Aalto University

Matrix algebra, w/ intro to Matlab

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Francesco Corona

Chemical and Metallurgical Engineering
School of Chemical Engineering

Matrix algebra

This tutorial notes overview some fundamental concepts in matrix algebra

- Matrix and vectors (definitions) and main matrix operators
- Determinant and rank, linear equations, and inverse
- Eigendecomposition, eigenvalues and eigenvectors

Matrices, vectors and scalars

Matrix algebra

Matrices, vectors, and scalars

Definition

A matrix

A **matrix** A of dimension $(m \times n)$ is a table of elements

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,j} & \cdots & a_{m,n} \end{bmatrix}$$

- m **rows**
- n **columns**

The notation $A = \{a_{i,j}\}$ indicates that matrix A has elements $a_{i,j}$

- At the intersection of row i with column j

We will consider (mostly) real matrices, in which element $a_{i,j} \in \mathcal{R}$

To indicate a matrix, we use upper-case letters $A, B, C, \dots$

- $A^{m \times n}$ indicates a matrix A of dimension $(m \times n)$

Matrices and vectors (cont.)

Example

Create a $m \times n$ matrix A of random values, with $m = 5$ and $n = 3$

```
m = 5;                                  % Set the number of rows, m
n = 3;                                  % Set the number of columns, n

A = rand(m,n);                          % Generates a random matrix A, size (m x n)
                                        % For details, 'help rand' or 'doc rand'
                                        %
                                        % Every time you run the command,
                                        % the elements of A will change (randomly)
```

```
>> A                                    % Check content of A
                                        %
A =                                     %
                                        %
    0.8147    0.0975    0.1576          % A is generated randomly
    0.9058    0.2785    0.9706          %
    0.1270    0.5469    0.9572          %
    0.9134    0.9575    0.4854          %
    0.6324    0.9649    0.8003          %
```

Matrices and vectors (cont.)

Example

Consider the (2×3) matrix

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

The elements of the matrix

- $\rightsquigarrow$ $a_{1,1} = 1$
- $\rightsquigarrow$ $a_{1,2} = 3.5$
- $\rightsquigarrow$ $a_{1,3} = 2$
- $\rightsquigarrow$ $a_{2,1} = 0$
- $\rightsquigarrow$ $a_{2,2} = 1$
- $\rightsquigarrow$ $a_{2,3} = 3$

```
A = [1, 3.5, 2; 0, 1, 3];          % Manually input matrix A, row-after-row
                                   % Semi-column separates rows
```

```
>> size(A)                         % Check dimensions of A
                                   % For details, 'help size' or 'doc size'
ans =                              %
                                   % 3 rows, 3 columns, as expected
     2     3                       %
```

Matrices and vectors (cont.)

```
>> A                                    % Check the content of variable A
                                        %
A =                                     %
                                        %
    1.0000   3.5000   2.0000            %
    0        1.0000   3.0000            %

>> A(1,1)                               % Checks element (1,1)
                                        %
ans =                                   % Element exists, return its value
                                        %
     1                                  %

>> A(3,1)                               % Check element (3,1)
Index exceeds matrix dimensions.        % Element (3,1) does not exist
                                        % Return error, matrix is (2 x 3)

>> A(1,3)                               % Check element (1,3)
                                        %
ans =                                   % Element (1,3) exists
                                        % Return its value
     2                                  %
```

Matrices and vectors

Definition

A scalar and a vector

A **scalar** is a matrix of dimension (1×1)

$$A = \begin{bmatrix} a_{1,1} \end{bmatrix}$$

A **vector** is a matrix in which one of the dimensions is one

$\rightsquigarrow$ **Column-vector**, a $(m \times 1)$ matrix (a column)

$$A = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{i,1} \\ \vdots \\ a_{m,1} \end{bmatrix}$$

$\rightsquigarrow$ **Row-vector**, $(1 \times n)$ matrix (a row)

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j} & \cdots & a_{1,n} \end{bmatrix}$$

Matrices and vectors (cont.)

To indicate a vector, we use lower-case letters

$$\rightsquigarrow \quad x, y, z, \ldots$$

$x \in \mathcal{R}^m$ indicates a column-vector x of dimension $(m \times 1)$

Matrices and vectors (cont.)

Example

Consider the 2 vectors

$$x = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 2 & 3 & 0 & 1.4 \end{bmatrix}$$

The type of vectors

- Vector x has dimension (3×1), a column-vector with 3 components
- Vector y has dimension (1×4), a row-vector with 4 components

Matrices and vectors (cont.)

```
x = [1; 0; 2];                  % Define x as column vector (3 x 1)
                                % x is a 1D tall matrix

y = [2, 3, 0, 1.4];             % Define y as row vector (1 x 4)
                                % y is a 1D wide matrix
```

```
>> size(x)                      % Check dimensions of X
                                %
ans =                           %
                                %
     3     1                    %

>> x = x';                      % Redefine x as equal to its transpose
                                % Symbol ' computes the transpose

>> size(x)                      % Check dimensions of new x
                                %
ans =                           %
                                %
     1     3                    %
```

Matrices and vectors (cont.)

A $(m \times n)$ matrix is understood as consisting of n $(m \times 1)$ column-vectors

$$\rightsquigarrow \quad A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

$\rightsquigarrow$ a_i is the i-th column

```
n = 10                          % Define the number n of column vectors a_in
m = 15                          % Define the number m of their elements

A = zeros(m,n)                  % Create a (m x n) A matrix of zeros (Initialise)

for in = 1:n                    % FOR each integer index 'in' between 1 and n,
    a = rand(m,1);              % create a random m-vector 'a', size (m x 1)
                                %
    A(:,in) = a;                % Place column-vector 'a' in the n-th column
                                % of matrix 'A' (overwriting the zeros)
end                             % Close the FOR-loop

doc for                         % Extended documentation about FOR-loops
help for                        % Quick documentation about FOR-loops
```

```
>> whos A                       % Return information about variables A
  Name          Size                Bytes  Class     Attributes

  A            15x10                 1200  double
```

Matrices and vectors (cont.)

A $(m \times n)$ matrix is understood as consisting of m $(1 \times n)$ row-vectors

$$\rightsquigarrow \quad A = \begin{bmatrix} -- & b_1 & -- \\ -- & b_2 & -- \\ & \vdots & \\ -- & & -- \\ -- & b_n & -- \end{bmatrix} \quad (b_i \text{ is the } i\text{-th row})$$

```
n = 10                          % Define the number n of column vectors a_in
m = 15                          % Define the number m of their elements

A = zeros(m,n)                  % Create a (n x m) matrix A of zeros (Initialise)

for im = 1:m                    % FOR each integer index 'in' between 1 and m,
    b = rand(1,n);              % create a random n-vector 'a', size (1 x n)
                                %
    A(im,:) = b;                % Place row-vector 'a' in the m-th row
                                % of matrix 'A' (overwriting the zeros)
end                             % Close the FOR-loop
```

The same code using the '**cell-array**' data structure for storing the vectors **a**

```
n = 10; m = 15                  % Define the dimensions of A
A = zeros(m,n)                  % Create a (m x n) A matrix of zeros

for in = 1:n                    % FOR integer index 'in' between 1 and n,
    a{in} = rand(m,1);          % create a m-vector 'a{in}', size (m x 1)
                                % as in-th cell of a cell array variables
    A(:,in) = a{in};            % Place 'a{in}' in the n-th column of 'A'
end                             % Close the FOR-loop
```

Matrices and vectors (cont.)

Example

Consider the (2×3) matrix

$$A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Extract columns and rows (that is, create column- and row-vectors from A)

```
>> A = [1, 3.5, 2; 0, 1, 3];     % Create matrix A
```

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 3.5 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{(As component columns)}$$

```
a1 = A(:,1);                     % Extract vector a1 from A: 1st column, all rows
a2 = A(:,2);                     % Extract vector a2 from A: 2nd column, all rows
a3 = A(:,3);                     % Extract vector a3 from A: 3rd column, all rows
```

$$b_1 = \begin{bmatrix} 1 & 3.5 & 2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \quad \text{(As component rows)}$$

```
b1 = A(1,:);                     % Extract vector b1 from A: 1st row, all columns
b2 = A(2,:);                     % Extract vector b2 from A: 2nd row, all columns
```

Example

Create a $m \times n$ matrix A of random values, with $m = 2$ and $n = 5$

1. Display matrix A and check its size
2. Extract element $a_{2,3}$ and element $a_{3,2}$
3. Extract the 4-th column and the 1-st row of A

Repeat the previous steps on a new matrix B with $m = n = 5$

Matrices and vectors (cont.)

Definition

A square matrix

A matrix A is said to be a **square matrix** if its dimension is $(n \times n)$

- The number of rows equals the number of columns

The **diagonal** of a square matrix A of order n is the set of elements

$$\{a_{1,1}, a_{2,2}, \cdots, a_{n,n}\}$$

They have the same row- and column-number

Matrices and vectors (cont.)

```
n = 5;                                                  % Set number of rows/columns
A = rand(n,n);                                          % Create a (n x n) matrix A
```

```
>> A                                                    % Check elements of matrix A

A =

    0.9631    0.6241    0.0377    0.2619    0.1068
    0.5468    0.6791    0.8852    0.3354    0.6538
    0.5211    0.3955    0.9133    0.6797    0.4942
    0.2316    0.3674    0.7962    0.1366    0.7791
    0.4889    0.9880    0.0987    0.7212    0.7150

>> diag(A)                                              % Show diagonal elements of A
                                                        % It is a column vector
ans =                                                   % The size is (n x 1)
                                                        %
    0.9631                                              % Type 'help diag'
    0.6791                                              % Type 'doc diag'
    0.9133                                              %
    0.1366                                              %
    0.7150                                              %
```

Matrices and vectors (cont.)

Example

Consider the order 4 square matrix $A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 4 & 3 \\ 3 & 2 & 6 \end{bmatrix}$, its diagonal elements $\{1, 4, 6\}$

```
A = [1, 3.5, 2; ...          % Create the (3 x 3) matrix A
     0,   4, 3; ...          % Use triple dots to continue
     3,   2, 6];             % onto next line
```

```
>> diag(A)                   % Show the diagonal elements of A

ans =

     1
     4
     6

>> d = diag(A);              % Create a vector d, size (3 x 1)
                             % with the diagonal elements of A
```

Matrices and vectors (cont.)

Definition

Square matrices

Diagonal

- All off-diagonal elements are zero

Identity matrix

- A diagonal matrix whose diagonal elements are equal to one, I or I_n

Block-diagonal

- All elements are zero except for some square blocks along the diagonal

Lower- (upper-) triangular

- All elements above (below) the diagonal are zero

Matrices and vectors (cont.)

```
>> m = 5;                               % Define some scalar, (1 x 1) matrix, m

>> d = 1:m;                             % Create a n-vector 'd', size (m,1)
                                        % The elements of d are (1,2,...,5)

                                        % Find out how to use ':' in Matlab

>> D = diag(d)                          % Create a diagonal matrix D based
                                        % on vector 'd', size (5 x 5)

D =

     1     0     0     0     0
     0     2     0     0     0
     0     0     3     0     0
     0     0     0     4     0
     0     0     0     0     5

>> I = eye(m)                           % Create an identity matrix I of order 'm'
                                        % Try 'help eye' and 'doc eye'
I =

     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0
     0     0     0     0     1
```

Matrices and vectors (cont.)

```
>> na = 2; A = rand(na,na);                % Create (na x na) matrix A
>> nb = 1; B = rand(nb,nb);                % Create (nb x nb) matrix B
>> nc = 3; C = rand(nc,nc);                % Create (nc x nc) matrix C

>> D = blkdiag(A,B,C)                      % Create a block-diagonal matrix D,
                                           % from A, B and C
 D =

    0.6490    0.4538         0         0         0         0
    0.8003    0.4324         0         0         0         0
         0         0    0.8253         0         0         0
         0         0         0    0.0835    0.3909    0.0605
         0         0         0    0.1332    0.8314    0.3993
         0         0         0    0.1734    0.8034    0.5269
```

Matrices and vectors (cont.)

Example

Consider the order 4 square matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 6 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 2 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Matrix A is diagonal
- Matrix B is lower-triangular
- Matrix C is upper-triangular

- Matrix I is an identity of order 3

Matrices and vectors (cont.)

Matrix $\widetilde{A}$ is block-diagonal

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_1 & 0 & 0 \\ 0 & \widetilde{A}_2 & 0 \\ 0 & 0 & \widetilde{A}_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Three blocks, $\widetilde{A}_1$, $\widetilde{B}_2$ and $\widetilde{B}_3$, one of order 2 and 2 of order 1

Matrix $\tilde{A}$ is upper-block-triangular

$$\tilde{A} = \begin{bmatrix} \tilde{B}_1 & \tilde{B}_3 \\ 0 & \tilde{B}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

Two diagonal blocks, $\widetilde{B}_1$ and $\widetilde{B}_2$, both of order 2

Matrix operators

Matrix algebra

Transposition

Definition

Matrix transposition

Consider a matrix $A = \{a_{i,j}\}$ of dimension $(m \times n)$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

The **transpose** of A is the matrix $A' = \{a'_{i,j} = a_{j,i}\}$ of dimension $(n \times m)$

$$A' = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}$$

- On the j-th row of A', the elements of the j-th column of A
- On the i-th column of A', the elements of the j-th row of A

Transposition (cont.)

Example

Consider the (2×3) matrix A and its transpose A'

$$A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad A' = \begin{bmatrix} 1 & 0 \\ 3.5 & 1 \\ 2 & 3 \end{bmatrix}$$

```
A = [1 3.5 2;                                    % Create matrix A
     0 1   3];                                   %

At = A';                                         % Compute its transpose
                                                 % Alternativaly, At = transp(A)
                                                 % The two forms are equivalent
```

Transposition (cont.)

The following properties hold

- If D is a diagonal matrix, we have $D = D'$
- If A is lower-triangular, then A' is upper-triangular
- If A is upper-triangular, then A' is lower-triangular
- If A is a row-vector, A' is a column-vector
- If A is a column-vector, A' is a row-vector
- If $B = A'$, we have $B' = (A')'$

```
>> m = 4; d = rand(m,1); D = diag(d)           % Define dimension and a random vector
                                                % Use vector to create diagonal matrix

D =                                             % Show the diagonal matrix D

    0.8147         0         0         0
         0    0.9058         0         0
         0         0    0.1270         0
         0         0         0    0.9134

>> Dt = D'                                      % Compute the transpose of matrix D
                                                % Display Dt
Dt =

    0.8147         0         0         0
         0    0.9058         0         0
         0         0    0.1270         0
         0         0         0    0.9134

>> D == Dt                                      % Check whether D and D'are equal
                                                % The check is done elementwise
                                                % Return a (m x m) logical array
                                                %
ans =                                           %
                                                %
  4x4 logical array                             %
                                                %
   1   1   1   1                                %
   1   1   1   1                                % An alternative way of checking
   1   1   1   1                                % >> isequal(D,Dt)
   1   1   1   1                                % Return one logical variable
```

Transposition (cont.)

```
>> m = 4; A = rand(m,m); Au = triu(A)

Au =

    0.6324    0.9575    0.9572    0.4218
         0    0.9649    0.4854    0.9157
         0         0    0.8003    0.7922
         0         0         0    0.9595

>> Au'

ans =

    0.6324         0         0         0
    0.9575    0.9649         0         0
    0.9572    0.4854    0.8003         0
    0.4218    0.9157    0.7922    0.9595

>> (Au')'

ans =

    0.6324    0.9575    0.9572    0.4218
         0    0.9649    0.4854    0.9157
         0         0    0.8003    0.7922
         0         0         0    0.9595
```

Transposition (cont.)

```
>> m = 4; A = rand(m,m); Al = tril(A)

Al =

    0.8147         0         0         0
    0.9058    0.0975         0         0
    0.1270    0.2785    0.1576         0
    0.9134    0.5469    0.9706    0.1419

>> Al'

ans =

    0.8147    0.9058    0.1270    0.9134
         0    0.0975    0.2785    0.5469
         0         0    0.1576    0.9706
         0         0         0    0.1419

>> (Al')' == Al

  4x4 logical array

   1   1   1   1
   1   1   1   1
   1   1   1   1
   1   1   1   1
```

Transposition (cont.)

```
>> m = 4; a = rand(1,m)

a =

    0.2511    0.6160    0.4733    0.3517

>> at = a'

at =

    0.2511
    0.6160
    0.4733
    0.3517

>> (a')'

ans =

    0.2511    0.6160    0.4733    0.3517

>> isequal((a')',a)

ans =

  logical

   1
```

Sum and difference

Definition

Matrix sum and difference

Consider two matrices $A = \{a_{i,j}\}$ and $B = \{b_{i,j}\}$ both of dimension $(m \times n)$

Define the **sum** of A and B as the $(m \times n)$ matrix $S = \{c_{i,j} = a_{i,j} + b_{i,j}\}$

$$
\begin{aligned}
S &= A + B \\
&= \begin{bmatrix}
a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,j} + b_{1,j} & \cdots & a_{1,n} + b_{1,n} \\
a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,j} + b_{2,j} & \cdots & a_{2,n} + b_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i,1} + b_{i,1} & a_{i,2} + b_{i,2} & \cdots & a_{i,j} + b_{i,j} & \cdots & a_{i,n} + b_{i,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \cdots & a_{m,j} + b_{m,j} & \cdots & a_{m,n} + b_{m,n}
\end{bmatrix}
\end{aligned}
$$

- Element $c_{i,j}$ is equal to the sum of elements $a_{i,j}$ and $b_{i,j}$

Define the **difference** of A and B as the $(m \times n)$ matrix

$$D = A - B = \{d_{i,j} = a_{i,j} - b_{i,j}\}$$

Sum and difference (cont.)

Example

Consider the two (2×3) matrices A and B

$$A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Their sum

$$S = A + B = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} \underbrace{1+1}_{2} & \underbrace{3.5+2}_{5.5} & \underbrace{2+3}_{5} \\ \underbrace{0+4}_{4} & \underbrace{1+5}_{6} & \underbrace{3+6}_{9} \end{bmatrix}$$

Their difference

$$D = A - B = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} \underbrace{1-1}_{0} & \underbrace{3.5-2}_{1.5} & \underbrace{2-3}_{-1} \\ \underbrace{0-4}_{-4} & \underbrace{1-5}_{-4} & \underbrace{3-6}_{-3} \end{bmatrix}$$

Sum and difference (cont.)

```
>> A = [1, 3.5, 2; 0, 1, 3];
>> B = [1,   2, 3; 4, 5, 6];

>> S = A + B

S =

    2.0000    5.0000    5.0000
    4.0000    6.0000    9.0000

>> D = A - B

D =
         0    1.5000   -1.0000
   -4.0000   -4.0000   -3.0000
```

Matrix-scalar multiplication

Definition

Matrix-scalar product

Consider a number $s \in \mathcal{R}$ and a $(m \times n)$ matrix $A = \{a_{i,j}\}$

Define **matrix-scalar product** of A and s as the $(m \times n)$ matrix $B = sA$

$$B = sA = \begin{bmatrix} s \cdot a_{1,1} & \cdots & s \cdot a_{1,j} & \cdots & s \cdot a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s \cdot a_{i,1} & \cdots & s \cdot a_{i,j} & \cdots & s \cdot a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s \cdot a_{m,1} & \cdots & s \cdot a_{m,j} & \cdots & s \cdot a_{m,n} \end{bmatrix}$$

- Element $b_{i,j}$ is equal to the product of s and element $a_{i,j}$

Matrix-scalar multiplication (cont.)

Example

Let $s = 4$ and let $A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, we have $sA = 4 \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 14 & 8 \\ 0 & 4 & 12 \end{bmatrix}$

```
>> s = 4; A = [1, 3.5, 2; 0, 1 3];

>> s*A

ans =

     4    14     8
     0     4    12

>> A*s

ans =

     4    14     8
     0     4    12
```

■

Matrix-vector multiplication

We treated matrices and vectors as simple collection of numbers, or rectangular tables

More mathematically, matrices associate with linear transformations or functions

- A function is an operation that takes an input and returns an output
- (We often denote those as independent and dependent variables)

In matrix algebra, we consider transformations that map vectors into vectors

$$\rightsquigarrow \qquad y = A(x), \quad \text{(with } x \text{ and } y \text{ vectors and } A \text{ a transformation)}$$

Think of the usual $2D$ Cartesian space, A transforms a vector x into another vector y

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Understanding linear functions means understanding how bases vectors are transformed

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$$

Matrix-vector multiplication (cont.)

Consider a transformation A such that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$

For any vector $x = \big[[x_1 \quad x_2]\big]'$, we have its new (transformed) coordinates

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \end{bmatrix}$$

We collect the transformed bases vectors in a (2×2) matrix A

$$\rightsquigarrow \quad A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

We can write

$$\rightsquigarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matrix-vector multiplication

Definition

Matrix-vector multiplication

Let $A = \{a_{i,j}\}$ be a $(m \times n)$ matrix and let $b = \{b_{i,j}\}$ be a $(n \times 1)$ matrix (a vector)

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_n \end{bmatrix}$$

The product between A and b is defined as a $(m \times 1)$ matrix $c = \{c_i\}$ (a vector)

$$c = \{c_i = \sum_{k=1}^{n} a_{i,k} b_k\}$$

Matrix-vector multiplication (cont.)

Example

Let $A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, then let $b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $c = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

To compute the vector $d = Ab$ and $e = Ac$, we have

$$d = Ab = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 3.5 \cdot 3 + 2 \cdot 5 \\ 0 \cdot 1 + 1 \cdot 3 + 3 \cdot 5 \\ 0 \cdot 1 + 0 \cdot 3 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 21.5 \\ 18 \\ 5 \end{bmatrix}$$

$$e = Ac = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 3.5 \cdot 4 + 2 \cdot 6 \\ 0 \cdot 2 + 1 \cdot 4 + 3 \cdot 6 \\ 0 \cdot 2 + 0 \cdot 4 + 1 \cdot 6 \end{bmatrix} = \begin{bmatrix} 28 \\ 22 \\ 6 \end{bmatrix}$$

Matrix-vector multiplication (cont.)

```
>> A = [1 3.5 2; 0 1 3; 0 0 1];                   % Define the matrix the (3x3) matrix A

>> b = [1; 3; 5];                                  % Define the (3x1) vector b
>> c = [2; 4; 6];                                  % Define the (3x1) vector c

>> d = A*b                                         % Compute the (3x1) vector d
                                                   % Try b*A and comment
d =

   21.5000
   18.0000
    5.0000

>> e = A*c                                         % Compute the (3x1) vector e
                                                   % Try c*A and comment
e =

   28.0000
   22.0000
    6.0000
```

Matrix-vector multiplication (cont.)

Example

Let $A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, then let $b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $c = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

Compute the vectors $d = Ab$ and $e = Ac$ and comment on the result

$$d = Ab = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 3.5 \cdot 3 + 2 \cdot 5 \\ 0 \cdot 1 + 1 \cdot 3 + 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 21.5 \\ 18 \end{bmatrix}$$

$$e = Ac = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 3.5 \cdot 4 + 2 \cdot 6 \\ 0 \cdot 2 + 1 \cdot 4 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 28 \\ 22 \end{bmatrix}$$

Matrix-vector multiplication (cont.)

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$, then let $b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $c = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

Compute the vectors $d = Ab$ and $e = Ac$ and comment on the results

Matrix-matrix multiplication

Definition

Matrix-matrix multiplication

Let $A = \{a_{i,j}\}$ be a $(m \times n)$ matrix and let $B = \{b_{i,j}\}$ be a $(n \times p)$ matrix

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix}, \quad B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,j} & \cdots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}$$

The product between A and B is defined as a $(m \times p)$ matrix $C = \{c_{i,j}\}$

$$C = \{c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}\}$$

Matrix-matrix product (cont.)

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,j} & \cdots & c_{1,p-1} & c_{1,p} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,j} & \cdots & c_{2,p-1} & c_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{i,1} & c_{i,2} & \cdots & c_{i,j} & \cdots & c_{i,p-1} & c_{i,p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ c_{m-1,1} & c_{m-1,2} & \cdots & c_{m-1,j} & \cdots & c_{m-1,p-1} & c_{m-1,p} \\ c_{m,1} & c_{m,2} & \cdots & c_{m,j} & \cdots & c_{m,p-1} & c_{m,p} \end{bmatrix}$$

Element $c_{i,j}$ of matrix C is given by the scalar product between a_i' and b_j

$$c_{i,j} = a_i' b_j = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,k} & \cdots & a_{i,n} \end{bmatrix} \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{k,j} \\ \vdots \\ b_{n,j} \end{bmatrix}$$

$$= a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots + a_{i,n} b_{n,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

Matrix-matrix product (cont.)

Example

Let $A = \begin{bmatrix} 1 & 3.5 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, we have

$$C = AB = \begin{bmatrix} 1\cdot 1 + 3.5\cdot 3 + 2\cdot 5 & 1\cdot 2 + 3.5\cdot 4 + 2\cdot 6 \\ 0\cdot 1 + 1\cdot 3 + 3\cdot 5 & 0\cdot 2 + 1\cdot 4 + 3\cdot 6 \\ 0\cdot 1 + 0\cdot 3 + 1\cdot 5 & 0\cdot 2 + 0\cdot 4 + 1\cdot 6 \end{bmatrix} = \begin{bmatrix} 21.5 & 28 \\ 18 & 22 \\ 5 & 6 \end{bmatrix}$$

```
>> A = [1 3.5 2; 0 1 3; 0 0 1];
>> B = [1 2; 3 4; 5 6];

>> C = A*B

C =

   21.5000   28.0000
   18.0000   22.0000
    5.0000    6.0000
```

Matrix-matrix product (cont.)

Element $c_{i,j}$ of matrix C is given by the scalar product between a_i' and b_j

$$c_{i,j} = a_i' b_j = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,k} & \cdots & a_{i,n} \end{bmatrix} \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{k,j} \\ \vdots \\ b_{n,j} \end{bmatrix}$$

```
clear C

for i = 1:size(A,1)
    for j = 1:size(B,2)
        C(i,j) = A(i,:)*B(:,j);
    end
end
```

```
>> isequal(A*B,C)

ans =

  logical

   1
```

Matrix-matrix product (cont.)

For every $(m \times n)$ matrix A, we have

$$\underbrace{I_m}_{(m\times m)} \underbrace{A}_{(m\times n)} = \underbrace{A}_{(m\times n)} \underbrace{I_n}_{(n\times n)} = \underbrace{A}_{(m\times n)}$$

Right- and left-multiplication of matrix A by an identity matrix

---

Matrix product is not necessarily commutative, $AB \neq BA$

$$\underbrace{A}_{(m\times n)} \underbrace{B}_{(n\times p)} = \underbrace{C}_{(m\times p)}$$

$$= \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,k} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,j} & \cdots & b_{k,p} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}$$

The product BA is not defined

A and B must be both square and of the same order (necessary condition)

Matrix-matrix product (cont.)

A $(n \times n)$ diagonal matrix D commutes with any $(n \times n)$ matrix A

$$DA = AD$$

$$\underbrace{D}_{(n\times n)}\underbrace{A}_{(n\times n)} = \underbrace{C}_{(n\times n)}$$

$$= \begin{bmatrix} d_{1,1} & \cdots & d_{1,k} & \cdots & d_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{i,1} & \cdots & d_{i,k} & \cdots & d_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{n,1} & \cdots & d_{n,k} & \cdots & d_{n,n} \end{bmatrix} \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,j} & \cdots & a_{k,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix}$$

We have,

$$\rightsquigarrow c_{ij} = \cancel{d_{i,1}} a_{1,j} + \cdots + d_{i,k} a_{k,j} + \cdots + \cancel{d_{i,n}} a_{n,j} = d_{i,k} a_{k,j}$$

Matrix-matrix product (cont.)

$$\underbrace{A}_{(n \times n)} \underbrace{D}_{(n \times n)} = \underbrace{C}_{(n \times n)}$$

$$= \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,j} & \cdots & a_{k,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} d_{1,1} & \cdots & d_{1,k} & \cdots & d_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{i,1} & \cdots & d_{i,k} & \cdots & d_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{n,1} & \cdots & d_{n,k} & \cdots & d_{n,n} \end{bmatrix}$$

We have,

$$\rightsquigarrow c_{ij} = a_{k,1}\cancel{d_{1,k}} + \cdots + a_{k,j}\, d_{i,k} + \cdots + a_{k,n}\cancel{d_{n,k}} = a_{k,j}\, d_{i,k}$$

Matrix-matrix product (cont.)

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$, we have that $AB = \begin{bmatrix} 6 & 6 \\ 4 & 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 2 & 10 \end{bmatrix} = BA$

```
>> A = [1 2; 0 2]; B = [2 0; 2 3];        % Define A and B
>> A*B                                    % Compute and display A*B

ans =

     6     6
     4     6

>> B*A                                    % Compute and display B*A

ans =

     2     4
     2    10

>> isequal(A*B,B*A)                       % Uncomment , remove ‘;’, to see the output
>> A*B == B*A                             % Uncomment , remove ‘;’, to seethe output
```

Matrix-matrix product (cont.)

Proposition

Let A be a $(m \times n)$ matrix and et B be a $(n \times p)$ matrix

$$A = \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_m' \end{bmatrix}, \quad B = \begin{bmatrix} b_1 | b_2 | \cdots | b_p \end{bmatrix}$$

Let S and Z be order m and order p diagonal matrices

$$S = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & s_m \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & z_p \end{bmatrix}$$

We can state a number of identities

Matrix-matrix product (cont.)

$$AB = \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_m' \end{bmatrix} B = \begin{bmatrix} a_1' B \\ a_2' B \\ \vdots \\ a_m' B \end{bmatrix} = A \left[b_1 | b_2 | \cdots | b_p \right] = \left[Ab_1 | Ab_2 | \cdots | Ab_p \right]$$

$$SA = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & s_m \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ \vdots \\ a_m' \end{bmatrix} = \begin{bmatrix} s_1 a_1' \\ s_2 a_2' \\ \vdots \\ s_m a_m' \end{bmatrix}$$

$$BZ = \left[b_1 | b_2 | \cdots | b_p \right] \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & z_p \end{bmatrix} = \left[z_1 b_1 | z_2 b_2 | \cdots | z_p b_p \right]$$

Matrix powers

Definition

Powers of a matrix

Let A be a square matrix of order n

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix}$$

The k-th power of matrix A is defined as matrix A^k of order n

$$A^k = \underbrace{AA\cdots A}_{k \text{ times}}$$

Special cases,

- $A^{k=0} = I$
- $A^{k=1} = A$

Matrix powers (cont.)

Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

We have,

$$A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad A^1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}; \quad A^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}; \quad A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}; \quad \cdots$$

```
>> A = [1 2; 0 1];                     % Define matrix A

>> A0 = A^0;                           % Compute its zero-th power
>> A1 = A^1;                           % Compute its first power
>> A2 = A^2;                           % Compute its second power
>> A3 = A^3;                           % Compute its third power

>> a3 = A.^3;                          % Compute the third power of its elements
```

The matrix exponential

Let z be some scalar, by definition its exponential is a scalar

$$\rightsquigarrow \quad e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{(The series always converges)}$$

Definition

The matrix exponential Let A be a $(n \times n)$ matrix, by definition its exponential is a $(n \times n)$ matrix

$$\rightsquigarrow \quad e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad \text{(The series always converges)}$$

The matrix exponential (cont.)

Proposition

The matrix exponential of block-diagonal matrices

Consider a block-diagonal matrix A

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix}$$

We have,

$$\rightsquigarrow \quad e^A = \begin{bmatrix} e^{A_1} & 0 & \cdots & 0 \\ 0 & e^{A_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{A_q} \end{bmatrix}$$

The matrix exponential (cont.)

Proof

For all $k \in \mathcal{N}$, we have

$$A^k = \begin{bmatrix} A_1^k & 0 & \cdots & 0 \\ 0 & A_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q^k \end{bmatrix}$$

Thus,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{A_1^k}{k!} & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{A_2^k}{k!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{A_q^k}{k!} \end{bmatrix}$$

$\blacksquare$

The matrix exponential (cont.)

Proposition

The matrix exponential of diagonal matrixes

Consider a diagonal $(n \times n)$ matrix A

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

We have,

$$\rightsquigarrow \quad e^A = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}$$

The result is a special case of the previous proposition

The matrix exponential (cont.)

Example

Consider the (3×3) matrix A, we are interested in its matrix exponential

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

We have,

$$e^A = \begin{bmatrix} e^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{0.5} \end{bmatrix}$$

```
>> A = [-2 0 0; 0 0 0; 0 0 0.5];          % Compute matrix A

>> expA = expm(A);                         % Compute its exponential
>> Aexp = exp(A);                          % Compute the exponential of its elements

>> isequal(expA,Aexp);                     % Compare the results
>> expA == Aexp;                           %
```

Determinant

Matrix algebra

Determinant (cont.)

Definition

Matrix minors

Consider a square matrix A of order $n \geq 2$

The **minor** (i, j) of matrix A is a square matrix $A_{i,j}$ of order $(n-1)$

⇝ From A by deleting the i-th row and the j-th column

$$A_{i,j} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cancel{a_{1,j}} & \cdots & a_{1,p} \\ a_{2,1} & a_{2,2} & \cdots & \cancel{a_{2,j}} & \cdots & a_{2,p} \\ \vdots & \vdots & \ddots & \cancel{\vdots} & \ddots & \vdots \\ \cancel{a_{i,1}} & \cancel{a_{i,2}} & \cancel{\cdots} & \cancel{a_{i,j}} & \cancel{\cdots} & \cancel{a_{i,p}} \\ \vdots & \vdots & \ddots & \cancel{\vdots} & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & \cancel{a_{m,j}} & \cdots & a_{m,p} \end{bmatrix}$$

Determinant (cont.)

Example

Consider the (3×3) matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The minors of order 2

$$A_{1,1} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A_{1,2} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad A_{1,3} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$A_{2,1} = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}, \quad A_{2,2} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}, \quad A_{2,3} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

Determinant (cont.)

Definition

Matrix determinant

Consider a square matrix A of order n, the **determinant** of A is a real number

$$\rightsquigarrow \quad \det(A) = |A|$$

- For $n = 1$, let $A = [a_{1,1}]$, we have

$$\rightsquigarrow \quad \det(A) = a_{1,1}$$

- For $n \geq 2$, we have

$$\rightsquigarrow \quad \det(A) = a_{1,1}\,\hat{a}_{1,1} + a_{2,1}\,\hat{a}_{2,1} + \cdots + a_{n,1}\,\hat{a}_{n,1} = \sum_{i=1}^{n} a_{i,1}\,\hat{a}_{i,1}$$

$\hat{a}_{i,j}$, the **cofactor** of element (i,j), is the determinant of minor $A_{i,j}$ times $(-1)^{i+j}$

Determinant (cont.)

Example

Consider a matrix A of order $n = 2$, we are interested in computing its determinant

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

We have,

$$A_{1,1} = \begin{bmatrix} a_{2,2} \end{bmatrix}, \quad \rightsquigarrow \quad \hat{a}_{1,1} = a_{2,2}$$
$$A_{2,1} = \begin{bmatrix} a_{1,2} \end{bmatrix}, \quad \rightsquigarrow \quad \hat{a}_{2,1} = -a_{1,2}$$

The determinant

$$\det(A) = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}\, a_{2,2} - a_{2,1}\, a_{1,2}$$

Determinant (cont.)

Example

Consider a matrix A of order $n = 3$, we are interested in computing its determinant

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

The cofactors of the elements along the first column

$$\hat{a}_{1,1} = \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} = a_{2,2}\,a_{3,3} - a_{2,3}\,a_{3,2}$$

$$\hat{a}_{2,1} = (-1)\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} = -(a_{1,2}\,a_{3,3} - a_{1,3}\,a_{3,2})$$

$$\hat{a}_{3,1} = \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} = a_{1,2}\,a_{2,3} - a_{1,3}\,a_{2,2}$$

Sum the product of each element $a_{i,1}$ along the first column by cofactor $\hat{a}_{i,1}$

$$\det(A) = a_{1,1}(a_{2,2}\,a_{3,3} - a_{2,3}\,a_{3,2}) - a_{2,1}(a_{1,2}\,a_{3,3} - a_{1,3}\,a_{3,2}) + a_{3,1}(a_{1,2}\,a_{2,3} - a_{1,3}\,a_{2,2})$$

Determinant (cont.)

Example

Consider a matrix A of order n, we are interested in computing its determinant

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,j} & \cdots & a_{n,n} \end{bmatrix}$$

Computation of $\det(A)$ develops along the elements of A's first column

$$\det(A) = a_{1,1}\,\hat{a}_{1,1} + a_{2,1}\,\hat{a}_{2,1} + \cdots + a_{n,1}\,\hat{a}_{n,1} = \sum_{i=1}^{n} a_{i,1}\,\hat{a}_{i,1}$$

Determinant (cont.)

Analogous formulas develop along the elements of any column, so for column j we have

$$\det(A) = a_{1,j}\,\hat{a}_{1,j} + a_{2,j}\,\hat{a}_{2,j} + \cdots + a_{n,j}\,\hat{a}_{n,j} = \sum_{i=1}^{n} a_{i,j}\,\hat{a}_{i,j}$$

Similarly, formulas develop along the elements of any row, so for row i we have

$$\det(A) = a_{i,1}\,\hat{a}_{i,1} + a_{i,2}\,\hat{a}_{i,2} + \cdots + a_{i,n}\,\hat{a}_{i,n} = \sum_{j=1}^{n} a_{i,j}\,\hat{a}_{i,j}$$

Determinant (cont.)

Some relationships

The determinant of a diagonal or triangular matrix A

- It is equal to the product of the elements along the diagonal

$$\rightsquigarrow \quad \det(A) = a_{1,1} a_{2,2} \cdots a_{n,n}$$

The determinant of a block-diagonal or block-triangular matrix A

- It is equal to the product of the determinants of the blocks along the diagonal

$$\rightsquigarrow \quad \det(A) = \prod_{i=1}^{q} \det(\widetilde{A}_i)$$

The determinant of the product of square matrices $C = AB$

- It is equal to the product of the determinants

$$\rightsquigarrow \quad \det(C) = \det(A)\det(B)$$

If $\det(A) = 0$, then matrix A is said to be **singular**, otherwise it is called non-singular

- Understand the determinant of a matrix as the size of a transformation
- (Visually, think of it as the amount of applied stretching/shrinking)

Determinant (cont.)

Example

Consider the linear transformations $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0.5 & 1 & 1.5 \\ 1 & 0 & 1 \end{bmatrix}$

- Compute their determinant and comment on the results

```
>> A = [1 2; 3 6]; detA = det(A)

detA =

  -3.3307e-16

>> B = [1 0 1; 0.5 1 1.5; 1 0 1]; detB = det(B)

detB =

     0
```

Determinant (cont.)

Example

Consider the following collection of order-2 square matrices

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We are interested in the corresponding linear transformations

- Determine their size by computing their determinant

```
>> A = [3 0; 0 2];
>> detA = det(A);

>> B = [1 3; -2 0];
>> detB = det(B);

>> C = [3 2; -2 1];
>> detC = det(C);

>> D = [2 -2; 1 -1];
>> detI = det(I);

>> DI = eye(2);
>> detI = det(I);
```

Determinant (cont.)

Example

Consider the following collection of order-3 square matrices

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 6 & 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 2 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We are interested in the corresponding linear transformations

- Determine their size by computing their determinant

```
>> A = [4 0 0; 0 3 0; 0 0 4];
>> detA = det(A);

>> B = [4 0 0; 2 3 0; 6 0 4];
>> detB = det(B);

>> C = [4 2 6; 0 3 0; 0 0 4];
>> detC = det(C);

>> I = [1 0 0; 0 1 0; 0 0 1];
>> detI = det(I);
```

Rank and kernel

Matrix algebra

Rank and kernel

Definition

Matrix rank

The **rank** of a $(m \times n)$ matrix A, denoted $\text{rank}(A)$, is equal to the number of columns (or rows, equivalently) of matrix A that are linearly independent, a non-negative integer

The set of all possible vectors from transformation A is the **column space** of A

- The span of the new bases vectors (after they have been projected)
- (The projected bases vectors are the columns of A)

The rank of A is thus also defined as the number of dimension in the columns space

$\rightsquigarrow$ The dimension of the vectors from transformation A

Rank and kernel (cont.)

Example

Consider the square matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, we are interested in its rank

Matrix A has zero determinant, $\det(A) = 1 \cdot 4 - 2 \cdot 2 = 4 - 4 = 0$

- $\rightsquigarrow$ A is singular and thus its rank is smaller than 2
- $\rightsquigarrow$ The column space of A has dimension 1

```
>> A = [1 2; 2 4]
>> rank(A)

ans =

     1
```

Rank and kernel (cont.)

Definition

Matrix kernel or null space

Consider a $(m \times n)$ matrix A, we define the **null space** or **kernel** of matrix A

$$\rightsquigarrow \quad \ker(A) = \{x \in \mathcal{R}^n | Ax = 0\}$$

The set of all vectors $x \in \mathcal{R}^n$ that left-multiplied by A produce the null vector 0

$\rightsquigarrow$ The set is a vector space, its dimension is called the **nullity** of matrix A

$$\rightsquigarrow \quad \text{null}(A)$$

The null vector is always in $\ker(A)$ and if it is the only element, then $\text{null}(A) = 0$

For a matrix A with n columns we have $n = \text{rank}(A) + \text{null}(A)$

Systems of equations

Matrix algebra

Systems of equations

Proposition

Consider a system of n linear equations in n unknowns

$$Ax = b$$

- $\rightsquigarrow$ A is a $(n \times n)$ matrix of **coefficients**
- $\rightsquigarrow$ b is a $(n \times 1)$ vector of **known terms**
- $\rightsquigarrow$ x is a $(n \times 1)$ vector of **unknowns**

We are looking for a vector x which, after applying the transformation A, equals b

If matrix A is non-singular ($\det(A) \neq 0$), there is one and only one solution

If A is singular, let $M = [A|b]$ be a $[n \times (n+1)]$ matrix

- If $\text{rank}(A) = \text{rank}(M)$, system has infinite solutions
- If $\text{rank}(A) < \text{rank}(M)$, system has no solutions

Systems of equations (cont.)

Example

Consider a system of two equations and two unknowns

$$2x_1 + x_2 = 4$$
$$6x_1 + 4x_2 = 14$$

In matrix form, $Ax = b$

$$A = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad b = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

The determinant of matrix A, $\det(A) = 2$, one and only one solution

The system can be solved by substitution

$$\begin{cases} x_1 = 2 - 1/2x_2 \\ 6x_1 + 4x_2 = 14 \end{cases} \rightsquigarrow \begin{cases} x_1 = 2 - 1/2x_2 \\ x_2 = 2 \end{cases} \rightsquigarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases} \rightsquigarrow x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Systems of equations (cont.)

Example

Consider a system of two equations and two unknowns

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases} \quad \rightsquigarrow \quad \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{b}$$

This system of equations has not got any solution, as $\text{rank}([A|b]) > \text{rank}(A)$

- $\rightsquigarrow$ Matrix A is singular and rank 1
- $\rightsquigarrow$ Matrix $[A|b]$ is rank 2

Systems of equations (cont.)

Example

Consider the linear system of two equation and two unknowns

$$\begin{cases} 1 = x_1 + 2x_2 \\ 2 = 2x_1 + 4x_2 \end{cases} \quad \rightsquigarrow \quad \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

This system of equations has infinite solutions, as $\text{rank}([A|b]) = \text{rank}(A)$

- Matrix A is singular and rank 1
- Matrix $[A|b]$ is rank 1

Inverse

Matrix algebra

Inverse

Definition

Matrix inverse

Consider a square matrix A of order n

Define **inverse** of A as the square matrix A^{-1} of order n

$$\rightsquigarrow \quad A^{-1}A = AA^{-1} = I$$

The inverse of A exists if and only if A is non-singular

Inverse (cont.)

Cofactor matrix and adjunct matrix

Consider a square matrix A of order $n \geq 2$

The **cofactor matrix** of A is a square matrix of order n whose element (i,j) is the cofactor $\hat{a}_{i,j}$ of A

$$\rightsquigarrow \quad \hat{A} = \{\hat{a}_{i,j}\}$$

The **adjunct matrix** of A is a square matrix of order n obtained by transposition of the cofactors

$$\rightsquigarrow \quad \text{adj}(A) = \{\alpha_{i,j} = \hat{a}_{j,i}\}$$

Proposition

Consider a non-singular square matrix A of order n

- If $n = 1$, let $A = [a_{1,1}]$, we have $A^{-1} = [a_{1,1}^{-1}]$
- If $n \geq 2$, we have $A^{-1} = \dfrac{1}{\det A}\text{adj}(A)$

Systems of equations (cont.)

Proposition

Consider a system of n linear equations in n unknowns $Ax = b$

Suppose that matrix A is non-singular, we have

$$\rightsquigarrow \quad x = A^{-1}b$$

Proof

Left-multiply both sides of $b = Ax$ by A^{-1}

$$b = Ax \quad \rightsquigarrow A^{-1}b = A^{-1}Ax \quad \rightsquigarrow Ix = A^{-1}b \quad \rightsquigarrow x = A^{-1}b$$

■

Systems of equations (cont.)

Consider a non-singular diagonal matrix A

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \rightsquigarrow A^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n^{-1} \end{bmatrix}$$

$\rightsquigarrow$ Its inverse A^{-1} is obtained by inverting the diagonal elements

Consider a non-singular block-diagonal matrix A

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_n \end{bmatrix} \rightsquigarrow A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_n^{-1} \end{bmatrix}$$

$\rightsquigarrow$ Its inverse A^{-1} is obtained by inverting the diagonal blocks

Systems of equations (cont.)

Consider two non-singular matrices A and B of order n, we have

$$\rightsquigarrow \quad (AB)^{-1} = B^{-1}A^{-1}$$

Consider a non-singular matrix A of order n, we have

$$\rightsquigarrow \quad \det\left(A^{-1}\right) = \frac{1}{\det\left(A\right)}$$

Eigenvalues and eigenvectors

Matrix algebra

Eigenvalues and eigenvectors

Definition

Eigenvalues and eigenvectors

Let $\lambda \in \mathcal{R}$ be some scalar and let $v \neq 0$ be a $(n \times 1)$ column vector

Consider a square matrix A of order n, we have the identity

$$\rightsquigarrow \quad Av = \lambda v$$

- $\rightsquigarrow$ The scalar quantity λ is an **eigenvalue** of A
- $\rightsquigarrow$ Vector v is the associated **eigenvector**

Eigenvalues and eigenvectors (cont.)

Proposition

Eigenvalues/eigenvectors of triangular/diagonal matrices

Let $A = \{a_{i,j}\}$ be a triangular or a diagonal matrix

The eigenvalues of A are $\{a_{i,i}\}$, $i = 1, \dots, n$

$\rightsquigarrow$ The n diagonal elements of A

Eigenvalues and eigenvectors (cont.)

Example

Consider the following diagonal or triangular matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

We are interested in their eigenvalues

The eigenvalues of A_1

- $\lambda_1 = 1$
- $\lambda_2 = 1$
- $\lambda_3 = 3$

```
>> A1 = [1, 0, 0; 0, 1, 2; 0, 0, 2];

>> evalA1 = eig(A1)      % Type 'help eig'
                         % Type 'doc eig'
evalA1 =

     1
     1
     2
```

Eigenvalues and eigenvectors (cont.)

$$A_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

The eigenvalues of A_2

- $\lambda_1 = 1$
- $\lambda_2 = 2$
- $\lambda_3 = 3$

```
>> A2 = [1, 1, 2; 0, 2, 2; 0, 0, 3];

>> evalA2 = eig(A2)      % Type 'help eig'
                         % Type 'doc eig'
evalA2 =

     1
     2
     3
```

The eigenvalues of A_3

- $\lambda_1 = 1$
- $\lambda_2 = 3$
- $\lambda_3 = -2$

```
>> A3 = [1, 0, 0; 2, 3, 2; 3, 0, -2];

>> evalA3 = eig(A3)      % Type 'help eig'
                         % Type 'doc eig'
evalA3 =

     1
     3
    -2
```

Eigenvalues and eigenvectors (cont.)

Definition

Characteristic polynomial

The **characteristic polynomial** of a square matrix A of order n

The n-order polynomial in the variable s

$$\rightsquigarrow \quad P(s) = \det\left(sI - A\right)$$

Eigenvalues and eigenvectors (cont.)

Example

Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ We are interested in its characteristic polynomial

We first calculate the matrix $(sI - A)$

$$(sI - A) = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s-2 & -1 \\ -3 & s-4 \end{bmatrix}$$

$\rightsquigarrow$ The elements are function of s

The determinant of the matrix

$$\begin{aligned} \rightsquigarrow \quad \det(sI - A) &= (s-2)(s-4) - 3 \\ &= s^2 - 6s + 5 \end{aligned}$$

This is also the characteristic polynomial $P(s)$

Eigenvalues and eigenvectors (cont.)

Proposition

Eigenvalues as roots of the characteristic polynomial

The eigenvalues of a matrix A of order n are the roots of its characteristic polynomial

- That is, they are the solutions to the equation $P(s) = \det(sI - A) = 0$

Let λ be an eigenvalue of matrix A

Each eigenvector v associated to it is a non-trivial solution to the system

$$(\lambda I - A)v = 0$$

0 is a $(n \times 1)$ column-vector whose elements are all zero

Proof

An eigenvalue λ and an eigenvector v must satisfy $Av = \lambda v$, $(\lambda I - A)v = 0$ follows

The non-trivial solution $v \neq 0$ is admissible iff matrix $(\lambda I - A)$ is singular

$$\rightsquigarrow \quad \det(\lambda I - A) = 0$$

Thus, λ is root to the characteristic polynomial of matrix A

Eigenvalues and eigenvectors (cont.)

Example

Consider the matrix A and its eigenvalues

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \quad \rightsquigarrow \quad \lambda_{1|2} = \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm 4}{2} \quad \rightsquigarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 5 \end{cases}$$

We are interested in its eigenvectors

```
>> A = [2 1; 3 4]; [nr,nc] = size(A);          % Define matrix A

>> P = (A - s*eye(nr))                          % Characteristic polynomial P
                                                % Variable 's' must be symbolic
P =

s^2 - 6*s + 5

>> R = solve(P,s)                               % Find the roots of polynomial P
                                                % (The eigenvalues of A)
R =

 1
 5

>> [V,D] = eig(A);                              % An alternative, more Matlab-ish way
>> R_optional = diag(D);                        %
```

Eigenvalues and eigenvectors (cont.)

Consider the eigenvector

$$v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$$

Eigenvector v_1 corresponds to eigenvalue $\lambda_1 = 1$, it must satisfy $(\lambda_1 I - A)v_1 = 0$

$$(\lambda_1 I - A)v_1 = \begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \quad \begin{cases} 0 = -a - b \\ 0 = -3a - 3b \end{cases}$$

If the first equation is satisfied then also the second one will be

- $\rightsquigarrow$ The two equations are linearly dependent
- Always with $(\lambda I - A)v = 0$

We limit ourselves and consider only one equation, say, $b = -a$

The choice of the first component is arbitrary, then $b = -a$

Let $a = 1$, then we have

$$\rightsquigarrow \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues and eigenvectors (cont.)

Consider the eigenvector

$$v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

Eigenvector v_2 corresponds to eigenvalue $\lambda_2 = 5$, it must satisfy $(\lambda_2 I - A)v_2 = 0$

$$(\lambda_2 I - A)v_2 = \begin{bmatrix} 3 & -1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \quad \begin{cases} 0 = 3c - d \\ 0 = -3c + d \end{cases}$$

If the first equation is satisfied then also the second one will be

- Again, the two equations are linearly dependent

By considering only the first equation, we have $d = 3c$

As the choice of the first component is arbitrary, we set $c = 1$

$$\rightsquigarrow \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Eigenvalues and eigenvectors (cont.)

We have shown that the system $(\lambda I - A)v$ has an infinite number of solutions

- Eigenvectors are determined up to a multiplicative constant

$\rightsquigarrow$ We always select the non-trivial (non-null) solution

Let v be the eigenvector associated to eigenvalue λ

$\rightsquigarrow$ Then, also $y = rv$ is eigenvector for λ $(r \neq 0)$

$$Ay = A(rv) = r(Av) = r(\lambda v) = \lambda(rv) = \lambda y$$

Eigenvalues and eigenvectors (cont.)

Proposition

Let $v_1, v_2, \ldots, v_k$ be the eigenvectors of matrix A

Suppose that the corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct

It can be shown that $v_1, v_2, \ldots, v_k$ are linearly independent

Proposition

Let A be a matrix of order n with n distinct eigenvalues

It can be shown that there exists a set of n linearly independent eigenvectors

The eigenvectors are a base for $\mathcal{R}^n$

Eigenvalues and eigenvectors (cont.)

Definition

Multiplicity

Consider a square matrix A or order n

Suppose that A has $r \leq n$ distinct eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_r \qquad (\lambda_i \neq \lambda_j, \text{ for } i \neq j)$$

The characteristic polynomial can be written in the form

$$P(s) = (s - \lambda_1)^{\nu_1}(s - \lambda_2)^{\nu_2} \cdots (s - \lambda_r)^{\nu_r}, \quad \sum_{i=1}^{r} \nu_i = n$$

$\rightsquigarrow$ $\nu_i \in \mathcal{N}^+$ (**algebraic multiplicity**)

Define the **geometric multiplicity** of the eigenvalue λ_i

- Number ν_i of linearly independent eigenvectors associated to it

Eigenvalues and eigenvectors (cont.)

Proposition

Consider a square matrix A

Let λ be an eigenvalue with algebraic multiplicity ν

The geometric multiplicity μ of the eigenvalue

$$\rightsquigarrow \quad \mu = \text{null}(\lambda I - A) \leq \nu$$

Proof

For each eigenvector v associated to λ, we have that $(\lambda I - A)v = 0$

- $\rightsquigarrow$ v belongs to the null space of $(\lambda I - A)$
- $\rightsquigarrow$ Dimension of $(\lambda I - A)$ is $\text{null}(\lambda I - A)$

■

Eigenvalues and eigenvectors (cont.)

Example

Consider the matrix of order $n = 4$ and its characteristic polynmial

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \rightsquigarrow \quad P(s) = (s-2)^2(s-3)^3$$

The roots

- $\rightsquigarrow$ $\lambda_1 = 2$, algebraic multiplicity $\nu_1 = 2$
- $\rightsquigarrow$ $\lambda_2 = 3$, algebraic multiplicity $\nu_2 = 2$

We are interested in the geometric multiplicities

Eigenvalues and eigenvectors (cont.)

The geometric multiplicity of the first eigenvalue

$$\mu_1 = \text{null}(\lambda_1 I - A) = n - \text{rank}(\lambda_1 I - A) = 4 - \text{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right)$$

$$= 4 - 3 = 1 < \nu_1$$

Each eigenvector associated to λ_1 is a linear combination of a single vector

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}'$$

The geometric multiplicity of the second eigenvalue

$$\mu_2 = \text{null}(\lambda_2 I - A) = n - \text{rank}(\lambda_2 I - A) = 4 - \text{rank}\left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\right)$$

$$= 4 - 2 = 2 = \nu_2$$

Each eigenvector associated to λ_2 is a linear combination of two vectors

$$v_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}'$$
$$v_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}'$$