复变函数 B 作业 5

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第 三 章

Question 6

(2) $\int_{-1}^{i}(1+4iz^3)\,dz = (z+iz^4)|_{-1}^{i} = (i+i)-(-1+i) = 1+i.$

Question 17

$\frac{\partial u}{\partial x} = 3ax^2+2bxy+cy^2$, $\frac{\partial^2 u}{\partial x^2} = 6ax+2by$;

$\frac{\partial u}{\partial y} = bx^2+2cxy+3dy^2$, $\frac{\partial^2 u}{\partial y^2} = 2cx+6dy$.

由 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, 得 $6ax+2by+2cx+6dy=0$, 即

$$\begin{cases} c=-3a; \\ b=-3d. \end{cases}$$

Question 18

(2) 设 $f(z)=u(x,y)+iv(x,y)$, 则 $|f(z)|^2=u^2+v^2$.

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) \\
&= \frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}+\frac{\partial^2(v^2)}{\partial x^2}+\frac{\partial^2(v^2)}{\partial y^2} \\
&= \frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial x}\left(2v\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(2v\frac{\partial v}{\partial y}\right) \\
&= 2\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial y}\right)^2+2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial v}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+2\left(\frac{\partial v}{\partial y}\right)^2+2v\frac{\partial^2 v}{\partial y^2}.
\end{aligned}$$

由于 $f(z)$ 解析, 故

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0; \\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0; \\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}; \\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \end{cases}$$

因此

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right).$$

又因为

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},$$

则

$$|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2,$$

故

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

Question 19

(1) u 调和, 则 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, 所以

$$\begin{aligned}\frac{\partial^2(u^2)}{\partial x^2}+\frac{\partial^2(u^2)}{\partial y^2}&=\frac{\partial}{\partial x}\left(2u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(2u\frac{\partial u}{\partial y}\right)\\&=2\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial y}\right)^2+2u\frac{\partial^2 u}{\partial y^2}\\&=2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)\\&\neq 0.\end{aligned}$$

故 u^2 不是调和函数.

(2) $\frac{\partial f(u)}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}$, $\frac{\partial^2 f(u)}{\partial x^2}=\frac{d^2 f}{du^2}(\frac{\partial u}{\partial x})^2+\frac{df}{du}\frac{\partial^2 u}{\partial x^2}$;

$\frac{\partial f(u)}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}$, $\frac{\partial^2 f(u)}{\partial y^2}=\frac{d^2 f}{du^2}(\frac{\partial u}{\partial y})^2+\frac{df}{du}\frac{\partial^2 u}{\partial y^2}$.

要使 $f(u)$ 调和, 则 $\frac{\partial^2 f(u)}{\partial x^2}+\frac{\partial^2 f(u)}{\partial y^2}=0$, 有 $\frac{d^2 f}{du^2}((\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2)=0$, 即 $\frac{d^2 f}{du^2}=0$.

则 $\frac{df}{du}=\int\frac{d^2 f}{du^2}\,du=C_1$,

$u=\int\frac{df}{du}\,du=C_1u+C_2$.

Question 20

(2) 首先验证 $u(x,y)$ 是调和函数.

$\frac{\partial u}{\partial x}=e^x((x+1)\cos y-y\sin y)$,

$\frac{\partial u}{\partial y}=-e^x((x+1)\sin y+y\cos y)$,

$\frac{\partial^2 u}{\partial x^2}=e^x((x+2)\cos y-y\sin y)$,

$\frac{\partial^2 u}{\partial y^2}=e^x(y\sin y-(x+2)\cos y)$,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, 故 $u(x,y)$ 是调和函数.

$v(x,y) = \int_{(0,0)}^{(x,y)} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy + C$.

因此

$$
\begin{aligned}
v(x,y) &= \int_{(0,0)}^{(x,y)} e^x((x+1)\sin y + y\cos y)\,dx + e^x((x+1)\cos y - y\sin y)\,dy + C \\
&= \int_0^y e^x((x+1)\cos y - y\sin y)\,dy + C \\
&= e^x(x+1)\sin y - e^x\int_0^y y\sin y\,dy + C \\
&= e^x(x+1)\sin y + e^x\int_0^y y\,d\cos y + C \\
&= e^x(x+1)\sin y + e^x\left(y\cos y - \int_0^y \cos y\,dy\right) + C \\
&= e^x(x\sin y + y\cos y) + C.
\end{aligned}
$$

因为 $f(0) = 0$, 故 $C = 0$.

令 $x = z, y = 0$, 得 $f(z) = u(x,y) + iv(x,y) = ze^z$.

(3) 首先验证 $v(x,y)$ 是调和函数.

$\frac{\partial v}{\partial x} = \frac{2(1+x)y}{((1+x)^2+y^2)^2}$,

$\frac{\partial v}{\partial y} = \frac{2y^2}{((1+x)^2+y^2)^2} - \frac{1}{(1+x)^2+y^2}$,

$\frac{\partial^2 v}{\partial x^2} = -y\left(\frac{8(1+x)^2}{((1+x)^2+y^2)^3} - \frac{2}{((1+x)^2+y^2)^2}\right)$,

$\frac{\partial^2 v}{\partial y^2} = \frac{4y}{((1+x)^2+y^2)^2} - y\left(\frac{8y^2}{((1+x)^2+y^2)^3} - \frac{2}{((1+x)^2+y^2)^2}\right)$,

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, 故 $v(x,y)$ 是调和函数.

$u(x,y) = \int_{(0,0)}^{(x,y)} \frac{\partial v}{\partial y}\,dx - \frac{\partial v}{\partial x}\,dy + C$.

因此

$$
\begin{aligned}
u(x,y) &= \int_{(0,0)}^{(x,y)} \left(\frac{2y^2}{((1+x)^2+y^2)^2} - \frac{1}{(1+x)^2+y^2}\right)dx - \frac{2(1+x)y}{((1+x)^2+y^2)^2}\,dy + C \\
&= -\int_0^x \frac{1}{(1+x)^2}\,dx - \int_0^y \frac{2(1+x)y}{((1+x)^2+y^2)^2}\,dy + C \\
&= \frac{1}{1+x} - (1+x)\int_0^y \frac{1}{((1+x)^2+y^2)^2}\,dy^2 + C \\
&= \frac{1}{1+x} + (1+x)\left(\frac{1}{(1+x)^2+y^2} - \frac{1}{(1+x)^2}\right) + C \\
&= \frac{x+1}{(x+1)^2+y^2} + C.
\end{aligned}
$$

因为 $f(0)=2$, 故 $C=1$.

令 $x=z, y=0$, 得 $f(z)=u(x,y)+iv(x,y)=\frac{z+2}{z+1}$.

第 四 章

Question 1

记收敛半径为 R.

(1) $a_n=\frac{1}{n^2}, r=\lim_{n\to+\infty}|\frac{a_{n+1}}{a_n}|=1, R=\frac{1}{r}=1$.

$|z|=1$ 时, $|\sum_{n=1}^{+\infty}\frac{z^n}{n^2}|=\sum_{n=1}^{+\infty}\frac{1}{n^2}$ 绝对收敛.

(2) $a_n=1, r=\lim_{n\to+\infty}|\frac{a_{n+1}}{a_n}|=1, R=\frac{1}{r}=1$.

$|z|=1$ 时, $|z^n|=1$, 数列不趋于 0. 故级数在收敛圆周上发散.

(3) $a_n=\frac{1}{n}, r=\lim_{n\to+\infty}|\frac{a_{n+1}}{a_n}|=1, R=\frac{1}{r}=1$.

事实上, 在收敛圆周 $|z|=1$ 上发散点有且只有 $z=1$.

这是因为, 当 $z\neq 1$ 时, $\sum_{n=1}^{+\infty}z^n=\frac{z-z^{n+1}}{1-z}$, 故 $|\sum_{n=1}^{+\infty}z^n|=|\frac{1-z^n}{1-z}|\leq\frac{1}{|1-z|}+\frac{|z^n|}{|1-z|}=\frac{2}{|1-z|}$.

由 Dirichlet's test (见下图), 级数收敛.

The test states that if (a_n) is a sequence of real numbers and (b_n) a sequence of complex numbers satisfying

- (a_n) is monotonic
- $\lim_{n\to\infty} a_n = 0$
- $\left|\sum_{n=1}^{N} b_n\right| \leq M$ for every positive integer N

where M is some constant, then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

通过上述证明, 我们可以得到复分析中的 Abel's test.

Abel's test states that if a sequence of *positive real numbers* (a_n) is decreasing monotonically (or at least that for all n greater than some natural number m, we have $a_n \geq a_{n+1}$) with

$$\lim_{n\to\infty} a_n = 0$$

then the power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

converges everywhere on the closed unit circle, except when $z = 1$. Abel's test cannot be applied when $z = 1$, so convergence at that single point must be investigated separately. Notice that Abel's test implies in particular that the radius of convergence is at least 1. It can also be

Question 2

(4) 记收敛半径为 R.

$$\begin{aligned}\frac{1}{z^2-3z+2} &= \frac{1}{z-2}-\frac{1}{z-1}\\ &= -\frac{1}{2}\frac{1}{1-\frac{z}{2}}+\frac{1}{1-z}\\ &= -\frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n+\sum_{n=0}^{+\infty}z^n\\ &= \sum_{n=0}^{+\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n.\end{aligned}$$

$a_n=(1-\frac{1}{2^{n+1}}), r=\lim_{n\to+\infty}|\frac{a_{n+1}}{a_n}|=1, R=\frac{1}{r}=1.$

(6) 记收敛半径为 R.

$$\begin{aligned}\frac{z}{(1-z)^2} &= z\left(\frac{1}{1-z}\right)'\\ &= z\left(\sum_{n=0}^{+\infty}z^n\right)'\\ &= z\sum_{n=0}^{+\infty}nz^{n-1}\\ &= \sum_{n=0}^{+\infty}nz^n.\end{aligned}$$

$a_n=n, r=\lim_{n\to+\infty}|\frac{a_{n+1}}{a_n}|=1, R=\frac{1}{r}=1.$

(8) 记收敛半径为 R.

$$\begin{aligned}\int_0^z\frac{\sin z}{z}\,dz &= \int_0^z\sum_{n=0}^{+\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!z}\,dz\\ &= \sum_{n=0}^{+\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!(2n+1)}.\end{aligned}$$

$a_{2n}=0, a_{2n+1}=\frac{(-1)^n}{(2n+1)!(2n+1)}, r=\lim_{n\to+\infty}\sqrt[n]{|a_n|}=0, R=\frac{1}{r}=+\infty.$

注释: 上面需要计算 $\lim_{n\to+\infty}\sqrt[n]{n!}=+\infty$, 除了用 Stirling 公式外, 我们也可以用一些巧法. 注意到 $(n!)^2=(1\cdot n)(2\cdot(n-1))(3\cdot(n-2))\cdots(n\cdot 1)\geq n\cdot n\cdot n\cdots n=n^n$, 故 $\sqrt[n]{n!}\geq\sqrt{n}$.