

Consider a n -order homogeneous ordinary differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_2 \frac{dy}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0$$

Alternatively, we write

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0$$

Let $y(t) = e^{\lambda t}$ be the solution, then differentiate

$$\left\{ \begin{array}{l} y(t) = e^{\lambda t} \\ \dot{y}(t) = \lambda e^{\lambda t} \\ \ddot{y}(t) = \lambda^2 e^{\lambda t} \\ \vdots \\ y^{(n)}(t) = \lambda^n e^{\lambda t} \end{array} \right.$$

Substituting in the ODE, we get

$$\underbrace{e^{\lambda t} (a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0)}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0$$

↓

$$\text{CHARACTERISTIC EQUATION}$$

The equation is verified when $(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0)$ equals zero, we thus need to find its roots

THIS IS IN GENERAL SATISFIED BY n VALUES OF λ

$$\underbrace{\{\lambda_1, \lambda_2, \dots, \lambda_n\}}$$

- POSITIVE AND NEGATIVE

- REAL OR COMPLEX NUMBER (+ COMPL-CONJ)

IT RELATES TO THE

- SINGLE OR REPEATED

SPECTRUM OF SOME MATRIX

The solution to the differential equation can be written as

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t}$$

The coefficients must be determined by using n initial cond.

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n)}(0) \end{bmatrix} \xrightarrow{\text{TO GET}} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$$

A WEIGHTED COMBINATION
OF EXPONENTIAL FUNCTIONS

ALTERNATIVE APPROACH

1 START FROM GENERAL N-ORDER DIFFERENTIAL EQUATION

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y = 0$$

2 DIVIDE BY THE COEFFICIENT OF THE HIGHEST ORDER DERIVAT.
 ↳ (HERE a_n)

3 WE OBTAIN

$$y^{(n)} + \alpha_{n-1} y^{(n-1)} + \alpha_{n-2} y^{(n-2)} + \dots + \alpha_2 y^{(2)} + \alpha_1 y^{(1)} + \alpha_0 y = 0 \quad (*)$$

$$\text{WITH } \alpha_{n-1} = \frac{a_{n-1}}{a_n}; \quad \alpha_{n-2} = \frac{a_{n-2}}{a_n}; \quad \dots$$

4 WE NOW INTRODUCE A SET OF DUMMY VARIABLES

$$\begin{cases} x_1 = y \\ x_2 = y^{(1)} \\ x_3 = y^{(2)} \\ \vdots \\ x_n = y^{(n-1)} \\ x_{n+1} = y^{(n)} \end{cases}$$

WE CAN THEN COMPUTE THEIR FIRST DERIVATIVES

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = - \underbrace{\left[\alpha_n x_n + \alpha_{n-1} x_{n-1} + \dots + \alpha_2 x_3 + \alpha_1 x_2 + \alpha_0 x_1 \right]}_{\text{FROM THE ODE } (*)}$$

5 WRITE IT IN MATRIX FORM

FROM THE ODE (*)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$\underbrace{\qquad\qquad\qquad}_{A} \qquad \underbrace{\qquad\qquad\qquad}_{X}$

GENERAL
FORM

$$\dot{x} = Ax$$

Example

$$\ddot{y} + 3\dot{y} + 2y = 0 \quad w/ \quad \begin{cases} y(0) = 2 \\ \dot{y}(0) = -3 \end{cases}$$

Let $y(t) = e^{\lambda t}$ be the assumed solution

$$\text{Then } \dot{y}(t) = \lambda e^{\lambda t}$$

$$\ddot{y}(t) = \lambda^2 e^{\lambda t}$$

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0 \quad \Rightarrow e^{\lambda t} [\lambda^2 + 3\lambda + 2] = 0$$

$$\lambda^2 + 3\lambda + 2 = 0 \quad \text{when}$$

$$\boxed{\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -2 \end{array}}$$

The general solution is $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$$y(0) = 2 \quad \Rightarrow C_1 + C_2 = 2$$

$$\dot{y}(t) = -C_1 e^{-t} - 2C_2 e^{-2t} \quad \Rightarrow \dot{y}(0) = -C_1 - 2C_2 = -3$$

$$\left| \begin{array}{l} C_1 = 1 \\ C_2 = 1 \end{array} \right| \quad \Rightarrow \quad y(t) = e^{-t} + e^{-2t} \quad \left| \begin{array}{l} \text{STABLE SOLUTION} \\ (\text{both exponentials decay}) \end{array} \right.$$

THE OTHER APPROACH

$$\ddot{y}^{(2)} + 3\dot{y}^{(1)} + 2y^{(0)} = 0$$

1) WE INTRODUCE DYNAMIC VARIABLES

$$\left\{ \begin{array}{l} x_1 = y \\ x_2 = \dot{y}^{(1)} \\ x_3 = \ddot{y}^{(2)} \end{array} \right. \quad \xrightarrow{\text{derivatives}} \quad \left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -3x_2 - 2x_1 \\ \dot{x}_3 = \quad [2x_1 + 3x_2 - x_3] \end{array} \right.$$

matrix form
 $\xrightarrow{\quad}$

2) WE GET

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left\{ \begin{array}{l} \text{CHECK} \\ \det(A - \lambda I) = 0 \\ \text{or eig}(A) \end{array} \right.$$

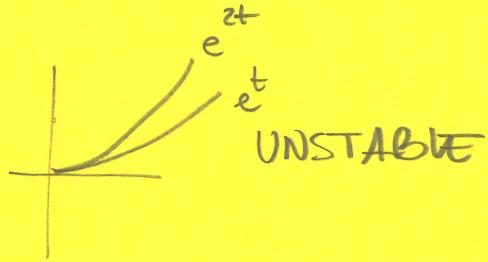
Example $y^{(2)} - 3y^{(1)} + 2y = 0$ w/ $\begin{cases} y(0) = 2 \\ \dot{y}(0) = 3 \end{cases}$

\uparrow CHANGED THIS ONLY

$$(x^2 - 3x + 2) \rightarrow (x-2)(x-1) = 0$$

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$$

$$c_1 e^t + c_2 e^{2t} = y(t)$$



Example $y^{(2)} + y^{(1)} - 2y = 0$ w/ $\begin{cases} y(0) = 3 \\ \dot{y}(0) = 0 \end{cases}$

$$(x^2 + x - 2) = 0 \rightarrow (x+2)(x-1) = 0$$

$$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -2 \end{cases}$$

$$c_1 e^t + c_2 e^{-2t}$$

THERE IS A CLOSE CONNECTION BETWEEN THE ROOT OF THE CHARACTERISTIC POLYNOMIAL AND THE EIGENVALUES OF THE STATE MATRIX

$\det(A - \lambda I)$ = characteristic polynomial

Easily checked for a 2×2 system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightsquigarrow A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} = -\lambda [x^2 + a_2\lambda + a_1] - 1[a_0] + 0$$

$$= \underbrace{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0}_\text{CHARACTERISTIC EQUATION} = 0$$

WE ARE INTERESTED IN SOLVING THE GENERAL SYSTEM $\dot{A}x = \dot{x}$

CASE 1 Uncoupled dynamics — MATRIX A is DIAGONAL

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \searrow & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

THIS MEANS THAT

$$\dot{x}_1 = \lambda_1 x_1 \quad \rightsquigarrow x_1(t) = e^{\lambda_1 t} x_1(0)$$

$$\dot{x}_2 = \lambda_2 x_2 \quad \rightsquigarrow x_2(t) = e^{\lambda_2 t} x_2(0)$$

|

$$\dot{x}_n = \lambda_n x_n \quad \rightsquigarrow x_n(t) = e^{\lambda_n t} x_n(0)$$

EQUivalently

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \searrow & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

$$x(t) = \underbrace{e^{At}}_{\text{Matrix exponential}} x(0)$$

THE MATRIX EXPONENTIAL OF STATE MATRIX A
(Easy because A is diagonal)

WE HAVE A GENERAL $\dot{x} = Ax$, WE ARE INTERESTED IN A CHANGE OF COORDINATES SUCH THAT $x = Tz$ THAT DIAGONALIZES THE ODE $\dot{z} = Dz$ WITH D DIAGONAL

WE START FROM $x = Tz \Rightarrow \dot{x} = T\dot{z} = Ax$

THAT IS, WE HAVE $T\dot{z} = Ax$ WE CAN LEFT MULTIPLY BY T^{-1}

$$\underbrace{T^{-1}T}_{I}\dot{z} = T^{-1}Ax \Rightarrow \dot{z} = \underbrace{T^{-1}AT}_{D}x$$

WE WANT THIS TO BE DIAGONAL

WE HAVE $D = T^{-1}AT$

WE CAN LEFT MULTIPLY BY T , TO GET $TD = \underbrace{T^{-1}T}_{I}AT$

SO THAT $AT = TD$ THIS IS THE EIGEN EQUATION IN MATRIX FORM

$$Av = \lambda v$$

$$A \begin{bmatrix} 1 \\ v_1 \\ \vdots \\ 1 \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ v_1 \\ \vdots \\ 1 \end{bmatrix}$$

THE SCALAR CASE

$$A \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} T$$

$\{\lambda_i\}$ $\{v_i\}$ EIGENVALUES AND EIGENVECTORS

From $D = T^{-1}AT$, WE CAN LEFT MULTIPLY BY T AND THEN WE CAN RIGHT MULTIPLY BY T^{-1}

$$TD = T \cancel{T^{-1}} \cancel{AT} \quad \text{I}$$

$$\Rightarrow TD = AT \quad = 1$$

$$TDT^{-1} = A + \cancel{T^{-1}}$$

$$\Rightarrow A = TDT^{-1}$$

THEN $A^2 = \underbrace{(TDT^{-1})(TDT^{-1})}_{I} = TD^2T^{-1}$

$$A^3 = \underbrace{(TDT^{-1})(TDT^{-1})(TDT^{-1})}_{I} = TD^3T^{-1}$$

$$A^n = TD^nT^{-1}$$

$$\begin{aligned}
 e^{At} &= I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots \\
 &= T \left[1 + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots \right] T^{-1} \\
 &= T e^{Dt} T^{-1}
 \end{aligned}$$

THE SOLUTION THEN BECOMES $y(t) = T e^{Dt} T^{-1} y(0)$