Momentum Diffusion

A Step Toward Understanding the Complexity of Navier–Stokes

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Conceptual Framework

The Navier–Stokes equations are the foundation of geophysical fluid dynamics, but due to their nonlinear and coupled nature, they are generally intractable in their full form. One strategy to gain insight is to isolate individual terms and analyze the resulting motion under idealized conditions.

In their horizontal momentum form (neglecting vertical structure and assuming constant density), the equations can be written as:

$$\underbrace{\frac{du}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{advection}} = \underbrace{fv}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\nu\nabla^2 u}_{\text{viscosity}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{advection}} = \underbrace{-fu}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{pressure}} + \underbrace{\nu\nabla^2 v}_{\text{viscosity}}$$

In this case, only the local acceleration and the viscous term are retained (highlighted in black), while advection, pressure gradient, and Coriolis forces are omitted (shown in gray). This configuration allows us to study how momentum diffuses under the influence of viscosity in the absence of external or advective forcing.

Before proceeding, let us clarify two fundamental concepts involved in this case:

- **Momentum**: In fluid dynamics, momentum refers to the product of mass and velocity. Since we work with velocity fields in a continuous medium, we typically refer to *momentum per unit volume*, given by ρu, where ρ is the fluid density and u is the velocity.

- **Viscosity**: Viscosity is a measure of a fluid's internal resistance to deformation. It quantifies how momentum is transferred between adjacent fluid layers. A high viscosity (like in honey) leads to strong damping and smooth velocity profiles, whereas a low viscosity (like in air or water) allows for more rapid and variable motion.

  In the Navier–Stokes equations, this effect is modeled by the diffusion term $\nu\nabla^2 u$, where ν is the *kinematic viscosity* (the dynamic viscosity μ divided by density: $\nu = \mu/\rho$).

To simplify the analysis, we consider a one-dimensional version of the system, where motion occurs only along the x-direction and all variables depend on x and time t.

Simplified Model

To make the problem analytically and numerically tractable, we consider the system in one spatial dimension. Assuming that velocity has only one component $u(x,t)$, and that it depends only on x and t, the governing equation becomes:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

This is the one-dimensional diffusion equation for momentum. It describes how an initial distribution of velocity is smoothed out over time due to the action of viscosity.

Step-by-Step Derivation

We begin from the full momentum equation, with all other terms removed:

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u \tag{2}$$

Since we are considering only one spatial direction x, the Laplacian reduces to a second spatial derivative:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Substituting equation (3) into equation (2), we obtain the one-dimensional diffusion equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{4}$$

This linear partial differential equation governs the time evolution of velocity when viscosity is the only active process. No pressure gradients, Coriolis terms, or nonlinear advection are present.

Physical Interpretation

The solution to equation (4) shows how any initial distribution of **velocity** is smoothed over time due to viscous effects. Sharp gradients are flattened, and kinetic energy is gradually dissipated as internal friction spreads motion through the fluid.

In this simulation, we start from a localized Gaussian pulse of velocity. As time progresses, the peak flattens and spreads symmetrically, without introducing net motion: the center of mass of the velocity distribution remains stationary.

It is important to note that diffusion does not depend directly on the magnitude of the velocity, but on its curvature in space. Regions with higher curvature (second derivative of velocity) experience more intense diffusion. In our initial condition, the center of the Gaussian has the steepest curvature, and thus, diffusion is strongest there. This explains why the highest part of the pulse decays fastest in the animation.

This case represents the purest expression of momentum diffusion and sets a clear benchmark for validating numerical methods that include viscous effects.

Code and Animation

- **Code available at:** https://bit.ly/OOM_momentum_diffusion
- **Animation available at:** https://youtu.be/BA4jnNMiRyI

Description

This simulation illustrates:

- How a localized velocity field evolves under the sole effect of viscosity.
- The smoothing action of diffusion, in the absence of pressure, advection or Coriolis forces.
- The difference between redistribution and transport: the momentum spreads, but does not propagate as a wave.

This is the third entry in our systematic breakdown of the Navier–Stokes momentum equations, where each case isolates a specific physical mechanism.