LINEARIZATION of NONLINEAR STATE-SPACE MODELS

MANY DYNAMICAL PROCESSES IN CHEMICAL ENGINEERING ARE MODELLED AS A SET OF NON-LINEAR EQUATIONS (FIRST-ORDER)

- FROM MATERIAL AND ENERGY CONSERVATION

MOST COMMON ANALYSIS AND CONTROL REQUIRE A LINEAR MODEL

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \longleftarrow \quad \begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x,u) \end{aligned}$$

Nonlinear systems can be linearized

GENERAL CASE — SINGLE VARIABLE $\quad dx/dt = f(x)$

Function $f(x)$ can be approximated using a Taylor Series around an equilibrium point $x_s \leftarrow f(x = x_s) = 0$

$$f(x) = f(x_s) + \frac{\partial f}{\partial x}\Big|_{x_s}(x - x_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_s}(x - x_s)^2 + \underbrace{O(x^3)}_{\text{higher order terms}}$$

IF WE NEGLECT THE QUADRATIC TERM AND HIGHER,

$$f(x) \approx f(x_s) + \frac{\partial f}{\partial x}\Big|_{x_s}(x - x_s)$$

WE NOTED THAT $\frac{dx}{dt}\Big|_{x_s} = f(x_s) = 0$

THUS, WE HAVE $f(x) \approx \underbrace{\cancel{f(x_s)}}_{=0} + \frac{\partial f}{\partial x}\Big|_{x_s}(x - x_s)$

$$\frac{dx}{dt} = f(x) \approx \frac{\partial f}{\partial x}\Big|_{x_s}(x - x_s)$$

Consider the variable $x' = (x - x_s)$ and compute its derivative

$$\frac{d(x - x_s)}{dt} = \frac{dx}{dt} - \underbrace{\cancel{\frac{dx_s}{dt}}}_{=0} = \frac{dx}{dt} \leadsto \frac{d(x - x_s)}{dt} \approx \frac{\partial f}{\partial x_s}\Big|_{x_s}(x - x_s)$$

$$\frac{d(x - x_s)}{dt} \approx \frac{\partial f}{\partial x}\Big|_{x_s}(x - x_s)$$

THIS REPRESENTATION ALLOW US TO DISCUSS 'DEVIATIONS WITH RESPECT TO AN EQUILIBRIUM POINT'

WE CAN WRITE THE SYSTEM LINEARISED AND WRT PERTURBATIONS

$$\frac{dx'}{dt} = \frac{\partial f}{\partial x}\Big|_{s} x' \quad \text{OR} \quad \dot{x}' = \alpha x' \quad \text{WITH} \quad \alpha = \frac{\partial f}{\partial x}\Big|_{x_s}$$

GENERAL CASE — SINGLE VARIABLE, SINGLE INPUT $\quad dx/dt = f(x,u)$

WE WRITE THE TAYLOR SERIES APPROXIMATION OF $f(x,u)$ AT (x_s, u_s)

$$\begin{aligned} f(x,u) = {} & f(x_s,u_s) + \frac{\partial f}{\partial x}\Big|_{x_s,u_s}(x - x_s) + \frac{\partial f}{\partial u}\Big|_{x_s,u_s}(u - u_s) + \\ & \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{x_s,u_s}(x - x_s)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}\Big|_{x_s u_s}(u - u_s)^2 + \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x \partial y}\Big|_{x_s,u_s}(u - u_s)(x - x_s) + O(x^3, u^3) \end{aligned}$$

IF WE NEGLECT THE QUADRATIC TERMS AND HIGHER,

$$f(x,u) \approx f(x_s,u_s) + \frac{\partial f}{\partial x}\Big|_{x_s u_s}(x - x_s) + \frac{\partial f}{\partial u}\Big|_{x_s,u_s}(u - u_s)$$

WE NOTED THAT $f(x_s,u_s) = \frac{dx}{dt}\Big|_{x_s u_s} = 0$

CONSIDER THE VARIABLE $x' = (x - x_s) \leadsto \frac{d(x - x_s)}{dt} = \frac{dx}{dt}$

WE HAVE

$$\frac{d(x - x_s)}{dt} \approx \overbrace{\cancel{f(x_s,u_s)}}^{=0} + \frac{\partial f}{\partial x}\Big|_{x_s u_s}(x - x_s) + \frac{\partial f}{\partial u}\Big|_{x_s u_s}(u - u_s)$$

LET $u' = (u - u_s)$, WE CAN WRITE $\frac{dx'}{dt} \approx \underbrace{\frac{\partial f}{\partial x}\Big|_{x_s u_s}}_{\alpha} x' + \underbrace{\frac{\partial f}{\partial u}\Big|_{x_s u_s}}_{\beta} u'$

ASSUME THAT THERE EXISTS AN OUTPUT VARIABLE $y = g(x,u)$

WE CAN AGAIN TAYLOR EXPAND AND TRUNCATE:

$$g(x,u) \approx g(x_s,u_s) + \left.\frac{\partial g}{\partial x}\right|_{x_s u_s}(x-x_s) + \left.\frac{\partial g}{\partial u}\right|_{x_s u_s}(u-u_s)$$

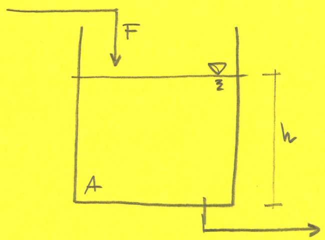
AS $g(x_s,u_s)$ IS THE STEADY STATE OF THE OUTPUT (y_s),

$$y \approx g(x_s,u_s) + \underbrace{\left.\frac{\partial g}{\partial x}\right|_{x_s u_s}}_{\gamma}(x-x_s) + \underbrace{\left.\frac{\partial g}{\partial u}\right|_{x_s u_s}}_{\delta}(u-u_s)$$

$$y - y_s = \gamma x' + \delta u'$$

Example SURGE TANK

$$\frac{dh}{dt} = \frac{F}{A} - \beta/A\sqrt{h}$$



A, β ARE PARAMETERS

h IS THE STATE VARIABLE

F IS THE INPUT VARIABLE

THE VECTOR FIELD IS $f(h,F)$, WE CAN WRITE

$$f(h,F) \approx f(h_s,F_s) + \left.\frac{\partial f}{\partial h}\right|_{h_s F_s}(h-h_s) + \left.\frac{\partial f}{\partial F}\right|_{h_s F_s}(F-F_s)$$

$$= \underbrace{\cancel{\left[\frac{F_s}{A} - \frac{\beta}{A}\sqrt{h_s}\right]}}_{=0 \ \text{(STEADY-STATE)}} + \left(-\frac{1}{2}(h)^{-1/2}\beta/A\right) + \frac{1}{A}(F-F_s)$$

WE CAN NOW LET $h' = h - h_s$, $u' = u - u_s$ AND WRITE

$$\frac{d(h-h_s)}{dt} \approx -\frac{\beta}{2A\sqrt{h_s}}(h-h_s) + \frac{1}{A}(F-F_s)$$

EQUIVALENTLY, WE HAVE

$$\frac{dh'}{dt} = \underbrace{-\frac{\beta}{2A\sqrt{h_s}}}_{\alpha} h' + \underbrace{\frac{1}{A}}_{\beta} u'$$

GENERAL CASE — MULTIPLE VARIABLES

Consider the two-state system
$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, u) \\ \dot{x}_2 = f_2(x_1, x_2, u) \\ y = g(x_1, x_2, u) \end{cases}$$

BY TAYLOR SERIES EXPANSIONS, WE HAVE

$$f_1(x_1, x_2, u) = f_1(x_1^s, x_2^s, u^s) + \left.\frac{\partial f_1}{\partial x_1}\right|_{ss} (x_1 - x_1^s) + \left.\frac{\partial f_1}{\partial x_2}\right|_{ss} (x_2 - x_2^s) + \left.\frac{\partial f_1}{\partial u}\right|_{ss} (u - u_s) + \text{higher order terms}$$

$$f_2(x_1, x_2, u) = f_2(x_1^s, x_2^s, u^s) + \left.\frac{\partial f_2}{\partial x_1}\right|_{ss} (x_1 - x_1^s) + \left.\frac{\partial f_2}{\partial x_2}\right|_{ss} (x_2 - x_2^s) + \left.\frac{\partial f_2}{\partial u}\right|_{ss} (u - u_s)$$

$$g(x_1, x_2, u) = g(x_1^s, x_2^s, u) + \left.\frac{\partial g}{\partial x_1}\right|_{ss} (x_1 - x_1^s) + \left.\frac{\partial g}{\partial x_2}\right|_{ss} (x_2 - x_2^s) + \left.\frac{\partial g}{\partial u}\right|_{ss} (u - u_s)$$

FROM THE EQUILIBRIUM CONDITIONS, WE HAVE THAT
$$\begin{cases} f_1(ss) = 0 \\ f_2(ss) = 0 \\ g(ss) = y_s \end{cases}$$

MOREOVER, WE CAN WRITE

$$\frac{d(x_1 - x_1^s)}{dt} = \frac{dx_1}{dt}$$

$$\frac{d(x_2 - x_2^s)}{dt} = \frac{dx_2}{dt}$$

THE JACOBIAN WRT X

$$\underbrace{\begin{bmatrix} \frac{d(x_1 - x_s)}{dt} \\ \frac{d(x_2 - x_s)}{dt} \end{bmatrix}}_{x'} = \underbrace{\begin{bmatrix} \partial f_1/\partial x_1|_{ss} & \partial f_1/\partial x_2|_{ss} \\ \partial f_2/\partial x_1|_{ss} & \partial f_2/\partial x_2|_{ss} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 - x_1^s \\ x_2 - x_2^s \end{bmatrix}}_{x'} + \rightsquigarrow$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} \partial f_1/\partial u|_{ss} \\ \partial f_2/\partial u|_{ss} \end{bmatrix}}_{B(b)} \underbrace{\begin{bmatrix} u - u_s \end{bmatrix}}_{u'}$$

AS FOR THE OUTPUT, WE CAN WRITE

$$[y - y_s] = \underbrace{\begin{bmatrix} \partial g/\partial x_1|_{ss} & \partial g/\partial x_2|_{ss} \end{bmatrix}}_{C(c)} \underbrace{\begin{bmatrix} x_1 - x_1^s \\ x_2 - x_2^s \end{bmatrix}}_{x'} + \underbrace{\begin{bmatrix} \partial g/\partial u|_{ss} \end{bmatrix}}_{D(d)} \underbrace{\begin{bmatrix} u - u_s \end{bmatrix}}_{u'}$$

GENERAL CASE — MULTIPLE VARIABLES, MULTIPLE INPUTS

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N_x} \end{bmatrix} = \begin{bmatrix} f_1(x, u) \\ f_2(x, u) \\ \vdots \\ f_{N_x}(x, u) \end{bmatrix} \rightsquigarrow \quad A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{ss} \qquad B_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{ss}$$

MOREOVER

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N_y} \end{bmatrix} = \begin{bmatrix} g_1(x, u) \\ g_2(x, u) \\ \vdots \\ g_{N_y}(x, u) \end{bmatrix} \rightsquigarrow \quad C_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{ss} \qquad D_{ij} = \left.\frac{\partial g_i}{\partial u_j}\right|_{ss}$$