

## LINEARISATION OF NON-LINEAR STATE-SPACE MODELS

MANY DYNAMICAL PROCESSES IN CHEMICAL ENGINEERING ARE MODELED AS A SET OF NON-LINEAR EQUATIONS (FIRST-ORDER)

- FROM MATERIAL AND ENERGY CONSERVATION

MOST CONTROL ANALYSIS AND CONTROL REQUIRE A LINEAR MODEL

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$\downarrow$

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x, u) \end{aligned}$$

Nonlinear systems can be linearized

GENERAL CASE - SINGLE VARIABLE

$$\frac{dx}{dt} = f(x)$$

Function  $f(x)$  can be approximated using a Taylor Series around an equilibrium point  $x_s \leftarrow f(x=x_s)=0$

$$f(x) = f(x_s) + \frac{\partial f}{\partial x} \Big|_{x_s} (x-x_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \Big|_{x_s} (x-x_s)^2 + O(x^3)$$

higher order terms

IF WE NEGLECT THE QUADRATIC TERM AND HIGHER,

$$f(x) \approx f(x_s) + \frac{\partial f}{\partial x} \Big|_{x_s} (x-x_s)$$

WE NOTED THAT  $\frac{dx}{dt} \Big|_{x_s} = f(x_s) = 0$

$$\text{THUS, WE HAVE } f(x) \approx f(x_s) + \frac{\partial f}{\partial x} \Big|_{x_s} (x-x_s) = 0$$

$$\frac{dx}{dt} = f(x) \approx \frac{\partial f}{\partial x} \Big|_{x_s} (x-x_s)$$

Consider the variable  $x' = (x-x_s)$  and compute its derivative

$$\frac{d(x-x_s)}{dt} = \frac{dx}{dt} - \frac{dx_s}{dt} = \frac{dx}{dt} \quad \text{and} \quad \frac{d(x-x_s)}{dt} \approx \frac{\partial f}{\partial x_s} \Big|_{x_s} (x-x_s)$$

$= 0$

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$$\frac{d(x-x_s)}{dt} \approx \frac{\partial f}{\partial x} \Big|_{x_s} (x-x_s)$$

THIS REPRESENTATION ALLOW US TO DISCUSS 'DEVIATIONS WITH RESPECT TO AN EQUILIBRIUM POINT'

WE CAN WRITE THE SYSTEM LINEARISED AND WRT PERTURBATIONS

$$\frac{dx'}{dt} = \frac{\partial f}{\partial x} \Big|_{x_s} x' \quad \text{OR} \quad x' = \alpha x \quad \text{WITH} \quad \alpha = \frac{\partial f}{\partial x} \Big|_{x_s}$$

GENERAL CASE - SINGLE VARIABLE, SINGLE INPUT  $\frac{dx}{dt} = f(x, u)$

WE WRITE THE TAYLOR SERIES APPROXIMATION OF  $f(x, u)$  AT  $(x_s, u_s)$

$$\begin{aligned} f(x, u) &= f(x_s, u_s) + \frac{\partial f}{\partial x} \Big|_{x_s, u_s} (x-x_s) + \frac{\partial f}{\partial u} \Big|_{x_s, u_s} (u-u_s) + \\ &\quad \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \Big|_{x_s, u_s} (x-x_s)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial u^2} \Big|_{x_s, u_s} (u-u_s)^2 + \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial x \partial u} \Big|_{x_s, u_s} (u-u_s)(x-x_s) + O(x^3, u^3) \end{aligned}$$

IF WE NEGLECT THE QUADRATIC TERM AND HIGHER,

$$f(x, u) \approx f(x_s, u_s) + \frac{\partial f}{\partial x} \Big|_{x_s, u_s} (x-x_s) + \frac{\partial f}{\partial u} \Big|_{x_s, u_s} (u-u_s)$$

$$\text{WE NOTED THAT } f(x_s, u_s) = \frac{dx}{dt} \Big|_{x_s, u_s} = 0$$

$$\text{CONSIDER THE VARIABLE } x' = (x-x_s) \quad \text{so} \quad \frac{d(x-x_s)}{dt} = \frac{dx}{dt}$$

WE HAVE

$$\frac{d(x-x_s)}{dt} \approx f(x_s, u_s) + \frac{\partial f}{\partial x} \Big|_{x_s, u_s} (x-x_s) + \frac{\partial f}{\partial u} \Big|_{x_s, u_s} (u-u_s)$$

$$\text{LET } u' = (u-u_s), \text{ WE CAN WRITE} \quad \frac{dx'}{dt} \approx \frac{\partial f}{\partial x} \Big|_{x_s, u_s} x' + \frac{\partial f}{\partial u} \Big|_{x_s, u_s} u'$$

$\alpha$        $\beta$

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ASSUME THAT THERE EXISTS AN OUTPUT VARIABLE  $y = g(x, u)$

WE CAN AGAIN TAYLOR EXPAND AND TRUNCATE:

$$g(x, u) \approx g(x_s, u_s) + \frac{\partial g}{\partial x} \Big|_{x=x_s, u=u_s} (x - x_s) + \frac{\partial g}{\partial u} \Big|_{x=x_s, u=u_s} (u - u_s)$$

AS  $g(x_s, u_s)$  IS THE STEADY STATE OF THE OUTPUT ( $y_s$ ),

$$y \approx y_s + \underbrace{\frac{\partial g}{\partial x} \Big|_{x=x_s, u=u_s} (x - x_s)}_r + \underbrace{\frac{\partial g}{\partial u} \Big|_{x=x_s, u=u_s} (u - u_s)}_s$$

$$y - y_s = r + s$$

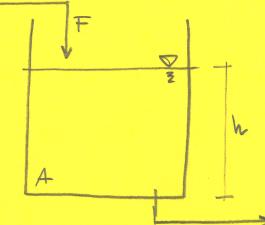
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Example SURGE TANK

$$\frac{dh}{dt} = \frac{F}{A} - \beta A \sqrt{h}$$

$h$  IS THE STATE VARIABLE

$F$  IS THE INPUT VARIABLE



$A, \beta$   
ARE PARAMETERS

THE VECTOR FIELD IS  $f(h, F)$ , WE CAN WRITE

$$\begin{aligned} f(h, F) &\approx f(h_s, F_s) + \frac{\partial f}{\partial h} \Big|_{h=h_s, F=F_s} (h - h_s) + \frac{\partial f}{\partial F} \Big|_{h=h_s, F=F_s} (F - F_s) \\ &= \left[ \frac{F_s}{A} - \frac{\beta}{A} \sqrt{h_s} \right] + \left( -\frac{1}{2} (h)^{-1/2} \beta / A \right) + \frac{1}{A} (F - F_s) \\ &= 0 \quad (\text{STEADY-STATE}) \end{aligned}$$

WE CAN NOW LET  $h' = h - h_s$ ,  $u = m - m_s$  AND WRITE

$$\frac{d(h-h_s)}{dt} \approx -\frac{\beta}{2A\sqrt{h_s}} (h-h_s) + \frac{1}{A} (F-F_s)$$

EQUIVALENTLY, WE HAVE

$$\frac{dh'}{dt} = \underbrace{-\frac{\beta}{2A\sqrt{h_s}}}_{\alpha} h' + \underbrace{\frac{1}{A} u}_{\beta}$$

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### GENERAL CASE — MULTIPLE VARIABLES

Consider the two-state system  $\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u) \\ \dot{x}_2 = f_2(x_1, x_2, u) \\ y = g(x_1, x_2, u) \end{cases}$

BY TAYLOR SERIES EXPANSIONS, WE HAVE

$$f_1(x_1, x_2, u) = f_1(x_1^s, x_2^s, u^s) + \frac{\partial f_1}{\partial x_1} \Big|_{ss} (x_1 - x_1^s) + \frac{\partial f_1}{\partial x_2} \Big|_{ss} (x_2 - x_2^s) + \\ + \frac{\partial f_1}{\partial u} \Big|_{ss} (u - u_s) + \text{higher order terms}$$

$$f_2(x_1, x_2, u) = f_2(x_1^s, x_2^s, u^s) + \frac{\partial f_2}{\partial x_1} \Big|_{ss} (x_1 - x_1^s) + \frac{\partial f_2}{\partial x_2} \Big|_{ss} (x_2 - x_2^s) + \\ + \frac{\partial f_2}{\partial u} \Big|_{ss} (u - u_s)$$

$$g(x_1, x_2, u) = g(x_1^s, x_2^s, u^s) + \frac{\partial g}{\partial x_1} \Big|_{ss} (x_1 - x_1^s) + \frac{\partial g}{\partial x_2} \Big|_{ss} (x_2 - x_2^s) + \\ + \frac{\partial g}{\partial u} \Big|_{ss} (u - u_s)$$

FROM THE EQUILIBRIUM CONDITIONS, WE HAVE THAT  $\begin{cases} f_1(ss) = 0 \\ f_2(ss) = 0 \\ g(ss) = y_s \end{cases}$

MOREOVER, WE CAN WRITE

$$\frac{d(x_1 - x_1^s)}{dt} = \frac{dx_1}{dt}$$

$$\frac{d(x_2 - x_2^s)}{dt} = \frac{dx_2}{dt}$$

$$\Rightarrow \underbrace{\left[ \begin{array}{c} \frac{\partial f_1}{\partial x_1} \Big|_{ss} \\ \frac{\partial f_2}{\partial x_2} \Big|_{ss} \end{array} \right]}_{B(b)} \underbrace{\left[ \begin{array}{c} u - u_s \\ u \end{array} \right]}_{u}$$

THE JACOBIAN WRT X

$$\left[ \begin{array}{c} \frac{d(x_1 - x_1^s)}{dt} \\ \frac{d(x_2 - x_2^s)}{dt} \end{array} \right] = \left[ \begin{array}{cc} \frac{\partial f_1}{\partial x_1} \Big|_{ss} & \frac{\partial f_1}{\partial x_2} \Big|_{ss} \\ \frac{\partial f_2}{\partial x_1} \Big|_{ss} & \frac{\partial f_2}{\partial x_2} \Big|_{ss} \end{array} \right] \cdot \left[ \begin{array}{c} x_1 - x_1^s \\ x_2 - x_2^s \end{array} \right] + \text{higher order terms} \quad (5)$$

AS FOR THE OUTPUT, WE CAN WRITE

$$[y - y_s] = \underbrace{\left[ \begin{array}{c} \frac{\partial g}{\partial x_1} \Big|_{ss} \\ \frac{\partial g}{\partial x_2} \Big|_{ss} \end{array} \right]}_{C(c)} \underbrace{\left[ \begin{array}{c} x_1 - x_1^s \\ x_2 - x_2^s \end{array} \right]}_{x'} + \underbrace{\left[ \begin{array}{c} \frac{\partial g}{\partial u} \Big|_{ss} \\ 0 \end{array} \right]}_{D(d)} \underbrace{\left[ \begin{array}{c} u - u_s \\ u \end{array} \right]}_u$$

### GENERAL CASE — MULTIPLE VARIABLES, MULTIPLE INPUTS

$$\left[ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N_x} \end{array} \right] = \left[ \begin{array}{c} f_1(x, u) \\ f_2(x, u) \\ \vdots \\ f_{N_x}(x, u) \end{array} \right] \quad \text{and} \quad A_{ij} = \frac{\partial f_i}{\partial x_j} \Big|_{ss}$$

$$B_{ij} = \frac{\partial f_i}{\partial u_j} \Big|_{ss}$$

MOREOVER

$$\left[ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{N_y} \end{array} \right] = \left[ \begin{array}{c} g_1(x, u) \\ g_2(x, u) \\ \vdots \\ g_{N_y}(x, u) \end{array} \right] \quad \text{and} \quad C_{ij} = \frac{\partial g_i}{\partial x_j} \Big|_{ss}$$

$$D_{ij} = \frac{\partial g_i}{\partial u_j} \Big|_{ss}$$

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