Signals and distributions

Stochastic algorithms

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Canonical signals

Signals and distributions

Unit step

We describe some **signals** or **functions** in the real variable t, time

$$f : \mathcal{R} \to \mathcal{C}$$

Such signals or functions are often discontinuous

- **Distribution**

A generalisation of function/signal

Unit step

Canonical signals

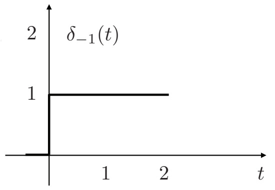
Unit step

Definition

Unit step

The unit step, denoted as $\delta_{-1}(t)$, is a function

$$\delta_{-1}(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases} \quad (1)$$



The function is continuous over the domain, except in the origin

- Discontinuity, size 1

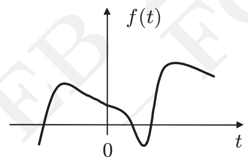
Unit step

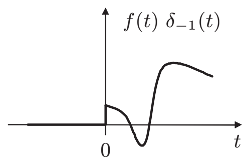
Consider a function $f : \mathcal{R} \rightarrow \mathcal{R}$

We can define the function

$$f(t)\delta_{-1}(t) = \begin{cases} 0, & \text{if } t < 0 \\ f(t), & \text{if } t \geq 0 \end{cases}$$

Graphically,





Ramps

Canonical signals

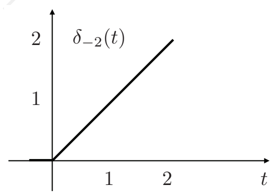
Ramps

Definition

Unit ramp

The integral of the unit step is called unit ramp, $\delta_{-2}(t)$

$$\delta_{-2}(t) = \int_{-\infty}^{t} \delta_{-1}(\tau)\mathrm{d}\tau = t\delta_{-1}(t) = \begin{cases} 0, & \text{if } t < 0 \\ t, & \text{if } t \geq 0 \end{cases} \quad (2)$$



Ramps (cont.)

Definition

Ramp functions

The family of ***ramp functions*** $\delta_{-k}(t)$ *can be be recursively defined for* $k > 2$

$$\delta_{-k}(t) = \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{k-1 \text{ times}} \delta_{-1}(\tau) d\tau = \frac{t^{k-1}}{(k-1)!}\delta_{-1}(t)$$
$$= \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{t^{k-1}}{(k-1)!}, & \text{if } t \geq 0 \end{cases}$$

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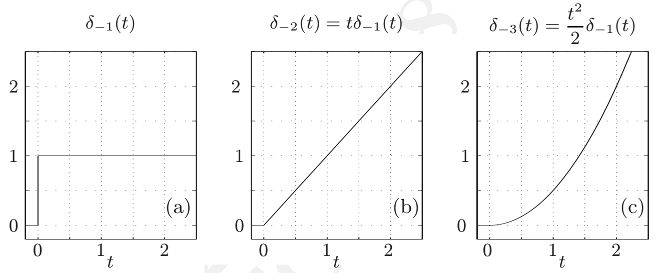
Ramps (cont.)



- **Quadratic ramp**, $k = 3$

$$\rightsquigarrow \quad \delta_{-3}(t) = \frac{t^2}{2!}\delta_{-1}(t)$$

- **Cubic ramp**, $k = 4$

$$\rightsquigarrow \quad \delta_{-4}(t) = \frac{t^3}{3!}\delta_{-1}(t)$$

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Ramps (cont.)

Definition

Exponential ramp

A generalisation of the ramp function is the ***exponential ramp****, or* ***cisoid***

It is defined in terms of two parameters $k \in \mathcal{N}$ *and* $a \in \mathcal{C}$

$$\frac{t^k}{k!}e^{at}\delta_{-1}(t) = \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{t^k}{(k)!}e^{at}, & \text{if } t \geq 0 \end{cases}$$

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Ramps (cont.)

Particular cases that can be generated from the exponential ramp

- $a = 0$ and $k = 0$, the unit ramp
- $a = 0$ and $k = 1, 2, \cdots$, the family of ramp functions

Linear combinations of ramps can be used for polynomial functions

$$c_2 t^2 + c_1 t + c_0$$

- $k = 0$ and $a \in \mathcal{R}$, exponential function e^{at}
- $k = 0$ and $a = j\omega \in \mathcal{I}$, a linear combinations of exponential ramps can be used to describe sinusoidal functions

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$
$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Impulse

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Impulse

We can extend the family of canonical signals

We consider the derivatives of the unit step

- The results of classical calculus cannot be used for the purpose
- The derivative of a discontinuous function is not defined

We can generalise the concept of function

- The distribution

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Impulse (cont.)

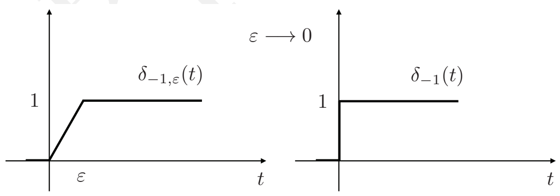
Let $\varepsilon > 0$ be some positive scalar

Define the function $\delta_{-1,\varepsilon}(t)$

$$\delta_{-1,\varepsilon}(t) = \begin{cases} 0, & \text{if } t < 0 \\ t/\varepsilon, & \text{if } t \in [0, \varepsilon) \\ 1, & \text{if } t \geq \varepsilon \end{cases}$$

This function is understood as a continuous approximation of the unit step

$$\lim_{\varepsilon \to 0} \delta_{-1,\varepsilon}(t) = \delta_{-1}(t)$$



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Impulse (cont.)

Definition

Finite impulse

Function $\delta_{-1,\varepsilon}(t)$ is continuous

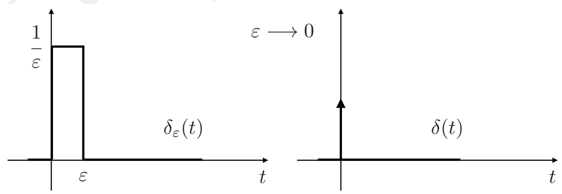
⇝ *It possesses a derivative*

$$\delta_{\varepsilon}(t) = \frac{d}{dt}\delta_{-1,\varepsilon} = \begin{cases} 1/\varepsilon, & \text{if } t \in [0, \varepsilon) \\ 0, & \text{otherwise} \end{cases}$$

Function $\delta_{\varepsilon}(t)$ is denoted as ***finite impulse*** *of base ε*

It is a rectangle with base ε and with height $1/\varepsilon$

- Area equal to 1, whatever the value of ε



Impulse (cont.)

We can define the derivative of the unit step

- **Unit impulse** or **Dirac function**

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t) = \frac{d}{dt}\lim_{\varepsilon\to 0}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon\to 0}\frac{d}{dt}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon\to 0}\delta_{\varepsilon}(t)$$

Such a definition is not formally correct in the sense of the classical calculus

- It is valid only if we accept the generalisation of a function
- (According to the distribution theory)

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Impulse (cont.)

The impulse $\delta(t)$ is not a function, it is a distribution

The following properties hold

- $\delta(t)$ is equal to zero everywhere except in the origin

$$\delta(t) = 0, \quad \text{if } t \neq 0$$

- $\delta(t)$ is equal to infinity in the origin

$$\delta(t) = \infty, \quad \text{if } t = 0$$

- The area of $\delta(t)$ is equal to 1

$$\int_{-\infty}^{+\infty}\delta(t)\,dt = \int_{0^-}^{0^+}\delta(t)\,dt = 1$$

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Impulse (cont.)

Theorem

Let $f(t)$ be some continuous function in $t = 0$

- *The product of $f(t)$ and the impulse $\delta(t)$*

$$\rightsquigarrow \quad f(t)\delta(t) = f(0)\delta(t)$$

Let $f(t)$ be some continuous function in $t = T$

- *The product of $f(t)$ and $\delta(t-T)$*

$$\rightsquigarrow \quad f(t)\delta(t-T) = f(T)\delta(t-T)$$

Proof

We have that $\delta(t) = 0$, for $t \neq 0$

The values taken by $f(t)$ for $t \neq 0$ are not significant (impulse is zero) ■

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Derivative of the impulse

Canonical signals

Derivative of the impulse

We can define higher-order derivatives of the impulse

- We use the limit reasoning

Derivative of the impulse

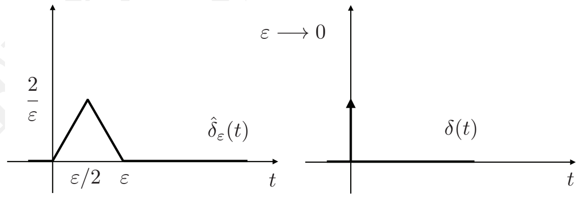
Definition

Consider the function $\hat{\delta}_\varepsilon(t)$

$$\hat{\delta}_\varepsilon(t) = \begin{cases} 4t/\varepsilon^2, & \textit{if } t \in [0, \varepsilon/2) \\ 4/\varepsilon - 4t/\varepsilon^2, & \textit{if } t \in [\varepsilon/2, \varepsilon) \\ 0, & \textit{otherwise} \end{cases}$$

The impulse can be re-defined

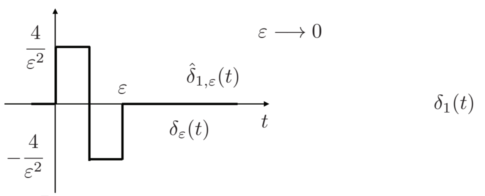
$$\delta(t) = \lim_{\varepsilon \to 0} \hat{\delta}_\varepsilon(t)$$



Derivative of the impulse (cont.)

We define the first-order derivative of the impulse

$$\delta_1(t) = \frac{d}{dt}\delta(t) = \frac{d}{dt}\lim_{\varepsilon \to 0} \hat{\delta}_\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{d}{dt}\delta_{1,\varepsilon}(t)$$



Derivative of the impulse (cont.)

The higher-order $(k > 1)$ derivatives of the impulse

$$\delta_k(t) = \frac{d^k}{dt^k}\delta(t) = \frac{d}{dt}\delta_{k-1}(t)$$

The family of canonical signals

Canonical signals

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The family of canonical signals

For $k \in \mathcal{Z}$, we can define a family of canonical signals, $\delta_k(t)$

- $\delta_0(t) = \delta(t)$, the impulse ($k = 0$)
- $k < 0$, the integrals of the impulse
- $k > 0$, the derivatives of the impulse

Such signals are linearly independent

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The family of canonical signals (cont.)

Definition

Linear dependence of scalar functions

Consider a set of scalar real functions $f_1(t), f_2(t), \dots, f_n(t)$, $f_i(t) : \mathcal{R} \rightarrow \mathcal{R}$

*Such functions are said to be **linearly dependent** over the interval $[t_1, t_2]$, if and only if there exist a set of real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ that are not all equal to zero and such that*

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots + \alpha_n f_n(t) = 0, \quad \forall t \in [t_1, t_2]$$

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The family of canonical signals (cont.)

Consider the function $f(t) = \sum_{k=-\infty}^{\infty} a_k \delta_k(t)$

If such a function is identically null over an interval $[a, b]$ with $a \neq b$, then we have that $a_k = 0$ for all $k \in \mathcal{Z}$

Derivatives of a discontinuous function

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Derivatives of a discontinuous function

We can formally calculate the derivative of discontinuous functions

- The theory of distributions

Such discontinuous signals/functions are common in systems analysis

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Derivatives of a discontinuous function (cont.)

Let $f(t)$ be a continuous function

We are interested in calculating the derivative of function $f(t)\delta_{-1}(t)$

- If $f(0) \neq 0$, then $f(t)\delta_{-1}(t)$ has a discontinuity in $t = 0$

The first-order derivative,

$$\begin{aligned}\frac{d}{dt}f(t)\delta_{-1}(t) &= \Big[\frac{d}{dt}f(t)\Big]\delta_{-1}(t) + f(t)\Big[\frac{d}{dt}\delta_{-1}(t)\Big] \\ &= \dot{f}(t)\delta_{-1}(t) + f(0)\delta(t)\end{aligned}$$

It is the first-order derivative of the original function multiplied by $\delta_{-1}(t)$, plus the impulse at the origin multiplied by $f(0)$

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Derivatives of a discontinuous function (cont.)

The second-order derivative,

$$\begin{aligned}\frac{d^2}{dt^2}f(t)\delta_{-1}(t) &= \Big[\frac{d}{dt}\dot{f}(t)\Big]\delta_{-1}(t) + f(t)\Big[\frac{d}{dt}\delta_{-1}(t)\Big] + f(0)\Big[\frac{d}{dt}\delta(t)\Big] \\ &= \ddot{f}(t)\delta_{-1}(t) + \dot{f}(0)\delta(t) + f(0)\delta_1(t)\end{aligned}$$

It is the second-order derivative of the original function multiplied by δ_{-1}, plus the impulse at the origin multiplied by $\dot{f}(0)$, plus $\delta_1(t)$ times $f(0)$

Derivatives of a discontinuous function (cont.)

Higher-order derivatives are calculated analogously,

$$\begin{aligned}\frac{d^k}{dt^k} f(t)\delta_{-1}(t) &= f^{(k)}\delta_1(t) + f^{(k-1)}(0)\delta(t) + \cdots + f(0)\delta_{k-1}(t)\\ &= f^{(k)}(t)\delta_{-1}(t) + \sum_{i=0}^{k-1} f^{(i)}(0)\delta_{k-1-i}(t)\end{aligned}$$

Derivatives of a discontinuous function (cont.)

Example

Consider the function

$$f(t) = \cos(t)\delta_{-1}(1)$$

We are interested in its derivatives

The first-order derivative,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\cos t\delta_{-1}(t) &= \Big[\frac{\mathrm{d}}{\mathrm{d}t}\cos(t)\Big] + \sin(0)\delta_t + \cos(0)\delta_1(t)\\ &= -\sin(t)\delta_{-1}(t) + \delta_1(t)\end{aligned}$$

The second-order derivative,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(t)\delta_{-1}(t) &= \Big[\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(t)\Big]\delta_{-1}(t) - \sin(0)\delta(t) + \cos(0)\delta_1(t)\\ &= -\cos(t)\delta_{-1}(t) + \delta_{-1}(t)\end{aligned}$$

■

Derivatives of a discontinuous function (cont.)

Example

Consider the cisoid function

$$f(t) = te^{(at)}\delta_{-1}(t)$$

We are interested in its derivatives

The first-order derivative,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} te^{(at)}\delta_{-1}(t) &= e^{(at)}\delta_{-1}(t) + ate^{(at)} + [te^{(at)}]_{t=0}\delta(t)\\ &= (1-at)e^{(at)}\delta_{-1}(t)\end{aligned}$$

The second-order derivative,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2} &= ae^{(at)}\delta_{-1}(t) + a(1+at)e^{(at)}\delta_{-1}(t) + [(1+at)e^{(at)}]_{t=0}\delta(t)\\ &= (2a + a^2 t)e^{(at)}\delta_{-1}(t) + \delta(t)\end{aligned}$$

■

Convolution integrals

Signals and distributions

Convolution integrals

Definition

Convolution

Consider the two functions

$$f, g : \mathcal{R} \to \mathcal{C}$$

The **convolution** *of f with g is a function $h : \mathcal{R} \to \mathcal{C}$ in the real variable t,*

$$h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau$$

Function $h(t)$ is built by using the operator **convolution integral**, $\star$

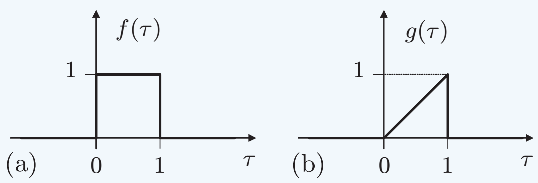
Convolution integrals (cont.)

Example

Consider the two functions

$$f(\tau) = \begin{cases} 1, & \text{if } \tau \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$g(\tau) = \begin{cases} \tau, & \text{if } \tau \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$



We want to calculate

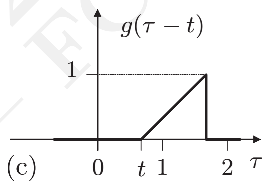
Convolution integrals (cont.)

To calculate

$$g(\tau - t) = \begin{cases} \tau - t, & \text{if } \tau \in [t, t+1] \\ 0, & \text{otherwise} \end{cases}$$

we need to shift $g(\tau)$ by a quantity t

- $t > 0$, to the right
- $t < 0$, to the left

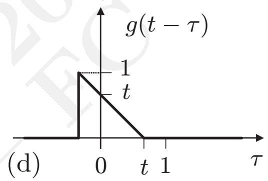


Convolution integrals (cont.)

To calculate

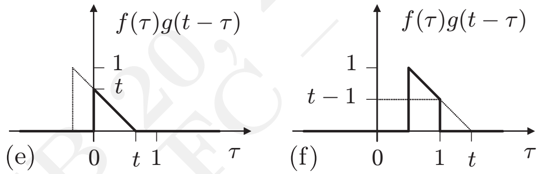
$$g(t - \tau) = \begin{cases} t - \tau, & \text{if } \tau \in [t - 1, t] \\ 0, & \text{otherwise} \end{cases}$$

we need to flip $g(\tau)$ around $\tau = t$



Convolution integrals (cont.)

To calculate $f(\tau)g(t-\tau)$



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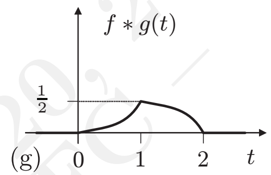

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Convolution integrals (cont.)

Theorem

The convolution operator is commutative

$$f \star g(t) = g \star f(t)$$

Proof

Let $\rho = t - \tau$, then write

$$f \star g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{+\infty} f(t-\rho)g(\rho)\mathrm{d}\rho = g \star f(t)$$

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Convolution integrals (cont.)

Definition

Consider the two functions $f, g : \mathbb{R} \to \mathbb{C}$

Let their derivatives be

$$\dot{f}(t) = \frac{d}{dt} f(t)$$

$$\dot{g}(t) = \frac{d}{dt} g(t)$$

Let their integrals be

$$\mathcal{F}(t) = \int_{-\infty}^{t} f(\tau) d\tau$$

$$\mathcal{G}(t) = \int_{-\infty}^{t} g(\tau) d\tau$$

Convolution integrals (cont.)

The following statements are true

(1) The derivative of the convolution between two functions is given by the convolution of one function with the derivative of the other function

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$$

(2) The integral of the convolution between two functions is given by the convolution of one function with the integral of the other function

$$\int_{-\infty}^{t} f \star g(\tau)\mathrm{d}\tau = f \star \mathcal{G}(t) = \mathcal{F} \star g(t)$$

(3) The integral of a convolution between two function does not change if one of the two operands is derived and the other one is integrated

$$f \star g(t) = \mathcal{F} \star \dot{g}(t) = \dot{f} \star \mathcal{G}(t)$$

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Convolution integrals (cont.)

Proof

To demonstrate (1), observe that we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{+\infty} f(\tau)\frac{\mathrm{d}}{\mathrm{d}t}g(t-\tau)\mathrm{d}\tau$$
$$= \int_{-\infty}^{+\infty} f(\tau)\dot{g}(t-\tau)\mathrm{d}\tau = f \star \dot{g}(t)$$

Because of the commutative property $f \star g(t) = g \star f(t)$, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = \frac{\mathrm{d}}{\mathrm{d}t} g \star f(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t} f(t-\tau)g(\tau)\mathrm{d}\tau$$
$$= \int_{-\infty}^{+\infty} \dot{f}(t-\tau)g(\tau)\mathrm{d}\tau = g \star \dot{f}(t) = \dot{f} \star g(t)$$

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Convolution integrals (cont.)

To demonstrate (2) where the three functions are identical, we use (1)

Observe that all three functions when evaluated for $t = -\infty$ are null

- Whereas their derivatives are equal, for all values of t

This is because of the definition of integral

$$\frac{d}{dt}\int_{-\infty}^{0} f * g(\tau)d\tau = f * g(t)$$

And, because

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star \mathcal{G}(t) = f \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}\Big](t) = f \star g(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F} \star g(t) = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\Big](t) = f \star g(t)$$

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Convolution integrals (cont.)

To demonstrate (3), we use (1) again

$\mathcal{F} \star \dot{g}(t)$ is obtained from (1)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F} \star g(t) = \mathcal{F} \star \Big[\frac{\mathrm{d}}{\mathrm{d}t} g\Big](t) = \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\Big] \star g(t) \quad \rightsquigarrow \quad \mathcal{F} \star \dot{g}(t) = f \star g(t)$$

$\dot{f} * \mathcal{G}(t)$ is obtained by differentiating $f \star \mathcal{G}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} f \star \mathcal{G}(t) = f \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}\Big](t) = \Big[\frac{\mathrm{d}}{\mathrm{d}t} f\Big] \star \mathcal{G}(t) \quad \rightsquigarrow \quad f \star g(t) = \dot{f} \star \mathcal{G}(t)$$

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Theorem

Convolution with the impulse

Consider a function $f : \mathcal{R} \rightarrow \mathcal{R}$ continuous in t

We have,

$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$$

For any interval (t_a, t_b) containing t, we have

$$f(t) = \int_{t_a}^{t_b} f(\tau)\delta(t-\tau)d\tau$$

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Convolution with canonical signals (cont.)

Proof

Observe that $\delta(t-\tau) = \delta(\tau - t)$ is an impulse centred in $\tau = t$

Thus,

$$\int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} \underbrace{f(t)\delta(t-\tau)}_{f(t)\delta(t-T)=f(T)\delta(t-T)} \mathrm{d}\tau$$
$$= f(t) \underbrace{\int_{-\infty}^{+\infty} \delta(t-\tau)\mathrm{d}\tau}_{\int_{-\infty}^{+\infty}\delta(t)\mathrm{d}t = \int_{0^-}^{0^+}\delta(t)dt = 1} = f(t)$$

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Convolution with canonical signals (cont.)

Theorem

Consider a function $f : \mathcal{R} \rightarrow \mathcal{R}$ continuous with k continuous derivatives

We have,

$$\frac{d^k}{dt^k} f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta_k(t-\tau)d\tau$$

■

Proof

Observe that $f(t) = f \star \delta(t)$

By repeatedly differentiating and using that $\frac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta(t) = f \star \Big[\frac{\mathrm{d}}{\mathrm{d}t}\delta\Big](t) = f \star \delta_1(t) \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta_1(t) = f \star \delta_2(t) \\
\cdots &= \cdots \\
\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta_{k-1}(t) = f \star \delta_k(t)
\end{aligned}$$

■