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On the observability of activated sludge plants

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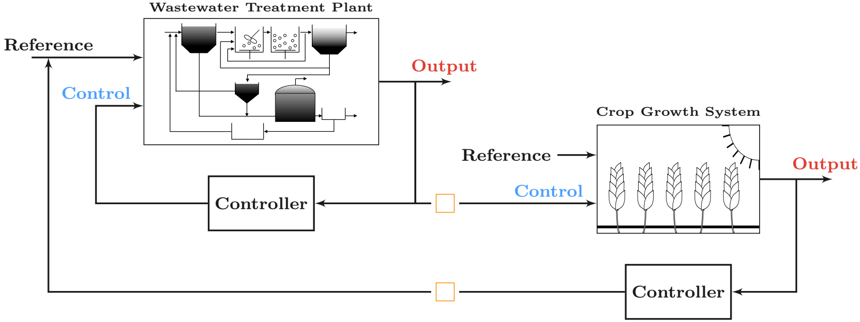
Introduction and motivation*

Control and Estimation for Wastewater Treatment Plants (WWTP)

- **Objective:** Sustainable reuse of wastewater
  - ⇝ A treatment-reclamation system for agricultural purposes

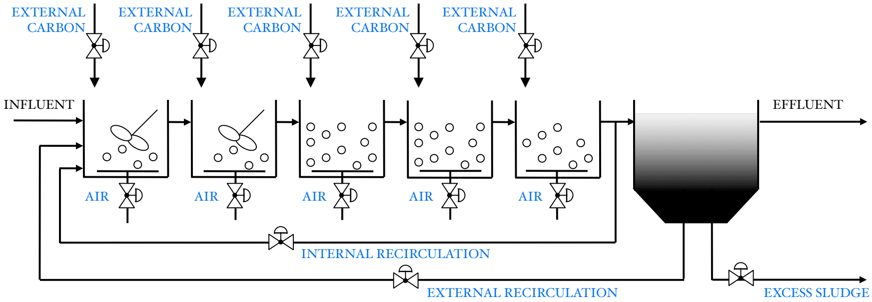


Motivation

Is it possible to operate the treatment plant to produce reusable wastewater of a specified quality, on demand?

* This study has been done within the international project Control4Reuse. We would like to thank FUNCAP (Project No JPI-00153-00004.01.00/19) for funding the Brazilian part of this project.

Activated Sludge Process, description



For the task, we considered a conventional Activated Sludge Process (ASP)

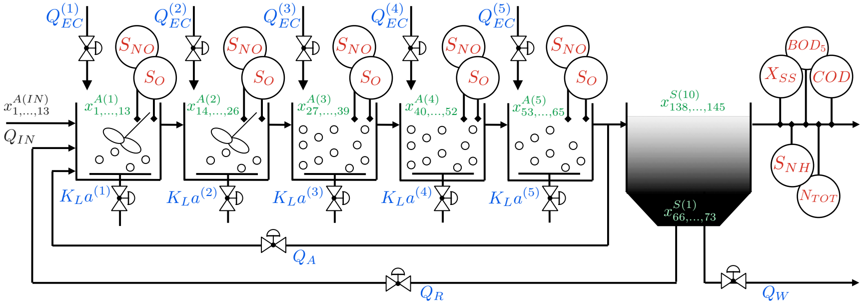
- The **Benchmark Simulation Model no. 1 (BSM1)**[1]

Plant layout

- 5 sequential bio-reactors (Activated Sludge Model no. 1)
- A non-reactive settler (10-layers double-exponential settling model)

[1] Gernaey, K., Jeppsson, U., Vanrolleghem, P., Copp, J., 2014. Benchmarking of Control Strategies for Wastewater Treatment Plants. IWA.

System-oriented description of the process



Reactor $(k = 1, \cdots, 5)$

State variables

- $x^{A(k)} \in \mathbb{R}^{13}_{\geq 0}$

Input variables

- $u^{A(k)} = [K_La^{(k)} \; Q_{EC}^{(k)}]$
- $[Q_A]$
- $d^{A(1)} = [Q_{IN} \quad x^{A(IN)}]$

Measurement variables

- $y^{A(k)} = [S_O^{A(k)} \; S_{NO}^{A(k)}]$

Settler $(m = 1, \cdots, 10)$

State variables

- $x^{S(m)} \in \mathbb{R}^{8}_{\geq 0}$

Input variables

- $[Q_R \; Q_W]$

Measurement variables

- $y^{S(10)} = [X_{SS}^{S(10)} \; S_{NH}^{S(10)} \; BOD_5^{S(10)} \; COD^{S(10)} \; N_{tot}^{S(10)}]$

State-space representation of the process

$$\dot{x}(t) = f(x(t), u(t), d(t)|\theta_x)$$
$$y(t) = g(x(t)|\theta_y)$$

- $x(t) = [x^{A(1)} \;\cdots\; x^{A(5)} \; x^{S(1)} \;\cdots\; x^{S(10)}]^T$
- $u(t) = [Q_A \; Q_R \; Q_W \; u^{A(1)} \;\cdots\; u^{A(5)}]^T$
- $y(t) = [y^{A(1)} \;\cdots\; y^{A(5)} \; y^{S(10)}]^T$
- $d(t) = [d^{A(1)}]^T$
- $\{\theta_x, \theta_y\}$: Model parameters

- An "expansion" of the state-vector compared to common representations
- All possible control and sensors that do not require changes in the plant layout
  - $N_x = 5 \times 13 + 10 \times 8 = 145$ state variables
  - $N_u = 3 + 5 \times 2 = 13$ controls
  - $N_y = 5 \times 2 + 5 = 15$ sensors

We try to address our initial question by studying two properties of this model

Full-state Controllability
Can we manipulate $u(t)$ to steer the state-vector $x(t)$ to a desired value?

Full-state Observability
Can we reconstruct the state-vector $x(t)$ from measurements $y(t)$?

Structural representation, definition

The system $\dot{x}(t) = f(\cdot|\theta_x)$, with $y(t) = g(\cdot|\theta_y)$, from a structural perspective

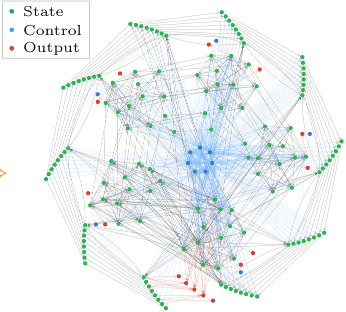
$$A_{i,j} = \frac{\partial f_i}{\partial x_j} \begin{cases} \neq 0 & (x_j \text{ affects } x_i) \\ = 0 & \text{o/w} \end{cases} \quad B_{i,k} = \frac{\partial f_i}{\partial u_k} \begin{cases} \neq 0 & (u_k \text{ affects } x_i) \\ = 0 & \text{o/w} \end{cases} \quad C_{k,j} = \frac{\partial g_k}{\partial x_j} \begin{cases} \neq 0 & (x_j \text{ affects } y_k) \\ = 0 & \text{o/w} \end{cases}$$

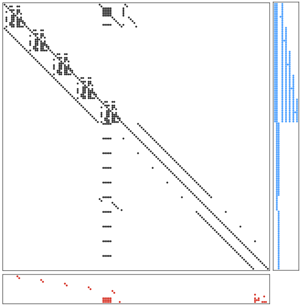
This structural system describes a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_A \cup \mathcal{V}_B \cup \mathcal{V}_C \\ &= \{x_1, \cdots, x_{N_x}\} \cup \{u_1, \cdots, u_{N_u}\} \\ &\quad \cup \{y_1, \cdots, y_{N_y}\} \end{aligned}$$

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_A \cup \mathcal{E}_B \cup \mathcal{E}_C \\ &= \{(x_j, x_i) | A_{i,j} \neq 0\} \cup \{(u_k, x_i) | B_{i,k} \neq 0\} \\ &\quad \cup \{(x_j, y_k) | C_{k,j} \neq 0\} \end{aligned}$$

Network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ $\Rightarrow$





$\Leftarrow$ **System** (A, B, C)

BSM1, structural controllability analysis

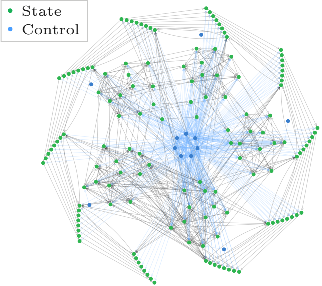
"The pair (A, B) is structurally controllable"

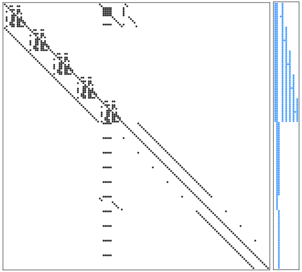
$\Updownarrow$

"Any realisation of (A, B) results in a controllable system (classical sense)"

i) **Accessibility**
There exists at least one path starting from any $u_k \in \mathcal{V}_B$ to each $x_i \in \mathcal{V}_A$

ii) **Dilation-free**
For every $\mathcal{S} \subseteq \mathcal{V}_A$, $|T_{in}(\mathcal{S})| \geq |\mathcal{S}|$, where $T_{in}(\mathcal{S})$ is the in-neighborhood set of $\mathcal{S}$





Pair (A, B) is **structurally controllable**
- Both accessibility and dilation-free are satisfied

$\rightsquigarrow$

The plant described by $\dot{x}(t) = f(\cdot|\theta_x)$ is **controllable** for almost all possible realisations of matrices A and B

BSM1, structural observability analysis

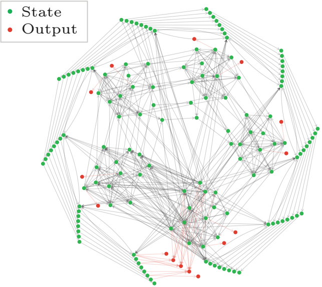
"The pair (A, C) is structurally observable"

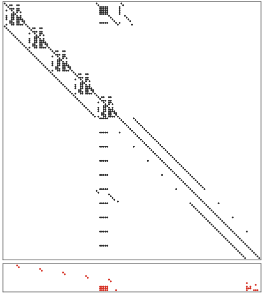
$\Updownarrow$

"Any realisation of (A, C) results in an observable system (classical sense)"

i) **Accessibility**
There exists at least one path starting from each $x_j \in \mathcal{V}_A$ to any $y_k \in \mathcal{V}_C$

ii) **Dilation-free**
For every $\mathcal{S} \subseteq \mathcal{V}_A$, $|T_{out}(\mathcal{S})| \geq |\mathcal{S}|$, where $T_{out}(\mathcal{S})$ is the out-neighborhood set of $\mathcal{S}$





Pair (A, C) is **structurally unobservable**
- Dilation-free satisfied but accessibility violated

$\rightsquigarrow$

The plant $\dot{x}(t) = f(\cdot|\theta_x)$ with $y(t) = g(\cdot|\theta_y)$ is **unobservable** for every possible realisation of matrices A and C

Conventional analysis, definitions

The benchmark suggests a linearisation for a steady-state $SS \equiv (x^{SS}, u^{SS}, d^{SS}, y^{SS})$

$$\dot{x}(t) = A^{SS}x(t) + B^{SS}u(t) + G^{SS}d(t) \qquad y(t) = C^{SS}x(t)$$

$\Longleftrightarrow$

- $\rightsquigarrow A^{SS} = (\partial f/\partial x)|_{SS}$
- $\rightsquigarrow B^{SS} = (\partial f/\partial u)|_{SS}$
- $\rightsquigarrow G^{SS} = (\partial f/\partial d)|_{SS}$
- $\rightsquigarrow C^{SS} = (\partial g/\partial x)|_{SS}$

- **Verify if the structural results hold for (A^{SS}, B^{SS}, C^{SS}) using classical tests**

Kalman's controllability test

The pair (A, B) is controllable IFF

$\rightsquigarrow$ $\mathcal{C} = [B \quad AB \quad \cdots \quad A^{N_x-1}B]$ is full-rank, i.e., $\texttt{rank}(\mathcal{C}) = N_x$

- Allows for a direct definition of
  - $\rightsquigarrow$ Controllable subspace
  - $\rightsquigarrow$ Uncontrollable subspace
- **Unpractical**: Numerical issues when state-space is high-dimensional

PBH controllability test

The pair (A, B) is controllable IFF

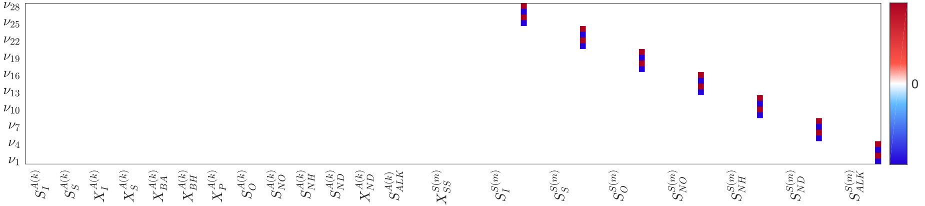
- $\rightsquigarrow$ $\texttt{rank}([\lambda I - A \quad B]) = N_x, \forall \lambda \in \mathbb{C}$
- $\rightsquigarrow$ $\texttt{rank}([\lambda_i I - A \quad B]) = N_x, \forall \lambda_i \in \sigma(A)$

- Relates $\nu_i(\lambda_i)$ to uncontrollable subspaces
  - $\rightsquigarrow$ If $\texttt{rank}([\lambda_i I - A \quad B]) < N_x$ then ν_i lies in the uncontrollable subspace
- **Scalable**: Requires at most N_x rank evaluations for a $N_x \times (N_x + N_u)$ matrix

Conventional analysis, PBH controllability test

The PBH test indicates that (A^{SS}, B^{SS}) is **uncontrollable in the classical sense**

- A real eigenvalue failing the test
  - Algebraic multiplicity: 28
  - Geometric multiplicity: 7
- The non-zero entries of associated eigenvectors $\nu_1, \cdots, \nu_{28}$ correspond to soluble matter in the effluent



For linearisation (A^{SS}, B^{SS}), we cannot control the effluent concentrations of soluble matter

Conventional analysis, contradiction with structural results

We found a contradiction between the controllability results

- (A, B) is **controllable in a structural sense**
- (A^{SS}, B^{SS}) is **uncontrollable in a classical sense**

Dilation-free condition: A known issue whenever some self-loops weights are identical

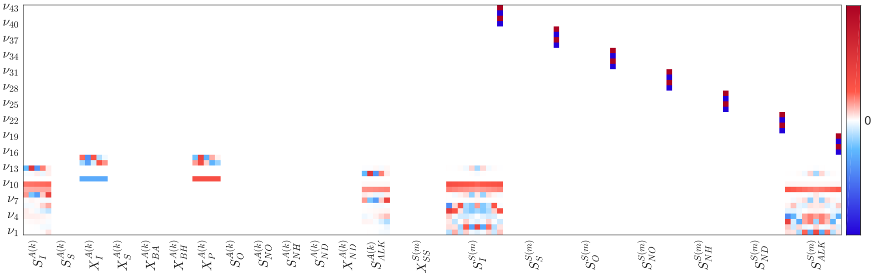
- **Non-reacting matter in reactors:** $S_a^{A(k)}$ $(a \in \{I, ALK\})$ and $X_b^{A(k)}$ $(b \in \{I, P\})$
  - $$\frac{\partial \dot{S}_a^{A(k)}}{\partial S_a^{A(k)}} = \frac{\partial \dot{X}_b^{A(k)}}{\partial X_b^{A(k)}} = -\frac{Q_A + Q_R + Q_{IN} + \sum_{j=1}^{k} Q_{EC}^{(j)}}{V_A^{(k)}}$$
- **Soluble matter in the settler:** $S_c^{S(m)}$ $(c \in \{I, S, O, NO, NH, ND, ALK\})$
  - For $m = 1, \cdots, 5$
    $$\frac{\partial \dot{S}_c^{S(m)}}{\partial S_c^{S(m)}} = \frac{-Q_R - Q_W}{V_S^{(m)}}$$
  - For $m = 6$
    $$\frac{\partial \dot{S}_c^{S(m)}}{\partial S_c^{S(m)}} = \frac{-Q_{IN} + Q_R}{V_S^{(m)}}$$
  - For $m = 7, \cdots, 10$
    $$\frac{\partial \dot{S}_c^{S(m)}}{\partial S_c^{S(m)}} = \frac{Q_W - Q_{IN}}{V_S^{(m)}}$$

> We can never control the full state-space for the model $\dot{x}(t) = f(\cdot|\theta_x)$, regardless of the linearisation being used

Conventional analysis, PBH observability test

The PBH test indicates that (A^{SS}, C^{SS}) is **unobservable in the classical sense** (as expected)

- 10 distinct eigenvalues failing the test
  - Including 5 complex pairs and 2 real values with multiplicity larger than 1
- Total of 43 eigenvectors $(\nu_1, \cdots, \nu_{43})$
- Non-zero entries correspond to
  - All non-reacting components
  - Soluble matter in the effluent



For linearisation (A^{SS}, C^{SS}), we cannot unequivocally determine the state-vector from a sequence of outputs over a finite time interval

Reduced models, definition

Goal: A reduced-order model consisting of most controllable/observable state-space directions

- Some state-space directions require more effort to control/observe than others

$$\begin{aligned}\dot{x} &= A^{SS}x + B^{SS}u\\ y &= C^{SS}x\end{aligned} \quad \overset{\widetilde{x}=Px}{\rightsquigarrow} \quad \begin{aligned}\dot{\widetilde{x}} &= \widetilde{A}^{SS}\widetilde{x} + \widetilde{B}^{SS}u\\ y &= \widetilde{C}^{SS}\widetilde{x}\end{aligned} \qquad (N_{\widetilde{x}} \leq N_x)$$

A Gramian-based quantitative analysis of controllability and observability

- $W_{\mathcal{C}} = \int_0^\infty e^{A^{SS}\tau} B^{SS} (B^{SS})^T e^{(A^{SS})^T\tau} d\tau \quad \rightsquigarrow \quad (\lambda_i(W_{\mathcal{C}}))^{-1} \propto$ Effort to control $\nu_i(\lambda_i)$
- $W_{\mathcal{O}} = \int_0^\infty e^{(A^{SS})^T\tau} (C^{SS})^T C^{SS} e^{A^{SS}\tau} d\tau \quad \rightsquigarrow \quad (\lambda_j(W_{\mathcal{O}}))^{-1} \propto$ Effort to observe $\nu_j(\lambda_j)$

Gramian-based controllability/observability metrics

We quantify controllability (or observability) using the metrics:

- $\rightsquigarrow$ $\mathtt{trace}(W)$
  Inversely proportional to the effort to control or observe the state-space
- $\rightsquigarrow$ $\mathtt{trace}(W^\dagger)$
  Proportional to the effort to control or observe the state-space
- $\rightsquigarrow$ $\lambda_{min}(W)$
  Inversely proportional to least controllable or observable state-space direction

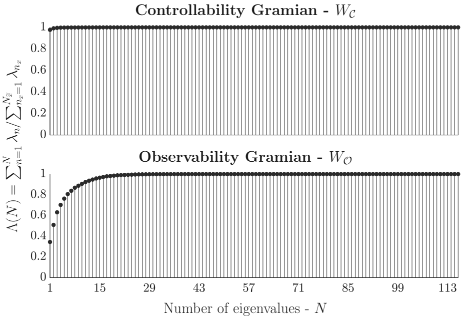
Reduced models, minimal realisation

We consider a minimal realisation for linearisation $(A^{SS}, B^{SS}, C^{SS}) \rightsquigarrow [N_{x_{co}} = 116]$

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co}^{SS} & 0 & A_{13}^{SS} & 0 \\ A_{21}^{SS} & A_{\bar{c}o}^{SS} & A_{23}^{SS} & A_{24}^{SS} \\ 0 & 0 & A_{c\bar{o}}^{SS} & 0 \\ 0 & 0 & A_{43}^{SS} & A_{\bar{c}\bar{o}}^{SS} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{\bar{c}o} \\ x_{c\bar{o}} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co}^{SS} \\ B_{\bar{c}o}^{SS} \\ 0 \\ 0 \end{bmatrix} u \quad \overset{x_{co} = Ux}{\rightsquigarrow} \quad \begin{aligned} \dot{x}_{co} &= A_{co}^{SS} x_{co} + B_{co}^{SS} u \\ y &= C_{co}^{SS} x_{co} \end{aligned}$$

$$y = \begin{bmatrix} C_{co}^{SS} & 0 & C_{\bar{c}o}^{SS} & 0 \end{bmatrix} \tilde{x}$$

Cumulative coverage of state-space



Gramian-based metrics

	$\mathtt{tr}(\widetilde{W})$	$\mathtt{tr}(\widetilde{W}^{\dagger})$	$\lambda_{\min}(\widetilde{W})$
$W_{\mathcal{C}}$	290.53	3.59E11	−1.55E−14
$W_{\mathcal{O}}$	0.521	4.00E14	1.14E−15

The control and estimation of minimal system $(A_{co}^{SS}, B_{co}^{SS}, C_{co}^{SS})$ are highly demanding tasks

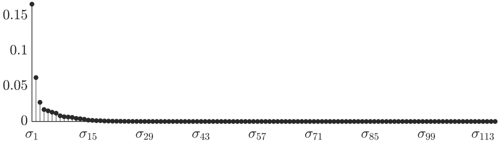
Reduced models, balanced truncation

We consider a balanced truncation for minimal $(A_{co}^{SS}, B_{co}^{SS}, C_{co}^{SS}) \rightsquigarrow$ [any chosen $N_{\tilde{x}_{co}} \leq 116$]

$$\begin{aligned} \dot{x}_{co} &= A_{co}^{SS} x_{co} + B_{co}^{SS} u \\ y &= C_{co}^{SS} x_{co} \end{aligned} \quad \underset{\rightsquigarrow}{\tilde{x}_{co} = S x_{co}} \quad \begin{aligned} \dot{\tilde{x}}_{co} &= \tilde{A}_{co}^{SS} \tilde{x}_{co} + \tilde{B}_{co}^{SS} u \\ y &= \tilde{C}_{co}^{SS} \tilde{x}_{co} \end{aligned}$$

Balanced realisation

- $\widetilde{W}_{\mathcal{C}} = \widetilde{W}_{\mathcal{O}} = \Sigma = \mathtt{diag}(\sigma_1, \cdots, \sigma_{N_{\tilde{x}_{co}}})$
  - $\rightsquigarrow$ $\sigma_i^{-1} \propto$ Effort to control/observe $\nu_i(\sigma_i)$



Gramian-based metrics

$N_{\tilde{x}_{co}}$	$\mathtt{tr}(\Sigma)$	$\mathtt{tr}(\Sigma^\dagger)$	$\lambda_{\min}(\Sigma)$
2	0.23	22.06	0.06
4	0.28	117.75	0.01
8	0.32	467.84	$8.12E{-3}$
16	0.35	$2.86E3$	$1.62E{-3}$
32	0.36	$1.51E5$	$3.20E{-5}$
64	0.36	$9.68E7$	$5.61E{-8}$
116	0.36	$3.13E17$	$1.46E{-17}$

A small number of state-space directions are sufficient to represent most of the input-output response of $(A_{co}^{SS}, B_{co}^{SS}, C_{co}^{SS})$

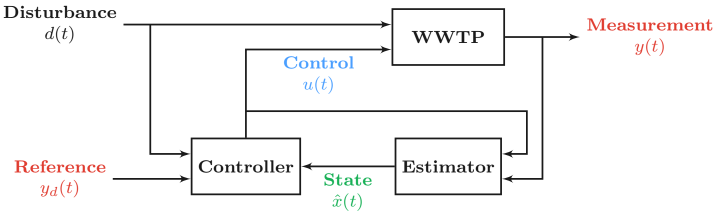
Final Remarks

The controllability and observability of the Activated Sludge Process were studied

Our results show that

- **Structural** (A, B, C): Controllable but unobservable in the structural sense
- **Linearisation** (A^{SS}, B^{SS}, C^{SS}): Uncontrollable and unobservable in the classical sense
  - $\rightsquigarrow$ A large portion of the state-space is still controllable (and observable)
- **Minimal** $(A^{SS}_{co}, B^{SS}_{co}, C^{SS}_{co})$: Control and estimation are highly demanding tasks
- **Balanced** $(\widetilde{A}^{SS}_{co}, \widetilde{B}^{SS}_{co}, \widetilde{C}^{SS}_{co})$: Original input-output response using few state-space directions

These results will be the backbone to the design of optimal controllers for the treatment-reclamation application initially described



Thank you!