Adaptive quadratures work well even for piecewise smooth functions(?)

Leszek Plaskota
University of Warsaw
leszekp@mimuw.edu.pl

Coauthor(s): Andrzej Kałuża

Adaptive quadratures are frequently used for computation of the integrals $\int_a^b f(x)\,\mathrm{d}x$, since they can adjust the subdivision of the initial integral to the behavior of the underlying function f and allow to return the value of the integral within a given accuracy ε. A theoretical justification for the practical use of adaptive quadratures assumes that the integrand is sufficiently smooth. Otherwise the quadratues are supplemented with special mechanisms that enable localization of singular points and proper approximation of the integral in their neighbourhoods.

It turns out however that adaptive quadraures work well even for integration of piecewise smooth functions, provided the subdivision strategy is properly chosen. We show this taking as an example adaptive Simpson quadratures. In this case the standard quadrature [1] does not work since recurssion does not terminate when f has discontinuities. Therefore we advocate the use the quadrature introduced in [2] and further developed in [3, 4] whose idea is to have the local errors in subintervals all equal. This is an optimal subdivision strategy and the corresponding quadrature asymptotically bahaves as though there were no singularities. Specifically, let F be a class of functions f that are piecewise four times continuously differentiable. Let μ be a 'natural' (essentially non-atomic) probability measure on F. Then, almost surely with respect to μ, the quadrature returns an ε-approximation to the integral, asymptotically as $\varepsilon \to 0$. Moreover, if $f^{(4)}$ does not change its sign in $[a, b]$ then the error asymptotically equals $\gamma \|f^{(4)}\|_{L^{1/5}} m^{-4}$, where γ is of order 1 and m is the number of subintervals in the final partition.

We believe that corresponding analysis can be done for more modern and generally preferred adaptive methods, like those based on, e.g., Clenshaw-Curtis or Gauss-Kronrod quadratures.

[1] J.N. Lyness: Notes on the adaptive Simpson quadrature routine. *Journal of the ACM* **16**, 483–495 (1969)

[2] L. Plaskota: Automatic integration using asymptotically optimal adaptive Simpson quadrature. *Numerische Mathematik* **131**, 173–198 (2015)

[3] L. Plaskota, P. Samoraj: Automatic approximation using asymptotically optimal adaptive interpolation. *Numerical Algorithms* **89**, 277–302 (2022)

[4] L. Plaskota, P. Przybyłowicz, Ł. Stępień: Monte Carlo integration of C^r functions with adaptive variance reduction: an asymptotic analysis. *BIT Numerical Mathematics* **63**, 32 (2023)