Numerical Analysis

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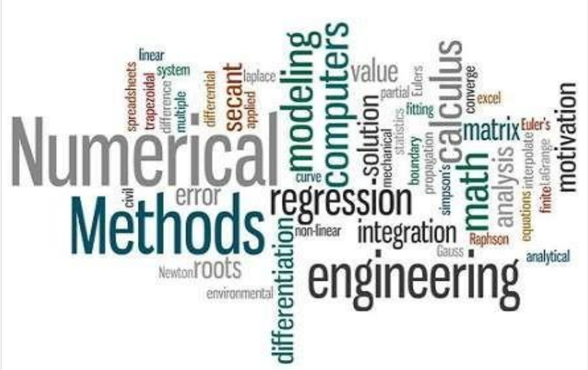
Week 5, Lecture-I-II



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Course Syllabus

- **Introduction:** Numerical analysis and backround, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.

- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis and order of convergence (Newton’s method and Secant method).

- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, LU-factorization, forward substitution, and crout factorization.

- **Polynomial:** Polynomial interpolation, piecewise linear interpolation, divided differences interpolation, cubic spline interpolation, and curve fitting in interpolation (Application: Regression).

- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration.

- **Ordinary Differential Equations:** Euler's Method, higher-order Taylor method, and Runge-Kutta methods.

False position method (Regula falsi method)

Probelm: $Solve\ the\ equation\ xe^{x} = \cos x\ \ using\ regula\ falsi\ method\ upto\ 4 - decimal\ places$

Let $F(x) = xe^{x} - \cos x = 0 \longrightarrow$ *eq (1)*

To find a and b

$f(0) = -1 \longrightarrow$ <0

$F(1) = 1e^{1} - \cos 1 = 2.177979 \longrightarrow$ >0

$F(0.5) = 0.5e^{0.5} - \cos 0.5 = -0.053221 \longrightarrow$ <0

$F(0.6) = 0.6e^{0.6} - \cos 0.6 = 0.267935 \longrightarrow$ >0

a= 0.5 b= 0.6

To find the first approximate root

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \qquad x_1 = \frac{0.5(0.267935) - 0.6(-0.053221)}{0.267935 - (-0.053221)} \qquad x_1 = 0.516571$$

False position method (Regula falsi method)

put x_1 in equation 1

$$F(x_1) = x_1 e^{x_1} - \cos x_1$$

$$F(0.0516571) = (0.0516571)e^{0.0516571} - \cos 0.0516571 = -0.003605 < 0$$

Root lies between b and x_1

$if\ f(x_1) < 0\ then\ \mathrm{a} = x_1$

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$

$$x_2 = 0.517678$$

$$F(x_2) = x_2 e^{x_2} - \cos x_2 = -0.000241 < 0$$

$if\ f(x_2) < 0\ then\ x_1 = x_2$

Root lies between b and x_2

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)} \qquad x_3 = 0.517751$$

False position method (Regula falsi method)

$$F(x_3) = x_3 e^{x_3} - \cos x_3 = -0.000019 < 0$$

$$if\ f(x_3) < 0\ then\ x_2 = x_3$$

Root lies between b and x_3

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$

$$x_4 = 0.517756$$

Hence the approximate root of the equation correct upto 4-decimal places is x=0.5177

- Order of convergence (Rate of Convergence)

What is rate of convergence

- To identify that how quickly we reached from 1st approximate to 2nd and from 3rd to 4rth.

- How is our procedure to reach from 1st approximate root to 2nd.

Let $\propto$ be the root and x_n be the nth approximation to the root. Define the error as

$$E_n = \propto - x_n$$

If for large *n* we have the approximate relationship

$$|E_{n+1}| = A|E_n|^K$$

with *A* a positive constant, then we say the root-finding numerical method is of order *K*. Larger values of *K* correspond to faster convergence to the root.

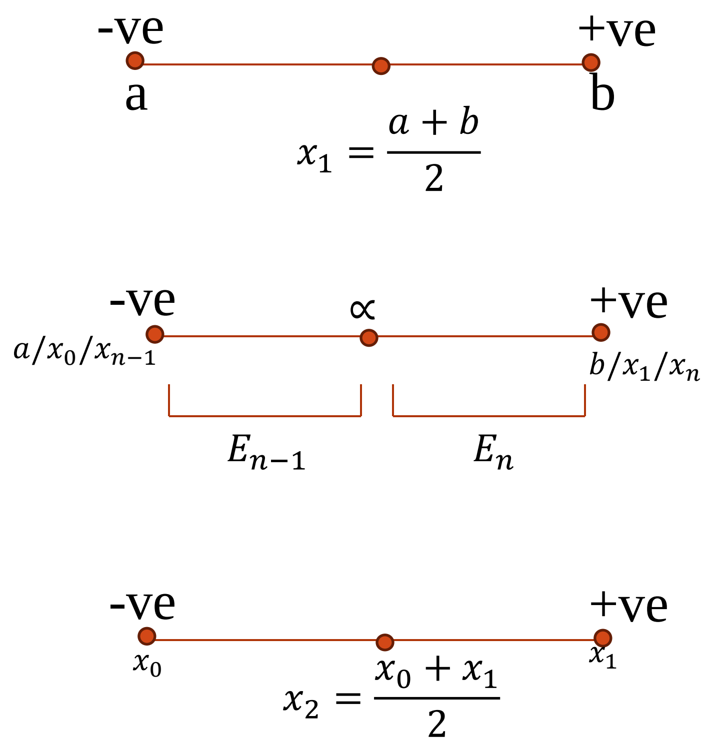
Order of convergence of bisection method

Order of convergence of bisection method

Let f(0)=0 is given equation and $x = \propto$ be the exact root of equation

$$x_{n-1} = E_{n-1} + \propto$$

$$x_n = E_n + \propto$$



The general formula of Bisection Method is

$$x_{n+1} = \frac{x_{n-1} + x_n}{2} \quad \text{——— eq (1)}$$

Put the value of x_n and x_{n-1}

$$E_{n+1} + \propto = \frac{(E_{n-1} + \propto) + (E_n + \propto)}{2}$$

$$E_{n+1} + \propto = \frac{E_{n-1} + E_n + 2 \propto}{2}$$

Order of convergence of bisection method

$$E_{n+1} + \propto = \frac{E_{n-1} + E_n + 2 \propto}{2}$$

$$E_{n+1} + \propto = \frac{E_{n-1} + E_n}{2} + \propto$$

$$E_{n+1} = \frac{E_{n-1} + E_n}{2}$$

$$E_{n+1} = \frac{E_n}{2}\left[1 + \frac{E_{n-1}}{E_n}\right]$$

This term will be very small, so it will be neglected

$$E_{n+1} = \frac{E_n}{2}$$

$$E_{n+1} = \frac{1}{2}(E_n)$$

$$E_{n+1} = A|E_n| \qquad \therefore \frac{1}{2} \; is \; constant \; (\frac{1}{2} = A)$$

This means that rate of convergence of Bisection Method is k=1 or linear

Order of convergence of Newton method

Order of convergence of newton method

Let f(0)=0 is given equation and $x = \propto$ be the exact root of equation

$$E_{n+1} = x_{n+1} - \propto$$

$$x_{n+1} = E_{n+1} + \propto$$

$$E_n = x_n - \propto$$

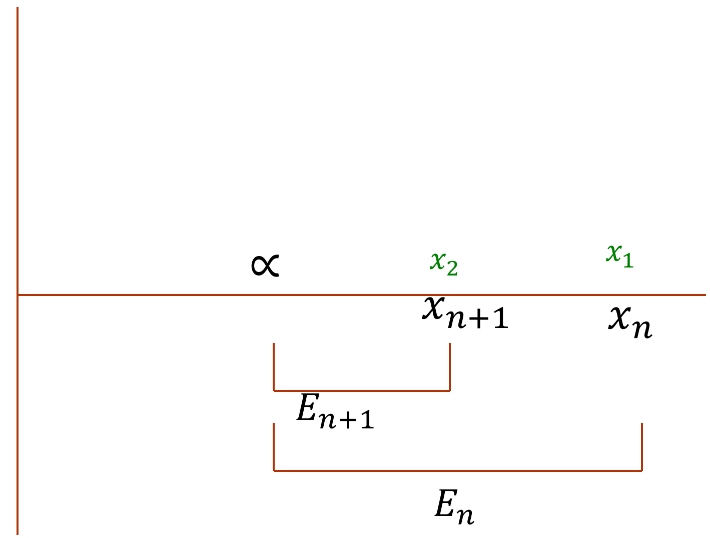
$$x_n = E_n + \propto$$

The general formula of Newton Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{eq (1)}$$

Put the value of x_n and x_{n+1}

$$E_{n+1} + \propto = E_n + \propto - \left[\frac{f(E_n + \propto)}{f'(E_n + \propto)}\right]$$



What is the propostionality of E_n and E_{n+1}

Order of convergence of newton method

$$E_{n+1} = E_n - \frac{f(E_n + \propto)}{f'(E_n + \propto)}$$

Apply Tayler Theorm

$$f(\propto + x) = f(\propto) + \frac{xf'(\propto)}{1!} + \frac{x^2 f''(\propto)}{2!} + \frac{x^3 f'''(\propto)}{3!} + \cdots, -\infty < x < \infty$$

$$E_{n+1} = E_n - \frac{f(\propto) + \frac{E_n f'(\propto)}{1!} + \frac{{E_n}^2 f''(\propto)}{2!} + \frac{{E_n}^3 f'''(\propto)}{3!} + \cdots, -\infty < x < \infty}{f'(\propto) + \frac{E_n f''(\propto)}{1!} + \frac{{E_n}^2 f'''(\propto)}{2!} + \frac{{E_n}^3 f''''(\propto)}{3!} + \cdots, -\infty < x < \infty}$$

$$E_{n+1} = E_n - \frac{E_n f'(\propto)}{f'(\propto) + E_n f''(\propto)}$$

$Since\ E_n\ power\ is\ very\ small\ quantities, so\ {E_n}^2, {E_n}^3 \ldots .\ neglect\ them$

$$f(\propto) = 0$$

$$E_{n+1} = \frac{E_n(f'(\propto) + E_n f''(\propto)) - E_n f'(\propto)}{f'(\propto) + E_n f''(\propto)}$$

Order of convergence of newton method

$$E_{n+1} = \frac{E_n f'(\propto) + {E_n}^2 f''(\propto) - E_n f'(\propto)}{f'(\propto) + E_n f''(\propto)}$$

$$E_{n+1} = \frac{{E_n}^2 f''(\propto)}{f'(\propto) + E_n f''(\propto)}$$

$$E_{n+1} = \frac{{E_n}^2 f''(\propto)}{f'(\propto)\left[1 + \frac{E_n f''(\propto)}{f'(\propto)}\right]}$$

$$E_{n+1} = \frac{{E_n}^2 f''(\propto)}{f'(\propto)}\left[1 + \frac{E_n f''(\propto)}{f'(\propto)}\right]^{-1}$$

$$E_{n+1} = \frac{{E_n}^2 f''(\propto)}{f'(\propto)}\left[1 - \frac{E_n f''(\propto)}{f'(\propto)} + {E_n}^2\left(\frac{f''(\propto)}{f'(\propto)}\right)^2 + \ldots\right]$$

Apply binomial Theorm

very small quantities, so neglect them

$$E_{n+1} = \frac{{E_n}^2 f''(\propto)}{f'(\propto)}$$

Order of convergence of newton method

$$E_{n+1} \approx E_n^{\,2}\left(\frac{f''(\propto)}{f'(\propto)}\right)$$

$$E_{n+1} \approx E_n^{\,2}(A)$$

Constant equal to A

$$where\ A = \frac{f''(\propto)}{f'(\propto)} \rightarrow Constant$$

$$E_{n+1} \approx AE_n^{\,2}$$

$$E_{n+1} \propto E_n^{\,2}$$

$Error \propto is\ proportional\ to\ 2$

This means that the rate of Convergence of Newton Raphson method is 2

or quadratic convergence

Order of convergence of iteration method

Order of convergence of Fixed point iteration method

Let f(0)=0 is given equation and $x = \propto$ be the exact root of equation

Let x_n be a differ from $\propto$ by a small quantity E_n and x_{n+1} by E_{n+1}

$$E_n = x_n - \propto$$

$$x_n = E_n + \propto$$

$$E_{n+1} = x_{n+1} - \propto$$

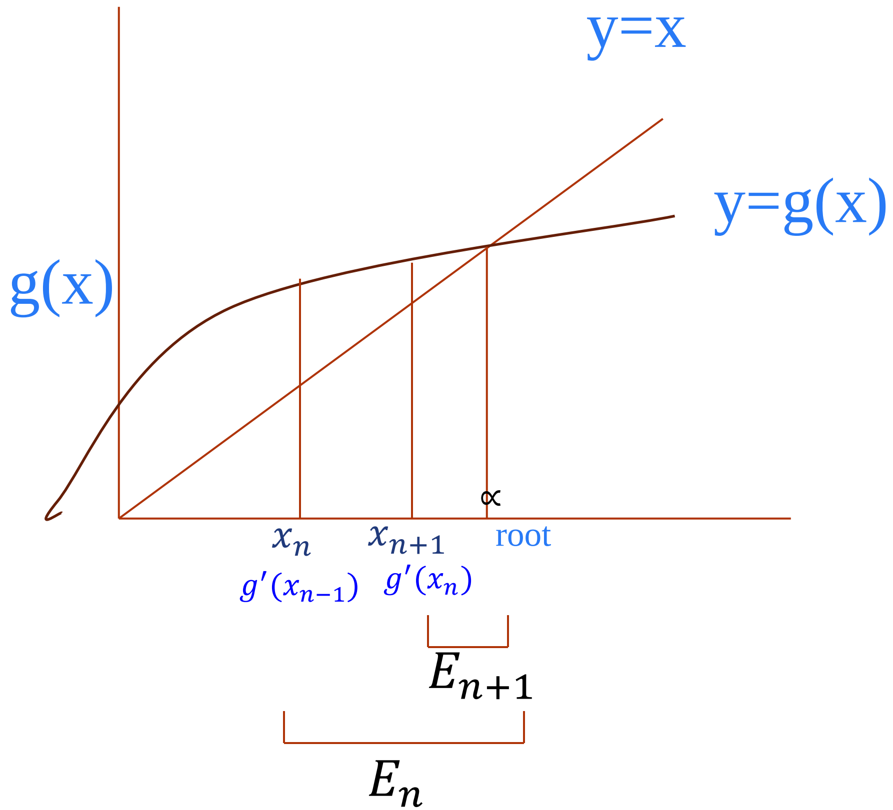
$$x_{n+1} = E_{n+1} + \propto$$

The general formula of iteration Method is

$$x_{n+1} = g(x_n)$$

$$E_{n+1} \propto E_n^{K}$$

k=1, 2, 3, ...



Order of convergence of ragula falsi method

Order of convergence of Ragula- falsi method

Let f(0)=0 is given equation and $x = \propto$ be the exact root of equation

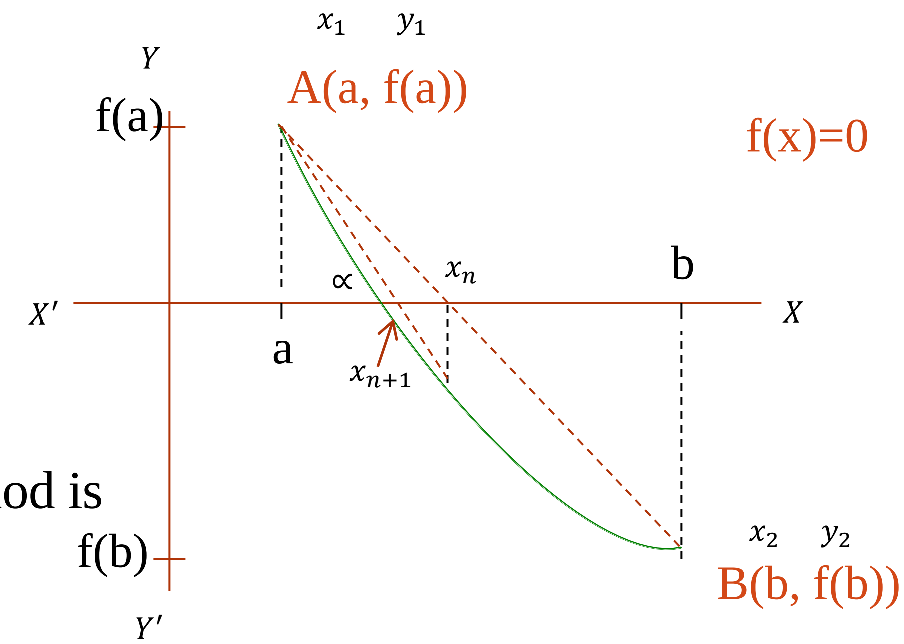
Let x_n be a differ from $\propto$ by a small quantity E_n and x_{n+1} by E_{n+1}

$$E_n = x_n - \propto$$

$$x_n = E_n + \propto$$

$$E_{n+1} = x_{n+1} - \propto$$

$$x_{n+1} = E_{n+1} + \propto$$



The general formula of Regula Falsi Method is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)} \qquad \text{eq (1)}$$

$$x_{n+2} = E_{n+2} + \propto$$

Order of convergence of Ragula- falsi method

$$E_{n+2}+\propto=\frac{(E_n+\propto)f(E_{n+1}+\propto)-(E_{n+1}+\propto)f(E_n+\propto)}{f(E_{n+1}+\propto)-f(E_n+\propto)}$$

$$E_{n+2}+\propto=\frac{E_nf(E_{n+1}+\propto)+\propto f(E_{n+1}+\propto)-E_{n+1}f(E_n+\propto)-\propto f(E_n+\propto)}{f(E_{n+1}+\propto)-f(E_n+\propto)}$$

$$E_{n+2}+\propto=\frac{E_nf(E_{n+1}+\propto)-E_{n+1}f(E_n+\propto)+\propto f(E_{n+1}+\propto)-\propto f(E_n+\propto)}{f(E_{n+1}+\propto)-f(E_n+\propto)}$$

$$E_{n+2}+\propto=\frac{E_nf(E_{n+1}+\propto)-E_{n+1}f(E_n+\propto)+\propto (f(E_{n+1}+\propto)-f(E_n+\propto)}{f(E_{n+1}+\propto)-f(E_n+\propto)}$$

$$E_{n+2}+\propto=\frac{E_nf(E_{n+1}+\propto)-E_{n+1}f(E_n+\propto)}{f(E_{n+1}+\propto)-f(E_n+\propto)}+\propto$$

Order of convergence of Ragula- falsi method

$$E_{n+2} = \frac{E_n f(E_{n+1} + \propto) - E_{n+1} f(E_n + \propto)}{f(E_{n+1} + \propto) - f(E_n + \propto)}$$

Expend this by Taylor theoram

$$E_{n+2} = \frac{E_n\left[f(\propto) + \frac{E_{n+1} f'(\propto)}{1!} + \frac{{E_{n+1}}^2 f''(\propto)}{2!} + \frac{{E_{n+1}}^3 f'''(\propto)}{3!} + \cdots\right] - E_{n+1}\left[f(\propto) + \frac{E_n f'(\propto)}{1!} + \frac{{E_n}^2 f''(\propto)}{2!} + \frac{{E_n}^3 f'''(\propto)}{3!} + \cdots\right]}{\left[f(\propto) + \frac{E_{n+1} f'(\propto)}{1!} + \frac{{E_{n+1}}^2 f''(\propto)}{2!} + \frac{{E_{n+1}}^3 f'''(\propto)}{3!} + \cdots\right] - \left[f(\propto) + \frac{E_n f'(\propto)}{1!} + \frac{{E_n}^2 f''(\propto)}{2!} + \frac{{E_n}^3 f'''(\propto)}{3!} + \cdots\right]}$$

$f(\propto) = 0, and\ neglect\ higher\ power\ {E_n}^3, {E_n}^4 \ldots.$

$$E_{n+2} = \frac{E_n\left[\frac{E_{n+1} f'(\propto)}{1!} + \frac{{E_{n+1}}^2 f''(\propto)}{2!}\right] - E_{n+1}\left[\frac{E_n f'(\propto)}{1!} + \frac{{E_n}^2 f''(\propto)}{2!}\right]}{\left[\frac{E_{n+1} f'(\propto)}{1!} + \frac{{E_{n+1}}^2 f''(\propto)}{2!}\right] - \left[\frac{E_n f'(\propto)}{1!} + \frac{{E_n}^2 f''(\propto)}{2!}\right]}$$

Order of convergence of Ragula- falsi method

$$E_{n+2} = \frac{E_n E_{n+1} f'(\propto) + \frac{E_n {E_{n+1}}^2 f''(\propto)}{2!} - E_{n+1} E_n f'(\propto) - \frac{E_{n+1} {E_n}^2 f''(\propto)}{2!}}{E_{n+1} f'(\propto) + \frac{{E_{n+1}}^2 f''(\propto)}{2!} - E_n f'(\propto) - \frac{{E_n}^2 f''(\propto)}{2!}}$$

$$E_{n+2} = \frac{\frac{E_n {E_{n+1}}^2 f''(\propto)}{2!} - \frac{E_{n+1} {E_n}^2 f''(\propto)}{2!}}{E_{n+1} f'(\propto) + \frac{{E_{n+1}}^2 f''(\propto)}{2!} - E_n f'(\propto) - \frac{{E_n}^2 f''(\propto)}{2!}}$$

$$E_{n+2} = \frac{\frac{E_n E_{n+1} f''(\propto)}{2!} (E_{n+1} - E_n)}{f'(\propto)(E_{n+1} - E_n) + \frac{f''(\propto)}{2!} ({E_{n+1}}^2 - {E_n}^2)}$$

$$E_{n+2} = \frac{E_n E_{n+1} f''(\propto)(E_{n+1} - E_n)}{2\, f'(\propto)(E_{n+1} - E_n) + \frac{f''(\propto)}{2!} (E_{n+1} - E_n)(E_{n+1} + E_n)}$$

$$E_{n+2} = \frac{E_n E_{n+1} f''(\propto)(E_{n+1} - E_n)}{2\, (E_{n+1} - E_n) \left[f'(\propto) + \frac{f''(\propto)}{2!} (E_{n+1} + E_n) \right]}$$

Order of convergence of Ragula- falsi method

$$E_{n+2} = \frac{E_n E_{n+1} f''(\propto)(E_{n+1} - E_n)}{2\,(E_{n+1} - E_n)\left[f'(\propto) + \frac{f''(\propto)}{2!}(E_{n+1} + E_n)\right]}$$

$$E_{n+2} = \frac{E_n E_{n+1} f''(\propto)}{2\left[f'(\propto) + \frac{f''(\propto)}{2!}(E_{n+1} + E_n)\right]}$$

$$E_{n+2} = \frac{E_n E_{n+1}}{2}\frac{f''(\propto)}{f'(\propto)}\frac{1}{\left[1 + \frac{f''(\propto)}{2f'(\propto)}(E_{n+1} + E_n)\right]}$$

$$E_{n+2} = \frac{E_n E_{n+1}}{2}\frac{f''(\propto)}{f'(\propto)}\left[1 + \frac{f''(\propto)}{2f'(\propto)}(E_{n+1} + E_n)\right]^{-1}$$

Apply binomial theorm

$$E_{n+2} = \frac{E_n E_{n+1}}{2}\frac{f''(\propto)}{f'(\propto)}\left[1 - \frac{f''(\propto)}{2f'(\propto)}(E_{n+1} + E_n) + \ldots\right]$$

Very small and will be neglected

Order of convergence of Ragula- falsi method

$$E_{n+2} = \frac{E_n E_{n+1}}{2} \frac{f''(\propto)}{f'(\propto)}$$

$\frac{f''(\propto)}{2f'(\propto)}$ *is constant and declared by M*

$$E_{n+2} = E_n E_{n+1} M$$ eq (2)

we know that the rate of convergence is

$$E_{n+1} = A\, {E_n}^K$$

$$\left(\frac{E_{n+1}}{A}\right)^{\frac{1}{k}} = E_n$$

Therefore

$$E_{n+2} = A\, {E_{n+1}}^K$$

From eq (2)

$$A\, {E_{n+1}}^K = E_n E_{n+1} M$$

$$A\, {E_{n+1}}^K = \left(\frac{E_{n+1}}{A}\right)^{\frac{1}{k}} E_{n+1} M$$

$$A\, {E_{n+1}}^K = A^{-\frac{1}{k}} {E_{n+1}}^{\frac{1}{k}} E_{n+1} M$$

$$A\, {E_{n+1}}^K = A^{-\frac{1}{k}} {E_{n+1}}^{\frac{1}{k}+1} M$$

$$K = \frac{1}{k} + 1$$

$$K = \frac{1+K}{k}$$

$$K.K = 1 + K$$

$$K^2 = 1 + K$$

$$K^2 - 1 - k = 0$$

$$K = \frac{1 \pm \sqrt{1+4}}{2}$$

$$K = \frac{1 \pm \sqrt{5}}{2}$$

$$K = 1.618$$

This means that the rate of Convergence of Ragula falsi method is 1.618

Root Finding

Problem: Find the approximate root $x^3 - 4x - 9$ by using bisection method and ragula falsi method upto four decimal places.

Bisection method

$x_1 = 2.5$
$x_2 = 2.75$
$x_3 = 2.625$
$x_4 = 2.6875$
$x_5 = 2.71875$
$x_6 = 2.7031$
$x_7 = 2.7109$
$x_8 = 2.707$
$x_9 = 2.7051$

$x_{10} = 2.7061$
$x_{11} = 2.7066$
$x_{12} = 2.7064$
$x_{13} = 2.7065$
$x_{14} = 2.7065$

Bisection method identified root in 14th iteration

Ragula Falsi method

$x_1 = 2.6$

$x_2 = 2.75$

$x_3 = 2.7106$

$x_4 = 2.7063$

$x_5 = 2.7065$

ragula Falsi method identified root in 5th iteration

Regula Falsi vs Newton vs Secant

- The Secant method and Newton's method are often used to refine an answer obtained by another technique (such as the Bisection Method). Both methods require good first approximations but generally give rapid acceleration.
- Use the method of False Position to find a solution to x = cos x , and compare the approximations with those given in a previous example which Newton's method and the Secant Method.
- To make a reasonable comparison we will use the same initial approximations as in the Secant method, that is, $x_0 = 0.5$ and $x_1 = \pi/4$.

Regula Falsi vs Newton vs Secant

	False Position	Secant	Newton
n	x_n	x_n	x_n
0	**0.5**	**0.5**	**0.7853981635**
1	**0.7853981635**	**0.7853981635**	**0.7395361337**
2	**0.7363841388**	**0.7363841388**	**0.7390851781**
3	**0.7390581392**	**0.7390581392**	**0.7390851332**
4	**0.7390848638**	**0.7390851493**	**0.7390851332**
5	**0.7390851305**	**0.7390851332**	
6	**0.7390851332**		

Note that the False Position and Secant approximations agree through x_3 and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method.

Regula Falsi vs Newton vs Secant

- The added insurance of the method of False Position commonly requires more calculation than the Secant method, . . .

- just as the simplification that the Secant method provides over Newton’s method usually comes at the expense of additional iterations.

- # Order of convergence (Rate of Convergence)

Bisection Method

- Bisection method is linear.
- Rate of convergence is 1.
- Bisection method is always convergent.

Newton Raphson Method

- It is quadratic.
- Rate of convergence is 2.

Regula-falsi and Secant Method

- It is linear.
- Rate of convergence is 1.618.

Thank You !