Reliable Error Estimation for Cubature Using Sobol' Sequences

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In honor of Ilya M. Sobol'

Joint work with Lluís Antoni Jiménez Rugama
speaking tomorrow, 11:15 am on lattice rules

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Automatic, Adaptive Quasi-Monte Carlo Cubature

option pricing, statistical physics, photon transport, . . .

$$\int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = ?$$

Automatic, Adaptive Quasi-Monte Carlo Cubature

$$\int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{n}\sum_{i=0}^{n-1} f(\boldsymbol{z}_i)$$

option pricing, statistical physics, photon transport, . . .

$\{\boldsymbol{z}_i\}$ chosen IID $\mathcal{U}[0,1)^d$, (Richtmyer, 1951), (Korobov, 1959), (Halton, 1960), (Sobol', 1967), (Faure, 1982), . . .

Automatic, Adaptive Quasi-Monte Carlo Cubature

$$\operatorname{err}(f, n, \{\boldsymbol{z}_i\}) := \left| \int_{[0,1)^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{n} \sum_{i=0}^{n-1} f(\boldsymbol{z}_i) \right| \leqslant \underbrace{D(n, \{\boldsymbol{z}_i\})}_{\text{quality of } \{\boldsymbol{z}_i\}} \; \underbrace{V(f)}_{\text{roughness of } f}$$

option pricing, statistical physics, photon transport, . . .

$\{\boldsymbol{z}_i\}$ chosen IID $\mathcal{U}[0,1)^d$, (Richtmyer, 1951), (Korobov, 1959), (Halton, 1960), (Sobol', 1967), (Faure, 1982), . . .

better points, tractability, multi-level (Hlawka, 1961), (Niederreiter, 1992), (Sloan and Joe, 1994), (Hickernell, 1998), (Dick and Pillichshammer, 2010), (Novak and Woźniakowski, 2010), (Dick et al., 2014) (Giles, 2014), . . .

Automatic, Adaptive Quasi-Monte Carlo Cubature

$$\operatorname{err}(f, n, \{\boldsymbol{z}_i\}) :=$$
$$\left| \int_{[0,1)^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{n} \sum_{i=0}^{n-1} f(\boldsymbol{z}_i) \right|$$
$$\leqslant \underbrace{D(n, \{\boldsymbol{z}_i\})}_{\text{quality of } \{\boldsymbol{z}_i\}} \quad \underbrace{V(f)}_{\text{roughness of } f}$$

How to automatically and adaptively choose n to ensure

$$\operatorname{err}(f, n, \{\boldsymbol{z}_i\}) \leqslant \varepsilon ?$$

(nonlinear algorithm,
non-convex set of integrands)

option pricing, statistical physics, photon transport, . . .

$\{\boldsymbol{z}_i\}$ chosen IID $\mathcal{U}[0,1)^d$, (Richtmyer, 1951), (Korobov, 1959), (Halton, 1960), (Sobol', 1967), (Faure, 1982), . . .

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guaranteed, automatic, adaptive Monte Carlo (Hickernell et al., 2014), trapezoidal rule (Clancy et al., 2014), GAIL (Choi et al., 2014)

Sobol' Points

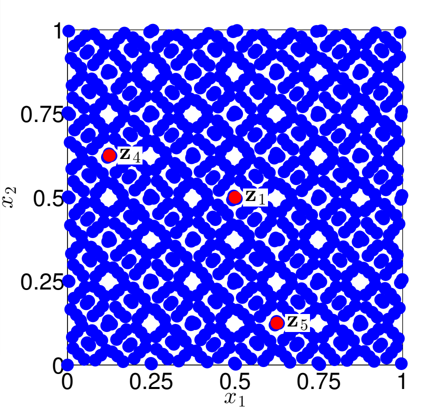
Let $\oplus$ denote binary digit by digit addition modulo 2:

$$\frac{1}{8} \oplus \frac{5}{8} = {}_2 0.001 \oplus {}_2 0.101 = {\oplus}_2 0.100 = \frac{1}{2}, \qquad 1 \oplus 5 = 001_2 \oplus 101_2 = 100_2 = 4$$

Sobol' points $\{\boldsymbol{z}_i\}$ satisfy

$$\boldsymbol{z}_0 = \boldsymbol{0}, \quad \boldsymbol{z}_i \oplus \boldsymbol{z}_\ell = \boldsymbol{z}_{i \oplus \ell} \quad \forall i, \ell \in \mathbb{N}_0$$

$$\begin{aligned} \boldsymbol{z}_1 \oplus \boldsymbol{z}_5 &= ({}_2 0.100, {}_2 0.100) \\ &\qquad \oplus ({}_2 0.101, {}_2 0.001) \\ &= ({}_2 0.001, {}_2 0.101) \\ &= \boldsymbol{z}_4 = \boldsymbol{z}_{1 \oplus 5} \end{aligned}$$

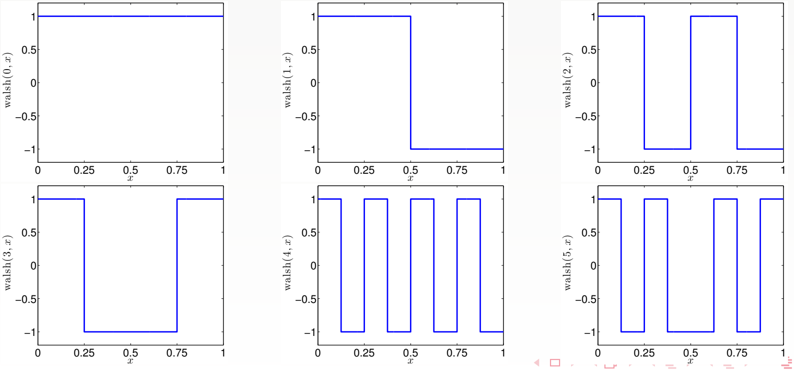


Walsh Functions

The base-2 Walsh function, $\text{walsh}(\cdot,\cdot) : (\boldsymbol{k},\boldsymbol{x}) \mapsto (-1)^{\langle \boldsymbol{k},\boldsymbol{x}\rangle}$, is defined in terms of the a bilinear function $\langle \cdot,\cdot\rangle : \mathbb{N}_0^d \times [0,1)^d \to \{0,1\}$:

$$\langle \boldsymbol{k},\boldsymbol{x}\rangle = \langle (k_1,\ldots,k_d),(x_1,\ldots,x_d)\rangle := \sum_{j=1}^{d} \langle k_j, x_j\rangle \bmod 2$$

$$\langle k_j, x_j\rangle = \langle (\cdots k_{j1}k_{j0})_2, {}_2 0.x_{j1}x_{j2}\cdots\rangle := k_{j0}x_{j1} + k_{j1}x_{j2} + \cdots \bmod 2$$



Sobol' Cubature Error via Fourier-Walsh Expansions

The Fourier-Walsh expansion of an integrand is given by

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} (-1)^{\langle \boldsymbol{k}, \boldsymbol{x} \rangle} \hat{f}(\boldsymbol{k}), \qquad \text{where } \hat{f}(\boldsymbol{k}) := \int_{[0,1)^d} f(\boldsymbol{x}) (-1)^{\langle \boldsymbol{k}, \boldsymbol{x} \rangle} \, \mathrm{d}\boldsymbol{x}.$$

Sobol' points integrate most Walsh functions perfectly, but some terribly:

$$\frac{1}{2^m} \sum_{i=0}^{2^m-1} (-1)^{\langle \boldsymbol{k}, \boldsymbol{z}_i \rangle} = \begin{cases} 1, & \boldsymbol{k} \in \mathcal{P}_m^\perp := \{\boldsymbol{k} \in \mathbb{N}_0^d : \langle \boldsymbol{k}, \boldsymbol{z}_i \rangle = 0, \ i = 0, \ldots 2^m - 1\}, \\ 0 & \boldsymbol{k} \notin \mathcal{P}_m^\perp \end{cases}$$

So Sobol' cubature error depends on the sizes of the Fourier-Walsh coefficients for $\boldsymbol{k}$ in the dual Sobol' set $\mathcal{P}_m^\perp$:

$$\mathrm{err}(f, 2^m, \{\boldsymbol{z}_i\}) := \left| \int_{[0,1)^d} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - \frac{1}{2^m} \sum_{i=0}^{2^m-1} f(\boldsymbol{z}_i) \right| = \left| \sum_{\boldsymbol{k} \in \mathcal{P}_m^\perp \setminus \{\boldsymbol{0}\}} \hat{f}(\boldsymbol{k}) \right|$$

How do we reliably bound this error based on the $f(\boldsymbol{z}_i)$?

Wavenumber Map

Define a bijective mapping $\tilde{\boldsymbol{k}} : \mathbb{N}_0 \to \mathbb{N}_0^d$ such that for all $m, \lambda \in \mathbb{N}_0$ and $\kappa = 0, \ldots, 2^m - 1$,

$$\tilde{\boldsymbol{k}}(0) = \boldsymbol{0}, \qquad \tilde{\boldsymbol{k}}(\kappa + \lambda 2^m) = \tilde{\boldsymbol{k}}(\kappa) \oplus \boldsymbol{l} \text{ for some } \boldsymbol{l} \in \mathcal{P}_m^{\perp},$$

$$\text{e.g., } \tilde{\boldsymbol{k}}(29) = \tilde{\boldsymbol{k}}(5) \oplus \boldsymbol{l} \text{ for some } \boldsymbol{l} \in \mathcal{P}_3^{\perp}, \text{ but not for some } \boldsymbol{l} \in \mathcal{P}_4^{\perp}$$

One can express the Fourier-Walsh expansion for the integrand and the error as

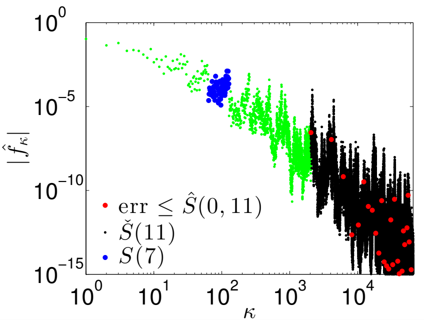
$$f(\boldsymbol{x}) = \sum_{\kappa=0}^{\infty} (-1)^{\langle \tilde{\boldsymbol{k}}(\kappa), \boldsymbol{x} \rangle} \hat{f}_{\kappa}, \qquad \hat{f}_{\kappa} := \hat{f}(\tilde{\boldsymbol{k}}(\kappa)), \qquad \operatorname{err}(f, 2^m, \{\boldsymbol{z}_i\}) = \left| \sum_{\lambda=1}^{\infty} \hat{f}_{\lambda 2^m} \right|$$

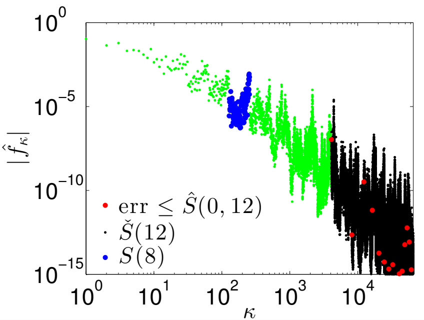
Large κ implies typically smaller $\hat{f}_{\kappa}$ in a way that is made explicit on the next slide. Moreover, the discrete Fourier-Walsh coefficients are aliased as follows:

$$\tilde{f}_{m,\kappa} := \frac{1}{2^m} \sum_{i=0}^{2^m-1} (-1)^{\langle \tilde{\boldsymbol{k}}(\kappa), \boldsymbol{z}_i \rangle} f(\boldsymbol{z}_i) \qquad \tilde{f}_{m,\kappa+\lambda 2^m} = \tilde{f}_{m,\kappa},$$

$$\text{e.g., } \tilde{f}_{3,29} = \tilde{f}_{3,5}, \text{ but generally } \hat{f}_{29} \neq \hat{f}_5 \text{ and } \tilde{f}_{4,29} \neq \tilde{f}_{4,5}$$

Cone Assumptions on Decay of Fourier Walsh Coefficients





Make cone assumptions on how the $\hat{f}_\kappa$ decay. There exist non-increasing $\widehat{\omega}$ and $\breve{\omega}$ such that for all $0 \leqslant \ell \leqslant m$

$$\widehat{S}(\ell,m) := \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\kappa+\lambda 2^m}\right|, \quad \breve{S}(m) := \sum_{\kappa=2^m}^{\infty} \left|\hat{f}_\kappa\right| \quad S(\ell) := \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \left|\hat{f}_\kappa\right|$$

$$\widehat{S}(\ell,m) \leqslant \widehat{\omega}(m-\ell)\breve{S}(m), \qquad \breve{S}(m) \leqslant \breve{\omega}(\ell) S(m-\ell) \quad \forall m \geqslant \ell + \ell_*.$$

Error Bound in Terms of Discrete Fourier-Walsh Coeff.

Cone conditions:

$$\widehat{S}(\ell,m) := \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\kappa+\lambda 2^m}\right|, \quad \check{S}(m) := \sum_{\kappa=2^m}^{\infty} \left|\hat{f}_\kappa\right| \quad S(\ell) := \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \left|\hat{f}_\kappa\right|$$

$$\widehat{S}(\ell,m) \leqslant \widehat{\omega}(m-\ell)\check{S}(m) \quad \forall \ell, \qquad \check{S}(m) \leqslant \check{\omega}(\ell) S(m-\ell) \quad \forall m \geqslant \ell + \ell_*.$$

Then we can bound the error as follows:

$$\begin{aligned}
\operatorname{err}(f, 2^m, \{\boldsymbol{z}_i\}) &= \left|\sum_{\lambda=1}^{\infty} \hat{f}_{\lambda 2^m}\right| \leqslant \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\lambda 2^m}\right| = \widehat{S}(0,m) \\
&\leqslant \widehat{\omega}(m)\check{S}(m) \leqslant \widehat{\omega}(m)\check{\omega}(\ell) S(m-\ell) \\
&\leqslant \frac{\widehat{\omega}(m)\check{\omega}(\ell)\widetilde{S}(m-\ell,m)}{1-\widehat{\omega}(\ell)\check{\omega}(\ell)}
\end{aligned}$$

proof

where

$$\widetilde{S}(\ell,m) := \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \left|\tilde{f}_{m,\kappa}\right|, \qquad \tilde{f}_{m,\kappa} := \frac{1}{2^m} \sum_{i=0}^{2^m-1} (-1)^{\left\langle \tilde{\boldsymbol{k}}(\kappa), \boldsymbol{z}_i \right\rangle} f(\boldsymbol{z}_i)$$

cubSobol_g—Guaranteed, Adaptive, Sobol' Cubature

Given an error tolerance $\varepsilon > 0$ and an integrand f, fix the lag $\ell \in \mathbb{N}$ and let

$$\mathfrak{C} = \frac{\breve{\omega}(\ell)}{1 - \widehat{\omega}(\ell)\breve{\omega}(\ell)}, \qquad m = \ell + \ell_*.$$

Step 1. Compute the sum of the (data-based) discrete Fourier-Walsh coefficients $\widetilde{S}(m-\ell, m)$.

Step 2. If the error tolerance is satisfied,

$$\mathfrak{C}\widehat{\omega}(m)\widetilde{S}(m-\ell, m) \leqslant \varepsilon,$$

then return the Sobol' cubature answer.

Step 3. Otherwise, increase m by one, and return to Step 1.

Theorem. If f satisfies the cone conditions on its Fourier-Walsh coefficients, then

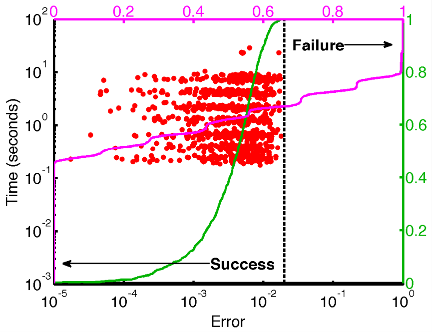
$$\left| \int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{2^m}\sum_{i=1}^{m} f(\boldsymbol{z}_i) \right| \leqslant \varepsilon$$

with $m \leqslant \min\{m' : \mathfrak{C}\widehat{\omega}(m')[1 + \widehat{\omega}(\ell)\breve{\omega}(\ell)]S(m') \leqslant \varepsilon\}$ proof.

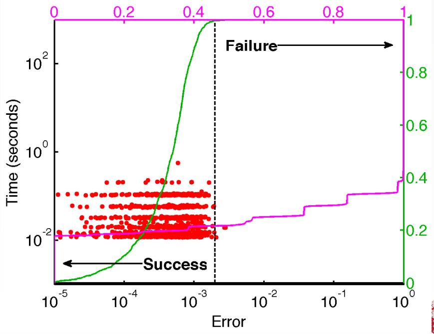
Asian Geometric Mean Call Option

$$S(0) = K = 100, \quad T = 1, \quad r = 0.03, \quad \text{volatility} \sim \mathcal{U}[0.1, 0.7],$$
$$d = \#\ \text{time steps} \sim \mathcal{U}\{1, 2, 4, 8, 16, 32, 64\}$$

IID, `cubMC_g`

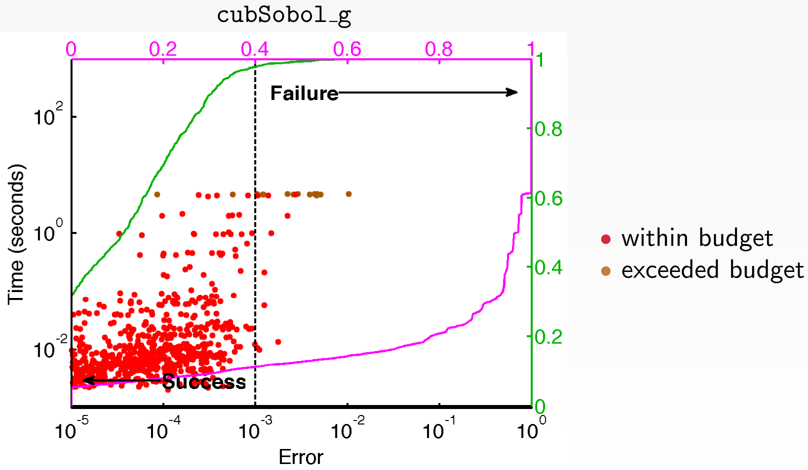


Sobol', `cubSobol_g`



Genz and Keister Examples

These examples come from Genz (1987) and Keister (1996)



What's Missing for `cubSobol_g`?

- Connections between our cone conditions and familiar spaces of integrands, such as Korobov spaces
- Lower bound on the computational complexity
- Relative error criterion
- Justification for our heuristic determination of the map κ

Why Not Replications to Estimate Error?

$$\left|\int_{[0,1)^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \hat{\mu}_n\right| \leqslant \mathfrak{C}\sqrt{\frac{1}{R-1}\sum_{r=1}^{R}(Y_r - \hat{\mu}_n)^2}, \qquad \hat{\mu}_n := \frac{1}{R}\sum_{i=0}^{R-1} Y_r$$

IID Replications $Y_r = \dfrac{R}{n}\displaystyle\sum_{i=0}^{n/R-1} f(\boldsymbol{z}_i^{(r)})$, where $\{\boldsymbol{z}_i^{(r)}\}$ are independent randomizations.

Internal Replications $Y_r = \dfrac{R}{n}\displaystyle\sum_{i=(r-1)n/R}^{rn/R-1} f(\boldsymbol{z}_i)$.

Want R small to take advantage of Sobol' point evenness, but need R large to ensure that sample variance of Y_r represents error (Deng, 2013; Hickernell et al., 2014).

Guaranteed Automatic Integration Library (GAIL) http://code.google.com/p/gail/

- Version 1.3 (February 14, 2014) (Choi et al., 2014) includes `integral_g.m`, `meanMC_g.m`, `cubMC_g.m`, and `funappx_g.m`
- Version 2.0 (exp. September 2014) will hopefully include some of the following:
  - quasi-Monte Carlo—`cubSobol_g.m`, `cubLattice_g.m`
  - Monte Carlo for Bernoulli—`meanBernoulli_g.m`
  - multi-level Monte Carlo—`meanMLMC_g.m`
  - multivariate function approximation
  - relative error criterion
  - higher order methods (for one dimensional problems)
  - local adaption (for one dimensional problems)
- Theory developed in Hickernell et al. (2014) and Clancy et al. (2014). Should apply to other problems where

$$\text{solution}(cf) = |c|\,\text{solution}(f).$$

References I

Choi, S.-C. T., Y. Ding, F. J. Hickernell, L. Jiang, and Y. Zhang. 2014. *GAIL: Guaranteed Automatic Integration Library (version 1.3)*.

Clancy, N., Y. Ding, C. Hamilton, F. J. Hickernell, and Y. Zhang. 2014. *The cost of deterministic, adaptive, automatic algorithms: Cones, not balls*, J. Complexity **30**, 21–45.

Deng, S. 2013. *An investigation of the quasi-standard error for quasi-monte carlo method*, Master's Thesis.

Dick, J., F. Kuo, and I. H. Sloan. 2014. *High dimensional integration — the Quasi-Monte Carlo way*, Acta Numer., 1–153.

Dick, J., F. Y. Kuo, G. W. Peters, and I. H. Sloan (eds.) 2014. *Monte Carlo and quasi-Monte Carlo methods 2012*, Springer-Verlag, Berlin.

Dick, J. and F. Pillichshammer. 2010. *Digital nets and sequences: Discrepancy theory and quasi-Monte Carlo integration*, Cambridge University Press, Cambridge.

Faure, H. 1982. *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41**, 337–351.

Genz, A. 1987. *A package for testing multiple integration subroutines*, Numerical integration: Recent developments, software and applications, pp. 337–340.

References II

Giles, M. 2014. *Multilevel Monte Carlo methods*, Monte Carlo and quasi-Monte Carlo methods 2012.

Halton, J. H. 1960. *On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals*, Numer. Math. **2**, 84–90.

Hickernell, F. J. 1998. *A generalized discrepancy and quadrature error bound*, Math. Comp. **67**, 299–322.

Hickernell, F. J., L. Jiang, Y. Liu, and A. B. Owen. 2014. *Guaranteed conservative fixed width confidence intervals via Monte Carlo sampling*, Monte Carlo and quasi-Monte Carlo methods 2012, pp. 105–128.

Hlawka, E. 1961. *Funktionen von beschränkter Variation in der Theorie der Gleichverteilung*, Ann. Mat. Pura Appl. **54**, 325–333.

Keister, B. D. 1996. *Multidimensional quadrature algorithms*, Computers in Physics **10**, 119–122.

Korobov, N. M. 1959. *The approximate computation of multiple integrals*, Dokl. Akad. Nauk. SSR **124**, 1207–1210. (Russian).

Niederreiter, H. 1992. *Random number generation and quasi-Monte Carlo methods*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia.

References III

Novak, E. and H. Woźniakowski. 2010. *Tractability of multivariate problems Volume ii: Standard information for functionals*, EMS Tracts in Mathematics, European Mathematical Society, Zürich.

Richtmyer, R. D. 1951. *The evaluation of definite integrals, and a Quasi-Monte-Carlo method based on the properties of algebraic numbers*, Technical Report LA-1342, Los Alamos Scientific Laboratory.

Sloan, I. H. and S. Joe. 1994. *Lattice methods for multiple integration*, Oxford University Press, Oxford.

Sobol', I. M. 1967. *The distribution of points in a cube and the approximate evaluation of integrals*, U.S.S.R. Comput. Math. and Math. Phys. **7**, 86–112.

Bounding $S(\ell)$ in Terms of $\widetilde{S}(\ell, m)$

$$
\begin{aligned}
S(\ell) &= \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \left|\hat{f}_\kappa\right| = \sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \left|\tilde{f}_{m,\kappa} - \sum_{\lambda=1}^{\infty} \hat{f}_{\kappa+\lambda 2^m}\right| \\
&\leqslant \underbrace{\sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \left|\tilde{f}_{m,\kappa}\right|}_{\widetilde{S}(\ell,m)} + \underbrace{\sum_{\kappa=\lfloor 2^{\ell-1}\rfloor}^{2^\ell-1} \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\kappa+\lambda 2^m}\right|}_{\widehat{S}(\ell,m)} \\
&\leqslant \widetilde{S}(\ell, m) + \widehat{\omega}(m-\ell)\breve{\omega}(m-\ell)S(\ell)
\end{aligned}
$$

$$
S(\ell) \leqslant \frac{\widetilde{S}(\ell, m)}{1 - \widehat{\omega}(m-\ell)\breve{\omega}(m-\ell)} \qquad \text{provided that } \widehat{\omega}(m-\ell)\breve{\omega}(m-\ell) < 1
$$

back

Bounding $\widetilde{S}(\ell, m)$ in Terms of $S(\ell)$

$$
\begin{aligned}
\widetilde{S}(\ell, m) &= \sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^\ell - 1} \left| \tilde{f}_{m,\kappa} \right| = \sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^\ell - 1} \left| \hat{f}_\kappa + \sum_{\lambda=1}^{\infty} \hat{f}_{\kappa+\lambda 2^m} \right| \\
&\leqslant \underbrace{\sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^\ell - 1} \left| \hat{f}_\kappa \right|}_{\widehat{S}(\ell)} + \underbrace{\sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^\ell - 1} \sum_{\lambda=1}^{\infty} \left| \hat{f}_{\kappa+\lambda 2^m} \right|}_{\widehat{S}(\ell, m)} \\
&\leqslant [1 + \widehat{\omega}(m-\ell)\mathring{\omega}(m-\ell)] S(\ell)
\end{aligned}
$$

back

Computing the Discrete Fourier Walsh Coefficients

$$\begin{aligned}
\widetilde{S}(\ell, m) &= \sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^{\ell}-1} \left|\tilde{f}_{m,\kappa}\right| = \sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^{\ell}-1} \left|\hat{f}_{\kappa} + \sum_{\lambda=1}^{\infty} \hat{f}_{\kappa+\lambda 2^m}\right| \\
&\leqslant \underbrace{\sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^{\ell}-1} \left|\hat{f}_{\kappa}\right|}_{\widehat{S}(\ell)} + \underbrace{\sum_{\kappa=\lfloor 2^{\ell-1} \rfloor}^{2^{\ell}-1} \sum_{\lambda=1}^{\infty} \left|\hat{f}_{\kappa+\lambda 2^m}\right|}_{\widehat{S}(\ell,m)} \\
&\leqslant [1 + \widehat{\omega}(m-\ell)\breve{\omega}(m-\ell)] S(\ell)
\end{aligned}$$