拉普拉斯变换法

The Method of LaplaceTransforms

一、拉氏变换

1、定义

$$F(p) = \int_0^{+\infty} f(t) e^{-pt} dt \qquad - \ f(t)\text{的拉氏变换}$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(p)\, e^{pt} dp \qquad - \ F(p)\text{的拉氏逆变换}$$

2、存在条件

(1) $f(t)$及导数除有限个第一类间断点外连续

(2) $\left| f(t) \right| \le Ke^{ct} (K, c \ge 0 \quad ; c\text{是增长指数})$

二、拉氏变换性质

1.线性：$L\left[\alpha f_1(t)+\beta f_2(t)\right]=\alpha L\left[f_1(t)\right]+\beta L\left[f_2(t)\right]$

2.位移：$L\left[e^{p_0 t}f(t)\right]=F(p-p_0)$ 记$L\left[f(t)\right]=F(p)$

3.延迟：$L\left[f(t-\tau)\right]=e^{-p\tau}F\left[p\right]$

4.相似：$L\left[f(at)\right]=\dfrac{1}{a}F\left(\dfrac{p}{a}\right), a>0$

5.微分：$L\left[f^{(n)}(t)\right]=p^nF(p)-p^{n-1}f(0)-p^{n-2}f'(0)-\ldots-f^{(n-1)}(0)$

6.积分：$L\left[\int_0^t f(\tau)d\tau\right]=\dfrac{1}{p}L\left[f(t)\right]$

7.卷积：$L\left[f_1(t)*f_2(t)\right]=L\left[f_1(t)\right]\cdot L\left[f_2(t)\right]$

$$f_1(t)*f_2(t)=\int_0^t f_1(\tau)f_2(t-\tau)d\tau$$

三、解数理方程

1. P190 习题8

$$\begin{cases} \dfrac{d^2 y}{dt^2} + \omega^2 y = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

记 $L[y(t)] = \int_0^{\infty} y(t) e^{-pt} dt = Y(p)$,

$L[f(t)] = \int_0^{\infty} f(t) e^{-pt} dt = F(p)$

则 $p^2 Y(p) - p y(0) - y'(0) + \omega^2 Y(p) = F(p)$

$$\therefore \; Y(p) = \frac{F(p)}{p^2 + \omega^2} = L\left[f(t) * \frac{1}{\omega} \sin \omega t \right]$$

$$\boxed{y(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega (t - \tau) d\tau}$$

2. 解偏微分方程混合问题

$$\begin{cases} u_{tt} = a^2 u_{xx}, 0 < x < \infty, t > 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 \\ u(0,t) = f(t), \lim\limits_{x\to\infty} u(x,t) = 0 \ (t \geq 0) \end{cases}$$

记 $L[u(x,t)] = U(x,p)$, $L[f(t)] = F(p)$.

则 $$\begin{cases} p^2 U(x,p) - pu(x,0) - u_t(x,0) = a^2 \dfrac{\partial^2}{\partial x^2} U(x,p) \\ U(0,p) = F(p), \quad \lim\limits_{x\to\infty} U(x,p) = 0 \end{cases}$$

即
$$\begin{cases} \dfrac{d^2}{dx^2}U(x,\ p) - \dfrac{p^2}{a^2}U(x,\ p) = 0 \\ U(0,\ p) = F(p) \\ \lim\limits_{x\to\infty} U(x,\ p) = 0 \end{cases}$$

$$U = c_1(p)e^{-\frac{p}{a}x} + c_2(p)e^{\frac{p}{a}x}$$

由 $U(0,\ p) = F(p) \Rightarrow c_1(p) + c_2(p) = F(p)$

由 $U(\infty,\ p) = 0 \Rightarrow c_2(p) = 0$

$$\therefore U(x,\ p) = F(p)e^{-\frac{p}{a}x} \Rightarrow u(x,t) = L^{-1}\left[F(p)\cdot e^{-\frac{p}{a}x}\right]$$

由延迟定理得 $\boxed{u(x,t) = L^{-1}L\left[f\left(t-\dfrac{x}{a}\right)\right] = f\left(t-\dfrac{x}{a}\right)}$