# 第一周习题解答.

8.31 星期三.

1.(2) $(x-i\sqrt{y})(-x-2i\sqrt{y}) = -x^2-2xi\sqrt{y}+xi\sqrt{y}-2y = (-x^2-2y)-xi\sqrt{y}$

(4) $\dfrac{5i}{\sqrt{2}-\sqrt{3}i} = \dfrac{5i\cdot(\sqrt{2}+\sqrt{3}i)}{(\sqrt{2}-\sqrt{3}i)(\sqrt{2}+\sqrt{3}i)} = -\sqrt{3}+\sqrt{2}i$

位于第三象限

2.(3) $|z| = \sqrt{(-\frac{1}{2})^2+(-\sqrt{3})^2} = \dfrac{\sqrt{13}}{2}$. $\varphi = \operatorname{arctg}\dfrac{-\sqrt{3}}{-\frac{1}{2}} - \pi + 2n\pi = \operatorname{arctg}2\sqrt{3}+(2n-1)\pi$. $n\in Z$.

三角式: $z=\dfrac{\sqrt{13}}{2}\left(-\dfrac{\sqrt{13}}{13}-\dfrac{2\sqrt{39}}{13}i\right)$ 指数式: $z=\dfrac{\sqrt{13}}{2}e^{i(\operatorname{arctg}2\sqrt{3}-\pi)}$

(4) ①若 $z=0 \Rightarrow \begin{cases}\cos\theta=1\\ \sin\theta=0\end{cases} \Rightarrow \theta=2k\pi\ (k\in Z)$. 此时幅角无意义.

②若 $z\neq 0 \Rightarrow z=2\sin^2\frac{\theta}{2}+2i\cdot\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+i\cdot\cos\frac{\theta}{2}\right)$

∴ 三角式: $z=2\sin\frac{\theta}{2}\left(\cos\frac{\pi-\theta}{2}+i\cdot\sin\frac{\pi-\theta}{2}\right)$

设 $\theta=\theta_0+2k\pi$. 指数式: $z=2\sin\frac{\theta_0}{2}e^{i(\frac{\pi-\theta_0}{2})}$. $\varphi=\frac{\pi-\theta_0}{2}+2n\pi$. $n\in Z$.

3.(2) $(\sqrt{3}+i)^{-3} = \left(\dfrac{1}{\sqrt{3}+i}\right)^3 = \left(\dfrac{\sqrt{3}-i}{4}\right)^3 = \left(\frac{1}{2}e^{-\frac{\pi}{6}i}\right)^3 = \frac{1}{8}e^{-\frac{\pi}{2}i} = -\frac{1}{8}i$.

7. 证明: 利用: $\displaystyle\sum_{k=1}^{n}e^{ik\theta} = \frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}} = \sum_{k=1}^{n}\cos k\theta + i\cdot\sum_{k=1}^{n}\sin k\theta$.

$$= \frac{e^{i\theta}(1-e^{in\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$ 运用公式: 和差化积.

$$= \frac{e^{i\theta}+e^{in\theta}-e^{i(n+1)\theta}-1}{2-(e^{i\theta}+e^{-i\theta})}$$

转成三角式后分离实虚部; 代入 $e^{i\theta}=\cos\theta+i\sin\theta$ 得:

$$\text{原式} = \frac{(\cos\theta+\cos n\theta-\cos(n+1)\theta-1)+i\cdot(\sin\theta+\sin n\theta-\sin(n+1)\theta)}{2-2\cos\theta}$$

实部: $\mathrm{Re}[\text{原式}] = \dfrac{\cos\theta-1-2\sin(\frac{2n+1}{2}\theta)\cdot\sin(-\frac{\theta}{2})}{2-2\cos\theta} = -\dfrac{1}{2}+\dfrac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} = \displaystyle\sum_{k=1}^{n}\cos k\theta$

虚部: $\mathrm{Im}[\text{原式}] = \dfrac{\sin\theta+\sin n\theta-\sin(n+1)\theta}{2-2\cos\theta} = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+2\cos(\frac{2n+1}{2}\theta)\cdot\sin(-\frac{\theta}{2})}{4\sin^2\frac{\theta}{2}}$

$$= \frac{1}{2}\operatorname{ctg}\frac{\theta}{2} - \frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} = \sum_{k=1}^{n}\sin k\theta$$

8. 解：证明：$|z_1+z_2|^2+|z_1-z_2|^2$

$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= |z_1|^2+|z_2|^2+2Re(z_1\bar{z}_2)+|z_1|^2+|z_2|^2-2Re(z_1\bar{z}_2)$

$= 2(|z_1|^2+|z_2|^2)$ 书本P5例2.

几何意义：平行四边形对角线平方和 = 四条边的平方和.

9. 解：由三角不等式：$|z^n+a| \le |z|^n+|a|$. 当且仅当 $\arg z^n = \arg a$ 时等号成立.

$= |z|^n+|a|$

由于 $|z|\le 1$. 故可知：$|z^n+a| \le 1+|a|$. $z = e^{i\frac{\arg a}{n}}$.

10. 解：(1) 证：由于 $1=|z|=z\cdot\bar{z}$. 也可以对式子平方后展开去证明.

$\therefore \left|\frac{z-a}{1-\bar{a}z}\right| = \left|\frac{z-a}{z\cdot\bar{z}-\bar{a}\cdot z}\right| = \left|\frac{z-a}{z(\bar{z}-\bar{a})}\right| = \frac{1}{|z|}\cdot\left|\frac{z-a}{\overline{z-a}}\right| = \frac{1}{1}\cdot 1 = 1$

(2) 证：$\left|\frac{z-a}{1-\bar{a}z}\right|^2 = \frac{|z|^2+|a|^2-a\bar{z}-\bar{a}z}{1+|a|^2|z|^2-a\bar{z}-\bar{a}z}$

由于 $|z|^2+|a|^2-1-|a|^2|z|^2 = -(|z|^2-1)(|a|^2-1) < 0$.

故可知 原式 $<1$. 得证

9.2. 星期五.

3.(3) $\sqrt[3]{1+i}$：$1+i = 2^{\frac{1}{2}}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

$\therefore \sqrt[3]{1+i} = 2^{\frac{1}{6}}(\cos\frac{\frac{\pi}{4}+2k\pi}{3}+i\sin\frac{\frac{\pi}{4}+2k\pi}{3})$ $k=0,1,2$

$$\therefore \sqrt[3]{1+i} = \begin{cases} 2^{\frac{1}{6}}(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}) & k=0 \\ 2^{\frac{1}{6}}(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i) & k=1 \\ 2^{\frac{1}{6}}(\cos\frac{17}{12}\pi+i\sin\frac{17}{12}\pi) & k=2 \end{cases}$$

4.(2) $z^3=-i=\cos(-\frac{\pi}{2})+i\cdot\sin(-\frac{\pi}{2})$

$\therefore z=\cos\frac{-\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{-\frac{\pi}{2}+2k\pi}{3}\quad k=0,1,2.$

$\therefore$ 解得: $z_1=\frac{\sqrt{3}}{2}-\frac{1}{2}i$. $z_2=i$. $z_3=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$.

16.(1) $a_n=(\frac{3+4i}{6})^n$. 由于 $|a_n|=(\frac{5}{6})^n$ 当 $n\to\infty$ 时. $(\frac{5}{6})^n\to 0$

$\therefore$ 该复数列存在极限. 为0.

18.(3) 设 $z=x+iy$. 则 $Re\frac{1}{z}=Re\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}=\alpha$.

$\therefore$ 曲线方程为: $\begin{cases}(x-\frac{1}{2\alpha})^2+y^2=\frac{1}{4\alpha^2} & \alpha\neq 0\\ x=0\quad y\neq 0 & \alpha=0\end{cases}$

$\therefore$ 图形为: $\alpha\neq0$ 时: 与 $y$ 轴相切于原点, 圆心位于 $x$ 轴上的圆
$\alpha=0$ 时: $y$ 轴. 去除原点.

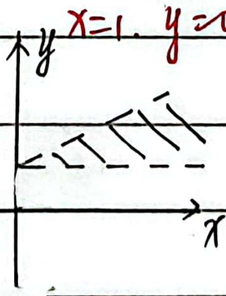
(5) 设 $z=x+iy$. $\left|\frac{z-1}{z+1}\right|=\left|\frac{(x-1)+iy}{(x+1)+iy}\right|=\alpha$.

$\Rightarrow (x-1)^2+y^2=\alpha^2[(x+1)^2+y^2]$

若 $\alpha=1$: $y$ 轴.
若 $\alpha\neq1$: $(x-\frac{1+\alpha^2}{1-\alpha^2})^2+y^2=\frac{4\alpha^2}{(1-\alpha^2)^2}$ 圆.
若 $\alpha=0$: $x=1$. $y=0$.

这题还有一个方法:
运用 $|z_1|=\lambda|z_2|$
$\Downarrow$
$|z_1-\lambda^2 z_2|=\lambda|z_1-z_2|$
可得: $\left|z-\frac{1+\alpha^2}{1-\alpha^2}\right|=\frac{2\alpha}{1-\alpha^2}$.
结论相同.

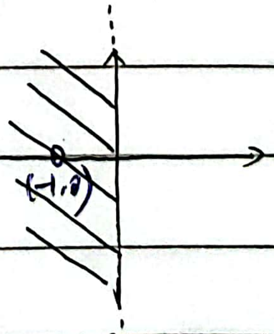
19. 解: (5)



无界开区域.
边界: $\{z|Re z\geq0, Im z=\frac{\sqrt{3}}{3}Re z+1\}\cup\{z|Re z\geq0, Im z=1\}$

(10) $\left|\frac{z-1}{z+1}\right|>1$

$z\neq(-1,0)$



无界开区域
边界: $\{z|Re z=0\}$ 及 $(-1,0)$.

21.(4) $z=t^2+\frac{i}{t^2}\ (t>0)\Rightarrow\begin{cases}x(t)=t^2\\ y(t)=\frac{1}{t^2}\end{cases}\Rightarrow y=\frac{1}{x}\ (x>0)$

22. 解: $(x+1)^2+y^2=2\Rightarrow|z+1|=\sqrt{2}$