Nonlinear Advection vs Pressure Gradient
Interplay Between Kinetic Energy and Pressure Forces in Navier–Stokes

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Conceptual Framework

The Navier–Stokes momentum equations describe the evolution of fluid velocity under the action of several physical mechanisms, each represented by a distinct term:

$$\underbrace{\frac{du}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{nonlinear advection}} = \underbrace{fv}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure gradient}} + \underbrace{\nu\nabla^2 u}_{\text{viscosity}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{nonlinear advection}} = \underbrace{-fu}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{pressure gradient}} + \underbrace{\nu\nabla^2 v}_{\text{viscosity}}$$

In this case, we isolate the nonlinear advection term and the pressure gradient term (highlighted in black). All other mechanisms are removed. We aim to understand how a fluid responds when the gain in kinetic energy through spatial acceleration is balanced solely by pressure forces.

To simplify the analysis, we reduce the problem to one spatial dimension, where the governing equation becomes:

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

This form expresses a balance between nonlinear advection (left-hand side) and pressure gradient (right-hand side). It is essentially a statement of energy conservation: the fluid accelerates spatially (changing kinetic energy) only when pressure forces act in the opposite direction.

In the absence of a pressure gradient, the velocity field would remain unchanged—the nonlinear advection term alone cannot alter the shape of a uniform flow. Conversely, any horizontal pressure gradient necessarily induces spatial changes in the velocity field.

In our simulation, we prescribe a linearly increasing pressure gradient and numerically compute the velocity field that satisfies equation (2). The result reveals how an increasing slope in the pressure field generates spatial changes in velocity.

To derive the final expression used in the numerical code, we start from the conservative form of the balance:

$$\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2}$$

Multiplying both sides by ρ and integrating with respect to x, we obtain:

$$\frac{u^2}{2} = -\frac{1}{\rho}p(x) + C \tag{3}$$

Solving for $u(x)$, we get:

$$u(x) = \sqrt{2\left(C - \frac{1}{\rho}p(x)\right)} \tag{4}$$

The integration constant C can be set according to boundary or reference conditions. In our simulation, we define it so that $u = 0$ where $p = p_{\max}$, ensuring that the square root remains real and non-negative across the domain.

To close the system, we prescribe a linear pressure profile across the domain:

$$p(x) = p_{\max}\left(1 - \frac{x}{L}\right) \tag{5}$$

where $p_{\max}$ is the pressure at the left boundary and L is the domain length. This choice creates a constant pressure gradient:

$$\frac{dp}{dx} = -\frac{p_{\max}}{L}$$

Substituting equation (5) into equation (4) gives the analytical expression for the velocity field:

$$u(x) = \sqrt{\frac{2p_{\max}}{\rho}\left(\frac{x}{L}\right)}$$

which reveals how increasing pressure slopes induce spatially varying velocities.

Animation

The animation illustrates how increasing the slope of the pressure field modifies the velocity distribution in space. Both pressure (dashed red line) and velocity (solid blue line) are plotted on the same panel.

As the pressure gradient strengthens, the velocity profile becomes more curved, indicating greater spatial acceleration. The two fields remain in perfect balance at each time step, satisfying the conservative form of the momentum equation.

- **Code available at:** https://bit.ly/NS_nonlinearadvection_pressure
- **Animation available at:** https://youtu.be/jjFHG5EJeFc

Interpretation

This experiment highlights a fundamental mechanism in fluid dynamics: the conversion of pressure potential energy into kinetic energy. Unlike previous cases, here the fluid is not evolving in time due to inertia or viscosity. Instead, the velocity field adjusts spatially to maintain equilibrium with the imposed pressure gradient.

This case is particularly relevant in steady-state flows through channels, pipes, or wind tunnels, where pressure gradients are used to drive flow and accelerate fluid particles.

This configuration is conceptually analogous to Bernoulli flow: a decrease in pressure corresponds to an increase in velocity, reflecting the conversion of pressure potential energy into kinetic energy. Although we do not solve Bernoulli's equation directly, the mechanism is similar—except here we prescribe the pressure gradient and solve for the velocity profile that maintains balance.

Unlike previous OpenOceanModels cases grounded in geophysical fluid dynamics, this scenario aligns more closely with classical fluid mechanics, where pressure gradients are key to accelerating flow through pipes, nozzles, and channels.