Construction of many irreducible Sobol' (0,2)-sequences in base $b > 2$

Victor Ostromoukhov
Université Claude Bernard Lyon 1/CNRS
victor.ostromoukhov@liris.cnrs.fr

Coauthor(s): Nicolas Bonneel, David Coeurjolly, Jean-Claude Iehl

When constructing Sobol' $(t, s)-$sequences from primitive polynomials [10,2,7] or irreducible polynomials [4,5], the t-value is related to the degrees of polyonomials ($t = \sum_{i=0}^{s-1}(e_i - 1)$) [9,3,4]. This limits the number of pairs of Sobol' polynomials (and their direction vectors), which produce $(0, 2)$-sequences in base b. In pactice, one can construct $(t^*, s)-$sequences with $t^* < t$ [3,4].

In this contribution, we explore the construction of irreducible Sobol' (0,2)-sequences in base $b > 2$. It is quite natural to have (0,2)-sequences using irreducible polynomials of degree 1. But, under certain conditions, (0,2)-sequences exist with polynomials of higher degree in base $b > 2$. First, we will demonstrate the evidence of existence of irreducible Sobol' (0,m,2)-nets in bases 3, 5 and 7, up to very large m (we have explored up to $m = 1000$). This allows us to conjecture that such particular polynomials, together with specific direction vectors, form *exceptional* (0,2)-sequences.

Our exploration of (0,m,2)-nets is based on the study of characteristic matrices used in several recent papers [8,6,1] For a pair of primitive of irreducible polynomials P_i and P_j, which produce generating NUT matrices M_i and M_j, the characteristic matrix C_{ij} can be expressed as $C_{ij} = M_i M_j^{-1}$. t-values of (t,m,2)-nets can be found by studying the characteristic matrix C_{ij}. On the other hand, we have observed self-similar nature of the characteristic matrices produced with exceptional (0,m,2)-nets that we studied. Based on this observation, we conjecture that such (0,m,2)-nets are (0,2)-sequences.

We try to generalise our result and establish a set of rules that would allow to find other irreducible Sobol' (0,2)-sequences of higher degrees, together with associated direction vectors, in different bases.

REFERENCES

[1] Abdalla GM Ahmed, Mikhail Skopenkov, Markus Hadwiger, and Peter Wonka. 2023. Analysis and synthesis of digital dyadic sequences. ACM Transactions on Graphics (TOG) 42, 6 (2023), 1–17.

[2] Paul Bratley and Bennett L. Fox. 1988. Algorithm 659: Implementing Sobol's quasirandom sequence generator. ACM Trans. Math. Softw. 14, 1 (1988), 88–100.

[3] Josef Dick and Harald Niederreiter. 2008. On the exact t-value of Niederreiter and Sobol' sequences. Journal of Complexity 24, 5 (2008), 572–581.

[4] Henri Faure and Christiane Lemieux. 2016. Irreducible Sobol'sequences in prime power bases. Acta Arithmetica 173, 1 (2016), 59–80.

[5] Henri Faure and Christiane Lemieux. 2019. Implementation of irreducible Sobol'sequences in prime power bases. Mathematics and Computers in Simulation 161 (2019), 13–22.

[6] Roswitha Hofer and Kosuke Suzuki. 2019. A Complete Classification of Digital (0,

3)-Nets and Digital (0, 2)-Sequences in Base 2. Uniform distribution theory 14, 1 (2019), 43–52.

[7] Stephen Joe and Frances Y Kuo. 2008. Constructing Sobol sequences with better two-dimensional projections. SIAM Journal on Scientific Computing 30, 5 (2008), 2635–2654.

[8] Hiroki Kajiura, Makoto Matsumoto, and Kosuke Suzuki. 2018. Characterization of matrices B such that (I, B, B2) generates a digital net with t-value zero. Finite Fields and Their Applications 52 (2018), 289–300.

[9] Harald Niederreiter. 1992. Random number generation and quasi-Monte Carlo methods. SIAM.

[10] Il'ya Meerovich Sobol'. 1967. On the distribution of points in a cube and the approximate evaluation of integrals. Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki 7, 4 (1967), 784–802.