第五次作业解答

P71~73

Date. No.

6. (2) $\int_{-1}^{i}(1+4iz^3)dz$ 易知 $f(z)=1+4iz^3$ 全平面解析.

故由牛-莱: 原式 $= z+iz^4\Big|_{-1}^{i} = 1+i$

17. 解: $\frac{\partial u}{\partial x}=3ax^2+2bxy+cy^2$ $\quad \frac{\partial u}{\partial y}=bx^2+2cxy+3dy^2$

$\frac{\partial^2 u}{\partial x^2}=6ax+2by$ $\quad \frac{\partial^2 u}{\partial y^2}=2cx+6dy$

又 ∵ $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ∴ $6ax+2by+2cx+6dy=0$

∴ $\begin{cases}6a+2c=0\\2b+6d=0\end{cases} \Rightarrow \begin{cases}c=-3a\\b=-3d.\end{cases}$

18. (2) 解: 证明: 设 $f(z)=u+iv$ $\quad |f(z)|^2=u^2+v^2$

∴ $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$

$=\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+\frac{\partial^2(v^2)}{\partial x^2}+\frac{\partial^2(v^2)}{\partial y^2}$

$=2\left(\frac{\partial u}{\partial x}\right)^2+2u\cdot\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial y}\right)^2+2u\cdot\frac{\partial^2 u}{\partial y^2}$

$+2\left(\frac{\partial v}{\partial x}\right)^2+2v\cdot\frac{\partial^2 v}{\partial x^2}+2\left(\frac{\partial v}{\partial y}\right)^2+2v\cdot\frac{\partial^2 v}{\partial y^2}$ (*)

由于 $f(z)$ 解析 ⇒ u、v 调和, 且有C-R方程 ⇒ $\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0\\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$

代入(*)式可得: (*) $=2\left(\frac{\partial u}{\partial x}\right)^2+2\left(-\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial x}\right)^2$

$=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$

又 ∵ $f'(z)=\frac{\partial u}{\partial x}+i\cdot\frac{\partial v}{\partial x}$ ⇒ $|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2$

综上: $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|$ 得证.

19. 解：由于 u 为调和，则有 $\Delta u=0$．$\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\right)$

(1) $\frac{\partial^2 u^2}{\partial x^2}=2\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}$，$\frac{\partial^2 u^2}{\partial y^2}=2\left(\frac{\partial u}{\partial y}\right)^2+2u\cdot\frac{\partial^2 u}{\partial y^2}$

$=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\neq 0$（$u$ 不恒为常数）

∴ u^2 不调和．

(2) $\frac{\partial f(u)}{\partial x}=\frac{df}{du}\cdot\frac{\partial u}{\partial x}$．$\Rightarrow \frac{\partial^2 f(u)}{\partial x^2}=\frac{d^2 f}{du^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2+\frac{df}{du}\cdot\frac{\partial^2 u}{\partial x^2}$

$\frac{\partial f(u)}{\partial y}=\frac{df}{du}\cdot\frac{\partial u}{\partial y}$．$\Rightarrow \frac{\partial^2 f(u)}{\partial y^2}=\frac{d^2 f}{du^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2+\frac{df}{du}\cdot\frac{\partial^2 u}{\partial y^2}$

要使 $f(u)$ 调和，则有 $\frac{\partial^2 f(u)}{\partial x^2}+\frac{\partial^2 f(u)}{\partial y^2}=0$

即：$\frac{d^2 f}{du^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2+\frac{d^2 f}{du^2}\left(\frac{\partial u}{\partial y}\right)^2+\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]\cdot\frac{df}{du}=0$

即：$\frac{d^2 f}{du^2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=0$

∴ $\frac{d^2 f}{du^2}=0$．即 $f(u)$ 关于 u 的二阶导为0

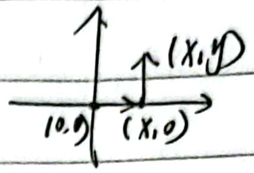
解该微分方程：$f''=0$

特征方程：$r^2=0$．$\Rightarrow r_1=r_2=0$．

∴ $f(u)=(C_1+C_2 x)\cdot e^{r_1 x}=C_1+C_2 x$.

20. 解: (1) $\frac{\partial^2 u}{\partial x^2} = 6x-12y \quad \frac{\partial^2 u}{\partial y^2} = -6x+12y \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

∴ $u(x,y)$ 调和, 可作为 $f(z)$ 的实部.



∴ $v(x,y) = \int_{(0,0)}^{(x,y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy + C$

$= \int_{(0,0)}^{(x,y)} (6x^2+3xy-6y^2)dx + (3x^2-12xy-3y^2)dy + C$

$= \int_0^x 6x^2 dx + \int_0^y (3x^2-12xy-3y^2)dy + C$

$= 2x^3\Big|_0^x + 3x^2y-6xy^2-y^3\Big|_0^y + C$

$= 2x^3+3x^2y-6xy^2-y^3+C$

∴ $f(z) = u+iv = (x^3-6x^2y-3xy^2+2y^3) + i\cdot(2x^3+3x^2y-6xy^2-y^3+C)$

$f(0) = i\cdot C = 0 \Rightarrow C=0$

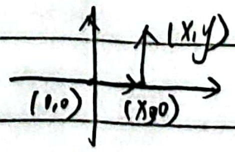
令 $x=z$, $y=0$. 故 $f(z) = z^3\cdot(1+2i)$

(3) $\frac{\partial v}{\partial x} = \frac{2y\cdot(x+1)}{((x+1)^2+y^2)^2} \quad \frac{\partial^2 v}{\partial x^2} = \frac{2y[(x+1)^2+y^2]^2 - 2y(x+1)\cdot 2[(x+1)^2+y^2]\cdot 2(x+1)}{((x+1)^2+y^2)^4}$

$\frac{\partial v}{\partial y} = \frac{-[(x+1)^2+y^2]+2y^2}{((x+1)^2+y^2)^2} \quad \frac{\partial^2 v}{\partial y^2} = \frac{2y\cdot[(x+1)^2+y^2]^2 - [-(x+1)^2+y^2]\cdot 2[(x+1)^2+y^2]\cdot 2y}{((x+1)^2+y^2)^4}$

∴ $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = \frac{2y[(x+1)^2+y^2] - 8y(x+1)^2}{((x+1)^2+y^2)^3} + \frac{2y[(x+1)^2+y^2] - 4y[-(x+1)^2+y^2]}{((x+1)^2+y^2)^3} = 0$

∴ $v(x,y)$ 调和, 可作为 $f(z)$ 虚部.



∴ $u(x,y) = \int_{(0,0)}^{(x,y)} \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy + C$

$= \int_0^x \frac{-(x+1)^2}{(x+1)^4}dx - \int_0^y \frac{2y(x+1)}{((x+1)^2+y^2)^2}dy + C$

$$= \int_0^x \frac{-1}{(x+1)^2}dx + \int_0^y \frac{2y(x+1)}{((x+1)^2+y^2)^2}dy + C$$

$$= \frac{1}{x+1}\Big|_0^x + \frac{(x+1)}{(x+1)^2+y^2}\Big|_0^y + C$$

$$= \frac{1}{x+1} + \frac{x+1}{(x+1)^2+y^2} - \frac{1}{x+1} + C$$

$$= \frac{x+1}{(x+1)^2+y^2} + C$$

$\therefore f(z) = u + iv = \frac{x+1}{(x+1)^2+y^2} + C + i\left(-\frac{y}{(x+1)^2+y^2}\right)$

由 $f(0)=2$ 可知：$\frac{1}{1+0} + C + i\cdot 0 = 2 \Rightarrow C=1$

令 $x=z, y=0$ 可得：$f(z) = \frac{z+1}{(z+1)^2} + 1 = \frac{1}{z+1} + 1 = \frac{z+2}{z+1}$.

P98 1. 解：(1) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{n^2}}} = 1$

$|z|=1$ 时，$\left|\sum_{n=1}^{+\infty} \frac{z^n}{n^2}\right| \le \sum_{n=1}^{+\infty} \frac{|z|^n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$ 收敛.

$\therefore$ 在收敛圆周上，该级数点点绝对收敛.

(2) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{1}} = 1$

$|z|=1$ 时，$|z|^n = 1$，$\therefore$ 一般项 z^n 不可能以0为极限，$\therefore$ 发散.

$\therefore$ 在收敛圆周上，该级数点点发散.

(3) $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{n}}} = 1$

$|z|=1$ 时：① $z=1$，则 $\sum_{n=1}^{+\infty} \frac{1}{n}$ 发散

② $z=-1$，则 $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ 收敛

$\therefore$ 有收敛点，也有发散点.

2. 解: (1) $\frac{1}{1-z}+e^{z}=\sum_{n=0}^{+\infty}z^{n}+\sum_{n=0}^{+\infty}\frac{z^{n}}{n!}$ Date. No.

$$=\sum_{n=0}^{+\infty}\left(1+\frac{1}{n!}\right)z^{n}$$

$$r=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{1+\frac{1}{(n+1)!}}{1+\frac{1}{n!}}=\lim_{n\to\infty}\frac{(n+1)!+1}{(n+1)!+(n+1)}=1$$

$\therefore R=1$

(3) $$\sin^2 z=\frac{1-\cos 2z}{2}=\frac{1}{2}-\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}(2z)^{2n}$$

$$=\frac{1}{2}-\frac{1}{2}\left(1+\sum_{n=1}^{+\infty}\frac{(-1)^n}{(2n)!}(2z)^{2n}\right)$$

$$=\frac{1}{2}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{(2n)!}(2z)^{2n}$$

$$r=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{4^{n+1}/(2(n+1))!}{4^{n}/(2n)!}=0$$

$\therefore R=\frac{1}{r}=+\infty.$

(4) $$\frac{1}{z^2-3z+2}=\frac{1}{z-2}-\frac{1}{z-1}=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}+\frac{1}{1-z}$$

$$=-\frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{+\infty}z^n$$

$$=\sum_{n=0}^{+\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n.$$

$$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{1-\frac{1}{2^{n+1}}}}=1$$

(6) $\because \frac{1}{(1-z)^2}=\left(\frac{1}{1-z}\right)'$

$\frac{1}{1-z}=\sum_{n=0}^{+\infty}z^n \Rightarrow \therefore \frac{z}{(1-z)^2}=z\cdot\sum_{n=0}^{+\infty}n\cdot z^{n-1}=\sum_{n=0}^{+\infty}n\cdot z^n$

$$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}=1$$

(8) $$\int_0^z\frac{\sin z}{z}dz=\int_0^z\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{z^{2n+1}}{z}dz$$

$$=\int_0^z\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}dz=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)!\,(2n+1)}\cdot z^{2n+1}.$$

$$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}}=+\infty$$