Analysis of the Sierpinski Triangle

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Introduction

The Sierpinski Triangle is a geometrical object named after the Polish mathematician Waclaw Sierpinski. It's an equilateral triangle which is recursively divided into smaller equilateral triangles. The first equilateral triangle (level-0 Sierpinski triangle) is an equilateral triangle with unit length sides. In the first iteration three new equilateral triangles with side length half of the unit are created and placed so that there is a triangular gap in the middle. This gap has the same side length as the other three.

This could also be seen as dividing the triangle into 4 sub-triangles of equal length. Then you would colour in all of the triangles except the middle one, leaving it to stand out (assuming the original triangle wasn't coloured at all). In the second iteration this process is repeated with each of the "couloured" triangles, creating four equilateral triangles of half the length of the previous one and colouring all but the middle one. This uncoloured triangle acts as a "gap". This process is iterated continously leaving the original triangle with more gaps in each iteration.

The purpose of this project is to come up with recurrence equations for the number of holes, perimeter and surface of the original triangle in each iteration and to solve them.

Number of holes

We want to find a recurrence equation $h(n)$ that gives us the number of gaps in the main triangle in each iteration. In the level 0 Sierpinski Triangle there are 0 gaps. After the first iteration 3 new triangles are created leaving 1 gap. In the second iteration each of these 3 triangles produce 3 more new triangles and 3 new gaps. Notice that the number of current gaps gives us an idea of the number of triangles and therefore the number of gaps that will be produced on the next iteration. For example, in the first iteration $h(1)$ we have only one gap, every gap is surrounded by 3 triangles which will each produce a new gap in the next iteration, so 3 new gaps. However we have to add the original big gap that was already there, so $h(2) = 4$. For $h(3)$ if we now look at the three new gaps formed, each of these are surrounded by three triangles, which will create one more gap each, so 9 new gaps. To these we have to add the 4 gaps that we had previously.

$h(3) = 13$. Following this process to find the number of gaps in iteration n we have to take into account the number of new gaps that appeared in the last iteration, for each of those we will get 3 new gaps and then we have to add the gaps that were already there. We come to the conclusion that $h(n) = 3(h(n-1)-h(n-2))+h(n-1) = 4h(n-1)-3h(n-2)$.

Lets solve this recurrence equation:
The characteristic equation would be $r^2 - 4r + 3 = (r-3)(r-1) = 0$. Therefore the roots are $r_1 = 3$ and $r_2 = 1$ which means that $h(n) = \alpha 3^n + \beta$. Taking into account that $h(0) = 0$ and $h(1) = 1$ we obtain $h(n) = \frac{3^n}{2} + -\frac{1}{2} = \frac{3^n-1}{2}$. It's clear that $\lim_{n\to\infty} \frac{3^n-1}{2} = \infty$. This makes sense as the number of gaps does not only increase after every iteration but so does the number of new gaps from one iteration to the next.

Perimeter

For simplicity we shall use the following notation, we will represent the unit length of the side of the original equilateral triangle with the number 1. This means that the perimeter of the original triangle is 3, $p(0) = 3$. In the first iteration the outward perimeter stays the same but there is now a triangular gap in the middle, meaning we have to take into account the sides of the triangles that border with this gap. This is the same as finding the perimeter of the triangular gap which will be $\frac{3}{2}$ as each side of this gap is $\frac{1}{2}$. Therefore $p(1) = \frac{9}{2}$. In the second iteration 3 new triangular gaps of side length $\frac{1}{4}$ are created, bringing the total perimeter to $p(2) = \frac{27}{4}$.

In general, in each iteration we take into account the perimeter of the previous iteration and then add the length of the sides that the new gaps create. In each iteration the number of new gaps increases by a factor of 3 but the length size of these decrease by a factor of a half. This results in the perimeter increasing by a factor of $\frac{3}{2}$ after each iteration. We obtain $p(n) = \frac{3}{2}p(n-1)$.

Clearly this recurrence equation has one root, $\frac{3}{2}$. Therefore $p(n) = \alpha(\frac{3}{2})^n$. As $p(0) = 3$ we get that $p(n) = 3(\frac{3}{2})^n$. When n tends to infinity clearly so does $p(n)$ meaning the perimeter of the fractal won't stop increasing.

Surface

Once again, for simplicity we are going to call the area of the original triangle $s(0) = A$. This area would actually be $\frac{a^2\sqrt{3}}{4}$ where a is the unit length of the sides. In the first iteration the triangle is divided into 4 equivalent parts, one of which is a gap. Therefore $s(1) = \frac{3}{4}A$. In the second iteration each of the 3 triangles remaining are divided once again into 4 equivalent parts, with one of them being a gap. Each of the 3 triangles that had an area of $\frac{1}{4}A$ get their area reduced by three quarters so that their area is now $\frac{3}{16}$. This brings the total area to $s(2) = \frac{9}{16}$.

It is easy to see that in general in each iteration we are reducing the area of each triangle by three quarters, but as these triangles make up for all of the area of the fractal, the total area is reduced by a factor of three quarters every time. In conclusion, $s(n) = \frac{3}{4}p(n-1)$. Solving this equation we obtain $s(n) = \alpha(\frac{3}{4})^n$. Using that $s(0) = A$ the final solution is $s(n) = A(\frac{3}{4})^n$. As $\frac{3}{4} < 1$, $\lim_{n\to\infty} A(\frac{3}{4})^n = 0$. Meaning that the total area of the fractal tends to zero as the number of gaps increases.