USEFUL INEQUALITIES

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There are some inequalities that get used over and over by me. Here they are summarized for easy reference.

1. Hölder's Inequality and Its Generalizations

Let $\{a_i\}_{i\in\mathcal{I}}$, and $\{b_i\}_{i\in\mathcal{I}}$ be two finite or countably infinite sequences of complex numbers. Hölder's inequality states that

$$\left|\sum_{i\in\mathcal{I}} a_i b_i\right| \le \sum_{i\in\mathcal{I}} |a_i b_i| = \left\|(a_i b_i)_{i\in\mathcal{I}}\right\|_1 \le \left\|(a_i)_{i\in\mathcal{I}}\right\|_p \left\|(b_i)_{i\in\mathcal{I}}\right\|_q, \qquad 1 \le p, q \le \infty, \quad \frac{1}{p}+\frac{1}{q}=1, \text{ i.e., } q=\frac{p}{p-1}. \tag{1a}$$

Moreover, in the above inequality,

$$\text{equality holds if } |a_i|^{p-1}|b_i|^{-1} \text{ is constant } \forall i \in \mathcal{I}. \tag{1b}$$

This inequality can be generalized. Choose any $r > 1$. It follows from that

$$\begin{aligned}
\left\|(a_i b_i)_{i\in\mathcal{I}}\right\|_r &= \left\|(a_i^r b_i^r)_{i\in\mathcal{I}}\right\|_1^{1/r} \le \left\|(a_i^r)_{i\in\mathcal{I}}\right\|_p^{1/r} \left\|(b_i^r)_{i\in\mathcal{I}}\right\|_q^{1/r} \\
&= \left\|(a_i)_{i\in\mathcal{I}}\right\|_{pr} \left\|(b_i)_{i\in\mathcal{I}}\right\|_{qr}, \qquad 1 \le p, q \le \infty, \quad \frac{1}{p}+\frac{1}{q}=1, \text{ i.e., } q=\frac{p}{p-1}.
\end{aligned}$$

By change of notation, this may be re-written as

$$\left\|(a_i b_i)_{i\in\mathcal{I}}\right\|_r \le \left\|(a_i)_{i\in\mathcal{I}}\right\|_p \left\|(b_i)_{i\in\mathcal{I}}\right\|_q, \qquad 1 \le r \le p, q \le \infty, \quad q = \frac{pr}{p-r}. \tag{2a}$$

Moreover, in the above inequality,

$$\text{equality holds if } |a_i|^{p-1}|b_i|^{-1} \text{ is constant } \forall i \in \mathcal{I}. \tag{2b}$$

This provides an bound on an r-norm of a vector whose components are products of two terms as a product of the p-norm of the vector of one factor and the q-norm of the vector of the other factors, where p and q are both no smaller than r.

2. Opposite Direction Inequality

For any $r \ge 1$, define the function $f(a) = (1+a)^r - 1 - a$. Since $f(0) = 0$, and $f'(a) = r[(1+a)^{r-1} - a^{r-1}] \ge 0$ for all $a \ge 0$, it follows that $f(a) \ge 0$ and so

$$1 + a^r \le (1+a)^r \quad \forall a \ge 0, \ r \ge 1.$$

This can be generalized to

$$a^r + b^r \le (a+b)^r \quad \forall a, b \ge 0, \ r \ge 1.$$

Now we prove

$$a_1^r + \cdots + a_n^r \le (a_1 + \cdots + a_n)^r \quad \forall a_i \ge 0, \ r \ge 1,$$

by induction. It is already true for $n = 2$. Suppose it is true for $n = N$, and consider $n = N + 1$. It follows that

$$\begin{aligned} a_1^r + \cdots + a_{N+1}^r &\le a_1^r + \cdots + a_{N-1}^r + b^r, \quad b = a_N + a_{N+1} \\ &\le (a_1 + \cdots + a_{N-1} + b)^r \\ &= (a_1 + \cdots + a_{N-1} + a_N + a_{N+1})^r, \end{aligned}$$

thus completing the proof. Therefore, we have

$$\|(a_i)_{i\in\mathcal{I}}\|_r \le \|(a_i)_{i\in\mathcal{I}}\|_p, \qquad 1 \le p \le r \le \infty. \tag{3}$$

Moreover, in the above inequality, equality holds if exactly one a_i is nonzero. Going further, we can can conclude that

$$\|(a_i b_i)_{i\in\mathcal{I}}\|_r \le \|(a_i)_{i\in\mathcal{I}}\|_p \|(b_i)_{i\in\mathcal{I}}\|_\infty, \qquad 1 \le p \le r \le \infty, \tag{4}$$

with equality holding if exactly one $a_i = 0$ for all $i \ne j$ and $b_j = \|(b_i)_{i\in\mathcal{I}}\|_\infty$. Combining this inequality with (2a) it yields

$$\|(a_i b_i)_{i\in\mathcal{I}}\|_r \le \|(a_i)_{i\in\mathcal{I}}\|_p \|(b_i)_{i\in\mathcal{I}}\|_q, \qquad 1 \le r, p \le \infty, \quad q = \frac{pr}{\max(p - r, 0)}. \tag{5}$$

3. Jensen's Inequality and Its Generalizations

Let ϕ be a convex function, i.e.,

$$\phi((1-\lambda)a + \lambda b) \le (1-\lambda)\phi(a) + \lambda\phi(b) \qquad \forall a, b, \lambda \text{ with } 0 \le \lambda \le 1.$$

It then follows by induction that

$$\begin{aligned} &\phi(\lambda_1 a_1 + \cdots + \lambda_n a_n) \le \lambda_1 \phi(a_1) + \cdots + \lambda_n \phi(a_n) \\ &\qquad\qquad \forall a_1, \ldots, a_n, \lambda_1, \ldots, \lambda_n \text{ with } 0 \le \lambda_i \le 1 \;\&\; \lambda_1 + \cdots + \lambda_n = 1. \end{aligned} \tag{6}$$

For example, if $-\infty < p \le q < \infty$, define $a_i = b_i^p$, where $b_i > 0$, and note that $\phi : x \mapsto x^{q/p}$ is convex. It follows from (6) that

$$\begin{aligned} &(\lambda_1 b_1^p + \cdots + \lambda_n b_n^p)^{1/p} \le (\lambda_1 b_1^q + \cdots + \lambda_n b^q)^{1/q} \\ &\qquad\qquad \forall b_1, \ldots, b_n, \lambda_1, \ldots, \lambda_n \text{ with } b_i > 0,\ 0 \le \lambda_i \le 1 \;\&\; \lambda_1 + \cdots + \lambda_n = 1. \end{aligned} \tag{7}$$

This means that the weighted p-mean is no greater than the weighted q-mean. If all the b_i are the same, then the two means are equal.

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