

LINEARIZATION OF NON-LINEAR STATE-SPACE MODELS

MANY DYNAMICAL PROCESSES IN CHEMICAL ENGINEERING ARE MODELLED AS A SET OF NON-LINEAR EQUATIONS (FIRST-ORDER)

- FROM MATERIAL AND ENERGY CONSERVATION

MOST COTATION ANALYSIS AND CONTROL REQUIRE A LINEAR MODEL

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \quad \leftarrow \quad \begin{array}{l} \dot{x} = f(x, u) \\ y = g(x, u) \end{array}$$

Nonlinear systems can be linearized

GENERAL CASE - SINGLE VARIABLE $\frac{dx}{dt} = f(x)$

Function $f(x)$ can be approximated using a Taylor Series around an equilibrium point $x_s \leftarrow f(x=x_s) = 0$

$$f(x) = f(x_s) + \frac{\partial f}{\partial x} \Big|_{x_s} (x - x_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \Big|_{x_s} (x - x_s)^2 + \underbrace{O(x^3)}_{\text{higher order terms}}$$

IF WE NEGLECT THE QUADRATIC TERM AND HIGHER,

$$f(x) \approx f(x_s) + \frac{\partial f}{\partial x} \Big|_{x_s} (x - x_s)$$

$$\text{WE NOTED THAT } \frac{dx}{dt} \Big|_{x_s} = f(x_s) = 0$$

$$\text{THUS, WE HAVE } f(x) \approx f(x_s) + \frac{\partial f}{\partial x} \Big|_{x_s} (x - x_s) \\ = 0$$

$$\frac{dx}{dt} = f(x) \approx \frac{\partial f}{\partial x} \Big|_{x_s} (x - x_s)$$

Consider the variable $\chi = (x - x_s)$ and compute its derivative

$$\frac{d(x - x_s)}{dt} = \frac{dx}{dt} - \frac{dx_s}{dt} = \frac{dx}{dt} \quad \text{and} \quad \frac{d(x - x_s)}{dt} \approx \frac{\partial f}{\partial x_s} \Big|_{x_s} (x - x_s) \\ = 0$$

$$\frac{d(x-x_s)}{dt} \approx \left. \frac{\partial f}{\partial x} \right|_{x_s} (x-x_s)$$

THIS REPRESENTATION ALLOWS US TO
DISCUSS 'DEVIATIONS WITH RESPECT TO THE
EQUILIBRIUM POINT'

WE CAN WRITE THE SYSTEM LINEARISED AND WRT PERTURBATIONS

$$\frac{dx'}{dt} = \left. \frac{\partial f}{\partial x} \right|_s x' \quad \text{OR} \quad x' = \alpha x' \quad \text{WITH} \quad \alpha = \left. \frac{\partial f}{\partial x} \right|_{x_s}$$

GENERAL CASE - SINGLE VARIABLE, SINGLE INPUT $dx/dt = f(x,u)$

WE WRITE THE TAYLOR SERIES APPROXIMATION OF $f(x,u)$ AT (x_s, u_s)

$$f(x,u) = f(x_s, u_s) + \left. \frac{\partial f}{\partial x} \right|_{x_s, u_s} (x-x_s) + \left. \frac{\partial f}{\partial u} \right|_{x_s, u_s} (u-u_s) + \\ \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_s, u_s} (x-x_s)^2 + \frac{1}{2} \left. \frac{\partial^2 f}{\partial u^2} \right|_{x_s, u_s} (u-u_s)^2 + \\ + \frac{1}{2} \left. \frac{\partial^2 f}{\partial x \partial u} \right|_{x_s, u_s} (u-u_s)(x-x_s) + O(x^3, u^3)$$

IF WE NEGLECT THE QUADRATIC, TERTIAC AND HIGHER,

$$f(x,u) \approx f(x_s, u_s) + \left. \frac{\partial f}{\partial x} \right|_{x_s, u_s} (x-x_s) + \left. \frac{\partial f}{\partial u} \right|_{x_s, u_s} (u-u_s)$$

$$\text{WE NOTED THAT } f(x_s, u_s) = \left. \frac{dx}{dt} \right|_{x_s, u_s} = 0$$

$$\text{CONSIDER THE VARIABLE } x' = (x-x_s) \quad \text{so} \quad \frac{d(x-x_s)}{dt} = \frac{dx}{dt}$$

WE HAVE

$$\frac{d(x-x_s)}{dt} \approx f(x_s, u_s) + \left. \frac{\partial f}{\partial x} \right|_{x_s, u_s} (x-x_s) + \left. \frac{\partial f}{\partial u} \right|_{x_s, u_s} (u-u_s)$$

$$\text{LET } u' = (u-u_s), \text{ WE CAN WRITE} \quad \underbrace{\frac{dx'}{dt}}_{\alpha} \approx \left. \frac{\partial f}{\partial x} \right|_{x_s, u_s} x' + \underbrace{\left. \frac{\partial f}{\partial u} \right|_{x_s, u_s} u'}_{\beta}$$

ASSUME THAT THERE EXISTS AN OUTPUT VARIABLE $y = g(x, u)$

WE CAN AGAIN TAYLOR EXPAND AND TRUNCATE :

$$g(x, u) \approx g(x_s, u_s) + \frac{\partial g}{\partial x} \Big|_{x_s, u_s} (x - x_s) + \frac{\partial g}{\partial u} \Big|_{x_s, u_s} (u - u_s)$$

AS $g(x_s, u_s)$ IS THE STEADY STATE OF THE OUTPUT (y_s),

$$y \approx g(x_s, u_s) + \underbrace{\frac{\partial g}{\partial x} \Big|_{x_s, u_s} (x - x_s)}_r + \underbrace{\frac{\partial g}{\partial u} \Big|_{x_s, u_s} (u - u_s)}_s$$

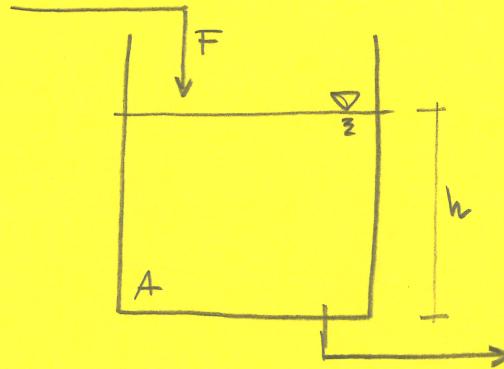
$$y - y_s = r x' + s u'$$

Example

SURFACE TANK

$$\frac{dh}{dt} = \frac{F}{A} - \beta / A \sqrt{h}$$

h IS THE STATE VARIABLE
 F IS THE INPUT VARIABLE



A, β
 ARE PARAMETERS

THE VECTOR FIELD IS $f(h, F)$, WE CAN WRITE

$$f(h, F) \approx f(h_s, F_s) + \frac{\partial f}{\partial h} \Big|_{h_s, F_s} (h - h_s) + \frac{\partial f}{\partial F} \Big|_{h_s, F_s} (F - F_s)$$

$$= \left[\frac{F_s}{A} - \frac{\beta}{A} \sqrt{h_s} \right] + \left(-\frac{1}{2} (h)^{-1/2} \beta / A \right) + \frac{1}{A} (F - F_s)$$

$$= 0 \quad (\text{STEADY-STATE})$$

WE CAN NOW LET $h' = h - h_s$, $u = u - u_s$ AND WRITE

$$\frac{d(h-h_s)}{dt} = -\frac{\beta}{2A\sqrt{h_s}} (h-h_s) + \frac{1}{A} (F-F_s)$$

EQUIVALENTLY, WE HAVE

$$\frac{dh'}{dt} = \underbrace{-\frac{\beta}{2A\sqrt{h_s}}}_{\alpha} h' + \underbrace{\frac{1}{A} u}_{\beta}$$

GENERAL CASE — MULTIPLE VARIABLES

Consider the two-state system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u) \\ \dot{x}_2 = f_2(x_1, x_2, u) \\ y = g(x_1, x_2, u) \end{cases}$$

BY TAYLOR SERIES EXPANSIONS, WE HAVE

$$f_1(x_1, x_2, u) = f_1(x_1^s, x_2^s, u^s) + \frac{\partial f_1}{\partial x_1} \Big|_{ss} (x_1 - x_1^s) + \frac{\partial f_1}{\partial x_2} \Big|_{ss} (x_2 - x_2^s) + \\ + \frac{\partial f_1}{\partial u} \Big|_{ss} (u - u_s) + \text{higher order terms}$$

$$f_2(x_1, x_2, u) = f_2(x_1^s, x_2^s, u^s) + \frac{\partial f_2}{\partial x_1} \Big|_{ss} (x_1 - x_1^s) + \frac{\partial f_2}{\partial x_2} \Big|_{ss} (x_2 - x_2^s) + \\ + \frac{\partial f_2}{\partial u} \Big|_{ss} (u - u_s)$$

$$g(x_1, x_2, u) = g(x_1^s, x_2^s, u) + \frac{\partial g}{\partial x_1} \Big|_{ss} (x_1 - x_1^s) + \frac{\partial g}{\partial x_2} \Big|_{ss} (x_2 - x_2^s) + \\ + \frac{\partial g}{\partial u} \Big|_{ss} (u - u_s)$$

FROM THE EQUILIBRIUM CONDITIONS, WE HAVE THAT

$$\begin{cases} f_1(ss) = 0 \\ f_2(ss) = 0 \\ g(ss) = y_s \end{cases}$$

MOREOVER, WE CAN WRITE

THE JACOBIAN WRT X

$$\frac{d(x_1 - x_1^s)}{dt} = \frac{dx_1}{dt}$$

$$\frac{d(x_2 - x_2^s)}{dt} = \frac{dx_2}{dt}$$

$$\left[\begin{array}{c} \frac{d(x_1 - x_1^s)}{dt} \\ \frac{d(x_2 - x_2^s)}{dt} \end{array} \right] = \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} \Big|_{ss} & \frac{\partial f_1}{\partial x_2} \Big|_{ss} \\ \frac{\partial f_2}{\partial x_1} \Big|_{ss} & \frac{\partial f_2}{\partial x_2} \Big|_{ss} \end{array} \right] \underbrace{\begin{bmatrix} x_1 - x_1^s \\ x_2 - x_2^s \end{bmatrix}}_A$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u} \Big|_{ss} \\ \frac{\partial f_2}{\partial u} \Big|_{ss} \end{bmatrix}}_B(b) \underbrace{\begin{bmatrix} u - u_s \end{bmatrix}}_u$$

$$\cdot \begin{bmatrix} x_1 - x_1^s \\ x_2 - x_2^s \end{bmatrix} + \text{error}$$

AS FOR THE OUTPUT, WE CAN WRITE

$$[y - y_s] = \underbrace{[\frac{\partial g}{\partial x_1} \Big|_{ss} \quad \frac{\partial g}{\partial x_2} \Big|_{ss}]}_{C(c)} \begin{bmatrix} x_1 - x_1^s \\ x_2 - x_2^s \end{bmatrix} + \underbrace{[\frac{\partial g}{\partial u} \Big|_{ss}]}_{D(d)} \underbrace{[u - u_s]}_{w}$$

GENERAL CASE - MULTIPLE VARIABLES, MULTIPLE INPUTS

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N_x} \end{bmatrix} = \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \\ \vdots \\ f_{N_x}(x, u) \end{bmatrix} \quad \rightsquigarrow \quad A_{ij} = \frac{\partial f_i}{\partial x_j} \Big|_{ss}$$

$$B_{ij} = \frac{\partial f_i}{\partial u_j} \Big|_{ss}$$

MOREOVER

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N_y} \end{bmatrix} = \begin{bmatrix} g_1(x, u) \\ g_2(x, u) \\ \vdots \\ g_{N_y}(x, u) \end{bmatrix} \quad \rightsquigarrow \quad C_{ij} = \frac{\partial g_i}{\partial x_j} \Big|_{ss}$$

$$D_{ij} = \frac{\partial g_i}{\partial u_j} \Big|_{ss}$$