Numerical Analysis

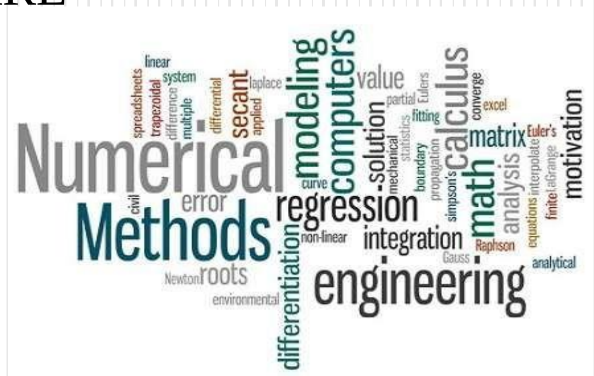
Dr. Farman Ali

Assistant Professor

DEPARTMENT OF SOFTWARE

SEJONG UNIVERSITY

Week 4, Lecture-I-II



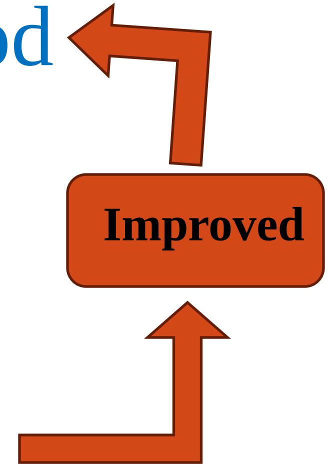
farmankanju@sejong.ac.kr

Course Syllabus

- **Introduction:** Numerical analysis and backround, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and error analysis, and order of convergence (Newton’s method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, LU-factorization, forward substitution, and crout factorization.
- **Polynomial:** Polynomial interpolation, piecewise linear interpolation, divided differences interpolation, cubic spline interpolation, and curve fitting in interpolation (Application: Regression).
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration.
- **Ordinary Differential Equations:** Euler's Method, higher-order Taylor method, and Runge-Kutta methods.

Root Finding

1. False position method



2. The secant method

Root finding

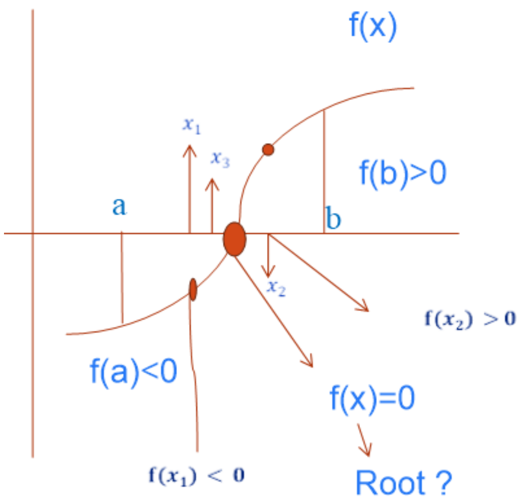
Bisection

Fixed point iteration

Newton Raphson

Root finding

Bisection

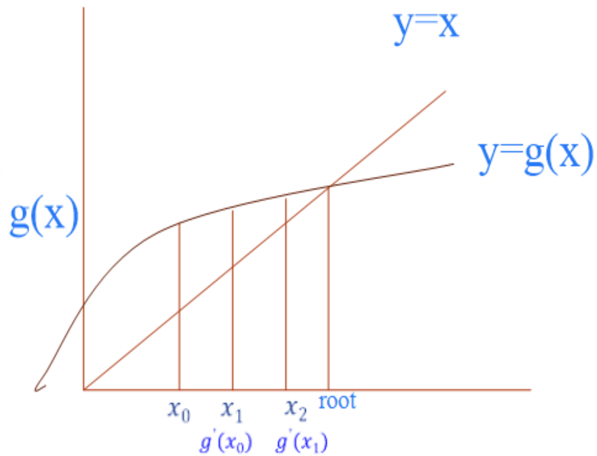


$$x_1 = \frac{a+b}{2}$$

$$x_2 = \frac{x_1+b}{2}$$

$$x_3 = \frac{x_1+x_2}{2}$$

Fixed point iteration



Identify a and b

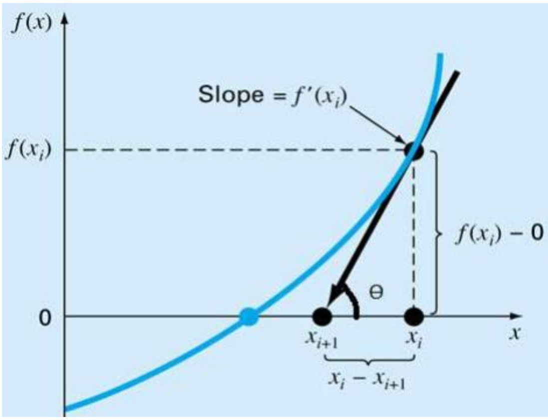
take x_0 *b/w a and b*

Rewrite the eq as x=g(x)

Calculate $g'(x)$ *if* $|g'(x)| < 1$*, then* eq_n *as iteration formula*

$$\mathbf{x_{n+1} = g(x_n)}$$

Newton Raphson



Identify x_0 $\quad f(x_0) \sim 0$

find $f(x_0)$ *and* $f'(x_0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

find $f(x_1)$ *and* $f'(x_1)$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

False position method (Regula falsi method)

- The **Regula-Falsi Method** (sometimes called the False Position Method) is a method used to find a numerical estimate of an equation.

- In bisection method, if f(x1)<0 then the root lies between x1 and b, if if f(x1)>0 then the root lies between a and x1.

- To refine the bisection method, we can choose a ‘false-position’ instead of the midpoint.

- The idea for the Regula-Falsi method is to connect the points (a,$f(a)$) and (b,$f(b)$) with a straight line.

- The intersection of this line with the x axis represents an improved estimate of the root.

False position method (Regula falsi method)

False position method (Regula falsi method)

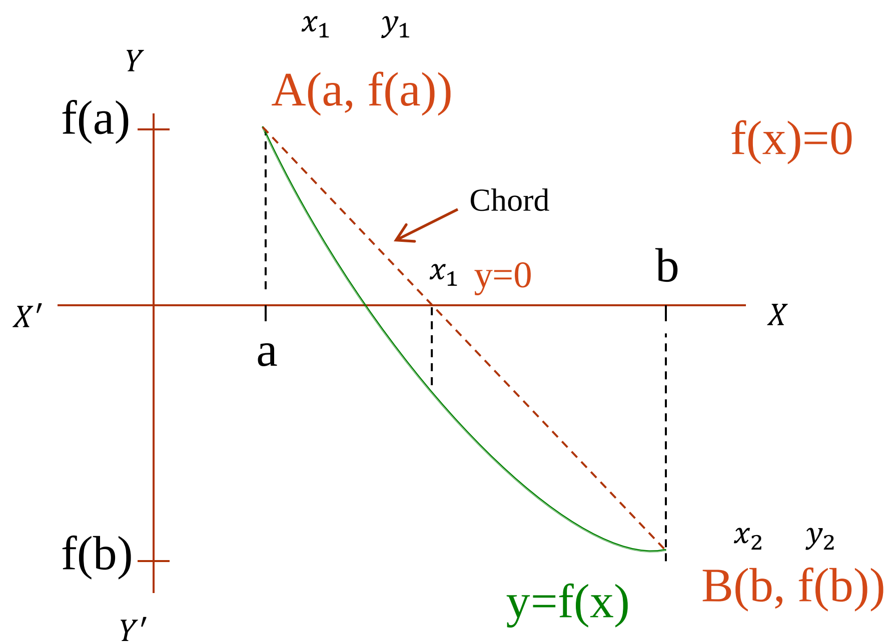
Consider the equation f(x)=0

Let f(a) > 0, F (b) < 0

This means root lies between a and b

The general equation for joining the two points is basically

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$



$$\frac{y-f(a)}{x-a}=\frac{f(b)-f(a)}{b-a} \qquad \text{eq (1)}$$

The chord meets the x-axis at the point x_1

$$\frac{0-f(a)}{x_1-a}=\frac{f(b)-f(a)}{b-a}$$

Put $x = x_1$ *and* $y = 0$ *in equation* 1

False position method (Regula falsi method)

$$(-f(a)(b-a) = (f(b)-f(a))\,(x_1-a)$$

$$-bf(a)+af(a) = x_1 f(b) - x_1 f(a) - af(b) + af(a)$$

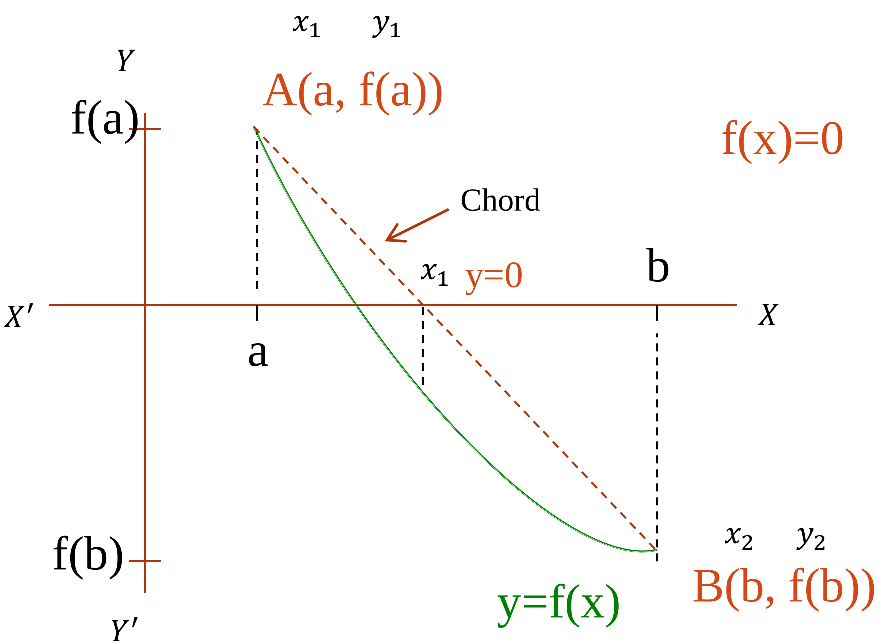
$$-bf(a)+\cancel{af(a)} + af(b) - \cancel{af(a)} = x_1 f(b) - x_1 f(a)$$

$$af(b) - bf(a) = x_1[f(b)-f(a)]$$

$$\frac{af(b)-bf(a)}{f(b)-f(a)} = x_1$$

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

This is first approximate root

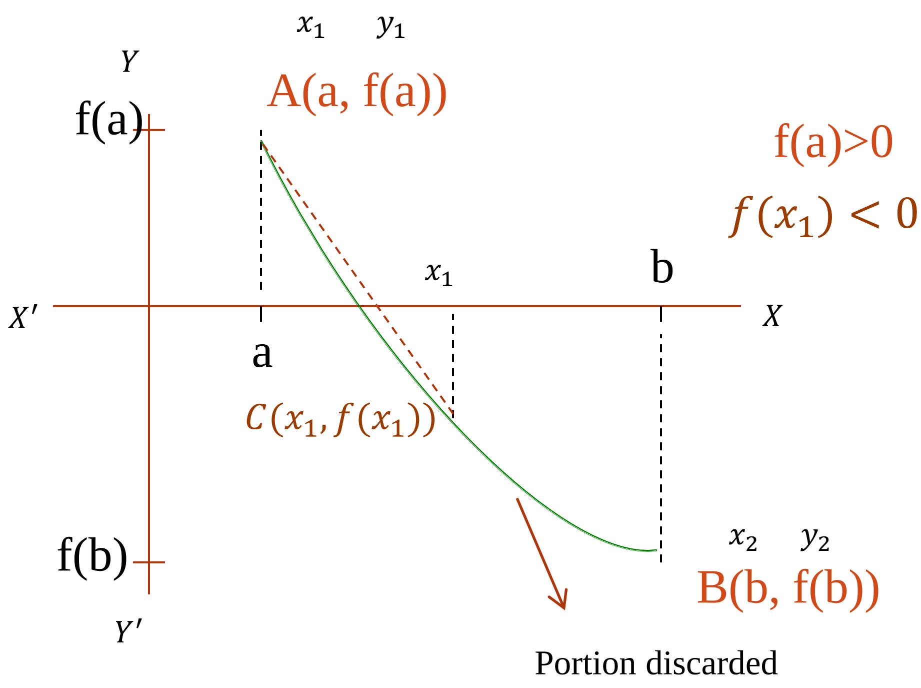


False position method (Regula falsi method)

For finding 2nd approximate root:

$if\ f(a)\ and\ f(x_1)\ are\ of\ opposite\ signs$

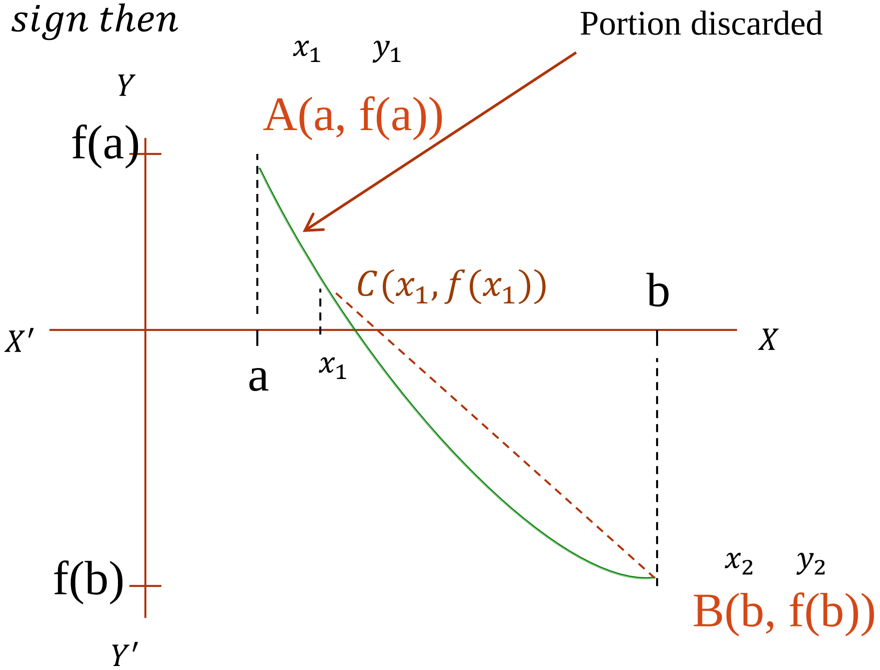
$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$



False position method (Regula falsi method)

Suppose if $f(x_1)$ *and* $f(b)$ *are of opposite sign then*

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$



In the same way we get

$x_3, x_4 \ldots\ldots.$

till we get required root nearer to the root of equation f(x)=0

False position method (Regula falsi method)

Rules/ Steps

Steps 1:

Steps 2:

Steps 3:

Steps 3.1:

Steps 3.2:

False position method (Regula falsi method)

Rules/ Steps

Steps 1: Given equation f(x)=0

find a and b f (a)>0, f(b) <0 *This means root lies between a and b*

Steps 2: Find the first approximate root by Regula falsi method

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Steps 3: $Find\ f(x_1)\ and\ examine\ its\ sign$

Steps 3.1: $if\ f(x_1)\ < 0\ then\ b = x_1$

Steps 3.2: $if\ f(x_1)\ > 0\ then\ a = x_1$

Find $f(x_2)$ and repeat step 3 untill the required accurate root.

False position method (Regula falsi method)

$Probelm$: $Find\ the\ real\ root\ of\ equation\ x^3 - 4x + 1 = 0\ using\ regula\ falsi\ method\ upto\ 3 - decimal\ places$

Let $F(x) = x^3 - 4x + 1 \rightarrow$ *eq (1)*

$f(0) = (0)^3 - 4(0) + 1 = 1 \longrightarrow >0$

$f(1) = (1)^3 - 4(1) + 1 = -2 \longrightarrow <0$

Since f(0) and f(1) are of opposite sign the root lies between 0 and 1

a= 0 b= 1

To find the first approximate root

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{0(-2) - 1(1)}{-2 - 1} \qquad x_1 = 0.3333$$

False position method (Regula falsi method)

$x_1 = 0.333$

put x_1 in equation 1

$$f(0.333) = (0.333)^3 - 4(0.333) + 1 \;= -0.2962$$

<0

$if\ f(x_1) < 0\ then\ b = x_1$

Root lies between a and x_1

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

$$x_2 = \frac{0(-0.2962) - 0.333(1)}{(-0.2962) - (1)}$$

f (a)	$f(x_1)$	f(b)
+ve	-ve	-ve

$x_2 = 0.2571$

False position method (Regula falsi method)

put x_2 in equation 1

$$f(0.2571) = (0.2571)^3 - 4(0.2571) + 1 \quad = -0.0115$$

<0

$if\ f(x_2)\ < 0\ then\ b = x_2$

Root lies between a and x_2

$$x_3 = \frac{af(x_2) - x_2 f(a)}{f(x_2) - f(a)}$$

$$x_3 = \frac{0(-0.0115) - 0.2571(1)}{(-0.0115) - (1)}$$

$$x_3 = 0.2541$$

f (a)	$f(x_2)$	$f(x_1)$	f(b)
+ve	-ve	-ve	-ve

False position method (Regula falsi method)

put x_3 in equation 1

$$f(0.2541) = (0.2541)^3 - 4(0.2541) + 1 \quad = 0.00001$$

$if\ f(x_3) > 0\ then\ a = x_3$

>0

Root lies between x_3 and x_2

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)}$$

$$x_4 = \frac{(0.2541)(-0.0115) - 0.2571(0.00001)}{(-0.0115) - (0.00001)}$$

$$x_4 = 0.2541$$

Hence the root is 0.2541

False position method (Regula falsi method)

Probelm: $Solve\ the\ equation\ xe^x = \cos x\ \ using\ regula\ falsi\ method$
$upto\ 4 - decimal\ places$

Let $F(x) = xe^x - \cos x = 0 \longrightarrow$ *eq (1)*

To find a and b

$f(0) = -1 \longrightarrow$ <0

$F(1) = 1e^1 - \cos 1 = 2.177979 \longrightarrow$ >0

$F(0.5) = 0.5e^{0.5} - \cos 0.5 = -0.053221 \longrightarrow$ <0

$F(0.6) = 0.6e^{0.6} - \cos 0.6 = 0.267935 \longrightarrow$ >0

a= 0.5 b= 0.6

To find the first approximate root

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} \qquad x_1 = \frac{0.5(0.267935) - 0.6(-0.053221)}{0.267935 - (-0.053221)} \qquad x_1 = 0.516571$$

False position method (Regula falsi method)

put x_1 in equation 1

$$F(x_1) = x_1 e^{x_1} - \cos x_1$$

$$F(0.0516571) = (0.0516571)e^{0.0516571} - \cos 0.0516571 = -0.003605 < 0$$

Root lies between b and x_1

$if\ f(x_1) < 0\ then\ \mathrm{a} = x_1$

$$x_2 = \frac{x_1 f(b) - b f(x_1)}{f(b) - f(x_1)}$$

$x_2 = 0.517678$

$$F(x_2) = x_2 e^{x_2} - \cos x_2 = -0.000241 < 0$$

$if\ f(x_2) < 0\ then\ x_1 = x_2$

Root lies between b and x_2

$$x_3 = \frac{x_2 f(b) - b f(x_2)}{f(b) - f(x_2)}$$

$x_3 = 0.517751$

False position method (Regula falsi method)

$F(x_3) = x_3 e^{x_3} - \cos x_3 = -0.000019 < 0$ $\quad$ $if\ f(x_3) < 0\ then\ x_2 = x_3$

Root lies between b and x_3

$$x_4 = \frac{x_3 f(b) - b f(x_3)}{f(b) - f(x_3)}$$

$$x_4 = 0.517756$$

Hence the approximate root of the equation correct upto 4-decimal places is x=0.5177

False position method (Programming Algo)

1. Start
2. Read values of a and b
   *Here a and b are the two initial guesses
3. Computer function values f(a) and f(b)
4. Check whether the product of f(a) and f(b) is negative or not.
   If it is positive take another initial guesses.
   If it is negative then goto step 5.
5. Determine:
   x =[a*f(b) – b*f(a)] / (f(b) – f(a))
6. Check whether the sign of f(x) is negative or not.
   If it is negative, then assign b=x;
   If it is positive, assign a =x;
7. Check whether the value of f(x) is greater than 0.00001 or not.
   If yes, goto step 5.
   If no, goto step 8.
8. Display the root as x.
9. Stop

False position method (Programming C)

```
#include<stdio.h>
#include<conio.h>
# define eqn x*x-5
float f(float x)
{
  float ans;
  ans=eqn;
  return(ans);
}
void main()
{
  float a,b,x=0,x1=0;
  int i=0;
  printf("Enter the a and b\n");
  scanf("%f%f",&a,&b);
if(f(a)*f(b)>=0)
  {
    printf("The interval is wrong\n");
    getch();
    return ;}
 do
  {
    x=a-((b-a)/(f(b)-f(a)))*f(a);

printf("\ta=%f\tb=%f\tX%d=%f\tf(x%d)=%f\n",a,b,i,x,i,f(x)); if(x==x1)
    {
      printf("\n\nThe root is %f\n",x);
      break;
    }
```

False position method (Programming C)

```
x1=x;
  i++;
    if(f(a)*f(x)<0
      ) b=x;
    else
      a=x;
  }while(f(a)-f(b)!=0.000);
  printf("\nThe final ans is %f",x);
  getch();
}
```

```
Enter the a and b  2
2.5
     a=2.000000   b=2.500000  X0=2.222222
f(x0)=-0.061728
     a=2.222222   b=2.500000     X1=2.235294
f(x1)=-0.003460
     a=2.235294   b=2.500000     X2=2.236025
f(x2)=-0.000193
     a=2.236025   b=2.500000     X3=2.236066
f(x3)=-0.000010
     a=2.236066   b=2.500000     X4=2.236068
f(x4)=-0.000001
     a=2.236068   b=2.500000     X5=2.236068
f(x5)=0.000000

     The final ans is 2.236068
```

Problem: Find the approximate root $x^3 - 4x - 9$ by using bisection method and ragula falsi method upto four decimal places.

Bisection method

$x_1 = 2.5$
$x_2 = 2.75$
$x_3 = 2.625$
$x_4 = 2.6875$
$x_5 = 2.71875$
$x_6 = 2.7031$
$x_7 = 2.7109$
$x_8 = 2.707$
$x_9 = 2.7051$

$x_{10} = 2.7061$
$x_{11} = 2.7066$
$x_{12} = 2.7064$
$x_{13} = 2.7065$
$x_{14} = 2.7065$

Bisection method identified root in 14th iteration

Ragula Falsi method

$x_1 = 2.6$

$x_2 = 2.75$

$x_3 = 2.7106$

$x_4 = 2.7063$

$x_5 = 2.7065$

ragula Falsi method identified root in 5th iteration

Secant method

- Secant method is improved form of regula fasli method.
- It converges faster than a linear rate so it is more rapidly convergent.
- It doesn’t require use of derivative of a given function because in some practical cases, derivatives become very hard to find.
- It requires only one function evaluation per iteration as compared to Newton’s method which requires two.
- Secant method is one of the analytical procedure available to earthquake engineers for predicting earthquake performance and structures.

Secant method

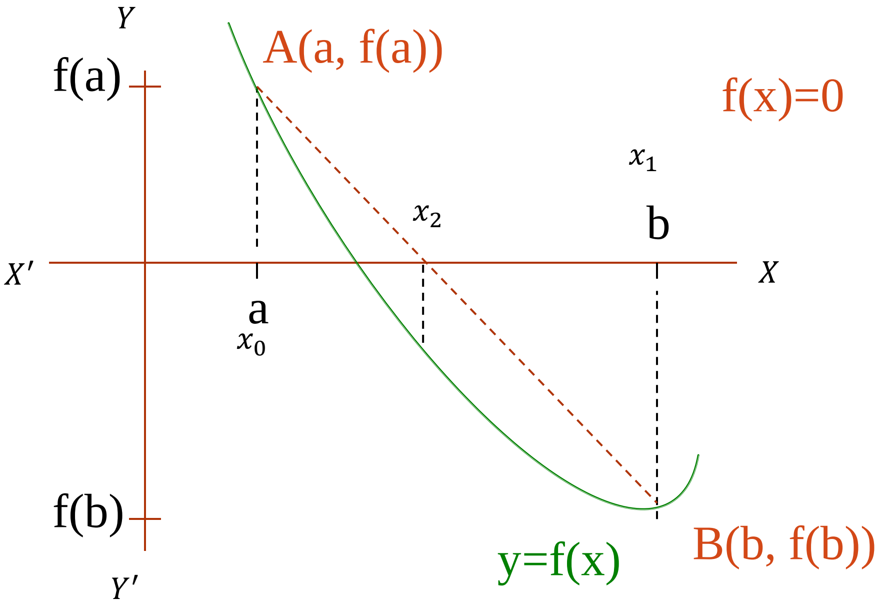
Secant method

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$x1 = x_2 \quad a = x_0 \quad b = x_1$

OR

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$



Secant method

Rules/ Steps

Steps 1: Given equation f(x)=0 $\longrightarrow$ *eq (1)*

Steps 2: $Find\ x_0\ and\ x_1\ such\ that\ f(x_0)\ < 0\ and\ f(x_1) > 0$ $this\ means\ root\ lies\ \ x_0\ and\ x_1$

Steps 3: Find the first approximate root by secant method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \qquad Find\ f(x_2)$$

Steps 4: Find the 2nd approximate root by secant method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \qquad Find\ f(x_3)$$

Steps 5: Find the 3nd approximate root by secant method

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \qquad Find\ f(x_4)$$

General formula

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Repeat the above process untill the required accurate root.

Secant method

Probelm: $Find\ the\ real\ root\ of\ equation\ x^3 - x - 1 = 0\ using$
$Secant\ method\ upto\ 4 - decimal\ places$

Let $F(x) = x^3 - x - 1 \longrightarrow$ *eq (1)*

$To\ find\ x_0\ and\ x_1$

$$f(0) = (0)^3 - (0) - 1 = -1 < 0$$

$$f(1) = (1)^3 - (1) - 1 = -1 < 0$$

$$f(2) = (2)^3 - (2) - 1 = 5 > 0$$

$$f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875 > 0$$

$$f(1.4) = (1.4)^3 - (1.4) - 1 = 0.343 > 0$$

$$f(1.3) = (1.3)^3 - (1.3) - 1 = -0.103 < 0$$

$$f(x_0) = -0.103$$

$$f(x_1) = 0.343$$

$Since\ f(1.4)\ and\ f(1.3)\ are\ of\ opposite\ sign, root\ lies\ b|w\ x_0 = 1.3\ and\ x_1 = 1.4$

Secant method

The first approximate root

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \qquad x_2 = \frac{(1.3)(0.343) - (1.4)(-0.103)}{0.343 - (-0.103)}$$

$$x_2 = 1.323042 \qquad f(x_2) = -0.007136 < 0$$

The 2nd approximate root

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \qquad x_3 = \frac{(1.4)(-0.007136) - (1.323042)(0.343)}{(-0.007136) - (0.343)}$$

$$x_3 = 1.324605 \qquad f(x_3) = -0.000481 < 0$$

The 3rd approximate root

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \qquad x_4 = \frac{(1.323042)(-0.000481) - (1.324605)(-0.007136)}{(-0.000481) - (-0.007136)}$$

$$x_4 = 1.324717 \qquad f(x_4) = -0.000004 < 0$$

Hence the approximate root of the equation correct upto 4-decimal places is x=1.324717

Secant method

Probelm: $Solve\ the\ equation\ e^{-x} - x\ using\ secant\ method\ upto\ 4 - decimal\ places$

Let $F(x) = e^{-x} - x = 0$ $\longrightarrow$ *eq (1)*

$To\ find\ x_0\ and\ x_1$

$$F(0) = e^{-0} - 0 = 1$$

$$F(1) = e^{-1} - 1 = -0.63212$$

$Since\ f(0)\ and\ f(1)\ are\ of\ opposite\ sign, root\ lies\ b|w\ x_0 = 0\ and\ x_1 = 1$

$x_0 = 0$ $\quad x_1 = 1$ $\quad f(x_0) = 1$ $\quad f(x_1) = -0.63212$

The first approximate root

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \qquad x_2 = \frac{0(-0.63212) - (1)(1)}{(-0.63212) - (1)}$$

$x_2 = 0.61270$ $\qquad f(x_2) = -0.070814$

Secant method

The 2nd approximate root

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \qquad x_3 = \frac{(1)(-0.070814) - (0.61270)(-0.63212)}{(-0.070814) - (-0.63212)}$$

$$x_3 = 0.5638187 \qquad f(x_3) = 0.005213$$

The 3rd approximate root

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \qquad x_4 = \frac{(0.61270)(0.005213) - (0.5638187)(-0.070814)}{(0.005213) - (-0.070814)}$$

$$x_4 = 0.5671 \qquad f(x_4) = -0.00001$$

Hence the approximate root of the equation correct upto 4-decimal places is x=0.5671

Thank You !