

# 第七次作业解答.

P100-101.

Date.

No.

14. 解: (2) 由于  $\lim_{z \rightarrow 0} \frac{z^2+4}{e^z}$  极限不存在, 故为本性奇点.

$$(4) \frac{1-\cos z}{z^n} = \frac{1}{z^n} - \sum_{m=0}^{+\infty} \frac{(-1)^m \cdot z^{2m}}{(2m)!} \cdot \frac{1}{z^n}$$

$$= \frac{1}{z^n} - \sum_{m=-n}^{+\infty} \frac{(-1)^{m+n} \cdot z^{2m}}{(2m+2n)!}$$

由于有无穷个非零项, 故为本性奇点.

$$(7) \sin \frac{1}{z} = \sum_{n=0}^{+\infty} \frac{(-1)^n (\frac{1}{z})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n-1}}{(2n+1)!}$$

由于有无穷个非零项, 故为可去奇点.

P131.

1. 解: (3)  $\frac{1}{e^z-1}$ :  $e^z-1=0 \Rightarrow z_k = 2k\pi i$  ( $k=0, \pm 1, \dots$ )

$$\text{由于 } (e^z-1)'|_{z=2k\pi i} = e^z|_{z=2k\pi i} = 1 \neq 0.$$

故  $z_k = 2k\pi i$  为  $\frac{1}{e^z-1}$  的一级极点.

$$\therefore \text{Res} \left[ \frac{1}{e^z-1}, z_k \right] = \frac{1}{\frac{d}{dz}(e^z-1)|_{z=2k\pi i}} = \frac{1}{e^{2k\pi i}} = 1$$

$$(4) \frac{1-e^{2z}}{z^4} = \frac{-(2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \dots)}{z^4}$$

$$= - \left( \frac{2}{z^3} + \frac{2}{z^2} + \frac{\frac{8}{3}}{z} + \frac{16}{4!} + \dots \right)$$

$$\therefore \text{Res} \left[ \frac{1-e^{2z}}{z^4}, 0 \right] = -\frac{8}{3!} = -\frac{8}{3 \times 2} = -\frac{4}{3}$$

(8) 易知  $z=0$  为一级极点.  $\text{Res}[f(z), 0] = \lim_{z \rightarrow 0} z \cdot f(z) = n.$

$$(9) \text{由 } f(z) = -\frac{1}{1-(z+1)} \cdot \left[ \frac{1}{z+1} + \dots + \frac{1}{(z+1)^n} \right]$$

$$= - \left[ \sum_{m=0}^{+\infty} (z+1)^m \right] \left[ \frac{1}{z+1} + \dots + \frac{1}{(z+1)^n} \right]$$

$$\text{则: } a_{-1} = - \underbrace{(1+1+\dots+1)}_{n \text{ 个 } 1} = -n. \therefore \text{Res}[f(z), -1] = -n.$$



3. 解: (2) C:  $x^2+y^2=2x \Rightarrow (x-1)^2+y^2=1$

由于  $z^4+1=0$ , 解得  $z = \frac{\sqrt{2}}{2}(1-i), \frac{\sqrt{2}}{2}(-1-i), \frac{\sqrt{2}}{2}(1+i), \frac{\sqrt{2}}{2}(-1+i)$

其中落在C内的有:  $z_1 = \frac{\sqrt{2}}{2}(1-i), z_2 = \frac{\sqrt{2}}{2}(1+i)$

$\therefore I = 2\pi i \cdot \{ \text{Res}[f(z), z_1] + \text{Res}[f(z), z_2] \}$

$= 2\pi i \cdot \left[ \lim_{z \rightarrow z_1} (z-z_1) \cdot f(z) + \lim_{z \rightarrow z_2} (z-z_2) \cdot f(z) \right]$

$= 2\pi i \cdot \left( \frac{1}{2\sqrt{2}(1-i)} + \frac{1}{2\sqrt{2}(1+i)} \right) = -\frac{\sqrt{2}}{2}\pi i$

13) 易知: 由于  $(z^2-1)(z^3+1)=0$ , 可解得如下极点:

①  $z^2-1=0 \Rightarrow z = \pm 1$ .

②  $z^3+1=0 \Rightarrow z = -1, z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore z_1=1$  一级极点,  $z_2=-1$  二级极点,  $z_{3,4} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$  一级极点

① 若  $r < 1$ : 则此时没有极点位于C内部.

$\therefore I = 0$

② 若  $r > 1$ : 此时  $z_1, z_2, z_3, z_4$  均在C内.

$\therefore I = 2\pi i \cdot \{ \text{Res}[f(z), z_1] + \text{Res}[f(z), z_2] + \text{Res}[f(z), z_3] + \text{Res}[f(z), z_4] \}$

$= 2\pi i \cdot \left[ \lim_{z \rightarrow z_1} \frac{1}{(z+1)^2(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} + \lim_{z \rightarrow z_2} [(z-z_2)^2 f(z)]' \right]$

$+ \lim_{z \rightarrow z_3} \frac{1}{(z-1)(z+1)^2(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i)} + \lim_{z \rightarrow z_4} \frac{1}{(z-1)(z+1)^2(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i)}$

$= 2\pi i \cdot \left[ \frac{1}{4} + \left( \frac{1}{z^3-2z^2+2z-1} \right)' \Big|_{z=-1} + \frac{1}{-3\sqrt{3}i} + \frac{1}{3\sqrt{3}i} \right]$

$= 2\pi i \cdot \left( \frac{1}{4} - \frac{1}{4} + \frac{1}{-3\sqrt{3}i} + \frac{1}{3\sqrt{3}i} \right)$

$= 0$

综上:  $I = 0$





(15) 修改  $L$  为  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}$

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由分母为0可得极点:  $z_1 = 1, z_2 = -1, z_3 = 3$

易知:  $z_1 = 1, z_2 = -1, z_3 = 3$  均为二级极点, 且均在  $L$  内部.

$$\therefore I = 2\pi i \cdot \left[ \lim_{z \rightarrow z_1} [(z-z_1)^2 f(z)]' + \lim_{z \rightarrow z_2} [(z-z_2)^2 f(z)]' + \lim_{z \rightarrow z_3} [(z-z_3)^2 f(z)]' \right]$$

$$= 2\pi i \cdot \left[ -2 \cdot \frac{2z-2}{(z+1)^2(z-3)^3} \Big|_{z=1} + (-2) \cdot \frac{2z-4}{(z-1)^2(z-3)^3} \Big|_{z=-1} + (-2) \cdot \frac{2z}{(z+1)^2(z-1)^3} \Big|_{z=3} \right]$$

$$= 2\pi i \cdot \left( 0 + \frac{3}{128} - \frac{3}{128} \right) = 0.$$

附加题: (1)  $\int_{|z|=3} \frac{z^{2021}}{z^{2022}-1} dz$

解: 由分母为0, 解得  $z_k = \left( \sqrt[2022]{1} \right)_k = \exp(i \cdot \frac{\arg 1 + 2k\pi}{2022}) = \exp(i \cdot \frac{k\pi}{1011}), k=0, 1, \dots, 2021$

由于  $|z_k|=1$ , 故  $z_k$  均在  $|z|=3$  内部, 均为一级极点.

$$\int_{|z|=3} f(z) dz = 2\pi i \cdot \sum_{k=0}^{2021} \text{Res}[f(z), z_k]$$

$$= 2\pi i \cdot \sum_{k=0}^{2021} \frac{z_k^{2021}}{\frac{d}{dz}(z^{2022}-1) \Big|_{z=z_k}}$$

$$= 2\pi i \cdot \sum_{k=0}^{2021} \frac{z_k^{2021}}{2022 \cdot z_k^{2021}}$$

$$= 2\pi i \cdot \sum_{k=0}^{2021} \frac{1}{2022} = 2\pi i$$

(2)  $\int_{|z|=3} e^{\frac{2022}{z}} dz$

解: 易知  $z=0$  为本性奇点,  $f(z) = \sum_{n=0}^{+\infty} \left(\frac{2022}{z}\right)^n / n! = 1 + \frac{2022}{z} + \dots$

$$\therefore \text{Res}[f(z), 0] = a_{-1} = 2022$$

$$\therefore I = 2022 \cdot 2\pi i = 4044\pi i$$



4. 解法2)  $\int_0^{2\pi} \frac{r - \cos \theta}{1 - 2r \cos \theta + r^2} d\theta \quad (r \neq 1 \text{ 补充条件})$

令  $z = e^{i\theta}$  可得:  $\int_{|z|=1} \frac{r-z}{i(z-1)(r-z)} dz$  其实部为所求积分

即:  $I = \operatorname{Re} \int_{|z|=1} \frac{r-z}{i(z-1)(r-z)} dz = \operatorname{Re} I_1$

下面讨论  $z = \frac{1}{r}$ ,  $z = r$  极点落在  $|z|=1$  内情况.

① 若  $|r| > 1$ , 则:  $I_1 = \frac{-1}{ri} \int_{|z|=1} \frac{r-z}{(z-\frac{1}{r})(z-r)} dz$

此时极点为  $z = \frac{1}{r}$ ,  $z = r$ , 只有  $z = \frac{1}{r}$  在  $|z|=1$  内.

$\therefore I_1 = 2\pi i \cdot \operatorname{Res} [f(z), \frac{1}{r}]$

$= 2\pi i \cdot \frac{-1}{ri} \cdot (-1) = \frac{2\pi}{r} \quad I = \operatorname{Re} I_1 = \frac{2\pi}{r}$

② 若  $|r| < 1$ , 则此时极点  $z = r$  在  $|z|=1$  内.

$\therefore I_1 = 2\pi i \cdot \operatorname{Res} [f(z), r] = 0$

$\therefore I = \operatorname{Re} I_1 = 0$

综上:  $|r| < 1$  时,  $I = 0$ ;  $|r| > 1$  时,  $I = \frac{2\pi}{r}$

13)  $I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{a^2 + \sin^2 \theta}$ , 由于  $\frac{1}{a^2 + \sin^2 \theta}$  为偶函数, 故  $I = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a^2 + \sin^2 \theta}$

$\therefore I = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a^2 + \frac{1-\cos \theta}{2}} \stackrel{\tilde{\theta} = 2\theta}{=} \frac{1}{2} \int_{-\pi}^{\pi} \frac{\frac{1}{2} d\tilde{\theta}}{a^2 + \frac{1-\cos \tilde{\theta}}{2}} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\tilde{\theta}}{2a^2 + 1 - \cos \tilde{\theta}}$

令  $z = e^{i\tilde{\theta}}$ , 可得  $I = \frac{1}{2} \int_{|z|=1} \frac{dz}{i \cdot (2a^2 + 1 - \frac{z+\frac{1}{z}}{2}) \cdot z}$

$= \frac{1}{-i} \int_{|z|=1} \frac{dz}{z^2 - (2a^2 + 2)z + 1}$





别极值可由分母=0得出:  $z_{1,2} = \frac{4a^2+2 \pm \sqrt{(4a^2+2)^2-4}}{2} = 2a^2+1 \pm 2a\sqrt{a^2+1}$ , 一级

由于  $a>0$ , 且  $2a^2+1+2a\sqrt{a^2+1} > 1$

$$2a^2+1-2a\sqrt{a^2+1} = 1-2a(\sqrt{a^2+1}-a) < 1$$

$\therefore$  在  $|z|=1$  内的极点为  $z_2 = 2a^2+1-2a\sqrt{a^2+1}$

$$\therefore I = 2\pi i \cdot \text{Res}[f(z) \text{ 在 } z_2]$$

$$= 2\pi i \cdot \left( -\frac{1}{(z^2-(4a^2+2)z+1)'} \right) \Big|_{z=z_2}$$

$$= -2\pi i \cdot \frac{1}{2z-(4a^2+2)} \Big|_{z=z_2} = \frac{\pi}{2a\sqrt{a^2+1}}$$

补充题: 11)  $\int_{|z-1|=2} \frac{|dz|}{1+|z|^2}$

解:  $|z-1|=2$ . 取  $z = 2e^{i\theta} + 1$ .  $dz = 2ie^{i\theta} d\theta$

则:  $|dz| = |2ie^{i\theta} d\theta| = 2d\theta$

$$|z|^2 = |2\cos\theta + 2i\sin\theta + 1|^2 = 4\cos^2\theta + 1 + 4\cos\theta + 4\sin^2\theta = 5 + 4\cos\theta$$

$$\therefore I = \int_0^{2\pi} \frac{2d\theta}{5+4\cos\theta} = \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$$

则取  $z = e^{i\theta}$   $I = \int_{|z|=1} \frac{dz}{iz(z+\frac{1}{z})}$

$$= \int_{|z|=1} \frac{dz}{i(z^2+3z+1)}$$

极点为  $z_{1,2} = \frac{-3 \pm \sqrt{5}}{2}$ , 其中  $z_1 = \frac{\sqrt{5}-3}{2}$  在  $|z|=1$  内.

$$\therefore \text{原积分 } I = 2\pi i \cdot \frac{1}{i} \cdot \frac{1}{(z^2+3z+1)'} \Big|_{z=z_1} = 2\pi \cdot \frac{1}{\sqrt{5}} = \frac{2\sqrt{5}}{5}\pi$$



$$(2) I = \text{Dato.} \int_0^{\pi} \frac{1 - \cos 2x}{5 - 4 \cos x} dx$$

由于  $f(x) = \frac{1 - \cos 2x}{5 - 4 \cos x}$  为偶函数. 故  $I = \frac{1}{2} \int_{-\pi}^{\pi} \frac{1 - \cos 2x}{5 - 4 \cos x} dx$

取  $z = e^{ix}$ . 则  $I = \text{Re} \left\{ \int_{|z|=1} \frac{1 - z^2}{i(5 - 2(z + \frac{1}{z}))z} dz \right\}$

其中:  $\int_{|z|=1} \frac{1 - z^2}{i(5 - 2(z + \frac{1}{z}))z} dz = \frac{1}{-2i} \int_{|z|=1} \frac{1 - z^2}{z^2 - \frac{5}{2}z + 1} dz$

$$= -\frac{1}{2i} \int_{|z|=1} \frac{1 - z^2}{(z - \frac{1}{2})(z - 2)} dz$$

其中, 只有  $z = \frac{1}{2}$  这一级极点存在  $|z|=1$  内.

故上式  $= 2\pi i \cdot \frac{1}{-2i} \cdot \frac{1 - z^2}{z - 2} \Big|_{z=\frac{1}{2}} = -\pi \cdot \frac{1 - \frac{1}{4}}{-\frac{3}{2}} = \frac{\pi}{2}$

$\therefore I = \text{Re} \left\{ \frac{\pi}{2} \right\} = \frac{\pi}{2}$

也可以直接用书 P109 页结论.  $I = \int_0^{\pi} \frac{\cos mx}{5 - 4 \cos x} dx = \frac{\pi}{3 \cdot 2^{m-1}}$ ,

则本题结果为  $\frac{\pi}{3 \cdot 2^{0-1}} - \frac{\pi}{3 \cdot 2^{2-1}} = \frac{\pi}{2}$

