Some Bounds for the Energy of Graphs

Fatemeh Koorepazan-Moftakhar

Comenius University, Bratislava, Slovakia

Sharif University of Technology, Tehran, Iran

Jonit work with Algebraic Graph Theory Group of SUT

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Definition

The singular values of a matrix A are defined as the square roots of the eigenvalues of A^*A, and the energy of A denoted by $\mathcal{E}(A)$ is the sum of its singular values. The energy of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of the eigenvalues of its adjacency matrix.

- I. Gutman, The energy of a graph, Ber. Math. –Statist. Sekt. Forschungsz. Graz. 103 (1978) 1–22.

Definition

A *factor* of a graph G, is a spanning subgraph of G. A k-regular factor is called a *k-factor*. Let $0 \leq a \leq b$ are two integers. An $\{a, b\}$*-factor* is a factor all of whose components are a-regular or b-regular.

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Theorem

Let G be a graph of order n. If G has a $\{1, 2\}$-factor, then $\mathcal{E}(G) \geq n$.

- A. Aashtab, S. Akbari, E. Ghasemian, A.H. Ghodrati, M.A. Hosseinzadeh, F. Koorepazan-Moftakhar,

On the minimum energy of regular graphs. Linear Algebra Appl. 581 (2019) 51–71.

Proof.

Since G has a $\{1,2\}$-factor, there is a permutation matrix $B = [b_{ij}]$ such that for every i, j, if $b_{ij} = 1$, then $a_{ij} = 1$. If we denote the singular values of A and B with $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ and $\sigma_1(B) \geq \cdots \geq \sigma_n(B)$, respectively, then by Von Neumann's trace theorem, we have:

$$\sum_{i=1}^{n} \sigma_i(A)\sigma_i(B) \geq \operatorname{tr}(AB^T) = n.$$

Also since $B^T B = I$, $\sigma_1(B) = \cdots = \sigma_n(B) = 1$. Therefore,

$$\mathcal{E}(G) = \sum_{i=1}^{n} \sigma_i(A) \geq n.$$

□

Definition

A graph G with n vertices is said to be *hypoenergetic* if $\mathcal{E}(G) < n$. Graphs for which $\mathcal{E}(G) \geq n$, are said to be *non-hypoenergetic*, which extensively studied by Gutman.

- I. Gutman, Hyperenergetic and hypoenergetic graphs, in: D. Cvetković, I. Gutman (Eds.), Selected Topics on Applications of Graph Spectra, Math. Inst., Belgrade, 2011, pp. 113–135.

Theorem

If the graph G is regular of any non-zero degree, then G is non-hypoenergetic.

- I. Gutman, S. Zare Firoozabadi, J. A. de la Pena, J. Rada, On the energy of regular graphs, MATCH Commun. Math. Comput. Chem. 57 (2007) 435–442.

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Theorem 1

If the graph G is non-singular (i.e., no eigenvalue of G is equal to zero), then G is non-hypoenergetic.

- I. Gutman, On graphs whose energy exceeds the number of vertices, Linear Algebra Appl. 429 (2008) 2670–2677.

Next theorem is an improvement of Theorem (1) for regular graphs whose adjacency matrices are non-singular.

Theorem

Let G be a k-regular graph whose adjacency matrix is non-singular. Then $\mathcal{E}(G) \geq n + k - 1$.

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Theorem

For every positive integer k, there exists a positive integer N, such that for every hamiltonian subcubic graph G of order $n \geq N$, $\mathcal{E}(G) \geq n + k$.

- A. Aashtab, S. Akbari, E. Ghasemian, A.H. Ghodrati, M.A. Hosseinzadeh, F. Koorepazan-Moftakhar,

On the minimum energy of regular graphs. Linear Algebra Appl. 581 (2019) 51–71.

Conjecture

There exists N such that for every connected cubic graph of order $n \geq N$, $\mathcal{E}(G) \geq 1.24n$.

- A. Aashtab, S. Akbari, E. Ghasemian, A.H. Ghodrati, M.A. Hosseinzadeh, F. Koorepazan-Moftakhar, On the minimum energy of regular graphs. Linear Algebra Appl. 581 (2019) 51–71.

Table 1: Minimum energy of traceable subcubic graph of small order.

n	4	5	6	7	8	9
$\frac{min(\mathcal{E}(P_n^*))}{n}$	1	0.979	1	1.081	1.142	1.13
n	10	11	12	13	14	15
$\frac{min(\mathcal{E}(P_n^*))}{n}$	1.129	1.117	1.15	1.144	1.167	1.173

Lemma

Let G be a traceable subcubic graph of order $n \geq 8$, then $\mathcal{E}(G) > 1.1n$.

• A. Aashtab, S. Akbari, E. Ghasemian, A.H. Ghodrati, M.A. Hosseinzadeh, F. Koorepazan-Moftakhar, On the minimum energy of regular graphs. Linear Algebra Appl. 581 (2019) 51–71.

Lemma

Let G be a hamiltonian subcubic graph of order $n \geq 11$, then $\mathcal{E}(G) > n + 2$.

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Theorem

Let G be a connected subcubic graph of order $n \geq 8$. Then $\mathcal{E}(G) \geq 1.01n$.

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Theorem

Let G be a cubic graph of order n. Then the following holds:

i) If $n = 6k$, then $\mathcal{E}(G) \geq n$ and the equality holds if and only if $G = K_{3,3}^{\frac{n}{6}}$.
ii) If $n = 6k + 4$, then $\mathcal{E}(G) \geq n + 2$ and the equality holds if and only if $G = K_{3,3}^{\frac{n-4}{6}} \cup K_4$.

• A. Aashtab, S. Akbari, E. Ghasemian, A.H. Ghodrati, M.A. Hosseinzadeh, F. Koorepazan-Moftakhar, On the minimum energy of regular graphs. Linear Algebra Appl. 581 (2019) 51–71.

Conjecture

Let G be a cubic graph of order n. If $n = 6k + 8$, then $\mathcal{E}(G) \geq n + 2\sqrt{5} + \sqrt{17} - 5$, and the equality holds if $G = K_{3,3}^{\frac{n-8}{6}} \cup H$, where H is the graph depicted in the following figure.

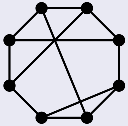

Figure 1: Cubic graph H of order 8 with minimum energy $\mathcal{E}(H) = 2\sqrt{5} + \sqrt{17} + 3 \approx 11.59$.

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Theorem

Let A be a Hermitian matrix in the following block form:

$$\begin{pmatrix} B & D \\ D^* & C \end{pmatrix}.$$

Then, $\mathcal{E}(A) \geq 2\mathcal{E}(D)$.

• S. Akbari, A.H. Ghodrati, M.A. Hosseinzadeh, Some lower bounds for the energy of graphs, Linear Algebra Appl. 591 (2020), 205–214.

For every proper non-empty subset $X \subset V(G)$, the subset of edges which connect a vertex of X to a vertex of $V(G) \setminus X$ is called an edge cut of G.

Theorem

Let G be a graph and H be a spanning subgraph of G such that $E(H)$ is an edge cut of G. Then, $\mathcal{E}(H) \leq \mathcal{E}(G)$.

- S. Akbari, A.H. Ghodrati, M.A. Hosseinzadeh, Some lower bounds for the energy of graphs, Linear Algebra Appl. 591 (2020), 205–214.

J. Day, W. So, Graph energy change due to edge deletion, Linear Algebra Appl. 428 (2008) 2070–2078.

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S. Akbari, E. Ghorbani, M.R. Oboudi, Edge addition, singular values, and energy of graphs and matrices, Linear Algebra Appl. 430(8–9) (2009) 2192–2199.

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Corollary

Adding any number of edges to each part of a bipartite graph, does not decrease its energy.

- S. Akbari, A.H. Ghodrati, M.A. Hosseinzadeh, Some lower bounds for the energy of graphs, Linear Algebra Appl. 591 (2020), 205–214.

Lemma

Let G be a graph whose cycles have odd lengths and are vertex disjoint. If from each cycle of G, we remove one edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.

• F. Ashraf, Energy, matching number and odd cycles of graphs, *Linear Algebra Appl.* **577** (2019) 159–167.

Lemma

Let G be a graph all of whose cycles are vertex disjoint and the length of every cycle is $2 \ (mod\ 4)$. If from each cycle of G, we remove one edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.

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Theorem

Let G be a graph all of whose cycles are vertex disjoint and the length of each cycle is not 0, modulo 4. Suppose that T is an arbitrary spanning tree of G, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.

- S. Akbari, K. C. Das, M. Ghahremani, F. Koorepazan-Moftakhar, E. Raoufi, Energy of Graphs Containing Disjoint Cycles, MATCH Commun. Math. Comput. Chem. 86(3) (2021) 543-547.

Conjecture

Let G be a C_4-free graph whose cycles are vertex disjoint. If from each cycle of G, we remove an arbitrary edge to obtain a tree T, then $\mathcal{E}(G) \geq \mathcal{E}(T)$.

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Remark

By means of a computer-aided search “Sage”, for connected C_4-free unicyclic graphs of order up to 14, we see this conjecture is correct.

Theorem

For any graph G, we have $\mathcal{E}(G) \geq 2\nu(G)$. Moreover, the equality holds if and only if G is a union of some complete bipartite graphs with parts of equal sizes and possibly some isolated vertices.

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Theorem

Let G be a graph whose cycles are vertex-disjoint. If we denote the number of odd cycles of G with length at least 5 by $c_o(G)$, then

$$\mathcal{E}(G) \geq 2\nu(G) + c_o(G).$$

- F. Ashraf, Energy, matching number and odd cycles of graphs, Linear Algebra Appl. 577 (2019) 159–167.

Theorem

Let G be a connected graph with k cycles all of whose cycles are vertex disjoint. If $|E(G)| = m$, then

$$\mathcal{E}(G) \leq 2\sqrt{(\nu + \frac{k}{2})m},$$

where ν is the matching number of graph G.

- S. Akbari, K. C. Das, M. Ghahremani, F. Koorepazan-Moftakhar, E. Raoufi, Energy of Graphs Containing Disjoint Cycles, MATCH Commun. Math. Comput. Chem. 86(3) (2021) 543-547.

Theorem

Let G be a graph of order n and size m. Then the following holds:

$$\mathcal{E}(G) \geq \frac{1}{2}\left(r(n-2) + \sqrt{r^2(n-2)^2 + 16m}\right),$$

which $r = \min\{|\lambda| : \lambda \in Spec(G)\}$.

Theorem

Let G be a connected graph of order n and size m with the adjacency matrix A. If $G \notin \{K_3, K_4, K_5, K_6\}$, then the following holds:

$$\mathcal{E}(G) \geq \sqrt{4m + n(n-2)|detA|^{\frac{2}{n}}}.$$

- S. Akbari, A.H. Ghodrati, M.A. Hosseinzadeh, Some lower bounds for the energy of graphs, Linear Algebra Appl. 591 (2020), 205–214.

B. Zhou studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters. He proved his result for quadrangle-free graphs.

- B. Zhou, Lower bounds for the energy of quadrangle-free graphs, MATCH Commun. Math. Comput. Chem. 55 (2006) 91–94.

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Recently, it is shown that for every graph G, $\mathcal{E}(G) \geq 2\delta(G)$, where $\delta(G)$ is the minimum degree of G, and the equality holds if and only if G is a complete multipartite graph with equal size of parts.

• X. Ma, A low bound on graph energy in terms of minimum degree, MATCH Commun. Math. Comput. Chem. 81 (2019) 393–404.

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- X. Ma, A low bound on graph energy in terms of minimum degree, MATCH Commun. Math. Comput. Chem. 81 (2019) 393–404.

Akbari and Hosseinzadeh provided a short proof for this result. Also, they gave an affirmative answer to a problem proposed in this paper.

- S. Akbari, M. A. Hosseinzadeh, A Short Proof for Graph Energy is at Least Twice of Minimum Degree, MATCH Commun. Math. Comput. Chem. 83(3) (2020) 631-633.

Conjecture

For every graph with maximum degree $\Delta(G)$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph.

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Theorem

They proved the validity of this conjecture for planar graphs, triangle-free graphs and quadrangle-free graphs.

- S. Akbari, M. Ghahremani, M. A. Hosseinzadeh, S. K. Ghezelahmad, H. Rasouli, A. Tehranian, A Lower Bound for Graph Energy in Terms of Minimum and Maximum Degrees, MATCH Commun. Math. Comput. Chem. 86(3) (2021) 549-558.

Conjecture 1 is valid for a number of nonsingular graphs, in particular, for those that satisfy either $n \geq \Delta + \delta$, or $|detA| \geq \lambda_1$, or $\sqrt{2m + n(n-1)} \geq \Delta + \delta$, or $\lambda_1 - \ln \lambda_1 \geq \delta$, or $\Delta \leq (n-1)^{1-\frac{1}{n}}$.

- S. Al-Yakoob, S. Filipovski, D. Stevanovic: Proofs of a Few Special Cases of a Conjecture on Energy of Non-Singular Graphs, MATCH Commun. Math. Comput. Chem. 68(3) (2021) 577-586.

They presented a proof of this conjecture for hyperenergetic graphs, and they proved an inequality that appears to support the conjectured inequality.

- S. Filipovski, R. Jajcay, Bounds for the Energy of Graphs, Mathematics 9(14) (2021) 1687.


Photo: Linn Haglund

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Fourth International Conference on Computational Algebra, Computational Number Theory and Its Application, February 2023, Kashan, Iran Ali Reza Ashrafi ashrafi@kashanu.ac.ir