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Generalized tractability for multivariate problems Part I Linear tensor product problems and linear information

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Abstract

Many papers study *polynomial* tractability for multivariate problems. Let $n(\varepsilon, d)$ be the minimal number of information evaluations needed to reduce the initial error by a factor of ε for a multivariate problem defined on a space of d-variate functions. Here, the initial error is the minimal error that can be achieved without sampling the function. Polynomial tractability means that $n(\varepsilon, d)$ is bounded by a polynomial in ε^{-1} and d and this holds for all $(\varepsilon^{-1}, d) \in [1, \infty) \times \mathbb{N}$. In this paper we study *generalized* tractability by verifying when $n(\varepsilon, d)$ can be bounded by a power of $T(\varepsilon^{-1}, d)$ for all $(\varepsilon^{-1}, d) \in \Omega$, where Ω can be a proper subset of $[1, \infty) \times \mathbb{N}$. Here T is a *tractability* function, which is non-decreasing in both variables and grows slower than exponentially to infinity. In this article we consider the set $\Omega = [1, \infty) \times \{1, 2, \ldots, d^*\} \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$ for some $d^* \geqslant 1$ and $\varepsilon_0 \in (0, 1)$. We study linear tensor product problems for which we can compute arbitrary linear functionals as information evaluations. We present necessary and sufficient conditions on T such that generalized tractability holds for linear tensor product problems. We show a number of examples for which polynomial tractability does *not* hold but generalized tractability *does*.

Keywords: Tractability; Multivariate problem; Tensor product problem; Information-based complexity; Linear information

1. Introduction

Tractability of multivariate problems has become a popular research problem in information-based complexity, see [9] where this concept was defined. There are too many tractability papers to cite. A survey of results for multivariate integration can be found in [3]. Tractability is the

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study of the difficulty of approximating operators defined on spaces of d-variate functions with the emphasis on large d. Problems with d in the hundreds and thousands occur in numerous applications such as in financial mathematics, physics and chemistry, see [6] for the discussion of this point. Tractability can be studied in various settings. In this paper we study the worst-case setting, in which the error of an algorithm is defined by its worst performance over a given class of functions.

Let $n(\varepsilon, d)$ denote the minimal number of information evaluations needed to reduce the initial error by a factor of $\varepsilon \in (0, 1]$. By an information evaluation, we mean the evaluation of a general linear functional or a function at some point. The initial error is defined as the minimal error that can be achieved without sampling the function. For linear operators, the initial error is simply the norm of the linear operator.

The essence of tractability study is to find necessary and sufficient conditions on spaces of functions and operators for which $n(\varepsilon, d)$ does *not* depend exponentially on ε^{-1} and on d. There are various ways of measuring the lack of exponential dependence. So far, tractability has been studied for the *polynomial* case, in which we insist that $n(\varepsilon, d)$ is bounded by a polynomial in ε^{-1} and d for all $\varepsilon \in (0, 1]$ and all $d \in \mathbb{N}$. Many results have been obtained for polynomial tractability. A typical result is that for classical spaces in which all variables play the same role, we do *not* have tractability since $n(\varepsilon, d)$ grows faster than any polynomial in ε^{-1} or d. In particular, this holds for linear tensor product problems. We note in passing that these negative results motivated the study of *weighted* spaces, in which variables or groups of variables play different roles and are moderated by weights. For weighted spaces, a typical result is that polynomial tractability holds for sufficiently small weights. Furthermore, we may even have *strong* tractability, in which $n(\varepsilon, d)$ is bounded by a polynomial in ε^{-1} independently of d, see [4], which was probably the first paper on tractability of weighted spaces, and [3] for a survey of results.

This is the first paper in a series studying *generalized* tractability for multivariate problems. Generalized tractability may differ from polynomial tractability in two ways. The first is the domain of (ε, d). For polynomial tractability, ε and d are independent, and $\left(\varepsilon^{-1}, d\right) \in [1, \infty) \times \mathbb{N}$. For some applications, as in mathematical finance, d is huge but we are only interested in a rough approximation, so that ε is not too small. There may be also problems for which d is relatively small and we are interested in a very accurate approximation which corresponds to a very small ε. For generalized tractability, we assume that $(\varepsilon^{-1}, d) \in \Omega$, where

$$[1, \infty) \times \{1, 2, \ldots, d^*\} \cup \left[1, \varepsilon_0^{-1}\right) \times \mathbb{N} \subseteq \Omega \subseteq [1, \infty) \times \mathbb{N} \tag{1}$$

for some non-negative integer d^* and some $\varepsilon_0 \in (0, 1]$ such that

$$d^* + (1 - \varepsilon_0) > 0.$$

The importance of the case $d^* = 0$ will be explained later.

The essence of (1) is that for all such Ω, we know that at least one of the parameters (ε^{-1}, d) may go to infinity but not necessarily both of them. Hence, for generalized tractability we assume that $(\varepsilon^{-1}, d) \in \Omega$ and we may choose an arbitrary Ω satisfying (1) for some d^* and ε_0.

The second way in which generalized tractability may differ from polynomial tractability is how we measure the lack of exponential dependence. We define a *tractability* function

$$T : [1, \infty) \times [1, \infty) \to [1, \infty),$$

which is non-decreasing in both variables and which grows to infinity slower than any exponential function a^x for x tending to infinity with $a > 1$. More precisely, for a given Ω satisfying (1), we assume that $T(x, y)/a^{x+y}$ tends to zero for $(x, y) \in \Omega$ as $x + y$ approaches infinity. This is equivalent to assuming that

$$\lim_{(x,y)\in\Omega, x+y\to\infty} \frac{\ln T(x, y)}{x + y} = 0. \tag{2}$$

With Ω satisfying (1) and T satisfying (2), we study generalized tractability which holds if there are positive numbers C and t such that

$$n(\varepsilon, d) \leqslant CT\left(\varepsilon^{-1}, d\right)^t \quad \text{for all } \left(\varepsilon^{-1}, d\right) \in \Omega.$$

We also have generalized strong tractability if we replace $T\left(\varepsilon^{-1}, d\right)$ above by $T\left(\varepsilon^{-1}, 1\right)$. In both cases, we are interested in the smallest exponents t; these are called the exponents of (generalized) tractability and strong tractability. The precise definitions are given in Section 2. Note that generalized tractability coincides with polynomial tractability if we take $\Omega = [1, \infty) \times \mathbb{N}$ and $T(x, y) = xy$.

We are mainly interested in how the choice of Ω and T affects the class of tractable problems. Some promising results were already obtained in [10] with $\Omega = [1, \infty) \times \mathbb{N}$ and $T(x, y) = f_1(x) f_2(y)$ for $f_i(t) = \exp\left(\ln^{1+\alpha_i} t\right)$ and non-negative α_i. Namely, it was proved that linear tensor product problems with polynomially decaying eigenvalues are tractable iff $\alpha_1\alpha_2 \geqslant 1$. Hence, these problems are *not* polynomial tractable, since this corresponds to $\alpha_1 = \alpha_2 = 0$, but are tractable if, for example, $\alpha_1 = \alpha_2 = 1$.

In this first paper on generalized tractability, we study linear tensor product problems for which we can use arbitrary bounded linear functionals. This type of information is called linear information, which explains the subtitle of our paper. Linear tensor product problems are fully characterized by eigenvalues $\lambda = \{\lambda_j\}$ for $d = 1$ that are ordered and normalized so that $1 = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_j \geqslant 0$ and $\lim_{j\to\infty} \lambda_j = 0$. In particular, we study eigenvalues with exponential, polynomial and logarithmic rates of convergence.

We also choose the "smallest" set,

$$\Omega = \Omega^{\text{res}} = [1, \infty) \times \{1, 2, \ldots, d^*\} \cup \left[1, \varepsilon_0^{-1}\right) \times \mathbb{N},$$

which is called the restricted tractability domain.

We provide necessary and sufficient conditions on the tractability function T such that generalized tractability holds for Ω^{res}. These conditions depend on the parameters d^* and ε_0, as well as on the sequence λ. In particular, the following results hold. Assume that $d^* \geqslant 1$ and $\varepsilon_0 < 1$. If the largest eigenvalue has multiplicity at least two, i.e., if $\lambda_2 = 1$, then generalized tractability does *not* hold, no matter how we choose the tractability function T.

Assume then that $\lambda_2 < 1$ and that we have a polynomial rate of convergence of the eigenvalues, i.e., $\lambda_j = \Theta\left(j^{-\beta}\right)$ for a positive β. This case is typical and corresponds to many classical Sobolev or Korobov tensor product spaces of smooth functions whose smoothness is measured by the parameter β.

Assume first that $\varepsilon_0^2 < \lambda_2$. Then generalized strong tractability does *not* hold, no matter how we choose T. Generalized tractability holds iff $\liminf_{x\to\infty} (\ln T(x, 1))/\ln x \in (0, \infty]$ and

$$\liminf_{d\to\infty} \inf_{\varepsilon\in[\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T\left(\varepsilon^{-1}, d\right)}{\alpha(\varepsilon) \ln d} \in (0, \infty],$$

where $\alpha(\varepsilon) = \lceil 2\ln(1/\varepsilon)/\ln(1/\lambda_2)\rceil - 1$. In particular, if we take $T(x, y) = xy$ then generalized tractability *holds* with the exponent

$$t^{\text{tra}} = \max\left\{\frac{2}{\beta}, \alpha(\varepsilon_0)\right\}.$$

Note that t^{tra} goes to $\max\{2/\beta, 1\}$ as $\lambda_2 - \varepsilon_0^2$ tends to zero, and t^{tra} goes to infinity as ε_0 tends to zero.

Assume now that $\lambda_2 \leqslant \varepsilon_0^2$. Then generalized strong tractability holds iff $\liminf_{x\to\infty}(\ln T(x, 1))/\ln x \in (0, \infty]$. For $T(x, y) = xy$, this holds and the exponent of generalized strong tractability is $t^{\text{str}} = 2/\beta$.

We end this introduction by a note on future research. We plan to study linear tensor product problems with linear information for yet more general Ω, including $\Omega = [1, \infty) \times \mathbb{N}$. Next, we want to study standard information, meaning that we can only compute function values. We want to verify which results on generalized tractability for linear information also hold for standard information. Finally, we plan to study weighted spaces and to verify how conditions on weights may be relaxed for generalized tractability.

2. Preliminaries

In this section we define multivariate problems and generalized tractability. Let m be a given positive integer. The multivariate problem will be given as a sequence of (dm)-variate problems defined on spaces of functions f of dm variables. Here, $d = 1, 2, \ldots,$ and our main emphasis will be on large d. Usually $m = 1$, but there are natural multivariate problems for which $m \geqslant 2$, see [1,2,7]. We wish to compute an ε-approximation for each d, and measure the difficulty of the computation by the minimal number of information evaluations needed for such an approximation. An information evaluation may be given by a function value at some point or, more generally, by a general linear functional. The essence of tractability is to assure that the minimal number of information evaluations is *not* exponentially dependent either on d or on ε^{-1}. In the tractability work so far it has been assumed that we want to guarantee that the minimal number of information evaluations is polynomial in d and ε^{-1}, see [9]. Generalized tractability studies arbitrary non-exponential functions, and how the tractability of a multivariate problem depends on the choice of these non-exponential functions.

2.1. Multivariate problems

For $m, d \in \mathbb{N}$, let F_d be a normed linear space of functions

$$f : D_d \subseteq \mathbb{R}^{dm} \to \mathbb{R}$$

and let G_d be a normed linear space. We consider a sequence $S = \{S_d\}$ of operators $S_d : F_d \to G_d$. We stress that S_d can be a non-linear operator, although all the technical results in this paper will be obtained for bounded linear operators. We call S a *multivariate problem*.

Let $\Lambda_d \subseteq F_d^*$ be a class of admissible continuous linear functionals. Two examples of Λ_d are mainly studied. The first class $\Lambda_d^{\text{all}} = F_d^*$ is the set of all continuous linear functionals defined on F_d and is called *linear information*. The second class Λ_d^{std} is called *standard information*, and is given by function evaluations. More precisely, we assume that $L_x(f) = f(x)$, for all $f \in F_d$, is a continuous linear functional for all $x \in D_d$. Then $\Lambda_d^{\text{std}} = \{L_x \mid x \in D_d\}$.

We consider algorithms that use finitely many admissible information evaluations. An algorithm $A_{n,d}$ has the form

$$A_{n,d}(f) = \phi(L_1(f), L_2(f), \dots, L_n(f)) \tag{3}$$

for some $L_i \in \Lambda_d$ and some mapping $\phi : \mathbb{R}^n \to G_d$. Adaptive choice of the functionals L_i is also allowed, as explained in, e.g., [5].

In this paper we restrict ourselves to the worst-case setting although settings such as the average case, randomized and probabilistic one could also be studied. The *worst-case error* of the algorithm $A_{n,d}$ is defined as

$$e^{\text{wor}}(A_{n,d}) = \sup_{f \in F_d, \|f\|_{F_d} \leqslant 1} \|S_d(f) - A_{n,d}(f)\|_{G_d}. \tag{4}$$

For $n = 0$ we do not sample the function f, and $A_{0,d}(f) = g$ is a constant mapping with $g \in G_d$. By the initial error we mean the minimal error of constant algorithms, which is defined as

$$e^{\text{init}}(S_d) = \inf_{g \in G_d} \sup_{f \in F_d, \|f\|_{F_d} \leqslant 1} \|S_d(f) - g\|_{G_d}.$$

It is clear that if S_d is a bounded linear operator, then the initial error is achieved for $g = 0$, and thus

$$e^{\text{init}}(S_d) = \|S_d\| = e^{\text{wor}}\left(A^*_{0,d}\right),$$

where $A^*_{0,d} = 0$ is the zero algorithm. Let

$$n(\varepsilon, S_d, \Lambda_d) = \min\left\{n \mid \exists A_{n,d} : e^{\text{wor}}(A_{n,d}) \leqslant \varepsilon e^{\text{init}}(S_d)\right\} \tag{5}$$

denote the minimal number of admissible information evaluations from Λ_d needed to reduce the initial error by a factor ε. Without loss of generality we may assume that $\varepsilon \in (0, 1]$ since we obviously have $n(\varepsilon, S_d, \Lambda_d) = 0$ for $\varepsilon > 1$.

We say that the algorithm $A_{n,d}$ computes an ε-approximation of S_d if its worst-case error is at most $\varepsilon e^{\text{init}}(S_d)$. This can only happen if $n \geqslant n(\varepsilon, S_d, \Lambda_d)$.

The number $n(\varepsilon, S_d, \Lambda_d)$ is called the information complexity of the problem S_d. It is also of interest to study the total complexity of the problem S_d which is defined as the minimal cost of computing an ε-approximation. For many linear problems, it follows from general results that the total complexity is roughly proportional to the information complexity $n(\varepsilon, S_d, \Lambda_d)$. This is also true for non-linear problems, if there exists an algorithm that computes an ε-approximation whose combinatory cost is of order of the information complexity, see [5] for more details. For simplicity, we only consider information complexity in this paper. More specifically, we study when $n(\varepsilon, S_d, \Lambda_d)$ does not depend exponentially on ε^{-1} and d. Obviously, this will lead to necessary conditions that the total complexity is also not dependent exponentially on ε^{-1} and d.

2.2. Generalized tractability

The essence of tractability is to guarantee that $n(\varepsilon, S_d, \Lambda_d)$ does *not* depend exponentially on ε^{-1} and d. There are various ways to measure the lack of exponential dependence. First of all, we must agree how the parameters ε and d vary. In previous work on tractability, it was assumed that ε and d are independent, and $\varepsilon \in (0, 1]$, $d \in \mathbb{N}$. In particular, it was assumed that both ε^{-1} and d

may go to infinity. For some applications, as in finance, we are interested in huge d and relatively small ε^{-1}. For instance, d may be in the hundreds or thousands, however, we may have $\varepsilon \geqslant 0.01$. The reason is that since financial models are weak, depending for instance on future re-financing rates, there is no merit in a more accurate solution. In such cases, the assumption that both ε^{-1} and d may go to infinity is too demanding.

That is why we assume that (ε^{-1}, d) belongs to Ω, where the domain Ω is, in general, a proper subset of $[1, \infty) \times \mathbb{N}$. Obviously, the domain Ω should be big enough to properly model the essence of multivariate problems.

We use the notation $[n] := \{1, \ldots, n\}$ for any integer n. In particular, $[n] = \emptyset$ if $n \leqslant 0$. We assume that

$$[1, \infty) \times [d^*] \cup \left[1, \varepsilon_0^{-1}\right) \times \mathbb{N} \ \subseteq \Omega \tag{6}$$

for some $d^* \in \mathbb{N}_0$ and some $\varepsilon_0 \in (0, 1]$ such that $d^* + (1 - \varepsilon_0) > 0$. Condition (6) is the only restriction we impose on Ω. The constraint $d^* + (1 - \varepsilon_0) > 0$ excludes the case $d^* = 0$ and $\varepsilon_0 = 1$ corresponding to no restriction on Ω.

Tractability for multivariate problems has so far been defined by demanding that $n(\varepsilon, S_d, \Lambda_d)$ is bounded by a polynomial in ε^{-1} and d. Obviously, there are different ways of guaranteeing that $n(\varepsilon, S_d, \Lambda_d)$ does not depend exponentially on ε^{-1} and d.

For instance, in theoretical computer science, tractability for discrete problems is usually understood by demanding that the cost bound of an algorithm is a polynomial in $k = \lceil \log_2(1 + \varepsilon^{-1}) \rceil$. That is, we can compute k correct bits of the solution in time polylog in ε^{-1}. We note in passing that if one adopts this definition of tractability then most multivariate problems become intractable since even for the univariate case, $d = 1$, we typically find that $n(\varepsilon, S_1, \Lambda_1)$ is a polynomial in ε^{-1}. For example, this holds for integration of univariate Lipschitz functions $S_1 f = \int_0^1 f(t)\,dt$ with $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [0, 1]$, for which $n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{std}}\right) = \lceil 4/\varepsilon \rceil$, see e.g., [6].

One may also take a point of view opposite to the one presented above, and consider a problem to be tractable when $n(\varepsilon, S_d, \Lambda_d)$ can be bounded by a function of ε^{-1} and d that grows faster than polynomials. This has been partially done in [10] by demanding that $n(\varepsilon, S_d, \Lambda_d)$ is bounded by a multiple of powers of $f_1\left(\varepsilon^{-1}\right)$ and $f_2(d)$ with functions f_i such as $f_i(x) = \exp\left(\ln^{1+\alpha_i}(x)\right)$ with $\alpha_i > 0$. Indeed, such functions grow faster than any polynomial as x tends to infinity, but slower than any exponential function a^x with $a > 1$. It was shown in [10] that the class of tractable multivariate problems is larger for such functions f_i than the tractability class studied before.

The approach of [10] is not fully general. Its notion of tractability decouples the parameters ε^{-1} and d since functions f_i depend only on one of these parameters. For some multivariate problems, such as tensor product problems, this restriction is essential. It is therefore better not to insist on separate dependence on ε^{-1} and d, and study tractability without assuming this property.

Hence in this paper, we study tractability defined by a function T of two variables, using a multiple of a power of $T\left(\varepsilon^{-1}, d\right)$ in the definition of generalized tractability. Obviously, we need to assume that T satisfies several natural properties. First of all, the problem of computing an ε-approximation usually becomes harder as ε decreases. Furthermore, with a proper definition of the operators S_d, the problem should become harder when d increases. That is why we assume that the function T is non-decreasing in both its arguments. Moreover, to rule out the exponential behavior of T, we assume that $T(x, y)/a^{x+y}$ tends to zero as $x + y$ tends to infinity for any $a > 1$. This is equivalent to assuming that $\ln T(x, y)/(x + y)$ tends to zero as $x + y$ approaches infinity. As we shall see in a moment, it will be convenient to define the domain of T as the set

$[1, \infty) \times [1, \infty)$. In particular, this domain allows us to say that T is non-decreasing, and will be useful for the concept of generalized strong tractability. This discussion motivates the following definitions.

A function $T : [1, \infty) \times [1, \infty) \to [1, \infty)$ is a *tractability function* if T is non-decreasing in x and y and

$$\lim_{(x,y)\in\Omega, x+y\to\infty} \frac{\ln T(x, y)}{x + y} = 0. \tag{7}$$

The multivariate problem $S = \{S_d\}$ is (T, Ω)*-tractable in the class* $\Lambda = \{\Lambda_d\}$ if there exist non-negative numbers C and t such that

$$n(\varepsilon, S_d, \Lambda_d) \leqslant CT\left(\varepsilon^{-1}, d\right)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \tag{8}$$

The *exponent* t^{tra} *of* (T, Ω)*-tractability* in the class Λ is defined as the infimum of all non-negative t for which there exists a $C = C(t)$ such that (8) holds.

Let $\varepsilon_0 < 1$. Then it is easy to see that if

$$n(\varepsilon_0, S_d, \Lambda_d) \geqslant \kappa^d \quad \text{for almost all } d \in \mathbb{N} \text{ with } \kappa > 1, \tag{9}$$

then S is *not* (T, Ω)-tractable in the class Λ for an arbitrary tractability function T and an arbitrary domain Ω satisfying (6). Indeed, suppose on the contrary that S is (T, Ω)-tractable in the class Λ. Then

$$\frac{\ln C + t \ln T(\varepsilon_0^{-1}, d)}{\varepsilon_0^{-1} + d} \geqslant \frac{d \ln \kappa}{\varepsilon_0^{-1} + d},$$

implying $\liminf_{d\to\infty} \ln T\left(\varepsilon_0^{-1}, d\right) / \left(\varepsilon_0^{-1} + d\right) \geqslant t^{-1} \ln \kappa > 0$, which contradicts (7).

Similarly, if $d^* \geqslant 1$ and there exist $d \in [d^*]$ and $\kappa > 1$ such that

$$n(\varepsilon, S_d, \Lambda_d) \geqslant \kappa^{1/\varepsilon} \quad \text{for sufficiently small } \varepsilon, \tag{10}$$

then S is *not* (T, Ω)-tractable in the class Λ for an arbitrary tractability function T and an arbitrary domain Ω satisfying (6). As before, this follows from the fact that $\liminf_{\varepsilon\to 0} \ln T\left(\varepsilon^{-1}, d\right) / \left(\varepsilon_0^{-1} + d\right) \geqslant t^{-1} \ln \kappa > 0$, which contradicts (7).

For some multivariate problems, it has been shown that $n(\varepsilon, S_d, \Lambda_d)$ is bounded by a multiple of some power of ε^{-1} that does not depend on d. This property is called *strong tractability*. In our case, we can define generalized strong tractability by insisting that the bound in (8) is independent of d. Formally, we replace $T\left(\varepsilon^{-1}, d\right)$ by $T\left(\varepsilon^{-1}, 1\right)$. We stress that (ε^{-1}, d) from Ω does not necessarily imply that $\left(\varepsilon^{-1}, 1\right)$ is in Ω. Nevertheless, due to the more general domain of T, the value $T\left(\varepsilon^{-1}, 1\right)$ is well defined, and since T is monotonic, we have $T\left(\varepsilon^{-1}, 1\right) \leqslant T\left(\varepsilon^{-1}, d\right)$.

The multivariate problem S is *strongly* (T, Ω)*-tractable* in the class $\Lambda = \{\Lambda_d\}$ if there exist non-negative numbers C and t such that

$$n(\varepsilon, S_d, \Lambda_d) \leqslant CT(\varepsilon^{-1}, 1)^t \quad \text{for all } (\varepsilon^{-1}, d) \in \Omega. \tag{11}$$

The *exponent* t^{str} *of strong* (T, Ω)*-tractability* in the class Λ is defined as the infimum of all non-negative t for which there exists a $C = C(t)$ such that (11) holds.

Clearly, strong (T, Ω)-tractability in the class Λ implies (T, Ω)-tractability in the class Λ. Furthermore, $t^{\text{tra}} \leqslant t^{\text{str}}$. For some multivariate problems the exponents t^{tra} and t^{str} are the same, and for some they are different. We shall see such examples also in this paper.

When it will cause no confusion, we simplify our notation and terminology as follows. If Ω and Λ are clear from the context, we say that S is T-tractable or strongly T-tractable. If T is also clear from the context, we say that S is tractable or strongly tractable. Finally, we talk about generalized tractability or generalized strong tractability if we consider various T, Ω and Λ.

We note in passing what happens if two tractability functions T_1 and T_2 are such that $T_1 = T_2^{\alpha}$ for some positive α. It is clear that the concepts of T_i-tractability are then essentially the same, with the obvious changes of their exponents. Therefore, we can obtain substantially different tractability results for T_1 and T_2 only if they are *not* polynomially related.

We now introduce a couple of specific cases of generalized tractability depending on the domain Ω and the form of the function T. We begin with two examples of Ω which seem especially interesting.

- *Restricted tractability domain*: Let

  $$\Omega^{\text{res}} = [1, \infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

  for some $d^* \in \mathbb{N}_0$ and $\varepsilon_0 \in (0, 1]$ with $d^* + (1 - \varepsilon_0) > 0$. This corresponds to the smallest set Ω used for tractability study. This case is called the *restricted tractability domain* independently of the function T.
  We may consider the special subcases where $d^* = 0$ or $\varepsilon_0 = 1$. If $d^* = 0$ then $\varepsilon_0 < 1$ and we have $\Omega^{\text{res}} = \left[1, \varepsilon_0^{-1}\right) \times \mathbb{N}$. Hence, we now want to compute an ε-approximation for only $\varepsilon \in (\varepsilon_0, 1]$ and for all d. We call this subcase *restricted tractability in* ε.
  If $\varepsilon_0 = 1$ then $d^* \geqslant 1$ and we have $\Omega^{\text{res}} = [1, \infty) \times [d^*]$. Hence, we now want to compute an ε-approximation for all $\varepsilon \in (0, 1]$ but only for $d \leqslant d^*$. We call this subcase *restricted tractability in* d.
- *Unrestricted tractability domain*: Let

  $$\Omega^{\text{unr}} = [1, \infty) \times \mathbb{N}.$$

  This corresponds to the largest set Ω used for tractability study. This case is called the *unrestricted tractability domain* independently of the function T.

We now present several examples of generalized tractability in terms of specific functions T that we think are of a particular interest.

- *Polynomial tractability*: Let

  $$T(x, y) = xy.$$

  In this case (T, Ω^{unr})-tractability coincides with tractability previously studied. For this function T, independently of Ω, tractability means that $n(\varepsilon, S_d, \Lambda_d)$ is bounded by a polynomial in ε^{-1} and d, explaining the name.
- *Separable tractability*: Let

  $$T(x, y) = f_1(x) f_2(y)$$

  with non-decreasing functions $f_1, f_2 : [1, \infty) \to [1, \infty)$. To guarantee (7) we assume that

  $$\lim_{x \to \infty} \frac{\ln f_i(x)}{x} = 0 \quad \text{for } i = 1, 2.$$

In this case $(T, \Omega^{\mathrm{unr}})$-tractability coincides with the notion of (f_1, f_2)-tractability studied in [10]. For this T, independently of Ω, the roles of ε^{-1} and d are separated, explaining the name. Observe that polynomial tractability is a special case of separable tractability for $f_1(x) = f_2(x) = x$.

For separable tractability, we can modify condition (8) by taking possibly different exponents of ε^{-1} and d. That is, the problem S is (T, Ω)-tractable in the class Λ if there are non-negative numbers C, p and q such that

$$n(\varepsilon, S_d, \Lambda_d) \leqslant C f_1\left(\varepsilon^{-1}\right)^p f_2(d)^q \quad \text{for all } \left(\varepsilon^{-1}, d\right) \in \Omega. \tag{12}$$

The exponents p and q are called the ε-exponent and the d-exponent. We stress that, in general, they do not need to be uniquely defined. Note that we obtain (8) from (12) by taking $t = \max\{p, q\}$. Similarly, the notion of strong (T, Ω)-tractability in the class Λ is obtained if $q = 0$ in the bound above, and the exponent t^{str} is the infimum of p satisfying the bound above with $q = 0$. Again for $f_1(x) = f_2(x) = x$ these notions coincide with the notions of polynomial tractability previously studied.

- *Separable restricted tractability*: Let

$$T(x, y) := \begin{cases} f_1(x) & \text{if } (x, y) \in [1, \infty) \times [1, d^*], \\ f_2(y) & \text{if } (x, y) \in [1, \varepsilon_0^{-1}) \times \mathbb{N} \setminus [1, d^*], \\ \max\{f_1(x), f_2(y)\} & \text{otherwise}, \end{cases}$$

where f_1, f_2 are as above with $f_2(d^*) \geqslant f_1\left(\varepsilon_0^{-1}\right)$.

It is easy to check that T is indeed a generalized tractability function. Suppose that the function T is considered on the restricted tractability domain Ω^{res}. Then $(T, \Omega^{\mathrm{res}})$-tractability corresponds to the smallest set Ω and we have a separate dependence on ε and d, explaining the same. As already discussed, such generalized tractability seems especially relevant for the case when for huge d we are only interested in a rough approximation to the solution.

- *Non-separable symmetric tractability*: Let

$$T(x, y) = \exp\left(f(x) f(y)\right) \tag{13}$$

with a non-decreasing function $f : [1, \infty) \to \mathbb{R}_+$. To guarantee (7) we need to assume that $\lim_{x+y\to\infty} f(x)f(y)/(x+y) = 0$. This holds, for example, if $f(x) = x^\alpha$ with $\alpha < \frac{1}{2}$ or if $f(x) = \ln^{1+\alpha}(x+1)$ with a positive α. The tractability function corresponding to $f(x) = \ln^{1+\alpha}(x+1)$ will be useful in the study of linear tensor product problems.

It is easy to see that this tractability function is *not* separable if f is not a constant function. Indeed, assume to the contrary that $T(x, y) = f_1(x) f_2(y)$ for some functions f_1 and f_2. For $x = 1$, we get $f_2(y) = f_1(1)^{-1} \exp(f(1)f(y))$, and similarly by taking $y = 1$, we obtain $f_1(x) = f_2(1)^{-1} \exp(f(1)f(x))$. Hence,

$$\exp(f(x)f(y)) = [f_1(1)f_2(1)]^{-1} \exp(f(1)(f(x) + f(y))).$$

Now $f_1(1)f_2(1) = \exp(f^2(1))$. Taking $x = y$, we obtain $f^2(x) = 2f(1)f(x) - f^2(1)$, which leads to the conclusion that $f(x) = f(1)$ for all x. This contradicts the requirement that f is not a constant function. Thus, T is not separable. Since the roles of ε^{-1} and d are the same, this motivates the name of this generalized tractability.

We finish this subsection by an example of a function T that is *not* a tractability function. Consider $T(x, y) = \exp(y^{1-1/x})$. This function is bounded in x for fixed y and increases sub-exponentially in y for fixed x. Nevertheless,

$$\limsup_{x+y\to\infty} \frac{\ln T(x, y)}{x + y} \geqslant \lim_{x=y\to\infty} \frac{x^{1-1/x}}{2x} = \frac{1}{2},$$

proving that T is not a tractability function. This example shows that the notion of tractability functions does not admit functions that increase asymptotically as fast as an exponential function in some direction.

3. Linear tensor product problems

In this section we consider multivariate problems defined as linear tensor product problems and study generalized tractability.

Let F_1 be a separable Hilbert space of real-valued functions defined on $D_1 \subseteq \mathbb{R}^m$, and let G_1 be an arbitrary separable Hilbert space. Let $S_1 : F_1 \to G_1$ be a compact linear operator. Then the non-negative self-adjoint operator

$$W_1 := S_1^* S_1 : F_1 \to F_1$$

is also compact. Let $\{\lambda_i\}$ denote the sequence of non-increasing eigenvalues of W_1, or equivalently, the sequence of squares of singular values of S_1. If $k = \dim(F_1)$ is finite, then W_1 has just finitely many eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then we formally put $\lambda_j = 0$ for $j > k$. In any case, the eigenvalues λ_j converge to zero.

There exist orthonormal bases $\{\zeta_i\}$, $\{\eta_i\}$ of F_1 and G_1, respectively, such that

$$S_1 f = \sum_{i\in\mathbb{N}} \sqrt{\lambda_i} \langle f, \zeta_i \rangle_{F_1} \eta_i \quad \text{for all } f \in F_1.$$

Without loss of generality, we assume that S_1 is not the zero operator, and normalize the problem by assuming that $\lambda_1 = 1$. Hence,

$$1 = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0.$$

This implies that $\|S_1\| = 1$ and the initial error is also one.

For $d \geqslant 2$, let

$$F_d = F_1 \otimes \cdots \otimes F_1$$

be the complete d-fold tensor product Hilbert space of F_1 of real-valued functions defined on $D_d = D_1 \times \cdots \times D_1 \subseteq \mathbb{R}^{dm}$. Similarly, let $G_d = G_1 \otimes \cdots \otimes G_1$, d times.

The linear operator S_d is defined as the tensor product operator

$$S_d = S_1 \otimes \cdots \otimes S_1 : F_d \to G_d.$$

We have $\|S_d\| = \|S_1\|^d = 1$, so that the initial error is one for all d. We call the multivariate problem $S = \{S_d\}$ a *linear tensor product problem.*

In this paper we analyze the problem S for the class of linear information $\Lambda^{\text{all}} = \{\Lambda_d^{\text{all}}\}$, leaving the case of standard information for future study. For linear information, we can compute arbitrary inner products. In particular, we can compute $\langle f, \zeta_{i_1} \otimes \zeta_{i_2} \otimes \cdots \otimes \zeta_{i_d} \rangle_{F_d}$. It is known, see e.g., [5], that the algorithm

$$\sum_{(i_1,i_2,\ldots,i_d):\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_d}>\varepsilon^2} \sqrt{\lambda_{i_1}\ldots\lambda_{i_d}}\langle f, \zeta_{i_1} \otimes \cdots \otimes \zeta_{i_d}\rangle_{F_d} \eta_{i_1} \otimes \cdots \otimes \eta_{i_d}$$

computes an ε-approximation of S_d and

$$n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right) = \left|\left\{(i_1, \ldots, i_d) \in \mathbb{N}^d \mid \lambda_{i_1} \ldots \lambda_{i_d} > \varepsilon^2\right\}\right|, \tag{14}$$

with the convention that the cardinality of the empty set is zero. Note that $n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right)$ is finite for all $\varepsilon \in (0, 1]$ and all d since $\lim_{j\to\infty} \lambda_j = 0$. For $d \geqslant 2$, we have

$$\begin{aligned} n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right) &= \sum_{j=1}^{\max\{i:\lambda_i>\varepsilon^2\}} n\left(\varepsilon/\sqrt{\lambda_j}, S_{d-1}, \Lambda_{d-1}^{\mathrm{all}}\right) \\ &= \sum_{j=1}^{n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right)} n(\varepsilon/\sqrt{\lambda_j}, S_{d-1}, \Lambda_{d-1}^{\mathrm{all}}). \end{aligned}$$

Since $n\left(\varepsilon/\sqrt{\lambda_j}, S_{d-1}, \Lambda_{d-1}^{\mathrm{all}}\right) \leqslant n\left(\varepsilon, S_{d-1}, \Lambda_{d-1}^{\mathrm{all}}\right)$ we obtain for all $d \geqslant 2$,

$$n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) \leqslant n\left(\varepsilon, S_{d-1}, \Lambda_{d-1}^{\mathrm{all}}\right) \leqslant n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right) \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right)^d. \tag{15}$$

We now show a simple lemma relating generalized tractability to the eigenvalues $\{\lambda_i\}$.

Lemma 3.1. *Let T be an arbitrary tractability function, Ω be a domain satisfying* (6) *with $\varepsilon_0 < 1$, $S = \{S_d\}$ be a linear tensor product problem defined as above, and $\Lambda = \{\Lambda_d\}$ be an arbitrary class of information evaluations.*

- *Let $\lambda_2 = 1$. Then S is not (T, Ω)-tractable in the class Λ.*
- *Let $\varepsilon_0^2 < \lambda_2 < 1$. Then S is not strongly (T, Ω)-tractable in the class Λ.*
- *Let $\lambda_2 = 0$. Then S is strongly (T, Ω)-tractable in the class Λ^{all} since $n(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}) = 1$ for all $(\varepsilon^{-1}, d) \in \Omega$ with $\varepsilon < 1$, and $t^{\mathrm{str}} = 0$.*

Proof. Since $\Lambda_d \subseteq \Lambda_d^{\mathrm{all}}$ we have $n(\varepsilon, S_d, \Lambda_d) \geqslant n(\varepsilon, S_d, \Lambda_d^{\mathrm{all}})$. If $\lambda_2 = 1$, then we can take $i_1, i_2, \ldots, i_d \in \{1, 2\}$ to conclude from (14) that

$$n\left(\varepsilon_0, S_d, \Lambda_d^{\mathrm{all}}\right) \geqslant 2^d \quad \text{for all } d.$$

Hence (9) holds with $\kappa = 2$, and S is not (T, Ω)-tractable in the class Λ.

If $\varepsilon_0^2 < \lambda_2 < 1$, then we take $d - 1$ values of $i_j = 1$ and one value of $i_j = 2$. Since we have at least d products of eigenvalues λ_{i_j} equal to λ_2, we get

$$n\left(\varepsilon_0, S_d, \Lambda_d^{\mathrm{all}}\right) \geqslant d \quad \text{for all } d.$$

This contradicts strong (T, Ω)-tractability in the class Λ, since $n\left(\varepsilon_0, S_d, \Lambda_d^{\mathrm{all}}\right)$ cannot be bounded by $CT\left(\varepsilon_0^{-1}, 1\right)^t$ for all d.

Finally, if $\lambda_2 = 0$ then S_1, as well as S_d, is a bounded linear functional, which can be computed exactly using only one information evaluation. This completes the proof. $\square$

In what follows we will need a simple bound for $n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right)$, which was proved in [8, Remark 3.1]. For the sake of completeness, we restate the short proof of this bound.

Lemma 3.2.

- *For $\varepsilon \in (0,1)$ and $\lambda_2 \in (0,1)$, let*

$$\alpha(\varepsilon) := \lceil 2\ln(1/\varepsilon)/\ln(1/\lambda_2)\rceil - 1,$$

$\beta(\varepsilon) := n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right)$, *and* $a := \min\{\alpha(\varepsilon), d\}$. *Then*

$$\binom{d}{a} \leqslant n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant \binom{d}{a}\beta(\varepsilon)^a. \tag{16}$$

- *If $\lambda_2 \leqslant \varepsilon_0^2 < 1$ then*

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) = 1 \quad \textit{for all } \varepsilon \in [\varepsilon_0, 1) \textit{ and for all } d \in \mathbb{N}. \tag{17}$$

Proof. Let us consider a product $\lambda_{i_1} \dots \lambda_{i_d}$ of eigenvalues of W_1 such that $\lambda_{i_1} \dots \lambda_{i_d} > \varepsilon^2$. Let k denote the number of indices i_j, where $j \in [d]$, with $i_j \geqslant 2$. Then necessarily $\lambda_2^k > \varepsilon^2$, which implies $k \leqslant \alpha(\varepsilon)$. Consequently, we have at most a indices that are not one. From (14), it follows that $\beta(\varepsilon) = \left|\left\{j \mid \lambda_j > \varepsilon^2\right\}\right|$, which implies that $i_j \leqslant \beta(\varepsilon)$ for all $j \in [d]$. This leads to (16). For $\lambda_2 \leqslant \varepsilon_0^2$, we may assume without loss of generality that $\lambda_2 > 0$, and we have $\alpha(\varepsilon) = a = 0$ for all $\varepsilon \in [\varepsilon_0, 1)$ and for all d. Then (16) implies (17). $\square$

4. Restricted tractability domain

In this section, we study generalized tractability for the linear tensor product problem S and the restricted tractability domain

$$\Omega^{\text{res}} = [1,\infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

for $d^* \in \mathbb{N}_0$ and $\varepsilon_0 \in (0,1]$ with $d^* + (1-\varepsilon_0) > 0$.

As before, $\lambda = \{\lambda_j\}$ denotes the sequence of non-increasing eigenvalues of the compact operator W_1 with $\lambda_1 = 1$. We first treat the two subcases of restricted tractability in ε and in d. We will see that in the first case, when $d^* = 0$, the second largest eigenvalue λ_2 is the only eigenvalue that effects tractability, while in the second case, when $\varepsilon_0 = 1$, the convergence rate of the sequence λ is the important criterion for tractability. Then we consider the case of the restricted tractability domain with $d^* \geqslant 1$ and $\varepsilon_0 < 1$.

4.1. Restricted tractability in ε

We now provide necessary and sufficient conditions for restricted tractability in ε, which we then illustrate for several tractability functions. In this subsection $\varepsilon_0 < 1$, and from Lemma 3.1 we see that we can restrict our attention to the case when $\lambda_2 < 1$.

Theorem 4.1. *Let $\varepsilon_0 < 1$ and $d^* = 0$, so that*

$$\Omega^{\text{res}} = [1, \varepsilon_0^{-1}) \times \mathbb{N}.$$

Let S be a linear tensor product problem with $\lambda_2 < \lambda_1 = 1$.

- *S is strongly (T, Ω^{res})-tractable in the class of linear information iff $\lambda_2 \leqslant \varepsilon_0^2$. If this holds, then $n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) = 1$ for all $(\varepsilon, d) \in [\varepsilon_0, 1) \times \mathbb{N}$, and the exponent of strong restricted tractability is $t^{\text{str}} = 0$.*
- *Let $\lambda_2 > \varepsilon_0^2$. Then S is (T, Ω^{res})-tractable in the class of linear information iff*

$$B := \liminf_{d\to\infty} \inf_{\varepsilon\in[\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T\left(\varepsilon^{-1}, d\right)}{\alpha(\varepsilon) \ln d} \in (0, \infty],$$

where, as in Lemma 3.2, $\alpha(\varepsilon) = \lceil 2\ln(1/\varepsilon)/\ln(1/\lambda_2)\rceil - 1$. If this holds then the exponent of restricted tractability is $t^{\text{tra}} = 1/B$.

Proof. The first part of the lemma follows directly from Lemmas 3.1 and 3.2. Before we verify the second part, we present an estimate of $n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right)$. Let $\varepsilon \in [\varepsilon_0, 1)$. For $d \geqslant \alpha(\varepsilon)$, we get from (16) of Lemma 3.2,

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant \frac{\beta(\varepsilon)^{\alpha(\varepsilon)}}{\alpha(\varepsilon)!} d(d-1)\dots(d-\alpha(\varepsilon)+1) \leqslant C_1 d^{\alpha(\varepsilon)}, \tag{18}$$

where C_1 depends only on ε_0 and S_1.

Let now $B \in (0, \infty]$. We want to show the existence of some positive C and t such that

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant C T(\varepsilon^{-1}, d)^t \quad \text{for all } (\varepsilon, d) \in [\varepsilon_0, 1) \times \mathbb{N}. \tag{19}$$

Let $\{B_n\}$ be a sequence in $(0, B)$ that converges to B. Then we find for each $n \in \mathbb{N}$ a number $d_n \in \mathbb{N}$ such that

$$\inf_{\varepsilon\in[\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \geqslant B_n \quad \text{for all } d \geqslant d_n.$$

Due to (18), to prove (19) it is sufficient to show that $C_1 d^{\alpha(\varepsilon)} \leqslant C T(\varepsilon^{-1}, d)^t$, which is equivalent to

$$\frac{\ln(C_1/C)}{t \ln d} + \frac{\alpha(\varepsilon)}{t} \leqslant \frac{\ln T(\varepsilon^{-1}, d)}{\ln d}.$$

If $C \geqslant C_1$ and $1/t = B_n$, then for all $d \geqslant d_n$ and all $\varepsilon \in [\varepsilon_0, 1]$, we have

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant C T\left(\varepsilon^{-1}, d\right)^t.$$

To make the last estimate hold for every $(\varepsilon, d) \in [\varepsilon_0, 1] \times \mathbb{N}$, we only have to increase the number C if necessary. Letting n tend to infinity, we see that $t^{\text{tra}} \leqslant 1/B$.

Now let (19) hold for some positive C and t. To prove that $B \in (0, \infty]$, we apply (16) of Lemma 3.2 for $\varepsilon \in \left[\varepsilon_0, \sqrt{\lambda_2}\right)$ and $d \geqslant \alpha(\varepsilon_0)$. Then

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \geqslant \binom{d}{\alpha(\varepsilon)} \geqslant \left(\frac{d}{\alpha(\varepsilon)}\right)^{\alpha(\varepsilon)} \geqslant C_2 d^{\alpha(\varepsilon)}$$

with[1] $C_2 = \alpha(\varepsilon_0)^{-\alpha(\varepsilon_0)}$. Thus for $\varepsilon \in \left[\varepsilon_0, \sqrt{\lambda_2}\right)$ we have

$$C_2 d^{\alpha(\varepsilon)} \leqslant C T(\varepsilon^{-1}, d)^t \quad \forall d \geqslant \alpha(\varepsilon_0),$$

which is equivalent to

$$\frac{\ln T\left(\varepsilon^{-1}, d\right)}{\ln d} \geqslant \frac{\alpha(\varepsilon)}{t} + \frac{\ln(C_2/C)}{t \ln d}.$$

The condition $\varepsilon^2 < \lambda_2$ implies $\alpha(\varepsilon) \geqslant 1$, and we get

$$\liminf_{d\to\infty} \inf_{\varepsilon\in\left[\varepsilon_0, \sqrt{\lambda_2}\right)} \frac{\ln T\left(\varepsilon^{-1}, d\right)}{\alpha(\varepsilon)\ln d} \geqslant \frac{1}{t}.$$

This proves that $B > 0$ and $t^{\mathrm{tra}} \geqslant 1/B$, and completes the proof. □

We illustrate Theorem 4.1 for a number of tractability functions *T*, assuming that $\lambda_2 \in \left(\varepsilon_0^2, 1\right)$. In this case we do *not* have strong tractability. However, tractability depends on the particular function *T*.

- Polynomial tractability, $T(x, y) = xy$. Then $(T, \Omega^{\mathrm{res}})$-tractability in the class of linear information holds with the exponent $t^{\mathrm{tra}} = 1/B$ with

$$B = \frac{1}{\alpha(\varepsilon_0)} = \frac{1}{\lceil 2\ln(1/\varepsilon_0)/\ln(1/\lambda_2)\rceil - 1}.$$

- Separable restrictive tractability, $T(x, y) = f_2(y)$ for $x, y \in \left[1, \varepsilon_0^{-1}\right] \times \mathbb{N}$, and with a non-decreasing function $f_2 : [1, \infty) \to [1, \infty)$ such that $\lim_{y\to\infty} (\ln f_2(y))/y = 0$. Then $(T, \Omega^{\mathrm{res}})$-tractability in the class of linear information holds iff

$$B_1 := \liminf_{d\to\infty} \frac{\ln f_2(d)}{\ln d} \in (0, \infty];$$

in this case we get $t^{\mathrm{tra}} = 1/B$, where

$$B = \frac{B_1}{\alpha(\varepsilon_0)} = \frac{B_1}{\lceil 2\ln(1/\varepsilon_0)/\ln(1/\lambda_2)\rceil - 1}.$$

Note that $B_1 > 0$ iff $f_2(d)$ is at least of order d^β for some positive β. Hence, if we take $f_2(d) = \lceil \ln(d+1)\rceil$ then we do not have tractability. On the other hand, if $f_2(d) = d^\beta$ for a positive β then $B_1 = \beta$. For $f_2(d) = \exp\left(\ln^{1+\beta}(d)\right)$ with $\beta > 0$, we obtain $B_1 = \infty$ and $t^{\mathrm{tra}} = 0$. This means that in this case for an arbitrarily small positive t we have

$$n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right) = o\left(T(\varepsilon^{-1}, d)^t\right) \quad \text{for all } \varepsilon \in [\varepsilon_0, 1],\ d \in \mathbb{N}.$$

[1] Here, we use the inequality $\binom{d}{k} \geqslant (d/k)^k$ for $d \geqslant k$, which can be easily checked by induction on d.

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). Then (T, Ω^{res})-tractability in the class of linear information holds iff

$$B_2 = \liminf_{d\to\infty} \frac{f(d)}{\ln d} \in (0, \infty],$$

and the exponent $t^{\text{tra}} = 1/B$, with

$$B = B_2 \inf_{\varepsilon\in[\varepsilon_0,\sqrt{\lambda_2})} \frac{f(x)}{\alpha(x)}.$$

Note that $B_2 > 0$ iff $f(d)$ is at least of order $\ln(d)$. For example, if $f(x) = \beta \ln d$ for a positive β then $B_2 = \beta$, whereas $f(d) = d^\alpha$ with $\alpha > 0$ yields $B_2 = \infty$ and $t^{\text{tra}} = 0$.

4.2. Restricted tractability in d

We now assume that $d^* \geqslant 1$ and $\varepsilon_0 = 1$ so that

$$\Omega^{\text{res}} = [1, \infty) \times [d^*].$$

We provide necessary and sufficient conditions for restricted tractability in d in terms of the sequence of eigenvalues $\lambda = \{\lambda_j\}$ of the compact operator $W_1 = S_1^* S_1$. Assume first that W_1 has a finite number of positive eigenvalues λ_j. Then

$$\lim_{\varepsilon\to 0} n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) < \infty$$

and (15) yields that

$$\lim_{\varepsilon\to 0} n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant \left(\lim_{\varepsilon\to 0} n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right)\right)^d < \infty$$

for all d. In our case, we have $d \leqslant d^*$. Hence, the problem is strongly (T, Ω^{res})-tractable with $t^{\text{str}} = 0$ for all tractability functions T, since

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant C := \left(\lim_{\varepsilon\to 0} n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right)\right)^{d^*} \quad \text{for all } (\varepsilon, d) \in \Omega^{\text{res}}.$$

Assume then that W_1 has infinitely many positive eigenvalues λ_j which is equivalent to assuming that $\lim_{\varepsilon\to 0} n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) = \infty$. In this case we have the following theorem.

Theorem 4.2. *Let*

$$\Omega^{\text{res}} = [1, \infty) \times [d^*] \quad \textit{with } d^* \geqslant 1 \textit{ and } \lim_{\varepsilon\to 0} n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) = \infty.$$

Then the following three statements are equivalent:

(i)

$$A := \liminf_{\varepsilon\to 0} \frac{\ln T\left(\varepsilon^{-1}, 1\right)}{\ln n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right)} \in (0, \infty];$$

(ii) *S is (T, Ω^{res})-tractable in the class of linear information*;

(iii) *S is strongly (T, Ω^{res})-tractable in the class of linear information.*

If (i) *holds then the exponent of strong* $(T, \Omega^{\rm res})$*-tractability and the exponent of* $(T, \Omega^{\rm res})$*-tractability satisfy*

$$\frac{1}{A} \leqslant t^{\rm tra} \leqslant t^{\rm str} \leqslant \frac{d^*}{A}.$$

Proof. It is enough to show that (iii)⇒(ii)⇒(i)⇒(iii).

(iii)⇒(ii) is obvious.

(ii)⇒(i). For $d = 1$ we now know that

$$C\,T(\varepsilon^{-1}, 1)^t \geqslant n\left(\varepsilon, S_1, \Lambda_1^{\rm all}\right)$$

for some positive C and t with $t \geqslant t^{\rm tra}$. Taking logarithms we obtain

$$\frac{\ln T\left(\varepsilon^{-1}, 1\right)}{\ln n\left(\varepsilon, S_1, \Lambda_1^{\rm all}\right)} \geqslant \frac{1}{t} + \frac{\ln C^{-1}}{t \ln n\left(\varepsilon, S_1, \Lambda_1^{\rm all}\right)}.$$

Since $n\left(\varepsilon, S_1, \Lambda_1^{\rm all}\right)$ goes to infinity, we conclude that $A \geqslant 1/t > 0$, as claimed. Furthermore, $t \geqslant 1/A$ and since t can be arbitrarily close to $t^{\rm tra}$, we have $t^{\rm tra} \geqslant 1/A$.

(i)⇒(iii). We now know that for any $\delta \in (0, A)$ there exists a positive ε_δ such that

$$n\left(\varepsilon, S_1, \Lambda_1^{\rm all}\right) \leqslant T\left(\varepsilon^{-1}, 1\right)^{1/(A-\delta)} \quad \forall \varepsilon \in (0, \varepsilon_\delta].$$

Hence, there is a constant $C_\delta \geqslant 1$ such that

$$n\left(\varepsilon, S_1, \Lambda_1^{\rm all}\right) \leqslant C_\delta T\left(\varepsilon^{-1}, 1\right)^{1/(A-\delta)} \quad \forall \varepsilon \in (0, 1].$$

From (15) we obtain that

$$n\left(\varepsilon, S_d, \Lambda_d^{\rm all}\right) \leqslant C_\delta^{d^*} T\left(\varepsilon^{-1}, 1\right)^{d^*/(A-\delta)} \quad \forall \varepsilon \in (0, 1]\ \forall d \in [d^*].$$

This proves strong tractability with the exponent at most $d^*/(A-\delta)$. Since δ can be arbitrarily small, $t^{\rm tra} \leqslant t^{\rm str} \leqslant d^*/A$, which completes the proof. □

Theorem 4.2 states that $(T, \Omega^{\rm res})$-tractability is equivalent to $A > 0$, where A depends only on the behavior of the eigenvalues for $d = 1$. The condition $A > 0$ means that $\ln T(\varepsilon^{-1}, 1)$ goes to infinity at least as fast as $\ln n(\varepsilon, S_1, \Lambda_1^{\rm all})$. Note that for a finite positive A and for $d^* > 1$, we do not have sharp bounds on the exponents. We shall see later that both bounds in Theorem 4.2 may be attained for some specific multivariate problems and tractability functions T. It may also happen that $A = \infty$. In this case $t^{\rm tra} = t^{\rm str} = 0$, which means that for all $d \in [d^*]$, and all positive t we have

$$n\left(\varepsilon, S_d, \Lambda_d^{\rm all}\right) = o(T(\varepsilon^{-1}, d)^t) \quad \text{as } \varepsilon \to 0.$$

To verify the condition $A > 0$ and find better bounds on the exponents of tractability, we study the different rates of convergence of the sequence $\lambda = \{\lambda_j\}$. We consider exponential, polynomial and logarithmic rates of λ. That is, we assume:

- Exponential rate: λ_j is of order $\exp(-\beta j)$ for some positive β, or a little more generally, λ_j is of order $\exp(-\beta j^\alpha)$ for some positive α and β.
- Polynomial rate: λ_j is of order $j^{-\beta} = \exp(-\beta \ln j)$, or a little more generally, λ_j is of order $\exp\left(-\beta (\ln j)^\alpha\right)$ for some positive α and β.
- Logarithmic rate: λ_j is of order $(\ln j)^{-\beta} = \exp(-\beta \ln\ln j)$ for some positive β.

Note that for $\alpha < 1$, we have sub-exponential or sub-polynomial behavior of the eigenvalues, whereas for $\alpha > 1$, we have super-exponential or super-polynomial behavior of the eigenvalues. For the sake of simplicity we omit the prefixes sub and super and talk only about exponential or polynomial rates.

As we shall see, tractability will depend on some limits. We will denote these limits using the subscripts indicating the rate of convergence of λ. Hence, the subscript e indicates an exponential rate, the subscript p a polynomial rate, and the subscript l a logarithmic one.

4.2.1. Exponential rate

Theorem 4.3. *Let $\Omega^{\text{res}} = [1, \infty) \times [d^*]$ with $d^* \geqslant 1$. Let S be a linear tensor product problem with $\lambda_1 = 1$ and with exponentially decaying eigenvalues λ_j, so that*

$$K_1 \exp\left(-\beta_1 j^{\alpha_1}\right) \leqslant \lambda_j \leqslant K_2 \exp\left(-\beta_2 j^{\alpha_2}\right) \quad \text{for all } j \in \mathbb{N}$$

for some positive numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, K_1$ and K_2.

Then S is (T, Ω^{res})-tractable (as well as strongly (T, Ω^{res})-tractable due to Theorem 4.2) in the class of linear information iff

$$A_e := \liminf_{x \to \infty} \frac{\ln T(x, 1)}{\ln \ln x} \in (0, \infty].$$

If $A_e > 0$ then the exponent of (T, Ω^{res})-tractability satisfies

$$\frac{1}{\alpha_1} \max_{d \in [d^*]} \frac{d}{A_{e,d}} \leqslant t^{\text{tra}} \leqslant \frac{1}{\alpha_2} \max_{d \in [d^*]} \frac{d}{A_{e,d}},$$

where

$$A_{e,d} = \liminf_{x \to \infty} \frac{\ln T(x, d)}{\ln \ln x},$$

(clearly, $A_{e,d} \geqslant A_{e,1} = A_e > 0$), and the exponent of strong (T, Ω^{res})-tractability satisfies

$$\frac{d^*}{\alpha_1 A_e} \leqslant t^{\text{str}} \leqslant \frac{d^*}{\alpha_2 A_e}.$$

Proof. We have

$$n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) = \min\left\{ j \mid \lambda_{j+1} \leqslant \varepsilon^2 \right\}.$$

Using the estimates of λ_j we obtain

$$\min\left\{ j \mid g_1(j) \leqslant \varepsilon^2 \right\} \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) \leqslant \min\left\{ j \mid g_2(j) \leqslant \varepsilon^2 \right\},$$

where $g_i(j) = K_i \exp\left(-\beta_i (j+1)^{\alpha_i}\right)$. This yields

$$\left(\frac{1}{\beta_1} \ln\left(K_1 \varepsilon^{-2}\right)\right)^{1/\alpha_1} - 1 \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) \leqslant \left(\frac{1}{\beta_2} \ln\left(K_2 \varepsilon^{-2}\right)\right)^{1/\alpha_2}.$$

For small ε this leads to

$$\frac{\ln\ln \varepsilon^{-1}}{\alpha_1}(1+o(1)) \leqslant \ln n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) \leqslant \frac{\ln\ln \varepsilon^{-1}}{\alpha_2}(1+o(1)).$$

Therefore, A from (i) of Theorem 4.2 satisfies $\alpha_2 A_e \leqslant A \leqslant \alpha_1 A_e$. Hence, $A > 0$ iff $A_e > 0$, and (i) of Theorem 4.2 yields the first part of Theorem 4.3.

We now find bounds on the exponents assuming that $A_e > 0$. First, we estimate $n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right)$. With $x := \ln\left(\left(K_2^d \varepsilon^{-2}\right)^{1/\beta_2}\right)$ we use (14) to obtain

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant m_e(x,d) := \left|\left\{(i_1,\dots,i_d) \,\middle|\, \sum_{j=1}^{d} i_j^{\alpha_2} < x\right\}\right|.$$

We now prove that

$$\kappa_{d,x}\left(\left(\frac{x}{d}\right)^{1/\alpha_2} - 1\right)^d \leqslant m_e(x,d) \leqslant x^{d/\alpha_2}, \tag{20}$$

where $\kappa_{d,x} = 1$ for $x \geqslant d$, and $\kappa_{d,x} = 0$ for $x < d$. We prove (20) by induction on d. Let $\alpha = \alpha_2$. For $d = 1$ we have $m(x,1) = |\{i \mid i < x^{1/\alpha}\}|$, i.e., $x^{1/\alpha} - 1 \leqslant m(x,1) < x^{1/\alpha}$. For $d > 1$, we have

$$m_e(x,d) = \sum_{k<x^{1/\alpha}} m_e(x-k^\alpha, d-1).$$

From our induction hypothesis we get

$$m_e(x,d) \leqslant \sum_{k<x^{1/\alpha}} (x-k^\alpha)^{(d-1)/\alpha} \leqslant \sum_{k<x^{1/\alpha}} x^{(d-1)/\alpha} \leqslant x^{d/\alpha}.$$

To prove a lower bound, we can assume that $x > d$, so that

$$\begin{aligned} m_e(x,d) &\geqslant \sum_{k:\, k^\alpha + d - 1 \leqslant x} \left(\left(\frac{x-\xi^\alpha}{d-1}\right)^{1/\alpha} - 1\right)^{d-1} \\ &\geqslant \int_1^{(x+1-d)^{1/\alpha}} \left(\left(\frac{x-\xi^\alpha}{d-1}\right)^{1/\alpha} - 1\right)^{d-1} d\xi. \end{aligned}$$

Since $x + 1 - d \geqslant x/d$ and $(x-\xi^\alpha)/(d-1) \geqslant x/d$ for $\xi \in \left[1, (x/d)^{1/\alpha}\right]$, we have

$$m_e(x,d) \geqslant \int_1^{(x/d)^{1/\alpha}} \left(\left(\frac{x}{d}\right)^{1/\alpha} - 1\right)^{d-1} d\xi = \left(\left(\frac{x}{d}\right)^{1/\alpha} - 1\right)^d,$$

as claimed.

Consequently, we have

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant \left(\frac{\ln\left(K_2^d \varepsilon^{-2}\right)}{\beta_2}\right)^{d/\alpha_2} \tag{21}$$

for all $\varepsilon \in (0, 1]$ and $d \in [d^*]$. Take $C^* := \sup\{(1/\beta_2)^{d/\alpha_2} | d \in [d^*]\}$. It is easy to see that $K_2 \geqslant 1$. We want to show the existence of some positive C and t such that

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant C^* \ln\left(K_2^{d^*} \varepsilon^{-2}\right)^{d^*/\alpha_2} \leqslant CT(\varepsilon^{-1}, 1)^t \tag{22}$$

for all $\varepsilon \in (0, 1]$. The right-hand inequality is equivalent to

$$\frac{\ln(C^*/C)}{t \ln\ln \varepsilon^{-1}} + \frac{d^*}{\alpha_2 t} \frac{\ln\ln\left(K_2^{d^*} \varepsilon^{-2}\right)}{\ln\ln \varepsilon^{-1}} \leqslant \frac{\ln T(\varepsilon^{-1}, 1)}{\ln\ln \varepsilon^{-1}}. \tag{23}$$

Let $\{A_n\}$ be a sequence in $(0, A_e)$ converging to A_e. Hence for every n there exists a positive ε_n such that

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\ln\ln \varepsilon^{-1}} \geqslant A_n \quad \text{for all } \varepsilon \in (0, \varepsilon_n].$$

Therefore, decreasing ε_n if necessary, we obtain (23) for all $\varepsilon \in (0, \varepsilon_n]$ as long as we choose $C \geqslant C^*$ and $t > d^*/(\alpha_2 A_n)$. To establish (22) for all $\varepsilon \in (\varepsilon_n, 1]$, we can keep the same t and, if necessary, increase C. Hence, we have strong tractability with the exponent $t^{\text{str}} \leqslant d^*/(\alpha_2 A_n)$, and with n tending to infinity, we conclude that $t^{\text{str}} \leqslant d^*/(\alpha_2 A_e)$.

We know that the problem is also tractable. To obtain an upper bound on the exponent of tractability, we use (21) and we find positive C and t for which

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant \left(\frac{\ln(K_2^{d^*} \varepsilon^{-2})}{\beta_2}\right)^{d/\alpha_2} \leqslant CT(\varepsilon^{-1}, d)^t \quad \forall d \in [d^*].$$

Proceeding as before, we conclude that $t^{\text{tra}} \leqslant \max_{d\in[d^*]} d/(\alpha_2 A_{e,d})$.

To obtain lower bounds on the exponents, we use the estimate

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \geqslant \tilde{m}_e(z, d) := \left|\left\{(i_1, \dots, i_d) \,\middle|\, \sum_{j=1}^{d} i_j^{\alpha_1} < z\right\}\right|,$$

where $z = z(\varepsilon, d) := \ln\left(\left(K_1^d \varepsilon^{-2}\right)^{1/\beta_1}\right)$. For sufficiently small ε, we can use the left-hand side of (20) with α_2 replaced by α_1 which yields

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \geqslant cz^{d/\alpha_1} = c\left(\ln\left(\left(K_1^d \varepsilon^{-2}\right)^{1/\beta_1}\right)\right)^{d/\alpha_1}$$

for all $d \in [d^*]$, where c is independent of ε and d. Thus, for all $t > t^{\text{str}}$ there exists a $C > 0$ such that for small ε we have the inequality

$$CT\left(\varepsilon^{-1}, 1\right)^t \geqslant c\left(\ln\left(\left(K_1^{d^*} \varepsilon^{-2}\right)^{1/\beta_1}\right)\right)^{d^*/\alpha_1},$$

which is equivalent to

$$\frac{\ln T(\varepsilon^{-1},1)}{\ln\ln\varepsilon^{-1}} \geqslant \frac{\ln(c/C)}{t\ln\ln\varepsilon^{-1}} + \frac{d^*}{\alpha_1 t}\frac{\ln\left(\beta_1^{-1}\ln\left(K_1^{d^*}\varepsilon^{-2}\right)\right)}{\ln\ln\varepsilon^{-1}}.$$

This implies $A_e \geqslant d^*/(\alpha_1 t)$, and $t^{\mathrm{str}} \geqslant d^*/(\alpha_1 A_e)$.

For tractability, we know that there are positive C and t such that

$$CT\left(\varepsilon^{-1},d\right)^t \geqslant c\left(\ln\left(\left(K_1^d\varepsilon^{-2}\right)^{1/\beta_1}\right)\right)^{d/\alpha_1} \quad \forall d\in[d^*].$$

Proceeding as before, we conclude that $t^{\mathrm{tra}} \geqslant \max_{d\in[d^*]} d/(\alpha_1 A_{e,d})$. This concludes the proof. $\square$

For exponentially decaying eigenvalues, Theorem 4.3 states that strong tractability (and tractability) are equivalent to the condition $A_e > 0$. If we know the precise order of convergence of λ, i.e., when $\alpha_1 = \alpha_2 = \alpha > 0$, then we know the exponents of tractability and strong tractability,

$$t^{\mathrm{tra}} = \frac{1}{\alpha}\max_{d\in[d^*]}\frac{d}{A_{e,d}},$$

$$t^{\mathrm{str}} = \frac{1}{\alpha}\frac{d^*}{A_e}.$$

As we shall see it may happen that $t^{\mathrm{str}} > t^{\mathrm{tra}}$.

We now illustrate Theorem 4.3 for a number of tractability functions T.

- Polynomial tractability, $T(x,y) = xy$. Then $A_{e,d} = A_e = \infty$, and we have strong tractability with $t^{\mathrm{tra}} = t^{\mathrm{str}} = 0$.
- Separable restrictive tractability, $T(x,y) = f_1(x)$ for $(x,y) \in \Omega^{\mathrm{res}}$ and a non-decreasing function

$$f_1 : [1,\infty) \to [1,\infty) \quad \text{with} \ \lim_{x\to\infty}\frac{\ln f_1(x)}{x} = 0.$$

Then strong $(T,\Omega^{\mathrm{res}})$-tractability holds iff

$$A_{e,d} = A_e = \liminf_{x\to\infty}\frac{\ln f_1(x)}{\ln\ln x} \in (0,\infty].$$

Note that $A_e > 0$ iff $f_1(x)$ is at least of order $(\ln x)^\beta$ for some positive β. If we take $f(x) = \lceil\ln(x+1)\rceil$ then we have strong tractability with $A_e = 1$. For $\alpha_1 = \alpha_2 = \alpha > 0$, the exponents are $t^{\mathrm{str}} = t^{\mathrm{tra}} = d^*/\alpha$.
- Non-separable symmetric tractability, $T(x,y) = \exp(f(x)f(y))$ with f as in (13). Then $(T,\Omega^{\mathrm{res}})$-tractability holds iff

$$A_{e,d} = f(d)\liminf_{x\to\infty}\frac{f(x)}{\ln\ln x} \in (0,\infty].$$

Hence, $A_e = A_{e,1} > 0$ iff $f(x)$ is at least of order $\beta\ln\ln x$ for some positive β. For example, if we take $f(x) = \ln^{1+\alpha}(x+1)$ for $\alpha > -1$, then $A_{e,d} = \infty$ and $t^{\mathrm{str}} = t^{\mathrm{tra}} = 0$. For $f(x) = \beta\ln\ln(x+c)$ with $c > \exp(1)-1$ and a positive β, we have $f(1) > 0$ and

$$A_{e,d} = f(d)\beta = \beta^2\ln\ln(d+c).$$

For $\alpha_1 = \alpha_2 = \alpha > 0$, we now have

$$t^{\mathrm{str}} = \frac{d^*}{\alpha\beta^2 \ln\ln(1+c)}.$$

Assume for simplicity that $d^* = 2$ and take c close to $\exp(1) - 1$. Then the maximum of the function $d/\ln\ln(d+c)$ is attained for $d = 1$, and we have

$$t^{\mathrm{tra}} = \frac{1}{\alpha\beta^2 \ln\ln(1+c)} = \frac{t^{\mathrm{str}}}{d^*}.$$

4.2.2. Polynomial rate

Theorem 4.4. *Let $\Omega^{\mathrm{res}} = [1,\infty) \times [d^*]$ with $d^* \geqslant 1$. Let S be a linear tensor product problem with $\lambda_1 = 1$ and with polynomially decaying eigenvalues λ_j, so that*

$$K_1 \exp\left(-\beta_1 (\ln j)^{\alpha}\right) \leqslant \lambda_j \leqslant K_2 \exp\left(-\beta_2 (\ln j)^{\alpha}\right) \quad \textit{for all } j \in \mathbb{N}$$

for some positive numbers α, β_1, β_2, K_1 and K_2.

Then S is $(T,\Omega^{\mathrm{res}})$-tractable (as well strongly (T,Ω)-tractable due to Theorem 4.2) in the class of linear information iff

$$A_p := \liminf_{x\to\infty} \frac{\ln T(x,1)}{(\ln x)^{1/\alpha}} \in (0,\infty].$$

If $\alpha \in (0,1]$ and $A_p > 0$ then the exponents of $(T,\Omega^{\mathrm{res}})$-tractability satisfy

$$\left(\frac{2}{\beta_1}\right)^{1/\alpha} A_p^{-1} \leqslant t^{\mathrm{tra}} \leqslant t^{\mathrm{str}} \leqslant \left(\frac{2}{\beta_2}\right)^{1/\alpha} A_p^{-1}.$$

If $\alpha \in (1,\infty)$ and $A_p > 0$ then the exponent of $(T,\Omega^{\mathrm{res}})$-tractability satisfies

$$\left(\frac{2}{\beta_1}\right)^{1/\alpha} \max_{d\in[d^*]} \frac{d^{1-1/\alpha}}{A_{p,d}} \leqslant t^{\mathrm{tra}} \leqslant \left(\frac{2}{\beta_2}\right)^{1/\alpha} \max_{d\in[d^*]} \frac{d^{1-1/\alpha}}{A_{p,d}},$$

where

$$A_{p,d} = \liminf_{x\to\infty} \frac{\ln T(x,d)}{(\ln x)^{1/\alpha}},$$

(clearly, $A_{p,d} \geqslant A_{p,1} = A_p > 0$), and the exponent of strong $(T,\Omega^{\mathrm{res}})$-tractability satisfies

$$(d^*)^{1-1/\alpha} \left(\frac{2}{\beta_1}\right)^{1/\alpha} A_p^{-1} \leqslant t^{\mathrm{str}} \leqslant (d^*)^{1-1/\alpha} \left(\frac{2}{\beta_2}\right)^{1/\alpha} A_p^{-1}.$$

Proof. We now have

$$\min\left\{ j \mid g_1(j) \leqslant \varepsilon^2 \right\} \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) \leqslant \min\left\{ j \mid g_2(j) \leqslant \varepsilon^2 \right\}$$

with $g_i(j) = K_i \exp\left(-\beta_i (\ln(j+1))^{\alpha}\right)$. This yields

$$\exp\left(\left(\beta_1^{-1} \ln\left(K_1 \varepsilon^{-2}\right)\right)^{1/\alpha}\right) - 1 \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) \leqslant \exp\left(\left(\beta_2^{-1} \ln\left(K_2 \varepsilon^{-2}\right)\right)^{1/\alpha}\right).$$

For small ε this leads to

$$\left(\frac{2\ln\varepsilon^{-1}}{\beta_1}\right)^{1/\alpha}(1+o(1)) \leqslant \ln n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) \leqslant \left(\frac{2\ln\varepsilon^{-1}}{\beta_2}\right)^{1/\alpha}(1+o(1)).$$

Hence, A from (i) of Theorem 4.2 satisfies $(\beta_2/2)^{1/\alpha}A_p \leqslant A \leqslant (\beta_1/2)^{1/\alpha}A_p$. Hence, $A>0$ iff $A_p>0$, and (i) of Theorem 4.2 yields the first part of Theorem 4.4, and the bound $t^{\mathrm{tra}} \geqslant (2/\beta_1)^{1/\alpha}A_p^{-1}$.

We now find bounds on the exponents assuming that $A_p > 0$. First, we estimate $n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right)$. With $x = x(\varepsilon, d) := \ln\left(\left(K_2^d\varepsilon^{-2}\right)^{1/\beta_2}\right)$, we have

$$n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right) \leqslant m_p(x,d) := \left|\left\{(i_1,\dots,i_d)\,\middle|\,\sum_{j=1}^{d}(\ln i_j)^{\alpha} < x\right\}\right|.$$

We now prove the following estimates on $m_p(x,d)$. Let $s>1$. If $\alpha\in(0,1]$ then there exists a positive number $C(s,d)$ such that

$$\exp\left(x^{1/\alpha}\right) - 1 \leqslant m_p(x,d) \leqslant C(s,d)\exp\left(sx^{1/\alpha}\right). \tag{24}$$

If $\alpha\in[1,\infty)$ then there exists a positive number $C(s,d)$ such that

$$\left(\exp\left(\left(\frac{x}{d}\right)^{1/\alpha}\right) - 1\right)^{d} \leqslant m_p(x,d) \leqslant C(s,d)\exp\left(sd^{1-1/\alpha}x^{1/\alpha}\right). \tag{25}$$

For $d=1$ we have $m_p(x,1) = \left|\left\{j \mid j < \exp\left(x^{1/\alpha}\right)\right\}\right|$ and $\exp\left(x^{1/\alpha}\right) - 1 \leqslant m_p(x,1) < \exp\left(x^{1/\alpha}\right)$.

We start with $\alpha\in(0,1]$. The lower bound is already proved since $m_p(x,d)\geqslant m_p(x,1)$. To obtain an upper bound on $m_p(x,d)$, we modify an argument from the proof of Theorem 3.1(ii) in [8]. Let $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ denote the Riemann zeta function. We show by induction on d that

$$m_p(x,d) \leqslant \zeta(s)^{d-1}\exp\left(sx^{1/\alpha}\right).$$

Clearly, this holds for $d=1$. Assume that our claim holds for d. Then

$$\begin{aligned} m_p(x,d+1) &= \sum_{k<\exp(x^{1/\alpha})} m_p\left(x-(\ln k)^{\alpha}, d\right) \\ &\leqslant \zeta(s)^{d-1}\sum_{k<\exp(x^{1/\alpha})}\exp\left(s\left(x-(\ln k)^{\alpha}\right)^{1/\alpha}\right). \end{aligned}$$

Since $(a-b)^{1/\alpha} \leqslant a^{1/\alpha} - b^{1/\alpha}$ for all $a\geqslant b\geqslant 0$ and $\alpha\in(0,1]$, we obtain

$$\begin{aligned} m_p(x,d+1) &\leqslant \zeta(s)^{d-1}\sum_{k<\exp(x^{1/\alpha})}\exp\left(sx^{1/\alpha}\right)\exp\left(-s\ln k\right) \\ &= \zeta(s)^{d-1}\exp\left(sx^{1/\alpha}\right)\sum_{k<\exp(x^{1/\alpha})}k^{-s} \leqslant \zeta(s)^{d}\exp\left(sx^{1/\alpha}\right). \end{aligned}$$

Let now $\alpha \in [1, \infty)$. Again we proceed by induction on d. The estimate (25) clearly holds for $d = 1$. Assume that our claim holds for d. Again we have

$$m_p(x, d+1) = \sum_{k<\exp(x^{1/\alpha})} m_p\left(x - (\ln k)^\alpha, d\right).$$

To get a lower bound on $m_p(x, d+1)$, we obtain

$$\begin{aligned} m_p(x, d+1) &\geqslant \int_1^{\exp(x^{1/\alpha})} \left(\exp\left(\left(\frac{x-(\ln\xi)^\alpha}{d}\right)^{1/\alpha}\right) - 1\right)^d d\xi \\ &\geqslant \int_1^{\exp((x/(d+1))^{1/\alpha})} \left(\exp\left(\left(\frac{x}{d+1}\right)^{1/\alpha}\right) - 1\right)^d d\xi \\ &\geqslant \left(\exp\left(\left(\frac{x}{d+1}\right)^{1/\alpha}\right) - 1\right)^{d+1}. \end{aligned}$$

We now obtain an upper bound on $m_p(x, d+1)$. Let $r = (1+s)/2$. Since $r > 1$, we can use the upper bound on $m_p(x, d)$ and obtain

$$\begin{aligned} m_p(x, d+1) \leqslant C(r, d) \Bigg\{ &\exp\left(r d^{1-1/\alpha} x^{1/\alpha}\right) \\ &+ \int_1^{\exp(x^{1/\alpha})} \exp\left(r d^{1-1/\alpha}\left(x - (\ln\xi)^\alpha\right)^{1/\alpha}\right) d\xi \Bigg\}. \end{aligned}$$

The substitution $z = \ln\xi$ leads to

$$\int_1^{\exp(x^{1/\alpha})} \exp\left(r d^{1-1/\alpha}\left(x - (\ln\xi)^\alpha\right)^{1/\alpha}\right) d\xi \leqslant \int_0^{x^{1/\alpha}} \exp(rh(z))\, dz,$$

where $h(z) = d^{1-1/\alpha}(x - z^\alpha)^{1/\alpha} + z$. Since

$$h'(z) = 1 - d^{1-1/\alpha}\left(\frac{x}{z^\alpha} - 1\right)^{1/\alpha - 1},$$

the function h takes its maximum at $z = (x/(d+1))^{1/\alpha}$, and we get

$$\begin{aligned} m_p(x, d+1) &\leqslant C(r, d)\left\{\exp\left(r d^{1-1/\alpha} x^{1/\alpha}\right) + x^{1/\alpha}\exp\left(r(d+1)^{1-1/\alpha} x^{1/\alpha}\right)\right\} \\ &\leqslant C(r, d)\left(1 + x^{1/\alpha}\right)\exp\left(r(d+1)^{1-1/\alpha} x^{1/\alpha}\right). \end{aligned}$$

Since

$$\begin{aligned} a :&= \sup_{x>0}\, (1 + x^{1/\alpha})\exp\left(-(s-r)(d+1)^{1-1/\alpha} x^{1/\alpha}\right) \\ &= \sup_{x>0}\, (1 + x^{1/\alpha})\exp\left(-(s-1)(d+1)^{1-1/\alpha} x^{1/\alpha}/2\right) < \infty \end{aligned}$$

we take $C(s, d+1) = aC(r, d)$ and conclude that

$$m_p(x, d+1) \leqslant C(s, d+1)\exp\left(s(d+1)^{1-1/\alpha} x^{1/\alpha}\right),$$

as claimed.

Let $\gamma := \max\{0, 1 - 1/\alpha\}$. Then (24) and (25) yield that for every $s > 1$ there exists a positive C_s such that

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant C_s \exp\left(s d^{\gamma} \left(\ln \varepsilon^{-2/\beta_2}\right)^{1/\alpha}\right) \tag{26}$$

for all $\varepsilon \in (0, 1]$ and $d \in [d^*]$. Knowing that $A_p > 0$, we want to show that

$$C_s \exp\left(s (d^*)^{\gamma} \left(\ln \varepsilon^{-2/\beta_2}\right)^{1/\alpha}\right) \leqslant C T\left(\varepsilon^{-1}, 1\right)^t \tag{27}$$

for some positive C and t. Let $\{A_n\}$ be a sequence in $(0, A_p)$ converging to A_p. Then for every n there exists a positive ε_n such that

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\left(\ln \varepsilon^{-1}\right)^{1/\alpha}} \geqslant A_n \quad \text{for all } \varepsilon \in (0, \varepsilon_n].$$

Observe that (27) is equivalent to

$$\frac{s(d^*)^{\gamma}}{t}\left(\frac{2}{\beta_2}\right)^{1/\alpha} + \frac{\ln(C_s/C)}{t\left(\ln \varepsilon^{-1}\right)^{1/\alpha}} \leqslant \frac{\ln T\left(\varepsilon^{-1}, 1\right)}{\left(\ln \varepsilon^{-1}\right)^{1/\alpha}}.$$

This holds for all $\varepsilon \in (0, \varepsilon_n]$ if $t \geqslant s(d^*)^{\gamma}(2/\beta_2)^{1/\alpha} A_n^{-1}$ and $C \geqslant C_s$. For $\varepsilon > \varepsilon_n$ we can keep the same t and, if necessary, increase C. Hence (27) holds with $t = s(d^*)^{\gamma}(2/\beta_2)^{1/\alpha} A_n^{-1}$. Thus, S is strongly (T, Ω^{res})-tractable. Taking s arbitrarily close to 1 and letting n tend to infinity, we conclude that $t^{\text{str}} \leqslant (d^*)^{\gamma}(2/\beta_2)^{1/\alpha} A_p^{-1}$.

We now show that in the case $\alpha \in (1, \infty)$ the exponent of strong tractability satisfies $t^{\text{str}} \geqslant (d^*)^{1-1/\alpha}(2/\beta_1)^{1/\alpha} A_p^{-1}$. Here, we use the estimate

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \geqslant m_p(z, d),$$

where $z = z(\varepsilon, d) := \ln\left(\left(K_1^d \varepsilon^{-2}\right)^{1/\beta_1}\right)$. For small ε, the left-hand side of (25) implies that there is a positive $c(d)$ such that

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \geqslant c(d) \exp\left(d^{1-1/\alpha}\left(\frac{1}{\beta_1} \ln\left(K_1^d \varepsilon^{-2}\right)\right)^{1/\alpha}\right). \tag{28}$$

Thus for all $t > t^{\text{str}}$ there exists a $C > 0$ such that for small ε, we have

$$C T(\varepsilon^{-1}, 1)^t \geqslant c(d^*) \exp\left((d^*)^{1-1/\alpha}\left(\frac{1}{\beta_1} \ln\left(K_1^d \varepsilon^{-2}\right)\right)^{1/\alpha}\right),$$

which is equivalent to

$$\frac{\ln T\left(\varepsilon^{-1}, 1\right)}{\left(\ln \varepsilon^{-1}\right)^{1/\alpha}} \geqslant \frac{\ln(c(d^*)/C)}{t\left(\ln \varepsilon^{-1}\right)^{1/\alpha}} + \frac{(d^*)^{1-1/\alpha}}{t}\left(\frac{1}{\beta_1} \frac{\ln\left(K_1^d \varepsilon^{-2}\right)}{\ln \varepsilon^{-1}}\right)^{1/\alpha}.$$

Taking the limit inferior as $\varepsilon \to 0$, we obtain $A_p \geqslant (d^*)^{1-1/\alpha}(2/\beta_1)^{1/\alpha} t^{-1}$, and $t^{\text{str}} \geqslant (d^*)^{1-1/\alpha}$ $(2/\beta_1)^{1/\alpha} A_p^{-1}$.

We finally find estimates of the exponent of tractability for $\alpha \in (1, \infty)$. We proceed similarly as before and assume that

$$CT\left(\varepsilon^{-1}, d\right)^t \geqslant n\left(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}\right) \quad \forall d \in [d^*].$$

By (28), this implies that

$$\frac{t \ln T\left(\varepsilon^{-1}, d\right)}{\left(\ln \varepsilon^{-1}\right)^{1/\alpha}} \geqslant d^{1-1/\alpha}\left(\frac{2}{\beta_1}\right)^{1/\alpha}(1+o(1))$$

for small ε. This yields that $t^{\mathrm{tra}} \geqslant (2/\beta_1)^{1/\alpha} \max_{d\in[d^*]} d^{1-1/\alpha}/A_{p,d}$.

To get an upper bound on t^{tra}, we use (26), and conclude that it is enough to find positive C and t such that

$$C_s \exp\left(s d^{1-1/\alpha}\left(\ln \varepsilon^{-2/\beta_2}\right)^{1/\alpha}\right) \leqslant CT(\varepsilon^{-1}, d)^t \quad \forall d \in [d^*].$$

This holds for $t \geqslant s \max_{d\in[d^*]} d^{1-1/\alpha}(2/\beta_2)^{1/\alpha} A_{p,d}^{-1}$. Since s can be arbitrarily close to one, we get that $t^{\mathrm{tra}} \leqslant \max_{d\in[d^*]} d^{1-1/\alpha}(2/\beta_2)^{1/\alpha} A_{p,d}^{-1}$, which completes the proof. □

For polynomially decaying eigenvalues, Theorem 4.4 states that strong tractability (and tractability) are equivalent to the condition $A_p > 0$. If we know the precise order of convergence of λ, so that $\beta_1 = \beta_2 = \beta > 0$, then we know the exponents of tractability. For $\alpha \in (0, 1]$ we have

$$t^{\mathrm{tra}} = t^{\mathrm{str}} = \left(\frac{2}{\beta}\right)^{1/\alpha} A_p^{-1},$$

whereas for $\alpha \in (1, \infty)$ we have

$$t^{\mathrm{tra}} = \left(\frac{2}{\beta}\right)^{1/\alpha} \max_{d\in[d^*]} \frac{d^{1-1/\alpha}}{A_{p,d}},$$

$$t^{\mathrm{str}} = (d^*)^{1-1/\alpha}\left(\frac{2}{\beta}\right)^{1/\alpha} A_p^{-1}.$$

As before, it may happen that $t^{\mathrm{str}} > t^{\mathrm{tra}}$.

We now illustrate Theorem 4.4 for a number of tractability functions T.

- Polynomial tractability, $T(x, y) = xy$. Then $A_{p,d} = A_p$ and its value depends on α. We have $A_p = 0$ for $\alpha < 1$, and $A_p = 1$ for $\alpha = 1$, and $A_p = \infty$ for $\alpha > 1$. Hence, we have strong tractability (and tractability) iff $\alpha \geqslant 1$. For $\alpha > 1$, we have $t^{\mathrm{tra}} = t^{\mathrm{str}} = 0$, whereas for $\alpha = 1$ and $\beta_1 = \beta_2 = \beta > 0$, we have $t^{\mathrm{tra}} = t^{\mathrm{str}} = 2/\beta$.
- Separable restrictive tractability, $T(x, y) = f_1(x)$ with f_1 as for exponential decaying eigenvalues. Then strong $(T, \Omega^{\mathrm{res}})$-tractability holds iff

$$A_{p,d} = A_p = \liminf_{x\to\infty} \frac{\ln f_1(x)}{(\ln x)^{1/\alpha}} \in (0, \infty].$$

Note that $A_p > 0$ iff $f_1(x)$ is at least of order $\exp\left(\eta(\ln x)^{1/\alpha}\right)$ for some positive η. If we take $f_1(x) = \exp\left(\eta(\ln x)^{1/\alpha}\right)$ then we have strong tractability with $A_p = \eta$. For $\beta_1 = \beta_2 = \beta > 0$, the exponents are $t^{\mathrm{str}} = t^{\mathrm{tra}} = (d^*)^{(1-1/\alpha)_+}(2/\beta)^{1/\alpha}\eta^{-1}$.

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). Then $(T, \Omega^{\mathrm{res}})$-tractability holds iff

$$A_{p,d} = f(d) \liminf_{x\to\infty} \frac{f(x)}{(\ln x)^{1/\alpha}} \in (0, \infty].$$

Hence, $A_p = A_{p,1} > 0$ iff $f(x)$ is at least of order $\eta(\ln x)^{1/\alpha}$ for some positive η. For example, if we take $f(x) = \eta(\ln(x + c))^{1/\alpha}$ with a positive c, then $A_{p,d} = f(d)\eta$. For a given $\alpha \in [1, \infty)$, $\beta_1 = \beta_2 = \beta > 0$, and sufficiently small c, the maximum of the function $d^{1-1/\alpha}/A_{p,d}$ is attained for $d = 1$, and we have

$$t^{\mathrm{tra}} = \frac{2^{1/\alpha}}{\eta^2(\beta \ln(1 + c))^{1/\alpha}} = \frac{t^{\mathrm{str}}}{(d^*)^{1-1/\alpha}}.$$

4.2.3. Logarithmic rate

Theorem 4.5. *Let $(\Omega^{\mathrm{res}} = [1, \infty) \times [d^*]$ with $d^* \geqslant 1$. Let S be a linear tensor product problem with $\lambda_1 = 1$ and with logarithmically decaying eigenvalues λ_j, so that*

$$K_1 \exp(-\beta \ln(\ln(j) + 1)) \leqslant \lambda_j \leqslant K_2 \exp(-\beta \ln(\ln(j) + 1)) \quad \text{for all } j \in \mathbb{N}$$

for some positive numbers β, K_1 and K_2.

Let $\beta \leqslant 2$. Then S is not $(T, \Omega^{\mathrm{res}})$-tractable in the class of linear information.

Let $\beta > 2$. Then S is $(T, \Omega^{\mathrm{res}})$-tractable (as well strongly $(T, \Omega^{\mathrm{res}})$-tractable due to Theorem 4.2) *in the class of linear information iff*

$$A_l := \liminf_{x\to\infty} \frac{\ln T(x, 1)}{x^{2/\beta}} \in (0, \infty].$$

If $\beta > 2$ and $A_l > 0$ then the exponent of $(T, \Omega^{\mathrm{res}})$-tractability satisfies

$$\max_{d\in[d^*]} \frac{K_1^{d/\beta}}{A_{l,d}} \leqslant t^{\mathrm{tra}} \leqslant \max_{d\in[d^*]} \frac{K_2^{d/\beta}}{A_{l,d}},$$

where

$$A_{l,d} := \liminf_{x\to\infty} \frac{\ln T(x, d)}{x^{2/\beta}},$$

(*clearly*, $A_{l,d} \geqslant A_{l,1} = A_l > 0$), *and the exponent of strong $(T, \Omega^{\mathrm{res}})$-tractability satisfies*

$$\frac{K_1^{1/\beta}}{A_l} \leqslant t^{\mathrm{str}} \leqslant \frac{K_2^{d^*/\beta}}{A_l}.$$

(*Note that the numbers K_1 and K_2 must satisfy $K_1 \leqslant 1 \leqslant K_2$. Thus, if $K_1 = K_2 = 1$, we have also $K_1^{1/\beta} = K_2^{d^*/\beta}$, and the last inequality becomes an equality.*)

Proof. We now have

$$\min\left\{ j \mid g_1(j) \leqslant \varepsilon^2 \right\} \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) \leqslant \min\left\{ j \mid g_2(j) \leqslant \varepsilon^2 \right\}$$

with $g_i(j) = K_i \exp(-\beta \ln(\ln(j + 1) + 1))$. This yields

$$\exp\left(K_1^{1/\beta}\varepsilon^{-2/\beta} - 1\right) - 1 \leqslant n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) \leqslant \exp\left(K_2^{1/\beta}\varepsilon^{-2/\beta} - 1\right).$$

For small ε this leads to

$$K_1^{1/\beta}\varepsilon^{-2/\beta}(1+o(1)) \leqslant \ln n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right) \leqslant K_2^{1/\beta}\varepsilon^{-2/\beta}(1+o(1)).$$

Assume first that $\beta \leqslant 2$. Then

$$\liminf_{\varepsilon\to 0} \frac{\ln T\left(\varepsilon^{-1},1\right)}{\ln n\left(\varepsilon, S_1, \Lambda_1^{\text{all}}\right)} \leqslant K_1^{-1/\beta} \liminf_{\varepsilon\to 0} \frac{\ln T\left(\varepsilon^{-1},1\right)}{\varepsilon^{-1}+1}\,\frac{\varepsilon^{-1}+1}{\varepsilon^{-2/\beta}} = 0$$

due to (7). Therefore A from (i) of Theorem 4.2 is zero, and we do not have tractability, as claimed.

Assume then that $\beta > 2$. Then $K_2^{-1/\beta}A_l \leqslant A \leqslant K_1^{-1/\beta}A_l$. Hence, $A > 0$ iff $A_l > 0$, and (i) of Theorem 4.2 yields the first part of Theorem 4.4, and that $t^{\text{str}} \geqslant K_1^{1/\beta}A_l^{-1}$.

We now find bounds on the exponents, assuming that $A_l > 0$. First, we estimate $n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right)$. With $x = x(\varepsilon, d) := \ln\left(\left(K_2^d/\varepsilon^2\right)^{1/\beta}\right)$ we get

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant m_l(x,d) := \left|\left\{(i_1,\ldots,i_d)\,\middle|\,\sum_{j=1}^{d}\ln(\ln(i_j)+1) < x\right\}\right|.$$

We prove that for every $s > 1$, there exists a positive number $C(s,d)$ such that

$$\exp\left(\exp(x)-1\right)-1 \leqslant m_l(x,d) \leqslant C(s,d)\exp\left(s\left(\exp(x)-1\right)\right). \tag{29}$$

Let $\eta := \exp(x)$. Clearly, we have $m_l(x,1) = |\{j \mid j < \exp(\eta-1)\}|$, which implies that

$$\exp(\eta-1)-1 \leqslant m_l(x,1) \leqslant \exp(\eta-1).$$

Let now $d \geqslant 1$ and assume that (29) holds for d. Then

$$m_l(x,d+1) = \sum_{k<\exp(\eta-1)} m_l\left(x-\ln(\ln(k)+1), d\right).$$

Thus, we get the trivial lower bound estimate

$$m_l(x,d+1) \geqslant m_l(x,d) \geqslant \exp(\eta-1)-1.$$

We now obtain an upper bound on $m_l(x,d+1)$. Let $r = (1+s)/2$. Then

$$m_l(x,d+1) \leqslant C(r,d)\left\{\exp(r(\eta-1)) + \int_1^{\exp(\eta-1)} \exp\left(r\left(\exp\left(x-\ln(\ln(\xi)+1)\right)-1\right)\right)d\xi\right\}.$$

The last integral is of the form

$$\int_1^{\exp(\eta-1)} \exp\left(r\left(\frac{\eta}{\ln(\xi)+1}-1\right)\right)d\xi = \int_1^{\eta}\exp(rh(z))\,dz,$$

where $z = \ln(\xi) + 1$, and $h : [1, \eta] \to \mathbb{R}$ with $h(z) = \eta/z + z/r - (1 + 1/r)$. It is easy to check that h takes its maximum $\eta - 1$ at the point $z = 1$. So we have

$$\int_1^\eta \exp(rh(z))\,dz \leqslant (\eta - 1)\exp(r(\eta - 1)).$$

This implies that

$$\begin{aligned} m_l(x, d+1) &\leqslant C(r,d)\eta\exp(r(\eta-1)) \\ &= C(r,d)\eta\exp\left((r-s)(\eta-1)\right)\exp\left(s(\eta-1)\right) \\ &\leqslant C(r,d)\left(\sup_{\xi \geqslant 1} \xi\exp\left(-(s-1)(\xi-1)/2\right)\right)\exp\left(s(\eta-1)\right) \\ &\leqslant C(s, d+1)\exp(s(\eta-1)) \end{aligned}$$

for suitably large $C(s, d+1)$, as claimed.

Due to (29), we conclude that

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \leqslant C(s,d)\exp\left(s\left(K_2^{d/\beta}\varepsilon^{-2/\beta} - 1\right)\right).$$

For $A_l \in (0, \infty]$, $\varepsilon \in (0, 1]$ and $d \in [d^*]$, we want to show that

$$C(s,d)\exp\left(s\left(K_2^{d/\beta}\varepsilon^{-2/\beta} - 1\right)\right) \leqslant C\,T(\varepsilon^{-1}, 1)^t \tag{30}$$

for some positive C and t. Therefore, let $\{A_n\}$ be a sequence in $(0, A_l)$ converging to A_l. Thus, for every n there exists a positive ε_n such that

$$\frac{\ln T(\varepsilon^{-1}, 1)}{\varepsilon^{-2/\beta}} \geqslant A_n \quad \text{for all } \varepsilon \in (0, \varepsilon_n].$$

Then (30) is equivalent to

$$\frac{sK_2^{d/\beta}}{t} + \frac{\ln(C(s,d)/C) - s}{t\varepsilon^{-2/\beta}} \leqslant \frac{\ln T(\varepsilon^{-1}, 1)}{\varepsilon^{-2/\beta}}.$$

This holds for all $\varepsilon \in (0, \varepsilon_n]$ if $t \geqslant sK_2^{d/\beta}A_n^{-1}$ and $C \geqslant C(s,d)$. For $\varepsilon \in (\varepsilon_n, 1]$ we can keep the same t and, if necessary, increase C. Letting s tend to 1 and n tend to infinity, we conclude $t^{\text{str}} \leqslant K_2^{d^*/\beta}A_l^{-1}$.

We can similarly show bounds on t^{tra}, since $\left(\ln T\left(\varepsilon^{-1}, d\right)\right)/\varepsilon^{-2/\beta}$ is arbitrarily close to $A_{l,d}$ for small ε. This leads to $t^{\text{tra}} \leqslant \max_{d\in[d^*]} K_2^{d/\beta}/A_{l,d}$. To get a lower bound on t^{tra}, we use the left-hand side inequality in (29) to conclude that

$$n\left(\varepsilon, S_d, \Lambda_d^{\text{all}}\right) \geqslant \exp\left(K_1^{d/\beta}\varepsilon^{-2/\beta} - 1\right) - 1.$$

This yields that $t^{\text{tra}} \geqslant \max_{d\in[d^*]} K_1^{d/\beta}/A_{l,d}$, and completes the proof. □

For logarithmically decaying eigenvalues, Theorem 4.5 states that for $\beta \leqslant 2$, we do not have tractability. This means that the eigenvalues λ_j converge to zero too slowly, no matter how we choose the tractability function T. For $\beta > 2$, strong tractability (and tractability) are equivalent to the condition $A_l > 0$. In this case, and for $K_1 = K_2 = 1$, we know the exponents of tractability satisfy

$$t^{\text{tra}} = t^{\text{str}} = A_l^{-1}.$$

We now illustrate Theorem 4.5 for a number of tractability functions T.

- Polynomial tractability, $T(x, y) = xy$. Then for $\beta > 2$, we have $A_{l,d} = A_l = 0$. Hence, strong tractability (and tractability) does *not* hold.
- Separable restrictive tractability, $T(x, y) = f_1(x)$ with f_1 as for exponential decaying eigenvalues. Let $\beta > 2$. Then strong (T, Ω^{res})-tractability holds iff

$$A_{l,d} = A_l = \liminf_{x\to\infty} \frac{\ln f_1(x)}{x^{2/\beta}} \in (0, \infty].$$

  Note that $A_l > 0$ iff $\ln f_1(x)$ is at least of order x^α with $\alpha \in [2/\beta, 1)$. If we take $f_1(x) = \exp\left(x^\alpha\right)$ then we have strong tractability with $A_l = 0$ for $\alpha \in (2/\beta, 1)$, and then $t^{\text{str}} = t^{\text{tra}} = 0$, whereas $A_l = 1$ for $\alpha = 2/\beta$ and $t^{\text{str}} = t^{\text{tra}} = 1$ for $K_2 = K_1 = 1$.
- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). For $\beta > 2$, (T, Ω^{res})-tractability holds iff

$$A_{l,d} = f(d) \liminf_{x\to\infty} \frac{f(x)}{x^{2/\beta}} \in (0, \infty].$$

  Hence, $A_l = A_{l,1} > 0$ iff $f(x)$ is at least of order $x^{2/\beta}$. For example, if we take $f(x) = x^{2/\beta}$ then $A_{l,d} = f(d)$. For $K_2 \leqslant \exp(1/d^*)$, the maximum of the function $K_2^{d/\beta}/f(d)$ is attained for $d = 1$, and $t^{\text{tra}} \leqslant K_2^{1/\beta}$ and $t^{\text{str}} \leqslant \exp(1)^{1/\beta}$.

4.3. Restricted tractability with $d^* \geqslant 1$ and $\varepsilon_0 < 1$

Based on the results for restricted tractability in ε and d, it is easy to study restricted tractability with $d^* \geqslant 1$ and $\varepsilon \in (0, 1)$. In this subsection we let

$$\Omega^{\text{res}} = \Omega^{\text{res}}(\varepsilon_0, d^*) = [1, \infty) \times [d^*] \cup [1, \varepsilon_0^{-1}) \times \mathbb{N}$$

for $d^* \in \mathbb{N}_0$ and $\varepsilon_0 \in (0, 1]$.

Hence, restricted tractability in ε corresponds to $\Omega^{\text{res}}(\varepsilon_0, 0) = [1, \varepsilon_0^{-1}) \times \mathbb{N}$ with $\varepsilon_0 \in (0, 1)$, and restricted tractability in d corresponds to $\Omega(1, d^*) = [1, \infty) \times [d^*]$ with $d^* \geqslant 1$.

Since $\Omega^{\text{res}}(\varepsilon_0, d^*) = \Omega^{\text{res}}(\varepsilon_0, 0) \cup \Omega^{\text{res}}(1, d^*)$, it is obvious that strong tractability and tractability for $d^* \geqslant 1$ and $\varepsilon_0 \in (0, 1)$ are equivalent to restricted strong tractability and tractability in ε and d, respectively. We summarize this simple fact in the following lemma.

Lemma 4.6. *Let $d^* \geqslant 1$ and $\varepsilon_0 \in (0, 1)$. Let S be a linear tensor product problem with $\lambda_1 = 1$. Then*

- *S is strongly $(T, \Omega^{\text{res}}(\varepsilon_0, d^*))$-tractable in the class of linear information iff S is strongly $(T, \Omega^{\text{res}}(\varepsilon_0, 0))$- and strongly $(T, \Omega^{\text{res}}(1, d^*))$-tractable in the class of linear information.*
- *S is $(T, \Omega^{\text{res}}(\varepsilon_0, d^*))$-tractable in the class of linear information iff S is $(T, \Omega^{\text{res}}(\varepsilon_0, 0))$- and $(T, \Omega^{\text{res}}(1, d^*))$-tractable in the class of linear information.*
- *The exponents of strong tractability and tractability for $\Omega^{\text{res}}(\varepsilon_0, d^*)$ are the respective maxima of the exponents for $\Omega^{\text{res}}(\varepsilon_0, 0)$ and $\Omega^{\text{res}}(1, d^*)$.*

We now combine the results of the previous subsections and present two theorems on the tractability of S for $\Omega^{\mathrm{res}}(\varepsilon_0, d^*)$. In these theorems, strong tractability of S means that S is strongly $(T, \Omega^{\mathrm{res}}(\varepsilon_0, d^*))$-tractable in the class of linear information, and tractability of S means that S is $(T, \Omega^{\mathrm{res}}(\varepsilon_0, d^*))$-tractable in the class of linear information.

Theorem 4.7. *Let $d^* \geqslant 1$ and $\varepsilon_0 \in (0, 1)$. Let S be a linear tensor product problem with $\lambda_1 = 1$.*

- *Let $\lambda_2 = 1$. Then S is not tractable.*
- *Let $\varepsilon_0^2 < \lambda_2 < 1$. Then S is not strongly tractable, and S is tractable iff*

$$A = \liminf_{\varepsilon \to 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\mathrm{all}})} \in (0, \infty],$$

$$B = \liminf_{d \to \infty} \inf_{\varepsilon \in [\varepsilon_0, \sqrt{\lambda_2})} \frac{\ln T(\varepsilon^{-1}, d)}{\alpha(\varepsilon) \ln d} \in (0, \infty],$$

  where $\alpha(\varepsilon) = \lceil 2 \ln(1/\varepsilon) / \ln(1/\lambda_2) \rceil - 1$.
  If $A > 0$ and $B > 0$ then

$$\max\left(A^{-1}, B^{-1}\right) \leqslant t^{\mathrm{tra}} \leqslant \max\left(d^* A^{-1}, B^{-1}\right).$$

- *Let $0 < \lambda_2 \leqslant \varepsilon_0^2$.*
  Let $\lim_{\varepsilon \to 0} n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) < \infty$. Then S is strongly tractable and $t^{\mathrm{str}} = 0$.
  Let $\lim_{\varepsilon \to 0} n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) = \infty$. Then S is strongly tractable iff S is tractable iff

$$A = \liminf_{\varepsilon \to 0} \frac{\ln T(\varepsilon^{-1}, 1)}{\ln n(\varepsilon, S_1, \Lambda_1^{\mathrm{all}})} \in (0, \infty].$$

  If $A > 0$ then

$$A^{-1} \leqslant t^{\mathrm{tra}} \leqslant t^{\mathrm{str}} \leqslant d^* A^{-1}.$$

- *Let $\lambda_2 = 0$. Then $n(\varepsilon, S_d, \Lambda_d^{\mathrm{all}}) = 1$ for all $(\varepsilon, d) \in \Omega(\varepsilon_0, d^*)$, and S is strongly tractable with $t^{\mathrm{str}} = 0$.*

Proof. For $\lambda_2 = 1$, it is enough to apply the first part of Lemma 3.1.

Let $\varepsilon_0^2 < \lambda_2 < 1$. The lack of strong tractability follows from the second part of Lemma 3.1. Tractability in ε holds iff $B \in (0, \infty]$ due to the second part of Theorem 4.1. Let $\lim_{\varepsilon \to 0} n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) < \infty$. Then tractability in d holds and, in this case, $A \in (0, \infty]$, due to the reasoning before Theorem 4.2. Let $\lim_{\varepsilon \to 0} n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) = \infty$. Then tractability in d holds iff $A \in (0, \infty]$ due to Theorem 4.2. Hence, Lemma 4.6 implies that S is tractable iff both $A, B \in (0, \infty]$. The bounds on t^{tra} now follow from Theorems 4.1 and 4.2 along with Lemma 4.6.

For $0 < \lambda_2 \leqslant \varepsilon_0^2$ and $\lim_{\varepsilon \to 0} n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) < \infty$, we conclude that S is strongly tractable due to the first part of Theorem 4.1, the reasoning before Theorem 4.2 and Lemma 4.6. In this case, $t^{\mathrm{str}} = 0$.

For $0 < \lambda_2 \leqslant \varepsilon_0^2$ and $\lim_{\varepsilon \to 0} n\left(\varepsilon, S_1, \Lambda_1^{\mathrm{all}}\right) = \infty$, strong tractability in ε holds with $t^{\mathrm{str}} = 0$ due to the first part of Theorem 4.1, and strong tractability in d is equivalent to tractability in d

and equivalent to $A \in (0, \infty]$ due to Theorem 4.2. This and Lemma 4.6 yield that S is strongly tractable iff S is tractable iff $A \in (0, \infty]$. The bounds on t^{tra} and t^{str} follow from Theorem 4.2.

For $\lambda_2 = 0$, the problem is trivial due to the last part of Lemma 3.1. □

We now summarize tractability conditions for $\Omega(\varepsilon_0, d^*)$, assuming the specific rates of convergence of the eigenvalues $\lambda = \{\lambda_j\}$ as discussed in Theorems 4.3–4.5.

Theorem 4.8. *Let $d^* \geqslant 1$ and $\varepsilon_0 \in (0, 1)$. Let S be a linear tensor product problem with $\lambda_2 < \lambda_1 = 1$.*

- *Let $\lambda_j = \Theta\left(\exp\left(-\beta j^{\alpha}\right)\right)$ converge to zero with an exponential rate for some positive α* and *β.*
  - *Let $\varepsilon_0^2 < \lambda_2$. Then S is not strongly tractable, and S is tractable iff $A_e = A_{e,1} \in (0, \infty]$ and $B \in (0, \infty]$ with*
  $$A_{e,d} = \liminf_{x\to\infty} \frac{\ln T(x,d)}{\ln \ln x} \in (0, \infty],$$
  and B as in Theorem 4.7. If $A_e > 0$ and $B > 0$ then
  $$t^{\mathrm{tra}} = \max\left(\frac{1}{\alpha} \max_{d\in[d^*]} \frac{d}{A_{e,d}}, \frac{1}{B}\right).$$
  - *Let $\lambda_2 \leqslant \varepsilon_0^2$. Then S is strongly tractable iff $A_e \in (0, \infty]$. If $A_e > 0$ then*
  $$t^{\mathrm{str}} = \frac{d^*}{\alpha A_e} \quad \text{and} \quad t^{\mathrm{tra}} = \frac{1}{\alpha} \max_{d\in[d^*]} \frac{d}{A_{e,d}}.$$
- *Let $\lambda_j = \Theta\left(\exp\left(-\beta(\ln j)^{\alpha}\right)\right)$ converge to zero with a polynomial rate for some positive α and β.*
  - *Let $\varepsilon_0^2 < \lambda_2$. Then S is not strongly tractable, and S is tractable iff $A_p = A_{p,1} \in (0, \infty]$ and $B \in (0, \infty]$ with*
  $$A_{p,d} = \liminf_{x\to\infty} \frac{\ln T(x,d)}{(\ln x)^{1/\alpha}} \in (0, \infty],$$
  and B as in Theorem 4.7. If $A_p > 0$ and $B > 0$ then
  $$t^{\mathrm{tra}} = \max\left(\left(\frac{2}{\beta}\right)^{1/\alpha} \max_{d\in[d^*]} \frac{d^{(1-1/\alpha)_+}}{A_{p,d}}, \frac{1}{B}\right).$$
  - *Let $\lambda_2 \leqslant \varepsilon_0^2$. Then S is strongly tractable iff $A_p \in (0, \infty]$. If $A_p > 0$ then*
  $$t^{\mathrm{str}} = \left(\frac{2}{\beta}\right)^{1/\alpha} \frac{(d^*)^{(1-1/\alpha)_+}}{A_p} \quad \text{and} \quad t^{\mathrm{tra}} = \left(\frac{2}{\beta}\right)^{1/\alpha} \max_{d\in[d^*]} \frac{d^{(1-1/\alpha)_+}}{A_{p,d}}.$$
- *Let $\lambda_j = \exp\left(-\beta\left(\ln(\ln(j)+1)\right)\right)$ converge to zero with a logarithmic rate for some positive β. For $\beta \leqslant 2$, S is not tractable. For $\beta > 2$, we have the following.*
  - *Let $\varepsilon_0^2 < \lambda_2$. Then S is not strongly tractable, and S is tractable iff $A_l \in (0, \infty]$ and $B \in (0, \infty]$ with*
  $$A_l = \liminf_{x\to\infty} \frac{\ln T(x,1)}{x^{2/\beta}} \in (0, \infty]$$

and B as in Theorem 4.7. If $A_l > 0$ and $B > 0$ then

$$t^{\mathrm{tra}} = \max\left(\frac{1}{A_l}, \frac{1}{B}\right).$$

∘ *Let $\lambda_2 \leqslant \varepsilon_0^2$. Then S is strongly tractable iff $A_l \in (0, \infty]$. If $A_l > 0$ then*

$$t^{\mathrm{str}} = t^{\mathrm{tra}} = \frac{1}{A_l}.$$

Proof. For the exponential rate and $\varepsilon_0^2 < \lambda_2$, the lack of strong tractability follows from Theorem 4.7, whereas tractability is equivalent to $A_e, B \in (0, \infty]$ due to Theorems 4.3 and 4.1. The formula for t^{tra} also follows from these two theorems and Lemma 4.6.

For the exponential rate and $\lambda_2 \leqslant \varepsilon_0^2$, strong tractability in ε trivially holds, and strong tractability in d holds iff $A_e > 0$ due to Theorem 4.3. The formulas for t^{str} and t^{tra} are also from Theorem 4.3.

For the polynomial and logarithmic rates, we proceed in the same way and use Theorem 4.4 for the polynomial case, and Theorem 4.5 for the logarithmic case, instead of Theorem 4.3. □

We illustrate Theorems 4.7 and 4.8 for a number of tractability functions T.

- Polynomial tractability, $T(x, y) = xy$. Then $A_{e,d} = A_{p,d} = \infty$ for $\alpha > 1$, whereas $A_{p,d} = 1$ if $\alpha = 1$, and $A_{p,d} = 0$ for $\alpha < 1$. Finally, $A_{l,d} = 0$ for $\beta > 2$. Hence, for logarithmically and polynomially decaying eigenvalues with $\alpha < 1$, S is not tractable.
  Let $\varepsilon_0^2 < \lambda_2$. We have $B = 1/\alpha(\varepsilon_0)$. Then for exponentially and polynomially decaying eigenvalues with $\alpha > 1$, S is not strongly tractable but is tractable with the exponent

  $$t^{\mathrm{tra}} = \alpha(\varepsilon_0) = \lceil 2\ln(1/\varepsilon_0)/\ln(1/\lambda_2)\rceil - 1.$$

  For polynomially decaying eigenvalues with $\alpha = 1$, S is not strongly tractable but is tractable with the exponent

  $$t^{\mathrm{tra}} = \max\left(\frac{2}{\beta}, \alpha(\varepsilon_0)\right).$$

  Let $\lambda_2 \leqslant \varepsilon_0^2$. Then for exponentially and polynomially decaying eigenvalues with $\alpha > 1$, S is strongly tractable with $t^{\mathrm{str}} = 0$. For polynomially decaying eigenvalues with $\alpha = 1$, S is strongly tractable with $t^{\mathrm{str}} = 2/\beta$.
- Separable restrictive tractability, $T(x, y) = f_1(x)$ for $(x, y) \in \Omega(1, d^*)$, and $T(x, y) = f_2(y)$ for $(x, y) \in \Omega(\varepsilon_0, 0)$ with non-decreasing f_1 and f_2 such that $\lim_{t\to\infty}(\ln f_i(t))/t = 0$.
  For simplicity, let us take $f_i(t) = \exp\left((\ln t)^{\alpha_i}\right)$ for some positive α_i. Then $A_{e,d} = \infty$, whereas $A_{p,d} = \infty$ if $\alpha_1 > 1/\alpha$, and $A_{p,d} = 1$ if $\alpha_1 = 1/\alpha$, and $A_{p,d} = 0$ if $\alpha_1 < 1/\alpha$. Finally, $A_{l,d} = 0$. Hence, for polynomially decaying eigenvalues with $\alpha_1 < 1/\alpha$, and for logarithmically decaying eigenvalues S is not tractable.
  Let $\varepsilon_0^2 < \lambda_2$. If $\alpha_2 < 1$, then $B = 0$ and S is not tractable. Let $\alpha_2 \geqslant 1$. Then for exponentially and polynomially decaying eigenvalues with $\alpha_1 > 1/\alpha$, S is not strongly tractable but S is tractable. The exponent of tractability is $t^{\mathrm{tra}} = \alpha(\varepsilon_0)$ if $\alpha_2 = 1$ and $t^{\mathrm{tra}} = 0$ if $\alpha_2 > 1$. For polynomially decaying eigenvalues with $\alpha_1 = 1/\alpha$, S is not strongly tractable but is tractable with exponent

  $$t^{\mathrm{tra}} = \max\left\{\left(\frac{2}{\beta}\right)^{1/\alpha}(d^*)^{(1-1/\alpha)_+}, \alpha(\varepsilon_0)\right\} \quad \text{if } \alpha_2 = 1$$

and

$$t^{\mathrm{tra}} = \left(\frac{2}{\beta}\right)^{1/\alpha} (d^*)^{(1-1/\alpha)_+} \quad \text{if } \alpha_2 > 1.$$

Let $\lambda_2 \leqslant \varepsilon_0^2$. Then for exponentially and polynomially decaying eigenvalues with $\alpha_1 > 1/\alpha$, S is strongly tractable and $t^{\mathrm{str}} = 0$. For polynomially decaying eigenvalues with $\alpha_1 = 1/\alpha$, S is strongly tractable with

$$t^{\mathrm{tra}} = t^{\mathrm{str}} = \left(\frac{2}{\beta}\right)^{1/\alpha} (d^*)^{(1-1/\alpha)_+}.$$

- Non-separable symmetric tractability, $T(x, y) = \exp(f(x)f(y))$ with f as in (13). For simplicity, let us take $f(x) = (\ln(x+1))^{\eta}$ for some positive η. Then $A_{e,d} = \infty$, whereas $A_{p,d} = \infty$ for $\eta > 1/\alpha$, and $A_{p,d} = f(d)$ for $\eta = 1/\alpha$, and $A_{p,d} = 0$ for $\eta < 1/\alpha$. Finally, $A_{l,d} = 0$. Hence, S is not tractable for logarithmically and polynomially decaying eigenvalues with $\eta < 1/\alpha$.
  Let $\varepsilon_0^2 < \lambda_2$. If $\eta < 1$, then $B = 0$ and S is not tractable. Let $\eta \geqslant 1$. Then for exponentially and polynomially decaying eigenvalues with $\eta > 1/\alpha$, S is not strongly tractable but S is tractable. In the case $\eta > 1$ we have $t^{\mathrm{tra}} = 0$. For polynomially decaying eigenvalues with $\eta = 1/\alpha$, S is not strongly tractable but tractable. If we have $\eta > 1$, then $\alpha \in (0, 1)$ and

$$t^{\mathrm{tra}} = \left(\frac{2}{\beta \ln 2}\right)^{1/\alpha}.$$

  Let $\lambda_2 \leqslant \varepsilon_0^2$. Then for exponentially and polynomially decaying eigenvalues with $\eta > 1/\alpha$, S is strongly tractable and $t^{\mathrm{str}} = 0$. For polynomially decaying eigenvalues with $\eta = 1/\alpha$, S is strongly tractable with

$$t^{\mathrm{str}} = (d^*)^{(1-1/\alpha)_+} \left(\frac{2}{\beta \ln 2}\right)^{1/\alpha}, \quad t^{\mathrm{tra}} = \left(\frac{2}{\beta}\right)^{1/\alpha} \max_{d \in [d^*]} \frac{d^{(1-1/\alpha)_+}}{(\ln(d+1))^{1/\alpha}}.$$

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