$$\dot{x}(t) = Ax(t) + Bu(t) + \underbrace{B_v v(t)}_{}$$ DETERMINISTIC DISTURBANCE

$$y(t) = Cx(t) + \underbrace{w(t)}_{\text{MEASUREMENT NOISE}}$$

We want to establish linear feedback, a linear feedback law

$$u(t) = -\underbrace{K}_{} x(t) + \underbrace{r(t)}_{}$$ REFERENCE INPUT (N_u)

FEEDBACK OR GAIN MATRIX $N_u \times N_x$

As we are assuming that all the state variables are measured the resulting feedback system is called full-state feedback

→ IT IS INTENDED THAT THE OUTPUT OBJECT SHOULD FOLLOW THE REFERENCE INPUT, IN SOME SENSE

SUBSTITUTING $u = -Kx + r$ INTO $\dot{x} = Ax + Bu$ WE GET

$$\dot{x} = Ax + B(-Kx + r) = \underbrace{Ax - BKx}_{} + Br$$ STATE EQUATION FOR THE CLOSED LOOP LINEAR SYS

→ THE SYSTEM IS ASYMPTOPTICALLY STABLE BASED ON THIS MATRIX

→ UNDER SOME CONDITIONS, IT IS POSSIBLE TO ARBITRARILY ASSIGN THE EIGENVALUES OF THIS MATRIX (By setting matrix K)

1

ONE OF THE DRAWBACK OF STATE FEEDBACK CONTROL IS THAT THE OUTPUT OF THE SYSTEM IS NOT DIRECTLY INVOLVED IN THE CONTROL

⇝ We cannot assign a predetermined relashionship between the inputs and the outputs

CONSIDER THE CLOSED-LOOP SYSTEM $\dot{x} = A_K x + Br$

IT IS ASSUMED THAT THE SYSTEM IS IN STATIONARY STATE

⇝ THE DERIVATIVE OF THE STATE VECTOR IS ZERO

$$0 = A_K x_0 + B r_0$$

ASSUMING THAT MATRIX A_K IS NONSINGULAR, WE CAN COMPUTE THE STATIONARY STATE, SOLVING FOR x_0

$$x_0 = -A_K^{-1} B r_0$$

THE STATIONARY OUTPUT WITH THIS STATE IS $y_0 = Cx_0$
$= -CA_K^{-1} B r_0$

IF THE SYSTEM HAS THE SAME NUBER OF INPUTS AND OUTPUTS, THEN MATRIX $-CA_K^{-1} B r_0$ WILL BE SQUARE

THE REFERENCE VECTOR FOR A GIVEN OUTPUT

$$r_0 = -\underbrace{(CA_K^{-1} B)^{-1}}_{\text{MUST BE NONSINGULAR}} y_0$$

The basis for linear feedback control is a linear model, typically found by linearization of a nonlinear model around some stationary operating point

THIS MEANS THAT THIS REFERENCE SIGNAL IS USEFULL ONLY IF THE SYSTEM STATE IS EXACTLY THE STATIONARY STATE OF THE LINEARISATION

2

BASIC CONCEPTS RELATED TO STABILITY OF $\dot{x} = Ax$

$$x(t) = T e^{Dt} T^{-1} x(0)$$

WITH $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}$

CLEARLY IF ANY $e^{\lambda_i t}$ IS UNSTABLE, THEN $x(t)$ WILL GROW AS TIME $\to \infty$

THIS IS ESPECIALLY TRUE IF WE HAVE $x(0)$ WITH ALL COMPONENTS THAT ARE NONZERO

LET $\lambda = a + ib$ THEN $e^{\lambda t}$ $\rightsquigarrow$

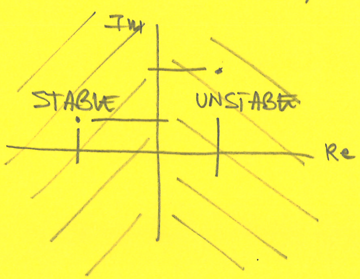
$= e^{at}\left[\cos(bt) + i\sin(bt)\right]$

BUT WE WE NEED TO GET A REAL SIGNAL $\rightarrow$ THIS GETS SORTED OUT BECAUSE FOR EACH $\lambda_i \in \mathbb{C}$ ALSO ITS CONJUGATE COMPLEX IS A SOLUTION OF THE CHARACTERISTIC EQUATION

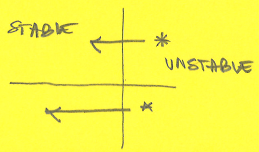
$\rightsquigarrow$ THE COMPLEX PART WILL CANCEL OUT

IF $a > 0$, THEN e^{at} WILL GROW EXPONENTIALLY (UNSTABLE)

IF $a < 0$, THEN e^{at} WILL DECAY EXPONENTIALLY (STABLE)



THE WHOLE IDEA OF CONTROL IS TO ACT ON THE SYSTEM IN SUCH A WAY THAT WHENEVER WE HAVE UNSTABLE EIGENVALUES WE CHANGE THEM TO BE STABLE $\longrightarrow$ THUS STABILIZING THE SYSTEM



WE MOMENTARILY CONSIDER THE DISCRETE-TIME VERSION OF $\dot{x} = Ax$

$x_{k+1} = \tilde{A} x_k$, WITH $x_k = x(k\Delta t)$ FOR SOME Δt

THIS IS THE DISCRETE VERSION OF MATRIX A

$\tilde{A} = e^{A\Delta t}$ (Related by the matrix exponential)

IN DISCRETE TIME, THE NOTION OF STABILITY

$\rightsquigarrow$ GIVEN $x(0)$, THEN

$$x(1) = \tilde{A} x_0$$
$$x(2) = \tilde{A}(x_1) = \tilde{A}^2 x_0$$
$$x(3) = \tilde{A}(x_2) = \tilde{A}^3 x_0$$
$$\vdots$$
$$x(N) = \tilde{A}^N x(0)$$

ANOTHER DIFFICULTY WITH STATE FEEDBACK CONTROL ARISES IF THE SYSTEM IS SUBJECTED TO DISTURBANCES

$\rightsquigarrow$ THE BASIC CONTROLLER DOES NOT TAKE A NON ZERO DISTURBANCE VECTOR INTO ACCOUNT

The solution to these limitation is found in augmentation with integrators

WE CAN INCLUDE INTEGRATION IN SEVERAL WAYS

WE DEFINE THE AUGMENTED STATE $x_a = [\underset{Nx}{x} \ \underset{Ni}{x_i}]^T$

$$\dot{x} = Ax + Bu + B_v v$$
$$u = -Kx + K_i x_i$$
$$\dot{x}_i = -Cx + r$$
$$y = Cx$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} x \\ x_i \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_1} u + \underbrace{\begin{bmatrix} 0 \\ I_r \end{bmatrix}}_{B_r} r + \underbrace{\begin{bmatrix} B_v \\ 0 \end{bmatrix}}_{B_{v1}} v$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_1} \begin{bmatrix} x \\ x_1 \end{bmatrix}$$

OR, IN TERMS OF THE AUGMENTED STATE VECTOR

$$\begin{cases} V\dfrac{d[A]}{dt} = F([A]_f - [A]) - VK_1[A]^2 \\ V\dfrac{d[B]}{dt} = F([B]_f - [B]) + V(K_1[A]^2 - K_3[B]^{1/2}) \end{cases}$$

$$\begin{cases} \dfrac{d[A]}{dt} = \dfrac{F}{V}([A]_f - [A]) - K_1[A]^2 \\ \dfrac{d[B]}{dt} = \dfrac{F}{V}([B]_f - [B]) + (K_1[A]^2 - K_3[B]^{1/2}) \end{cases}$$

$$\begin{cases} \dot{x}_1 = \overbrace{\dfrac{F}{V}(u_1 - x_1) - K_1 x_1^2}^{f_1(x_1, x_2, u_1, u_2)} \\ \dot{x}_2 = \underbrace{\dfrac{F}{V}(u_2 - x_2) + K_1 x_1^2 - K_3 x_2^{1/2}}_{f_2(x_1, x_2, u_1, u_2)} \end{cases}$$

$\left| \begin{matrix} x_1^{ss} \\ x_2^{ss} \\ u_1^{ss} \\ u_2^{ss} \end{matrix} \right.$ STEADY STATE VALUES

$$f_1(x_1, x_2, u_1, u_2) = \underbrace{f_1(x_1^{ss}, x_2^{ss}, \cancel{u_1^{ss}}, u_2^{ss})}_{=0} + \frac{\partial f_1}{\partial x_1}\Big|_{ss}(x_1 - x_1^{ss}) + \frac{\partial f_1}{\partial x_2}\Big|_{ss}(x_2 - x_2^{ss}) + \frac{\partial f_1}{\partial u_1}\Big|_{ss}(u_1 - u_1^{ss}) + \frac{\partial f_1}{\partial u_2}\Big|_{ss}(u_2 - u_2^{ss})$$

$$f_2(x_1, x_2, u_1, u_2) = \underbrace{f_2(x_1^{ss}, x_2^{ss}, \cancel{u_1^{ss}}, u_2^{ss})}_{=0} + \frac{\partial f_2}{\partial x_1}\Big|_{ss}(x_1 - x_1^{ss}) + \frac{\partial f_2}{\partial x_2}\Big|_{ss}(x_2 - x_2^{ss}) + \frac{\partial f_2}{\partial u_1}\Big|_{ss}(u_1 - u_1^{ss}) + \frac{\partial f_2}{\partial u_2}\Big|_{ss}(u_2 - u_2^{ss})$$

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = \begin{bmatrix} x_1 \dot{-} x_1^{ss} \\ x_2 \dot{-} x_2^{ss} \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_2 & \partial f_2/\partial x_2 \end{bmatrix}_{ss} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} + \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{bmatrix}_{ss} \begin{bmatrix} u_1 - u_1^{ss} \\ u_2 - u_2^{ss} \end{bmatrix}$$

$$\begin{cases} \partial f_1/\partial x_1 = -\frac{F}{V} - 2K_1 x_1 & \partial f_1/\partial x_2 = 0 \\ \partial f_2/\partial x_1 = 2K_1 x_1 & \partial f_2/\partial x_2 = -F/V - 1/2\, K_3 x_2^{-1/2} \end{cases}$$

$\rightarrow$ At SS $\rightarrow$ A

$$\begin{cases} \partial f_1/\partial u_1 = F/V & \partial f_1/\partial u_2 = 0 \\ \partial f_2/\partial u_1 = 0 & \partial f_2/\partial u_2 = F/V \end{cases}$$

$\rightarrow$ At SS $\rightarrow$ B

$$\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = \begin{bmatrix} -F/V - 2K_1 x_1^{ss} & 0 \\ 2K_1 x_1^{ss} & -F/V - 1/2\, K_3 (x_2^{ss})^{-1/2} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} +$$

$$+ \begin{bmatrix} F/V & 0 \\ 0 & F/V \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$