Numerical Analysis

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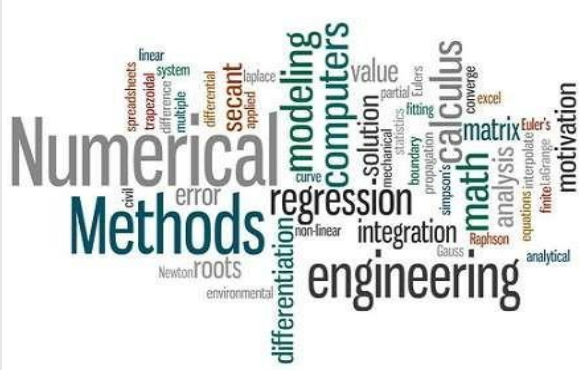
Week 3, Lecture I-II



Course Syllabus

- **Introduction:** Numerical analysis and background, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis, and order of convergence (Newton’s method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, and PA=LU-factorization,….. .
- **Polynomial:** Polynomial interpolation, piecewise linear interpolation, divided differences interpolation, cubic spline interpolation, and curve fitting in interpolation (Application: Regression).
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration.
- **Ordinary Differential Equations:** Euler's Method, higher-order Taylor method, and Runge-Kutta methods….

Root Finding

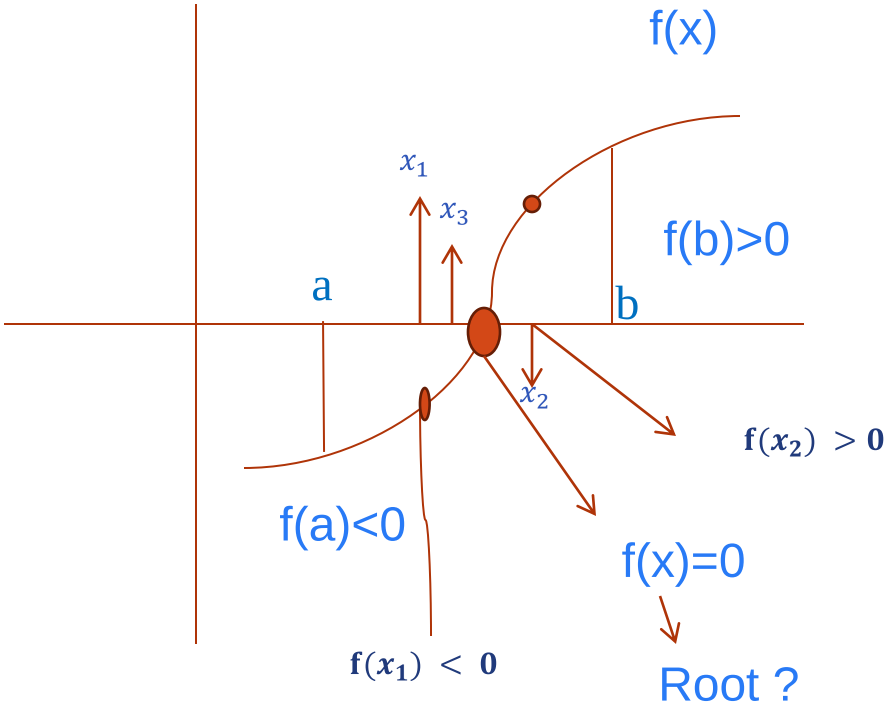
1. Bisection method

2. Fixed-point iteration method

3. Newton's method

Bisection method

Concept



$$x_1 = \frac{a+b}{2}$$

$$x_2 = \frac{x_1+b}{2}$$

$$x_3 = \frac{x_1+x_2}{2}$$

Fixed Point Iteration Method

Fixed point iteration method

- Fixed point of given function $g: R \longrightarrow R$ is value x such that

$$x = g(x)$$

- Many iterative methods for solving nonlinear equation use fixed-point iteration scheme of form

$$x_{n+1} = g(x_n)$$

where fixed points for g are solutions for f(x)=0

- Also called functional iteration, since function g is applied repeatedly to initial starting value x_0

- For given equation f(x)=0, there may be many equivalent fixed-point problems x= g(x) with different choices for g

Fixed point iteration method

Rules/ Steps:

Step 1:Let f(x)=0 ⟶ *eq (1)*

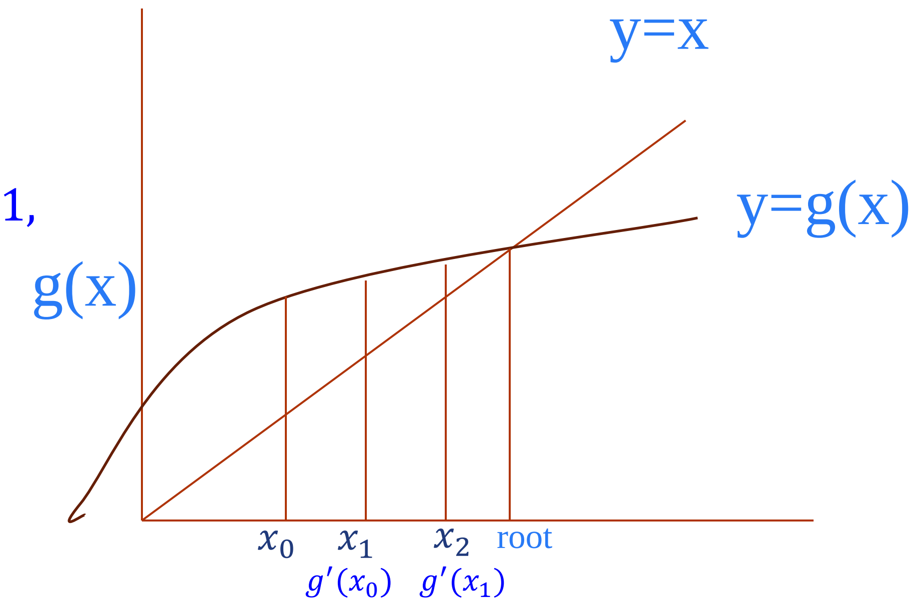
Step 2: Identify a and b if not given and take x_0 *as initial root of f(x)=0*

take x_0 *b/w a and b*

Step 3: Rewrite the eq(1) as x=g(x)

Step 4: Calculate $g'(x)$ *if* $|g'(x)| < 1$, *then* eq_n *as iteration formula*

$$\mathbf{x_{n+1} = g(x_n)}$$ *n= 0, 1, 2, 3,4....*

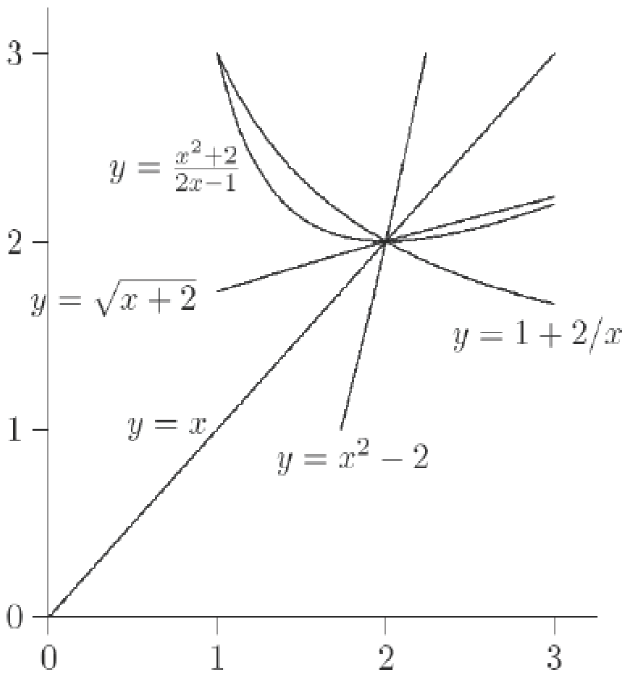


Fixed point iteration method

If $f(x) = x^2 - x - 2$, then fixed points of each of functions

- $g(x) = x^2 - 2$
- $g(x) = \sqrt{x+2}$
- $g(x) = 1 + 2/x$
- $g(x) = \dfrac{x^2+2}{2x-1}$

are solutions to equation $f(x) = 0$



Fixed point iteration method

Problem: Find the real root of equation $x^3 - 9x + 1 = 0$, correct upto 3-decimal places.

Let $F(x) = x^3 - 9x + 1$

$f(0) = (0)^3 - 9(0) + 1 \;\; = 1$

$f(1) = (1)^3 - 9(1) + 1 \;\; = -7$

$f(2) = (2)^3 - 9(2) + 1 \;\; = -9 \longrightarrow$ <0

$f(3) = (3)^3 - 9(3) + 1 \;\; = 1 \longrightarrow$ >0

Since root of f(x)=0 lies between 2 and 3, so we should take the initial root is $x_0 = 2.7$

Fixed point iteration method

Case 1: Rewrite f(x)=0 as

$$x^3 - 9x + 1 = 0$$

$$x(x^2 - 9) + 1 = 0$$

$$x = \frac{-1}{x^2 - 9} = \frac{1}{9 - x^2} = g(x)$$

$$g'(x) = \frac{2x}{(9 - x^2)^2} \qquad \left(\frac{d}{dx} g(x)\right)$$

$$x_0 = 2.7$$

$$g'(2.7) = \frac{2(2.7)}{(9 - (2.7)^2)^2}$$

$$g'(2.7) = 1.8467 > 1$$

Fixed point iteration method

Case 2: Rewrite f(x)=0 as

$$x^3 - 9x + 1 = 0$$

$$x = (9x-1)^{\frac{1}{3}} = g(x)$$

$$g'(x) = \frac{3}{(9x-1)^{\frac{2}{3}}} \qquad \left(\frac{d}{dx}g(x)\right)$$

$$x_0 = 2.7$$

$$g'(2.7) = \frac{3}{(9(2.7)-1)^{\frac{2}{3}}}$$

$$g'(2.7) = \frac{3}{8.15}$$

$$g'(2.7) = 0.3680 < 1$$

Fixed point iteration method

Case 3: Rewrite f(x)=0 as

$$x^3 - 9x + 1 = 0$$

$$9x = x^3 + 1$$

$$x = \frac{x^3 + 1}{9} = g(x)$$

$$g'(x) = \frac{3x^2}{9} = \frac{x^2}{3} \qquad \left(\frac{d}{dx} g(x)\right)$$

$$x_0 = 2.7$$

$$g'(2.7) = \frac{(2.7)^2}{3}$$

$$g'(2.7) = \frac{7.29}{3}$$

$$g'(2.7) = 2.43 > 1$$

Fixed point iteration method

Since in case 2, satisfy the condition $|g'(x)| < 1$ so taking

$$x = (9x - 1)^{\frac{1}{3}}$$ *(Iteration formula)*

$x_0 = 2.7$ *if zero substitute*

$n = 0$

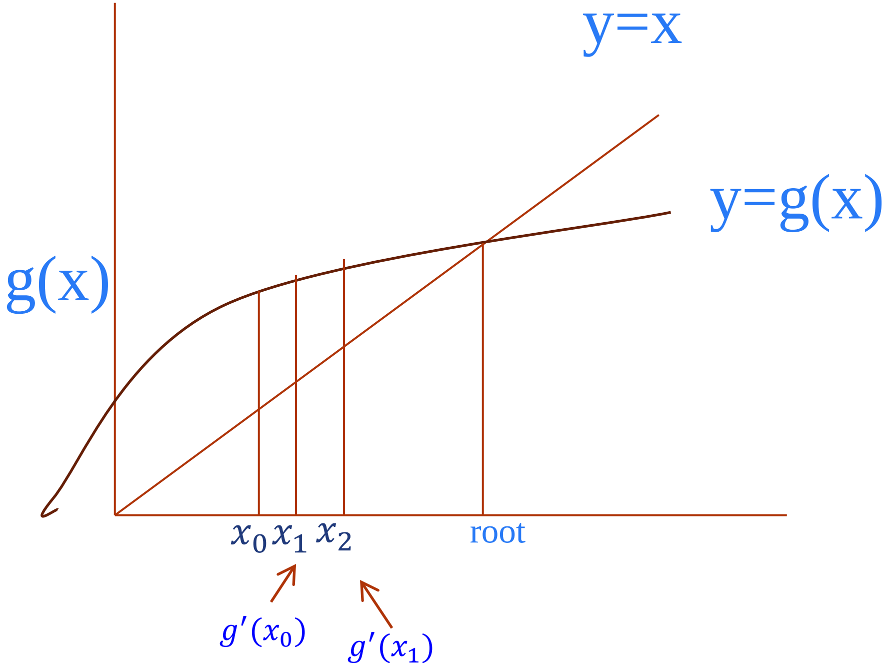
$$x_1 = (9x_0 - 1)^{\frac{1}{3}} = ((9 * 2.7) - 1)^{\frac{1}{3}}$$

$$x_1 = 2.8562$$

$n = 1$

$$x_2 = (9x_1 - 1)^{\frac{1}{3}} = ((9 * 2.8562) - 1)^{\frac{1}{3}}$$

$$x_2 = 2.9125$$



Fixed point iteration method

$n = 2$

$$x_3 = (9x_2 - 1)^{\frac{1}{3}} = ((9 * 2.9125) - 1)^{\frac{1}{3}}$$

$$x_3 = 2.9322$$

$n = 3$

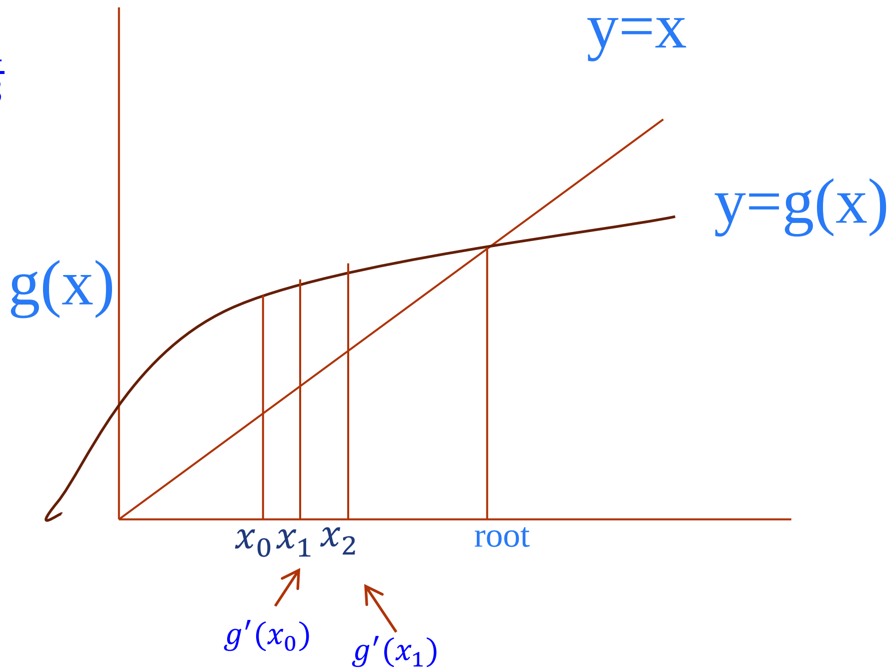
$$x_4 = (9x_3 - 1)^{\frac{1}{3}} = ((9 * 2.9322) - 1)^{\frac{1}{3}}$$

$$x_4 = 2.9391$$

$n = 4$

$$x_5 = (9x_4 - 1)^{\frac{1}{3}} = ((9 * 2.9391) - 1)^{\frac{1}{3}}$$

$$x_5 = 2.9415$$



Fixed point iteration method

$n = 5$

$$x_6 = (9x_5 - 1)^{\frac{1}{3}} = ((9 * 2.9415) - 1)^{\frac{1}{3}}$$

$$x_6 = 2.9423$$

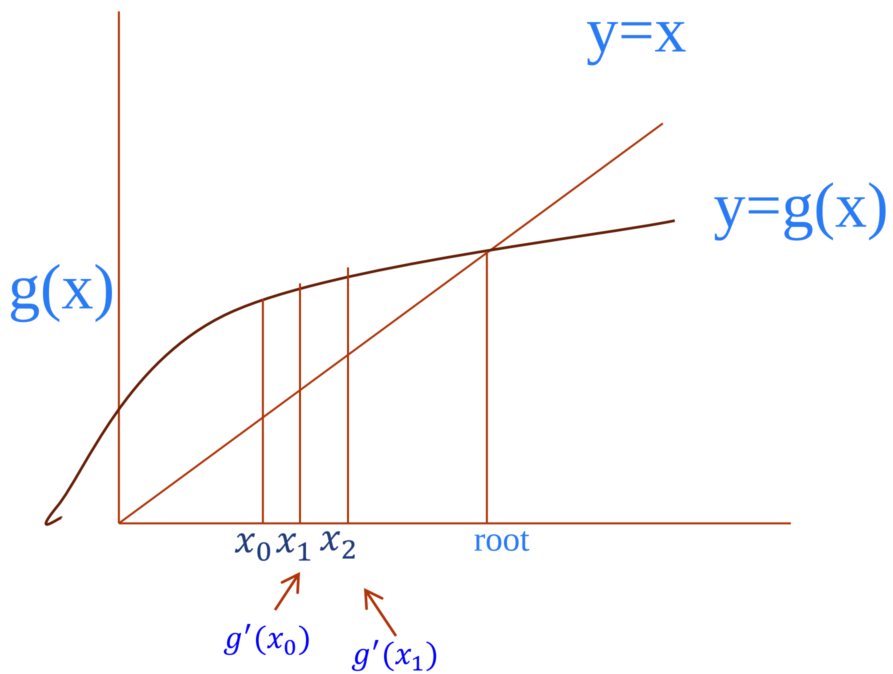
$n = 6$

$$x_7 = (9x_6 - 1)^{\frac{1}{3}} = ((9 * 2.9423) - 1)^{\frac{1}{3}}$$

$$x_7 = 2.9426$$

Hence the approximate root is

$$x = 2.942$$



Newton Method

Newton Raphson Method

Newton Method

- Newton`s method, also known as theNewton-Raphson Method, is the simplest and fastest approach to find the root of a function.

- Newton Raphson method is a numerical technique which is used to find the roots of Algebraic Equations .

- It is an open bracket method and requires only one initial guess.

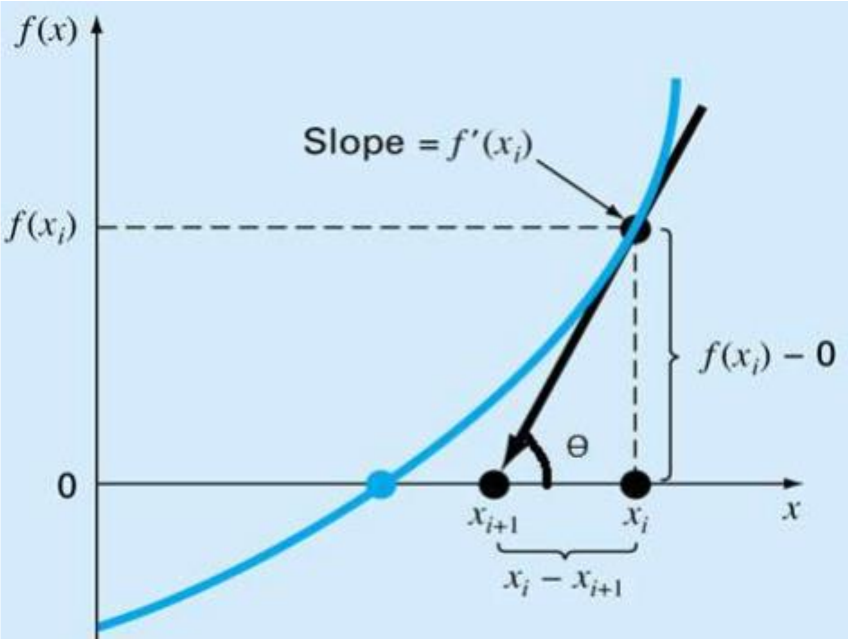
- It is often used to improve the result or value of the root obtained from other methods.

Newton Rahpson Method

Concept:

- Let us consider an equation f(x)=0 having graphical representation as
- f(x) =0 ,is an given equation
- Starting from an initial point x_0
- Determine the slope of f(x) at x= x_0

  Call it $f'(x_0)$.

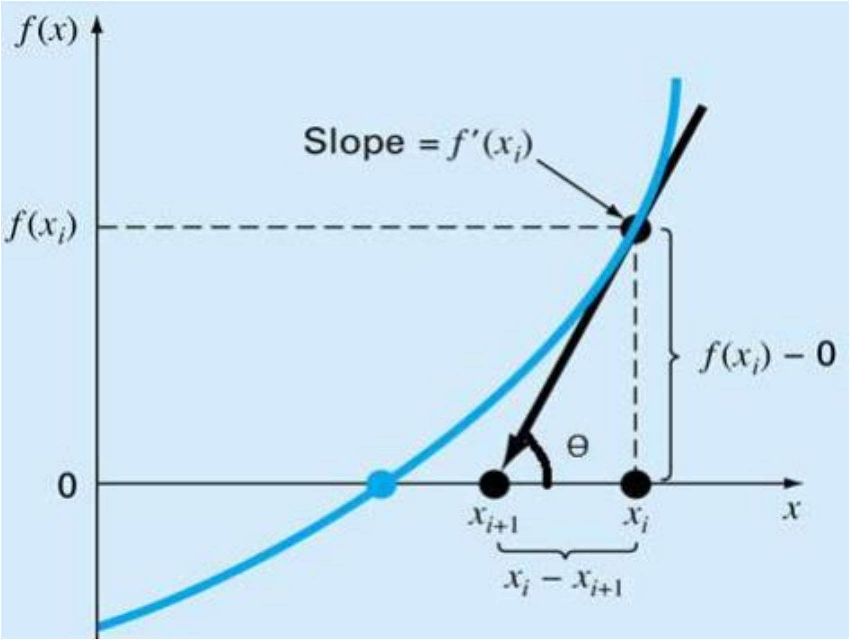


$$Slope = \tan\theta = \frac{f(x_i) - 0}{x_i - x_{i+1}} = f'(x_i).$$

From here we get

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

Newton Rahpson Method



Hence;

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

OR

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

OR

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

This is the general formula of Newton Raphson Method

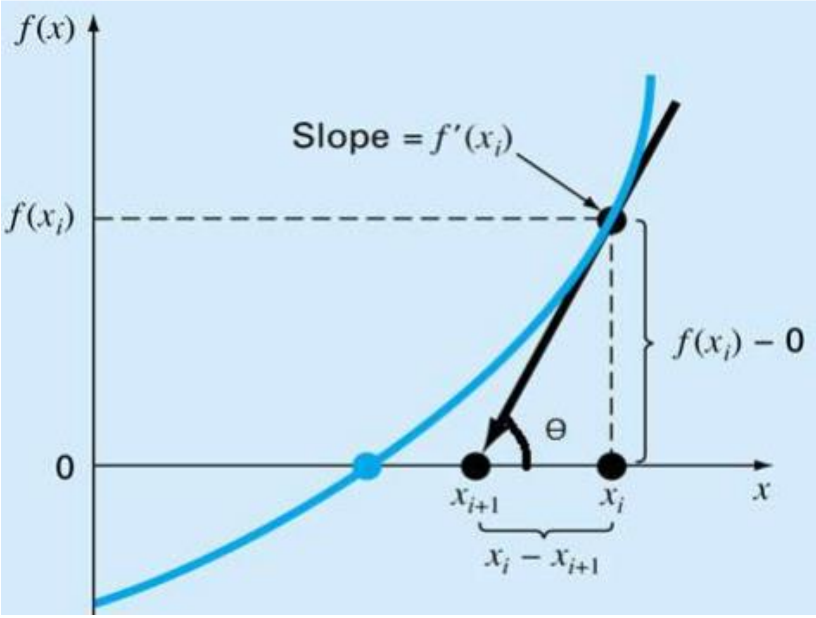
Newton Rahpson Method

Rules/ Steps:

Step 1:Let f(x)=0 $\longrightarrow$ *eq (1)*

Identify initial root x_0 *Such that*

$f(x_0) \sim 0$ *(i.e* x_0 *is near to root of eq (1))*



Step 2: *find* $f(x_0)$ *and* $f'(x_0)$

Step 3: find first approximate root by newton-Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

find $f(x_1)$ *and* $f'(x_1)$

Newton Rahpson Method

Rules/ Steps:

Step 4: find 2nd approximate root by newtwon-Raphson method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$find\ f(x_2)\ and\ f'(x_2)$

Step 5: find 3rd approximate root by newtwon-Raphson method

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$find\ f(x_3)\ and\ f'(x_3)$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

general formula

Newton Rahpson Method

Problem: Find the real root of equation $x^3 - 3x + 1 = 0$ using newton Rahpson method, correct upto 3-decimal places.

Let $F(x) = x^3 - 3x + 1 \rightarrow$ *eq (1)*

$f(0) = (0)^3 - 3(0) + 1 = 1 \longrightarrow >0$

$f(1) = (1)^3 - 3(1) + 1 = -1 \longrightarrow <0$

$f(0.5) = (0.5)^3 - 3(0.5) + 1 = -0.375 \longrightarrow <0$

$f(0.4) = (0.4)^3 - 3(0.4) + 1 = -0.136 \longrightarrow <0$

$f(0.3) = (0.3)^3 - 3(0.3) + 1 = 0.127 \longrightarrow >0$

$Choosing\ x_0 = 0.3, f(x_0) = 0.127$

$f'(x) = 3x^2 - 3 \longrightarrow$ *eq (2)*

$f'(x_0) = -2.73$

Newton Rahpson Method

First approximate root by newton-Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.3 - \frac{0.127}{(-2.73)} = 0.346520$$

$$f(x_1) = 0.002048, f'(x_1) = -2.6397$$

2nd approximate root by newton-Raphson method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.3465 - \frac{0.002048}{(-2.639771)}$$

$$x_2 = 0.347295$$

$$f(x_2) = 0.00035 , f'(x_2) = -2.6381$$

Newton Rahpson Method

3nd approximate root by newton-Raphson method

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 0.347295 - \frac{0.00035}{(-2.6381)}$$

$$x_3 = 0.3474$$

$$f(x_3) = -0.00027\ ,\ f'(x_3) = -2.637$$

hence the approximate root correct upto 3-decimal places is x=0.347

Thank You !