复变函数 B 作业 1

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Question 1

(2) $$\begin{aligned}(x-i\sqrt{y})(-x-2i\sqrt{y}) &= -x^2-2ix\sqrt{y}+ix\sqrt{y}-2y \\ &= -x^2-2y-ix\sqrt{y}.\end{aligned}$$

(3) $\frac{3-4i}{4+3i}=\frac{(3-4i)(4-3i)}{25}=\frac{12-12-25i}{25}=-i.$

Question 2

(3) $$\begin{aligned}z &= \frac{\sqrt{13}}{2}e^{i(-\pi+\arctan 2\sqrt{3})} \\ &= \frac{\sqrt{13}}{2}(\cos(-\pi+\arctan 2\sqrt{3})+i\sin(-\pi+\arctan 2\sqrt{3})),\end{aligned}$$

$\mathrm{Arg} z=\arg z+2k\pi=-\pi+\arctan 2\sqrt{3}+2k\pi, k\in\mathbb{Z}.$

(4) 若 $\theta=2k\pi, k\in\mathbb{Z}$, 则 $z=0$, 辐角无意义.

否则设 $\theta=2k\pi+\theta_0, \theta_0\in(0,2\pi)$, 则

$$\begin{aligned}z &= 2\sin\frac{\theta_0}{2}e^{i(\frac{\pi-\theta_0}{2})} \\ &= 2\sin\frac{\theta_0}{2}(\cos\frac{\pi-\theta_0}{2}+i\sin\frac{\pi-\theta_0}{2}),\end{aligned}$$

$\mathrm{Arg} z=\arg z+2k\pi=\frac{\pi-\theta_0}{2}+2k\pi, k\in\mathbb{Z}.$

注释:

1. 很多人没有讨论 z 是否为 0.
2. 如果不约束 $\theta_0\in(0,2\pi)$, 而是直接用 θ 表示, 需要注意 $\sqrt{2-2\cos\theta}=\sqrt{4\sin^2\frac{\theta}{2}}=2|\sin\frac{\theta}{2}|$.

Question 3

(3) $$\begin{aligned}\sqrt[3]{1+i} &= \left(\sqrt{2}e^{i(\frac{\pi}{4}+2k\pi)}\right)^{\frac{1}{3}} \\ &= \sqrt[6]{2}e^{i(\frac{\pi}{12}+\frac{2k\pi}{3})}, k=0,1,2.\end{aligned}$$

Question 4

(2) $z = \left(e^{i(-\frac{\pi}{2}+2k\pi)}\right)^{\frac{1}{3}}$
$= e^{i(-\frac{\pi}{6}+\frac{2k\pi}{3})}, k = 0, 1, 2.$

(3) $z = \left(e^{i\pi+2k\pi}\right)^{\frac{1}{4}}$
$= e^{i(\frac{\pi}{4}+\frac{k\pi}{2})}, k = 0, 1, 2, 3.$

Question 6

两边平方得 $x^2 - y^2 + 2ixy = a + ib$, 故 $\begin{cases} x^2 - y^2 = a; \\ 2xy = b. \end{cases}$

$2xy = b \Rightarrow 4x^2y^2 = b^2 \Rightarrow 4x^2(x^2 - a) = b^2$, 解得 $\begin{cases} x^2 = \frac{a+\sqrt{a^2+b^2}}{2}; \\ y^2 = \frac{-a+\sqrt{a^2+b^2}}{2}. \end{cases}$

再次注意到 $2xy = b$, 所以 $\begin{cases} x = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}; \\ y = \pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}. \end{cases}$

其中 $b > 0$ 时 x, y 同号, $b < 0$ 时 x, y 异号, $b = 0$ 时 $y = 0$.

Question 7

令 $z = \cos\theta + i\sin\theta$, 则

$$\sum_{k=1}^{n} z^k = \sum_{k=1}^{n} \cos k\theta + i\sum_{k=1}^{n} \sin k\theta.$$

只需求

$$\sum_{k=1}^{n} z^k = \begin{cases} z\frac{z^n-1}{z-1} & \text{if } z \neq 1; \\ n & \text{if } z = 1. \end{cases}$$

的实部和虚部即可. 如 $z \neq 1$,

$$\begin{aligned} z - 1 &= (\cos\theta - 1) + i\sin\theta \\ &= -2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ &= -2\sin\frac{\theta}{2}\left(\cos(\frac{\pi}{2} - \frac{\theta}{2}) - i\sin(\frac{\pi}{2} - \frac{\theta}{2})\right) \\ &= -2\sin\frac{\theta}{2}e^{i(\frac{\theta}{2}-\frac{\pi}{2})}. \end{aligned}$$

同理,

$$z^n - 1 = -2\sin\frac{n\theta}{2}e^{i(\frac{n\theta}{2}-\frac{\pi}{2})}.$$

故

$$z\frac{z^n-1}{z-1} = e^{i\theta}\frac{-2\sin\frac{n\theta}{2}e^{i(\frac{n\theta}{2}-\frac{\pi}{2})}}{-2\sin\frac{\theta}{2}e^{i(\frac{\theta}{2}-\frac{\pi}{2})}} = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}e^{i\frac{n+1}{2}\theta} = \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\left(\cos\frac{n+1}{2}\theta + i\sin\frac{n+1}{2}\theta\right).$$

积化和差

$$\begin{cases}\sin\frac{n\theta}{2}\cos\frac{n+1}{2}\theta=\frac{1}{2}\left(\sin(n+\frac{1}{2})\theta-\sin\frac{\theta}{2}\right);\\ \sin\frac{n\theta}{2}\sin\frac{n+1}{2}\theta=\frac{1}{2}\left(\cos\frac{\theta}{2}-\cos(n+\frac{1}{2})\theta\right).\end{cases}$$

所以

$$\sum_{k=1}^{n}\cos k\theta=\begin{cases}-\frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} & \text{if } \theta\neq 2m\pi;\\ n & \text{if } \theta=2m\pi.\end{cases}$$

$$\sum_{k=1}^{n}\sin k\theta=\begin{cases}\frac{1}{2}\cot\frac{\theta}{2}-\frac{\cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} & \text{if } \theta\neq 2m\pi;\\ n & \text{if } \theta=2m\pi.\end{cases}$$

其中 $m\in\mathbb{Z}$.

Question 8

$$\begin{aligned}|z_1+z_2|^2+|z_1-z_2|^2&=(z_1+z_2)(\overline{z_1}+\overline{z_2})+(z_1-z_2)(\overline{z_1}-\overline{z_2})\\&=2z_1\overline{z_1}+2z_2\overline{z_2}\\&=2(|z_1|^2+|z_2|^2).\end{aligned}$$

几何意义: 一个平行四边形的两条对角线长度的平方和, 等于它四边长度的平方和.

Question 9

$$|z^n+a|\leq|z^n|+|a|=|z|^n+|a|\leq 1+|a|.$$

当且仅当 $|z|=1$ 且 z^n 与 a 在复平面上方向相同时取得最大值.

注释: 当 $a=0$ 时其辐角无意义, 故取等条件不宜直接写作 $z=e^{i\frac{\arg a}{n}}$.

Question 10

(1) $\left|\frac{z-a}{1-\overline{a}z}\right|=\left|\frac{(z-a)\overline{z}}{(1-\overline{a}z)\overline{z}}\right|=\left|\frac{(z-a)\overline{z}}{\overline{z}-\overline{a}}\right|=\frac{|z-a||\overline{z}|}{|\overline{z}-\overline{a}|}=1.$

(2) 平方, 只需证 $|z-a|^2<|1-\overline{a}z|^2$,

只需证 $|z|^2+|a|^2-\overline{a}z-a\overline{z}<1+|az|^2-\overline{a}z-a\overline{z}$,

只需证 $|z|^2+|a|^2<1+|az|^2$,

只需证 $(|z|^2-1)(|a|^2-1)>0$,

由于 $|z|<1,|a|<1$, 命题得证.

Question 16

(2) 令 $z_0=0$, 则 $\lim_{n\to+\infty}|z_n-z_0|=\lim_{n\to+\infty}\frac{1}{n}=0$. 故复数列有极限, 值为 0.

Question 18

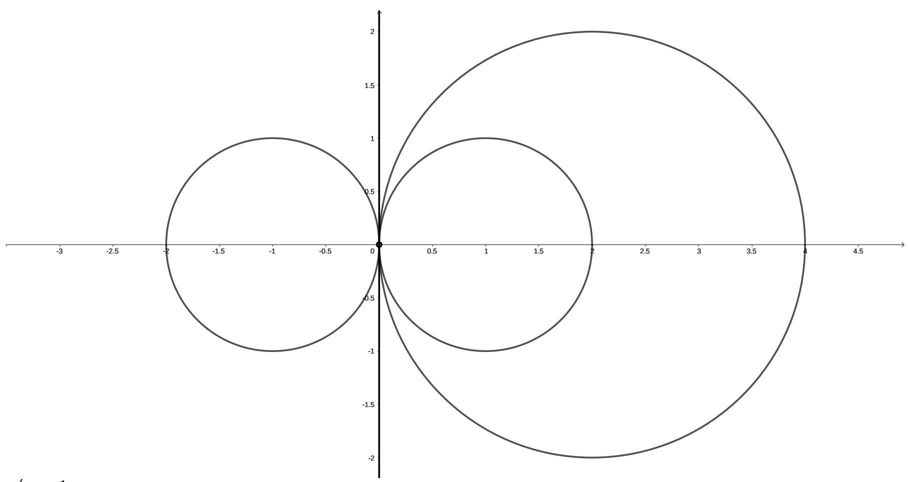
设 $z = x + iy$.

(3) $z \neq 0, \Re(\frac{1}{z}) = \Re(\frac{x-iy}{x^2-y^2}) = \frac{x}{x^2+y^2} = \alpha$.

若 $\alpha = 0$, 得 $x = 0$, 是不包含原点的虚轴.

若 $\alpha \neq 0$, 得 $(x - \frac{1}{2\alpha})^2 + y^2 = (\frac{1}{2\alpha})^2$, 是不包含原点, 与虚轴相切于原点的圆族.

注释: 圆心为 $(\frac{1}{2\alpha}, 0)$, 半径为 $\lvert\frac{1}{2\alpha}\rvert$. 部分同学没写绝对值.



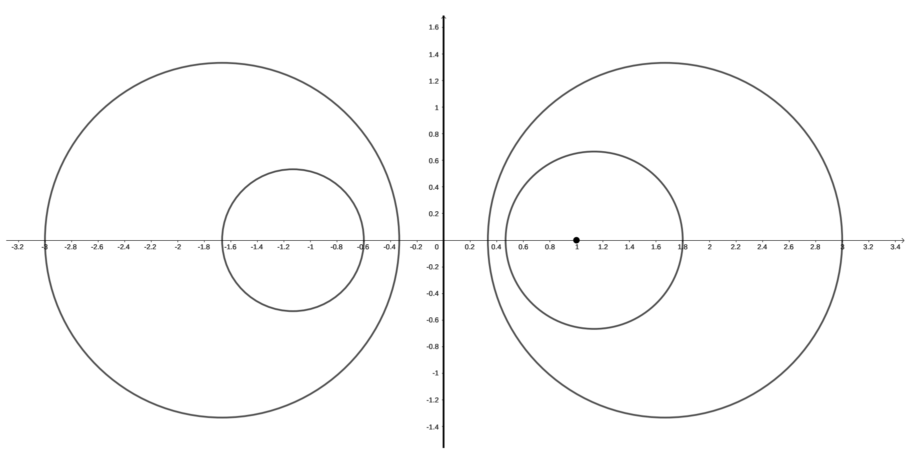
(5) α 非负, $z \neq -1$.

$|\frac{z-1}{z+1}|^2 = \frac{(x-1)^2+y^2}{(x+1)^2+y^2} = \alpha^2$.

若 $\alpha = 0$, 得 $x = 1, y = 0$, 是点 $(1, 0)$.

若 $\alpha = 1$, 得 $x = 0$, 是虚轴.

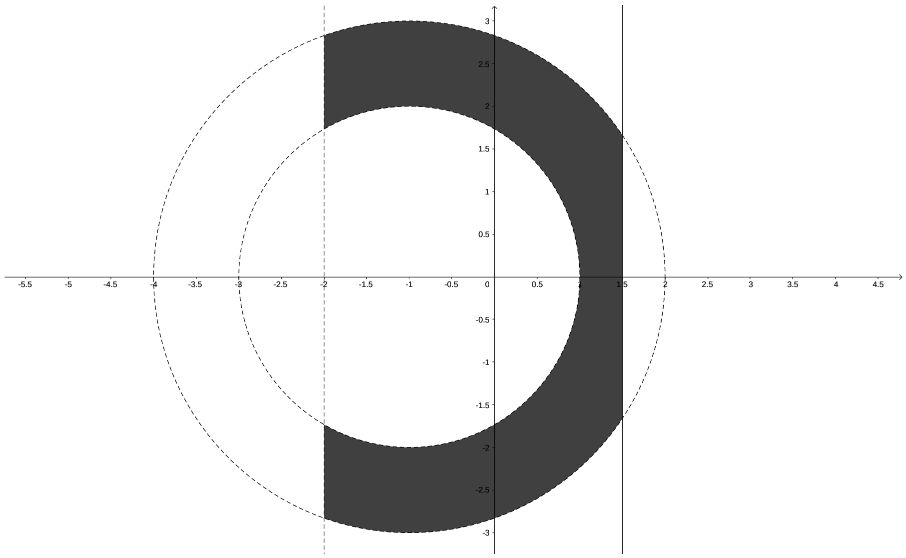
否则, 得 $\left(x - \frac{1+\alpha^2}{1-\alpha^2}\right)^2 + y^2 = \left(\frac{2\alpha}{1-\alpha^2}\right)^2$, 是以点 $(\pm 1, 0)$ 为对称点的 Apollonius 圆族.[1]



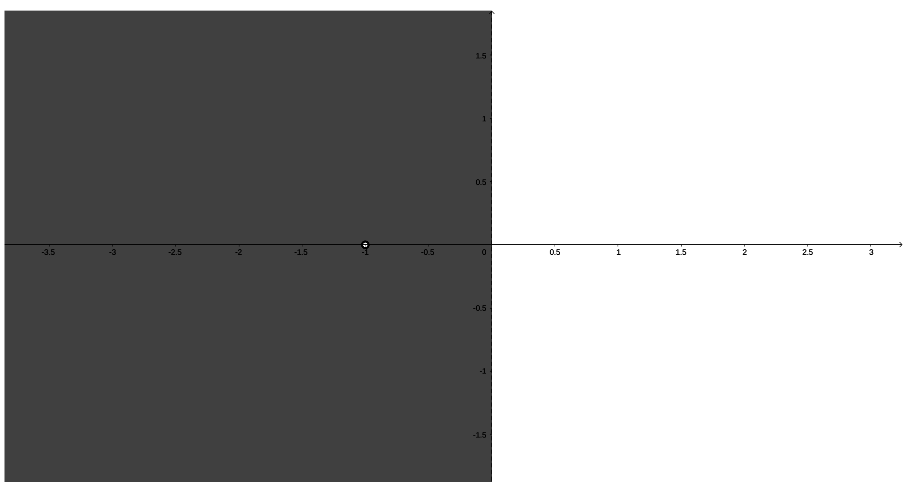
[1] https://en.wikipedia.org/wiki/Apollonian_circles.

Question 19

(6) 不是区域, 边界为 $\{(x,y) \mid x=-2, \sqrt{3} \le |y| \le 2\sqrt{2}\} \cup \{(x,y) \mid x=\frac{3}{2}, |y| \le \frac{\sqrt{11}}{2}\} \cup \{(x,y) \mid (x+1)^2+y^2=4, x \ge -2\} \cup \{(x,y) \mid (x+1)^2+y^2=9, -2 \le x \le \frac{3}{2}\}$.



(10) 是区域, 边界为 $\{(x,y) \mid x=0\}$.



注释: 这题很多人错, 请自行阅读教材 1.3 节.

Question 21

(3) 设 $z=x+iy$, 则 $xy=1$, 是双曲线.

Question 22

配方, 得 $(x+1)^2+y^2=2$.

设 $z=x+iy$, 则 $|z+1|^2=2$, 即 $|z+1|=\sqrt{2}$.

注释: 通法是代入 $x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2}$ 暴力计算.