—here, α is the 'risk limit'. To save time and resources, it is desirable to stop an RLA as soon as possible, and this can sometimes occur at unexpectedly small sample sizes, making it difficult to place much trust in asymptotics. We have already seen martingale-based methods and CSs have important applications for RLAs (see Spertus & Stark, 2022; Stark, 2020; Waudby-Smith, Stark, et al., 2021 and the papers of Philip Stark and colleagues more generally).

However, even when resorting to asymptotics, anytime-valid procedures such as 'asymptotic confidence sequences' (Waudby-Smith, Arbour, et al., 2021) may be desirable since they do away with needing to pre-specify maximum sample sizes and/or interim analysis times. While such pre-specifications may fit naturally in settings with strong governmental regulations, we still find it worthwhile to develop new methods for other applications that allow for more flexibility.

We thank Hien Nguyen for the comparison to universal inference (UI) in the parametric setting of mean estimation for Beta random variables. Similar to Nguyen, we were somewhat surprised to see that betting CSs managed to outperform UI in some but not all cases, despite the latter directly exploiting the parametric structure of the problem. One explanation for this phenomenon may be that UI constructs a two-dimensional confidence set for the (α, β) parameters of a Beta distribution, from which an implied one-dimensional set for the mean can be deduced, while our martingale-based confidence sets target the mean directly. Nevertheless, it is interesting to see how these methods compare in practice, and we thank Nguyen for running these simulations.

Li, Li, and Dai make an empirical comparison between our confidence sequences to a sequence of L geometrically spaced bootstrap CIs, perhaps as a sanity check to compare our method's performance against a simple competitor. Specifically, they compute one bootstrap $(1-\alpha/L)$-CI C_N^B for each sample size $N \in \{2^l : l = 1, \ldots, L\}$, and set $C_n^B \leftarrow C_N^B$ for all $N \leqslant n < 2N$. They find that betting-based confidence sets are sometimes tighter and sometimes looser than these sequential bootstrap CIs, but the latter become very loose for large sample sizes. It is important to also keep in mind that even for a fixed sample size n, bootstrap CIs are only valid *asymptotically* (i.e. as $n \to \infty$) while we only focused on non-asymptotic confidence sets that enjoy guarantees of the form $\mathbb{P}(\mu \in \dot{C}_n) \geqslant 1-\alpha$ or $\mathbb{P}(\forall t, \mu \in \bar{C}_t) \geqslant 1-\alpha$ for CIs $\dot{C}_n$ and CSs $(\bar{C}_t)_{t=1}^{\infty}$, respectively. While we focused on non-asymptotics in this paper, there do exist CSs that trade non-asymptotic validity for tightness and versatility—for example, Waudby-Smith, Arbour, et al. (2021) derive the so-called 'asymptotic confidence sequences' that only make finite moment assumptions similar to the central limit theorem, and unlike the geometrically spaced bootstrap CIs, these remain tight in large samples (and indeed shrink to a zero width in almost all cases).

We thank Martin Larsson and Johannes Ruf for their crisp summary of the core of our methods using intuitive financial terms. Larsson and Ruf also show how the diversified Kelly process $(K_t^{\mathrm{dKelly}})_{t=1}^{\infty}$—which is itself an average of several capital processes obtained via individual betting strategies—has an alternative representation as a *single* process with a capital-reweighted betting strategy. The mentioned connections to the portfolio theory of Cover (1991) are illuminating, and it would be interesting to explore similar representations of other betting strategies and combinations thereof.

We are grateful to Philip Stark for his kind words as well as his excellent historical overview of the area. Stark mentions that it may be of interest to compare our bounds to Phan et al. (2021) and indeed we have a simulation comparing to it in the supplement (Waudby-Smith & Ramdas, 2023, Figure 16) where we find that our methods produces shorter intervals than theirs across sample sizes and different simulations (but their bound is quite competitive for low-variance continuous distributions). Further, their method requires an i.i.d. assumption, is much more computationally expensive than ours, and it does not have a time-uniform nor without-replacement analogue. Nevertheless, they use an interesting and rather different proof technique that deserves further exploration. We are also pleased to see how (Spertus & Stark, 2022) have used properties of martingales and e-values to derive union-intersection tests with stratified sampling.

Rong Jiang and Keming Yu discuss the choices of λ in Hoeffding and our Predictable Plug-in empirical Bernstein (PrPl-EB) CIs, namely noting that the former involve a single data-independent choice while the latter involve several data-dependent tuning-parameters. While it is true that Hoeffding's CI has a single data-independent tuning parameter, this is only because it *cannot* adapt to features of the distribution other than the bounds on the support—indeed, one could have derived a predictable plug-in Hoeffding CI with several tuning parameters, but the Hoeffding supermartingale would not benefit from this. Our empirical Bernstein CIs on the other hand, are built from a different process that can adapt to the unknown variance σ^2 using a value of λ that depends on σ^2. Since σ^2 is unknown to us, we estimate it in a careful way—i.e. through predictable plugins—using the fact that the empirical Bernstein process nevertheless forms a non-negative supermartingale, yielding non-asymptotic validity for any sample size n (including for any $n < 100$ as queried by Jiang and Yu). The choice of $c \in (0, 1)$ has some effect on finite sample tightness (but not on validity), but one can inspect $\lambda_t^{\mathrm{PrPl-EB}}$ and see that for sufficiently large t, the value of c will be of no importance. This is additionally illustrated by the fact that in the i.i.d. case, our PrPl-EB CIs attain an asymptotic width of $2\sigma\sqrt{2\log(2/\alpha)/n}$ a.s. regardless of how c is chosen.

Thus $c = 1/2$ is simply a default practical suggestion, and it is unlikely that changing c will be able to improve much on this default.

While Jiang and Yu correctly point out that H. Wang and Ramdas (2023) derive CSs for the mean under weaker assumptions than us (such as a known $(1 + \delta)$th moment), it is important to keep in mind that their bounds are not adaptive to the unknown variance (and provably cannot be without further assumptions), while that was a focal point of our work. Moreover, their use of the PrPl technique in the unbounded case was inspired by our work.

We thank Ruodu Wang for his kind words and for the interesting discussion of admissibility. We fully agree with Wang that this *e*-testing approach is an incredibly versatile (model-free) tool that can be used to derive tests and confidence sets for other functionals, and as discussed in a previous paragraph, we are quite enthusiastic about his clean and elegant application to risk measures among other quantities (Q. Wang et al., 2022). This is also directly relevant to Jiang and Yu's interest in studying other functionals such as the mode, expectile, and extremile, as is the work on elicitable functionals by Casgrain et al. (2022), and the earlier mentioned works of Howard and Ramdas (2022) and Manole and Ramdas (2023) for quantiles and CDFs.

Wang brings up a sharp point about some betting strategies using features of the distribution—such as the mean, variance, etc.—to arrive at $(\lambda_t)_{t=1}^{\infty}$, and that this may be tailored towards settings with some degree of stationarity. We have two responses. (1) Firstly, some betting strategies such as an average of wealths from constant strategies (e.g. hgKelly) are agnostic to any distributional features and yet seem to do well in practice. (2) Second and finally, inherent in the problem statement itself is a 'stationarity' condition, i.e. the fact that $\mathbb{E}(X_t \mid X_1, \ldots, X_{t-1}) = \mu$ for some *fixed and unchanging* μ. There are other works that derive CSs for functionals like the *running average mean so far* $\widetilde{\mu}_t := \frac{1}{t}\sum_{i=1}^{t}\mu_i$, where $\mu_i = \mathbb{E}[X_i \mid X_1, \ldots, X_{i-1}]$; see Howard et al. (2021), Waudby-Smith, Arbour, et al. (2021), and Waudby-Smith et al. (2022).

We thank Steve Howard—whose own prior work on time-uniform confidence sequences forms an important foundation on which we build—for his kind words. We echo his desire to see more collaborations between the fields of statistics, information theory, online learning, and finance; our other works have already benefited from such collaborations. Howard raises three important questions regarding vector-valued functional estimation, comparisons to CLT-based CIs, and non-stationarity.

First, notice that it is in principle straightforward to extend our martingales to vector-valued observations and parameters by writing $\mathcal{K}_t(\mu) := \prod_{i=1}^{t}(1 + \lambda_i^{\top}(\mathbf{X}_i - \mu))$ where $\lambda_i, \mathbf{X}_i, \mu \in \mathbb{R}^d$. While this may enable computationally efficient *testing* procedures, computing the set $\{m \in [0, 1]^d : \mathcal{K}_t(m) < 1/\alpha\}$ appears expensive, and we do not know of neat shortcuts. Some progress towards efficient variance-adaptive confidence sets in higher dimensions has recently been made in Whitehouse et al. (2023), where the authors derive multivariate generalisations of the empirical-Bernstein technique of Howard et al. (2021), amongst other self-normalised time-uniform bounds.

As for comparisons to CLT-based CIs, we are in agreement that non-asymptotics are sometimes a hard sell, but whether or not this ought to be the case is a separate philosophical matter altogether. Nevertheless, one can derive non-asymptotic CIs with CLT-like asymptotic performance via Berry–Esseen bounds (assuming an upper bound on the third moment, which is the case in our [0, 1]-bounded setting) and a recent preprint by Austern and Mackey (2022) seems to have taken this further. We share Howard's enthusiasm for these types of results.

Finally, Howard asks whether there are counterexamples showing that betting-based CSs break down in non-stationary settings where (conditional) means vary over time. Theoretically, Waudby-Smith and Ramdas (2023, Prop. 2) tells us that our capital processes being test martingales is *equivalent* to the observations being bounded with conditional mean μ, and hence this proposition breaks down in the face of time-varying means. For an empirical illustration, Waudby-Smith et al. (2022, Figure 4) shows precisely this phenomenon, where an empirical Bernstein-style CS (inspired by Howard et al., 2021 but for bandit problems) manages to cover the running mean while a betting-based CS fails to. When means *are* stationary however, the same betting-based CSs are substantially tighter in practice.

We share the viewpoints of Philip Thomas, Erik Learned-Miller, and My Phan that assumption-light confidence intervals will form important building blocks for downstream machine learning applications including safety and fairness. Thomas et al. mention a conjectured (with-replacement) confidence interval due to Gaffke (2005) whose proof has eluded the community for years (see also Learned-Miller & Thomas, 2019). We are also in agreement regarding the impressive empirical performance of these bounds, and we believe that a proof of its correctness would be an important step forward.

We would like to thank Volodya Vovk not only for his discussion of our paper, but also for laying many of the foundations in game-theoretic probability and statistics on which our work builds. Vovk provides historical context surrounding the method of mixtures, drawing connections to fundamental results in Bayesian statistics, measure- and game-theoretic probability, Thomas Cover's universal portfolio, online learning, and his (now-canonical) aggregating algorithm (Vovk, 1990).

Ryan Martin brings up a fascinating idea of improving our confidence intervals (and maybe even sequences?) using tools from imprecise probability such as credal sets and possibility measures. Whether they are practically implementable for non-parametric problems like ours remains to be seen, but it is certainly an intriguing possibility (pun intended).

Anthony Davison and Igor Rodionov explore a new (and unanticipated to us) application of our Proposition 3 to fitting a generalised Pareto distribution. We were initially unfamiliar with this application, and thus we enjoyed reading about their explorations. We hope that their ideas lead to something fruitful in theory or in practice (or both).

We now finally turn to the votes of thanks by Peter Grünwald and Gergely Neu. Let us start with Grünwald whose comments on the i.i.d. setting are particularly interesting. We begin by thanking him for his generous words at the beginning of the discussion. We share his enthusiasm for applications of betting and confidence sequences to PAC-Bayesian theory, and indeed recent work of Chugg et al. (2023) and concurrent work by Jang et al. (2023) explores such connections. The latter focuses on the bounded setting using betting-type concentration arguments, and the former giving a rather general treatment considering both bounded and unbounded settings. We enjoy Grünwald's neat analogy to quantum mechanics and are in full agreement that classical fixed-n procedures are not only invalidated by early stopping, but also whether early stopping *may have occurred* had the observed data been different, but did not actually occur. In light of the emphasised importance of pre-specified stopping criteria, confidence sequences are particularly attractive due to their validity under *arbitrary* stopping rules.

Grünwald also makes some thoughtful comments on the difficulties of Bayesian inference in non-parametric problems. Indeed, for a setting as 'simple' as estimating means of bounded random variables, a Bayesian approach would proceed by choosing an infinite-dimensional prior distribution that appears difficult to construct because it must encompass all possible continuous and discrete distributions with any support within [0, 1] (and their mixtures). In contrast, non-asymptotic frequentist guarantees are enjoyed by our simple and flexible betting martingales (among other e-value-based confidence intervals). This phenomenon was described crisply in a quote by Wasserman (2007):

> 'The idea that statistical problems do not have to be solved as one coherent whole is anathema to Bayesians but is liberating for frequentists. To estimate a quantile, an honest Bayesian needs to put a prior on the space of all distributions and then find the marginal posterior. The frequentist need not care about the rest of the distribution and can focus on much simpler tasks'.

Grünwald's comments regarding optimality of GRO, REGROW, and KLinf are extremely prescient: in a recent preprint, Shekhar and Ramdas (2023) show (via lower bounds and near-matching upper bounds) that several betting-based confidence intervals and sequences of the present paper are near-optimal in the sense of an 'effective width', in which the KLinf is central.

Finally, to the seconder of the vote of thanks. We are grateful to Gergely Neu for his kind words about our paper and we share his keen interest in looking forward to the vistas the betting framework may (continue to) open. We share his enthusiasm for the follow-up work of Orabona and Jun (2021)—which applies only to the time-uniform, with-replacement setting, and not the other three in our paper—although we are quite puzzled by Neu's repeated use of the word 'concurrent' to describe a paper that appeared more than a year after ours. We remark that an earlier paper by these authors (Jun & Orabona, 2019) did not make explicit connections to testing nor to confidence sequences, nor did it use the wealth processes directly as we do. Furthermore, it is absent of citations to Ville, the sequential testing and estimation literature led by Wald, Robbins, Lai and Siegmund, and the game theoretic probability literature pioneered by Shafer and Vovk. In sum, their 2021 arXiv preprint was their first work which explicitly derives time-uniform with-replacement CSs that mirror the form appearing in our paper. (This is not meant to downplay their contributions in any way, and we are excited about their new advances. We simply wanted to add necessary context to the discussion of concurrency.)

Neu sees three limitations in our paper, and while we certainly agree with some of them in spirit, we will highlight some caveats in detail below.

1. Neu suggests that we could have used betting strategies with more formal guarantees on their growth rate, ultimately writing 'it remains unclear what betting algorithm one should use to, say, minimise the confidence width'. (As an initial point of clarification, Neu claims that we 'criticize the use of principled betting strategies' but we never intended to criticise them and are completely fine with using any strategy that works well for our statistical ends, and indeed point out that any strategy does result in a valid confidence sequence.)

   Nevertheless, we agree that it is not always obvious which betting strategy one should use in every scenario. However, despite our own use of such principles, there is no general theorem according to which maximising the growth rate of a generic test supermartingale guarantees sharper widths of the derived confidence sets. For an explicit counterexample, one need only look to the 1-sub-Gaussian supermartingale whose resulting (fixed-n) CI width is minimised a.s. by a constant betting strategy $\lambda := \sqrt{2\log(1/\alpha)/n}$ which itself does not maximise the growth rate (indeed, it depends on the type-I error α and not on the true mean μ). For this reason, many of our strategies were derived with tightness of width in mind, and not always the growth rate. For example, our Predictable Plug-in Empirical Bernstein confidence intervals attain an asymptotic width whose first-order term of $2\sigma\sqrt{2\log(2/\alpha)/n}$ matches that of Bernstein confidence intervals *exactly* including constants (Waudby-Smith & Ramdas, 2023, Equation 17), and this was the only fully empirical confidence interval that we were aware of with such a property proved about it. After substantial theoretical analysis, recent work of Shekhar and Ramdas (2023) have also shown that our betting confidence intervals with the same (not necessarily growth-rate-optimal!) strategy are at least as tight. It is not known whether the same guarantee holds when using growth-rate-optimal strategies in the derivation of fixed-n confidence intervals. To summarise, confidence intervals and sequences should use different betting strategies, and in general the ones optimising width (especially for intervals) may not be the ones maximising the growth rate.

   Neu points to Orabona and Jun (2021) to highlight betting strategies that have growth-rate guarantees, but only one of their three algorithms has an asymptotic width guarantee (and it is empirically the loosest of the ones they propose for moderate sample sizes). However, the aforementioned work of Shekhar and Ramdas (2023) now shows that a wide class of capital-weighted mixture strategies enjoy certain 'effective width'-optimality properties, drawing a concrete connection between the growth rate of the wealth and the appearance of the KLinf in the width, as alluded to in our reply to Grünwald.
2. Quoting Neu, 'The confidence intervals depend on the order in which the data is presented to the algorithm'. Neu again points to Orabona and Jun (2021) as having examples of permutation-invariant confidence sequences but as written, their Algorithms 1 and 2 do depend on the order of the data since they take the running intersection of the confidence sequence at each step. Slightly modifying their algorithms to not take running intersections, however, their bounds would then indeed be permutation-invariant and retain their satisfying growth-rate guarantees. We quickly clarify that not all of our bounds depend on the order of the data: one confidence sequence that can be found in several plots is that resulting from the 'hgKelly' strategy found in Waudby-Smith and Ramdas (2023, Section B.6) which is permutation-invariant and does not depend on the order of the data (as long as the running intersection is not taken).

   Neu feels that our derandomisation scheme for CIs—averaging the final wealth on permutations of the data—is somewhat conceptually unsatisfying. We agree, and may rather opt for a betting strategy that is permutation-invariant from the start if that is an important aesthetic criterion.

However, while Neu finds non-permutation-invariant procedures 'hard to justify when the data is i.i.d.', we are much more comfortable with this. That is, we do not view the asymmetry of the algorithm as a no-go, if it also comes with upsides. For example, in a completely different context, modern semiparametric causal inference is largely possible due to sample splitting, so that the resulting estimates and intervals do not treat the data symmetrically (Kennedy, 2022). The same can be said about the split likelihood ratio test of Wasserman et al. (2020), which can be made symmetric in the same way as ours by averaging over different permutations, and even of ubiquitous algorithms such as cross-validation (except for leave-one-out). In many of these cases, some sort of concentration of measure kicks in to ensure that the practical results differ little across permutations of the data; we have found this to be true in our experiments as well, so that the mathematical dependence on the order of the data appears to be practically inconsequential. Going further, Ramdas and Manole (2023) show how external randomisation or asymmetric methods can yield strictly more powerful procedures. Thus, we do not see asymmetry as something to be avoided at all costs, and in some cases, it could even be embraced if it enables powerful procedures (like in the latter paper), and especially if the asymmetry has little practical effect.

3. Finally, Neu highlights that the present paper was focused on bounded random variables. While we agree that the bounded case was our primary focus, it is certainly not true that '[t]he method is limited to random variables bounded almost surely'. In particular, boundedness is not a requirement needed to exploit the techniques, intuitions, or formalities of betting. We were merely engrossed by how many interesting avenues there were to explore within this one classical problem of estimating means of bounded random variables. As we mention in Waudby-Smith and Ramdas (2023, Proposition 3), however, we provide a universal representation of *any* test (super)martingale, and we explicitly mention that this applies to possibly unbounded distributions. That is, Proposition 3 states that a process $(M_t)_{t=0}^{\infty}$ with $M_0 = 1$ forms a test (super)martingale *if and only if* it is given by

$$M_t = \prod_{i=1}^{t} (1 + \lambda_i Z_i) \tag{1}$$

for some $Z_i \geqslant -1$ so that $\mathbb{E}_P(Z_i \mid \mathcal{F}_{i-1}) = (\leqslant)\ 0$ and some predictable $(\lambda_t)_{t=1}^{\infty}$ so that $\lambda_i Z_i \geqslant -1$. One can immediately see ideas from betting being applicable to the choice of $(\lambda_t)_{t=1}^{\infty}$. The martingales we developed for mean estimation are just one particular instance of (1) with quantile estimation for (unbounded) distributions discussed briefly in Waudby-Smith and Ramdas (2023, Section 7). More generally, the prior works of Robbins and Siegmund (1974), Robbins (1970), and Howard et al. (2021) (among others) used ideas from betting implicitly, all in potentially unbounded settings, and all falling under the general representation of (1).

It is hopefully clear that we found the comments of Grünwald and Neu—and indeed all the wonderful discussants—very stimulating, and we hope our responses add nuance to the discussion. As has been demonstrated by past and recent works, the scope of betting ideas (test supermartingales, e-processes, confidence sequences, etc.) in statistics extends far beyond the bounded case studied in this paper, but we hope that the foundations we set forth in studying this basic problem provide mathematical guidance and intuition in pursuit of solutions for other problems.

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