Nonlinear Advection

A Step Toward Understanding the Complexity of Navier–Stokes

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Conceptual Framework

The Navier–Stokes equations describe the full dynamics of geophysical flows, combining acceleration, advection, pressure gradients, Coriolis forces, and viscous diffusion. Each term represents a physical mechanism that drives or modifies motion.

As in previous cases, we isolate a specific term to understand its role in shaping fluid behavior. Here we focus exclusively on the nonlinear advection term:

$$\underbrace{\frac{du}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{advection}} = \underbrace{fv}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\nu\nabla^2 u}_{\text{viscosity}}$$
$$\underbrace{\frac{dv}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{advection}} = \underbrace{-fu}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{pressure}} + \underbrace{\nu\nabla^2 v}_{\text{viscosity}}$$

Only the acceleration and nonlinear advection terms are retained (in black); all others are grayed out.

Nonlinear advection means that the velocity field itself controls the speed at which different fluid parcels move. Regions with higher velocities travel faster than regions with lower velocities, resulting in a distortion of the original shape of any imposed disturbance.

For simplicity, we restrict our analysis to one spatial dimension. This allows us to isolate the essential features of nonlinear advection without additional complexity from multidimensional interactions.

Importantly, this term cannot initiate motion on its own: if the system starts from rest, it will remain at rest. The model requires an initial perturbation to evolve—such as the step profile we impose at the start of the simulation.

To better handle such nonlinearity and guarantee momentum conservation, we rewrite the model in conservative form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{1}$$

This form is especially important when dealing with sharp gradients or discontinuities, as it ensures consistency in the numerical treatment of fluxes across cell boundaries.

In this case study, we observe how an initial velocity step evolves: faster parts of the fluid outpace the slower ones, transforming the sharp front into a progressively smoother ramp. This emergent structure is a direct consequence of nonlinear advection and illustrates how momentum is redistributed in the absence of external forces.

Numerical Experiment

We implement a numerical simulation using the Lax–Friedrichs scheme to solve equation (1). The initial condition is a step function:

$$u(x,0) = \begin{cases} 1, & x < x_0 \\ 0, & x \geq x_0 \end{cases}$$

This setup initiates a nonlinear front that evolves due to the self-advection of momentum.

Physical Interpretation

Unlike linear advection, where all parts of the velocity profile propagate at the same speed, nonlinear advection causes different parts of the front to travel at different speeds. Higher values of u travel faster, leading to a distortion of the initial step into a sloping ramp.

In the animation, we focus exclusively on the evolving velocity front. The initial step profile gradually transforms into a ramp as the high-velocity region advances faster than the lower one. This illustrates the intrinsic deformation introduced by nonlinear advection, in the absence of viscosity or external forces.

Code and Animation

- **Code available at:** https://github.com/OpenOceanModels/NS_nonlinear_advection
- **Animation available at:** https://www.youtube.com/watch?v=VTZKAsie4QE

Description

This experiment illustrates:

- The nonlinear nature of self-advection.
- The evolution of a discontinuous initial condition into a sloping front.
- The importance of conservative formulations when dealing with non-linear systems.

This is the fourth case in our systematic breakdown of the momentum equations from Navier–Stokes.