Acoustic Wave

A Step Toward Understanding the Complexity of Navier–Stokes

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Conceptual Framework

The Navier–Stokes equations are the foundation of geophysical fluid dynamics, but due to their nonlinear and coupled nature, they are generally intractable in their full form. One strategy to gain insight is to isolate individual terms and analyze the resulting motion under idealized conditions.

In their horizontal momentum form (neglecting vertical structure and assuming constant density), the equations can be written as:

$$\underbrace{\frac{du}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{advection}} = \underbrace{fv}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\nu\nabla^2 u}_{\text{viscosity}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{advection}} = \underbrace{-fu}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{pressure}} + \underbrace{\nu\nabla^2 v}_{\text{viscosity}}$$

In this case, only the local acceleration and pressure gradient terms are retained (highlighted in black), while advection, Coriolis, and viscosity are omitted (shown in gray). This reduced system isolates the role of pressure gradients as the only source of motion, allowing us to study how they directly induce acceleration in a compressible fluid.

To simplify the analysis, we consider a one-dimensional version of the system. The goal is to examine how a small pressure perturbation evolves in a compressible medium when no other physical processes are present. The simplified momentum equation reads:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1}$$

However, this equation alone is not sufficient to determine the evolution of the system, because it contains three unknowns: the velocity $u(x,t)$, the pressure $p(x,t)$, and the density $\rho(x,t)$, which may all vary in time and space. Therefore, additional equations are needed to close the system.

We introduce:

- The **continuity equation** for a compressible fluid:

$$\frac{\partial \rho}{\partial t} + \rho_0\frac{\partial u}{\partial x} = 0 \tag{2}$$

- A **linearized equation of state**:

$$p = c^2\rho \tag{3}$$

To express pressure as the sole unknown, we first differentiate the equation of state (3) with respect to time:

$$\frac{\partial p}{\partial t} = c^2\frac{\partial \rho}{\partial t} \tag{4}$$

Then, substituting equation (4) into the continuity equation (2), we obtain:

$$\frac{\partial p}{\partial t} = -c^2 \rho_0 \frac{\partial u}{\partial x} \tag{5}$$

Now, we differentiate equation (5) with respect to time:

$$\frac{\partial^2 p}{\partial t^2} = -c^2 \rho_0 \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) \tag{6}$$

To proceed analytically, we assume that the density variations are small compared to a constant reference value ρ_0, i.e., $\rho = \rho_0 + \rho'$ with $\rho' \ll \rho_0$. Under this assumption, we linearize the momentum equation (1) by approximating $1/\rho \approx 1/\rho_0$, yielding:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \tag{7}$$

Differentiating equation (7) with respect to space:

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) = -\frac{1}{\rho_0} \frac{\partial^2 p}{\partial x^2} \tag{8}$$

Substituting equation (8) into (6) gives:

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{9}$$

Equation (9) is the classical wave equation, describing the propagation of acoustic disturbances in a compressible medium.

It is worth noting that the wave equation (9) arises under the assumption of a compressible fluid. While the atmosphere satisfies this condition and supports the propagation of acoustic waves with significant amplitude and speed, the ocean behaves as a nearly incompressible medium under most circumstances. In oceanic contexts, pressure variations due to acoustic waves are typically negligible compared to hydrostatic or geostrophic balances, and the compressibility effects are often filtered out in large-scale models. However, analyzing how pressure acts as the sole driver of motion in a compressible regime helps clarify the individual role of each term in the Navier–Stokes equations and illustrates the physical behavior that emerges when other mechanisms are absent.

Code and Animation

- **Code available at:** https://bit.ly/OOM_acoustic_wave
- **Animation available at:** https://youtu.be/1aJauxYwTDc

Description

This simulation demonstrates the propagation of a small-amplitude pressure pulse in a 1D compressible fluid:

- The model retains only the local acceleration and pressure gradient terms from the Navier–Stokes equations.
- The resulting wave equation has a known analytical solution: pressure disturbances propagate symmetrically at the speed of sound.
- The animation shows how a Gaussian pulse splits into left- and right-propagating waves that reflect off the domain boundaries.

This case isolates the fundamental mechanism behind acoustic propagation and serves as a canonical example of inertial-pressure interaction in the absence of other forces or nonlinearities.