复变函数 B 作业 4

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Question 9

解：$\int_{|z|=1}\dfrac{e^z}{z}dz = 2\pi i\cdot e^z|_{z=0} = 2\pi i$

证明过程如下：

$\int_{-\pi}^{0} e^{\cos\theta}\cos(\sin\theta)d\theta \overset{\hat{\theta}=-\theta}{=} \int_{\pi}^{0} e^{\cos\hat{\theta}}\cos(\sin(-\hat{\theta}))d(-\hat{\theta}) = \int_{0}^{\pi} e^{\cos\hat{\theta}}\cos(\sin\hat{\theta})d\hat{\theta}$

$\Rightarrow \int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \int_{0}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta + \int_{-\pi}^{0} e^{\cos\theta}\cos(\sin\theta)d\theta = 2\int_{0}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$

$\therefore \int_{0}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \frac{1}{2}\int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)d\theta = \frac{1}{2}\int_{-\pi}^{\pi} e^{\cos\theta}\frac{e^{i\sin\theta}+e^{-i\sin\theta}}{2}d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\frac{e^{\cos\theta+i\sin\theta}+e^{\cos\theta-i\sin\theta}}{2}d\theta$

令 $z = e^{i\theta} = \cos\theta + i\sin\theta$, 则原式 $=\frac{1}{4}\int_{|z|=1}\frac{e^z+e^{-z}}{iz}dz = \frac{1}{4}\cdot 2\pi(e^z|_{z=0} + e^{-z}|_{z=0}) = \frac{1}{4}\cdot 2\pi(1+1) = \pi$. 证毕

Question 10

(2) 解：$\int_C \dfrac{e^z}{1+z^2}dz = \int_{|z+i|=1}\dfrac{e^z}{(z+i)(z-i)}dz = 2\pi i\cdot\dfrac{e^z}{z-i}|_{z=-i} = -\pi e^{-i}$

Question 11

解：奇点：$z=0$（二阶），$z=-1, z=1$（一阶）

①$r>1$: 此时奇点均在域内，原式 $=2\pi i((+\frac{1}{z^2(z+1)}|_{z=1} + \frac{1}{z^2(z-1)}|_{z=-1}) = 2\pi i(0+\frac{1}{2}-\frac{1}{2}) = 0$

②$0<r<1$: 此时域内只有二阶奇点 $z=0$，原式 $=2\pi i(\frac{1}{(z+1)(z-1)})'|_{z=0} = 0$

综上所述：原积分 $=0$

Question 12

解：(2)$f(z) = \dfrac{-z}{(z^2-9)(z+i)} = \dfrac{-z}{(z+3)(z-3)(z+i)}$, 则奇点为：$z=3, z=-3, z=i$ 均为一阶且均在 $|z| = \frac{10}{3}$ 内。

$$\text{原式} = 2\pi i\left(\frac{-z}{(z+3)(z+i)}|_{z=3} + \frac{-z}{(z-3)(z+i)}|_{z=-3} + \frac{-z}{(z+3)(z-3)}|_{z=-i}\right)$$

$$= 2\pi i\cdot\left(\frac{-3}{6(3+i)} + \frac{3}{-6(-3+i)} + \frac{i}{-10}\right)$$

$$= 0$$

Question 13

解：(1) 证：由柯西积分公式：$f(z_0)=\frac{1}{2\pi i}\int_{|z|=2}\frac{f(z)}{z-z_0}dz$

取 $f(z)=2z^2-z+1$，则 $2\pi i f(z_0)=\int_{|z|=2}\frac{2z^2-z+1}{z-z_0}dz=g(z_0)$

$\therefore g(1)=2\pi i f(1)=2\pi i\cdot(2z^2-z+1)|_{z=1}=4\pi i$

(2)$|z_0|>2$ 时，积分闭域内无奇点，全解析

由柯西积分定理 $\Rightarrow g(z_0)=0$

Question 14

解：$((z+i)(z-i))^2=0\Rightarrow$ 奇点为 $z=i, z=-i$, 均为二阶，且仅 $z=i$ 位于积分域内

$\therefore$ 原积分 $=2\pi i(\frac{z^2}{(z+i)^2})'|_{z=i}=2\pi i\frac{2iz}{(z+i)^3}|_{z=i}=\frac{\pi}{2}$

Question 15

证：由题意可知：$\dfrac{P'(z)}{P(z)}=\dfrac{1}{z-a_1}+\dfrac{1}{z-a_2}+...+\dfrac{1}{z-a_n}=\sum\limits_{i=1}^{n}\dfrac{1}{z-a_i}$

不妨假设 C 内有 k 个零点，分别为 $a_{(1)},a_{(2)},...,a_{(k)}(k\leq 0)$ (k) 表示按下标从小到大排序，$(1)\neq 1$

对奇点位于 C 内的项：$\int_C\frac{1}{z-a_{(i)}}dz=2\pi i\times 1|_{z=a_{(i)}}=2\pi i$

对奇点位于 C 外的项：$\frac{1}{z-a_i}$ 在积分域内全解析，积分 =0

$\therefore$ 原式 $=\frac{1}{2\pi i}\sum\limits_{i=1}^{n}\int_C\frac{1}{z-a_i}dz=\dfrac{1}{2\pi i}\cdot(2\pi i\cdot k+0\cdot(n-k))=k$

$\therefore$ 积分等于闭路内 $P(z)$ 的零点个数，证毕