Francisco Javier Sáez Maldonado
José Antonio Álvarez Ocete

# Ejercicios Programación Lineal
## Optimización

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**Exercise 1**
Show that if $S$ is an open set, its complement $S^c$ is closed, and viceversa.

**Exercise 2**
If $S_1, S_2$ are convex subsets, prove that the following are also convex sets:

$$
\begin{aligned}
S_1 \cap S_2 &= \{x \ : \ x \in S_1 \text{ and } x \in S_2\} \\
S_1 + S_2 &= \{x + x' \ : \ x \in S_1, x' \in S_2\} \\
S_1 - S_2 &= \{x - x' \ : \ x \in S_1, x0 \in S_2\}
\end{aligned}
$$

**Exercise 3**
If $f : S \to \mathbb{R}$ is a convex function on the convex set $S$, the set $\{x : \ x \text{ is a minimum of f}\}$ is a convex set.

**Exercise 4**
Given a quadratic form $q(w) = w^T Q w + bw + c$, with $Q$ a symmetric $d \times d$ matrix, $w, b$ being $d \times 1$ vectors and $c$ a real number, derive its gradient and Hessian

$$
\nabla q(w) = Qw + b, \quad Hq(w) = Q
$$

*Hint: expand* $q(w) = \sum_{i=1}^d \sum_{j=1}^d Q_{ij} w_i w_j + \sum_1^d b_i w_i + c$ *and take the partials* $\frac{\partial q}{\partial w_i}$ *and* $\frac{\partial^2 q}{\partial w_i \partial w_j}$.

**Exercise 5**
If $(p_1, \ldots, p_n)$ is a probability distribution, prove that its entropy $H(p_1, \ldots, p_n) = -\sum_{i=1}^n p_i \log p_i$ is a concave function. Show also that its maximum is $\log n$, attained when $p_i = \frac{1}{n}$ for all $i$.

**Exercise 6**
We want to solve the following constrained restriction problem:

$$
\begin{aligned}
\min \quad & x^2 + 2y^2 + 4xy \\
\text{s.t} \quad & x + y = 1 \\
& x, y \geq 0.
\end{aligned}
$$

1. Write its Lagrangian with $\alpha, \beta$ the multipliers of the inequality constraints.
2. Write the KKT conditions.
3. Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0)$, $(\alpha > 0, \beta = 0)$, $(\alpha = 0, \beta > 0)$, $(\alpha > 0, \beta > 0)$ cases.

**Exercise 11**
If $Q$ is a symmetric, positive definite $d \times d$ matrix, show that $f(x) = x^T Q x, \ x \in \mathbb{R}^d$, is a convex function.

**Exercise 12**
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function and assume that $epi(f) \subset \mathbb{R}^d \times \mathbb{R}$ is convex. Prove that then $f$ is convex.

**Exercise 13**

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Prove that $\mathrm{epi}(f)$ is a closed set and that $(x, f(x)) \in \partial\,\mathrm{epi}(f)$ .

**Exercise 14**
Prove that if $f$ is strictly convex, it has a unique global minimum.

**Exercise 15**
Let $f, g : S \subset \mathbb{R}^d \to \mathbb{R}$ be two convex functions on the convex set $S$ . Prove that, as subsets, $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$ for any $x \in S$.

**Exercise 16**
Compute the proximal of $f(x) = 0$ and of $g(x) = \frac{1}{2}\|x\|^2$.

**Exercise 17**
Assume that $f$ is convex. Prove that for any $\lambda > 0, \partial(\lambda f)(x) = \lambda \partial f(x)$ as subsets.

**Exercise 19**
Compute the proximals of the hinge $f(x) = max\{0, -x\}$ and the $\epsilon$-insensitive $g(x) = max\{0, |x| - \epsilon\}$ loss functions.