FEEDBACK CONTROL & CONTROLLABILITY

WE STUDIED THE HOMOGENEOUS SYSTEM IN DETAIL $\dot{x} = Ax$

- THE FORCE-FREE RESPONSE $\underline{x_u(t) = e^{At}x(0)}$ (NO INPUT)
- STABILITY / EIGENVALUES $\underline{\det(\lambda I - A) = 0}$
- EIGENVALUES / DIAGONALIZATION

WE ALSO COMPUTED THE FORCED-RESPONSE (ZERO INITIAL STATE)

$$\underline{x_f = \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau}$$

(A WEIGHTED SUM OF THE INPUT SIGNAL, BY WEIGHTING FUNCTION $e^{A(t-\tau)}B$, A CONVOLUTION)

$\dot{x} = Ax + Bu$

BECAUSE THE SYSTEM IS LINEAR, THE COMPLETE SYSTEM RESPONSE IS GIVEN BY THE <u>SUPERPOSITION</u> OF THE RESPONSES

$$\boxed{x(t) = x_u(t) + x_f(t)}$$

GIVEN $x(0) = x_0$
AND $u(t)$ for $t \geq 0$

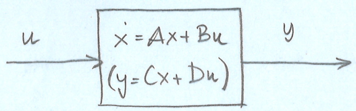
The idea of adding the controls means that we are interested in designing a function $u(t)$ such that we can transfer the system from any initial state to any final state

$\dot{x} = Ax + Bu$ WITH THE USUAL DIMENSIONS $\begin{cases} x \in \mathbb{R}^{N_x} \\ A \in \mathbb{R}^{N_x \times N_x} \end{cases}$

$(y = Cx + Du)$ $\begin{cases} u \in \mathbb{R}^{N_u} \\ B \in \mathbb{R}^{N_x \times N_u} \end{cases}$

$\hookrightarrow$ the density of a mixture as weighted sum of the densities of the pure components with weights the concentrations in the mixture

$\begin{cases} y \in \mathbb{R}^{N_y} \\ C \in \mathbb{R}^{N_y \times N_x} \end{cases}$



OUR NEXT QUESTIONS

- IS THE SYSTEM CONTROLLABLE? (we only checked stability)
- IF YES, HOW TO DESIGN $u(t)$ OPTIMALLY?

ONE, FORMAL, DEFINITION OF CONTROLLABILITY

"A linear and time invariant system $\dot{x}(t) = Ax(t) + Bu(t)$ is said to be controllable if and only if, it is possible to transfer the system from any arbitrary initial state $x(t_0) = x_0$ to any other final state $x(t_f)$, in a finite time $(t_f < \infty)$, only by choosing an appropriate control $u(t)$"

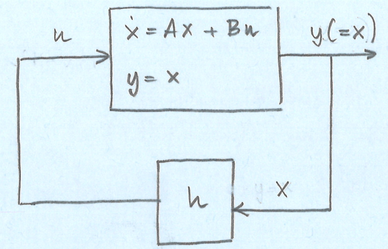
READ THIS AS "the control $u(t)$ is capable of influencing all the states through the integral $\int_{t_0}^{t} e^{A(t-\tau)} B u(t)\, d\tau$"

WE WILL MAKE A SIMPLIFYING ASSUMPTION AND ASSUME THAT WE CAN MEASURE ALL OF OUR STATE VARIABLES → THAT IS $y = x$, OR $y = Ix$ WITH MATRIX $C = I$

WE ARE ALSO ASSUMING THAT THERE IS NO INPUT FEEDTHROUGH → THAT IS $D = [0]$ (the input does not affect the measures)

The main idea is to design an input (a control) $u(t)$ which is a function of the state itself

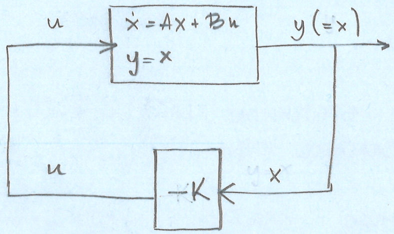
→ $u(t) = h(x(t))$



STATE FEEDBACK CONTROL

→ the question, how?

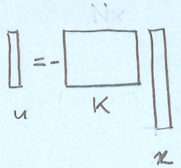
→ How to define / choose h, in such a way that $u(t) = h(x(t))$ is optimal



→ Function $h = -K$ is optimal in some sense for linear systems

$u(t) = -Kx(t)$

DIMENSIONS OF K?



→ $K \in \mathbb{R}^{N_u \times N_x}$

SO AMONG ALL POSSIBLE FUNCTIONS h THAT CAN TRANSFORM THE STATE TO MAKE AN INPUT SIGNAL, IT TURNS OUT THAT A SIMPLE MATRIX IS OPTIMAL FOR LINEAR SYSTEMS

→ ONLY ONE CONDITIONS 'CONTROLLABILITY'

THE FEEDBACK CONTROL LAW IS GIVEN A MATRIX $K \in \mathbb{R}^{N_u \times N_x}$

→ In general matrix K depends on time ($K = K(t)$)

→ For the zero-set-point case, we have $u(t) = \underbrace{-K(t)}x(t)$

↓

$K(t)$ IS THE CLOSED LOOP GAIN MATRIX

THERE EXIST SEVERAL PROCEDURES TO DETERMINE MATRIX $K(t)$

→ THEY DEPEND ON THE SPECIFIC CONTROL OBJECTIVE

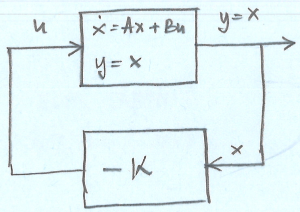
→ The first objective could be: IMPOSE PREDETERMINED DYNAMICS TO THE CLOSED-LOOP SYSTEM

↓

BY CHOOSING ITS EIGENVALS

↓ This leads to

$$\boxed{\text{A CONSTANT DYNAMIC MATRIX } K \neq K(t)}$$



$$\dot{x} = Ax + Bu$$

$$u = -Kx \quad \text{for some } K$$

By substituting we get

$$\boxed{\begin{aligned}\dot{x} &= Ax + B(\underbrace{-Kx}_{u}) \\ &= Ax - BKx \\ &= (A - BK)x\end{aligned}}$$

THIS IS AN HOMOGENEOUS SYSTEM, WE KNOW HOW TO TREAT IT

WE HAVE A 'NEW' A MATRIX, FOR THE SYSTEM WITH FEEDBACK CONTROL

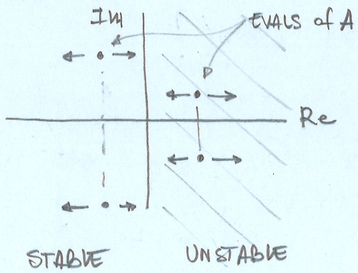
$$A_{FB} = (A - BK)$$

GIVEN

GIVEN

WE GET TO CHOOSE IT

WE CAN SELECT K SO THAT WE CAN PLACE THE EIGENVALUES OF A_{FB} WHERE EVER WE WANT THEM



– If the original system was unstable, we use K to make it stable

– If the original system was already stable, we use K to make it faster to reach steady-state

ALL THIS, AGAIN, ASSUMING THAT THE SYS IS CONTROLLABLE

CASE 1

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

STABLE AND CONTROLLABLE

or

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

UNSTABLE AND CONTROLLABLE (WE CAN STABILIZE)

CASE 1

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}[u]$$

STABLE AND UNCONTROLLABLE

or

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}[u]$$

UNSTABLE AND UNCONTROLLABLE

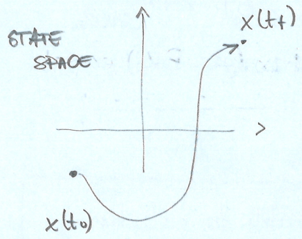
CASE 3

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & +1 \\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}[u]$$

??

IN SUMMARY, WITH MATRIX K (the controller) WE CHANGE THE ~~DYNAMIC~~ DYNAMICS OF THE SYSTEM

"CONTROLLABILITY": The pair (A,B) is said to be controllable (THE SYSTEM IS CONTROLLABLE) if for any initial condition x_0 and for any final condition $x(t_f)$, WE CAN compute a $u(t)$ that transfers the system from $x_0 \to x(t_f)$ in finite time, and the design of u is based on some matrix K



FOR SIMPLICITY, CONTROLLABILITY MEANS THAT WE CAN CHOOSE K SO THAT WE CAN PLACE THE EIGENVALUES OF A_{FB} ANY WHERE

→ WHEN IS THE SYSTEM CONTROLLABLE?

→ HOW TO TEST FOR CONTROLLABILITY?

whether a system is controllable only depends on the pair A and B

- ON THE DYNAMICS OF THE HOMOGENEOUS SYSTEM (ITS STABILITY PROPERTIES), VIA MATRIX A
- ON HOW THE CONTROLS WERE BUILT, VIA MATRIX B

$$A = \begin{bmatrix} -0.4 & 0 \\ 0.2 & -0.2 \end{bmatrix} \qquad B = \begin{bmatrix} 0.5 & 0.2 \\ -0.5 & 0 \end{bmatrix}$$

```
>> help ctrb
```
(COMPUTES THE CONTROLLABILITY MATRIX)

```
>> ctrb(A,B)
```
$([B \;\; AB \;\; A^2B \;\; \dots \;\; A^{N_x-1}B])$

THE SYSTEM (A,B) IS CONTROLLABLE IFF THE CONTROLLABILITY MATRIX IS FULL RANK

```
>> isequal(rank(ctrb(A,B)), 2)
```


ONE, FORMAL, DEFINITION OF CONTROLLABILITY

$x \in \mathbb{R}^{N_x}$, $u \in \mathbb{R}^{N_u}$

"The system $\dot{x}(t) = Ax(t) + Bu(t)$ is said to be controllable if and only if, given any arbitrary set of N_x real o complex conjugate pairs of numbers, $\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_{N_x}$, there exists a feedback dynamic matrix $K \in \mathbb{R}^{N_u \times N_x}$ such that the eigenvalues of the closed loop matrix $(A-BK)$ equal $\{\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_{N_x}\}$"

→ Thus, controllability means that it is possible to arbitrarily assign the eigenvalues of the closed-loop system, by a constant feedback of the state

THE CONTROLLABILITY MATRIX

$$\mathcal{C} = [\,B \;\; AB \;\; A^2B \;\; A^3B \;\; \dots \;\; A^{N_x-1}B\,]$$

APPEND MORE COLUMNS

DIMENSIONS?

THIS MATRIX NEED BE FULL COLUMN RANK → IT MUST HAVE N_x INDEPENDENT COLUMNS FOR THE SYSTEM TO BE CONTROLLABLE

→ SYSTEM IS CONTROLLABLE IF AND ONLY IF $\text{rank}(\mathcal{C}) = N_x$

→ IT IS A SIMPLE CRITERION

→ ON/OFF (Not quantitative)
- system is controllable
- system is not controllable

If this is not the case, it means that there exist some directions in $\mathbb{R}^{N_x}$ that cannot be reached

⇝ HENCE, THE UNCONTROLLABILITY

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ⇝ $\mathcal{C}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ⇝ $\mathcal{C}$

* BY HAND

* IN MATLAB

---

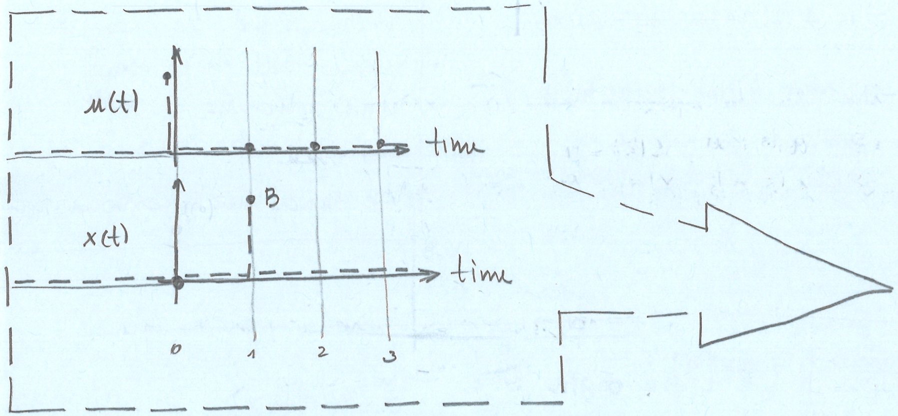
1. THE SYSTEM IS CONTROLLABLE
2. ARBITRARY PLACE THE EIGENVALUES OF $(A-KB)$ (Closed-loop) (There exists a matrix K that allows to do that)
3. REACHEABILITY ⇝ CAN TRANSFER THE STATE FROM ANY INITIAL VALUE $x(0)$ TO ANY FINAL VALUE $x(t_f)$ IN FINITE TIME $t_f < \infty$

$$R_t = \{ \xi \in \mathbb{R}^{N_x} \mid \exists \text{ an input such that } x(t) = \xi \} \equiv \mathbb{R}^{N_x}$$

The system is controllable if the control $u(t)$ can influence all the states through the integral $\int_{t_0}^{t} e^{A(t-\tau)} B u(t)\, d\tau$

↓

THE INTEGRAL MUST ALLOW THE INFLUENCE OF $u(t)$ TO REACH ALL POSSIBLE STATE VALUES $x(t) \in \mathbb{R}^{N_x}$



$$x_{k+1} = \tilde{A} x_k + \tilde{B} u_k$$

$k=0$ $\quad x_0 = 0,\ u_0 = 1$ ⇝ $x_1 = A\cancel{x_0}^{0} + B\cancel{u_0}^{1} = B$

$k=1$ $\quad x_1 = B,\ u_1 = 0$ ⇝ $x_2 = A\cancel{x_1}^{B} + B\cancel{u_1}^{0} = AB$

$k=2$ $\quad x_2 = AB,\ u_2 = 0$ ⇝ $x_3 = A\cancel{x_2}^{AB} + B\cancel{u_2}^{0} = A^2B$

$\mathbb{C} = [B \; AB \; A^2B \; \cdots \; A^{N_x-1} B]$ The controllability matrix

WE SWITCH TO BUILD INTUITION ON $\mathbb{C}$ TO DISCRETE TIME

$\dot{x} = Ax + Bu \rightsquigarrow \boxed{x_{k+1} = \tilde{A} x_k + \tilde{B} u_k}$

DISCRETE-TIME SYSTEM MATRIXES

$$\tilde{A} = e^{A\Delta t}$$

$$\tilde{B} = \int_0^{\Delta t} e^{At} B \, dt$$

THEY DIFFER FROM THE ORIGINAL (CONTINUOUS-TIME) MATRIXES A AND B

We start with the system a time 0, $x(0) = 0$, then ...

$u(0) = 1$	$u(1) = 0$	$u(2) = 0$	$u(3) = 0$	$\ldots$	$u(n) = 0$
$x(0) = 0$	$x(1) = B$	$x(2) = AB = Ax_1$	$x(3) = AAB = Ax_2$	$\ldots$	$x(n) = A^{n-1}B$

GRAPHICALLY

The system starts at the origin, then moves along B, then along AB, then along A^2B, ..., COVERING THE SPACE

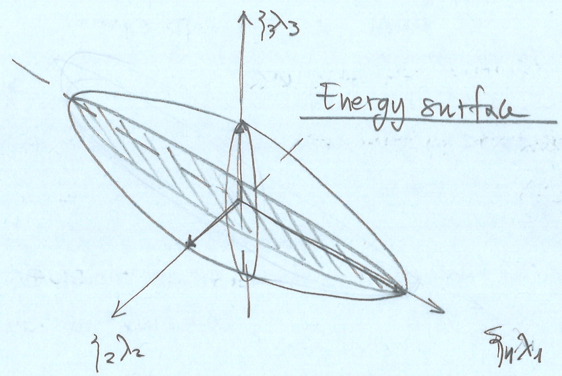
IF IT HAS MOVED OVER ALL DIRECTIONS, OR IF $\mathbb{C}$ IS FULL RANK, THEN WE CAN CLAIM THAT THE SYSTEM IS CONTROLLABLE

→ This explains why a controllability matrix has that form $\mathbb{C} = [B \; AB \; A^2B \; \cdots \; A^{N_x-1} B]$

$\gg [U, \Sigma, V] = \mathrm{svd}(\underbrace{\mathrm{ctrb}(A,B)}_{\mathbb{C}})$

DIAGONAL MATRIX OF SINGULAR VALUES (largest to smallest)

COLUMNS ARE SINGULAR VECTORS (most controllable to least)



THE CONTROLLABILITY TEST IS ON/OFF, WE DISCUSS SOME ALTERNATIVES

CONSIDER THE SOLUTION $x(t) = \underbrace{e^{At}x(0)}_{\text{unforced response}} + \underbrace{\int_{t_0=0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{forced response}}$

$$\boxed{\text{THE CONTROLLABILITY GRAMIAN} \quad W_c(0,t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau}\, d\tau}$$

DIMENSIONS?

$N_x \times N_x$, SHOW THIS!

MATRIX $W_c(0,t)$ IT IS REAL VALUED AND SYMMETRIC

⇝ REAL AND POSITIVE EIGENVALUES

WE ARE INTERESTED IN THE EIGENDECOMPOSITION OF $W_c(0,t)$

>> eig($W_c(0,t)$)   (EIGENVALUES AND EIGENVECTORS)

$$\boxed{W_c \xi = \lambda \xi}$$

WE CAN SORT THE EIGENVALUES

⇝ THE EIGENVECTORS ASSOCIATED TO THE LARGEST EIGENVALUES ARE THE MOST CONTROLLABE DIRECTIONS IN STATE SPACE

⇝ THE SYSTEM MOVES MORE WITH THE SAME AMOUNT OF CONTROL ENERGY

FOR DISCRETE-TIME

$\underbrace{W_t \approx CC^T}$

THE EMPIRICAL GRAMIAN

⇝ ITS EIGENVALUES AND EIGENVECTOR ARE THE SINGULAR VALUES AND VECTORS OF THE CONTROLLABILITY MATRIX

$\underbrace{[A - \lambda I \quad B]}$ MUST HAVE N_x LINEARLY INDEPENDENT COLUMNS FOR CONTROLLABILITY OF (A,B) TO HOLD, FOR ANY $\lambda \in \mathbb{C}$

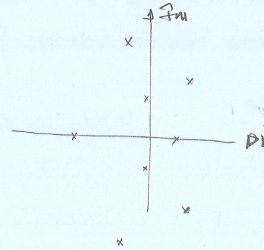
↓ That means that the colums must span $\mathbb{R}^{N_x}$

THE DETERMINANT OF $(A-\lambda I)$ EQUAL TO ZERO IS THE DEFINITION OF THE EIGENVALUES OF A

THUS $(A-\lambda I)$ IS ALWAYS FULL RANK EXCEPT FOR THOSE VALUES OF λ THAT ARE EIGENVALUES OF A

WHEN $\lambda \equiv$ EIGENVALUE THEN THE RANK DROPS

→ THIS IMPLIES THAT THE CONDITION CAN BE CHECKED ONLY FOR $\lambda \equiv$ EIGENVALUES (as they are the only responsible for rank drop)



No need to test the entire complex plane

→ N_x EIGENVALUES

→ CHECK RANK N_x TIMES

WE ALSO GAIN INSIGHT ABOUT CONTROLLABLE EIGENDIRECTIONS

→ NEXT PAGES (BACK)

POPOV - BELEVITCH - HAUTUS CONTROLLABILITY TEST

→ (A,B) IS CONTROLLABLE IFF RANK $\left[\underbrace{(A-\lambda I)}_{N_x \times N_x} \;\; \underbrace{B}_{N_x \times N_u}\right] = N_x$ for all $\lambda \in \mathbb{C}$

CONSIDER MATRIX $(A-\lambda I)$, ITS DETERMINANT IS $\det(A-\lambda I)$ AND IT IS ZERO FOR ALL LAMBDAS THAT ARE EIGENVALUES OF A

⇝ ITS RANK IS ALWAYS N_x, EXCEPT WHEN λ IS AN EVALUE OF A

SO, WE CAN ~~REPHRASE~~ STATE THE CONTROLLABILITY TEST

→ (A,B) IS CONTROLLABLE IFF $\text{rank}\left[(A-\lambda I) \;\; B\right] = N_x$ for all $\lambda \in \sigma(A)$

(with $\sigma(A)$ the spectrum of A)

WHEN $\lambda_i \in \sigma(A)$, THE RANK OF $(A-\lambda I)$ DROPS (BY 1, OR MORE)

HOW DO I REINSTATE FULL RANK?

→ NOTE THAT $(A-\lambda I)$ IS RANK DEFICIENT IN THE EIGENVECTOR $\underbrace{\text{DIRECTION ASSOCIATED}}_{\text{why?}}$ TO λ_i (IT'S FROM THE NULL SPACE)

$\text{rank}(A) + \text{null}(A) = n$

→ MATRIX B NEEDS TO HAVE AT LEAST ONE COMPONENT IN THE DIRECTION OF THAT EIGENVECTOR, WHILE STILL BEING LINEARLY INDEPENDENT ON THE OTHER COLUMNS

⇝ IF B IS A RANDOM VECTOR, THEN (A,B) IS CONTROLLABLE WITH HIGH PROBABILITY (ALMOST SURELY)
(WHO SAID THAT?)

→ From the PBH test, we get information on the minimum n° of controls that are needed to make the system controllable. (MULTIPLICITY OF THE EVALUES OF A)

SUPPOSE THAT $[(A-\lambda I) \;\; B]$ IS RANK DEFICIENT BY 1

⇝ $\text{rank}([(A-\lambda I) \;\; B]) = N_x - 1$ (happens for evalues with multiplicity one)

HOW DO WE MAKE UP FOR THAT DROP IN RANK?

⇝ B MUST PROVIDE AT LEAST ONE COLUMN THAT IS LINEARLY INDEPENDENT OF THE COLUMNS OF $(A-\lambda I)$ ⇝ THUS, INCREASE RANK BY 1

⇝ B MUST NOT HAVE COLUMNS IN THE RANGE OF $(A-\lambda I)$

$\boxed{B \notin \text{range}(A-\lambda I)}$

↓ SAME AS COLUMN SPACE

IF WE MAKE THIS CONDITION FOR ALL EIGENVALUES, THAT IS

$B \notin \text{range}(A-\lambda_1 I)$

$B \notin \text{range}(A-\lambda_2 I)$

$\vdots$

$B \in \text{range}(A-\lambda_{N_x} I)$

$\boxed{⇝ B \notin \bigcup_{i=1}^{N_x} \text{range}(A-\lambda_i I)}$

THIS IS THE SPACE SPANNED BY THE COLUMNS OF $A-\lambda_i I$

$(A-\lambda I)v = 0$ ← EIGENVECTORS OF A ARE IN THE NULL SPACE

WE MUST FIND A BASIS FOR THIS SPACE ('orth' IN MATLAB)

↑ Generates an orthonormal basis for this space