Sampling numbers of smoothness classes via ℓ^1-minimization

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Special session: Function recovery and discretization problems

We study the recovery problem for functions $\Omega \to \mathbb{C}$ belonging to a given quasi-Banach smoothness space $\mathcal{F}$ given only m function values. The worst-case L^2-approximation error in this setup, i.e.,

$$\varrho_m(\mathcal{F})_{L^2} = \inf_{t_1,\dots,t_m\in\Omega} \inf_{R:\mathbb{C}^m\to L^2} \sup_{\|f\|_{\mathcal{F}}\leq 1} \|f - R(f(t_1),\dots,f(t_m))\|_{L^2},$$

is called the mth sampling number of $\mathcal{F}$. We derive new upper bounds for the sampling numbers through an explicit nonlinear recovery map R which is based on ℓ^1-minimization (basis pursuit denoising). In relevant cases such as mixed and isotropic weighted Wiener spaces or mixed-smoothness Sobolev spaces, sampling numbers in L^2 can be upper bounded by best n-term trigonometric widths in L^∞, i.e.,

$$\sigma_n(f;\mathcal{B})_{L^\infty} = \inf_{\substack{J\subseteq I,\#J\leq n,\\ (c_j)_{j\in J}\in\mathbb{C}^J}} \left\| f - \sum_{j\in J} c_j b_j \right\|_{L^\infty}$$

with $\mathcal{B} = (b_j)_{j\in I}$ being the Fourier basis.

With this method, a significant gain in the rate of convergence compared to recently developed linear recovery methods is achieved. In this deterministic worst-case setting we see an additional speed-up of $n^{-1/2}$ compared to linear methods in case of weighted Wiener spaces. For their quasi-Banach counterparts even arbitrary polynomial speed-up is possible. Surprisingly, our approach allows to recover mixed-smoothness Sobolev functions from S^r_pW on the d-torus with a logarithmically better rate of convergence than any linear method can achieve when $1 < p < 2$ and d is large.

[1] T. Jahn, T. Ullrich, F. Voigtlaender: Sampling numbers of smoothness classes via ℓ^1-minimization, J. Complexity 79:101786, 2023.