$$\dot{x}(t) = Ax(t) + Bu(t) + \underbrace{B_v v(t)}_{\text{DETERMINISTIC DISTURBANCE}}$$
$$y(t) = Cx(t) + \underbrace{w(t)}_{\text{MEASUREMENT NOISE}}$$

We want to establish linear feedback, a linear feedback law

$$u(t) = -\underbrace{K}_{\text{FEEDBACK OR GAIN MATRIX } N_u \times N_x} x(t) + \underbrace{r(t)}_{\text{REFERENCE INPUT } (N_u)}$$

As we are assuming that all the state variables are measured the resulting feedback system is called full-state feedback

→ IT IS INTENDED THAT THE OUTPUT OBJECT SHOULD FOLLOW THE REFERENCE INPUT, IN SOME SENSE

SUBSTITUTING $u = -Kx + r$ INTO $\dot{x} = Ax + Bu$ WE GET

$$\dot{x} = Ax + B(-Kx + r) = \underbrace{Ax - BKx}_{} + Br$$

STATE EQUATION FOR THE CLOSED LOOP LINEAR SYS

→ THE SYSTEM IS ASYMPTOTICALLY STABLE BASED ON THIS MATRIX

→ UNDER SOME CONDITIONS, IT IS POSSIBLE TO ARBITRARILY ASSIGN THE EIGENVALUES OF THIS MATRIX (By setting matrix K)

ONE OF THE DRAWBACK OF STATE ~~FEEDBACK~~ CONTROL IS THAT THE OUTPUT OF THE SYSTEM IS NOT DIRECTLY INVOLVED IN THE CONTROL

⇝ we cannot assign a predetermined relashionship between the inputs and the outputs

CONSIDER THE CLOSED-LOOP SYSTEM $\dot{x} = A_K x + Br$

IT IS ASSUMED THAT THE SYSTEM IS IN STATIONARY STATE

⇝ THE DERIVATIVE OF THE STATE VECTOR IS ZERO

$$0 = A_K x_0 + B r_0$$

ASSUMING THAT MATRIX A_K IS NONSINGULAR, WE CAN COMPUTE THE STATIONARY STATE, SOLVING FOR x_0

$$x_0 = -A_K^{-1} B r_0$$

THE STATIONARY OUTPUT WITH THIS STATE IS $y_0 = Cx_0$
$= -CA_K^{-1} B r_0$

IF THE SYSTEM HAS THE SAME NUBER OF INPUTS AND OUTPUTS, THEN MATRIX $-CA_K^{-1} B r_0$ WILL BE SQUARE

THE REFERENCE VECTOR FOR A GIVEN OUTPUT

$$r_0 = -\underbrace{(CA_K^{-1} B)}_{\text{MUST BE NONSINGULAR}}{}^{-1} y_0$$

The basis for linear feedback control is a linear model, typically found by linearization of a nonlinear model along some stationary operating point

THIS MEANS THAT THIS REFERENCE SIGNAL IS USEFULL ONLY IF THE SYSTEM STATE IS EXACTLY THE STATIONARY STATE OF THE LINEARISATION

BASIC CONCEPTS RELATED TO STABILITY OF $\dot{x} = Ax$

$$x(t) = T e^{Dt} T^{-1} x(0)$$

WITH $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \ e^{\lambda_n t} \end{bmatrix}$

CLEARLY IF ANY $e^{\lambda_i t}$ IS UNSTABLE, THEN $x(t)$ WILL GROW AS TIME $\to \infty$

THIS IS ESPECIALLY TRUE IF WE HAVE $x(0)$ WITH ALL COMPONENTS THAT ARE NONZERO

LET $\lambda = a + ib$ THEN $e^{\lambda t} \rightsquigarrow$

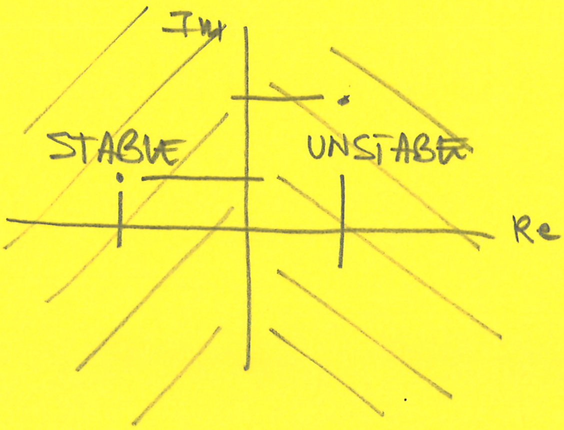
$$= e^{at} \left[\cos(bt) + i \sin(bt) \right]$$

BUT WE WE NEED TO GET A REAL SIGNAL $\to$ THIS GETS SORTED OUT BECAUSE FOR EACH $\lambda_i \in \mathbb{C}$ ALSO ITS CONJUGATE COMPLEX IS A SOLUTION OF THE CHARACTERISTIC EQUATION

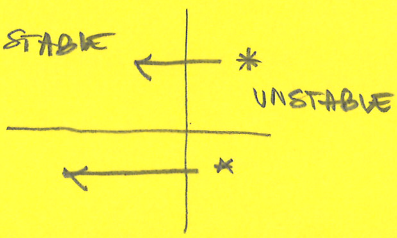
$\rightsquigarrow$ THE COMPLEX PART WILL CANCEL OUT

IF $a > 0$, THEN e^{at} WILL GROW EXPONENTIALLY (UNSTABLE)

IF $a < 0$, THEN e^{at} WILL DECAY EXPONENTIALLY (STABLE)



THE WHOLE IDEA OF CONTROL IS TO ACT ON THE SYSTEM IN SUCH A WAY THAT WHENEVER WE HAVE UNSTABLE EIGENVALUES WE CHANGE THEM TO BE STABLE $\longrightarrow$ THUS STABILIZING THE SYSTEM



WE MOMENTARILY CONSIDER THE DISCRETE-TIME VERSION OF $\dot{x} = Ax$

$x_{k+1} = \tilde{A} x_k$, WITH $x_k = x(k\Delta t)$ FOR SOME Δt

THIS IS THE DISCRETE VERSION OF MATRIX A

$\tilde{A} = e^{A\Delta t}$ (Related by the matrix exponential)

IN DISCRETE TIME, THE NOTION OF STABILITY

$\rightsquigarrow$ GIVEN $x(0)$, THEN

$$
\begin{aligned}
x(1) &= \bar{A} x_0 \\
x(2) &= \tilde{A}(x_1) = A^2 x_0 \\
x(3) &= \tilde{A}(x_2) = A^3 x_0 \\
&\vdots \\
x(N) &= \tilde{A}^N x(0)
\end{aligned}
$$

ANOTHER DIFFICULTY WITH STATE FEEDBACK CONTROL ARISES IF THE SYSTEM IS SUBJECTED TO DISTURBANCES

$\rightsquigarrow$ THE BASIC CONTROLLER DOES NOT TAKE A NON ZERO DISTURBANCE VECTOR INTO ACCOUNT

the solution to these limitation is found in augmentation with integrators

WE CAN INCLUDE INTEGRATION IN SEVERAL WAYS

WE DEFINE THE AUGMENTED STATE $x_a = [\underset{N_x}{x} \quad \underset{N_i}{x_i}]^T$

$$\dot{x} = Ax + Bu + B_v v$$
$$u = -Kx + K_i x_i$$
$$\dot{x}_i = -Cx + r$$
$$y = Cx$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \overbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}^{A_1} \begin{bmatrix} x \\ x_i \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_1} u + \underbrace{\begin{bmatrix} 0 \\ I_r \end{bmatrix}}_{B_r} r + \underbrace{\begin{bmatrix} B_v \\ 0 \end{bmatrix}}_{B_{v1}} v$$

$$y = \underbrace{[C \quad 0]}_{C_1} \begin{bmatrix} x \\ x_i \end{bmatrix}$$

OR, IN TERMS OF THE AUGMENTED STATE VECTOR

$$\begin{cases} V\dfrac{d[A]}{dt} = F([A]_f - [A]) - VK_1[A]^2 \\ V\dfrac{d[B]}{dt} = F([B]_f - [B]) + V(K_1[A]^2 - K_3[B]^{1/2}) \end{cases}$$

—

$$\begin{cases} \dfrac{d[A]}{dt} = \dfrac{F}{V}([A]_f - [A]) - K_1[A]^2 \\ \dfrac{d[B]}{dt} = \dfrac{F}{V}([B]_f - [B]) + (K_1[A]^2 - K_3[B]^{1/2}) \end{cases}$$

—

$$\begin{cases} \dot{x}_1 = \overbrace{\dfrac{F}{V}(u_1 - x_1) - K_1 x_1^2}^{f_1(x_1, x_2, u_1, u_2)} \\ \dot{x}_2 = \underbrace{\dfrac{F}{V}(u_2 - x_2) + K_1 x_1^2 - K_3 x_2^{1/2}}_{f_2(x_1, x_2, u_1, u_2)} \end{cases}$$

$$\begin{cases} x_1^{ss} \\ x_2^{ss} \\ u_1^{ss} \\ u_2^{ss} \end{cases}$$ STEADY STATE VALUES

$$f_1(x_1, x_2, u_1, u_2) = \underbrace{f_1(x_1^{ss}, x_2^{ss}, \cancel{u_1^{ss}}, u_2^{ss})}_{=0} + \frac{\partial f_1}{\partial x_1}\Big|_{ss}(x_1 - x_1^{ss}) + \frac{\partial f_1}{\partial x_2}\Big|_{ss}(x_2 - x_2^{ss})$$
$$+ \frac{\partial f_1}{\partial u_1}\Big|_{ss}(u_1 - u_1^{ss}) + \frac{\partial f_1}{\partial u_2}\Big|_{ss}(u_2 - u_2^{ss})$$

$$f_2(x_1, x_2, u_1, u_2) = \underbrace{f_2(x_1^{ss}, x_2^{ss}, \cancel{u_1^{ss}}, u_2^{ss})}_{=0} + \frac{\partial f_2}{\partial x_1}\Big|_{ss}(x_1 - x_1^{ss}) + \frac{\partial f_2}{\partial x_2}\Big|_{ss}(x_2 - x_2^{ss})$$
$$+ \frac{\partial f_2}{\partial u_1}\Big|_{ss}(u_1 - u_1^{ss}) + \frac{\partial f_2}{\partial u_2}\Big|_{ss}(u_2 - u_2^{ss})$$

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = \begin{bmatrix} \dot{x}_1 - \dot{x}_1^{ss} \\ \dot{x}_2 - \dot{x}_2^{ss} \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_2 & \partial f_2/\partial x_2 \end{bmatrix}_{ss} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} +$$
$$+ \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{bmatrix}_{ss} \begin{bmatrix} u_1 - u_1^{ss} \\ u_2 - u_2^{ss} \end{bmatrix}$$

$$\begin{cases} \partial f_1/\partial x_1 = -\frac{F}{V} - 2K_1 x_1 & \partial f_1/\partial x_2 = 0 \\ \partial f_2/\partial x_1 = 2K_1 x_1 & \partial f_2/\partial x_2 = -F/V - 1/2\, K_3 x_2^{-1/2} \end{cases}$$

$\rightarrow$ At SS $\rightarrow$ A

$$\begin{cases} \partial f_1/\partial u_1 = F/V & \partial f_1/\partial u_2 = 0 \\ \partial f_2/\partial u_1 = 0 & \partial f_2/\partial u_2 = F/V \end{cases}$$

$\rightarrow$ At SS $\rightarrow$ B

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = \begin{bmatrix} -F/V - 2K_1 x_1^{ss} & 0 \\ 2K_1 x_1^{ss} & -F/V - 1/2\, K_3 (x_2^{ss})^{-1/2} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} +$$

$$+ \begin{bmatrix} F/V & 0 \\ 0 & F/V \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$