A Dynamic Theory of Learning and Relationship Lending

Yunzhi Hu   Felipe Varas*

March 16, 2020

Abstract

This paper shows that in a dynamic lending relationship, zombie lending is inevitable. An entrepreneur borrows from a relationship bank or the market. Bank financing incurs a higher cost of capital but produces private information over time. While the entrepreneur accumulates reputation in the lending relationship, asymmetric information is also developed between the bank/entrepreneur and the market. We show that once the entrepreneur gets sufficiently reputable, the bank will roll over loans even after bad news, for the prospect of future market financing. This incentive of zombie lending gets mitigated when the entrepreneur faces financial constraints. Finally, the bank stops producing information before the entrepreneur gets sufficiently reputable if information production is costly.

Keywords: private learning, experimentation, reputation, relationship banking, information monopoly, debt rollover, extend and pretend, adverse selection, dynamic games.

1 Introduction

Zombie firms – firms whose operating cash flows persistently fall below their interest payments – are prevalent in the real world. According to a recent study (Banerjee and Hofmann, 2018), zombie firms consist of about 12% of all publicly-traded firms across 14 advanced economies. These firms are detrimental to the real economy: they crowd out credit to their healthy competitors, thereby reducing aggregate productivity and hindering investment. It has been widely perceived that zombie lending was the main reason behind Japan's "lost decade" in the 1990s (Caballero et al., 2008; Peek and Rosengren, 2005). More recently, Acharya et al. (2019) and Blattner et al. (2019) show Europe's economic recovery from the debt crisis has also been plagued by banks' lending to low-quality firms with whom they have pre-existing lending relationships. Why do banks extend loans to firms that are known to be in distress and very likely cannot repay their loan obligations?

One explanation is through the lens of bank capital (see Bruche and Llobet (2013) for example). By extending "evergreening" loans to their impaired borrowers, banks in distress gamble for resurrection, hoping that their borrowing firms regain solvency, or, at least, they can delay taking a balance sheet hit. However, why wouldn't these banks lend to healthy firms instead? On the other hand, why do well-capitalized banks also lend to zombie firms? Moreover, why do policies that aim to reduce banks' cost of capital could breed even more zombie lending (see (Giannetti and Simonov, 2013; Banerjee and Hofmann, 2018))?

In this paper, we build a dynamic model of relationship lending and argue that even absent of regulatory concerns on bank capital, zombie lending, or extend and pretend,[1] is *inevitable* but yet *self-limiting*. Our explanation hinges on the assumption that banks and private lenders have an informational advantage relative to market-based lenders. Consequently, a borrower's reputation typically grows with the length of its lending relationship. This growth of reputation in turn gives a bank incentives to roll over bad loans – evergreening – prior to passing the buck to the market. Therefore, zombie lending is inevitable. However, if the bank rolls over bad loans all the time, it can destroy the reputation benefits in the lending relationship and hence the bank's incentive to engage in zombie lending in the first place. Therefore, zombie lending must also be self-limiting.

Let us be more specific. We model an entrepreneur investing in a long-term, illiquid project whose quality is either good or bad. A good project should continue to be financed, whereas a bad project should be liquidated immediately. Initially, the quality of the project is unknown to everyone, including the entrepreneur herself. She can raise funding from either the competitive financial market or a bank that will develop

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[1] We sometimes use "zombie lending", "extend and pretend", and "evergreening" interchangeably. They all refer to the lending decisions to borrowers that are known to be in distress. For most part of the paper, we will use "extend and pretend".

into a relationship. Market financing takes the form of arm's-length debt so that lenders only need to break even given their beliefs about the project's quality. Under market financing, no information is ever produced. By contrast, if the entrepreneur borrows from a bank, screening and monitoring will produce "news" about the project's quality. We model news arrival as a Poisson event and assume it is only observed by the entrepreneur and the bank. In other words, the bank and the entrepreneur learn *privately* about the project's quality as time goes by. Meanwhile, all agents, including lenders in the financial market, can observe the time since the initialization of the project, which will turn out to be the important state variable. When the bank loan matures, the bank and the entrepreneur decide to roll it over, to liquidate the project, or to refinance with the market-based lenders. These decisions depend crucially on the level of the state variable, and they are the central analysis that we will focus on.

We show the equilibrium is characterized by two thresholds in time and therefore includes three stages. If the bank loan matures in the first stage, a bad project is liquidated, whereas all other types' matured loans will be rolled over. During this period, the average quality of borrowers who remained with banks improves because the informed-bad types get liquidated and exit funding. Equivalently, remaining borrowers gain reputation from the liquidation decisions of the informed-bad types. These liquidation decisions are socially efficient, and therefore we name this stage after *efficient liquidation.* If the bank loan matures in the second stage, however, it will be rolled over irrespective of the quality of the project. In particular, the relationship bank will roll over the loan matured during this stage even if it has known that the project is bad. In other words, an informed-bad bank keeps *extending* the loan to *pretend* no bad news has occurred yet, which is inefficient. This result on banks' rolling over bad loans can be interpreted as zombie lending. Finally, in the second stage, all entrepreneurs will refinance with the market upon their bank loans maturing – the *market-financing* stage.

The intuitions for these results are best explained backwards in time. When time elapsed gets sufficiently long, all entrepreneurs will get sufficiently reputable and switch to market financing, driven by the assumption that market-based lenders are competitive and have lower costs of capital. This is the equilibrium outcome in the last stage. Now, imagine a scenario in which bad news arrives shortly before the last stage. The relationship-bank could liquidate the project, in which case, it receives a low liquidation value. Alternatively, it can roll over the loan and pretend no bad news has arrived yet. By hiding bad news *today*, the bank helps the borrower accumulate reputation so that she will be able to refinance with the market in the *future*. Such "extending and pretending" is preferred because very soon, the bank will get fully repaid after the entrepreneur refinances with the market. In this case, the expected loss will most likely be born by the market-based lenders. On the other hand, if negative news arrives early on, "extending and pretending" is much more costly, due to both large time discounting and the

high probability that before the time reaches the last stage, the project may mature, in which case the loss is entirely born by the relationship bank. In this case, liquidating the project is the more preferred option. Note that the extend-and-pretend stage lasts for a significant period, implying zombie lending is inevitable once the entrepreneur gets sufficiently reputable. However, this period is necessary to incentivize informed-bad types to liquidate and exit in the first stage, which enables the entrepreneur to accumulate reputation early on during a lending relationship. Moreover, it implies that zombie lending is self-limiting: the reputation benefit in the lending relationship will vanish if the bank extends and pretends all the time.

We show the financial constraint mitigates the inefficiency induced by extend and pretend. Specifically, the financial constraint requires that at each rollover date, the newly negotiated loan payment cannot exceed the level of the interim cash flow. Equivalently, this constraint limits the transfer that the entrepreneur can make to the bank in the Nash bargaining game, and as a result, it leads to scenarios in which a bad project gets liquidated even though the liquidation value falls below the joint surplus if both parties choose to roll it over.

Our interpretation of learning is the process of bank screening and monitoring, which generates useful information on the entrepreneur's business prospect but yet cannot be shared with others in the financial market. While we endogenize learning as a costly decision, we show the bank ceases to learn during the efficient-liquidation region. Intuitively, the benefit of learning (or equivalently the source of convexity in the bank's payoff function) arises because an informed-bad bank could liquidate a bad project and receive the liquidation value. This learning benefit vanishes once time passes the efficient-liquidation region, because a bad project will no longer be liquidated. This result highlights a new type of hold-up problem in a lending relationship: the bank under-invests its effort in producing information because it anticipates the borrower will refinance with the market in the future. Therefore, even if the relationship bank has all the bargaining power in the lending relationship, it is unable to capture all the surplus – including the current and the future – generated from information production. Knowing so, the bank under-supplies its effort in producing information.

Related Literature

We build on the previous literature on dynamic signaling and private learning (Janssen and Roy, 2002; Kremer and Skrzypacz, 2007; Daley and Green, 2012; Grenadier et al., 2014; Atkeson et al., 2015; Martel et al., 2018; Hwang, 2018). In our model, news is private, whereas in Daley and Green (2012), news is publicly observable. Martel et al. (2018) and Hwang (2018) also study problems in which sellers get gradually informed of an asset's quality. Besides the specific application to relationship banking and zombie

lending, our model has different theoretical implications. First, sellers in these two papers only choose the time of trading, whereas in our model, the bank is also endowed the option of liquidation. This additional option, which is natural in the banking context, generates very different dynamics. In our paper, bad types initially choose to separate through gradual liquidation and only pool with other types once the reputation is sufficiently high. Second, we study the problem in which information production is costly and endogenous and show how the incentives of learning are affected by reputation and asymmetric information. By doing so, we are able to discover a new type of hold-up problem in bank's information production.

Our paper is among the first ones that introduce dynamic learning in the context of banking (also see Halac and Kremer (2018) and Hu (2017)). We extend previous work by Diamond (1991b); Rajan (1992); Boot and Thakor (1994, 2000); Parlour and Plantin (2008) among others, by studying the impact of dynamic learning and adverse selection on lending relationships.[2] For example, in Diamond (1991a), borrowers are financed with arm's length debt, and lenders decisions are myopic, implying that lenders will never have incentives to roll over bad loans and increase the borrower's reputation. Rajan (1992) studies the tradeoff between relationship-based lending and arm's-length debt, without an explicit role of the borrower's reputation. Chemmanur and Fulghieri (1994a) and Chemmanur and Fulghieri (1994b) emphasize the role of lenders' reputation in borrowers' choices between bank versus market financing.

Our paper is also related to previous work on debt rollover by He and Xiong (2012); Brunnermeier and Oehmke (2013); He and Milbradt (2016). One related paper is Geelen (2019), which models the dynamic tradeoff of debt issuance and rollover under asymmetric information. Our main distinction from this literature is, while most of the papers study competitive lenders, we model one lender that gets gradually informed – the bank – together with competitive lenders – the market.

Our explanation for zombie lending differs from existing theories that largely rely on loan officers' career concerns (Rajan, 1994) or on regulatory capital requirements that are triggered if banks write off bad loans (Caballero et al., 2008; Peek and Rosengren, 2005). We offer a complementary explanation that is based on the prospect of market financing in the future, and that it is related to previous work on dynamic adverse selection. In particular, our mechanism is related to work by Fuchs and Skrzypacz (2015) that study how suspensions and delays in trading can promote efficiency in markets plagued by adverse selection.

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[2]There is also a literature that uses a dynamic-contracting approach to study similar issues, including Sanches (2010), Boot and Thakor (1994) and Verani (2018). Our approach is derived from dynamic games and therefore does not require any commitment.

2 Model

We consider a continuous-time model with an infinite horizon. An entrepreneur (she) invests in a long-term project with unknown quality. She borrows from either a bank that will develop into a relationship or the competitive financial market. Compared to market financing, bank financing has the advantage of producing valuable information but with the downside of a higher cost of capital and the possibility of information monopoly. Below, we describe the model in detail.

2.1 Project

We consider a long-term project that generates a constant stream of interim cash flows $c\,dt$ over a period $[t, t+dt]$. The project matures at a random time τ_ϕ, which arrives at an exponential time with intensity $\phi > 0$. Upon maturity, the project produces some random final cash flows $\tilde{R}$, depending on its type. A good (g) project produces cash flows $\tilde{R} = R$ with certainty, whereas a bad (b) project produces $\tilde{R} = R$ with probability $\theta < 1$. With probability $1 - \theta$, a matured bad project fails to produce any final cash flows, that is, $\tilde{R} = 0$. In addition to the failure in generating the final cash flows, a bad project may also fail prematurely, in which case it stops generating any cash flows, including both the interim ones and the final ones. The premature failure event arrives at an independent exponential time τ_η, where $\eta > 0$ is the arrival intensity. We will sometimes refer to this premature failure as the *public* news.

Initially, no agent, including the entrepreneur herself, knows the exact type of the project; all agents share the same public belief that q_0 is the probability of the project's type being good. Obviously, if the project fails prematurely, all agents will learn that the project is bad with certainty. At any time before the final cash flows are produced and the premature failure, the project can be terminated with a liquidation value $L > 0$. Later in in Assumption 1, we impose a parametric assumption that L is higher than the value of discounted future cash flows generated by a bad project. As a result, it will be socially valuable for a bank to learn about the project's type and liquidate it if bad. Note the liquidation value is independent of the project's quality, so it should be understood as the liquidation of the physical asset used in production. For example, one can think of L as the value from alternative use of the asset in production, which exceeds the value of cash flows generated from a bad project.

Let $r > 0$ be the entrepreneur's discount rate; therefore, the fundamental value of the project to the entrepreneur at $t = 0$ is given by the discounted value of its future cash flows:

$$PV_r^g = \frac{c + \phi R}{r + \phi}, \quad PV_r^b = \frac{c + \phi \theta R}{r + \phi + \eta}, \quad PV_r^u = q_0 PV_r^g + (1 - q_0)PV_r^b. \tag{1}$$

Note that the denominator of NPV_r^b contains an additional term η, which accounts for the premature failure event.

Remark 1. While we do not explicitly model the initial investment, one can imagine a fixed investment scale I is needed at $t = 0$ to initialize the project. In subsection 3.3.1, we will derive the maximum amount that an entrepreneur is able to raise at the initial date and therefore discuss the minimum contribution needed from the entrepreneur.

2.2 Agents and debt financing

The borrower has no initial wealth and needs to borrow through debt contracts. The use of debt contracts is not crucial and can be justified by non-verifiable final cash flows (Townsend, 1979). One can also interpret these contracts as equity shares with different control rights and therefore think of the entrepreneur as a manager of a start-up venture. We consider two types of debt, offered by banks and market-based lenders, respectively. First, the entrepreneur can take out a loan from a banker (he), who has the same discount rate r. For tractability reasons, we assume a bank loan lasts for a random period and matures at a random time τ_m, upon the arrival of an independent Poisson event with intensity $\frac{1}{m} > 0$. m can therefore be interpreted as the expected maturity of the loan. In most of the analysis, we will study the limiting case of instantly-maturing loans, i.e., $m \to 0$. Subsection 3.4 solves the case with general m, where results stay qualitatively unchanged.

The second type of debt is provided by the market and thus can be considered as public bonds. In particular, we consider a competitive financial market in which lenders have discount rate δ satisfying $\delta \in (0, r)$. In other words, market financing is cheaper than bank financing, so that if the project's type were publicly known, the entrepreneur would strictly prefer to borrow from the market. Regarding (1), let us define the value of the project to the market as

$$PV_\delta^g = \frac{c + \phi R}{\delta + \phi}, \quad PV_\delta^b = \frac{c + \phi \theta R}{\delta + \phi + \eta}, \quad PV_\delta^u = q_0 PV_\delta^g + (1 - q_0)PV_\delta^b. \tag{2}$$

The assumption $\delta < r$ captures the realistic feature that banks have a higher cost of capital than the market, which can be justified by either regulatory requirements or the skin in the game needed to monitor borrowers (see Holmstrom and Tirole (1997) for example and Schwert (2018) for recent empirical evidence). As will be clear shortly, the maturity of the public debt does not matter, and for simplicity, we assume it only matures with the project.

Both types of debt share the same exogenously-specified face value: $F \in (L, R)$. $F > L$ guarantees that debt is risky, whereas $F < R$ captures the wedge between a project's maximum income and its pledgeable income (Holmström and Tirole, 1998). Note

that we take F as given: our paper intends to study the tradeoff between relationship borrowing and public debt, rather than the optimal leverage.[3] At $t = 0$, the entrepreneur chooses between public debt and a bank loan that will develop into a relationship. Once the bank loan matures, she can still replace it with a public bond. Alternatively, she could roll over the loan with the same bank which may have an information advantage over the project's quality.[4] In this case, the two parties bargain over y_t, the rate of the loan that is prevalent until the next rollover date. For simplicity, we assume the bank always has all the bargaining power. The results under interior bargaining power only differ quantitatively. The allocation of bargaining power together with the financial constraint $y_t \leq c$ naturally lead to the result that $y_t \equiv c$. As we will show shortly, this financial constraint limits the size of the transfer that the entrepreneur can make to the bank; therefore, the Nash bargaining outcome is sometimes not the one that maximizes the joint surplus of the two parties.

Because market financing is competitive and market-based lenders have a lower cost of capital, the entrepreneur will always prefer to take as high leverage as possible once she borrows from the market. Therefore, the coupon payments associated with the public bond are naturally *cdt*.

Remark 2. We have assumed the entrepreneur is only allowed to take one type of debt. In other words, we have ruled out the possibility of the entrepreneur using more sophisticated capital structure to signal her type. See Leland and Pyle (1977) and DeMarzo and Duffie (1999) for these issues.

2.3   Learning and information structure

The quality of the project is initially unknown, with $q_0 \in (0, 1)$ being the commonly-shared belief that it is good. If the entrepreneur finances with the bank, that is, if she takes out a loan, the entrepreneur-bank pair can *privately* learn the true quality of the project through "news." Private news arrives at a random time τ_λ, modeled as an independent Poisson event with intensity $\lambda > 0$. Upon arrival, the news fully reveals the project's true type. In practice, one can think of the news process as information learned during bank screening and monitoring, which includes due diligence and covenant violations. We assume such news can only be observed by the two parties and no committable mechanism is available to share it with third parties, such as credit bureaus and market participants. In this sense, the news can be understood as soft information on project

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[3]The maximum pledgeable cash flow can be microfounded by some unobservable action taken by the entrepreneur (e.g., cash diversion) shortly before the final cash flows are produced (Tirole, 2010). All our results will go through if $F \equiv R$ and there are some non-pledgeable control rents accrued to the entrepreneur if the project matures.

[4]We assume without loss of generality that the entrepreneur would never want to switch to a different banks upon the loan's maturity. Intuitively, the market has a lower cost of capital than an outsider bank and the same information structure.

quality (Petersen, 2004). For instance, one can think of this news as the information that banks acquire upon covenant violation, which includes details on the business prospect, collateral quality, and financial soundness of the borrower.

Although the public-market participants do not observe the private news, they can observe 1) the public news – whether the project has failed prematurely, 2) t – the project's time since initialization, and 3) whether the project has been liquidated. Therefore, the public can make inference about the project's quality based on these observations. Let $i \in \{u, g, b\}$ denote the type of the bank/entrepreneur, where u, g, and b refer to the uninformed, informed-good, and informed-bad types, respectively. Let μ_t be the (naive) belief on the project's quality if the market lenders solely learn from the fact that the project has not failed prematurely. A standard filtering result implies

$$\dot{\mu}_t = \eta \mu_t (1 - \mu_t), \tag{3}$$

where $\mu_0 = q_0$. Note that the public news could only be bad – which occurs if the project has failed prematurely. This assumption captures the realistic feature that in the banking context, the structure of debt contracts enables more public information to be produced under distress scenarios.

Let us first describe the *private*-belief process; that is, the belief held by the bank and the entrepreneur. If the private news hasn't arrived yet, the private belief remains at μ_t. Upon news arrival at t_λ, the private belief jumps to 1 in the case of good news and 0 if bad. To characterize the public-belief process, we introduce a belief system $\left\{\pi_t^u, \pi_t^g, \pi_t^b\right\}$, where π_t^u is the public's belief at time t that the private news hasn't arrived yet, and π_t^g (π_t^b) is the public belief that the private news has arrived and is good (bad). In any equilibrium where the belief is rational, π_t^i is consistent with the actual probability that the bank and the entrepreneur are of type $i \in \{u, g, b\}$. Given $\left\{\pi_t^u, \pi_t^g, \pi_t^b\right\}$, the public belief that the project is good is[5]

$$q_t = \pi_t^u \mu_t + \pi_t^g. \tag{4}$$

For the remainder of this paper, we sometimes refer to q_t as the average quality or the average belief.

2.4 Rollover

When the loan matures, the entrepreneur and the bank have three options: liquidate the project for L, switch to market financing, or continue the relationship by rolling over the loan. Control rights are assigned to the bank if the loan is not fully repaid, and renegotiation could potentially be triggered. Let $O_t^i \equiv O_{Et}^i + O_{Bt}^i$, $i \in \{u, g, b\}$ be

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[5]To simplify notation, we abuse notation and use $\left\{\pi_t^i, q_t\right\}$ to denote $\left\{\pi_{t-}^i, q_{t-}\right\}$.

the maximum joint surplus to the two parties if the loan is not rolled over, where O^i_{Et} and O^i_{Bt} are the value accrued to the entrepreneur and the bank, respectively. Because $F > L$, in the case of liquidation, the bank receives the entire liquidation value L and the entrepreneur receives nothing, that is, $O^i_{Bt} = L$, and $O^i_{Et} = 0$. If the two parties are able to switch to market financing, the bank receives full payment $O^i_{Bt} = F$, whereas the entrepreneur receives the remaining surplus $O^i_{Et} = \bar{V}^i_t - F$, where

$$\bar{V}^g_t = D_t + \frac{\phi\,(R - F)}{r + \phi}, \qquad \bar{V}^b_t = D_t + \frac{\phi\theta\,(R - F)}{r + \phi + \eta}, \qquad \bar{V}^u_t = \mu_t \bar{V}^g_t + (1 - \mu_t)\,\bar{V}^b_t. \quad (5)$$

In (5),

$$D_t = \hat{q}_t D^g + (1 - \hat{q}_t)\,D^b \tag{6}$$

is competitive price of a bond at time t, where $D^g = \frac{c + \phi F}{\delta + \phi}$ and $D^b = \frac{c + \phi\theta F}{\delta + \phi + \eta}$ are the price for the bond backed a good- and bad-type project, respectively. $\hat{q}_t$ is the average quality of the project conditional on refinancing with the market, and in the case that all types choose to refinance, $\hat{q}_t = q_t$. The second terms in (5) are the value of the final cash flows that the entrepreneur $i \in \{u, g, b\}$ receives upon the project's maturity. Because the entrepreneur is financially constrained, the bond price D_t must be at least F, implying

$$\hat{q}_t \geq \underline{q} \equiv \frac{F - D^b}{D^g - D^b}. \tag{7}$$

Two conditions need to be satisfied for a loan to be rolled over. First, $V^i_t > \max\left\{L, \bar{V}^i_t\right\}$ so that rolling over is indeed the decision that maximizes the joint surplus. Second, the bank needs to prefer rolling over the loan and receiving the entire interim cash flow *cdt* to liquidating the project and receiving L. Otherwise, the bank, endowed with control rights over the asset if not fully repaid, will choose to liquidate the asset.

2.5   Strategies and equilibrium

The public history $\mathcal{H}_t$ consists of 1) time t, 2) whether the project has failed prematurely, and 3) the entrepreneur's and the bank's actions up to t. Specifically, it includes at any time $s \leq t$, whether the entrepreneur borrows from the bank or the market and whether the project has been liquidated. For any public history, the price of market debt D_t summarizes the market strategy. Given that the market is competitive, the price of debt satisfies (6).

The private history h_t consists of the public history $\mathcal{H}_t$, the rollover event, as well as the Poisson event on the private news arrival and, of course, the content of the news. Essentially, the strategy of the entrepreneur and the bank is to choose an optimal stopping time, and at the stopping time whether to liquidate the project or refinance with the market. This choice is subject to the additional constraint that at the stopping time,

the bank's continuation value is at least (weakly) greater than L, the liquidation value of the project. Let V_t^i be the joint value of the entrepreneur and the bank in the lending relationship, and B_t^i be the continuation value of the bank.[6] Let τ^i, $i \in \{u, g, b\}$ be the (realized) stopping time of type i,[7] we have

$$
V_t^u = \max_{\substack{\tau^u \geq t, \\ s.t.\ B_{\tau^u}^u \geq L}} \mathbb{E}_{t^-} \left\{ \int_t^{\tau^u} e^{-r(s-t)} c\, ds + e^{-r(\tau^u - t)} \left[\mathbb{1}_{\tau^u \geq \tau_\phi} \left[\mu_{\tau_\phi} + \left(1 - \mu_{\tau_\phi}\right) \theta \right] R + \mathbb{1}_{\tau^u \geq \tau_\eta} \cdot 0 \right. \right.
$$

$$
\left. \left. + \mathbb{1}_{\tau^u \geq \tau_\lambda} \left[\mu_{\tau_\lambda} V_{\tau_\lambda}^g + \left(1 - \mu_{\tau_\lambda}\right) V_{\tau_\lambda}^b \right] + \mathbb{1}_{\tau^u < \min\{\tau_\phi, \tau_\lambda, \tau_\eta\}} \max\{L, \bar{V}_{\tau^u}^u\} \right] \right\},
$$

$$(8)$$

and

$$
B_t^u = \mathbb{E}_{t^-} \left\{ \int_t^{\tau^u} e^{-r(s-t)} c\, ds + e^{-r(\tau^u - t)} \left[\mathbb{1}_{\tau^u \geq \tau_\phi} \left[\mu_{\tau_\phi} + \left(1 - \mu_{\tau_\phi}\right) \theta \right] F + \mathbb{1}_{\tau^u \geq \tau_\eta} 0 \right. \right.
$$

$$
\left. \left. + \mathbb{1}_{\tau^u \geq \tau_\lambda} \left[\mu_{\tau_\lambda} B_{\tau_\lambda}^g + (1 - \mu_{\tau_\lambda}) B_{\tau_\lambda}^b \right] + \mathbb{1}_{\tau^u < \min\{\tau_\phi, \tau_\lambda\}} \max\left\{L, \min\left\{\bar{V}_{\tau^u}^u, F\right\}\right\} \right] \right\}.
$$

$$(9)$$

In (8), τ^u is the stopping time of the entrepreneur and the bank if both are uninformed. The first term, $\int_t^{\tau^u} e^{-r(s-t)} c\, ds$, is the value of interim cash flows until τ^u. The project matures and pays off the final cash flows if $\tau^u \geq \tau_\phi$. If $\tau^u \geq \tau_\eta$, the project fails prematurely, with continuation payoff being zero. If $\tau^u \geq \tau_\lambda$, private news arrives, after which the two parties become informed. Finally, if $\tau^u < \min\{\tau_\phi, \tau_\eta, \tau_\lambda\}$, the bank and the entrepreneur choose to stop before any of the above events arrives, upon which they either liquidate the project for L or refinance with the market for $\bar{V}_{\tau^u}^u$. The decision is made subject to the constraint that $B_{\tau^u}^u \geq L$. Equation (9) can be interpreted similarly. The value functions of type g and b are similar but contain fewer terms. Therefore, we omit them to avoid redundancy.

We look for a perfect Bayesian equilibrium of this game.

Definition 1. *An equilibrium of the game satisfies the following:*

1. *Optimality: The rollover decisions are optimal for the bank and the entrepreneur, given the belief processes $\{\pi_t^i, \mu_t, q_t\}$.*

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[6]We use the standard notation $\mathbb{E}_{t^-}[\cdot] = \mathbb{E}[\cdot | h_{t^-}]$, to indicate the expectations is conditional on the history before the realization of the stopping time τ.

[7]Formally, let t^u be the optimal stopping time to liquidate/refinance chosen by type u, then $\tau^u = \min\{t^u, \tau_\phi, \tau_\eta, \tau_\lambda\}$.

2. *Belief Consistency: For any history on the equilibrium path, the belief process $\left\{\pi_t^u, \pi_t^g, \pi_t^b\right\}$ is consistent with Bayes' rule.*

3. *Market Breakeven: The price of the public bond satisfies* (6).

4. *No (unrealized) Deals:[8] For any $t > 0$ and $i \in \{u, g, b\}$,*

$$V_t^g \geq \mathbb{E}\left[D^i | \mathcal{H}_t, \ D^i \leq D^g\right] + \frac{\phi\left(R - F\right)}{r + \phi}$$

$$V_t^u \geq \mathbb{E}\left[D^i | \mathcal{H}_t, \ D^i \leq D^u\right] + \mu_t \frac{\phi\left(R - F\right)}{r + \phi} + (1 - \mu_t)\frac{\phi\theta\left(R - F\right)}{r + \phi + \eta},$$

where

$$D^u = \mu_t D^g + (1 - \mu_t) D^b.$$

5. *Belief monotonicity: continued bank financing is never perceived a (strictly) negative signal, implying $\dot{q}_t \geq \eta q_t\left(1 - q_t\right)$.*

The first three conditions are standard. The *No-Deals* condition follows Daley and Green (2012), reflecting the requirement that the market cannot profitably deviate by making an offer that the entrepreneur and the bank will accept. Note that the second terms on the right-hand-side of the No-Deals condition reflect the fact that even after market refinancing, the entrepreneur's continuation payoff is still type-specific.

As is standard in the literature, we use a refinement to rule out unappealing equilibria that arise due to unreasonable beliefs. Specifically, we impose a belief-monotonicity refinement which requires that continued bank financing is never perceived a (strictly) negative signal. As a result, the public belief about the project's quality conditional on bank financing is weakly higher than the naive belief process that is only updated from the public news that no premature failure has occurred yet. Effectively, this condition eliminates equilibria that can arise due to threatening beliefs. For example, suppose the belief is that a project that does not refinance with the market at time $\hat{t}$ is treated as a bad type, then under some parametric conditions, all types will be forced to refinance at time $\hat{t}$.

2.6 Parametric assumptions

We make the following parametric assumptions to make the problem non-trivial. Appendix A.2 presents the first-best benchmark under these assumptions.

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[8]We offer a micro-foundation as follows. In each period, two short-lived market-based lenders simultaneously enter and make private offers to all entrepreneurs. This microfoundation will give rise to the *No-Deals* condition as in Daley and Green (2012).

Assumption 1 (Liquidation value)**.**

$$PV_\delta^b < L < PV_r^g \tag{10}$$

According to Assumption 1, the discounted cash flows of a good project to the bank and the entrepreneur is above its liquidation value, which in turn is above the discounted cash flows of a bad project to the market. Therefore, it is socially optimal to liquidate a bad project but continue a good project.

Assumption 2 (Risky loan)**.**

$$F > \max\left\{\theta R, L, D^b\right\}. \tag{11}$$

Assumption 2 assumes the face value of the debt is above the liquidation value, and the expected repayment, and the value of the bond of a bad type; otherwise, the loan is effectively riskless as the risk-neutral bank is always repaid F fully in expectation.

Assumption 3 (interim cash flow)**.**

$$c \geq rF. \tag{12}$$

Assumption 3 guarantees the size of the interim cash flow c is large enough to compensate the lenders' cost of capital. Otherwise, the face value of the loan F needs to grow during rollover dates.

Assumption 4 (Optimal bank financing)**.**

$$D^b < \frac{\delta + \phi}{r + \phi} D^g \tag{13}$$

$$PV_\delta^b < \frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta}. \tag{14}$$

This final assumption imposes an restriction such that at least some level of bank financing will be used in the first-best benchmark. (13) requires that the price of a bad-type bond to be lower than the value of a good-type loan. Similarly, (14) requires the present value of a bad-type project to fall below the present value of the bad-type project if it is financed with the bank. Note that the right-hand side of (14) takes into account that after the private news arrives, the project will be liquidated for L in the case of bank financing.

3 Equilibrium

We solve the model in this section. In subsection 3.1, we study an economy without the financial constraints $y_t \leq c$ and (7). The main result is that there exists an extend-and-pretend region $[t_b, t_g]$ during which the bank will always roll over the loan, even if it has already known the borrower's project is bad. Subsection 3.2 studies the equilibrium with a formal treatment of the financial constraints. We show that the equilibrium structure is similar to the one in subsection 3.1, but financial constraints reduces the length of the extend-and-pretend region. We present a special case without premature failure in subsection 3.3, where all results are derived in simple and closed form. Subsection 3.4 further extends the analysis to loans with general maturity and studies how results varies with loan maturity.

3.1 Benchmark: No financial constraints

The economy is characterized by state variables in private and public beliefs. It turns out that all public beliefs (without liquidation and public news) are deterministic functions of the time elapsed. Therefore, we use time t as the state variable. Specifically, we will construct an equilibrium characterized by two thresholds $\{t_b, t_g\}$, as illustrated by Figure 1. If $t \in [0, t_b]$, the bank and the entrepreneur will liquidate the project upon the arrival of bad news – *efficient-liquidation* region. Loans for other types (good and unknown) will be rolled over. If $t \in [t_b, t_g]$, all types of loans will be rolled over, including the bad ones – *extend-and-pretend* region. Finally, if $t \in [t_g, \infty)$, the two entities will always refinance with the market upon loan maturity – *market financing*.
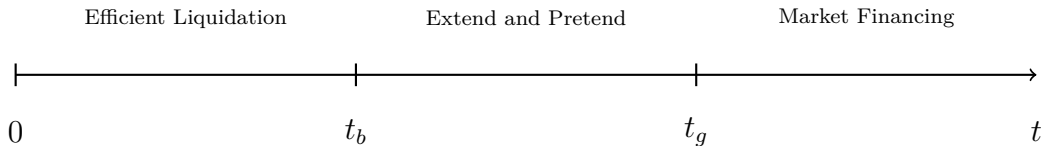

Efficient Liquidation   Extend and Pretend   Market Financing

0   t_b t_g t

Figure 1: Equilibrium regions

Given the equilibrium conjecture, the evolution of beliefs follow Lemma 1.

Lemma 1. *In a monotone equilibrium with thresholds* $\{t_b, t_g\}$*, the belief on a project's average quality evolves according to:*

$$\dot{q}_t = \begin{cases} (\lambda + \eta) q_t (1 - q_t) & t \leq t_b \\ \eta q_t (1 - q_t) & t > t_b \end{cases} \tag{15}$$

with the initial condition q_0.

Heuristically, before t reaches t_b, q_t evolves as if the premature failure arrives at rate $\lambda + \eta$, since a project will be immediately liquidated following bad private news. After t

14

reaches t_b, however, q_t evolves as if there is no private news at all, since a privately-known bad project will never be liquidated.

For the remainder of this subsection, we treat the bank and the entrepreneur as one entity and characterize the continuation value in different equilibrium regions as well as the boundary conditions. These three equilibrium regions differ in the decisions when the loan matures. To better explain the economic intuition, we describe the results backwards in the time elapsed.

Market Financing: $[t_g, \infty)$. In this region, $V_t^i = \bar{V}_t^i$, $i \in \{u, g, b\}$, where $\{\bar{V}_t^u, \bar{V}_t^g, \bar{V}_t^b\}$ have been defined in (5) and $\hat{q}_t = q_t$. Let us offer the economic intuition. Ultimately, if the entrepreneur gets sufficiently reputable, market financing is cheaper because market lenders are competitive, and they have lower cost because $\delta < r$. As a result, all types will replace their loans with public bonds. The threshold in reputation is obtained as the public belief q_t reaches $\bar{q}$. As we will show below, this is accomplished since in equilibrium, bad types would have failed prematurely or get liquidated. The absence of both premature failure and liquidation helps the entrepreneur accumulate reputation.

Extend and Pretend: $[t_b, t_g)$. Working backwards, we now consider the region $[t_b, t_g)$ during which all types of loans, including bad ones, are rolled over. Mathematically, the value functions of all three types satisfy the following Hamilton–Jacobi–Bellman (HJB) equation system:

$$(r + \phi + \lambda + (1 - \mu_t)\eta) V_t^u = \dot{V}_t^u + c + \phi [\mu_t + (1 - \mu_t)\theta] R + \lambda [\mu_t V_t^g + (1 - \mu_t) V_t^b] \tag{16a}$$

$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R \tag{16b}$$

$$(r + \phi + \eta) V_t^b = \dot{V}_t^b + c + \phi \theta R, \tag{16c}$$

The first term on the right-hand side (16a) is the change in valuation due to time; the second term captures the benefits of interim cash flow, and the third term corresponds to the event of project maturing, which arrives at rate ϕ. In this case, the bank and the entrepreneur receive the final cash flows R with probability $\mu_t + (1 - \mu_t)\theta$. The fourth term stands for the arrival of private news at rate λ. Following the news, the bank and the entrepreneur become informed. Equations (16b) and (16c) can be interpreted in a similar vein.

When time gets close to t_g, the bank and entrepreneur find it optimal to wait until t_g and refinance with the market, even if bad news has arrived. Intuitively, rolling over bad loans allows the bank to be fully repaid and share the expected losses with market lenders. When time is close to t_g, this decision can be optimal compared to liquidating the project for a low valuation L. In this region, even though no project is liquidated, the

entrepreneur's reputation keeps growing as long as the project does not fail prematurely.

We show that $t_g - t_b > 0$, implying that extend and pretend is inevitable in a dynamic lending relationship. Equilibrium in this region is clearly inefficient. A bad project should be liquidated, but instead, the bank and the entrepreneur roll it over in the hope of sharing the losses with the market lenders at t_g. As we are going to see the next, by not liquidating between 0 and t_b, they have accumulated a good reputation; therefore, extend and pretend can be sustained in equilibrium.

Efficient Liquidation: $[0, t_b)$ Finally, we turn to the first region $[0, t_b)$, where bad loans are not rolled over but instead liquidated. Mathematically, V_t^u and V_t^g are still described by (16a) and (16b), whereas $V_t^b = L$. At the early stage of the lending relationship, only the uninformed and informed-good types roll over maturing loans. By contrast, banks that have learned the project is bad choose to liquidate. Assumption 1 guarantees that liquidation possesses a higher value than continuing the project until the final date t_ϕ. By continuity, liquidation still has a higher payoff if type b needs to wait for a long time (until t_g in this case) to refinance. As a result, extend and pretend is suboptimal because t_g is far into the future: the firm could likely default or fail prematurely before it receives the opportunity of market financing. The equilibrium is socially efficient in this region. The result $t_b > 0$ implies the bank cannot extend and pretend all the time. Therefore, extend and pretend is self-limiting.

Boundary Conditions: The following two boundary conditions are needed to pin down $\{t_b, t_g\}$:

$$V_{t_b}^b = L \tag{17a}$$

$$\dot{V}_{t_g}^g = \dot{V}_{t_g}^g = \left(D^g - D^b\right) \eta q_{t_g} \left(1 - q_{t_g}\right). \tag{17b}$$

(17a) is the indifference condition for the bad type to liquidate at t_b, which is the traditional value-matching condition in optimal stopping problems. In this case, rolling over brings the same payoff L, and thus by continuity and monotonicity, she prefers liquidating when $t < t_b$ and rolling over when $t > t_b$. The second condition, smooth pasting, comes from the *No-Deals* condition and the belief monotonicity refinement. We show in the Appendix that if this condition fails, type g will have strictly higher incentives to switch to market financing before t_g, which constitutes a strictly profitable opportunity for market lenders. Intuitively, since a bad project's present value falls below the liquidation value, the equilibrium decision of refinancing with the market must be one with pooling. Given the pooling structure in market refinancing, the smooth-pasting condition solves the optimal-stopping time problem for the good types. Essentially, the smooth-pasting condition picks the earliest t_g for the good entrepreneur to refinance with the market.

With the boundary conditions, we can uniquely pin down $\{t_b, t_g\}$, given by the following proposition.

Proposition 1. *There exists a $\bar{\eta}$ such that, if $\eta < \bar{\eta}$ and $V_0^u \geq \max\left\{L, \bar{V}_0^u\right\}$, an unique monotone equilibrium exists in the absence of financial constraints and is characterized by thresholds t_b and t_g, where*

$$t_b = \frac{1}{\lambda + \eta}\left[\log\left(\frac{1 - q_0}{q_0}\frac{\bar{q}}{1 - \bar{q}}\right) - \eta(t_g - t_b)\right] \tag{18}$$

$$t_g - t_b = \frac{1}{r + \phi + \eta}\log\left(\frac{\bar{V}_{t_g}^b - PV_r^b}{L - PV_r^b}\right), \tag{19}$$

and $\bar{q}$ solves

$$\bar{q}^2 - \left(1 - \frac{r + \phi}{\eta}\right)\bar{q} + \frac{r + \phi}{\eta}\left(\frac{D^b}{D^g - D^b} - \frac{\delta + \phi}{r + \phi}\frac{D^g}{D^g - D^b}\right) = 0. \tag{20}$$

Proposition 1 shows that the length of extend-and-pretend period (Equation (19)) is sufficiently long to deter bad types from mimicking others at t_b: while $\bar{V}_{t_g}^b - PV_r^b$ captures the additional benefit of extend-and-pretend until t_g, the denominator term in the logarithm function $L - PV_r^b$ captures the relative benefit of liquidating the project at t_b. Equation (22a) shows that the length of the efficient-liquidation period t_b gets shorter as either the length of the subsequent period $t_g - t_b$ gets longer or the arrival rate of public news η gets higher. Intuitively, when public news will reveal the project's type, the reputation of the project grows faster. Therefore, the length of the initial efficient-liquidation stage, during which reputation grows without liquidation, is necessarily shorter.

The condition $\eta < \bar{\eta}$ is not necessary but helps simplify the expositions. Intuitively, as η gets sufficiently low, the No-Deals condition is always slack for the uninformed type after $t = 0$ so that they would never be interested in refinancing with bad types only.[9] The other condition, $V_0^u \geq \max\left\{L, \bar{V}_0^u\right\}$, requires the uninformed type to choose bank financing at $t = 0$: the continuation value exceeds both the value of immediate market financing and liquidation. In the appendix, we provide a close-form expression for V_0^u which allows us to write the condition $V_0^u \geq \max\left\{L, \bar{V}_0^u\right\}$ in term of primitives.

Remark 3. We specify a pessimistic belief during $[t_b, t_g)$: any entrepreneur that seeks market financing during this period will be treated as a bad one and will be unable to refinance with the market. As in other signaling models, there are multiple off-equilibrium beliefs that could sustain the equilibrium outcome. The pessimistic belief is one of them, and perhaps the most common one in these models. In subsection 5.1, we consider an

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[9]If η gets very high, the average belief on the uninformed type increases very fast after $t = 0$ so that the No-Deals condition for type u may bind after $t = 0$ even if it holds at $t = 0$. In other words, the uninformed types' incentives to pool with bad types can be non-monotonic or even increase over time.

extension in which the lending relationships may break up exogenously, so there are always some entrepreneurs who seek market financing on the equilibrium path, and hence there is no need to specify off-equilibrium beliefs. The structure of the equilibrium is similar in this case, and we show that the equilibrium outcome converges to the one in our model when the probability of exogenous break ups goes to zero. We can show that market belief in the limit is the one that makes the bad type indifferent between rolling over bad loans and immediate financing with the market, and there is no discontinuity in beliefs. That is, the refinement selects an off equilibrium belief that is continuous in time. That said, throughout the paper, we continue to use the pessimistic off-equilibrium belief as this is commonly used in the literature.

3.2 Equilibrium under financial constraints

Our benchmark case applies to a scenario in which the Coase Theorem holds, so that frictionless bargaining and negotiation will lead to the efficient allocation between the entrepreneur and the bank. As a result, all rollover decisions are made to maximize the joint surplus of the bank and the entrepreneur. Specifically, at each rollover date, a loan will be rolled over if the joint surplus is above the liquidation value L. In this subsection, we formally analyze the model with financial constraints. Clearly, the Coase Theorem clearly no longer applies, and we need to study the incentives of the bank and the entrepreneur separately.

There are a total of two financial constraints. First, at rollover dates before t_g, the negotiated loan rate y cannot exceed the rate of interim cash flow c. Second, at rollover dates at t_g when the entrepreneur intends to refinance with the market, the price of the market debt D_t must be sufficient to cover the face value of the loan F, or equivalently (7) holds. This condition turns out always slack so that we will only focus on the effect of $y_t \leq c$ for the remainder of this subsection.[10]

The HJBs for the value function $\{V_t^i, \ i \in \{u, g, b\}\}$ remain unchanged from those in subsection 3.1. Again, we can use two thresholds $\{t_b, t_g\}$ to characterize the equilibrium solutions. Let us now turn to the boundary conditions under financial constraints. First, the smooth-pasting condition continues to hold, because it selects the equilibrium in which a good-type entrepreneur chooses to refinance with the market as early as possible. The smooth-pasting condition pins down $\bar{q}$, implying that the average quality financed by the market remains unchanged under financial constraints. The second boundary condition – value matching at t_b – is affected by the financial constraint $y_t \leq c$. In particular, because the entrepreneur is financially constrained and cannot repay its loan before t_g, the bank has the right to decide whether to liquidate the project. It only chooses not to do so if its

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[10]As shown in the next subsection, $\bar{q}$ increases with η. One can easily verify that for $\eta = 0$, $\bar{q} > \underline{q}$.

continuation value lies above c. As a result, the value-matching condition at t_b becomes

$$B_{t_b}^b = L, \tag{21}$$

instead of $V_{t_b}^b = L$.

Proposition 2. *If $B_0^u \geq L$, with financial constraints and under the same parametric conditions as in Proposition 1, the equilibrium is characterized by the two thresholds $\{t_b, t_g\}$*

$$t_b = \frac{1}{\lambda + \eta} \left[\log \left(\frac{1 - q_0}{q_0} \frac{\bar{q}}{1 - \bar{q}} \right) - \eta(t_g - t_b) \right] \tag{22a}$$

$$t_g - t_b = \frac{1}{r + \phi + \eta} \log \left(\frac{F - \frac{c + \phi \theta F}{r + \phi + \eta}}{L - \frac{c + \phi \theta F}{r + \phi + \eta}} \right), \tag{22b}$$

where $\bar{q}$ remains unchanged from Proposition 1.

Comparing Proposition 1, 2, and the result that $\bar{V}_{t_g}^b > L$, we show that financial constraints mitigate the inefficiency arising from extend and pretend.

Corollary 1. *The length of the extend-and-pretend period $t_g - t_b$ gets shorter under the financial constraint. t_b gets larger so that the period of efficient liquidation gets longer.*

Intuitively, the financial constraint $y_t \leq c$ limits the size of transfers that the entrepreneur is able to make to the bank until the next rollover date. Therefore, the constraint makes it possible that the bank's continuation value falls below the liquidation value L, even though the joint surplus is still above L. As a result, the bad project is liquidated more often, compared to the case without financial constraint. Consequently, the length of the zombie-lending period $t_g - t_b$ gets shorter. This result has empirical implications for the relationship between the borrow firm's financial constraints and credit quality. In particular, it highlights the role of financial constraints and their interaction with asymmetric information among different types of lenders. Whereas the benchmark case shows that zombie lending is inevitable, results in this subsection show the firm's financial constraints can mitigate this source of inefficiency. While most of the existing empirical research on zombie lending has emphasized the effect of a financially-constrained bank, our theory offers a new testable implication on the effect of financially-constrained firms in the lending relationship. In particular, our results imply that as firms get more financially constrained, zombie lending could be mitigated.

Numerical Example Figure 2 plots the value function of all three types: while the left panel shows the joint valuations of the entrepreneur and the bank, the right one only shows those of the bank. In this example, the equilibrium $t_b = 1.2921$ and $t_g = 2.1006$.

In the figure, the green, blue, and red lines stand for the value functions of the informed-good, the uninformed, and the informed-bad, respectively. The dashed horizontal line marks the levels of L. Before t reaches t_b, the bad-type's value function stays at L, and all the continuation value accrues to the bank. Note that at t_b, the bad-type entrepreneur's value function experience a discontinuous jump, whereas there is no such a jump in the bank's value function. This contrast is due to the fact that the bank is authorized the right to liquidate the project, and financial constraint limits the size of transfers that the entrepreneur could make to the bank. Indeed, without the financial constraint, both value functions are smooth.
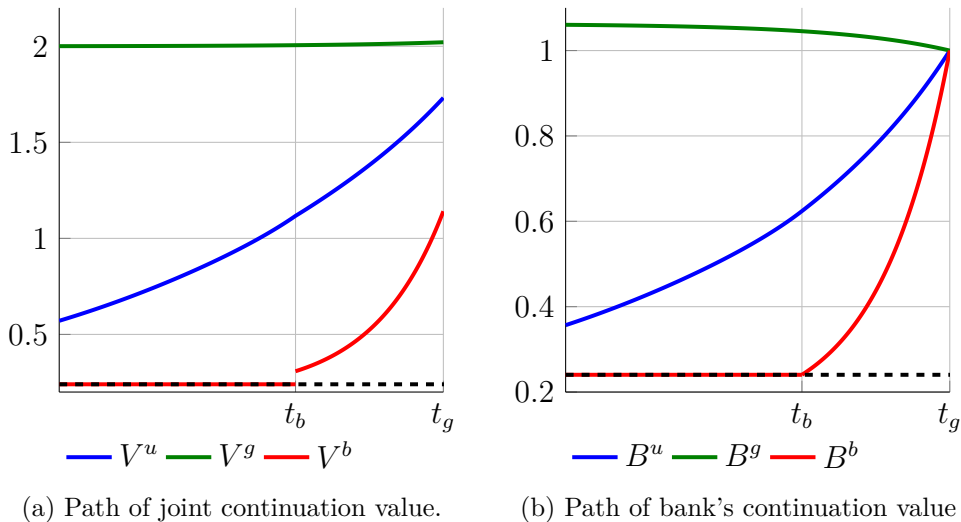

(a) Path of joint continuation value. (b) Path of bank's continuation value

Figure 2: Value functions

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.02$, $m \to 0$, $F = 1$, $\phi = 1.5$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.25 \times NPV_r^b$, $\lambda = 2$, $\eta = 1$, and $q_0 = 0.2$.

Interestingly, the value function of a good-type bank (green line of the right panel) decreases over time. This pattern illustrates the dynamics of the bank's ability to extract rents in a lending relationship. As time approaches t_g, this ability to extract excessive rents from a good-type entrepreneur gets more and more limited. Therefore, the continuation value of a good-type bank declines. This result highlights an important distinction of our paper from the literature on loan sales and securitization.[11] In loan sales and securitization, banks with good loans choose to retain a bigger share of the loans (or more junior tranches) for a longer period of time to signal the loans' quality. The benefit is by doing so, they receive more proceeds by selling the loans at higher prices. In our context, however, a good bank has the opposite incentive: it does not want to signal that

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[11]In practice, 60% of the loans are first sold within one month of loan origination and nearly 90% are sold within one year (Drucker and Puri, 2008). As Gande and Saunders (2012) argue, a special role of banks is to create an active secondary loan market while still producing information.

its borrower is good. Due to information monopoly, the good bank prefers to extract surplus in the lending relationship as long as possible. Once the borrower refinances with the market, the bank does not receive any extra proceeds above the full repayment of the loan, and therefore, a good-type bank prefers to keep its borrower in the lending relationship, as opposed to selling or securitizing the loan.

We end this subsection with a remark on the information technology.

Remark 4. Note that learning and the arrival of private news require joint input from both the entrepreneur and the bank. Therefore, we can think of learning as exploration and understanding of the underlying business prospect, which requires the entrepreneur's experimentation and the bank's previous experience in financing related businesses. In this sense, our model could also be applied to study venture capital firms. Alternatively, we can interpret learning as a process that solely relies on the entrepreneur's input which is independent of the source of financing, whereas only the bank gets to observe the news content through monitoring. Put differently, even without bank financing, the entrepreneur will still be able to learn from news about the quality of her project over time. Our results are identical in this alternative setting, because in the lending relationship, the bank and the entrepreneur are always equally informed. The asymmetric information lies between the bank and the entrepreneur on one side, versus the market-based lenders on the other side.

3.3   No Premature Failure

In this subsection, we study a special case of our model in which there is no premature failures, i.e., $\eta \equiv 0$. As a result, μ_t, the (naive) belief update from no premature failure will always stay at q_0. Proposition 3 shows the results, in which we obtain simple and closed-form solutions for $\bar{q}$ and $t_g - t_b$.

Proposition 3. *If $\eta = 0$ so that there is no premature failure, the equilibrium is characterized by thresholds $\{\bar{q}, t_b, t_g\}$, where*

$$\bar{q} = \frac{\frac{\delta+\phi}{r+\phi}D^g - D^b}{D^g - D^b} \tag{23}$$

$$t_g - t_b = \frac{1}{r+\phi} \log \left(\frac{F - \frac{c+\phi\theta F}{r+\phi}}{L - \frac{c+\phi\theta F}{r+\phi}} \right). \tag{24}$$

Compared to the case without premature failure ($\eta = 0$), the case with premature failure ($\eta > 0$) has a higher $\bar{q}$ and lower $t_g - t_b$.

This result implies the existence of premature failure as public news leads to higher credit quality $\bar{q}$. Intuitively, if a good-type entrepreneur has the additional option to signal her type through a long history without any premature failure, she is less inclined

21

to refinance with the market early, unless the price of the bonds is sufficiently high, or equivalently $\bar{q}$ gets very high. On the other hand, the presence of premature failure renders extend-and-pretend by bad types more costly, because it is possible that the project could fail during this period. As a result, the period of extend-and-pretend gets shorter. Empirically, Proposition 3 implies for firms with more transparent governance and accounting system, the concern for extend and pretend is mitigated.

Our next corollary shows some interesting comparative static results on the amount of extend and pretend and credit quality with respective to several primitive variables.

Corollary 2. *In the case of $\eta = 0$, $\bar{q}$ increases with δ, decreases with r and θ and is unaffected by either λ or L. Moreover, $t_g - t_b$ decreases with r, L, and θ, and is unaffected by δ or λ.*

Let us offer some explanations to the results on r and δ. Note that the role of the extend-and-pretend period is to discourage bad types from mimicking other types at $t = t_b$. This is clearly seen in (24): while $F - \frac{c + \phi \theta F}{r + \phi}$ captures the additional benefit of extend-and-pretend until t_g, the denominator term in the logarithm function $L - \frac{c + \phi \theta F}{r + \phi}$ captures the relative benefit of liquidating the project at t_b.

Intuitively, lower δ is associated with cheaper market financing so that the entrepreneur has lower incentives to remain in the lending relationship. Ceteris paribus, a good-type entrepreneur is willing to accept market financing at a lower level of credit quality $\bar{q}$, which compensates for the decrease in δ. By contrast, if the cost of bank financing r gets cheaper, credit quality $\bar{q}$ gets lower. The reason is, the cost for bad types to mimic others during the extend-and-pretend region gets lower when bank financing becomes cheaper. As a result, an informed-bad type bank liquidates the project less often, resulting in a lower credit quality. However, if r gets even lower than δ so that bank financing is cheaper than the market financing, then the entrepreneur will never refinance with the market. Knowing so, an informed bad bank will always liquidate the project and never engage in extend and pretend.

The contrasting results of a small decrease in r versus a large decrease in r have implications on the real effects of bank bailouts. It is widely perceived that the size of capital injection into the banking sector is crucial for the success of bank bailouts. Using Japan's lost decade in 1990s as a laboratory, Giannetti and Simonov (2013) shows that small size of capital injections encouraged more zombie lending and failed to increase overall credit supply. By contrast, capital injections that are large enough actually increase credit supply and spur investment. In the context of our model, a small decrease in r could lead to a longer period of zombie lending (bigger $t_g - t_b$). However, if r gets so low such that $r \leq \delta$, there is no advantage associated with market financing and therefore no zombie lending as well.

3.3.1 Initial Borrowing

Under the financial constraints, the entrepreneur is able to raise B_0^u at $t = 0$. Therefore, if the entrepreneur needs to invest I at $t = 0$, she needs to contribute at least $\max\{I - B_0^u, 0\}$. Proposition 4 describes the closed-form expression of B_0^u.

Proposition 4. *In the case of $\eta = 0$, the entrepreneur borrowing capacity at $t = 0$ is*

$$B_0^u = q_0 \left[\frac{c + \phi F}{r + \phi} + e^{-(r+\phi)t_b} \left(B_{t_b}^g - \frac{c + \phi F}{r + \phi} \right) \right]$$
$$+ (1 - q_0) \left[\frac{c + \phi\theta F + \lambda L}{r + \phi + \lambda} + e^{-(r+\phi+\lambda)t_b} \left(L - \frac{c + \phi\theta F}{r + \phi + \lambda} \right) \right], \quad (25)$$

where

$$B_{t_b}^g = \frac{c + \phi F}{r + \phi} + e^{-(r+\phi)(t_g - t_b)} \left(F - \frac{c + \phi F}{r + \phi} \right). \quad (26)$$

Intuitively, B_0^u in (25) has two components. With probability q_0, the project is good, in which case, the bank is able to receive payments $\frac{c+\phi F}{r+\phi}$ until t_g, after which it gets fully repaid. With probability $1 - q_0$, the project turns out bad, and the bank has the option to liquidate it if the bad private news arrives before t_b.

An increase in the cost of bank financing r may increase or decrease B_0^u. On one hand, all the payments (interim and final repayments) are more heavily discounted when r goes up. On the other hand, both $\bar{q}$ and $t_g - t_b$ gets lower because the incentive to extend and pretend is lower. As a result, the entrepreneur is able to refinance with the market (in which case the bank gets fully repaid) earlier. The overall effect thus depends on the relative magnitude of these two effects.

An increase in δ may also increase or decrease the initial borrowing amount B_0^u. When market financing gets more expensive, $\bar{q}$ gets higher and so is t_b and t_g. However, there are two counter-veiling effects in B_0^u. First, if the project turns out good, the bank is able to extract excessive rents for a longer period of time, which increases the amount that it is willing to lend upfront. Second, for the fixed payments, the bank needs to wait longer to get fully repaid, which decreases the amount that it is willing to lend upfront. In the proof in the Appendix, we offer details conditions that characterize the monotonicity, where we show that in general, an increase in δ first decreases then increases B_0^u.

3.4 General Maturity

Our analysis so far has focused on the case of instantly-maturing loans ($m \to 0$). In this subsection, we describe the results for the general case where loans have expected maturity m. We will show that all our previous results go through. Moreover, we will

show how t_b, $t_g - t_b$ and $\bar{q}$ vary with loan maturity m. For simplicity, we focus on the case without premature failure by taking $\eta = 0$.

When loans mature gradually, bad projects are also liquidated gradually as their loans mature during $[0, t_b]$. In Appendix A.1.5, Lemma 2 describes the evolution of the public beliefs without liquidation. Under general loan maturity, for $t \in [t_b, t_g]$, we can generalize the HJB equation systems into

$$(r + \phi) V_t^u = \dot{V}_t^u + c + \phi \left[q_0 + (1 - q_0) \theta \right] R \tag{27a}$$
$$+ \lambda \left[q_0 V_t^g + (1 - q_0) V_t^b - V_t^u \right] + \frac{1}{m} \mathcal{R}(V_t^u, \bar{V}_t^u)$$

$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R + \frac{1}{m} \mathcal{R}(V_t^g, \bar{V}_t^g) \tag{27b}$$

$$(r + \phi + \eta) V_t^b = \dot{V}_t^b + c + \phi \theta R + \frac{1}{m} \mathcal{R}(V_t^b, \bar{V}_t^b), \tag{27c}$$

where

$$\mathcal{R}(V_t^i, \bar{V}_t^i) \equiv \max \left\{ 0, \bar{V}_t^i - V_t^i, L - V_t^i \right\}. \tag{28}$$

Note the equations systems are identical to those in (16a)-(16c), except for the last terms, which account for the event that the loan matures. In this case, the bank and the entrepreneur choose between rolling over the debt (0 in equation (28)), replacing the loan with the market bond ($\bar{V}_t^i - V_t^i$ in (28)), and liquidating the project ($L - V_t^i$ in (28)). Note that under general maturity, the entrepreneur in general does not get to refinance immediately after t reaches t_g. Therefore, the expressions for $\bar{V}_t^i$ are different from (5), and we supplement them in Appendix A.1.5.

The boundary conditions are unchanged. Again, we characterize the equilibrium in three regions.

Proposition 5. *If the loan has general maturity m, the liquidation threshold is given by*

$$t_b = \min \left\{ t > 0 : \frac{q_0 \left(1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{\lambda} - 1} e^{\frac{1}{m} t}}{1 + \frac{1}{m} \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda) s} ds} = \bar{q} \right\} \tag{29a}$$

$$\bar{q} = \frac{\delta + \phi}{r + \phi} \frac{D^g}{D^g - D^b} - \frac{D^b}{D^g - D^b}. \tag{29b}$$

1. *Without financial constraints,*

$$t_g - t_b = \frac{1}{r + \phi} \log \left(\frac{V_{t_g}^b - PV_r^b}{L - PV_r^b} \right). \tag{30}$$

2. Under financial constraints,

$$t_g - t_b = \frac{1}{r + \phi} \log \left(\frac{\frac{r+\phi}{r+\phi+1/m}(c + \phi\theta F + \frac{1}{m}F) - (c + \phi\theta F)}{(r + \phi)L - (c + \phi\theta F)} \right). \tag{31}$$

A simple comparison with the results in subsection 3.3 shows that when m gets higher, $\bar{q}$ stays unchanged, t_b gets higher, whereas $t_g - t_b$ becomes lower. Intuitively, $\bar{q}$ is determined as the lowest average quality at which a good-type entrepreneur is willing to refinance with the market, conditional on the loan matures. In this case, the condition of market financing is such that good types get the identical payoff from staying with the bank and refinancing with the market. Thus, $\bar{q}$ doesn't vary with the maturity of the loan.[12] In order for the average quality to reach $\bar{q}$ from q_0, it takes longer when the maturity m gets longer, because bad projects are liquidated less frequently. Therefore, t_b gets higher. Finally, after t reaches t_g, it takes longer for bad types to refinance with the market, and therefore, $V_{t_g}^b$ decreases with m. Therefore, $t_g - t_b$ needs to be shorter to deter bad types from mimicking at t_b.

4 Endogenous Learning

Our analysis has assumed learning and information production as an exogenous process, which happens as long as the entrepreneur has an outstanding loan from the bank. In this section, we analyze the model in which learning and information production are endogenously chosen by the bank as a costly decision. We will show that the equilibrium structure is still captured by thresholds $\{t_b, t_g\}$. An interesting result is, even if the cost of learning is very small, the bank will stop producing information before t reaches t_b. Note that this result goes through even if the bank has *all* the bargaining power in the lending relationship. Therefore, our analysis highlights a new type of hold-up problem in relationship banking: the bank under-supplies its effort in producing valuable information.

Throughout this section, we assume away the premature failure event by taking $\eta = 0$. We will present the results with financial constraints, and the case without the constraints yields qualitatively similar results. The structure of the model stays unchanged from section 2, except that the private news must be learned by banks by choosing a rate $a_t \in [0, 1]$. Given a_t, private news arrives at Poisson rate λa_t, and our previous analysis corresponds to the case that $a_t \equiv 1$. Clearly, a higher rate leads to an earlier arrival of private news in expectation. Meanwhile, learning incurs a flow cost ψa_t so that a higher rate is also more costly to the bank. Heuristically, within a short period $[t, t + dt]$, learning benefit is $\lambda a_t \left[q_0 B_t^g + (1 - q_0) B_t^b - B_t^u \right] dt$: with probability $\lambda a_t dt$, private news

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[12]Mathematically, this condition is determined by the smooth pasting condition $\dot{V}_{t_g}^g = 0$.

arrives, upon which the bank receives a continuation payoff B_t^g with probability q_0 and B_t^b with probability $1 - q_0$ as opposed to B_t^u. The cost of learning is approximately $\psi a_t dt$ during the same period. Given the linear structure, the bank's learning decision follows a bang-bang structure. Specifically, it chooses to learn maximally ($a_t = 1$) if and only if

$$\lambda \left[q_0 B_t^g + (1 - q_0) B_t^b - B_t^u \right] \geq \psi. \tag{32}$$

Otherwise, it chooses not to learn at all, and $a_t = 0$.

Proposition 6. *1. If $\frac{\psi}{\lambda} < \frac{\frac{1}{m}(1 - q_0)}{r + \phi + \frac{1}{m}} \left(L - \frac{c + \phi \theta F}{r + \phi} \right)$, there is an equilibrium characterized by $\{t_a, t_b, t_g\}$ and $t_a < t_b < t_g$. The bank learns if and only if $t < t_a$.*

2. Otherwise, the bank never learns, and the entrepreneur never borrows from the bank.

While $\{t_a, t_b, t_g\}$ are given by the solution to the system of equations (A.32) in the appendix, let us offer some intuitions behind Proposition 6. When the bad project no longer gets liquidated, the value of an uninformed bank is a linear combination of an informed-good one and an informed-bad one, i.e., $B_t^u = q_0 B_t^g + (1 - q_0) B_t^b$, which is the case after t reaches t_b. In other words, after t reaches t_b the benefits of learning is zero, implying that in any equilibrium, banks may only learn for $t \leq t_b$. During $[0, t_b)$, when the bad projects still get liquidated, the value of becoming informed is positive since liquidation avoids the expected loss generated from a bad project (see Figure 3 for a graphical illustration.), i.e., $q_0 B_t^g + (1 - q_0) L - B_t^u > 0$. In this case, information is valuable. The above proposition shows that if the cost of learning is sufficiently low, the bank learns until t_a. If the cost is relatively high, however, the bank will never learn, and consequently, entrepreneurs will never choose bank financing.
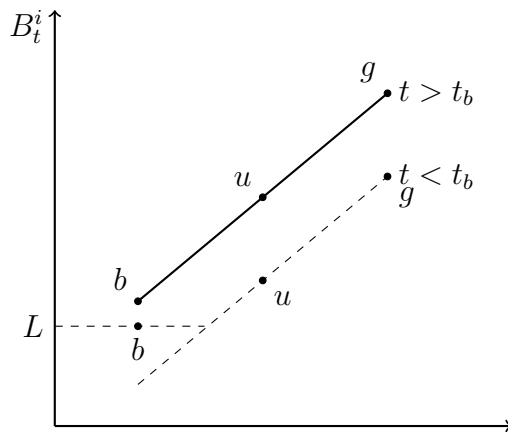

Figure 3: Graphical illustration of learning benefits

How does endogenous learning affect $\bar{q}$, t_b and $t_g - t_b$? A simple comparison between Proposition 3 and 6 shows that under endogenous learning, $\bar{q}$ and $t_g - t_b$ stay unchanged, whereas t_b gets higher. This is because $\bar{q}$ is determined by the good-type's indifference

condition between bank financing and market financing, whereas $t_g - t_b$ is the length of extend-and-pretend period that is just sufficient to deter bad types from mimicking others. Since both $\bar{q}$ and $t_g - t_b$ are determined by types that area already informed, they are unaffected when producing information becomes endogenous and costly. Finally, since less information is produced when learning becomes costly, bad types are liquidated less often, and it takes longer for the average quality q_0 to reach $\bar{q}$, resulting in a higher t_b.

Proposition 6 highlights a new type of hold-up problem in relationship lending that only emerges in the dynamic setup. Rajan (1992) shows that in a lending relationship, the entrepreneur has incentive to underinvest her effort due to the prospect of renegotiation following private news. Our paper shows that the relationship bank will also underinvest in its effort in producing information, even if the bank has *all* the bargaining power and the cost of producing information is infinitesimal (but still positive). This is because the prospect of future market refinancing prevents the bank from capturing all the rents generated from information production, even though it has all the bargaining power. Knowing so, the bank under-supplies its effort in producing information.

This result is reminiscent of some anecdotal stories in the venture capital (VC) industries.[13] On Sep 30, 2019, WeWork, a private company that provides shared workspaces for startups, withdrew its S-1 filing and postponed the IPO. Masayoshi Son, the founder of SoftBank, WeWork's largest investor, admitted he had turned a "blind eye" to governance lapses at WeWork.[14]

Numerical Example   Under the same set of parameters as in subsection 3.2 (except for $\eta = 0$), with the additional parameter that $\psi = 0.06$ and $m = 1$, we can get $t_a = 3.8350$, $t_b = 4.7861$, and $t_g = 5.1352$.

Figure 4 illustrates the effect of loan maturity on equilibrium results. In general, there are two effects when the loan maturity gets longer. First, the bank has more incentives to learn, because it needs to wait (in expectation) longer to liquidate a bad project. Ceteris paribus, the bank has higher incentive to screen out bad-type borrowers. Second, long maturity reduces the option value of new information, since the bank must wait until the next rollover date to change the loan rate. As a result, the incentive to produce information is also reduced. In our numerical exercise, the first effect dominates so that t_a, the boundary that bank stops learning increases as loan maturity increases. Therefore, B_0^u, the amount of initial borrowing, gets lower to compensate the increased learning cost. The bottom two panel shows that as a result, both t_b and t_g get higher, whereas the difference, $t_g - t_b$ stays unchanged. When maturity gets further longer, the effect becomes non-monotonic. In the extreme case where the loan never matures, private news

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[13]We don't explicitly model the difference between VCs and banks, but one can think of the debt contracts in our model as equities with different voting rights.

[14]https://www.ft.com/content/3694a074-0061-11ea-b7bc-f3fa4e77dd47

is useless, and $t_a = 0$. This latter pattern is captured by the second case of Proposition 6 in which, bank financing is not used in equilibrium (equivalently ψ gets very high).
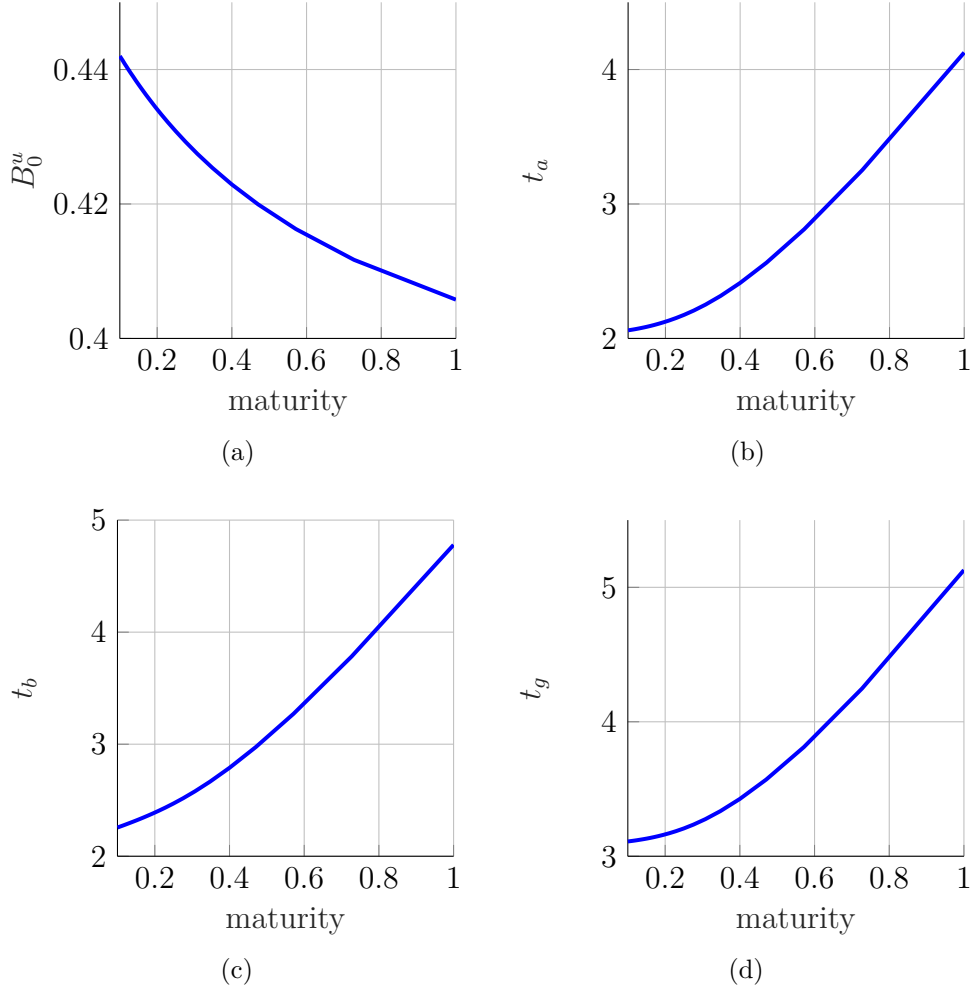

Figure 4: Comparative Statics with Endogenous Learning

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.02$, $F = 1$, $\phi = 1.5$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.25 \times NPV_r^b$, $\lambda = 2$, and $q_0 = 0.2$.

5 Extensions and Robustness

We conduct two extensions that serve as robustness checks. Throughout this section, we always assume that $\eta = 0$.

5.1 Lending Relationship Breakups

In practice, lending relationship may break up for reasons independent of the underlying project's quality. For instance, the relationship may have to be terminated if the bank experiences shocks that dry up its funding. In this subsection, we modify the

model setup by assuming that with rate $\chi > 0$, the lending relationship breaks up, upon which the entrepreneur is forced to refinance with the market; otherwise, the project gets liquidated immediately. We show that the equilibrium of this modified game converges to the one in subsection 3.1 as $\chi \to 0$.[15] For simplicity, let us focus on the case without financial constraints.

We construct an equilibrium that has the same qualitative features as the one in subsection 3.1. The equilibrium in this modified game, however, necessarily involves bad types using mixed strategies. In the appendix we show that the equilibrium is characterized by three thresholds, t_ℓ, t_b and t_g. Figure 5 illustrates the equilibrium strategies. When $t \in [0, t_\ell]$, a bad project is immediately liquidated. When $t \in [t_\ell, t_b]$, a bad project is liquidated with probability $\ell_t \in (0, 1)$ and with complementary probability, the entrepreneur refinances with the market. As a result, the overall rate of projects being liquidated is $\lambda (1 - q_0) \pi_t^u \ell_t$. Panel 5a plots the probability of liquidation ℓ_t, which decreases with t during $[t_\ell, t_b]$.

When $t \in [t_b, t_g]$, while the good and uninformed types remain in the lending relationship without the exogenous breakup, the bad types play a mixed strategy between bank financing and market financing. Let α_t be the rate of bad types seeking financing without the exogenous breakup. Panel 5b plots $\gamma_t = \alpha_t \pi_t^b$, the overall rate of bad types that voluntarily seek market financing without the breakup.[16] As time increases, γ_t decreases because waiting until t_g becomes more attractive.

Panel 5c plots the bond price for borrowers who seek market financing between $[0, t_g]$. The price pattern is such so that the bad types will indeed use a mixed-strategy in equilibrium. Between 0 and t_ℓ, the price increases as bad types liquidate their projects. Note that this price is a constant between t_ℓ and t_b, just to make the bad type indifferent between liquidation and financing with the market. After t reaches t_b and approaches t_g, the bond price needs to grow in order to make bad types still indifferent between bank and market financing.

As $\chi \to 0$, the equilibrium of this game converges to the one in subsection 3.3. The average quality of the entrepreneurs that seek financing between t_b and t_g, however, does not converge to 0. Instead, it converges to a positive level q_{t+}. In the game that $\chi \equiv 0$ and the off-equilibrium belief during $t \in [t_b, t_g]$ is that $q_t = q_{t+}$, only the bad entrepreneurs will choose to voluntarily refinance with the market. Therefore, this modified game can serve as a micro-foundation to justify the discontinuity in beliefs in the game that $\eta = 0$ and $\chi = 0$.

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[15]This limit can be interpreted as a refinement of the equilibrium in the case without public news in the spirit of trembling-hand perfect equilibria (Fudenberg and Tirole, 1991).

[16]The total flow of bad types is $\chi \pi_t^b + \gamma_t$.

(a) Probability of liquidation, ℓ_t

(b) Voluntary flow of bad projects into market, $\gamma_t = \alpha_t \pi_t^b$.
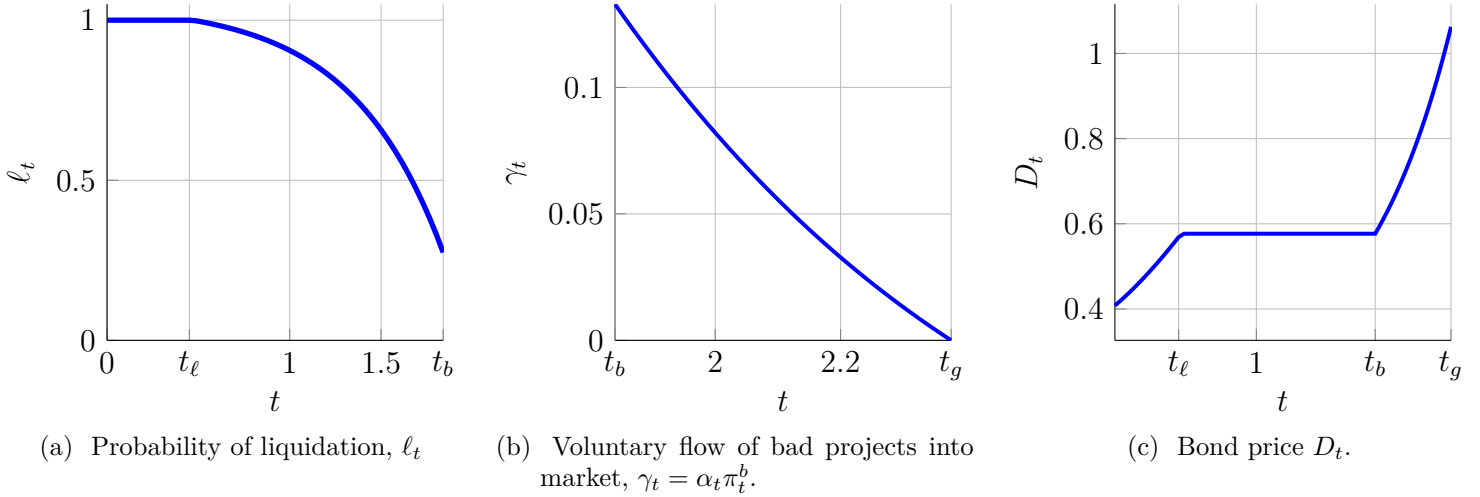
(c) Bond price D_t.

Figure 5: Equilibrium with Exogenous Breakups.

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.02$, $m \to 0$, $F = 1$, $\phi = 1.5$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV_r^b$, $\lambda = 2$, and $q_0 = 0.2$. The equilibrium thresholds are $t_\ell = 0.45$, $t_b = 1.84$, and $t_g = 2.37$.

5.2 Observable Rollover and Deterministic Maturity

In the baseline model, we have assumed rolling over a loan is unobservable to the market participants. Moreover, we have modeled the maturing event as a Poisson event to simplify the solution. Both assumptions have been made for simplicity. In this subsection, we introduce two modifications to the model. First, whenever a loan matures, we assume that whether the bank decides to roll over or liquidate is observable. Second, the maturity of loans is publicly known to be fixed at m. We show the model is essentially identical to one in a discrete-time framework. Thus, as in most discrete-time models with a binary type of asymmetric information, the equilibrium will, in general, involve mixed strategies due to the integer problem. In the remainder of this subsection, we construct an equilibrium that has features like those described in Proposition 1.

Let $n \in \{1, 2, 3 \cdots\}$ be the sequence of rollover events. The date associated with the n-th rollover event is $t_n = n \cdot m$, and the time between two rollover dates is m. As before, we can construct an equilibrium with two thresholds: t_b and t_g. However, with deterministic rollovers, specifying the two thresholds in term of the rollover events is notationally more convenient: n_b and n_g. The following proposition describes the equilibrium.

Proposition 7. *If loan maturity is fixed at m and rollover is observable, then there exists $\{n_b, n_g\}$ such that*

    1. Efficient liquidation

(a) *For $n < n_b$, bad projects are liquidated, whereas other projects are rolled over;*

2. *Extend and pretend*

   (a) *For $n = n_b$, a fraction $\alpha_b \leq 1$ of the bad projects are liquidated, whereas other projects are rolled over.*

   (b) *When $n \in (n_b, n_g - 1)$, all loans are rolled over.*

3. *Market Financing*

   (a) *When $n = n_g$, market lenders make an offer at D_{τ_m} ($q_{\tau_m} = \bar{q}$) with probability $\alpha_g \leq 1$.*

   (b) *When $n = n_g + 1$, all entrepreneurs refinance with the market.*

4. *If $m \to 0$, the equilibrium converges to the one in Proposition 1.*

Under observable rollovers, the public belief about the loan quality q_t stays unchanged during any two rollover events. At the rollover event $n < n_b$ or equivalently $t < n_b \cdot m$, the belief will experience a discrete jump. If the project is liquidated, clearly q_t jumps to 0. If the loan is rolled over, q_t jumps upwards to the level described in the instantly maturing debt, as shown in (15). Note an equivalence in beliefs under fixed maturity and instantly maturing exponential debt, because the event of maturing is occurring with certainty at $t = n \cdot m$. q_t stays unchanged at $\bar{q}$ after $t > n_b \cdot m$.

6 Concluding Remarks

This paper offers a novel explanation towards the prevalent phenomenon of zombie lending. In particular, we introduce private learning into a banking model and argue that in a dynamic lending relationship, zombie lending is inevitable but yet self-limiting. We show how the length of zombie-lending period is affected by various factors such as the cost of bank and market financing, as well as entrepreneur's financial constraints. Moreover, we show that in the dynamic lending relationship, the bank has incentives to under-supply its effort in producing information.

To focus on dynamic learning and the resulting asymmetric information between the relationship bank and market-base lenders, we have not explicitly modeled interbank competition (Boot and Thakor, 2000) and secured lending (Boot and Thakor, 1994). Interbank competition does not change any result in the context of our model, because the new bank is as uninformed as market-based lenders. Broadly speaking, the liquidation value of the asset L can be interpreted as collateral value. That said, both issues deserve more careful examinations in future research. Moreover, zombie lending emerges in our paper due to potential competition between the relationship bank and market-based

lenders. An interesting extension is to introduce complementarity between banks and markets as in Song and Thakor (2010) and study how the degree of zombie lending changes.

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A Appendix

A.1 Proofs

A.1.1 Proof of Proposition 1

First, the No Deals condition and belief monotonicity requirement imply smooth pasting at $t = t_g$. That is,

$$\dot{V}^g_{t_g} = \dot{D}_t,$$

where

$$\dot{D}_t = \dot{q}_t(D^g - D^b).$$

The proof follows closely the proof of Theorem 5.1 in Daley and Green (2012). No Deals and value matching condition immediately implies that $\dot{V}^g_{t_g} \leq \dot{D}_t = \dot{q}_t \left(D^g - D^b \right)$. Suppose that $\dot{V}^g_{t_g} < \dot{q}_t \left(D^g - D^b \right)$ instead. In this case, consider a deviation in which the good types wait until $t_g + \epsilon$ to refinance with the market, where ϵ is sufficiently small. Belief monotonicity implies that $q_{t_g+\epsilon}$ is at least (approximately) $q_{t_g} + \eta q_{t_g} \left(1 - q_{t_g} \right) \epsilon$, which shows that the good types have strict incentives to wait until $t_g + \epsilon$.

Recall that μ_t is the belief held by the uninformed types

$$\dot{\mu}_t = \eta \mu_t \left(1 - \mu_t \right).$$

Under the conjectured equilibrium structure,

$$\dot{q}_t = \begin{cases} (\lambda + \eta) \, q_t \left(1 - q_t \right) & t \leq t_b \\ \eta q_t \left(1 - q_t \right) & t > t_b \end{cases}.$$

Between $[0, t_b]$, a bad type will be liquidated immediately so it is as if bad news arrives at rate $\lambda + \eta$. After t_b, bad news arrives at rate η only since there is no liquidation so learning only occurs through exogenous news.

By applying the smooth pasting and value matching conditions for the type g at the market financing time $t = t_g$ we get

$$\dot{V}^g_t = \left(D^g - D^b \right) \eta q_t \left(1 - q_t \right).$$

Using the HJB for the type g during the zombie lending region $[t_b, t_g]$, and setting $\bar{q} = q_{t_g}$,

we obtain the following quadratic equation for $\bar{q}$

$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R$$

$$\Rightarrow \phi (R - F) + (r + \phi) \left[\bar{q} D^g + (1 - \bar{q}) D^b \right] = \left(D^g - D^b \right) \eta \bar{q} (1 - \bar{q}) + c + \phi R.$$

Next, we show that given the Assumption 4 there is only one root on $[0, 1]$, which corresponds to the maximal root of the quadratic equation. First, we evaluate $LHS - RHS$ at $\bar{q} = 0$:

$$(r + \phi) \left(D^b - \frac{c + \phi F}{r + \phi} \right) = (r + \phi) \left(\frac{c + \phi \theta F}{\delta + \phi + \eta} - \frac{c + \phi F}{r + \phi} \right).$$

Next, we evaluate $LHS - RHS$ at $\bar{q} = 1$:

$$(r + \phi) \left(D^g - \frac{c + \phi F}{r + \phi} \right) = (r + \phi) \left(\frac{c + \phi F}{\delta + \phi} - \frac{c + \phi F}{r + \phi} \right) > 0.$$

So we can conclude that there is only one root on $[0, 1]$. Next, we rewrite the quadratic equation for $\bar{q}$ as

$$\bar{q}^2 - \left(1 - \frac{r + \phi}{\eta} \right) \bar{q} + \frac{r + \phi}{\eta} \left(\frac{D^b}{D^g - D^b} - \frac{\delta + \phi}{r + \phi} \frac{D^b}{D^g - D^b} \right) = 0.$$

As a result, the minimum of the quadratic function is attained at

$$q^{\min} \equiv \frac{1}{2} \left(1 - \frac{r + \phi}{\eta} \right), \tag{A.1}$$

and that $\bar{q} > q^{\min}$. We will use the observation that $\bar{q} > q^{\min}$ later to verify the optimality decisions by different types in equilibrium.

The next step is to solve for the length of the zombie lending region, $t_g - t_b$. At $t = t_g$, $q_t = \bar{q}$, implying

$$D_{t_g} = \bar{q} D^g + (1 - \bar{q}) D^b = \bar{D}.$$

and

$$\bar{V}_{t_g}^b = \bar{D} + \frac{\phi \theta (R - F)}{r + \phi + \eta}.$$

Using the boundary condition for the bad type at time t_b, $V_{t_b}^b = L$, together with the type-b's HJB equation on $[t_b, t_g]$

$$(r + \phi + \eta) V_t^b = \dot{V}_t^b + c + \phi \theta R,$$

we obtain the equation

$$\frac{\bar{V}^b_{t_g} - PV^b_r}{L - PV^b_r} = \exp\left[(r + \phi + \eta)(t_g - t_b)\right]$$

$$\Rightarrow t_g - t_b = \frac{1}{r + \phi + \eta} \log\left(\frac{\bar{V}^b_{t_g} - PV^b_r}{L - PV^b_r}\right).$$

From here, we can find t_b using the equation

$$\bar{q} = \frac{q_0}{q_0 + (1 - q_0)e^{-(\lambda+\eta)t_b}e^{-\eta(t_g-t_b)}},$$

which yields

$$t_b = \frac{1}{\lambda + \eta}\left[\log\left(\frac{1 - q_0}{q_0}\frac{\bar{q}}{1 - \bar{q}}\right) - \eta(t_g - t_b)\right]$$

Optimality the Good Type's Strategy. We need to verify that it is indeed optimal for the type g to obtain market financing at time t_g. The HJB equation for the high type on $[0, t_g)$.

$$(r + \phi)V^g_t = \dot{V}^g_t + c + \phi R.$$

To verify that it is not optimal to delay market financing, we need to verify that the following inequality holds for any $t > t_g$

$$(r + \phi)\bar{V}^g_t \geq \dot{\bar{V}}^g_t + c + \phi R.$$

On the other hand, to verify that it is not optimal for the good type to seek market financing before time t_g, we need to verify that for any $t < t_g$.

$$V^g_t \geq \bar{V}^g_t.$$

Next, we proceed to verify each of these inequalities. First, we verify the optimality for $t > t_g$. Let's define

$$G_t = (r + \phi)\bar{V}^g_t - \dot{\bar{V}}^g_t - c - \phi R$$

$$= (r + \phi)\left(D_t + \frac{\phi(R - F)}{r + \phi}\right) - \dot{D}_t - c - \phi R$$

$$= (r + \phi)D_t - \dot{D}_t - c - \phi F$$

By construction, $G_{t_g} = 0$ so it is enough to show that $\dot{G}_t \geq 0$ for $t > t_g$. This amounts to verify that

$$\dot{G}_t = (r + \phi)\dot{D}_t - \ddot{D}_t \geq 0$$

Substituting the expressions for D_t, we get

$$\dot{G}_t = (D^g - D^b)\left[(r + \phi)\dot{q}_t - \ddot{q}_t\right].$$

In the last region where $t > t_g$, we have $\dot{q}_t = \eta q_t(1 - q_t)$ so we get that

$$\dot{G}_t = (D^g - D^b)\left[r + \phi - \eta(1 - 2q_t)\right]\dot{q}_t.$$

The conclusion that $\dot{G}_t > 0$ follows from the inequality $\bar{q} > q^{\min}$, where $q^{\min}$ is defined in equation (A.1). Next, we verify the optimality for $t < t_g$. Let's define $H_t \equiv V_t^g - \bar{V}_t^g$. The first step is to show that H_t single crosses zero from above. We have that

$$
\begin{aligned}
\dot{H}_t &= \dot{V}_t^g - \dot{\bar{V}}_t^g \\
&= (r + \phi)V_t^g - c - \phi R - (D^g - D^b)\dot{q}_t \\
&= (r + \phi)H_t + (r + \phi)\bar{V}_t^g - c - \phi R - (D^g - D^b)\dot{q}_t.
\end{aligned}
$$

Hence, a sufficient condition is

$$\dot{H}_t\Big|_{H_t=0} = (r + \phi)\bar{V}_t^g - c - \phi R - (D^g - D^b)\dot{q}_t < 0$$

on $(0, t_g)$, which requires that

$$(r + \phi)[q_t D^g + (1 - q_t)D^b] < (D^g - D^b)\dot{q}_t + c + \phi F.$$

Since we have that $\dot{q}_t \geq \eta q_t(1 - q_t)$ and $\bar{q} > q^{\min}$, it follows that

$$(r + \phi)[q D^g + (1 - q)D^b] < (D^g - D^b)\eta q(1 - q) + c + \phi F,$$

for all $0 < q < \bar{q}$, which means that $\dot{H}_t\Big|_{H_t=0} < 0$ for $t < t_g$. From here we can conclude that $V_t^g \geq \bar{V}_t^g$ for $t < t_g$.

Optimality of the Bad Type's Strategy. By a similar reasoning to the one used to verify the optimality of the good type strategy, the strategy of the low type is optimal if for any $t \geq t_b$ we have that $V_t^b \geq L$, and for any $t < t_b$ we have that

$$(r + \phi + \eta)L \geq c + \phi\theta R.$$

Note here we have plugged in the condition that in equilibrium, $V_t^b = L$ for $t < t_b$.

First, we verify the previous inequality for $t \in (0, t_b)$. If we substitute $V_t^b = L$, the previous inequality reduces to

$$L \geq PV_r^b,$$

which follows from the assumption that liquidation of bad projects is efficient. To verify that $V_t^b \geq L$, for $t < t_b$, notice that on (t_b, t_g), the value function satisfies

$$(r + \phi + \eta) V_t^b = \dot{V}_t^b + c + \phi \theta R.$$

This equation can be written as

$$(r + \phi + \eta)(V_t^b - L) = \dot{V}_t^b + c + \phi \theta R - (r + \phi + \eta) L.$$

Letting $G_t = V_t^b - L$, we obtain the equation

$$\dot{G}_t = (r + \phi + \eta)\left(G_t + L - PV_r^b\right), \quad G_{t_b} = 0.$$

Clearly, $\dot{G}_t\big|_{G_t=0} > 0$ so that $G_t = V_t^b - L \geq 0$ for all $t \geq t_b$.

Optimality of the Uninformed Type's Strategy. Next, we verify that the uninformed is better off rolling over at time $t < t_b$ rather than liquidating. First, we solve for the continuation value of the uninformed at any time $t < t^b$. For $t \in (0, t_b)$, we have that

$$(r + \phi + \lambda + (1 - \mu_t)\eta) V_t^u = \dot{V}_t^u + c + \phi\left[\mu_t + (1 - \mu_t)\theta\right] R$$
$$+ \lambda\left[\mu_t V_t^g + (1 - \mu_t) L\right]$$
$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R.$$

Solving backward starting at time t_b we get

$$V_t^u = \int_t^{tb} e^{-(r+\phi+\lambda)(s-t) - \int_t^s \eta(1-\mu_u) du} \left(c + \phi\left[\mu_s + (1 - \mu_s)\theta\right] R + \lambda\left[\mu_s V_s^g + (1 - \mu_s) L\right]\right) ds$$
$$+ e^{-(r+\phi+\lambda)(t_b-t) - \int_t^{t_b} \eta(1-\mu_u) du} V_{t_b}^u$$

Substituting the relation

$$\int_t^s \eta(1 - \mu_s) ds = \int_t^s \frac{\dot{\mu}_s}{\mu_s} ds = \log(\mu_s/\mu_t),$$

and the continuation value of the good type

$$V_t^g = \frac{c + \phi R}{r + \phi}\left(1 - e^{-(r+\phi)(t_b-t)}\right) + e^{-(r+\phi)(t_b-t)} V_{t_b}^g,$$

we obtain

$$V_t^u = \mu_t \left[PV_r^g + e^{-(r+\phi)(t_b-t)} \left(V_{t_b}^g - PV_r^g \right) \right]$$
$$+ (1-\mu_t) \left[\frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta} + e^{-(r+\phi+\lambda+\eta)(t_b-t)} \left(L - \frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta} \right) \right]. \quad (A.2)$$

It is convenient to express the continuation value of the uninformed in terms of the uninformed's belief μ_t. Let $t(\mu)$ be the time at which the belief is μ, which is given by

$$t(\mu) = -\frac{1}{\eta} \log \left(\frac{q_0}{1-q_0} \frac{1-\mu}{\mu} \right)$$

Let μ_b be given by $t(\mu_b) = t_b$ so

$$t(\mu_b) - t(\mu) = -\frac{1}{\eta} \log \left(\frac{1-\mu_b}{\mu_b} \frac{\mu}{1-\mu} \right).$$

Substituting $t(\mu_b) - t(\mu)$ we get

$$V^u(\mu) = \mu \left[PV_r^g + \left(\frac{1-\mu_b}{\mu_b} \frac{\mu}{1-\mu} \right)^{\frac{r+\phi}{\eta}} \left(V_{t_b}^g - PV_r^g \right) \right]$$
$$+ (1-\mu) \left[\frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta} + \left(\frac{1-\mu_b}{\mu_b} \frac{\mu}{1-\mu} \right)^{1+\frac{r+\phi+\lambda}{\eta}} \left(L - \frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta} \right) \right]. \quad (A.3)$$

Letting $z \equiv \mu/(1-\mu)$, we have that $V^u(\mu) \geq L$ if

$$z \left[PV_r^g - L + \left(\frac{z}{z_b} \right)^{\frac{r+\phi}{\eta}} \left(V_{t_b}^g - PV_r^g \right) \right] - \frac{r + \phi + \eta}{r + \phi + \lambda + \eta} \left[1 - \left(\frac{z}{z_b} \right)^{1+\frac{r+\phi+\lambda}{\eta}} \right] \left(L - \frac{c + \phi\theta R}{r + \phi + \lambda} \right) > 0.$$

The LHS is increasing in z (so $V^u(\mu)$ in increasing in μ), hence $V^u(\mu) \geq L$ for all $\mu \in [q_0, \mu_b]$ only if $V^u(q_0) \geq L$.

No Deals Uninformed. Finally, we need to verify that the no deals condition holds for the uninformed type. This is immediate when $\eta = 0$, but requires verification when $\eta > 0$. No deals requires that

$$V_t^u \geq \tilde{D}_t + \mu_t \frac{\phi(R-F)}{r+\phi} + (1-\mu_t) \frac{\phi\theta(R-F)}{r+\phi+\eta},$$

where $\tilde{D}_t$ is the value of debt if the uninformed is pooled with the bad type. In particular,

$$\tilde{D}_t = \tilde{q}_t D^g + (1-\tilde{q}_t) D^b,$$

where $\tilde{q}_t$ is the beliefs conditional on being either uninformed or bad. For $t < t_b$, the probability of being bad is zero so the probability of the project being good conditional on being either bad or uninformed is given by

$$\tilde{q}_t = \mu_t = \frac{q_0}{q_0 + (1-q_0)e^{-\eta t}}.$$

For $t \in (t_b, t_g)$ we gave

$$\tilde{q}_t = \mu_t \frac{\pi_t^u}{1 - \pi_t^g}$$

where

$$\pi_t^u = \frac{\left(q_0 + (1-q_0)e^{-\eta t}\right)e^{-\lambda t}}{q_0 + (1-q_0)e^{-(\lambda+\eta)t_b}e^{-\eta(t-t_b)}}$$

$$\pi_t^g = \frac{q_0(1 - e^{-\lambda t})}{q_0 + (1-q_0)e^{-(\lambda+\eta)t_b}e^{-\eta(t-t_b)}}$$

so

$$\frac{\pi_t^u}{1 - \pi_t^g} = \frac{\left(q_0 + (1-q_0)e^{-\eta t}\right)e^{-\lambda t}}{q_0 e^{-\lambda t} + (1-q_0)e^{-(\lambda+\eta)t_b}e^{-\eta(t-t_b)}}$$

and

$$\tilde{q}_t = \frac{q_0 e^{-\lambda t}}{q_0 e^{-\lambda t} + (1-q_0)e^{-(\lambda+\eta)t_b}e^{-\eta(t-t_b)}} = \frac{q_0}{q_0 + (1-q_0)e^{-\lambda t_b}e^{(\lambda-\eta)t}}.$$

From here we get that $\tilde{D}_t$ is decreasing in time only if $\lambda > \eta$. It can be verified solving the HJB equation for the uninformed type, than for any $t \in [t_b, t_g]$, the continuation value fo the uninformed is given by

$$V_t^u = \mu_t V_t^g + (1 - \mu_t)V_t^b.$$

So the no-deals condition on (t_b, t_g) can be written as

$$\mu_t \left(V_t^g - \frac{\phi(R-F)}{r+\phi} \right) + (1 - \mu_t) \left(V_t^b - \frac{\phi\theta(R-F)}{r+\phi+\eta} \right) \geq \tilde{D}_t$$

The LHS in the previous inequality is increasing in t.

Claim 1.

$$\mu_t \left(V_t^g - \frac{\phi(R-F)}{r+\phi} \right) + (1 - \mu_t) \left(V_t^b - \frac{\phi\theta(R-F)}{r+\phi+\eta} \right)$$

is increasing in time.

Proof. To show that the expression in the proposition is increasing in time, it is enough to show that to show that

$$V_t^g - V_t^b \geq \frac{\phi(R-F)}{r+\phi} - \frac{\phi\theta(R-F)}{r+\phi+\eta}.$$

At time t_g we have

$$V_t^g - V_t^b = \frac{\phi(R-F)}{r+\phi} - \frac{\phi\theta(R-F)}{r+\phi+\eta},$$

while at any time $t < t^g$ we have

$$\dot{V}_t^g - \dot{V}_t^b = (r+\phi)(V_t^g - V_t^b) - \eta V_t^b - \phi(1-\theta)R.$$

Solving backward in time starting at t_g we get

$$V_t^g - V_t^b = \frac{\eta}{r+\phi}\frac{c+\phi\theta R}{r+\phi+\eta}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}\left(1 - e^{-\eta(t_g-t)}\right)\left(\bar{V}_{t_g}^b - \frac{c+\phi\theta R}{r+\phi+\eta}\right)$$
$$+ \frac{\phi(1-\theta)R}{r+\phi}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}\left(\frac{\phi(R-F)}{r+\phi} - \frac{\phi\theta(R-F)}{r+\phi+\eta}\right).$$

Hence, we get that

$$V_t^g - V_t^b - \left(\frac{\phi(R-F)}{r+\phi} - \frac{\phi\theta(R-F)}{r+\phi+\eta}\right) = e^{-(r+\phi)(t_g-t)}\left(1 - e^{-\eta(t_g-t)}\right)\left(\bar{V}_{t_g}^b - \frac{c+\phi\theta R}{r+\phi+\eta}\right)$$
$$+ \frac{\left(1 - e^{-(r+\phi)(t_g-t)}\right)}{r+\phi+\eta}\left(\frac{\eta}{r+\phi}c + \phi(1-\theta)F + \frac{\eta}{r+\phi}\phi F\right) > 0.$$

It follows immediately from the fact that μ_t, V_t^b and V_t^g are increasing in time that

$$\mu_t\left(V_t^g - \frac{\phi(R-F)}{r+\phi}\right) + (1-\mu_t)\left(V_t^b - \frac{\phi\theta(R-F)}{r+\phi+\eta}\right)$$

is also increasing in time. $\qquad\square$

If $\lambda > \eta$, then the previous claim implies that the it enough to verify the uninformed no deal's condition at time t_b to guarantee that it is satisfied for all $t \in [t_b, t_g]$. In which case, we only need to verify that

$$\mu_{t_b}\left(V_{t_b}^g - \frac{\phi(R-F)}{r+\phi}\right) + (1-\mu_{t_b})\left(L - \frac{\phi\theta(R-F)}{r+\phi+\eta}\right) \geq \tilde{D}_{t_b}. \qquad \text{(A.4)}$$

At time t_b, we have that

$$\tilde{q}_{t_b} = \frac{q_0}{q_0 + (1-q_0)e^{-\eta t_b}} = \mu_{t^b},$$

thus, we can write condition (A.4) as

$$\mu_{t_b}\left(V_{t_b}^g - D^g - \frac{\phi(R-F)}{r+\phi}\right) + (1-\mu_{t_b})\left(L - D^b - \frac{\phi\theta(R-F)}{r+\phi+\eta}\right) \geq 0. \qquad \text{(A.5)}$$

Next, we look at the no-deals condition for $t \in (0, t_b)$. Because $\tilde{q}_t = \mu_t$ on $(0, t_b)$, the

no-deals condition for the uninformed on $(0, t_b)$ condition amounts to verify that

$$V_t^u \geq \mu_t \left(D^g + \frac{\phi(R - F)}{r + \phi} \right) + (1 - \mu_t) \left(D^b + \frac{\phi\theta(R - F)}{r + \phi + \eta} \right).$$

Using equation (A.3), we can write the uninformed's no-deal condition as

$$F(z) \equiv z \left(\frac{z}{z_b} \right)^{\frac{r+\phi}{\eta}} \left(V_{t_b}^g - PV_r^g \right) + \left(\frac{z}{z_b} \right)^{1 + \frac{r+\phi+\lambda}{\eta}} \left(L - \frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta} \right)$$
$$- z \left(D^g + \frac{\phi(R - F)}{r + \phi} - PV_r^g \right) + \frac{c + \phi\theta R + \lambda L}{r + \phi + \lambda + \eta} - \frac{\phi\theta(R - F)}{r + \phi + \eta} - D^b \geq 0.$$

It can be easily verified that $F(z)$ is convex and that its first derivate is given by

$$F'(z) = \left(1 + \frac{r + \phi}{\eta} \right) \left(\frac{z}{z_b} \right)^{\frac{r+\phi}{\eta}} \left[\left(V_{t_b}^g - PV_r^g \right) + \frac{1}{z_b} \left(L - PV_r^b \right) \right]$$
$$- \left(D^g + \frac{\phi(R - F)}{r + \phi} - PV_r^g \right).$$

To verify the no-deals condition we need to consider the case in which the minimum of $F(z)$ is on the boundary of $[z_0, z_b]$ and the case in which it is in the interior. Because $F(z)$ is convex, the previous three cases correspond to: 1) if $F'(z_0) \geq 0$, then F is increasing on $[z_0, z_b]$ so it is sufficient to check that $F(z_0) \geq 0$; 2) if $F'(z_b) \leq 0$, then $F(z)$ is decreasing on $[z_0, z_b]$ so it is sufficient to check that $F(z_b) \geq 0$; and 3) if $F'(z_0) < 0 < F'(z_b)$, then F attains its minimum at $z^{\min}$ in the interior of $[z_0, z_b]$, and we need to verify that $F(z^{\min}) \geq 0$. Notice that $F'(z) > 0$ when $\eta \to 0$, so for η sufficiently small, the uninformed no deals condition reduces to $F(z_0) \geq 0$.

Case 1: $F'(z_0) \geq 0$. This is the case when

$$\left(\frac{z_0}{z_b} \right)^{\frac{r+\phi}{\eta}} \geq \frac{\eta}{r + \phi + \eta} \frac{\left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g \right)}{\left(V_{t_b}^g - PV_r^g \right) + \frac{1}{z_b} \left(L - PV_r^b \right)}$$

On the other hand, if $F'(z_0) \geq 0$, then the uninformed no deals condition is satisfied if $F(z_0) \geq 0$. $F'(z_0) \geq 0$ if

$$\left(\frac{z_0}{z_b} \right)^{\frac{r+\phi}{\eta}} > \frac{\eta}{r + \phi + \eta} \frac{\left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g \right)}{\left(V_{t_b}^g - PV_r^g \right) + \frac{1}{z_b} \left(L - PV_r^b \right)}.$$

A9

In this case, the uninformed's no-deals condition is

$$\left(\frac{z_0}{z_b}\right)^{\frac{r+\phi}{\eta}} (V_{t_b}^g - PV_r^g) + \left[\left(\frac{z_0}{z_b}\right)^{1+\frac{r+\phi+\lambda}{\eta}} - 1\right] \frac{r+\phi+\eta}{r+\phi+\lambda+\eta} (L - PV_r^b) \geq$$

$$z_0 \left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g\right) + D^b + \frac{\phi\theta(R-F)}{r+\phi+\eta} - L.$$

Case 2: $F'(z_b) \leq 0$. This is the case when

$$1 \leq \frac{\eta}{r+\phi+\eta} \frac{\left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g\right)}{\left(V_{t_b}^g - PV_r^g\right) + \frac{1}{z_b} (L - PV_r^b)}$$

If the previous inequality is satisfied, the uninformed no-deals condition reduces to $F(z_b) \geq 0$, which can be written as

$$V_{t_b}^g - PV_r^g \geq z_b \left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g\right) + D^b + \frac{\phi\theta(R-F)}{r+\phi+\eta} - L.$$

Case 2: $F'(z_0) < 0 < F'(z_b)$ Finally, if

$$\left(\frac{z_0}{z_b}\right)^{\frac{r+\phi}{\eta}} < \frac{\eta}{r+\phi+\eta} \frac{\left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g\right)}{\left(V_{t_b}^g - PV_r^g\right) + \frac{1}{z_b} (L - PV_r^b)} < 1$$

then, $F(z)$ attains its minimum in the interior of (z_0, z_b), and we need to check the no-deals condition at its minimize. Solving for the first order condition, we find that $z^{\min} = \arg\min_z F(z)$ is

$$\left(\frac{z^{\min}}{z_b}\right) = \left[\frac{\left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g\right)}{\left(1 + \frac{r+\phi}{\eta}\right)\left[\left(V_{t_b}^g - PV_r^g\right) + \frac{1}{z_b} (L - PV_r^b)\right]}\right]^{\frac{\eta}{r+\phi}}.$$

Substituting $z^{\min}$ in $F(z)$ we find that the no-deals condition for the uninformed in this case is

$$\frac{(r+\phi)\left(V_{t_b}^g - PV_r^g\right) - \frac{(r+\phi+\eta)(r+\phi+\lambda)}{r+\phi+\lambda+\eta} \frac{1}{z_b} (L - PV_r^b)}{\left(1 + \frac{r+\phi}{\eta}\right)^{1+\frac{\eta}{r+\phi}}\left[\left(V_{t_b}^g - PV_r^g\right) + \frac{1}{z_b} (L - PV_r^b)\right]^{1+\frac{\eta}{r+\phi}}} \geq \frac{\eta}{z_b} \frac{D^b + \frac{\phi\theta(R-F)}{r+\phi+\eta} - \frac{c+\phi\theta R+\lambda L}{r+\phi+\lambda+\eta}}{\left(D^g + \frac{\phi(R-F)}{r+\phi} - PV_r^g\right)^{1+\frac{\eta}{r+\phi}}}.$$

A.1.2 Proof of Proposition 2

The financial constraint only becomes relevant for the rollover decision and does not affect the market financing decision. The bank will be willing to rollover only if $B_t^b \geq L$. In the previous equilibrium, $B_{t_b}^b < V_{t_b}^b = L$, which means that the rollover decision violates

A10

the low type bank incentive constraint. Hence, in the presence of financial constraints, the boundary condition for t_b is replaced by $B^b_{t_b}$. By direct computation, we get that the bank's continuation value at time t_b is given by

$$B^b_{t_b} = \frac{c + \phi\theta F}{r + \phi + \eta} \left(1 - e^{-(r+\phi+\eta)(t_g-t_b)}\right) + e^{-(r+\phi+\eta)(t_g-t_b)} F$$

Solving the boundary condition $B^b_{t_b} = L$ we get

$$t_g - t_b = \frac{1}{r + \phi + \eta} \log\left(\frac{F - \frac{c+\phi\theta F}{r+\phi+\eta}}{L - \frac{c+\phi\theta F}{r+\phi+\eta}}\right).$$

The no-deals conditions for the good and uninformed type are the same as the unconstrained case. Hence, the only step left is to analyze the optimality of the rollover strategy. First, we look at the problem of the for the low type. In this case, we need to verify that $B^b_t \geq L$ for $t > t_b$, and that it is not optimal to delay liquidation before time t_b. To verify that $B^b_t \geq L$ on (t_b, t_g), notice that

$$(r + \phi + \eta) B^b_t = \dot{B}^b_t + c + \phi\theta F,$$

so it follows

$$\dot{B}^b_t\Big|_{B^b_t=L} = (r + \phi + \eta)\left(L - \frac{c + \phi\theta F}{r + \phi + \eta}\right) > (r + \phi + \eta)\left(L - PV^b_r\right) > 0$$

which immediately implies that $B^b_t \geq L$ for $t > t_b$. To verify that it is not optimal to delay liquidation on $(0, t_b)$, notice that

$$\dot{B}^b_t + c + \phi\theta F - (r + \phi + \eta) B^b_t = c + \phi\theta F - (r + \phi + \eta) L < 0,$$

which implies that it is optimal to liquidate for $t < t_b$.

Next, we need to verify that the uninformed is willing to rollover the loan at time $t \in (0, t_b)$. The continuation value the uninformed satisfies the equation

$$(r + \phi + \lambda + (1 - \mu_t)\eta) B^u_t = \dot{B}^u_t + c + \phi\left[\mu_t + (1 - \mu_t)\theta\right] F$$
$$+ \lambda\left[\mu_t B^g_t + (1 - \mu_t) L\right]$$
$$(r + \phi)B^g_t = \dot{B}^g_t + c + \phi F.$$

Solving backward in time starting at t_b, we get that for any $t \in [0, t_b]$, the uninformed

continuation value is

$$
B_t^u = \mu_t \left[\frac{c + \phi F}{r + \phi} + e^{-(r+\phi)(t_b - t)} \left(B_{t_b}^g - \frac{c + \phi F}{r + \phi} \right) \right]
$$
$$
+ (1 - \mu_t) \left[\frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda + \eta} + e^{-(r+\phi+\lambda+\eta)(t_b - t)} \left(L - \frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda + \eta} \right) \right]. \quad (A.6)
$$

Making a change of variables to express the continuation value as a function of the bank's belief we get

$$
B^u(\mu) = \mu \left[\frac{c + \phi F}{r + \phi} + \left(\frac{1 - \mu_b}{\mu_b} \frac{\mu}{1 - \mu} \right)^{\frac{r+\phi}{\eta}} \left(B_{t_b}^g - \frac{c + \phi F}{r + \phi} \right) \right]
$$
$$
+ (1 - \mu) \left[\frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda + \eta} + \left(\frac{1 - \mu_b}{\mu_b} \frac{\mu}{1 - \mu} \right)^{1 + \frac{r+\phi+\lambda}{\eta}} \left(L - \frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda + \eta} \right) \right]. \quad (A.7)
$$

The condition $B_t^u \geq L$ can be written in terms of the likelihood ratio $z \equiv \mu/(1 - \mu)$ as

$$
z \left[\frac{c + \phi F}{r + \phi} - L + \left(\frac{z}{z_b} \right)^{\frac{r+\phi}{\eta}} \left(B_{t_b}^g - \frac{c + \phi F}{r + \phi} \right) \right]
$$
$$
- \frac{r + \phi + \eta}{r + \phi + \lambda + \eta} \left[1 - \left(\frac{z}{z_b} \right)^{1 + \frac{r+\phi+\lambda}{\eta}} \right] \left(L - \frac{c + \phi \theta F}{r + \phi + \eta} \right) \geq 0.
$$

The left hand side is increasing in z, so it is enough to verify that $B_0^u \geq L$.

A.1.3 Proof of Proposition 3

Proof. The result on $\bar{q}$ naturally follows by plugging $\frac{\frac{\delta+\phi}{r+\phi} D^g - D^b}{D^g - D^b}$ into (20). It is easily shown that this expression is negative.

The result on $t_g - t_b$ follows Assumption 1 and Assumption 4. $\qquad \square$

A.1.4 Proof of Proposition 4

Proof. Given the boundary conditions, we can show that

$$
B_{t_b}^g = \frac{c + \phi F}{r + \phi} + e^{-(r+\phi)(t_g - t_b)} \left(F - \frac{c + \phi F}{r + \phi} \right),
$$

whereas $B_{t_b}^b = L$. When $t \in [0, t_b]$, the HJB satisfies

$$
(r + \phi + \lambda) B_t^u = \dot{B}_t^u + c + \phi \left[q_0 + (1 - q_0) \theta \right] F + \lambda \left[q_0 B_t^g + (1 - q_o) L \right].
$$

Solving this ODE, we can get the result. We can also write B_0^u in terms of primitives

$$B_0^u = q_0 \left[\frac{c + \phi F}{r + \phi} + \left(\frac{1 - \bar{q}}{\bar{q}} \frac{q_0}{1 - q_0} \right)^{\frac{r + \phi}{\lambda}} \left(\frac{L - \frac{c + \phi \theta F}{r + \phi}}{F - \frac{c + \phi \theta F}{r + \phi}} \right) \frac{rF - c}{r + \phi} \right]$$
$$+ (1 - q_0) \left[\frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda} + \left(\frac{1 - \bar{q}}{\bar{q}} \frac{q_0}{1 - q_0} \right)^{1 + \frac{r + \phi}{\lambda}} \left(L - \frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda} \right) \right].$$

Finally,

$$\frac{dB_0^u}{d\frac{1 - \bar{q}}{\bar{q}}} = q_0^{1 + \frac{r + \phi}{\lambda}} (1 - q_0)^{-\frac{r + \phi}{\lambda}} \times \left(\frac{1 - \bar{q}}{\bar{q}} \right)^{\frac{r + \phi}{\lambda} - 1}$$
$$\times \left(\frac{L - \frac{c + \phi \theta F}{r + \phi}}{F - \frac{c + \phi \theta F}{r + \phi}} \right) \frac{rF - c}{r + \phi} \frac{r + \phi}{\lambda} + \left(L - \frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda} \right) \left(1 + \frac{r + \phi}{\lambda} \right) \left(\frac{1 - \bar{q}}{\bar{q}} \right),$$

which is positive if $\left(L - \frac{c + \phi \theta F + \lambda L}{r + \phi + \lambda} \right) \left(1 + \frac{r + \phi}{\lambda} \right) \left(\frac{1 - \bar{q}}{\bar{q}} \right) > \left(\frac{L - \frac{c + \phi \theta F}{r + \phi}}{F - \frac{c + \phi \theta F}{r + \phi}} \right) \frac{c - rF}{r + \phi} \frac{r + \phi}{\lambda}$. Since $\bar{q}$ increases with δ, $\frac{dB_0^u}{d\frac{1 - \bar{q}}{\bar{q}}} > 0$ and equivalently $\frac{dB_0^u}{d\delta} < 0$ if δ is sufficiently small, whereas $\frac{dB_0^u}{d\delta} < 0$ if δ gets sufficiently large. $\qquad\square$

A.1.5 Proof of Proposition 5

Lemma 2. *Without liquidation, the public beliefs $(\pi_t^u, \pi_t^g, \pi_t^b)$ satisfy the following differential equation:*

$$\dot{\pi}_t^u = -\lambda \pi_t^u + \mathbb{1}_{t \leq t_b} \frac{1}{m} \pi_t^u \pi_t^b \tag{A.8a}$$

$$\dot{\pi}_t^g = \lambda \pi_t^u q_0 + \mathbb{1}_{t \leq t_b} \frac{1}{m} \pi_t^g \pi_t^b \tag{A.8b}$$

$$\dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0) - \mathbb{1}_{t \leq t_b} \frac{1}{m} \pi_t^b (1 - \pi_t^b). \tag{A.8c}$$

Proof. The evolution of beliefs can be derived directly in continuous time using the filtering formulas for counting processes in Lipster and Shiryaev (2001), examples 2 and 3 in Chapter 19. Here, we provide a heuristic derivation starting in discrete time and taking the continuous time limit. Using Baye's rule we get

$$\pi_t^g = \frac{\pi_{t-h}^g + \pi_{t-h}^u q_0 \lambda h}{\pi_{t-h}^g + \pi_{t-h}^u q_0 \lambda h + \pi_{t-h}^u (1 - \lambda h) + \left(\pi_{t-h}^b + \pi_{t-h}^u (1 - q_0) \lambda h \right) \left(1 - \frac{1}{m} h \right)},$$

so

$$\frac{\pi_t^g - \pi_{t-h}^g}{h} = \frac{\lambda q_0 \pi_{t-h}^u + \frac{1}{m} \pi_{t-h}^g \pi_{t-h}^b}{\pi_{t-h}^g + \pi_{t-h}^u q_0 \lambda h + \pi_{t-h}^u (1 - \lambda h) + \left(\pi_{t-h}^b + \pi_{t-h}^u (1 - q_0) \lambda h \right) \left(1 - \frac{1}{m} h \right)} + \frac{O(h^2)}{h}.$$

Taking the limit when $h \to 0$, we get

$$\dot{\pi}_t^g = \lambda q_0 \pi_t^u + \frac{1}{m} \pi_t^g \pi_t^b. \tag{A.9}$$

Similarly, we have that

$$\pi_t^u = \frac{(1 - \lambda h) \pi_{t-h}^u}{\pi_{t-h}^g + \pi_{t-h}^u q_0 \lambda h + \pi_{t-h}^u (1 - \lambda h) + \left(\pi_{t-h}^b + \pi_{t-h}^u (1 - q_0) \lambda h \right) (1 - \frac{1}{m} h)},$$

so

$$\frac{\pi_t^u - \pi_{t-h}^u}{h} = \frac{-\lambda \pi_{t-h}^u + \frac{1}{m} \pi_{t-h}^u \pi_{t-h}^b}{\pi_{t-h}^g + \pi_{t-h}^u q_0 \lambda h + \pi_{t-h}^u (1 - \lambda h) + \left(\pi_{t-h}^b + \pi_{t-h}^u (1 - q_0) \lambda h \right) (1 - \frac{1}{m} h)} + \frac{O(h^2)}{h}.$$

Taking the limit as $h \to 0$ we get

$$\dot{\pi}_t^u = -\lambda \pi_t^u + \frac{1}{m} \pi_t^u \pi_t^b. \tag{A.10}$$

Finally, using equations (A.9) and (A.10), together with $\pi_t^b + \pi_t^u + \pi_t^b = 1$, we get

$$\dot{\pi}_t^b = \lambda (1 - q_0) \pi_t^u - \frac{1}{m} \pi_t^b (1 - \pi_t^b). \tag{A.11}$$

$\square$

Under general maturity,

$$\bar{V}_t^g = \frac{c + \phi R}{r + \phi + \frac{1}{m}} + \frac{\frac{1}{m}}{r + \phi + \frac{1}{m}} \left[D_t + \frac{\phi (R - F)}{r + \phi} \right]$$

$$\bar{V}_t^b = \frac{c + \phi \theta R}{r + \phi + \frac{1}{m}} + \frac{\frac{1}{m}}{r + \phi + \frac{1}{m}} \left[D_t + \frac{\phi \theta (R - F)}{r + \phi} \right]$$

$$\bar{V}_t^u = q_0 \bar{V}_t^g + (1 - q_0) \bar{V}_t^b.$$

The final step is to find the solution for q_t in the interval $[0, t_b]$. Let us define in this proof $z_t = \frac{\pi_t^g}{\pi_t^u}$, then,

$$
\begin{aligned}
\dot{z}_t &= \frac{\dot{\pi}_t^g \pi_t^u - \pi_t^g \dot{\pi}_t^u}{(\pi_t^u)^2} = \frac{\dot{\pi}_t^g}{\pi_t^u} - z_t \frac{\dot{\pi}_t^u}{\pi_t^u} \\
&= \lambda q_0 + \frac{1}{m} z_t (1 - \pi_t^g - \pi_t^u) - z_t \left(-\lambda + \frac{1}{m} (1 - \pi_t^g - \pi_t^u) \right) \\
&= \lambda (q_0 + z_t).
\end{aligned}
$$

Therefore, we have the solution

$$z_t = q_0 \left(e^{\lambda t} - 1 \right)$$
$$\Rightarrow \pi_t^g = q_0 \left(e^{\lambda t} - 1 \right) \pi_t^u. \tag{A.12}$$

Since $\pi_t^u + \pi_t^g + \pi_t^b = 1$, we also have

$$\pi_t^b = 1 - \left(q_0 e^{\lambda t} + 1 - q_0 \right) \pi_t^u. \tag{A.13}$$

Substituting (A.12) and (A.13) into the ODE system, we get a first-order ODE for π_t^u

$$\dot{\pi}_t^u = \left(\frac{1}{m} - \lambda \right) \pi_t^u - \frac{1}{m} \left(q_0 e^{\lambda t} + 1 - q_0 \right) (\pi_t^u)^2 ,$$

which corresponds to a continuous-time Riccati equation. This equation can be transformed it into a second-order ODE. In this proof, let $v_t = -\frac{1}{m} \left(q_0 e^{\lambda t} + 1 - q_0 \right) \pi_t^u$ and $Q_t = q_0 e^{\lambda t} + 1 - q_0$,

$$\dot{v}_t = v_t^2 + \frac{v_t}{Q_t} \left[q_0 e^{\lambda t} \lambda + Q_t \left(\frac{1}{m} - \lambda \right) \right]. \tag{A.14}$$

Further, if we let $v_t = -\frac{\dot{\nu}_t}{\nu_t} \Rightarrow \dot{v}_t = -\frac{\ddot{\nu}_t}{\nu_t} + (v_t)^2$, then we can transform equation (A.14) into the following second-order ODE

$$\ddot{\nu}_t = \frac{\dot{\nu}_t}{Q_t} \left[q_0 e^{\lambda t} \lambda + Q_t \left(\frac{1}{m} - \lambda \right) \right]$$

From here, we get that

$$\dot{\nu}_t = \dot{\nu}(0) e^{\int_0^t \lambda \frac{1}{1 + \frac{1-q_0}{q_0} e^{-\lambda s}} ds + (\frac{1}{m} - \lambda) t}$$

Moreover,

$$\int_0^t \frac{1}{1 + \frac{1-q_0}{q_0} e^{-\lambda s}} ds = \frac{1}{\lambda} \log \left(1 - q_0 + q_0 e^{\lambda t} \right)$$

so

$$\dot{\nu}_t = \dot{\nu}(0) \left(1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda) t}$$

Integrating one more time, we get

$$\nu_t = \nu(0) + \dot{\nu}(0) \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda) s} ds.$$

Using the definition of v_t and ν_t, we have

$$\dot{\nu}_0 = -v_0 \nu(0) = \frac{1}{m} \nu(0)$$

A15

so

$$\nu_t = \nu(0) \left(1 + \frac{1}{m} \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda)s} ds \right).$$

Using the definition of v_t we get

$$v_t = -\frac{\frac{1}{m} \left(1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda)t}}{1 + \frac{1}{m} \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda)s} ds}$$

so

$$\pi_t^u = \frac{\left(1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{\lambda} - 1} e^{(\frac{1}{m} - \lambda)t}}{1 + \frac{1}{m} \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda)s} ds} \tag{A.15}$$

Thus, substituting (A.15) in the definition for q_t, we get

$$\begin{aligned}
q_t &= \pi_t^g + q_0 \pi_t^u \\
&= q_0 \left(e^{\lambda t} - 1 \right) \pi_t^u + q_0 \pi_t^u \\
&= q_0 e^{\lambda t} \pi_t^u \\
&= \frac{q_0 \left(1 - q_0 + q_0 e^{\lambda t} \right)^{\frac{1}{\lambda} - 1} e^{\frac{1}{m} t}}{1 + \frac{1}{m} \int_0^t \left(1 - q_0 + q_0 e^{\lambda s} \right)^{\frac{1}{\lambda}} e^{(\frac{1}{m} - \lambda)s} ds}
\end{aligned} \tag{A.16}$$

Because q_t is monotone, the solution for t_b and t_g is unique.

A.1.6 Proof of Proposition 6

We offer the proof for both the case with and without financial constraints. Define $\underline{t}_a \equiv \frac{1}{\lambda} \log \left(\frac{\bar{q}}{1 - \bar{q}} \frac{1 - q_0}{q_0} \right)$.

Unconstrained Case The proof is divided in two parts. First, we show that the equilibrium can be characterized by three thresholds, $\{t_a, t_b, t_g\}$, and then we derive equations determining $\{t_a, t_b, t_g\}$.

The proof for the existence of the zombie lending and market financing region region, and the determination of the boundary conditions determining t_b, t_g follows the one without endogenous learning. It only remains to determine the equilibrium learning policy. The first step is to show that the bank never learns after time t_b, that is we need to show that $a_t = 0$ for all $t > t_b$.

Solving the HJB equation for V_t^u backward starting at t_g, we find that for any $t \in (t_b, t_g)$, the continuation value of the uninformed is

$$V_t^u = q_0 V_t^g + (1 - q_0) V_t^b - \int_t^{t_g} e^{-(r + \phi)(s - t) - \int_t^s \lambda a_u du} \psi a_s ds.$$

It follows that there is no benefit of learning on $[t_b, t_g]$, so in any equilibrium characterized

A16

by $\{t_b, t_g\}$, the bank never learn after t_b. Let's define

$$\Psi_t \equiv \int_t^{t_g} e^{-(r+\phi)(s-t) - \int_t^s \lambda a_u du} \psi a_s ds$$

$$\Gamma_t \equiv q_0 B_t^g + (1 - q_0) B_t^b - B_t^u$$

Looking for a contradiction, suppose that the bank learns during (t_b, t_g). For any $t \in (t_a, t_g)$, the HJB equation is

$$\left(r + \phi + \frac{1}{m} \right) B_t^u = \dot{B}_t^u + y_t F - \psi a_t + \phi[q_0 + (1 - q_0)\theta]F + \frac{1}{m} V_t^u + \lambda a_t \Gamma_t \qquad \text{(A.17a)}$$

$$\left(r + \phi + \frac{1}{m} \right) B_t^g = \dot{B}_t^g + y_t F + \phi F + \frac{1}{m} V_t^g \qquad \text{(A.17b)}$$

$$\left(r + \phi + \frac{1}{m} \right) B_t^b = \dot{B}_t^b + y_t F + \phi \theta F + \frac{1}{m} V_t^b. \qquad \text{(A.17c)}$$

Using the HJB equations we get the following ODE for Γ_t

$$\left(r + \phi + \frac{1}{m} + \lambda a_t \right) \Gamma_t = \dot{\Gamma}_t + \psi a_t + \frac{1}{m}(q_0 V_t^g + (1 - q_0)V_t^b - V_t^u) = \dot{\Gamma}_t + \psi a_t + \frac{1}{m} \Psi_t,$$

Since $\Gamma_{t_g} = 0$, it implies that $\lambda \Gamma_t \leq 0 < \psi$ for $\forall t \in [t_b, t_g]$. Therefore, there can only be learning in $[0, t_b)$.

Next, we prove that it if learning happens at all, then it must happen on $[0, t_a]$ for some $t_a < t_b$. Let $t_a = \sup\{t \leq t_b : \lambda \Gamma_t = \psi\}$. Noticing that $\Gamma_{t_b} = 0$, we can conclude that $t_a < t_b$. We want to show that the optimal policy is $a_t = \mathbf{1}_{t < t_a}$. Suppose not, then there is t_a' such that $\lambda \Gamma_t < \psi$ on $(t_a' - \epsilon, t_a')$. In particular, consider $t_a' = \sup\{t < t_a : \lambda \Gamma_t < \psi\}$. Consider the regions (t_a', t_a), in this region, the bank's HJB equation is

$$\left(r + \phi + \frac{1}{m} \right) B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F - \psi + \frac{1}{m} V_t^u + \lambda \Gamma_t$$

$$\left(r + \phi + \frac{1}{m} \right) B_t^g = \dot{B}_t^g + y_t F + \phi F + \frac{1}{m} V_t^g$$

$$\left(r + \phi + \frac{1}{m} \right) B_t^b = \dot{B}_t^b + y_t F + \phi \theta F + \frac{1}{m} L,$$

so

$$\left(r + \phi + \frac{1}{m} + \lambda \right) \Gamma_t = \dot{\Gamma}_t + \frac{1}{m}(q_0 V_t^g + (1 - q_0)L - V_t^u) + \psi \qquad \text{(A.18)}$$

A17

Let $H_t \equiv (1 - q_0)L + q_0 V_t^g - V_t^u$, we get

$$\left(r + \phi + \frac{1}{m} + \lambda\right) \Gamma_t = \dot{\Gamma}_t + \frac{1}{m} H_t + \psi, \ t \in (t_a', t_a)$$

$$\left(r + \phi + \frac{1}{m}\right) \Gamma_t = \dot{\Gamma}_t + \frac{1}{m} H_t, \ t \in (t_a, t_b).$$

Taking the left and right limit at t_a' we get $\dot{\Gamma}_{t_a-} = \dot{\Gamma}_{t_a+}$, so Γ_t is differentiable at t_a. It follows from the ODE for Γ_t that if $\dot{H}_t \leq 0$ on (t_a', t_b), then Γ_t is a quasi-convex function of t on (t_a', t_b). To show that $\dot{H}_t \leq 0$, we write an ODE for H_t using the HJB equations for V_t^b and V_t^g, which is given by

$$(r + \phi + \lambda) H_t = \dot{H}_t + (r + \phi)(1 - q_0)L - (1 - q_0)(c + \phi \theta R) \tag{A.19}$$

$$+ \psi - \lambda(1 - q_0)(V_t^b - L), \ t \in (t_a', t_a) \tag{A.20}$$

$$(r + \phi) H_t = \dot{H}_t + (r + \phi)(1 - q_0)L - (1 - q_0)(c + \phi \theta R), \ t \in (t_a, t_b), \tag{A.21}$$

where $H_{t_b} = (1 - q_0)V_{t_b}^b + q_0 V_{t_b}^g - V_{t_b}^u = 0$. Assumption 1 implies that $\dot{H}_{t_b} < 0$. Differentiating equation (A.20) and (A.21) we get

$$(r + \phi + \lambda) \dot{H}_t = \ddot{H}_t - \lambda(1 - q_0) \dot{V}_t^b, \ t \in (t_a', t_a)$$

$$(r + \phi) \dot{H}_t = \ddot{H}_t, \ t \in (t_a, t_b).$$

It immediately follows that $\dot{H}_t = 0 \Rightarrow \ddot{H}_t \geq 0$ since $\dot{V}_t^b \geq 0$. Hence, $\dot{H}_t$ single crosses 0 from negative to positive, so $\dot{H}_{t_b} < 0 \Rightarrow \dot{H}_t < 0, \forall t \in (t_a', t_b)$.

Since, Γ_t is quasi-convex on (t_a', t_b), $\Gamma_{t_b} = 0$ and $\Gamma_{t_a} = \psi/\lambda$. It must be the case that $\Gamma_{t_a'} > \psi/\lambda$, which provides the desired contractions. Thus, it must be the case that $\lambda \Gamma_t \geq \psi$ for all $t < t_a$

Having shown that the optimal policy is characterized by $\{t_a, t_b, t_g\}$, we provide a solution and derive parametric assumptions needed to validate it. Note that in the equilibrium characterized by $\{t_a, t_b, t_g\}$, beliefs evolve on $t \in (t_a, t_b)$ according to

$$\dot{\pi}_t^u = \frac{1}{m} \pi_t^u \pi_t^b$$

$$\dot{\pi}_t^g = \frac{1}{m} \pi_t^g \pi_t^b$$

$$\dot{\pi}_t^b = -\frac{1}{m} \pi_t^b \left(1 - \pi_t^b\right),$$

which means that the average quality evolves according to

$$\dot{q}_t = \frac{1}{m} q_t \pi_t^b.$$

A18

solving the previous equation starting at time t_a, we obtain that for any $t > t_a$ the average belief is

$$q_t = q_{t_a} e^{\frac{1}{m} \int_{t_a}^t \pi_s^b ds} \tag{A.22}$$

The differential equation for π_t^b is decoupled from the one for π_t^u and π_t^g so it can be solved independently to get

$$\pi_t^b = \frac{\pi_{t_a}^b}{\pi_{t_a}^b + (1 - \pi_{t_a}^b) e^{\frac{1}{m}(t-t_a)}}$$

$$e^{\frac{1}{m} \int_{t_a}^t \pi_s^b ds} = \frac{1 - \pi_t^b}{1 - \pi_{t_a}^b}.$$

Substituting in equation (A.22) we get

$$q_t = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-\frac{1}{m}(t-t_a)}} q_{t_a}. \tag{A.23}$$

To find $\pi_{t_a}^b$, we use equations (A.9), (A.10), and (A.11) to get:

$$\pi_{t_a}^b = 1 - \frac{q_{t_a}}{q_0} \left(q_0 + (1 - q_0) e^{-\lambda t_a} \right).$$

Taking the limit of q_t in equation (A.23) as $t \to \infty$ we get that $q_t \to \frac{q_{t_a}}{1 - \pi_{t_a}^b}$, so $\lim_{t \to \infty} q_t > \bar{q}$ only if

$$t_a \geq \underline{t}_a \equiv \frac{1}{\lambda} \log \left(\frac{\bar{q}}{1 - \bar{q}} \frac{1 - q_0}{q_0} \right). \tag{A.24}$$

Having stablished a lower bound for t_a, we proceed to derive a system of equations for t_a, t_b, t_g. On (t_a, t_b), the bank's continuation value satisfies equation (A.17) evaluated at $a_t = 0$. We can solve for Γ_t using the terminal condition $\Gamma_{t_b} = 0$ to get

$$\Gamma_t = \int_t^{t_b} e^{-(r + \phi + \frac{1}{m})(s-t)} \frac{1}{m} \left[(1 - q_0) L + q_0 V_s^g - V_s^u \right] ds. \tag{A.25}$$

Te continuation value of the good and the uninformed types can be solved in close form and are given by for V_t^u

$$q_0 V_t^g = \frac{q_0 c + q_0 \phi R}{r + \phi} (1 - e^{-(r+\phi)(t_g - t)}) + e^{-(r+\phi)(t_g - t)} q_0 V_{t_g}^g$$

$$V_t^u = \frac{c + \phi \left[q_0 + (1 - q_0) \theta \right] R}{r + \phi} (1 - e^{-(r+\phi)(t_g - t)}) + e^{-(r+\phi)(t_g - t)} V_{t_g}^u.$$

A19

Thus, we get that

$$(1-q_0)L + q_0 V_t^g - V_t^u = (1-q_0)\left[L - \frac{c+\phi\theta R}{r+\phi}(1 - e^{-(r+\phi)(t_g-t)})\right] + e^{-(r+\phi)(t_g-t)}(q_0 V_{t_g}^g - V_{t_g}^u)$$

$$= (1-q_0)\left[L - \frac{c+\phi\theta R}{r+\phi} + e^{-(r+\phi)(t_g-t)}\left(\frac{c+\phi\theta R}{r+\phi} - V_{t_g}^b\right)\right]$$

Substituting in equation (A.25) we get

$$\Gamma_t = \frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - \frac{c+\phi\theta R}{r+\phi}\right)\left(1 - e^{-(r+\phi+\frac{1}{m})(t_b-t)}\right) +$$

$$(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi} - V_{t_g}^b\right)e^{-(r+\phi)(t_g-t)}\left(1 - e^{-\frac{1}{m}(t_b-t)}\right).$$

After substituting $V_{t_g}^b$, we get the following equation for t_a:

$$\frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - \frac{c+\phi\theta R}{r+\phi}\right)\left(1 - e^{-(r+\phi+\frac{1}{m})(t_b-t_a)}\right) +$$

$$(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi} - \frac{c+\phi\theta R + \frac{1}{m}\bar{V}^b}{r+\phi+\frac{1}{m}}\right)e^{-(r+\phi)(t_g-t_a)}\left(1 - e^{-\frac{1}{m}(t_b-t_a)}\right) = \frac{\psi}{\lambda}. \quad \text{(A.26)}$$

Combining equations (A.26) and (A.23), together with the incentive compatibility condition determining $t_g - t_b$ in equation (30), we obtain three equations to characterize the thresholds: $\{t_a, t_b, t_g\}$

$$\bar{q} = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-\frac{1}{m}(t_b-t_a)}} q_{t_a} \quad \text{(A.27a)}$$

$$\frac{\psi}{\lambda} = \frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - PV_r^b\right)\left(1 - e^{-(r+\phi+\frac{1}{m})(t_b-t_a)}\right)$$

$$+ (1-q_0)\left(PV_r^b - \frac{c+\phi\theta R + \frac{1}{m}\bar{V}^b}{r+\phi+\frac{1}{m}}\right)e^{-(r+\phi)(t_g-t_a)}\left(1 - e^{-\frac{1}{m}(t_b-t_a)}\right) \quad \text{(A.27b)}$$

$$t_g = t_b + \frac{1}{r+\phi}\log\left(\frac{V_{t_g}^b - PV_r^b}{L - PV_r^b}\right). \quad \text{(A.27c)}$$

The final step for the equilibrium to find conditions for an equilibrium with learning (i.e. $t_a > 0$). Let $\bar{t}_a$ be the threshold the first time $q_t = \bar{q}$ in the benchmark model in which $\psi = 0$, which is the same as if $t_a = t_b$. On the other hand, if $t_a = \underline{t}_a \equiv \frac{1}{\lambda}\log\left(\frac{\bar{q}}{1-\bar{q}}\frac{1-q_0}{q_0}\right)$ we have that $\inf\{t > t_a : q_t = \bar{q}\} = \infty$. We have already shown that if $t_a = \bar{t}_a$, then $\Gamma_{t_a} = 0 < \psi/\lambda$. Thus, in any equilibrium $t_a < \bar{t}_a$. Hence, it is sufficient to show that if

A20

$t_a = \underline{t}_a$, then $\Gamma_{t_a} > \psi/\lambda$. We have that $\lim_{t_a \to \underline{t}_a} t_g = \lim_{t_a \to \underline{t}_a} t_b = \infty$, which means that

$$\frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}} \left(L - PV_r^b\right) \left(1 - e^{-(r+\phi+\frac{1}{m})(t_g-t_a)}\right)$$

$$+(1-q_0)\left(PV_r^b - \frac{c+\phi\theta R + \frac{1}{m}\bar{V}^b}{r+\phi+\frac{1}{m}}\right) e^{-(r+\phi)(t_g-t_a)} \left(1 - e^{-\frac{1}{m}(t_g-t_a)}\right) \to \frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - PV_r^b\right)$$

Hence, there is $t_a \in (\underline{t}_a, \bar{t}_a)$ such that $\lambda\Gamma_t = \psi$ if and only if

$$\frac{\psi}{\lambda} < \frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - PV_r^b\right).$$

Finally, we can verify that if the previous condition is not satisfied, then there is no learning in equilibrium. Suppose that the firm never learns and never goes to the market. In this case, we have the value of the project being

$$V^u = PV_r^u = \frac{c + \phi(q_0 + (1-q_0)\theta)R}{r+\phi},$$

so that the value of bank at loan rate y is

$$B^u = \frac{yF + \frac{1}{m}V^u + \phi(q_0 + (1-q_0)\theta)R}{r+\phi+\frac{1}{m}}.$$

Next, suppose that the bank becomes informed (which only occurs off the equilibrium path). In this case, for any loan rate y, the continuation value fo the good and bad types are

$$B^b = \frac{y + \frac{1}{m}L + \phi\theta F}{r+\phi+\frac{1}{m}}$$

$$B^g = \frac{y + \frac{1}{m}V^g + \phi F}{r+\phi+\frac{1}{m}},$$

where

$$V^g = PV_r^g.$$

Combining the previous expressions, we get that

$$q_0 B^g + (1-q_0)B^b - B^u = \frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - PV_r^b\right),$$

which means that not learning is optimal if

$$\frac{\psi}{\lambda} \geq \frac{\frac{1}{m}(1-q_0)}{r+\phi+\frac{1}{m}}\left(L - PV_r^b\right)$$

A21

Constrained Case Next, we consider the case in which the entrepreneur is financially constrained. By similar arguments as the ones in the unconstrained case, the bank never learns after time t_b. Hence, we can restrict attention to the case in which $t_a < t_b$. If $y_t = c$, the value function of the bank satisfies the following HJB equation on (t_a, t_b)

$$(r + \phi) B_t^u = \dot{B}_t^u + c + \phi[q_0 + (1 - q_0)\theta]F \tag{A.28a}$$

$$(r + \phi) B_t^g = \dot{B}_t^g + c + \phi F \tag{A.28b}$$

$$\left(r + \phi + \frac{1}{m}\right) B_t^b = \dot{B}_t^b + c + \phi\theta F + \frac{1}{m}L. \tag{A.28c}$$

It follows that Γ_t satisfies the following ODE

$$(r + \phi)\Gamma_t = \dot{\Gamma}_t + \frac{1}{m}(1 - q_0)(L - B_t^b)$$

$$= \dot{\Gamma}_t - \frac{1}{m}(1 - q_0)\frac{c + \phi\theta F - (r + \phi)L}{r + \phi + \frac{1}{m}}\left(1 - e^{-(r+\phi+\frac{1}{m})(t_b-t)}\right) \tag{A.29}$$

Solving equation (A.29) backward in time starting at t_b we get

$$\Gamma_t = \int_t^{t_b} e^{-(r+\phi)(s-t)}\frac{1}{m}(1 - q_0)\frac{(r+\phi)L - c - \phi\theta F}{r + \phi + \frac{1}{m}}\left(1 - e^{-(r+\phi+\frac{1}{m})(t_b-s)}\right) ds$$

$$= \frac{\frac{1}{m}(1 - q_0)}{r + \phi}\frac{(r+\phi)L - c - \phi\theta F}{r + \phi + \frac{1}{m}}\left(1 - e^{-(r+\phi)(t_b-t)}\right)$$

$$- (1 - q_0)\frac{(r+\phi)L - c - \phi\theta F}{r + \phi + \frac{1}{m}}e^{-(r+\phi)(t_b-t)}\left(1 - e^{-\frac{1}{m}(t_b-t)}\right)$$

Substituting in the optimality condition $\lambda\Gamma_{t_a} = \psi$ we get

$$\frac{\psi}{\lambda} = (1 - q_0)\frac{(r+\phi)L - c - \phi\theta F}{r + \phi + \frac{1}{m}}\left(\frac{\frac{1}{m}}{r + \phi}\left(1 - e^{-(r+\phi)(t_b-t_a)}\right)\right.$$

$$\left. - e^{-(r+\phi)(t_b-t_a)}\left(1 - e^{-\frac{1}{m}(t_b-t_a)}\right)\right) \tag{A.30}$$

As before, when t_a converges to $\frac{1}{\lambda}\log\left(\frac{\bar{q}}{1-\bar{q}}\frac{1-q_0}{q_0}\right)$ the threshold t_b converges to infinity. Thus, as in the proof for the unconstrained case, a solution exists if and only if

$$\frac{\psi}{\lambda} < \frac{\frac{1}{m}(1 - q_0)}{r + \phi + \frac{1}{m}}\left(L - \frac{c + \phi\theta F}{r + \phi}\right) \tag{A.31}$$

A22

Finally, we need to verify that Γ_t single crosses ψ/λ. Because B_t^b is increasing on $(0, t_b)$, it follows that Γ_t is quasi-convex on (t_a, t_b). On $(0, t_a)$, Γ_t satisfies

$$(r + \phi + \lambda)\Gamma_t = \dot{\Gamma}_t - \frac{1}{m}(1 - q_0)(B_t^b - L) + \psi.$$

Again, we can verify that Γ_t is quasi-convex on $(0, t_a)$. Moreover, as $\dot{\Gamma}_{t_a-} = \dot{\Gamma}_{t_a+}$, we can conclude that Γ_t is quasi-convex on $[0, t_b]$. From here we can conclude that if $\Gamma_{t_a} = \psi/\lambda > 0$ and $\Gamma_{t_b} = 0$, then it must be the case that $\Gamma_t \leq \psi/\lambda$ on (t_a, t_b). On the other hand, if there is $\tilde{t}_a < t_a$ such that $\Gamma_{\tilde{t}_a} < \psi/\lambda$, then it must be that case that Γ_t has local maximum on $[0, t_b]$. However, this cannot be the case as Γ_t is quasi-convex. We can conclude that $\Gamma_t \geq \psi/\lambda$ for all $t < t_a$.

The rest of the equilibrium is determined as in the case with exogenous learning. The threshold t_a must be such $q_{t_b} = \bar{q}$, where q_t is given by equation (A.23), while $t_g - t_b$ is given by equation (31). Summarizing, the equilibrium thresholds are given by the solution to equations

$$\bar{q} = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-\frac{1}{m}(t_b - t_a)}} q_{t_a} \tag{A.32a}$$

$$\frac{\psi}{\lambda} = (1 - q_0)\frac{(r + \phi)L - c - \phi\theta F}{r + \phi + \frac{1}{m}}\left(\frac{\frac{1}{m}}{r + \phi}\left(1 - e^{-(r+\phi)(t_b - t_a)}\right)\right.$$

$$\left. - e^{-(r+\phi)(t_b - t_a)}\left(1 - e^{-\frac{1}{m}(t_b - t_a)}\right)\right) \tag{A.32b}$$

$$t_g = t_b + \frac{1}{r + \phi}\log\left(\frac{\frac{r+\phi}{r+\phi+\frac{1}{m}}(c + \phi\theta F + \frac{1}{m}F) - (c + \phi\theta F)}{(r + \phi)L - (c + \phi\theta F)}\right). \tag{A.32c}$$

A.1.7 Proof of Proposition 7

Proof. With some abuse of notation, let V_n^i be the continuation value at the n-th rollover time (i.e. at time nm. In that case, we have

$$V_{n_g-1}^g = \int_0^m e^{-(r+\phi)s}(c + \phi R)\,ds + e^{-(r+\phi)m}V_{n_g}^g.$$

To simplify notation, let

$$\nu^g \equiv \int_0^m e^{-(r+\phi)s}(c + \phi R)\,ds$$

be the flow payoff between two rollover events, which is time independent. We can rewrite $V_{n_g-1}^g = \nu^g + e^{-rm}V_{n_g}^g$ which has to be greater or equal than $\bar{V}$. On the other hand, the *No Deals* condition also requires $V_{n_g-1}^g \geq \bar{V}$. Combining these two conditions we find that

$$\nu^g + e^{-rm}V_{n_g}^g = V_{n_g-1}^g \geq \bar{V} \geq \nu^g + e^{-rm}\bar{V}.$$

This means that $q_{n_g m} = \bar{q}$. Because q_t is constant after the n_b-th rollover date, it has to be the case that $q_{n_b m} = \bar{q}$. Let

$$\hat{q}(n) = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda nm}}$$

be the beliefs after n rolled over period given the market conjecture that a bad loan is not rolled over, and let $\hat{n} \equiv \min\{n : \hat{q}(n) \geq \bar{q}\}$. If $\hat{n}m$ is an integer, then $n_b = \bar{n}$ and $\alpha_b = 0$. However, if $\hat{n}m$ is not an integer then $n_b = \hat{n} - 1$, and fraction α_b of the bad projects is liquidated so that the belief conditional on a rollover at n_b is $\bar{q}$. In this case, α_b satisfies

$$\bar{q} = \frac{(1 - \alpha_b)\left(1 - q_{(n_b - 1)m}\right)}{q_{(n_b - 1)m} + (1 - \alpha_b)\left(1 - q_{(n_b - 1)m}\right)}. \tag{A.33}$$

The definition of $\hat{n}$ together with equation (A.33) uniquely determine n_b and α_b. It is only left to determine n_g and α_g. We do this by turning our attention to the bad type incentive compatibility constraint. The bad type has to be indifferent between liquidating and continue rolling over after n_b. Let $\hat{V}(n', n_b)$ be the payoff if the bad type rolls over until period n' and receives a payoff $\bar{V}^b$, which is given by:

$$\hat{V}^b(n', n_b) \equiv \int_0^{(n' - n_b)m} e^{-(r + \phi)s} \left(c + \phi R\right) ds + e^{-(r + \phi)(n' - n_b)m} \bar{V}^b$$

For fixed n_b, the function $\hat{V}(n', n_b)$ is increasing in n'. Let's define $\tilde{n} \equiv \max\{n' : \hat{V}^b(n', n_b) \geq L\}$. If $\hat{V}^b(\tilde{n}, n_b) = L$, then we can set $n_g = \tilde{n}$ and $\alpha_g = 1$. Otherwise, we have that $\hat{V}^b(\tilde{n}, n_b) > L$ and $\hat{V}^b(\tilde{n} + 1, n_b) < L$ so a mixed strategy is required. In particular, if we set $n_g = \tilde{n}$ and choose α_g such that

$$\alpha_g \hat{V}^b(n_g, n_b) + (1 - \alpha_g)\hat{V}^b(n_g + 1, n_b) = L,$$

we get that $V_{n_b}^b = L$ so the low type is indifferent between liquidating and rolling over. Finally, because the market investors make zero profit, they are willing to mix between the two debt prices at the rollover period n_g. Moreover, we have the following corollary $\quad\square$

A.2 First-best outcome

The first-best outcome is achieved if the private news could be publicly observable. Assumption 1 leads to the result that any (known) good project will immediately receive financing from the market, whereas a (known) bad project will be liquidated upon news arrival. Proposition 8 describes the time-0 financing decision of an unknown project with belief q_0.

Proposition 8. *There exists a unique pair $\left\{\underline{\mu}_{FB}, \bar{\mu}_{FB}\right\}$ such that in the first-best bench-*

mark,

1. If $q_0 \leq \underline{\mu}_{FB}$, the unknown project is liquidated at $t = 0$.

2. If $q_0 \in \left(\underline{\mu}_{FB}, \bar{\mu}_{FB}\right)$, the unknown project is financed with the bank at $t = 0$.

3. If $q_0 \geq \bar{\mu}_{FB}$, the unknown project is financed with the market at $t = 0$.

In the case that $q_0 \in \left(\underline{\mu}_{FB}, \bar{\mu}_{FB}\right)$, the project receives financing from the bank at $t = 0$. Over time, either news or the premature failure event may arrive, upon which the project receives immediate market financing following good news and gets immediate liquidation following bad news. If neither the news nor the premature failure event arrives, the belief on the project follows (3). In this case, the project will also be financed with the market once μ_t reaches $\bar{\mu}_{FB}$.

Proof. If news is publicly observable, then the belief follows (3), and there is a one-to-one mapping between time t and the public belief μ_t in the case without any news. Therefore, we prove this proposition in the time domain.

If a project is financed with a bank, its continuation value satisfies the following HJB equation

$$(r + \phi + \lambda + (1 - \mu_t)\eta) V_{FB}^u(t) = \dot{V}_{FB}^u(t) + c + \phi \left[\mu_t + (1 - \mu_t)\theta\right] R$$
$$+ \lambda \left[\mu_t \frac{c + \phi R}{\delta + \phi} + (1 - \mu_t) L\right].$$

Two boundary conditions characterize $\bar{\mu}_{FB} = \mu_{t=t_{FB}}$, which is to switch to market financing. First, the value matching condition holds so that

$$V_{FB}^u(t_{FB}) = \mu_{t_{FB}} PV_\delta^g + (1 - \mu_{t_{FB}}) PV_\delta^b. \tag{A.34}$$

Second, the smooth-pasting condition holds as t_{FB} is optimally chosen

$$\dot{V}_{FB}^u(t_{FB}) = \left(PV_\delta^g - PV_\delta^b\right) \eta \mu_{t_{FB}} (1 - \mu_{t_{FB}}). \tag{A.35}$$

These two conditions pin down the unique solution

$$\bar{\mu}_{FB} = \frac{c + \phi\theta R + \lambda L - (r + \phi + \lambda + \eta) PV_\delta^b}{(r + \phi) PV_\delta^g - (r + \phi + \lambda + \eta) PV_\delta^b - \phi(1 - \theta)R + \lambda L}.$$

Assumption 4 guarantees that $\bar{\mu}_{FB} > 0$. Otherwise, bank financing is never used. The condition $PV_\delta^g > \frac{c + \phi R}{r + \phi} \equiv PV_r^g$ guarantees that $\bar{\mu}_{FB} < 1$. Finally, $V_{FB}^u(\underline{t}_{FB}) = L$ pins down the unique solution $\underline{t}_{FB}$ at which $\mu_{t=\underline{t}_{FB}} = \underline{\mu}_{FB}$.

For the remainder of the proof, we verify that a policy that seeks market financing at t_{FB} is optimal. In optimal stopping problems, the value function has to satisfy the

following variational inequality

$$\max \left\{ \begin{array}{l} \dot{V}^u_{FB}(t) + c + \phi \left[\mu_t + (1 - \mu_t) \, \theta \right] R + \lambda \left[\mu_t PV^g_\delta + (1 - \mu_t) \, L \right] \\ \quad - (r + \phi + \lambda + (1 - \mu_t)\eta) \, V^u_{FB}(t), \mu_t PV^g_\delta + (1 - \mu_t) \, PV^b_\delta - V^u_{FB}(t) \end{array} \right\} = 0$$

In particular, we need to verify for $t < t_{FB}$,

$$(r + \phi + \lambda + (1 - \mu_t)\eta) \, V^u_{FB}(t) = \dot{V}^u_{FB}(t) + c + \phi \left[\mu_t + (1 - \mu_t) \, \theta \right] R$$
$$+ \lambda \left[\mu_t PV^g_\delta + (1 - \mu_t) \, L \right]$$
$$V^u_{FB}(t) \geq \mu_t PV^g_\delta + (1 - \mu_t) \, PV^b_\delta,$$

while for $t > t_{FB}$

$$(r + \phi + \lambda + (1 - \mu_t)\eta) \, V^u_{FB}(t) \geq \dot{V}^u_{FB}(t) + c + \phi \left[\mu_t + (1 - \mu_t) \, \theta \right] R$$
$$+ \lambda \left[\mu_t PV^g_\delta + (1 - \mu_t) \, L \right]$$
$$V^u_{FB}(t) = \mu_t PV^g_\delta + (1 - \mu_t) \, PV^b_\delta.$$

Step 1: Optimality for $t > t_{FB}$. Let's define

$$G(t) \equiv \dot{\mu}_t (PV^g_\delta - PV^b_\delta) + c + \phi \left[\mu_t + (1 - \mu_t) \, \theta \right] R + \lambda \left[\mu_t PV^g_\delta + (1 - \mu_t) \, L \right]$$
$$- (r + \phi + \lambda + (1 - \mu_t)\eta) \left(\mu_t PV^g_\delta + (1 - \mu_t) \, PV^b_\delta \right)$$

Smooth pasting plus value matching imply that $G(t_{FB}) = 0$, and we need to verify that $G(t) \leq 0$ for $t > t_{FB}$. Thus, it is enough to show that $G(t) = 0 \Rightarrow \dot{G}(t) \leq 0$. Substituting

$$\dot{\mu}_t = \eta \mu_t (1 - \mu_t)$$

in $G(t)$ we get

$$G(t) = c + \phi \left[\mu_t + (1 - \mu_t) \, \theta \right] R + \lambda \left[\mu_t \frac{c + \phi R}{\delta + \phi} + (1 - \mu_t) \, L \right]$$
$$- (r + \phi + \lambda) \left(\mu_t PV^g_\delta + (1 - \mu_t) \, PV^b_\delta \right) - \eta(1 - \mu_t) PV^b_\delta.$$

Differentiating $G(t)$ we get

$$\dot{G}(t) = \left[\phi(1 - \theta)R + \lambda \left[\frac{c + \phi R}{\delta + \phi} - L \right] - (r + \phi + \lambda) \left(PV^g_\delta - PV^b_\delta \right) + \eta PV^b_\delta \right] \dot{\mu}_t,$$

and given that $\dot{\mu}_t > 0$, the inequality $\dot{G}(t) \leq 0$ reduces to the following condition

$$\phi(1-\theta)R + \lambda \left[\frac{c + \phi R}{\delta + \phi} - L \right] - (r + \phi + \lambda)\left(PV_\delta^g - PV_\delta^b\right) + \eta PV_\delta^b \leq 0$$

that we can rewrite as

$$0 \leq (r + \phi) PV_\delta^g - (r + \phi + \lambda + \eta)PV_\delta^b - \phi(1-\theta)R + \lambda L,$$

which holds following Assumption 4.

Step 2: Optimality for $t < t_{FB}$. Next, we need to show that for $t < t_{FB}$

$$V_{FB}^u(t) \geq \mu_t PV_\delta^g + (1 - \mu_t) PV_\delta^b.$$

By construction
$$V_{FB}^u(t_{FB}) = \mu_{t_{FB}} PV_\delta^g + (1 - \mu_{t_{FB}}) PV_\delta^b;$$

hence, it is enough to show that

$$V_{FB}^u(t) - \mu_t PV_\delta^g - (1 - \mu_t) PV_\delta^b$$

single crosses zero from above for $t < t_{FB}$. Let $H(t) \equiv V_{FB}^u(t) - \mu_t PV_\delta^g - (1 - \mu_t) PV_\delta^b$ we get

$$\dot{H}_t = \dot{V}_{FB}^u(t) - \dot{\mu}_t \left(PV_\delta^g - PV_\delta^b\right).$$

Substituting V_{FB}^u and $\dot{\mu}_t$ we get

$$\dot{H}_t = (r + \phi) V_{FB}^u(t) - c - \phi \left[\mu_t + (1 - \mu_t)\theta\right] R - \lambda \left[\mu_t PV_\delta^g + (1 - \mu_t) L - V_{FB}^u(t)\right]$$
$$- \eta(1 - \mu_t) \left[\mu_t \left(PV_\delta^g - PV_\delta^b\right) - V_{FB}^u(t)\right]$$

Using the definition of H_t we get

$$\dot{H}_t = (r + \phi + \lambda + \eta(1 - \mu_t)) H_t + (r + \phi) \left(\mu_t PV_\delta^g + (1 - \mu_t) PV_\delta^b\right)$$
$$- c - \phi \left[\mu_t + (1 - \mu_t)\theta\right] R - \lambda (1 - \mu_t) (L - PV_\delta^b) + \eta(1 - \mu_t)PV_\delta^b$$

Thus,

$$\dot{H}_t \Big|_{H_t=0} = (r + \phi) \left(\mu_t PV_\delta^g + (1 - \mu_t) PV_\delta^b\right)$$
$$- c - \phi \left[\mu_t + (1 - \mu_t)\theta\right] R - \lambda (1 - \mu_t) (L - PV_\delta^b) + \eta(1 - \mu_t)PV_\delta^b$$

Rearranging terms we get that $\dot{H}_t\big|_{H_t=0} < 0$ if and only if

$$\mu_t < \bar{\mu}_{FB}.$$

Moreover, at time t_{FB} we have $\dot{H}_t = 0$. The only step left then is to verify that $\ddot{H}_{t^{FB}} > 0$ so we can conclude that for any $t < t_{FB}$ we have $\dot{H}_t\big|_{H_t=0} < 0$, which then implies that $H_t > 0$ for all $t < t_{FB}$. Taking the second derivative we get

$$\ddot{H}_t\big|_{H_t=0,\dot{H}_t=0} = \left[(r+\phi)PV_\delta^g - (r+\phi+\lambda+\eta)PV_\delta^b - \phi(1-\theta)R + \lambda L\right]\dot{\mu}_t > 0$$

$\square$

A.3 Lending Relationship Breakups

In this section, we offer a comprehensive analysis of the model when the lending relationship may break up exogenously. In this case, there is always some probability that good types and informed seek market financing, so the price of the bond is higher than D^b on the equilibrium path. Because there is always some market financing on the equilibrium path, there is no need to specify off equilibrium believes in this case. The equilibrium we construct is characterized by three thresholds, $\{t_\ell, t_b, t_g\}$. Because there is always some good types financing with the market, the bad-type's strategy must involve the use of mixed-strategies in equilibrium. For $t \in [0, t_\ell]$, the project is immediately liquidate upon learning that it is a bad type. For, $t \in [t_\ell, t_b]$, upon obtaining negative information, the project is liquidated with probability less than one and otherwise the firm obtains market financing. For $t \in [t_b, t_g]$, the good and uninformed types will remain in the lending relationship unless it breaks up, the bad type mixes between continuing with the bank or financing with the market.

Equilibrium Beliefs

The first step in the construction of the equilibrium is to specify the beliefs given our conjectured equilibrium strategies. For any $t \in [0, t_b]$, conditional on the lending relationship continuing, the market knows for sure that the entrepreneur is not bad, and that it can only be a good type or an uninformed (that is, $\pi_t^b = 0$, $\forall t \in [0, t_b]$). After an entrepreneur knows her type, she is indifferent between liquidating or refinancing with the market. Let ℓ_t be the probability of liquidating the project upon learning that it is bad at time $t \in [0, t_b]$, so that $1 - \ell_t$ is the probability that the bad type chooses to refinance with the market when they learn that the project is bad. Because different types finance with the market with different probability, the probability of being good conditional on market financing is different that the probability of being good conditional

A28

on continuing with the bank. Let q_{t+} be the average quality of borrower conditional on refinancing with the market. One can easily verify using Baye's rule that

$$q_{t+} = \frac{\chi}{\chi + \lambda\left(1 - q_t\right)\left(1 - \ell_t\right)} q_t.$$

Next, let us turn to the belief on $t \in [t_b, t_g]$, and let $\alpha_t \in [0, \infty)$ be the intensity at which the bad type seeks market financing at time $t \in (t_b, t_g)$ if the lending relationship doesn't break up exogenously. On $t \in (t_b, t_g)$, conditional on staying with the bank, beliefs evolve according to

$$\dot{\pi}_t^u = -\lambda\pi_t^u + \alpha_t\pi_t^u\pi_t^b$$
$$\dot{\pi}_t^g = \lambda\pi_t^u q_0 + \alpha_t\pi_t^g\pi_t^b$$
$$\dot{\pi}_t^b = \lambda\pi_t^u\left(1 - q_0\right) - \alpha_t\pi_t^b\left(1 - \pi_t^b\right).$$
$$\dot{q}_t = \dot{\pi}_t^g + q_0\dot{\pi}_t^u = \alpha_t\pi_t^b q_t.$$

To find, the beliefs conditional on market financing, $\left\{\pi_{t+}^u, \pi_{t+}^g, \pi_{t+}^b, q_{t+}\right\}$, notice that the good and the uninformed type only refinance with the market if the lending relationship breaks up exogenously, which happens at a rate χ, whereas the bad type also refinance voluntarily so its financing rate is $\chi + \alpha_t$. This implies that

$$\pi_{t+}^u = \frac{\chi\pi_t^u}{\chi + \alpha_t\pi_t^b}$$
$$\pi_{t+}^g = \frac{\chi\pi_t^g}{\chi + \alpha_t\pi_t^b}$$
$$\pi_{t+}^b = \frac{(\chi + \alpha_t)\pi_t^b}{\chi + \alpha_t\pi_t^b}$$
$$q_{t+} = \pi_{t+}^g + q_0\pi_{t+}^u = \frac{\chi\pi_t^g}{\chi + \alpha_t\pi_t^b} + \frac{\chi\pi_t^u q_0}{\chi + \alpha_t\pi_t^b} = \frac{\chi}{\chi + \alpha_t\pi_t^b} q_t.$$

As time gets close to t_g, the bad type has no incentives to finance with the market prematurely, which means that $\alpha_{t_g} = 0$, so $q_{t_g^+} = q_{t_g} = \bar{q}$. It is convenient to specify the equilibrium in terms of the total financing rate of bad types $\gamma_t \equiv \alpha_t\pi_t^b$ instead of the specifying the equilibrium in terms of α_t. The price of the bond at time t given γ_t and q_t is

$$\bar{D}_t = D^b + (D^g - D^b)\frac{\chi}{\chi + \gamma_t} q_t = D^b + (D^g - D^b)\frac{\chi}{\chi + \gamma_t} q_t$$

at time t.

Having specified the market beliefs and the price of bonds, we can look at the optimality of the bank's financing strategy.

Bank's Problem, Value Function, and Equilibrium

As we did before, we solve for the equilibrium working backward, and starting the extend and pretend region $t \in [t_b, t_g]$. In this region, the continuation value V_t^i satisfies the following HJB equation

$$(r + \phi + \chi)V_t^g = \dot{V}_t^g + c + \phi R + \chi \bar{V}_t^g$$
$$(r + \phi + \chi)V_t^b = \dot{V}_t^b + c + \phi \theta R + \chi \bar{V}_t^b + \alpha_t(\bar{V}_t^b - V_t^b)$$
$$(r + \phi + \lambda + \chi)V_t^u = \dot{V}_t^u + c + \phi[q_0 + (1 - q_0)\theta]R + \lambda[q_0 V_t^g + (1 - q_0)V_t^u] + \chi \bar{V}_t^u.$$

In equilibrium, the bad type is mixing between rolling over bank loans and financing with the market (that is $\alpha_t \in (0, \infty)$), so the indifference condition $V_t^b = \bar{V}_t^b$, must be satisfied for all $t \in [t_b, t_g]$. Substituting this indifference condition in the HJB equation of the bad type we get

$$(r + \phi)\bar{V}_t^b = \dot{\bar{V}}_t^b + c + \phi \theta R$$
$$\bar{V}_t^b = \bar{D}_t + \frac{\phi \theta (R - F)}{r + \phi}.$$

As in the model without exogenous breaks, the threshold belief $\bar{q}$ can be determined using the smooth pasting and value matching conditions at t_g, which gives

$$\bar{q} = \frac{\frac{c + \phi F}{r + \phi} - D^b}{D^g - D^b}.$$

Once again, we can solve for the length of the zombie lending region $t_g - t_b$ using the indifference condition for liquidation $V_{t_b}^b = L$. Finally, the value of γ_t must be such the bank is indifferent to finance with the market and rolling over the loan:

$$V_t^b = \left(1 - e^{-(r+\phi)(t_g - t)}\right) PV_r^b + e^{-(r+\phi)(t_g - t)} \bar{V}_{t_g}^b$$
$$= D^b + (D^g - D^b)\frac{\chi}{\chi + \gamma_t}q_t + \frac{\phi \theta (R - F)}{r + \phi},$$

where the payoff that the bad type gets if it finance with the market at time t_g is given by

$$\bar{V}_{t_g}^b = D^b + \bar{q}\left(D^g - D^b\right) + \frac{\phi \theta (R - F)}{r + \phi}.$$

Having specified the equilibrium in the zombie lending region, we can now move to the liquidation region $[0, t_b]$. In this region, the value functions satisfy

$$(r + \phi + \chi)V_t^g = \dot{V}_t^g + c + \phi R + \chi \bar{V}_t^g$$
$$V_t^b = L$$
$$(r + \phi + \lambda + \chi)V_t^u = \dot{V}_t^u + c + \phi[q_0 + (1 - q_0)\theta]R + \lambda[q_0 V_t^g + (1 - q_0)L] + \chi \bar{V}_t^u.$$

Since the bad type is indifferent between liquidating and financing with the market on $t \in (0, t_b)$, it must be the case that $\bar{V}_t^b = L$, which means that implies

$$L = D^b + (D^g - D^b)q_{t+} + \frac{\phi\theta(R - F)}{r + \phi}$$
$$= D^b + (D^g - D^b)\frac{\chi}{\chi + \lambda(1 - q_t)(1 - \ell_t)}q_t + \frac{\phi\theta(R - F)}{r + \phi}.$$

Letting $\underline{q} \equiv \frac{L - D^b - \frac{\phi\theta(R-F)}{r+\phi}}{D^g - D^b}$, we can write the indifference condition as

$$\frac{\chi}{\chi + \lambda(1 - q_t)(1 - \ell_t)}q_t = \underline{q} \Rightarrow 1 - \ell_t = \frac{\chi}{\lambda(1 - q_t)}\left(\frac{q_t}{\underline{q}} - 1\right),$$

where

$$q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}}, \quad \forall t \in [0, t_b].$$

The solution to ℓ_t is interior only if $q_t > \underline{q}$. If this condition is not satisfied, the price of bonds is so low that the bad type prefers to liquidate even if he can fully pool with the good and uninformed types. In this case, the bad type liquidates with probability 1. Since q_t increases over time, if $q_0 < \underline{q}$, then there is $0 < t_\ell < t_b$ such that $\ell_t = 1$ for all $t < t_\ell$, and $\ell_t \in (0, 1)$ for $t \in (t_\ell, t_b)$.

Finally, if $q_t < \frac{(\chi + \lambda)\underline{q}}{\chi + \lambda \underline{q}}$, the bad type has a strict preference for market financing. In this case, the equilibrium has only two regions, and it is fully characterized by the threshold t_g. For any $t \in [0, t_g]$, the bad type is indifferent between financing with the market immediately and waiting until time t_g.

We can summarize the previous discussion as follows: The equilibrium is characterized by $\{t_\ell, t_b, t_g, \gamma_t, \ell_t\}$, where

- The thresholds t_ℓ, t_b, t_g satisfy

$$\frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t_\ell}} = \underline{q}$$
$$\frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t_b}} = \bar{q},$$

and

$$L = \left(1 - e^{-(r+\phi)(t_g - t_b)}\right) PV_r^b + e^{-(r+\phi)(t_g - t_b)} \bar{V}_{t_g}^b.$$

- The liquidation probability on $[0, t_b]$ is

$$\ell_t = \min\left\{1 - \frac{\chi}{\lambda(1-q_t)}\left(\frac{q_t}{\underline{q}} - 1\right), 1\right\}$$

- The market financing rate is given by

$$\frac{\chi}{\chi + \gamma_t} q_t = \frac{\left(1 - e^{-(r+\phi)(t_g - t)}\right) PV_r^b + e^{-(r+\phi)(t_g - t)} \bar{V}_{t_g}^b - D^b \frac{\phi\theta(R-F)}{r+\phi}}{D^g - D^b}$$

One final remark. Notice that in the limit when $\chi \to 0$, the equilibrium converges to the one in the model without exogenous breaks (that is, $\gamma_t \to 0$ and $\ell_t \to 1$). Interestingly, the off-path price of bonds is not given by D^b, but is rather then one that makes the bad type indifferent. We can interpret this limit as a refinement of the equilibrium in the spirit of trembling hand perfection.