复变函数 B 期中考试

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Question 1

设 $z = x + iy$, 则 $\frac{z-i}{z+i} = \frac{x+i(y-1)}{x+i(y+1)} = \frac{x^2+y^2-1-2xi}{x^2+(y+1)^2}$. (10 分)

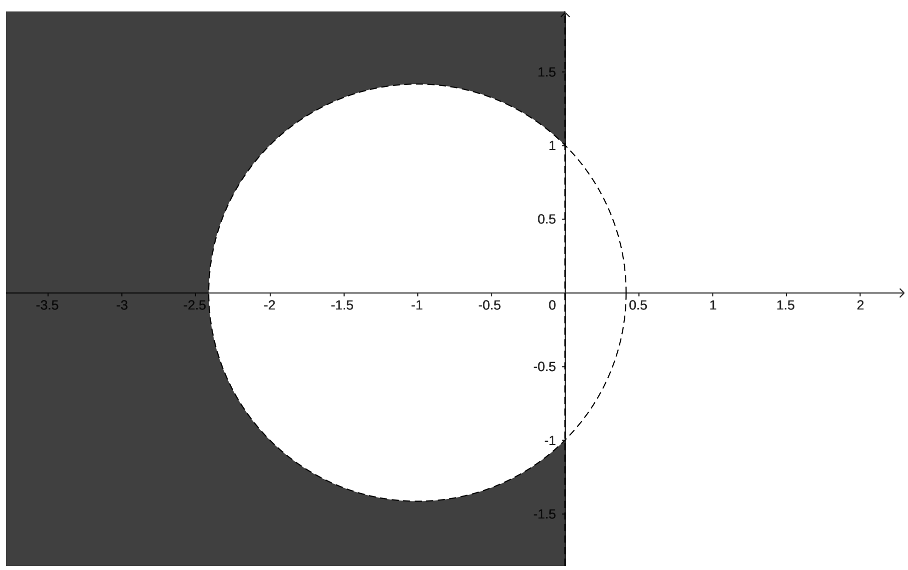
由于 $0 < \arg \frac{z-i}{z+i} < \frac{\pi}{4}$, 故

$$\begin{cases} x^2 + y^2 - 1 > 0; \\ -2x > 0; \\ x^2 + y^2 - 1 > -2x. \end{cases}$$

即

$$\begin{cases} x < 0; \\ (x+1)^2 + y^2 > 2. \end{cases}$$

虚轴左侧, 以 -1 为圆心 $\sqrt{2}$ 为半径的圆的外部区域. (20 分)



注释: 习题课已经讲过, arg 和 arctan 不等价. 很多人直接 $0 < \frac{-2x}{x^2+y^2-1} < 1$, 算出两个区域的并.

Question 3

1.5.16. Before we begin, notice that the statement $au(x,y) + bv(x,y) = c$ with not all of a, b, c equal to zero is equivalent to saying that the values of $f(z) = u(x,y) + iv(x,y)$ lie on a straight line.

(a) We write $u = u(x,y)$, $v = v(x,y)$ etc. to simplify notation. If we apply $\partial/\partial x$ and $\partial/\partial y$ to the equation $au + bv = c$ we obtain

$$\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= 0 \\ a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y} &= 0. \end{aligned}$$

Since f is analytic we can apply the Cauchy-Riemann equations in the second equality above to obtain the system

$$\begin{aligned} a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} &= 0 \\ b\frac{\partial u}{\partial x} - a\frac{\partial v}{\partial x} &= 0 \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{matrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix}.$$

The determinant of the coefficient matrix is $-(a^b + b^2)$ which cannot be zero because a, b, c are real and not all zero. Hence, the only solution to the system is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0.$$

Therefore

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

and, since A is connected, we conclude that f is constant.

(b) If a, b, c are complex, then the proof above does not apply. However, if we write $a = a_1 + ia_2$, $b = b_1 + ib_2$ and $c = c_1 + ic_2$ with $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, then the equation $au + bv = c$ is equivalent to the pair of equations

$$\begin{aligned} a_1 u + b_1 v &= c_1 \\ a_2 u + b_2 v &= c_2. \end{aligned}$$

Since not all of a, b, c are zero, it must be that in at least one of these equations not all of the constants are zero. We may then apply part (a) to that equation to conclude that f is constant. In other words, the statement is valid for complex a, b, c as well.

Source: link.

Question 5

首先验证 $v(x,y)$ 是调和函数.

$\frac{\partial v}{\partial x} = -\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}$, (2 分)

$\frac{\partial v}{\partial y} = \frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}$, (4 分)

$\frac{\partial^2 v}{\partial x^2} = -\frac{\left(-1+\frac{x}{\sqrt{x^2+y^2}}\right)^2}{4\left(-x+\sqrt{x^2+y^2}\right)^{3/2}} + \frac{-\frac{x^2}{(x^2+y^2)^{3/2}}+\frac{1}{\sqrt{x^2+y^2}}}{2\sqrt{-x+\sqrt{x^2+y^2}}}$, (5 分)

$\frac{\partial^2 v}{\partial y^2} = -\frac{y^2}{4(x^2+y^2)\left(-x+\sqrt{x^2+y^2}\right)^{3/2}} - \frac{y^2}{2(x^2+y^2)^{3/2}\sqrt{-x+\sqrt{x^2+y^2}}} + \frac{1}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}$, (6 分)

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, 故 $v(x,y)$ 是调和函数. (9 分)

$u(x,y) = \int_{(0,0)}^{(x,y)} \frac{\partial v}{\partial y}\,dx - \frac{\partial v}{\partial x}\,dy + C$. (10 分)

令 $x, y \geq 0$, 有

$$
\begin{aligned}
u(x,y) &= \int_{(0,0)}^{(x,y)} \frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}\,dx + \frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy + C \\
&= \int_{(0,0)}^{(x,y)} \frac{\sqrt{x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dx + \frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy + C \\
&= \int_0^x \frac{\sqrt{x+\sqrt{x^2}}}{2\sqrt{x^2}}\,dx + \int_0^y \frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy + C \\
&= \sqrt{2x} + \int_0^y \frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy + C.
\end{aligned}
$$

(12 分)

令 $t = \sqrt{x^2+y^2}$, 则 $y = \sqrt{t^2-x^2}, dy = \frac{t}{\sqrt{t^2-x^2}}\,dt$, 于是

$$
\begin{aligned}
u(x,y) &= \sqrt{2x} + \int_0^y \frac{\sqrt{-x+\sqrt{x^2+y^2}}}{2\sqrt{x^2+y^2}}\,dy + C \\
&= \sqrt{2x} + \int_x^{\sqrt{x^2+y^2}} \frac{\sqrt{-x+t}}{2t}\frac{t}{\sqrt{t^2-x^2}}\,dt + C \\
&= \sqrt{2x} + \int_x^{\sqrt{x^2+y^2}} \frac{1}{2\sqrt{t+x}}\,dt + C \\
&= \sqrt{x+\sqrt{x^2+y^2}} + C.
\end{aligned}
$$

(15 分)

因为 $f(0)=0$, 故 $C=0$. (16 分)

令 $x=z\geq 0, y=0$, 得 $f(z)=u(x,y)+iv(x,y)=(2z)^{\frac{1}{2}}, z\in\mathbb{C}$. (20 分)

注释: 习题课已经讲过, 这种题必须先验证调和函数, 最后结果也要用 z 表示. 我们先得到 $f(z)$ 在实轴非负半轴处的函数表达式, 由唯一性定理即可推广至全复平面. 当然, 很多人这里积分没算对, 所有人都认为 $y=0$ 时 $\frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}=0$, 而没注意到此时分母也是 0. 如果不化简 $\frac{y}{2\sqrt{x^2+y^2}\sqrt{-x+\sqrt{x^2+y^2}}}$, 对于这种分子分母都为 0 的情形, 我们不能取 $y=0$ 积分, 但可以取 $y=1$ 积分, 此时积分路径变为 $(0,1)\to(x,1)\to(x,y)$. 或者, 也可以取积分路径 $(0,0)\to(0,y)\to(x,y)$. 另一个需要换元的积分算出来的寥寥无几, 为避免麻烦, 我们这里也可以用第 6 题的结论, 在极坐标下处理.
设 $z=re^{i\theta}, \theta\in[0,2\pi)$, 则 $v=\sqrt{r-r\cos\theta}=\sqrt{2r\sin^2\frac{\theta}{2}}=\sqrt{2r}\sin\frac{\theta}{2}$.
由极坐标系下的 Laplace 方程, 我们可以验证 $v(r,\theta)$ 是调和函数, 即满足:

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0.$$

因为

$$\begin{cases}\frac{\partial v}{\partial r}=\frac{1}{\sqrt{2r}}\sin\frac{\theta}{2};\\ \frac{\partial v}{\partial \theta}=\frac{\sqrt{2r}}{2}\cos\frac{\theta}{2}.\end{cases}$$

所以由第 6 题结论有

$$\begin{cases}\frac{\partial u}{\partial r}=\frac{1}{\sqrt{2r}}\cos\frac{\theta}{2};\\ \frac{\partial u}{\partial \theta}=-\sqrt{\frac{r}{2}}\sin\frac{\theta}{2}.\end{cases}$$

于是

$$\begin{aligned}u(r,\theta)&=\int_{(0,0)}^{(r,\theta)}\frac{\partial u}{\partial r}\,dr+\frac{\partial u}{\partial \theta}\,d\theta+C\\&=\int_0^r\frac{1}{\sqrt{2r}}\,dr-\int_0^\theta\sqrt{\frac{r}{2}}\sin\frac{\theta}{2}\,d\theta+C\\&=\sqrt{2r}-2\sqrt{\frac{r}{2}}(\cos\frac{\theta}{2}-1)+C\\&=\sqrt{2r}\cos\frac{\theta}{2}.\end{aligned}$$

因为 $f(0)=0$, 故 $C=0$.
令 $r=z, \theta=0$, 得 $f(z)=u(r,\theta)+iv(r,\theta)=(2z)^{\frac{1}{2}}, z\in\mathbb{C}$.
由此可见极坐标下处理更为简单. 若取积分路径为 $(0,0)\to(0,\theta)\to(r,\theta)$, 甚至只需计算一个积分.

Question 6

设 $\begin{cases}x=r\cos\theta;\\ y=r\sin\theta.\end{cases}$

则 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix}$.

由于 $\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$,

故 $\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \cos\theta & sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

那么 $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{bmatrix} \begin{bmatrix} \cos\theta & sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} & \frac{\partial u}{\partial r} sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \\ \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} & \frac{\partial v}{\partial r} sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

因此

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} = \frac{\partial v}{\partial r} sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}. \tag{1}$$

$$\frac{\partial u}{\partial r} sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} = -\frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r}. \tag{2}$$

$(1)\cos\theta + (2)\sin\theta$ 得: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$(1)\sin\theta - (2)\cos\theta$ 得: $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial v}{\partial \theta}$.

故极坐标下的柯西–黎曼方程是

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \end{cases}$$

注释: 有一些同学貌似是提前知道答案, 利用配凑的方法, 将 $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ 用 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ 表示, 凑出最后的结论. 我们这里的思路是将 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ 用 $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}$ 表示, 并代入直角坐标下的柯西–黎曼方程.