# FUNDAMENTAL THEOREM OF ORDINARY DIFFERENTIAL EQ$^{ns}$

We discuss properties of ordinary differential equations that guarantee that ODEs have a solution and that that solution is unique (for a given initial condition)

— PRESENT AND DISCUSS THE THEOREM

**Th** (FUNDAMENTAL THEOREM) We consider a differential equation

$$\dot{x} = f(x, u, t) = f(x(t), u(t), t),$$

with $\begin{cases} x(t) \in \mathbb{R}^n \\ u(t) \in \mathbb{R}^{n_i} \end{cases}$

$$x(t) = [x_1(t), x_2(t), \_\_\_\_, x_n(t)]^T$$
$$u(t) = [u_1(t), u_2(t), \_\_\_\_, u_{n_i}(t)]^T$$

$$f = \begin{bmatrix} f_1(x, u, t) \\ f_2(x, u, t) \\ \vdots \\ f_n(x, u, t) \end{bmatrix}$$

$$f(\cdot, \cdot) = \mathbb{R}^n \times \mathbb{R}^{n_i} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$$

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More simply $\dot{x}(t) = f(x, t)$

Under what conditions, this ODE has a solution and that solution is unique

$x(t)$ would be the solution (for some initial condition) $x(t_0) = x_0$

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We are interested in the solution to that differential equation in some specific time interval $[t_0, t_1]$

$\longrightarrow$ $x(t)$, a trajectory in time that satisfies both $\dot{x} = f(x, t)$ and $x(t_0) = x_0$

THERE EXIST AN UNIQUE SOLUTION IFF :

- $f(x, \cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ MUST BE PIECE-WISE CONTINUOUS, FOR ALL $x$
- $f(\cdot, t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ MUST BE LIPSCHITZ CONTINUOUS, FOR ALL $t$

→ Then, ∃ a unique function $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ which is differentiable almost everywhere, and this function satisfies $\dot{x} = f(x,t)$ and $x(t_0) = x_0$

$$\begin{cases} \dot{\phi}(t) = f(\phi(t), t) \\ \phi(t_0) = x_0 \end{cases}$$

, $\forall t$ in $[t_0, t_1] \setminus D$ where $f(x,\cdot)$ is not continuous (as a function of time)

THE CONDITIONS ARE RELATIVE TO THE RIGHT HAND SIDE OF THE DIFF. EQ (given the initial conditions)

IF WE CHANGE THE INITIAL CONDITIONS WE WILL GET A DIFFERENT SOLUTION

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WE MORE CLOSELY LOOK AT THE MEANING OF THE STATED CONDITIONS

— Continuity (and piecewise continuity and Lipschitz continuity)

A function $f(\cdot)$ is continuous if for $\forall \varepsilon$ $\exists \delta$ with $\varepsilon, \delta \in \mathbb{R}_+$ such that if $\underbrace{\| x_1 - x_2 \| < \delta}_{1}$ then $\underbrace{\| f(x_1) - f(x_2) \| < \varepsilon}_{2}$

IF 1 IS SMALLER OF SOME $\delta$, THEN WE CAN MAKE 2 TO BE SMALLER THAN SOME $\varepsilon$

(NOTE THE NORMS, WHICH IMPLIES THAT ALL SPECIFIC SPACES ARE VECTOR SPACES EQUIPPED WITH A NORM.. )

" IF THE DISTANCE BETWEEN $x_1$ AND $x_2$ IS SMALL, THEN THAT WILL BE MAINTAINED ALSO IN THE RANGE, BETWEEN $f(x_1)$ AND $f(x_2)$ "

— Piecewise continuity (PC) : this is continuity everywhere, except a certain number of points (discontinuity points)

But those points need to be well behaved

$f(x,\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ IS SAID TO BE PIECEWISE CONTINUOUS IN TIME IF IT IS CONTINUOUS EXCEPT AT POINTS OF DISCONTINUITY (there can only be finitely many of these points in any closed and bounded (compact) interval of time)

— Lipschitz continuity (LC) :

$f(\cdot,t): \mathbb{R}_+ \to \mathbb{R}^n$ IS SAID TO BE LIPSCHITZ CONTINUOUS IF THERE EXIST A PIECEWISE CONTINUOUS FUNCTION $K(\cdot)$ OF TIME $K(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ SUCH THAT

FUNCTION $f$ CANNOT GO OFF TO INFINITY OVER THE TIME INTERVAL THAT WE ARE INTERESTED IN (closed and bounded time intervals)

$$\| f(x_1,t) - f(x_2,t) \| \leq K(t) \, \| x_1^{(t)} - x_2^{(t)} \|$$

$\forall x_1, x_2 \in \mathbb{R}^n$

$\forall t \in \mathbb{R}_+$

LIPSCHITZ INEQUALITY

LIPSCHITZ FUNCTION

A LIPSCHITZ CONTINUOUS FUNCTION IS ALWAY CONTINUOUS, BUT A CONTINUOUS FUNCTION IS NOT NECESSARILY LIPSCHITZ CONTINUITY

→ lipschitz continuity is a stricter condition than basic continuity

→ Useful to construct a solution to a differential equation

→ It is hard to show that a function is LC

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IT IS DIFFICULT TO DERIVE A LIPSCHITZ FUNCTION $K(\cdot)$

→ CANDIDATES, THEN CHECKS

A norm on the Jacobian of f is usually a good candidate for the LF

$\| \underbrace{D_x}_{} f \|$

Derivative of f wrt. x

A generalization of the mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\| f(x_1 t) - f(x_2, t) \| \leq \underbrace{\| D_x f \|}_{}, \| x_1(t) - x_2(t) \|, \quad \forall t$$

induced norm (by the used vector norms)

Depends on x, depending on the chosen norm

$$D_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$