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Ejercicios Programación Lineal
Optimización

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Exercise 1
Show that if S is an open set, its complement S^c is closed, and viceversa.

TO BE COMPLETED

$\boxed{\Rightarrow}$

Let S be an open set. Clearly, $cl(S^c) \subset S^c$.

Now, let x be an element of S^c. Then,

$$\forall \delta > 0, \quad B(x, \delta) \cap S^c \neq \emptyset$$

$\boxed{\Leftarrow}$

Let us see that $S^c = cl(S^c)$ since, if this happens, S is open.

Let $x \in S$. Then, $x \notin S^c$, which implies $x \notin cl(S^c)$. This implies that

$$\exists \lambda > 0 \text{ such that } B(x, \lambda) \cap S^c = \emptyset \implies B(x, \delta) \subset S \implies S \text{ is open.}$$

Exercise 2
If S_1, S_2 are convex subsets, prove that the following are also convex sets:

$$\begin{aligned} S_1 \cap S_2 &= & \{x \ : \ x \in S_1 \text{ and } x \in S_2\} \\ S_1 + S_2 &= & \{x + x' \ : \ x \in S_1, x' \in S_2\} \\ S_1 - S_2 &= & \{x - x' \ : \ x \in S_1, x0 \in S_2\} \end{aligned}$$

- Consider $S_1 - S_2 = \{x - z \ : \ x \in S_1, z \in S_2$. Now, let $x - z \in S_1 - S_2$ and $x' - z' \in S_1 - S_2$. Also, consider the segment

$$\begin{aligned} \lambda(x - z) + (1 - \lambda)(x' - z') &= \lambda x - \lambda z + (1 - \lambda)x' - (1 - \lambda z') \\ &= \underbrace{\lambda x + (1 - \lambda)x'}_{x'' \in S_1} - \underbrace{(\lambda z + (1 - \lambda)z')}_{z'' \in S_2} \end{aligned}$$

Where the underbraces are true due to the convexity of S_1 and S_2, so we have

$$x'' - z'' \in S_1 - S_2$$

so the segment is in the difference set, so the set $S_1 - S_2$ is convex.

Exercise 3
If $f : S \to \mathbb{R}$ is a convex function on the convex set S, the set $\{x : \; x \text{ is a minimum of f}\}$ is a convex set.

Exercise 4
Given a quadratic form $q(w) = w^T Q w + bw + c$, with Q a symmetric $d \times d$ matrix, w, b being $d \times 1$ vectors and c a real number, derive its gradient and Hessian

$$\nabla q(w) = Qw + b, \quad Hq(w) = Q$$

Hint: expand $q(w) = \sum_{i=1}^{d} \sum_{j=1}^{d} Q_{ij} w_i w_j + \sum_1^d b_i w_i + c$ *and take the partials* $\frac{\partial q}{\partial w_i}$ *and* $\frac{\partial^2 q}{\partial w_i \partial w_j}$.

Exercise 5
If $(p_1, \ldots, p_n)$ is a probability distribution, prove that its entropy $H(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log p_i$ is a concave function. Show also that its maximum is $\log n$, attained when $p_i = \frac{1}{n}$ for all i.

Exercise 6
We want to solve the following constrained restriction problem:

$$\begin{aligned} \min \quad & x^2 + 2y^2 + 4xy \\ \text{s.t} \quad & x + y = 1 \\ & x, y \geq 0. \end{aligned}$$

1. Write its Lagrangian with α, β the multipliers of the inequality constraints.
2. Write the KKT conditions.
3. Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0)$, $(\alpha > 0, \beta = 0)$, $(\alpha = 0, \beta > 0)$, $(\alpha > 0, \beta > 0)$ cases.

Exercise W
e have worked out the dual problem for the soft SVC problem. Do the same for the simpler **hard** SVC problem

$$\min_{w,b} \frac{1}{2} \|w\|^2$$

subject to $y^p \left(w \cdot x^p + b\right) \geq 1$. What are here the KKT conditions?

Firstly, consider the Lagrangian for this problem

$$\begin{aligned} L(w, b, \lambda) &= \frac{1}{2} \|w\|^2 - \sum \alpha_p \left[y^p \left(w \cdot x^p + b \right) - 1 \right] \\ &= \frac{1}{2} w \cdot w - w \sum \lambda_p y^p x^p - b \sum \lambda_p y_p + \sum \lambda_p \\ &= w \left(\frac{1}{2} w - \sum \lambda_p y^p x^p \right) b \sum \lambda_p y_p + \sum \lambda_p \end{aligned}$$

Then, we have to compute the gradient of the Lagrangian

$$\nabla_w L(w, b, \lambda) = w - \sum \lambda_p y^p x^p = 0 \implies w = \sum \lambda_p y^p x^p$$

$$\frac{\partial L}{\partial b} = -\sum \lambda_p y^p = 0$$

Lastly, we have to use the optimalas that we have found in the expression of the Lagrangian:

$$\begin{aligned} L(w, b, \alpha) &= \sum \alpha_p - \frac{1}{2} \left(\sum_p \alpha_p y^p x^p \right) \left(\sum_q \alpha_q y^q x^q \right) \\ &= \sum \alpha_p - \frac{1}{2} \sum_{p,q} \alpha_p y^p \alpha_q y^q x^p x^q = -\alpha \end{aligned}$$

Our optimization problem has turned today

$$\max_\alpha \sum \alpha^p - \frac{1}{2} \sum \alpha_p \alpha_q y^p y^q x^p x^q$$

subject to $\alpha^p, \sum \alpha_p y^p = 0$.

Let us state the KKT conditions for this problem.

Exercise 8
A typical Linear Programming (LP) problem can be stated as the following constrained optimization problem:

$$\min_x c \cdot x \quad s.t. \quad x \geq 0, Ax \leq b$$

with $x \in \mathbb{R}^d$, A an $m \times d$ matrix and $b \in \mathbb{R}^m$. A tool often used in LP is to study the so called dual problem, which in this case is

$$\min_z b \cdot z \quad s.t. \quad z \geq 0, A^t z \leq -c$$

with now $z \in \mathbb{R}^m$. Apply our Lagrangian dual construction technique to show that this is indeed the dual formulation of the initial LP problem

Firstly, we have to write the Lagrangian for this problem:

$$L(x, \lambda, \mu) = c \cdot x - \sum_{i=1}^{d} \lambda_i x_i + \sum_{j=1}^{m} \mu_j (a_j \cdot x - b_j)$$

Hence, the gradient respect to x of the lagrangian is:

$$\nabla_x L = c - \lambda + A^t \mu$$

CHECK!!!
(this Lagrangian must have dimension d).

Exercise 11
If Q is a symmetric, positive definite $d \times d$ matrix, show that $f(x) = x^T Q x,\ x \in \mathbb{R}^d$, is a convex function.

Exercise 12
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function and assume that $epi(f) \subset \mathbb{R}^d \times \mathbb{R}$ is convex. Prove that then f is convex.

Exercise 13

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Prove that $\text{epi}(f)$ is a closed set and that $(x, f(x)) \in \partial\, \text{epi}(f)$.

Exercise 14
Prove that if f is strictly convex, it has a unique global minimum.

Exercise 15
Let $f, g : S \subset \mathbb{R}^d \to \mathbb{R}$ be two convex functions on the convex set S . Prove that, as subsets, $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$ for any $x \in S$.

Exercise 16
Compute the proximal of $f(x) = 0$ and of $g(x) = \frac{1}{2}\|x\|^2$.

Exercise 17
Assume that f is convex. Prove that for any $\lambda > 0, \partial(\lambda f)(x) = \lambda \partial f(x)$ as subsets.

Exercise 19
Compute the proximals of the hinge $f(x) = max\{0, -x\}$ and the ϵ-insensitive $g(x) = max\{0, |x| - \epsilon\}$ loss functions.