复变函数 B 作业 2

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2023 年 10 月 15 日

Question 1

$w = u + iy = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} \therefore v = \frac{x}{x^2+y^2}, u = \frac{-y}{x^2+y^2}$, 且 $z \neq 0$

(3)$y = x \Rightarrow u = -v$, 即 $u + v = 0$

$\therefore$ 变成 w 平面上的正比例函数 (不包括原点)$\leftarrow$ 这一条件很多同学漏写了；

(4)$x^2 + y^2 = 4 \Rightarrow u^2 + v^2 = \frac{x^2+y^2}{(u^2+v^2)^2} = \frac{1}{4}$

$\therefore$ 变为 w 平面上以 $(0,0)$ 为圆心，以 $\frac{1}{4}$ 为半径的圆周；

(5)$\begin{cases} u + iv = \dfrac{1}{x+iy} \\ (x-1)^2 + y^2 = 5 \end{cases} \Rightarrow \begin{cases} x = \dfrac{4u}{1-2u} \\ y = \dfrac{4v}{2u-1} \end{cases}$

代回 $(x-1)^2 + y^2 = 5$ 并整理得：$(u+\frac{1}{4})^2 + v^2 = \frac{5}{16}$

$\therefore$ 变为 w 平面上以 $(-\frac{1}{4}, 0)$ 为圆心，以 $\frac{\sqrt{5}}{4}$ 为半径的圆周；

Question 2

令 $z = x + iy, f(z) = \frac{1}{2i}(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}) = \frac{1}{2i}\frac{z^2 - \overline{z}^2}{|z|^2} = \frac{2xy}{x^2+y^2}$

$\because \lim\limits_{\substack{z=x+ikx \\ x\to 0}} f(z) = \frac{2k}{1+k^2}$，具体取值与 k 有关 $\therefore$ 极限不存在

Question 3

与 Question 2 类似，令 $y = kx$ 可得 $z \to 0$ 时极限不存在 $\therefore$ 不连续

Question 5

(2)$f(z) = u + iv = x + y \therefore u = x + y, v = 0$

$\therefore \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$，在全平面均不符合 C-R 方程

(3)①$z = 0$ 时，函数无定义

②$z \neq 0$: 令 $z = x + iy$，$f(z) = \frac{1}{\overline{z}} = \frac{x+iy}{x^2+y^2} = u + iv$

$$\therefore \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{y^2 - x^2}{(x^2+y^2)^2} \\ \dfrac{\partial u}{\partial y} = \dfrac{-2xy}{(x^2+y^2)^2} \\ \dfrac{\partial v}{\partial x} = \dfrac{-2xy}{(x^2+y^2)^2} \\ \dfrac{\partial v}{\partial y} = \dfrac{x^2 - y^2}{(x^2+y^2)^2} \end{cases} \Rightarrow \text{C-R 方程在 } z \neq 0 \text{ 时处处不成立}$$

$\therefore$ 全平面上处处不可导

Question 6

(2)①$z < 1$ 时，$f(z) = |z|z = \sqrt{x^2+y^2}(x+iy)$

$$\frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y}{\sqrt{x^2+y^2}}, \frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}}, \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$$

C-R 方程仅在 $x = y = 0$ 处成立，$(0,0)$ 的邻域内始终存在不解析的点, 则在 $|z| < 1$ 时不解析；

②$z \geq 1$ 时：$f(z) = z^2$ 为解析函数

对于 $|z| = 1$ 上的任一点，$\because |z| < 1$ 时不解析，则其邻域内始终存在不解析的点，$\therefore |z| = 1$ 不解析（很多同学没有考虑到这一点）

$\therefore$ 解析区域为 $|z| > 1$（可以通过观察结果是否为区域来判断正确性，解析区域不可能包含边界点和孤立点）

Question 7

(2)$u = e^x(x\cos y - y\sin y), v = e^x(y\cos y + x\sin y)$

$$\begin{cases} \dfrac{\partial u}{\partial x} = e^x(x\cos y - y\sin y + \cos y) = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -e^x(x\sin y + \sin y + y\cos y) = -\dfrac{\partial v}{\partial x} \end{cases} \text{在全平面成立，则在全平面解析}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y + \cos y) + ie^x(x\sin y + \sin y + y\cos y) = e^z(z+1)$$

Question 8

(4) 令 $f(z) = u + iv$, $Imf(z) = v = C_1 \therefore \dfrac{\partial v}{\partial x} = 0 = -\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y} = 0 = \dfrac{\partial u}{\partial x}$，$\therefore Ref(z)$ 也为常数 C_2

$\therefore f(z) = C_2 + iC_1$ 为复常数

(6)$\arg f(z)$ 为常数 $\therefore v = ku$,k 为实常数

$$\frac{\partial v}{\partial x} = k\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} = k\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

$\therefore u, v$ 均为常数，即 $f(z)$ 为复常数；

Question 10

有理函数可直接判断解析域，利用求导法则求导即可

(1) $\dfrac{1}{z^2-3z+2} = \dfrac{1}{z-1} + \dfrac{1}{z-2}$

$\therefore$ 解析区域为 $\{z|z \neq 1, z \neq 2\}$,$f'(z) = \dfrac{3-2z}{(z^2-3z+2)^2}$

(2) 解析区域为 $\{z|z \neq a^{1/3}e^{i(\pi/3+2k\pi/3)}\}, k = 0, 1, 2$

$f'(z) = \dfrac{-3z^2}{(z^3+a)^2}$