THE BELLMAN – GRONWALL LEMMA

We discuss the existence and uniqueness theorem of solution of ordinary differential equation ⟶ focus on uniqueness using the BELLMAN – GRONWALL LEMMA

We consider an ordinary differential equation $\dot{x} = f(x,t)$ with some initial condition $x(t_0) = x_0$, with $x(t) \in \mathbb{R}^n$

$$f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$$

IF FUNCTION $f(\cdot,\cdot)$ IS LIPSCHITZ CONTINUOUS IN x AND PIECEWISE CONTINUOUS IN TIME t, THEN THERE EXISTS AN UNIQUE SOLUTION FOR THAT INITIAL CONDITION

almost everywhere (at some point the derivative does not exist)

WE ARE NOT GOING TO PROVE THIS, BUT WE WILL DISCUSS SOME OF THE TOOLS USED IN THE PROOF

⟶ The machinery used to prove uniqueness is called the Bellman – Gronwall lemma

Lemma – Suppose we have functions $\underbrace{u(\cdot) \text{ and } k(\cdot)}$

BOTH ARE REAL-VALUED, PIECEWISE CONTINUOUS (PC) POSITIVE (≥ 0 on $\mathbb{R}_+$)

- We also have some constant $c_1 \geq 0$ and some initial time t_0

IF $u(t)$ CAN BE BOUNDED $u(t) \leq c_1 + \int_{t_0}^{t} k(\tau)\, u(\tau)\, d\tau$ THEN WE HAVE THE FOLLOWING BOUND HOLDS

$$\leadsto \quad u(t) \leq c_1 e^{\int_{t_0}^{t} k(\tau)\, d\tau}$$

THE LEMMA CAN BE EASILY PROVEN

- Let assume that $t > t_0$ and define $U(t) = C_1 + \int_{t_0}^{t} k(\tau) u(\tau) dt$

$\rightsquigarrow u(t) \le U(t)$

- We then multiply this inequality on both size by function

$$k(t) e^{-\int_{t_0}^{t} k(\tau) dt}$$

$$\rightsquigarrow u(t) \underbrace{\underbrace{k(t)}_{\text{non negative}} \underbrace{e^{-\int_{t_0}^{t} k(\tau) d\tau}}_{\text{nonnegative}}}_{\text{nonnegative}}$$

nonnegative ($\rightsquigarrow$ will not change the sign of the inequality)

- We get $u(t) k(t) e^{+\int_{t_0}^{t} k(\tau) dt} \le U(t) k(t) e^{-\int_{t_0}^{t} k(\tau) d\tau}$

- Rearranging $u(t) k(t) e^{+\int_{t_0}^{t} k(\tau) d\tau} - U(t) k(t) e^{-\int_{t_0}^{t} k(\tau) d\tau}$

$$\rightsquigarrow \frac{d}{dt}\left[U(t) e^{-\int_{t_0}^{t} k(\tau) d\tau}\right] \le 0$$

which can be integrated between to and t

$u(t) \le U(t) \le C_1 e^{+\int_{t_0}^{t} k(\tau) d\tau}$ which ends the proof.

The use of the Bellman-Gronwall lemma in proving the uniqueness of ordinary differential equation with given initial conditions

$\dot{x} = f(x,t)$ with $x(t_0) = x_0$

Suppose we derived a solution $\phi(t)$ (a function that satisfies both the differential eqn. and the initial condition)

How to show that there is no other such function?

We can assume that such function is NOT unique and see if a contradiction can be derived

$\exists$ another function $\psi(t)$ that satisfies both the differential equation and the initial condition (thus, a solution)

We know that:

$$\dot{\phi}(t) = f(\phi(t), t) \;,\; \phi(t_0) = x_0 \rightsquigarrow \phi(t) = x_0 + \int_{t_0}^{t} f(\phi(\tau), \tau)\, dt$$

$$\dot{\psi}(t) = f(\psi(t), t) \;,\; \psi(t_0) = x_0 \rightsquigarrow \underbrace{\psi(t) = x_0 + \int_{t_0}^{t} f(\psi(\tau), \tau)\, d\tau}_{\text{By rewriting the differential equations in integral form}}$$

We can take the difference between the two, to get

$$\|\phi(t) - \psi(t)\| = \left\| \int_{t_0}^{t} f(\phi(\tau), \tau)\, d\tau - \int_{t_0}^{t} f(\psi(\tau), \tau)\, d\tau \right\|$$

$$= \left\| \int_{t_0}^{t} [f(\phi(\tau), \tau) - f(\psi(\tau), \tau)]\, d\tau \right\|$$

$$= \int_{t_0}^{t} \| f(\phi(\tau), \tau) - f(\psi(\tau), \tau) \|\, d\tau$$

$$\leq \bar{K} \int_{t_0}^{t} \underbrace{\|\phi(t) - \psi(t)\|}_{u(t)}\, d\tau$$

$$\underbrace{\bar{K} \geq K(t)}_{\text{the supremum}}$$

CHECK STEPS (UNCLEAR)

BY LIPSCHITZ CONTINUITY OF f WRT x

The Bellman – Gronwall Lemma

The distance between $\phi(t)$ and $\psi(t)$ is less or equal to $0 \cdot \cancel{e^{\int_{t_0}^{t} \bar{K}\, dt}} = 0$

→ Thus, by definition of the norm this can only be true if $\phi(t) = \psi(t)$ (which completes the proof)

$$\underbrace{\|\phi(t) - \psi(t)\| = 0}_{\text{the two different functions are equal}}$$