Numerical Analysis

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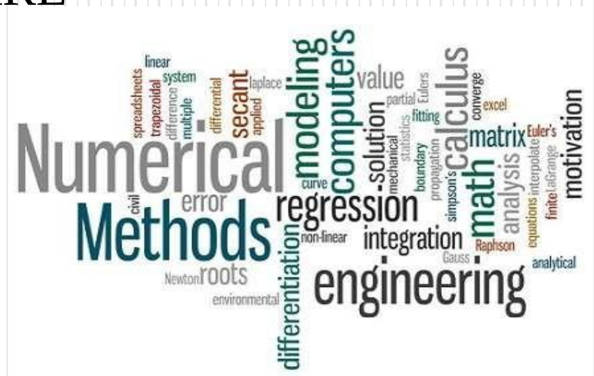
Week 13, Lecture-I-II



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Course Syllabus

- **Introduction:** Numerical analysis and background, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis and order of convergence (Newton's method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, and PA=LU-factorization,…..
- **Polynomial:** Polynomial interpolation, Lagrange interpolation, Piecewise interpolation, divided differences interpolation, and curve fitting in interpolation (Application: Regression)…...
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration…….
- **Ordinary Differential Equations:** Euler's Method, and Runge-Kutta methods…..

- **Numerical Integration**
- **Trapezoid Rule**
- **SIMPSON'S Rule**
- **Mid-Point Rule**

❖**Numerical Integration**

❖**Trapezoid Rule**

❖**SIMPSON'S Rule**

❖**Mid-Point Rule**

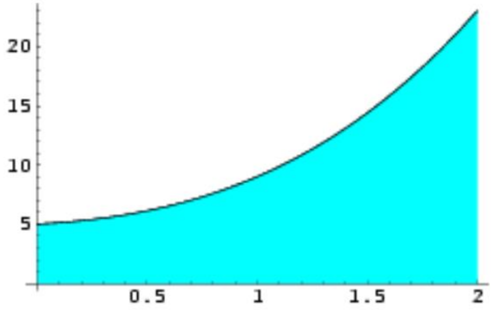
Numerical Integration

❑ Indefinite integral

$$\int f(x)dx = most\ general\ antiderivative\ for\ f(x)$$

❑ Definite integral

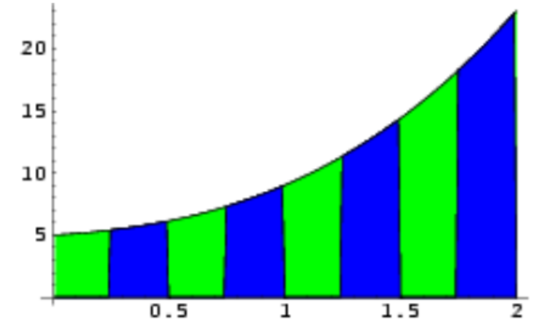
- Antiderivative of f(x) may not exist, or exist but very difficult to find out.
- This is related to summation (it is a limit of sums of a certain kind). The integral sign $\int$ was originally invented as a modified S (for sum).
- $\int_a^b f(x) \quad dx$ (read integral from a to b of the function *f*).
- $\int_a^b f(x) \quad dx$ is the area of the region in the plane bounded by the graph *y* = *f*(*x*), the *x*-axis and the vertical lines *x* = *a*, *x* = *b*.



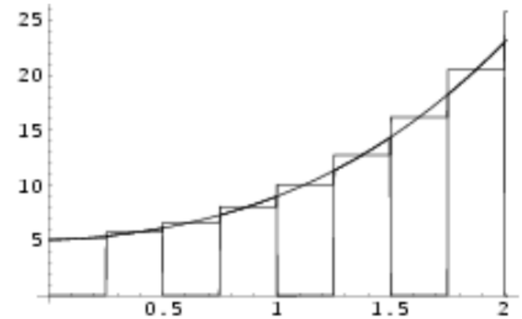
Numerical Integration

❑ Definite integral

- We need a way to compute the area of such a shape.



- Dealing with $\int_a^b f(x) \quad dx$
- n subdivisions of interval $[a, b]$
- Number division point $a = x_0 < x_1 < x_2 < \cdots < x_n = b$
- Heights of rectangles $y_j = f(x_j^*),\ x_{j-1} \le x_j^* \le x_j$



- Area" of jth rectangle = width × height $=(x_j - x_{j-1})y_j$
- Total "area" = sum of these

$$= \quad y_1(x_1 - x_0) + y_2(x_2 - x_1) + \cdots + y_n(x_n - x_{n-1})$$
$$= \quad f(x_1^*)(x_1 - x_0) + f(x_2^*)(x_2 - x_1) + \cdots + f(x_n^*)(x_n - x_{n-1})$$

(called *Riemann sums* for the integral)

- $\int_a^b f(x)dx$ = limit of these Riemann sums as $n \to \infty$ and max width $\to 0$.

Numerical Differentiation and Integration

Numerical Integration

❑ What is numerical integration?

- Numerous definite integrals an engineer may encounter in practice cannot be calculated by hand. Some others can, but the calculations are extremely tedious. In both cases, it seems reasonable to use some numerical technique instead, which delivers an approximate value of the integral.

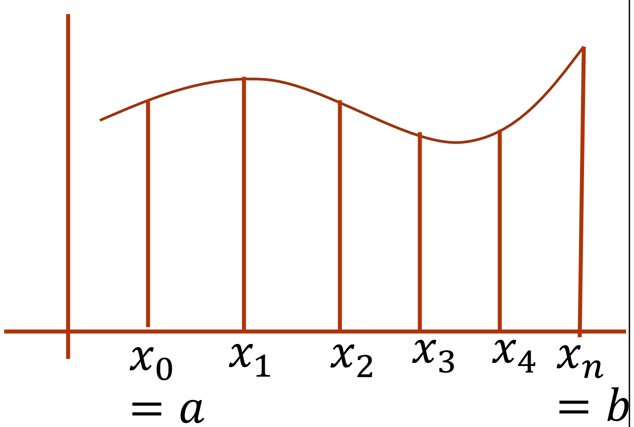
- Sometimes the antiderivative of f(x) may not exist or very difficult to find out.

- The limit makes sense if f (x) is continuous on the finite closed interval [a, b] (including end points).

$$\int_a^b f(x) \quad dx$$

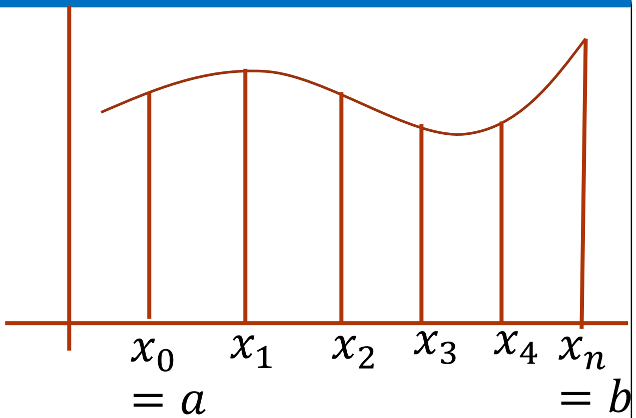
$$\int_a^b f(x)dx = b - a \lim_{n\to\infty} \frac{1}{n}[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$$

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i\, f(x_i)$$



- The coefficients c_i depends on the particular method.

Numerical Integration



- The points where the function is evaluated are called nodes

There are two parts of numerical integration

1. Nodes are evenly spread

$\underbrace{x_0 \quad x_1}_{h},\ \underbrace{x_1 \quad x_2}_{h},\ \dots\dots x_n$

Newton-cotes formulas

- Newton-cotes closed formulas → 1. Trapezoid Rule 2. Simpson rule

  (use the end points of interval a and b)

  $$\int_a^b f(x) \quad dx$$

- Newton-cotes open formulas → 3. Mid-point rule

  (not use the end points of interval a and b)

2. Nodes are chosen

Gaussian Quadrature

Note: The formulas for each of these Newton-cotes methods can be derived by approximating the function to be integrated by its Lagrange interpolating polynomial and then integrating the polynomial exactly.

- **Numerical Integration**

- **Trapezoid Rule**

- **SIMPSON'S Rule**

- **Mid-Point Rule**

Trapezoid Rule

- The word Trapezoid is based on American English and it means the same as Trapezium.

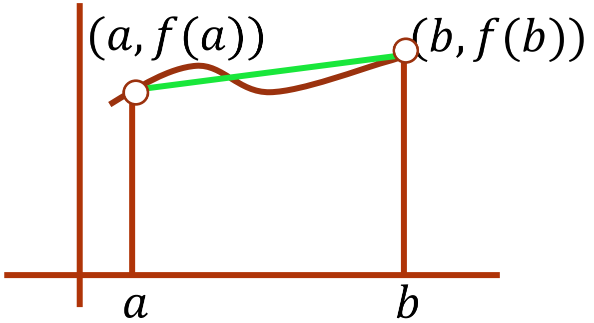
- The trapezoid rule is based on an estimation of the area under a curve using trapezoids. First the interval $[a\,,b]$ is divided into subintervals according to the partition $P = \{x_0 \ < x_1 < x_2 < \cdots < x_n = \}$.

- Suppose we want to find the integral of f(x) or any curve

$$\int_a^b f(x) \quad dx$$

- The trapezoid rule approximates the curve by a straight line passing through the points $(a, f(a)), (b, f(b))$

$x_0 = a \quad x_1 = b \qquad h = b - a$



The trapezoid rule is

$$\int_a^b f(x)\, dx \approx \frac{h}{2}[f(x_0) + f(x_1)]$$

Trapezoid Rule

Example: Apply the trapezoid rule to find the integral of the function $f(x) = e^{-x^2}$

Solution: We have to find

$$\int_0^2 e^{-x^2} dx = ?$$

Here, we have $x_0 = \text{a} = 0, \quad x_1 = b = 2, h = 2 - 0 = 2$

Apply the trapezoid rule

$$\int_0^2 f(x)\, dx \approx \frac{h}{2}[f(x_0) + f(x_1)] \quad = \frac{2}{2}\left[e^{-0^2} + e^{-2^2}\right] \quad = 1[1 + e^{-4}] \quad = 1.0183$$

Homework: Find the approximate value of

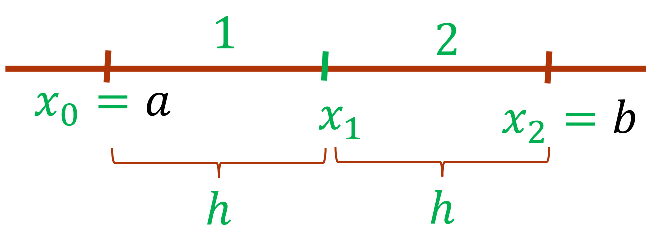
$$I = \int_0^1 \frac{1}{1 + x^2} dx$$

using trapezoid rule

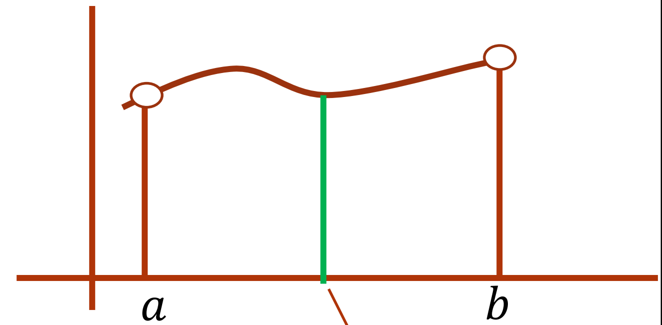
Trapezoid Rule

❑ Composite Integration

- The easiest method of improving the accuracy of numerical integration is to apply one of the lower order methods repeatedly on several subintervals. This is known as composite integration.
- Suppose we insert one point



$$h = x_1 - x_0 = \frac{b-a}{2}$$



$$x_1 = \frac{a+b}{2}$$

The trapezoid rule after one point insertion is

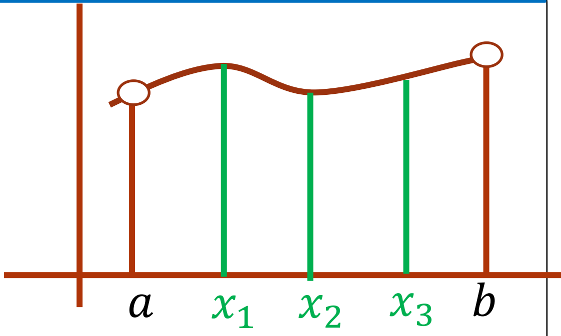
$$\int_a^b f(x) \quad dx = \int_a^{x_1} f(x) \quad dx + \int_{x_1}^b f(x) \quad dx$$

$$\approx \frac{h}{2}[f(a) + f(x_1)] + \frac{h}{2}[f(x_1) + f(b)]$$

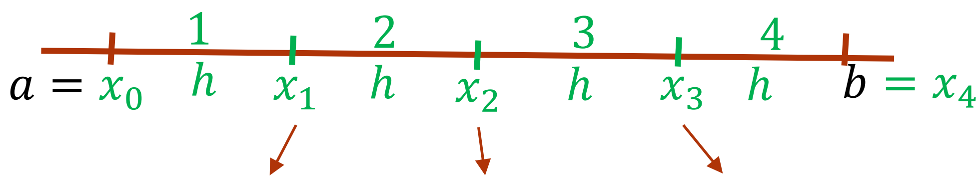
$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2f(x_1) + f(b)]}$$

Trapezoid Rule

❑ Composite Integration



- Suppose we insert three points



$$\frac{3a+b}{4} \qquad \frac{a+b}{2} \qquad \frac{a+3b}{4}$$

$$h = \frac{b-a}{4}$$

The trapezoid rule after three point insertion is

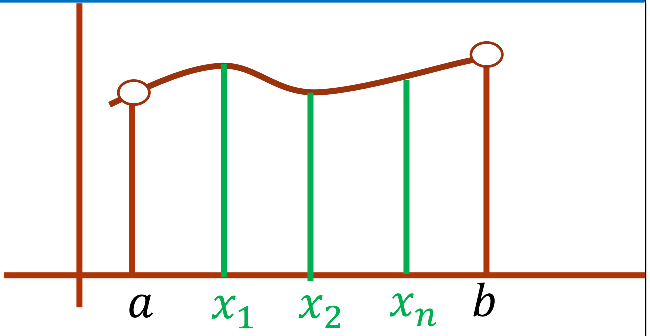
$$\int_a^b f(x) \quad dx = \int_a^{x_1} f(x) \quad dx + \int_{x_1}^{x_2} f(x) \quad dx + \int_{x_2}^{x_3} f(x) \quad dx + \int_{x_3}^{b} f(x) \quad dx$$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + f(x_1)] + \frac{h}{2}[f(x_1) + f(x_2)] + \frac{h}{2}[f(x_2) + f(x_3)] + \frac{h}{2}[f(x_3) + f(b)]$$

Trapezoid Rule

❑ Composite Integration

- Suppose we insert nth points



$$= \frac{a+b}{2}$$

$$h = \frac{b-a}{n}$$

The trapezoid rule after nth point insertion is

$$\boxed{\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)]}$$

Trapezoid Rule

Question: Find $\int_1^2 \frac{1}{x}\,dx$ for $n = 1, 2, and\ 4$ points

Solution: Here we have

$$f(x) = \frac{1}{x}$$

Basic Trapezoidal rule is

$For\ n = 1$ $\quad x_0 = a = 1 \quad x_1 = b = 2 \quad h = 2 - 1 = 1$

$$I_0 = \int_1^2 \frac{1}{x}\,dx \approx \frac{h}{2}[f(x_0) + f(x_1)] = \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{3}{4} = 0.75$$

$For\ n = 2$ $\quad x_0 = 1 \quad x_1 = \frac{1+2}{2} = \frac{3}{2} \quad x_2 = 2 \quad h = \frac{2-1}{2} = \frac{1}{2}$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(b)] = \frac{1}{4}\left[1 + 2\frac{1}{\frac{3}{2}} + \frac{1}{2}\right] = \frac{1}{4}\left[\frac{17}{6}\right] = \frac{17}{24} = 0.7083$$

Trapezoid Rule

$For\ n = 4 \quad x_0 = 1 \quad x_1 = \frac{3 * 1 + 2}{4} = \frac{5}{4} \quad x_2 = \frac{3}{2} \quad x_3 = \frac{1 + 3(2)}{4} = \frac{7}{4} \quad x_4 = 2$

$$\int_1^2 \frac{1}{x}\, dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(b)] \qquad h = \frac{1}{4}$$

$$= \frac{1}{8}\left[1 + 2.\frac{1}{\frac{5}{4}} + 2.\frac{1}{\frac{3}{2}} + 2.\frac{1}{\frac{7}{4}} + \frac{1}{2}\right]$$

$$\approx 0.6970$$

Now what the exact value of the given function

$$\int_1^2 \frac{1}{x}\, dx = \ln|_1^2 = \ln|2| - \ln|1| = \ln|2| \approx 0.693147$$

- **Numerical Integration**

- **Trapezoid Rule**

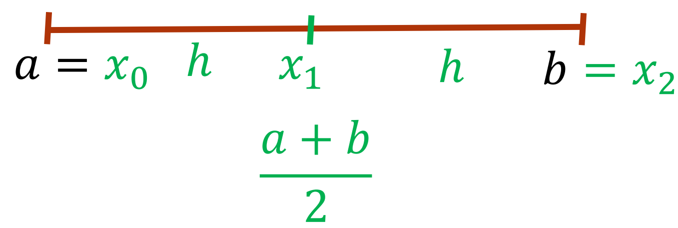
- **SIMPSON’S Rule**

- **Mid-Point Rule**

SIMPSON'S Rule (SIMPSON'S $\frac{1}{3}$ RULE)

- Simpsons Rule is the next most sophisticated method after the trapezoidal rule.
- With the trapezoidal rule we used bits of straight lines.
- Now we use bits of quadratic graphs $y = ax^2 + bx + c$.
- This method is based on approximating the function (to be integrated) by a quadratic polynomial.
- The first problem is that, while 2 points determine a line, we need 3 points to pin down a quadratic graph. Then we also need a formula for the 'area under a quadratic graph'.

$$\int_a^b f(x) \quad dx$$

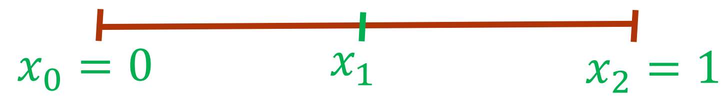


The approximating integral is given by

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[f(a) + 4f(x_1) + f(b)]$$

SIMPSON'S Rule (SIMPSON'S $\frac{1}{3}$ RULE)

Question: Find $\int_0^1 \frac{1}{1+x^2}\, dx$ using Simpson's rule



Solution: The given function is

$$f(x) = \frac{1}{1+x^2}$$

$$\frac{1+0}{2} = \frac{1}{2}$$

$$h = \frac{1-0}{2} = \frac{1}{2}$$

Apply the Simpson's rule

$$\int_a^b \frac{1}{1+x^2}\, dx \approx \frac{h}{3}\left[f(a) + 4f(x_1) + f(b)\right]$$

$$\overset{x_0}{f(0)} = \frac{1}{1+0^2} = 1$$

$$\overset{x_1}{f}\left(\frac{1}{2}\right) = \frac{1}{1+\frac{1}{2}^2} = \frac{4}{5}$$

$$\int_0^1 \frac{1}{1+x^2}\, dx \approx \frac{h}{3.2}\left[1 + 4.\frac{4}{5} + \frac{1}{2}\right] = \frac{1}{6}\left[\frac{47}{10}\right] \approx 0.7833$$

$$\overset{x_2}{f(1)} = \frac{1}{1+1^2} = \frac{1}{2}$$

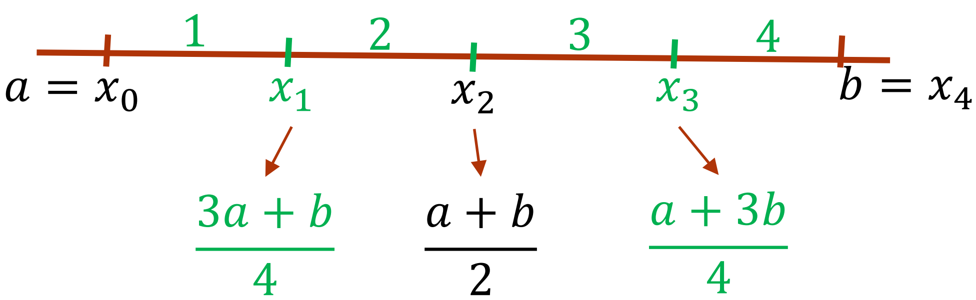
$$tan^{-1}(x)|_0^1 = tan^{-1}(1) = \frac{\pi}{4} \approx 0.785$$

SIMPSON'S Rule (SIMPSON'S $\frac{1}{3}$ RULE)

❑ Composite Integration

- [a, b] is divide into two subintervals, we have

$$h = \frac{b-a}{4}$$



The Simpson rule after points insertion is $\int_a^b f(x) \quad dx = \int_a^{x_2} f(x) \quad dx + \int_{x_2}^b f(x) \quad dx$

$$\approx \frac{h}{3}[f(a) + 4f(x_1) + f(x_2)] + \frac{h}{3}[f(x_2) + 4f(x_3) + f(b)]$$

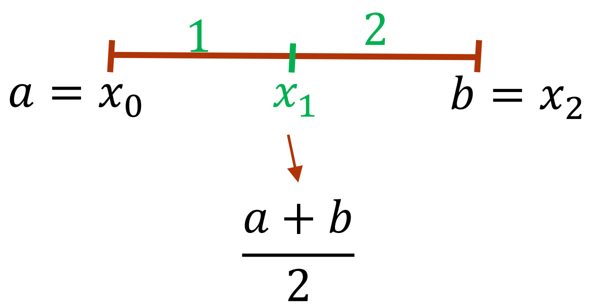
$$\approx \frac{h}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(b)]$$

In general, for even n, we have

$$\boxed{\int_a^b f(x)\, dx \approx \frac{h}{3}[f(a) + f(b) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots]}$$

SIMPSON'S Rule (SIMPSON'S $\frac{1}{3}$ RULE)

Question: Find $\int_0^2 e^{-x^2}\,dx$ using Simpson's rule



Solution: The given function is

$$f(x) = e^{-x^2}$$

For n=2

$$x_0 = 0 \qquad x_1 = \frac{0+2}{2} = 1 \qquad x_2 = b = 2 \qquad h = \frac{2-0}{2} = 1$$

Apply the Simpson's rule

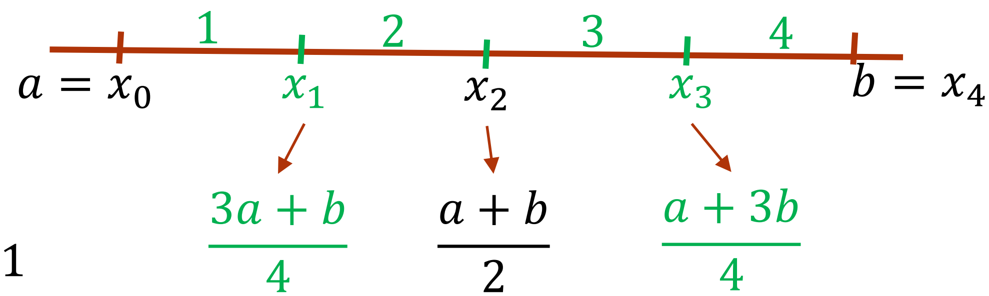
$$\int_0^2 e^{-x^2}\,dx \approx \frac{h}{3}\left[f(a) + 4f(x_1) + f(b)\right]$$

$$= \frac{1}{3}\left[1 + 4e^{-1} + e^{-4}\right] = 0.8299$$

SIMPSON'S Rule (SIMPSON'S $\frac{1}{3}$ RULE)

Question: Find $\int_0^2 e^{-x^2}\,dx$ using Simpson's rule

Solution:



For n=4

$x_0 = 0$ $\quad x_1 = \frac{3.0+2}{4} = \frac{1}{2}$ $\quad x_2 = 1$

$x_3 = \frac{0+6}{4} = \frac{3}{2}$ $\quad x_4 = b = 2$ $\quad h = \frac{2-0}{4} = \frac{1}{2}$

Apply the Simpson's rule

$$\int_0^2 e^{-x^2}dx \approx \frac{h}{3}[f(a) + f(b) + 4f(x_1) + 2f(x_2) + 4f(x_3)]$$

$$\int_0^2 e^{-x^2}dx \approx \frac{1}{6}\left[1 + e^{-4} + 4.e^{-\frac{1}{2}^2} + 2.e^{-1} + 4.e^{-\left(\frac{3}{2}\right)^2}\right]$$

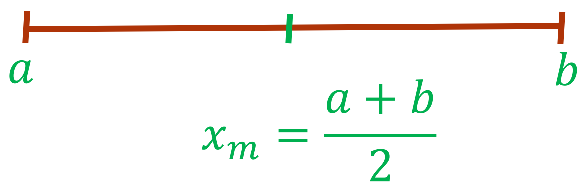
$$= 0.8818$$

- **Numerical Integration**
- **Trapezoid Rule**
- **SIMPSON'S Rule**
- $\boxed{\textbf{Mid-Point Rule}}$

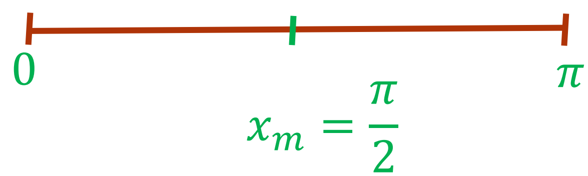
Mid-Point Rule

- In mid-point rule, end points are not used
- This formula uses only one function evaluation (so n=1) at the mid-point of the interval i.e. $x_m = \frac{a+b}{2}$

$$\int_a^b f(x)dx \approx (b-a)f(x_m) \quad \approx (b-a)\left(\frac{a+b}{2}\right)$$



Question: Using the mid-point rule to approximate the integral



$$\int_0^\pi \frac{\sin x}{x} dx$$

Solution: by mid-point rule, we have $a = 0 \quad x_m = \frac{0+\pi}{2} \quad b = \pi$

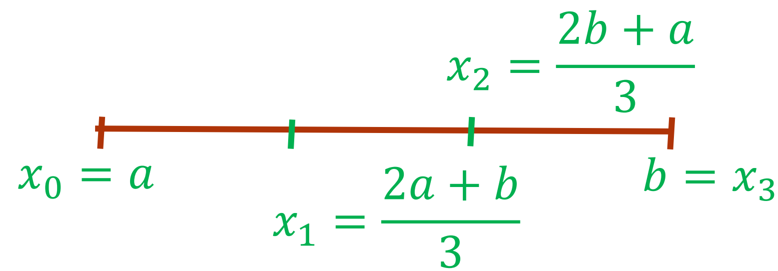
$$\int_0^\pi \frac{\sin x}{x} dx = (\pi - 0)f(x_m) = \pi f(\frac{\pi}{2})$$

$$= \pi \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} \quad = 2\sin\frac{\pi}{2} = 2$$

Exact value=1.851937

Mid-Point Rule

Two point formula

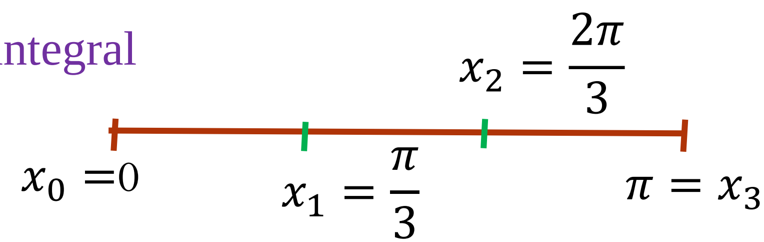


- We have taken two mid-points

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(x_1) + f(x_2)]$$

Question: Using the mid-point rule to approximate the integral

$$\int_0^\pi \frac{\sin x}{x} dx$$



Solution: by mid-point rule, we have

$$\int_0^\pi \frac{\sin x}{x} dx = \frac{\pi - 0}{2}\left[f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right)\right] = \frac{\pi}{2}\left[\frac{\sin\frac{\pi}{3}}{\frac{\pi}{3}} + \left(\frac{\sin\frac{2\pi}{3}}{\frac{2\pi}{3}}\right)\right]$$

$$= \frac{\pi}{2}\left[\frac{3\sqrt{3}}{2.\pi} + \frac{3\sin\pi - \frac{\pi}{3}}{2\pi}\right] = \frac{\pi}{2}.\frac{1}{2\pi}\left[3\sqrt{3} + 3.\frac{\sqrt{3}}{2}\right] = 1.94856$$

Numerical analysis

Thank You !