Numerical Analysis

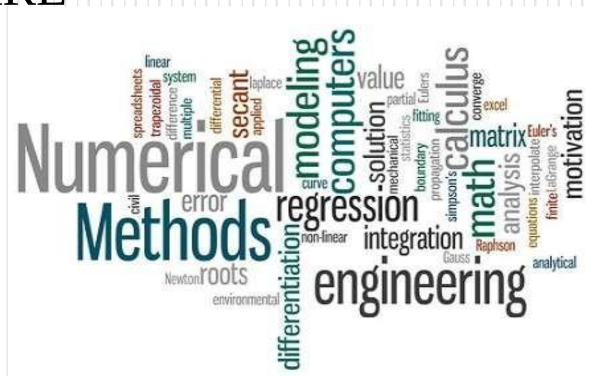
Dr. Farman Ali

Assistant Professor

DEPARTMENT OF SOFTWARE

SEJONG UNIVERSITY

Week 10, Lecture-I-II



farmankanju@sejong.ac.kr

Course Syllabus

- **Introduction:** Numerical analysis and background, definitions of computer number systems, floating point representation, representation of numbers in different bases, and round-off errors.
- **Root Finding:** Bisection method, fixed-point iteration, Newton's method, the secant method and their error analysis and order of convergence (Newton’s method and Secant method).
- **Direct Methods for Solving Linear Systems:** Gaussian elimination, LU decomposition, pivoting strategies, and PA=LU-factorization,…..
- **Polynomial:** Polynomial interpolation, Lagrange interpolation, Piecewise interpolation, divided differences interpolation, and curve fitting in interpolation (Application: Regression)…...
- **Integration:** Numerical differentiation, numerical integration, and composite numerical integration…….
- **Ordinary Differential Equations:** Euler's Method, and Runge-Kutta methods…..

- **Polynomial Interpolation**
- **Lagrange interpolation**
- **Piecewise linear interpolation**

Data and functions

- This lecture is about working with data.
- Example: A census of the population of the US is taken every 10 years.
- If we want to know the population of the US in year 1965 or year 2010, we have to fit a function through the given data.

Year	Population (in thousands)
1940	132,165
1950	151,326
1960	179,323
1970	203,302
1980	226,542
1990	249,633

Goal: To fit functions through data and find missing values.

Interpolation and polynomial

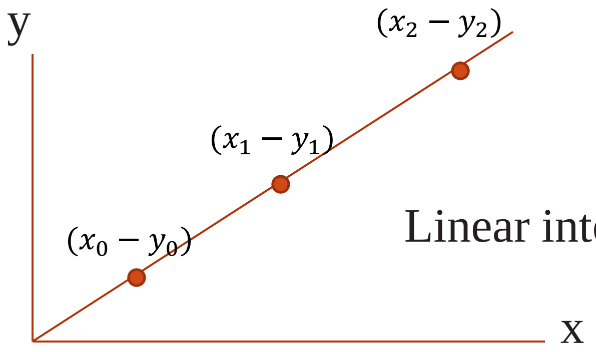
- Definition: The process of fitting a function through given data is called **interpolation.**

- Usually when we have data, we don't know the function f(x) that generated the data. So we fit a certain class of functions.

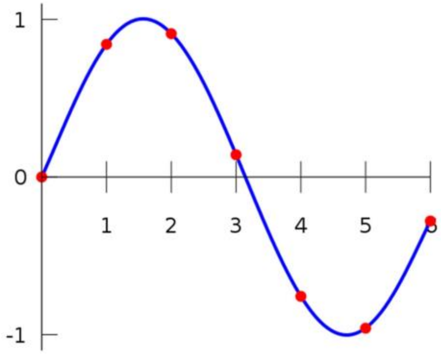
- The most usual class of functions fitted through data are **polynomials**.

- **Polynomials** are often used because they have the property of approximating any continuous function.

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n



Linear interpolation $y = a_0 + a_1 x$

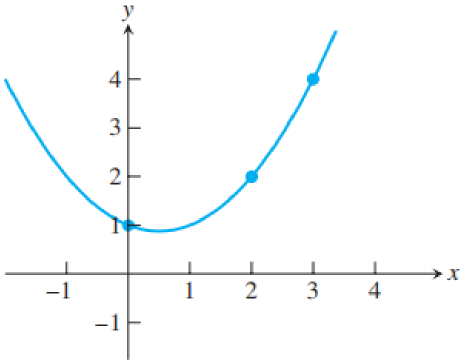


Data and interpolating functions

- Definition: The process of fitting a polynomial through given data is called **polynomial interpolation.**

- For example, a set of (x,y) data points, such as (0,1),(2,2), and (3,4). There is a function that passes through the three points.

- This function is called the degree 2 interpolating polynomial passing through the three points.



- The points (0,1), (2,2), and (3,4) are interpolated by the function P(x) = $\frac{1}{2}x^2 - \frac{1}{2}x + 1$.
- Definition: The function $y = P(x)$ **interpolates** the data points $(x_1, y_1), \ldots, (x_n, y_n)$ if $P(x_i) = y_i$ for each $1 \leq i \leq n$.

Lagrange Interpolation

Lagrange interpolation

- When n=2, the problem becomes: Given:

$$\begin{array}{cc} x_0 & x_1 \\ f_0 & f_1 \end{array}$$

- Find a polynomial of degree one such that

$$P(x_0) = f_0 \qquad\qquad P(x_1) = f_1$$

$$y = f_0 \frac{x - x_1}{x_0 - x_1} + f_1 \frac{x - x_0}{x_1 - x_0}$$

Denote by $L_0(x)$ and $L_1(x)$ the two first degree polynomials:

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

So we can rewrite the polynomial that fits the data in the form:

$$P(x) = f_0 L_0(x) + f_1 L_1(x)$$

Lagrange interpolation

- Consider the points with unequal interval;
- Given:
  - x_0, x_1, x_2, x_3
  - $f_0,\ f_1,\ f_2,\ f_3$

General Formula for any x, f(x) is given by

$$f(x) = \underbrace{f_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}}_{f_0L_0(x)} + \underbrace{f_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}}_{f_1L_1(x)}$$

$$+\underbrace{f_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}}_{f_2L_2(x)} + \underbrace{f_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}}_{f_3L_3(x)}$$

$$P(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x) + f_3L_3(x)$$

Called Lagrange interpolation formula

Lagrange interpolation

- Now we generalize the approach to n+1 points.
- Given:
  - $x_0, x_1, x_2 \dots., x_n$
  - $f_0\, f_1\, f_2, \dots., f_n$
- In general, suppose that we are presented with n points $(x_0, f_0), \dots. (x_n, f_n)$. For each k between 0 and n, define the degree n + 1 polynomial

$$L_k(x) = \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)}$$

- The interesting property of L_k is that $L_k(x_k) = 1$, while $L_k(x_j) = 0$, where x_j is any of the other data points.
- $L_k(x)$ is called basic Lagrange polynomial of degree n.
- Then $P_{n-1}(x) = f_0 L_0(x) + f_1 L_1(x) + \dots. f_n L_n(x)$
- We would have $P(x_k) = f_k$

P(x) is called the nth Lagrange interpolating polynomial.

Lagrange interpolation

- **Example:** Consider the following table of functional values

i	x_i	f_i
0	0.40	-0.916291
1	0.50	-0.693147
2	0.70	-0.356675
3	0.80	-0.223144

Find $\mathbf{f(0.60)}$.

Solution: The Lagrange interpolation formula is

$$f(x) = f_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$+f_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Lagrange interpolation

$$f_0L_0(x) = -0.916291 \cdot \frac{(0.60-0.50)(0.60-0.70)(0.60-0.80)}{(0.40-0.50)(0.40-0.70)(0.40-0.80)} = 0.1527151667$$

$$f_1L_1(x) = -0.693147 \cdot \frac{(0.60-0.40)(0.60-0.70)(0.60-0.80)}{(0.50-0.40)(0.50-0.70)(0.50-0.80)} = -0.462098$$

$$f_2L_2(x) = -0.356675 \cdot \frac{(0.60-0.40)(0.60-0.50)(0.60-0.80)}{(0.70-0.40)(0.70-0.50)(0.70-0.80)} = -0.23778333$$

$$f_3L_3(x) = -0.223144 \cdot \frac{(0.60-0.40)(0.60-0.50)(0.60-0.70)}{(0.80-0.40)(0.80-0.50)(0.80-0.70)} = 0.0371906667$$

$$\mathrm{P}(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x) + f_3L_3(x)$$

$$\mathrm{P}(x) = -0.509976$$

Lagrange interpolation

- **Example:** Consider the following table of functional values

i	x_i	f_i
0	0.40	-0.916291
1	0.50	-0.693147
2	0.70	-0.356675
3	0.80	-0.223144

Apply inverse Lagrange's method to find the value of x when f(x)= -0.509976

Solution: The Lagrange interpolation formula is

$$x = x_0 \frac{(f-f_1)(f-f_2)(f-f_3)}{(f_0-f_1)(f_0-f_2)(f_0-f_3)} + x_1 \frac{(f-f_0)(f-f_2)(f-f_3)}{(f_1-f_0)(f_1-f_2)(f_1-f_3)}$$

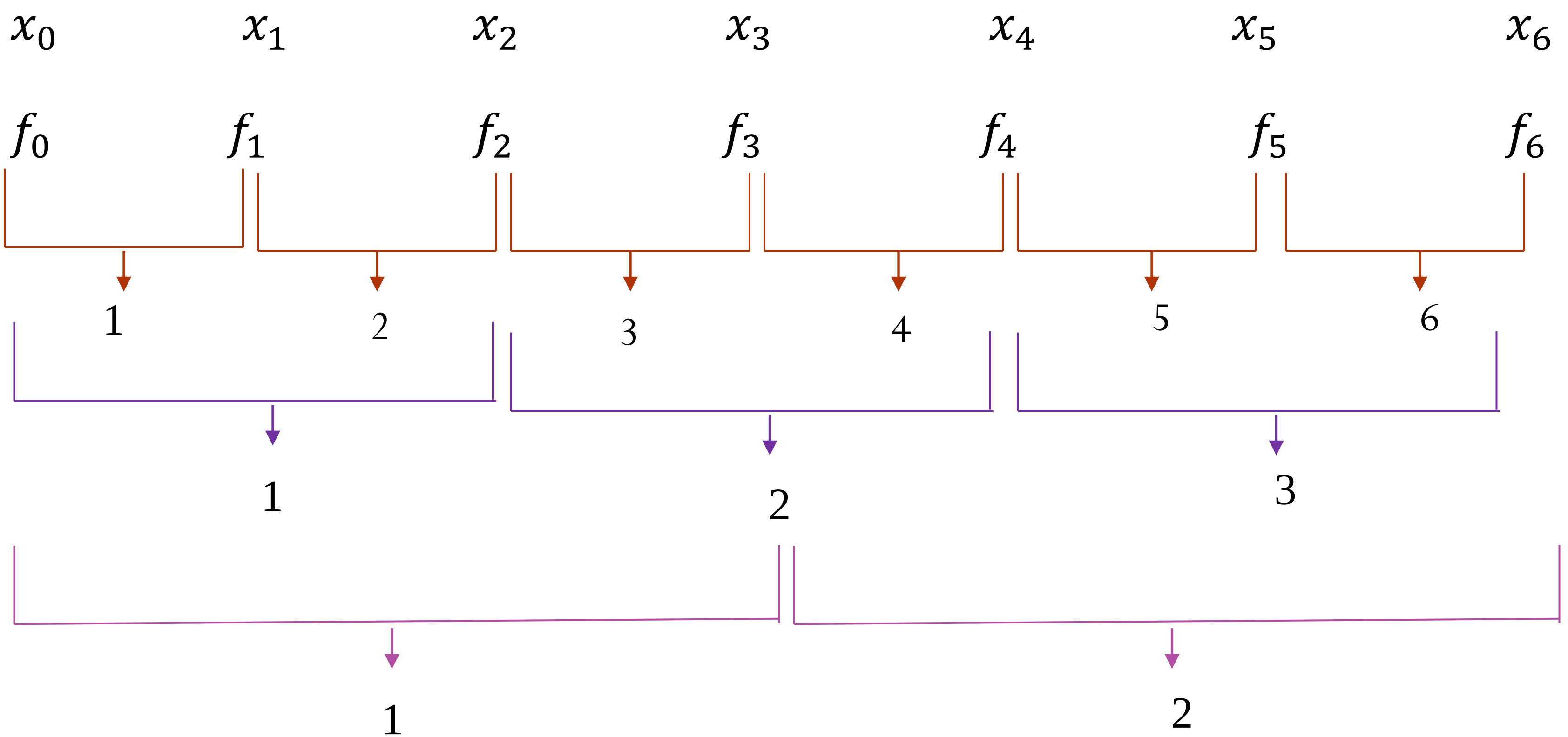
$$+x_2 \frac{(f-f_0)(f-f_1)(f-f_3)}{(f_2-f_0)(f_2-f_1)(f_2-f_3)} + x_3 \frac{(f-f_0)(f-f_1)(f-f_2)}{(f_3-f_0)(f_3-f_1)(f_3-f_2)}$$

Piecewise linear interpolation

Why do we need piecewise interpolation?

- When data is large, a polynomial of higher degree is generated whose computations becomes costly.
- The computed results become unreliable because of roundoff errors.
- In order to keep the degree of interpolating polynomial small and also to achieve accurate results we use piecewise interpolation.
- In piecewise interpolation, we subdivide the given intervals [a, b] into a number of sub intervals $[x_{i-1}, x_i]$, i=1,2,….n and approximate the function by some lower degree polynomial in each subinterval .
- For example, we subdivide the interval [a, b] where a $= x_0 < x_1 < x_2, \dots \dots < x_n = b$ in to a number of non-overlapping subintervals each containing 2 or 4 nodal points.

Piecewise linear interpolation



- Piecewise linear polynomial
- Piecewise quadratic polynomial
- Piecewise cubic polynomial

Piecewise linear interpolation

Recall: Lagrange interpolation for two points

- Suppose we have

x_0	x_1
f_0	f_1

- $P(x) = f_0 L_0(x) + f_1 L_1(x)$

This formula we will use in each subintervals

$$\Rightarrow \mathrm{P}(x) = f_0 \frac{(x - x_1)}{(x_0 - x_1)} + f_1 \frac{(x - x_0)}{(x_1 - x_0)}$$

Definition: Suppose we have n+1 distinct nodal/ data points a $= x_0, x_1, x_2, \ldots \ldots, x_n = b$ we can subdivide [a, b] into n subintervals say $x \in [x_{i-1}, x_i]$ using linear Lagrange interpolation

- Suppose we need to interpolate at $x \in [x_{i-1}, x_i]$ using linear Lagrange interpolation

x_{i-1}	x_i
f_{i-1}	f_i

$$\Rightarrow P_i(x) = f_{i-1} \frac{(x - x_i)}{(x_{i-1} - x_i)} + f_i \frac{(x - x_{i-1})}{(x_i - x_{i-1})}, i = 1,2,..n$$

Piecewise linear interpolation

Question: Obtain the piecewise linear interpolation polynomial for the function f(x) defined by the data

x	1	2	4	8
f	3	7	21	73

Hence, estimate the value of f(3) and f(7)

Solution:

In the interval [1, 2], we have

$$P_1(x) = f_0 \frac{(x - x_1)}{(x_0 - x_1)} + f_1 \frac{(x - x_0)}{(x_1 - x_0)}$$

$$P_1(x) = 3\frac{(x - 2)}{(1 - 2)} + 7\frac{(x - 1)}{(2 - 1)}$$

$$P_1(x) = -3x + 6 + 7x - 7$$

$$P_1(x) = 4x - 1$$

In the interval [2, 4], we have

$$P_2(x) = f_1 \frac{(x - x_2)}{(x_1 - x_2)} + f_2 \frac{(x - x_1)}{(x_2 - x_1)}$$

$$P_2(x) = 7\frac{(x - 4)}{(2 - 4)} + 21\frac{(x - 2)}{(4 - 2)}$$

$$P_2(x) = 7x - 7$$

Piecewise linear interpolation

In the interval [4, 8], we have

$$P_2(x) = f_2 \frac{(x - x_3)}{(x_2 - x_3)} + f_3 \frac{(x - x_2)}{(x_3 - x_2)}$$

$$P_2(x) = 21 \frac{(x - 8)}{(4 - 8)} + 73 \frac{(x - 4)}{(8 - 4)}$$

$$P_2(x) = 13x - 31$$

Hence, the piecewise linear interpolating polynomials are

$$\mathrm{P(x)} = \begin{cases} 4x - 1 & 1 \le x \le 2 \\ 7x - 7 & 2 \le x \le 4 \\ 13x - 31 & 4 \le x \le 8 \end{cases}$$

Here, the generated interpolating polynomials are small degree polynomials (means linear or single degree) and therefore, this method is very useful.

Now we need to find f(3) and f(7)

$3 \in [2,4]$, Therefore $7x - 7$ will be used

$$f(3) = 7(3) - 7 = 14$$

$7 \in [4,8]$, Therefore $13x - 31$ will be used

$$f(7) = 13(7) - 31 = 60$$

Thank You !