复变函数 B 作业 9

2023 年 11 月 22 日

1. 求下列函数的像函数

(9) $\mathcal{L}\{te^{5t}\} = -\left(\frac{1}{p-5}\right)' = \frac{1}{(p-5)^2}$

(12) $\mathcal{L}\left\{\frac{d^2}{dt^2}(e^{-at}\sin wt)\right\} = p^2\mathcal{L}[e^{-at}\sin wt] - p\cdot f(+0) - f'(+0)$

$= p^2\cdot\frac{w}{(p+a)^2+w^2} - w$

(14) $\mathcal{L}\left\{\int_0^t te^{2t}dt\right\} = \frac{\mathcal{L}\{te^{2t}\}}{p} = \frac{-\left(\frac{1}{p-2}\right)'}{p} = \frac{1}{p(p-2)^2}$

(16) $\mathcal{L}\left\{\int_0^t (t-\tau)^n e^{-a\tau}\cos w\tau\, d\tau\right\} = \mathcal{L}\{t^n * e^{-at}\cos wt\}$

$= \mathcal{L}\{t^n\}\mathcal{L}\{e^{-at}\cos wt\}$

$= \frac{n!}{p^{n+1}}\cdot\frac{p+a}{(p+a)^2+w^2} = \frac{n!\,(p+a)}{p^{n+1}\cdot((p+a)^2+w^2)}$

(18) $\mathcal{L}\{\cos w(t-\varphi)\,h(t-2\varphi)\}$

$= \mathcal{L}\{\cos w((t-2\varphi)+\varphi)\,h(t-2\varphi)\}$

$= e^{-2p\varphi}\mathcal{L}\{\cos w(t+\varphi)\}$

$= e^{-2p\varphi}\cdot\mathcal{L}\{\cos wt\cdot\cos w\varphi - \sin wt\cdot\sin w\varphi\}$

$= e^{-2p\varphi}\cdot\left[\frac{p\cos w\varphi}{p^2+w^2} - \frac{w\sin w\varphi}{p^2+w^2}\right]$

6. 求下列像函数的本函数：

(16)$\mathcal{L}^{-1}\{\frac{3p+7}{p^2+2p+1+a^2}\} = \mathcal{L}^{-1}\{\frac{3(p+1)}{(p+1)^2+a^2} + \frac{4}{a}\frac{a}{(p+1)^2+a^2}\} = e^{-t}(3\cos at + \frac{4}{a}\sin at)$

(14) $\mathcal{L}^{-1}\{\frac{1}{p(p+a)}\} = \mathcal{L}^{-1}\{\frac{1}{a}(\frac{1}{p} - \frac{1}{p+a})\} = \frac{1}{a}(1-e^{-at})$

(16) $\mathcal{L}^{-1}\{\frac{1}{(p^2+1)(p^2+3)}\} = \mathcal{L}^{-1}\{\frac{1}{2}(\frac{1}{p^2+1} - \frac{1}{p^2+3})\} = \frac{1}{2}(\sin t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t)$

(18) $\mathcal{L}^{-1}\{\frac{1}{p(p-2)^2}\} = \mathcal{L}^{-1}\{\frac{1}{4p} - \frac{1}{4(p-2)} + \frac{1}{2(p-2)^2}\} = \frac{1}{4}(1-e^{2t}+2te^{2t})$

(18) $\mathcal{L}^{-1}\{\frac{1-p}{(p+1)(p^2+1)}e^{-10p}\} = \mathcal{L}^{-1}\{e^{-10p}(\frac{1}{p+1} - \frac{p}{p^2+1})\} = (e^{-(t-10)} - \cos(t-10))h(t-10)$

7. 利用拉氏变换求解下列方程：

解：(3) 设 $\mathcal{L}\{y(t)\}=Y(p)$. 则 $\mathcal{L}\{y''(t)\}=p^2Y-py(0)-y'(0)=p^2Y-1$

$\mathcal{L}\{y'(t)\}=pY-y(0)=pY$.

∴ 对原方程两边作拉氏变换：

$(p^2Y-1)-(a+b)pY+abY=0.\Rightarrow Y=\frac{1}{(p-a)(p-b)}$

$\therefore\ y(t)=\mathcal{L}^{-1}\{\frac{1}{(p-a)(p-b)}\}=\frac{1}{b-a}(e^{at}-e^{bt})$

(5) 设 $\mathcal{L}\{y(t)\}=Y(p)$. 则 $\mathcal{L}\{y''(t)\}=p^2Y-py(0)-y'(0)=p^2Y+p+2$

$\mathcal{L}\{4\sin t\}=\frac{4}{p^2+1}.\quad \mathcal{L}\{5\cos 2t\}=\frac{5p}{p^2+4}$

∴ 对原方程两边作拉氏变换：

$(p^2Y+p+2)-Y=\frac{4}{p^2+1}+\frac{5p}{p^2+4}\Rightarrow Y=-\frac{p^3+2p^2+p+8}{(p^2+1)(p^2+4)}=-(\frac{2}{p^2+1}+\frac{p}{p^2+4})$

$\therefore\ y(t)=\mathcal{L}^{-1}\{-(\frac{2}{p^2+1}+\frac{p}{p^2+4})\}=-\cos 2t-2\sin t$

(7) 设 $\mathcal{L}\{x(t)\}=X(p)$. $\Rightarrow$ $Y(p)$. $Z(p)$ 同理.

则 $\mathcal{L}\{x'(t)\}=pX.\quad \mathcal{L}\{y'(t)\}=pY.\quad \mathcal{L}\{z'(t)\}=pZ.\quad \mathcal{L}\{t\}=\frac{1}{p^2}$

$$\therefore \begin{cases} pX-pY=0 \\ pY+pZ=\frac{1}{p} \\ pX-pZ=\frac{1}{p^2} \end{cases}\Rightarrow \begin{cases} X=\frac{1}{2p^2}+\frac{1}{2p^3} \\ Y=\frac{1}{2p^2}+\frac{1}{2p^3} \\ Z=\frac{1}{2p^2}-\frac{1}{2p^3} \end{cases}\Rightarrow \begin{cases} x(t)=\frac{1}{2}t+\frac{t^2}{4} \\ y(t)=\frac{1}{2}t+\frac{t^2}{4} \\ z(t)=\frac{1}{2}t-\frac{t^2}{4} \end{cases}$$

(9) 如上设：$x(0)=0,y(0)=1\Rightarrow\mathcal{L}\{x'\}=pX,\mathcal{L}\{y'\}=pY-1$

$$\begin{cases} x'-2y'=\sin t \\ x'+y'=\cos t \end{cases}\Rightarrow\begin{cases} pX-2pY+2=\frac{1}{p^2+1} \\ pX+pY-1=\frac{p}{p^2+1} \end{cases}\Rightarrow\begin{cases} X=\frac{2p+1}{3p(p^2+1)}=\frac{2}{3(p^2+1)}+\frac{1}{3p}-\frac{p}{3(p^2+1)} \\ Y=\frac{3p^2+p+2}{3p(p^2+1)}=\frac{1}{3(p^2+1)}+\frac{p}{3(p^2+1)}+\frac{2}{3p} \end{cases}$$

$$\Rightarrow\begin{cases} x=\frac{2}{3}\sin t+\frac{1}{3}(1-\cos t) \\ y=\frac{1}{3}(\cos t+\sin t)+\frac{2}{3} \end{cases}$$