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The existence of good extensible rank-1 lattices ☆

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Abstract

Extensible integration lattices have the attractive property that the number of points in the node set may be increased while retaining the existing points. It is shown here that there exist generating vectors, **h**, for extensible rank-1 lattices such that for $n = b, b^2, \ldots$ points and dimensions $s = 1, 2, \ldots$ the figures of merit R_α, P_α and discrepancy are all small. The upper bounds obtained on these figures of merit for extensible lattices are some power of $\log n$ worse than the best upper bounds for lattices where **h** is allowed to vary with n and s.

Keywords: Discrepancy; Figures of merit; Extensible lattices; Lattice rules; Quasi-Monte Carlo integration

1. Introduction

Multidimensional integrals over the unit cube are often approximated by taking the average over a well-chosen set of points

$$\int_{[0,1)^s} f(\mathbf{x})\, d\mathbf{x} \approx \frac{1}{n} \sum_{i=0}^{n-1} f(\mathbf{z}_i).$$

Quasi-Monte Carlo methods [9] choose the set $\{\mathbf{z}_i\}$ to be evenly distributed over the unit cube. One popular choice of points are the node sets of rank-1 integration

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lattices (see [5]; [9, Chapter 5]; [12]). Such sets may be expressed as

$$\{\{i\mathbf{h}/n + \mathbf{\Delta}\}\colon\ i = 0, 1, \dots, n-1\}, \tag{1}$$

where $\mathbf{h}$ is an s-dimensional integer generating vector, $\mathbf{\Delta}$ is an s-dimensional shift, and $\{\cdot\}$ denotes the fractional part, i.e., $\{\mathbf{x}\} = \mathbf{x}\,(\mathrm{mod}\ 1)$. The quality of the set defined in (1) depends mainly on the choice of $\mathbf{h}$ and somewhat on the choice of $\mathbf{\Delta}$. There are several quality measures, including R_α, P_α, and the discrepancy, which will be defined and discussed later in this article.

One disadvantage of lattice rules for numerical integration has been that the generating vector depends on n and s. If one changes either of these, then one must also change $\mathbf{h}$. Recently, lattice node sets have been proposed that can be extended in n (for this reason they are called *extensible lattices*). For a fixed integer base $b \geqslant 2$, let any integer $i \geqslant 0$ be expressed as $i = i_1 + i_2 b + i_3 b^2 + \cdots$, where the digits i_j are in $\{0, 1, \dots, b-1\}$. Then define the radical inverse function as

$$\Phi_b(i) = i_1 b^{-1} + i_2 b^{-2} + i_3 b^{-3} + \cdots .$$

The sequence

$$\{\mathbf{z}_i = \{\mathbf{h}\Phi_b(i) + \mathbf{\Delta}\}\colon\ i = 0, 1, \dots\}$$

has the property that any finite piece with $n = b^m$ points,

$$\begin{aligned}&\{\mathbf{z}_i = \{\mathbf{h}\Phi_b(i + lb^m) + \mathbf{\Delta}\} = \{\mathbf{h}\Phi_b(i) + \Phi_b(l)\mathbf{h}b^{-m} + \mathbf{\Delta}\}\colon\ i = 0, 1, \dots, b^m - 1\},\\ &\quad l = 0, 1, \dots,\ \ m = 1, 2, \dots\end{aligned} \tag{2}$$

is the node set of a shifted rank-1 lattice [2,3], just like the pieces of a (t, s)-sequence are (t, m, s)-nets [9, Chapter 4]. Good generating vectors $\mathbf{h}$ that work simultaneously for a range of m and s have been found experimentally [3]. The purpose of this article is to prove that there exist generating vectors $\mathbf{h}$, dependent on b but independent of m and s, that give node sets of form (1) and (2) with figures of merit that are nearly as good as the best known upper bounds for node sets where the generating vector is allowed to depend on m and s.

The remainder of this introduction reviews several figures of merit for lattice rules. The next section gives an upper bound on the value of R_α for extensible lattices. This is used to give upper bounds on P_α and discrepancy in the following two sections.

In this article it is assumed that the integrand, f, is a function of $\mathbf{x} = (x_1, x_2, \dots) \in [0, 1)^\infty$, and that the integration domain is $[0, 1)^\infty$. Integration problems over $[0, 1)^s$ just assume that f depends only on the first s variables. Let $1 : s$ denote the set $\{1, \dots, s\}$, and for any $u \subset 1 : \infty$, let $\mathbf{x}_u$ denote the vector indexed by the elements of u. Let $|u|$ denote the cardinality of u. Throughout this article it is assumed that u is a finite set.

Suppose that the integrand can be written as an absolutely summable Fourier series,

$$f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^\infty} \tilde{f}(\mathbf{k}) e^{2\pi \imath \mathbf{k}^T \mathbf{x}},$$

where $\imath = \sqrt{-1}$. For any parameter $\boldsymbol{\gamma} \in [0, \infty)^\infty$ and $\mathbf{k} \in \mathbb{Z}^\infty$ let

$$\tilde{r}(\mathbf{k}, \boldsymbol{\gamma}) = \prod_{j=1}^{\infty} r(k_j, \gamma_j), \quad \text{where } r(k_j, \gamma_j) = \begin{cases} 1, & k_j = 0, \\ \gamma_j^{-1}|k_j|, & k_j \neq 0. \end{cases}$$

For any $\alpha > 1$, this quantity can be used to define a Banach space of functions:

$$\mathscr{F}_\alpha = \{f \in \mathscr{L}_2([0,1)^\infty): \, ||f||_{\mathscr{F}_\alpha} < \infty\},$$

where

$$||f||_{\mathscr{F}_\alpha} = \sup_{\mathbf{k} \in \mathbb{Z}^\infty} (\tilde{r}(\mathbf{k}, \boldsymbol{\gamma})^\alpha |\tilde{f}(\mathbf{k})|).$$

If $\gamma_j = 0$, then functions in $\mathscr{F}_\alpha$ are assumed not to depend on the coordinate x_j, i.e., $\tilde{f}(\mathbf{k}) = 0$ for all $\mathbf{k}$ with $k_j \neq 0$. Let $\tilde{\mathbb{Z}}^u = \{\mathbf{k} \in \mathbb{Z}^\infty: \, k_j = 0 \ \forall j \notin u\}$. Then the subspace of $\mathscr{F}_\alpha$ containing functions depending only on the coordinates indexed by u is $\mathscr{F}_{u,\alpha} = \{f \in \mathscr{F}_\alpha: \, \tilde{f}(\mathbf{k}) = 0 \ \forall \mathbf{k} \notin \tilde{\mathbb{Z}}^u\}$.

For any positive integer n and any $\mathbf{h} \in \mathbb{Z}^\infty$, the dual lattice consists of all wave numbers $\mathbf{k}$ with $\mathbf{k}^T\mathbf{h} = 0 \bmod n$. The set $B(\mathbf{h}, n, u) = \{\mathbf{0} \neq \mathbf{k} \in \tilde{\mathbb{Z}}^u: \ \mathbf{k}^T\mathbf{h} = 0 \bmod n\}$ consists of all nonzero wave numbers in the dual lattice whose nonzero components are in the directions indexed by u. Let

$$P_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) = \sum_{\mathbf{k} \in B(\mathbf{h},n,u)} \tilde{r}(\mathbf{k}, \boldsymbol{\gamma})^{-\alpha}. \tag{3}$$

Then one may derive the following tight worst-case integration error bound for lattice rules using the node set (1) [1]:

$$\left| \int_{[0,1)^\infty} f(\mathbf{x}) \, d\mathbf{x} - \frac{1}{n} \sum_{i=0}^{n-1} f(\mathbf{z}_i) \right| \leqslant P_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) ||f||_{\mathscr{F}_\alpha} \quad \forall f \in \mathscr{F}_{u,\alpha}. \tag{4}$$

For band-limited integrands one may derive a similar error bound. Let $\tilde{\mathbb{Z}}^u_n = \tilde{\mathbb{Z}}^u \cap (-n/2, n/2]^\infty$ and $\mathscr{F}_{n,u,\alpha} = \{f \in \mathscr{F}_{u,\alpha}: \, \tilde{f}(\mathbf{k}) = 0 \ \forall \mathbf{k} \notin \tilde{\mathbb{Z}}^u_n\}$. Then the analogous error bound to (4) is

$$\left| \int_{[0,1)^\infty} f(\mathbf{x}) \, d\mathbf{x} - \frac{1}{n} \sum_{i=0}^{n-1} f(\mathbf{z}_i) \right| \leqslant R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) ||f||_{\mathscr{F}_\alpha} \quad \forall f \in \mathscr{F}_{n,u,\alpha}, \tag{5}$$

where

$$R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) = \sum_{\mathbf{k} \in \tilde{B}(\mathbf{h},n,u)} \tilde{r}(\mathbf{k}, \boldsymbol{\gamma})^{-\alpha} \tag{6}$$

and $\tilde{B}(\mathbf{h}, n, u) = B(\mathbf{h}, n, u) \cap (-n/2, n/2]^\infty$. Note that whereas one must require $\alpha > 1$ for $P_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u)$ to be defined and (4) to make sense, the error bound (5) is well defined for $\alpha \geqslant 0$. Note also that both $P_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u)$ and $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u)$ do not depend on the shift vector $\boldsymbol{\Delta}$.

Lattice rules are not only used for periodic integrands, but also for non-periodic ones. Let $\mathscr{W}_p$ denote the Banach space of functions that are absolutely continuous on $[0,1]^\infty$ with $\mathscr{L}_p$-integrable mixed partial derivatives of up to order one in each coordinate direction. Let $\gamma_u = \prod_{j\in u}\gamma_j$. The norm for this space is defined as

$$||f||_{\mathscr{W}_p} = \left\{ \sum_{\substack{u\subseteq 1:\infty\\ |u|<\infty}} \gamma_u^{-p} \int_{[0,1]^u} \left| \left. \frac{\partial^{|u|} f}{\partial \mathbf{x}_u} \right|_{\mathbf{x}_{1:\infty\setminus u}=\mathbf{1}} \right|^p d\mathbf{x}_u \right\}^{1/p}, \quad 1\leqslant p<\infty,$$

$$||f||_{\mathscr{W}_\infty} = \sup_{\substack{u\subseteq 1:\infty\\ |u|<\infty}} \gamma_u^{-1} \sup_{\mathbf{x}_u\in[0,1]^u} \left| \left. \frac{\partial^{|u|} f}{\partial \mathbf{x}_u} \right|_{\mathbf{x}_{1:\infty\setminus u}=\mathbf{1}} \right|.$$

Again, if $\gamma_j = 0$, then the functions in $\mathscr{W}_p$ are assumed not to depend on x_j. As above let $\mathscr{W}_{u,p}$ denote the subspace of $\mathscr{W}_p$ whose elements do not depend on x_j for $j\notin u$. The integration error bound for this space is [4]

$$\left| \int_{[0,1)^\infty} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{n}\sum_{i=0}^{n-1} f(\mathbf{z}_i) \right| \leqslant D_q^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,u)||f||_{\mathscr{W}_p}$$
$$\forall f\in\mathscr{W}_{u,p}, \ \frac{1}{p}+\frac{1}{q}=1, \tag{7}$$

where the (weighted) $\mathscr{L}_q$-star discrepancy is defined as

$$D_q^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,u) = \left\{ \sum_{\emptyset\subset v\subseteq u} \gamma_v^q \int_{[0,1]^v} |\mathrm{disc}_v(\mathbf{0},\mathbf{x},\{\mathbf{z}_i\},n)|^q\, d\mathbf{x}_v \right\}^{1/q},$$
$$1\leqslant q<\infty, \tag{8a}$$

$$D_\infty^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,u) = \sup_{\emptyset\subset v\subseteq u} \gamma_v \sup_{\mathbf{x}_v\in[0,1]^v} |\mathrm{disc}_v(\mathbf{0},\mathbf{x},\{\mathbf{z}_i\},n)|. \tag{8b}$$

The discrepancy function, $\mathrm{disc}_u(\mathbf{y},\mathbf{x},\{\mathbf{z}_i\},n)$, measures the difference between the volume of the box $[\mathbf{y}_u,\mathbf{x}_u)$ and the proportion of sample points inside it:

$$\mathrm{disc}_u(\mathbf{y},\mathbf{x},\{\mathbf{z}_i\},n) = \prod_{j\in u}(x_j-y_j) - \frac{1}{n}\sum_{i=0}^{n-1}\prod_{j\in u}(1_{(z_{ij},\infty)}(x_j) - 1_{(z_{ij},\infty)}(y_j)), \tag{8c}$$

where $y_j\leqslant x_j$ for all $j\in u$. Here $1_{\{\cdot\}}$ denotes the characteristic function.

One may also define an $\mathscr{L}_q$-unanchored discrepancy:

$$D_q(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,u)$$
$$= \left\{ \sum_{\emptyset\subset v\subseteq u} 2^{|v|}\gamma_v^q \int_{\substack{[0,1]^{2v}\\ \mathbf{y}_v\leqslant\mathbf{x}_v}} |\mathrm{disc}_v(\mathbf{y},\mathbf{x},\{\mathbf{z}_i\},n)|^q\, d\mathbf{y}_v\, d\mathbf{x}_v \right\}^{1/q}, \quad 1\leqslant q<\infty, \tag{9a}$$

$$D_\infty(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,u) = \sup_{\emptyset\subset v\subseteq u} \gamma_v \sup_{\mathbf{0}\leqslant\mathbf{y}_v\leqslant\mathbf{x}_v\leqslant\mathbf{1}} |\mathrm{disc}_v(\mathbf{y},\mathbf{x},\{\mathbf{z}_i\},n)|. \tag{9b}$$

Both the star and the unanchored discrepancies may be bounded as follows:

$$D_q^*(\{\mathbf{z}_i\}, \boldsymbol{\gamma}, n, u), D_q(\{\mathbf{z}_i\}, \boldsymbol{\gamma}, n, u) \leqslant \sum_{\emptyset \subset v \subseteq u} \gamma_v \sup_{\mathbf{0} \leqslant \mathbf{y}_v \leqslant \mathbf{x}_v \leqslant \mathbf{1}} |\mathrm{disc}_v(\mathbf{y}, \mathbf{x}, \{\mathbf{z}_i\}, n)|, \quad 1 \leqslant q \leqslant \infty. \tag{10}$$

2. Existence of extensible lattices with small R_α

The existence of lattices with small $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s)$ for fixed n and s has been shown using averaging arguments. After computing an upper bound on the average of $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s)$ over some set of $\mathbf{h}$, one argues that there exist at least some $\mathbf{h}$ with an $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s)$ at least as small as that upper bound. In the following, an averaging argument is also used, but with a difference. For extensible lattice rules it will be argued that there exists some $\mathbf{h}$, independent of n and s, such that $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s)$ is not too much worse than the upper bound on the average for all n and s.

For extensible lattices one needs to allow generating vectors, $\mathbf{h}$, whose coordinates are generalizations of integers. For any integer $b \geqslant 2$ let $\mathbb{Z}_b$ be the set of all b-adic integers $i = \sum_{l=1}^{\infty} i_l b^{l-1}$, where $i_l \in \{0, 1, \ldots, b-1\}$ for all $l \geqslant 1$; see [7] for the theory of b-adic integers. Note that $\mathbb{Z}_b \supset \mathbb{Z}$. The set $\mathbb{Z}_b$ contains too many bad numbers, so define

$$H_b = \{i \in \mathbb{Z}_b: \ \gcd(i_1, b) = 1\}.$$

Then $H_b^\infty = H_b \times H_b \times \cdots$ is the set of candidates for $\mathbf{h}$, an ∞-vector. The set $\mathbb{Z}_b$ has a probability measure such that the set of all $i \in \mathbb{Z}_b$ with specified first l digits has measure b^{-l}. This probability measure conditional on H_b is denoted by μ_b. The Cartesian product H_b^∞ has the product probability measure μ_b^∞. The following upper bound on the average of $R_1(\mathbf{h}, \mathbf{1}, n, u)$ taken over H_b^∞ is implied by the proof of [8, Theorem 1]. For $\mathbf{h} \in H_b^\infty$ we define $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u)$ with $n = b, b^2, \ldots$ in the obvious way, namely by (6) with $\mathbf{h}$ considered modulo n.

Lemma 1. *For any finite $u \subset 1:\infty$ we have*

$$\int_{H_b^\infty} R_1(\mathbf{h}, \mathbf{1}, n, u)\, d\mu_b^\infty(\mathbf{h}) \leqslant n^{-1}(\beta_1 + \beta_2 \log n)^{|u|}$$

for some positive absolute constants β_1 and β_2 and $n = b, b^2, \ldots$.

Theorem 2. *Suppose we are given a fixed integer $b \geqslant 2$, a fixed $\boldsymbol{\gamma} \in [0, \infty)^\infty$, a fixed $\alpha \geqslant 1$, and a fixed $\varepsilon > 0$.*

(i) *There exist a μ_b^∞-measurable $\tilde{G}_b \subset H_b^\infty$ and some constant $C_R(\alpha, \boldsymbol{\gamma}, \varepsilon, s)$ such that for all $\mathbf{h} \in H_b^\infty \setminus \tilde{G}_b$,*

$$R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s) \leqslant C_R(\alpha, \boldsymbol{\gamma}, \varepsilon, s) n^{-\alpha} (\log n)^{\alpha(s+1)} [\log\log(n+1)]^{\alpha(1+\varepsilon)},$$
$$n = b, b^2, \ldots, \quad s = 1, 2, \ldots \; . \tag{11}$$

Furthermore, one may make $\mu_b^\infty(\tilde{G}_b)$ *arbitrarily close to zero by choosing* $C_R(\alpha, \boldsymbol{\gamma}, \varepsilon, s)$ *large enough.*

(ii) *If* $\sum_{j=1}^\infty \gamma_j^a j[\log(j+1)]^{1+\varepsilon} < \infty$ *for some* $a \in [1, \alpha]$, *then for any fixed* $\delta > 0$ *there exists some* $\tilde{C}_R(\alpha, a, \boldsymbol{\gamma}, \delta)$ *such that for all* $\mathbf{h} \in H_b^\infty \setminus \tilde{G}_b$ *we have*

$$R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s) \leqslant \tilde{C}_R(\alpha, a, \boldsymbol{\gamma}, \delta) n^{-\alpha/a+\delta}, \quad n = b, b^2, \ldots, \quad s = 1, 2, \ldots \; . \tag{12}$$

Again one may make $\mu_b^\infty(\tilde{G}_b)$ *arbitrarily close to zero by choosing the above leading constant large enough.*

(iii) *If* $\sum_{j=1}^\infty \gamma_j^a < \infty$ *for some* $a \in [1, \alpha]$, *then for any fixed* $\delta > 0$ *there exists a* μ_b^∞-*measurable* $G_b \subset H_b^\infty$ *such that for all* $\mathbf{h} \in H_b^\infty \setminus G_b$ (12) *is satisfied, but* (11) *may not be. Here too,* $\mu_b^\infty(G_b)$ *can be made arbitrarily close to zero.*

Proof. Define the quantities

$$\tilde{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s) = \sum_{\{s\} \subseteq u \subseteq 1:s} \gamma_u^\alpha R_\alpha(\mathbf{h}, \mathbf{1}, n, u),$$

$$\hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s) = \sum_{\emptyset \subset u \subseteq 1:s} \gamma_u^\alpha R_\alpha(\mathbf{h}, \mathbf{1}, n, u) = \sum_{d=1}^{s} \tilde{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, d). \tag{13}$$

Note that $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s) \leqslant \hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s)$. The proof here actually shows the above conclusions for $\hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s)$ rather than $R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, 1:s)$ because these are needed for Theorem 7.

There are two important relations among $\hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s)$ for different parameters α that are used in this proof. Because $R_\alpha(\mathbf{h}, \mathbf{1}, n, u)$ is non-increasing for α increasing, it follows that

$$\begin{aligned}\hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s) &= \sum_{\emptyset \subset u \subseteq 1:s} \gamma_u^\alpha R_\alpha(\mathbf{h}, \mathbf{1}, n, u) \leqslant \sum_{\emptyset \subset u \subseteq 1:s} \gamma_u^\alpha R_a(\mathbf{h}, \mathbf{1}, n, u) \\ &= \hat{R}_a(\mathbf{h}, \boldsymbol{\gamma}^{\alpha/a}, n, s)\end{aligned} \tag{14}$$

for $\alpha \geqslant a$, where $\boldsymbol{\gamma}^{\alpha/a}$ denotes the vector obtained by raising each component of $\boldsymbol{\gamma}$ to the power α/a. Furthermore, the definitions of R_α and $\hat{R}_\alpha$ together with Jensen's inequality imply that

$$\begin{aligned}\hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s) &= \sum_{\emptyset \subset u \subseteq 1:s} \gamma_u^\alpha R_\alpha(\mathbf{h}, \mathbf{1}, n, u) \leqslant \sum_{\emptyset \subset u \subseteq 1:s} [\gamma_u^a R_a(\mathbf{h}, \mathbf{1}, n, u)]^{\alpha/a} \\ &\leqslant \left[\sum_{\emptyset \subset u \subseteq 1:s} \gamma_u^a R_a(\mathbf{h}, \mathbf{1}, n, u)\right]^{\alpha/a} = [\hat{R}_a(\mathbf{h}, \boldsymbol{\gamma}, n, s)]^{\alpha/a}\end{aligned} \tag{15}$$

for $\alpha \geqslant a$.

Lemma 1 implies the following upper bound on the average value of $\tilde{R}_1(\mathbf{h}, \boldsymbol{\gamma}, n, s)$:

$$\int_{H_b^\infty} \tilde{R}_1(\mathbf{h}, \boldsymbol{\gamma}, n, s)\, d\mu_b^\infty(\mathbf{h}) \leqslant \tilde{M}(\boldsymbol{\gamma}, n, s) \quad \text{for } n = b, b^2, \ldots,$$

where

$$\tilde{M}(\boldsymbol{\gamma}, n, s) := n^{-1}\gamma_s(\beta_1 + \beta_2 \log n) \prod_{j=1}^{s-1} [1 + \gamma_j(\beta_1 + \beta_2 \log n)].$$

For any $\varepsilon > 0$ and $j = 1, 2, \ldots$ define $c_j := c_j(\varepsilon) := c_0(\varepsilon) j [\log(j+1)]^{1+\varepsilon}$, where $c_0(\varepsilon)$ is chosen to be larger than $\sum_{j=1}^{\infty} j^{-1}[\log(j+1)]^{-1-\varepsilon}$. A set of bad $\mathbf{h}$ for a particular $n = b^m$ and s is then defined as

$$\tilde{G}_{bms} = \{\mathbf{h} \in H_b^\infty :\ \tilde{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) > c_m c_s \tilde{M}(\boldsymbol{\gamma}, b^m, s)\},$$
$$m = 1, 2, \ldots,\quad s = 1, 2, \ldots\ .$$

These sets cannot be too large. In fact, $\mu_b^\infty(\tilde{G}_{bms}) \leqslant 1/(c_m c_s)$ since

$$\begin{aligned}\mu_b^\infty(\tilde{G}_{bms}) c_m c_s \tilde{M}(\boldsymbol{\gamma}, b^m, s) &\leqslant \int_{\tilde{G}_{bms}} \tilde{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, s)\, d\mu_b^\infty(\mathbf{h}) \\ &\leqslant \int_{H_b^\infty} \tilde{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) \mu_b^\infty(\mathbf{h}) \leqslant \tilde{M}(\boldsymbol{\gamma}, b^m, s).\end{aligned}$$

The set of all bad $\mathbf{h}$ is defined as the union of these sets:

$$\tilde{G}_b = \bigcup_{m=1}^{\infty} \bigcup_{s=1}^{\infty} \tilde{G}_{bms}.$$

It follows that this set is also not too large, namely,

$$\mu_b^\infty(\tilde{G}_b) \leqslant \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{c_m c_s} = \left\{ \frac{1}{c_0(\varepsilon)} \sum_{j=1}^{\infty} \frac{1}{j[\log(j+1)]^{1+\varepsilon}} \right\}^2 < 1.$$

Thus, $\mu_b^\infty(H_b^\infty \setminus \tilde{G}_b) > 0$, i.e., there exist some $\mathbf{h}$ that are not in $\tilde{G}_b$. In fact, $\mu_b^\infty(H_b^\infty \setminus \tilde{G}_b)$ may be made arbitrarily close to 1 by choosing $c_0(\varepsilon)$ large enough.

For any $\mathbf{h} \in H_b^\infty \setminus \tilde{G}_b$ one may derive the following upper bound on $\hat{R}_1$:

$$\begin{aligned}\hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) &= \sum_{j=1}^{s} \tilde{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, j) \leqslant c_m \sum_{j=1}^{s} c_j \tilde{M}(\boldsymbol{\gamma}, b^m, j) \\ &\leqslant c_0(\varepsilon) m [\log(m+1)]^{1+\varepsilon} b^{-m} \\ &\quad \times \prod_{j=1}^{s} [1 + \gamma_j c_0(\varepsilon) j \{\log(j+1)\}^{1+\varepsilon} (\beta_1 + \beta_2 m \log b)]\end{aligned}$$

for $m = 1, 2, \ldots$ and $s = 1, 2, \ldots$. Applying the inequality $\hat{R}_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, s) \leqslant [\hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}, n, s)]^\alpha$ from (15) implies part (i) of Theorem 2.

If $\sum_{j=1}^{\infty} \gamma_j^a j[\log(j+1)]^{1+\varepsilon} < \infty$, then by (14) above and Lemma 3 below it follows that for any $\delta > 0$ and any $\mathbf{h} \in H_b^{\infty} \setminus \tilde{G}_b$,

$$\hat{R}_a(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) \leqslant \hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}^a, b^m, s) \leqslant \hat{C}_R(\alpha, a, \boldsymbol{\gamma}, \delta) b^{m(-1+\delta)},$$
$$m = 1, 2, \ldots, \quad s = 1, 2, \ldots, \quad a \geqslant 1,$$

for some constant $\hat{C}_R(\alpha, a, \boldsymbol{\gamma}, \delta)$. Then applying (15) completes the proof of part (ii) of Theorem 2.

Lemma 1 also implies an upper bound on the average value of $\hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}, n, s)$:

$$\int_{H_b^{\infty}} \hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}, n, s)\, d\mu_b^{\infty}(\mathbf{h}) \leqslant M(\boldsymbol{\gamma}, n) \quad \text{for } n = b, b^2, \ldots,$$

where it is assumed that $\sum_{j=1}^{\infty} \gamma_j < \infty$ and $M(\boldsymbol{\gamma}, n)$ is defined as

$$M(\boldsymbol{\gamma}, n) := n^{-1} \prod_{j=1}^{\infty} [1 + \gamma_j(\beta_1 + \beta_2 \log n)].$$

With c_m as above, a set of bad $\mathbf{h}$ for a particular $n = b^m$ and s is then defined as

$$G_{bms} = \{\mathbf{h} \in H_b^{\infty} : \hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) > c_m M(\boldsymbol{\gamma}, b^m)\}, \quad m = 1, 2, \ldots, \quad s = 1, 2, \ldots .$$

By a similar argument to that above, $\mu_b^{\infty}(G_{bms}) \leqslant 1/c_m$ for all m and s. Moreover, since $\hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}, b^m, s)$ is a non-decreasing function of s, it follows that $G_{bm1} \subseteq G_{bm2} \subseteq \cdots$. Letting $G_{bm} = \bigcup_{s=1}^{\infty} G_{bms}$, it follows that $\mu_b^{\infty}(G_{bm}) \leqslant 1/c_m$. If $G_b = \bigcup_{m=1}^{\infty} G_{bm}$, then $\mu_b^{\infty}(G_b) < 1$ by the choice of the c_m. Thus, for all $\mathbf{h} \in H_b^{\infty} \setminus G_b$, it follows from (14) and by substituting $\boldsymbol{\gamma}^a$ for $\boldsymbol{\gamma}$ that

$$\hat{R}_a(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) \leqslant \hat{R}_1(\mathbf{h}, \boldsymbol{\gamma}^a, b^m, s) \leqslant c_m M(\boldsymbol{\gamma}^a, b^m)$$
$$= c_0(\varepsilon) m [\log(m+1)]^{1+\varepsilon} b^{-m} \prod_{j=1}^{\infty} [1 + \gamma_j^a(\beta_1 + \beta_2 m \log b)]$$

for all $a \geqslant 1$, $m \geqslant 1$, and $s \geqslant 1$. Again using Lemma 3 and (15), it follows that if $\sum_{j=1}^{\infty} \gamma_j^a < \infty$ for some $a \in [1, \alpha]$, then for any fixed $\delta > 0$ there exists a $\hat{C}_R(\alpha, a, \boldsymbol{\gamma}, \delta)$ such that for all $\mathbf{h} \in H_b^{\infty} \setminus G_b$,

$$\hat{R}_{\alpha}(\mathbf{h}, \boldsymbol{\gamma}, b^m, s) \leqslant [\hat{R}_a(\mathbf{h}, \boldsymbol{\gamma}, b^m, s)]^{\alpha/a} \leqslant \hat{C}_R(\alpha, a, \boldsymbol{\gamma}, \delta) b^{m(-\alpha/a + \delta)},$$
$$m = 1, 2, \ldots, \quad s = 1, 2, \ldots .$$

This completes the proof of part (iii) of Theorem 2. $\square$

The following lemma was used in the above proof. A similar result was proved in [4].

Lemma 3. *Given a fixed* $\tilde{\boldsymbol{\gamma}} \in [0, \infty)^{\infty}$, *let*

$$S(\tilde{\boldsymbol{\gamma}}, m) = \prod_{j=1}^{\infty} [1 + \tilde{\gamma}_j m], \quad m = 1, 2, \ldots .$$

If $\sum_{j=1}^{\infty} \tilde{\gamma}_j < \infty$, *then for any* $\delta > 0$ *it follows that* $S(\tilde{\boldsymbol{\gamma}}, m) \leqslant \tilde{C}(\tilde{\boldsymbol{\gamma}}, \delta) b^{\delta m}$ *for* $m = 1, 2, \ldots$.

Proof. Let $\sigma_d = \sum_{j=d+1}^{\infty} \tilde{\gamma}_j, \ d = 0, 1, \ldots$. Note that by increasing d one may make σ_d arbitrarily small. We can assume w.l.o.g. that all $\sigma_d > 0$. Then applying some relatively elementary inequalities yields

$$\begin{aligned}
\log[S(\tilde{\boldsymbol{\gamma}}, m)] &= \sum_{j=1}^{\infty} \log[1 + \tilde{\gamma}_j m] \\
&\leqslant \sum_{j=1}^{d} \log[1 + \sigma_d^{-1} + \tilde{\gamma}_j m] + \sum_{j=d+1}^{\infty} \log[1 + \tilde{\gamma}_j m] \\
&= d \log(1 + \sigma_d^{-1}) + \sum_{j=1}^{d} \log[1 + \tilde{\gamma}_j m/(1 + \sigma_d^{-1})] \\
&\quad + \sum_{j=d+1}^{\infty} \log[1 + \tilde{\gamma}_j m] \\
&\leqslant d \log(1 + \sigma_d^{-1}) + \sigma_d m \sum_{j=1}^{d} \tilde{\gamma}_j + m \sum_{j=d+1}^{\infty} \tilde{\gamma}_j \\
&\leqslant d \log(1 + \sigma_d^{-1}) + \sigma_d(\sigma_0 + 1) m.
\end{aligned}$$

Thus, it follows that

$$S(\tilde{\boldsymbol{\gamma}}, m) \leqslant (1 + \sigma_d^{-1})^d b^{m\sigma_d(\sigma_0+1)/\log b}.$$

Choosing d large enough to make $\sigma_d \leqslant \delta(\log b)/(\sigma_0 + 1)$ completes the proof. $\square$

Theorem 2 shows the existence of good extensible lattices with $R_1(\mathbf{h}, \boldsymbol{\gamma}, n, 1 : s) = O(n^{-1}(\log n)^{s+1}[\log\log(n+1)]^{1+\varepsilon})$ for any s with n tending to infinity. This is slightly worse than the result of [9, Theorem 5.10], which shows the existence of lattices with $R_1 = O(n^{-1}(\log n)^s)$ for fixed s and n, and the lower bound of [6] of the same order. Thus, the price for an extensible lattice (in both s and n) is an extra factor of the order $(\log n)[\log\log(n+1)]^{1+\varepsilon}$.

3. Existence of extensible lattices with small P_α

Upper bounds on P_α follow from the upper bounds on R_α derived in the previous section. The theorem below follows immediately from the following lemma, which is a minor generalization of [9, Theorem 5.5].

Lemma 4. *For any integer* $n \geqslant 2$, *finite set* $u \subset 1:\infty$, $\boldsymbol{\gamma} \in [0,\infty)^\infty$, $\alpha > 1$, *and* $\mathbf{h} = (h_j)_{j=1}^\infty \in \mathbb{Z}^\infty$ *with* $\gcd(h_j, n) = 1$ *for all* $j \in u$ *we have*

$$P_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) < R_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) - 1 + \exp\left(2\zeta(\alpha) n^{-\alpha} \sum_{j \in u} \gamma_j^\alpha\right) + n^{-1} \exp\left(2\zeta(\alpha) \sum_{j \in u} \gamma_j^\alpha\right) \times \left\{-1 + \exp\left[2^\alpha \zeta(\alpha) n^{1-\alpha} \sum_{j \in u} \gamma_j^\alpha / (1 + 2\gamma_j^\alpha \zeta(\alpha))\right]\right\},$$

where $\zeta(\alpha) = \sum_{k=1}^\infty k^{-\alpha}$ *is the Riemann zeta function.*

Proof. The proof here follows that in [9, Theorem 5.5]. Starting from (3) the definition of P_α may be written as a sum of several pieces:

$$\begin{aligned} P_\alpha(\mathbf{h}, \boldsymbol{\gamma}, n, u) &= -1 + \sum_{\mathbf{l} \in \tilde{\mathbb{Z}}^u} \tilde{r}(n\mathbf{l}, \boldsymbol{\gamma})^{-\alpha} + \sum_{\mathbf{k} \in \tilde{B}(\mathbf{h}, n, u)} \sum_{\mathbf{l} \in \tilde{\mathbb{Z}}^u} \tilde{r}(\mathbf{k} + n\mathbf{l}, \boldsymbol{\gamma})^{-\alpha} \\ &= -1 + \prod_{j \in u} \left\{\sum_{l \in \mathbb{Z}} r(nl, \gamma_j)^{-\alpha}\right\} \\ &\quad + \sum_{\mathbf{k} \in \tilde{B}(\mathbf{h}, n, u)} \prod_{j \in u} \left\{\sum_{l \in \mathbb{Z}} r(k_j + nl, \gamma_j)^{-\alpha}\right\}. \end{aligned} \tag{16}$$

The second term in this equation may be simplified as

$$\begin{aligned} \prod_{j \in u} \left\{\sum_{l \in \mathbb{Z}} r(nl, \gamma_j)^{-\alpha}\right\} &= \prod_{j \in u} [1 + 2\gamma_j^\alpha \zeta(\alpha) n^{-\alpha}] \\ &= \exp\left[\sum_{j \in u} \log(1 + 2\gamma_j^\alpha \zeta(\alpha) n^{-\alpha})\right] \\ &\leqslant \exp\left(2\zeta(\alpha) n^{-\alpha} \sum_{j \in u} \gamma_j^\alpha\right). \end{aligned}$$

The inner sum in the last term in (16) may be written as

$$\begin{aligned}\sum_{l\in\mathbb{Z}} r(k_j+nl,\gamma_j)^{-\alpha} &= r(k_j,\gamma_j)^{-\alpha}+\gamma_j^{\alpha}\left[\sum_{l=1}^{\infty}(k_j+nl)^{-\alpha}+\sum_{l=1}^{\infty}(-k_j+nl)^{-\alpha}\right]\\ &\leqslant r(k_j,\gamma_j)^{-\alpha}+\gamma_j^{\alpha}\left[\sum_{l=1}^{\infty}(nl)^{-\alpha}+\sum_{l=1}^{\infty}(-n/2+nl)^{-\alpha}\right]\\ &= r(k_j,\gamma_j)^{-\alpha}+2^{\alpha}\gamma_j^{\alpha}n^{-\alpha}\left[\sum_{l=1}^{\infty}(2l)^{-\alpha}+\sum_{l=1}^{\infty}(2l-1)^{-\alpha}\right]\\ &= r(k_j,\gamma_j)^{-\alpha}+2^{\alpha}\gamma_j^{\alpha}\zeta(\alpha)n^{-\alpha}.\end{aligned}$$

Thus, the last term in (16) satisfies

$$\begin{aligned}&\sum_{\mathbf{k}\in\tilde{B}(\mathbf{h},n,u)}\prod_{j\in u}\left\{\sum_{l\in\mathbb{Z}} r(k_j+nl,\gamma_j)^{-\alpha}\right\}\\ &\quad\leqslant \sum_{\mathbf{k}\in\tilde{B}(\mathbf{h},n,u)}\prod_{j\in u}\{r(k_j,\gamma_j)^{-\alpha}+2^{\alpha}\gamma_j^{\alpha}\zeta(\alpha)n^{-\alpha}\}\\ &\quad< R_{\alpha}(\mathbf{h},\boldsymbol{\gamma},n,u)\\ &\qquad+\sum_{v\subset u}\left\{\prod_{j\in u\setminus v}[2^{\alpha}\gamma_j^{\alpha}\zeta(\alpha)n^{-\alpha}]\sum_{\substack{\mathbf{k}\in\tilde{\mathbb{Z}}_n^u\\ \mathbf{k}^T\mathbf{h}=0 \bmod n}}\prod_{j\in v} r(k_j,\gamma_j)^{-\alpha}\right\}.\end{aligned}$$

Using the same argument as in the proof of [9, Theorem 5.5], it follows that

$$\sum_{\substack{\mathbf{k}\in\tilde{\mathbb{Z}}_n^u\\ \mathbf{k}^T\mathbf{h}=0 \bmod n}}\prod_{j\in v} r(k_j,\gamma_j)^{-\alpha}<n^{|u|-|v|-1}\prod_{j\in v}[1+2\gamma_j^{\alpha}\zeta(\alpha)].$$

Thus, the last term in (16) has the following upper bound:

$$\begin{aligned}&\sum_{\mathbf{k}\in\tilde{B}(\mathbf{h},n,u)}\prod_{j\in u}\left\{\sum_{l\in\mathbb{Z}} r(k_j+nl,\gamma_j)^{-\alpha}\right\}\\ &\quad< R_{\alpha}(\mathbf{h},\boldsymbol{\gamma},n,u)+n^{-1}\sum_{v\subset u}\left\{\prod_{j\in u\setminus v}[2^{\alpha}\gamma_j^{\alpha}\zeta(\alpha)n^{1-\alpha}]\prod_{j\in v}[1+2\gamma_j^{\alpha}\zeta(\alpha)]\right\}\end{aligned}$$

$$
\begin{aligned}
&= R_\alpha(\mathbf{h},\boldsymbol{\gamma},n,u) - n^{-1}\prod_{j\in u}[1+2\gamma_j^\alpha\zeta(\alpha)] \\
&\quad + n^{-1}\prod_{j\in u}[1+2\gamma_j^\alpha\zeta(\alpha)+2^\alpha\gamma_j^\alpha\zeta(\alpha)n^{1-\alpha}] \\
&\leqslant R_\alpha(\mathbf{h},\boldsymbol{\gamma},n,u) \\
&\quad + n^{-1}\exp\left(2\zeta(\alpha)\sum_{j\in u}\gamma_j^\alpha\right) \\
&\quad \times\left\{-1+\exp\left[2^\alpha\zeta(\alpha)n^{1-\alpha}\sum_{j\in u}\gamma_j^\alpha/(1+2\gamma_j^\alpha\zeta(\alpha))\right]\right\}.
\end{aligned}
$$

Combining this bound with that for the second term in (16) completes the proof of this lemma. □

The lemma above implies that $P_\alpha(\mathbf{h},\boldsymbol{\gamma},n,u)<R_\alpha(\mathbf{h},\boldsymbol{\gamma},n,u)+O(n^{-\alpha})$, where the leading constant in the O term depends on u. This relationship is uniform in u if $\sum_{j=1}^\infty \gamma_j^\alpha<\infty$. Thus, Theorem 2 and Lemma 4 imply the following existence theorem for extensible lattices with good P_α.

Theorem 5. *Theorem 2 also holds if one replaces R_α by P_α, allowing for a change of constants as well.*

For fixed s the quantity $P_\alpha(\mathbf{h},n,1:s)$ is known to have the following lower and upper bounds if the generating vector is allowed to depend on n and s [10,11]:

$$C_1(\alpha,s)n^{-\alpha}(\log n)^{s-1}\leqslant P_\alpha(\mathbf{h},n,1:s)\leqslant C_2(\alpha,s,\varepsilon)n^{-\alpha}(\log n)^{\alpha(s-1)+1+\varepsilon}.$$

The lower bound holds for all $\mathbf{h}$ and the upper bound holds for suitable $\mathbf{h}$. The price of an extensible lattice is that the upper bound has now increased by roughly a factor of $(\log n)^{2\alpha-1}$. Eq. (11) applied to P_α improves upon [13, Theorem 2.1] because the right-hand side decays more quickly with n, and the same $\mathbf{h}$ works for all $n=b^m$ as well as all s. Eq. (12) applied to P_α improves upon [14, Theorem 3] because the same $\mathbf{h}$ works for all $n=b^m$ and all s.

4. Existence of extensible lattices with small discrepancy

The unanchored discrepancy defined in (9) is related to R_1 [9, Chapter 5]. The following lemma, a slight generalization of [9, Theorems 3.10 and 5.6], makes this relationship explicit.

Lemma 6. *For any integer $n=b^m$, $m=1,2,\dots$, any finite subset $u\subset 1:\infty$, any $\mathbf{h}\in\mathbb{Z}^\infty$, and any corresponding node set of a shifted lattice, $\{\mathbf{z}_i=\{\mathbf{h}\Phi_b(i)+\boldsymbol{\Delta}\}: i=0,1,\dots,b^m-1\}$, the unanchored $\mathscr{L}_\infty$-discrepancy of these points projected into the* *coordinates indexed by u has the following upper bound*:

$$\sup_{\mathbf{0}\leqslant \mathbf{y}_u\leqslant \mathbf{x}_u\leqslant \mathbf{1}} |\mathrm{disc}_u(\mathbf{y},\mathbf{x},\{\mathbf{z}_i\},n)| \leqslant 1-(1-1/n)^{|u|}+\frac{1}{2}R_1(\mathbf{h},\mathbf{1},n,u).$$

Proof. It suffices to note that the proof of [9, Theorem 3.10] works also for the discrepancy extended over all intervals modulo 1 and that this discrepancy is invariant under shifts of the node set modulo 1. □

Lemma 6 implies the following theorem for the existence of extensible lattice rules with small discrepancy.

Theorem 7. *Suppose we are given a fixed integer $b\geqslant 2$, a fixed $\boldsymbol{\gamma}\in[0,\infty)^\infty$, and a fixed $\varepsilon>0$.*

(i) *There exist a μ_b^∞-measurable $\tilde{G}_b\subset H_b^\infty$ and some constant $C_D(\boldsymbol{\gamma},\varepsilon,s)$ such that for all $\mathbf{h}\in H_b^\infty\setminus\tilde{G}_b$ and all $\boldsymbol{\Delta}\in[0,1)^\infty$, the node set of the shifted lattice given in* (2) *satisfies*

$$\begin{aligned} &D_q^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s), D_q(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s)\\ &\quad\leqslant C_D(\boldsymbol{\gamma},\varepsilon,s)n^{-1}(\log n)^{s+1}[\log\log(n+1)]^{1+\varepsilon},\\ &\qquad n=b,b^2,\dots,\quad s=1,2,\dots,\quad 1\leqslant q\leqslant\infty. \end{aligned} \tag{17}$$

Furthermore, one may make $\mu_b^\infty(\tilde{G}_b)$ arbitrarily close to zero by choosing $C_D(\boldsymbol{\gamma},\varepsilon,s)$ large enough.

(ii) *If $\sum_{j=1}^\infty \gamma_j j[\log(j+1)]^{1+\varepsilon}<\infty$, then for any fixed $\delta>0$ there exists some $\tilde{C}_D(\boldsymbol{\gamma},\delta)$ such that for all $\mathbf{h}\in H_b^\infty\setminus\tilde{G}_b$ and all $\boldsymbol{\Delta}\in[0,1)^\infty$, the node set of the resulting shifted lattice satisfies*

$$\begin{aligned} &D_q^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s), D_q(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s)\leqslant \tilde{C}_D(\boldsymbol{\gamma},\delta)n^{-1+\delta},\\ &\quad n=b,b^2,\dots,\quad s=1,2,\dots,\quad 1\leqslant q\leqslant\infty. \end{aligned} \tag{18}$$

Again one may make $\mu_b^\infty(\tilde{G}_b)$ arbitrarily close to zero by choosing the above leading constant large enough.

(iii) *If $\sum_{j=1}^\infty \gamma_j<\infty$, then for any fixed $\delta>0$ there exists a μ_b^∞-measurable $G_b\subset H_b^\infty$ such that for all $\mathbf{h}\in H_b^\infty\setminus G_b$ and all $\boldsymbol{\Delta}\in[0,1)^\infty$* (18) *is satisfied, but* (17) *may not be. Here too, $\mu_b^\infty(G_b)$ can be made arbitrarily close to zero.*

Proof. From (10) and Lemma 6 it follows that the star and unanchored discrepancies have the following upper bound:

$$
\begin{aligned}
&D_q^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s), D_q(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s)\\
&\quad\leqslant \sum_{\emptyset\subset u\subseteq 1:s}\gamma_u \sup_{\mathbf{0}\leqslant \mathbf{y}_u\leqslant \mathbf{x}_u\leqslant \mathbf{1}} |\mathrm{disc}_u(\mathbf{y},\mathbf{x},\{\mathbf{z}_i\},n)|\\
&\quad\leqslant \sum_{\emptyset\subset u\subseteq 1:s}\gamma_u\left[1-(1-1/n)^{|u|}+\frac{1}{2}R_1(\mathbf{h},\mathbf{1},n,u)\right]\\
&\quad=\prod_{j=1}^{s}(1+\gamma_j)-\prod_{j=1}^{s}[1+\gamma_j(1-1/n)]+\frac{1}{2}\hat{R}_1(\mathbf{h},\boldsymbol{\gamma},n,s)\\
&\quad=\prod_{j=1}^{s}(1+\gamma_j)\left[1-\prod_{j=1}^{s}\left(1-\frac{\gamma_j}{n(1+\gamma_j)}\right)\right]+\frac{1}{2}\hat{R}_1(\mathbf{h},\boldsymbol{\gamma},n,s),
\end{aligned}
$$

where $\hat{R}$ was defined in (13). Using the fact that $\log(1-x)\geqslant x(\log(1-a))/a$ for $0\leqslant x\leqslant a<1$, we obtain a lower bound on the second product:

$$
\begin{aligned}
\log\left(\prod_{j=1}^{s}\left(1-\frac{\gamma_j}{n(1+\gamma_j)}\right)\right)&=\sum_{j=1}^{s}\log\left(1-\frac{\gamma_j}{n(1+\gamma_j)}\right)\\
&\geqslant \log(1-1/n)\sum_{j=1}^{s}\frac{\gamma_j}{1+\gamma_j}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&D_q^*(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s), D_q(\{\mathbf{z}_i\},\boldsymbol{\gamma},n,1:s)\\
&\quad\leqslant \prod_{j=1}^{s}(1+\gamma_j)\left[1-\exp\left(\log(1-1/n)\sum_{j=1}^{s}\frac{\gamma_j}{1+\gamma_j}\right)\right]+\frac{1}{2}\hat{R}_1(\mathbf{h},\boldsymbol{\gamma},n,s)\\
&\quad=\prod_{j=1}^{s}(1+\gamma_j)\left[1-\left(1-\frac{1}{n}\right)^{\sum_{j=1}^{s}\gamma_j/(1+\gamma_j)}\right]+\frac{1}{2}\hat{R}_1(\mathbf{h},\boldsymbol{\gamma},n,s).
\end{aligned}
$$

Since the star and unanchored discrepancies are both bounded above by $\hat{R}_1(\mathbf{h},\boldsymbol{\gamma},n,s)/2$ plus a term that is $O(n^{-1})$ uniformly in s assuming that the γ_j are summable, the conclusions of this theorem follow immediately from the proof of Theorem 2. □

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