The L_2-discrepancy of latin hypercubes

Nicolas Nagel
Chemnitz University of Technology
nicolas.nagel@mathematik.tu-chemnitz.de

Coauthor(s): Dmitriy Bilyk

Special session: Stochastic Computation and Complexity

Motivated by [1] we study the L_2-discrepancy of point sets generated by permutations and their higher dimensional analogues. In [2] it has been observed that the extremal and periodic L_2-discrepancies of certain N-point sets $X \subseteq [0,1)^2$ (Hammersley/van der Corput point sets and rational lattices) fulfil a precise relation of the form

$$L_2^{\mathrm{per}}(X)^2 = 4L_2^{\mathrm{extr}}(X)^2 + \frac{N^2+1}{18N^2}.$$

We prove a much broader result about generalized energies of weighted point sets on the discretized torus, which gives us an analogous relation for general point sets constructed from latin hypercubes in all dimensions $d \geq 2$, the case $d = 2$ being permutations and including the point sets for which the relation was already noted in [2].

If $X \subseteq [0,1)^d$ is constructed from a latin hypercube then this relation immediately yields $L_2^{\mathrm{per}}(X) \gtrsim (\#X)^{\frac{d-2}{d-1}}$. A more detailed analysis of the introduced energies gives

$$L_2^{\mathrm{per}}(X) \geq \left(\frac{d}{2 \cdot 3^d}\right)^{1/2} (\#X)^{\frac{d-2}{d-1}},$$

with a similar bound of the same asymptotic rate holding for the extremal L_2-discrepancy. Random constructions yield (even in expectation) that this bound is optimal in the rate $\frac{d-2}{d-1}$ for $d \geq 3$ and even in the constant factor for $d \geq 4$. In particular these constructions can only yield point sets of low L_2-discrepancy in dimension $d = 2$.

[1] A. Hinrichs, J. Oettershagen. Optimal point sets for quasi-Monte Carlo integration of bivariate periodic functions with bounded mixed derivatives. In *Monte Carlo and Quasi-Monte Carlo Methods: MCQMC, Leuven, Belgium, April 2014* (pp. 385-405). Springer International Publishing. (2016)

[2] A. Hinrichs, R. Kritzinger, F. Pillichshammer. Extreme and periodic L_2 discrepancy of plane point sets. *Acta Arithmetica.* (2021)