复变函数 B 作业 3

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第二章

Question 11

设 $z = x + iy$.

(2) 令 $y = 0$, 则 $\lim_{z\to 0} z\sin\frac{1}{z} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$.

令 $x = 0$, 则 $\lim_{z\to 0} z\sin\frac{1}{z} = \lim_{y\to 0} iy\frac{e^{\frac{1}{y}}-e^{-\frac{1}{y}}}{2i} = \lim_{y\to 0}\frac{y}{2}(e^{\frac{1}{y}} - e^{-\frac{1}{y}}) = +\infty$.

故 $\lim_{z\to 0} z\sin\frac{1}{z}$ 不存在.

(3) 令 $y = 0$, 则 $\lim_{z\to 1}\frac{ze^{\frac{1}{z-1}}}{e^z-1} = \lim_{x\to 1}\frac{xe^{\frac{1}{x-1}}}{e^x-1}$.

$\lim_{x\to 1^+}\frac{xe^{\frac{1}{x-1}}}{e^x-1} = +\infty, \lim_{x\to 1^-}\frac{xe^{\frac{1}{x-1}}}{e^x-1} = 0$.

故 $\lim_{z\to 1}\frac{ze^{\frac{1}{z-1}}}{e^z-1}$ 不存在.

Question 13

(1) $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = 2$.

令 $t = e^{iz}$, 上式化为 $t^2 - 4it - 1 = 0$, 解得 $t = e^{iz} = 2i \pm \sqrt{3}i$.

故 $z = \frac{1}{i}\operatorname{Ln}(2i \pm \sqrt{3}i) = \frac{1}{i}\left(\ln(2\pm\sqrt{3}) + i(\frac{\pi}{2} + 2k\pi)\right) = (2k + \frac{1}{2})\pi - i\ln(2\pm\sqrt{3}), k \in \mathbb{Z}$.

(3) $z = \operatorname{Ln} A = \ln|A| \pm i(\arg A + 2k\pi), k \in \mathbb{Z}$.

Question 14

(1) $e^z + 1 \neq 0$, 故解析区域为 $\{z \mid z \neq i(2k+1)\pi, k \in \mathbb{Z}\}$.

$f'(z) = -\frac{e^z}{(1+e^z)^2}$.

(3) 解析区域为 $\{z \mid z \neq 1\}$.

$f'(z) = e^{\frac{1}{z-1}}\left(1 - \frac{z}{(z-1)^2}\right)$.

Question 16

设 $z=x+iy$, 则 $\cos z=\frac{1}{2}(e^{iz}+e^{-iz})$

$$=\frac{1}{2}(e^{ix-y}+e^{y-ix})$$

$$=\frac{1}{2}\left(e^{-y}(\cos x+i\sin x)+e^{y}(\cos x-i\sin x)\right).$$

$\Im(\cos z)=0$, 即 $e^{-y}\sin x-e^{y}\sin x=0$, 解得 $y=0$ 或 $x=k\pi,k\in\mathbb{Z}$.

故 $\cos z$ 在实轴及直线族 $\Re(z)=k\pi,k\in\mathbb{Z}$ 上取实数值.

Question 17

以下 $k\in\mathbb{Z}$.

(2) $1^{\sqrt{2}}=e^{\sqrt{2}\,\mathrm{Ln}\,1}=e^{i2\sqrt{2}k\pi}$.

$(-2)^{\sqrt{2}}=e^{\sqrt{2}\,\mathrm{Ln}(-2)}=e^{\sqrt{2}(\ln 2+i(\pi+2k\pi))}=e^{\sqrt{2}\ln 2+i\sqrt{2}(2k+1)\pi}$.

$2^{i}=e^{i\,\mathrm{Ln}\,2}=e^{i(\ln 2+i2k\pi)}=e^{-2k\pi+i\ln 2}$.

$(3-4i)^{1+i}=e^{(1+i)\,\mathrm{Ln}(3-4i)}=e^{(\ln 5+\arctan\frac{4}{3}-2k\pi)+i(\ln 5-\arctan\frac{4}{3}+2k\pi)}=e^{(\ln 5+\arctan\frac{4}{3}-2k\pi)+i(\ln 5-\arctan\frac{4}{3})}$.

注释: 有同学喜欢把 $\arctan\frac{4}{3}$ 写成 $53°$, 请不要这么写, 这二者不相等且我们习惯使用弧度制.

第 三 章

Question 1

(2) 令 $z=2e^{i\theta}$, 则 $dz=2ie^{i\theta}d\theta$.

$\int_C\frac{2z-3}{z}\,dz=\int_{-\pi}^{0}\frac{4e^{i\theta}-3}{2e^{i\theta}}2ie^{i\theta}\,d\theta=\int_{-\pi}^{0}(4ie^{i\theta}-3i)\,d\theta=4e^{i\theta}|_{-\pi}^{0}-3i\theta|_{-\pi}^{0}=8-3i\pi$.

Question 2

(2) 令 $z=e^{i\theta}$, 则 $dz=ie^{i\theta}d\theta$.

$\int_C|z|\,dz=-\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}ie^{i\theta}\,d\theta=-e^{i\theta}|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=2i$.

Question 3

(2) $f(z)=x^2+iy^2$, 则 $|f(z)|=\sqrt{x^4+y^4}\le\sqrt{(x^2+y^2)^2}=1$.

积分路径长度为 π, 由长大不等式, $|\int_C f(z)\,dz|\le\pi$.

Question 4

$f(z)=\frac{1}{z^2}$, 则 $|f(z)|=\frac{1}{|z|^2}\le 1$.

积分路径长度为 2, 由长大不等式, $|\int_C f(z)\,dz|\le 2$.

Question 7

本题的证明完全类似于书本例 3.

对任意 $\epsilon > 0$, 存在 $R_0 > 0$, 当 $|z| > R_0$ 时, 有 $|zf(z) - A| < \epsilon$.

再注意到 $\int_{C_R} \frac{dz}{z} = \int_0^\alpha \frac{iRe^{i\theta}}{Re^{i\theta}}\, d\theta = i\alpha$.

于是取 $R > R_0$, 由长大不等式, 即得下面的估计:

$$\begin{aligned}\left|\int_{C_R} f(z)\, dz - iA\alpha\right| &= \left|\int_{C_R} f(z)\, dz - \int_{C_R} \frac{A}{z}\, dz\right| \\ &= \left|\int_{C_R} \frac{zf(z) - A}{z}\, dz\right| \\ &< \frac{\epsilon}{R} R\alpha \\ &= \epsilon\alpha.\end{aligned}$$

这就证明了 $\lim_{R\to+\infty} \int_{C_R} f(z)\, dz = iA\alpha$.

Question 8

由于 $Q(z)$ 比 $P(z)$ 高 2 次, 则 $\lim_{z\to\infty} \frac{zP(z)}{Q(z)} = 0$. 应用上题结论, $\lim_{R\to+\infty} \int_{|z|=R} \frac{P(z)}{Q(z)}\, dz = 0$.

直接使用长大不等式也可证明, 注意到 $\frac{P(z)}{Q(z)} = \frac{1}{z^2} M(z)$, $M(z)$ 存在有限的最大值 M, 故 $|\int_{|z|=R} \frac{P(z)}{Q(z)}\, dz| = \int_{|z|=R} \frac{M(z)}{z^2}\, dz \le \frac{M}{R^2} 2\pi R$, 故 $\lim_{R\to+\infty} \int_{|z|=R} \frac{P(z)}{Q(z)}\, dz = 0$.

附 加

Question 1

由于 $\lim_{z\to\infty} f(z) = 0$, 则对任意 $\epsilon > 0$, 存在 $R_0 > 0$, 当 $|z| > R_0$ 时, 有 $|f(z)| < \epsilon$.

令 $z = Re^{i\theta}$, 则 $|dz| = |Rie^{i\theta} d\theta| = Rd\theta$, $|e^{imz}| = e^{\Re(imz)} = e^{-Rm\sin\theta}$.

$$\begin{aligned}\left|\int_{C_R} f(z)e^{imz}\, dz\right| &\le \int_0^\pi |f(z)|e^{-Rm\sin\theta} R\, d\theta \\ &< \epsilon R \int_0^\pi e^{-Rm\sin\theta}\, d\theta \\ &= 2\epsilon R \int_0^{\frac{\pi}{2}} e^{-Rm\sin\theta}\, d\theta \\ &\le 2\epsilon R \int_0^{\frac{\pi}{2}} e^{-Rm\frac{2}{\pi}\theta}\, d\theta \\ &= -\frac{\epsilon\pi}{m} e^{-Rm\frac{2}{\pi}\theta}\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\epsilon\pi}{m}(1 - e^{-Rm}).\end{aligned}$$

故 $\lim_{R\to+\infty}\int_{C_R} f(z)e^{imz}\,dz = 0$.

Question 2

设 $z = x + iy$. 在复平面上取闭路 $C = \{(x,y) \mid -R \le x \le R, y = 0\} \cup \{(x,y) \mid x = -R, 0 \le y \le b\} \cup \{(x,y) \mid -R \le x \le R, y = b\} \cup \{(x,y) \mid x = R, 0 \le y \le b\}$.

由柯西积分定理,

$$\int_C f(z)\,dz = \int_R^{-R} f(z)\,dz + \int_{-R}^{-R+ib} f(z)\,dz + \int_{-R+ib}^{R+ib} f(z)\,dz + \int_{R+ib}^{R} f(z)\,dz = 0.$$

四个积分在闭路的四个部分上.

对于第一部分,

$$\begin{aligned}\lim_{R\to+\infty}\int_R^{-R} f(z)\,dz &= -\int_{-\infty}^{+\infty} e^{-ax^2}\,dx \\ &= -\sqrt{\frac{\pi}{a}}\ (\text{这是高斯积分}).\end{aligned}$$

对于第二部分, $z = -R + iy$,

$$\begin{aligned}\left|\int_{-R}^{-R+ib} f(z)\,dz\right| &= \left|\int_0^b e^{-a(-R+iy)^2} i\,dy\right| \\ &= \int_0^b e^{-aR^2+ay^2}\,dy \\ &= e^{-aR^2}\int_0^b e^{ay^2}\,dy \\ &\le e^{-aR^2}\int_0^b e^{ab^2}\,dy \\ &= e^{-aR^2} b e^{ab^2},\end{aligned}$$

故

$$\lim_{R\to+\infty}\int_{-R}^{-R+ib} f(z)\,dz = 0.$$

对于第四部分, $z = R + iy$,

$$\begin{aligned}\left|\int_{R}^{R+ib} f(z)\,dz\right| &= \left|\int_0^b e^{-a(R+iy)^2} i\,dy\right| \\ &= \int_0^b e^{-aR^2+ay^2}\,dy \\ &\le e^{-aR^2} b e^{ab^2},\end{aligned}$$

故

$$\lim_{R\to+\infty}\int_{R+ib}^{R} f(z)\,dz = 0.$$

题目中的积分为第三部分, $\int_{-\infty+ib}^{+\infty+ib} e^{-az^2}\,dz = \lim_{R\to+\infty}\int_{-R+ib}^{R+ib} f(z)\,dz = \sqrt{\frac{\pi}{a}}$.