

# 拉普拉斯变换法

The Method of Laplace Transforms

# 一、拉氏变换

## 1、定义

$$F(p) = \int_0^{+\infty} f(t) e^{-p t} dt \quad - f(t) \text{的拉氏变换}$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(p) e^{p t} dp \quad - F(p) \text{的拉氏逆变换}$$

## 2、存在条件

(1)  $f(t)$ 及导数除有限个第一类间断点外连续

(2)  $|f(t)| \leq K e^{ct} (K, c \geq 0 ; c \text{是增长指数})$

## 二、拉氏变换性质

1. 线性 :  $L[\alpha f_1(t) + \beta f_2(t)] = \alpha L[f_1(t)] + \beta L[f_2(t)]$
2. 位移 :  $L[e^{p_0 t} f(t)] = F(p - p_0)$  记  $L[f(t)] = F(p)$
3. 延迟 :  $L[f(t - \tau)] = e^{-p\tau} F[p]$
4. 相似 :  $L[f(at)] = \frac{1}{a} F\left(\frac{p}{a}\right), a > 0$
5. 微分 :  $L[f^{(n)}(t)] = p^n F(p) - p^{n-1} f(0) - p^{n-2} f'(0) - \dots - f^{(n-1)}(0)$
6. 积分 :  $L\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{p} L[f(t)]$
7. 卷积 :  $L[f_1(t) * f_2(t)] = L[f_1(t)] \cdot L[f_2(t)]$   
$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau$$

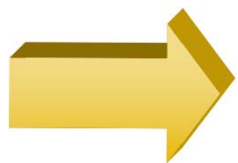
### 三、解数理方程

1. P190  
习题8

$$\begin{cases} \frac{d^2 y}{dt^2} + \omega^2 y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \quad \text{记} \quad \begin{aligned} L[y(t)] &= \int_0^\infty y(t)e^{-pt} dt = Y(p), \\ L[f(t)] &= \int_0^\infty f(t)e^{-pt} dt = F(p) \end{aligned}$$

则  $p^2 Y(p) - py(0) - y'(0) + \omega^2 Y(p) = F(p)$

$$\therefore Y(p) = \frac{F(p)}{p^2 + \omega^2} = L\left[f(t) * \frac{1}{\omega} \sin \omega t\right]$$



$$y(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega(t - \tau) d\tau$$

## 2. 解偏微分方程混合问题

$$\begin{cases} u_{tt} = a^2 u_{xx}, 0 < x < \infty, t > 0 \\ u(x, 0) = 0, \quad u_t(x, 0) = 0 \\ u(0, t) = f(t), \lim_{x \rightarrow \infty} u(x, t) = 0 \quad (t \geq 0) \end{cases}$$

记  $L[u(x, t)] = U(x, p)$ ,  $L[f(t)] = F(p)$ .

则 
$$\begin{cases} p^2 U(x, p) - pu(x, 0) - u_t(x, 0) = a^2 \frac{\partial^2}{\partial x^2} U(x, p) \\ U(0, p) = F(p), \quad \lim_{x \rightarrow \infty} U(x, p) = 0 \end{cases}$$

即

$$\begin{cases} \frac{d^2}{dx^2} U(x, p) - \frac{p^2}{a^2} U(x, p) = 0 \\ U(0, p) = F(p) \\ \lim_{x \rightarrow \infty} U(x, p) = 0 \end{cases}$$

$$U = c_1(p)e^{-\frac{p}{a}x} + c_2(p)e^{\frac{p}{a}x}$$

$$\text{由 } U(0, p) = F(p) \Rightarrow c_1(p) + c_2(p) = F(p)$$

$$\text{由 } U(\infty, p) = 0 \Rightarrow c_2(p) = 0$$

$$\therefore U(x, p) = F(p)e^{-\frac{p}{a}x} \Rightarrow u(x, t) = L^{-1} \left[ F(p) \cdot e^{-\frac{p}{a}x} \right]$$

由延迟定理得

$$u(x, t) = L^{-1} L \left[ f \left( t - \frac{x}{a} \right) \right] = f \left( t - \frac{x}{a} \right)$$