Rigged Hilbert space

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In mathematics, a **rigged Hilbert space** (**Gelfand triple**, **nested Hilbert space**, **equipped Hilbert space**) is a construction designed to link the distribution and square-integrable aspects of functional analysis. Such spaces were introduced to study spectral theory in the broad sense.[vague] They bring together the 'bound state' (eigenvector) and 'continuous spectrum', in one place.

Motivation [edit]

A function such as

$$x \mapsto e^{ix},$$

is an eigenfunction of the differential operator

$$-i\frac{d}{dx}$$

on the real line $\mathbf{R}$, but isn't square-integrable for the usual Borel measure on $\mathbf{R}$. To properly consider this function as an eigenfunction requires some way of stepping outside the strict confines of the Hilbert space theory. This was supplied by the apparatus of Schwartz distributions, and a *generalized eigenfunction* theory was developed in the years after 1950.

Functional analysis approach [edit]

The concept of rigged Hilbert space places this idea in an abstract functional-analytic framework. Formally, a rigged Hilbert space consists of a Hilbert space H, together with a subspace Φ which carries a finer topology, that is one for which the natural inclusion

$$\Phi \subseteq H$$

is continuous. It is no loss to assume that Φ is dense in H for the Hilbert norm. We consider the inclusion of dual spaces H^* in Φ^*. The latter, dual to Φ in its 'test function' topology, is realised as a space of distributions or generalised functions of some sort, and the linear functionals on the subspace Φ of type

$$\phi \mapsto \langle v, \phi \rangle$$

for v in H are faithfully represented as distributions (because we assume Φ dense).

Now by applying the Riesz representation theorem we can identify H^* with H. Therefore, the definition of *rigged Hilbert space* is in terms of a sandwich:

$$\Phi \subseteq H \subseteq \Phi^*.$$

The most significant examples are those for which Φ is a nuclear space; this comment is an abstract expression of the idea that Φ consists of test functions and Φ^* of the corresponding distributions. Also, a simple example is given by Sobolev spaces: Here (in the simplest case of Sobolev spaces on $\mathbb{R}^n$)

$$H = L^2(\mathbb{R}^n),\ \Phi = H^s(\mathbb{R}^n),\ \Phi^* = H^{-s}(\mathbb{R}^n),$$

where $s > 0$.

Formal definition (Gelfand triple) [edit]

A **rigged Hilbert space** is a pair (H, Φ) with H a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous.

Identifying H with its dual space H^*, the adjoint to i is the map

$$i^* : H = H^* \to \Phi^*.$$

The duality pairing between Φ and Φ^* is then compatible with the inner product on H, in the sense that:

$$\langle u, v \rangle_{\Phi \times \Phi^*} = (u, v)_H$$

whenever $u \in \Phi \subset H$ and $v \in H = H^* \subset \Phi^*$. In the case of complex Hilbert spaces, we use a Hermitian inner product; it will be complex linear in u (math convention) or v (physics convention), and conjugate-linear (complex anti-linear) in the other variable.

The triple $(\Phi,\ H,\ \Phi^*)$ is often named the "Gelfand triple" (after the mathematician Israel Gelfand).

Note that even though Φ is isomorphic to Φ^* (via Riesz representation) if it happens that Φ is a Hilbert space in its own right, this isomorphism is *not* the same as the composition of the inclusion i with its adjoint i^*

$$i^* i : \Phi \subset H = H^* \to \Phi^*.$$

References [edit]

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