Numerical Analysis

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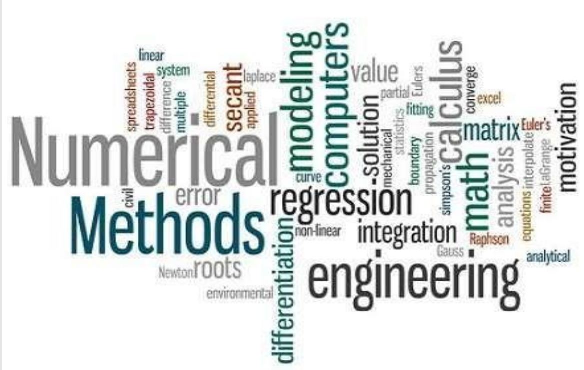

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Course Syllabus

- **Introduction:** Numerical analysis and background, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis and order of convergence (Newton’s method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, and PA=LU-factorization,…..
- **Polynomial:** Polynomial interpolation, Lagrange interpolation, Piecewise interpolation, divided differences interpolation, and curve fitting in interpolation (Application: Regression)…...
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration…….
- **Ordinary Differential Equations:** Euler's Method, and Runge-Kutta methods…..

- **Differential Equations**
- **Euler's Method**
- **Runge Kutta Method**

Differential Equations

STATIC

DYNAMIC

Rate of change

$$\frac{dx}{dt}$$

1. Linear Programming problem
2. Quadratics equation

$$Speed = \frac{change\ in\ distance}{change\ in\ time}$$

- A differential equation is a mathematical equation that relates some function of one or more variables with its derivatives. Differential equations arise whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated.

- Ordinary differential equations are DEs whose unknowns are functions of a single variable; they arise most commonly in the study of dynamical systems and electrical networks.

Differential Equations

Dependent and independent variable

$y = f(x)$

Y is dependent variable

x is independent variable

- Another definition: An equation containing dependent, independent variables and their derivatives is called differential equation (DE).

Order: It is the highest order derivative of a D.E.

$$\frac{dy}{dx} + x = 9 \longrightarrow \text{First order derivatives}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3x = 9y \longrightarrow \text{Second order derivatives}$$

$$\frac{d^5y}{dx^5} + \frac{d^2y}{dx^2} = 3x$$

Differential Equations

Degree: The power of the highest order derivative is called Degree of a D.E

$$\left(\frac{d^5y}{dx^5}\right)^1 + \frac{d^2y}{dx^2} = 3x$$

Order=5

Degree=1

$$\left(\frac{d^5y}{dx^5}\right)^{10} + \frac{dy}{dx} = 9$$

First order differential equations

$$\frac{dy}{dx} + f(x) = f(y)$$

$$\frac{dy}{dx} + f(x, y) = 0$$

Differential Equations

Separable equations

$$\frac{dy}{dx} = H(x, y)$$

$$\frac{dy}{dx} = \quad \mathrm{g}(x) \qquad \mathrm{h}(y)$$

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \qquad \mathrm{f}(y) = \frac{1}{h(y)}$$

$$f(y)dy = dx\, g(x) \quad or \quad f(y)dy = \ g(x)dx$$

Integrating on both sides, we get

$$\int f(y)dy = \int g(x)\, dx$$

$$\mathrm{F(y) = G\ (x) + C}$$

Example

Differential Equations

Example: A simple example of a differential equation (DE) is

$$\frac{dy}{dx} = \lambda y$$

where λ is a constant. The unknown is y and the independent variable is x. The equation involves both the unknown y as well as the unknown dy/dx; and for this reason is called a *differential equation*. We know from calculus that

$$y(x) = c\,\mathrm{e}^{\lambda x}, \;\; c = \text{constant},$$

satisfies this equation since

$$\frac{dy}{dx} = \frac{d}{dx} c\,\mathrm{e}^{\lambda x} = c\,\lambda \mathrm{e}^{\lambda x} = \lambda y(x).$$

- Differential Equations

- Euler's Method

- Runge Kutta Method

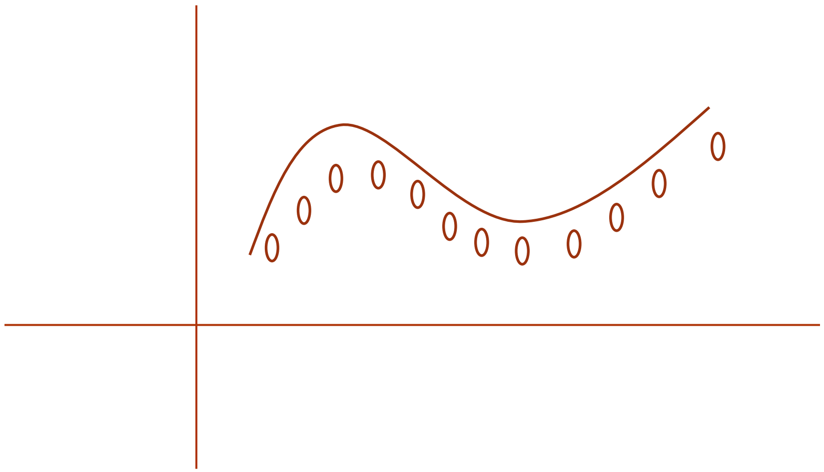
Euler's Method

Solution of Ordinary Differential Equations

$$\frac{dy}{dx} + f(x, y) = 0 \qquad \text{First order}$$

If there is first order differential equation given, then we can solve

- Directly with the help of integration
- Power Series method

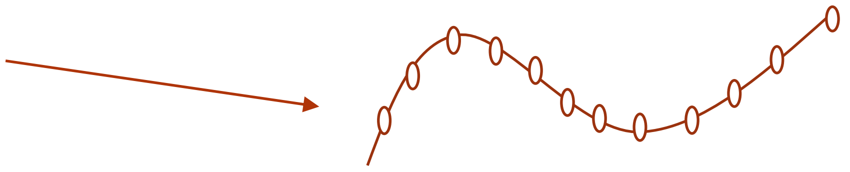


Why Euler's Method

- To solve first order O.D.E

Suppose $\frac{dy}{dx} = f(x, y)$ be an ODE

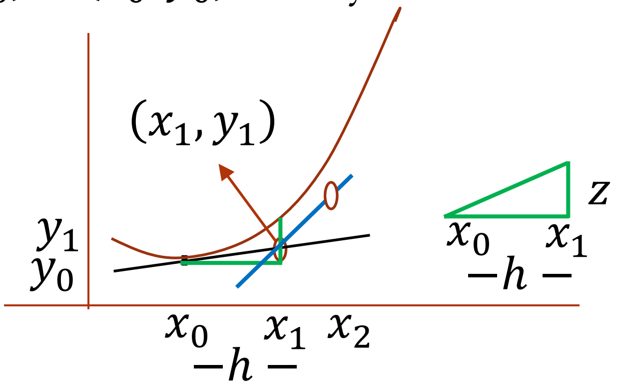
Give us rough estimated curve



Euler's Method

$$\frac{dy}{dx} + f(x,y)$$

We start with an initial value say $y_0 = y(x_0)$or (x_0, y_0) and try to find out next value



Start

$x = x_0$ $\qquad$ $y = y_0$

$x_1 = x_0 + h$ $\qquad$ $y_1 = y_0 + h\,f(x_0, y_0)$

$x_2 = x_1 + h$ $\qquad$ $y_2 = y_1 + h\,f(x_1, y_1)$

$$\boxed{x_n = x_{n-1} + h \qquad y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})}$$

Slope at $(x_0, y_0) = \frac{z}{h}$

$$f(x_0, y_0) = \frac{z}{h} \Rightarrow z = h\,f(x_0, y_0)$$

Euler's method is a numerical method for solving initial value problems. Euler's method is based on the insight that some differential equations (which are the ones we can solve using Euler's method) provide us with the slope of the function (at all points), while an initial value provides us with a point on the function. Using this information we can approximate the function with a tangent line at the point given by the initial value. As we have seen, the tangent line is only a good approximation over a small interval. Thus, after moving a small interval, we will want to construct a new tangent line.

Euler's Method

Example Approximate $y(0.4)$ using Euler's Method

$$\frac{dy}{dx} + x + 2y, \qquad y(0) = 0 \qquad \text{Take step size } h = 0.1$$

Solution We have given

$$f(x,y) = x + 2y \qquad x_0 = 0 \qquad y_0 = 0 \qquad h = 0.1 \qquad y(0.4)$$

Start

$$\mathrm{y} = y_0 = 0$$

$$x = x_0 = 0$$

$$x_1 = x_0 + h$$

$$x_1 = 0 + 0.1 = 0.1$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_1 = 0 + 0.1(x_0 + 2y_0)$$

$$y_1 = 0 + 0.1(0 + 2(0))$$

$$y_1 = 0$$

Euler's Method

$x_2 = x_1 + h$

$x_2 = 0.1 + 0.1 = 0.2$

$y_2 = y_1 + h\,f(x_1, y_1)$

$y_2 = 0 + 0.1\,(x_1 + 2y_1)$

$y_2 = 0 + 0.1\,(0.1 + 2(0))$

$y_2 = 0.1\,(0.1) = 0.01$

$x_3 = x_2 + h$

$x_3 = 0.2 + 0.1$

$x_3 = 0.3$

$y_3 = y_2 + h\,f(x_2, y_2)$

$y_3 = 0.01 + 0.1\,(x_2 + 2y_2)$

$y_3 = 0.01 + 0.1\,(0.2 + 2(0.01))$

$y_3 = 0.01 + 0.1\,(0.2 + 0.02))$

$y_3 = 0.032$

$x_4 = x_3 + h$

$x_4 = 0.3 + 0.1$

$x_4 = 0.4$

$y_4 = y_3 + h\,f(x_3, y_3)$

$y_4 = 0.032 + 0.1\,(x_3 + 2y_3)$

$y_4 = 0.032 + 0.1\,(0.3 + 2(0.032))$

$y_4 = 0.032 + 0.1\,(0.364)$

$\boxed{y_4 = 0.0684}$

- **Differential Equations**
- **Euler's Method**
- **Runge Kutta Method**

Runge Kutta Methods

Definition

- The Euler method and Mid Point method, as well as the improved and modified Euler methods are all examples on Runge-Kutta methods (RK). That is why we can call it big boss.

- The ODE can be solved using Taylor Series. But there are too many higher derivatives are involved.

- RK Methods achieve the accuracy of Taylor Series approach without requiring the calculations of higher derivatives .

Runge Kutta Methods

General Form of RK methods

$$y_{i+1} = y_i + \varphi h$$

Where φ is called an increment function and can be interpreted as slope over the interval

$$\varphi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

Where a_i's are constant and k_i's are

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, \quad y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_2 h, \quad y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$k_n$$

Where $p's$ and $q's$ are constants

This is called nth order RK method

Runge Kutta Methods

First order of Runge Kutta Method

If we put $n = 1$

$$y_{i+1} = y_i + \varphi h \quad \text{Where } \varphi = a_1 k_1 \text{ and } k_1 = f(x_i, y_i)$$

$$y_{i+1} = y_i + a_1 k_1 h$$

If we take $a_1 = 1$

$$y_{i+1} = y_i + f(x_i, y_i)h \longrightarrow \text{Euler Method}$$

So the first order of RK method is in fact Euler Method

Second order of Runge Kutta Method

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h \text{ ——— Eq 1}$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, \quad y_i + q_{11} k_1 h)$$

Runge Kutta Methods

To evaluate a_1 , a_2 , p_1, and q_{11}, we will equate equation 1 with second order Taylor Series

we will get

$$a_1 + a_2 = 1 \Rightarrow a_1 = 1 - a_2$$

$$a_2 p_1 = \frac{1}{2} \Rightarrow p_1 = \frac{1}{2a_2}$$

$$a_2 q_{11} = \frac{1}{2} \Rightarrow q_{11} = \frac{1}{2a_2}$$

There are 4 unknowns and 3 equations so we have infinite No. of Second-order RK methods. Generally, every version would yield exactly the same results.

Let us take $a_2 = \frac{1}{2}$ $\quad a_1 = \frac{1}{2}, p_1 = q_{11} = 1$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$y_{i+1} = y_i + (\frac{1}{2} k_1 + \frac{1}{2} k_2)h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

HEUN's Method with iteration

Runge Kutta Methods

Let us take $a_2 = 1$ $\quad a_1 = 0, p_1 = \frac{1}{2} = q_{11}$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$y_{i+1} = y_i + (0\, k_1 + 1 k_2)h$$

$$y_{i+1} = y_i + k_2 h$$

Where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, \qquad y_i + q_{11} k_1 h)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

The Mid-point Method

Thank You !