Digital Signal Processing 2/ Advanced Digital Signal Processing
Lecture 3,
SNR, non-linear Quantisation

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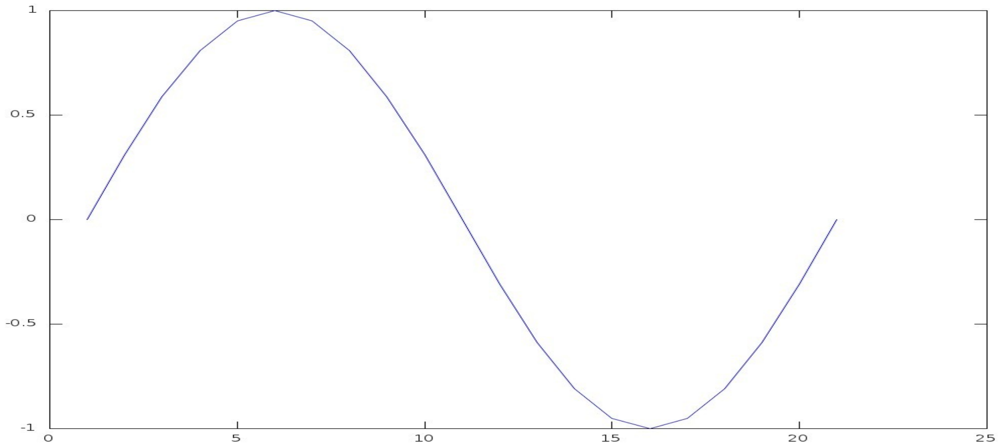
What is our SNR if we have a sinusoidal signal? What is its pdf? Basically it is its normalized histogram, such that its integral becomes 1, to obtain a probability distribution.



If we look at the signal, and try to see how probable it is for the signal to be in a certain small interval on the y axis, we see that the signal stays longest around +1 and -1, because there the signal slowly turns around. Hence we would expect a pdf, which has peaks at +1 and -1.

If you calculate the pdf of a sine wave, x=sin(t), with t being continous and with a range larger than 2pi, then the result is

$$p(x)=\frac{1}{\pi\cdot\sqrt{1-x^2}}$$

This results from the derivative of the inverse sine function (arcsin). This derivation can be found for instance on Wikipedia. For our pdf we need to know how fast a signal x passes through a given bin in x. This is what we obtain if we compute the inverse function $x=f^{-1}(y)$, and then its derivative $d f^{-1}(x)/dy$.

Here we can see that p(x) indeed becomes infinite at x=+/-1! We could now use the same approach as before to obtain the expectation of the power, multiplying it with x^2 and integrating it. But this seems to be somewhat tedious. But since we now have a deterministic signal, we can also try an **alternative** solution.

We can simply directly compute the power of our sine signal over t, and then take the average over at least one period of the sine function.

$$E(x^2)=\frac{1}{2\pi}\int_{t=0}^{2\pi}\sin^2(t)\,dt=\frac{1}{2\pi}\int_{t=0}^{2\pi}(1-\cos(2t))/2\,dt$$

the cosine integrated over complete periods becomes 0, hence we get

$$=\frac{1}{2\pi}\int_{t=0}^{2\pi}1/2\,dt=\frac{1}{2\pi}\cdot\pi=\frac{1}{2}$$

What do we get for a sinusoid with a different amplitude, say $A/2\cdot\sin(t)$?

The expected power is

$$E(x^2)=\frac{A^2}{8}$$

So this leads to an SNR of

$$SNR = \frac{A^2/8}{\Delta^2/12} = \frac{3 \cdot A^2}{2 \cdot \Delta^2}$$

Now assume again we have a A/D converter with N bits, and the sinusoid is at full range for this converter. Then

$$A = 2^N \cdot \Delta$$

We can plug in this result into the above equation, and get

$$SNR = \frac{3 \cdot 2^{2N} \cdot \Delta^2}{2 \cdot \Delta^2} = 1.5 \cdot 2^{2N}$$

In dB this will now be

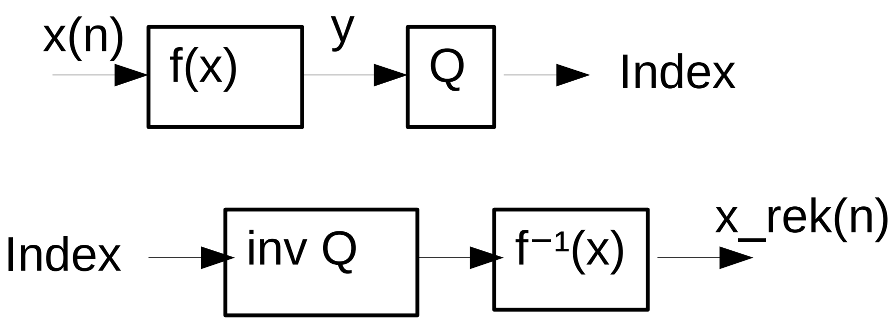
$$10 \cdot \log_{10}(SNR) = 10 \cdot \log_{10}(1.5) + N \cdot 20 \cdot \log_{10}(2) =$$
$$= 1.76\, dB + N \cdot 6.02\, dB$$

Here we can see now, that using a sinusoidal signal instead of a uniformly distributed signal gives us a **boost of 1.76 dB** in SNR. This is because it is more likely to have larger values!

Companding

This is a scheme to make the SNR less dependent on the signal size.
This is a synonym for compression and expanding. Uniform quantization can be seen as a

quantization value which is constant on the absolute scale. Non-uniform quantization, using companding, can be seen as having step sizes which stay constant relative to the amplitude, their step size grows with the amplitude.
We obtain this non-uniform quantization by first applying a non-linear function to the signal (to boost small values), and then apply a uniform quantizer. On the decoding side we first apply the inverse quantizer, and then the inverse non-linear function (we reduce small values again to restore their original size).



The range of (index) values is compressed, smaller values become larger, large values don't grow as fast. The following functions are standardized as “ μ -Law” and “A-Law”:

EQUATION 22-1
Mu law companding. This equation provides the nonlinearity for µ255 law companding. The constant, µ, has a value of 255, accounting for the name of this standard.

$$y = \frac{\ln(1+\mu x)}{\ln(1+\mu)} \qquad \text{for } 0 \le x \le 1$$

EQUATION 22-2
"A" law companding. The constant, *A*, has a value of 87.6.

$$y = \frac{1+\ln(Ax)}{1+\ln(A)} \qquad \text{for } 1/A \le x \le 1$$

$$y = \frac{Ax}{1+\ln(A)} \qquad \text{for } 0 \le x \le 1/A$$

(From: http://www.dspguide.com/ch22/5.htm, also below)

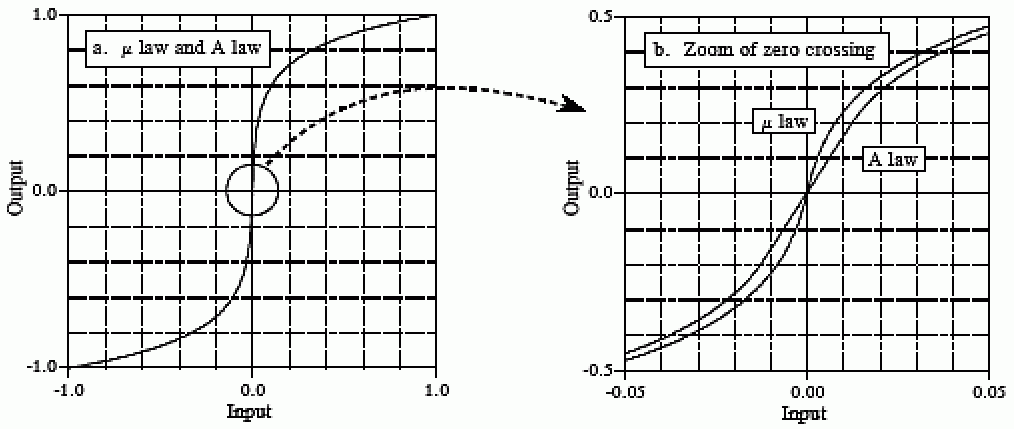


FIGURE 22-7
Companding curves. The µ255 law and "A" law companding curves are nearly identical, differing only near the origin. Companding increases the amplitude when the signal is small, and decreases it when it is large.

This "**compression**" function is applied **before a uniform quantizer** in the **encoder**. In the decoder, after uniform reverse quantisation, the inverse function is applied, turning y back into x. Observe that these equations assume that we first normalize our signal and keep its sign separate.

Example: For $\mu=255$ (which is used in the standard) and an x with a maximum amplitude of A, hence $-A \leq x \leq A$ we obtain:

$$y=sign(x)\cdot\frac{\ln\left(1+255\cdot\left|(x/A)\right|\right)}{\ln(1+255)}$$

In the example of 8-bit mu-law PCM the quantization **index** is then

$$index=round\left(2^7\cdot y\right)$$

since we take 7 bits for the magnitude of the signal and 1 bit for its sign. This index is then encoded as a 7+1= 8 bit **codeword**.

In the **decoder** we compute the inverse quantized y including its sign from the index,

$$y=index/\left(2^7\right)$$

and we compute the inverse compression function, the “**expanding**” function (hence the name “**companding**”)

We obtain the inverse through the following steps,

$$\left|(y)\right|=\frac{\ln\left(1+255\cdot\left|(x/A)\right|\right)}{\ln(1+255)}$$

$$|y|\cdot\ln(256)=\ln\left(1+255\cdot|x/A|\right)$$

$$\rightarrow e^{\ln(256)\cdot|y|}=1+255\cdot|x|/A$$

$$\rightarrow \frac{\left(256^{|y|}-1\right)}{255}\cdot A=|x|$$

$$\rightarrow x=sign(y)\frac{\left(256^{|y|}-1\right)}{255}\cdot A$$

This x is now our **de-quantized value** or signal. Observe that with this companding, the effective quantisation step size remains approximately constant **relative** to the **signal amplitude**. Large signal components have large step sizes and hence larger quantisation errors, small signals have smaller quantisation step sizes and hence smaller quantisation errors. In this way we get a more or less **constant SNR** over a wide **range of signal amplitudes**.

Important point to remember: this approach is **identical** to having **non-uniform quantization** step sizes, smaller step sizes at smaller signal values, and larger step sizes at larger signal values. The compression and expanding of the signal makes the uniform step sizes “look” relatively smaller to the signal, it has more quantization steps to cover. And this has the same effect as a smaller signal with smaller quantization steps.

Python example to mu-law quantizer:

First listen to the uniform mid-tread quantizer using our python script, and make sure the part for the mid-tread quantizer is un-commented:

```
python pyrecplay_quantizationblock.py
```

Observe that the voice dissapear, is rounded to zero, if the speaker is at some distance from the microphone.

Now use mu-law quantization with:

```
python pyrecplay_mulawquantizationblock.py
```

Observe: At the distance at which the speech disappeared in the uniform case, the speech is now there.

At closer distance there is not much difference.

Lloyd-Max Quantizer

This is a type of non-uniform quantizer, which is adapted to the signals pdf. It basically minimizes the expectation of the quanization power (its second moment), given the pdf of the signal to quantize.

Let's call our Quantisation function Q(x) (this is quantization followed by inverse quantization).

You can also think of non-uniform quantization as first applying this non-linear function and then to use uniform quantization. Then the expectation of our quantization power is

$$D=E\left(\left(x-Q(x)\right)^2\right)$$

Observe that we use the square here, and not for instance the magnitude of the error, because the square leads to an easier solution for minimum, which we would like to find.