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A new characterization of (s, t)-weak tractability[☆]

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Dedicated to the memory of Joseph F. Traub (1932–2015)

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ABSTRACT

Siedlecki and Weimar (2015) defined the notion of (s, t)-weak tractability for linear multivariate problems, which holds if the information complexity of the multivariate problem is not exponential in d^t and ε^{-s}, where d is the number of variables and ε is the error threshold with positive s and t. For Hilbert spaces, they were able to characterize (s, t)-weak tractability in terms of how quickly the corresponding ordered singular values decay. Using this result, they studied the embedding of $H^r(\mathbb{T}^d)$ into $L_2(\mathbb{T}^d)$, where $\mathbb{T}^d$ is the d-dimensional torus, determining precisely when this problem is (s, t)-tractable for a given d and r. Their proof is based on deep results of Kühn et al. (2014), which are complicated by the difficulty of ordering the singular values. In this paper, we provide a new characterization of (s, t)-weak tractability of multivariate problems over Hilbert spaces, which does not require us to order the singular values. This allows us to obtain a new, and somewhat simpler, proof of the Siedlecki and Weimar (2015) result that does not need to use the results of Kühn et al. (2014).

1. Introduction

Tractability of multivariate problems, first studied in [12], has become one of the most fruitful areas of research in information-based complexity (IBC) theory. Hundreds of papers have been written on the topic, along with several books, most notably the three-volume monograph series [8–10].

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In classical discrete complexity theory [3], problems that can be solved in polynomial time are considered to be tractable, whereas those requiring exponential time are deemed intractable; in the famous phrase coined by Richard Bellman, the latter suffer from the "curse of dimensionality". Of course, many important problems of discrete complexity theory are NP-complete. Therefore it is *unknown* whether or not these problems can be solved in polynomial time, the resolution of this question awaiting the solution of the P-versus-NP problem.

On the other hand, it is *known* that the standard formulations of many important classical problems of IBC are not polynomially tractable, the earliest example being Bakhvalov's result (original article [1], English translation [2]) showing that multivariate integration of r-times continuously differentiable functions is not polynomially tractable. But a problem's non-solvability in polynomial time does not necessarily imply that it suffers from the curse of dimensionality. (After a hiatus of 55 years, the aforementioned integration problem was eventually shown to suffer from the curse of dimensionality in [5].) In any case, if we are going to talk about the tractability of multivariate problems, then we need weaker notions of tractability.

One such notion, introduced in [11], is (s,t)-weak tractability. Roughly speaking, a problem is (s,t)-weakly tractable for some positive s and t if the number of information operations needed to compute an ε-accurate approximation is not exponential in $(1/\varepsilon)^s$ or in d^t. Hence (s,t)-weak tractability generalizes the concepts of weak tractability (which is $(1,1)$-weak tractability) and uniform weak tractability (which is (s,t)-weak tractability for all positive s and t) as special cases.

Now if arbitrary continuous linear information is admissible, then the various notions of tractability can be characterized by how quickly the singular values of the solution operator decay. For some of these notions (e.g., quasi-polynomial, polynomial and strong polynomial) this is fairly straightforward, whereas for other notions (e.g., (s,t)-weak tractability) this is somewhat more complicated. The common aspect of all these characterizations is that we need to order the singular values of the solution operator. This can be an onerous task for multivariate problems.

However, the older notions of tractability have alternate characterizations that do not need the singular values to be ordered. Typically, these involve convergence of an infinite series whose terms are all the singular values, but raised to some power, see the material in [8, Section 5.1]. In this spirit, we offer a new characterization of (s,t)-weak tractability (Theorem 3.1) that does not require us to order the singular values of the solution operator. (The paper [6] presents another approach to this problem.) We then use this result to revisit a topic covered in [11], namely the (s,t)-tractability of $L_2(\mathbb{T}^d)$-approximation of functions from the Sobolev space $H^r(\mathbb{T}^d)$ defined over the d-torus. We present a new proof of the result found in [11], namely, that this problem is (s,t)-weakly tractable iff $t>1$ or $rs>2$. Whereas the proof in [11] requires us to know the ordering of the singular values of the approximation operator, our proof does not. Our main idea is to replace sums of powers of singular values by multidimensional integrals. We then find estimates of these integrals that are tight enough for us to reach the desired conclusions; although based on some fairly-standard techniques, this is arguably the most difficult part of the analysis.

The paper is organized as follows. In Section 2, we precisely define the notion of (s,t)-tractability, and briefly review some well-known results of IBC that deal with nth minimal errors and algorithms achieving this error. In Section 3, we state and prove our new criterion for (s,t)-weak tractability. In Section 4, we describe the L_2-approximation problem for H^r-functions defined over the d-dimensional torus. Finally, in Section 5, we characterize (s,t)-weak tractability for the approximation problem defined in Section 4.

2. Preliminaries

Let $S=\{S_d\}_{d\in\mathbb{N}}$, where $S_d: F_d \to G_d$ is a linear operator between Hilbert spaces F_d and G_d. To make our notation easier we assume that F_d is infinite-dimensional.

We approximate $S_d(f)$ by algorithms $A_n(f)$ using n linear functionals on f, so that

$$A_n(f)=\phi_n(L_1(f),L_2(f),\dots,L_n(f)),$$

where $\phi_n:\mathbb{C}^n\to G_d$ can be an arbitrary mapping and the $L_j: F_d\to\mathbb{C}$ are linear functionals that can be chosen adaptively. That is, the choice of any L_j may depend on the previously-computed values

$L_1(f), L_2(f), \dots, L_{j-1}(f)$. We consider the worst case setting, in which the error of A_n is defined as

$$e(A_n) = \sup_{\|f\|_{F_d}\le 1} \|S_d(f) - A_n(f)\|_{G_d}.$$

It is well-known that we can find algorithms A_n such that $\lim_{n\to\infty} e(A_n) = 0$ iff S_d is compact. Thus we shall assume that S_d is compact in the remainder of this paper. Then the operator

$$W_d = S_d^* S_d : F_d \to F_d$$

is compact, self-adjoint, and non-negative. Let $\{(\lambda_{d,n}, e_{d,n})\}_{n\in\mathbb{N}}$ denote its eigensystem, so that

$$W_d e_{d,n} = \lambda_{d,n} e_{d,n},$$

where the $e_{d,n}$ are orthonormal and the $\lambda_{d,n}$ are non-increasing. We normalize the problems S_d by assuming that $\lambda_{d,1} = 1$, and so

$$\lambda_{d,1} = 1 \ge \lambda_{d,2} \ge \cdots \ge \lambda_{d,n} \ge 0, \quad \text{with } \lim_{n\to+\infty} \lambda_{d,n} = 0.$$

It is also known that the linear algorithm of the form

$$A_n^*(f) = \sum_{k=1}^{n} \langle f, e_{d,k} \rangle_{F_d} S_d(e_{d,k}) \tag{2.1}$$

minimizes the worst case error among all algorithms (linear *and* non-linear) using n adaptively chosen linear functionals on f, and that

$$e(A_n^*) = \sqrt{\lambda_{d,n+1}}.$$

The information complexity $n(\varepsilon, S_d)$ is the minimal n for which the error of $e(A_n^*) \le \varepsilon$, where $\varepsilon \in (0, 1)$. Hence,

$$n(\varepsilon, S_d) = \left|\left\{n \in \mathbb{N} : \lambda_{d,n} > \varepsilon^2\right\}\right|.$$

For positive s and t, we say that our problem is (s,t)-*weakly tractable* (or "(s,t)-WT" for short) if $n(\varepsilon, S_d)$ is *not* exponential in d^t and ε^{-s}, i.e., if

$$\lim_{\varepsilon^{-1}+d\to+\infty} \frac{\ln n(\varepsilon, S_d)}{d^t + \varepsilon^{-s}} = 0. \tag{2.2}$$

Our problem is *weakly tractable* (or "WT" for short) if it is $(1,1)$-WT, whereas it is *uniformly weakly tractable* (or "UWT" for short) if it is (s,t)-WT for all positive s and t.

3. Criterion for (s,t)-weak tractability

Theorem 3.1. *Let $S = \{S_d\}_{d\in\mathbb{N}}$ be defined as in Section 2. For $c > 0$, define*

$$\sigma(c, d, s) := \sum_{j=1}^{\infty} \exp\left(-c\lambda_{d,j}^{-s/2}\right). \tag{3.1}$$

Then S is $(s\,t)$-WT iff

$$\mu(c, s, t) := \sup_{d\in\mathbb{N}} \sigma(c, d, s) \exp\left(-c\, d^t\right) < +\infty \quad \forall\, c > 0. \tag{3.2}$$

Proof. [$\Longrightarrow$] Assume that S is (s,t)-WT. Then for any $c > 0$, there exists a positive integer number $C = C(c, s, t)$ such that

$$n := n(\varepsilon, S_d) \le \left\lfloor \exp\left(c(\varepsilon^{-s} + d^t)\right)\right\rfloor$$

whenever $\varepsilon^{-1} + d \geq C$. For any $d \in \mathbb{N}$, choose ε such that $\varepsilon^{-1} \geq \max\{1, C - d\}$. Since the eigenvalues $\lambda_{d,n}$ are non-increasing in n, it follows that

$$\lambda_{d,\lfloor \exp(c(\varepsilon^{-s}+d^t))\rfloor+1} \leq \varepsilon^2$$

for any such ε. Let $n = \left\lfloor \exp\left(c(\varepsilon^{-s} + d^s)\right)\right\rfloor + 1$ and $n^* = \left\lfloor \exp\left(c(\max\{1, (C-d)^s\} + d^t)\right)\right\rfloor + 1$. By varying ε^{-1} in the interval $[\max\{1, C-d\}, +\infty)$, we can make n assume any integer value greater than or equal to n^*. Furthermore, we have

$$n - 1 \leq \exp\left(c(\varepsilon^{-s} + d^t)\right),$$

which is equivalent to

$$\varepsilon^2 \leq \left(\frac{c}{\ln\left((n-1)/\exp(cd^t)\right)}\right)^{2/s}$$

for $n > \exp(cd^t) + 1$. Therefore for $n \geq n^{**} = \max\{n^*, \lfloor \exp(cd^t)\rfloor\} + 2$, we have

$$\lambda_{d,n} \leq \left(\frac{c}{\ln\left((n-1)/\exp(cd^t)\right)}\right)^{2/s},$$

which is equivalent to

$$\exp\left(-2c\left(\lambda_{d,n}^{-s/2} + d^t\right)\right) \leq \frac{1}{(n-1)^2}.$$

Therefore

$$\sum_{n=1}^{\infty} \exp\left(-2c\,\lambda_{d,n}^{-s/2}\right) \cdot \exp(-2c\,d^t) \leq \exp(-2c\,d^t)(n^{**} - 1) + \sum_{n=n^{**}}^{\infty} \frac{1}{(n-1)^2}.$$

Clearly, the last sum is at most $\pi^2/6$ and

$$\exp(-2c\,d^t)(n^* - 1) \leq \exp\left(-2c\,d^t + c(C^s + d^t)\right) \leq \exp(cC^s).$$

Moreover,

$$\exp(-2c\,d^t)\left(1 + \exp(cd^t)\right) \leq 2.$$

Hence it follows that for any $c > 0$, we have

$$\begin{aligned}\mu(c,s,t) &= \sup_{d\in\mathbb{N}} \sigma(c,d,s)\exp\left(-c\,d^t\right) \leq \exp(-2c\,d^t)\sum_{n=1}^{\infty}\exp\left(-2c\,\lambda_{d,n}^{-s/2}\right) \\ &\leq \max\left\{\exp(cC^s), 2\right\} + \frac{\pi^2}{6} < +\infty,\end{aligned}$$

establishing the first part of the proof.

[$\Longleftarrow$] Assume now that $\mu(c,s,t)$ is finite for all positive c. Without loss of generality we also assume that $\mu(c,s,t) \geq 1$. Since the terms of the series in the definition of $\mu(c,s,t)$ are nonincreasing, we have

$$\exp(-c\,d^t)\, n\, \exp\left(-c\,\lambda_{d,n}^{-s/2}\right) \leq \mu(c,s,t).$$

Then for $n > \mu(c,s,t)\exp(cd^t)$ we have

$$\sqrt{\lambda_{d,n}} \leq \left(\frac{c}{\ln\left(n/\left(\mu(c,s,t)\exp(cd^t)\right)\right)}\right)^{1/s}.$$

Hence for

$$n = \left\lceil \mu(c, s, t)\ \exp\left(c\,(\varepsilon^{-s} + d^t)\right)\right\rceil,$$

we have $\sqrt{\lambda_{d,n+1}} \le \varepsilon$ and $n(\varepsilon, S_d) \le n$. Note that the argument x of the ceiling function in the previous line is at least one, so that $\lceil x\rceil \le 2x$. Hence,

$$n(\varepsilon, S_d) \le \exp\left(c\left(\frac{\ln(2\mu(c,s,t))}{c} + \varepsilon^{-s} + d^t\right)\right).$$

Clearly, there exists a positive $C = C(c, s, t)$ such that

$$\varepsilon^{-1} + d \ge C \Longrightarrow \varepsilon^{-s} + d^t \ge \frac{\ln(2\mu(c,s,t))}{c}.$$

Then we finally have

$$n(\varepsilon, S_d) \le \exp\left(2c(\varepsilon^{-s} + d^t)\right).$$

Since this holds for all positive c, we conclude that S is (s, t)-WT, which completes the proof. $\square$

Remark 3.1. Note the following:

1. Theorem 3.1 yields necessary and sufficient conditions for WT if we take $s = t = 1$. We see that S is WT iff $\mu(c, 1, 1) < +\infty$ for all positive c.
2. Similarly, Theorem 3.1 yields necessary and sufficient conditions for UWT. Namely, S is UWT iff $\mu(c, s, t) < +\infty$ for all positive c, s and t. $\square$

4. Sobolev spaces over the torus

We now look at spaces defined on the d-torus $\mathbb{T}^d = [0, 2\pi]$, identifying opposite points, so that for any $f : Td \to \mathbb{C}$, we have $f(\boldsymbol{x}) = f(\boldsymbol{y})$ whenever $\boldsymbol{x} - \boldsymbol{y} \in 2\pi\,\mathbb{Z}^d$. In other words, functions on the d-torus are 2π-periodic in each component.

Let $L_2(\mathbb{T}^d)$ denote the space of complex-valued square-integrable functions over $\mathbb{T}^d$, under the usual inner product

$$\langle f, g\rangle_{L_2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} \quad \forall f, g \in L_2(\mathbb{T}^d).$$

The following facts are well-known:

1. For $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$, let

$$e_{d,\mathbf{k}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}}\ \exp(\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x}) \quad \forall \boldsymbol{x} = (x_1, x_2, \dots, x_d) \in \mathbb{T}^d, \tag{4.1}$$

   where $\mathrm{i} = \sqrt{-1}$ and $\mathbf{k}\cdot\boldsymbol{x} = \sum_{j=1}^d k_j x_j$ Then $\{e_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is an orthonormal basis for $L_2(\mathbb{T}^d)$.
2. For any $f \in L_2(\mathbb{T}^d)$, we have

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{d,\mathbf{k}}(f)\, e_{d,\mathbf{k}},$$

   where

$$c_{d,\mathbf{k}}(f) = \langle f, e_{d,\mathbf{k}}\rangle_{L_2(\mathbb{T}^d)} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} f(\boldsymbol{x})\ \exp(-\mathrm{i}\,\mathbf{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

   is the $\mathbf{k}$th Fourier coefficient.
3. We have

$$\|f\|_{L_2(\mathbb{T}^d)} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{d,\mathbf{k}}(f)|^2\right)^{1/2}.$$

Let $r > 0$. Following [7,11], we define the space

$$H^r(\mathbb{T}^d) = \{f \in L_2(\mathbb{T}^d) : \|f\|_{H^{r,+}(\mathbb{T}^d)} < +\infty\},$$

where

$$\langle f, g\rangle_{H^{r,+}(\mathbb{T}^d)} = \sum_{\mathbf{k}\in\mathbb{Z}^d} \left(1 + \|\mathbf{k}\|^2_{\ell_2(\mathbb{R}^d)}\right)^r c_{d,\mathbf{k}}(f)\, \overline{c_{d,\mathbf{k}}(g)} \quad \forall f, g \in H^r(\mathbb{T}^d), \tag{4.2}$$

is the *natural inner product*, with

$$\|\boldsymbol{x}\|_{\ell_2(\mathbb{R}^d)} = \left(\sum_{j=1}^d x_j^2\right)^{1/2} \quad \forall \boldsymbol{x} \in \mathbb{R}^d \tag{4.3}$$

being the usual $\ell_2(\mathbb{R}^d)$-norm.

Note that

$$\|f\|_{L_2(\mathbb{T}^d)} \le \|f\|_{H^{r,+}(\mathbb{T}^d)} \quad \forall f \in H^r(\mathbb{T}^d). \tag{4.4}$$

Furthermore, the last bound is sharp, since it holds with equality for $f \equiv 1$.

We are ready to define the multivariate approximation problem as the embedding

$$\mathrm{App}_d : Hr \to L_2(\mathbb{T}^d) \quad \text{given by} \qquad \mathrm{App}_d f = f \quad \forall f \in H^r(\mathbb{T}^d).$$

From the sharp estimate (4.4), we find that

$$\|\,\mathrm{App}_d\,\| = 1 \quad \forall d \in \mathbb{N}.$$

Hence the approximation problem is well-normalized.

Remark 4.1. Note that when $r \in \mathbb{N}_0$, we typically study $H^r(\mathbb{T}^d)$ under its usual inner product

$$\langle f, g\rangle_{H^r(\mathbb{T}^d)} = \sum_{|\mathbf{m}|\le r} \langle D^{\mathbf{m}} f, D^{\mathbf{m}} g\rangle_{L_2(\mathbb{T}^d)} \quad \forall f, g \in H^r(\mathbb{T}^d),$$

where $D^{\mathbf{m}}$ denotes the distributional derivative and $|\mathbf{m}| = \sum_{j=1}^d m_j$ for $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}_0^d$. As pointed out in [7], we can use the multinomial identity to see that

$$\frac{1}{\sqrt{r!}}\|\cdot\|_{H^{r,+}(\mathbb{T}^d)} \le \|\cdot\|_{H^r(\mathbb{T}^d)} \le \|\cdot\|_{H^{r,+}(\mathbb{T}^d)}.$$

Since the equivalence constants for these two norms are independent of d, we see the level of tractability is the same for both norms. So we use the natural norm in this paper, rather than the usual norm, so that our results will hold for non-integer values of r. $\square$

We easily see that the eigensystem of $W_d = \mathrm{App}_d^* \mathrm{App}_d$ is given by $\left\{\left(\lambda_{d,\mathbf{k}}, e^{\mathrm{nor}}_{d,\mathbf{k}}\right)\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, where

$$\lambda_{d,\mathbf{k}} = \|e_{d,\mathbf{k}}\|^{-2}_{H^{r,+}(\mathbb{T}^d)} = \left(1 + \|\mathbf{k}\|^2_{\ell_2(\mathbb{R}^d)}\right)^{-r} \quad \forall \mathbf{k} \in \mathbb{Z}^d \tag{4.5}$$

and $e_{d,\mathbf{k}}$ is given by (4.1). The algorithm A_n^* given by (2.1) now takes the form

$$A_n^*(f) = \sum_{k\in K_n} \langle f, e^{\mathrm{nor}}_{d,\mathbf{k}}\rangle_{H^{r,+}(\mathbb{T}^d)}\, e^{\mathrm{nor}}_{d,\mathbf{k}} = \sum_{k\in K_n} c_{d,\mathbf{k}}(f)\, e_{d,\mathbf{k}} \quad \forall f \in H^r(\mathbb{T}^d),$$

where the set K_n consists of any n largest elements of the set $\{\lambda_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$. The error $e(A_n^*)$ is the square root of any $(n+1)$st largest element of the set $\{\lambda_{d,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$.

The information complexity $n(\varepsilon, \mathrm{App}_d)$ is now

$$n(\varepsilon, \mathrm{App}_d) = \left|\left\{\mathbf{k} \in \mathbb{Z}^d : \lambda_{d,\mathbf{k}} > \varepsilon^2\right\}\right|.$$

Assume for a moment that $r = 0$. Then $\lambda_{d,\mathbf{k}} = 1$ for all $\mathbf{k} \in \mathbb{Z}^d$, and so $n(\mathrm{App}_d, \varepsilon) = +\infty$ for all $\varepsilon \in (0, 1)$. This should not come as a surprise: when $r = 0$, the mapping App_d is the identity operator on the infinite-dimensional space $L_2(\mathbb{T}^d)$, which cannot be approximated with error less than 1 by finite-rank operators such as A_n^*. Hence in the sequel, we shall assume that $r > 0$. Then App_d is a compact operator and the information complexity $n(\varepsilon, \mathrm{App}_d)$ is finite for any $\varepsilon \in (0, 1)$.

5. (s, t)-weak tractability for App

As mentioned in the Introduction, [11] gave conditions that were necessary and sufficient for App to be (s,t)-WT. Using Theorem 3.1, we provide a new proof of their result, which does not require us to order the eigenvalues of W_d for $d \in \mathbb{N}$.

The main idea here be to replace sums by integrals

$$\iota(c,n,\alpha) := \int_{\mathbb{R}^n} \exp\left(-c\left(1+\|\boldsymbol{x}\|^2_{\ell_2(\mathbb{R}^n)}\right)^{\alpha/2}\right)\,\mathrm{d}\boldsymbol{x}, \tag{5.1}$$

which will be easier to handle. To do this, we first establish a few preliminary results.

Lemma 5.1. *For $c > 0$, $n \in \mathbb{N}$, $\alpha > 0$, and $S \subseteq \mathbb{Z}$, let*

$$\sigma(c,n,\alpha,S) = \begin{cases} \sum_{\mathbf{k}\in S} \exp\left(-c\left(1+\|\mathbf{k}\|^2_{\ell_2(\mathbb{R}^n)}\right)^{\alpha/2}\right), & \text{if } n>0, \\ \exp(-c) & \text{if } n=0. \end{cases} \tag{5.2}$$

Then

$$\mu(c,s,t) = \sup_{d\in\mathbb{N}^d} \left[\sigma(c,d,rs,\mathbb{Z})\exp\left(-c\,d^t\right)\right]. \tag{5.3}$$

Moreover, we have

$$\sigma(c,d,\alpha,\mathbb{Z}) = \sum_{n=0}^{d} \binom{d}{n} 2^n\,\sigma(c,n,\alpha,\mathbb{N}) \tag{5.4}$$

and

$$2^d\sigma(c,d,\alpha,\mathbb{N}) \le \iota(c,d,\alpha) \le 2^d\sigma(c,d,\alpha,\mathbb{N}_0). \tag{5.5}$$

Proof. Clearly we have $\sigma(c,d,s) = \sigma(c,d,rs,\mathbb{Z})$, where $\sigma(c,d,s)$ is defined by (3.1). Eq. (5.3) now follows from (3.2), (4.5), and (5.2).

To establish (5.4), note that the terms of $\sigma(c,d,\alpha,\mathbb{Z})$ do not depend on the signs or the ordering of the components of the $\mathbf{k}$ appearing in the sum (5.2). For each $n \in \{0,1,\dots,d\}$, we separate the indices $\mathbf{k} \in \mathbb{Z}^d$ with $d-n$ components equal to zero, finding that

$$\sigma(c,d,\alpha,\mathbb{Z}) = \sum_{n=0}^{d}\binom{d}{d-n}2^n\,\sigma(c,n,\alpha,\mathbb{N}) = \sum_{n=0}^{d}\binom{d}{n}2^n\,\sigma(c,n,\alpha,\mathbb{N}),$$

as required.

We now establish (5.5). Let $\mathbf{k}\in\mathbb{N}^d$, and write $\mathbf{k}-\mathbf{1} = (k_1-1,\dots,k_d-1)$. Since $x_j \in [0,+\infty) \mapsto \exp\left(-c(1+\|\boldsymbol{x}\|^2_{\ell_2(\mathbb{R}^d)})^{\alpha/2}\right)$ is a decreasing function for any $j\in\{1,\dots,d\}$, we have

$$\begin{aligned}\exp\left(-c(1+\|\mathbf{k}\|^2_{\ell_2(\mathbb{R}^d)})^{\alpha/2}\right)\,\mathrm{d}\boldsymbol{x} &\le \int_{[\mathbf{k}-\mathbf{1},\mathbf{k})} \exp\left(-c(1+\|\boldsymbol{x}\|^2_{\ell_2(\mathbb{R}^d)})^{\alpha/2}\right)\,\mathrm{d}\boldsymbol{x} \\ &\le \exp\left(-c(1+\|\mathbf{k}-\mathbf{1}\|^2_{\ell_2(\mathbb{R}^d)})^{\alpha/2}\right)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Summing over all such $\mathbf{k}$, we find that

$$\sigma(c,d,\alpha,\mathbb{N}) \le \int_{[0,+\infty)^d} \exp\left(-c(1+\|\boldsymbol{x}\|^2_{\ell_2(\mathbb{R}^d)})^{\alpha/2}\right)\,\mathrm{d}\boldsymbol{x} \le \sigma(c,d,\alpha,\mathbb{N}_0).$$

By the symmetry of the integrand, it is clear that

$$\iota(c,d,\alpha) = 2^d \int_{[0,+\infty)^d} \exp\left(-c(1+\|\boldsymbol{x}\|^2_{\ell_2(\mathbb{R}^d)})^{\alpha/2}\right)\,\mathrm{d}\boldsymbol{x}.$$

The inequality (5.5) follows immediately from these last two results. $\square$

We now find bounds on the integral ι.

Lemma 5.2. *Let* $\alpha > 0$ *and* $n \in \mathbb{N}$. *For any* $c > 0$, *we have*

$$\frac{2\pi^{n/2}}{\Gamma(n/2)} \left[\frac{\Gamma\big(n/\alpha\big)}{2^{n/2} c^{n/\alpha} \alpha} - \frac{1}{n} \right] \le \iota(c, n, \alpha) \le \frac{2}{\alpha} \left(\frac{\pi^{1/2}}{c^{1/\alpha}} \right)^n \frac{\Gamma\big(n/\alpha\big)}{\Gamma(n/2)}.$$

Proof. Since the integrand in $\iota(c, n, \alpha)$ depends only the norm $\|\boldsymbol{x}\|_{\ell_2(\mathbb{R}^d)}$, we may apply formula 4.642 from [4], which is based on polar coordinates, to get

$$\iota(c, n, \alpha) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \rho^{n-1} \exp\left(-c(1+\rho^2)^{\alpha/2}\right) \mathrm{d}\rho. \tag{5.6}$$

We first establish the upper bound on $\iota(c, n, \alpha)$. By dropping the 1 in the exponent of the integrand in (5.6), we get

$$\iota(c, n, \alpha) \le \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \rho^{n-1} \exp\left(-c\rho^\alpha\right) \mathrm{d}\rho. \tag{5.7}$$

Changing variables $y = -c\rho^\alpha$, or using Mathematica™, we find that

$$\int_0^\infty \rho^{n-1} \exp\left(-c\rho^\alpha\right) \mathrm{d}\rho = \frac{\Gamma\big(n/\alpha\big)}{\alpha c^{n/\alpha}}. \tag{5.8}$$

The desired upper bound now follows from (5.7) and (5.8).

We now find a lower bound on $\iota(c, n, \alpha)$ by shrinking the interval of integration in (5.6) to $[1, +\infty)$. For $\rho \in [1, +\infty)$, we have $1 + \rho^2 \le 2\rho^2$ and so

$$\begin{aligned} \iota(c, n, \alpha) &\ge \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_1^\infty \rho^{n-1} \exp\left(-c(1+\rho^2)^{\alpha/2}\right) \mathrm{d}\rho \\ &\ge \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_1^\infty \rho^{n-1} \exp\left(-c\, 2^{\alpha/2}\rho^\alpha\right) \mathrm{d}\rho \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \left[\int_0^\infty \rho^{n-1} \exp\left(-c\, 2^{\alpha/2}\rho^\alpha\right) \mathrm{d}\rho \right. \\ &\qquad \left. - \int_0^1 \rho^{n-1} \exp\left(-c\, 2^{\alpha/2}\rho^\alpha\right) \mathrm{d}\rho \right). \end{aligned} \tag{5.9}$$

Again changing variables $y = c2^{\alpha/2}\rho^\alpha$ or using Mathematica, we find that

$$\int_0^\infty \rho^{n-1} \exp\left(-c\, 2^{\alpha/2}\rho^\alpha\right) \mathrm{d}\rho = \frac{\Gamma\big(n/\alpha\big)}{2^{n/2} c^{n/\alpha} \alpha}. \tag{5.10}$$

On the other hand,

$$\int_0^1 \rho^{n-1} \exp\left(-c\, 2^{\alpha/2}\rho^\alpha\right) \mathrm{d}\rho \le \int_0^1 \rho^{n-1} \,\mathrm{d}\rho = \frac{1}{n}. \tag{5.11}$$

The desired lower bound now follows from (5.9)–(5.11). □

For $\alpha > 2$, we need a sharper upper bound on $\sigma(c, d, \alpha, \mathbb{N})$.

Lemma 5.3. *Let* $\alpha > 2$ *and* $c \in (0, 1]$. *Then*

$$\sigma(c, n, \alpha, \mathbb{N}) \le 2^{-n} \zeta_{\alpha,c} \exp\left(-\tfrac{1}{2} c n^{\alpha/2}\right) \quad \forall\, n \in \mathbb{N}_0, \tag{5.12}$$

where

$$C_\alpha := \sup_{m\in\mathbb{N}} \frac{\Gamma(m/\alpha)}{\Gamma(m/2)} \in [1,+\infty) \quad \textit{and} \quad \zeta_{\alpha,c} := C_\alpha \sup_{n\in\mathbb{N}_0} \frac{\left((2+\sqrt{\pi}/c)\right)^n}{\exp\left(\frac{1}{2}cn^{\alpha/2}\right)} < +\infty. \tag{5.13}$$

Proof. We first show that (5.13) holds. Stirling's approximation tells us that

$$\ln \frac{\Gamma(m/\alpha)}{\Gamma(m/2)} = -\left(\frac{1}{2}-\frac{1}{\alpha}\right) m \ln m\big(1+o(1)\big).$$

Since $\alpha > 2$, we see that $C_\alpha < +\infty$. Moreover, note that $\Gamma(1/\alpha)$ is an increasing function of $\alpha \in (2,+\infty)$, and so

$$C_\alpha \geq \frac{\Gamma(1/\alpha)}{\Gamma(1/2)} = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{\alpha}\right) \geq \frac{1}{\sqrt{\pi}}\Gamma\left(\tfrac{1}{2}\right) = 1,$$

as claimed. Finally, since $(2+\sqrt{\pi}/c)^n = o\left(\exp\left(\frac{1}{2}cn^{\alpha/2}\right)\right)$ as $n\to\infty$, we see that $\zeta_{\alpha,c} < +\infty$.

We now wish to show that (5.12) holds. As in the proof of Lemma 5.1, for each $m \in \{0, 1 \dots, n\}$, we separate out the indices $\mathbf{k} \in \mathbb{Z}^d$ having $n-m$ components equal to one. Then

$$\begin{aligned}\sigma(c,n,\alpha,\mathbb{N}) \leq{}& \exp\big(-c\,(1+n)^{\alpha/2}\big)\\ &+ \sum_{m=1}^{n} \binom{n}{n-m} \sum_{\mathbf{k}\in[2,+\infty)^m} \exp\left[-c\left(1+n+\sum_{j=1}^{m}(k_j^2-1)\right)^{\alpha/2}\right].\end{aligned}$$

Note that for $k_j \geq 2$ we have $k_j^2 - 1 \geq (k_j-1)^2$ and $k_j - 1 \in \mathbb{N}$. Since $\alpha/2 \geq 1$, we have $(a+b)^{\alpha/2} \geq a^{\alpha/2} + b^{\alpha/2}$ for any positive a and b. Hence we have

$$\begin{aligned}\sigma(c,d,\alpha,\mathbb{N}) \leq{}& \exp\big(-c\,(1+n)^{\alpha/2}\big) + \sum_{m=1}^{n}\binom{n}{m}\\ &\times \sum_{\mathbf{k}\in[2,+\infty)^m} \exp\big(-c(1+n)^{\alpha/2} - c\,\|\mathbf{k}-\mathbf{1}\|^{\alpha}_{\ell_2(\mathbb{R}^m)}\big)\\ \leq{}& \exp\left(-cn^{\alpha/2}\right)\left[1+\sum_{m=1}^{n}\binom{n}{m}\sum_{\mathbf{k}\in\mathbb{N}^m}\exp\left(-c\|\mathbf{k}\|^{\alpha}_{\ell_2(\mathbb{R}^m)}\right)\right].\end{aligned}$$

As in the proof of Lemma 5.1, we find that

$$\begin{aligned}\sum_{\mathbf{k}\in\mathbb{N}^m}\exp\left(-c\|\mathbf{k}\|^{\alpha}_{\ell_2(\mathbb{R}^m)}\right) &\leq \int_{[0,+\infty)^m}\left(1+\exp\left(-c\|\boldsymbol{x}\|^{\alpha}_{\ell_2(\mathbb{R}^m)}\right)\right)\,\mathrm{d}\boldsymbol{x}\\ &= 2^{-m}\int_{\mathbb{R}^m}\exp\left(-c\|\boldsymbol{x}\|^{\alpha}_{\ell_2(\mathbb{R}^m)}\right)\,\mathrm{d}\boldsymbol{x}\\ &= 2^{-m}\frac{2\pi^{m/2}}{\Gamma(m/2)}\int_0^\infty \rho^{m-1}\exp\left(-c\rho^\alpha\right)\,\mathrm{d}\rho\\ &= 2^{-m}\frac{2\pi^{m/2}}{\Gamma(m/2)}\frac{\Gamma(m/\alpha)}{\alpha c^{m/\alpha}} \leq C_\alpha\frac{2}{\alpha}\left(\frac{\pi^{1/2}}{2c^{1/\alpha}}\right)^m \leq C_\alpha\left(\frac{\pi^{1/2}}{2c^{1/\alpha}}\right)^m.\end{aligned}$$

Since $C_\alpha \geq 1$, we have

$$\begin{aligned}\sigma(c,n,\alpha,\mathbb{N}) &\leq \exp\left(-cn^{\alpha/2}\right)\left[1+\sum_{m=1}^{n}\binom{n}{m}\sum_{\mathbf{k}\in\mathbb{N}^m}\exp\left(-c\|\mathbf{k}\|_{\ell_2(\mathbb{R}^m)}^{\alpha}\right)\right]\\ &\leq \exp\left(-cn^{\alpha/2}\right)\left(1+C_\alpha\sum_{m=1}^{n}\binom{n}{m}\left(\frac{\pi^{1/2}}{2c^{1/\alpha}}\right)^m\right)\\ &\leq C_\alpha\exp\left(-cn^{\alpha/2}\right)\left(1+\left(1+\frac{\pi^{1/2}}{2c^{1/\alpha}}\right)^n\right).\end{aligned}$$

Since $\alpha > 2$ and $c \in (0, 1]$, we have

$$1+\frac{\pi^{1/2}}{2c^{1/\alpha}} \leq \frac{2+\sqrt{\pi}}{2c},$$

and so

$$1+\left(1+\frac{\pi^{1/2}}{2c^{1/\alpha}}\right)^n \leq 1+\left(\frac{2+\sqrt{\pi}}{2c}\right)^n \leq 2\left(\frac{2+\sqrt{\pi}}{2c}\right)^n.$$

Hence

$$\begin{aligned}\sigma(c,n,\alpha,\mathbb{N}) &\leq C_\alpha\exp\left(-cn^{\alpha/2}\right)\cdot 2\left(\frac{2+\sqrt{\pi}}{2c}\right)^n\\ &= 2^{-n}\cdot 2C_\alpha\frac{\left((2+\sqrt{\pi}/c)\right)^n}{\exp\left(\frac{1}{2}cn^{\alpha/2}\right)}\cdot\exp\left(-\tfrac{1}{2}cn^{\alpha/2}\right)\\ &\leq 2^{-n}\zeta_{\alpha,c}\exp\left(-\tfrac{1}{2}cn^{\alpha/2}\right),\end{aligned}$$

as required. □

We are now ready to prove the main result of this section:

Theorem 5.1. *For a fixed positive r, let* App $= \{\mathrm{App}_d\}_{d\in\mathbb{N}}$, *where* App_d *is the* $L_2(\mathbb{T}^d)$ *approximation problem for the Sobolev space* $H^r(\mathbb{T}^d)$.

- *Let $t > 1$. Then* App *is (s, t)-WT for all $s > 0$.*
- *Let $t \in (0, 1]$. Then* App *is (s, t)-WT iff $rs > 2$.*

Proof. Using Theorem 3.1 and Lemma 5.1, we see that App is (s, t)-WT iff

$$\ln\mu(c,s,t) = \sup_{d\in\mathbb{N}}\left[\ln\sigma(c,d,rs,\mathbb{Z}) - c\,d^t\right] < +\infty \quad \forall c > 0. \tag{5.14}$$

Since $\mu(\cdot, s, t)$ is a decreasing function, it suffices to check condition (5.14) for small values of c. Moreover since (s, t)-WT implies (s_1, t)-WT for any $s_1 > s$, it also suffices to check condition (5.14) for small values of s when establishing a positive result.

First, suppose that $t > 1$. Without loss of generality, we can assume that d is an even integer. Using Lemma 5.1, we see that

$$\begin{aligned}\sigma(c,d,rs,\mathbb{Z}) &= \sum_{n=0}^{d}\binom{d}{n}2^n\sigma(c,n,rs,\mathbb{N}) \leq \sum_{n=0}^{d}\binom{d}{n}\iota(c,n,rs)\\ &\leq (d+1)\binom{d}{d/2}\max_{0\leq n\leq d}\iota(c,n,rs).\end{aligned}\tag{5.15}$$

Now

$$\ln\binom{d}{d/2} = \ln d! - 2\ln(d/2)!) \sim d\ln d - 2\left(\frac{d}{2}\ln\frac{d}{2}\right) = d\ln 2. \tag{5.16}$$

Let $\beta_c = \pi^{1/2}/c^{1/\alpha}$. From Lemma 5.2, we see that

$$\begin{aligned} \ln \iota(c,n,rs) &\le \ln \frac{2}{rs} + n \ln \beta_c + \ln \frac{\Gamma(n/(rs))}{\Gamma(n/2)} \sim n \ln \beta_c + \left(\frac{1}{rs} - \frac{1}{2}\right) n \ln n \\ &\le d \ln \beta_c + \left(\frac{1}{rs} - \frac{1}{2}\right) d \ln d = \mathcal{O}(d \ln d). \end{aligned} \tag{5.17}$$

(This is an over-estimate if $rs < 2$, but it is good enough for our purposes.) Taking logarithms in (5.15), and using (5.16) and (5.17), we see that

$$\ln \sigma(c,d,rs,\mathbb{Z}) = \mathcal{O}(d \ln d).$$

Since $t > 1$, we know that $d \ln d = o(d^t)$, and so $\ln \mu(c,s,t) < +\infty$ for all $c > 0$. Hence App is (s,t)-WT whenever $t > 1$.

We next suppose that $t \in (0,1]$ and that $rs \le 2$. Let $\theta_c = c^{-1/(rs)}\sqrt{\pi/2}$, so that using Lemma 5.2, we see that

$$\begin{aligned} \ln \iota(c,d,rs) &\ge \ln \left[\theta_c^d \frac{2}{rs} \frac{\Gamma(d/(rs))}{\Gamma(d/2)}\right] \bigl(1+o(1)\bigr) \sim d \ln \theta_c + \ln \frac{2}{rs} + \ln \frac{\Gamma(d/(rs))}{\Gamma(d/2)} \\ &\sim d \ln \theta_c + \left(\frac{1}{rs} - \frac{1}{2}\right) d \ln d. \end{aligned}$$

Since $\mathbb{N}_0 \subset \mathbb{Z}$, we may now use Lemma 5.1 to see that

$$\begin{aligned} \ln \sigma(c,d,rs,\mathbb{Z}) - cd^t &\ge \ln \sigma(c,d,rs,\mathbb{N}_0) - cd^t \ge \ln\bigl(2^{-d}\iota(c,d,rs)\bigr) - cd^t \\ &\ge \left[\left(\frac{1}{rs} - \frac{1}{2}\right) d \ln d + d \ln \tfrac{1}{2}\theta_c\right] \bigl(1+o(1)\bigr) - cd^t \\ &\ge d\left(\ln \tfrac{1}{2}\theta_c - c\right)\bigl(1+o(1)\bigr). \end{aligned}$$

Let $c_0 = \bigl(\exp(-2)\sqrt{\pi/8}\bigr)^{rs}$, so that $\ln \frac{1}{2}\theta_{c_0} = 2$. Thus for any $c \in (0,c_0]$, we have $\ln \frac{1}{2}\theta_c - c \ge 1$, and hence $\ln \mu(c,s,t) = +\infty$ for $c \in (0,c_0]$. Hence App is not (s,t)-WT for the case $t \in (0,1]$ when $rs \le 2$.

Finally, we consider the one remaining case, in which $t \in (0,1]$ and $rs > 2$. Let $c > 0$. Using (5.4) and Lemma 5.3, we see that

$$\sigma(c,d,rs,\mathbb{Z}) \le \zeta_{\alpha,c}\,(d+1) \max_{0 \le n \le d} f(n), \tag{5.18}$$

where

$$f(n) = \binom{d}{n} \exp\left(-\tfrac{1}{2}cn^{rs/2}\right).$$

Since $rs > 2$, we have

$$(n+1)^{rs/2} - n^{rs/2} = n^{rs/2}\left[\left(1+\frac{1}{n}\right)^{rs/2} - 1\right] \ge n^{rs/2-1},$$

and so for $n \le d-1$ we have

$$f(n+1) = f(n)\frac{d-n}{n+1} \exp\left[-\tfrac{1}{2}c\left((n+1)^{rs/2} - n^{rs/2}\right)\right] \le f(n)\frac{d-n}{n+1} \exp\left(-\tfrac{1}{2}c\,n^{rs/2-1}\right).$$

Thus

$$n \ge \left(\frac{2\ln d}{c}\right)^{2/(rs-2)} \implies \exp\left(\tfrac{1}{2}cn^{rs-1}\right) \ge d \implies f(n+1) \le f(n).$$

This means that we can confine our search for the maximum in (5.18) to $n \in \{0, \ldots, n^*\}$, where

$$n^* = \left\lfloor \left(\frac{2\ln d}{c}\right)^{2/(rs-2)} \right\rfloor.$$

Hence

$$\sigma(c, d, rs, \mathbb{Z}) \le \zeta_{rs,c}(d+1)d^{n^*},$$

so that

$$\ln \sigma(c, d, rs, \mathbb{Z}) \le \ln \zeta_{rs,c} + \ln(d+1) + \left(\frac{2}{c}\right)^{2rs/(rs-2)} (\ln d)^{2rs/(rs-2)+1} = \Theta\left((\ln d)^{2rs/(rs-2)+1}\right).$$

Since $t > 0$, it follows that

$$\lim_{d\to\infty} \ln \sigma(c, d, rs, \mathbb{Z}) - cd^t = -\infty,$$

and so $\mu(c, s, t) < +\infty$. Since this is true for any positive c, it follows that App is (s, t)-WT in this case. □

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