NUMERICAL INTEGRATION of DYNAMIC SYSTEMS

6b

Our ability to solve optimal control methods for systems with non-trivial dynamics relies on our ability to perform efficient and accurate simulations

→ THIS IN TURN DEPENDS ON OUR ABILITY TO BUILD DISCRETISATIONS

BUT FIRST THINGS FIRST : the existence of a solution to an ordinary differential equation and its initial conditions

→ INITIAL VALUE PROBLEMS (IVP)

THIS IS GUARANTEED AS LONG AS f IS CONTINUOUS WRT x AND t (PEANO)

EXISTENCE PLUS UNIQUENESS IS MORE INTERESTING (PICARD + LINDELÖF)

Th. (EXISTENCE + UNIQUENESS OF IVPs) Consider the general IVP $\dot{x}(t) = f(x(t), t)$ FOR $t \in [0, T]$ AND $x(0) = x_0$

ASSUME THAT f IS CONTINUOUS WRT x AND t

ASSUME THAT f IS LIPSCHITZ WRT x

$\rightsquigarrow \exists L : \| f(x,t) - f(y,t) \| \leq L \| x - y \| \quad , \forall x, y$

THEN → THERE EXISTS A UNIQUE SOLUTION $x(t)$ OF THE IVP IN THE NEIGHBORHOOD OF $(x_0, 0)$

→ The theorem can be extended → $f(x,t)$ discontinuous

NUMERICAL INTEGRATION : Obtain an approximated solution to a well-posed IVP (with unique solution)

→ ONE-STEP METHODS
→ MULTI-STEP METHODS
} FIRST POSSIBLE CHARACTERIZ.

→ EXPLICIT
→ IMPLICIT
} SECOND POSSIBLE CHARACTERIZATION

The general setup :

- ARBITRARY TIME INTERVAL $[t_0, t_f]$

CONSIDER SOME $f(x,t)$ WITH FINITELY MANY DISCONTINUITIES WITH RESPECT TO t

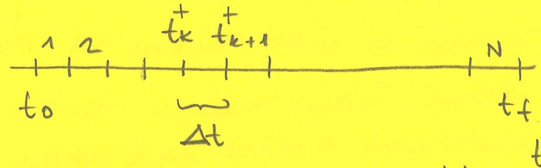
$\rightsquigarrow$ THE SOLUTIONS TO THE IVP ARE STILL UNIQUE

BECAUSE OF THE EXISTENCE OF UNIQUE SOLUTIONS WE HAVE

→ WE CAN SOLVE OPTIMAL CONTROL PROBLEMS, THOUGH WE OBTAIN DISCONTINUOUS CONTROL TRAJECTORIES

→ DIRECT METHODS EXIST, SO THAT WE FIRST DISCRETIZE THE CONTROLS (PIECEWISE CONSTANT)

1. DISCRETIZE THE STATE TRAJECTORY OVER A TIME GRID ON $[t_0, t_f]$ $\rightsquigarrow$ THE GRID IS OFTEN ASSUMED TO BE UNIFORM



N EQUALLY SIZED INTERVALS, $\Delta t = \dfrac{t_f - t_0}{N}$

$t_k \triangleq t_0 + k\Delta t$, with $k = 0, 1, \_\_\_, N$

EACH INTERVAL IS THUS $[t_k, t_{k+1}]$

2. THE SOLUTION IS APPROXIMATE ON THE GRID POINTS t_k BY VALUES S_k SUCH THAT $S_k \approx x(t_k)$, WITH $k = 0, 1, \_\_\_, N$ (THE EXACT SOLUTION IS $x(t)$)

$\boxed{\text{WHAT DIFFERS AMONG DIFFERENT METHODS IS HOW THE APPROXIMATION IS PERFORMED}}$

ONE STEP, EXPLICIT (most basic)

ONE STEP INTEGRATION METHODS ARE BASED ON A MAP ϕ THAT GENERATES THE SEQUENCE $x_0, x_1, \_\_\_, x_N$, BY RECURSION

$$\boxed{x_{k+1} = \phi_{RK}(x_k, \underbrace{t_k, \Delta t}) \quad , \quad \text{for } k = 0,1,\_\_\_, N-1}$$

HERE, FOR GENERALITY
(IT OFTEN SUFFICES TO SAY $\phi(x_k)$)

- EXPLICIT EULER
- EXPLICIT RUNGE-KUTTA
- IMPLICIT EULER
- IMPLICIT RUNGE-KUTTA
- ....

Examples of one-step integrators

THERE IS ALSO ANOTHER GROUP OF METHODS THAT FALLS UNDER THIS CLASS

- COLLOCATION METHODS

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EXPLICIT EULER INTEGRATOR

Let $f(x_k) = f(x(t_k)) = f_k$, then the simplest integrator

$\rightsquigarrow x_{k+1} = x_k + \Delta t f_k$

Example (1D): $\dot{x}(t) = -50[x(t) - \cos(t)]$ (time-dependent)

6d

6e

DIRECT APPROACHES

WE DISCUSS DIRECT APPROACHES TO CONTINUOUS-TIME OPTIMAL CONTROL

→ Finitely parameterize the infinite dimensional decision variables

→ Approximate the original problem by a finite dimensional NLP

* WE ALREADY DISCUSSED HOW TO DISCRETIZE AN ODE
* WE ALREADY DISCUSSED HOW TO SOLVE A DISCRETE TIME OPTIMAL CONTROL

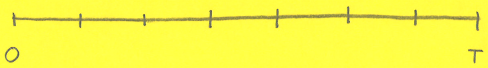
We start again from the Optimal Control Problem in Continuous time

$$\boxed{\begin{array}{llll} \underset{x(\cdot),\,u(\cdot)}{\text{minimise}} & \int_0^T \mathcal{L}(x,u)\,dt + E(x) & & \\ \text{subject to} & \dot{x} = f(x,u) \quad [0,T] & x = x(t) & \text{Dynamics} \\ & h(x,u) \leq 0 \quad '' & u = u(t) & \text{Path const.} \\ & r(x(T)) \leq 0 & & \text{Terminal constraints} \end{array}}$$

→ SINGLE SHOOTING

→ MULTIPLE SHOOTING

– WE START WITH A DISCRETIZATION OF THE TIME INTERVAL FOR THE CONTROLS

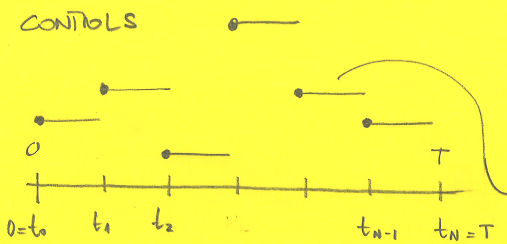


Could be unevenly separated

FOR EACH INTERVAL WE CHOOSE A CONTROL, USUALLY PIECEWISE CONSTANT

→ IT IS A PARAMETERIZATION USING POLINOMIALS (ZERO ORDER POLYNOMIALS)

→ $u(t,q)$, q is the parameter



DIRECT METHODS DIFFER ON HOW THE TRANSLATE THE ORIGINAL CONTROL PROBLEM INTO A FINITE NLP

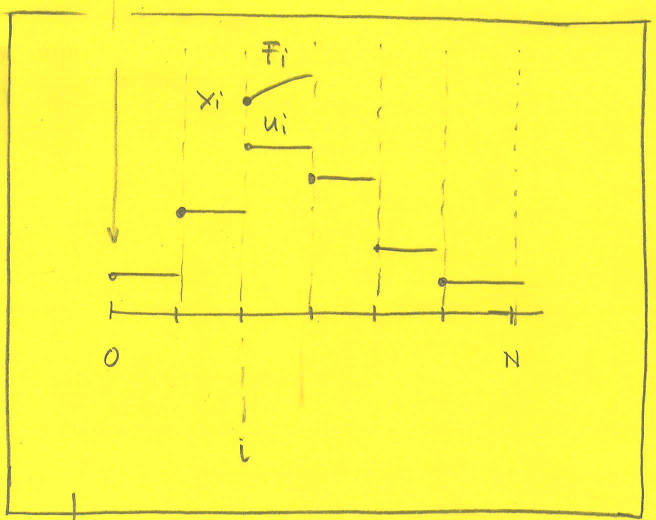
$u(t,q) = q_k \,, t \in [t_k, t_{k+1}]$

$$\min_{x,u} \sum_{i=0}^{N-1} L_i(x_i,u_i) + E(x_N)$$

$$\text{s.t.} \quad x_{i+1} - F(x_i,u_i) = 0 \qquad i = 0,1,\_\_\_, N-1$$

$$H_i(x_i,u_i) \leq 0 \qquad i = 0,1,\_\_\_, N-1$$

$$r(x_N) \leq 0$$



THE ODE SOLVER IS EMBEDDED IN ORDER TO ELIMINATE THE CONTINUOUS TIME DYNAMICS

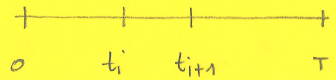
DIRECT APPROACHES

→ SINGLE SHOOTING
→ SEQUENTIAL
(Eliminate all the states)

→ MULTIPLE SHOOTING
→ SIMULTANEOUS

ABOUT FUNCTIONS F_i, L_i, and H_i

* We consider first an integral between times t_i and t_{i+1}

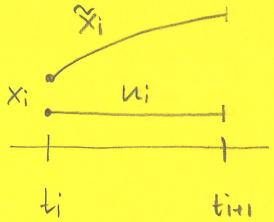


* WE HAVE A SOLUTION MAP THAT DEFINES THE APPROXIMATED SOLUTION OF THE ODE WITH SOME INTEGRATION METHOD

* NEEDS TO BE SPECIFIED

We can define the TRAJECTORY $\tilde{x}_i$

- STARTS AT t_i WITH VALUE x_i
- ENDS AT t_{i+1}
- RECEIVES THE CONTROL u_i AS INPUT



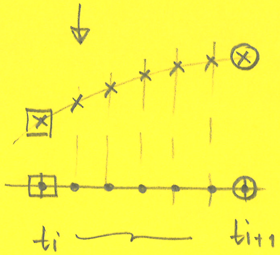
$\tilde{x}_i(t \mid x_i, u_i)$ ← THE TRAJECTORY SHOULD SATISFY THE DIFFERENTIAL EQUATION

$$\frac{d\tilde{x}_i(\cdot)}{dt} = f(\tilde{x}_i(\cdot), u_i)$$

WITH INITIAL CONDITION

$$\tilde{x}_i(t_i \mid x_i, u_i) = x_i$$

IN THE INTERVAL $t \in [t_i, t_{i+1}]$



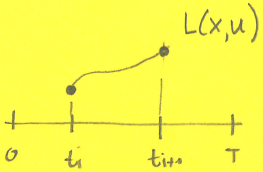
↳ Internal integrator's steps

DA 3

FUNCTION $F_i(x_i,u_i) \triangleq \underbrace{x_i(t_{i+1} | x_i, u_i)}$

THIS IS THE LAST POINT IN THE INTEGRATION OF THE ODE IN t_i AND t_{i+1} $\boxtimes \cdots \otimes$ ← this one!

* Then $\underbrace{\int_{t_i}^{t_{i+1}} \mathcal{L}(\tilde{x}_i(t | x_i, u_i), u_i)\, dt}$



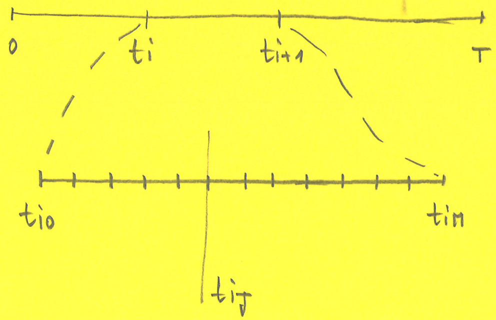
THIS NEEDS TO BE APPROXIMATED IN THE DISCRETIZATION GRID

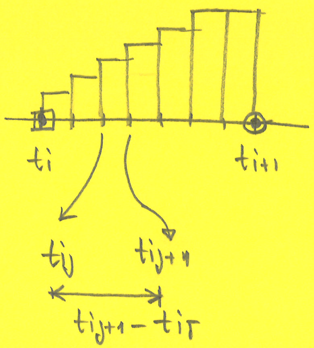
~> RIEMANN SUM IN THE INTERVAL AT TIME GRID $t_{i0}, t_{i1}, \_\_\_\_, t_{in}$ OF EVALUATION POINTS

SUCH THAT

- $t_{i0} = t_i$
- $t_{in} = t_{i+1}$

$$\mathcal{L}_i(x_i,u_i) = \sum_{j=0}^{n-1} L(\tilde{x}_i(t_{ij} | x_i, u_i), u_i) \times (t_{i,j+1} - t_{ij})$$



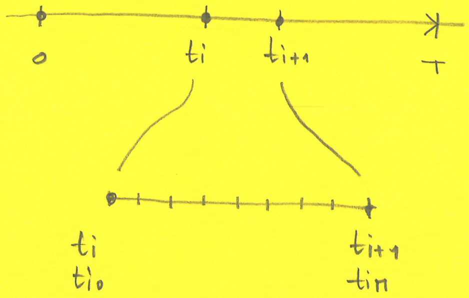


An alternatie approach to define $L_i(x_i, u_i)$ is to add a QUADRATURE STATE

DA 4a

* we then define H_i functions

* WE CAN CONSIDER THE SAME TIME GRID IN $t_i \longrightarrow t_{i+1}$
  (for simplicity)



$$H_i(x_i,u_i) = \begin{bmatrix} h(x_i,u_i) \\ h(\tilde{x}_i(t_{i1}|x_i,u_i),u_i) \\ \vdots \\ h(\tilde{x}_i(t_{in-1}|x_i,u_i),u_i) \end{bmatrix}$$

ELEMENTS OF A DIRECT METHOD THAT MUST CHOSEN

1. Select the integrator (to get $\tilde{x}$)
   (RK4, ... , etc ... as long that $\tilde{x}$ is differentiable)

2. Step size (also 1a. maybe)

3. Integral discretization

4. Path constraint discretization

5. NLP solver (solution method)

PIECEWISE CONSTANT PARAMETRIZATION OF THE CONTROLS IS ADVANTAGEUS IN TERMS OF SIMPLICITY (SPARSITY)

* OTHER PIECEWISE PARAMETRIZATIONS
  * ORDER - 1
  * ORDER - 2
  * . . .

The elimination of the states reduces the decision variables to be the controls

DA6