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Ejercicios Programación Lineal
Optimización

March 25, 2022

Exercise 1
Show that if S is an open set, its complement S^c is closed, and viceversa.

$\boxed{\Rightarrow}$

Let S be an open set. We know that S^c is a closed set if and only if for any sequence of elements $\{x_n\} \subset S^c$ such that $\{x_n\} \longrightarrow x$, then $x \in S^c$.

Let us use this characterization to prove that S^c is closed. Let $\{x_n\} \subset S^c$ such that $\{x_n\} \longrightarrow x$. Suppose that x is not in S^c. Then x must be in its complementary, S. Since S is an open set we know that $\exists \epsilon > 0$ such that $B(x, \epsilon) \subset S$.

Since $\{x_n\} \longrightarrow x$, for any $\delta > 0 \exists n \in \mathbb{N}$ such that $\| x - x_n \| < \delta$. In particular, for $\delta = \epsilon$ there is a element of the succesion x_n in $B(x, \epsilon) \subset S$, but $\{x_n\} \subset S^c$. This contradiction implies that x is, in fact, in S^c.

$\boxed{\Leftarrow}$

Let S be a set such that its complement S^c is closed (that is, $S^c = cl(S^c)$). Let us show that S is open.

Let $x \in S$. Then, $x \notin S^c$, which implies $x \notin cl(S^c)$. This implies that

$$\exists \epsilon > 0 \text{ such that } B(x, \epsilon) \cap S^c = \emptyset \implies B(x, \epsilon) \subset S$$

Thus, S is open. □

Exercise 2
If S_1, S_2 are convex subsets, prove that the following are also convex sets:

$$S_1 \cap S_2 = \{x \ : \ x \in S_1 \text{ and } x \in S_2\}$$
$$S_1 + S_2 = \{x + x' \ : \ x \in S_1, x' \in S_2\}$$
$$S_1 - S_2 = \{x - x' \ : \ x \in S_1, x' \in S_2\}$$

- Consider $S_1 \cap S_2 = \{x \ : \ x \in S_1 \text{ and } x \in S_2\}$. Let $x, x' \in S_1 \cap S_2$ and consider the segment

  $$z \equiv \lambda x + (1 - \lambda) x', \lambda \in [0, 1]$$

  Since x is in in S_1, which is a convex set, z will also be in S_1 for any $\lambda \in [0, 1]$. Likewise, z will be in S_2, and thus in the intersection $S_1 \cap S_2$. Since the segment is contained in the intersection, $S_1 \cap S_2$ is a convex subset.

- Consider $S_1 + S_2 = \{x + z \ : \ x \in S_1, z \in S_2\}$. Now, let $x + z \in S_1 + S_2$ and $x' + z' \in S_1 + S_2$. Also, consider the segment

$$\begin{aligned}\lambda(x+z) + (1-\lambda)(x'+z') &= \lambda x + \lambda z + (1-\lambda)x' + (1-\lambda z') \\ &= \underbrace{\lambda x + (1-\lambda)x'}_{x'' \in S_1} + \underbrace{\lambda z + (1-\lambda)z'}_{z'' \in S_2}\end{aligned}$$

Where the underbraces are true due to the convexity of S_1 and S_2, so we have

$$x'' + z'' \in S_1 + S_2$$

so the segment is in the sum set, and thus, the set $S_1 + S_2$ is convex.

- We use the same process done in the previous set. Consider $S_1 - S_2 = \{x - z \ : \ x \in S_1, z \in S_2\}$. Now, let $x - z \in S_1 - S_2$ and $x' - z' \in S_1 - S_2$. Also, consider the segment

$$\begin{aligned}\lambda(x-z) + (1-\lambda)(x'-z') &= \lambda x - \lambda z + (1-\lambda)x' - (1-\lambda z') \\ &= \underbrace{\lambda x + (1-\lambda)x'}_{x'' \in S_1} - \underbrace{\left(\lambda z + (1-\lambda)z'\right)}_{z'' \in S_2}\end{aligned}$$

Where the underbraces are true due to the convexity of S_1 and S_2, so we have

$$x'' - z'' \in S_1 - S_2$$

so the segment is in the difference set, and thus, the set $S_1 - S_2$ is convex.

Exercise 3
If $f : S \to \mathbb{R}$ is a convex function on the convex set S, the set $S_{min} = \{x : \ x \text{ is a minimum of f}\}$ is a convex set.

We omit the case where $S_{min} = \emptyset$, since the empty set is convex. Now, let y be the minimum of $f(x)$: $y = \min_x f(x)$. Then $S_{min} = \{x \in S : \ f(x) = y\}$. We need need to show that for all $x, x' \in S_{min}$ and for all $\lambda \in [0, 1]$:

$$z \equiv \lambda x + (1-\lambda)x' \in S_{min} \leftrightarrow f(z) = y$$

Since S is convex, z is in S, and since f is also convex:

$$\begin{aligned}f(z) &= f\left(\lambda x + (1-\lambda)x'\right) \\ &\leq \lambda f(x) + (1-\lambda)f(x') \\ &= \lambda y + (1-\lambda)y \\ &= y\end{aligned}$$

where we used that $x, x' \in S_{min}$. But since y is the minimum of f, the equality holds $f(z) = y$. This means that z is in S_{min}, therefore S_{min} is a convex set.

Exercise 4
Given a quadratic form $q(w) = w^T Q w + bw + c$, with Q a symmetric $d \times d$ matrix, w, b being $d \times 1$ vectors and c a real number, derive its gradient and Hessian.

$$\nabla q(w) = Qw + b, \quad Hq(w) = Q$$

Hint: expand $q(w)$ and take the partials with respect to w_i and w_i, w_j.

Let us start by unrolling the quadratic form expression:

$$q(w) = \sum_{i,j=1}^{d} Q_{ij} w_i w_j + \sum_{i=1}^{d} b_i w_i + c,$$

and compute the partial derivative over the $k-th$ component:

$$\frac{\partial q}{\partial w_k}(w) = \sum_{i=1}^{d} Q_{ik} w_i + \sum_{j=1}^{d} Q_{kj} w_j + b_k$$

where $k \in \{1, \ldots, d\}$. By using that Q is symmetric we obtain:

$$\frac{\partial q}{\partial w_k}(w) = 2 \sum_{j=1}^{d} Q_{kj} w_j + b_k. \tag{1}$$

That is, we are multiplying the $k-th$ row of the Q matrix and multiplying it by w. We can obtain gradient as a product of matrices using the previous expression:

$$\nabla q(w) = \begin{pmatrix} \frac{\partial q}{\partial w_1}(w) \\ \vdots \\ \frac{\partial q}{\partial w_d}(w) \end{pmatrix} = \begin{pmatrix} 2 \sum_{j=1}^{d} Q_{1j} w_j + b_1 \\ \vdots \\ 2 \sum_{j=1}^{d} Q_{dj} w_j + b_d \end{pmatrix} = 2 \begin{pmatrix} \sum_{j=1}^{d} Q_{1j} w_j \\ \vdots \\ \sum_{j=1}^{d} Q_{dj} w_j \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} = 2\, Qw + b$$

In order to obtain the Hessian, we take partial derivatives over the $l-th$ component in 1:

$$\begin{aligned} \frac{\partial^2 q}{\partial w_k \partial w_l}(w) &= \frac{\partial q}{\partial w_l}\left(\frac{\partial q}{\partial w_k}\right)(w) \\ &= \frac{\partial q}{\partial w_l}\left(2 \sum_{j=1}^{d} Q_{kj} w_j + b_k\right)(w) \\ &= 2\, Q_{kl} \end{aligned}$$

Hence, the Hessian matrix of q will have $2Q_{kl}$ in position (k, l). That is:

$$\text{Hess}\, q(w) = 2Q$$

Exercise 5
If $(p_1, \ldots, p_K)$ is a probability distribution, prove that its entropy $H(p_1, \ldots, p_K) = -\sum_{i=1}^{K} p_i \log p_i$ is a concave function. Show also that its maximum is $\log K$, attained when $p_i = \frac{1}{K}$ for all i.

In this problem, since we are dealing with probabilities, two new constraints appear:

$$\sum_{i=1}^{K} p_i = 1$$
$$p_i \geq 0 \quad \forall i = 1, \dots, n$$

They will be used later.

Let us compute the gradient and Hessian of H to see that it is concave. Firstly, we have that

$$\frac{\partial H}{\partial p_i} = -\log(p_i) - 1, \quad \forall i = 1, \dots, K$$

and, hence,

$$\frac{\partial^2 H}{\partial p_i \partial p_j} = -\frac{\lambda_{ij}}{p_i}$$

Lastly, since $p_i \geq 0$, we have that the Hessian is a negative-definite diagonal matrix, so H is concave.

In order to find the minimum entropy, we have to solve the following optimization problem:

$$\begin{aligned} &\max_{(p_1,\dots,p_K)} H(p_1, \dots, p_K) \\ &\text{s.t.} \\ &\sum_{i=1}^{K} p_i - 1 = 0 \\ &p_i \geq 0 \quad \forall i = 1, \dots, K \end{aligned}$$

Consider the lagrangian of this problem:

$$L\left(\{p_i\}_{i=1}^K, \lambda\right) = -\sum_{i=1}^{K} p_i \log(p_i) + \lambda \left(\sum_{i=1}^{K} p_i - 1\right).$$

We can obtain its gradient derivating with respect to each variable

$$\frac{\partial L}{\partial p_i} = -\log p_i - 1 + \lambda, \quad \frac{\partial L}{\partial \lambda} = \sum_{i=1}^{K} p_i - 1.$$

Using this derivatives, we have to equate them to zero, which is solving the following equations system:

$$\begin{cases} \log p_i = \lambda - 1, \quad i = 1, \dots, K \\ \sum_{i=1}^{K} p_i = 1 \end{cases}$$

Looking at the first equation, the p_i have no interdependencies, they are constant and have the same value. Also, since they have to add 1, the only possible solution is that each $p_i = \frac{1}{K}$. Lastly, we can compute the maximum value of the Entropy:

$$H\left(\{p_i\}_{i=1}^K\right) = -\sum_{i=1}^{K} \frac{1}{K} \log\left(\frac{1}{K}\right) = -\frac{1}{K}\left(\sum_{i=1}^{K} \log 1 - \log K\right) = \frac{1}{K} \cdot K \log K = \log K,$$

as we wanted to see. $\square$

Exercise 6
We want to solve the following constrained restriction problem:

$$\begin{aligned} \min \quad & x^2 + 2y^2 + 4xy \\ \text{s.t} \quad & x + y = 1 \\ & x, y \geq 0. \end{aligned}$$

1. Write its Lagrangian with α, β the multipliers of the inequality constraints.
2. Write the KKT conditions.
3. Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0)$, $(\alpha > 0, \beta = 0)$, $(\alpha = 0, \beta > 0)$, $(\alpha > 0, \beta > 0)$ cases.

Writing the **Lagrangian** in terms of α, β, λ is pretty straightforward:

$$L(x, y, \alpha, \beta, \lambda) = x^2 + 2y^2 + 4xy + \lambda(x + y - 1) + \alpha x + \beta y.$$

Now, to write the KKT conditions. As a very brief summarization, the KKT conditions are: the gradient of the Lagrangian equals to zero and the inequality restrictions (multiplied by its corresponding constant) also equal to zero. In our case, the **KKT conditions** are:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2x + 4y + \lambda + \alpha = 0 \\ \frac{\partial L}{\partial y} &= 4y + 4x + \lambda + \beta = 0 \\ \alpha x &= 0 \\ \beta y &= 0 \end{aligned}$$

Now, we want to solve this equations system to see if we can find a minimum of our problem. We have the following cases:

- Case $\alpha = \beta = 0$.
  In this case, the system is

  $$\begin{aligned} 2x + 4y + \lambda &= 0 \\ 4y + 4x + \lambda &= 0 \end{aligned}$$

  If we substitute $4y$ from the first equation into the second one, we obtain

  $$-2x - \lambda + 4x + \lambda = 0 \implies 2x = 0 \implies x = 0,$$

  and, since $x + y = 1$, we obtain that our *KKT point* is $(0, 1)$.
- Case $\alpha, \beta > 0$.
  In this case, we obtain from the KKT conditions that $x = y = 0$, which does not match our initial conditions $x + y = 1$, so no *KKT points* are obtained.
- Case $\alpha > 0, \beta = 0$.
  Looking at our KKT conditions, since $\alpha > 0$, we have that $x = 0$, resulting in $y = 1$ and a *KKT point* $(0, 1)$, which is the same that we obtained in the first case.

- Case $\alpha = 0, \beta > 0$.
  Using the same reasoning, we obtain $(1, 0)$ as a new *KKT* point.

Until now, we have two candidates to be the optimal one: $\{(0,1),(1,0)\}$. Now, we make use of the following theorem:

Theorem 1 *If in a minimization problem with restrictions $g_i(x), h_j(x) \in C^1$, if we assume f to be convex and h_j to be affine, then a KKT point x^* is an optimum of this problem. (Slide 18)*

So, we can evaluate the function on our KKT points to find the minimum. We obtain that $f(1,0) = 1, f(0,1) = 2$, so the minimum is reached in $(1,0)$ with optimal value 1.

Exercise 7
We have worked out the dual problem for the soft SVC problem. Do the same for the simpler **hard** SVC problem

$$\min_{w,b} \frac{1}{2}\|w\|^2$$

subject to $y^p\left(w \cdot x^p + b\right) \geq 1$. What are here the KKT conditions?

Firstly, consider the Lagrangian for this problem

$$\begin{aligned} L(w,b,\alpha) &= \frac{1}{2}\|w\|^2 - \sum_p \alpha_p \left[y^p\left(w \cdot x^p + b\right) - 1\right] \\ &= \frac{1}{2} w \cdot w - w \sum_p \alpha_p y^p x^p - b \sum_p \alpha_p y^p + \sum_p \alpha_p \\ &= w \left(\frac{1}{2} w - \sum_p \alpha_p y^p x^p\right) - b \sum_p \alpha_p y^p + \sum_p \alpha_p \end{aligned}$$

Then, we have to compute the gradient of the Lagrangian

$$\nabla_w L(w,b,\alpha) = w - \sum_p \alpha_p y^p x^p = 0 \implies w = \sum_p \alpha_p y^p x^p$$

$$\frac{\partial L}{\partial b} = -\sum_p \alpha_p y^p = 0$$

Lastly, we have to use these equalities in the expression of the Lagrangian:

$$\begin{aligned} L(w,b,\alpha) &= \sum \alpha_p - \frac{1}{2}\left(\sum_p \alpha_p y^p x^p\right)\left(\sum_q \alpha_q y^q x^q\right) \\ &= \sum \alpha_p - \frac{1}{2}\sum_{p,q} \alpha_p \alpha_q y^p y^q x^p x^q \end{aligned}$$

By defining the matrix Q with value $y^p y^q x^p x^q$ in position (p,q), ur dual optimization problem gets simplified into

$$\begin{cases} \max_\alpha \sum_p \alpha^p - \frac{1}{2}\alpha^T Q\alpha \\ \text{s.t.}\alpha^p, \sum \alpha_p y^p = 0 \end{cases}$$

At this point we may realize that we have completely removed the dependency b from our problem. Let us state the KKT conditions for this problem. There are:

$$\begin{aligned} \lambda_w f(x^*) + \sum_i \lambda_i \lambda g_i(x^*) &= 0 \\ \lambda_i g_i(x^*) &= 0 \end{aligned}$$

For our particular problem we obtain:

$$\begin{aligned} w + \sum_p \alpha_p y^p x^p &= 0 \\ TODOLASIGUEITNTELINEA \lambda_i g_i(x^*) &= 0 \end{aligned}$$

Exercise 8
A typical Linear Programming (LP) problem can be stated as the following constrained optimization problem:

$$\min_x c \cdot x \quad s.t. \quad x \geq 0, Ax \leq b$$

with $x \in \mathbb{R}^d$, A an $m \times d$ matrix and $b \in \mathbb{R}^m$. A tool often used in LP is to study the so called dual problem, which in this case is

$$\min_z b \cdot z \quad s.t. \quad z \geq 0, A^t z \leq -c$$

with now $z \in \mathbb{R}^m$. Apply our Lagrangian dual construction technique to show that this is indeed the dual formulation of the initial LP problem

Firstly, we have to write the Lagrangian for this problem:

$$L(x, \lambda, \mu) = c \cdot x - \sum_{i=1}^{d} \lambda_i x_i + \sum_{j=1}^{m} \mu_j (a_j \cdot x - b_j)$$

Hence, the gradient respect to x of the lagrangian is:

$$\nabla_x L = c - \lambda + A^t \mu$$

CHECK!!!
(this Lagrangian must have dimension d).

Exercise 9
We know that, theoretically, the minimum SVC primal f^* and the maximum SVC dual q^* are equal. Check this in this case by writing q^* and f^* in terms of the α_p^* and checking that both

expressions coincide.

Exercise 10
We want to apply out Lagrangian theory to solve the homogeneous constrained Ridge problem (i.e., with a model $w \cdot x$

$$\arg\min_{w} \ \mathrm{mse}(w) = \frac{1}{n}\sum_{p=1}^{n}(t^p - w \cdot x^p)^2, \quad \text{s.t.} \quad \|w\|_2^2 \le \rho^2.$$

Write its Lagrangian and, using the lecture slides, the detailed formulation of the KKT conditions at an optimal w^* and multiplier λ^*.
Assuming that $\lambda^* > 0$, use the gradient KKT condition to show that w^* also solves a standard Ridge regression problem for the optimal value λ^* of the regularization parameter.
Assuming now that $\lambda^* = 0$, use again the slides to write down the solution in this case and use this solution to get a lower bound for ρ.

HECHO EN CLASE

Exercise 11
If Q is a symmetric, positive definite $d \times d$ matrix, show that $f(x) = x^T Q x,\ x \in \mathbb{R}^d$, is a convex function.

HECHO EN CLASE

Exercise 12
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function and assume that $epi(f) \subset \mathbb{R}^d \times \mathbb{R}$ is convex. Prove that then f is convex.

Consider the set

$$\mathrm{epi}(f) = \{(x,t) \in \mathbb{R}^d \times R \ : \ t \ge f(x)\}.$$

This set is, by hypothesis, convex. That is, for any $(x,t),(x',t') \in \mathrm{epi}(f)$, we have

$$\lambda(x,t) + (1-\lambda)(x',t') = \big(\lambda x + (1-\lambda)x', \lambda t + (1-\lambda)t'\big) \in \mathrm{epi}(f) \quad \forall \lambda \in [0,1].$$

This implies that

$$f(\lambda x + (1-\lambda)x') \le \lambda t + (1-\lambda)t', \quad \forall \lambda \in [0,1]. \tag{2}$$

Also, since each of the points belongs to $\mathrm{epi}(f)$, we have that:

$$\lambda f(x) + (1-\lambda) f(x') \le \lambda t + (1-\lambda)t', \quad \forall \lambda \in [0,1] \tag{3}$$

Lastly, if we substract Equation (3) from Equation (2) we obtain:

$$\begin{aligned} & f(\lambda x + (1-\lambda)x') - \big(\lambda f(x) + (1-\lambda) f(x')\big) \le 0, && \forall \lambda \in [0,1] \\ \Longrightarrow\ & f(\lambda x + (1-\lambda)x') \le \lambda f(x) + (1-\lambda) f(x'), && \forall \lambda \in [0,1]. \end{aligned}$$

Lastly, recalling that $(x, f(x)) \in \mathrm{epi}(f)$ for all $x \in S$, we obtain that f is convex. □

Exercise 13
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Prove that $\mathrm{epi}(f)$ is a closed set and that $(x, f(x)) \in \partial\, \mathrm{epi}(f)$.

Exercise 14
Prove that if f is strictly convex, it has a unique global minimum.

HECHO EN CLASE

Exercise 15
Let $f, g : S \subset \mathbb{R}^d \to \mathbb{R}$ be two convex functions on the convex set S . Prove that, as subsets, $\partial f(x) + \partial g(x) \subset \partial(f+g)(x)$ for any $x \in S$.

We already know that $\xi \in \partial f(x)$ implies that $f(x') > f(x) + \xi(x - x')$ for all $x' \in S$. Let us apply this definition to obtain the result.
Consider $\xi_1 \in \partial f(x)$ and $\xi_2 \in \partial g(x)$. Then, $\xi_1 + \xi_2 \in \partial f(x) + \partial g(x)$. Now, using the definition for each of the ξ_i with $i = 1, 2$, we obtain:

$$f(x') > f(x) + \xi_1(x - x'), \quad g(x') > g(x) + \xi_2(x - x')$$

And, if we add both inequalitys:

$$\begin{aligned} f(x') + g(x') &> f(x) + g(x) + (\xi_1 + \xi_2)(x - x') \\ (f+g)(x') &> (f+g)(x) + (\xi_1 + \xi_2)(x - x') \end{aligned}$$

which means that $\xi_1 + \xi_2 \in \partial (f+g)(x)$, as we wanted to see. $\square$

Exercise 16
Compute the proximal of $f(x) = 0$ and of $g(x) = \frac{1}{2}\|x\|^2$.

HECHO EN CLASE

Exercise 17
Assume that f is convex. Prove that for any $\lambda > 0, \partial(\lambda f)(x) = \lambda \partial f(x)$ as subsets.

HECHO EN CLASE

Exercise 18
Prove that the $\epsilon-$insensitive loss function $\ell_\epsilon(z) = \max\{0, |z - \epsilon|\}$ is convex. Give also its subgradient $\partial \ell_\epsilon(x)$ at any $x \in \mathbb{R}$

Exercise 19
Compute the proximals of the hinge $f(x) = max\{0, -x\}$ and the ϵ-insensitive $g(x) = max\{0, |x| - \epsilon\}$ loss functions.

HECHO EN CLASE

Exercise 20
We have seen that we can solve the constrained Ridge problem by a Projected Gradient algorithm. Using the lecture slides, write down in as much detail as you can the computations needed at each iteration of the algorithm.