第四周作业解答

高涛

7. 解：证明：由 $\lim\limits_{z\to\infty} zf(z)=A$ 可知，

$\forall \varepsilon>0$，$\exists R_0>0$，当 $|z|>R_0$ 时，$|zf(z)-A|<\varepsilon$.

又：$\int_{C_R}\frac{1}{z}dz=\int_0^{\alpha}\frac{iRe^{i\theta}d\theta}{Re^{i\theta}}=i\alpha$.

取 $R>R_0$，$\left|\int_{C_R}f(z)dz-iA\alpha\right|=\left|\int_{C_R}f(z)dz-\int_{C_R}\frac{A}{z}dz\right|$

$=\left|\int_{C_R}\frac{zf(z)-A}{z}dz\right|$

$\leq\int_{C_R}\left|\frac{zf(z)-A}{z}\right|ds$

$=\frac{\varepsilon}{R}\cdot R\alpha=\varepsilon\alpha$. 得证. $\lim\limits_{R\to+\infty}\int_{C_R}f(z)dz=iA\alpha$

8. 解：证明：由于 $Q(z)$ 比 $P(z)$ 高 2 次. 则 $\lim\limits_{z\to\infty}\frac{zP(z)}{Q(z)}=0$.

由 7 结论可知：$\lim\limits_{R\to+\infty}\int_{|z|=R}\frac{P(z)}{Q(z)}dz=0$，得证.

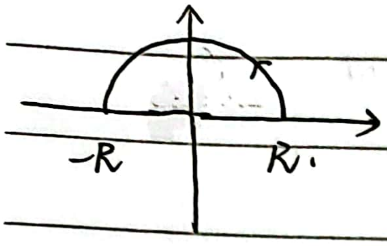
(或者：$\left|\frac{P(z)}{Q(z)}\right|=\frac{1}{|z|^2}\cdot|M(z)|$. 取 R 充分大. $\left|\int_{C_R}\frac{P(z)}{Q(z)}dz\right|\leq\frac{M}{R^2}2\pi R=\frac{2\pi M}{R}\to 0$.

有界. 最大值 M.

$\therefore\lim\limits_{R\to+\infty}\int_{|z|=R}\frac{P(z)}{Q(z)}dz=0$. 得证)

(*)

附加题 Dai解证明：由 $\lim\limits_{z\to\infty} f(z)=0$ 可知：$\forall \varepsilon>0, \exists R_0>0$, 当 $|z|>R_0$, $|f(z)|<\varepsilon$



则：$\left|\int_{C_R} f(z) e^{imz} dz\right|$ 由长大不等式

$$\le \int_{C_R} |f(z)|\cdot|e^{imz}|\cdot|dz|$$

由于 $z=Re^{i\theta}$ $(0\le \arg z\le \pi)$. 则 $|e^{imz}|=e^{-Rm\sin\theta}$.

$dz=Rie^{i\theta}d\theta \Rightarrow |dz|=Rd\theta$.

$$\therefore |原积分| \le \int_0^{\pi}|f(z)|\cdot e^{-Rm\sin\theta}\cdot Rd\theta$$

$$=\int_0^{\frac{\pi}{2}}|f(z)|\cdot e^{-Rm\sin\theta}\cdot Rd\theta+\int_{\frac{\pi}{2}}^{\pi}|f(z)|e^{-Rm\sin\theta}\cdot Rd\theta$$

$$=2R\int_0^{\frac{\pi}{2}}|f(z)|e^{-Rm\sin\theta}d\theta$$

由于 $\theta\in[0,\frac{\pi}{2}]$ 时，$\frac{2}{\pi}\theta\le\sin\theta$

$$\therefore |原积分| \le 2R\int_0^{\frac{\pi}{2}}|f(z)|e^{-Rm\frac{2}{\pi}\theta}d\theta$$

$$=-\frac{m}{\pi}e^{-Rm\frac{2}{\pi}\theta}|f(z)|\Big|_0^{\frac{\pi}{2}}=|f(z)|\left(1-\frac{m}{\pi}\cdot e^{-Rm}\right)$$

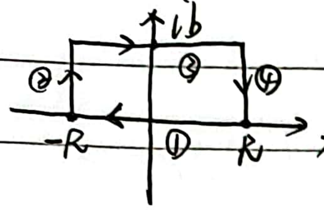
$$<|f(z)|$$

而由(*)式可知：$\forall\varepsilon>0, \exists R_0>0$. 取 $|z|>R_0$.

则此时 $|原积分| <|f(z)|<\varepsilon$.

$$\therefore \lim_{R\to\infty}\int_{C_R} f(z)e^{imz}dz=0. \text{ 得证}$$

附加题2：解：由于在该闭域内 $f(z)=e^{-az^2}$ 解析



由柯西积分定理：$\oint_C f(z)\,dz=0$.

即：$\int_{R}^{-R} f(z)\,dz + \int_{-R}^{-R+ib} f(z)\,dz + \int_{-R+ib}^{R+ib} f(z)\,dz$

$+\int_{R+ib}^{R} f(z)\,dz = 0$.

对①：$R\to\infty$ 时：$\int_{+\infty}^{-\infty} e^{-az^2}dz = -\int_{-\infty}^{+\infty} e^{-ax^2}dx = -\sqrt{\frac{\pi}{a}}$.（概率积分）

对②：设 $z=-R+iy$. 则：$\int_{-R}^{-R+ib} f(z)\,dz = \int_0^b e^{-a(-R+iy)^2}\,i\,dy = i\int_0^b e^{-aR^2+ay^2+2aRiy}\,dy$

∴ $\left|\int_{-R}^{-R+ib} f(z)\,dz\right| = \int_0^b e^{-aR^2}\cdot e^{ay^2}\,dy$

由于 $\int_0^b e^{ay^2}dy = ye^{ay^2}\Big|_0^b - \underbrace{\int_0^b 2ay^2 e^{ay^2}dy}_{>0}$.

∴ $\int_0^b e^{ay^2}dy < ye^{ay^2}\Big|_0^b = be^{ab^2}$

∴ $\left|\int_{-R}^{-R+ib} f(z)\,dz\right| < e^{-aR^2}\cdot be^{ab^2}$.

当 $R\to+\infty$ 时：$e^{-aR^2}\to 0$. ∴ $\left|\int_{-R}^{-R+ib} f(z)\,dz\right| \to 0$.

对④：令 $z=R+iy$：则：$\int_{R+ib}^{R} f(z)\,dz = \int_b^0 e^{-a(R+iy)^2}\,i\,dy = -i\int_0^b e^{-aR^2+ay^2-2aRiy}\,dy$

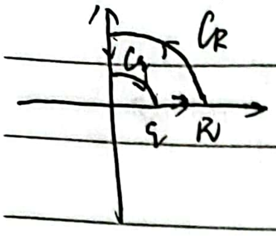
∴ $\left|\int_{R+ib}^{R} f(z)\,dz\right| = \int_0^b e^{-aR^2}\cdot e^{ay^2}\,dy$

同上：可知 $\left|\int_{R+ib}^{R} f(z)\,dz\right| < e^{-aR^2}\cdot be^{ab^2}$.

当 $R\to+\infty$ 时：$\left|\int_{R+ib}^{R} f(z)\,dz\right| \to 0$

综上所述：$\int_{-\infty+ib}^{\infty+ib} f(z)\,dz = \sqrt{\frac{\pi}{a}}$.

附加题3的解：证明：令 $f(z)=z^{s-1}e^{-z}$，在该闭域内解析.



$$\therefore \int_C f(z)\,dz=0.$$

$$\therefore \underset{①}{\int_{C_\varepsilon} f(z)\,dz}+\underset{②}{\int_\varepsilon^R f(z)\,dz}+\underset{③}{\int_{C_R} f(z)\,dz}+\underset{④}{\int_{iR}^{i\varepsilon} f(z)\,dz}=0.$$

对①：$\left|\int_{C_\varepsilon} f(z)\,dz\right| < \int_{C_\varepsilon}|f(z)||z'(\theta)|\,d\theta$ $\begin{cases}|f(z)|=\varepsilon^{s-1}\cdot e^{-\varepsilon\cos\theta}\\ |z'(\theta)|=\varepsilon\end{cases}$

取 $z=\varepsilon e^{i\theta}$

$$=\int_{\frac{\pi}{2}}^{0}\varepsilon^s\cdot e^{-\varepsilon\cos\theta}\,d\theta$$

令 $\theta=\frac{\pi}{2}-\theta$

$$=\int_0^{\frac{\pi}{2}}\varepsilon^s\cdot e^{-\varepsilon\sin\theta}\,(-d\theta)$$

由于 $\theta\in[0,\frac{\pi}{2}]$，$\sin\theta\geq\frac{2\theta}{\pi}$.

$$\leq -\varepsilon^s\int_0^{\frac{\pi}{2}} e^{-\varepsilon\frac{2\theta}{\pi}}\,d\theta$$

$$=\frac{\varepsilon^{s-1}}{2\pi}e^{-\varepsilon\frac{2\theta}{\pi}}\Big|_0^{\frac{\pi}{2}}=\frac{\varepsilon^{s-1}}{2\pi}(e^{-\varepsilon}-1)$$

由于 $\varepsilon\to0$ 时：$e^{-\varepsilon}-1\sim-\varepsilon$，$\lim\limits_{\varepsilon\to0}$ 该式 $=\lim\limits_{\varepsilon\to0}\frac{-2\varepsilon^s}{\pi}=0$.

对③：$\left|\int_{C_R} f(z)\,dz\right| < \int_{C_R}|f(z)||z'(\theta)|\,d\theta$ $\begin{cases}|f(z)|=R^{s-1}\cdot e^{-R\cos\theta}\\ |z'(\theta)|=R\end{cases}$

$$=\int_0^{\frac{\pi}{2}}R^s\cdot e^{-R\cos\theta}\,d\theta$$

令 $\theta=\frac{\pi}{2}-\theta$

$$=\int_{\frac{\pi}{2}}^{0}R^s\cdot e^{-R\sin\theta}\,(-d\theta)$$

$$=\int_0^{\frac{\pi}{2}}R^s e^{-R\sin\theta}\,d\theta$$

$$\leq R^s\int_0^{\frac{\pi}{2}}e^{-R\frac{2\theta}{\pi}}\,d\theta$$

$$=-\frac{R^{s-1}}{2\pi}e^{-R\frac{2\theta}{\pi}}\Big|_0^{\frac{\pi}{2}}=-\frac{R^{s-1}}{2\pi}\cdot(e^{-R}-1)$$

$$=\frac{1-e^{-R}}{2\pi R^{1-s}}\to0.\ 当R\to+\infty时.$$

∴ $\int_{\varepsilon}^{R} f(z) + \int_{R}^{\varepsilon} f(z)\,dz = 0$. 令 $\varepsilon \to 0$, $R \to +\infty$.

即: $\underbrace{\int_0^{+\infty} z^{s-1} e^{-z}\,dz}_{取\ z=x} + \underbrace{\int_{+\infty}^{0} z^{s-1} e^{-z}\,dz}_{取\ z=iy} = 0$

∴ $\int_0^{+\infty} x^{s-1} e^{-x}\,dx = \int_0^{+\infty} (iy)^{s-1} \cdot e^{-iy} (i\,dy)$

$\Rightarrow \int_0^{+\infty} x^{s-1} e^{-x}\,dx = i^{s} \int_0^{+\infty} y^{s-1} e^{-iy}\,dy$

$= e^{i\frac{\pi}{2}s} \left[\int_0^{+\infty} y^{s-1} (\cos y - i\sin y)\,dy \right]$

∴ $e^{-i\frac{\pi s}{2}} \Gamma(s) = \int_0^{+\infty} y^{s-1} \cos y\,dy - i \int_0^{+\infty} y^{s-1} \sin y\,dy$

$\Rightarrow \begin{cases} \mathrm{Re}\,左 = \Gamma(s)\cdot\cos\frac{\pi s}{2} = \int_0^{+\infty} t^{s-1}\cos t\,dt. = \mathrm{Re}\,右. \\ -\mathrm{Im}\,左 = \Gamma(s)\cdot\sin\frac{\pi s}{2} = -\int_0^{+\infty} t^{s-1}\sin t\,dt. = -\mathrm{Im}\,右. \end{cases}$

∴ $\begin{cases} \int_0^{+\infty} t^{s-1}\cos t\,dt = \Gamma(s)\cos\frac{\pi s}{2} \\ \int_0^{+\infty} t^{s-1}\sin t\,dt = \Gamma(s)\sin\frac{\pi s}{2}. \end{cases}$ 得证.

P71-72

9. 解：$\int_{|z|=1} \frac{e^z}{z} dz = 2\pi i \cdot e^z \Big|_{z=0} = 2\pi i$.

证：由于 $\tilde{\theta} = -\theta$ 时，原式 $= \int_0^{-\pi} e^{\cos\theta} \cos(\sin\theta)(-d\theta) = \int_{-\pi}^{0} e^{\cos\theta} \cos(\sin\theta) d\theta$

$\therefore \int_0^{\pi} e^{\cos\theta} \cos(\sin\theta) d\theta + \int_{-\pi}^{0} e^{\cos\theta} \cos(\sin\theta) d\theta = 2\int_0^{\pi} e^{\cos\theta} \cos(\sin\theta) d\theta$

$\therefore$ 原式 $= \frac{1}{2} \int_{-\pi}^{\pi} e^{\cos\theta} \cos(\sin\theta) d\theta$

$= \frac{1}{2} \int_{-\pi}^{\pi} e^{\cos\theta} \frac{e^{i\sin\theta} + e^{-i\sin\theta}}{2} d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{\cos\theta + i\sin\theta} + e^{\cos\theta - i\sin\theta}}{2} d\theta$

令 $z = e^{i\theta}$. $dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$.

$\therefore$ 原式 $= \frac{1}{4} \int_{|z|=1} \left(\frac{e^z + e^{\bar{z}}}{iz}\right) dz$

$= \frac{1}{4i} \cdot 2\pi i \left(e^z \Big|_{z=0} + e^{\bar{z}} \Big|_{z=0} \right)$

$= \frac{1}{4i} \cdot 2\pi i \cdot (1+1) = \pi$. 得证.

10. 解：(2) $\int_C \frac{e^z}{1+z^2} dz = \int_{|z+i|=1} \frac{e^z}{(z+i)(z-i)} dz$

$= 2\pi i \cdot \frac{e^z}{z-i} \Big|_{z=-i}$

$= -\pi e^{-i}$

11. 解：奇点：$z=0$.（二阶） $z=-1$. $z=1$（一阶）

① $r>1$. 原式 $= 2\pi i \cdot \left[\left(\frac{1}{(z+1)(z-1)}\right)' \Big|_{z=0} + \frac{1}{z^2(z+1)} \Big|_{z=1} + \frac{1}{z^2(z-1)} \Big|_{z=-1} \right]$

奇点均在域内

$= 2\pi i \cdot \left(0 + \frac{1}{2} - \frac{1}{2}\right) = 0$.

②$0<r<1$时：原式 $= 2\pi i\cdot\left[\frac{1}{(z+1)(z-1)}\right]'\Big|_{z=0} = 0.$ Date. No.

只有$z=0$在内

综上：原积分 $=0.$

12. 解：(2) 奇点：$z=3$. $z=-3$. $z=-i$. (一阶).

$|z|=\frac{10}{3}>3$. 故均在内. $f(z)=\frac{-z}{(z^2-9)(z+i)}=\frac{-z}{(z+3)(z-3)(z+i)}$

$$\therefore 原式 = 2\pi i\left[\frac{-z}{(z+3)(z+i)}\Big|_{z=3}+\frac{-z}{(z-3)(z+i)}\Big|_{z=-3}+\frac{-z}{(z+3)(z-3)}\Big|_{z=-i}\right]$$

$$=2\pi i\cdot\left(\frac{-3}{6(3+i)}+\frac{3}{-6(-3+i)}+\frac{i}{-10}\right)$$

$$=0$$

13. 解：(1) 证：由柯西积分公式可知：$f(z_0)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-z_0}dz.$

取 $f(z)=2z^2-z+1$. 则 $f(z_0)=\frac{1}{2\pi i}\oint_C\frac{2z^2-z+1}{z-z_0}dz$

$\therefore g(z_0)=2\pi i\,f(z_0)$

$\therefore g(1)=2\pi i\,f(1)=(2-1+1)\cdot 2\pi i=4\pi i$. 得证.

(2) $|z_0|>2$时，积分闭域内无奇点，全解析.

故由柯西积分定理可知：$g(z_0)=0$.

14. 解：奇点：$[(z+i)(z-i)]^2 \Rightarrow z=i \quad z=-i.$

$$\therefore 原积分 = 2\pi i\cdot\left[\frac{z^2}{(z+i)^2}\right]'\Big|_{z=i}$$

$$=2\pi i\cdot\frac{2iz}{(z+i)^3}\Big|_{z=i}$$

$$=2\pi i\cdot\frac{1}{4i}$$

$$=\frac{\pi}{2}$$

1). 解：证明：由题可知. $\frac{P'(z)}{P(z)} = \frac{1}{z-a_1}+\cdots+\frac{1}{z-a_n} = \sum_{i=1}^{n} \frac{1}{z-a_i}$.

不妨设C内有零点k个，分别为 $a_{(1)}, a_{(2)}, \cdots a_{(k)}$. $k\le n$

(k) 表示按下标从小至大排序. $(1)\neq 1$.

由柯西积分公式：原式 $=\frac{1}{2\pi i}\sum_{i=1}^{n}\int_C \frac{1}{z-a_i}\,dz$

$$= \frac{1}{2\pi i}\left[2\pi i\times\left(1\Big|_{z=a_{(1)}}+1\Big|_{z=a_{(2)}}+\cdots+1\Big|_{z=a_{(k)}}\right) + \underline{0\times(n-k)}\right]$$

另外 $n-k$ 个零点不在C内.

则积分为0.

$= k$.

∴ $\frac{1}{2\pi i}\int_C\frac{P'(z)}{P(z)}\,dz = k$ 即C内零点个数. 得证.