1.5.16. Before we begin, notice that the statement $au(x, y) + bv(x, y) = c$ with not all of a, b, c equal to zero is equivalent to saying that the values of $f(z) = u(x, y) + iv(x, y)$ lie on a straight line.

(a) We write $u = u(x, y)$, $v = v(x, y)$ etc. to simplify notation. If we apply $\partial/\partial x$ and $\partial/\partial y$ to the equation $au + bv = c$ we obtain

$$\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= 0 \\ a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y} &= 0. \end{aligned}$$

Since f is analytic we can apply the Cauchy-Riemann equations in the second equality above to obtain the system

$$\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= 0 \\ b\frac{\partial u}{\partial x} - a\frac{\partial v}{\partial x} &= 0 \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{matrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix}.$$

The determinant of the coefficient matrix is $-(a^b + b^2)$ which cannot be zero because a, b, c are real and not all zero. Hence, the only solution to the system is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0.$$

Therefore

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

and, since A is connected, we conclude that f is constant.

(b) If a, b, c are complex, then the proof above does not apply. However, if we write $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $c = c_1 + ic_2$ with $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, then the equation $au + bv = c$ is equivalent to the pair of equations

$$\begin{aligned} a_1 u + b_1 v &= c_1 \\ a_2 u + b_2 v &= c_2. \end{aligned}$$

Since not all of a, b, c are zero, it must be that in at least one of these equations not all of the constants are zero. We may then apply part (a) to that equation to conclude that f is constant. In other words, the statement is valid for complex a, b, c as well.