Aalto University
School of Chemical
Engineering

Approximations to the solution of the Kushner-Stratonovich equation for the stochastic chemostat

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The chemostat

F_{in}: influent flow rate

F_{out}: outgoing flow rate

V: volume of the vessel

s_{in}: incoming flow of substrate

b: outgoing concentration of biomass

s: outgoing concentration of substrate

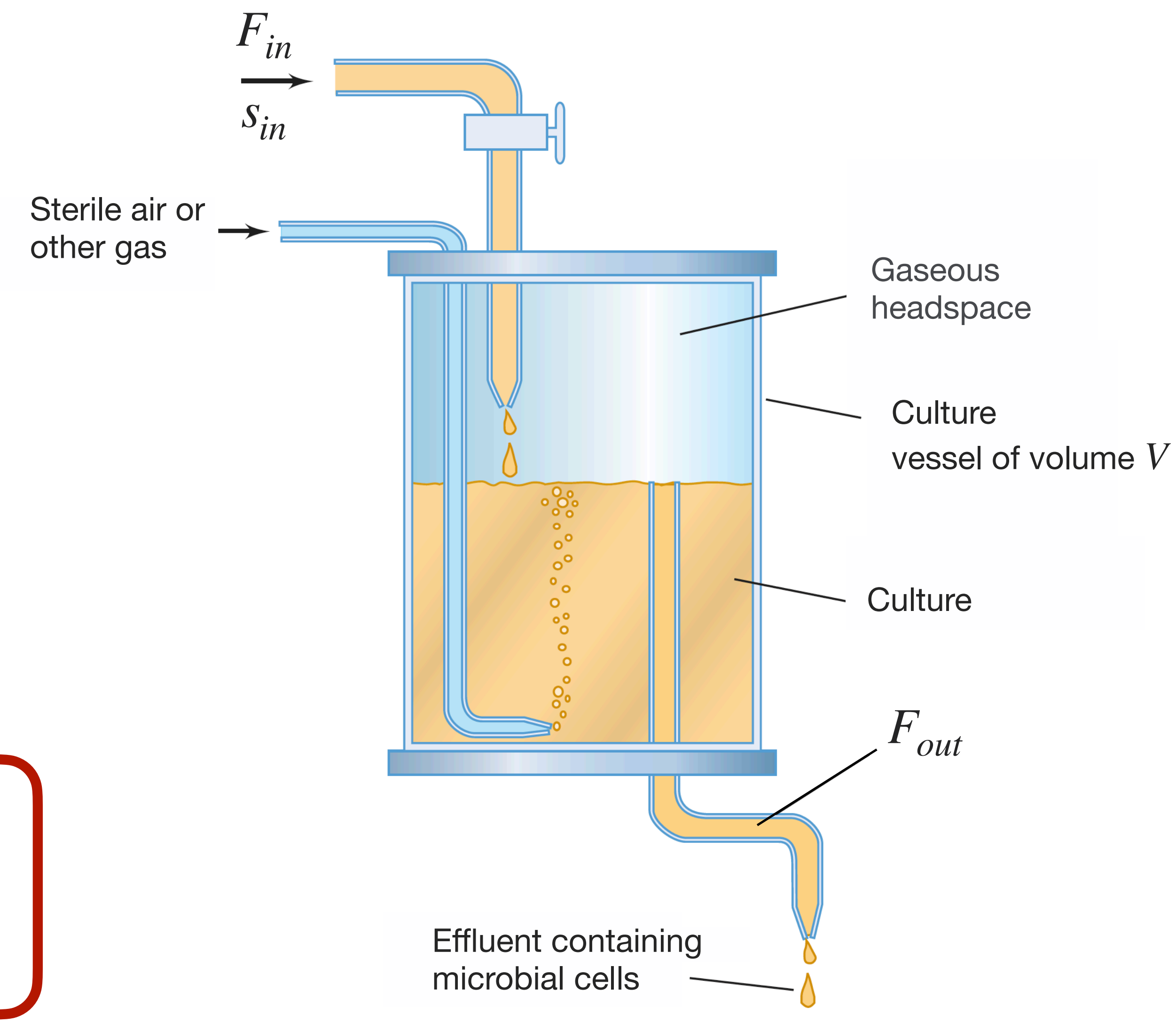

Figure 1: schematic for the chemostat.

The chemostat

$F_{in} = F_{out} = F$: flow rate $[Lh^{-1}]$ $D = F/V$: dilution rate $[h^{-1}]$ $\mu(\cdot)$: specific growth function

$$d \begin{bmatrix} b(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} (\mu(b(t), s(t)) - D)b(t) \\ D(s_{in} - s(t)) - \kappa\mu(b(t), s(t))b(t) \end{bmatrix} dt,$$

with $b(0) = b_0, s(0) = s_0$.

The chemostat

$$d \underbrace{\begin{bmatrix} b(t) \\ s(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} (\mu(b(t), s(t)) - D)b(t) \\ D(s_{in} - s(t)) - \kappa\mu(b(t), s(t))b(t) \end{bmatrix}}_{f(t, x(t))} dt,$$

with $x(0) = (b(0), s(0))^\top$.

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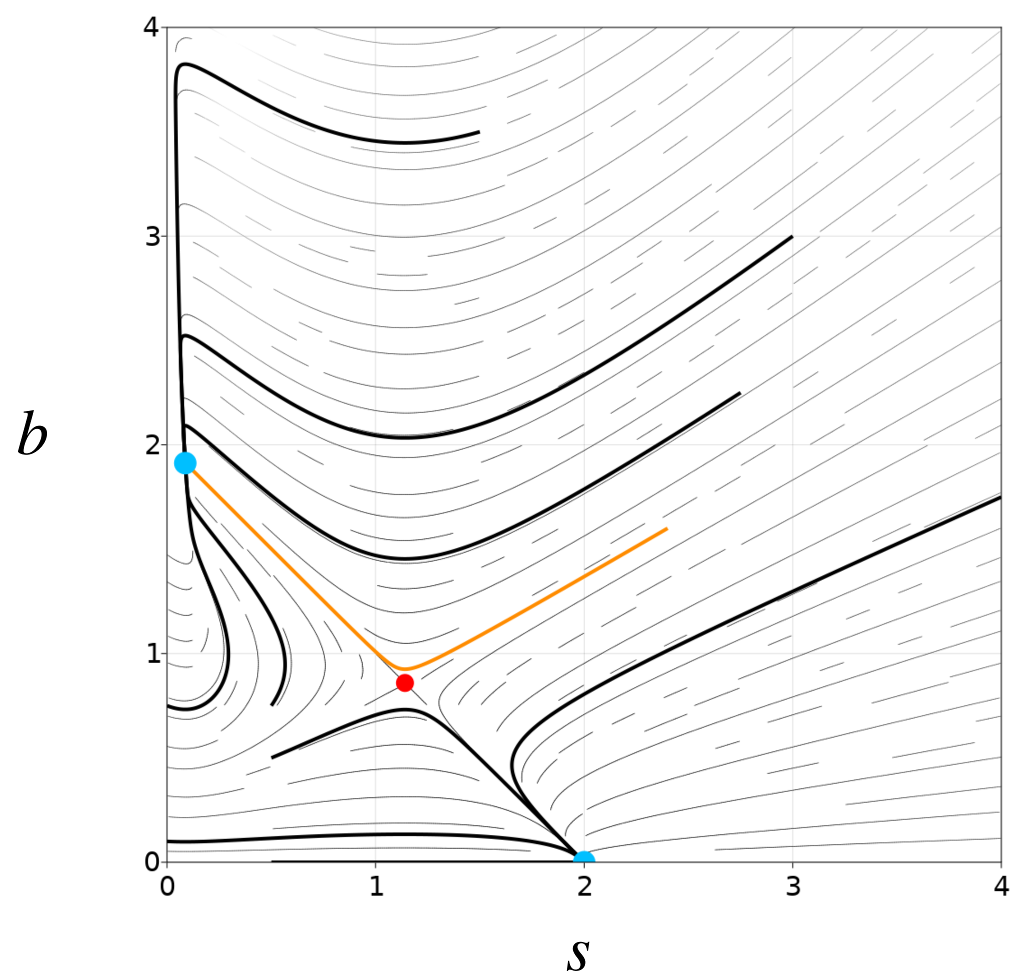

Figure 2: phase portrait for the deterministic model.

The stochastic chemostat

$$\underbrace{d\begin{bmatrix} b(t) \\ s(t) \end{bmatrix}}_{X_t} = \underbrace{\begin{bmatrix} (\mu(b(t),s(t)) - D)b(t) \\ D(s_{in} - s(t)) - \kappa\mu(b(t),s(t))b(t) \end{bmatrix}}_{f(t,X_t)} dt + \underbrace{\begin{bmatrix} \omega_b b(t) & 0 \\ 0 & \omega_s s(t) \end{bmatrix}}_{g(t,X_t)} d\underbrace{\begin{bmatrix} B_t^b \\ B_t^s \end{bmatrix}}_{B_t^x},$$

with initial condition X_0.

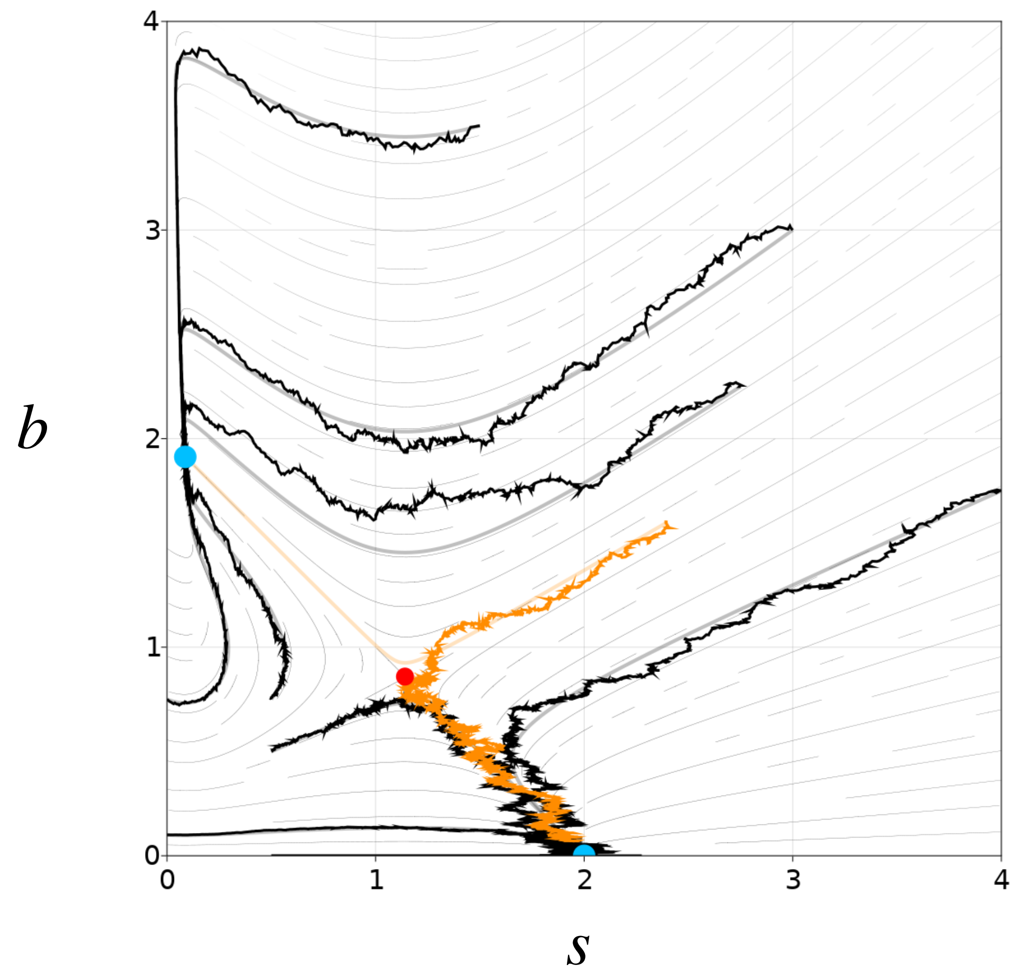

Figure 3: trajectories from the stochastic model.

The Kolmogorov Forward Equation

Also known as the Fokker-Planck equation

Evolution of the probability distribution of the signal $\{X_t\}$:

$$\frac{\partial}{\partial t} p(t,x) = -\sum_{d_x=1}^{D_x} \frac{\partial}{\partial x_{d_x}} [f_{d_x}(t,x) p(t,x)] + \sum_{d_x=1}^{D_x} \sum_{d'_x=1}^{D_x} \frac{\partial^2}{\partial x_{d_x} \partial x_{d'_x}} [G_{d_x d'_x}(t,x) p(t,x)],$$

with diffusion terms $G_{d_x d'_x}(t,x) = \frac{1}{2} \sum_{m=1}^{M} g_{d_x m}(t,x) g_{d'_x m}(t,x)$,

and initial condition $p(0,x)$.

$dX_t = f(t, X_t)dt + g(t, X_t)dB_t^x$

The Kolmogorov Forward Equation

Also known as the Fokker-Planck equation

Evolution of the probability distribution of the signal $\{X_t\}$:

Change in probability

Advection

Diffusion

$$\frac{\partial}{\partial t} p(t,x) = -\sum_{d_x=1}^{D_x} \frac{\partial}{\partial x_{d_x}} [f_{d_x}(t,x)p(t,x)] + \sum_{d_x=1}^{D_x} \sum_{d'_x=1}^{D_x} \frac{\partial^2}{\partial x_{d_x} \partial x_{d'_x}} [G_{d_x d'_x}(t,x)p(t,x)],$$

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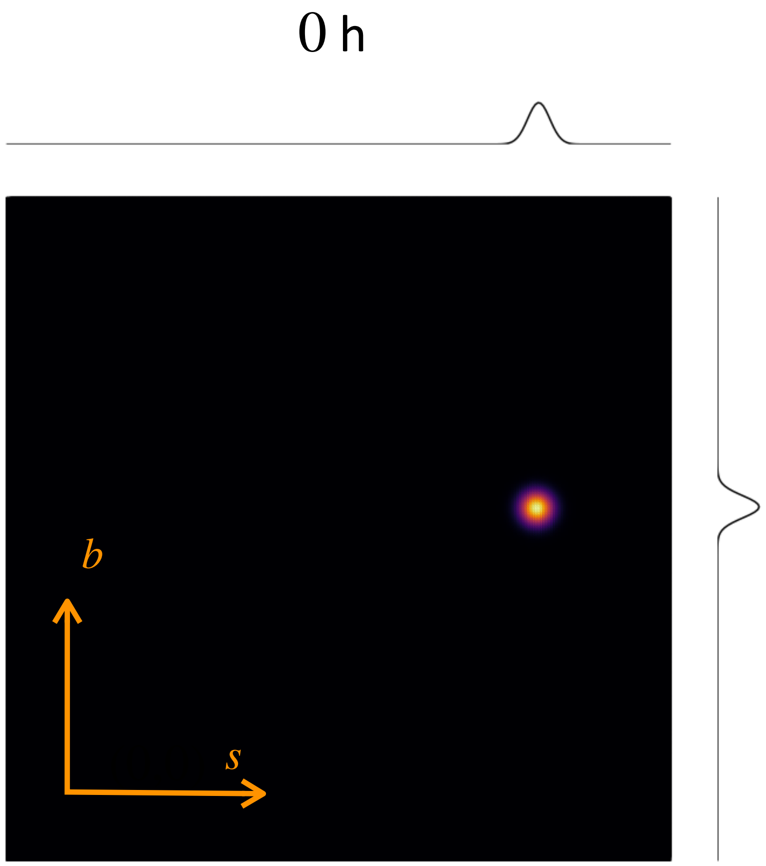
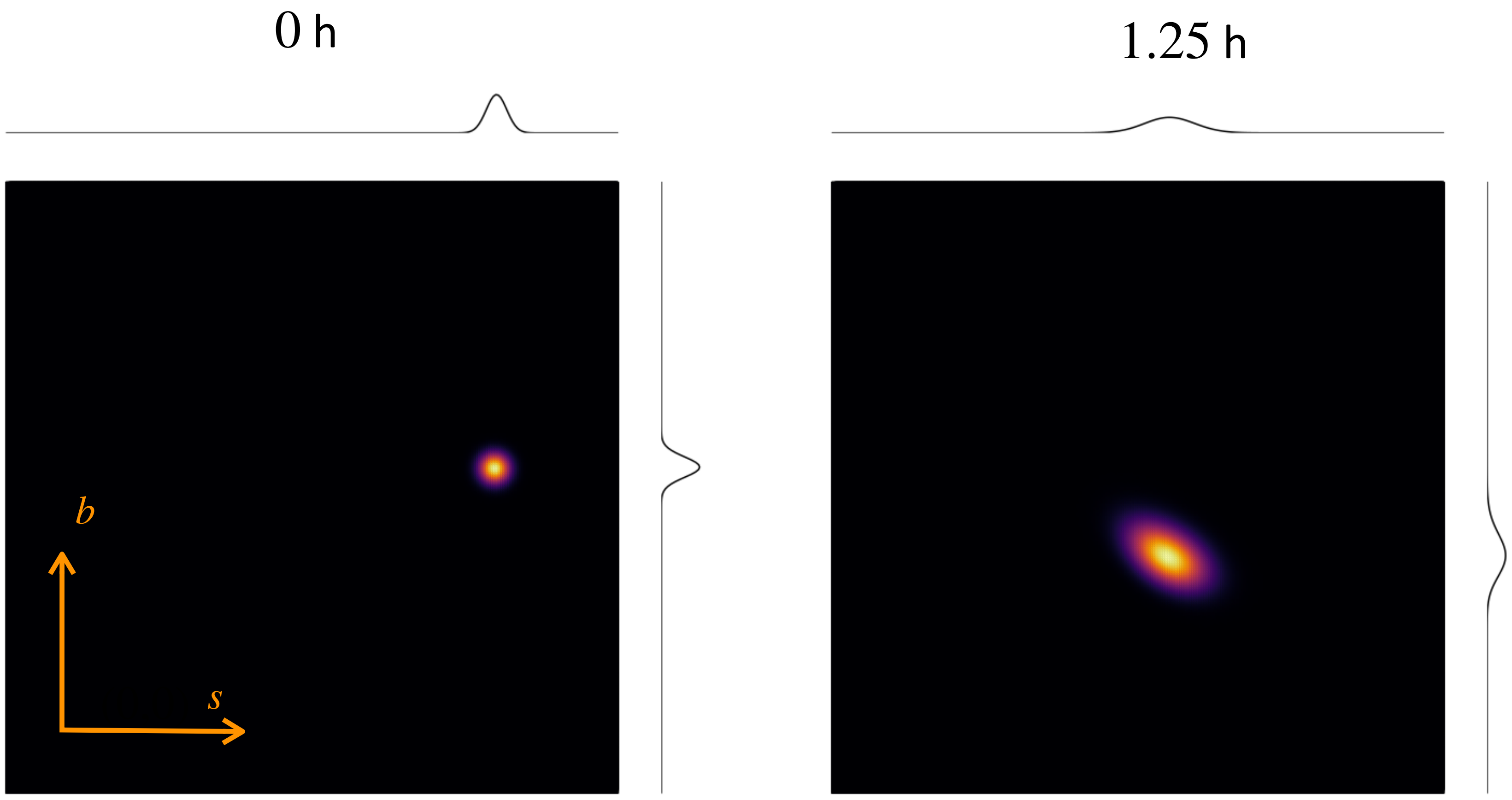


Figure 4: solution to the Fokker-Planck equation.



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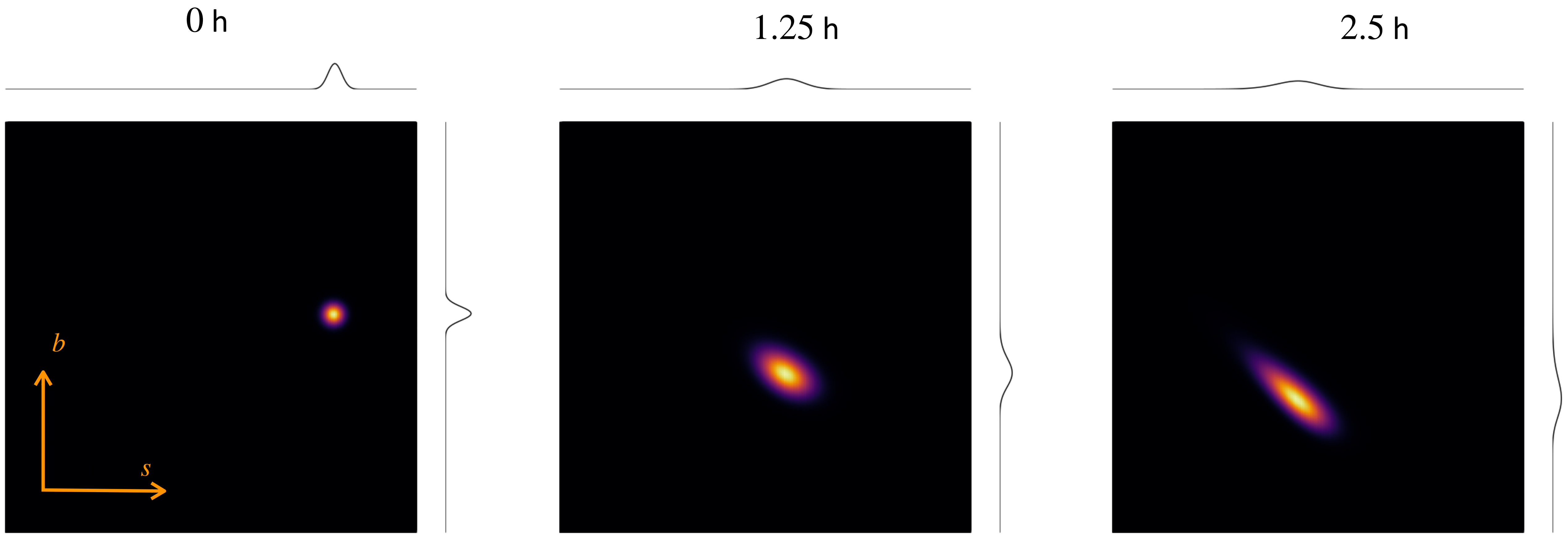

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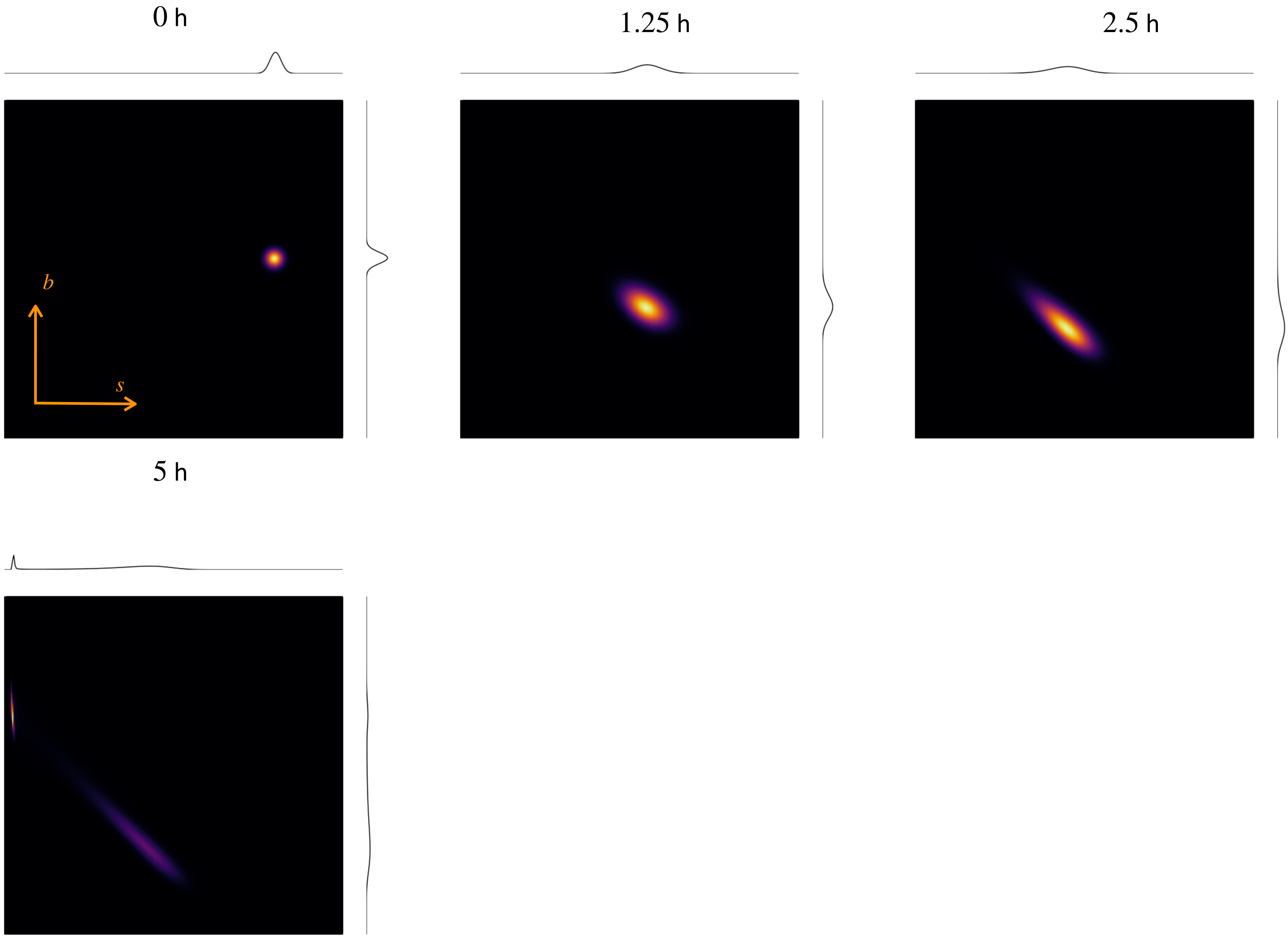

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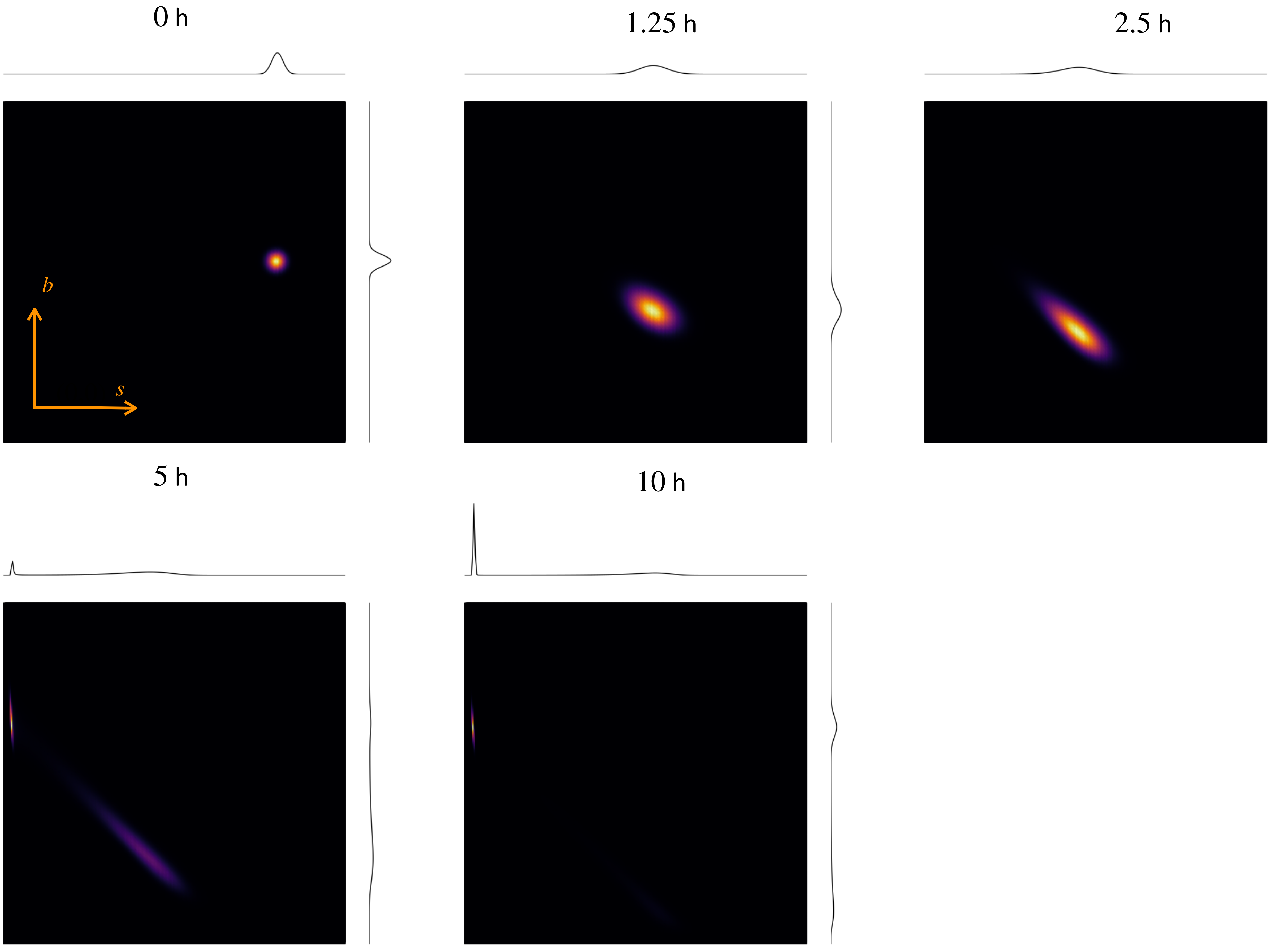

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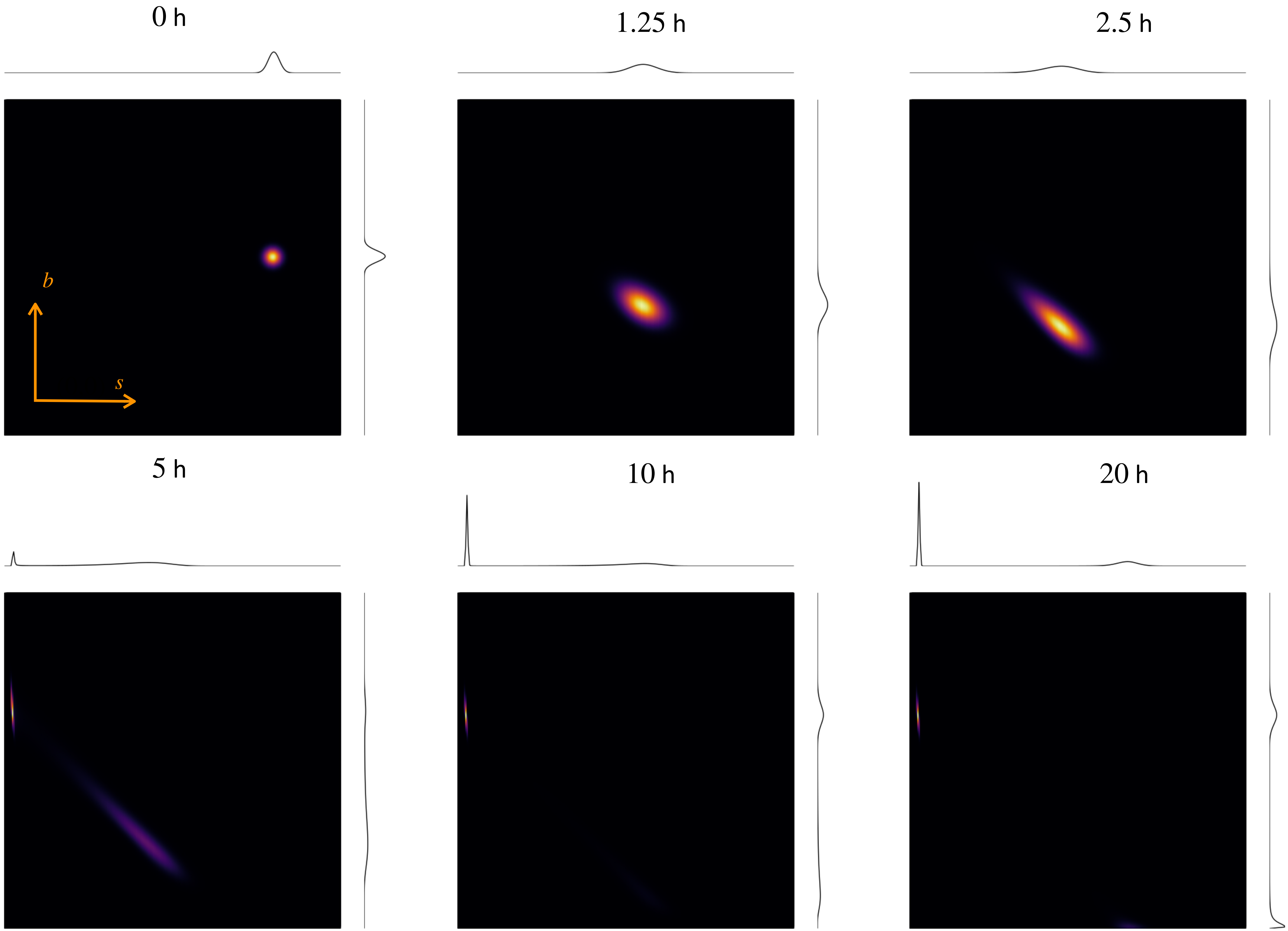

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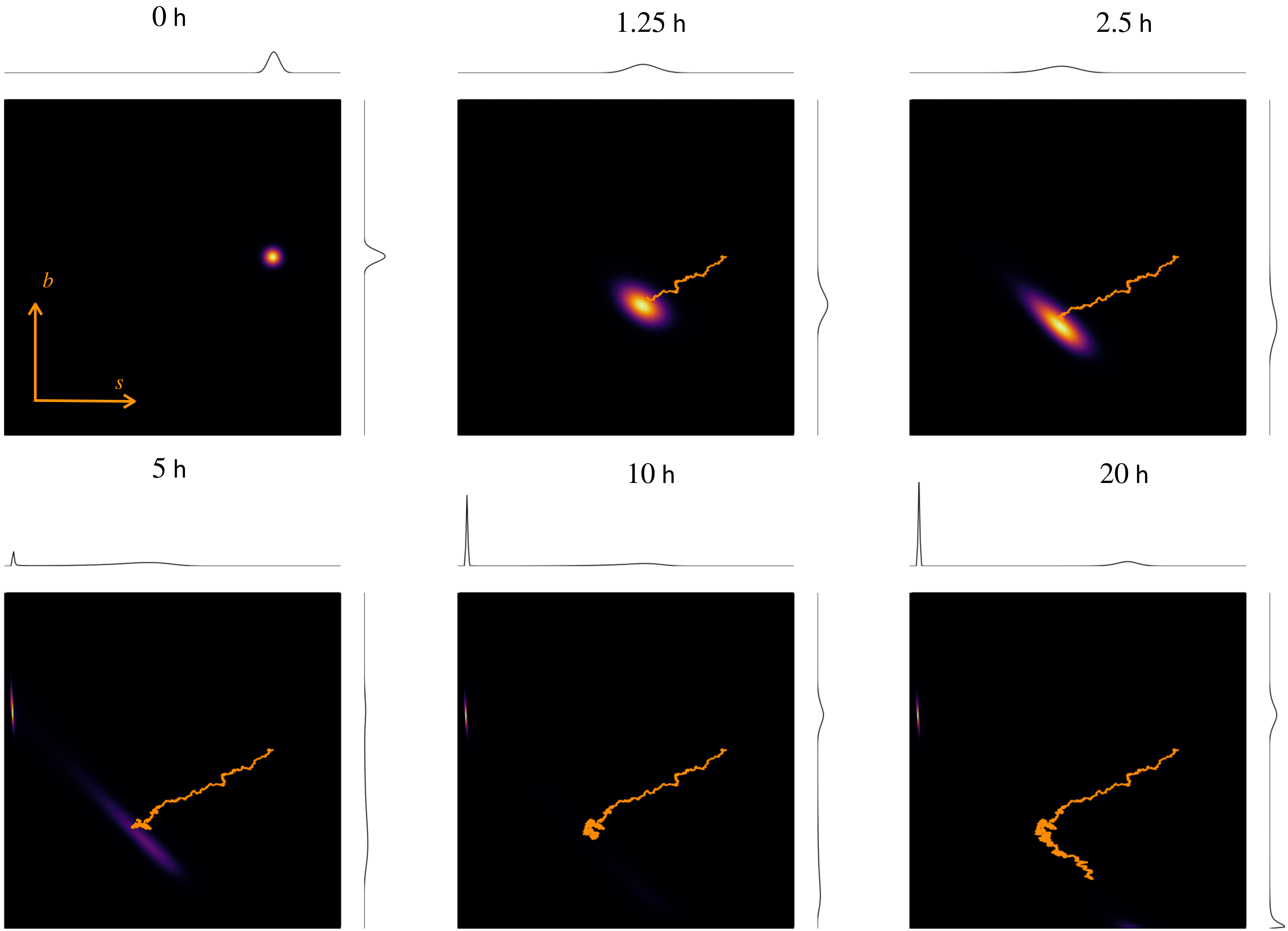

Figure 4: solution to the Fokker-Planck equation.

$$\dot{Q}(t) \quad = \quad V\mu(b(t), s(t))b(t) \quad + \quad \varepsilon$$

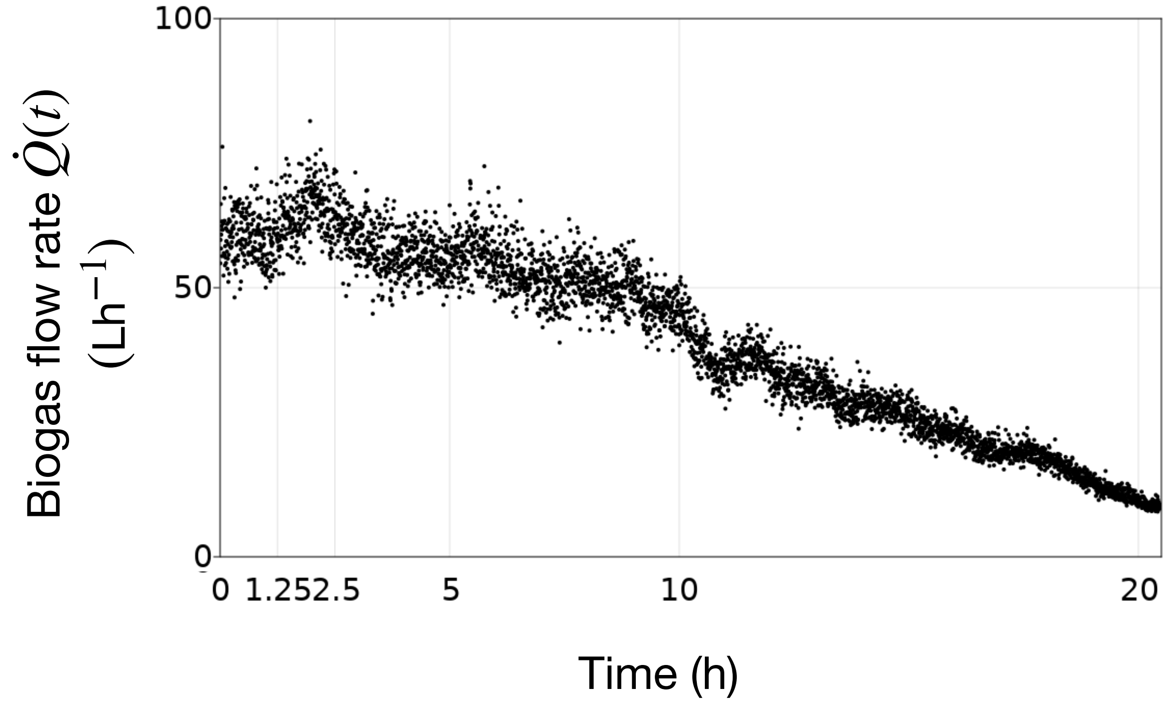

Figure 5: measurements corrupted by noise.

$$\underline{\dot{Q}(t)} \quad = \quad \underline{V\mu(b(t), s(t))b(t)} \quad + \quad \varepsilon$$

$$Y_t \qquad\qquad h(t, X_t)$$

Observation function   Gaussian noise

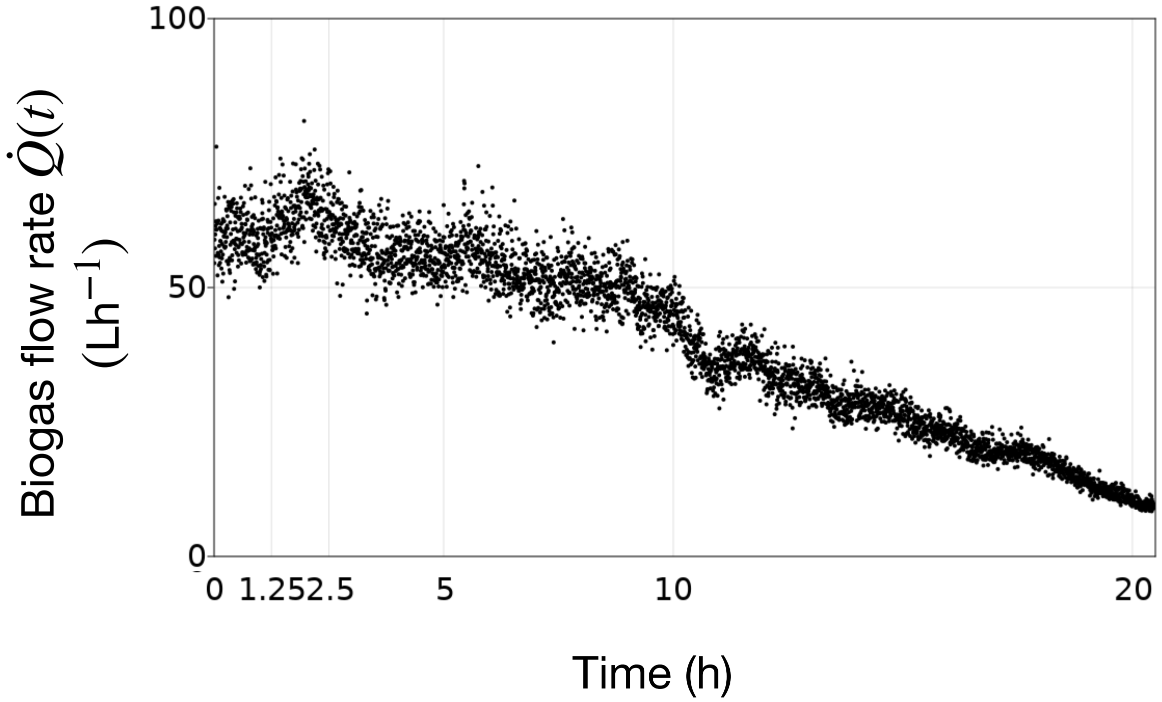

Figure 5: measurements corrupted by noise.

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t^x,$$

$$Y_t = h(t, X_t) + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, k^2(t))$$

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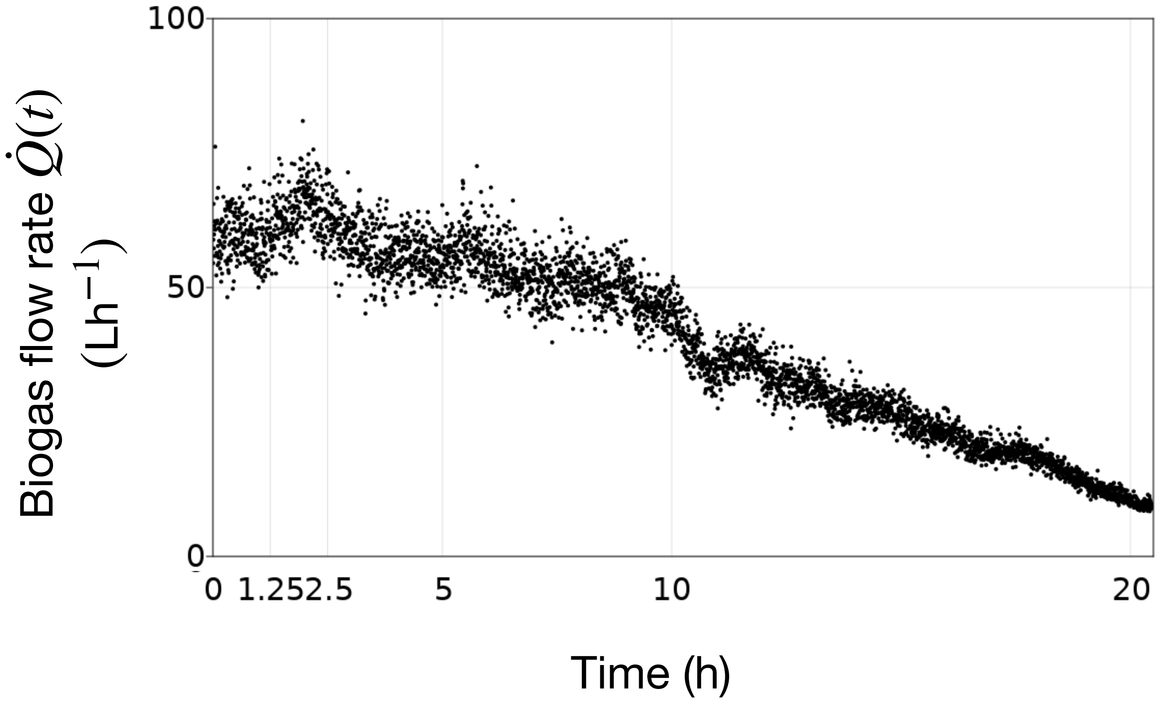

Figure 5: measurements corrupted by noise.

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t^x,$$

$$Y_t = h(t, X_t) + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, k^2(t))$$

Goal: find the distribution of the signal $\{X_t\}$ given some information F_τ^y from measurements, for $\tau \leq t$,

i.e. $p_t := p(t, x \mid F_\tau^y)$.

The Kushner-Stratonovich Equation

Kushner (1964, 1967) and Stratonovich (1968)

$$\frac{\partial}{\partial t} p_t = -\sum_{d_x=1}^{D_x} \frac{\partial}{\partial x_{d_x}} [f_{d_x}(t,x) p_t] + \sum_{d_x=1}^{D_x} \sum_{d'_x=1}^{D_x} \frac{\partial^2}{\partial x_{d_x} \partial x_{d'_x}} [G_{d_x d'_x}(t,x) p_t]$$

$$+ \quad p_t \times (h(t,x) - \mathbb{E}_t[h(t,x)])^\top (k^2(t))^{-1} \times (dY_t - \mathbb{E}_t[h(t,x)] dt)$$

$$p_t := p(t, x \,|\, F^y_\tau)$$

$$dX_t = f(t, X_t)dt + g(t, X_t) dB^x_t$$

$$Y_t = h(t, X_t) + \varepsilon$$

The Kushner-Stratonovich Equation

Kushner (1964, 1967) and Stratonovich (1968)

Partial derivatives in a D_x-dimensional grid

$$\frac{\partial}{\partial t} p_t = -\sum_{d_x=1}^{D_x} \frac{\partial}{\partial x_{d_x}}[f_{d_x}(t,x)p_t] + \sum_{d_x=1}^{D_x}\sum_{d_x'=1}^{D_x} \frac{\partial^2}{\partial x_{d_x}\partial x_{d_x'}}[G_{d_x d_x'}(t,x)p_t]$$

$$+ \quad p_t \times (h(t,x) - \mathbb{E}_t[h(t,x)])^\top (k^2(t))^{-1} \times (dY_t - \mathbb{E}_t[h(t,x)]dt)$$

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$$+ \quad p_t \times (h(t,x) - \mathbb{E}_t[h(t,x)])^\top (k^2(t))^{-1} \times (dY_t - \mathbb{E}_t[h(t,x)]dt)$$

Nonlinearity on p_t :

$$\mathbb{E}_t[h(t,x)] = \int_{\mathbb{R}_{D_x}} h(t,x) p(t,x \,|\, F_\tau^y) dx$$

$p_t := p(t,x \,|\, F_\tau^y)$

$dX_t = f(t,X_t)dt + g(t,X_t)dB_t^x$

$Y_t = h(t,X_t) + \varepsilon$

The Kushner-Stratonovich Equation

Kushner (1964, 1967) and Stratonovich (1968)

Partial derivatives in a D_x-dimensional grid

$$\frac{\partial}{\partial t} p_t = - \sum_{d_x=1}^{D_x} \frac{\partial}{\partial x_{d_x}} [f_{d_x}(t, x) p_t] + \sum_{d_x=1}^{D_x} \sum_{d'_x=1}^{D_x} \frac{\partial^2}{\partial x_{d_x} \partial x_{d'_x}} [G_{d_x d'_x}(t, x) p_t]$$

Stochastic process

$$+ \quad p_t \times (h(t, x) - \mathbb{E}_t[h(t, x)])^\top (k^2(t))^{-1} \times (dY_t - \mathbb{E}_t[h(t, x)]dt)$$

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Nonlinearity on p_t :

$$\mathbb{E}_t[h(t,x)] = \int_{\mathbb{R}_{D_x}} h(t,x) p(t,x \mid F_\tau^y) dx$$

This is a nonlinear stochastic partial integral differential equation!

$$p_t := p(t,x \mid F_\tau^y)$$

$$dX_t = f(t,X_t)dt + g(t,X_t)dB_t^x$$

$$Y_t = h(t,X_t) + \varepsilon$$

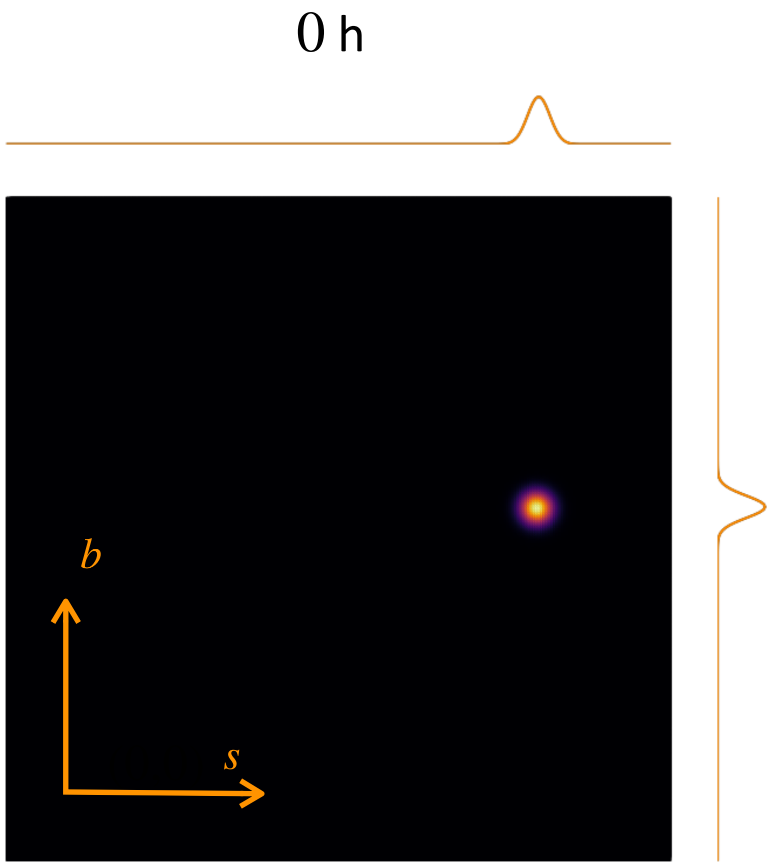
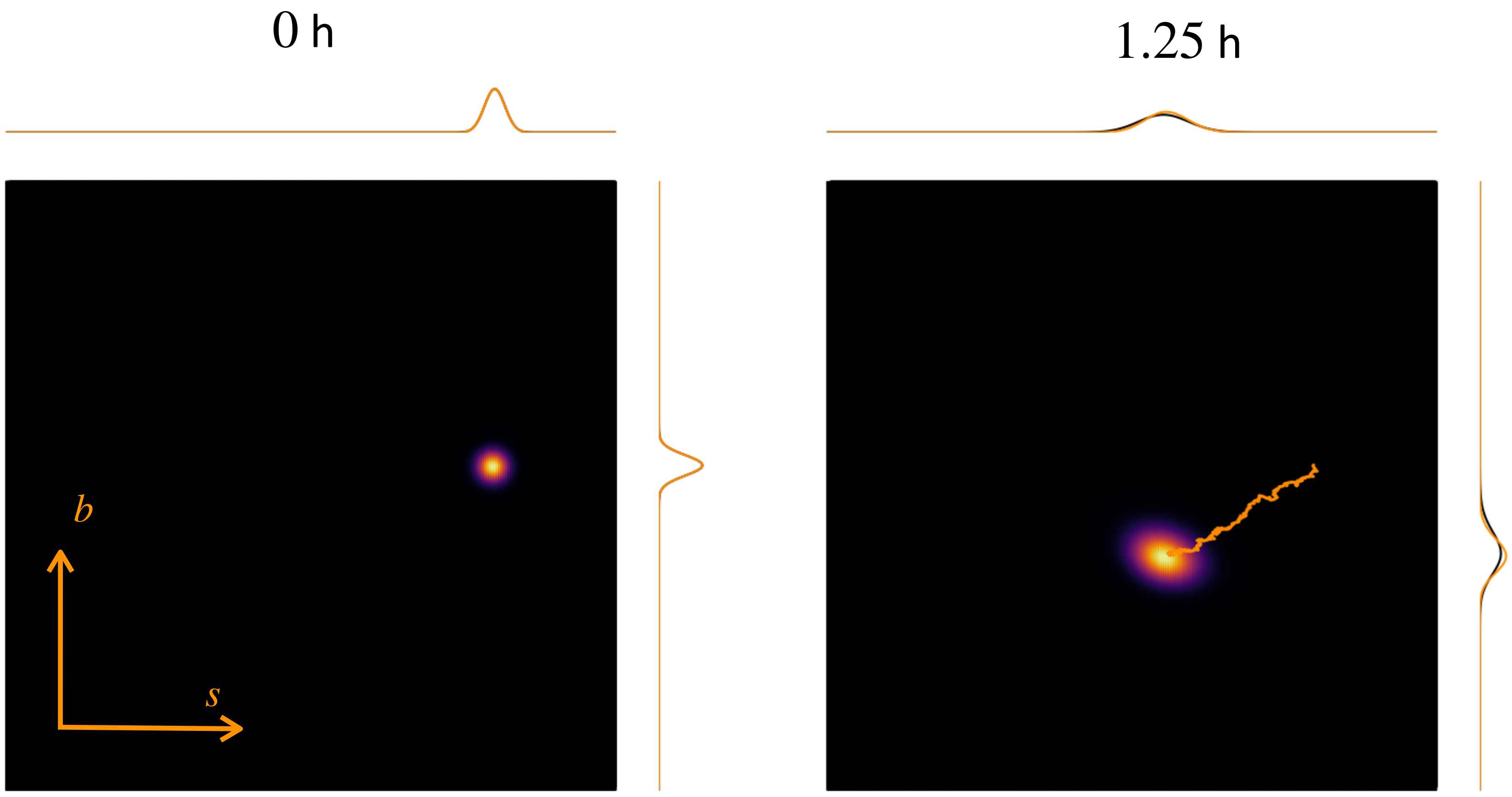


Figure 6: approximated solution to the Kushner-Stratonovich equation.



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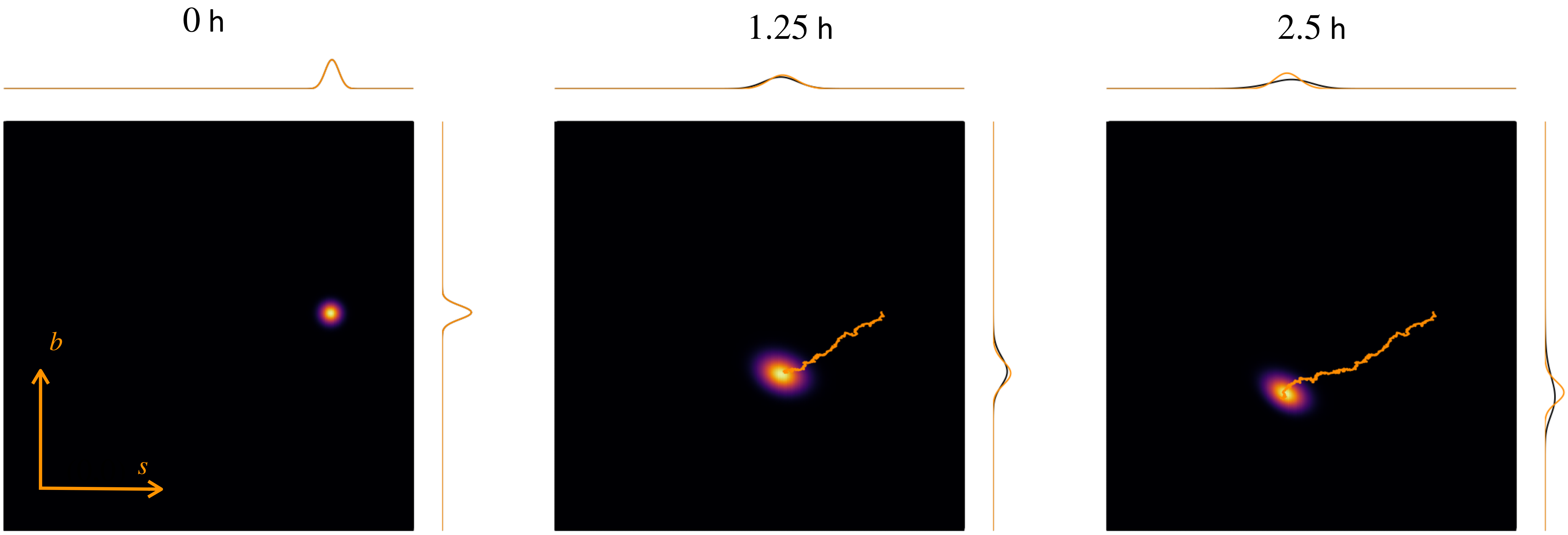

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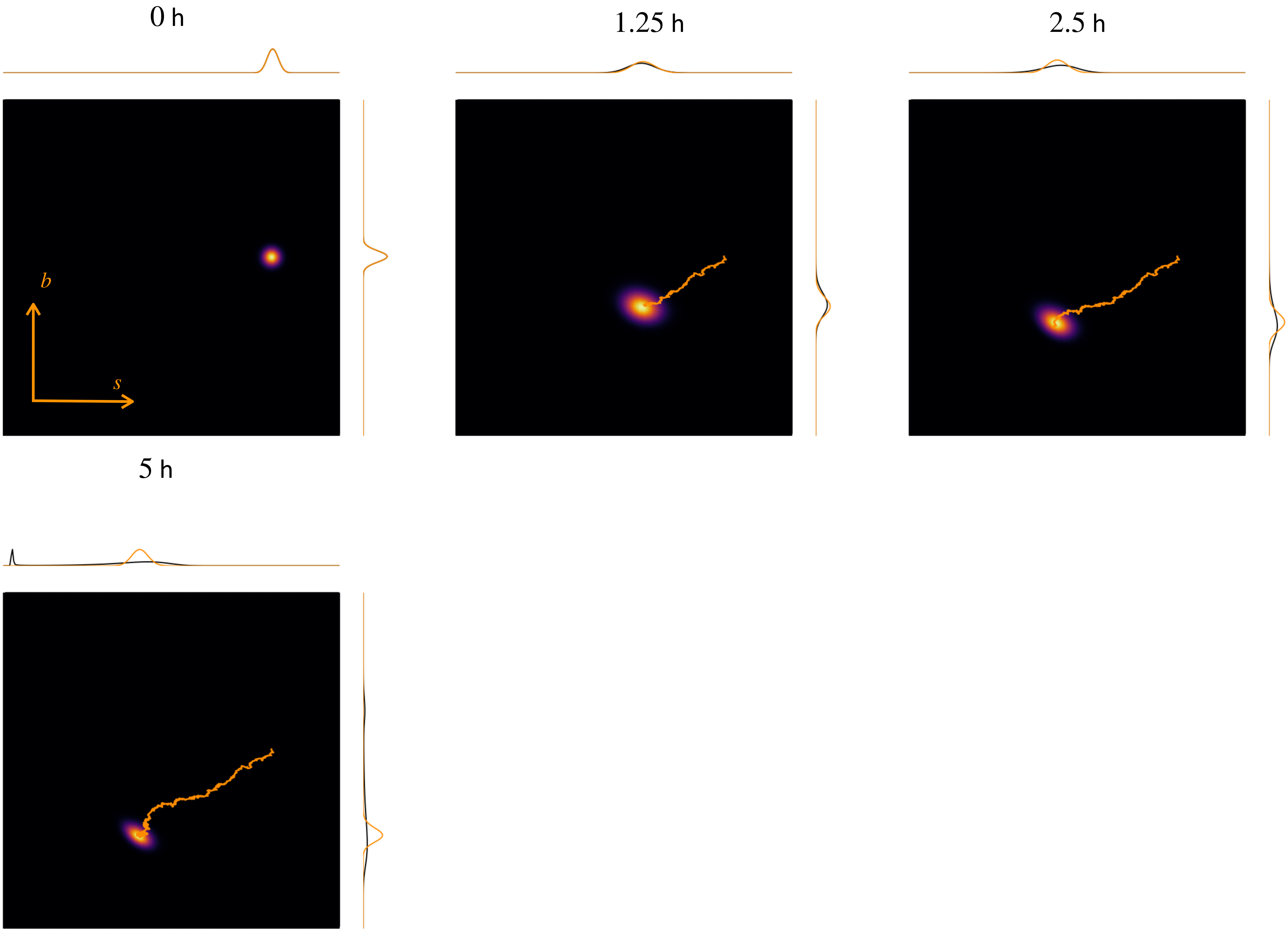

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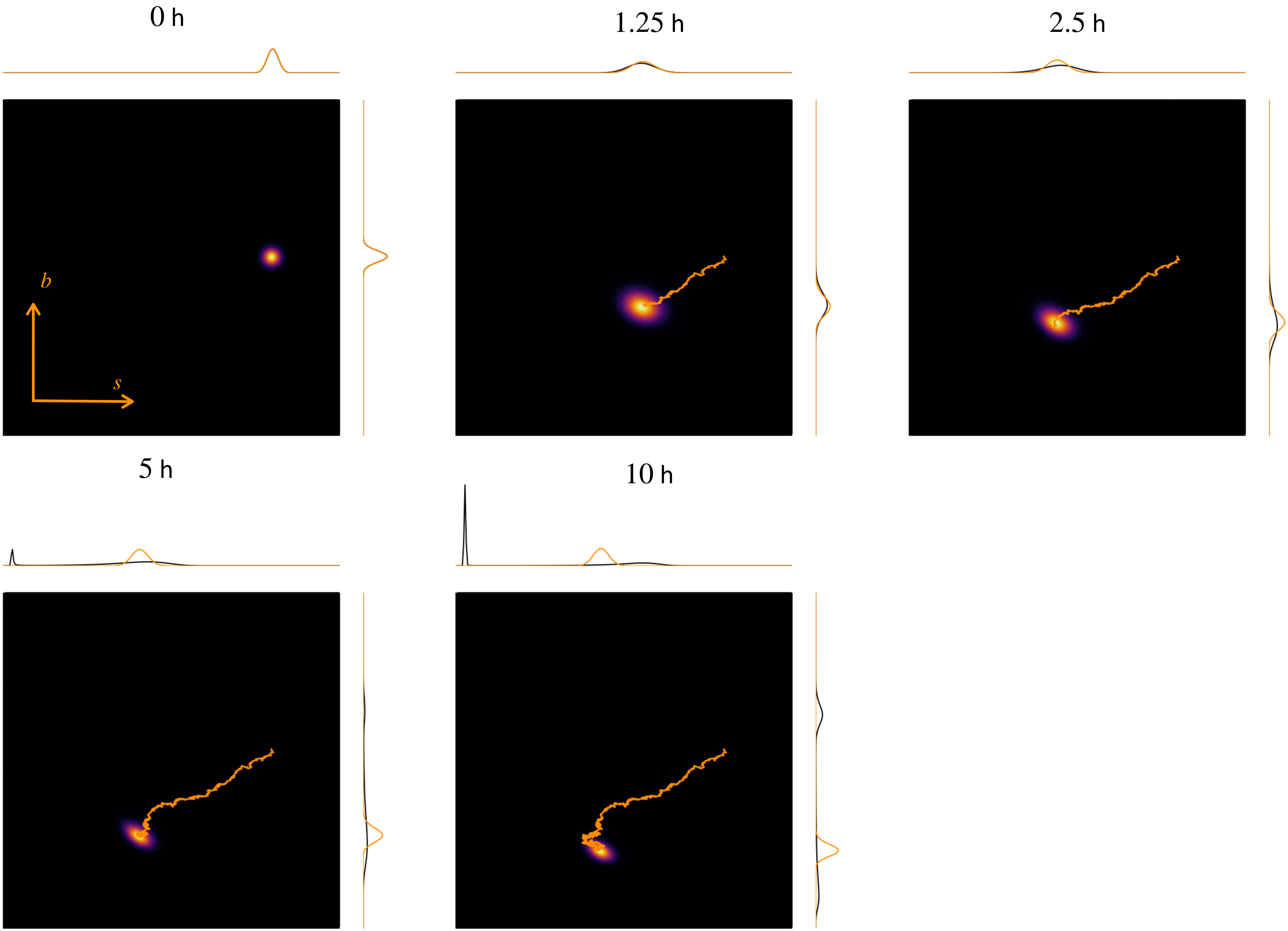

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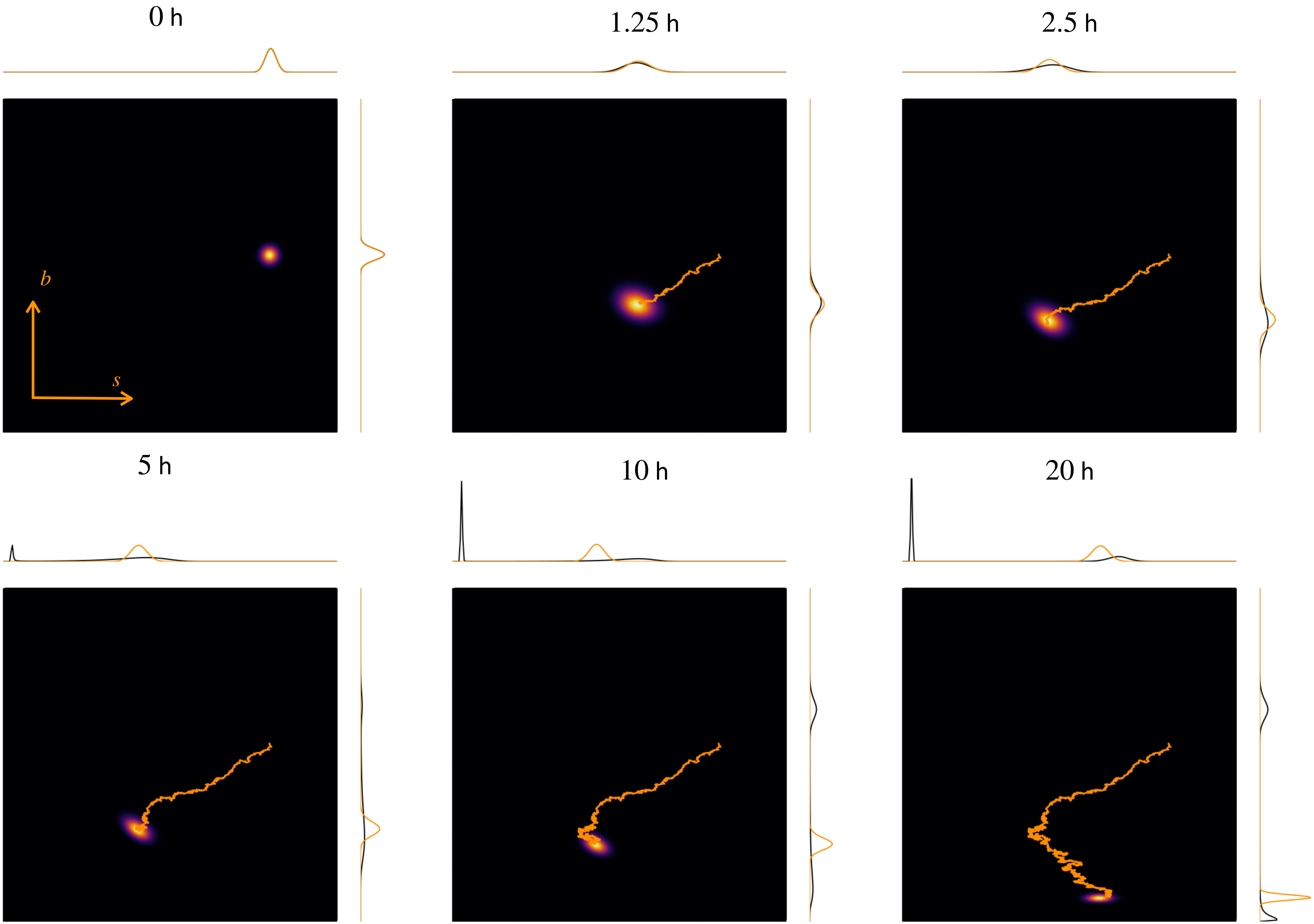

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Summary

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Prior knowledge

(Fokker-Planck equation)

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Outlook

How to obtain a solution to the Kushner-Stratonovich equation?

- Methods for Partial Differential Equations (PDEs)
- Sequential Monte Carlo (SMC) methods

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Statistical properties of some SMC methods

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