Inertial Oscillations

A Step Toward Understanding the Complexity of Navier–Stokes

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Conceptual Framework

The Navier–Stokes equations are the foundation of geophysical fluid dynamics, but due to their nonlinear and coupled nature, they are generally intractable in their full form. One strategy to gain insight is to isolate individual terms and analyze the resulting motion under idealized conditions.

In their horizontal momentum form (neglecting vertical structure and assuming constant density), the equations can be written as:

$$\underbrace{\frac{du}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{advection}} = \underbrace{fv}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{pressure}} + \underbrace{\nu\nabla^2 u}_{\text{viscosity}}$$

$$\underbrace{\frac{dv}{dt}}_{\text{acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{advection}} = \underbrace{-fu}_{\text{Coriolis}} - \underbrace{\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{pressure}} + \underbrace{\nu\nabla^2 v}_{\text{viscosity}}$$

In this case, only the local acceleration and Coriolis terms are retained (highlighted in black), while advection, pressure horizontal gradient, and viscosity are omitted (shown in gray).

The resulting simplified equations become:

$$\frac{du}{dt} = fv,$$

$$\frac{dv}{dt} = -fu.$$

The solution describes circular inertial oscillations, a fundamental mode of motion in rotating fluids.

The inability to find analytical solutions to the full Navier–Stokes equations stems from the complex interactions among these terms. By exploring such reduced systems, we gain intuition into their individual roles and the challenges of solving the full system.

Code and Animation

- **Code available at:** https://bit.ly/OOM_inertial_oscillation
- **Animation available at:** https://www.youtube.com/watch?v=Vw8OAoujJm0

Description

This simulation demonstrates inertial oscillations in two scenarios:

- With no background translation: fluid parcels follow circular orbits centered at their starting position.

- With background translation: the same circular motion is superimposed on a linear drift toward the southwest.

The animation emphasizes the nature of purely inertial dynamics and sets the stage for future comparisons with other simplified terms of the Navier–Stokes equations (e.g., pressure gradient, advection, or viscosity).