Numerical Analysis

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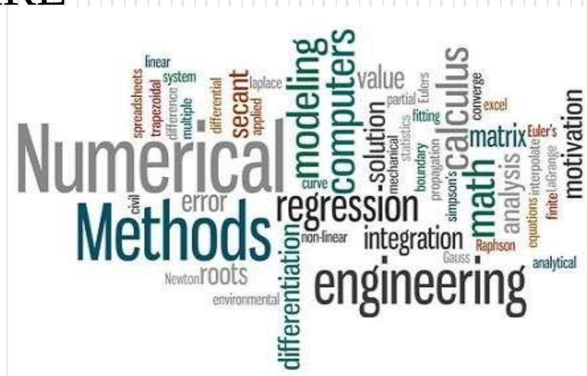

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Course Syllabus

- **Introduction:** Numerical analysis and background, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis and order of convergence (Newton’s method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, and PA=LU-factorization,…..
- **Polynomial:** Polynomial interpolation, piecewise linear interpolation, cubic spline interpolation, and curve fitting in interpolation (Application: Regression)…...
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration…….
- **Ordinary Differential Equations:** Euler's Method, and Runge-Kutta methods…..

❖ System of equations

- System of linear equations
- General form of linear equations
- Vector equation and matrix equation

❖ Gaussian elimination

❖ Gauss Jordan elimination method

Direct methods for solving linear systems

- In Root findings, we studied methods for solving a single equation in a single variable.

- In direct methods for solving linear systems, we consider the problem of solving serval simultaneous equations in several variables.

- A set of equations is called a **System of equations**

- The **solution** must satify each equation in the system.

- A **linear equation in n unknowns** has the form of $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$

- If all equations in a system are linear, the system is a linear system (or a system of linear equations).

Solving a linear system

There are several algorithms for solving a system of linear equations.

A system of linear equations can have:

1. Exactly one solution
2. No solutions
3. Infinitely many solutions

Elementary example

The simplest kind of linear system involves two equations and two variables:

1

$$2x + 3y = 6$$
$$4x + 9y = 15$$

One method for solving such a system is as follows. First, solve the top equation for x in terms of y:

$$X = 3 - \frac{3}{2}y.$$

Direct methods for solving linear systems

Now substitute this expression for x into the bottom equation:

$$4\left(3-\frac{3}{2}y\right)+9y=15$$

This results in a single equation involving only the variable y. Solving gives y = 1, and substituting this back into the equation for x yields x = 3/2.

2

$$2x+y=5$$
$$3x-y=5$$

How we can find the value of x and y, if we have two variables then we need two equations to Solve it.

$$2x+\cancel{y}=5$$
$$3x\cancel{-y}=5$$

$$5x=10$$

$$x=2$$

Put in equation

$$2(2)+y=5$$

$$y=1$$

(2,1)

3

$$5x+4y=22$$
$$7x+6y=32$$

$$3(5x+4y=22)$$
$$-2(7x+6y=32)$$

$$(15x+12y=66)$$
$$(-14x-12y=-64)$$

$$x=2$$

Put x=2 in equation

$$5(2)+4y=22$$

$$10+4y=22$$

$$4y=12$$

$$y=3$$

(2,3)

General form

A general system of m linear equations with n unknowns can be written as

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n &= b_2 \\ \vdots \qquad \quad \vdots \qquad \qquad \quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Here $x_1, x_2, \dots \dots, x_n$ are the unknowns, $a_{11}, a_{12}, \dots ., a_{mn}$ are the coefficients of the system, and $b_1, b_2, \dots . b_m$ are the constant terms.

Often the coefficients and unknowns are real or complex numbers, but integers and rational numbers are also seen.

Vector equation

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix equation

The vector equation is equivalent to a matrix equation of the form

$$Ax = b$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is a column vector with n entries, and $\mathbf{b}$ is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Gaussian elimination

In **row reduction**, the linear system is represented as an augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right].$$

This matrix is then modified using elementary row operations until it reaches reduced row echelon form. There are three types of elementary row operations:

Type 1: Swap the positions of two rows.
Type 2: Multiply a row by a nonzero scalar.
Type 3: Add to one row a scalar multiple of another.

Because these operations are reversible, the augmented matrix produced always represents a linear system that is equivalent to the original.

There are several specific algorithms to row-reduce an augmented matrix, the simplest of which are Gaussian elimination and Gauss-Jordan method.

Gaussian elimination

Working Rule

1. Consider the system of equation

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Ax=B

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

2. Find Augmented matrix for given system.

$$C = [A{:}\,B]$$

3. Transform augmented matrix 'C' into upper triangular form/ echelon form

$$\left[\begin{array}{ccc:c} 1 & {b_1}' & {C_1}' & {d_1}' \\ 0 & 1 & {C_2}' & {d_2}' \\ 0 & 0 & 1 & {d_3}' \end{array}\right]$$

4. Find equations corresponding to upper triangular matrix/ echelon matrix

5. Using back substitution, find solution of given system of equations

Gaussian elimination

Q. Solve the system of equation by Gauss elimination method.

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2$$

2nd and 3rd equation Co-officiant is 1

Solution: The given system of equation can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$$

Now the augmented matrix for given system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 9 \end{array}\right]$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$$

Gaussian elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & -3 & 1 & -3 \end{array}\right]$$

$$R_3 \rightarrow 2R_3 - 3R_2$$

$$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

This is upper triangular form of matrix

$$x + y + z = 6$$

$$-2y = -4 \Rightarrow y = 2$$

$$x + 2 + 3 = 6$$

$$2Z = 6 \Rightarrow Z = 6/2 = 3$$

$$x + 5 = 6$$

$$x = 1$$

Gaussian elimination

Ex.1 Solve the following equations by Gauss Elimination or Backward Substitution.

$2y + z = -8$

$x - 2y - 3z = 0$

$-x + y + 2z = 3$

Solution:

Augmented Matrix

$$\begin{bmatrix} 0 & 2 & 1 & | & 8 \\ 1 & -2 & -3 & | & 0 \\ -1 & 1 & 2 & | & 3 \end{bmatrix}$$

Interchange R_1& R_2

$$\begin{bmatrix} 1 & -2 & -3 & | & 0 \\ 0 & 2 & 1 & | & -8 \\ -1 & 1 & 2 & | & 3 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & -3 & | & 0 \\ 0 & 2 & 1 & | & -8 \\ 0 & -1 & -1 & | & 3 \end{bmatrix}$$

Interchange R_2& R_3

$$\begin{bmatrix} 1 & -2 & -3 & | & 0 \\ 0 & -1 & -1 & | & 3 \\ 0 & 2 & 1 & | & -8 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & -2 & -3 & | & 0 \\ 0 & -1 & -1 & | & 3 \\ 0 & 0 & -1 & | & -2 \end{bmatrix}$$

Gaussian elimination

Therefore,

$$x - 2y - 3z = 0 \ldots (1)$$
$$-y - z = 3 \ldots (2)$$
$$-z = -2 \Rightarrow \boldsymbol{z = 2 \ldots (3)}$$

By using (3) in (2)

$$-y - 2 = 3 \Rightarrow -y = 5 \Rightarrow \boldsymbol{y = -5 \ldots (4)}$$

By using (3), (4) in (1)

$$x - 2(-5) - 3(2) = 0$$
$$\Rightarrow x + 10 - 6 = 0$$
$$\Rightarrow \boldsymbol{x = -4 \ldots (5)}$$

Thus the solution of given equations

$$\begin{aligned} 2y + z &= -8 \\ x - 2y - 3z &= 0 \\ -x + y + 2z &= 3 \end{aligned}$$

are

$$\boldsymbol{x = -4,}$$
$$\boldsymbol{y = -5}$$
$$\&$$
$$\boldsymbol{z = 2}$$

Gaussian elimination

Ex.1 Solve the following equations by Gauss Elimination or Backward Substitution.

$$\begin{aligned} y + 7z + w &= 5 \\ -4x + 4y + z - 2w &= 2 \\ x - y + 3z + w &= 1 \\ 2x + y + \omega &= 3 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ -4 & 4 & 1 & -2 & 2 \\ 2 & 1 & 0 & 1 & 3 \end{array}\right]$$

$R_3 \to \mathbb{R}_3 + 4R_1, R_4 \to R_4 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ 0 & 0 & 13 & 2 & 6 \\ 0 & 3 & -6 & -1 & 1 \end{array}\right]$$

$R_4 \to R_4 - 3R_2$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ 0 & 0 & 13 & 2 & 6 \\ 0 & 0 & -27 & -4 & 14 \end{array}\right]$$

$R_4 \to R_4 + 2R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ 0 & 0 & 13 & 2 & 6 \\ 0 & 0 & -1 & 0 & -2 \end{array}\right]$$

$R_4 \to -R_4, R_3 \Leftrightarrow R_4$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 13 & 2 & 6 \end{array}\right]$$

Gaussian elimination

$$R_4 \to R_4 - 13R_3$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & -20 \end{array}\right]$$

$$R_4 \to \frac{1}{2} R_4$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 1 \\ 0 & 1 & 7 & 1 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -10 \end{array}\right]$$

$Z = 2$

$W = -10$

End of Gaussian Elimination

backward substitution

$y + 7z + w = 5 \longrightarrow$ Eq 1

Put the value of z and w in eq 1

$$y + 7(2) + (-10) = 5$$

$$y + 14 - 10 = 5 \longrightarrow y$$

Put the value of y, z and w in eq 2

$x - \ y + \ 3z \ + w = 1 \longrightarrow$ Eq 2

$x = 6$

Gauss Jordan method

Advance method of gaussian elimination

Working Rule

1. Consider the system of equation

$$Ax=B \longrightarrow \text{Eq (1)}$$

2. Find Augmented matrix for given system.

$$C = [A:B]$$

3. Transform augmented matrix 'C' into normal form

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & d_1' \\ 0 & 1 & 0 & d_2' \\ 0 & 0 & 1 & d_3' \end{array}\right]$$

To convert diagonal elements of matrix into unity elements

4. find solution of given system of equations

Gauss Jordan method

Solve the following equations by Gauss-Jordan method

$$6x - y + z = 13$$
$$x + y + z = 9$$
$$10x + y - z = 19$$

Solution: The augmented matrix for a given equation system is

$$C = [A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 6 & -1 & 1 & 13 \\ 10 & 1 & -1 & 19 \end{array}\right]$$

$$R_2 \to R_2 - 6R_1, \qquad R_3 \to R_3 - 10R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -7 & -5 & -41 \\ 0 & -9 & -11 & -71 \end{array}\right]$$

$$R_3 \to R_3 - R_2, \qquad R_3 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -7 & -5 & -41 \\ 0 & -2 & -6 & -30 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -2 & -6 & -30 \\ 0 & -7 & -5 & -41 \end{array}\right]$$

Gauss Jordan method

$$R_2 \to R_2/(-2)$$

$$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 & 15 \\ 0 & -7 & -5 & -41 \end{array}\right]$$

$$R_3 \to R_3 + 7R_2$$

$$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 & 15 \\ 0 & 0 & 16 & 64 \end{array}\right]$$

$$R_3 \to R_3/16$$

$$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 & 15 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$R_2 \to R_2 - 3R_3, \quad R_1 \to R_1 - R_3$$

$$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$R_1 \to R_1 - R_2$$

$$C \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

X=2

y=3

z=4

Gauss Jordan method

Ex.1 Solve the following equations by Gauss Jordan Method.

$$2y + z = -8$$
$$x - 2y - 3z = 0$$
$$-x + y + 2z = 3$$

Solution:

Augmented Matrix

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

Interchange R_1& R_2

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

$R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right]$$

Interchange R_2& R_3

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

Gauss Jordan method

$R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -6 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$R_2 \to R_2 - R_3, R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

Multiply R_2 & R_3 both by (-1)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, $\boldsymbol{x = -4, y = -5}$ & $\boldsymbol{z = 2}$

Thus the solution of given equations

$$2y + z = -8$$
$$x - 2y - 3z = 0$$
$$-x + y + 2z = 3$$

are

$$\boldsymbol{x = -4,}$$
$$\boldsymbol{y = -5}$$
$$\&$$
$$\boldsymbol{z = 2}$$

Gauss Jordan method

Solve the following equations by Gauss-Jordan method

$$\begin{aligned} x + 2y - 3z - t &= 0 \\ -3y + 2z + 6t &= -8 \\ -3x - y + 3z + t &= 0 \\ 2x + 3y + 2z - t &= -8 \end{aligned}$$

Solution: The augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & -3 & 2 & 6 & -8 \\ -3 & -1 & 3 & 1 & 0 \\ 2 & 3 & 2 & -1 & -8 \end{array}\right)$$

We perform elemental operations in the rows to obtain the reduced row echelon form

$R_3 \to R_3 + 3R_1 \qquad R_4 \to R_4 - 2R_1$

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & -3 & 2 & 6 & -8 \\ 0 & 5 & -6 & -2 & 0 \\ 0 & -1 & 8 & 1 & -8 \end{array}\right)$$

Gauss Jordan method

$R_1 \to R_1 + 2R_4, \quad R_2 \to R_2 - 3R_4, \quad R_3 \to R_3 + 5R_4$

$$\left(\begin{array}{cccc|c} 1 & 0 & 13 & 1 & -16 \\ 0 & 0 & -22 & 3 & 16 \\ 0 & 0 & 34 & 3 & -40 \\ 0 & -1 & 8 & 1 & -8 \end{array}\right)$$

$R_4 \to -1R_4 \qquad R_4 \leftrightarrow R_2$

$$\left(\begin{array}{cccc|c} 1 & 0 & 13 & 1 & -16 \\ 0 & 1 & -8 & -1 & 8 \\ 0 & 0 & 34 & 3 & -40 \\ 0 & 0 & -22 & 3 & 16 \end{array}\right)$$

$R_3 \to 1/34R_3 \qquad R_4 \to -1/22R_4$

$$\left(\begin{array}{cccc|c} 1 & 0 & 13 & 1 & -16 \\ 0 & 1 & -8 & -1 & 8 \\ 0 & 0 & 1 & 3/34 & -40/34 \\ 0 & 0 & 1 & -3/22 & -8/11 \end{array}\right)$$

$R_4 \to R_4 - R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & 13 & 1 & -16 \\ 0 & 1 & -8 & -1 & 8 \\ 0 & 0 & 1 & 3/34 & -40/34 \\ 0 & 0 & 0 & -42/187 & 84/187 \end{array}\right)$$

Gauss Jordan method

$$R_4 \to -\frac{187}{42} R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 13 & 1 & -16 \\ 0 & 1 & -8 & -1 & 8 \\ 0 & 0 & 1 & 3/34 & -40/34 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_1 \to R_1 + (-13)R_3 \qquad R_2 \to R_2 + 8R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -5/34 & -12/17 \\ 0 & 1 & 0 & -5/17 & -24/17 \\ 0 & 0 & 1 & 3/34 & -40/34 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$R_1 \to R_1 + \frac{5}{34} R_4, \quad R_2 \to R_2 + \frac{5}{17} R_4, \quad R_3 \to R_3 + \frac{-3}{34} R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

Therefore, the solution is

X= -1
Y= -2
Z= -1
T= -2

Lu decomposition method

- LU Decomposition is another method to solve a set of simultaneous linear equations

- In linear algebra, the **LU decomposition** is a matrix decomposition which writes a matrix as the product of a lower triangular matrix and an upper triangular matrix. The product sometimes includes a permutation matrix as well. This decomposition is used in numerical analysis to solve systems of linear equations or calculate the determinate.

- Let A be a $n \times n$ square matrix. The LU decomposition is the technique of factoring a matrix A as a product of Lower triangular matrix (L) and upper triangular matrix (U). That is, $A=LU$ where L and U have same dimension of A.

$$[A] = [L][U]$$

where
$[L]$ = lower triangular matrix
$[U]$ = upper triangular matrix

Lu decomposition method

Working Rule

1. Consider the system of equation

$AX=B$…………………(1)

Where

$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

means that this is three variables of equation, we have to do substitution.

Let $A=LU$…………(2)

Where

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Direct methods for solving linear systems

Lu decomposition method

Working Rule

By (1) & (2), we get,

LUX=B……………….(3)

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Put ***Ux=y*** ,Where y= $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ …………(4)

Then (3) Equation Become ***Ly=B***…………(5)

Solving (5) From y,Put the value of y into (4) & solve It for x.

Forward substitution:

Given **[L]** and **[B]** find **[Y]**

$$[L][Y] = [B] \quad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Backward substitution

Given **[U]** and **[Y]** find **[X]**

$$[U][X] = [Y] \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Lu decomposition method

Solve the following system of equations using LU Decomposition method

$$3x + y + z = 4$$
$$\mathrm{x} + 2\mathrm{y} + 2\mathrm{z} = 3$$
$$2\mathrm{x} + \mathrm{y} + 3\mathrm{z} = 4$$

Solution In matrix form, the given system of equation can be written as,

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Which is of the form AX=B $\longrightarrow$ Eq (1)

Let A=LU $\longrightarrow$ Eq (2)

Which implies

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Lu decomposition method

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}l_{21} & U_{12}l_{21}+U_{22} & U_{13}l_{21}+U_{23} \\ U_{11}l_{31} & U_{12}l_{31}+U_{22}l_{32} & U_{13}l_{31}+U_{23}l_{32}+U_{33} \end{bmatrix}$$

We should do comparison (column wise)

$$U_{11} = 3$$

$$U_{11}l_{21} = 1 \Rightarrow 3l_{21} = 1 \Rightarrow l_{21} = {}^{1}/_{3}$$

$$U_{11}l_{31} = 2 \Rightarrow 3l_{31} = 2 \Rightarrow l_{31} = {}^{2}/_{3}$$

$$U_{12} = 1$$

$$U_{12}l_{21} + U_{22} = 2 \Rightarrow 1\left({}^{1}/_{3}\right) + U_{22} = 2 \Rightarrow U_{22} = 2 - {}^{1}/_{3} = {}^{5}/_{3}$$

$$U_{22} = {}^{5}/_{3}$$

$$U_{12}l_{31} + U_{22}l_{32} = 1 \Rightarrow 1\left({}^{2}/_{3}\right) + \left({}^{5}/_{3}\right)l_{32} = 1 \Rightarrow {}^{5}/_{3}\, l_{32} = 1 - \frac{2}{3}$$

$$l_{32} \Rightarrow \left({}^{1}/_{3}\right){}^{3}/_{5} = \frac{1}{s} \Rightarrow l_{32} = {}^{1}/_{5}$$

Lu decomposition method

$u_{13} = 1$

$$U_{13}l_{21} + U_{23} = 2 \Rightarrow 1\left({}^{1}/_{3}\right) + U_{23} = 2 \Rightarrow U_{23} = 2 - {}^{1}/_{3} = {}^{5}/_{3}$$

$U_{23} = {}^{5}/_{3}$

$$U_{13}l_{31} + U_{23}l_{32} + U_{33} = 3 \Rightarrow 1\left({}^{2}/_{3}\right) + \left({}^{5}/_{3}\right)\left({}^{1}/_{5}\right) + U_{33} = 3$$

$$\frac{2}{3} + {}^{1}/_{3} + U_{33} = 3 \Rightarrow U_{33} = 3 - \frac{2}{3} - {}^{1}/_{3} \Rightarrow \frac{9-2-1}{3} = {}^{6}/_{3} = 2$$

$U_{33} = 2$

Therefore, we get

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{5} & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & 0 & 2 \end{bmatrix}$$

Lu decomposition method

By equation 1 and 2

LUX=B $\longrightarrow$ Eq (3)

Now,Let ***Ux=y,***then ***Ly=b*** implies

Let UX=Y $\longrightarrow$ Eq (4) $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then eq (3) become

LY=B $\longrightarrow$ Eq (5)

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$y_1 = 4$$

$$\frac{y_1}{3} + y_2 = 3 \Rightarrow {}^4/_3 + y_2 = 3 \Rightarrow y_2 = 3 - {}^4/_3 \Rightarrow y_2 = \frac{9-4}{3} \Rightarrow y_2 = {}^5/_3$$

$$\frac{2y_1}{3} + {}^{y_2}/_5 + y_3 = 4 \Rightarrow {}^{2(4)}/_3 + {}^{\left(\frac{5}{3}\right)}/_5 + y_3 = 4 \Rightarrow {}^8/_3 + \frac{1}{3} + y_3 = 4$$

$$y_3 = 4 - {}^8/_3 - {}^1/_3 \Rightarrow \frac{12-8-1}{3} \Rightarrow y_3 = \frac{3}{3} \Rightarrow y_3 = 1$$

Lu decomposition method

By equation 4

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{5}{3} & \frac{5}{3} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

By back substitution, we get

$we\ have\ to\ start\ from\ R_3 to\ multiply\ with\ X\ column$

$$2z = 1 \Rightarrow z = {}^{1}/_{2}$$

$${}^{5}/_{3}\,y + {}^{5}/_{3}\,z = {}^{5}/_{3} \Rightarrow {}^{5}/_{3}\,(y) + {}^{5}/_{3}\,({}^{1}/_{2}) = \frac{5}{3} \Rightarrow {}^{5}/_{3}\,(y) + {}^{5}/_{6} = {}^{5}/_{3}$$

$$\Rightarrow y = \left(\frac{5}{3} - {}^{5}/_{6}\right)\frac{3}{5} \Rightarrow y = \left(\frac{10-5}{6}\right){}^{3}/_{5} = {}^{1}/_{2} \qquad y = {}^{1}/_{2}$$

$$3x + y + z = 4 \Rightarrow 3x + \frac{1}{2} + {}^{1}/_{2} \Rightarrow 3x = 4 - \frac{1}{2} - {}^{1}/_{2} \Rightarrow 3x = \frac{8-1-1}{2} = {}^{6}/_{2}$$

$$\Rightarrow 3x = 3 \Rightarrow x = {}^{3}/_{3} \Rightarrow x = 1$$

Hence the solution of equation is

$$x = 1, \qquad y = {}^{1}/_{2}, \qquad z = {}^{1}/_{2}$$

Numerical analysis

Thank You !