第三周作业解答

第二章

11.(2) 令 $z=iy$，则 $\lim\limits_{z\to0} z\sin\frac{1}{z} = \lim\limits_{y\to0} iy\,\dfrac{e^{\frac{i}{iy}}-e^{\frac{-i}{iy}}}{2i}$

$= \lim\limits_{y\to0} \frac{y}{2}\left(e^{\frac{1}{y}}-e^{-\frac{1}{y}}\right)$

由于 $y\to0$ $\begin{cases} y\to0^+ & \lim\limits_{y\to0^+} e^{\frac{1}{y}}=\infty. \\ y\to0^- & \lim\limits_{y\to0^-} e^{\frac{1}{y}}=0. \end{cases}$ 故 $\lim\limits_{y\to0}\frac{y}{2}\left(e^{\frac{1}{y}}-e^{-\frac{1}{y}}\right)\to\infty.$

∴极限不存在

(3) 令 $z=1+x$. $x\to0$. 则 $\lim\limits_{z\to1}\dfrac{ze^{\frac{1}{z-1}}}{e^z-1} = \lim\limits_{x\to0}\dfrac{(1+x)\cdot e^{\frac{1}{x}}}{e^{1+x}-1}$

因为 $\lim\limits_{x\to0} e^{\frac{1}{x}}$ 不存在，故极限不存在.

13.(1) $\sin z=2 \Rightarrow \sin z = \dfrac{e^{iz}-e^{-iz}}{2i}=2$，令 $t=e^{iz}$. 结果取 $2\pm\sqrt{3}$

则：方程化为 $t-\frac{1}{t}=4i$. $\Rightarrow t=\dfrac{4i\pm\sqrt{-12}}{2}=2i\pm\sqrt{3}i$

∵ $t=e^{iz} \Rightarrow Ln(t)=iz$.

∴ $z=\dfrac{Ln(t)}{i}=\dfrac{\ln|2\pm\sqrt{3}|+(\frac{\pi}{2}+2k\pi)i}{i}$

$=(\frac{\pi}{2}+2k\pi)-i\cdot\ln(2\pm\sqrt{3})$

(3) $e^z=A$. $\Rightarrow z=Ln A=\ln|A|+(2k\pi+\arg A)i$

14.(2) $\dfrac{1}{\sin z-2}$. $\Rightarrow \sin z\neq2$. 由13(1)可知. $z\neq(\frac{\pi}{2}+2k\pi)-i\cdot\ln|2\pm\sqrt{3}|$

∴解析区域为 $z\neq(\frac{\pi}{2}+2k\pi)-i\cdot\ln(2\pm\sqrt{3})$.

$\left(\dfrac{1}{\sin z-2}\right)'=\dfrac{-\cos z}{(\sin z-2)^2}$

(3) $ze^{\frac{1}{z-1}}$. $\Rightarrow z\neq1$ 即可.

$\left(ze^{\frac{1}{z-1}}\right)'=e^{\frac{1}{z-1}}+z\cdot\frac{-1}{(z-1)^2}e^{\frac{1}{z-1}}=e^{\frac{1}{z-1}}\left(1-\frac{z}{(z-1)^2}\right)$

16. 解：$\cos z = \cos x \,\mathrm{ch} y - i\sin x \,\mathrm{sh} y$ = 实数.

则：$\sin x \,\mathrm{sh} y = 0$. $\begin{cases} x = k\pi. \\ y = 0. \end{cases}$ ⇒ 实轴及直线族 $\mathrm{Re}\, z = k\pi$.
$(k = 0, \pm 1 \cdots)$

17. 解 (2) $1^{\sqrt{2}} = e^{\sqrt{2}\,\mathrm{Ln} 1} = e^{\sqrt{2}(\ln 1 + i\cdot 2k\pi)} = e^{i\cdot 2\sqrt{2}k\pi}$

$(-2)^{\sqrt{2}} = e^{\sqrt{2}\,\mathrm{Ln}(-2)} = e^{\sqrt{2}(\ln 2 + i\cdot(2k\pi+\pi))} = e^{\sqrt{2}\ln 2 + i\cdot\sqrt{2}(2k+1)\pi}$.

$2^{i} = e^{i\,\mathrm{Ln} 2} = e^{i(\ln 2 + i 2k\pi)} = e^{-2k\pi + i\cdot\ln 2}$

$(3-4i)^{1+i} = e^{(1+i)\mathrm{Ln}(3-4i)} = e^{(1+i)[\ln 5 + i\cdot(\arg(3-4i)+2k\pi)]}$

$= e^{[\ln 5 - \mathrm{arctg}(-\frac{4}{3}) - 2k\pi] + i\cdot[\ln 5 + \mathrm{arctg}(-\frac{4}{3}) + 2k\pi]}$

(3) $\cos(2+i) = \cos 2 \,\mathrm{ch} 1 - i\sin 2 \,\mathrm{sh} 1$

$\sin 2i = \sin 0 \cdot \mathrm{ch} 2 + i\cdot\cos 0 \cdot \mathrm{sh} 2 = i\,\mathrm{sh} 2$

$\mathrm{ctg}(\frac{\pi}{4} - i\ln 2) = \dfrac{\cos(\frac{\pi}{4} - i\ln 2)}{\sin(\frac{\pi}{4} - i\ln 2)}$

$= \dfrac{\cos\frac{\pi}{4}\cdot\mathrm{ch}(-\ln 2) - i\cdot\sin\frac{\pi}{4}\cdot\mathrm{sh}(-\ln 2)}{\sin\frac{\pi}{4}\cdot\mathrm{ch}(-\ln 2) + i\cdot\cos\frac{\pi}{4}\cdot\mathrm{sh}(-\ln 2)}$

$= \dfrac{\frac{\sqrt{2}}{2}[\mathrm{ch}(\ln 2) + i\,\mathrm{sh}(\ln 2)]}{\frac{\sqrt{2}}{2}[\mathrm{ch}(\ln 2) - i\,\mathrm{sh}(\ln 2)]}$

分母有理化 $= \dfrac{[\mathrm{ch}(\ln 2)]^2 - [\mathrm{sh}(\ln 2)]^2 + 2i\cdot\mathrm{sh}(\ln 2)\cdot\mathrm{ch}(\ln 2)}{[\mathrm{ch}(\ln 2)]^2 + [\mathrm{sh}(\ln 2)]^2}$

$= \dfrac{1 + i\cdot\mathrm{sh}(\ln 4)}{[\mathrm{ch}(\ln 2)]^2 + [\mathrm{sh}(\ln 2)]^2}$

$= \dfrac{1 + i\cdot\frac{1}{2}(e^{\ln 4} - e^{-\ln 4})}{[\frac{1}{2}(e^{\ln 2} + e^{-\ln 2})]^2 + [\frac{1}{2}(e^{\ln 2} - e^{-\ln 2})]^2}$

$= \dfrac{8+15i}{17}$

Date. No.

$$\operatorname{cth}(2+i)=\frac{\operatorname{ch}(2+i)}{\operatorname{sh}(2+i)}=\frac{\operatorname{ch}2\cdot\operatorname{ch}i-\operatorname{sh}2\cdot\operatorname{sh}i}{\operatorname{sh}2\cdot\operatorname{ch}i+\operatorname{ch}2\cdot\operatorname{sh}i}$$

$$=\frac{\operatorname{ch}2\cdot\cos 1+i\operatorname{sh}2\cdot\sin 1}{\operatorname{sh}2\cdot\cos 1+i\cdot\operatorname{ch}2\cdot\sin 1}$$

分母有理化 $\underline{\underline{\quad}}$

$$\frac{\operatorname{ch}2\cdot\operatorname{sh}2\cdot\cos^2 1-i\cdot\operatorname{ch}^2 2\cdot\sin 1\cdot\cos 1+i\cdot\operatorname{sh}^2 2\cdot\sin 1\cdot\cos 1+\operatorname{sh}2\cdot\operatorname{ch}2\cdot\sin^2 1}{(\operatorname{sh}2\cdot\cos 1)^2+(\operatorname{ch}2\cdot\sin 1)^2}$$

$$=\frac{\operatorname{ch}2\cdot\operatorname{sh}2(\cos^2 1+\sin^2 1)+i\cdot\frac{1}{2}\sin 2\cdot(\operatorname{sh}^2 2-\operatorname{ch}^2 2)}{(\operatorname{sh}2\cdot\cos 1)^2+(\operatorname{ch}2\cdot\sin 1)^2}$$

$$=\frac{\frac{1}{2}\operatorname{sh}4+\frac{1}{2}i\cdot\sin 2\cdot(-1)}{\operatorname{sh}^2 2\cdot\cos^2 1+\operatorname{ch}^2 2\cdot\sin^2 1}$$

$$=\frac{\operatorname{sh}4-i\sin 2}{2(\operatorname{sh}^2 2\cdot(1-\sin^2 1)+\operatorname{ch}^2 2\cdot\sin^2 1)}$$

$$=\frac{\operatorname{sh}4-i\cdot\sin 2}{2(\operatorname{sh}^2 2+\sin^2 1)}$$

第三章.

1. (2) $\int_C \frac{2z-3}{z}dz$. C为$|z|=2$的下半圆. -2到2. $\Rightarrow$ 取$z=2e^{i\theta}$, $dz=2ie^{i\theta}d\theta$

$\therefore$ 原积分 $=\int_{\pi}^{2\pi}\frac{4e^{i\theta}-3}{2e^{i\theta}}\cdot 2ie^{i\theta}d\theta$

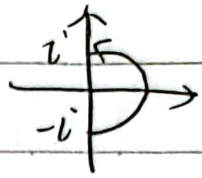
$$=\int_{\pi}^{2\pi}(4ie^{i\theta}-3i)d\theta$$

$$=4e^{i\theta}\Big|_{\pi}^{2\pi}-3i\theta\Big|_{\pi}^{2\pi}$$

$$=4(1-(-1))-3i\cdot(2\pi-\pi)$$

$$=8-3\pi i.$$

2.(3) $\int_{-i}^{i}|z|dz$ 积分路径: $|z|=1$ 的右半圆.



令 $z=e^{i\theta}$. $|z|=1$. $dz=ie^{i\theta}d\theta$

∴ 原积分 $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot ie^{i\theta}d\theta = e^{i\theta}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = i-(-i)=2i$.

3.(2) 证明: $f(z)=x^2+iy^2$. $\Rightarrow |f(z)|=\sqrt{x^4+y^4}$.

积分路径: $|z|=1$ 右半圆.

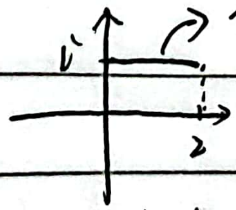
则在曲线 C 上: $\max_{C}|f(z)|=1.=M$

由长大不等式: $\left|\int_{-i}^{i}(x^2+iy^2)dz\right|\le M\cdot l$.

其中 $M=\max_{C}|f(z)|=1$. $l=\pi r=\pi$

∴ $\left|\int_{-i}^{i}(x^2+iy^2)dz\right|\le \pi$, 得证

4. 证明:



积分路径 l, 长度 $l=2$

则: $\max_{C}\left|\frac{1}{z^2}\right|=1$.

∴ $\left|\int_{i}^{2+i}\frac{dz}{z^2}\right| \le \max_{C}\left|\frac{1}{z^2}\right|\cdot l = 1\times 2=2$. 得证.