

## Signals and distributions

UFC/DC  
SA (CK0191)  
2018.1

### Canonical signals

Unit step  
Ramps  
Impulse  
Derivative of the impulse  
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

## Signals and distributions Stochastic algorithms

Francesco Corona

Department of Computer Science  
Federal University of Ceará, Fortaleza

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## Canonical signals

We describe some **signals** or **functions** in the real variable  $t$ , time

$$f : \mathcal{R} \rightarrow \mathcal{C}$$

In our studies, such functions are often discontinuous

- We introduce a generalisation of function
- The **distribution**

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## Unit step Canonical signals

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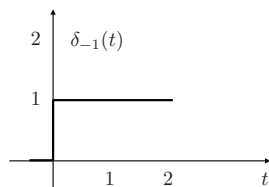
## Unit step

### Definition

#### Unit step

The **unit step**, denoted as  $\delta_{-1}(t)$ , is a function

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



The function is continuous over the domain, except for the origin

- Discontinuity, size 1



## Unit step (cont.)

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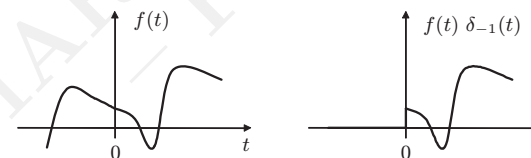
We can use the unit step to define new functions

Consider some function  $f(t) : \mathcal{R} \rightarrow \mathcal{R}$ , we have

$$f(t)\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

Values of  $f(t)$  for  $t < 0$  have been set to zero

Graphically,



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## Ramps

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## Ramps

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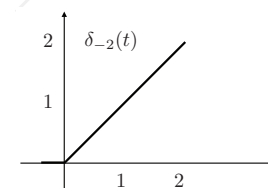
#### Convolution with canonical signals

### Definition

#### Unit ramp

The integral of the unit step is called the **unit ramp**,  $\delta_{-2}(t)$

$$\begin{aligned} \delta_{-2}(t) &= \int_{-\infty}^t \delta_{-1}(\tau) d\tau \\ &= \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases} \\ &= t\delta_{-1}(t) \end{aligned}$$



## Ramps (cont.)

### Definition

#### Ramp functions

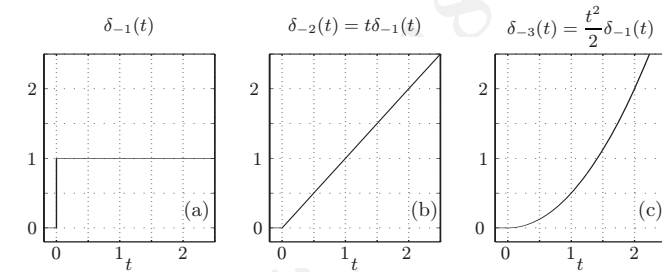
The family of **ramp functions**  $\delta_{-k}(t)$  can be recursively defined for  $k > 2$

$$\delta_{-k}(t) = \underbrace{\int_{-\infty}^t \cdots \int_{-\infty}^t}_{k-1 \text{ times}} \delta_{-1}(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \frac{t^{k-1}}{(k-1)!}, & t \geq 0 \end{cases}$$

$$= \frac{t^{k-1}}{(k-1)!} \delta_{-1}(t)$$

■

## Ramps (cont.)



- **Quadratic ramp**,  $k = 3$

$$\rightsquigarrow \delta_{-3}(t) = \frac{t^2}{2!} \delta_{-1}(t)$$

- **Cubic ramp**,  $k = 4$

$$\rightsquigarrow \delta_{-4}(t) = \frac{t^3}{3!} \delta_{-1}(t)$$

## Ramps (cont.)

### Definition

#### Exponential ramp

A generalisation of the ramp function is the **exponential ramp**, or **cisoid**

$$\frac{t^k}{k!} e^{at} \delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ \frac{t^k}{(k)!} e^{at}, & t \geq 0 \end{cases}$$

It is defined in terms of two parameters

$$\rightsquigarrow k \in \mathcal{N}$$

$$\rightsquigarrow a \in \mathcal{C}$$

■

## Ramps (cont.)

$$\frac{t^k}{k!} e^{at} \delta_{-1}(t)$$

Particular cases that can be generated from the exponential ramp

$$\rightsquigarrow a = 0 \text{ and } k = 1, 2, \dots, \text{ the family of ramp functions}$$

$$\rightsquigarrow a = 0 \text{ and } k = 0, \text{ the unit ramp}$$

$$\rightsquigarrow k = 0 \text{ and } a \in \mathcal{R}, \text{ exponential function } e^{at}$$

$$\rightsquigarrow k = 0 \text{ and } a = j\omega \in \mathcal{I}, \text{ a linear combinations of exponential ramps can be used to describe sinusoidal functions}$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

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## Ramps (cont.)

Linear combinations of ramps can be used for polynomial functions

$$c_2 t^2 + c_1 t + c_0$$

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**Impulse**  
**Canonical signals**

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## Impulse

We can extend the family of canonical signals

We consider the derivatives of the unit step (so far, we integrated it)

- The results of classical calculus cannot be used for the purpose
- The derivative of a discontinuous function is not defined

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## Impulse (cont.)

We need to generalise the concept of function

↪ The distribution

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## Impulse (cont.)

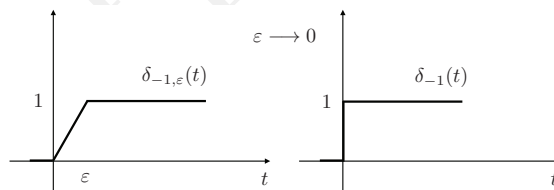
Let  $\varepsilon > 0$  be some positive scalar

Define the function  $\delta_{-1,\varepsilon}(t)$

$$\delta_{-1,\varepsilon}(t) = \begin{cases} 0, & t < 0 \\ t/\varepsilon, & t \in [0, \varepsilon) \\ 1, & t \geq \varepsilon \end{cases}$$

This function is understood as a continuous approximation of the unit step

$$\rightsquigarrow \lim_{\varepsilon \rightarrow 0} \delta_{-1,\varepsilon}(t) = \delta_{-1}(t)$$



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## Impulse (cont.)

### Definition

#### Finite impulse

Function  $\delta_{-1,\varepsilon}(t)$  is continuous, it thus possesses a derivative

$$\delta_{\varepsilon}(t) = \frac{d}{dt} \delta_{-1,\varepsilon} = \begin{cases} 1/\varepsilon, & t \in [0, \varepsilon) \\ 0, & \text{elsewhere} \end{cases}$$

Function  $\delta_{\varepsilon}(t)$  is denoted as **finite impulse** of base  $\varepsilon$

Function  $\delta_{\varepsilon}(t)$  is a rectangle with base  $\varepsilon$  and with height  $1/\varepsilon$

- Area equal to 1, whatever the value of  $\varepsilon$

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## Impulse (cont.)

We can define the derivative of the unit step

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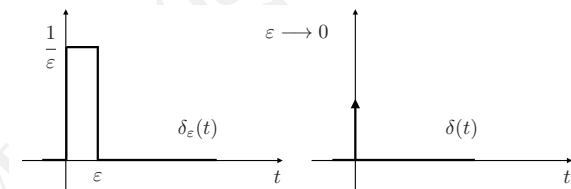
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## Impulse (cont.)

### Definition

#### Unit impulse or Dirac function

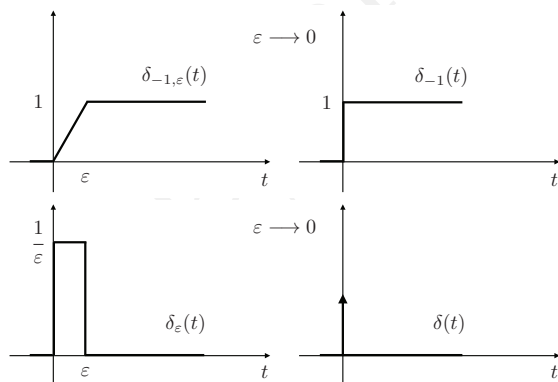
$$\delta(t) = \frac{d}{dt} \delta_{-1}(t) = \frac{d}{dt} \lim_{\varepsilon \rightarrow 0} \delta_{-1,\varepsilon}(t) = \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \delta_{-1,\varepsilon}(t) = \lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon}(t)$$



Such a definition is not formally correct in the sense of the classical calculus

- It is valid only if we accept the generalisation of a function
- The impulse  $\delta(t)$  is not a function, it is a distribution

## Impulse (cont.)



## Impulse (cont.)

The following properties hold

↪  $\delta(t)$  is equal to zero everywhere, except the origin

$$\delta(t) = 0, \quad \text{if } t \neq 0$$

↪  $\delta(t)$  at the origin is equal to infinity

$$\delta(t) = \infty, \quad \text{if } t = 0$$

↪ The area under  $\delta(t)$  is equal to 1

$$\int_{-\infty}^{+\infty} \delta(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

## Impulse (cont.)

### Theorem

Let  $f(t)$  be some continuous function in  $t = 0$

- The product of  $f(t)$  and the impulse  $\delta(t)$

$$\rightsquigarrow f(t)\delta(t) = f(0)\delta(t)$$

Let  $f(t)$  be some continuous function in  $t = T$

- The product of  $f(t)$  and  $\delta(t - T)$

$$\rightsquigarrow f(t)\delta(t - T) = f(T)\delta(t - T)$$

### Proof

We have that  $\delta(t) = 0$ , for  $t \neq 0$

The values taken by  $f(t)$  for  $t \neq 0$  are not significant (impulse is zero)



## Derivative of the impulse

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## Derivative of the impulse

We use the limit reasoning to define higher-order derivatives of the impulse

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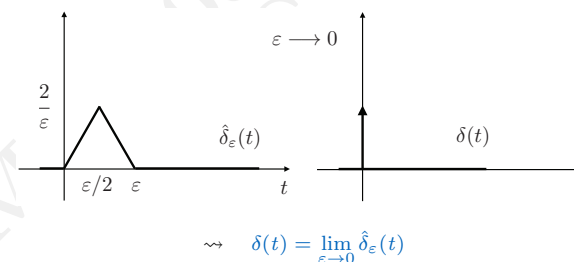
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## Derivative of the impulse (cont.)

Consider the function  $\hat{\delta}_\varepsilon(t)$

$$\hat{\delta}_\varepsilon(t) = \begin{cases} 4t/\varepsilon^2, & t \in [0, \varepsilon/2) \\ 4/\varepsilon - 4t/\varepsilon^2, & t \in [\varepsilon/2, \varepsilon) \\ 0, & \text{elsewhere} \end{cases}$$

The impulse can be re-defined



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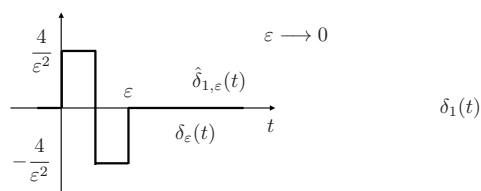
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## Derivative of the impulse (cont.)

We define the first-order derivative of the impulse



$$\delta_1(t) = \frac{d}{dt}\delta(t) = \frac{d}{dt} \lim_{\varepsilon \rightarrow 0} \hat{\delta}_\varepsilon(t) = \lim_{\varepsilon \rightarrow 0} \frac{d}{dt} \hat{\delta}_{1,\varepsilon}(t)$$

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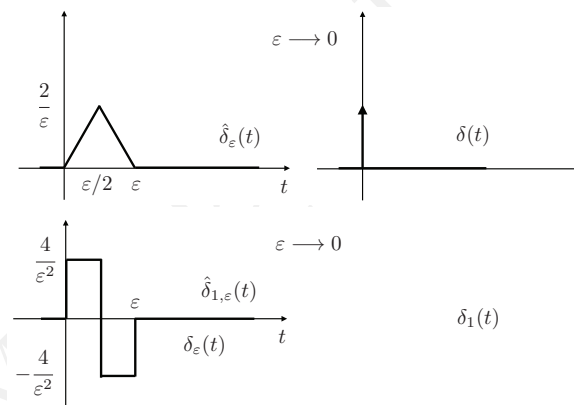
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## Derivative of the impulse (cont.)



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## Derivative of the impulse (cont.)

The higher-order ( $k > 1$ ) derivatives of the impulse

$$\rightsquigarrow \delta_k(t) = \frac{d^k}{dt^k} \delta(t) = \frac{d}{dt} \delta_{k-1}(t)$$



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## The family of canonical signals

For  $k \in \mathbb{Z}$ , we can define a family of canonical signals,  $\delta_k(t)$

- $\rightsquigarrow \delta_0(t) = \delta(t)$ , the impulse ( $k = 0$ )
- $\rightsquigarrow k < 0$ , the integrals of the impulse
- $\rightsquigarrow k > 0$ , the derivatives of the impulse

Such signals are linearly independent

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## The family of canonical signals (cont.)

### Definition

#### Linear dependence of scalar functions

Consider a set of scalar real functions  $f_1(t), f_2(t), \dots, f_n(t)$

$$f_i(t) : \mathcal{R} \rightarrow \mathcal{R}$$

Such functions are said to be **linearly dependent** over the interval  $[t_1, t_2]$ , if and only if there exist a set of real numbers  $\alpha_1, \alpha_2, \dots, \alpha_n$  that are not all equal to zero and such that

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \dots + \alpha_n f_n(t) = 0, \quad \forall t \in [t_1, t_2]$$





## The family of canonical signals (cont.)

### Example

Consider two functions  $f_1(t)$  and  $f_2(t)$

$$f_1(t) = t, \quad t \in (-\infty, \infty)$$

$$f_2(t) = |t| = \begin{cases} -t, & t \in (-\infty, 0) \\ t, & t \in (0, \infty) \end{cases}$$

The two functions are linearly dependent on each interval  $[t_1, t_2]$  with  $t_2 \leq 0$

- Let  $\alpha_1 = \alpha_2 \neq 0$ , we have  $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$ , for every  $t \in [t_1, t_2]$

The two functions are linearly dependent on each interval  $[t_1, t_2]$  with  $t_1 \geq 0$

- Let  $\alpha_1 = \alpha_2 \neq 0$ , we have  $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$ , for every  $t \in [t_1, t_2]$

## The family of canonical signals (cont.)

The two functions are linearly independent on  $[t_1, t_2]$ ,  $t_1 < 0$  and  $t_2 > 0$

## The family of canonical signals (cont.)

### Remark

Two or more functions can be linearly dependent in an interval

- Yet, they can be linearly independent in a larger interval

Conversely, linear independence in a given interval implies linear independence in any larger interval of which the initial interval is a subset

## The family of canonical signals (cont.)

Consider the function

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \delta_k(t)$$

Suppose that such a function is identically null over  $[a, b]$ , with  $a \neq b$

↪ Then,  $a_k = 0$  for all  $k \in \mathbb{Z}$

# Derivatives of a discontinuous function

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## Derivatives of a discontinuous function

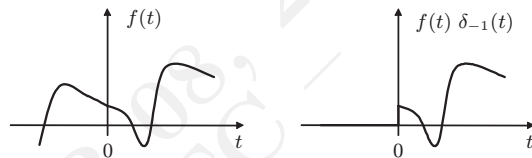
We can formally calculate the derivative of discontinuous functions

Discontinuous functions are common in systems analysis

- Zero for  $t < 0$  and continuous for  $t \geq 0$
- Discontinuity in the origin

## Derivatives of a discontinuous function (cont.)

Let  $f(t)$  be a continuous function



We are interested in calculating the derivative of function  $f(t)\delta_{-1}(t)$

- If  $f(0) \neq 0$ , then  $f(t)\delta_{-1}(t)$  has a discontinuity in  $t = 0$

## Derivatives of a discontinuous function (cont.)

The first-order derivative

$$\begin{aligned} \frac{d}{dt}f(t)\delta_{-1}(t) &= \left[\frac{d}{dt}f(t)\right]\delta_{-1}(t) + f(t)\underbrace{\left[\frac{d}{dt}\delta_{-1}(t)\right]}_{\delta(t)} \\ &\approx \dot{f}(t)\delta_{-1}(t) + f(0)\delta(t) \end{aligned}$$

It is the first-order derivative of the original function multiplied by  $\delta_{-1}(t)$

- plus the impulse at the origin multiplied by  $f(0)$

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## Derivatives of a discontinuous function (cont.)

The second-order derivative

$$\begin{aligned}\frac{d^2}{dt^2}f(t)\delta_{-1}(t) &= \left[\frac{d}{dt}\dot{f}(t)\right]\delta_{-1}(t) + f(t)\left[\frac{d}{dt}\delta_{-1}(t)\right] + f(0)\left[\frac{d}{dt}\delta(t)\right] \\ &\rightsquigarrow = \ddot{f}(t)\delta_{-1}(t) + \dot{f}(0)\delta(t) + f(0)\delta_1(t)\end{aligned}$$

It is the second-order derivative of the original function multiplied by  $\delta_{-1}$

- plus the impulse at the origin multiplied by  $\dot{f}(0)$
- plus  $\delta_1(t)$  times  $f(0)$

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## Derivatives of a discontinuous function (cont.)

Higher-order derivatives are calculated analogously

$$\begin{aligned}\frac{d^k}{dt^k}f(t)\delta_{-1}(t) &= f^{(k)}\delta_1(t) + f^{(k-1)}(0)\delta(t) + \dots + f(0)\delta_{k-1}(t) \\ &\rightsquigarrow = f^{(k)}(t)\delta_{-1}(t) + \sum_{i=0}^{k-1} f^{(i)}(0)\delta_{k-1-i}(t)\end{aligned}$$

We used  $\delta_0(t) = \delta(t)$

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## Derivatives of a discontinuous function (cont.)

### Example

Consider the function

$$f(t) = \cos(t)\delta_{-1}(1)$$

We are interested in its derivatives

The first-order derivative,

$$\begin{aligned}\frac{d}{dt}\cos t\delta_{-1}(t) &= \left[\frac{d}{dt}\cos(t)\right]\delta_{-1}(t) + \sin(0)\delta(t) + \cos(0)\delta_1(t) \\ &= -\sin(t)\delta_{-1}(t) + \delta_1(t)\end{aligned}$$

The second-order derivative,

$$\begin{aligned}\frac{d^2}{dt^2}\cos(t)\delta_{-1}(t) &= \left[\frac{d^2}{dt^2}\cos(t)\right]\delta_{-1}(t) - \sin(0)\delta(t) + \cos(0)\delta_1(t) \\ &= -\cos(t)\delta_{-1}(t) + \delta_{-1}(t)\end{aligned}$$



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## Derivatives of a discontinuous function (cont.)

### Example

Consider the cisoid function

$$f(t) = te^{(at)}\delta_{-1}(t)$$

We are interested in its derivatives

The first-order derivative,

$$\begin{aligned}\frac{d}{dt}te^{(at)}\delta_{-1}(t) &= e^{(at)}\delta_{-1}(t) + ate^{(at)} + [te^{(at)}]_{t=0}\delta(t) \\ &= (1 - at)e^{(at)}\delta_{-1}(t)\end{aligned}$$

The second-order derivative,

$$\begin{aligned}\frac{d^2}{dt^2} &= ae^{(at)}\delta_{-1}(t) + a(1 + at)e^{(at)}\delta_{-1}(t) + [(1 + at)e^{(at)}]_{t=0}\delta(t) \\ &= (2a + a^2t)e^{(at)}\delta_{-1}(t) + \delta(t)\end{aligned}$$



# Convolution integrals

## Signals and distributions

# Convolution integrals

## Definition

### Convolution

Consider the two functions

$$f, g : \mathcal{R} \rightarrow \mathcal{C}$$

The **convolution** of  $f$  with  $g$  is a function  $h : \mathcal{R} \rightarrow \mathcal{C}$  in the real variable  $t$ ,

$$h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)d\tau$$

Function  $h(t)$  is built by using the operator **convolution integral**,  $\star$

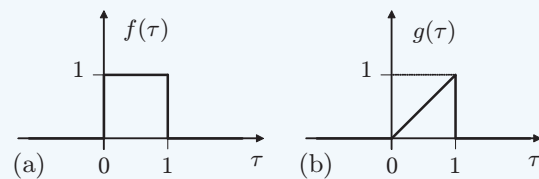
## Convolution integrals (cont.)

### Example

Consider the two functions

$$f(\tau) = \begin{cases} 1, & \text{if } \tau \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$g(\tau) = \begin{cases} \tau, & \text{if } \tau \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$



We want to calculate

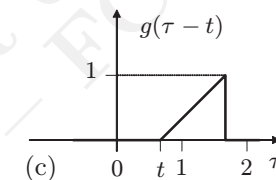
## Convolution integrals (cont.)

To calculate

$$g(\tau - t) = \begin{cases} \tau - t, & \text{if } \tau \in [t, t + 1] \\ 0, & \text{otherwise} \end{cases}$$

we need to shift  $g(\tau)$  by a quantity  $t$

- $t > 0$ , to the right
- $t < 0$ , to the left



## Signals and distributions

UFC/DC  
SA (CK0191)  
2018.1

### Canonical signals

Unit step  
Ramps  
Impulse  
Derivative of the impulse  
The family of canonical signals

Derivatives of a discontinuous function

### Convolution integrals

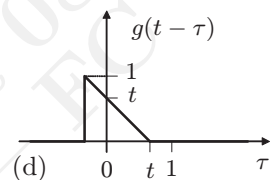
Convolution with canonical signals

## Convolution integrals (cont.)

To calculate

$$g(t - \tau) = \begin{cases} t - \tau, & \text{if } \tau \in [t - 1, t] \\ 0, & \text{otherwise} \end{cases}$$

we need to flip  $g(\tau)$  around  $\tau = t$



## Signals and distributions

UFC/DC  
SA (CK0191)  
2018.1

### Canonical signals

Unit step  
Ramps  
Impulse  
Derivative of the impulse  
The family of canonical signals

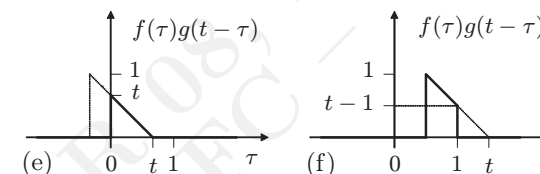
Derivatives of a discontinuous function

### Convolution integrals

Convolution with canonical signals

## Convolution integrals (cont.)

To calculate  $f(\tau)g(t - \tau)$



## Signals and distributions

UFC/DC  
SA (CK0191)  
2018.1

### Canonical signals

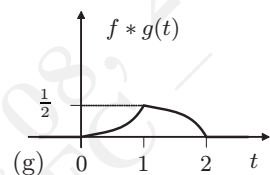
Unit step  
Ramps  
Impulse  
Derivative of the impulse  
The family of canonical signals

Derivatives of a discontinuous function

### Convolution integrals

Convolution with canonical signals

## Convolution integrals (cont.)



■

## Signals and distributions

UFC/DC  
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2018.1

### Canonical signals

Unit step  
Ramps  
Impulse  
Derivative of the impulse  
The family of canonical signals

Derivatives of a discontinuous function

### Convolution integrals

Convolution with canonical signals

## Convolution integrals (cont.)

### Theorem

The convolution operator is commutative

$$f \star g(t) = g \star f(t)$$

### Proof

Let  $\rho = t - \tau$ , then write

$$f \star g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)d\tau = \int_{-\infty}^{+\infty} f(t - \rho)g(\rho)d\rho = g \star f(t)$$

■

## Convolution integrals (cont.)

### Definition

Consider the two functions  $f, g: \mathbb{R} \rightarrow \mathbb{C}$

Let their derivatives be

$$\dot{f}(t) = \frac{d}{dt}f(t)$$

$$\dot{g}(t) = \frac{d}{dt}g(t)$$

Let their integrals be

$$\mathcal{F}(t) = \int_{-\infty}^t f(\tau) d\tau$$

$$\mathcal{G}(t) = \int_{-\infty}^t g(\tau) d\tau$$

## Convolution integrals (cont.)

The following statements are true

- (1) The derivative of the convolution between two functions is given by the convolution of one function with the derivative of the other function

$$\frac{d}{dt}f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$$

- (2) The integral of the convolution between two functions is given by the convolution of one function with the integral of the other function

$$\int_{-\infty}^t f \star g(\tau) d\tau = f \star \mathcal{G}(t) = \mathcal{F} \star g(t)$$

- (3) The integral of a convolution between two function does not change if one of the two operands is derived and the other one is integrated

$$f \star g(t) = \mathcal{F} \star \dot{g}(t) = \dot{f} \star \mathcal{G}(t)$$

## Convolution integrals (cont.)

### Proof

To demonstrate (1), observe that we can write

$$\begin{aligned} \frac{d}{dt}f \star g(t) &= \frac{d}{dt} \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{+\infty} f(\tau) \frac{d}{dt}g(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} f(\tau)\dot{g}(t-\tau)d\tau = f \star \dot{g}(t) \end{aligned}$$

Because of the commutative property  $f \star g(t) = g \star f(t)$ , we also have

$$\begin{aligned} \frac{d}{dt}f \star g(t) &= \frac{d}{dt}g \star f(t) = \int_{-\infty}^{+\infty} \frac{d}{dt}f(t-\tau)g(\tau)d\tau \\ &= \int_{-\infty}^{+\infty} \dot{f}(t-\tau)g(\tau)d\tau = g \star \dot{f}(t) = \dot{f} \star g(t) \end{aligned}$$

## Convolution integrals (cont.)

To demonstrate (2) where the three functions are identical, we use (1)

Observe that all three functions when evaluated for  $t = -\infty$  are null

- Whereas their derivatives are equal, for all values of  $t$

This is because of the definition of integral

$$\frac{d}{dt} \int_{-\infty}^0 f \star g(\tau) d\tau = f \star g(t)$$

And, because

$$\frac{d}{dt}f \star \mathcal{G}(t) = f \star \left[ \frac{d}{dt}\mathcal{G} \right](t) = f \star g(t)$$

$$\frac{d}{dt}\mathcal{F} \star g(t) = \left[ \frac{d}{dt}\mathcal{F} \right](t) = f \star g(t)$$

## Convolution integrals (cont.)

To demonstrate (3), we use (1) again

$\mathcal{F} \star \dot{g}(t)$  is obtained from (1)

$$\frac{d}{dt} \mathcal{F} \star g(t) = \mathcal{F} \star \left[ \frac{d}{dt} g \right](t) = \left[ \frac{d}{dt} \mathcal{F} \right] \star g(t) \rightsquigarrow \mathcal{F} \star \dot{g}(t) = f \star g(t)$$

$\dot{f} \star \mathcal{G}(t)$  is obtained by differentiating  $f \star \mathcal{G}(t)$

$$\frac{d}{dt} f \star \mathcal{G}(t) = f \star \left[ \frac{d}{dt} \mathcal{G} \right](t) = \left[ \frac{d}{dt} f \right] \star \mathcal{G}(t) \rightsquigarrow f \star g(t) = \dot{f} \star \mathcal{G}(t)$$

■

## Convolution with canonical signals

Signals and distributions

## Convolution with canonical signals

### Theorem

#### Convolution with the impulse

Consider a function  $f : \mathcal{R} \rightarrow \mathcal{R}$  continuous in  $t$

We have,

$$f(t) = \int_{-\infty}^{+\infty} f(\tau) \delta(t - \tau) d\tau$$

For any interval  $(t_a, t_b)$  containing  $t$ , we have

$$f(t) = \int_{t_a}^{t_b} f(\tau) \delta(t - \tau) d\tau$$

■

## Convolution with canonical signals (cont.)

### Proof

Observe that  $\delta(t - \tau) = \delta(\tau - t)$  is an impulse centred in  $\tau = t$

Thus,

$$\begin{aligned} \int_{-\infty}^{+\infty} f(\tau) \delta(t - \tau) d\tau &= \int_{-\infty}^{+\infty} \underbrace{f(t) \delta(t - \tau)}_{f(t) \delta(t - T) = f(T) \delta(t - T)} d\tau \\ &= f(t) \underbrace{\int_{-\infty}^{+\infty} \delta(t - \tau) d\tau}_{\int_{-\infty}^{+\infty} \delta(t) dt = \int_0^{0+} \delta(t) dt = 1} = f(t) \end{aligned}$$

■

## Convolution with canonical signals (cont.)

### Theorem

Consider a function  $f : \mathcal{R} \rightarrow \mathcal{R}$  continuous with  $k$  continuous derivatives

We have,

$$\frac{d^k}{dt^k} f(t) = \int_{-\infty}^{+\infty} f(\tau) \delta_k(t - \tau) d\tau$$

### Proof

Observe that  $f(t) = f \star \delta(t)$

By repeatedly differentiating and using that  $\frac{d}{dt} f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$ ,

$$\frac{d}{dt} f(t) = \frac{d}{dt} f \star \delta(t) = f \star \left[ \frac{d}{dt} \delta \right](t) = f \star \delta_1(t)$$

$$\frac{d^2}{dt^2} f(t) = \frac{d}{dt} f \star \delta_1(t) = f \star \delta_2(t)$$

$$\dots = \dots$$

$$\frac{d^k}{dt^k} f(t) = \frac{d}{dt} f \star \delta_{k-1}(t) = f \star \delta_k(t)$$