

STATE ESTIMATION, OBSERVABILITY, STATE OBSERVATION

WE CONSIDERED LINEAR AND TIME-INVARIANT PROCESS MODELS OF THE FORM $\dot{x}(t) = Ax(t) + Bu(t)$

$$x \in \mathbb{R}^{N_x}, u \in \mathbb{R}^{N_u} \text{ AND } A \in \mathbb{R}^{N_x \times N_x}, B \in \mathbb{R}^{N_x \times N_u}$$

MAYBE (PROBABLY) THE MODEL RESULTS FROM A LINEARIZATION OF SOME NONLINEAR MODEL (THUS IT IS EXPRESSED IN DEVIATION VARIABLES)

GIVEN THE MODEL WE LEARNED HOW TO DESIGN A CONTROLLER THAT STABILIZES THE DYNAMICS OF THE AUTONOMOUS (CLOSED LOOP) SYSTEM, THIS BRINGING IT BACK TO ITS STEADY-STATE

WE DISCUSSED TWO TECHNIQUES FOR THE TASK

1. EIGENVALUE PLACEMENT (ARBITRARY, BUT OKAY AS A STARTING POINT)

2. OPTIMAL CONTROL USING THE LINEAR QUADRATIC REGULATOR CAN, ARBITRARY BUT REASONABLE, OBJECTIVE FUNCTION DEFINES AN OPTIMALITY CRITERION, AND THERE ARE SOME PARAMETERS TO BE DEFINED, ARBITRARILY AGAIN, BY THE USER, MATRICES Q_x AND Q_u)

→ IN BOTH CASES, WE CAN COMPUTE A MATRIX $K \in \mathbb{R}^{N_u \times N_x}$ SO THAT $u(t) = -Kx(t)$ FOR STATE FEEDBACK CONTROL

We must know the state at time t to be able to compute the control action

(value of the variables)

LAGRANGE'S FORMULA

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau$$

$\rightarrow y = Cx + Du$, thus

$$\begin{aligned} y(t) &= C \left[e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau \right] + D u(t) \\ &= \underbrace{C e^{A(t-t_0)} x(t_0)}_{\text{KNOWN}} + \underbrace{C \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau}_{\text{KNOWN}} + \underbrace{D u(t)}_{\text{KNOWN, WE 'CHOOSE' THIS}} \end{aligned}$$

↓

KNOWN,
 IF $x(t_0)$
 IS KNOWN

↓

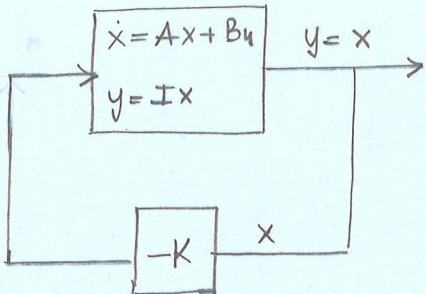
STATE AT t , $x(t)$

↓

KNOWN, BECAUSE MEASURED

↓

THIS IS KNOWN IN ITS ENTIRITY



THE STATE VARIABLE AT TIME t
MUST BE KNOWN TO COMPUTE
THE CONTROL ACTION AT TIME t

- We always assumed we could do this, we always assumed the state was fully measurable

$$\dot{x} = Ax + Bu$$

$$\Rightarrow Ax - BKx = \underbrace{(A - BK)x}_{\text{WE CHOOSE } K}$$

FOR THIS AUTONOMOUS
SYSTEM TO BEHAVE
AS DESIRED

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N_y} \end{bmatrix} = Cx = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_x} \end{bmatrix}$$

IDENTITY MATRIX C

- It is rarely possible to be able to measure the full state
- \Rightarrow ONLY SOME STATE VARIABLE ARE ACTUALLY MEASURABLE
 - \Rightarrow OR, WE CAN ONLY MEASURE SOME COMBINATION (linear mixture)

This information about what and how we measure is encoded in the matrix C

$$y = Cx \quad (y \in \mathbb{R}^{N_y}, C \in \mathbb{R}^{N_y \times N_x})$$

WHETHER

WE ARE INTERESTED IN UNDERSTANDING $\vec{x}(t)$ WITH THE AVAILABLE MEASUREMENTS y IT IS POSSIBLE TO KNOW (DETERMINE) THE VALUE OF ALL THE STATE VARIABLES

THIS CONCEPT IS KNOWN AS OBSERVABILITY
(IT IS THE DUAL OF CONTROLLABILITY)

"CAN WE ESTIMATE ANY STATE VALUE FROM MEASUREMENTS $y(t)$?"

THIS PROPERTY DEPENDS ON THE PAIR (A, C)

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- STABILITY
- CONTINUABILITY (TESTS)
- FIND SOME K (EIGENVALUE PLACEMENT, LQR)
- OBSERVABILITY (TESTS)

$$\Phi = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{full rank}$$

LET $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

- STABILITY, UNCHANGED
- CONTINUABILITY, UNCHANGED
- TUNE K (UNCHANGED)
- OBSERVABILITY?

$$\Phi = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{rank deficient}$$

Example

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

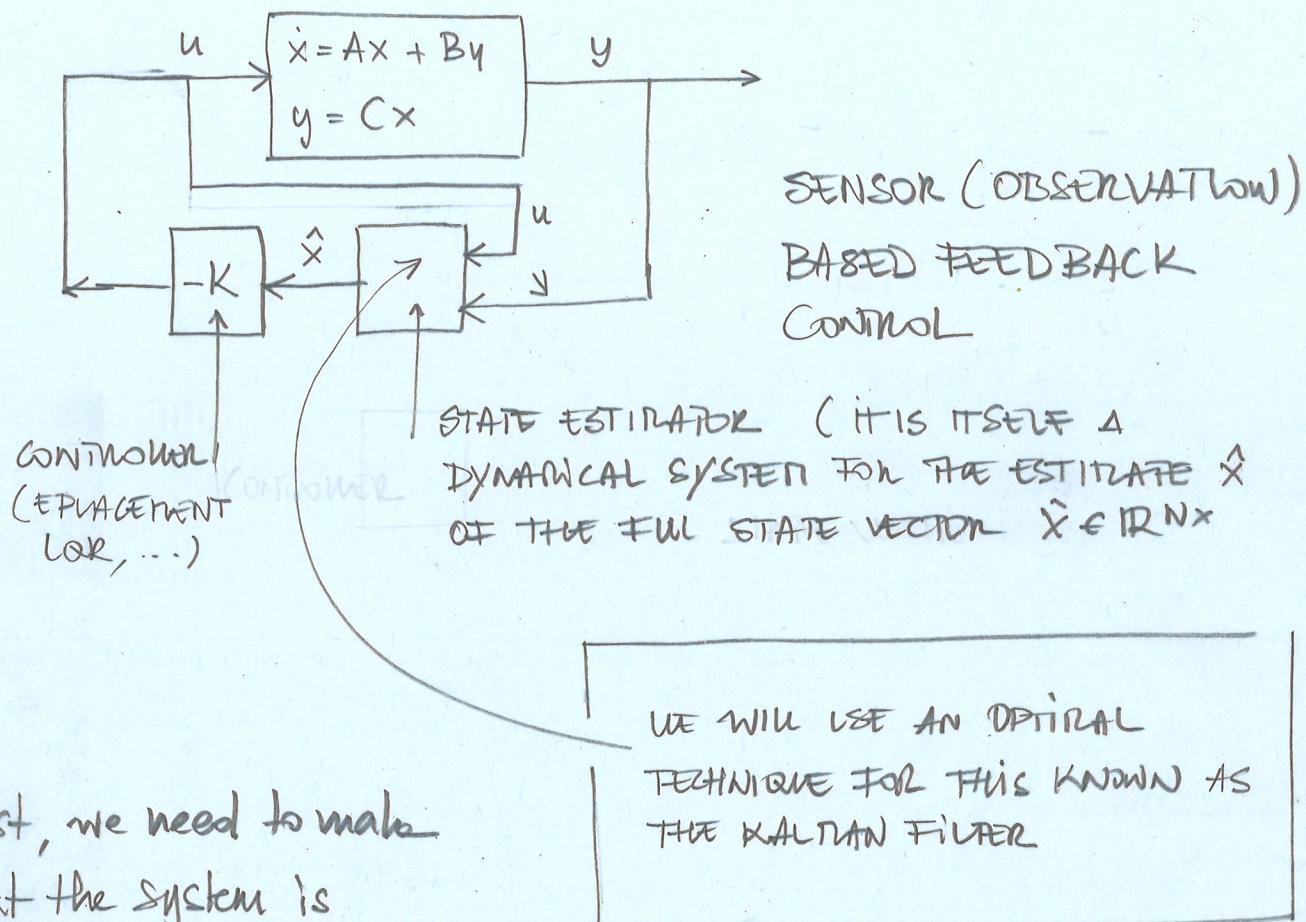
- STABILITY
- CONTINUABILITY
- TUNE K
- OBSERVABILITY

IF THE SYSTEM IS OBSERVABLE, IT IS POSSIBLE TO ESTIMATE $x(t)$ FROM MEASUREMENT $y(t)$

→ IF WE HAVE AN ESTIMATE OF $x(t)$, WE CAN USE IT IN A FEEDBACK CONTROL SCHEME

FULL-STATE

1. WE NEED TO VERIFY WHETHER A SYSTEM IS OBSERVABLE
2. WE NEED TO DESIGN A DEVICE THAT ESTIMATES THE STATE



But, first, we need to make sure that the system is OBSERVABLE

(WE APPLIED AN IDENTICAL REASONING WHEN WE CHECKED (TEST) FOR THE CONTINUABILITY)

→ OBSERVABILITY TEST(S)

→ LIMITED MEASUREMENT INFORMATION

OBSERVABILITY (A, C)

\rightsquigarrow OBSERVABILITY MATRIX Φ MUST BE FULL RANK

$$\Phi = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_{x-1}} \end{bmatrix}$$

$$\rightsquigarrow \left\{ \begin{array}{l} C = [B \ AB \ A^2B \ A^3B \ \dots \ A^{N_{x-1}}B] \\ \text{THE OBSERVABILITY MATRIX} \end{array} \right.$$

- THE SYSTEM, THE PAIR (A, C) , IS OBSERVABLE IF THE COLUMN SPACE OF Φ IS FULL RANK $\text{rank}(\Phi) = N_x$
- POPOV, BELYAKOV, TRAUTS $\text{rank}[(\lambda I - A) \ C] = N_x \text{ for all } \lambda_i \in \Gamma(A)$
- OBSERVABILITY GRAMIAN IS NON SINGULAR $W_o(t) = \int_0^t e^{\tilde{A}t} C^T C e^{\tilde{A}t} dt$
for all $t > 0 \det(W_o(t)) \neq 0$

IN MATLAB $\gg \text{obsv}(A, C)$ BUILD THE OBSERVABILITY MATRIX

$$C = [B \ AB \ A^2B \ \dots \ A^{N_{x-1}}B]$$

$$\Phi^T = [C^T \ A^T C^T \ (A^T)^2 C^T \ \dots \ (A^T)^{N_{x-1}} C^T]$$

$$\Phi = \begin{bmatrix} C \\ AC \\ A^2C \\ \vdots \\ A^{N_{x-1}}C \end{bmatrix}$$

OBSERVABILITY IS THUS EQUIVALENT TO THE CONTROLLABILITY OF SOME 'DUAL SYSTEM' $\dot{x} = A^T x + C^T u$

$\sim // \sim // \sim$

Example

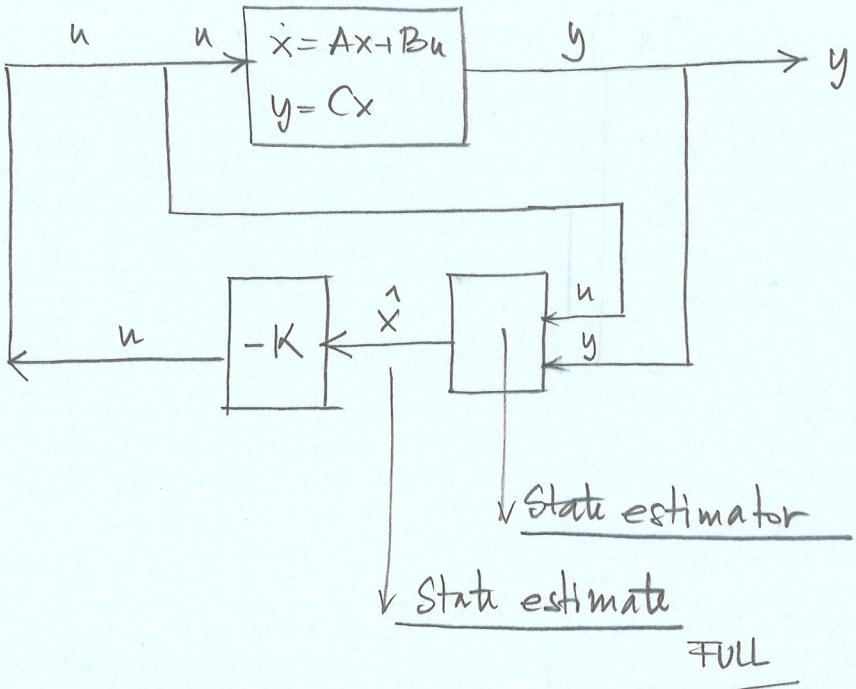
$$A = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.1 & -0.8 & 2.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = [0 \ 1 \ 2]$$

- STUDY ME!

Example

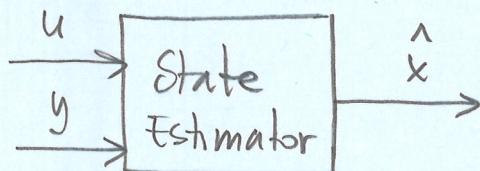
$$A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \ 1]$$

- STUDY ME!



THE KALMAN FILTER IS AN OPTIMAL STATE ESTIMATOR

→ REQUIRES OBSERVABILITY



IT IS IMPORTANT TO UNDERSTAND
THE STATE ESTIMATOR AS A
DYNAMICAL SYSTEM ITSELF
(it evolves alongside the process)

* INPUTS TO THIS SYSTEM :

→ CONTROLS, u(t)

→ MEASUREMENTS y

* OUTPUTS TO THIS SYSTEM

→ ESTIMATED MEASUREMENTS \hat{y}

$$\frac{d\hat{x}}{dt} = A\hat{x} + B u + \underbrace{K_0(y - \hat{y})}_{\hat{y} = C\hat{x}}$$

It performs a comparison between the actual observation y and what the estimator predicts (\hat{y})

- whenever there is a difference, we stimulate the estimation

→ to correct the estimate \hat{x} towards x so that y and \hat{y} match

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \text{THE SYSTEM}$$

$$\begin{cases} \frac{d\hat{x}}{dt} = \hat{Ax} + Bu + K_b(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases} \quad \text{THE STATE ESTIMATOR}$$

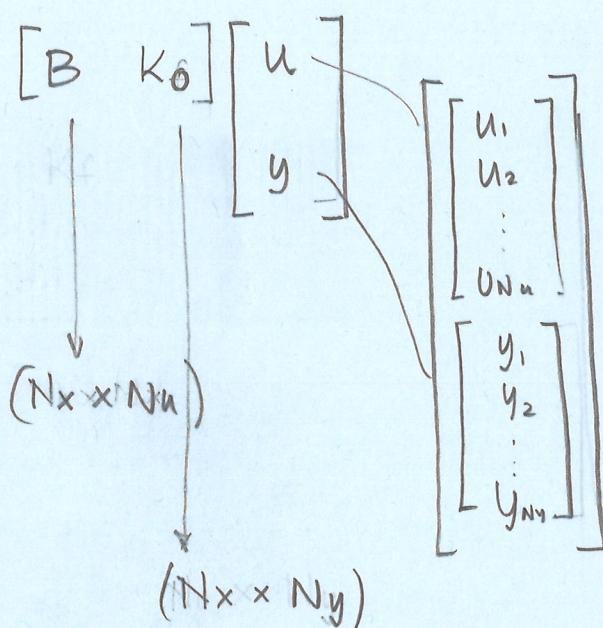
By substituting \hat{y} , we get

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \hat{Ax} + Bu + K_b(y - \hat{y}) \\ &= (A - K_b C)\hat{x} + Bu + K_b y \end{aligned}$$

↓ THE 'STATE' MATRIX FOR THE STATE ESTIMATES (THE DY MATRICES OF IT)

Also reminds of the full-state feedback controller

"WE CAN TUNE IT BY CHOOSING ITS EIGENVALUES"



$[B \ K_b]$ IS UNDERSTOOD AS THE 'INPUT' MATRIX FOR STATE ESTIMATION

- WE CAN COMPUTE THE STABILITY
- WE CAN COMPUTE THE CONTROLLABILITY
- WE CAN COMPUTE THE OBSERVABILITY

WE CAN TUNE K_b BY PLACING THE EIGENVALUES OF $(A - K_b C)$

SUPPOSE THAT $e = x - \hat{x}$ FULL STATE
 ~> THIS IS THE ERROR THAT OUR ESTIMATOR MAKES AT ANY t
 ~> WE ARE INTERESTED IN MINIMISING IT

WE CONSIDER THE DYNAMICS OF THIS ERROR $\frac{de}{dt}$

~> WE WANT THAT $\frac{de}{dt} \rightarrow 0$

~> THAT THE ESTIMATOR IS STABLE

} ASYMPTOTICALLY,
AS $t \rightarrow \infty$

$$\frac{de}{dt} = \frac{dx}{dt} - \frac{d\hat{x}}{dt}$$

$$Ax + Bu$$

$$A\hat{x} + Bu + K_0 y - K_0 \hat{y}$$

$$= (Ax + Bu) - A\hat{x} - B\hat{u} + K_0 y - K_0 \hat{y}$$

$$= Ax - A\hat{x} + K_0 y + K_0 \hat{y}$$

$$= A(x - \hat{x}) + K_0(y - \hat{y})$$

with $y = Cx$

$$= A(x - \hat{x}) + K_0(C\hat{x} - Cx)$$

$$\hat{y} = C\hat{x}$$

$$= A(x - \hat{x}) - K_0 C(x - \hat{x})$$

$$= (A - K_0 C)(x - \hat{x})$$

$$= (A - K_0 C)e$$

THE Luenberger
(STATE) OBSERVER

the dynamics of the error

~> stable / unstable

~> stabilisable using K_0 (place its evals)
or $\text{LQR } C$

OBSERVABILITY MUST BE VERIFIED

FULL STATE FEED BACK CONTROL WITH A STATE OBSERVER

$$1. \quad \dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$2. \quad u(t) = -Kx(t)$$

WITH

$$\left\{ \begin{array}{l} x \in \mathbb{R}^{N_x} \\ u \in \mathbb{R}^{N_u} \\ y \in \mathbb{R}^{N_y} \\ A \in \mathbb{R}^{N_x \times N_x} \\ B \in \mathbb{R}^{N_x \times N_u} \\ C \in \mathbb{R}^{N_y \times N_x} \end{array} \right.$$

$$K \in \mathbb{R}^{N_u \times N_x}$$

$$3. \quad \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_o[y(t) - \hat{y}(t)], \quad \text{with } \hat{y} = C\hat{x}$$

$$= A\hat{x}(t) + Bu(t) + K_o y(t) - K_o \hat{y}(t)$$

$$= (A - K_o C)\hat{x}(t) + Bu(t) + K_o y(t)$$

THE RESULTING CLOSED LOOP HAS ORDER $2N_x$ AND ITS EIGENVALUES ARE GIVEN BY $\{\sigma(A-BK) \cup \sigma(A - K_o C)\}$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ K_o C & A - BK - K_o C \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

$$\text{LET } P = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix} = \bar{P}^1$$

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - K_o C \end{bmatrix} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$$

FULL STATE FEEDBACK CONTROL WITH A KALMAN FILTER

