Numerical Analysis

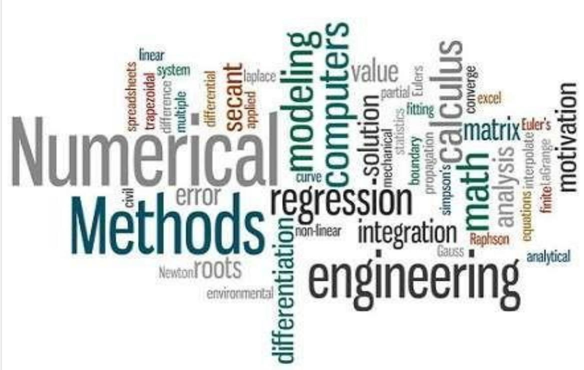
Dr. Farman Ali

Assistant Professor

DEPARTMENT OF SOFTWARE

SEJONG UNIVERSITY

Week 9, Lecture-I-II



farmankanju@sejong.ac.kr

Course Syllabus

- **Introduction:** Numerical analysis and background, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis and order of convergence (Newton’s method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, and PA=LU-factorization,…..
- **Polynomial:** Polynomial interpolation, piecewise linear interpolation, cubic spline interpolation, and curve fitting in interpolation (Application: Regression)…...
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration…….
- **Ordinary Differential Equations:** Euler's Method, and Runge-Kutta methods…..

Direct methods for solving linear systems

- **Jacobi method**

- **Gauss-Seidel Method**

- **Cholesky method**

Iterative Methods for Solving Linear Systems

- Gaussian elimination is a finite sequence of $O(n^3)$floating point operations that result in a solution. For that reason, Gaussian elimination is called a direct method for solving systems of linear equations.

- Direct methods, in theory, give the exact solution within a finite number of steps.

- Direct methods stand in contrast to the root-finding methods, which are iterative in form.

- So-called iterative methods also can be applied to solving systems of linear equations.

- Similar to Fixed-Point Iteration, the methods begin with an initial guess and refine the guess at each step, converging to the solution vector.

Gauss-Jacobi Method

- The Jacobi Method is a form of fixed-point iteration (FPI) for a system of equations.
- In FPI the first step is to rewrite the equations, solving for the unknown.
- The first step of the Jacobi Method is to do this in the following standardized way: Solve the ith equation for the ith unknown.
- Then, iterate as in Fixed-Point Iteration, starting with an initial guess.

Gauss-Jacobi Method

Steps of Jacobi Method

1. The system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

Has a unique solution.

2. The coefficient matrix has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are non zeros.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

If any of the diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

Gauss-Jacobi Method

Steps of Jacobi Method

3. To begin the Jacobi method, solve the first equation for x_1 the second equation for x_2 and so on, as follows.

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$
$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$
$$\vdots$$
$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Iteration 1

Then make an initial guess of the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots\ldots x_n^{(0)})$.
Substitute these values into the right hand side of the rewritten equations to obtain the first approximation, $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots\ldots x_n^{(1)})$. This accomplishes one iteration.

Gauss-Jacobi Method

Steps of Jacobi Method

Iteration 2

In the same way, the second approximation $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots\ldots x_n^{(2)})$ is computed by substituting the first approximation's x-values into the right hand side of the rewritten equations.

Iteration k

By repeated iterations, we form a sequence of approximations $x^k =$
$= (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots\ldots x_n^{(k)})^t$, K= 1, 2, 3…….

Gauss-Jacobi Method

Example

Use the Jacobi method to approximate the solution of the following system of linear equations.

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

To begin, rewrite the system

$$\begin{aligned} x_1 &= \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3 \\ x_2 &= \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3 \\ x_3 &= -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2 \end{aligned}$$

Gauss-Jacobi Method

Example

Because you do not know the actual solution, choose

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0$$ Initial approximation

as a convenient initial approximation. So, the first approximation is

$$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$
$$x_2^{(1)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$
$$x_3^{(1)} = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

Continuing this procedure, you obtain the sequence of approximations shown in Table

Gauss-Jacobi Method

Example

n	0	1	2	3	4	5	6	7
x_1	0.000	-0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0.000	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

Because the last two columns in Table are identical, you can conclude that to three significant digits the solution is

$$x_1 = 0.186, \qquad x_2 = 0.331, \qquad x_3 = -0.423.$$

For the system of linear equations given in Example, the Jacobi method is said to **converge.** That is, repeated iterations succeed in producing an approximation that is correct to three significant digits. As is generally true for iterative methods, greater accuracy would require more iterations.

Gauss-Jacobi Method in matrix form

Consider to solve an size system of linear equations $Ax = b$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ -a_{21} & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = D - L - U$$

$A\boldsymbol{x} = \boldsymbol{b}$ is transformed into $(D - L - U)\boldsymbol{x} = \boldsymbol{b}$

$$D\boldsymbol{x} = (L + U)\boldsymbol{x} + \boldsymbol{b}$$

Gauss-Jacobi Method in matrix form

Assume D^{-1} exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$\boldsymbol{x} = D^{-1}(L + U)\boldsymbol{x} + D^{-1}\boldsymbol{b}$$

The matrix form of Jacobi iterative method is

$$\boldsymbol{x}^{(k)} = D^{-1}(L + U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} \quad k = 1,2,3,\dots$$

Note that the simplicity of this method is both good and bad: good, because it is relatively easy to understand and thus is a good first taste of iterative methods; bad, because it is not typically used in practice. Still, it is a good starting point for learning about more useful, but more complicated, iterative methods.

Gauss-Seidel Method

- The modification of the Jacobi method called the Gauss-Seidel method. This modification is no more difficult to use than the Jacobi method.

- It often requires fewer iterations to produce the same degree of accuracy.

- With the Jacobi method, the values of x_i obtained in the nth approximation remain unchanged until the entire $(n + 1)th$ approximation has been calculated.

- With the Gauss Seidel method, on the other hand, you use the new values of each x_i as soon as they are known.

- That is, once you have determined x_1 from the first equation, its value is then used in the second equation to obtain the new x_2.

- Similarly, the new x_1 and x_2 are used in the third equation to obtain the new x_3 and so on.

Gauss-Seidel Method

Steps of Gauss-Seidel Method

1. The system given by

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n
\end{gathered}
$$

Has a unique solution.

2. The coefficient matrix has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are non zeros.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

If any of the diagonal entries $a_{11}, a_{22}, \dots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

Gauss-Seidel Method

Steps of Gauss-Seidel Method

3. To begin the Gauss seidel method, solve the first equation for x_1 the second equation for x_2 and so on, as follows.

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$
$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$
$$\vdots$$
$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Iteration 1

- Then make an initial guess of the solution $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots\ldots x_n^{(0)})$.
- After getting the value of x_1, put it in equation 2 to get the value of x_2
- After getting the value of x_1 and x_2, Put it in equation 3 to get the value of $x_3$and so on.

Gauss-Seidel Method

Steps of Gauss-Seidel Method

Iteration 2

- In second approximation, $x_1^{(2)}$ is computed by putting $x_2 = x_2^{(1)}$and $x_3 = x_2^{(1)}$ in equation 1.
- $x_2^{(2)}$ is computed by putting $x_1 = x_1^{(2)}$and $x_3 = x_2^{(1)}$ in equation 2.
- $x_3^{(2)}$ is computed by putting $x_1 = x_1^{(2)}$and $x_2 = x_2^{(2)}$ in equation 3.

Iteration k

Repeated the above steps until the required solution is achieved.

Gauss-Seidel Method

Example

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations given

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

To begin, rewrite the system

$$x_1 = \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$
$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

Gauss-Seidel Method

Example

The first computation is identical to that given in Jacobi example. That is, using $(x_1, x_2,\ x_3) = (0,0,0)$ as the initial approximation, you obtain the following new value for x_1.

$$x_1 = -\tfrac{1}{5} + \tfrac{2}{5}(0) - \tfrac{3}{5}(0) = -0.200$$

Now that you have a new value for x_1, however, use it to compute a new value for x_2 That is,

$$x_2 = \tfrac{2}{9} + \tfrac{3}{9}(-0.200) - \tfrac{1}{9}(0) \approx 0.156.$$

Similarly, use $x_1 = -0.200$ and $x_2 = 0.156$ to compute a new value for x_3. That is,

$$x_3 = -\tfrac{3}{7} + \tfrac{2}{7}(-0.200) - \tfrac{1}{7}(0.156) \approx -0.508.$$

So the first approximation is $x_1 = -0.200$, $x_2 = 0.156$, and $x_3 = -0.508$. Continued iterations produce the sequence of approximations shown in Table.

Gauss-Seidel Method

Example

n	0	1	2	3	4	5
x_1	0.000	−0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	−0.508	−0.429	−0.422	−0.423	−0.423

Note that after only five iterations of the Gauss-Seidel method, you achieved the same accuracy as was obtained with seven iterations of the Jacobi method

The iterative methods presented in this section always converges. That is, it is possible to apply the Jacobi method or the Gauss-Seidel method to a system of linear equations and obtain a divergent sequence of approximations. In such cases, it is said that the method **diverges.**

Cholesky Method

- Symmetric matrices hold a favored position in linear systems analysis because of their special structure, and because they have only about half as many independent entries as general matrices.

- It raises the question whether a factorization like the LU can be realized for half the computational complexity, and using only half the memory locations.

- For symmetric positive-definite matrices, this goal can be achieved with the Cholesky factorization.

- The $n \times n$ matrix A is symmetric if $A^T = A$. The matrix A is positive-definite if $x^T Ax > 0$ for all vectors $x \neq 0$.

Cholesky Method

every positive definite matrix

$$A \times = b \longrightarrow \text{Eq 1}$$

can be factored as

$$A = LL^T \longrightarrow \text{Eq 2}$$

where L is lower triangular with positive diagonal elements

$$\mathrm{L} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Algorithm

1. Determine $l_{11}, l_{21}, l_{22}, l_{31}, l_{32}$, and l_{33}

2. By eq 1 and eq 2

$$LL^T \times = B \longrightarrow \text{Eq 3}$$

Put $L^T \times = \mathrm{Y}$ in eq 3

$LY = B$ *Solve it and find y*

$L^T \times = \mathrm{Y}$ *Solve it and find x*

Cholesky Method

Example Solve the system by Cholesky method

$$x + 2y + 3z = 5, \qquad 2x + 8y + 22z = 6, \qquad 3x + 22y + 82z = -10$$

Solve Let the given system is $A \times = b \longrightarrow$ Eq 1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

Let $A = LL^T \longrightarrow$ Eq 2

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} {l_{11}}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & {l_{21}}^2 + {l_{22}}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & {l_{31}}^2 + {l_{32}}^2 + {l_{33}}^2 \end{bmatrix}$$

Cholesky Method

we have to do equating (equating both sides rows wise), then

Equate row 1

$$l_{11}^{\;2} = 1$$
$$\Rightarrow l_{11} = 1$$

$$l_{11}l_{21} = 2$$
$$(1)l_{21} = 2$$
$$l_{21} = 2$$

$$l_{11}l_{31} = 3$$
$$(1)l_{31} = 3$$
$$l_{31} = 3$$

Equate row 2

$$l_{11}l_{21} = 2$$
$$(1)l_{21} = 2$$
$$l_{21} = 2$$

$$l_{21}^{\;2} + l_{22}^{\;2} = 8$$
$$2^2 + l_{22}^{\;2} = 8$$
$$l_{22}^{\;2} = 8 - 4 \Rightarrow 4$$
$$l_{22} = 2$$

$$l_{21}l_{31} + l_{22}l_{32} = 22$$
$$2 * 3 + 2 * l_{32} = 22$$
$$l_{32} = 8$$

Equate row 3

$$l_{11}l_{31} = 3$$
$$(1)l_{31} = 3$$
$$l_{31} = 3$$

$$l_{21}l_{31} + l_{22}l_{32} = 22$$
$$2 * 3 + 2 * l_{32} = 22$$
$$l_{32} = 8$$

$$l_{31}^{\;2} + l_{32}^{\;2} + l_{33}^{\;2} = 82$$
$$3^2 + 8^2 + l_{33}^{\;2} = 82$$
$$9 + 64 + l_{33}^{\;2} = 82$$
$$l_{33} = 3$$

Cholesky Method

The L matrix become

$$\therefore \mathrm{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

By eq 1 and eq 2

$$LL^T \times = B \longrightarrow \text{Eq 3}$$

Put $L^T \times = \mathrm{Y}$ in eq 3, where $\mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \longrightarrow$ Eq 4

Then equation 3 become

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$y_1 = 5$

$2y_1 + 2y_2 = 6$

$y_2 = -2$

$3y_1 + 8y_2 + 3y_3 = -10$

$y_3 = -3$

By eq 4, we will get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

Simplify it from down side

$$3z = -3$$

$$z = -1$$

$$2y + 8z = -2$$

$$y = 3$$

$$\mathrm{x} + 2y + 3z = 5$$

$$\mathrm{x} = 2$$

Direct methods for solving linear systems

- **Jacobi method**

- **Gauss-Seidel Method**

- **Cholesky method**

Thank You !