Constructions of Small Regular Graphs of Given Degree and Girth Using Voltage Lift Graphs and Computer Searches

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The Cage Problem

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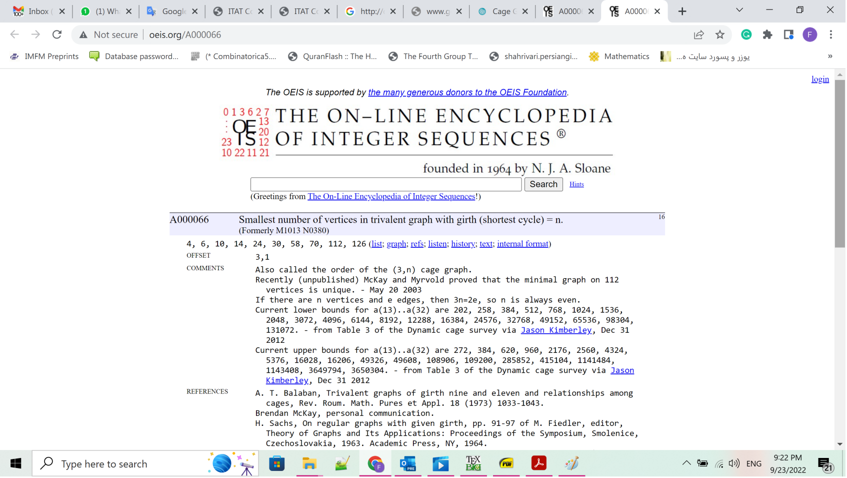
- $n(k, g) =$ the order of a smallest (k, g)-graph.
- A (k, g)-graph of the smallest order is called a (k, g)-cage

g	$n(3, g)$	$n(4, g)$	$n(5, g)$	$n(6, g)$	$n(7, g)$
Sloane	A000066				
3	4	5	6	7	8
4	6	8	10	12	14
5	10	19	30	40	50
6	14	26	42	62	90
7	24	67	152	294	
8	30	80	170	312	
9	58	275			
10	70	384			
11	112				
12	126	728	2730	7812	



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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000066 Smallest number of vertices in trivalent graph with girth (shortest cycle) = n. (Formerly M1013 N0380)

4, 6, 10, 14, 24, 30, 58, 70, 112, 126 (list; graph; refs; listen; history; text; internal format)

OFFSET 3,1

COMMENTS

Also called the order of the (3,n) cage graph.

Recently (unpublished) McKay and Myrvold proved that the minimal graph on 112 vertices is unique. - May 20 2003

If there are n vertices and e edges, then 3n=2e, so n is always even.

Current lower bounds for a(13)..a(32) are 202, 258, 384, 512, 768, 1024, 1536, 2048, 3072, 4096, 6144, 8192, 12288, 16384, 24576, 32768, 49152, 65536, 98304, 131072. - from Table 3 of the Dynamic cage survey via Jason Kimberley, Dec 31 2012

Current upper bounds for a(13)..a(32) are 272, 384, 620, 960, 2176, 2560, 4324, 5376, 16028, 16206, 49326, 49608, 108906, 109200, 285852, 415104, 1141484, 1143408, 3649794, 3650304. - from Table 3 of the Dynamic cage survey via Jason Kimberley, Dec 31 2012

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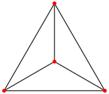
A. T. Balaban, Trivalent graphs of girth nine and eleven and relationships among cages, Rev. Roum. Math. Pures et Appl. 18 (1973) 1033-1043.

Brendan McKay, personal communication.

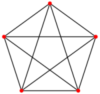
H. Sachs, On regular graphs with given girth, pp. 91-97 of M. Fiedler, editor, Theory of Graphs and Its Applications: Proceedings of the Symposium, Smolenice, Czechoslovakia, 1963. Academic Press, NY, 1964.

(v, g)	counts	named cages (or references)
(3, 3)	1	complete graph K_4
(3, 4)	1	complete bipartite graph $K_{3,3}$
(3, 5)	1	Petersen graph
(3, 6)	1	Heawood graph
(3, 7)	1	McGee graph
(3, 8)	1	Levi graph
(3, 9)	18	Biggs and Hoare (1980), Brinkmann et al. (1995)
(3, 10)	3	Balaban 10-cage, Harries graph, Harries-Wong graph
(3, 11)	1	Balaban 11-cage; Balaban (1973), Myrvold and McKay
(3, 12)	1	Tutte 12-cage; Polster et al. (1998, p. 179)

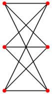
(3,3)-cage
tetrahedral graph



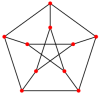
(3,3)-cage
pentatope graph



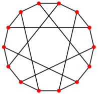
(3,4)-cage
utility graph



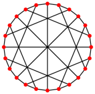
(3,5)-cage
Petersen graph



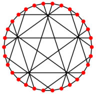
(3,6)-cage
Heawood graph



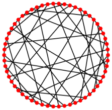
(3,7)-cage
McGee graph



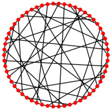
(3,8)-cage
Levi graph



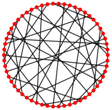
(3,9)-cage 1



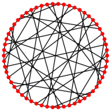
(3,9)-cage 2



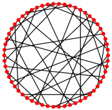
(3,9)-cage 3



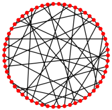
(3,9)-cage 4



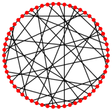
(3,9)-cage 5



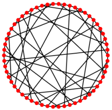
(3,9)-cage 6



(3,9)-cage 7



(3,9)-cage 8



Moore bound

$$n(k,g) \geq M(k,g) = \begin{cases} \frac{k(k-1)^{\frac{g-1}{2}}-2}{k-2}, & g \text{ odd}, \\ \\ \frac{2(k-1)^{\frac{g}{2}}-2}{k-2}, & g \text{ even}. \end{cases}$$

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Moore Graphs

(k,g)-graphs whose order is equal to $M(k,g)$.

Finding cages requires two steps:

- A good construction of a (k, g)-graph
- A proof that there is no smaller (k, g)-graph

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- A proof that there is no smaller (k, g)-graph

Since proving the minimality is usually very hard to get by, good constructions lead to **Record Graphs**.

Dynamic Cage Survey

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Abstract

A (k, g)-cage is a k-regular graph of girth g of minimum order. In this survey, we present the results of over 50 years of searches for cages. We present the important theorems, list all the known cages, compile tables of current record holders, and describe in some detail most of the relevant constructions.

Record Cubic Cages

Girth g	Lower bound	Upper bound	Due to
5	10	10	Petersen
6	14	14	Heawood
7	24	24	McGee
8	30	30	Tutte
9	58	58	Brinkmann-McKay-Saager
10	70	70	O'Keefe-Wong
11	112	112	McKay-Myrvold; Balaban
12	126	126	Benson
13	**202**	**272**	**McKay-Myrvold; Hoare**
14	258	384	McKay; Exoo
15	384	620	Biggs
16	512	960	Exoo
17	768	2176	Exoo
18	1024	2560	Exoo
19	1536	4324	Hoare, H(47)

Record Cubic Cages

Girth g	Lower bound	Upper bound	Due to
20	2048	5376	Exoo
21	3072	16028	Exoo
22	4096	16206	Biggs-Hoare, S(73)
23	6144	49326	Exoo
24	8192	49608	Bray-Parker-Rowley
25	12288	1089	Exoo
26	16384	109200	Bray-Parker-Rowley
27	24576	285852	Bray-Parker-Rowley
28	32768	368640	Erskine-Tuite
29	49152	805746	Erskine-Tuite
30	65536	806736	Erskine-Tuite
31	98304	1440338	Erskine-Tuite
32	131072	1441440	Erskine-Tuite

Voltage Graph Construction

- Γ, an undirected graph; possibly with parallel edges and loops
- each edge replaced by a pair of opposing arcs; denoted by $D(\Gamma)$

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voltage assignment

A **voltage assignment** on Γ is any mapping α from $D(\Gamma)$ into a group G that satisfies the condition $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for all $e \in D(\Gamma)$.

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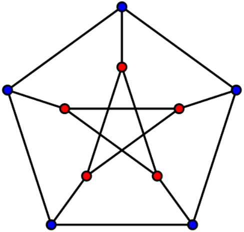
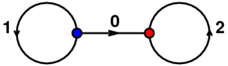
The **derived regular cover (lift)** of Γ with respect to the voltage assignment α is the graph denoted by $\Gamma^{\alpha,G}$:

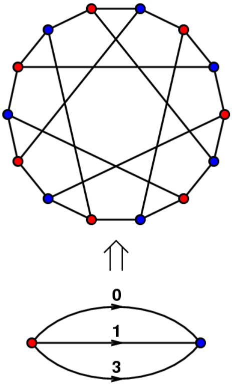
- $V(\Gamma^{\alpha,G}) = V(\Gamma) \times G$
- u_g and v_f are adjacent iff $e = (u,v) \in D(\Gamma)$ and $f = g \cdot \alpha(e)$



$\Uparrow$



$\mathbb{Z}_5$



$\mathbb{Z}_7$

Girth g	Lower bound	Upper bound	Due to
14	258	384	McKay; Exoo
16	512	960	Exoo
17	768	2176	Exoo
18	1024	2560	Exoo
20	2048	5376	Exoo

Gonet, Sophia Rose, "Jacobians of Finite and Infinite Voltage Covers of Graphs" (2021). Graduate College Dissertations and Theses. 1366.

Theorem

Let G be a base graph of girth g. Then there exists a voltage graph lift of G of girth $2g$.

- Exoo, G., Jajcay, R.: On the girth of voltage graph lifts. European J. Combin. **32**(4) (2011) 554–562.

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- $11 = \lceil \frac{258}{24} \rceil \leq |G| < \lfloor \frac{384}{24} \rfloor = 16$
  - $\mathbb{Z}_{11}$
  - $\mathbb{Z}_3 : \mathbb{Z}_4$, $\mathbb{Z}_{12}$, $\mathbb{A}_4$, $\mathbb{D}_{12}$, $\mathbb{Z}_6 \times \mathbb{Z}_2$
  - $\mathbb{Z}_{13}$
  - D_{14}, $\mathbb{Z}_{14}$
  - $\mathbb{Z}_{15}$

It is known that without loss of generality one may choose any spanning tree $\mathcal{T}$ of the base graph G and assign the identity 1_Γ of the voltage group Γ to all edges of $\mathcal{T}$ (every voltage lift is isomorphic to a lift whose spanning tree has been assigned the identity voltage).

• J.L. Gross, T.W. Tucker, Topological Graph Theory, Dover, Mineola, New York, 2001.

Definition

Any closed walk $\mathcal{W}$ in the base graph G lifts to a walk $\mathcal{W}^\alpha$ in the derived graph G^α. If $\mathcal{W}$ consists of the sequence of darts $e_1, e_2, \ldots, e_n$, the *net voltage* $\alpha(W)$ of W is the product $\alpha(e_1)\alpha(e_2)\cdots\alpha(e_n)$.

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Lemma

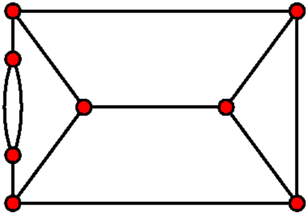
Let G be a finite graph and $\alpha : G \to \Gamma$ be a voltage assignment of G. The girth of the voltage graph lift G^α is equal to the length of a shortest closed non-reversing walk $\mathcal{W}$ in G of net voltage 1_Γ.

Algorithm 1: Creating all non-reversing closed walks of given length (Sage)

```
Input: A list of graph edges E(G) in the triple format of length l
Output: A list of all non-reversing closed walks of given length l
walk_list = []

main()
  {
  -E = Input a list of edges of graph G.
  -L = [E[1]].
  -#E[1] is the first edge of the list E.
  -max_depth = Input a depth for searching .
  -DFS_LFC(L,0).
  -Print(walk_list).
  }

DFS_LFC(path_edges,depth)
  {
  -If depth == max_depth:
    -Return path_edges
  -For all incident edge e of the last element f of path_edges:
    -If e is not the backward of f:
       -Append e to path_edges.
       -If initial vertex of the path_edge[1] equals to terminal vertex of e:
            -Add path_edges to walk_list whether there is not any permutation
             of path_edges and reverse of path_edges in the walk_list.
        -DFS_LFC(path_edges, depth+1).
   }
```



$$G = \mathrm{SmallGroup}(48, 18)$$

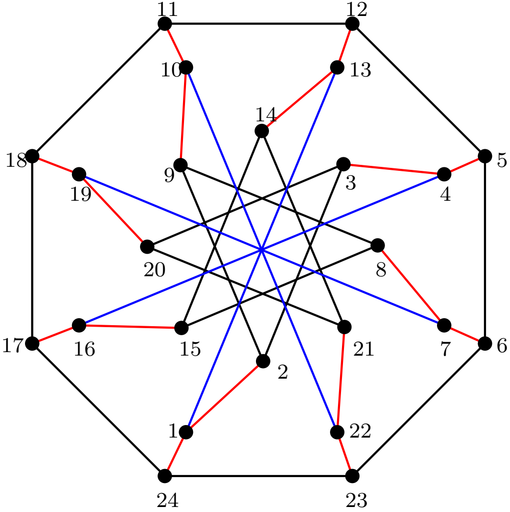


Figure 1. The McGee graph and its edge-orbits.

The order of the automorphism group of the McGee graph is 32 and the group is isomorphic $\mathbb{Z}_8 : (\mathbb{Z}_2 \times \mathbb{Z}_2)$. If we denote the vertices of the McGee graph the way they are denoted in Figure 1, this automorphism group is generated by the following three permutations:

β=(3,9)(4,10)(5,11)(6,18)(7,19)(8,20)(15,21)
(16,22)(17,23)
δ=(2,24)(3,17)(4,16)(5,15)(6,8)(9,23)(10,22)
(11,21)(12,14)(18,20)
λ=(1,4,19,22,13,16,7,10)(2,3,20,21,14,15,8,9)
(5,18,23,12,17,6,11,24)

The induced action of this group on the set of the edges of the graph has three orbits; of lengths 16, 4, and 16. These edge orbits are depicted in Figure 1 colored by different colors. We also present them as lists.

$$\begin{aligned} O_1 = [&[1,2],[13,14],[19,20],[7,8],[1,24],[12,13],[18,19],[6,7],\\ &[3,4],[15,16],[21,22],[9,10],[4,5],[16,17],[22,23],[10,11]](red) \end{aligned}$$

$$O_2 = [[1,13],[7,19],[4,16],[10,22]] \text{ (blue)}$$

$$\begin{aligned} O_3 = [&[2,3],[14,15],[2,9],[14,21],[3,20],[8,15],[20,21],[8,9],\\ &[17,24],[5,12],[23,24],[11,12],[17,18],[5,6],[11,18],[6,23]]\text{(black)} \end{aligned}$$

Let $\Gamma_1 = \langle \lambda \rangle$. In this case, the orbits O_1 and O_3 each split into two orbits, resulting in the action of Γ_1 on the edge set of the McGee graph having 5 orbits as follows:

$$\begin{aligned}
L_1 = [&[1,2],[13,14],[19,20],[7,8],[3,4],[15,16],\\
&[21,22],[9,10]],\\
L_2 = [&[1,13],[7,19],[4,16],[10,22]],\\
L_3 = [&[1,24],[12,13],[18,19],[6,7],[4,5],[16,17],\\
&[22,23],[10,11]],\\
L_4 = [&[2,3],[14,15],[20,21],[8,9],[3,20],[8,15],\\
&[14,21],[2,9]],\\
L_5 = [&[5,6],[17,18],[23,24],[11,12],[11,18],\\
&[6,23],[5,12],[17,24]]
\end{aligned}$$

- Start with one edge, assign a voltage and look whether there is a problem. If there is no problem (net voltage $<>$ () for all nRCW), assign voltage to another edge.

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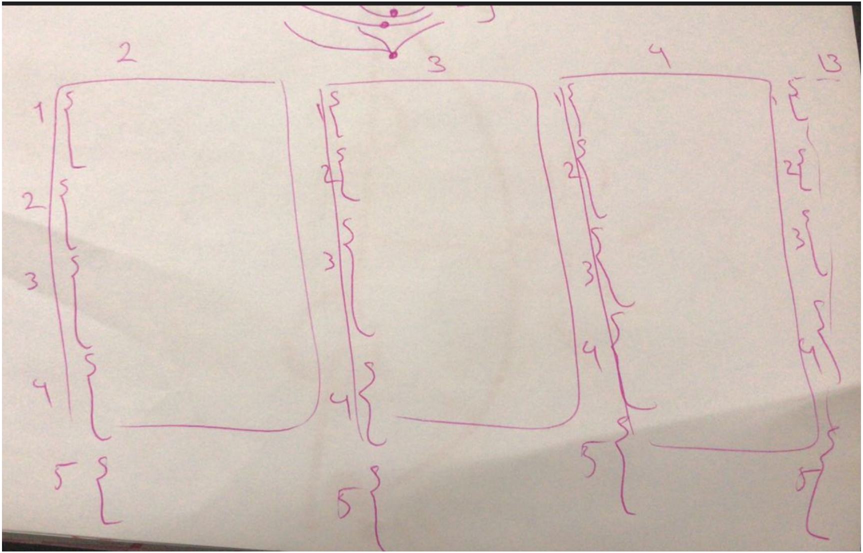
- **List the nRCW by the number of appearances of distinct non-tree edges. Start with one edge and assign possible voltages to this one edge and check all the cycles that only contain that edge. If any of them get identity net-voltage, you never try that assignment again. From now on you will only use the one that survived with the next edge. Remember which ones do not work and don't use them again.**

- $\underline{a}\ \underline{b}$ s.t. $b \neq a^{-1}$
- $\underline{a}\ \underline{b}\ \underline{c}$ s.t. $c \neq a^{-1}$ and $c \neq b^{-1}$ $[a * b * c = ()\ 888]$
- you only keep the ones that do not create a short cycle.
- **[our condition for surviving is non of the cycles which contains only those nRCW is identity.]** and most of them will be identity. You don't use those.
- he says you make the list of the once that don't create the identity as a voltage. You don't try all the cycles, just the cycles that contain this tree and if one of them is identity, forget that assignment and then you pick another one that shows up with this tree or those two edges.

Table 1: The McGee graph

number of non-tree edge	number of nRCW
1	13
2	65
3	144
4	178
5	137
6	77
7	25
8	18
9	5
10	0
11	0
12	0
13	0


2
3
4
13
1
2
3
4
5

Thank you!