The Γ, χ^2 and β distributions

Useful distributions

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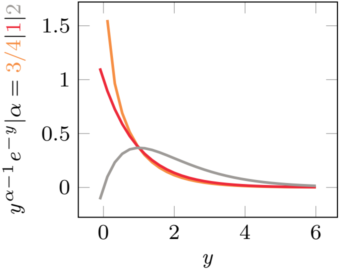
It can be shown that the following integral exists for $\alpha > 0$



$$\int_0^\infty y^{(\alpha-1)} e^{(-y)} \mathrm{d}y$$

The value of the integral is a positive number

The integral is called the **gamma function** of α

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-y) \mathrm{d}y$$

The Γ, χ^2 and β distributions (cont.)

Let $\alpha = 1$, then

$$\Gamma(1) = \int_0^\infty e^{(-y)} \mathrm{d}y = 1$$

Let $\alpha > 1$, by an integration by parts

$$\Gamma(\alpha) = (\alpha - 1) \int_0^\infty y^{\alpha-2} e^{(-y)} \mathrm{d}y = (\alpha - 1)\Gamma(\alpha - 1)$$

Let α be a positive integer greater than 1, then

$$\Gamma(\alpha) = (\alpha - 1)(\alpha - 2) \cdots (3)(2)(1)\Gamma(1) = (\alpha - 1)!$$

The Γ, χ^2 and β distributions (cont.)

Consider the definition of $\Gamma(\alpha)$

$$\Gamma(\alpha) = \int_0^{\infty} y^{(\alpha-1)} e^{(-y)} \mathrm{d}y$$

We introduce a new variable $y = x/\beta$, for some $\beta > 0$

$$\Gamma(\alpha) = \int_0^{\infty} (\underbrace{x/\beta}_{y})^{(\alpha-1)} \exp\left(-\underbrace{x/\beta}_{y}\right) \underbrace{(1/\beta)\mathrm{d}x}_{\mathrm{d}y}$$

Equivalently,

$$1 = \int_0^{\infty} \frac{1}{\Gamma(a)\beta^{\alpha}} x^{(\alpha-1)} e^{(-x/\beta)} \mathrm{d}x$$

The Γ, χ^2 and β distributions (cont.)

Since $\alpha > 0$, $\beta > 0$ and $\Gamma(\alpha) > 0$,

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{(\alpha-1)} e^{(-x/\beta)}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases} \tag{1}$$

This is the PDF of a random variable of the continuous type

A RV X with the PDF of this form is said to have a **gamma distribution**

- α and β are the parameters

The Γ, χ^2 and β distributions (cont.)

We write that X has a $\Gamma(\alpha, \beta)$ distribution

$$X \sim \Gamma(\alpha, \beta)$$

The Γ, χ^2 and β distributions (cont.)

The MGF of a gamma distribution

Since

$$M(t) = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{(\alpha-1)} e^{(-x/\beta)} \mathrm{d}x$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{(\alpha-1)} e^{[-x(1-\beta t)/\beta]} \mathrm{d}x$$

Set $y = x(1-\beta t)/t$ for $t < 1/\beta$, or equivalently, $x = \beta y/(1-\beta t)$,

$$M(t) = \int_0^\infty \frac{\beta/(1-\beta t)}{\Gamma(\alpha)\beta^\alpha} \Big(\frac{\beta y}{1-\beta t}\Big)^{(\alpha-1)} e^{-y} \mathrm{d}x$$

The Γ, χ^2 and β distributions (cont.)

That is,

$$M(t) = \Big(\frac{1}{1-\beta t}\Big)^{\alpha} \underbrace{\int_0^{\infty} \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} \mathrm{d}y}_{1}$$

$$= \frac{1}{(1-\beta t)^{\alpha}}, \quad t < 1/\beta$$

The Γ, χ^2 and β distributions (cont.)

Thus,

$$M'(t) = (-\alpha)(1-\beta t)^{(-\alpha-1)}(-\beta)$$
$$M''(t) = (-\alpha)(-\alpha-1)(1-\beta t)^{(-\alpha-2)}(-\beta)^2$$

Hence, we have

$$\rightsquigarrow \quad \mu = M'(0) = \alpha\beta$$
$$\rightsquigarrow \quad \sigma^2 = M''(0) - \mu^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

The Γ, χ^2 and β distributions (cont.)

Suppose that X has a gamma distribution with parameters $\alpha = a$ and $\beta = b$

- `pgamma(x,shape=a,scale=b)`, $P(X \leq x)$
- `dgamma(x,shape=a,scale=b)`, the PDF of X at x

The Γ, χ^2 and β distributions (cont.)

Example

Let X be a random variable with the m-order moment

$$E(X^m) = \frac{(m+3)!}{3!} 3^m, \quad m = 1, 2, 3, \dots$$

The MGF of X is given by the series

$$M(t) = 1 + \frac{4!}{3!1!} 3t + \frac{5!}{3!2!} 3^2 t^2 + \frac{6!}{3!3!} 3^3 t^3 + \cdots$$

This is the Maclaurin's series for $(1-3t)^{-4}$

- Provided that $-1 < 3t < 1$

Thus, X has a gamma distribution with parameters $\alpha = 4$ and $\beta = 3$

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The Γ, χ^2 and β distributions (cont.)

Consider a special case of the gamma distribution

- Let $\alpha = r/2$, for some positive integer r
- Let $\beta = 2$

Let X be a RV of the continuous type with the PDF

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(r/2)2^{(r/2)}} x^{(r/2-1)} e^{(-x/2)}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases} \tag{2}$$

and with the MGF

$$M(t) = (1-2t)^{(-r/2)}, \quad t < 1/2$$

X is said to have a **chi-square** (χ^2) **distribution**, with parameter r

The Γ, χ^2 and β distributions (cont.)

Any $f(x)$ of this form is called a **chi-square** (χ^2) **PDF**

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(r/2)2^{(2/r)}} x^{(r/2-1)} e^{(-x/2)}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

The mean and the variance of a chi-square distribution

$$\mu = \alpha\beta = (r/2)2 = r$$
$$\sigma^2 = \alpha\beta = (r/2)^2 = 2r$$

Parameter r is called the number of degrees of freedom of the distribution

- (or of the chi-square PDF)

The Γ, χ^2 and β distributions (cont.)

For a RV X that has a chi-square distribution with r degrees of freedom,

$$X \sim \chi^2(r)$$

The Γ, χ^2 and β distributions (cont.)

Example

Let X be a random variable of the continuous type with the PDF

$$f(x) = \begin{cases} 1/4xe^{-x/2}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

That is, X is $\chi^2(4)$

Hence, $\mu = 4$, $\sigma^2 = 8$

$M(t) = (1-2t)^{-2}$ with $t < 1/2$

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The Γ, χ^2 and β distributions (cont.)

Example

Let X e a random variable with the MGF

$$M(t) = (1-2t)^{-8}, \text{ for } t \leq 1/2$$

Then, X is $\chi^2(16)$

The Γ, χ^2 and β distributions (cont.)

If a random variabe is $\chi^2(r)$, then with $c_1 < c_2$, we have

$$P(c_1 < X < c_2) = P(X \leq c_2) - P(X \leq c_1)$$

since $P(X = c_2) = 0$

Such a probability is computed from the integral

$$P(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2) 2^{(r/2)}} w^{(r/2-1)} e^{(-w/2)} \mathrm{d}w$$

Tables can be found for selected values of r and x

The Γ, χ^2 and β distributions (cont.)

- **`pchisq(x,r)`**, $P(X \leq x)$
- **`dchisq(x,r)`**, the PDF of X at x

The Γ, χ^2 and β distributions (cont.)

Theorem

Let X have a $\chi^2(r)$ distribution

If $k > -r/2$, then $E(X^k)$ exists

$$E(X^k) = \frac{2^k \Gamma(r/2 + k)}{\Gamma(r/2)}, \quad \text{if } k > -r/2 \tag{3}$$

Proof

Note that

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma(r/2) 2^{(r/2)}} x^{[(r/2)+k-1]} e^{(-x/2)} \mathrm{d}x$$

The Γ, χ^2 and β distributions (cont.)

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma(r/2)2^{(r/2)}} x^{[(r/2)+k-1]} e^{(-x/2)} \mathrm{d}x$$

Make the change of variable $u = x/2$ in the integral, to get

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma(r/2)2^{[(r/2)-1]}} x^{[(r/2)+k-1]} e^{-u} \mathrm{d}u$$

This is the desired result, provided that $k > -(r/2)$

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The Γ, χ^2 and β distributions (cont.)

Note that $k > -(r/2)$ always holds true for k a non-negative integer

$\rightsquigarrow$ Hence, all moments of a χ^2 distribution exist

The k-th moment

$$\rightsquigarrow \quad E(X^k) = \frac{2^k \Gamma(r/2 + k)}{\Gamma(r/2)}$$

The Γ, χ^2 and β distributions (cont.)

Example

Let the RV X have a gamma distribution with $\alpha = r/2$ and $\beta > 0$

- Let r be a positive integer

Define the random variable $Y = 2X/\beta$

We are interested in the PDF of Y

The Γ, χ^2 and β distributions (cont.)

The CDF of Y

$$G(y) = P(Y \leq y) = P(X \leq \beta y/2)$$

- If $y \geq 0$, then

$$G(y) = 0$$

- If $y > 0$, then

$$G(y) = \int_0^{\beta y/2} \frac{1}{\Gamma(r/2)\beta^{(r/2)}} x^{(r/2-1)} e^{(-x\beta)} \mathrm{d}x$$

The Γ, χ^2 and β distributions (cont.)

Accordingly, the PDF of Y

$$
\begin{aligned}
g(y) = G'(y) &= \frac{\beta/2}{\Gamma(r/2)2^{(r/2)}}(\beta y/2)^{(r/2-1)} e^{(-y/2)} \\
&= \frac{1}{\Gamma(r/2)2^{(r/2)}} y^{(r/2-1)} e^{(-y/2)}, \text{ for } y > 0
\end{aligned}
$$

That is, X is $\chi^2(r)$

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The Γ, χ^2 and β distributions (cont.)

Theorem

Let $X_1, \dots, X_n$ be independent random variables

Suppose, for $i = 1, \dots, n$, that X_i has a $\Gamma(\alpha_i, \beta)$ distribution

Let $Y = \sum_{i=1}^{n} X_i$

Then, Y has a $\Gamma(\sum_{i=1}^{n} \alpha_i, \beta)$ distribution

Proof

Because of the assumed independence and the MGF of a gamma distribution

$$M_Y(t) = \prod_{i=1}^{n} (1 - \beta t)^{-\alpha_i} = (1 - \beta t)^{\sum_{i=1}^{n} \alpha_i}$$

for $t < 1/\beta$

This is the MGF of a $\Gamma(\sum_{i=1}^{n} \alpha_i, \beta)$ distribution

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The Γ, χ^2 and β distributions (cont.)

We can use this result for the χ^2 distribution

- $\beta = 2$
- $\sum \alpha_i = \sum r_i/2$

Corollary

Let $X_1, \dots, X_n$ be independent random variables

Suppose, for $i = 1, \dots, n$, that X_i has a $\chi^2(r_i)$ distribution

Let $Y = \sum_{i=1}^n X_i$

Then, Y has a $\chi^2(\sum_{i=1}^n r_i)$ distribution

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The Γ, χ^2 and β distributions (cont.)

The **beta** distribution can be derived from two independent gamma RVs

Let X_1 and X_2 be two independent RVs with gamma distribution

For $\alpha > 0$ and $\beta > 0$, let the joint PDF

$$h(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{(\alpha-1)} x_2^{(\beta-1)} e^{(-x_1-x_2)}, \quad 0 < x_1, x_2 < \infty$$

(zero elsewhere)

The Γ, χ^2 and β distributions (cont.)

Let

- $Y_1 = X_1 + X_2$
- $Y_2 = X_1/(X_1 + X_2)$

It can be shown that Y_1 and Y_2 are independent

The Γ, χ^2 and β distributions (cont.)

Space $\mathcal{S}$ is (points on axes excluded) the 1-st quadrant of the $x_1 - x_2$ plane

$$y_1 = u_1(x_1, x_2) = x_1 + x_2$$

$$y_2 = u_2(x_1, x_2) = \frac{x_1}{x_1 + x_2}$$

Equivalently,

$$x_1 = y_1 y_2$$

$$x_2 = y_1(1 - y_2)$$

The Jacobian of the inverse transformation,

$$J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1 \neq 0$$

The transformation is one-to-one

It maps $\mathcal{S}$ onto $\mathcal{T} = \{(y_1, y_2) : 0 < y_1 < \infty, 0 < y_2 < 1\}$ in the $y_1 - y_2$ plane

The Γ, χ^2 and β distributions (cont.)

The joint PDF of Y_1 and Y_2

$$g(y_1, y_2) = (y_1)\frac{1}{\Gamma(\alpha)\Gamma(\beta)}(y_1 y_2)^{(\alpha-1)}\big[y_1(1-y_2)\big]^{(\beta-1)}e^{(-y_1)}$$
$$= \begin{cases} \dfrac{y_2^{(\alpha-1)}(1-y_2)^{(\beta-1)}}{\Gamma(\alpha)\Gamma(\beta)} y_1^{(\alpha+\beta-1)} e^{(-y_1)}, & 0 < y_1 < \infty, 0 < y_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

The random variables are independent, $g(y_1, y_2) \equiv g_1(y_1)g_2(y_2)$

The Γ, χ^2 and β distributions (cont.)

- The marginal PDF of Y_2

$$
\begin{aligned}
g_2(y_2) &= \frac{y_2^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} y_1^{\alpha+\beta-1} \mathrm{d}y_1 \\
&= \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1-y_2)^{\beta-1}, & 0 < y_2 < \infty \\ 0, & \text{elsewhere} \end{cases}
\end{aligned}
$$

This PDF is that of the **beta distribution** with parameters α and β

The Γ, χ^2 and β distributions (cont.)

- The marginal PDF of Y_1

$$g_1(y_1) = \begin{cases} \dfrac{1}{\Gamma(\alpha+\beta)} y_1^{\alpha+\beta-1} e^{-y_1}, & 0 < y_1 < \infty \\ 0, & \text{elsewhere} \end{cases}$$

This is the PDF of a gamma distribution with parameters $\alpha + \beta$ and 1

The Γ, χ^2 and β distributions (cont.)

The mean and the variance of Y_2

$$\mu = \frac{\alpha}{\alpha + \beta}$$

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$$

The Γ, χ^2 and β distributions (cont.)

Probabilities for the beta distribution with parameters $\alpha = a$ and $\beta = b$

- `pbeta(x,a,b)`, $P(X \leq x)$
- `dbeta(x,a,b)`, the PDF of X at x

The Γ, χ^2 and β distributions (cont.)

Example

Dirichlet distribution

Let $X_1, X_2, \dots, X_{k+1}$ be independent random variables

Let each X_i have a gamma distribution with $\beta = 1$

The joint PDF of these variables

$$h(x_1, x_2, \dots, x_{k+1}) = \begin{cases} \prod_{i=1}^{k+1} \dfrac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i}, & 0 < x_i < \infty \\ 0, & \text{elsewhere} \end{cases}$$

The Γ, χ^2 and β distributions (cont.)

Let

$$Y_i = \frac{X_i}{X_1 + X_2 + \cdots + X_{k+1}}, \quad i = 1, 2, \dots, k$$

Let Y_{k+1} denote the $k+1$ random variables

$$Y_{k+1} = X_1 + X_2 + \cdots + X_{k+1}$$

The Γ, χ^2 and β distributions (cont.)

The associated transformation maps set $\mathcal{A}$ onto set $\mathcal{B}$

$$\mathcal{A} = \big\{(x_1, \ldots, x_{k+1}) : 0 < x_i < \infty, i = 1, \ldots, k+1\big\}$$
$$\mathcal{B} = \big\{(y_1, \ldots, y_{k+1}) : 0 < y_i, i = 1, \ldots, k, y_1 + \cdots + y_k < 1, 0 < y_{k+1} < \infty\big\}$$

The Γ, χ^2 and β distributions (cont.)

The single-valued inverse functions

$$\begin{aligned} x_1 &= y_1 y_{k+1} \\ \vdots &= \vdots \\ x_k &= y_k y_{k+1} \\ x_{k+1} &= y_{k+1}(1 - y_1 - \cdots - y_k) \end{aligned}$$

The Jacobian of the inverse transformation,

$$J = \begin{vmatrix} y_{k+1} & 0 & \cdots & 0 & y_1 \\ 0 & y_{k+1} & \cdots & 0 & y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & y_{k+1} & y_k \\ -y_{k+1} & -y_{k+1} & \cdots & -y_{k+1} & (1 - y_1 - \cdots - y_k) \end{vmatrix} = y_{k+1}^k$$

The Γ, χ^2 and β distributions (cont.)

Hence, the joint PDF of $Y_1, \dots, Y_k, Y_{k+1}$

$$\frac{y^{\alpha_1+\cdots+a_{k+1}-1} y^{\alpha_1-1} \cdots y^{\alpha_k-1}(1-y_1-\cdots-y_k)^{\alpha_{k+1}-1} e^{-y_{k+1}}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_k)\Gamma(\alpha_{k+1})}$$

For $(y_1, \dots, y_k, y_{k+1}) \in \mathcal{B}$ and zero elsewhere

The Γ, χ^2 and β distributions (cont.)

Integrating out y_{k+1}, the joint PDF of $Y_1, \dots, Y_k$

$$g(y_1, \dots, y_k) \\ = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{k+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{k+1})} y^{\alpha_1 - 1} \cdots y^{\alpha_k - 1} (1 - y_1 - \cdots - y_k)^{\alpha_{k+1} - 1}$$

For $0 < y_1, i = 1, \dots, k,\ y_1 + \cdots + y_k < 1$ and zero elsewhere

The Γ, χ^2 and β distributions (cont.)

RV $Y_1, \ldots, Y_k$ that have a joint PDF of this form have a **Dirichlet PDF**

In the special case of $k = 1$, the Dirichlet PDF becomes a beta PDF

Y_{k+1} has a gamma distribution, parameters $\alpha_1 + \cdots + \alpha_k + \alpha_{k+1}$ and $\beta = 1$

- Y_{k+1} is independent of Y_1, $Y_2, \ldots,$ Y_k

We see this from the joint PDF of $Y_1, \ldots,$ Y_k, Y_{k+1}

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