Uniform distribution via lattices: from point sets to sequences

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Special session:

In this talk I present a version of my main result in [1]. I will show how to construct computationally simple sequences $S = S^{\square}_{b,d}$ in the d-dimensional hypercube with base b. For $d = 1$ these will be (generalized) van der Corput sequences. Further, I will introduce the notion of f-*subadditivity* and use it to define a very general notion of discrepancy function $\mathcal{D}$ which serves as an umbrella term that covers the L^p-discrepancy, Wasserstein p-distance, and many more methods to compare empirical measures to an underlying base measure. Thus by definition

$$\mathcal{D}\Big(\sum_{j=1}^{n} E_j\Big) \;\le\; f(n)\cdot\sum_{j=1}^{n}\mathcal{D}(E_j), \qquad \forall n\in\mathbb{N},\ \forall E_1,\dots,E_n \text{ empirical measures.}$$

Define $\ell_j^d + v := \big(b^{-j}\mathbb{Z}^d + v\big)\cap[0,1)^d$ for $v\in\mathbb{R}^d$ and let $P : [0,1)^d \to \mathbb{M}$ be any map to an arbitrary set $\mathbb{M}$ such that there is a sequence $\{h_j\}_{j=0}^{\infty}\subset\mathbb{R}$ that satisfies for each $v\in\mathbb{R}^d$ and $j\in\mathbb{N}_0$

$$\mathcal{D}\big(E_{P(\ell_j^d+v)}\big)\le h_j,$$

where E_W denotes the empirical measure with atoms at points from the (multi)set $W\subset\mathbb{M}$, i.e.

$$E_W := \sum_{x\in W}\delta_x.$$

Theorem (DF 2023) Let S_N denote the first N elements of the sequence S and let $N = \sum_{j=0}^{n}\alpha_j b^{dj}$ for $\alpha_j\in\{0,1,\dots,b^d-1\}$, then with the notation and assumptions as above

$$\mathcal{D}\big(E_{P(S_N)}\big) \;\le\; f\Big(\sum_{j=0}^{n}\alpha_j\Big)\cdot\sum_{j=0}^{n}\alpha_j\ h_j.$$

We apply this theorem in $d = 1$ to obtain bounds for the L^p–discrepancy of (generalized) van der Corput sequences for all $p\in(0,\infty]$ where $P = \mathrm{id}$, $f(n) = n^{p-1}$ if $p\ge 1$ and $f\equiv 1$ for $0<p<1$, $h_j = (p+1)^{-1}$.

Further, for $d = 2$, this construction in combination with a previous result of mine [2] yields many sequences on the two-sphere such that all their initial segments have low spherical cap discrepancy.

1. "Uniform distribution via lattices: from point sets to sequences" by D. Ferizović, https://arxiv.org/abs/2308.13297
2. "Spherical Cap Discrepancy of Perturbed Lattices Under the Lambert Projection" by D. Ferizović, Discrete Comput Geom (2023). https://doi.org/10.1007/s00454-023-00547-4