Ordinary differential equations

Stochastic algorithms

Francesco Corona

Department of Computer Science
Federal University of Ceará, Fortaleza

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Ordinary differential equations

Ordinary differential equations are equations in some unknown quantity

- The unknown quantity is a function

The equations involve the derivatives of the unknown function

We provide some general background on ODEs

MAR 09, 2018 _ FC _

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General concepts

Ordinary differential equations

MAR 09, 2018 _ FC _

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General concepts

Interrelated changing entities are commonplace in systems modelling

- Changing entities are called **variables**

The rate of change of one variable with respect to another is a **derivative**

Relations among variables and their derivatives are **differential equations**

We are interested in knowing how the variables are related

MAR 09, 2018 _ FC _

Origins

General concepts

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins

Example

Consider the problem of determining the age of a bonfire

⇝ From the remains of charcoal

We know a few things, from common sense and notions

- Charcoal is burned wood
- Wood is organic matter
- Organic matter is C
- C has two isotopes

⇝ C^{14} and C^{12}

In living organisms, the $[\mathrm{C}^{12}]/[\mathrm{C}^{14}]$ ratio is constant

- C^{14} is radioactive
- C^{12} is stable

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins (cont.)

When organic matter dies its composition changes, with time

- C^{14} lost by radiation is not replaced

⇝ $[\mathrm{C}^{14}]$ and $[\mathrm{C}^{12}]/[\mathrm{C}^{14}]$ change

The changing entities of this problem are $[C^{14}]$ and time t

- The changing entities are related to each other

The relation between them requires the use of derivatives

- The relation is a differential equation

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins (cont.)

Let t be the time elapsed since the wood was chopped off its tree

Let $x(t)$ be the amount of C^{14} in the dead chops/charcoal

- At any time t

The instantaneous rate at which C^{14} decomposes is $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$

Origins (cont.)

We assume that the rate of decomposition varies linearly with $x(t)$

$$\rightsquigarrow \quad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = -kx(t)$$

- $k > 0$, proportionality constant
- $-$ sign, $[C^{14}]$ is decreasing

Instantaneous rate of decomposition of C^{14} is k-times the amount of C^{14}

- According to this relationship (a differential equation)

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins (cont.)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -kx(t)$$

For instance, let us suppose that $k = 0.01$ and let t be measured in years

- $\rightsquigarrow$ For $x(t)|_{t_1} = 200$ [units], we have $\mathrm{d}x(t)/\mathrm{d}t|_{t_1} = 2$ [units/year]
- $\rightsquigarrow$ For $x(t)|_{t_2} = 50$ [units], we have $\mathrm{d}x(t)/\mathrm{d}t|_{t_2} = 1/2$ [units/year]

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins (cont.)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -kx(t)$$

Next task, try to determine a functionality between x and time t, $x(t)$

We multiply both sides of the differential equation by $\mathrm{d}t/x(t)$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}\frac{\mathrm{d}t}{x(t)} = -kx(t)\frac{\mathrm{d}t}{x(t)}$$

$$\rightsquigarrow \quad \frac{\mathrm{d}x(t)}{x(t)} = -k\mathrm{d}t$$

By integration,

$$\rightsquigarrow \quad \log\big[x(t)\big] = -kt + c$$

c is an arbitrary constant

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins (cont.)

$$\log\left[x(t)\right] = -kt + c$$

By the definition of logarithm, we get

$$\rightsquigarrow \quad x(t) = e^{(-kt+c)} = e^c e^{(-kt)} = Ae^{(-kt)}$$

This is nearly the answer we are after[1]

- We need values for A and k

[1] A relation between the variable quantity x and the variable time t.

Origins (cont.)

$$x(t) = Ae^{(-kt)}$$

At time $t = 0$, by substitution, we know we had $x(t = 0) = A$ units of C^{14}

From chemistry, we know $\sim 99,876\%$ of A is still present after 10 years

- For $t = 10$, we have $x(t = 10) = 0.99876A$

Thus,

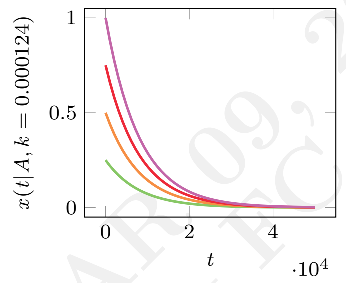
$$\begin{aligned} 0.99876A &= Ae^{(-10k)} \\ 0.99876 &= e^{(-10k)} \\ \log(0.99876) &= -10k \\ -0.00124 &= -10k \end{aligned}$$

$$\rightsquigarrow \quad k = 0.000124$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Origins (cont.)



For $k = 0.000124$

$$\rightsquigarrow \quad x(t) = Ae^{-0.000124t}$$

We need to determine the value of A

- The initial amount of C^{14}

MAR 09, 2018 — FC

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

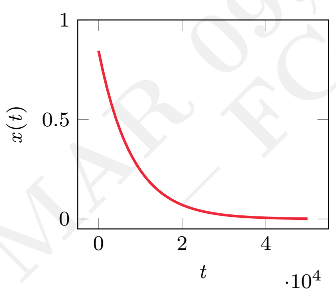
Origins (cont.)

By chemical analysis of charcoal, we can measure $[C^{14}]/[C^{12}]$

- Living wood (known) and bonfire (measured)

At time t (now), 85.5% of $[C^{14}]$ had decomposed

$\rightsquigarrow$ 14.5% remained ($0.145A$)



$$\begin{aligned} 0.145A &= Ae^{-0.000124t} \\ 0.145 &= e^{-0.000124t} \\ \log(0.145) &= -0.000124t \\ -1.9310 &= -0.000124t \end{aligned}$$

$$\rightsquigarrow \quad t = 15573$$

MAR 09, 2018 — FC

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Definitions

General concepts

MAR 09, 2018 — FC

Definitions

In calculus, we studied methods for differentiating elementary functions

Example

Consider the function $y(x) = \log(x)$

We have the successive derivatives

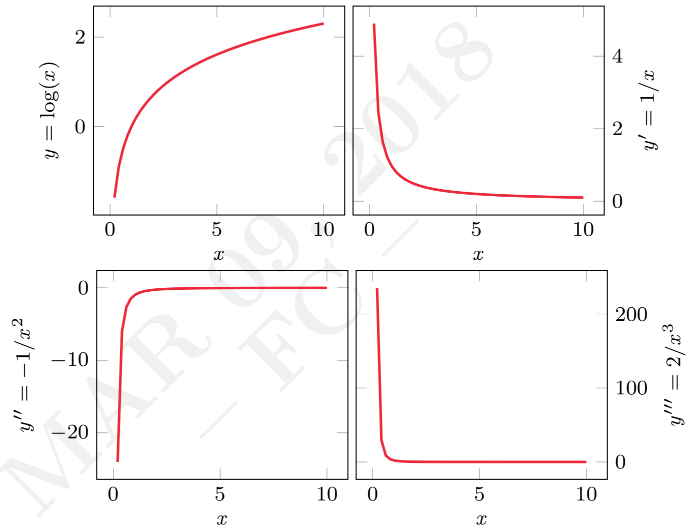
$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} y(x) &= \frac{1}{x} = y' \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2} y(x) &= \frac{-1}{x^2} = y'' \\
\frac{\mathrm{d}^3}{\mathrm{d}x^3} y(x) &= \frac{2}{x^3} = y''' \\
\cdots &= \cdots
\end{aligned}
$$

The equations involve variables and their derivatives

- One independent variable x

They are called ordinary differential equations

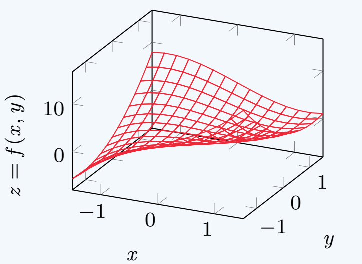
Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem



Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Definitions (cont.)

Example



Consider the function

$$z(x, y) = x^3 - 3xy + 2y^2$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Definitions (cont.)

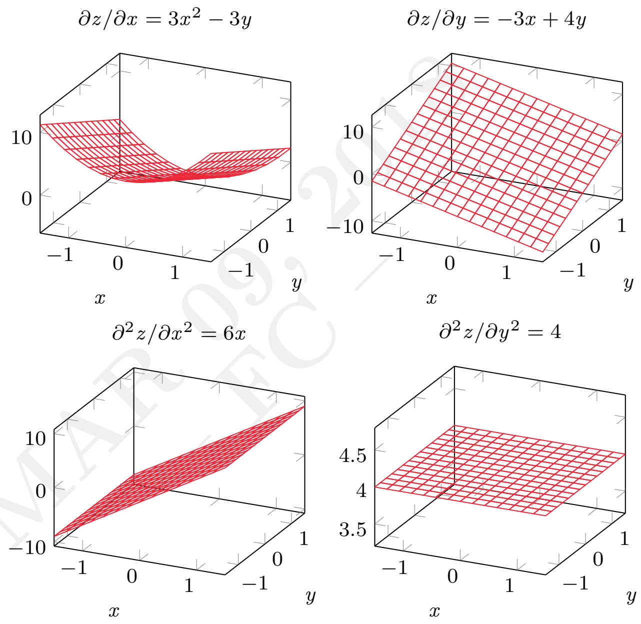
The partial derivatives with respect to x and y

$$
\begin{aligned}
\frac{\partial}{\partial x} z(x, y) &= 3x^2 - 3y \\
\frac{\partial}{\partial y} z(x, y) &= -3x + 4y \\
\frac{\partial^2}{\partial x^2} z(x, y) &= 6x \\
\frac{\partial^2}{\partial y^2} z(x, y) &= 4 \\
\cdots &= \cdots
\end{aligned}
$$

The equations involves variables and their derivatives

- Two independent variables x and y

They are called partial differential equations



Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Definitions (cont.)

Definition

Ordinary differential equation

Let $f(x)$ be a function of x defined over some interval, $\mathcal{I}: a < x < b$

*By **ordinary differential equation**, we mean an equation involving x, the function $f(x)$ and one or more of the derivatives of $f(x)$*

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Definitions (cont.)

Habits

Common custom in writing differential equations uses $f(x)$ for $y(x)$ or y

For example,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) + x \cdot \big[f(x)\big]^2 = 0 \quad \rightsquigarrow \quad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x} y(x) & = x \cdot [y(x)]^2 = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}x} y & = xy^2 = 0 \\ y' & = xy = 0 \end{cases}$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Definitions (cont.)

Definition

Order of a differential equation

*The **order of a differential equation** is the order of the highest derivative involved in the equation*

Solution

Ordinary differential equations

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Solution

Consider the algebraic equation

$$x^2 - 2x - 3 = 0$$

If x is replaced by 3, the equality holds true

- We say that $x = 3$ is a solution

We mean that $x = 3$ satisfies the equation

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Solution (cont.)

Consider the differetial equation

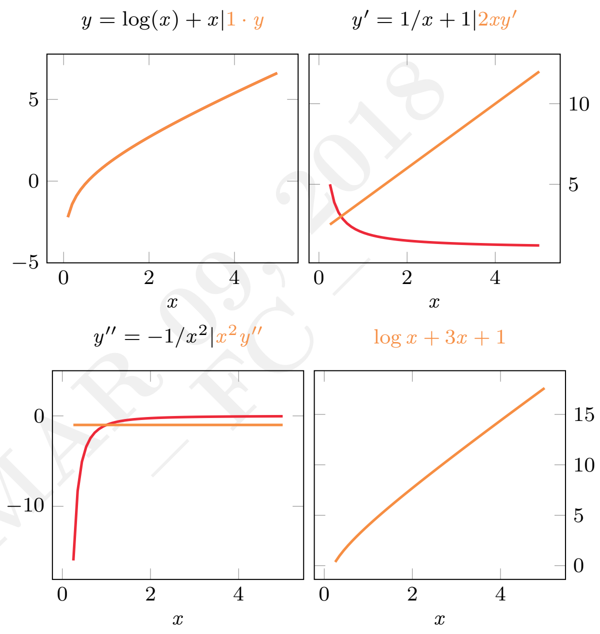
$$x^2 y'' + 2xy' + y = \log(x) + 3x + 1, \text{ with } x > 0$$

Function $f(x) = \log(x) + x$ is a solution of the differential equation $(x > 0)$

$f(x)$ and its first and second derivatives can be substituted in y, y' and y''

- The equality will hold true

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem



Solution (cont.)

Two things are worth noting

Values of x for which function $f(x)$ is defined had been clearly specified

- Though they would have been tacitly assumed
- $\log(x)$ is undefined for $x \leq 0$

We specified the interval in which the differential equation makes sense

- Redundant, because of the presence of $\log(x)$

Generalities

Solution

Explicit solution

Definition

Explicit solution

Let $y = f(x)$ define y as a function of x over an interval, $\mathcal{I} : a < x < b$

We say that function $f(x)$ is an explicit solution of an ordinary differential equation involving x, $f(x)$ and derivatives, if it satisfies the equation $\forall x \in \mathcal{I}$

Function $f(x)$ is a solution of the differential equation

$$F\big[x, y, y', \cdots, y^{(n)}\big] = 0,$$

if

$$F\big[x, f(x), f'(x), \cdots, f(x)^{(n)}\big] = 0, \quad \text{for every } x \text{ in } \mathcal{I}$$

Explicit solution (cont.)

We can replace y by $f(x)$, y' by $f'(x)$, y'' by $f''(x)$, ..., $y^{(n)}$ by $f^{(n)}(x)$

$\rightsquigarrow$ The differential equation reduces to an identity in x

Explicit solution (cont.)

Habits

We use expressions like ‘solve’ or ‘find a solution’ of a differential equation

$\rightsquigarrow$ ‘Find a function which is solution of the differential expression’

We may refer to a certain equation as the solution of a differential equation

$\rightsquigarrow$ We mean, the function defined by the equation is the solution

Explicit solution (cont.)

Remark

An equation that does not define a function, cannot be a solution

- You may still show that the differential equation is satisfied

A function

Suppose that to each element of an independent variable x on a set E (the set must be specified) there corresponds one and only one (real) value of a dependent variables y

We say that the dependent variable y is a **function** of the independent variable x on the set E

Explicit solution (cont.)

Example

Equation $y = \sqrt{-(1+x^2)}$ does not define a (real) function

Meaningless to say that it is a solution of the differential equation $x + yy' = 0$

Though, by formal substitution, we obtain an identity

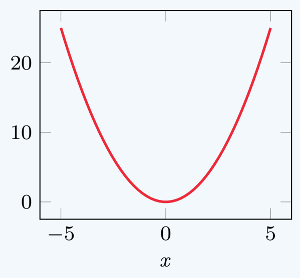
$\rightsquigarrow$ $y = \sqrt{-(1+x^2)}$

$\rightsquigarrow$ $y' = -x\sqrt{-(1+x^2)}$

Explicit solution (cont.)

Example

$y = x^2$



Consider the function

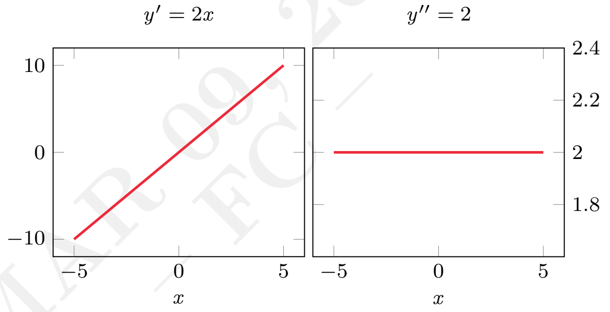
$y = x^2$, with $-\infty < x < \infty$

Verify that it is a solution to the differential equation

$$(y'')^3 + (y')^2 - y - 3x^2 - 8 = 0$$

Explicit solution (cont.)

Together with $y = f(x) = x^2$, we have $y' = f'(x) = 2x$ and $y'' = f''(x) = 2$



Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Explicit solution (cont.)

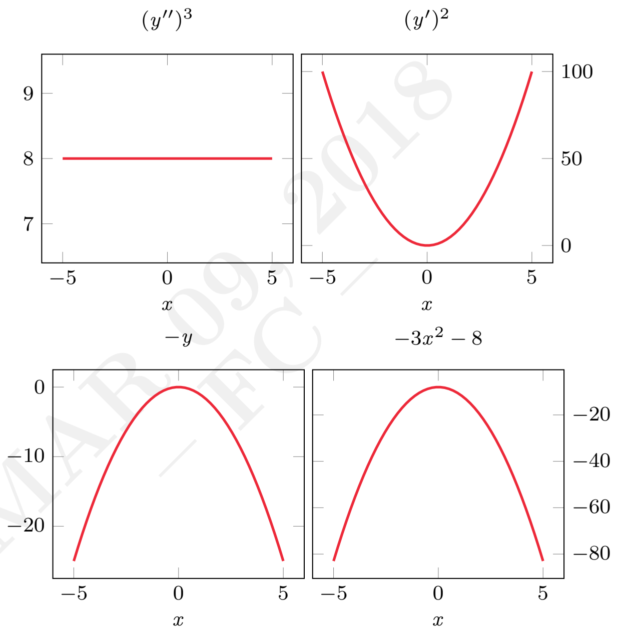
$$\underbrace{(y'')}_{f''(x)=2}{}^3 + \underbrace{(y')}_{f'(x)=2x}{}^2 - \underbrace{y}_{f(x)=x^2} - 3x^2 - 8 = 0$$

Substituting these values, we obtain

$$\underbrace{8 + (4x^2 - x^2) - 3x^2 - 8}_{F\left[x, f(x), f'(x), f''(x)\right]} = 0$$

LHS is zero, $y = x^2$ is an explicit solution

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

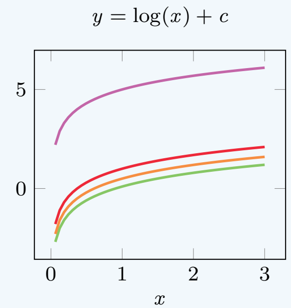


■

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Explicit solution (cont.)

Example



Consider function

$$y = \log(x) + c, \text{ with } x > 0$$

Verify that it is a solution to the differential equation

$$y' = 1/x$$

Explicit solution (cont.)

Together with $y = f(x) = \log(x) + c$, we have $y' = f'(x) = 1/x$, for $x > 0$

By substitution of these values, we obtain an identity in the variable x

- $y = \log(x) + c$ is a solution of $y' = 1/x$, for all $x > 0$

■

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Explicit solution (cont.)

Example

■

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution

We can test if an implicit solution defined by $f(x, y) = 0$ is a solution

- The procedure is much more involved

Not always easy to solve the equation $f(x, y) = 0$ for y in terms of x

Suppose that it can be shown that an implicit function $y = g(x)$ satisfies a given differential equation over an interval $\mathcal{I} : a < x < b$

- Relation $f(x, y) = 0$ is an implicit solution of the differential equation

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution (cont.)

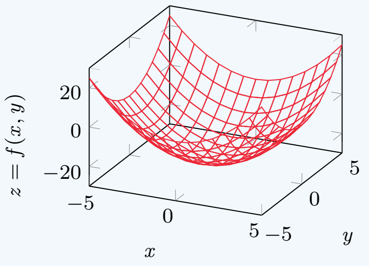
Implicit function

The relation $f(x, y) = 0$ defines y as an **implicit function** of x over the interval $\mathcal{I} : a < x < b$, if there exists a function $y = g(x)$ defined over $\mathcal{I}$

$$\rightsquigarrow \quad f\big[x, g(x)\big] = 0, \text{ for every } x \in \mathcal{I}$$

Implicit solution (cont.)

Example



Consider the relationship

$$x^2 + y^2 - 25 = 0$$

Does it define a function?

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution (cont.)

Let $x > +5$ or $x < -5$

- The formula will not determine a value of y
- (If $x = 7$, no y can satisfy the relation)

Let $-5 \leq x \leq +5$

- We solve the relation for y
- $y = \pm\sqrt{25 - x^2}$

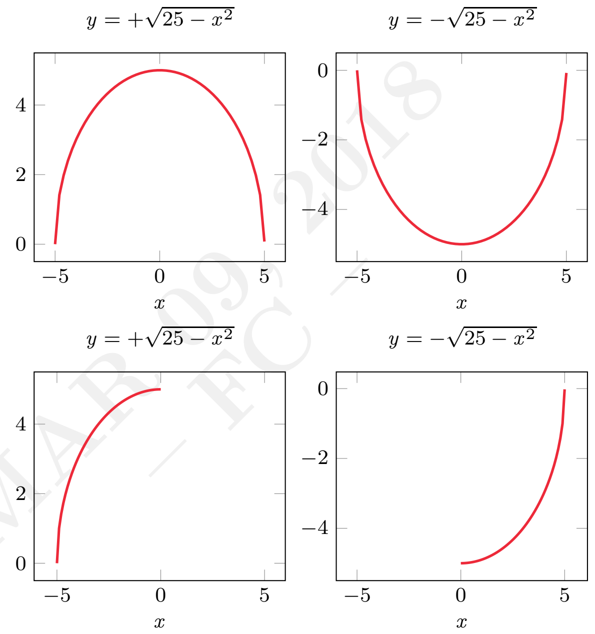
It does not uniquely define $y(x)$

$$y = +\sqrt{25 - x^2}, \quad (x \in [-5, +5])$$
$$y = -\sqrt{25 - x^2}, \quad (x \in [-5, +5])$$
$$y = +\sqrt{25 - x^2}, \quad (x \in [-5, 0])$$
$$y = -\sqrt{25 - x^2}, \quad (x \in (0, +5))$$

Each formula defines a proper function

- We can *choose* any of them

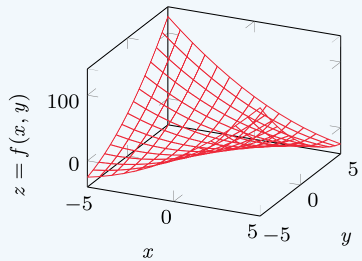
Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem



Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution (cont.)

Example



Consider the relationship

$$x^2 + y^2 - 3xy = 0$$

Does it define a function?

If it does, for what values of x will it uniquely determine a value of y?

It is not easy to solve the relation for y in terms of x

Implicit solution (cont.)

Definition

Implicit solution

*A relation $f(x,y)=0$ is an **implicit solution** of the differential equation*

$$F\big[x, y, y', y'', \cdots, y^{(n)}\big] = 0, \text{ with } x \in \mathcal{I} = (a,b)$$

if

1. *$f(x,y)$ defines y as an implicit function of x on $\mathcal{I}$ (there exists a function $y = g(x)$ defined over $\mathcal{I}$ such that $f[x, g(x)] = 0$ for every $x \in \mathcal{I}$)*
2. *$g(x)$ satisfies the differential equation*

$$F\big[x, g(x), g(x)', g(x)'', \cdots, g(x)^{(n)}\big] = 0, \text{ for every } x \in \mathcal{I} = (a,b)$$

■

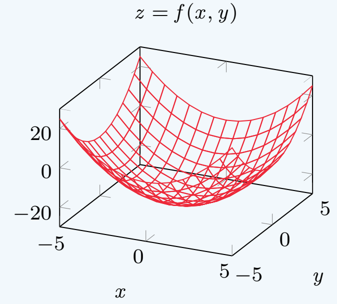
Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution (cont.)

Example



Consider the relation

$$f(x,y) = x^2 + y^2 - 25 = 0$$

Check whether $f(x,y)=0$ is an implicit solution of the differential equation

$$F(x,y,y') = yy' + x = 0, \quad \text{with } \mathcal{I}: -5 < x < 5$$

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution (cont.)

Function $f(x,y) = x^2 + y^2 - 25$ defines y as an implicit function of $x \in \mathcal{I}$

⇝ There is a function $g(x)$ defined on $\mathcal{I}$ such that

$$f\big[x, g(x)\big] = 0, \quad \forall x \in \mathcal{I}$$

Specifically, let $g(x) = y = \sqrt{25 - x^2}$ for $-5 \leq x \leq +5$

Then, $f(x,y) = f\big[x, g(x)\big] = x^2 + \underbrace{\big[\sqrt{25-x^2}\big]}_{y}{}^2 - 25 = 0$ is satisfied

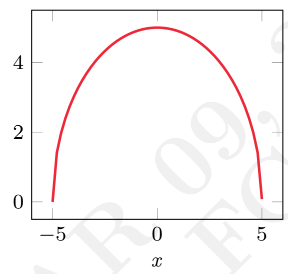
Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

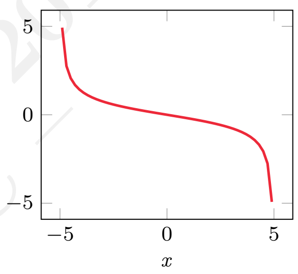
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Implicit solution (cont.)

$g(x) = \sqrt{25 - x^2}$



$g'(x) = -x/\sqrt{25 - x^2}$



By substituting $g(x)$ for y and $g'(x)$ for y' in $F(x,y,y') = yy' + x = 0$

$$\rightsquigarrow \quad f\big[x, g(x), g'(x)\big] = \sqrt{25-x^2}\Big(-\frac{x}{\sqrt{25-x^2}}\Big) + x = 0$$

■

General solution

In calculus, we studied methods for integrating elementary functions

- It was the same as solving simple differential equations

$$\rightsquigarrow \quad y'(x) = f(x)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Example

Consider the differential equation

$$y'(x) = e^x$$

Its solution, by integration

$$\rightsquigarrow \quad y(x) = e^x + c$$

c can take any arbitrary numerical value

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

If $y''(x) = e^x$, then its solution by double integration

$$\rightsquigarrow \quad y(x) = e^x + c_1 x + c_2$$

c_1 and c_2 can take any numerical values

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

If $y'''(x) = e^x$, then its solution by triple integration

$$\rightsquigarrow \quad y(x) = e^x + c_1 x^2 + c_2 x + c_3$$

c_1, c_2 and c_3 can take any numerical values

■

General solution (cont.)

Two important (yet false) conjectures seem to stem from this example

If a differential equation has a solution, it has infinitely many solutions
- ⇝ As many as there are values of c

If a differential equation is first order, then there is only one constant
- ⇝ If it is second order, two constants
- ⇝ If it is third order, three constants
- ⇝ ...

If a differential equation is n-th order, the solution has n constants

General solution (cont.)

Example

Consider the first-order differential equation

$$(y')^2 + y^2 = 0$$

Consider the second-order differential equation

$$(y'')^2 + y^2 = 0$$

Both differential equations admit only one solution

$$\rightsquigarrow \quad y(x) = 0$$

General solution (cont.)

Example

Consider the first-order differential equation

$$|y'| + 1 = 0$$

Consider the second-order differential equation

$$|y''| + 1 = 0$$

Both differential equations have no solution

General solution (cont.)

Example

Consider the first-order differential equation

$$xy' = 1$$

The equation has no solution if $x \in \mathcal{I} = (-1, +1)$

The differential equation can be formally solved

$$y(x) = \log|x| + c$$

The function is discontinuous at the origin $x = 0$
- Not okay in $\mathcal{I}$

General solution (cont.)

If $x < 0$, we have $y(x) = \log(-x) + c_1$

- This is a valid solution in $x < 0$

If $x > 0$, we have $y(x) = \log(x) + c_2$

- This is a valid solution in $x > 0$

There is no valid solution at $x = 0$

■

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Example

Consider the first-order differential equation

$$(y' - y)(y' - 2y) = 0$$

The solution to this differential equation

$$(y - c_1 e^x)(y - c_2 e^2 x) = 0$$

- Two arbitrary constants (not one)

■

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

The examples warn that not all differential equations have a solution

- Also, the number of constants is not the order of the equation

The conjectures are true for a large class of differential equations

Consider a solution that contains n constants $c_1, c_2, \dots, c_n$

- ⇝ It is called a **n-parameter family of solutions**
- ⇝ $c_1, c_2, \dots, c_n$ are thus the **parameters**

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Definition

Consider the family of functions in the $(n+1)$ variables $x, c_1, c_2, \dots, c_n$

$$y = f(x, c_1, c_2, \dots, c_n)$$

*Such functions are called a **n-parameter family of solutions** of the n-order differential equation*

$$F\big[x, y, y', \cdots, y^{(n)}\big] = 0$$

if for each choice of a set of values $c_1, c_2, \dots, c_n$ the resulting function $f(x)$ (a function of x alone) is such that

$$F\big[x, f(x), f'(x), \cdots, f^{(n)}(x)\big] = 0$$

■

General solution (cont.)

Example

Consider the functions

$$y = f(x, c_1, c_2) = 3 + 2x + c_1 e^{x} + c_2 e^{2x}$$

A 2-parameter family of solutions of second-order differential equation

$$F\big[x, y, y', y''\big] = y'' - 3y' + 2y - 4x = 0$$

Let a, b be any two values of c_1, c_2

Then, as a function of only x

$$y = f(x) = 3 + 2x + ae^{x} + be^{2x}$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

The first and second derivatives of $y = f(x)$

$$y' = f'(x) = 2 + ae^{x} + 2be^{2x}$$
$$y'' = f''(x) = ae^{x} + 4be^{2x}$$

Substituting the values of f, f' and f'' for y, y' and y'', we get

$$\rightsquigarrow \quad F(x, f, f', f'') = ae^{x} + 4be^{2x} - 6 - 3ae^{x} - 6be^{2x} + 4x + 6 + 2ae^{x} + 2be^{2x} - 4x = 0$$

■

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

We now discuss the inverse problem of finding a differential equation

- ⇝ Given that its n-parameter family solutions is known

The family will contain the requisite number of n arbitrary constants

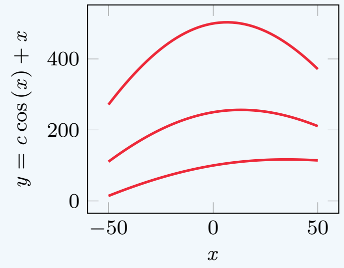
- The n-order equation does not contain them
- The constants need be eliminated (not easy)

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Example

Find a differential equation for the 1-parameter family of solutions



This family has one constant

$$y(x) = c\cos(x) + x$$

We seek a first-order equation

By differentiating the family of solutions, we obtain

$$y' = -c\sin(x) + 1$$

General solution (cont.)

$$y' = -c\sin(x) + 1$$

This differential equation still contains the constant

- It is not the searched one

We eliminate it by the following multiplications

To be completed as exercise?

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Example

Find a differential equation for the 2-parameter family of solutions

$$y = c_1 e^{x} + c_2 e^{-x}$$

First-order equation

To be completed as exercise?

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Consider the second order differential equation of a forced spring

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x(t)}{\mathrm{d}} + \nu^2 x(t) = w(t)$$

γ and ν are constants

Force $w(t)$ is some given function (may/may not depend on time)

- Position $x(t)$ is the dependent variable
- Time t is the independent variable

The equation is called inhomogeneous, because of the forcing term

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution

The solution to the differential equation is defined as **particular solution**

- It satisfies the ordinary differential equation
- Does not contain arbitrary constants

A **general solution** contains every possible particular solutions

- Parameterised by some free constants

General solution (cont.)

To solve the equation, we tie together general solution and initial conditions

We need to know the spring position $x(t_0)$ and velocity $\mathrm{d}x(t_0)/\mathrm{d}t$

- At some fixed initial time t_0

Given the initial conditions, there is a unique solution to the equation

- (provided that $w(t)$ is continuous)

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

It is common to omit dependencies of x and w on t

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}t}x(t) + \nu^2 x(t) = w(t)$$

Time derivatives are often denoted using dot (Newtonian) notation

$$\ddot{x}(t) + \gamma\dot{x}(t) + \nu^2 x(t) = w(t)$$

$\rightsquigarrow$ $\ddot{x}(t) = \mathrm{d}^2 x(t)/\mathrm{d}t$
$\rightsquigarrow$ $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$

MAR 09, 2018 _FC_

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

Differential equations of arbitrary order n can almost always be converted

$\rightsquigarrow$ Vector differential equations of order one

MAR 09, 2018 _FC_

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma\frac{\mathrm{d}}{\mathrm{d}}x(t) + \nu^2 x(t) = w(t)$$

For the spring model, we can define the **state variable** $\mathbf{x}(t)$

$$\rightsquigarrow \quad \mathbf{x}(t) = \big[x_1(t), x_2(t)\big] = \big[x(t), \mathrm{d}x(t)/\mathrm{d}t\big]$$

We re-write the original equation as a first-order equation

$$\rightsquigarrow \quad \underbrace{\begin{bmatrix} \mathrm{d}x_1(t)/\mathrm{d}t \\ \mathrm{d}x_2(t)/\mathrm{d}t \end{bmatrix}}_{\mathrm{d}\mathbf{x}(t)/\mathrm{d}t} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{f}[\mathbf{x}(t)]} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{L}} w(t)$$

MAR 09, 2018 _FC_

General solution (cont.)

The more general form

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\big[\mathbf{x}(t), t\big] + \mathbf{L}\big[\mathbf{x}(t), t\big]\mathbf{w}(t)$$

- The vector values function $\mathbf{x}(t) \in \mathcal{R}^n$ is called the state of the system
- The vector valued function $\mathbf{w}(t) \in \mathcal{R}^s$ is the forcing (input) function

It is possible to absorb the second term in the RHS into the first one

We get,

$$\rightsquigarrow \quad \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\big[\mathbf{x}(t), t\big]$$

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

The first-order vector representation of a n-order differential equation

$\rightsquigarrow$ The state-space representation

We develop the theory and solution methods for first-order equations

MAR 09, 2018 _ FC

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General solution (cont.)

The spring model is a special case of linear differential equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{L}(t)\mathbf{w}(t)$$

It is an important class of differential equations

$\rightsquigarrow$ We can actually solve these equations

MAR 09, 2018 _ FC

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant

Solution

MAR 09, 2018 _ FC

Linear and time-invariant

Consider a scalar linear homogeneous differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = fx(t), \quad \text{given } x(0)$$

f is a constant (time-independent) scalar

The equation can be solved by variable separation

$$\rightsquigarrow \quad \frac{\mathrm{d}x(t)}{x(t)} = f\mathrm{d}t$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

$$\frac{\mathrm{d}x(t)}{x(t)} = f\mathrm{d}t$$

We integrate the LHS from $x(0)$ to $x(t)$ and the RHS from 0 to t

We get,

$$\ln\left[x(t)\right] - \ln\left[x(0)\right] = ft \quad \rightsquigarrow \quad x(t) = x(0)e^{(ft)}$$

MAR 09, 2018 — FC

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

Another way of solving the equation consists of integrating both sides

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = fx(t), \quad \text{given } x(0)$$

Integrating from 0 to t, we get $\int_0^t \mathrm{d}x/\mathrm{d}t = x(t) - x(0)$

$$\rightsquigarrow \quad x(t) = x(0) + \int_0^t \mathrm{d}\tau f x(\tau)$$

We can now substitute the RHS of the equation for $x(t)$ in the integral

$$\begin{aligned} x(t) &= x(0) + \int_0^t \mathrm{d}\tau f\Big[x(0) + \int_0^\tau \mathrm{d}\tau f x(\tau)\Big] \\ &= x(0) + fx(0)\int_0^\tau \mathrm{d}\tau + \int_0^t\int_0^t \mathrm{d}\tau^2 f x(\tau) \\ &= x(0) + fx(0)t \int_0^t\int_0^t \mathrm{d}\tau f^2 x(\tau) \end{aligned}$$

MAR 09, 2018 — FC

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

The same procedure can be performed again on the last integral

$$\begin{aligned} x(t) &= x(0) + fx(0)t \int_0^t\int_0^t \mathrm{d}\tau^2 f^2\Big[x(0) + \int_0^\tau \mathrm{d}\tau f x(\tau)\Big] \\ &= x(0) + fx(0)t + f^2x(0)\int_0^t\int_0^t \mathrm{d}\tau^2 + \int_0^t\int_0^t\int_0^t f^3 x(\tau)\mathrm{d}\tau^3 \end{aligned}$$

It is easy to repeat the same procedure

$$\begin{aligned} x(t) &= x(0) + fx(0)t + f^2x(0)\frac{t^2}{2} + f^3x(0)\frac{t^3}{6} + \cdots \\ \rightsquigarrow &= \underbrace{\Big(1 + ft + \frac{f^2t^2}{2!} + \frac{f^3t^3}{3!} + \cdots\Big)}_{e^{(ft)}} x(0) \end{aligned}$$

As the Taylor expansion of $e^{(ft)}$ converges, we have

$$\rightsquigarrow \quad x(t) = e^{(ft)}x(0)$$

MAR 09, 2018 — FC

Linear and time-invariant (cont.)

The multivariate generalisation of homogeneous linear differential equations

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}\mathbf{x}(t), \quad \text{given } \mathbf{x}(0)$$

$\mathbf{F}$ is a constant (time-independent) matrix

We cannot use variable separation

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

We can use the expansion-based type solution

$$\rightsquigarrow \quad \mathbf{x}(t) = \underbrace{\Big(\mathbf{I} + \mathbf{F}t + \frac{\mathbf{F}^2 t^2}{2!} + \frac{\mathbf{F}^3 t^3}{3!} + \cdots\Big)}_{e^{(\mathbf{F}t)}} \mathbf{x}(0)$$

The series (always) converges [To the matrix exponential $e^{(\mathbf{F}t)}$]

$$\rightsquigarrow \quad \mathbf{x}(t) = e^{(\mathbf{F}t)}\mathbf{x}(0)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

The matrix exponential can be evaluated analytically

- Taylor series expansion
- Laplace transform
- Fourier transform
- Cayley-Hamilton
- $\cdots$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

Consider the linear differential equation, with inhomogeneous term

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}\mathbf{x}(t) + \mathbf{L}\mathbf{w}(t), \quad \text{given } \mathbf{x}(0)$$

$\mathbf{F}$ and $\mathbf{L}$ are constant (time-independent) matrices

These equations can be solved using the integrating factor method

Linear and time-invariant (cont.)

We first move $\mathbf{F}\mathbf{x}(t)$ to the LHS and then we multiply by $e^{(-\mathbf{F}t)}$

$$\rightsquigarrow \quad e^{(-\mathbf{F}t)}\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} - e^{(-\mathbf{F}t)}\mathbf{F}\mathbf{x}(t) = e^{(-\mathbf{F}t)}\mathbf{L}(t)\mathbf{w}(t)$$

From the definition of matrix exponential, we derive

$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\big[e^{(-\mathbf{F}t)}\big] = -e^{(-\mathbf{F}t)}\mathbf{F}$$

We have,

$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\big[e^{(-\mathbf{F}t)}\mathbf{x}(t)\big] = e^{(-\mathbf{F}t)}\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} - e^{(-\mathbf{F}t)}\mathbf{F}\mathbf{x}(t)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

Thus, we can re-write

$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\big[e^{(-\mathbf{F}t)}\mathbf{x}(t)\big] = e^{(-\mathbf{F}t)}\mathbf{L}(t)\mathbf{w}(t)$$

By integrating between t_0 and t, we get

$$\rightsquigarrow \quad e^{(-\mathbf{F}t)}\mathbf{x}(t) - e^{(-\mathbf{F}t_0)}\mathbf{x}(t_0) = \int_{t_0}^{t} \mathrm{d}\tau\, e^{(-\mathbf{F}\tau)}\mathbf{L}(\tau)\mathbf{w}(\tau)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Linear and time-invariant (cont.)

The complete solution

$$\rightsquigarrow \quad \mathbf{x}(t) = e^{\left[-\mathbf{F}(t-t_0)\right]}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{d}\tau\, e^{\left[\mathbf{F}(t-\tau)\right]}\mathbf{L}(\tau)\mathbf{w}(\tau)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General linear

Solution

General linear

Consider the general time-varying linear differential equations

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}(t)\mathbf{x}(t), \quad \text{given } \mathbf{x}(t_0)$$

- The solution in terms of matrix exponential is not valid

We can still formulate the implicit solution

$$\rightsquigarrow \quad \mathbf{x}(t) = \mathbf{\Psi}(t, t_0)\mathbf{x}(t_0)$$

- $\mathbf{\Psi}(t, t_0)$ is the **transition matrix**

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General linear (cont.)

The properties that define the transition matrix $\mathbf{\Psi}(t, t_0)$

$$\begin{aligned}
\partial\mathbf{\Psi}(\tau, t)/\partial\tau &= \mathbf{F}(\tau)\mathbf{\Psi}(\tau, t)\\
\partial\mathbf{\Psi}(\tau, t)/\partial t &= -\mathbf{\Psi}(\tau, t)\mathbf{F}(t)\\
\mathbf{\Psi}(\tau, t) &= \mathbf{\Psi}(\tau, s)\mathbf{\Psi}(s, t)\\
\mathbf{\Psi}(t, \tau) &= \mathbf{\Psi}^{-1}(\tau, t)\\
\mathbf{\Psi}(t, t) &= \mathbf{I}
\end{aligned}$$

Given the transition matrix, we can build the solution

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

General linear (cont.)

Consider the inhomogeneous case

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{L}(t)\mathbf{w}(t), \quad \text{given } \mathbf{x}(t_0)$$

The solution is analogous to the time-invariante case

- The integrating factor is $\mathbf{\Psi}(t_0, t)$

We obtain the solution,

$$\rightsquigarrow \quad \mathbf{x}(t) = \mathbf{\Psi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{d}\tau \mathbf{\Psi}(t, \tau)\mathbf{L}(\tau)\mathbf{w}(\tau)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Transforms

Ordinary differential equations

Fourier transforms

Transforms

Fourier transforms

Useful for solving inhomogeneous linear time-invariant differential equations

The **Fourier transform** of a function $g(t)$

$$\rightsquigarrow \quad G(i\omega) = \mathcal{F}\big[g(t)\big] = \int_{-\infty}^{\infty} \mathrm{d}t g(t) e^{(-i\omega t)}$$

The corresponding inverse transform

$$\rightsquigarrow \quad g(t) = \mathcal{F}^{-1}\big[G(i\omega)\big] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega G(i\omega) e^{(-i\omega t)}$$

Fourier transforms (cont.)

The main usefulness comes from the following property

$$\rightsquigarrow \quad \mathcal{F}\big[\mathrm{d}^n g(t)/\mathrm{d}t^n\big] = (i\omega)^n \mathcal{F}\big[h(t)\big]$$

- Differentiation transformed into multiplication by $(i\omega)$

Also convolution can be transformed into multiplication

$$\rightsquigarrow \quad \mathcal{F}\big[g(t) \star h(t)\big] = \mathcal{F}\big[g(t)\big]\mathcal{F}\big[h(t)\big]$$

- This is known as the convolution theorem[2]

[2] Convolution is defined as

$$g(t) \star (t) = \int_{-\infty}^{\infty} \mathrm{d}\tau g(t-\tau)h(\tau).$$

Fourier transforms (cont.)

The above properties require that the initial conditions are zero

- Not an actual restriction

Fourier transforms (cont.)

Consider the spring model

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \nu^2 = w(t)$$

By taking the Fourier transform, we get

$$\rightsquigarrow \quad (i\omega)^2 X(i\omega) + \gamma(i\omega) X(i\omega) + \nu^2 X(i\omega) = W(i\omega)$$

- $X(i\omega)$ is the Fourier transform of $x(t)$
- $W(i\omega)$ is the Fourier transform of $w(t)$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Fourier transforms (cont.)

$$(i\omega)^2 X(i\omega) + \gamma(i\omega) X(i\omega) + \nu^2 X(i\omega) = W(i\omega)$$

We first solve for $X(i\omega)$, we get

$$X(i\omega) = \frac{W(i\omega)}{(i\omega)^2 + \gamma(i\omega)\nu^2}$$

We take the inverse-transform, we get

$$\rightsquigarrow \quad x(t) = \mathcal{F}^{-1}\Big[\frac{W(i\omega)}{(i\omega)^2 + \gamma(i\omega) + \nu^2}\Big]$$

This is the solution

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Fourier transforms (cont.)

For a general $w(t)$, we note that the RHS is a product

$$\frac{W(i\omega)}{(i\omega)^2 + \gamma(i\omega) + \nu^2} = \frac{1}{(i\omega)^2 + \gamma(i\omega) + \nu^2} W(i\omega) = H(i\omega) W(i\omega)$$

This product can be converted into a convolution

We compute the impulse response function

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\Big[\frac{1}{(i\omega)^2 + \gamma(i\omega) + \nu^2}\Big] \\ &= b^{-1} e^{(-at)} \sin(bt) u(t) \end{aligned}$$

- $\rightsquigarrow$ We have $a = \gamma/2$ and $b = \sqrt{\nu^2 - \gamma^2/4}$
- $\rightsquigarrow$ $u(t)$, the Heaviside step function

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Fourier transforms (cont.)

Then, we get the full solution

$$\rightsquigarrow \quad x(t) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, h(t-\tau) w(\tau)$$

We construct $x(t)$ by feeding the signal $w(t)$ through a linear system

- (a filter) with impulse responses $h(t)$

Fourier transforms (cont.)

We can use the Fourier transform for general LTI equations

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}\mathbf{x}(t) + \mathbf{L}\mathbf{w}(t)$$

By taking the Fourier transform, we get

$$\rightsquigarrow \quad (i\omega)\mathbf{X}(i\omega) = \mathbf{F}\mathbf{X}(i\omega) + \mathbf{L}\mathbf{W}(i\omega)$$

By solving for $\mathbf{X}(i\omega)$, we get

$$\rightsquigarrow \quad \mathbf{X}(i\omega) = \big[(i\omega)\mathbf{I} - \mathbf{F}\big]^{-1}\mathbf{L}\mathbf{W}(i\omega)$$

Fourier transforms (cont.)

$$\mathbf{X}(i\omega) = \big[(i\omega)\mathbf{I} - \mathbf{F}\big]^{-1}\mathbf{L}\mathbf{W}(i\omega)$$

We compare it with the solution

$$\mathbf{x}(t) = \mathbf{\Psi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{d}\tau \mathbf{\Psi}(t, \tau)\mathbf{L}(\tau)\mathbf{w}(\tau)$$

We obtain the useful identify

$$\rightsquigarrow \quad \mathcal{F}^{-1}\Big\{\big[(i\omega)\mathbf{I} - \mathbf{F}\big]^{-1}\Big\} = e^{(\mathbf{F}t)}u(t)$$

This is a valid way of computing matrix exponentials

Laplace transforms

Transforms

Laplace transforms

Another transform commonly used for solving LTI equations

The **Laplace transform** of a function $f(t)$

$$\rightsquigarrow \quad F(s) = \mathcal{L}\big[f(t)\big](s) = \int_0^{\infty} \mathrm{d}t f(t) e^{(-st)}, \quad \text{for } t \geq 0$$

The corresponding inverse transform

$$\rightsquigarrow \quad f(t) = \mathcal{L}^{-1}[F(s)](t)$$

Numerical integration

Ordinary differential equations

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration

Consider the nonlinear differential equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}\big[\mathbf{x}(t), t\big], \quad \text{given } \mathbf{x}(t_0)$$

We cannot derive an analytical solution

- ⇝ We resort to a numerical solution
- ⇝ An approximation

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}[\mathbf{x}(t), t]$$

We integrate the equation from t to $t + \Delta t$

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \int_{t}^{t+\Delta t} \mathrm{d}\tau \mathbf{f}\big[\mathbf{x}(\tau), \tau\big]$$

We generate the solution at time steps $t_0, t_1 = t_0 + \Delta t, t_2 = t_0 + 2\Delta t, \cdots$

- We must know how to calculate the integral

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

$$\begin{aligned}
\mathbf{x}(t_0 + \Delta t) &= \mathbf{x}(t_0) + \int_{t_0}^{t_0+\Delta t} \mathrm{d}\tau \mathbf{f}\big[\mathbf{x}(\tau), \tau\big] \\
\mathbf{x}(t_0 + 2\Delta t) &= \mathbf{x}(t_0) + \int_{t_0+\Delta t}^{t_0+2\Delta t} \mathrm{d}\tau \mathbf{f}\big[\mathbf{x}(\tau), \tau\big] \\
\mathbf{x}(t_0 + 3\Delta t) &= \mathbf{x}(t_0) + \int_{t_0+2\Delta t}^{t_0+3\Delta t} \mathrm{d}\tau \mathbf{f}\big[\mathbf{x}(\tau), \tau\big] \\
\cdots &= \cdots
\end{aligned}$$

Different approximations of the integral lead to different numerical methods

Numerical integration (cont.)

Euler's method

Use the integral approximation

$$\int_{t}^{t+\Delta\tau} d\tau \mathbf{f}\big[\mathbf{x}(\tau),\tau\big] \approx \mathbf{f}\big[\mathbf{x}(t),t\big]\Delta t$$

Start from $\hat{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$ and divide the integration interval $[t_0, t]$

- $\rightsquigarrow$ n steps, $t_0 < t_1 < \cdots < t_n = t$
- $\rightsquigarrow$ $\Delta t = t_{k+1} - t_k$

At each step k, we approximate the solution

$$\rightsquigarrow \quad \hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}\big[\hat{\mathbf{x}}(t_k), t_k\big]\Delta t$$

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

The global order of a numerical method

It is defined to be the smallest exponent p such that if we numerically solve an ODE using $n = 1/\Delta t$ steps of length Δt, there is a constant K such that

$$\rightsquigarrow \big|\hat{\mathbf{x}}(t_n) - \mathbf{x}(t_n)\big| \leq K(\Delta t)^p$$

$\hat{\mathbf{x}}(t_n)$ is the approximation of $\mathbf{x}(t_n)$, the true solution

The error of integrating over $1/\Delta t$ steps is proportional to Δt

- The first discarded term is order $(\Delta t)^2$

Thus, the Euler method is order $p = 1$

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

We can improve this approximation by using a trapezoidal approximation

$$\int_{t}^{t+\Delta t} d\tau \mathbf{f}[\mathbf{x}(\tau),\tau] \approx \frac{\Delta t}{2}\{\mathbf{f}[\mathbf{x}(t),t] + \mathbf{f}[\mathbf{x}(t+\Delta t), t+\Delta t]\}$$

The resulting approximation integration rule

$$\rightsquigarrow \quad \mathbf{x}(t_{k+1}) \approx \mathbf{x}(t_k) + \frac{\Delta t}{2}\Big\{\mathbf{f}\big[\mathbf{x}(t_k), t_k\big] + \mathbf{f}\big[\mathbf{x}(t_{k+1}), t_{k+1}\big]\Big\}$$

This is an implicit recursion rule [$\mathbf{x}(t_{k+1})$ appears also on the RHS]

- We must solve a nonlinear system of equation to use this rule
- (At each iteration step, heavy for large $\mathbf{x}$)

Ordinary differential equations

UFC/DC
SA (CK0191)
2018.1

General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

Heun's method

Replace the the RHS of the solution with its Euler's approximation

Start from $\hat{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$ and divide the integration interval $[t_0, t]$

- $\rightsquigarrow$ n steps, $t_0 < t_1 < \cdots < t_n = t$
- $\rightsquigarrow$ $\Delta t = t_{k+1} - t_k$

At each step k, we approximate the solution

$$\rightsquigarrow \quad \tilde{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}\big[\hat{\mathbf{x}}(t_k), t_k\big]\Delta t$$

$$\rightsquigarrow \quad \hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \frac{\Delta t}{2}\Big\{\mathbf{f}\big[\hat{\mathbf{x}}(t_k), t_k\big] + \mathbf{f}\big[\tilde{\mathbf{x}}(t_{k+1}), t_{k+1}\big]\Big\}$$

The method has global order $p = 2$

Numerical integration (cont.)

Another useful class of methods are the Runge-Kutta methods

- We consider the classical 4-th order case

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

Runge-Kutta method (4-th order)

Start from $\hat{\mathbf{x}}(t_0) = \mathbf{x}(t_0)$ and divide the integration interval $[t_0, t]$

$\rightsquigarrow$ n steps, $t_0 < t_1 < \cdots < t_n = t$

$\rightsquigarrow$ $\Delta t = t_{k+1} - t_k$

At each step k, we approximate the solution

$$\begin{aligned}
\Delta\mathbf{x}_k^1 &= \mathbf{f}\big[\hat{\mathbf{x}}(t_k), t_k\big]\Delta t\\
\Delta\mathbf{x}_k^2 &= \mathbf{f}\big[\hat{\mathbf{x}}(t_k) + \frac{\Delta\mathbf{x}_k^1}{2}, t_k + \frac{\Delta t}{2}\big]\Delta t\\
\Delta\mathbf{x}_k^3 &= \mathbf{f}\big[\hat{\mathbf{x}}(t_k) + \frac{\Delta\mathbf{x}_k^2}{2}, t_k + \frac{\Delta t}{2}\big]\Delta t\\
\Delta\mathbf{x}_k^4 &= \mathbf{f}\big[\hat{\mathbf{x}}(t_k) + \Delta\mathbf{x}_k^3, t_k + \Delta t\big]\Delta t
\end{aligned}$$

$$\rightsquigarrow \quad \hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \frac{1}{6}(\Delta\mathbf{x}_k^1 + 2\Delta\mathbf{x}_k^2 + 2\Delta\mathbf{x}_k^3 + \Delta\mathbf{x}_k^4)$$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

The method can be derived by writing the Taylor expansion for the solution

- Select coefficient so that lower-order terms cancel out

The method has global order $p = 4$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Numerical integration (cont.)

There is a wide class of methods for integrating ordinary differential forms

The methods that we have overviewed have fixed step length

- There exists various variable step size methods

Picard-Lindelöf theorem

Ordinary differential equations

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Picard-Lindelöf theorem

It is important to know whether a solution to a ODE exists and is unique

We consider a general equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}\big[\mathbf{x}(t), t\big]$$

- Function $\mathbf{f}(\mathbf{x}(t), t)$ is given

Suppose that function $f \mapsto \mathbf{f}\big[\mathbf{x}(t), t\big]$ is integrable in the Reimann sense

- We can integrate both sides of the equation from t_0 to t

$$\rightsquigarrow \quad \mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathrm{d}\tau \mathbf{f}\big[\mathbf{x}(\tau), \tau\big]$$

The identity can be used to find approximate solutions

- **Picard's iteration**

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Picard-Lindelöf theorem (cont.)

Picard's algorithm

Start with an initial guess $\varphi_0(t) = \mathbf{x}_0$

Then, compute the approximations $\varphi_1 t$,$\varphi_2(t)$,$\varphi_3(t)$,$\ldots$,

$$\rightsquigarrow \quad \varphi_{n+1}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathrm{d}\tau \mathbf{f}\big[\varphi_n(t), t\big]$$

- Same recursion used for linear differential equations

The procedure converges to the unique (around $t = t_0$) solution

$$\rightsquigarrow \quad \lim_{n\rightarrow\infty} \varphi_n(t) = \mathbf{x}(t)$$

$\mathbf{f}(\mathbf{x}, t)$ must be continuous in $\mathbf{x}$ and t, and Lipschitz continuous in $\mathbf{x}$

Ordinary differential equations
UFC/DC
SA (CK0191)
2018.1
General concepts
Origins
Definitions
Solution
Generalities
Linear and time-invariant
General linear
Transforms
Fourier transforms
Laplace transforms
Numerical integration
Picard-Lindelöf theorem

Picard-Lindelöf theorem (cont.)

The Picard-Lindelöf theorem

Under the above continuity conditions, the differential equation has a solution and that solution is unique in a certain interval around $t = t_0$