A Dynamic Theory of Learning and Relationship Lending

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Abstract

This paper studies the dynamics of relationship-bank lending and market financing. Bank lending facilitates private learning over time but introduces an information monopoly. We show the entrepreneur starts with bank financing and subsequently refinances with the market. Asymmetric information is developed over time, while the entrepreneur also accumulates reputation. For sufficiently reputable entrepreneurs, banks will extend loans even after bad news, for the prospect of future market financing. Moreover, this incentive to pretend gets mitigated when the entrepreneur faces financial constraints. We further endogenize learning as the bank's costly decision and show banks stop learning when entrepreneurs become sufficiently reputable.

Keywords: private learning, experimentation, reputation, relationship banking, information monopoly, debt rollover, extend and pretend, adverse selection, dynamic games.

1 Introduction

How do lending relationships evolve over time? How do firms choose *dynamically* between bank and market financing? How does a firm's reputation interact with its financial constraint in determining its access to finance? Why do banks sometimes roll over loans that are known to be insolvent? To answer these questions, we introduce a dynamic framework in the context of relationship lending.

Researchers have widely documented that bank loans contain important information about borrowers that is not available to market-based lenders (Addoum and Murfin, 2017; James, 1987; Gustafson et al., 2017). Moreover, Lummer and McConnell (1989) suggest such information is not produced upon a bank's first contact with a borrower, but, instead, through repeated interactions during prolonged lending relationships that involve substantive screening and monitoring. On the other hand, Rajan (1992) shows learning provides an information advantage to the relationship bank and thus increases the information-monopoly cost so that, ultimately, the borrower may switch to lenders in the financial market. When should borrowers switch from relationship lending to market financing? How do entrepreneurs balance the tradeoff between learning and the information-monopoly cost? How does loan maturity affect these decisions?

To answer these questions, we introduce private learning into a dynamic model of relationship lending. Specifically, we model an entrepreneur investing in a long-term, illiquid project whose quality is either good or bad. Only a good project has positive net present value (NPV) and should be financed. A bad project should be liquidated immediately. Initially, the quality of the project is unknown to everyone, including the entrepreneur herself. She can raise funding from either the competitive financial market or a bank that will develop into a relationship. Market financing takes the form of arm's-length debt so that lenders only need to break even given their beliefs about the project's quality. Under market financing, no information is ever produced; therefore, the maturity of the market debt is irrelevant. By contrast, if the entrepreneur borrows from a bank, screening and monitoring will produce "news" about the project's quality. We model news arrival as a Poisson event and assume it is only observed by the entrepreneur and the bank. In other words, the bank and the entrepreneur learn *privately* about the project's quality as time goes by. Meanwhile, all agents, including lenders in the financial market, can observe the time since the initialization of the project, which will turn out to be the state variable.

Given the structure of learning, the bank and the borrower possess one of the three types of private information after time 0: (1) news has arrived and implies the project is good – the informed-good type g; (2) news has arrived and implies the project is bad – the informed-bad

type b; and (3) no news has arrived yet – the uninformed type u. Upon the maturing of the bank loan, the bank and the entrepreneur jointly determine whether to roll it over, to liquidate the project, or to switch to market-based financing. These decisions are modeled as a Nash bargaining problem with the financial constraint that the entrepreneur has no personal wealth.

By solving the model in closed form, we characterize the equilibrium with two thresholds $\{t_g, t_b\}$ in the time since project initialization. Consequently, the equilibrium is characterized into three stages. If the bank loan matures between 0 and t_b, an informed-bad type's project will be liquidated. All other types' matured loans will be rolled over. During this period, the average quality of borrowers who remained with banks drifts up because the informed-bad types get liquidated and exit funding. Equivalently, remaining borrowers gain reputation from the liquidation decisions of others. These liquidation decisions are socially efficient, and therefore we name this stage after *efficient liquidation*. If the bank loan matures between t_b and t_g, however, it will be rolled over irrespective of the quality of the project. In particular, the relationship bank will roll over the loan matured between t_b and t_g even if bad news has arrived. In other words, the bank keeps *extending* the loan to *pretend* no bad news has occurred yet. Clearly, this rollover decision is inefficient. This result on banks' rolling over bad loans can be linked to zombie lending. Finally, after time passes t_g, all entrepreneurs will refinance with the market upon their bank loans maturing – the *market-financing* stage.

The intuitions for these results can be best explained backwards in time. When time elapsed gets sufficiently long, all entrepreneurs will ultimately switch to market financing, driven by the assumption that market-based lenders are competitive and have lower costs of capital. Threshold t_g captures this effect. Now, imagine a scenario in which bad news arrives shortly before t_g. The relationship-bank could liquidate the project, in which case, it receives a fixed payoff. Alternatively, it can roll over the loan and pretend no bad news has arrived yet. Essentially, by hiding bad news *today*, the bank helps the borrower accumulate reputation so that the loan could be sold to the market in the *future*. Such "extending and pretending" incurs relatively low costs because shortly afterwards, these bad loans will be sold to the lenders in the market, and the loss will be shared. On the other hand, if negative news arrives early on, "extending and pretending" is much more costly, due to both large time discounting and the high probability that before t_g, the project may mature and the loss will be entirely born by the relationship bank. In this case, liquidating the project is the more profitable option. The threshold t_b captures the time at which an informed-bad bank is indifferent between liquidating and rolling over. Note a significant gap exists between t_b and t_g so that the extend and pretend stage lasts for a significant period. During this period, the average quality of borrowers stays unchanged. However, this period is necessary to incentivize

informed-bad types to liquidate and exit before t_b, which leads to an improvement in the average quality during the efficient-liquidation stage.

We show the financial constraint mitigates the inefficiency induced by "extend and pretend." Specifically, the financial constraint requires that at each rollover date, the newly negotiated loan rate cannot exceed the rate of the interim cash flow. Equivalently, this constraint limits the transfer that the entrepreneur can make to the bank in the Nash bargaining game, and as a result, it leads to certain scenarios in which a bad project gets liquidated even though the liquidation value falls below the joint surplus if both parties roll it over.

Our interpretation of learning is the process of bank screening and monitoring, which generates useful information on the entrepreneur's business prospect but cannot be shared with others. In the last part of the paper, we endogenize learning as a costly decision and study the bank's tradeoff in choosing its effort in learning. We show the bank ceases to learn during the efficient-liquidation region. One may conclude the information created via learning is useless in a risk-neutral world, because it does not change the expected value of the project: the value of an uninformed project is simply a linear combination of the value of a good one and a bad one. However, once informed, the bank could liquidate a bad project and receive the liquidation value, which exceeds the NPV of the bad project. This option creates convexity to the net benefit of learning and therefore leads to the result that the bank stops to learn after time passes the efficient-liquidation region, because a bad project will no longer be liquidated. Moreover, we show that financial constraints, by expanding the region of efficient liquidation, also promote more bank learning that benefits the entire society.

Related Literature

Our model builds on the literature on private learning, reputation, and experimentation (Daley and Green, 2012; Grenadier et al., 2014; Martel et al., 2018; Che and Hörner, 2017; Akcigit and Liu, 2015; Kremer et al., 2014; Hwang, 2018). We make a novel contribution by modeling both a gradually informed buyer (the relationship bank) and a competitive set of uninformed buyers (market-based lenders), which leads to new and interesting equilibrium dynamics. This innovation is particularly relevant in the setup to study the dynamics of relationship lending and market-based financing.

Our paper is among the first ones that introduce dynamic learning in the context of banking (also see Halac and Kremer (2018) and Hu (2017)). We extend previous work by Diamond (1991b); Rajan (1992); Boot and Thakor (1994, 2000); Parlour and Plantin (2008) among others, by studying the impact of strategic learning and adverse selection on lending

relationships.[1] For example, in Diamond (1991a), borrowers are financed with arm's length debt, and lenders decisions are myopic, implying that lenders will never have incentives to roll over bad loans and increase the borrower's reputation (that is, lenders do not have incentives to extend and pretend). Rajan (1992) studies the tradeoff between relationship-based lending and arm's-length debt, without an explicit role of the borrower's reputation. Our paper is also related to previous work on debt rollover by Brunnermeier and Oehmke (2013); He and Xiong (2012); He and Milbradt (2016). Unlike this literature that has primarily focused on competitive and uninformed financiers, we consider a large creditor (the bank) who gradually obtains an informational advantage over the market through private learning.

Our explanation for extend and pretend differs from existing theories of zombie lending that largely rely on loan officers' career concerns (Rajan, 1994) or on regulatory capital requirements that are triggered if banks write off bad loans (Caballero et al., 2008; Peek and Rosengren, 2005). We offer a complementary explanation that is based on the prospect of future loan sales, and that it is related to previous work on dynamic adverse selection. In particular, our mechanism is related to work by Kremer and Skrzypacz (2007) and Fuchs and Skrzypacz (2015) that study how suspensions and delays in trading can promote efficiency in markets plagued by adverse selection.

2   Model

We consider a continuous-time model with an infinite horizon. An entrepreneur (she) invests in a long-term project with unknown quality. She borrows from either a bank that will develop into a relationship or the competitive financial market. Compared to market financing, bank financing has the advantage of producing valuable information but with the downside of a higher cost of capital and the possibility of information monopoly. Below, we describe the model in detail.

2.1   Project

We consider a long-term project that generates a constant stream of interim cash flows cdt over a period $[t, t+dt]$. The project matures at a random time τ_ϕ, which arrives at an exponential time with intensity $\phi > 0$. Upon maturity, the project produces some random final cash flows $\tilde{R}$, depending on its type. A good (g) project produces cash flows $\tilde{R} = R$ with certainty, whereas a bad (b) project produces $\tilde{R} = R$ with probability $\theta < 1$. With probability $1 - \theta$, a matured bad project fails to produce any final cash flows, that is, $\tilde{R} = 0$.

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[1]Boot and Thakor (1994) study similar issues using a dynamic-contracting approach.

Initially, no agent, including the entrepreneur herself, knows the exact type of the project; all agents share the same public belief that q_0 is the probability of the project's type being good. At any time before the final cash flows are produced, the project can be terminated with a liquidation value $L > 0$. Note the liquidation value is independent of the project's quality, so it should be understood as the liquidation of the physical asset used in production. Let $r > 0$ be the entrepreneur's discount rate; therefore, the fundamental value of the project to the entrepreneur is given by the discounted value of its future cash flows:

$$NPV_r^g = \frac{c + \phi R}{r + \phi}, \quad NPV_r^b = \frac{c + \phi \theta R}{r + \phi}, \quad NPV_r^u = q_0 NPV_r^g + (1 - q_0) NPV_r^b. \quad (1)$$

2.2 Agents and debt financing

The borrower has no initial wealth and needs to borrow through debt contracts.[2] The use of debt contracts can be justified by non-verifiable final cash flows (Townsend, 1979). One can therefore think of the entrepreneur as a manager of a start-up venture who faces financial constraints. We consider two types of debt, offered by banks and market-based lenders, respectively. First, the entrepreneur can take out a loan from a banker (he), who has the same discount rate r. Following Leland (1998), we assume a bank loan lasts for a random period and matures at a random time τ_m, upon the arrival of an independent Poisson event with intensity $m > 0$. The assumption of an exponentially maturing loan simplifies the analysis, because at any time before the loan matures, the expected remaining maturity is always $\frac{1}{m}$. In section 4, we study the case with deterministic maturity and show all the results carry over.

The second type of debt is provided by the market and thus can be considered as public bonds. In particular, we consider a competitive financial market in which lenders have discount rate δ satisfying $\delta \in (0, r)$. As a result, market financing is cheaper than bank financing, so that if the project's type were publicly known, the entrepreneur would strictly prefer to borrow from the market. Regarding (1), let us define the NPV of the project to the market as

$$NPV_\delta^g = \frac{c + \phi R}{\delta + \phi}, \quad NPV_\delta^b = \frac{c + \phi \theta R}{\delta + \phi}, \quad NPV_\delta^u = q_0 NPV_\delta^g + (1 - q_0) NPV_\delta^b. \quad (2)$$

The assumption $\delta < r$ captures the realistic feature that banks have a higher cost of capital than the market, which can be justified by either regulatory requirements or the skin in the game needed to monitor borrowers (see Holmstrom and Tirole (1997) for example and

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[2]We derive the maximum amount that she can raise at the initial date after solving the model, in which case, we can discuss the minimum net worth needed to finance a project with a fixed investment scale.

Schwert (2018) for recent empirical evidence).[3] As will be clear shortly, the maturity of the public debt does not matter, and for simplicity, we assume it only matures with the project.

Both types of debt share the same exogenously-specified face value: $F \in (L, R)$. $F > L$ guarantees debt is risky, whereas $F < R$ captures the wedge between a project's income and its pledgeable income (Holmström and Tirole, 1998).[4] Note we take F as given: our paper intends to study the tradeoff between relationship borrowing and public debt, rather than the optimal leverage. At $t = 0$, the entrepreneur chooses between public debt and a bank loan that will develop into a relationship. Once the bank loan matures at time τ_m, she can still replace it with a public bond. Alternatively, she could roll over the loan with the same bank that may have an information monopoly over the project's quality.[5] In this case, the two parties bargain over y_{τ_m}, the rate of the loan that is prevalent from τ_m until the next rollover date. Specifically, we follow Rajan (1992) and model the decision of y_{τ_m} as a Nash bargaining game, with $(\beta, 1 - \beta)$ being the entrepreneur's and the bank's bargaining power. Due to the financial constraint, the entrepreneur cannot promise any rate y_{τ_m} above c, the maximum level of the interim cash flow. As we show shortly, this constraint limits the size of the transfer that the entrepreneur can make to the bank at rollover dates; therefore, the Nash bargaining outcome is sometimes not the one that maximizes the joint surplus of the two parties.

Because market financing is competitive and market-based lenders have a lower cost of capital, the entrepreneur will always prefer to take as high leverage as possible. Therefore, the coupon payments associated with the public bond are *cdt*.

Remark 1. We have assumed the entrepreneur is only allowed to take one type of debt. In other words, we have ruled out the possibility of the entrepreneur using more sophisticated capital structure to signal her type. See Leland and Pyle (1977) and DeMarzo and Duffie (1999) for these issues.

2.3 Learning and information structure

The quality of the project is initially unknown, with $q_0 \in (0, 1)$ being the belief that it is good. This belief is based on public information and is commonly shared by all agents in the economy. If the entrepreneur finances with the bank, that is, if she takes out a loan, the

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[3]The entire model can be written as one with $r = \delta$, but a transaction cost is associated with rolling over bank loans.

[4]The maximum pledgeable cash flow can be microfounded by some unobservable action taken by the entrepreneur (e.g., cash diversion) shortly before the final cash flows are produced (Tirole, 2010).

[5]We assume without loss of generality that the entrepreneur would never want to switch to a different banks upon the loan's maturity. Intuitively, the market has a lower cost of capital than an outsider bank and the same information structure.

entrepreneur-bank pair can *privately* learn the true quality of the project through "news." News arrives at a random time τ_λ, modeled as an independent Poisson event with intensity $\lambda > 0$. Upon arrival, the news fully reveals the project's true type. In practice, one can think of the news process as information learned during bank screening and monitoring, which includes due diligence and covenant violations. We assume such news can only be observed by the two parties and no committable mechanism is available to share it with third parties, such as credit bureaus and market participants. In this sense, the news can be understood as soft information on project quality (Petersen, 2004). For instance, one can think of this news as the information that banks acquire upon covenant violation, which includes details on the business prospect, collateral quality, and financial soundness of the borrower.

Remark 2. Note that learning and news arrival require joint input from both the entrepreneur and the bank. Therefore, we can think of learning as exploration and understanding of the underlying business prospect, which requires the entrepreneur's experimentation and the bank's previous experience in financing related businesses. In this regard, our model could also be applied to study venture capital firms. Alternatively, we can model learning as a process that solely relies on the entrepreneur's input, whereas the bank simply observes the news content through monitoring. Put differently, even without bank financing, the entrepreneur will still be able to learn news about the quality of her project over time. Our results are identical in this alternative setting.

Although the public-market participants do not observe the news, they can observe t – the project's time since initialization–and therefore make inference about the project's quality. Clearly, they form beliefs based on the time elapsed, as well as the decisions during the (random) rollover events. In the benchmark model, we assume the realization of each rollover event τ_m is unobservable to market participants. In section 4, we relax this assumption and show all results continue to hold qualitatively. Let $i \in \{u, g, b\}$ denote the type of the bank/entrepreneur, where u, g, and b refer to the uninformed, informed-good, and informed-bad types, respectively. We assume any failure to roll over the loan is publicly observable because in this case, either the project will be liquidated or the entrepreneur will seek market financing. In other words, the market cannot observe when the bank loan has been rolled over with the same bank but can observe whether the firm still has bank loans on its balance sheet. Throughout the paper, we assume the loan contract signed between the bank and the entrepreneur, that is, the loan rate y_t, is not observable by the third party. One should therefore interpret y_t not just as the interest-rate payments made by the entrepreneur, but also fees, administrative costs, and so on.[6]

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[6] The equilibrium under which past loan rates are observable will involve players using mixed strategies.

Given the unique feature of Poisson learning, the *private*-belief process; that is, the belief held by the bank and the entrepreneur, is straightforward. If news hasn't arrived yet, the belief remains at $\mu_t^u = q_0$. In this case, no news is simply no news. Upon news arrival at t_λ, the private belief jumps to $\mu_t^g = 1$ in the case of good news and $\mu_t^b = \theta$ if bad. For the remainder of this paper, we suppress the time subscripts for private belief and simply use $\{\mu^u, \mu^g, \mu^b\}$ without loss of generality. To characterize the public-belief process, we introduce a belief system $\{\pi_t^u, \pi_t^g, \pi_t^b\}$, where π_t^u is the public's belief at time t that news hasn't arrived yet, and π_t^g (π_t^b) is the public belief that the news has arrived and is good (bad). In any equilibrium where the belief is rational, π_t^i is consistent with the actual probability that the bank and the entrepreneur are of type $i \in \{u, g, b\}$. Given $\{\pi_t^u, \pi_t^g, \pi_t^b\}$, the public belief that the project is good is

$$q_t = \pi_t^u q_0 + \pi_t^{g}.^7 \tag{3}$$

2.4 Rollover

When the loan matures at τ_m, the entrepreneur and the bank have three options: liquidate the project for L, switch to market financing, or continue the relationship by rolling the loan over. Control rights are assigned to the bank if the loan is not fully repaid, and renegotiation could potentially be triggered. Let $O_{\tau_m}^i \equiv O_{E\tau_m}^i + O_{B\tau_m}^i$, $i \in \{u, g, b\}$ be the maximum joint surplus to the two parties if the loan is not rolled over, where $O_{E\tau_m}^i$ and $O_{B\tau_m}^i$ are the value accrued to the entrepreneur and the bank, respectively. Because $F > L$, in the case of liquidation, the bank receives the entire liquidation value L and the entrepreneur receives nothing, that is, $O_{B\tau_m}^i = L$, and $O_{E\tau_m}^i = 0$. If the two parties choose to switch to market financing, the bank receives full payment $O_{B\tau_m}^i = F$, whereas the entrepreneur receives the remaining surplus $O_{E\tau_m}^i = \bar{V}_{\tau_m}^i - F$, where

$$\bar{V}_{\tau_m}^i = D_{\tau_m} + \frac{\phi \mu^i (R - F)}{r + \phi}. \tag{4}$$

In (4),

$$D_{\tau_m} = \frac{c + \phi [q_{\tau_m} + (1 - q_{\tau_m}) \theta] F}{\delta + \phi} \tag{5}$$

is the amount of proceeds that the entrepreneur raises from the market at time τ_m by issuing a bond with coupon cdt and face value F due whenever the project matures. The two components, $\frac{c}{\delta + \phi}$ and $\frac{\phi [q_{\tau_m} + (1 - q_{\tau_m}) \theta] F}{\delta + \phi}$, correspond to the present value of the coupon

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See Hörner and Jamison (2008) for a solution.

[7] To simplify notation, we abuse notation and use $\{\pi_t^i, q_t\}$ to denote $\{\pi_{t-}^i, q_{t-}\}$. We state them differently whenever they cause confusions.

payments and final payoff, respectively. The second term in (4) is the expected final cash flows that the entrepreneur receives upon the project's maturity. Because the entrepreneur is financially constrained, the bond price D_{τ_m} must be at least F, implying

$$q_{\tau_m} \geq q_{\min} \equiv 1 - \frac{c - \delta F}{\phi F (1 - \theta)}. \tag{6}$$

If the entrepreneur and the bank decide to roll over the loan at time τ_m, the two parties bargain over the loan rate y_{τ_m} until the next rollover date. To simplify notation, we sometimes drop the subscripts for loan rate unless doing so causes confusion. With some abuse of notation, let $B_{\tau_m}(y)$ and $E_{\tau_m}(y)$ be the continuation value for the bank and the entrepreneur if y is the loan rate decided by the bargaining.[8] Let $V_{\tau_m}^i \equiv B_{\tau_m}^i(y) + E_{\tau_m}^i(y)$. Specifically, the Nash bargaining problem can be written as

$$y_{\tau_m}^i = \arg\max_{y \leq c} \left\{ \left(B_{\tau_m}^i(y) - O_{B\tau_m}^i\right)^{1-\beta} \left(E_{\tau_m}^i(y) - O_{E\tau_m}^i\right)^{\beta} : B_{\tau_m}^i(y) \geq O_{B\tau_m}^i, E_{\tau_m}^i(y) \geq O_{E\tau_m}^i \right\}. \tag{7}$$

Note that in principle, Nash bargaining enables the entrepreneur and the bank to always pursue the option that maximizes their joint surplus. However, this result requires the solution to be implementable by some loan rate y below c. The financial constraint $y \leq c$ therefore results in scenarios in which the maximal joint surplus may not be implementable. If the solution is interior, that is, $y_{\tau_m}^i < c$, the bank value at the rollover rate is given by the conventional rule for the division of surplus:

$$B_{\tau_m}^i(y) = O_{B\tau_m}^i(y) + (1 - \beta)(V_{\tau_m}^i - O_{\tau_m}^i). \tag{8}$$

If the solution is a corner one, that is, $y_{\tau_m}^i = c$, the entrepreneur is financially constrained from making a higher transfer to the bank. In this case, $B_{\tau_m}^i < O_{B\tau_m}^i + (1 - \beta)(V_{\tau_m}^i - O_{\tau_m}^i)$, and the constraint limits the transfer from the entrepreneur to the bank. In both cases, it is convenient to write the continuation value of the bank in two parts:

$$B_{\tau_m}^i(y) = B_{\tau_m}^i(rF) + T(y), \tag{9}$$

where $B_{\tau_m}^i(rF)$ is the continuation value of the bank with loan rate $y = rF$, and

$$T(y) = \mathbb{E}\left[\int_0^{\tau_m \wedge \tau_\phi} e^{-r(s-t)}(y - rF)\,ds\right] = \frac{y - rF}{r + m + \phi}$$

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[8]We assume the two parties do not bargain over the face value F, with the underlying microfoundation where F is the maximum pledgeable income of the project.

is the discounted value of all the interim payments in excess of rF until either the loan or the project matures. Clearly, (9) makes clear that by negotiating the loan rate, the entrepreneur effectively makes a (possibly negative) transfer to the bank at the rollover date.

Two conditions must be satisfied for a loan to be rolled over. First, $V_{\tau_m}^i \geq \max\left\{L, \bar{V}_{\tau_m}^i\right\}$ so that rolling over is indeed the decision that maximizes the joint surplus. Second, it must be that $B_{\tau_m}^i(c) \geq L$ so that the bank prefers rolling over the loan and receiving the entire interim cash flow to liquidating the project and receiving L. Otherwise, the bank, endowed with control rights over the asset once not fully repaid, will choose to liquidate the asset.

2.5 Strategies and equilibrium

The public history $\mathcal{H}_t$ consists of time t and the entrepreneur's and the bank's actions up to t. Specifically, it includes at any time $s \leq t$, whether the entrepreneur borrows from the bank or the market and whether the project is liquidated. For any public history, the price of market debt D_{τ_m} summarizes the market strategy. Given that the market is competitive, the price of debt at which it breaks even satisfies (5).[9]

The private history h_t consists of the public history $\mathcal{H}_t$ and the rollover event, as well as the Poisson event on news arrival and, of course, the content of news. Essentially, a strategy of the entrepreneur is a stopping time that determines the time to refinance with the market. The strategy of the bank specifies whether to roll over the loan at each rollover date τ_m or to liquidate it, if it does not receive the full payment F. V_t^i – the joint value of the entrepreneur

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[9]We offer a micro-foundation as follows. In each period, two short-lived market-based lenders simultaneously enter and make private offers to all entrepreneurs. Those whose loans have matured, that is, $t = \tau_m$ may choose to accept the offer. This microfoundation will give rise to the *No-Deals* condition as in Daley and Green (2012).

and the bank in the lending relationship – satisfies the following Bellman equations:[10]

$$V_t^u = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}\left[q_0 + (1-q_0)\,\theta\right]R \right.\right.$$
$$\left.\left. + \mathbb{1}_{\tau=\tau_\lambda}\left[q_0 V_\tau^g + (1-q_0)\,V_\tau^b\right] + \mathbb{1}_{\tau=\tau_m}\max_{B_\tau^u(c)\geq L}\left\{V_\tau^u, L, \bar{V}_\tau^u\right\}\right]\right\}$$
(10a)

$$V_t^g = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}R + \mathbb{1}_{\tau=\tau_m}\max_{B_\tau^g(c)\geq L}\left\{V_\tau^g, L, \bar{V}_\tau^g\right\}\right]\right\}$$
(10b)

$$V_t^b = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}cds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}\theta R + \mathbb{1}_{\tau=\tau_m}\max_{B_\tau^b(c)\geq L}\left\{V_\tau^b, L, \bar{V}_\tau^b\right\}\right]\right\}.$$
(10c)

With some abuse of notation, in the first equation, we let $\tau = \min\{\tau_\phi, \tau_\lambda, \tau_m\}$, whereas in the last two equations, we let $\tau = \min\{\tau_\phi, \tau_m\}$. The first term in all three equations, cds, is the value of interim cash flows over time $[s, s+ds]$. The project matures and pays off the final cash flows if $\tau = \tau_\phi$. If $\tau = \tau_m$, the bank loan matures and the two parties choose from rolling over, liquidating, or switching to market financing. If the pair is uninformed, news arrives at random time τ_λ, after which they become informed. The maximization at time τ_m is subject to the additional constraint whereby the value of the bank at any rollover date has to be greater than L. Otherwise, the continuation value at the rollover date is L.

Recall that we have defined $E_t^i(y_t)$ and $B_t^i(y_t)$ as the continuation value of the entrepreneur and the bank at time t, where y_t is the prevalent loan rate. Sometimes, we also refer to $E_t^i(y_t)$ as equity value. By definition, for type $i \in \{g, b\}$,

$$E_t^i(y_t) = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}(c-y_t)ds + e^{-r(\tau-t)}\left[\max\left\{\tilde{R}-F, 0\right\} + \mathbb{1}_{\tau=\tau_m}\left(\mathbb{1}_{\text{rollover}}E_{\tau_m}^i(y_{\tau_m}) \right.\right.\right.$$
$$\left.\left.\left. + \mathbb{1}_{\text{market}}\bar{E}_{\tau_m}^i(y_{\tau_m}) \right)\right]\right\}$$

$$B_t^i(y_t) = \mathbb{E}_{t-}\left\{ \int_t^\tau e^{-r(s-t)}y_t ds + e^{-r(\tau-t)}\left[\mathbb{1}_{\tau=\tau_\phi}\min(\tilde{R}, F) + \mathbb{1}_{\tau=\tau_m}\left(\mathbb{1}_{\text{rollover}}B_{\tau_m}^i(y_{\tau_m}) \right.\right.\right.$$
$$\left.\left.\left. + \mathbb{1}_{\text{market}}F + \mathbb{1}_{\text{liquidation}}L \right)\right]\right\},$$

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[10]We use the standard notation $\mathbb{E}_{t-}[\cdot] = \mathbb{E}[\cdot|h_{t-}]$, to indicate the expectations is conditional on the history before the realization of the stopping time τ.

where $\bar{E}^i_{\tau_m} = \bar{V}^i_{\tau_m} - F$ is the continuation value of the entrepreneur once she finances with the market, with $\bar{V}^i_{\tau_m}$ defined in (4). As before, the indicator variables imply whether the loan is rolled over, the project is liquidated, or the entrepreneur obtains market financing. The value of type u is similar, with the additional term of becoming informed.

We look for a perfect Bayesian equilibrium of this game.

Definition 1. *An equilibrium of the game satisfies the following:*

1. *Optimality: The rollover decisions are optimal for the bank and the entrepreneur, given the beliefs $\{\pi^i_t, \mu^i_t, q_t\}$. The rate of the loan at rollover dates solves the Nash bargaining problem (7).*

2. *Belief Consistency: For any history on the equilibrium path, the belief process $\left\{\pi^u_t, \pi^g_t, \pi^b_t\right\}$ is consistent with Bayes' rule.*

3. *Market Breakeven: The price of the public debt satisfies (5).*

4. *No (unrealized) Deals: For any $t > 0$ and $i \in \{u, g, b\}$, $V^i_t \geq \mathbb{E}\left[\bar{V}_t \big| \mathcal{H}_t\right]$.*

The first three conditions are standard. The *No-Deals* condition follows Daley and Green (2012), reflecting the requirement that the market cannot profitably deviate by making an offer that the entrepreneur and the bank will accept.

As is standard in the literature, we use a refinement to rule out unappealing equilibria that arise due to unreasonable beliefs. Specifically, continued bank financing is never perceived negatively, so the public belief about the project's quality conditional on bank financing is non-decreasing. Effectively, this condition guarantees that once the good type refinances her loan with the market at some time t, all other types will do so for any $t > $ t.

Definition 2. *Belief monotonicity is satisfied if, conditional on the entrepreneur still borrowing from the bank, the public's belief that the project is good, q_t, is non-decreasing in t. An equilibrium that satisfies belief monotonicity is referred to as a monotonic equilibrium.*

We will show a unique monotone equilibrium exists. Moreover, we show that if the maturity of the loan is sufficiently long, the equilibrium is unique even without the refinement, that is, any perfect Bayesian equilibrium satisfies belief monotonicity.

2.6 Parametric assumptions

We make the following parametric assumptions to make the problem non-trivial.

Assumption 1 (Liquidation value).

$$NPV_\delta^b < L < NPV_r^g \tag{11}$$

According to Assumption 1, the NPV of a good project to the bank and the entrepreneur is above its liquidation value, which is in turn above the NPV of a bad project to the market. Therefore, liquidating a bad project but continuing a good project is socially optimal.

Assumption 2 (Risky debt).

$$F > \max\{\theta R, L\}. \tag{12}$$

Assumption 2 assumes the face value of the debt is above both the liquidation value and the expected repayment; otherwise, both the bank loan and the public bond can be safe.

Finally, we assume the size of the interim cash flow c to be weakly higher than rF.

Assumption 3 (interim cash flow).

$$c \geq rF. \tag{13}$$

2.7 First-best outcome

Before formally characterizing the equilibrium, we present the first-best outcome, which is achieved if news could be publicly observable and loans mature instantly. Assumption 1 guarantees that any good project will immediately receive financing from the market, whereas a bad project will be liquidated upon news arrival. Let $NPV_{r\to\delta}^u$ be the time-0 valuation of the unknown project if it is financed with the bank and switches to market/liquidation upon good/bad news.

$$NPV_{r\to\delta}^u = \frac{c + \phi\left[q_0 + (1-q_0)\theta\right]R}{r + \phi + \lambda} + \frac{\lambda}{r + \phi + \lambda}\left[q_0\frac{c + \phi R}{\delta + \phi} + (1-q_0)L\right]. \tag{14}$$

Proposition 1. *In the first-best outcome, a good project is immediately financed by the market, whereas a bad project is immediately liquidated. If $\{NPV_\delta^u, NPV_{r\to\delta}^u, L\} = L$, an unknown project will be liquidated. If $\{NPV_\delta^u, NPV_{r\to\delta}^u, L\} = NPV_\delta^u$, it will be financed by the market. If $\{NPV_\delta^u, NPV_{r\to\delta}^u, L\} = NPV_{r\to\delta}^u$, it will be financed by the bank until news comes out.*

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3 Equilibrium

In this section, we solve the equilibrium. In subsection 3.1, we study a benchmark economy by ignoring the financial constraints, $y_{\tau_m} \leq c$ and $D_{\tau_m} \geq F$.[11] In subsection 3.2, we describe the equilibrium with a formal treatment of the financial constraints. The equilibrium will be similar to the one in subsection 3.1, except for the boundary conditions. Both subsection 3.1 and 3.2 assume learning is an exogenous process, whereas subsection 3.3 analyzes the case in which learning is a costly decision by banks.

3.1 Benchmark: No financial constraints

The economy is characterized by state variables in private and public beliefs $\{\mu_t^i, \pi_t^i\}$. All public beliefs turn out to be deterministic functions of the time elapsed. Therefore, we use time t as the state variable. Specifically, the equilibrium will be characterized by two thresholds $\{t_b, t_g\}$, as illustrated by Figure 1. If $t \in [0, t_b]$, the bank and the entrepreneur will liquidate the project upon loan maturity if bad news has arrived – *efficient-liquidation* region. Loans for other types (good and unknown) will be rolled over. If $t \in [t_b, t_g]$, all types of loans will be rolled over, including the bad ones – *extend-and-pretend* region. Finally, if $t \in [t_g, \infty)$, the two entities will always refinance with the market upon loan maturity – *market financing*.
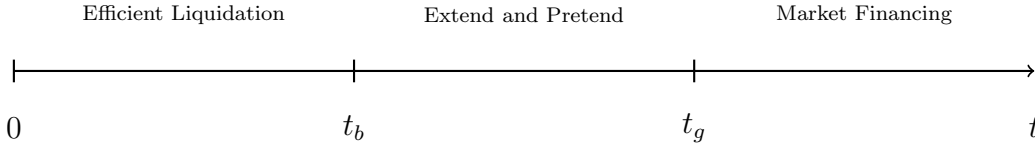

| Efficient Liquidation | Extend and Pretend | Market Financing |

| 0 | t_b | t_g | t |

Figure 1: Equilibrium regions

Given the equilibrium conjecture, the evolution of beliefs follow Lemma 1.

Lemma 1. *In a monotone equilibrium with threshold $\{t_b, t_g\}$, beliefs evolve as follows.*

1. *Without liquidation, the public beliefs $(\pi_t^u, \pi_t^g, \pi_t^b)$ satisfy the following differential equa-*

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[11]This case corresponds to an entrepreneur who is less financially constrained but still not deep-pocketed enough to finance the entire loan F. In other words, we assume she has enough funds to absorb the rollover losses but not enough to fully repay the bank loan. Alternatively, one can think of this case in the traditional trade-off framework with a deep-pocketed entrepreneur who uses debt to take advantage of the tax shields.

tion:

$$\dot{\pi}_t^u = -\lambda \pi_t^u + \mathbb{1}_{t \le t_b} m \pi_t^u \pi_t^b \tag{15a}$$

$$\dot{\pi}_t^g = \lambda \pi_t^u q_0 + \mathbb{1}_{t \le t_b} m \pi_t^g \pi_t^b \tag{15b}$$

$$\dot{\pi}_t^b = \lambda \pi_t^u (1 - q_0) - \mathbb{1}_{t \le t_b} m \pi_t^b (1 - \pi_t^b). \tag{15c}$$

2. *With liquidation, π_t^b jumps to 1, whereas π_t^u and π_t^g jump to 0.*

3. *Initially, $\pi_0^u = 1$ and $\pi_0^g = \pi_0^b = 0$.*

4. *For any $t < t_b$,*

$$q_t = \frac{q_0 \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda} - 1} e^{mt}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m - \lambda)s} ds}. \tag{16}$$

5. *For $t > t_b$, $q_t = \bar{q}$ is a constant.*

Figure 2 provides a graphical illustration of the public-belief systems for $t < t_g$. The left panel shows π_t^u, which decreases monotonically due to the arrival of news over time. By contrast, π_t^g keeps increasing, because the informed good type is discovered over time and keeps rolling over the loan. Finally, π_t^b evolves non-monotonically. During $[0, t_b]$, it increases initially as bad types are revealed (note they don't exit immediately due to the finite maturity of the loan). Ultimately, it starts to decline as more and more of the informed bad types are liquidated and exit funding. After t passes t_b (the dashed line), because no bad type will further liquidate her projects, π_t^b starts to increase again.
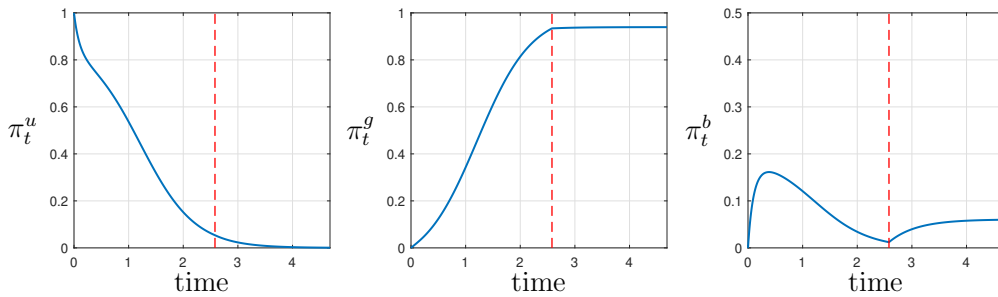

Figure 2: Public beliefs when $t < t_g$

This figure plots the public beliefs process with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV_r^b$, $\lambda = 2$, $q_0 = 0.1$, and $\beta = 0.2$.

The evolution of q_t is also straightforward. During $[0, t_b)$, q_t drifts up because bad entrepreneurs' projects are liquidated over time. After t goes above t_b, the average quality of

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borrowers remains unchanged from the market's perspective, that is,

$$\dot{q}_t = \dot{\pi}_t^g + q_0 \dot{\pi}_t^u = 0.$$

For the remainder of this subsection, we treat the bank and the entrepreneur as one entity and solve for t_b and t_g, the optimal choice of timing when they liquidate the project and when they switch to market finance. By considering the changes in valuation V_t^i, $i \in \{u, g, b\}$ over a small interval $[t, t+dt]$, we are able to derive the following Hamilton-Jacobi-Bellman (HJB) equation system:

$$(r + \phi) V_t^u = \dot{V}_t^u + c + \phi [q_0 + (1 - q_0) \theta] R \tag{17a}$$
$$+ \lambda [q_0 V_t^g + (1 - q_0) V_t^b - V_t^u] + m\mathcal{R}(V_t^u, \bar{V}_t^u)$$
$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R + m\mathcal{R}(V_t^g, \bar{V}_t^g) \tag{17b}$$
$$(r + \phi) V_t^b = \dot{V}_t^b + c + \phi\theta R + m\mathcal{R}(V_t^b, \bar{V}_t^b), \tag{17c}$$

where

$$\mathcal{R}(V_t^i, \bar{V}_t^i) \equiv \max\left\{0, \bar{V}_t^i - V_t^i, L - V_t^i\right\}. \tag{18}$$

The first term on the right-hand side $\dot{V}_t^u$ is the change in valuation; the second term captures the benefits of interim cash flow, and the third term corresponds to the project maturing, with arrival rate ϕ. In the latter case, the bank and the entrepreneur receive a payoff of R with probability $q_0 + (1 - q_0)\theta$. The fourth term stands for the arrival of news at rate λ. Following the news, the bank and the entrepreneur become informed. Finally, upon loan maturity which happens with an arrival rate m, the bank and the entrepreneur choose between rolling over the debt (0 in equation (18)), replacing the loan with the market bond ($\bar{V}_t^i - V_t^i$ in (18)), and liquidating the project ($L - V_t^i$ in (18)). Note we assume a project will be rolled over if $0 = \arg\max \mathcal{R}\left(V_t^i, \bar{V}_t^i\right)$, which will no longer be the case with financial constraints. Equations (17b) and (17c) can be interpreted in a similar vein.

The three equilibrium regions will differ in $\mathcal{R}\left(V_t^i, \bar{V}_t^i\right)$, that is, the decision when the loan matures. To better explain the economic intuition, we describe the equilibrium backwards in the time elapsed.

Market Financing: $[t_g, \infty)$. In this region, $\mathcal{R}(V_t^i, \bar{V}_t^i)$ is maximized by letting it equals $\bar{V}_t^i - V_t^i$. Also, $\dot{V}_t^i$, $i \in \{u, g, b\}$ is dropped in equations (17a)-(17c) because the belief q_t stays unchanged.

Ultimately, market financing is cheaper because market lenders have lower discount rates

$\delta < r$, and all types will therefore replace their loans with public bonds. Note the bond is set at a price that reflects the average quality of the project $\bar{q}$, which exceeds the initial quality q_0, because in equilibrium, some bad types would have liquidated their projects in the *efficient-liquidation* region.

Extend and Pretend: $\big[t_b, t_g\big)$**.** Working backward, we now consider the region $\big[t_b, t_g\big)$ during which all loans, including bad ones, are rolled over. When time is close to t_g, the bank finds it optimal to wait until t_g. Mathematically, on the right-hand side of equations (17a)-(17c), $\mathcal{R}(V_t^i, \bar{V}_t^i)$ is maximized by letting it equal 0. Intuitively, rolling over bad loans allows the bank to be fully repaid and transfer the losses from a bad loan to market lenders. When time is close to t_g, this decision can be optimal for the bank and the entrepreneur.

During $\big[t_b, t_g\big)$, the off-equilibrium belief for any entrepreneur who seeks financing from the market will be treated as bad with certainty.[12] Otherwise, the equilibrium is no longer viable. To see this, note that as in standard dynamic signaling games, the good type does not mix, and if the good type switches to the market, so will the other two types.[13] Therefore, if the good type refinances with the market right after t_b, the bad type will never liquidate before t_b. As a result, q_t will not increase up to $\bar{q}$, and the equilibrium is no longer viable.

Equilibrium in this region is clearly inefficient. A bad project should be liquidated, but instead, the bank and the entrepreneur roll it over in the hope of sharing the losses with the market lenders after t_g. By not liquidating between 0 and t_b, they have accumulated a "good" reputation; therefore, "extend and pretend" can be sustained in equilibrium.

Efficient Liquidation: $\big[0, t_b\big)$ Finally, we focus on the initial region $\big[0, t_b\big)$, where bad loans are not rolled over but instead liquidated. The equilibrium is socially efficient in this region. Mathematically, on the right-hand side of equations (17a)-(17c), $\mathcal{R}(V_t^b, \bar{V}_t^b)$ is maximized by letting it equal $L - V_t^b$, whereas $\mathcal{R}(V_t^g, \bar{V}_t^g)$ and $\mathcal{R}(V_t^u, \bar{V}_t^u)$ are still maximized by letting it equal 0. At the early stage of the lending relationship, only the uninformed and informed-good types roll over maturing loans. By contrast, banks that have learned the project is bad choose to liquidate. Assumption 1 guarantees that liquidation possesses a higher value than continuing the project until the final date t_ϕ. By continuity, liquidation still has a higher payoff if type b needs to wait for a long time (until t_g) to refinance. In this region, extend and pretend is suboptimal because t_g is far away: the firm could likely default before it receives the opportunity of market financing.

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[12]In principle, other off-equilibrium beliefs could support the same equilibrium structure. In fact, any off-equilibrium belief (e.g., below q_0) that deters the bad type from seeking market financing would support the equilibrium structure.

[13]See Lemma 5.1 of Daley and Green (2012) for a proof.

Boundary Conditions: The following two boundary conditions are needed to pin down $\{t_b, t_g\}$:

$$V_{t_b}^b = L \tag{19a}$$

$$\dot{V}_{t_g}^g = 0. \tag{19b}$$

(19a) is the indifference condition for the bad type to liquidate at t_b, which is the traditional value-matching condition in optimal stopping problems. In this case, rolling over brings the same payoff L, and thus by continuity and monotonicity, she prefers liquidating when $\tau_m < t_b$ and rolling over when $\tau_m > t_b$. The second condition, smooth pasting, comes from the *No-Deals* condition. We show in Lemma 4 of Appendix A.1.2 that if this condition fails, type g will have strictly higher incentives to switch to market financing before t_g, which constitutes a strictly profitable opportunity for market lenders. Essentially, the *No-Deals* condition guarantees the equilibrium will ultimately be one with pooling; therefore, the smooth-pasting condition solves the optimal-stopping time problem for the good types. The smooth-pasting condition picks the earliest t_g for the good entrepreneur to switch to refinance with the market. With the boundary conditions, we can uniquely pin down $\{t_b, t_g\}$, which is given by the following proposition.

Proposition 2. *In the absence of financial constraints, a unique monotone equilibrium exists and is characterized by rollover thresholds t_b and t_g:*

$$t_b = \min\{t : q_t = \bar{q}\}, \tag{20}$$

and

$$t_g = t_b + \frac{1}{r + \phi} \log \left(\frac{(r + \phi)V_{t_g}^b - (c + \phi\theta R)}{(r + \phi)L - (c + \phi\theta R)} \right), \tag{21}$$

where

$$\bar{q} = \frac{1}{(1 - \theta)} \left(\frac{\delta + \phi}{r + \phi} \frac{c + \phi F}{\phi F} - \frac{c}{\phi F} - \theta \right) \tag{22}$$

$$V_{t_g}^b = \frac{c + \phi R}{r + \phi} - \frac{\phi R(1 - \theta) + m \frac{\phi(R - F)(1 - \theta)}{r + \phi}}{r + \phi + m}. \tag{23}$$

Numerical Example Figure 3 plots the value function of all three types. In this example, the equilibrium $t_b = 2.5806$ and $t_g = 4.6861$. In all three panels, the blue solid lines stand for the value function, whereas the red dashed line shows the levels of L. Clearly, all three value functions stay constant after t passes t_g. In fact, as proven in Lemma 5 of Appendix

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A.1.2, V_g stays constant throughout the entire range. In other words, the informed-good types always expect the same continuation value. By contrast, the value of informed-bad types (right panel) exceeds L only after t passes t_b and then increases sharply until $t = t_g$.
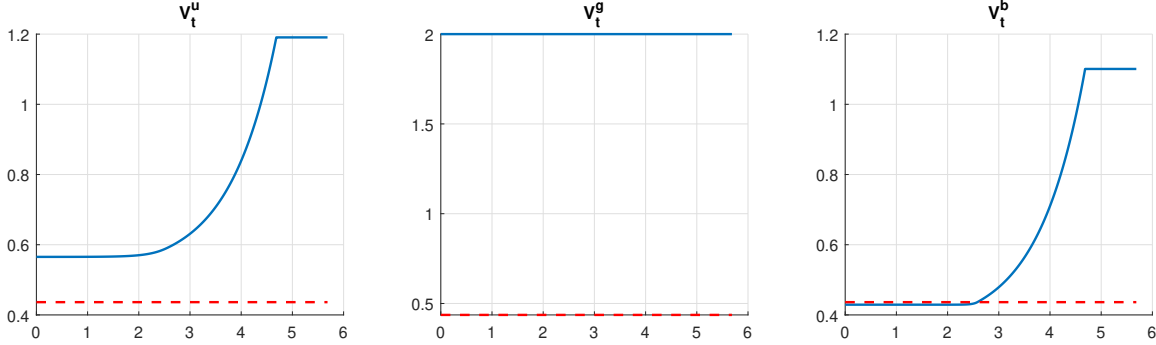

Figure 3: Value functions

This figure plots the value function with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV_r^b$, $\lambda = 2$, $q_0 = 0.1$, and $\beta = 0.2$.

We conclude this subsection by discussing the equilibrium uniqueness without the monotone belief refinement.

Proposition 3. *A unique pair $\{\underline{m}, \bar{m}\}$ exists satisfying $\underline{m} < \bar{m}$ such that the equilibrium described in Proposition 2 is unique if and only if $m \in (\underline{m}, \bar{m})$. If $m > \bar{m}$, the equilibrium is not unique without the monotone belief refinement. If $m < \underline{m}$, the unique equilibrium is one in which $t_b = t_g$.*

Intuitively, when m gets sufficiently low and equivalently the maturity of the loan gets long enough, an entrepreneur will need to wait for a long time after t_g until her loan matures, upon which she could refinance with the market. The prolonged waiting period resulting from loan maturity will be sufficient to deter bad types from mimicking others.

Remark 3. The results will stay unchanged if we allow the entrepreneur to renegotiate and prepay the bank loans. During $[0, t_g)$, renegotiation is never triggered. After t_g, all bank loans will immediately be renegotiated. As a result, for all three types, $V_{t_g}^i = \bar{V}_{t_g}^i$. According to Proposition 2, t_b stays unchanged, whereas t_g gets even higher.

3.2 Binding financial constraint

By relaxing financial constraints, we have assumed all rollover decisions are made to maximize the joint surplus of the bank and the entrepreneur. Specifically, at each rollover

date, a loan will be rolled over if the joint surplus is above the liquidation value L. In this subsection, we formally analyze the model with two financial constraints. First, at rollover dates before t_g, the negotiated loan rate y cannot exceed the rate of interim cash flow c. This constraint limits the transfer from the entrepreneur to the bank. Second, at rollover dates after t_g when the entrepreneur intends to refinance with the market, the price of the market debt D_{τ_m} must be sufficient to cover the face value of the loan F. As we show, the bank will sometimes liquidate the project to get L even though the joint surplus is higher.

The HJBs for the value function $\{V_t^i, \ i \in \{u, g, b\}\}$ are the same as those in subsection 3.1. Again, we can use two thresholds $\{t_b, t_g\}$ to characterize the equilibrium solutions. One may wonder whether the financial constraint could always be slack. Lemma 2 shows this case is never possible.

Lemma 2. *The financial constraint $y \leq c$ always binds at t_b.*

We offer a heuristic proof as follows. Suppose instead the constraint is always slack; the boundary condition at t_b, characterized by (19a), immediately shows that $B_{t_b}^b(y_{t_b}) = L + (1 - \beta)(V_{t_b}^b - L) = L$. This result implies that if a bad loan matures at t_b, the bank will receive a continuation value L, and the bad entrepreneur will receive a continuation value 0. However, the entrepreneur's continuation payoff at time t_b should be strictly positive: she can always wait until t_g to refinance with the market.

3.2.1 Equilibrium boundaries

Let us now turn to the the boundary conditions under financial constraints. First, the smooth-pasting condition $\dot{V}_{t_g}^g = 0$ continues to hold. As explained in subsection 3.1, this condition follows from the *No-Deals* condition, which essentially selects the equilibrium in which a good-type entrepreneur chooses to refinance with the market as early as possible. Note the smooth-pasting condition pins down $\bar{q}$, the average quality of entrepreneurs during $[t_b, t_g)$. Therefore, the average quality of entrepreneurs after t_b is identical to the case without financial constraint, and $t_b = \min \{t : q_t \geq \bar{q}\}$ also remains unchanged. Given $\bar{q}$, we can easily verify the second financial constraint whereby after t_g, the entrepreneur can borrow more than F from the market to repay the bank.

Lemma 3. *Under Assumption 3, $\bar{q} \geq q_{\min}$, and $D_{\tau_m} \geq F$. In other words, the entrepreneur can raise more than F from the market after time passes t_g.*

For the remainder of this subsection, we focus on the first financial constraint, $y \leq c$. According to Lemma 2, this constraint must alter the second boundary condition – the value-matching condition. In particular, because the entrepreneur is financially constrained

and cannot repay its loan before t_g, the bank has the right to decide whether to liquidate the project. Therefore, the value-matching condition at t_b becomes

$$B_{t_b}^b(c) = L. \tag{24}$$

Depending on parameters, the constraint $y \leq c$ may or may not bind at $t = t_g$ for the bad type. Proposition 4 summarizes the equilibrium whereby it also binds at t_g. The other case is described and proved in the appendix.[14]

Proposition 4. *If*

$$L - \frac{c + (\phi\theta + m)F}{r + \phi + m} + (1 - \beta)\left[\frac{c}{r + \phi} + \frac{\phi}{r + \phi}\left((1 - \theta)\frac{mF}{r + \phi + m} + \theta R\right) - L\right] > 0,$$

the equilibrium is characterized by two thresholds $\{t_b, t_g\}$, where $\{\bar{q}, t_b\}$ are identical to those in Proposition 2, and

$$t_g = t_b - \frac{1}{r + \phi}\log\left(\frac{(r + \phi)L - (c + \phi\theta F)}{\frac{r + \phi}{r + \phi + m}(c + \phi\theta F + mF) - (c + \phi\theta F)}\right). \tag{25}$$

In subsection 3.1, we show the development of private information over time gives rise to an equilibrium region in which bad types pretend and extend, which is socially suboptimal. Our next result shows financial constraints mitigate this inefficiency.

Corollary 1. *The length of the extend-and-pretend period $t_g - t_b$ gets shorter under the financial constraint.*

Proof. Because $E_{t_b}^b(c) > 0$, (24) implies at $t = t_b$, the joint surplus $V_{t_b}^b > L$. $\square$

This result has implications for the relationship between financial constraints and credit quality. In particular, it highlights the role of financial constraints and their interaction with asymmetric information across lenders. Whereas the benchmark case shows asymmetric information per se will endogenously generate incentives for banks to roll over bad loans, results in this subsection show financial constraints can mitigate this concern. Intuitively, the ability for bad types to mimic others is compromised under financial constraints, because the transfer that a bad entrepreneur can use to "bribe" the bank to not liquidate the project is limited.

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[14]In the appendix, we show the constraint binds monotonically. If $y \leq c$ binds at t_g, it binds everywhere at $t \in [t_b, t_g]$. Otherwise, there exists a $t_c \in [t_b, t_g]$ such that the constraint binds on $t \in [t_b, t_c]$ but not $t \in [t_c, t_g]$.

Moreover, note the financial constraint does not affect $\bar{q}$, the average quality of entrepreneurs that will be ultimately financed with the market. This result relies on Assumption 3: entrepreneurs will not refinance with the market until they have accumulated enough reputation such that their financial constraints no longer bind once they switch to the market. Without Assumption 3, the financial constraint will result in a higher $\bar{q}$, so that the credit quality financed by the market is higher under financial constraint.

Numerical Example Under the same set of parameter values, $t_b = 2.5806$ and $t_g = 3.8418$. A comparison with the benchmark case shows t_b stays unchanged, whereas t_g gets smaller, confirming our theoretical findings. Moreover, we can compute $B_0^u(c)$, the maximum amount of bank borrowing at time 0, which equals 0.4401. By contrast, in the benchmark case, the entrepreneur is able to borrow up to $B_0^u(c) = 0.5242$. As expected, the financial constraint during debt rollovers in the future decreases the amount of upfront borrowing.

3.2.2   Entrepreneur and bank value

In this subsection, we study how the joint surplus of the entrepreneur and the bank is distributed between the two parties. We mainly describe the results graphically and leave the analytical details to Appendix A.2, including the HJB equations.

Figure 4 plots the value functions at loan rate $y = rF$. A first prominent feature is the kink in the entrepreneur's value function at $t_b = 2.5806$. This finding is not surprising, because it is the bank that makes the liquidation decision. Second, although most of the value functions are monotonically increasing in time, a good-type bank's value function decreases with time. Intuitively, two forces are at work here. First, the value of a project is (weakly) increasing as time gets closer and closer to the market-financing stage. As a result, the surplus of rolling over the loan $V_t^i - L$ gets (weakly) larger. Ceteris paribus, both the entrepreneur's and the bank's value function should increase. However, a second, counterveiling force is at work. During the market-financing stage, the disagreement point in the Nash bargaining game is $(0, L)$: if the bargaining does not reach an agreement, the bank only receives the liquidation value L. During the market-financing stage, however, the bank will always be fully repaid and thus receive F. As time t gets closer to the market-financing stage, the bank's ability to "exploit" the entrepreneur becomes more limited because the likelihood that the next rollover event will occur during the market-financing stage is increasing. Therefore, the entrepreneur's value function increases, whereas the bank's decreases. For the good type, the first effect is muted, because the value function V_t^g is a constant over time. The second effect in this case dominates and leads to the monotonically decreasing pattern. For the other two types, the first effect dominates. Given the opposite effects of these two forces,
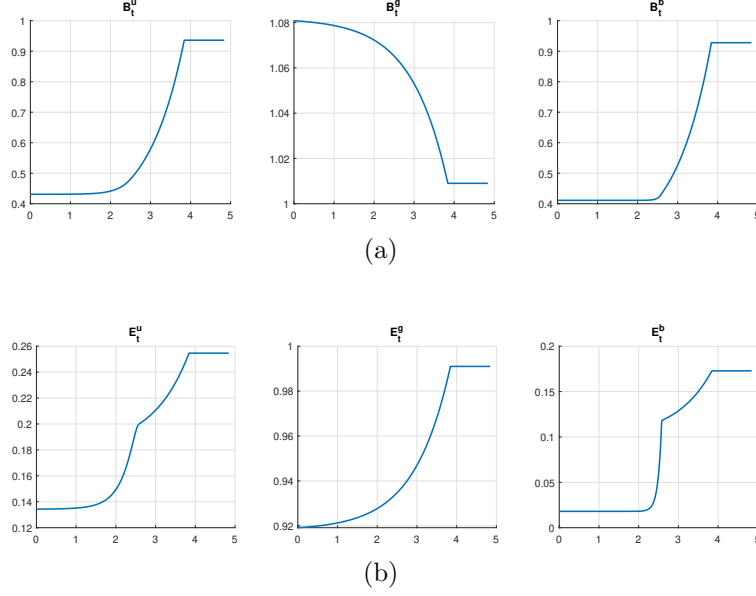
Figure 4: Bank and Entrepreneur Value Functions

This figure plots the bank's (a) and entrepreneur's (b) value function at loan rate $y = rF$ with the following parameter values: $r = 0.1$, $\delta = 0.05$, $m = 10$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV_r^b$, $\lambda = 2$, $q_0 = 0.1$, and $\beta = 0.2$.

the overall effect can be non-monotonic.[15]

3.3   Endogenous learning

Our analysis has so far assumed learning is an exogenous process, which happens as long as the entrepreneur borrows from the bank. In this subsection, we consider the situation in which learning is endogenously chosen by the bank as a costly decision. In particular, we assume the learning rate is chosen as $a_t \in [0, 1]$ by the bank, and for a given rate, the news arrives at Poisson intensity λa_t. In the meantime, learning incurs a flow cost ψa_t. Let us continue to look for a monotone equilibrium with rollover thresholds $\{t_b, t_g\}$.

Obviously, the bank never learns after $t > t_g$, because the entrepreneur will always refinance the loan with the market whenever the loan matures, that is, $a_t \equiv 0$, $\forall t \geq t_g$. At

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[15]As we show in Lemma 7 in the Appendix, the sign of E_t^b can only change sign at most once though.

any time $t \in (0, t_g)$, the bank's continuation value satisfies the HJB equation:

$$(r + \phi + m) B_t^u = y_t F + \phi[q_0 + (1 - q_0)\theta]F + \dot{B}_t^u$$
$$+ \max_{a_t \in [0,1]} \left\{ \lambda a_t \left[q_0 B_t^g + (1 - q_0)B_t^b - B_t^u - \frac{\psi}{\lambda} \right] \right\} + m[L + (1 - \beta)(V_t^u - L)].$$

Note we have used the result that at any rollover date, the bank's continuation value jumps to $m[L + (1 - \beta)(V_t^u - L)]$ following Nash bargaining. Clearly, the net benefit of learning is positive, which implies $a_t \equiv 1$ if and only if

$$q_0 B_t^g + (1 - q_0)B_t^b - B_t^u > \frac{\psi}{\lambda}.$$

We can show the bank always stops learning before time reaches t_b. Intuitively, when banks do not liquidate the bad project in equilibrium, the value of an uninformed bank is a linear combination of an informed-good and an informed-bad. In this case, the benefits of becoming informed is zero. If the bank liquidates bad projects in equilibrium, however, the value of an informed-bad bank is at least L, so that the overall payoff is convex in the type of information (see Figure 5 for a graphical illustration.). In this case, information is valuable and learning will be endogenously chosen if the cost is small enough. The next proposition characterizes the equilibrium.

Proposition 5. *In an equilibrium characterized by $\{t_b, t_g\}$, the bank never learns after t_b.*

1. If
$$\frac{\psi}{\lambda} < \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left(L - \frac{c + \phi\theta R}{r + \phi} \right),$$

an equilibrium exists in which the bank learns during $[0, t_a]$, where $t_a \equiv \frac{1}{\lambda} \log \left(\frac{\bar{q}}{1 - \bar{q}} \frac{1 - q_0}{q_0} \right) < t_a < t_b$. In this equilibrium, beliefs on $(0, t_a)$ are still given by (15a) - (15c), whereas on (t_a, t_b), beliefs evolve as

$$\dot{\pi}_t^u = m\pi_t^u \pi_t^b, \quad \dot{\pi}_t^g = m\pi_t^g \pi_t^b, \quad \dot{\pi}_t^b = -m\pi_t^b \left(1 - \pi_t^b \right).$$

2. If
$$\frac{\psi}{\lambda} > \frac{m(1 - \beta)(1 - q_0)}{r + \phi + m} \left(L - \frac{c + \phi\theta R}{r + \phi} \right),$$

the bank never learns.

Numerical Example Under the same set of parameters, with the additional parameter that $\psi = 0.06$, we can get $t_a = 2.4690$, $t_b = 3.4825$, and $t_g = 5.5879$. Note that because
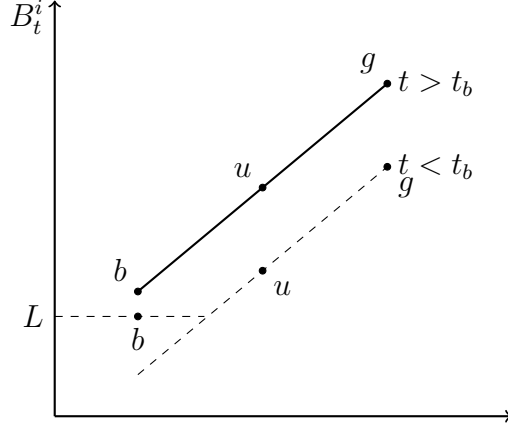
25

Figure 5: Graphical illustration of learning benefits

the bank stops learning after t_a, it takes longer for the average quality q_t to reach $\bar{q}$, which explains why t_b is higher than in the benchmark case. The length of the extend-and-pretend period, $t_g - t_b$, stays unchanged at 2.1055.

Finally, we look at the equilibrium with endogenous learning once we incorporate the constraint whereby $y_t \leq c$. To keep the analysis simple, we only consider the case in which the constraint is always binding $(y_t = c)$ for all types, as is the case if the bank's bargaining power is high enough (equivalently, β is close enough to 0). The next proposition characterizes the equilibrium in this case.

Proposition 6. *Suppose*

$$\frac{c + \phi\big((1 - \theta)F + \theta R\big)}{r + \phi} > \frac{c + (\phi + m)F}{r + \phi + m}.$$

Then if

$$\frac{\psi}{\lambda} < \frac{m(1 - q_0)}{r + \phi + m}\left(L - \frac{c + \phi\theta F}{r + \phi}\right),$$

there exists a $\bar{\beta}$ such that for any $\beta \leq \bar{\beta}$, an equilibrium exists in which $y_t^i = c$ for $i \in \{u, g, b\}$ and the bank learns during $[0, t_a]$, where $\underline{t}_a \equiv \frac{1}{\lambda}\log\left(\frac{\bar{q}}{1 - \bar{q}}\frac{1 - q_0}{q_0}\right) < t_a < t_b$. In this equilibrium, beliefs follow Proposition 5.

The first inequality guarantees that for β sufficiently close to 0 the constraint $y_t \leq c$ binds for all types. The second condition implies the cost of learning is low enough that the bank has incentives to learn. Notice the upper bound on ψ/λ in Proposition 6 is less stringent than the one in Proposition 5. This observation captures the idea that the bank has more incentives to learn when the entrepreneur is financially constrained.

Comparing the equilibrium in the case with financial constraints with the case without

financial constraints is intrusive. To guarantee the conditions in Proposition 6 always hold, we consider the special case where $\beta = 0$. All other parameters stay unchanged. Without the financial constraints, $t_a = 2.4691$, $t_b = 3.2101$, and $t_g = 5.3156$. With financial constraints, $t_a = 2.4739$, $t_b = 2.7983$, and $t_g = 4.0594$. Clearly, financial constraints expand the region in which banks choose to learn. The reason goes back to our previous result that financial constraints reduce the extend-and-pretend period (from 2.1055 to 1.2611 under given parameters). As a result, bad projects are liquidated more often under financial constraints, because the role of learning for a risk-neutral bank is to liquidate bad projects. Therefore, the information is more valuable, and learning persists for a longer period of time under financial constraints.

Figure 6 shows how t_a, t_b, and t_g vary with loan maturity $\frac{1}{m}$. The left panel clearly shows that for all maturities, the bank learns for a longer period of time under financial constraints, and the difference tends to increase with the maturity of loans. Moreover, in general, the length of loan maturity could have a non-monotonic effect on learning, due to the tradeoff between the value of information and strategic rollover. On one hand, longer maturity reduces the value of information, because the bank has to wait longer to liquidate after learning that project is bad. On the other hand, longer maturity reduces the incentives to roll over bad loans and increases the incentives to learn. The overall effect therefore depends on which effect dominates.
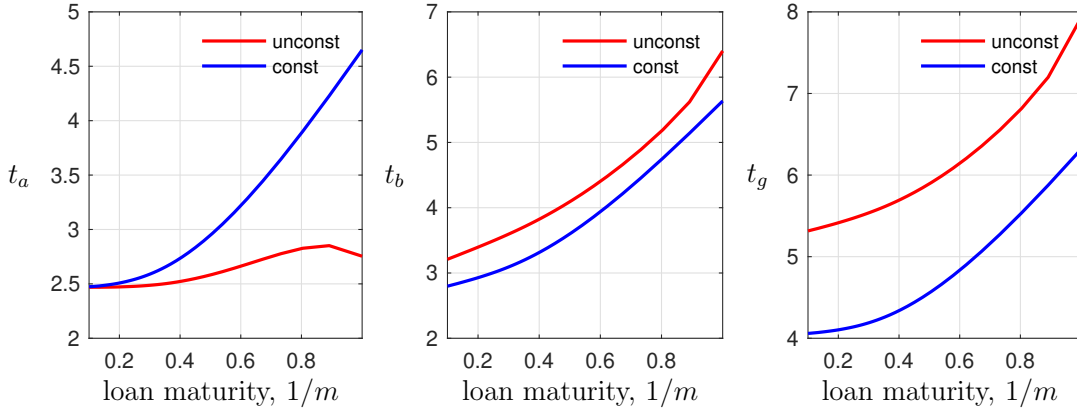

Figure 6: Endogenous learning with and without financial constraints

This figure plots t_a, t_b, and t_g with the following parameter values: $r = 0.1$, $\delta = 0.05$, $F = 1$, $\phi = 1$, $R = 2$, $c = 0.2$, $\theta = 0.1$, $L = 1.2 \times NPV_r^b$, $\lambda = 2$, $q_0 = 0.1$, and $\beta = 0$.

3.4 Special case: Instantly maturing loan

Next, we consider a special case in which loans mature instantly. This case corresponds to the limit whereby the intensity of the debt maturing m goes to infinity. By doing so, we obtain simple and intuitive closed-form solutions in primitives. The results follow naturally by taking limits to the propositions in the previous section; therefore, we omit the proofs.

Proposition 7 is the counterpart of Proposition 2, where we study the case without financial constraints.

Proposition 7. *If bank loans mature instantly and financial constraints are present, the equilibrium is given by*

$$q_t = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda t}} \quad \forall t < t_b \tag{26a}$$

$$t_b = \frac{1}{\lambda} \left[\log \left(\frac{\bar{q}}{1 - \bar{q}} \right) - \log \left(\frac{q_0}{1 - q_0} \right) \right] \tag{26b}$$

$$t_g = t_b + \frac{1}{r + \phi} \log \left(\frac{\phi (1 - \theta) F}{(r + \phi)L - (c + \phi\theta R)} \right) \tag{26c}$$

$$\bar{q} = \frac{1}{(1 - \theta)} \left(\frac{\delta + \phi}{r + \phi} \frac{c + \phi F}{\phi F} - \frac{c}{\phi F} - \theta \right). \tag{26d}$$

When bank loans mature instantly, a project is immediately liquidated once it is known as bad before t_b. In this case, t_b, the length of this efficient liquidation region, depends only on the speed of learning λ and the ultimate credit quality $\bar{q}$. $t_g - t_b$, the length of the extend-and-pretend period, can be written equivalently as $\frac{1}{r+\phi} \log \left(\frac{\frac{\phi(1-\theta)F}{r+\phi}}{L - NPV_r^b} \right)$. Indeed, if $L - NPV_r^b$ get higher so that liquidation becomes relatively more profitable, bad types find mimicking others to be less attractive. As a result, $t_g - t_b$ gets shorter. The numerator term, $\frac{\phi(1-\theta)F}{r+\phi}$, is the discounted NPV of the loss to the bad-type bank if it finances the borrower until the final cash flows are produced. If this value increases, mimicking other types and sharing the loss with market lenders become more appealing; therefore $t_g - t_b$, needs to be longer.

Next, we consider the equilibrium in Proposition 4 for the case in which the loan rate y is constrained by the interim cash flow c. If the financial constraint always binds for the bad type on $[t_b, t_g]$, we get the following results.

Proposition 8. *If*

$$L - F + (1 - \beta) \left[\frac{c}{r + \phi} + \frac{\phi}{r + \phi} ((1 - \theta)F + \theta R) - L \right] > 0,$$

the equilibrium is characterized by two thresholds $\{t_b, t_g\}$, where $\{\bar{q}, t_b\}$ are identical to those

in equation (26a), and

$$t_g = t_b + \frac{1}{r + \phi} \log \left(\frac{F - \frac{c + \phi\theta F}{r + \phi}}{L - \frac{c + \phi\theta F}{r + \phi}} \right). \tag{27}$$

Note we can write equivalently $t_g - t_b = \frac{1}{r+\phi} \log \left(\frac{F - NPV_r^b}{L - NPV_r^b} \right)$. Once again, the length of the extend-and-pretend period decreases with the liquidation value of the project, whereas it increases with the amount of loan payments due to the bank.

Finally, we consider the equilibrium in Proposition 5 under endogenous learning without financial constraints.

Proposition 9. *In an equilibrium characterized by $\{t_b, t_g\}$, the bank never learns after t_b.*

1. If

$$\frac{\psi}{\lambda} < (1 - \beta)(1 - q_0) \left(L - \frac{c + \phi\theta R}{r + \phi} \right), \tag{28}$$

an equilibrium exists in which the bank learns during $[0, t_a]$. The belief threshold $\bar{q}$ and the length of the extend-and-pretend period $t_g - t_b$ are the same as in Proposition 7. The thresholds t_a, t_b, and t_g are given by

$$t_a = \frac{1}{\lambda} \left[\log \left(\frac{\bar{q}}{1 - \bar{q}} \right) - \log \left(\frac{q_0}{1 - q_0} \right) \right]$$

$$t_b = t_a + \frac{1}{r + \phi} \log \left(\frac{(1 - \beta)(1 - q_0) \left(L - \frac{c + \phi\theta R}{r + \phi} \right)}{(1 - \beta)(1 - q_0) \left(L - \frac{c + \phi\theta R}{r + \phi} \right) - \frac{\psi}{\lambda}} \right)$$

$$t_g = t_a + \frac{1}{r + \phi} \log \left(\frac{\phi}{r + \phi} \frac{(1 - \beta)(1 - q_0)(1 - \theta)F}{(1 - \beta)(1 - q_0) \left(L - \frac{c + \phi\theta R}{r + \phi} \right) - \frac{\psi}{\lambda}} \right).$$

2. If

$$\frac{\psi}{\lambda} > (1 - \beta)(1 - q_0) \left(L - \frac{c + \phi\theta R}{r + \phi} \right),$$

the bank never learns.

Note the right-hand side of (28) can be rewritten as $(1 - \beta)(1 - q_0) \left(L - NPV_r^b \right)$, where $(1 - \beta)$ is the bank's bargaining power, $(1 - q_0)$ is the ex-ante probability of a bad project, and $\left(L - NPV_r^b \right)$ is the benefit from learning a project is bad. This expression makes clear that the expected benefit of learning to the bank is to liquidate bad projects and receive a liquidation value, rather than continue to offer financing.

4 Observable Rollover and Deterministic Maturity

In the baseline model, we have assumed rolling over a loan is unobservable to the market participants. Moreover, we have modeled the maturing event as a Poisson event to simplify the solution. Both assumptions have been made for simplicity. In this section, we introduce two modifications to the model. First, whenever a loan matures, we assume that whether the bank decides to roll over or liquidate is observable. Second, the maturity of loans is publicly known to be fixed at $1/m$. We show the model is essentially identical to one in a discrete-time framework. Thus, as in most discrete-time models with a binary type of asymmetric information, the equilibrium will, in general, involve mixed strategies due to the integer problem. In the remainder of this section, we construct an equilibrium that has features like those described in Proposition 2.

Let $n \in \{1, 2, 3 \cdots\}$ be the sequence of rollover events. The date associated with the n-th rollover event is $t_n = n/m$, and the time between two rollover dates is $1/m$. As before, we can construct an equilibrium with two thresholds: t_b and t_g. However, with deterministic rollovers, specifying the two thresholds in term of the rollover events is notationally more convenient: n_b and n_g. The following proposition describes the equilibrium.

Proposition 10. *If loan maturity is fixed at $1/m$ and rollover is observable, then there exists $\{n_b, n_g\}$ such that*

1. *Efficient liquidation*

    (a) For $n < n_b$, bad projects are liquidated, whereas other projects are rolled over;

2. *Extend and pretend*

    (a) For $n = n_b$, a fraction $\alpha_b \leq 1$ of the bad projects are liquidated, whereas other projects are rolled over.

    (b) When $n \in (n_b, n_g - 1)$, all loans are rolled over.

3. *Market Financing*

    (a) When $n = n_g$, market lenders make an offer at $D_{\tau_m}(q_{\tau_m} = \bar{q})$ with probability $\alpha_g \leq 1$.

    (b) When $n = n_g + 1$, all entrepreneurs refinance with the market.

4. *If $m \to \infty$, the equilibrium converges to the one in Proposition 7.*

Under observable rollovers, the public belief about the loan quality q_t stays unchanged during any two rollover events. At the rollover event $n < n_b$ or equivalently $t < n_b/m$, the belief will experience a discrete jump. If the project is liquidated, clearly q_t jumps to 0. If the loan is rolled over, q_t jumps upwards to the level described in the instantly maturing debt, as shown in (26a). Note an equivalence in beliefs under fixed maturity and instantly maturing exponential debt, because the event of maturing is occurring with certainty at $t = n/m$. q_t stays unchanged at $\bar{q}$ after $t > n_b/m$.

5 Concluding Remarks and Empirical Relevance

In this paper, we introduce private learning into a banking model and study the dynamic tradeoffs of relationship-based lending. Compared to market financing, bank financing enables learning about the quality of the project being financed, but is also subject to the cost of an information monopoly. We construct an equilibrium in which an entrepreneur starts with bank financing and subsequently refinances with the market. The novel result is that under information asymmetry, banks will endogenously roll over bad loans after some time. We characterize conditions for this extend-and-pretend and show how it is affected by factors such as the entrepreneur's financial constraints. We also study banks' incentives to learn over time.

To focus on dynamic learning and the resulting asymmetric information between the relationship bank and market-base lenders, we have not explicitly modeled interbank competition (Boot and Thakor, 2000) and secured lending (Boot and Thakor, 1994). Interbank competition does not change any result in our context, because the new bank is assumed to have the identical information as market-based lenders. Broadly speaking, the liquidation value of the asset L can be interpreted as collateral value. That said, both issues deserve more careful examinations in future research. Moreover, zombie lending emerges in our paper due to potential competition between the relationship bank and market-based lenders. An interesting extension is to introduce complementarity between banks and markets as in Song and Thakor (2010) and study how the degree of zombie lending changes.

Our paper is closely related to the empirical literature on relationship lending and loan sales. In practice, 60% of the loans are first sold within one month of loan origination and nearly 90% are sold within one year (Drucker and Puri, 2008). As Gande and Saunders (2012) argue, a special role of banks is to create an active secondary loan market while still producing information. A key assumption in our paper is that private information between a relationship bank and other credit suppliers is built up over time, which is consistent with the observation in Dahiya et al. (2003) and Jiang et al. (2013).

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A main prediction in our paper is banks have endogenous incentives to roll over bad loans. This result on "extend and pretend" is largely reminiscent of the popular discussions on how loan sales and securitization induce agency conflicts (Purnanandam, 2010; Keys et al., 2010). As documented by existing studies (Agarwal et al., 2011), mortgage lenders and loan servicers rarely write off losses shortly after borrowers become financially distressed. Dahiya et al. (2003) find firms that file bankruptcy after loan sales are not necessarily the worst-performing firms at the time of the loan sale.[16] Jiang et al. (2013) find that at the time of loan origination, loans with a high probability of sale have higher delinquency rates subsequently, whereas ex-post loans sold by banks have lower delinquency rates. Our paper implies reputation may not reduce the related agency conflicts. Griffin et al. (2014) show that highly reputable banks may produce complicated assets (CLO, MBS, ABS, and CDOs) that underperform subsequently, implying a bank's reputation may contribute negatively to the quality of its assets. To some extent, our results are related to "evergreening" loans and zombie lending, which in the literature have mostly been attributed to banks' capital requirements when they write off bad loans (Peek and Rosengren, 2005; Caballero et al., 2008).

Our paper has a unique prediction on how the rates on bank loans vary with the length of lending relationships. López-Espinosa et al. (2017), using Spanish data, find loan rates do not drop until the relationship extends beyond two years. By contrast, Ioannidou and Ongena (2010) find loan rates initially decrease but eventually ratchet up, which encourages firms to switch to other banks. Moreover, our model predicts that the concerns about extend and pretend would be less severe for less-reputable borrowers who are more financially constrained, and, specifically, borrowers with longer relationships with banks are more likely to receive waivers upon covenant violation. These tests will be interesting to conduct with careful empirical studies.

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[16]Interestingly, Dahiya et al. (2003) find a negative stock-price reaction to the announcement of loan sales using data from 1995-1998, whereas Gande and Saunders (2012) find the reaction to be positive using data post 2000s. The main difference is, as argued by Gande and Saunders (2012), many lenders terminated their lending relationships in Dahiya et al. (2003)'s sample. Therefore, they conclude the role of the secondary loan market has evolved from informed lenders off-loading troubled loans to banks creating liquidity.

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A   Appendix

A.1   Proofs

A.1.1   Proof of Lemma 1

Proof. The proof relies on the filtering formula for counting processes in Lipster and Shiryaev (Chapter 19). Let χ_t^i be the probability that a type $i \in \{g, b, u\}$ firm looks for external financing at time t and let ℓ_t^i be the probability that a type i firm liquidates at time t. Let L_t be the counting process associated to the liquidation time and and M_t be the counting process associated with going to the market. If we denote the type of the firm at time t by $i(t)$ then L_t has intensity $m\ell_t^{i(t)}$ while M_t has intensity $m\chi_t^{i(t)}$. The process $i(t)$ has transitions governed by the infinitesimal generator

$$
\Lambda \equiv \begin{pmatrix} -\lambda & \lambda q_0 & \lambda(1 - q_0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

Using Theorem 19.6 (and following similar calculations to the ones in Examples 2 and 3 therein) we get that

$$
d\pi_t^u = -\lambda \pi_t^u dt + \pi_t^u \left(\frac{(\ell_t^u - \ell_t^b)(1 - \pi_t^u) - (\ell_t^g - \ell_t^b)\pi_t^g}{\pi_t^u \ell_t^u + \pi_t^g \ell_t^g + \pi_t^b \ell_t^b} \right) \cdot [dL_t - m(\pi_t^u \ell_t^u + \pi_t^g \ell_t^g + \pi_t^b \ell_t^b)dt]
$$
$$
+ \pi_t^u \left(\frac{(\chi_t^u - \chi_t^b)(1 - \pi_t^u) - (\chi_t^g - \chi_t^b)\pi_t^g}{\pi_t^u \chi_t^u + \pi_t^g \chi_t^g + \pi_t^b \chi_t^b} \right) \cdot [dM_t - m(\pi_t^u \chi_t^u + \pi_t^g \chi_t^g + \pi_t^b \chi_t^b)dt]
$$

From here, we get that in absence of liquidation and market financing beliefs are given by

$$
\dot{\pi}_t^u = -\lambda \pi_t^u - m\pi_t^u \left((\ell_t^u + \chi_t^u - \ell_t^b - \chi_t^b)(1 - \pi_t^u) - (\ell_t^g + \chi_t^g - \ell_t^b - \chi_t^b)\pi_t^g \right)
$$

Suppose that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = \chi_t^u = \chi_t^g = \chi_t^b = 0$, then we have

$$
\dot{\pi}_t^u = -\lambda \pi_t^u + m\pi_t^u \pi_t^b
$$

Similarly, we get

$$
\dot{\pi}_t^g = \lambda q_0 \pi_t^u - m\pi_t^g \left[(\ell_t^g + \chi_t^g - \ell_t^b - \chi_t^b)(1 - \pi_t^g) - (\ell_t^u + \chi_t^u - \ell_t^b - \chi_t^b)\pi_t^u \right]
$$
$$
\dot{\pi}_t^b = \lambda(1 - q_0)\pi_t^u - m\pi_t^b \left[(\ell_t^b + \chi_t^b - \ell_t^g - \chi_t^g)(1 - \pi_t^b) - (\ell_t^u + \chi_t^u - \ell_t^g - \chi_t^g)\pi_t^u \right]
$$

so in the particular case that $\ell_t^b = 1$ and $\ell_t^u = \ell_t^g = \chi_t^u = \chi_t^g = \chi_t^b = 0$, then we get

$$\dot{\pi}_t^g = \lambda q_0 \pi_t^u + m \pi_t^g \pi_t^b$$
$$\dot{\pi}_t^b = \lambda(1 - q_0)\pi_t^u - m \pi_t^b(1 - \pi_t^b)$$

$\square$

A.1.2   Proof of Proposition 2

We start with two lemmas that will be useful for later proofs. Throughout, we sometimes use $\bar{V}^i$ to represent (4) evaluated at $q_{\tau_m} = \bar{q}$.

Lemma 4. *The No Deals condition implies the good type's value function must satisfy smooth-pasting at $t = t_g$. That is*

$$\dot{V}_{t_g}^g = 0.$$

Proof. We prove by contradiction. Suppose $\dot{V}_{t_g-}^g < 0$, then Equation (17b) in $[t_b, t_g]$ implies $V_t^g < \frac{c+\phi R}{(r+\phi)}$. However, this is impossible because $\frac{c+\phi R}{(r+\phi)}$ is the continuation value of the good types if they never finance with the market.

Next, let us assume $\dot{V}_{t_g-}^g > 0$. Under the constructed equilibrium, $\dot{q}_t = 0$ for any $t > t_b$. As a result, $\bar{V}_t^g$ – the continuation payoff when the good type financed with the market at time t also stays at a constant after t_b, which is denoted as $\bar{V}^g$. If $\dot{V}_{t_g-}^g > 0$, that implies that for ε sufficiently small, $V_{t_g-\varepsilon}^g < \bar{V}^g$ so that the *No Deals condition* fails. Note that this step relies on the fact that $\bar{V}_t^g$ stays a constant for $t \in [t_b, t_g]$. In the equilibrium without the zombie lending stage ($m < m^*$), this condition no longer holds so that in general, $\dot{V}_{t_g}^g \geq 0$.

$\square$

Lemma 5. *V_t^g stays at a constant in any equilibrium that is constructed under t_b and t_g.*[17]

Proof. We write out the HJB for the good types in all regions and the proof directly follows by plugging (30) into (29).

$$(r + \phi) V_t^g = \dot{V}_t^g + c + \phi R \qquad t \in [0, t_g] \qquad (29)$$
$$(r + \phi + m) V_t^g = \dot{V}_t^g + c + \phi R + m \bar{V}^g \qquad t \in [t_g, \infty). \qquad (30)$$

$\square$

---

[17]This is true under any equilibrium that we construct, which consists of thresholds $\{t_b, t_g\}$. However, it may not hold under any arbitrary equilibrium, which could exist when m gets very large.

Proof. By applying the smooth pasting condition

$$V_{t_g}^g = \frac{c + \phi R}{r + \phi} = \frac{c + \phi R + m\bar{V}^g}{r + \phi + m},$$

we get

$$\bar{q} = \frac{1}{(1-\theta)} \left(\frac{\delta + \phi}{r + \phi} \frac{c + \phi F}{\phi F} - \frac{c}{\phi F} - \theta \right)$$

after some derivations. Clearly, the equation system in the last region shows

$$V_{t_g}^g - V_{t_g}^b = \frac{\phi R \left(1 - \theta\right) + m \left(\bar{V}^g - \bar{V}^b\right)}{r + \phi + m} = \frac{\phi R \left(1 - \theta\right) + m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r + \phi + m}.$$

In that case, using the same smooth pasting condition, we get

$$V_{t_g}^b = \frac{c + \phi R}{r + \phi} - \frac{\phi R \left(1 - \theta\right) + m\frac{\phi(R-F)(1-\theta)}{r+\phi}}{r + \phi + m}.$$

Given that, let us solve for $t_g - t_b$ using the ODE system in region 2. In particular, for any $t \in [t_b, t_g]$,

$$V_t^b = e^{(r+\phi)(t-t_g)} V_{t_g}^b + \frac{c + \phi\theta R}{r + \phi} \left[1 - e^{(r+\phi)(t-t_g)}\right].$$

Using the boundary condition $V_{t_b}^b = L$, we can get

$$t_g - t_b = -\frac{1}{(r + \phi)} \log \left(\frac{L - \frac{c+\phi\theta R}{r+\phi}}{V_{t_g}^b - \frac{c+\phi\theta R}{r+\phi}} \right).$$

The threshold t_b is determine by the condition

$$t_b = \min\{t : q_t = \bar{q}\}.$$

The final step is to find the solution for q_t in the interval $[0, t_b]$. The ODE system in region 1 is

$$\begin{aligned} \dot{\pi}_t^u &= -\lambda \pi_t^u + m\pi_t^u \pi_t^b \\ \dot{\pi}_t^g &= \lambda \pi_t^u q_0 + m\pi_t^g \pi_t^b \\ \dot{\pi}_t^b &= \lambda \pi_t^u \left(1 - q_0\right) - m\pi_t^b \left(1 - \pi_t^b\right). \end{aligned}$$

Let us define $z_t = \frac{\pi_t^g}{\pi_t^u}$, then,

$$
\begin{aligned}
\dot{z}_t &= \frac{\dot{\pi}_t^g \pi_t^u - \pi_t^g \dot{\pi}_t^u}{(\pi_t^u)^2} = \frac{\dot{\pi}_t^g}{\pi_t^u} - z_t \frac{\dot{\pi}_t^u}{\pi_t^u} \\
&= \lambda q_0 + m z_t \left(1 - \pi_t^g - \pi_t^u\right) - z_t \left(-\lambda + m \left(1 - \pi_t^g - \pi_t^u\right)\right) \\
&= \lambda \left(q_0 + z_t\right).
\end{aligned}
$$

Therefore, we have the solution

$$
z_t = q_0 \left(e^{\lambda t} - 1\right)
$$
$$
\Rightarrow \pi_t^g = q_0 \left(e^{\lambda t} - 1\right) \pi_t^u. \tag{31}
$$

Since $\pi_t^u + \pi_t^g + \pi_t^b = 1$, we also have

$$
\pi_t^b = 1 - \left(q_0 e^{\lambda t} + 1 - q_0\right) \pi_t^u. \tag{32}
$$

Substituting (31) and (32) into the ODE system, we get a first-order ODE for π_t^u

$$
\dot{\pi}_t^u = (m - \lambda) \pi_t^u - m \left(q_0 e^{\lambda t} + 1 - q_0\right) (\pi_t^u)^2,
$$

which corresponds to a continuous-time Riccati equation. This equation can be transformed it into a second-order ODE. Let $v_t = -m \left(q_0 e^{\lambda t} + 1 - q_0\right) \pi_t^u$ and $Q_t = q_0 e^{\lambda t} + 1 - q_0$,

$$
\dot{v}_t = v_t^2 + \frac{v_t}{Q_t} \left[q_0 e^{\lambda t} \lambda + Q_t (m - \lambda)\right]. \tag{33}
$$

Further, if we let $v_t = -\frac{\dot{\nu}_t}{\nu_t} \Rightarrow \dot{v}_t = -\frac{\ddot{\nu}_t}{\nu_t} + (v_t)^2$, then we can transform equation (33) into the following second-order ODE

$$
\ddot{\nu}_t = \frac{\dot{\nu}_t}{Q_t} \left[q_0 e^{\lambda t} \lambda + Q_t (m - \lambda)\right]
$$

From here, we get that

$$
\dot{\nu}_t = \dot{\nu}(0) e^{\int_0^t \lambda \frac{1}{1 + \frac{1 - q_0}{q_0} e^{-\lambda s}} ds + (m - \lambda) t}
$$

Moreover,

$$
\int_0^t \frac{1}{1 + \frac{1 - q_0}{q_0} e^{-\lambda s}} ds = \frac{1}{\lambda} \log \left(1 - q_0 + q_0 e^{\lambda t}\right)
$$

so

$$
\dot{\nu}_t = \dot{\nu}(0) \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}} e^{(m - \lambda) t}
$$

Integrating one more time, we get

$$\nu_t = \nu(0) + \dot\nu(0) \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds.$$

Using the definition of v_t and ν_t, we have

$$\dot\nu_0 = -v_0\nu(0) = m\nu(0)$$

so

$$\nu_t = \nu(0) \left(1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds\right).$$

Using the definition of v_t we get

$$v_t = -\frac{m\left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)t}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}$$

so

$$\pi_t^u = \frac{\left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{(m-\lambda)t}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds} \tag{34}$$

Thus, substituting (34) in the definition for q_t, we get

$$\begin{aligned}
q_t &= \pi_t^g + q_0 \pi_t^u \\
&= q_0 \left(e^{\lambda t} - 1\right) \pi_t^u + q_0 \pi_t^u \\
&= q_0 e^{\lambda t} \pi_t^u \\
&= \frac{q_0 \left(1 - q_0 + q_0 e^{\lambda t}\right)^{\frac{1}{\lambda}-1} e^{mt}}{1 + m \int_0^t \left(1 - q_0 + q_0 e^{\lambda s}\right)^{\frac{1}{\lambda}} e^{(m-\lambda)s} ds}
\end{aligned} \tag{35}$$

Because q_t is monotone, the solution for t_b and t_g is unique.

Finally, we examine $t_g - t_b$, which equals

$$\frac{1}{r+\phi} \log\left(\frac{(r+\phi)V_{t_g}^b - (c + \phi\theta R)}{(r+\phi)L - (c + \phi\theta R)}\right).$$

A necessary condition for the equilibrium to be true is $t_g - t_b > 0$. However, if $m = 0$, this is clearly violated because Assumption 1 guarantees $V_{t_g}^b < L$. If $m \to \infty$,

$$V_{t_g}^b \to \frac{c + \phi R}{r+\phi} - \frac{\phi\left(R - F\right)\left(1 - \theta\right)}{r+\phi}$$

so that it exceeds L. Finally, a quick comparative static analysis shows that $\frac{dV_{t_g}^b}{dm} > 0$. Therefore, there exists a unique $\underline{m}$ so that such an equilibrium exists if and only if $m > \underline{m}$. $\square$

A.1.3 Proof of Proposition 3

Proof. We have already shown that the unique monotone equilibrium is the one in Proposition 2. It is only left to show that any equilibrium must be monotone if m is low enough. The proof for monotonicity follows the traditional skimming property in bargaining models. In our case, the skimming property is satisfied only if m is low enough. In particular, we show that

$$V_t^g - \bar{V}^g > V_t^u - \bar{V}^u > V_t^b - \bar{V}^b. \tag{36}$$

Let $x_t^i \in \{0, 1\}$ and $\ell_t^i \in \{0, 1\}$ be the rollover and liquidation decision, respectively. The expected payoff, given strategy (x^i, ℓ^i) is

$$V_t^u = \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)} cF ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} \left[q_0 + (1-q_0)\theta \right] R \right. \right.$$

$$\left. \left. + \mathbb{1}_{\tau=\tau_\lambda} \left[q_0 V_\tau^g + (1-q_0) V_\tau^b \right] + \mathbb{1}_{\tau=\tau_m} \left[x_\tau^u V_\tau^u + \ell_\tau^u L + (1 - x_\tau^u - \ell_\tau^u) \bar{V}^u \right] \right] \right\}$$

$$V_t^g = \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)} cF ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} R + \mathbb{1}_{\tau=\tau_m} \left[x_\tau^g V_\tau^g + \ell_\tau^g L + (1 - x_\tau^g - \ell_\tau^g) \bar{V}^g \right] \right] \right\}$$

$$V_t^b = \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)} cF ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} \theta R + \mathbb{1}_{\tau=\tau_m} \left[x_\tau^b V_\tau^b + \ell_\tau^b L + (1 - x_\tau^b - \ell_\tau^b) \bar{V}^b \right] \right] \right\}.$$

A good type can always mimic the strategy of a low type, hence the continuation payoff of a good type must at least as high as the payoff of mimicking the strategy of the bad type.

$$V_t^g \geq \mathbb{E}_t \left\{ \int_t^\tau e^{-r(s-t)} cF ds + e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} R + \mathbb{1}_{\tau=\tau_m} \left[x_\tau^b V_\tau^g + \ell_\tau^b L + (1 - x_\tau^b - \ell_\tau^b) \bar{V}^g \right] \right] \right\}.$$

Hence, for all $t < \tau$, we have

$$V_t^g - V_t^b \geq \mathbb{E}_t \left\{ e^{-r(\tau-t)} \left[\mathbb{1}_{\tau=\tau_\phi} (1-\theta) R + \mathbb{1}_{\tau=\tau_m} \left[x_\tau^b (V_\tau^g - V_\tau^b) + (1 - x_\tau^b - \ell_\tau^b)(\bar{V}^g - \bar{V}^b) \right] \right] \right\}.$$

Because the time $\tau = \min\{\tau_\phi, \tau_m\}$ is exponentially distributed with mean arrival rate $m + \phi$, we we can write the previous expression as

$$V_t^g - V_t^b \geq \frac{\phi(1-\theta)R}{r+\phi+m} + \int_t^\infty e^{-(r+m+\phi)(s-t)} m \left[x_\tau^b(V_s^g - V_s^b) + (1 - x_s^b - \ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds,$$

$$\bar{V}^g - \bar{V}^b = \frac{\phi(1-\theta)(R-F)}{r+\phi} > 0.$$

Letting $\Delta_t \equiv (V_t^g - \bar{V}^g) - (V_t^b - \bar{V}^b)$, we can write the previous inequality as

$$\Delta_t = A + \int_t^\infty e^{-(r+m+\phi)(s-t)} m \left[x_s^b \Delta_\tau + (1 - \ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds,$$

where

$$A \equiv \frac{\phi(1-\theta)}{(r+\phi+m)(r+\phi)} \left[(r+\phi)F - m(R-F) \right]$$

Differentiating Δ_t, we get the differential equation

$$\dot{\Delta}_t = \left(r + m(1 - x_t^b) + \phi \right) \Delta_t + m(1 - \ell_t^b)(\bar{V}^g - \bar{V}^b) - (r + m + \phi)A.$$

The solution to this equation is given by

$$\Delta_t = \int_t^\infty e^{-(r+\phi)(s-t)+\Psi(s,t)} \left[\phi(1-\theta) \left(F - \frac{m}{r+\phi}(R-F) \right) + \frac{m}{m+\phi}(1 - \ell_s^b)(\bar{V}^g - \bar{V}^b) \right] ds$$

where

$$\Psi(s,t) \equiv \int_t^s m(1 - x_u^b) du$$

From here we get that if

$$m < \bar{m} = (r+\phi)\frac{F}{R-F}$$

then $\Delta_t > 0$ for any policy ℓ_τ^b. Hence, in any equilibrium we have that

$$V_t^g - \bar{V}^g > V_t^b - \bar{V}^b,$$

which means that if there is a time $\tilde{t}_g$ at which the good type chooses market financing with positive probability, then the bad type chooses market financing for sure. Repeating the same calculations for the pairs $\{g, u\}$ and $\{u, b\}$, we can conclude that the skimming property (36) holds.

Using the definition $q_t = q_0 \pi_t^u + \pi_t^g$ and the evolution of beliefs in the proof of Lemma 1,

we get that the evolution of q_t is given by the following differential equation.

$$\dot{q}_t = mq_t \left[(\ell_t^b + x_t^b)(1 - \pi_t^u - \pi_t^g) - (\ell_t^u + x_t^u)(1 - \pi_t^u) + (\ell_t^g + x_t^g)\pi_t^g \right]$$
$$+ m\pi_t^g \left[(\ell_t^u + x_t^u) - (\ell_t^g + x_t^g) \right]$$

If $\ell_t^g + x_t^g = \ell_t^u + x_t^u = 0$ and $\ell_t^b + x_t^b = 1$ then $\dot{q}_t = mq_t(1 - \pi_t^u - \pi_t^g) > 0$. On the other hand, by the skimming property, we have that:

1. If $\ell_t^g + x_t^g = 1$, then $\ell_t^u + x_t^u = \ell_t^b + x_t^b = 1$ so $\dot{q}_t = 0$.

2. If $\ell_t^b + x_t^b = 0$, then $\ell_t^g + x_t^g = \ell_t^u + x_t^u = 0$ and also $\dot{q}_t = 0$.

3. If $\ell_t^g + x_t^g = 0$ and $\ell_t^u + x_t^u = 1$, then $\ell_t^b + x_t^b = 1$ so $\dot{q}_t = m\pi_t^g(1 - q_t) > 0$.

Hence, in any equilibrium, the trajectory of q_t must be non-decreasing in time so an equilibrium must be monotone.

$\square$

A.1.4   Proof of Proposition 4

Proof. Lemma 2 shows that the constraint $y \le c$ must bind at time t_b. However, depending on the parameters of the problem, the constraint might be slack at time t_g. In particular, we can show that the constraint on $[t_b, \infty)$ is monotonic, that is, there exists t_c such that $y_t^b = c$ on $[t_b, t_c]$ and $y_t^b < c$ after t_c. Note that if such a t_g satisfies $t_c > t_g$, then the financial constraint always binds until the borrower refinances with the market. As in the case in which we ignore the financial constraint, we have that

$$V_{t_g}^b = V_{t_g}^g + \frac{\phi R (\theta - 1) + m\frac{\phi(R-F)(\theta-1)}{r+\phi}}{r + \phi + m}$$
$$= \frac{c + \phi R}{r + \phi} + \frac{\phi R (\theta - 1) + m\frac{\phi(R-F)(\theta-1)}{r+\phi}}{r + \phi + m}.$$

Given the boundary, we solve for V_t^b

$$(r + \phi) V_t^b = \dot{V}_t^b + c + \phi\theta R$$
$$V_t^b = \frac{c + \phi R\theta}{r + \phi} + e^{(r+\phi)(t-t_g)} \left[V_{t_g}^b - \frac{c + \phi R\theta}{r + \phi} \right]$$

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At $t = t_c$, the financial constraint exactly binds so that it must be

$$B_t^b(c) = L + (1 - \beta)\left(V_t^b - L\right)$$

$$L + (1 - \beta)\left(V_t^b - L\right) - B_t^b(rF) \equiv \frac{c - rF}{r + \phi + m}, \tag{37}$$

where $B_t^b(rF)$ solves

$$(r + \phi + m)\, B_t^b(rF) = \dot{B}_t^b(rF) + rF + \phi\theta F + m\left[L + (1 - \beta)\left(V_t^b - L\right)\right], \qquad t \in (t_c, t_g).$$

Let's define $Z_t \equiv L + (1 - \beta)\left(V_t^b - L\right) - B_t^b(rF)$. Substituting the ODEs for V_t^b and $B_t^b(rF)$, and defining the constant

$$\Gamma_1 \equiv (r + \phi)\,\beta L + (1 - \beta)\,(c + \phi\theta R) - (r + \phi\theta)\,F,$$

we get the following ODE for Z_t on (t_c, t_g)

$$(r + \phi + m)\, Z_t = \dot{Z}_t + \Gamma_1 \qquad t \in (t_c, t_g). \tag{38}$$

At time t_g we have

$$B_{t_g}^b(rF) = \frac{\phi\theta + r + m}{r + \phi + m}F,$$

which means that we can solve Z_{t_g} up to primitives according to

$$Z_{t_g} = L + (1 - \beta)\left(V_{t_g}^b - L\right) - B_{t_g}^b(rF).$$

Solving (38) backward in time and combining with equation (37) we get

$$Z_{t_c} = \frac{\Gamma_1}{r + \phi + m} + e^{(r + \phi + m)(t_c - t_g)}\left[Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}\right] = \frac{c - rF}{r + \phi + m}.$$

From here, we can solve for $t_c - t_g$, which is given by

$$t_c - t_g = \frac{1}{r + \phi + m}\log\left(\frac{c - rF - \Gamma_1}{r + \phi + m}\frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}}\right).$$

Next, letting

$$\Gamma_2 \equiv rF + \phi\theta F + m\frac{c - rF}{r + \phi + m}$$

we can find $B_t^b(rF)$ for $t \in (t_b, t_c)$ solving the following ODE

$$(r + \phi) B_t^b = \dot{B}_t^b + \Gamma_2, \qquad t \in (t_b, t_c),$$

with initial condition

$$B_{t_b}^b(rF) + \frac{c - rF}{r + \phi + m} = L.$$

The solution at time t_c is

$$B_{t_c}^b(rF) = \frac{\Gamma_2}{r + \phi} + e^{(r+\phi)(t_c - t_b)} \left[L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi} \right]. \tag{39}$$

From equation (37), we also know that at t_c

$$B_{t_c}^b(rF) = L + (1 - \beta) \left(V_{t_c}^b - L \right) - \frac{c - rF}{r + \phi + m}, \tag{40}$$

where

$$V_{t_c}^b = \frac{c + \phi R\theta}{r + \phi} + \left[V_{t_g}^b - \frac{c + \phi R\theta}{r + \phi} \right] \left[\frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right]^{\frac{r + \phi}{r + \phi + m}}.$$

Combining equations (39) and (40) we get

$$\frac{\Gamma_2}{r + \phi} + e^{(r+\phi)(t_c - t_b)} \left[L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi} \right] = \Gamma_3$$

$$t_c - t_b = \frac{1}{r + \phi} \log \left(\frac{\Gamma_3 - \frac{\Gamma_2}{r + \phi}}{L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi}} \right),$$

where

$$\Gamma_3 \equiv L + (1 - \beta) \left(\frac{c + \phi R\theta}{r + \phi} + \left[V_{t_g}^b - \frac{c + \phi R\theta}{r + \phi} \right] \left[\frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right]^{\frac{r + \phi}{r + \phi + m}} - L \right) - \frac{c - rF}{r + \phi + m}$$

Thus, we get that

$$t_b = t_g + \frac{1}{r + \phi + m} \log \left(\frac{c - rF - \Gamma_1}{r + \phi + m} \frac{1}{Z_{t_g} - \frac{\Gamma_1}{r + \phi + m}} \right) - \frac{1}{r + \phi} \log \left(\frac{\Gamma_3 - \frac{\Gamma_2}{r + \phi}}{L - \frac{c - rF}{r + \phi + m} - \frac{\Gamma_2}{r + \phi}} \right).$$

The previous solution can be simplified significantly when, $y_t = c$ for all $t \in [t_b, t_g]$. This happens if

$$\beta L + (1 - \beta) V_{t_g}^b > B_{t_g}^b(rF) + \frac{c - rF}{r + \phi + m},$$

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which reduces to

$$L - \frac{c + (\phi\theta + m)F}{r + \phi + m} + (1 - \beta)\left[\frac{c}{r + \phi} + \frac{\phi}{r + \phi}\left((1 - \theta)\frac{mF}{r + \phi + m} + \theta R\right) - L\right] > 0$$

In this case, let $B^i_{\max}(t|t_g)$ be the maximum continuation value that a bank can obtain at time t given that the loan rate after time t is c and the firm switches to market financing after time t_g.

$$B^b_{\max}(t|t_g) = \frac{c + \phi\theta F}{r + \phi}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}\frac{c + (\phi\theta + m)F}{r + \phi + m}$$

In equilibrium, the thresholds $\{t_b, t_g\}$ must be such $B^b_{\max}(t|t_g) \geq L$ for all $t \in (t_b, t_g)$, and in particular

$$B^b_{\max}(t_b|t_g) = \frac{c + \phi\theta F}{r + \phi}\left(1 - e^{-(r+\phi)(t_g-t_b)}\right) + e^{-(r+\phi)(t_g-t_b)}\frac{c + (\phi\theta + m)F}{r + \phi + m} = L$$

which implies (25). $\qquad\square$

A.1.5 Proof of Proposition 5

The following Lemma is very useful for the proof:

Lemma 6. *For any* $t \in (t_b, t_g)$,

$$V^u_t = q_0 V^g_t + (1 - q_0)V^b_t$$

Proof. On $t_a \in (t_b, t_g)$, the continuation values V^i_t follow

$$(r + \phi)V^u_t = \dot{V}^u_t + c + \phi\left[q_0 + (1 - q_0)\theta\right]R$$
$$(r + \phi)V^g_t = \dot{V}^g_t + c + \phi R$$
$$(r + \phi)V^b_t = \dot{V}^b_t + c + \phi\theta R,$$

which implies

$$V^u_t = \frac{c + \phi\left[q_0 + (1 - q_0)\theta\right]R}{r + \phi}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}V^u_{t_g}$$
$$V^g_t = \frac{c + \phi R}{r + \phi}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}V^g_{t_g}$$
$$V^b_t = \frac{c + \phi\theta R}{r + \phi}\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}V^b_{t_g}.$$

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Therefore,

$$q_0 V_t^g + (1 - q_0)V_t^b - V_t^u = e^{-(r+\phi)(t_g-t)} \frac{m}{r + \phi + m} \left(q_0 \bar{V}^g + (1 - q_0)\bar{V}^b - \bar{V}^u \right),$$

where $\bar{V}^i$ is derived from (4) with $q_{\tau_m} = \bar{q}$:

$$\bar{V}^u = \frac{c + \phi \left[\bar{q} + (1 - \bar{q}) \theta \right] F}{\delta + \phi} + \frac{\phi[q_0 + (1 - q_0)\theta] (R - F)}{r + \phi}$$

$$\bar{V}^b = \frac{c + \phi \left[\bar{q} + (1 - \bar{q}) \theta \right] F}{\delta + \phi} + \frac{\phi\theta (R - F)}{r + \phi}$$

$$\bar{V}^g = \frac{c + \phi \left[\bar{q} + (1 - \bar{q}) \theta \right] F}{\delta + \phi} + \frac{\phi (R - F)}{r + \phi},$$

$q_0 \bar{V}^g + (1 - q_0)\bar{V}^b - \bar{V}^u = 0$, and therefore $q_0 V_t^g + (1 - q_0)V_t^b - V_t^u = 0$. $\qquad \square$

Let us first show that in any equilibrium characterized by $\{t_b, t_g\}$, banks never learn after t_b. To see this, suppose that instead the bank learns during $[t_b, t_a]$, where $t_a > t_b$, the HJB equations are as follows for $t \in [t_a, t_g]$

$$(r + \phi + m) B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F + m[L + (1 - \beta)(V_t^u - L)]$$

$$(r + \phi + m) B_t^g = \dot{B}_t^g + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]$$

$$(r + \phi + m) B_t^b = \dot{B}_t^b + y_t F + \phi\theta F + m[L + (1 - \beta)(V_t^b - L)].$$

From here, we get

$$(r + \phi + m) \Gamma_t = \dot{\Gamma}_t + m(1 - \beta)(q_0 V_t^g + (1 - q_0)V_t^b - V_t^u) = \dot{\Gamma}_t,$$

where $\Gamma_t \equiv q_0 B_t^g + (1 - q_0) B_t^b - B_t^u$, and the second equality follows from Lemma 6. Since $\Gamma_{t_g} = 0$, it implies that $\Gamma_t = 0$ for $\forall t \in [t_a, t_g]$. In this case, the net benefit of learning $\Omega_t = \Gamma_t - \frac{\psi}{\lambda}$ is always negative at t_a. Therefore, if learning happens in an interval, this interval must be a sub-interval of $[0, t_b]$.

Next, we prove that it if learning happens at all, it must be that the bank learns during $[0, t_a]$, where $t_a < t_b$. Note that the threshold t_b is given by the condition $q_{t_b} = \bar{q}$. Using the fact that $\Gamma_{t_b} = 0$, we can solve Γ_t backward in time and solve for t_a such that $\Gamma_{t_a} = \psi/\lambda$. Once we have solved for $\{t_a, t_b, t_g\}$, the only step left is to verify that Γ_t single-crosses ψ/λ

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from above at time t_a. Consider the regions $t < t_a$, in this region, we have

$$(r + \phi + m) B_t^u = \dot{B}_t^u + y_t F + \phi[q_0 + (1 - q_0)\theta]F - \psi + m[L + (1 - \beta)(V_t^u - L)] + \lambda\Gamma_t$$
$$(r + \phi + m) B_t^g = \dot{B}_t^g + y_t F + \phi F + m[L + (1 - \beta)(V_t^g - L)]$$
$$(r + \phi + m) B_t^b = \dot{B}_t^b + y_t F + \phi\theta F + mL,$$

so

$$(r + \phi + m + \lambda)\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)((1 - q_0)L + q_0 V_t^g - V_t^u) + \psi$$

Let $H_t \equiv (1 - q_0)L + q_0 V_t^g - V_t^u$, and combine the previous ODE with the ODE for Γ_t on (t_a, t_b) we get

$$(r + \phi + m + \lambda)\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)H_t + \psi, \ t \in [0, t_a]$$
$$(r + \phi + m)\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)H_t, \ t \in [t_a, t_b]$$

Take left and right limit at t_a and $\dot{\Gamma}_{t_a-} = \dot{\Gamma}_{t_a+}$. Let $\Omega_t \equiv \Gamma_t - \psi/\lambda$, and $\eta \equiv (r + \phi + m)\frac{\psi}{\lambda}$, so

$$(r + \phi + m + \lambda)\Omega_t = \dot{\Omega}_t + m(1 - \beta)H_t - \eta, \ t \in (0, t_a)$$
$$(r + \phi + m)\Omega_t = \dot{\Omega}_t + m(1 - \beta)H_t - \eta, \ t \in (t_a, t_b)$$

Differentiating Ω_t, we get

$$(r + \phi + m + \lambda)\dot{\Omega}_t = \ddot{\Omega}_t + m(1 - \beta)\dot{H}_t, \ t \in (0, t_a)$$
$$(r + \phi + m)\dot{\Omega}_t = \ddot{\Omega}_t + m(1 - \beta)\dot{H}_t, \ t \in (t_a, t_b).$$

If we can prove $\dot{H}_t \leq 0$ on $(0, t_b)$, then once $\dot{\Omega}_t = 0$, it immediately implies $\ddot{\Omega}_t \geq 0$. Hence, $\dot{\Omega}_t$ single crosses 0 from negative to positive. This implies that if $\dot{H}_t \leq 0$, then Ω_t is quasi-convex on $(0, t_b)$, which means that if $\Omega_{t_a} = 0$ and $\Omega_{t_b} < 0$ (which necessarily holds since $\Gamma_{t_b} = 0$) it must be the case that $\Omega_t \geq 0$ for $t < t_a$ and $\Omega_t \leq 0$ on (t_a, t_b). Therefore, it only remains to show to show that $\dot{H}_t \leq 0$. We have

$$(r + \phi + \lambda) H_t = \dot{H}_t + (r + \phi)(1 - q_0)L - (1 - q_0)(c + \phi\theta R) + \psi - \lambda(1 - q_0)(V_t^b - L), \ t \in [0, t_a]$$
$$(r + \phi) H_t = \dot{H}_t + (r + \phi)(1 - q_0)L - (1 - q_0)(c + \phi\theta R), \ t \in [t_a, t_b],$$

where $H_{t_b} = (1 - q_0)V_{t_b}^b + q_0 V_{t_b}^g - V_{t_b}^u = 0$. Under Assumption 1, this implies that $\dot{H}_{t_b} < 0$.

Differentiating the previous equation we get

$$(r + \phi + \lambda)\dot{H}_t = \ddot{H}_t - \lambda(1 - q_0)\dot{V}_t^b, \ t \in (0, t_a)$$
$$(r + \phi)\dot{H}_t = \ddot{H}_t, \ t \in (t_a, t_b).$$

Since $\dot{V}_t^b \geq 0$, we immediately get the result that $\ddot{H}_t \geq 0$ if $\dot{H}_t = 0$. Hence, $\dot{H}_t$ single crosses 0 from negative to positive, so $\dot{H}_{t_b} < 0 \Rightarrow \dot{H}_t < 0$, $\forall t \in (0, t_b)$.

Finally, we provide conditions to characterize the equilibrium and therefore the parametric assumptions needed to validate it. Note that in the equilibrium characterized by $\{t_a, t_b, t_g\}$, beliefs evolve on $t \in (t_a, t_b)$ according to

$$\dot{\pi}_t^u = m\pi_t^u \pi_t^b, \qquad \dot{\pi}_t^g = m\pi_t^g \pi_t^b, \qquad \dot{\pi}_t^b = -m\pi_t^b(1 - \pi_t^b).$$

In particular,

$$\dot{q}_t = mq_t\pi_t^b,$$

so

$$q_t = q_{t_a}e^{m\int_{t_a}^t \pi_s^b ds}$$

Solving for π_t^b we get

$$\pi_t^b = \frac{\pi_{t_a}^b}{\pi_{t_a}^b + (1 - \pi_{t_a}^b)e^{m(t-t_a)}}.$$

We have that

$$m\int_{t_a}^t \pi_s^b ds = \int_{t_a}^t \frac{-\dot{\pi}_s^b}{1 - \pi_s^b}ds$$
$$= \log(1 - \pi_s^b)\Big|_{t_a}^t$$
$$= \log\left(\frac{1 - \pi_t^b}{1 - \pi_{t_a}^b}\right)$$

so

$$e^{m\int_{t_a}^t \pi_s^b ds} = \frac{1 - \pi_t^b}{1 - \pi_{t_a}^b}$$
$$= \frac{e^{m(t-t_a)}}{\pi_{t_a}^b + (1 - \pi_{t_a}^b)e^{m(t-t_a)}}$$

so

$$q_t = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t-t_a)}}q_{t_a}.$$

To find $\pi^b_{t_a}$, we use equations (32), (34), and (35) in region $[0, t_a]$:

$$\pi^b_{t_a} = 1 - \frac{q_{t_a}}{q_0}\left(q_0 + (1 - q_0)e^{-\lambda t_a}\right).$$

Notice that as $t \to \infty$ we have that

$$\frac{1}{1 - \pi^b_{t_a} + \pi^b_{t_a}e^{-m(t-t_a)}}q_{t_a} \to \frac{q_{t_a}}{1 - \pi^b_{t_a}}$$

This limit is greater than $\bar{q}$ if and only if

$$
\begin{aligned}
q_{t_a} &\geq (1 - \pi^b_{t_a})\bar{q} = \frac{q_{t_a}}{q_0}\left(q_0 + (1 - q_0)e^{-\lambda t_a}\right)\bar{q}\\
\Rightarrow t_a &\geq \underline{t}_a \equiv \frac{1}{\lambda}\log\left(\frac{\bar{q}}{1 - \bar{q}}\frac{1 - q_0}{q_0}\right).
\end{aligned}
$$

Next, we derive a system of equations for t_a, t_b, t_g. For $t \in (t_a, t_b)$ we have the following ODE for B^i_t

$$
\begin{aligned}
(r + \phi + m)\, B^u_t &= \dot{B}^u_t + y_t F + \phi[q_0 + (1 - q_0)\theta]F + m[L + (1 - \beta)(V^u_t - L)]\\
(r + \phi + m)\, B^g_t &= \dot{B}^g_t + y_t F + \phi F + m[L + (1 - \beta)(V^g_t - L)]\\
(r + \phi + m)\, B^b_t &= \dot{B}^b_t + y_t F + \phi\theta F + mL.
\end{aligned}
$$

Thus, for $\Gamma_t \equiv q_0 B^g_t + (1 - q_0)\, B^b_t - B^u_t$,

$$(r + \phi + m)\,\Gamma_t = \dot{\Gamma}_t + m(1 - \beta)((1 - q_0)L + q_0 V^g_t - V^u_t)$$

with boundary $\Gamma_{t_b} = 0$. Thus,

$$\Gamma_t = \int_t^{t_b} e^{-(r+\phi+m)(s-t)}m(1 - \beta)\left[(1 - q_0)L + q_0 V^g_s - V^u_s\right] ds.$$

Next,

$$
\begin{aligned}
V^u_t &= \frac{c + \phi\left[q_0 + (1 - q_0)\,\theta\right]R}{r + \phi}(1 - e^{-(r+\phi)(t_g-t)}) + e^{-(r+\phi)(t_g-t)}V^u_{t_g}\\
q_0 V^g_t &= \frac{q_0 c + q_0\phi R}{r + \phi}(1 - e^{-(r+\phi)(t_g-t)}) + e^{-(r+\phi)(t_g-t)}q_0 V^g_{t_g}
\end{aligned}
$$

so

$$(1-q_0)L + q_0 V_t^g - V_t^u = (1-q_0)\left[L - \frac{c+\phi\theta R}{r+\phi}(1 - e^{-(r+\phi)(t_g-t)})\right] + e^{-(r+\phi)(t_g-t)}(q_0 V_{t_g}^g - V_{t_g}^u)$$

$$= (1-q_0)\left[L - \frac{c+\phi\theta R}{r+\phi} + e^{-(r+\phi)(t_g-t)}\left(\frac{c+\phi\theta R}{r+\phi} - V_{t_g}^b\right)\right]$$

Thus, we get

$$\Gamma_t = \frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L - \frac{c+\phi\theta R}{r+\phi}\right)\left(1 - e^{-(r+\phi+m)(t_b-t)}\right) +$$

$$(1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi} - V_{t_g}^b\right)e^{-(r+\phi)(t_g-t)}\left(1 - e^{-m(t_b-t)}\right).$$

Substituting $V_{t_g}^b$ we get the following equation for t_a:

$$\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L - \frac{c+\phi\theta R}{r+\phi}\right)\left(1 - e^{-(r+\phi+m)(t_b-t_a)}\right) +$$

$$(1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi} - \frac{c+\phi\theta R+m\bar{V}^b}{r+\phi+m}\right)e^{-(r+\phi)(t_g-t_a)}\left(1 - e^{-m(t_b-t_a)}\right) = \frac{\psi}{\lambda}. \quad (41)$$

Therefore, we have three equations to characterize the thresholds: $\{t_a, t_b, t_g\}$

$$\bar{q} = \frac{1}{1 - \pi_{t_a}^b + \pi_{t_a}^b e^{-m(t_b-t_a)}}q_{t_a}$$

$$\frac{\psi}{\lambda} = \frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L - \frac{c+\phi\theta R}{r+\phi}\right)\left(1 - e^{-(r+\phi+m)(t_b-t_a)}\right)$$

$$+ (1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi} - \frac{c+\phi\theta R+m\bar{V}^b}{r+\phi+m}\right)e^{-(r+\phi)(t_g-t_a)}\left(1 - e^{-m(t_b-t_a)}\right).$$

$t_g - t_b$ is unchanged from Proposition 2, which can be simplified to

$$t_g = t_b + \frac{1}{r+\phi}\log\left(\frac{m}{r+\phi+m}\frac{(1-\theta)\phi F}{(r+\phi)L - (c+\phi\theta R)}\right).$$

The final step for the equilibrium to hold is to verify conditions for bank's learning policy. Let $\bar{t}_a$ be the threshold the first time $q_t = \bar{q}$ in the benchmark model in which $\psi = 0$, which is the same as if $t_a = t_b$. On the other hand, recall that $\underline{t}_a \equiv \frac{1}{\lambda}\log\left(\frac{\bar{q}}{1-\bar{q}}\frac{1-q_0}{q_0}\right)$ in which case the we have that $\inf\{t > t_a : q_t = \bar{q}\} = \infty$. In particular, we need to verify that $t_a \in [\underline{t}_a, \bar{t}_a]$.

We have already shown that if $t_a = \bar{t}_a$, we have $\Gamma_{t_a} = 0 < \psi/\lambda$. Hence, we only need to show that if $t_a = \underline{t}_a$, then $\Gamma_{t_a} > \psi/\lambda$. In this case both $t_g \to \infty$ and $t_b \to \infty$, which means

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that

$$\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)\left(1-e^{-(r+\phi+m)(t_g-t_a)}\right)$$
$$+ (1-\beta)(1-q_0)\left(\frac{c+\phi\theta R}{r+\phi}-\frac{c+\phi\theta R+m\bar{V}^b}{r+\phi+m}\right)e^{-(r+\phi)(t_g-t_a)}\left(1-e^{-m(t_g-t_a)}\right)\rightarrow$$
$$\frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)$$

Hence, a solution exists if and only if

$$\frac{\psi}{\lambda} < \frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right)$$

Next, we show that if this condition is not satisfied, then there is no learning in equilibrium. Suppose that the firm never learns and never goes to the market. In this case, we have the value of the project being

$$V^u = NPV_r^u = \frac{c+\phi(q_0+(1-q_0)\theta)R}{r+\phi},$$

so that the value of bank at loan rate y is

$$B^u = \frac{yF + m(L+(1-\beta)(V^u-L)) + \phi(q_0+(1-q_0)\theta)R}{r+\phi+m}$$

Suppose that the bank is informed (which only occurs off the equilibrium path). In this case, for any loan rate y,

$$B^b = \frac{y+mL+\phi\theta F}{r+\phi+m}$$

and

$$B^g = \frac{y+m(L+(1-\beta)(V^g-L))+\phi F}{r+\phi+m},$$

where

$$V^g = \frac{c+\phi R}{r+\phi}.$$

Combining the previous expressions, we get that

$$q_0 B^g + (1-q_0)B^b - B^u = \frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L-\frac{c+\phi\theta R}{r+\phi}\right),$$

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which means that not learning is optimal if

$$\frac{\psi}{\lambda} \geq \frac{m(1-\beta)(1-q_0)}{r+\phi+m}\left(L - \frac{c+\phi\theta R}{r+\phi}\right)$$

A.1.6 Proof of Proposition 6

Proof. Next, we consider the case in which the entrepreneur is financially constrained. We consider the case in which the constraint $y_t F = c$ binds for all types. By similar arguments as the ones in the unconstrained case, the bank never monitors after time t_b. Hence, we can restrict attention to the case in which $t_a < t_b$. If $y_t = c$, the value function of the bank satisfies the following HJB equation on (t_a, t_b)

$$(r+\phi)\, B_t^u = \dot{B}_t^u + c + \phi[q_0 + (1-q_0)\theta]F$$
$$(r+\phi)\, B_t^g = \dot{B}_t^g + c + \phi F$$
$$(r+\phi+m)\, B_t^b = \dot{B}_t^b + c + \phi\theta F + mL.$$

It follows that Γ_t satisfies the following ODE

$$(r+\phi)\Gamma_t = \dot{\Gamma}_t + m(1-q_0)(L - B_t^b)$$
$$= \dot{\Gamma}_t + m(1-q_0)\left(L - \frac{c+\phi\theta F + mL}{r+\phi+m}\left(1 - e^{-(r+\phi+m)(t_b-t)}\right) - e^{-(r+\phi+m)(t_b-t)}L\right)$$
$$= \dot{\Gamma}_t - m(1-q_0)\frac{c+\phi\theta F - (r+\phi)L}{r+\phi+m}\left(1 - e^{-(r+\phi+m)(t_b-t)}\right)$$

From here, we get that

$$\Gamma_t = \int_t^{t^b} e^{-(r+\phi)(s-t)} m(1-q_0)\frac{(r+\phi)L - c - \phi\theta F}{r+\phi+m}\left(1 - e^{-(r+\phi+m)(t_b-s)}\right) ds$$
$$= \frac{m(1-q_0)}{r+\phi}\frac{(r+\phi)L - c - \phi\theta F}{r+\phi+m}\left(1 - e^{-(r+\phi)(t_b-t)}\right)$$
$$- (1-q_0)\frac{(r+\phi)L - c - \phi\theta F}{r+\phi+m}e^{-(r+\phi)(t_b-t)}\left(1 - e^{-m(t_b-t)}\right)$$

It follows that

$$\frac{\psi}{\lambda} = (1-q_0)\frac{(r+\phi)L - c - \phi\theta F}{r+\phi+m}\left(\frac{m}{r+\phi}\left(1 - e^{-(r+\phi)(t_b-t_a)}\right)\right.$$
$$\left. - e^{-(r+\phi)(t_b-t_a)}\left(1 - e^{-m(t_b-t_a)}\right)\right) \quad (42)$$

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As before, when t_a converges to $\frac{1}{\lambda}\log\left(\frac{\bar{q}}{1-\bar{q}}\frac{1-q_0}{q_0}\right)$ the threshold t_b converges to infinity. Thus, as in the proof of Proposition 5, a solution exists if

$$\frac{\psi}{\lambda} < \frac{m(1-q_0)}{r+\phi+m}\left(L - \frac{c+\phi\theta F}{r+\phi}\right) \tag{43}$$

Finally, we need to verify that Γ_t single crosse ψ/λ. Because B_t^b is increasing on $(0, t_b)$, it follows that Γ_t is quasi-convex on (t_a, t_b). On $(0, t_a)$, Γ_t satisfies

$$(r+\phi+\lambda)\Gamma_t = \dot{\Gamma}_t - m(1-q_0)(B_t^b - L) + \psi.$$

Again, we can verify that Γ_t is quasi-convex on $(0, t_a)$. Moreover, as $\dot{\Gamma}_{t_a-} = \dot{\Gamma}_{t_a+}$, we can conclude that Γ_t is quasi-convex on $[0, t_b]$. From here we can conclude that if $\Gamma_{t_a} = \psi/\lambda > 0$ and $\Gamma_{t_b} = 0$, then it must be the case that $\Gamma_t \leq \psi/\lambda$ on (t_a, t_b). On the other hand, if there is $\tilde{t}_a < t_a$ such that $\Gamma_{\tilde{t}_a} < \psi/\lambda$, then it must be that case that Γ_t has local maximum on $[0, t_b)$. However, this cannot be the case as Γ_t is quasi-convex. We can conclude that $\Gamma_t \geq \psi/\lambda$ for all $t < t_a$.

The rest of the equilibrium is determined as in the case with exogenous learning. The threshold t_a must be such $q_{t_b} = \bar{q}$, where q_t is given by the same expression as in the proof of Proposition 5, while $t_g - t_b$ is the same as in Proposition 4.

The final step is to find conditions such that the constraint $y_t \leq c$ is binding. Letting $z_t^i = \beta L + (1-\beta)V_t^i - B_t^i$, we get that the constraint is binding if $z_t^i > 0$. Using the equations for V_t^i and B_t^i we can derive an ODE for z_t^i. In particular,

$$\dot{z}_t^g = (r+\phi)z_t^g - (r+\phi)\beta L + \beta c - \phi\big[(1-\beta)R - F\big]$$

$$\dot{z}_t^b = (r+\phi+m\mathbb{1}_{t<t_b})z_t^b - (r+\phi)\beta L + \beta c - \phi\theta[(1-\beta)R - F]$$

$$\dot{z}_t^u = (r+\phi+\lambda\mathbb{1}_{t<t_a})z_t^u - (r+\phi)\beta L + \beta(c - \mathbb{1}_{t<t_a}\psi) - \phi(q_0 + (1-q_0)\theta)[(1-\beta)R - F]$$
$$\quad - \lambda\mathbb{1}_{t<t_a}[q_0 z_t^g + (1-q_0)z_t^b]$$

Solving the previous expressions we get

$$z_t^g = \frac{1}{r+\phi}\Big((r+\phi)\beta L + \phi\big[(1-\beta)R - F\big] - \beta c\Big)\left(1 - e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}z_{t_g}^g$$

$$z_t^b = \int_t^{t_g} e^{-(r+\phi)(s-t)-m(s\wedge t_b-t)}\Big((r+\phi)\beta L + \phi\theta[(1-\beta)R - F] - \beta c\Big)ds + e^{-(r+\phi)(t_g-t)-m(t_b-t)}z_{t_g}^b$$

$$z_t^u = \int_t^{t_g} e^{-(r+\phi)(s-t)-\lambda(s\wedge t_a-t)}\Big((r+\phi)\beta L + \phi(q_0 + (1-q_0)\theta)[(1-\beta)R - F]$$
$$\quad - \beta(c - \mathbb{1}_{s<t_a}\psi) + \lambda\mathbb{1}_{s<t_a}[q_0 z_t^g + (1-q_0)z_t^b]\Big)ds + e^{-(r+\phi)(t_g-t)-\lambda(t_a-t)}z_{t_g}^u$$

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When β goes to zero we get that the previous expressions converge to

$$z_t^g = \frac{1}{r+\phi}\phi(R-F)\left(1-e^{-(r+\phi)(t_g-t)}\right) + e^{-(r+\phi)(t_g-t)}z_{t_g}^g$$

$$z_t^b = \int_t^{t_g} e^{-(r+\phi)(s-t)-m(s\wedge t_b-t)}\phi\theta(R-F)ds + e^{-(r+\phi)(t_g-t)-m(t_b-t)}z_{t_g}^b$$

$$z_t^u = \int_t^{t_g} e^{-(r+\phi)(s-t)-\lambda(s\wedge t_a-t)}\Big(\phi(q_0 + (1-q_0)\theta)(R-F)$$

$$+ \lambda\mathbb{1}_{s<t_a}[q_0 z_s^g + (1-q_0)z_s^b]\Big)ds + + e^{-(r+\phi)(t_g-t)-\lambda(t_a-t)}z_{t_g}^u$$

Noting that $z_{t_g}^u = q_0 z_{t_g}^g + (1-q_0)z_{t_g}^b$ we find if $z_{t_g}^g$ and $z_{t_g}^b$ are positive then the previous limit is positive for all $t \in [0, t_g]$. As verified in the proof of Proposition 4, $z_{t_g}^b$ is positive if

$$L - \frac{c+(\phi\theta+m)F}{r+\phi+m} + (1-\beta)\left[\frac{c}{r+\phi} + \frac{\phi}{r+\phi}\left((1-\theta)\frac{mF}{r+\phi+m} + \theta R\right) - L\right] > 0$$

On the other hand, $z_{t_g}^g$ is positive if

$$\beta L + (1-\beta)\frac{c+\phi R}{r+\phi} > \frac{c+(\phi+m)F}{r+\phi+m}.$$

When β goes to zero, the previous inequalities converge to

$$\frac{c+\phi((1-\theta)F+\theta R)}{r+\phi} > \frac{c+(\phi+m)F}{r+\phi+m}$$

$$\frac{c+\phi R}{r+\phi} > \frac{c+(\phi+m)F}{r+\phi+m}$$

Which means that it is enough to verify that

$$\frac{c+\phi((1-\theta)F+\theta R)}{r+\phi} > \frac{c+(\phi+m)F}{r+\phi+m}$$

$\square$

A.1.7   Proof of Proposition 10

Proof. With some abuse of notation, let V_n^i be the continuation value at the n-th rollover time (i.e. at time n/m. In that case, we have

$$V_{n_g-1}^g = \int_0^{\frac{1}{m}} e^{-(r+\phi)s}(c+\phi R)ds + e^{-\frac{r+\phi}{m}}V_{n_g}^g.$$

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To simplify notation, let

$$\nu^g \equiv \int_0^{\frac{1}{m}} e^{-(r+\phi)s} (c + \phi R) \, ds$$

be the flow payoff between two rollover events, which is time independent. We can rewrite $V_{n_g-1}^g = \nu^g + e^{-\frac{r}{m}} V_{n_g}^g$ which has to be greater or equal than $\bar{V}$. On the other hand, the *No Deals* condition also requires $V_{n_g-1}^g \geq \bar{V}$. Combining these two conditions we find that

$$\nu^g + e^{-\frac{r}{m}} V_{n_g}^g = V_{n_g-1}^g \geq \bar{V} \geq \nu^g + e^{-\frac{r}{m}} \bar{V}.$$

This means that $q_{n_g/m} = \bar{q}$. Because q_t is constant after the n_b-th rollover date, it has to be the case that $q_{n_b/m} = \bar{q}$. Let

$$\hat{q}(n) = \frac{q_0}{q_0 + (1 - q_0)e^{-\lambda n/m}}$$

be the beliefs after n rolled over period given the market conjecture that a bad loan is not rolled over, and let $\hat{n} \equiv \min\{n : \hat{q}(n) \geq \bar{q}\}$. If $\hat{n}/m$ is an integer, then $n_b = \bar{n}$ and $\alpha_b = 0$. However, if $\hat{n}/m$ is not an integer then $n_b = \hat{n} - 1$, and fraction α_b of the bad projects is liquidated so that the belief conditional on a rollover at n_b is $\bar{q}$. In this case, α_b satisfies

$$\bar{q} = \frac{(1 - \alpha_b)\left(1 - q_{\frac{n_b-1}{m}}\right)}{q_{\frac{n_b-1}{m}} + (1 - \alpha_b)\left(1 - q_{\frac{n_b-1}{m}}\right)}. \tag{44}$$

The definition of $\hat{n}$ together with equation (44) uniquely determine n_b and α_b. It is only left to determine n_g and α_g. We do this by turning our attention to the bad type incentive compatibility constraint. The bad type has to be indifferent between liquidating and continue rolling over after n_b. Let $\hat{V}(n', n_b)$ be the payoff if the bad type rolls over until period n' and receives a payoff $\bar{V}^b$, which is given by:

$$\hat{V}^b(n', n_b) \equiv \int_0^{\frac{n'-n_b}{m}} e^{-(r+\phi)s} (c + \phi R) \, ds + e^{-(r+\phi)\frac{n'-n_b}{m}} \bar{V}^b$$

For fixed n_b, the function $\hat{V}(n', n_b)$ is increasing in n'. Let's define $\tilde{n} \equiv \max\{n' : \hat{V}^b(n', n_b) \geq L\}$. If $\hat{V}^b(\tilde{n}, n_b) = L$, then we can set $n_g = \tilde{n}$ and $\alpha_g = 1$. Otherwise, we have that $\hat{V}^b(\tilde{n}, n_b) > L$ and $\hat{V}^b(\tilde{n} + 1, n_b) < L$ so a mixed strategy is required. In particular, if we set $n_g = \tilde{n}$ and choose α_g such that

$$\alpha_g \hat{V}^b(n_g, n_b) + (1 - \alpha_g)\hat{V}^b(n_g + 1, n_b) = L,$$

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we get that $V_{n_b}^b = L$ so the low type is indifferent between liquidating and rolling over. Finally, because the market investors make zero profit, they are willing to mix between the two debt prices at the rollover period n_g. Moreover, we have the following corollary $\qquad\square$

A.2 Bank and Entrepreneur Value Function

In this subsection, we supplement the details in subsection 3.2.2. Below, we will describe the value function of the entrepreneur and the bank respectively in three different regions.

In the *Market Financing* region (t_g, ∞), the value of the equity depends on the loan rate determined at the last rollover date before t_g, which we denote by y.

$$E_t^u = \frac{\phi \left[q_0 + (1 - q_0) \, \theta \right] (R - F)}{r + \phi + \lambda + m} + \frac{\lambda \left[q_0 E_t^g + (1 - q_0) \, E_t^b \right]}{r + \phi + \lambda + m} \tag{45a}$$

$$+ \frac{(c - y) + m \left(\bar{D} - F \right) + m \frac{\phi [q_0 + (1 - q_0) \theta] (R - F)}{r + \phi}}{r + \phi + \lambda + m}$$

$$E_t^g = \frac{\phi (R - F) + (c - y) + m \left(\bar{D} - F \right) + m \frac{\phi (R - F)}{r + \phi}}{r + \phi + m} \tag{45b}$$

$$E_t^b = \frac{\phi \theta (R - F) + (c - y) + m \left(\bar{D} - F \right) + m \frac{\phi \theta (R - F)}{r + \phi}}{r + \phi + m}, \tag{45c}$$

where $\bar{D} = \bar{D}_{\tau_m}$ in (5) evaluated at $q_{\tau_m} = \bar{q}$. The value function of bank satisfy

$$B_t^u = \frac{rF + \phi \left[q_0 + (1 - q_0) \, \theta \right] F + \lambda \left[q_0 B_t^g + (1 - q_0) \, B_t^b \right] + mF}{r + \phi + \lambda + m} \tag{46a}$$

$$B_t^g = F \tag{46b}$$

$$B_t^b = \frac{rF + \phi \theta F + mF}{r + \phi + m}. \tag{46c}$$

Next, we study the bank's and the entrepreneur's value in the other two regions. In general, these values will depend on the loan rate that they have agreed on so that we will use $E_t^i(y)$ and $B_t^i(y)$ to denote the values at loan rate is y. Note this interim payment will continued to be made until either the project matures or the loan matures, that is, until $\tau = \min \{\tau_m, \tau_\phi\}$. Equivalently, we can write the present value of this interim payment as $T(y) = \frac{y - rF}{r + m + \phi}$ so that $B_t^i(y) = B_t^i(rF) + T(y)$ and $E_t^i(y) = E_t^i(rF) - T(y)$. For the remainder of this subsection, we will use E_t^i and B_t^i for $E_t^i(rF)$ and $B_t^i(rF)$. For type u and

g, when $t \in (0, t_g)$,

$$(r + \phi + m) B_t^u = \dot{B}_t^u + rF + \phi \left[q_0 + (1 - q_0) \theta \right] F + \lambda \left[q_0 B_t^g + (1 - q_0) B_t^b - B_t^u \right] \quad (47a)$$
$$+ m \left[L + (1 - \beta) (V_t^u - L) \right] \quad (47b)$$
$$(r + \phi + m) E_t^u = \dot{E}_t^u + (c - rF) + \phi \left[q_0 + (1 - q_0) \theta \right] (R - F) + m\beta (V_t^u - L) \quad (47c)$$
$$+ \lambda \left[q_0 E_t^g + (1 - q_0) E_t^b - E_t^u \right]$$
$$(r + \phi + m) B_t^g = \dot{B}_t^g + rF + \phi F + m \left[L + (1 - \beta) (V_t^g - L) \right] \quad (47d)$$
$$(r + \phi + m) E_t^g = \dot{E}_t^g + (c - rF) + \phi (R - F) + m\beta (V_t^g - L). \quad (47e)$$

In contrast, the value functions for a bad-type entrepreneur differ across the two regions.

$$(r + \phi + m) E_t^b = \dot{E}_t^b + (c - rF) + \phi\theta (R - F) \quad \forall t \in (0, t_b) \quad (48a)$$
$$(r + \phi + m) B_t^b = \dot{B}_t^b + rF + \phi\theta F + mL \quad (48b)$$
$$(r + \phi + m) E_t^b = \dot{E}_t^b + (c - rF) + \phi\theta (R - F) + m\beta (V_t^b - L) \quad \forall t \in (t_b, t_g) \quad (48c)$$
$$(r + \phi + m) B_t^b = \dot{B}_t^b + rF + \phi\theta F + m \left[L + (1 - \beta) (V_t^b - L) \right]. \quad (48d)$$

Intuitively, in the *efficient liquidation* region, a bad project gets liquidated when the loan matures, whereas in the *zombie lending* region, the same loan will get rolled over.

Finally, given that we have shown the value function E_t^b can be non-monotonic in $[t_b, t_g)$, Lemma 7 proves that in the region of (t_b, t_g), $\dot{E}_t^b$ will change sign at most once. Therefore, the value of E_t^b is either monotonically increasing, or first increases and then decreases.

Lemma 7. $\ddot{E}_t^b > 0$ for $t \in (t_b, t_g)$.

Proof. Take derivative to both sides of equation (48c), we can get

$$\ddot{E}_t^b = (r + \phi + m) \dot{E}_t^b - m\beta \dot{V}_t^b.$$

This implies any local extrema of E_t^b (which satisfies $\dot{E}_t^b = 0$) is a local maximum. if $\dot{V}_t^b > 0$. Therefore, if $\dot{V}_t^b > 0$ for any $t \in (t_b, t_g)$, E_t^b cannot change sign more than once over $t \in (t_b, t_g)$. To show this, let us take derivative to both sides of equation (17c) in region $t \in [t_b, t_g)$

$$\ddot{V}_t^b = (r + \phi) \dot{V}_t^b.$$

At $t = t_b$, $\dot{V}_{t_b}^b = (r + \phi) L - c - \phi\theta R > 0$ following Assumption 1.Therefore, since sign $\left(\dot{V}_t^b \right) =$ sign $\left(\ddot{V}_t^b \right)$ for any $t \in (t_b, t_g)$, that implies $\dot{V}_t^b > 0$ in this region as well.

$\square$