7.2.6 本函数积分公式

$$L\left[\int_0^t f(s)\,\mathrm{d}s\right]=\frac{1}{p}L[f(t)].$$

证明：$\dfrac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^t f(s)\,\mathrm{d}s\right\}=f(t)$，故由微分公式得

$$L[f(t)]=L\left[\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^t f(s)\,\mathrm{d}s\right\}\right]=pL\left[\int_0^t f(s)\,\mathrm{d}s\right]-\underline{\lim_{t\to0^+}\int_0^t f(s)\,\mathrm{d}s}$$

$$=pL\left[\int_0^t f(s)\,\mathrm{d}s\right].$$ 两边除以p 得出结论.# $=\int_0^0 f(s)\,\mathrm{d}s=0$

例 $L\left[\int_0^t s^2\mathrm{e}^{5s}\,\mathrm{d}s\right]=\dfrac{1}{p}L\left[t^2\mathrm{e}^{5t}\right]$，$L\left[t^2\right]=\dfrac{2!}{p^3}$，

$L\left[t^2\mathrm{e}^{5t}\right]=\dfrac{2}{(p-5)^3}$. 故 $L\left[\int_0^t s^2\mathrm{e}^{5s}\,\mathrm{d}s\right]=\dfrac{2}{p(p-5)^3}$.

7.2.7 延迟定理(P169)

设$\tau > 0$, $f(t-\tau)h(t-\tau)=\begin{cases} f(t-\tau), & t \ge \tau, \\ 0, & t < \tau. \end{cases}$

$f(t)$延迟τ长时刻，图像上由$f(t)$向右平移τ所得，

$$L[f(t-\tau)h(t-\tau)] = e^{-p\tau} L[f(t)].$$

证明 $L[f(t-\tau)h(t-\tau)] = \int_0^{+\infty} [f(t-\tau)h(t-\tau)] e^{-pt}\, dt$

$= \int_\tau^{+\infty} f(t-\tau) e^{-pt}\, dt$ (令$t_1 = t-\tau$, $t = t_1 + \tau$)

$= \int_0^{+\infty} f(t_1) e^{-p(t_1+\tau)}\, dt_1 = e^{-p\tau} \int_0^{+\infty} f(t_1) e^{-pt_1}\, dt_1$

$= e^{-p\tau} L[f(t)]$. #

$\Rightarrow L^{-1}\left[e^{-p\tau} L[f(t)]\right] = f(t-\tau)h(t-\tau).$

$$\boxed{L[f(t-\tau)h(t-\tau)] = e^{-p\tau} L[f(t)].}$$

例 $L\left[(t-2)^n h(t-2)\right] = e^{-2p} L\left[t^n\right] = \dfrac{n!}{p^{n+1}} e^{-2p}$.

例 $L\left[h(t-\varphi)\sin\omega(t-\varphi)\right] = e^{-p\varphi} L\left[\sin\omega t\right] = \dfrac{\omega}{p^2+\omega^2} e^{-p\varphi}$.

例 $L\left[h(t-3\varphi)\sin\omega(t-\varphi)\right] = L\left[h(t-3\varphi)\sin\omega\{(t-3\varphi)+2\varphi\}\right]$

$$= e^{-3\varphi p} L\left[\sin\omega(t+2\varphi)\right] \quad = e^{-3\varphi p} L\left[\cos 2\omega\varphi \sin\omega t + \sin 2\omega\varphi \cos\omega t\right]$$

$$= e^{-3\varphi p}\left(\cos 2\omega\varphi L\left[\sin\omega t\right] + \sin 2\omega\varphi L\left[\cos\omega t\right]\right)$$

$$= e^{-3\varphi p}\left(\frac{\omega}{p^2+\omega^2}\cos 2\omega\varphi + \frac{p}{p^2+\omega^2}\sin 2\omega\varphi\right).$$

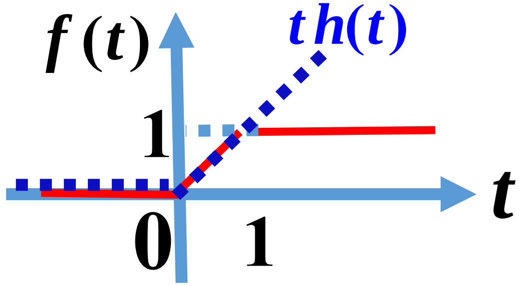
P187习题1(17)(18),2仿照此处例题做.

$$\boxed{\text{设} F(p) = L[f(t)],\ \text{则} L\left[e^{\lambda t} f(t)\right] = F(p-\lambda).}$$

$$\boxed{L^{-1}\left[e^{-p\tau} L[f(t)]\right] = f(t-\tau)h(t-\tau).}$$

分段函数(非周期函数)的拉氏变换

例 求如图所示波形的像函数.



解 如图, $t \le 1$时, $f(t)=t\ h(t)$,

$t>1$时, $f(t)=1=t\ h(t)-(t-1)$

$$=t\ h(t)-(t-1)\ h(t-1).$$

故 $f(t)=t\ h(t)-(t-1)\ h(t-1)$.

$$L[f(t)]=L\left[t\,h(t)\right]-L\left[(t-1)\ h(t-1)\right]$$

$$=L[t\,h(t)]-e^{-p}\,L[t\,h(t)]=\left(1-e^{-p}\right)L[t\,h(t)]=\frac{1-e^{-p}}{p^2}.$$

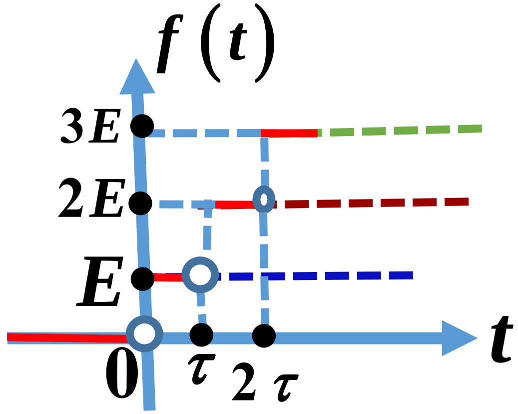
$$\boxed{L\left[t\,h(t)\right]=\frac{1}{p^2}.}$$

例 求阶梯函数$K(t)=\begin{cases}0, & t<0,\\ nE, & (n-1)\tau \le t < n\tau,\ n=1,2,3,\cdots\end{cases}$的像函数.

解 $K(t) = Eh(t) + Eh(t-\tau) + Eh(t-2\tau)$

$+\cdots + Eh(t-n\tau) + \cdots$

$= E\sum_{n=0}^{+\infty} h(t-n\tau).$



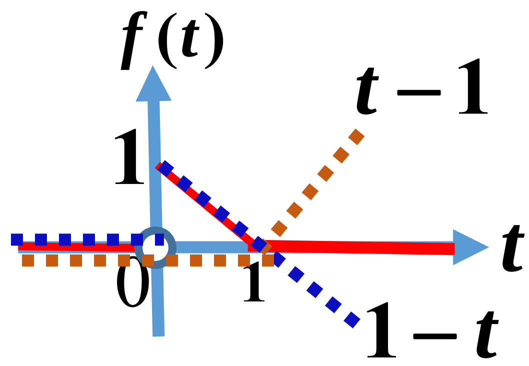
故$L[K(t)] = E\sum_{n=0}^{+\infty} L\left[h\left(t-n\tau\right)\right] = E\sum_{n=0}^{+\infty} e^{-n\tau p} L[h(t)] = \frac{E}{p}\sum_{n=0}^{+\infty}\left(e^{-p\tau}\right)^n,$

$\operatorname{Re} p > 0,\quad \left|e^{-p\tau}\right| = e^{-\tau \operatorname{Re} p} < 1,$ 故

$L[K(t)] = \frac{E}{p}\cdot\frac{1}{1-e^{-p\tau}}.$

$\boxed{L[H(t-\tau)f(t-\tau)] = e^{-p\tau} L[f(t)]}$ $\boxed{L[h(t)] = \frac{1}{p}}$

例 求函数 $f(t)=\begin{cases}0, & t<0,\\ 1-t, & 0<t<1,\\ 0, & t>1\end{cases}$ 像函数.



解 当 $t\le 1$时，$f(t)=(1-t)h(t)$.

当 $t>1$时, $f(t)=0=(1-t)h(t)-(1-t)=(1-t)h(t)+(t-1)$

故 $f(t)=(1-t)h(t)+(t-1)h(t-1)$,

$$L[f(t)]=L[h(t)]-L[t\,h(t)]+L[(t-1)\,h(t-1)]$$

$$=\frac{1}{p}-\frac{1}{p^2}+e^{-p}L[t\,h(t)]=\frac{1}{p}-\frac{1}{p^2}+\frac{e^{-p}}{p^2}.$$

$$\boxed{L[h(t)]=\frac{1}{p}}$$

$$\boxed{L[t\,h(t)]=\frac{1}{p^2}}$$

$$L^{-1}\left[e^{-p\tau}L[f(t)]\right]=f(t-\tau)h(t-\tau).$$

例 $L^{-1}\left[\dfrac{p}{p^2+6}e^{-5p}\right]=h(t-5)\cos\left(\sqrt{6}\right)(t-5)=\begin{cases}\cos\left(\sqrt{6}\right)(t-5), & t\geq 5,\\ 0, & t<5.\end{cases}$

例 $L^{-1}\left[\dfrac{1}{p}\left(e^{-5p}-1\right)\right]=L^{-1}\left[\dfrac{1}{p}e^{-5p}\right]-L^{-1}\left[\dfrac{1}{p}\right]=h(t-5)-h(t)=\begin{cases}0, & t\geq 5,\\ -1, & 0\leq t<5,\\ 0, & t<0.\end{cases}$

P188-189习题6(17)(18)(19)(20)仿照此处例题做.

7.2.8 卷积定理 ★★★

卷积定义：$f(x)*g(x)=\int_{-\infty}^{+\infty}f(x-\xi)g(\xi)\,d\xi\triangleq(f*g)(x).$

运算法则：

(1) (交换律) $f(x)*g(x)=g(x)*f(x)$;

(2) (结合律) $f(x)*\left(g_1(x)*g_2(x)\right)=\left(f(x)*g_1(x)\right)*g_2(x)$;

(3) (分配律) $f(x)*\left(g_1(x)+g_2(x)\right)=f(x)*g_1(x)+f(x)*g_2(x)$.

卷积定理：设$f_j(t)=h(t)f_j(t),\ j=1,2,$都满足定理1条件(1)和(2)：

(1) 在t轴任意有限区间$f_j(t)$逐段光滑，$j=1,2,$

(2) $\exists K_1,K_2>0,\ c_1,c_2\geq 0,$ 使得$\left|f_j(t)\right|\leq K_j e^{c_j t},\ j=1,2,\ \forall t\in[0,\ +\infty),$

则 $f_1(t)*f_2(t)=\begin{cases}\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau, & t\geq 0,\\ 0, & t<0,\end{cases}$ 满足(1),(2), $L[f_1*f_2]=L[f_1]L[f_2].$

证明 1)求f_1*f_2. 当$t<0$时，由卷积定义，

$$f_1(t)*f_2(t)=\int_{-\infty}^{0} f_1(t-\xi)\underbrace{f_2(\xi)}_{=0}d\xi+\int_0^{+\infty}\underbrace{f_1(t-\xi)}_{=0}f_2(\xi)d\xi=0.$$

当$t\geq 0$时，

$$f_1(t)*f_2(t)=\int_{-\infty}^{0} f_1(t-\xi)\underbrace{f_2(\xi)}_{=0}d\xi+\int_0^t f_1(t-\xi)f_2(\xi)d\xi+\int_t^{+\infty}\underbrace{f_1(t-\xi)}_{=0}f_2(\xi)d\xi$$

$$=\int_0^t f_1(t-\xi)f_2(\xi)d\xi.$$

因f_1和f_2都满足(1)，故f_1*f_2满足(1). 下证f_1*f_2满足(2).

当$t<0$时，$f_1(t)*f_2(t)=0$; 当$t\geq 0$时，$f_1(t)*f_2(t)=\int_0^t f_1(t-\xi)f_2(\xi)\,\mathrm{d}\xi$.

又因 f_1和f_2都满足(1)，故 f_1*f_2 满足(1). 下证 f_1*f_2 满足(2).

当$t\geq 0$时，记$c=\max\{c_1,c_2\}$，

$$\left|f_1(t)*f_2(t)\right|\leq\int_0^t\left|f_1(t-\xi)\right|\cdot\left|f_2(\xi)\right|\mathrm{d}\xi\leq\int_0^t K_1\,\mathrm{e}^{c(t-\xi)}\cdot K_2\,\mathrm{e}^{c\xi}\,\mathrm{d}\xi$$

$$\leq K_1K_2\,\mathrm{e}^{ct}\int_0^t 1\,\mathrm{d}\xi=K_1K_2\,\mathrm{e}^{ct}\,t\leq M_\varepsilon\,\mathrm{e}^{(c+\varepsilon)t}\quad\left(t\leq\frac{M_\varepsilon}{K_1K_2}\mathrm{e}^{\varepsilon t}\right),$$

$\varepsilon>0$是任意正实数，$M_\varepsilon>0$是与ε有关的正常数. 故f_1*f_2满足(2).

由定理1，$L[f_1*f_2]$有意义，在$\operatorname{Re}p>c$ 内解析. 下面求$L[f_1*f_2]$.

卷积定理：设$f_j(t)=h(t)f_j(t),\ j=1,2$,都满足定理1条件(1) 和(2)：

(1) 在t 轴任意有限区间 $f_j(t),\ f_j'(t),\ j=1,2$,除有限个第一类简断点外，处处连续；

(2) $\exists K_1,K_2>0,\ c_1,c_2\geq 0$, 使得 $\left|f_j(t)\right|\leq K_j\,\mathrm{e}^{c_j t},\ j=1,2,\ \forall t\in[0,\ +\infty)$,

则 $f_1*f_2=\begin{cases}\int_0^t f_1(t-\tau)f_2(\tau)\,\mathrm{d}\tau, & t\geq 0,\\ 0, & t<0,\end{cases}$ 满足(1),(2)，$L[f_1*f_2]=L[f_1]L[f_2]$.

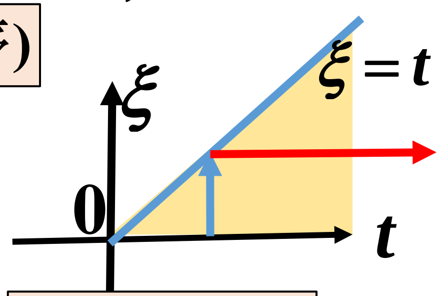
当$t<0$时，$f_1(t)*f_2(t)=0$; 当$t\geq 0$时，$f_1(t)*f_2(t)=\int_0^t f_1(t-\xi)f_2(\xi)\mathrm{d}\xi$.

又因 f_1和f_2都满足定理1中条件(1)和(2),

由定理1, $L\left[f_1*f_2\right]$有意义, 在$\mathrm{Re}\,p>c$ 内解析. 下面求$L\left[f_1*f_2\right]$.

$$L\left[f_1*f_2\right]=L\left[\int_0^t f_1(t-\xi)f_2(\xi)\mathrm{d}\xi\right]=\int_0^{+\infty}\left(\int_0^t f_1(t-\xi)f_2(\xi)\mathrm{d}\xi\right)\mathrm{e}^{-pt}\,\mathrm{d}t$$

(交换积分顺序)



令$\tau=t-\xi$

$$=\int_0^{+\infty}\left(\int_\xi^{+\infty} f_1(t-\xi)f_2(\xi)\mathrm{e}^{-pt}\,\mathrm{d}t\right)\mathrm{d}\xi$$

$$=\int_0^{+\infty}\underline{\left(\int_\xi^{+\infty} f_1(t-\xi)\mathrm{e}^{-p(t-\xi)}\,\mathrm{d}t\right)}f_2(\xi)\mathrm{e}^{-p\xi}\,\mathrm{d}\xi$$

$$=\int_0^{+\infty}\left(\int_0^{+\infty} f_1(\tau)\mathrm{e}^{-p\tau}\,\mathrm{d}\tau\right)f_2(\xi)\mathrm{e}^{-p\xi}\,\mathrm{d}\xi$$

$$=\left(\int_0^{+\infty} f_1(\tau)\mathrm{e}^{-p\tau}\,\mathrm{d}\tau\right)\int_0^{+\infty} f_2(\xi)\mathrm{e}^{-p\xi}\,\mathrm{d}\xi=L\left[f_1\right]L\left[f_2\right].\ \#$$

$L\left[f_1*f_2\right]=L\left[f_1\right]L\left[f_2\right].$ $\Rightarrow$ $L^{-1}\left[F_1(p)F_2(p)\right]=L^{-1}\left[F_1(p)\right]*L^{-1}\left[F_1(p)\right].$

$L\left[f_1*f_2\right]\neq L\left[f_1\right]*L\left[f_2\right].$ $L^{-1}\left[F_1(p)F_2(p)\right]\neq L^{-1}\left[F_1(p)\right]L^{-1}\left[F_1(p)\right].$

$$f_1(t)*f_2(t)=\begin{cases}\int_0^t f_1(t-\tau)f_2(\tau)\,\mathrm{d}\tau, & t\geq 0,\\ 0, & t<0,\end{cases}\qquad L[f_1*f_2]=L[f_1]L[f_2].$$

卷积定理

例(P187习题1(16)) $L\left[\int_0^t \underline{(t-\tau)^n}\,\mathrm{e}^{-a\tau}\cos\omega\tau\,\mathrm{d}\tau\right]=L\left[t^n*\left(\mathrm{e}^{-at}\cos\omega t\right)\right]$

$$=L[t^n]L\left[\mathrm{e}^{-at}\cos\omega t\right]=\frac{n!}{p^{n+1}}\cdot\frac{p+a}{(p+a)^2+\omega^2}=\frac{n!(p+a)}{p^{n+1}\left\{(p+a)^2+\omega^2\right\}}.$$

(位移定理)

$$L^{-1}[F_1(p)F_2(p)]=L^{-1}[F_1(p)]*L^{-1}[F_2(p)].$$

例 设$F(p)=L[f(t)]$，求$L^{-1}\left[\dfrac{pF(p)}{p^2+2}\right]$.

解 $L^{-1}\left[\dfrac{pF(p)}{p^2+2}\right]=L^{-1}\left[\dfrac{p}{p^2+2}\right]*L^{-1}[F(p)]$

$$=\left(h(t)\cos\left(\sqrt{2}\right)t\right)*(h(t)f(t))=\int_0^t f(\tau)\cos\left(\sqrt{2}\right)(t-\tau)\,\mathrm{d}\tau.$$

注意卷积定理与积分公式$L\left[\int_0^t f_1(s)f_2(s)\,\mathrm{d}s\right]=\dfrac{1}{p}L[f_1(t)f_2(t)]$的区别.

例 求(卷积型)积分方程 $y(t)=at+\int_0^t y(\tau)\sin(t-\tau)\,\mathrm{d}\tau$.

解 1)设$Y(p)=L[y(t)]$, 在方程两边作拉氏变换得

$$L[y(t)]=L\left[at+\int_0^t y(\tau)\sin(t-\tau)\,\mathrm{d}\tau\right]$$

$$=aL[t]+L\left[\underline{\int_0^t y(\tau)\sin(t-\tau)\,\mathrm{d}\tau}\right]=aL[t]+L[(\sin t)*y(t)]$$

$$=\frac{a}{p^2}+L[\sin t]L[y(t)]=\underline{\frac{a}{p^2}+\frac{1}{p^2+1}Y.\quad \text{左边}=L[y(t)]=Y.}$$

故$Y=\frac{a}{p^2}+\frac{1}{p^2+1}Y$. $\quad \frac{a}{p^2}=\left(1-\frac{1}{p^2+1}\right)Y=\frac{p^2}{p^2+1}Y$.

2)解得$Y(p)=\frac{a\left(p^2+1\right)}{p^4}=a\left(\frac{1}{p^2}+\frac{1}{p^4}\right)$.

3) $y(t)=L^{-1}[Y(p)]=aL^{-1}\left[\frac{1}{p^2}\right]+aL^{-1}\left[\frac{1}{p^4}\right]=a\,t+a\frac{t^3}{3!}=a\,t+\frac{at^3}{6}$.

P190 第9题 类似地求解.

7.3 拉氏变换反演公式

拉氏变换反演公式：设$f(t)$满足定理1条件(1), (2)，$F(p)=L[f(t)]$，

则在$f(t)$连续点处，$\forall \sigma > c$，$f(t)=\dfrac{1}{2\pi \mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} F(p)\mathrm{e}^{pt}\,\mathrm{d}p$，

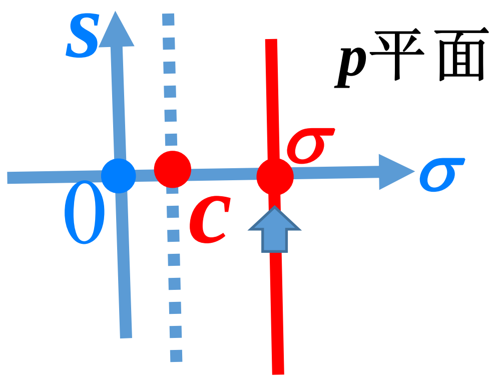
积分路径为从$\sigma-\mathrm{i}\infty$到$\sigma+\mathrm{i}\infty$的直线路径. (Fourier−Mellin公式)

证明　由定理1, 当$\mathrm{Re}\,p=\sigma>c$时, $F(p)$解析.

$$F(p)=F(\sigma+\mathrm{i}s)=\mathscr{F}\left[h(t)f(t)\mathrm{e}^{-\sigma t}\right],\quad \mathscr{F}\text{: Fourier变换.}$$

$$h(t)f(t)\mathrm{e}^{-\sigma t}=\mathscr{F}^{-1}\left[F(\sigma+\mathrm{i}s)\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\sigma+\mathrm{i}s)\mathrm{e}^{\mathrm{i}ts}\,\mathrm{d}s$$

$$\overset{p=\sigma+\mathrm{i}s}{=}\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}F(p)\mathrm{e}^{(p-\sigma)t}\,\mathrm{d}p,$$



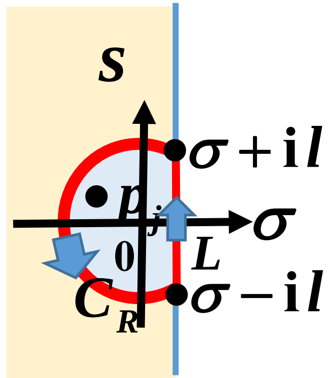
其中积分路径为从$\sigma-\mathrm{i}\infty$到$\sigma+\mathrm{i}\infty$的直线路径.

两边除以$\mathrm{e}^{-\sigma t}$得

$$f(t)h(t)=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}F(p)\mathrm{e}^{pt}\,\mathrm{d}p.\#$$

定理2(P175) 设$F(p)=L[f(t)]$在$\mathrm{Re}\,p\le\sigma$内有奇点$p_1,p_2,\cdots,p_n$，除此这些奇点外，$F(p)$在p平面处处解析，设$\lim\limits_{p\to\infty}F(p)\to 0$，

则 $f(t)=\sum\limits_{k=1}^{n}\mathrm{Res}\left[F(p)\mathrm{e}^{pt},p_k\right]$.



证明 由反演公式 $f(t)=\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}F(p)\mathrm{e}^{pt}\,\mathrm{d}p$.

故只需证 $\dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}F(p)\mathrm{e}^{pt}\,\mathrm{d}p=\sum_{k=1}^{n}\mathrm{Res}\left[F(p)\mathrm{e}^{pt},p_k\right]$. (Δ)

取$C_R=\{p\,\big|\,|p|=R,\ \mathrm{Re}\,p\le\sigma\}$，$L=\{p\,|\,\mathrm{Re}\,p=\sigma,-l\le\mathrm{Im}\,p\le l\}$，$C=C_R+L$.

取$R>0$充分大，使所有奇点$p_1,\cdots,p_n$都在C内. 由留数定理得

$$\int_{C_R}F(p)\mathrm{e}^{pt}\,\mathrm{d}p+\int_L F(p)\mathrm{e}^{pt}\,\mathrm{d}p=2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}\left[F(p)\mathrm{e}^{pt},p_k\right].\quad(\Delta\Delta)$$

令 $p=\mathrm{i}z$， $\displaystyle\int_{C_R}F(p)\mathrm{e}^{pt}\,\mathrm{d}p=\mathrm{i}\int_{\tilde{C}_R}F(\mathrm{i}z)\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}z$，

$z=-\mathrm{i}p$， $\mathrm{Im}\,z=-\mathrm{Re}\,p\ge-\sigma$， $\tilde{C}_R=\{z\,\big|\,|z|=R,\ \mathrm{Im}\,z\ge-\sigma\}$.

$\lim\limits_{p\to\infty}F(p)\to 0$，故由若当引理$(\lambda=t)$得 $\lim\limits_{R\to+\infty}\displaystyle\int_{C_R}F(p)\mathrm{e}^{pt}\,\mathrm{d}p=0$.

故在$(\Delta\Delta)$中令$R\to+\infty$后两边除以$2\pi\mathrm{i}$，得(Δ). #

定理2(P175)　设$F(p)=L[f(t)]$在$\mathrm{Re}\,p\le\sigma$内有奇点$p_1,p_2,\cdots,p_n$，除此这些奇点外，$F(p)$在p平面处处解析，设$\lim\limits_{p\to\infty}F(p)\to 0$，

则 $f(t)=\sum\limits_{k=1}^{n}\mathrm{Res}\left[F(p)\mathrm{e}^{pt},p_k\right]$.

- 若$F(p)$是有理式，且分母比分子次数高1次或以上，则 $\lim\limits_{p\to\infty}F(p)\to 0$.
- 若$F(p)$是有理式，且分母分子中系数都是实数，

如果复数 p_* 是$F(p)$的一个奇点，则 $\overline{p}_*$也是$F(p)$的奇点，且

$$\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\overline{p}_*\right]=\overline{\mathrm{Res}\left[F(p)\mathrm{e}^{pt},p_*\right]}.$$

例 求$F(p)=\dfrac{2p^2-4p}{(2p+1)(p^2+1)}$的本函数.

解 $F(p)$分母比分子次数高1次，故 $\lim\limits_{p\to\infty}F(p)\to 0$. $F(p)$有3个奇点$-\frac{1}{2},\mathrm{i},-\mathrm{i}$.

$$L^{-1}[F(p)]=\mathrm{Res}\left[F(p)\mathrm{e}^{pt},-\tfrac{1}{2}\right]+\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\mathrm{i}\right]+\boxed{\mathrm{Res}\left[F(p)\mathrm{e}^{pt},-\mathrm{i}\right]}$$

$$=\mathrm{Res}\left[F(p)\mathrm{e}^{pt},-\tfrac{1}{2}\right]+2\mathrm{Re}\left(\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\mathrm{i}\right]\right).\qquad =\overline{\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\mathrm{i}\right]}$$

例 求$F(p)=\dfrac{2p^2-4p}{(2p+1)(p^2+1)}$的本函数.

解 $F(p)$分母比分子次数高1次, 故 $\lim\limits_{p\to\infty}F(p)\to 0$. $F(p)$有3个奇点$-\frac{1}{2},\mathrm{i},-\mathrm{i}$.

$$L^{-1}[F(p)]=\mathrm{Res}\left[F(p)\mathrm{e}^{pt},-\tfrac{1}{2}\right]+\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\mathrm{i}\right]+\boxed{\mathrm{Res}\left[F(p)\mathrm{e}^{pt},-\mathrm{i}\right]}$$

$$=\mathrm{Res}\left[F(p)\mathrm{e}^{pt},-\tfrac{1}{2}\right]+2\,\mathrm{Re}\left(\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\mathrm{i}\right]\right).$$

$$=\overline{\mathrm{Res}\left[F(p)\mathrm{e}^{pt},\ \mathrm{i}\right]}$$

当$p=-\frac{1}{2},\mathrm{i}$时, 分子$2p^2-4p\neq 0$, $-\frac{1}{2},\mathrm{i}$都是1级极点.

$$L^{-1}[F(p)]=\left.\frac{(2p^2-4p)\mathrm{e}^{pt}}{\frac{\mathrm{d}}{\mathrm{d}p}\{(2p+1)(p^2+1)\}}\right|_{p=-\frac{1}{2}}+2\,\mathrm{Re}\left(\left.\frac{(2p^2-4p)\mathrm{e}^{pt}}{\frac{\mathrm{d}}{\mathrm{d}p}\{(2p+1)(p^2+1)\}}\right|_{p=\mathrm{i}}\right)$$

$$=\left.\frac{(2p^2-4p)\mathrm{e}^{pt}}{6p^2+2p+2}\right|_{p=-\frac{1}{2}}+2\,\mathrm{Re}\left(\left.\frac{(2p^2-4p)\mathrm{e}^{pt}}{6p^2+2p+2}\right|_{p=\mathrm{i}}\right)$$

$$=\mathrm{e}^{-\frac{1}{2}t}+2\,\mathrm{Re}\left(\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\right)=\mathrm{e}^{-\frac{1}{2}t}+2\,\mathrm{Re}\left(\mathrm{i}\cos t-\sin t\right)=\mathrm{e}^{-\frac{1}{2}t}-2\sin t.$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left\{(2p+1)(p^2+1)\right\}=\frac{\mathrm{d}}{\mathrm{d}p}(2p^3+p^2+2p+1)=6p^2+2p+2.$$

例 P189习题7(8)解方程组 $\begin{cases} y' + x' = 4y + 1, \\ y' + x = 3y + t^2, \\ y(0) = a, \quad x(0) = b. \end{cases}$

解 设 $X(p) = L[x(t)], \quad Y(p) = L[y(t)],$

则 $L[x'(t)] = pX - x(+0) = pX - b,$

$L[y'(t)] = pY - y(+0) = pY - a.$

故对方程组作拉氏变换得

$$\begin{cases} (pY - a) + (pX - b) = 4Y + L[1] = 4Y + \frac{1}{p}, \\ (pY - a) + X = 3Y + L[t^2] = 3Y + \frac{2!}{p^3}, \end{cases}$$

即 $\begin{cases} (p-4)Y + pX = \frac{1}{p} + a + b, & (1) \\ (p-3)Y + X = \frac{2}{p^3} + a. & (2) \end{cases}$

消去 X，得

$$Y = \frac{ap^3-(a+b)p^2-p+2}{(p-2)^2 p^2}.$$

设 $\dfrac{ap^3-(a+b)p^2-p+2}{(p-2)^2 p^2} = \dfrac{A}{p-2} + \dfrac{B}{(p-2)^2} + \dfrac{C}{p} + \dfrac{D}{p^2},$

$$Y=\frac{ap^3-(a+b)p^2-p+2}{(p-2)^2p^2}=\frac{A}{p-2}+\frac{B}{(p-2)^2}+\frac{C}{p}+\frac{D}{p^2},$$

$A=a-\frac{1}{4},\quad B=a-b,\quad C=\frac{1}{4},\quad D=\frac{1}{2}.$

$$y(t)=L^{-1}[Y]=AL^{-1}\left[\frac{1}{p-2}\right]+BL^{-1}\left[\frac{1}{(p-2)^2}\right]+CL^{-1}\left[\frac{1}{p}\right]+DL^{-1}\left[\frac{1}{p^2}\right]$$

$$=\left(A\mathrm{e}^{2t}+B\mathrm{e}^{2t}L^{-1}\left[\frac{1}{p^2}\right]+C+Dt\right)h(t)=\left(A\mathrm{e}^{2t}+B\mathrm{e}^{2t}t+C+Dt\right)h(t)$$

$$=\left\{\left(a-\frac{1}{4}\right)\mathrm{e}^{2t}+(a-b)t\,\mathrm{e}^{2t}+\frac{1}{4}+\frac{1}{2}t\right\}h(t).$$

由第二个方程得 $x(t)=-y'+3y+t^2=\cdots$

P189习题7(8) 解方程组 $\begin{cases} y'+x'=4y+1, \\ \underline{y'+x=3y+t^2}, \\ y(0)=a,x(0)=b. \end{cases}$

例 P188习题6(20) 求$F(p)=\dfrac{p}{(p^2+1)(1-e^{-\pi p})}$的本函数.

解 题目默认：$\text{Re}\,p>0$, 故$\left|e^{-\pi p}\right|=e^{-\pi\,\text{Re}\,p}<1$,

故$\dfrac{1}{1-e^{-\pi p}}=\sum_{n=0}^{+\infty}(e^{-\pi p})^n$,

$$F(p)=\frac{p}{p^2+1}\sum_{n=0}^{+\infty}e^{-n\pi p}=\sum_{n=0}^{+\infty}\frac{p}{p^2+1}e^{-n\pi p}.$$

故$L^{-1}[F(p)]=\sum_{n=0}^{+\infty}L^{-1}\left[\dfrac{p}{p^2+1}e^{-n\pi p}\right]=\sum_{n=0}^{+\infty}h(t-n\pi)\cos(t-n\pi)$.

(延迟定理)

例 求$F(p)=\dfrac{p+7}{(p-1)(p^2+2p+5)}$的本函数.

解 设$\dfrac{p+7}{(p-1)(p^2+2p+5)}=\dfrac{A}{p-1}+\dfrac{Bp+C}{p^2+2p+5}$，右边通分后再比较两边分子得

$p+7=A\left(p^2+2p+5\right)+(p-1)(Bp+C)$. 两边取$p=1$可得 $A=1$.

两边取$p=0$可得 $7=5A-C$,　　$C=5A-7=-2$.

比较两边p^2系数得 $A+B=0$, 故$B=-A=-1$. 故$F(p)=\dfrac{1}{p-1}-\dfrac{p+2}{p^2+2p+5}$.

故$L^{-1}[F(p)]=L^{-1}\left[\dfrac{1}{p-1}\right]-L^{-1}\left[\dfrac{p+2}{p^2+2p+5}\right]$

$$=e^t-L^{-1}\left[\frac{(p+1)+1}{(p+1)^2+4}\right]=e^t-e^{-t}L^{-1}\left[\frac{p+1}{p^2+4}\right]$$

$$=e^t-e^{-t}\left(L^{-1}\left[\frac{p}{p^2+2^2}\right]+L^{-1}\left[\frac{1}{p^2+2^2}\right]\right)=e^t-e^{-t}\left(\cos 2t+\frac{1}{2}\sin 2t\right).$$

$2^2-4\cdot1\cdot5<0$,　p^2+2p+5在实数域不可再分解.

例P188习题6(2) 求$F(p)=\dfrac{1-p}{p^3+p^2+p+1}$的本函数.

解 $p^3+p^2+p+1=p^2(p+1)+(p+1)=(p+1)(p^2+1).$

设$\dfrac{1-p}{p^3+p^2+p+1}=\dfrac{A}{p+1}+\dfrac{Bp+C}{p^2+1}$， 右边通分后, 比较两边分子得

$1-p=A(p^2+1)+(Bp+C)(p+1).$ 两边取$p=-1$得 $2=2A$, $A=1.$

两边取$p=0$得$1=A+C$, $C=1-A=0.$

两边比较p^2系数得$0=A+B$, $B=-A=-1.$

故$F(p)=\dfrac{1}{p+1}-\dfrac{p}{p^2+1}.$

$$L^{-1}[F(p)]=L^{-1}\left[\frac{1}{p+1}\right]-L^{-1}\left[\frac{p}{p^2+1}\right]=\mathrm{e}^{-t}-\cos t.$$

作业

P187　　1 (16), (18)

P188　　6 (16), (18)

7 (3)(5)(9)(10)

例. 求$L^{-1}\left[\dfrac{1}{\left(p^2+a^2\right)^2}\right]$.

解 因$\dfrac{\mathrm{d}}{\mathrm{d}p}\left(\dfrac{1}{p^2+a^2}\right)=-\dfrac{2p}{\left(p^2+a^2\right)^2}$,

故 $\dfrac{1}{\left(p^2+a^2\right)^2}=-\dfrac{1}{2p}\dfrac{\mathrm{d}}{\mathrm{d}p}\left(\dfrac{1}{p^2+a^2}\right)=-\dfrac{1}{2ap}\dfrac{\mathrm{d}}{\mathrm{d}p}\left(\dfrac{a}{p^2+a^2}\right)$.

故$L^{-1}\left[\dfrac{1}{\left(p^2+a^2\right)^2}\right]=\dfrac{1}{2a}\displaystyle\int_0^t L^{-1}\left[-\dfrac{\mathrm{d}}{\mathrm{d}p}\left(\dfrac{a}{p^2+a^2}\right)\right]\mathrm{d}t=\dfrac{1}{2a}\int_0^t tL^{-1}\left[\dfrac{a}{p^2+a^2}\right]\mathrm{d}t$

$$=\frac{1}{2a}\int_0^t t\sin at\,\mathrm{d}t=-\frac{1}{2a^2}\int_0^t t\,\mathrm{d}\cos at$$

$$=-\frac{1}{2a^2}\left(t\cos at-\int_0^t\cos at\,\mathrm{d}t\right)=-\frac{1}{2a^2}\left(t\cos at-\frac{1}{a}\sin at\right).$$

$$L\left[h(t-\tau)f(t-\tau)\right] = e^{-p\tau}L\left[f(t)\right]. \qquad L^{-1}\left[\frac{1}{p}\right] = h(t)$$

$$\Rightarrow L^{-1}\left[e^{-p\tau}L\left[f(t)\right]\right] = \left[h(t-\tau)f(t-\tau)\right].$$

例 求(1)$L^{-1}\left[\frac{1}{p}\left(e^{-5p}-1\right)\right]$; (2)$L^{-1}\left[\frac{e^{-8p}(p+1)}{p^2+2}\right]$.

解 (1)$L^{-1}\left[\frac{1}{p}\left(e^{-5p}-1\right)\right] = L^{-1}\left[\frac{e^{-5p}}{p}\right] - L^{-1}\left[\frac{1}{p}\right] = h(t-5) - h(t).$

(2)因$L^{-1}\left[\frac{p+1}{p^2+2}\right] = L^{-1}\left[\frac{p}{p^2+2}\right] + L^{-1}\left[\frac{1}{p^2+2}\right] = h(t)\left\{\cos\left(\sqrt{2}\right)t + \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\right)t\right\},$

故 $L^{-1}\left[\frac{e^{-8p}(p+1)}{p^2+2}\right] = h(t-8)\left\{\cos\left(\sqrt{2}\right)(t-8) + \frac{(\sqrt{2})}{2}\sin\left(\sqrt{2}\right)(t-8)\right\}.$

求L^{-1}时，根据7.1的规定，$h(t)$ 可以略去，
但是$\tau \neq 0$时，$h(t-\tau)$不可略去.

P188习题6(17),(18),(19),(20)可参考此例.