第二周习题解答.

$(x,y)\neq(0,0)$

9.7 星期三.

1. 解：设 $z=x+iy$ $w=u+iv$ $\Rightarrow w=\frac{1}{z}\Rightarrow u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}\Rightarrow\begin{cases}u=\frac{x}{x^2+y^2}\\ v=\frac{-y}{x^2+y^2}\end{cases}$ (*)

(1) $x=1$：代入 (*)：$u=\frac{1}{1+y^2}$，$v=\frac{-y}{1+y^2}$

$\Rightarrow y=-\frac{v}{u}$. 则 $u=\frac{1}{1+\frac{v^2}{u^2}}\Rightarrow u^2+v^2-u=0$

∴ 方程为：$(u-\frac{1}{2})^2+v^2=\frac{1}{4}$. w平面上以 $(\frac{1}{2},0)$ 为圆心，$\frac{1}{2}$ 为半径的圆

(3) $y=x$：代入 (*)：$u=\frac{x}{x^2+x^2}=\frac{1}{2x}$　$v=\frac{-x}{x^2+x^2}=-\frac{1}{2x}$

$\Rightarrow u=-v$　w平面上的直线 $u=-v$.

均去除原点.

(4) $x^2+y^2=4$：代入 *：$u=\frac{x}{4}$, $v=\frac{-y}{4}$

$\Rightarrow u^2+v^2=\frac{x^2+y^2}{16}=\frac{1}{4}$

∴ 方程为 $u^2+v^2=\frac{1}{4}$. w平面上以 $(0,0)$ 为圆心，$\frac{1}{2}$ 为半径的圆.

(5) $(x-1)^2+y^2=5$：$|z-1|=\sqrt{5}\Rightarrow|\frac{1}{w}-1|=\sqrt{5}$. $\Rightarrow(\frac{1}{w}-1)\cdot(\overline{\frac{1}{w}-1})=5$

即：$\frac{1-w}{w}\cdot\frac{\overline{1-w}}{\overline{w}}=5$

$\Rightarrow\frac{[(1-u)-iv][(1-u)+iv]}{u^2+v^2}=5$

$\Rightarrow(1-u)^2+v^2=5(u^2+v^2)$

$\Rightarrow(u+\frac{1}{4})^2+v^2=\frac{5}{16}$. w平面上以 $(-\frac{1}{4},0)$ 为圆心，$\frac{\sqrt{5}}{4}$ 为半径的圆.

2. 解：证明：$f(z)=\frac{1}{2i}(\frac{z}{\bar{z}}-\frac{\bar{z}}{z})$　设 $z=x+iy$

∴ $f(z)=\frac{1}{2i}\cdot\frac{x^2-y^2+2xyi-x^2+y^2+2xyi}{x^2+y^2}=\frac{1}{2i}\cdot\frac{4xyi}{x^2+y^2}=\frac{2xy}{x^2+y^2}$

∴ $u(x,y)=\frac{2xy}{x^2+y^2}$. $v(x,y)=0$.

∵ $\lim\limits_{(x,y)\to(0,0)}u(x,y)=\lim\limits_{\substack{x\to0\\y\to0}}\frac{2xy}{x^2+y^2}$

当 z 沿 $y=kx$ 趋于0时：$\lim\limits_{\substack{x\to0\\y=kx}}\frac{2xy}{x^2+y^2}=\lim\limits_{x\to0}\frac{2x\cdot kx}{x^2+(kx)^2}=\frac{2k}{1+k^2}$ 与k有关.

故可知：$\lim\limits_{(x,y)\to(0,0)}u(x,y)$ 不存在. ∴ $z\to0$ 时. $f(z)$ 极限不存在.

3. 解：证明：由题：$u(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & z\neq0\\ 0 & z=0\end{cases}$ $v(x,y)\equiv0$.

则：$\lim\limits_{(x,y)\to(0,0)} u(x,y)=\lim\limits_{\substack{x\to0\\y\to0}}\frac{xy}{x^2+y^2}\xrightarrow[\text{趋近}]{y=kx}\lim\limits_{\substack{x\to0\\y=kx}}\frac{kx^2}{x^2+k^2x^2}=\frac{k}{1+k^2}$ 与 k 有关.

故可知：$\lim\limits_{(x,y)\to(0,0)} u(x,y)\neq u(0,0)$.

∴ $f(z)$ 在 $z_0=0$ 处不连续.

9.9 星期三.

5. 解：(2) 证明：由于 $u(x,y)=x+y$，$v(x,y)=0$，

$\frac{\partial u}{\partial x}=1\neq\frac{\partial v}{\partial y}=0$，$\frac{\partial u}{\partial y}=1\neq-\frac{\partial v}{\partial x}=0$

由 C-R 方程可知：z 平面上处处不可导.

(3) 证明：$f(z)=\frac{1}{\bar z}=\frac{1}{x-iy}=\frac{x+iy}{x^2+y^2}$

∴ $u(x,y)=\frac{x}{x^2+y^2}$，$v(x,y)=\frac{y}{x^2+y^2}$

① $z\neq0$ 时：$\frac{\partial u}{\partial x}=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}$，$\frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}$，$\frac{\partial v}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}$

不满足 $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$，$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

故由 C-R 方程可知：$z\neq0$ 时，$f(z)$ 不可导.

② $z=0$ 时：函数无定义，不可导.

综①②可知：$f(z)$ 在 z 平面上处处不可导. 得证.

6.(2) 解: $f(z)=\begin{cases}\sqrt{x^2+y^2}\cdot(x+yi) & |z|<1\\ x^2-y^2+2xyi & |z|\geq 1\end{cases} \Rightarrow \begin{cases} u(x,y)=\begin{cases}x\cdot\sqrt{x^2+y^2} & |z|<1\\ x^2-y^2 & |z|\geq 1\end{cases}\\ v(x,y)=\begin{cases}y\cdot\sqrt{x^2+y^2} & |z|<1\\ 2xy & |z|\geq 1.\end{cases}\end{cases}$

Date. No.

①当$|z|<1$时: $\frac{\partial u}{\partial x}=\sqrt{x^2+y^2}+x\cdot\frac{x}{\sqrt{x^2+y^2}}=\frac{2x^2+y^2}{\sqrt{x^2+y^2}}$. $\frac{\partial v}{\partial y}=\sqrt{x^2+y^2}+y\cdot\frac{y}{\sqrt{x^2+y^2}}=\frac{x^2+2y^2}{\sqrt{x^2+y^2}}$

$\frac{\partial u}{\partial y}=\frac{xy}{\sqrt{x^2+y^2}}$. $\frac{\partial v}{\partial x}=\frac{xy}{\sqrt{x^2+y^2}}$

$\therefore$ 不满足 $\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$ 故不解析.

②当$|z|=1$时: $\lim\limits_{x^2+y^2\to 1}|z|\cdot z=\lim\limits_{x^2+y^2\to 1}\sqrt{x^2+y^2}\cdot(x+yi)=\begin{cases}\lim\limits_{\substack{x\to 1\\ y\to 0}}\sqrt{x^2+y^2}\cdot(x+yi)=1\\ \lim\limits_{\substack{x\to 0\\ y\to 1}}\sqrt{x^2+y^2}\cdot(x+yi)=i\end{cases}$ 不等. 不连续. 不解析.

故 $\lim\limits_{x^2+y^2\to 1}|z|\cdot z$ 不存在. 不解析($|z|=1$). 只需讨论边界三种可以. 也可以写成 $|z|<1$. $|z|=1$ 的邻域内

③当$|z|>1$时: $\frac{\partial u}{\partial x}=2x.=\frac{\partial v}{\partial y}=2x$ 故可知: $|z|>1$ 解析 均有不解析点. 故

$\frac{\partial u}{\partial y}=-2y=-\frac{\partial v}{\partial x}=-2y$. $|z|<1$, $|z|=1$ 不解析

7.(3) 解: 证明: $\cos z=\cos x\,\mathrm{ch}y-i\sin x\,\mathrm{sh}y$

$\mathrm{Re}(\cos z)=\cos x\,\mathrm{ch}y=u(x,y)$

$\mathrm{Im}(\cos z)=-\sin x\,\mathrm{sh}y=v(x,y)$.

$\therefore \frac{\partial u}{\partial x}=-\sin x\,\mathrm{ch}y$. $\frac{\partial u}{\partial y}=\cos x\cdot\mathrm{sh}y$

$\frac{\partial v}{\partial x}=-\cos x\cdot\mathrm{sh}y$ $\frac{\partial v}{\partial y}=-\sin x\,\mathrm{ch}y$ $\Rightarrow\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$

由C-R方程可知: 处处解析

设: $f(z)=\cos z$. $\therefore f'=-\sin z$

实部　虚部

8. 解：证明：$f(z) = u(x,y) + i v(x,y) \Rightarrow u(x,y).\ \ v(x,y)$

$\overline{f(z)} = u(x,y) - i v(x,y) \Rightarrow u(x,y).\ -v(x,y)$

由于 $f(z)$ 解析. 则由 C-R 方程. $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$ (*)

$\overline{f(z)}$ 解析：则由 C-R 方程. $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial (-v)}{\partial y} = -\dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial (-v)}{-\partial x} = \dfrac{\partial v}{\partial x} \end{cases}$ (**)

由 (*) 与 (**) 可知：$\dfrac{\partial v}{\partial y} = -\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$，$\dfrac{\partial v}{\partial x} = -\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y}$ $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$

∴ $f(z)$ 为常数. 得证

(2) 证：$\arg f(z) = \begin{cases} \arctan \frac{v}{u} & u>0. \\ \arctan \frac{v}{u} + \pi & u<0.\ v>0 \\ \arctan \frac{v}{u} - \pi & u<0.\ v<0. \end{cases} = \begin{cases} C_1 & u>0 \\ C_1 + \pi & u<0. v>0 \\ C_1 - \pi & u<0, v>0 \end{cases}$ C_1为常数

$f(z) = u(x,y) + i v(x,y)$

对 $\arg f(z)$ 求偏导：设 $\arg f(z) = g(z)$

则：$\dfrac{\partial g}{\partial x} = \dfrac{1}{1+\frac{v^2}{u^2}} \cdot \left(\dfrac{\frac{\partial v}{\partial x} \cdot u - v \cdot \frac{\partial u}{\partial x}}{u^2} \right) = \dfrac{u \cdot \frac{\partial v}{\partial x} - v \cdot \frac{\partial u}{\partial x}}{u^2+v^2} = 0$ (*)

$\dfrac{\partial g}{\partial y} = \dfrac{1}{1+\frac{v^2}{u^2}} \cdot \left(\dfrac{\frac{\partial v}{\partial y} \cdot u - v \cdot \frac{\partial u}{\partial y}}{u^2} \right) = \dfrac{u \cdot \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y}}{u^2+v^2} = 0$ (**)

结合 C-R 方程：$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$ 代入 (*) 与 (**) 中：

$\begin{cases} -u \cdot \dfrac{\partial u}{\partial y} - v \cdot \dfrac{\partial u}{\partial x} = 0 \\ u \cdot \dfrac{\partial u}{\partial x} - v \cdot \dfrac{\partial u}{\partial y} = 0 \end{cases}$ $\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$. 同理可知：$\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$

∴ $u.v$ 均为常数 ∴ $f(z) = C_1 + iC_2$ 为复常数.

也可以直接设 $u(x,y) = k\, v(x,y)$

这题求 $f'(z)$ 直接求导法则判断

10、解：(1) $\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)} \Rightarrow z\neq 1$ 且 $z\neq 2$

Date. No.

$f'(z)=\frac{-(2z-3)}{(z^2-3z+2)^2}$ → 有理函数，直接判断解析域.

(2) $\frac{1}{z^3+a} \Rightarrow z^3\neq -a.$ $z\neq \sqrt[3]{a}\cdot(\cos\frac{\pi+2k\pi}{3}+i\sin\frac{\pi+2k\pi}{3})$ $k=0,1,2$

$f'(z)=\frac{-3z^2}{(z^3+a)^2}$