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A System-Level Approach to Closed-Loop Best Response Dynamics

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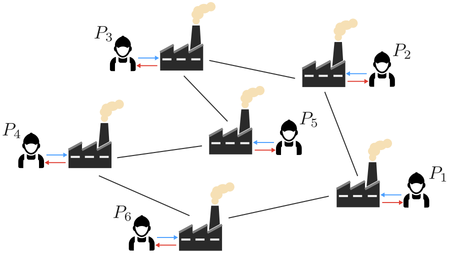
Intro, (large-scale) cyber-physical systems

- We are interested in the control of modern **Cyber-Physical Systems (CPS)**



- Large-scale decentralized network of subsystems
- Non-cooperative decision-making agents
- Information exchange often asymmetric

The centralized approach for controller design becomes unsuitable!

Game theory: A framework for computing competitive equilibria in multi-agent settings

(Almost) Complete theory:
2-players, finite-duration, unconstrained

Extremely challenging:
N-players, infinite-duration, constrained
(specially under dynamic information structures)

We study a (straightforward) approach to computing equilibrium strategies in N-players non-cooperative games

Intro, non-cooperative games and Nash equilibria

We consider N_P-players nonzero-sum stochastic difference-games (NZS-SDG)

$\rightsquigarrow$ Simultaneous play; closed-loop perfect state (CLPS) information; stage-additive costs

$$\forall p \in \underbrace{\{1,\dots,N_P\}}_{\mathcal{P}} : \begin{cases} \min\limits_{u_p := K_p(x)} \quad \lim\limits_{T\to\infty} \mathbb{E}\left[\sum\limits_{t=0}^{T} L_p(x(t), u_p(t), u_{-p}(t))\right] := J_p(u_1,\dots,u_{N_P}) \\ \text{s.t.} \quad x(t+1) = f(x(t), u_1(t),\dots,u_{N_P}(t)) + w(t), \quad x(0) \text{ given}, \\ \qquad x(t) \in \mathcal{X}, \quad u_p(t) \in \mathcal{U}_p \end{cases}$$

(w/ $\mathbb{E}[w(t)] = 0$ and $\mathbb{E}[w(t)w(t')^{\mathsf{T}}] = \delta(t-t')I_{N_x}$)

Nash Equilibrium (NE)

A strategy profile $K^\star = (K_1^\star,\dots,K_{N_P}^\star)$ is a **NE** iff

$$u_p^\star := K_p^\star(x)$$
$$\implies J_p(u_p^\star, u_{-p}^\star) \le J_p(u_p, u_{-p}^\star), \quad (\forall p \in \mathcal{P})$$

for all strategies $\{u_p(t) = K_p(x(0),\dots,x(t)) \in \mathcal{U}_p\}_{t=0}^{T}$ and resulting state-trajectories $\{x(t) \in \mathcal{X}\}_{t=0}^{T}$.

Issues: Existence $\rightarrow$ Uniqueness

Computation:

- [A]DP (coupled Ricatti equations);
- PMP (HJ-equation, coupled costates);
- Numerically? (open-loop, $T < \infty$);

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Intro, Learning dynamics and potential games

Algorithm: Best-Response Dynamics (BRD)

Initialize $K^{(0)} := (K_1^{(0)}, \dots, K_{N_P}^{(0)})$ and $t := 0$;

while $K^{(t)}$ *is not an ϵ-NE* **do**

```
/* Players observe x(t) and decide u_1(t), ..., u_{N_P}(t)   */
```

Choose a player $p \in \mathcal{P}$; // (e.g., "clockwise")

Update $K_p^{(t+1)} := BR_p(K_{-p}^{(t)}) = \arg\min_{K_p} J_p(K_p, K_{-p}^{(t)})$;

$t := t + 1$;

Convergence $\rightsquigarrow$ **NE**

- Fixed-point $K^\star = BR(K^\star)$
- **Relaxation:** ϵ-NE,

$$\epsilon > 0 \;\Rightarrow\; J_p^\star \le J_p + \epsilon \quad (\forall p)$$

Does it always converge?

Potential Games (PG): Class of games for which DBR converges

$\rightsquigarrow$ A game that admits a **potential** P such that $(\forall p, u_p, u_p', u_{-p})$

$$\frac{\Delta_p L_p}{\Delta_p P} := \frac{L_p(x, u_p, u_{-p}) - L_p(x, u_p', u_{-p})}{P(x, u_p, u_{-p}) - P(x, u_p', u_{-p})} = 1 \iff \underbrace{\frac{\partial L_p}{\partial x} = \frac{\partial P}{\partial x} \quad \text{and} \quad \frac{\partial L_p}{\partial u_p} = \frac{\partial P}{\partial u_p}}_{\text{(Sufficient conditions)}}$$

$\rightsquigarrow$ **Special case:** $L_p(x, u_p, u_{-p}) = L(x, u) + L_u^p(u_p)$, such that

$$P(x, u_p, u_{-p}) = L(x, u) + \sum_{p \in \mathcal{P}} L_u^p(u_p)$$

(e.g., LQ-games with $Q_1 = \cdots = Q_{N_P} = Q$ and $R_{\bar{p}p} = 0 \; (\bar{p} \neq p)$)

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SLS Approach to BRD

A System-Level Approach to Best Response Dynamics

March 18, 2022

System Level Synthesis, overview

- **System Level Synthesis (SLS)**: A novel approach for (robust + optimal) controller design

⇝ Consider centralized dynamics and state-feedback controller $\boldsymbol{K} \in \mathcal{RH}_\infty$

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$
$$u(t) = (K * x)(t)$$

Theorem. (SLP, [1])

Assume (A, B) stabilizable. The following are true:

a. $\begin{bmatrix} zI-A & B \end{bmatrix} \mathbf{\Phi} = I$ parametrizes all responses achievable by a stabilizing controller K;

b. A given response $\mathbf{\Phi} = (\mathbf{\Phi}_x, \mathbf{\Phi}_u)$ satisfying (a.) is achieved by the controller $K = \mathbf{\Phi}_u \mathbf{\Phi}_x{}^{-1}$.

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LQG (or $\mathcal{H}_2$) Control Problem via SLS

$$\min_{\mathbf{\Phi}} \quad \left\| \begin{bmatrix} Q^{\frac{1}{2}} & \\ & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \right\|_{\mathcal{H}_2}^2$$

$$\text{s.t.} \quad z\mathbf{\Phi}_x = I + A\mathbf{\Phi}_x + B\mathbf{\Phi}_u,$$
$$\mathbf{\Phi}_x \in \mathbf{C}_x, \ \mathbf{\Phi}_u \in \mathbf{C}_u$$

⇝ Design K by a LQR-like problem (Using $\mathbf{\Phi} = \sum_{n=1}^{\infty} \Phi(n) z^{-n}$)

⇝ Structure of K encoded by $(\mathbf{C}_x, \mathbf{C}_u)$ (e.g., spatiotemporal info. structures encoded by sparsity constraints)

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LQG (or $\mathcal{H}_2$) Control Problem via SLS

$$\min_{\Phi(\cdot)} \quad \sum_{n=1}^{\infty} \left(\left\| Q^{1/2} \Phi_x(n) \right\|_F^2 + \left\| R^{1/2} \Phi_u(n) \right\|_F^2 \right)$$
$$\text{s.t.} \quad \Phi_x(n+1) = A\Phi_x(n) + B\Phi_u(n), \quad \Phi_x(1) = I_{N_x}$$
$$\Phi_x(n) \in \mathcal{C}_x(n), \quad \Phi_u(n) \in \mathcal{C}_u(n)$$

- Design $\boldsymbol{K}$ by a LQR-like problem (Using $\boldsymbol{\Phi} = \sum_{n=1}^{\infty} \Phi(n) z^{-n}$)

- Structure of $\boldsymbol{K}$ encoded by $(\mathbf{C}_x, \mathbf{C}_u)$ (e.g., spatiotemporal info. structures encoded by sparsity constraints)

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SLS for Linear Quadratic Games

(Potential) Linear Quadratic Games - SLS Approach

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$$

- System-level parametrization (SLP)

$$
\begin{bmatrix} zI - A & B_1 \cdots B_{N_P} \end{bmatrix} \Phi = I
$$
$$
\Rightarrow z\Phi_x = I + A\Phi_x + \sum_{p\in\mathcal{P}} B_p \Phi_{u,p}
$$

- System-level constraints (SLCs)

$$
\begin{matrix} x(t) \in \mathcal{X} \\ u_p(t) \in \mathcal{U}_p \end{matrix} \Rightarrow \begin{matrix} (\Phi_x * w)(n) \in \mathcal{X} \\ (\Phi_{u,p} * w)(n) \in \mathcal{U}_p \end{matrix}
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- Finite-impulse response (FIR) approximation
(Given a horizon $\bar{N} < \infty$)

$$
\mathbf{C}_x := \mathbf{C}_x \cap \left\{ \Phi_x \in \mathcal{RH}_\infty : \sum_{n=1}^{\bar{N}} z^{-n} \Phi_x(n) \right\}
$$

- Relaxed "terminal" constraint
(Given weighting matrix $Q_f = Q_f^T \succ 0$)

$$
\Phi_x(\bar{N}+1) = 0 \Rightarrow \min J_p + \left\| Q_f^{1/2} \Phi_x(\bar{N}+1) \right\|_F^2
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(Potential) Linear Quadratic Games - SLS Approach

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Letting $z_{-p} = \sum_{\bar{p}\in\mathcal{P}\setminus\{p\}} B_{\bar{p}}\Phi_{u,\bar{p}}$ denote other players' strategies

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SLS for Linear Quadratic Games

(Potential) Linear Quadratic Games - SLS Approach

$$\forall p \in \mathcal{P} : \begin{cases} \min\limits_{\Phi_{u,p}(\cdot)} & \sum\limits_{n=1}^{\bar{N}} \left(\left\| Q^{1/2} \Phi_x(n) \right\|_F^2 + \left\| R_{pp}^{1/2} \Phi_{u,p}(n) \right\|_F^2 \right) \\ \text{s.t.} & \Phi_x(n+1) = A\Phi_x(n) + B_p \Phi_{u,p}(n) + z_{-p}(n), \quad \Phi_x(1) = I_{N_x}, \quad \Phi_x(\bar{N}+1) = 0 \\ & H_x(\Phi_x * w)(n) \leq h_x, \quad H_{u,p}(\Phi_{u,p} * w)(n) \leq h_{u,p}, \\ & \Phi_x(n) \in \mathcal{C}_x(n), \quad \Phi_{u,p}(n) \in \mathcal{C}_{u,p}(n) \end{cases}$$

Letting $z_{-p} = \sum_{\bar{p} \in \mathcal{P} \setminus \{p\}} B_{\bar{p}} \Phi_{u,\bar{p}}$ denote other players' strategies

- System-level parametrization (SLP)

$$\begin{bmatrix} zI - A & B_1 \cdots B_{N_P} \end{bmatrix} \Phi = I$$
$$\Rightarrow z\mathbf{\Phi}_x = I + A\mathbf{\Phi}_x + \sum\nolimits_{p \in \mathcal{P}} B_p \mathbf{\Phi}_{u,p}$$

- System-level constraints (SLCs)

$$\begin{matrix} x(t) \in \mathcal{X} \\ u_p(t) \in \mathcal{U}_p \end{matrix} \Rightarrow \begin{matrix} (\Phi_x * w)(n) \in \mathcal{X} \\ (\Phi_{u,p} * w)(n) \in \mathcal{U}_p \end{matrix}$$

- Finite-impulse response (FIR) approximation
  (Given a horizon $\bar{N} < \infty$)

$$\mathbf{C}_x := \mathbf{C}_x \cap \left\{ \mathbf{\Phi}_x \in \mathcal{RH}_\infty : \sum\nolimits_{n=1}^{\bar{N}} z^{-n} \Phi_x(n) \right\}$$

- Relaxed "terminal" constraint
  (Given weighting matrix $Q_f = Q_f^\top \gg 0$)

$$\Phi_x(\bar{N}+1) = 0 \Rightarrow \min J_p + \left\| Q_f^{1/2} \Phi_x(\bar{N}+1) \right\|_F^2$$

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SLS for Linear Quadratic Games, overview

Algorithm: System-Level Best-Response Dynamics (SL-BRD)

Initialize $\Phi_u^{(0)} := (\Phi_{u,1}^{(0)}, \dots, \Phi_{u,N_p}^{(0)})$, $K^{(0)} := \Phi_u^{(0)} \Phi_x^{(0)^{-1}}$ and $t := 0$;

while $K^{(t)}$ *is not an ϵ-NE* **do**

```
/* Players observe x(t) and decide u_1(t), ..., u_{N_P}(t)   */
```

Choose a player $p \in \mathcal{P}$; // (e.g., "clockwise")

Update $\Phi_{u,p}^{(t+1)} := BR_p(\Phi_{u,-p}^{(t)}) = \arg\min_{\Phi_{u,p}} J_p(\Phi_{u,p}, \Phi_{u,-p}^{(t)})$;

for $p \in \mathcal{P}$ **do**

Update $z_{-p} = \sum_{\bar{p} \in \mathcal{P} \setminus \{p\}} B_{\bar{p}} \Phi_{u,\bar{p}}$ and compute $\Phi_x^{(t+1)}$;

Update $K_p^{(t+1)} := \Phi_{u,p}^{(t+1)} \Phi_x^{(t+1)^{-1}}$;

$t := t + 1$;

- ✓ Closed-loop NE via open-loop game
- ✓ Operational and structural constraints
- ✓ Learning dynamics not affected by w

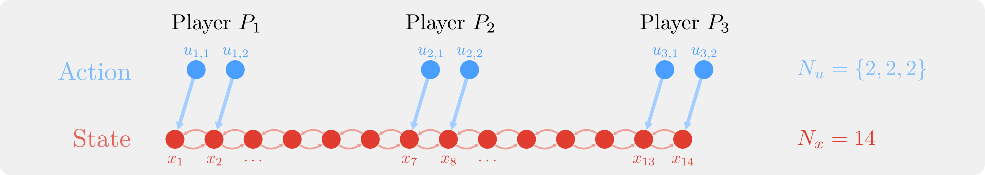
- × Slow convergence rates
- × $(\mathcal{X}, \mathcal{U})$-constraints as functions of w
- × Relaxed FIR approximation

Example

A System-Level Approach to Best Response Dynamics

March 18, 2022

Example, N-chain system w/ communication constraints



- **Dynamics:** $zx = Ax + Bu + w$,

$$A = \begin{bmatrix} 1 & 0.2 & & \\ -0.2 & \ddots & \ddots & \\ & \ddots & \ddots & 0.2 \\ & & -0.2 & 1 \end{bmatrix}_{N_x \times N_x}$$

$$B = \begin{bmatrix} \underbrace{I_2}_{B_1} & & \\ & \underbrace{I_2}_{B_2} & \\ & & \underbrace{I_2}_{B_3} \end{bmatrix}_{N_x \times N_u}$$

- **Cost parameters:**

$$Q = I_{N_x}, \quad Q_f = 10^3 I_{N_x}, \quad R_{pp} = I_{N_u^p}$$

- **Operational constraints:**

$$\mathcal{X} = \mathbb{R}^{N_x}, \quad \mathcal{U}_p = \mathbb{R}^{N_u^p} \ (\forall p \in \mathcal{P})$$

- **Structural constraints:**

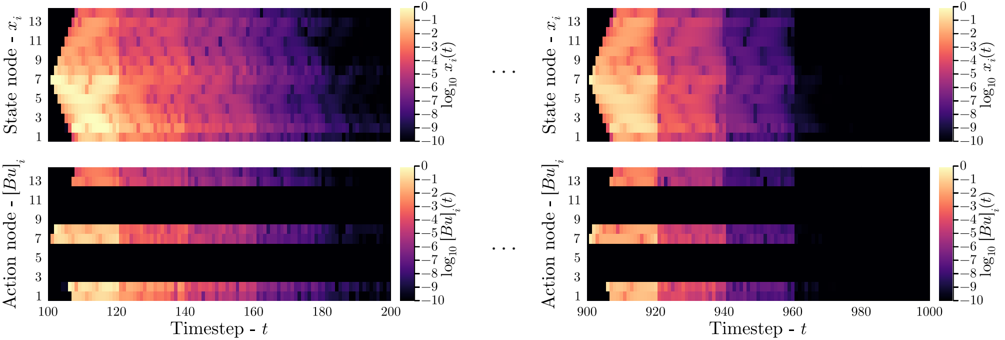
$$\bar{N} = 20 \quad \text{(FIR Horizon)}$$

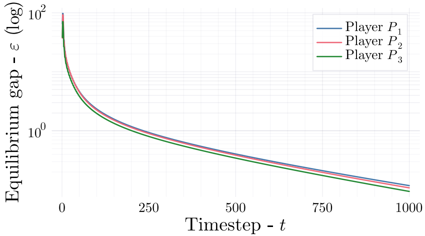
$$\mathcal{C}_x(n) = \{\Phi_x \in \mathbb{R}^{N_x \times N_x} : \text{sp } \Phi_x = \text{sp } A^{n-1}\}$$

$$\mathcal{C}_{u,p}(n) = \{\Phi_{u,p} \in \mathbb{R}^{N_u^p \times N_x} : \text{sp } \Phi_{u,p} = \text{sp } B_p^\top A^{n-1}\}$$

Considering $w(t) \overset{iid}{\sim} \mathcal{N}(0, I_{N_x})$, $t = 0, 1, \ldots$

Example, N-chain system w/ communication constraints (cont.)





- **Equilibrium gap:** $\epsilon = J_p - J_p^* \approx \{0.12, 0.11, 0.09\}$
- **FIR soft-constraint:** $\|\Phi_x(T+1)\|_F^2 \approx 10^{-5}$

The SLS-BRD yields a **closed-loop ϵ-NE** of control policies which are **stabilizing** and **satisfy dynamic information patterns**

Thank you!

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Questions?