复变函数 B 期中考试

张学涵 (xxhzhang@mail.ustc.edu.cn)

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Question 1

设 $z = x + iy$, 则 $\frac{z-i}{z+i} = \frac{x+i(y-1)}{x+i(y+1)} = \frac{x^2+y^2-1-2xi}{x^2+(y+1)^2}$. (10 分)

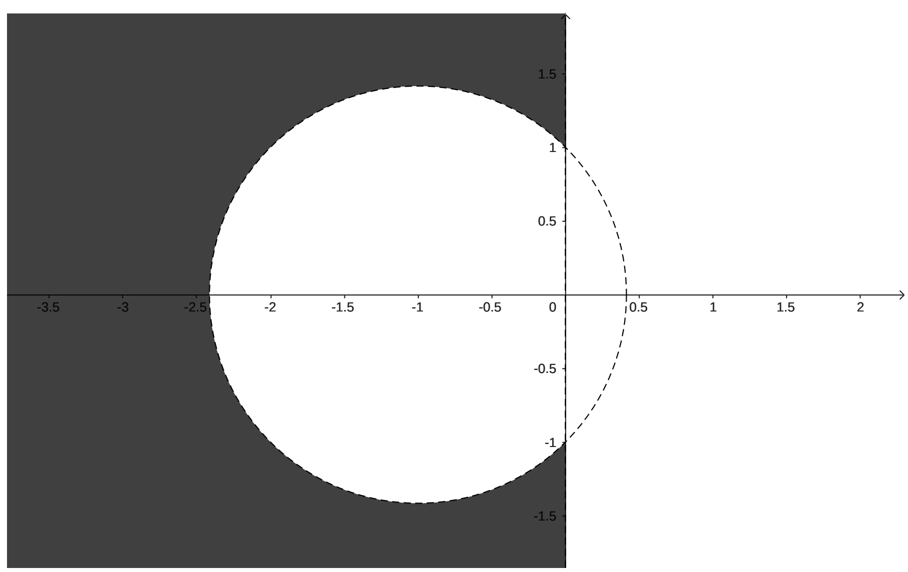
由于 $0 < \arg \frac{z-i}{z+i} < \frac{\pi}{4}$, 故

$$\begin{cases} x^2 + y^2 - 1 > 0; \\ -2x > 0; \\ x^2 + y^2 - 1 > -2x. \end{cases}$$

即

$$\begin{cases} x < 0; \\ (x+1)^2 + y^2 > 2. \end{cases}$$

虚轴左侧, 以 -1 为圆心 $\sqrt{2}$ 为半径的圆的外部区域. (20 分)



注释: 习题课已经讲过, arg 和 arctan 不等价. 很多人直接 $0 < \frac{-2x}{x^2+y^2-1} < 1$, 算出两个区域的并.

Question 5

首先验证 $v(x,y)$ 是调和函数.

$\frac{\partial v}{\partial x}=-\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}$, (2 分)

$\frac{\partial v}{\partial y}=\frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}$, (4 分)

$\frac{\partial^2 v}{\partial x^2}=-\frac{\left(-1+\frac{x}{\sqrt{x^2+y^2}}\right)^2}{4\left(-x+\sqrt{x^2+y^2}\right)^{3/2}}+\frac{-\frac{x^2}{(x^2+y^2)^{3/2}}+\frac{1}{\sqrt{x^2+y^2}}}{2\sqrt{-x+\sqrt{x^2+y^2}}}$, (5 分)

$\frac{\partial^2 v}{\partial y^2}=-\frac{y^2}{4(x^2+y^2)\left(-x+\sqrt{x^2+y^2}\right)^{3/2}}-\frac{y^2}{2(x^2+y^2)^{3/2}\sqrt{-x+\sqrt{x^2+y^2}}}+\frac{1}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}$, (6 分)

$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$, 故 $v(x,y)$ 是调和函数. (9 分)

$u(x,y)=\int_{(0,0)}^{(x,y)}\frac{\partial v}{\partial y}\,dx-\frac{\partial v}{\partial x}\,dy+C.$ (10 分)

令 $x,y\geq 0$, 有

$$
\begin{aligned}
u(x,y)&=\int_{(0,0)}^{(x,y)}\frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}\,dx+\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy+C\\
&=\int_{(0,0)}^{(x,y)}\frac{\sqrt{x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dx+\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy+C\\
&=\int_0^x\frac{\sqrt{x+\sqrt{x^2}}}{2\sqrt{x^2}}\,dx+\int_0^y\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy+C\\
&=\sqrt{2x}+\int_0^y\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy+C.
\end{aligned}
$$

(12 分)

令 $t=\sqrt{x^2+y^2}$, 则 $y=\sqrt{t^2-x^2}, dy=\frac{t}{\sqrt{t^2-x^2}}\,dt$, 于是

$$
\begin{aligned}
u(x,y)&=\sqrt{2x}+\int_0^y\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy+C\\
&=\sqrt{2x}+\int_x^{\sqrt{x^2+y^2}}\frac{\sqrt{-x+t}}{2t}\frac{t}{\sqrt{t^2-x^2}}\,dt+C\\
&=\sqrt{2x}+\int_x^{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{t+x}}\,dt+C\\
&=\sqrt{x+\sqrt{x^2+y^2}}+C.
\end{aligned}
$$

(15 分)

因为 $f(0)=0$, 故 $C=0$. (16 分)

令 $x=z\geq 0, y=0$, 得 $f(z)=u(x,y)+iv(x,y)=\sqrt{2z}, z\in\mathbb{C}$. (20 分)

注释: 习题课已经讲过, 这种题必须先验证调和函数, 最后结果也要用 z 表示. 我们先得到 $f(z)$ 在实轴非负半轴处的函数表达式, 由唯一性定理即可推广至全复平面. 当然, 很多人这里积分没算对, 所有人都认为 $y=0$ 时 $\frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}=0$, 而没注意到此时分母也是 0. 如果不化简 $\frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}$, 对于这种分子分母都为 0 的情形, 我们不能取 $y=0$ 积分, 但可以取 $y=1$ 积分, 此时积分路径变为 $(0,1)\to(x,1)\to(x,y)$. 或者, 也可以取积分路径 $(0,0)\to(0,y)\to(x,y)$. 另一个需要换元的积分算出来的寥寥无几, 为避免麻烦, 我们这里也可以用第 6 题的结论, 在极坐标下处理.
设 $z=re^{i\theta},\theta\in[0,2\pi)$, 则 $v=\sqrt{r-r\cos\theta}=\sqrt{2r\sin^2\frac{\theta}{2}}=\sqrt{2r}\sin\frac{\theta}{2}$.
由极坐标系下的 Laplace 方程, 我们可以验证 $v(r,\theta)$ 是调和函数, 即满足:

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0.$$

因为

$$\begin{cases}\frac{\partial v}{\partial r}=\frac{1}{\sqrt{2r}}\sin\frac{\theta}{2};\\ \frac{\partial v}{\partial \theta}=\frac{\sqrt{2r}}{2}\cos\frac{\theta}{2}.\end{cases}$$

所以由第 6 题结论有

$$\begin{cases}\frac{\partial u}{\partial r}=\frac{1}{\sqrt{2r}}\cos\frac{\theta}{2};\\ \frac{\partial u}{\partial \theta}=-\sqrt{\frac{r}{2}}\sin\frac{\theta}{2}.\end{cases}$$

于是

$$\begin{aligned}u(r,\theta)&=\int_{(0,0)}^{(r,\theta)}\frac{\partial u}{\partial r}\,dr+\frac{\partial u}{\partial \theta}\,d\theta+C\\&=\int_0^r\frac{1}{\sqrt{2r}}\,dr-\int_0^\theta\sqrt{\frac{r}{2}}\sin\frac{\theta}{2}\,d\theta+C\\&=\sqrt{2r}-2\sqrt{\frac{r}{2}}(\cos\frac{\theta}{2}-1)+C\\&=\sqrt{2r}\cos\frac{\theta}{2}.\end{aligned}$$

因为 $f(0)=0$, 故 $C=0$.
令 $r=z,\theta=0$, 得 $f(z)=u(r,\theta)+iv(r,\theta)=\sqrt{2z},z\in\mathbb{C}$.
由此可见极坐标下处理更为简单. 若取积分路径为 $(0,0)\to(0,\theta)\to(r,\theta)$, 甚至只需计算一个积分.

Question 6

设 $\begin{cases}x=r\cos\theta;\\ y=r\sin\theta.\end{cases}$

则 $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial r}&\frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r}&\frac{\partial y}{\partial \theta}\end{bmatrix}\begin{bmatrix}r\\ \theta\end{bmatrix}=\begin{bmatrix}\cos\theta&-r\sin\theta\\ \sin\theta&r\cos\theta\end{bmatrix}\begin{bmatrix}r\\ \theta\end{bmatrix}.$

由于 $\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}^{-1}=\begin{bmatrix}\cos\theta & sin\theta\\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r}\end{bmatrix}$,

故 $\begin{bmatrix}r\\ \theta\end{bmatrix}=\begin{bmatrix}\cos\theta & sin\theta\\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$.

那么 $\begin{bmatrix}u\\ v\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta}\end{bmatrix}\begin{bmatrix}r\\ \theta\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta}\\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta}\end{bmatrix}\begin{bmatrix}\cos\theta & sin\theta\\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} & \frac{\partial u}{\partial r}sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\\ \frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} & \frac{\partial v}{\partial r}sin\theta+\frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$.

因此

$$\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}=\frac{\partial v}{\partial r}sin\theta+\frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}. \tag{1}$$

$$\frac{\partial u}{\partial r}sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}=-\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}. \tag{2}$$

$(1)\cos\theta+(2)\sin\theta$ 得: $\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$.

$(1)\sin\theta-(2)\cos\theta$ 得: $\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial v}{\partial \theta}$.

故极坐标下的柯西–黎曼方程是

$$\begin{cases}\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta};\\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.\end{cases}$$

注释: 有一些同学貌似是提前知道答案, 利用配凑的方法, 将 $\frac{\partial u}{\partial r},\frac{\partial u}{\partial \theta}$ 用 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ 表示, 凑出最后的结论. 我们这里的思路是将 $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}$ 用 $\frac{\partial u}{\partial r},\frac{\partial u}{\partial \theta}$ 表示, 并代入直角坐标下的柯西–黎曼方程.