

Single state var

Single state var  
and single input

Multiple states  
and multiple  
inputs



Aalto University

# Linearisation of nonlinear state-space models

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# Linearisation of nonlinear state-space models

Many dynamical models used to describe processes in chemical engineering are given as a set of first-order ordinary differential equations, the equations are very often nonlinear

- From the application of material and energy conservation laws

A body of commonly used techniques for system analysis and control uses linear models

- To access such a technology we need to simplify common process models
- This will require approximating the general state-space representation
- The model approximations of our interest are Jacobian linearisation

$$\underbrace{\begin{cases} \dot{x}(t) = f(x(t), u(t) | \theta_x) \\ y(t) = g(x(t), u(t) | \theta_y) \end{cases}}_{\text{Nonlinear dynamics and read-out}} \rightsquigarrow \underbrace{\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}}_{\text{Linear dynamics and read-out}}$$

The main idea behind the linearisation of nonlinear process models in state-space form

↪ Approximate function  $f$  (nonlinear) with matrices  $A$  and  $B$

↪ Approximate function  $g$  (nonlinear) with matrices  $C$  and  $D$

# Linearisation of nonlinear state-space models

We discuss how to determine a linear approximation of a nonlinear state-space model

- ↪ We study a number of cases, of increasing complexity
- ↪ As a result, we will be able to linearise any<sup>1</sup> model

$$\underbrace{\begin{cases} \dot{x}(t) = f(x(t), u(t)|\theta_x) \\ y(t) = g(x(t), u(t)|\theta_y) \end{cases}}$$

Nonlinear dynamics and read-out

In general, we have a general state-space model with variables of arbitrary dimension

- ↪  $x(t) \in \mathcal{R}^{N_x}$
- ↪  $u(t) \in \mathcal{R}^{N_u}$
- ↪  $y(t) \in \mathcal{R}^{N_y}$

We start with  $N_x = 1$  (one state),  $N_u = 0$  (no inputs) and  $N_y = 1$  (one measurement)

- Then, we will add complexity (we add more variables)

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<sup>1</sup>We only require that model functions  $f$  and  $g$  are continuous and differentiable.

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inputs

## A single state variable, no inputs

$$\dot{x}(t) = f(x(t)), \quad \text{with } x(t) \in \mathcal{R}$$

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We suppose that function  $f(x(t))$  can be approximated by a Taylor series expansion

- We are interested in an approximation around an fixed point,  $x(t) = x_{SS}$
- At the steady-state point  $x_{SS}$ , time variations are zero  $\dot{x}(t) = 0$
- Thus, we also have that  $f(x_{SS}) = 0$ , whatever the time  $t$

We can perfectly represent function  $f(x)$  by using an infinite Taylor series expansion

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{df(x)}{dx}\bigg|_{x_{SS}}(x - x_{SS})}_{\text{linear in } x} + \underbrace{\frac{1}{2} \frac{d^2 f(x)}{dx^2}\bigg|_{x_{SS}}(x - x_{SS})^2}_{\text{quadratic in } x} + \underbrace{\mathcal{O}(x^3)}_{\text{H.O. terms}}$$

The expansion is a sum of polynomials of  $x$ , with the derivatives of  $f$  as coefficients

- Given this representation of  $f$ , we want to use it to approximate  $f$

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Note that steady-state, stationary, fixed, equilibrium points are all equivalent terms

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## A single state variable, no inputs (cont.)

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{df(x)}{dx}\bigg|_{x_{SS}} (x - x_{SS})}_{\text{linear in } x} + \underbrace{\frac{1}{2} \frac{d^2 f(x)}{dx^2}\bigg|_{x_{SS}} (x - x_{SS})^2}_{\text{quadratic in } x} + \underbrace{\mathcal{O}(x^3)}_{\text{H.O. terms}}$$

Suppose that we are interested in an approximation based only on first-order terms

- We accept to neglect (truncate out) second- and higher-order terms

$$f(x) = \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\frac{df(x)}{dx}\bigg|_{x_{SS}} (x - x_{SS})}_{\text{linear in } x} + \underbrace{\mathcal{O}(x^2)}_{\text{quadratic and H.O. terms}}$$

After truncation, we get an approximation of  $f$  which is a linear function of  $x$

$$f(x) \approx f(x_{SS}) + \frac{df(x)}{dx}\bigg|_{x_{SS}} (x - x_{SS})$$

---

Note the complete expression of  $f(x(t))$  with explicit dependencies also wrt time  $t$

$$f(x(t)) \approx f(x_{SS}) + \frac{df(x(t))}{dx}\bigg|_{x_{SS}} (x(t) - x_{SS})$$

## A single state variable, no inputs (cont.)

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$$f(x) \approx \underbrace{f(x_{SS})}_{\text{constant}} + \underbrace{\left. \frac{df(x)}{dx} \right|_{x_{SS}}}_{\text{linear in } x} (x - x_{SS})$$

Because  $x_{SS}$  is chosen to be a fixed-point, we have  $\left. \frac{dx(t)}{dt} \right|_{x_{SS}} = \dot{x}(t) \Big|_{x_{SS}} = f(x_{SS}) = 0$

Thus, we can write

$$f(x) \approx \underbrace{\cancel{f(x_{SS})}}_{=0} + \left. \frac{df(x)}{dx} \right|_{x_{SS}} (x - x_{SS})$$

$$\rightsquigarrow \dot{x}$$

## A single state variable, no inputs (cont.)

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$$\dot{x}(t) \approx \left. \frac{df(x(t))}{dx} \right|_{x_{SS}} (x(t) - x_{SS})$$

We can now introduce a **perturbation** or **deviation variable**  $x'(t) = x(t) - x_{SS}$

- Variable  $x'(t)$  encodes how far state variable  $x(t)$  is from steady-state  $x_{SS}$
- (And, because variable  $x(t)$  varies with time, also  $x'(t)$  varies)

Therefore, we can also compute the time-derivative of perturbation variable  $x'(t)$

- It describes the rate of change of variable  $x'(t)$  with respect to time
- Or, equivalently, its dynamics

By differentiating  $x'(t)$  to get  $dx'(t)dt$ , we have

$$\begin{aligned} \frac{d(x(t) - x_{SS})}{dt} &= \frac{dx(t)}{dt} - \underbrace{\frac{dx_{SS}}{dt}}_{=0} \\ &= \dot{x}(t) \\ &\rightsquigarrow f(x(t)) \end{aligned}$$

Deviation/perturbation variables and state variables have identical dynamics

## A single state variable, no inputs (cont.)

We obtained that  $\frac{d(x(t) - x_{SS})}{dt} = f(x(t))$  and  $f(x(t)) \approx \left. \frac{df(x(t))}{dx} \right|_{x_{SS}} (x(t) - x_{SS})$

We can equate the two terms and write

$$\frac{d(x(t) - x_{SS})}{dt} = f(x(t)) \approx \underbrace{\left. \frac{df(x(t))}{dx} \right|_{x_{SS}}}_{\text{constant}} (x(t) - x_{SS})$$

We have derived the approximated state equation for the deviation variable  $x'(t)$

$$\dot{x}'(t) \approx \left. \frac{df(x(t))}{dx} \right|_{x_{SS}} (x(t) - x_{SS})$$

This is a linear time-invariant approximation of the (perturbed) state equation

$$\dot{x}'(t) = \alpha x'(t), \quad \text{with constant } \alpha = \left. \frac{df(x)}{dx} \right|_{x_{SS}} \in \mathcal{R}$$

We also know how solve it for some initial condition  $x'(0)$

$$x'(t) = e^{\alpha t} x'(0)$$

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# A single state and a single input variable

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$$\dot{x}(t) = f(x(t), u(t)), \quad \text{with } x, u \in \mathcal{R}$$

We assume that function  $f(x, u)$  can be approximated by a Taylor series expansion

- We are interested in an approximation around some fixed-point point

$$(x = x_{SS}, u = u_{SS})$$

- At fixed points  $(x = x_{SS}, u = u_{SS})$ , time-variations are zero  $\dot{x}(t) = 0$

↪ (The state remains fixed at  $x_{SS}$  as long as the input is fixed, at  $u_{SS}$ )

Because at steady-state there is no evolution, the right hand-side is zero

$$\rightsquigarrow f(x = x_{SS}, u = u_{SS}) = 0$$

## A single state and a single input variable

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Suppose that we can perfectly represent  $f(x, u)$  by using an infinite Taylor expansion

$$\begin{aligned}
 f(x, u) = & \underbrace{f(x_{SS}, u_{SS})}_{\text{constant}} + \underbrace{\frac{\partial f(x, u)}{\partial x} \Big|_{x_{SS}, u_{SS}} (x - x_{SS})}_{\text{linear in } x} + \underbrace{\frac{\partial f(x, u)}{\partial u} \Big|_{x_{SS}, u_{SS}} (u - u_{SS})}_{\text{linear in } u} \\
 & + \underbrace{\frac{\partial^2 f(x, u)}{\partial x^2} \Big|_{x_{SS}, u_{SS}} (x - x_{SS})^2}_{\text{quadratic in } x} + \underbrace{\frac{\partial^2 f(x, u)}{\partial u^2} \Big|_{x_{SS}, u_{SS}} (u - u_{SS})^2}_{\text{quadratic in } u} \\
 & + \underbrace{\frac{\partial^2 f(x, u)}{\partial x \partial u} \Big|_{x_{SS}, u_{SS}} (x - x_{SS})(u - u_{SS})}_{\text{quadratic}} \\
 & + \text{H.O. terms}
 \end{aligned}$$

Suppose that we are interested in an approximation based only on first-order terms

- By truncation, we accept to neglect second- and higher-order terms

## A single state and a single input variable (cont.)

Single state var

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We can again obtain a linear, first-order, approximation of function  $f$  by truncation

$$f(x, u) \approx f(x_{SS}, u_{SS}) + \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_{SS}, u_{SS}} (x - x_{SS}) + \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_{SS}, u_{SS}} (u - u_{SS})$$

---

For completeness, again note the explicit dependencies

$$\begin{aligned} f(x(t), u(t)) \approx f(x_{SS}, u_{SS}) &+ \left. \frac{\partial f(x(t), u(t))}{\partial x} \right|_{x_{SS}, u_{SS}} (x(t) - x_{SS}) \\ &+ \left. \frac{\partial f(x(t), u(t))}{\partial u} \right|_{x_{SS}, u_{SS}} (u(t) - u_{SS}) \end{aligned}$$

## A single state and a single input variable (cont.)

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Since  $(x_{SS}, u_{SS})$  is chosen to be a fixed-point, we have  $\left. \frac{dx(t)}{dt} \right|_{x_{SS}, u_{SS}} = f(x_{SS}, u_{SS}) = 0$

Thus, we can write

$$f(x, u) \approx \underbrace{f(x_{SS}, u_{SS})}_{=0} + \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_{SS}, u_{SS}} (x - x_{SS}) + \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_{SS}, u_{SS}} (u - u_{SS})$$

$\rightsquigarrow \dot{x}$

We can define again perturbation/deviation variables  $x' = x - x_{SS}$  and  $u' = u - u_{SS}$

By computing the time-derivative of the (perturbed) state variables, we get

$$\begin{aligned} \frac{d(x(t) - x_{SS})}{dt} &= \frac{dx(t)}{dt} - \underbrace{\frac{dx_{SS}}{dt}}_{=0} \\ &= \dot{x}(t) \\ &\rightsquigarrow f(x(t), u(t)) \end{aligned}$$

## A single state and a single input variable (cont.)

Single state var

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As a result, we have

$$\begin{aligned}
 \dot{x}'(t) &= \left. \frac{\partial f(x(t), u(t))}{\partial x} \right|_{x_{SS}, u_{SS}} (x(t) - x_{SS}) + \left. \frac{\partial f(x(t), u(t))}{\partial u} \right|_{x_{SS}, u_{SS}} (u(t) - u_{SS}) \\
 &= \underbrace{\left. \frac{\partial f(x(t), u(t))}{\partial x} \right|_{x_{SS}, u_{SS}}}_{\alpha} x'(t) + \underbrace{\left. \frac{\partial f(x(t), u(t))}{\partial u} \right|_{x_{SS}, u_{SS}}}_{\beta} u'(t) \\
 &= \alpha x'(t) + \beta u'(t)
 \end{aligned}$$

with constants

$$\rightsquigarrow \alpha = \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_{SS}, u_{SS}} \in \mathcal{R}$$

$$\rightsquigarrow \beta = \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_{SS}, u_{SS}} \in \mathcal{R}$$

# A single state, a single input, and a single output variable

Now, suppose that there exists also a single measurement variable,  $y(t) = g(x(t), u(t))$

Single state var

Single state var  
and single input

Multiple states  
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We treat function  $g(x, u)$  similarly, by using a Taylor expansion around  $(x_{SS}, u_{SS})$

- Then, we truncate the expansion to keep only first-order terms

$$g(x, u) \approx \underbrace{g(x_{SS}, u_{SS})}_{y_{SS}} + \underbrace{\left. \frac{\partial g(x, u)}{\partial x} \right|_{x_{SS}, u_{SS}}}_{\gamma} (x - x_{SS}) + \underbrace{\left. \frac{\partial g(x, u)}{\partial u} \right|_{x_{SS}, u_{SS}}}_{\delta} (u - u_{SS})$$

$y_{SS} = g(x_{SS}, u_{SS})$  is the fixed-point  $y_{SS}$  of the measurement (not necessarily zero!)

$$y(t) \approx g(x_{SS}, u_{SS}) + \left. \frac{\partial g(x(t), u(t))}{\partial x} \right|_{x_{SS}, u_{SS}} (x(t) - x_{SS}) + \left. \frac{\partial g(x(t), u(t))}{\partial u} \right|_{x_{SS}, u_{SS}} (u(t) - u_{SS})$$

And, we can also introduce a perturbation variable for the measurements, to get

$$\underbrace{y(t) - y_{SS}}_{y'(t)} = \gamma x'(t) + \delta u'(t)$$

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## Multiple state and multiple input variables

We can easily generalise the procedure to process models of arbitrary dimensionality

- That is, with an arbitrary number of state, input and output variables

Consider a system with two state variables  $x = (x_1, x_2)'$ , one input  $u$ , one output  $y$

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t), u(t)) \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t), u(t)) \\ y(t) = g(x_1(t), x_2(t), u(t)) \end{cases} \rightsquigarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1(t), x_2(t), u(t)) \\ f_2(x_1(t), x_2(t), u(t)) \end{bmatrix}$$

We can linearise this system by using truncated Taylor series expansions of  $f$  and  $g$

- Around a fixed point  $(x_{SS}, u_{SS}) = (\underbrace{(x_1^{SS}, x_2^{SS})}_{x_{SS}}, u_{SS})$

Note that now function  $f$  is vector-valued (two values), it is two functions  $f = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}$

- They need to be treated (linearised) individually

↪ With respect to each state variable

↪ With respect to the input variable

## Multiple state and multiple input variables (cont.)

We start with  $f_1(x_1(t), x_2(t), u(t))$ , then after expanding and truncating we obtain

$$\begin{aligned}
 f_1(x_1(t), x_2(t), u(t)) &= \underbrace{f_1(x_1^{SS}, x_2^{SS}, u^{SS})}_{\text{constant}} \\
 &+ \underbrace{\frac{\partial f_1(x_1(t), x_2(t), u(t))}{\partial x_1} \bigg|_{x_{SS}, u_{SS}} (x_1(t) - x_1^{SS})}_{\text{linear in } x_1} \\
 &+ \underbrace{\frac{\partial f_1(x_1(t), x_2(t), u(t))}{\partial x_2} \bigg|_{x_{SS}, u_{SS}} (x_2(t) - x_2^{SS})}_{\text{linear in } x_2} \\
 &+ \underbrace{\frac{\partial f_1(x_1(t), x_2(t), u(t))}{\partial u} \bigg|_{x_{SS}, u_{SS}} (u(t) - u_{SS})}_{\text{linear in } u} \\
 &\quad + \text{H.O. terms}
 \end{aligned}$$

At any fixed-point the constant term is equal to zero, thus  $f_1(x_1^{SS}, x_2^{SS}, u_{SS}) = 0$

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inputs



## Multiple state and multiple input variables (cont.)

Single state var

Single state var  
and single inputMultiple states  
and multiple  
inputsBy retaining only first-order terms, we get the linear approximation of function  $f(x, u)$ 

$$\begin{aligned} f_1(x_1(t), x_2(t), u(t)) \approx & \underbrace{\frac{\partial f_1(x_1(t), x_2(t), u(t))}{\partial x_1} \Big|_{x_{SS}, u_{SS}}}_{a_{11}} (x_1(t) - x_1^{SS}) \\ & + \underbrace{\frac{\partial f_1(x_1(t), x_2(t), u(t))}{\partial x_2} \Big|_{x_{SS}, u_{SS}}}_{a_{12}} (x_2(t) - x_2^{SS}) \\ & + \underbrace{\frac{\partial f_1(x_1(t), x_2(t), u(t))}{\partial u} \Big|_{x_{SS}, u_{SS}}}_{b_1} (u(t) - u_{SS}) \end{aligned}$$

## Multiple state and multiple input variables (cont.)

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We can rewrite the linearised (perturbed) first state equation more compactly, to get

$$f_1(x_1, x_2, u) = a_{11} \underbrace{(x_1 - x_1^{SS})}_{x'_1} + a_{12} \underbrace{(x_2 - x_2^{SS})}_{x'_2} + b_1 \underbrace{(u - u_{SS})}_{u'}$$

The constants are the partials of  $f_1$  with respect to  $x_1$ ,  $x_2$ , and  $u$ , at  $(x_{SS}, u_{SS})$

$$\rightsquigarrow a_{11} = \left. \frac{\partial f_1}{\partial x_1} \right|_{x_{SS}, u_{SS}}$$

$$\rightsquigarrow a_{12} = \left. \frac{\partial f_1}{\partial x_2} \right|_{x_{SS}, u_{SS}}$$

$$\rightsquigarrow b_1 = \left. \frac{\partial f_1}{\partial u} \right|_{x_{SS}, u_{SS}}$$

## Multiple state and multiple input variables

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Similarly for function  $f_2(x_1(t), x_2(t), u(t))$ , by Taylor expansion and truncation we get

$$\begin{aligned}
 f_2(x_1(t), x_2(t), u(t)) &= \underbrace{f_2(x_1^{SS}, x_2^{SS}, u^{SS})}_{\text{constant}} \\
 &+ \underbrace{\left. \frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial x_1} \right|_{x_{SS}, u_{SS}} (x_1(t) - x_1^{SS})}_{\text{linear in } x_1} \\
 &+ \underbrace{\left. \frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial x_2} \right|_{x_{SS}, u_{SS}} (x_2(t) - x_2^{SS})}_{\text{linear in } x_2} \\
 &+ \left. \frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial u} \right|_{x_{SS}, u_{SS}} (u(t) - u_{SS}) \\
 &\quad + \text{H.O. terms}
 \end{aligned}$$

Because at some fixed-point, we have again a zero constant term  $f_2(x_1^{SS}, x_2^{SS}, u_{SS}) = 0$

## Multiple state and multiple input variables (cont.)

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Again, by retaining only the first-order terms we get the second linear approximation

$$\begin{aligned}
 f_2(x_1(t), x_2(t), u(t)) \approx & \underbrace{\frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial x_1} \Big|_{x_{SS}, u_{SS}}}_{a_{21}} \underbrace{(x_1(t) - x_1^{SS})}_{x'_1} \\
 & + \underbrace{\frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial x_2} \Big|_{x_{SS}, u_{SS}}}_{a_{22}} \underbrace{(x_2(t) - x_2^{SS})}_{x'_2} \\
 & + \underbrace{\frac{\partial f_2(x_1(t), x_2(t), u(t))}{\partial u} \Big|_{x_{SS}, u_{SS}}}_{b_2} \underbrace{(u(t) - u_{SS})}_{u'}
 \end{aligned}$$

## Multiple state and multiple input variables (cont.)

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We can rewrite the linearised (perturbed) first state equation more compactly, to get

$$f_2(x_1, x_2, u) = a_{21} \underbrace{(x_1 - x_1^{SS})}_{x'_1} + a_{22} \underbrace{(x_2 - x_2^{SS})}_{x'_2} + b_2 \underbrace{(u - u_{SS})}_{u'}$$

The constants are the partials of  $f_2$  with respect to  $x_1$ ,  $x_2$ , and  $u$ , at  $(x_{SS}, u_{SS})$

$$\rightsquigarrow a_{21} = \left. \frac{\partial f_2}{\partial x_1} \right|_{x_{SS}, u_{SS}}$$

$$\rightsquigarrow a_{22} = \left. \frac{\partial f_2}{\partial x_2} \right|_{x_{SS}, u_{SS}}$$

$$\rightsquigarrow b_2 = \left. \frac{\partial f_2}{\partial u} \right|_{x_{SS}, u_{SS}}$$

## Multiple state and multiple input variables (cont.)

By collecting the linear approximation results for function  $f_1$  and  $f_2$ , we have

$$f_1(x_1, x_2, u) = a_{11} \underbrace{(x_1 - x_1^{SS})}_{x_1'(t)} + a_{12} \underbrace{(x_2 - x_2^{SS})}_{x_2'(t)} + b_1 \underbrace{(u - u_{SS})}_{u'(t)}$$

$$\begin{aligned} &= \frac{dx_1(t)}{dt} \\ &= \frac{d(x_1(t) - x_1^{SS})}{dt} = \dot{x}_1'(t) \end{aligned}$$

$$\rightsquigarrow \dot{x}_1'(t) = a_{11}x_1'(t) + a_{12}x_2'(t) + b_1u'(t)$$

$$\rightsquigarrow \dot{x}_2'(t) = a_{21}x_1'(t) + a_{22}x_2'(t) + b_2u'(t)$$

$$\begin{aligned} &= \frac{dx_2(t)}{dt} \\ &= \frac{d(x_2(t) - x_2^{SS})}{dt} = \dot{x}_2'(t) \end{aligned}$$

$$f_2(x_1, x_2, u) = a_{21} \underbrace{(x_1 - x_1^{SS})}_{x_1'(t)} + a_{22} \underbrace{(x_2 - x_2^{SS})}_{x_2'(t)} + b_2 \underbrace{(u - u_{SS})}_{u'(t)}$$

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## Multiple state and multiple input variables (cont.)

We can combine the equations, to get the linearised state-space model

$$\underbrace{\begin{bmatrix} \dot{x}_1'(t) \\ \dot{x}_2'(t) \end{bmatrix}}_{\dot{x}'(t)} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}}_{x'(t)} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_b u'(t)$$

In matrix form, we get the compact formulation

$$\dot{x}'(t) = Ax'(t) + bu'(t)$$

---

Remember that matrix  $A$  and  $b$  contain the partials of  $f$  with respect to  $x$  and  $u$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_1, x_2, u)}{\partial x_1} & \frac{\partial f_1(x_1, x_2, u)}{\partial x_2} \\ \frac{\partial f_2(x_1, x_2, u)}{\partial x_1} & \frac{\partial f_2(x_1, x_2, u)}{\partial x_2} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(x_1, x_2, u)}{\partial u} \\ \frac{\partial f_2(x_1, x_2, u)}{\partial u} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

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## Multiple state and multiple input variables (cont.)

We proceed similarly to linearise the read-out function  $g$  around the point  $(x_{SS}, u_{SS})$

In the case of a single output measurement  $y \in \mathcal{R}$ , we have

$$y(t) = g(x_1(t), x_2(t), u(t))$$

The first-order approximation of  $g(x, u)$ ,

$$\begin{aligned} g(x_1(t), x_2(t), u(t)) &\approx \underbrace{g(x_1^{SS}, x_2^{SS}, u_{SS})}_{y_{SS}} \\ &+ \underbrace{\frac{\partial g(x_1(t), x_2(t), u(t))}{\partial x_1} \Big|_{x_{SS}, u_{SS}}}_{c_1} (x_1(t) - x_1^{SS}) \\ &+ \underbrace{\frac{\partial g(x_1(t), x_2(t), u(t))}{\partial x_2} \Big|_{x_{SS}, u_{SS}}}_{c_2} (x_2(t) - x_2^{SS}) \\ &+ \underbrace{\frac{\partial g(x_1(t), x_2(t), u(t))}{\partial u} \Big|_{x_{SS}, u_{SS}}}_d (u(t) - u_{SS}) \end{aligned}$$



## Multiple state and multiple input variables (cont.)

Single state var

Single state var  
and single inputMultiple states  
and multiple  
inputs

$$\underbrace{y(t) - y_{SS}}_{y'(t)} = \underbrace{c_1 (x_1(t) - x_1^{SS})}_{x'_1(t)} + \underbrace{c_2 (x_2(t) - x_2^{SS})}_{x'_2(t)} + \underbrace{d (u(t) - u_{SS})}_{u'(t)}$$

We can again get a compact formulation,

$$y'(t) = \underbrace{\begin{bmatrix} c_1 & c_2 \end{bmatrix}}_C \underbrace{\begin{bmatrix} x'_1(t) \\ x'_2(t) \end{bmatrix}}_{x'(t)} + d u'(t)$$

---

Remember that matrix  $C$  and  $d$  contain the partials of  $g$  with respect to  $x$  and  $u$

$$C = \begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial g(x_1, x_2, u)}{\partial x_1} & \frac{\partial g(x_1, x_2, u)}{\partial x_2} \end{bmatrix}_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

$$d = \begin{bmatrix} d \end{bmatrix} = \left[ \frac{\partial g(x_1, x_2, u)}{\partial u} \right]_{x_1^{SS}, x_2^{SS}, u_{SS}}$$

## Multiple state and multiple input variables (cont.)

Single state var

Single state var  
and single inputMultiple states  
and multiple  
inputsFor the general case where  $x \in \mathcal{R}^{N_x}$ ,  $u \in \mathcal{R}^{N_u}$  and  $y \in \mathcal{R}^{N_y}$ , we have $\rightsquigarrow$  State-space equation,

$$\underbrace{\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_{N_x}(t) \end{bmatrix}}_{\dot{x}'(t)} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{N_x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}}{\partial x_1} & \cdots & \frac{\partial f_{N_x}}{\partial x_{N_x}} \end{bmatrix}}_{\substack{A \\ (N_x \times N_x)}} \underbrace{\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_{N_x}(t) \end{bmatrix}}_{x'(t)} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_{N_u}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}}{\partial u_1} & \cdots & \frac{\partial f_{N_x}}{\partial u_{N_u}} \end{bmatrix}}_{\substack{B \\ (N_x \times N_u)}} \underbrace{\begin{bmatrix} u'_1(t) \\ \vdots \\ u'_{N_u}(t) \end{bmatrix}}_{u'(t)}$$

$$= \underbrace{\begin{bmatrix} a_{1,1} & \cdots & a_{1,N_x} \\ \vdots & \ddots & \vdots \\ a_{N_x,1} & \cdots & a_{N_x,N_x} \end{bmatrix}}_{(N_x \times N_x)} \underbrace{\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_{N_x}(t) \end{bmatrix}}_{(N_x \times 1)} + \underbrace{\begin{bmatrix} b_{1,1} & \cdots & b_{1,N_u} \\ \vdots & \ddots & \vdots \\ b_{N_x,1} & \cdots & b_{N_x,N_u} \end{bmatrix}}_{(N_x \times N_u)} \underbrace{\begin{bmatrix} u'_1(t) \\ \vdots \\ u'_{N_u}(t) \end{bmatrix}}_{(N_u \times 1)}$$

## Multiple state and multiple input variables (cont.)

Single state var

Single state var  
and single inputMultiple states  
and multiple  
inputsFor general case where  $x \in \mathcal{R}^{N_x}$ ,  $u \in \mathcal{R}^{N_u}$  and  $y \in \mathcal{R}^{N_y}$ , we have $\rightsquigarrow$  Read-out map,

$$\begin{aligned}
 \underbrace{\begin{bmatrix} y'_1(t) \\ \vdots \\ y'_{N_y}(t) \end{bmatrix}}_{y'(t)} &= \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{N_x}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{N_y}}{\partial x_1} & \cdots & \frac{\partial g_{N_y}}{\partial x_{N_x}} \end{bmatrix}}_{\substack{C \\ (N_y \times N_x)}} \underbrace{\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_{N_x}(t) \end{bmatrix}}_{\substack{x'(t) \\ (N_x \times 1)}} \underbrace{\quad}_{SS} + \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_{N_u}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{N_y}}{\partial u_1} & \cdots & \frac{\partial g_{N_y}}{\partial u_{N_u}} \end{bmatrix}}_{\substack{D \\ (N_y \times N_u)}} \underbrace{\begin{bmatrix} u'_1(t) \\ \vdots \\ u'_{N_u}(t) \end{bmatrix}}_{\substack{u'(t) \\ (N_u \times 1)}} \underbrace{\quad}_{SS} \\
 &= \underbrace{\begin{bmatrix} c_{1,1} & \cdots & c_{1,N_x} \\ \vdots & \ddots & \vdots \\ c_{N_y,1} & \cdots & c_{N_y,N_x} \end{bmatrix}}_{(N_y \times N_x)} \underbrace{\begin{bmatrix} x'_1(t) \\ \vdots \\ x'_{N_x}(t) \end{bmatrix}}_{(N_x \times 1)} + \underbrace{\begin{bmatrix} d_{1,1} & \cdots & d_{1,N_u} \\ \vdots & \ddots & \vdots \\ d_{N_y,1} & \cdots & d_{N_y,N_u} \end{bmatrix}}_{(N_y \times N_u)} \underbrace{\begin{bmatrix} u'_1(t) \\ \vdots \\ u'_{N_u}(t) \end{bmatrix}}_{(N_u \times 1)}
 \end{aligned}$$