STATE ESTIMATION, OBSERVABILITY, STATE OBSERVATION

WE CONSIDERED LINEAR AND TIME-INVARIANT PROCESS MODELS OF THE FORM $\dot{x}(t) = Ax(t) + Bu(t)$

$x \in \mathbb{R}^{N_x}$, $u \in \mathbb{R}^{N_u}$ AND $A \in \mathbb{R}^{N_x \times N_x}$, $B \in \mathbb{R}^{N_x \times N_u}$

MAYBE (MOSTLY) THE MODEL RESULTS FROM A LINEARIZATION OF SOME NONLINEAR MODEL (THUS IT IS EXPRESSED IN DEVIATION VARIABLES)

GIVEN THE MODEL WE LEARNED HOW TO DESIGN A CONTROLLER THAT STABLISES THE DYNAMICS OF THE AUTONOMOUS (CLOSED LOOP) SYSTEM, THUS BRINGING IT BACK TO ITS STEADY-STATE

WE DISCUSSED TWO TECHNIQUES FOR THE TASK

1. EIGENVALUE PLACEMENT (ARBITRARY, BUT OKAY AS A STARTING POINT)

2. OPTIMAL CONTROL USING THE LINEAR QUADRATIC REGULATOR (AN, ARBITRARY BUT REASONABLE, OBJECTIVE FUNCTION DEFINES AN OPTIMALITY CRITERION, AND THERE ARE SOME PARAMETERS TO BE DEFINED, ARBITRARILY :- AGAIN, BY THE USER, MATRICES Q_x AND Q_u)

→ IN BOTH CASES, WE CAN COMPUTE A MATRIX $K \in \mathbb{R}^{N_u \times N_x}$ SO THAT $u(t) = -Kx(t)$ FOR STATE FEEDBACK CONTROL

(value of the) (variables)

We must know the state at time t to be able to compute the control action

LAGRANGE'S FORMULA

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

→ $y = Cx + Du$, thus

$$y(t) = C\left[e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(t)\,d\tau\right] + Du(t)$$

$$= Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,dt + Du(t)$$

KNOWN

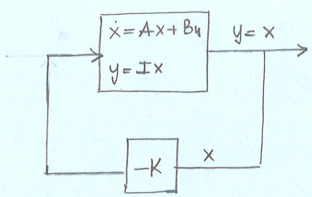
KNOWN, IF $x(t_0)$ IS KNOWN

STATE AT t, $x(t)$

KNOWN, BECAUSE MEASURED

KNOWN, WE 'CHOOSE' THEM

THIS IS KNOW IN ITS ENTIRITY



THE STATE VARIABLE AT TIME t MUST BE KNOW TO COMPUTE THE CONTROL ACTION AT TIME t

— We always assumed we could do this, we always assumed the state was fully measurable

$$\begin{bmatrix} y_1 \\ y_2 \\ | \\ y_{N_y} \end{bmatrix} = Cx = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ | & | & \ddots & | \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{\text{IDENTITY MATRIX } C} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_{N_x} \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$\rightsquigarrow Ax - BKx = \underbrace{(A - BK)}_{\substack{\text{WE CHOOSE K} \\ \text{FOR THIS AUTONOMOUS} \\ \text{SYSTEM TO BEHAVE} \\ \text{AS DESIRED}}} x$$

It is rarely possible to be able to measure the full state

⇝ ONLY SOME STATE VARIABLE ARE ACTUALLY MEASURABLE

⇝ OR, WE CAN ONLY MEASURE SOME COMBINATION (linear mixture)

This information about what and how we measure is encoded in the matrix C

$$y = Cx \qquad (y \in \mathbb{R}^{N_y},\ C \in \mathbb{R}^{N_y \times N_x})$$

WE ARE INTERESTED IN UNDERSTANDING WHETHER, WITH THE AVAILABLE MEASUREMENTS y IT IS POSSIBLE TO KNOW (DETERMINE) THE VALUE OF ALL THE STATE VARIABLES

⇝ THIS CONCEPT IS KNOWN AS OBSERVABILITY (IT IS THE DUAL OF CONTROLLABILITY)

"CAN WE ESTIMATE ANY STATE VALUE $x(t)$ FROM MEASUREMENTS $y(t)$?"

THIS PROPERTY DEPENDS ON THE PAIR (A, C)

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},\quad B = \begin{bmatrix} -2 \\ 3 \end{bmatrix},\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- STABILITY
- CONTROLLABILITY (TESTS)
- FIND SOME K (EIGENVALUE PLACEMENT, LQR)
- OBSERVABILITY (TESTS)

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{full rank}$$

LET $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

- STABILITY, UNCHANGED
- CONTROLLABILITY, UNCHANGED
- TUNE K (UNCHANGED)
- OBSERVABILITY ?

$$\mathcal{O} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{rank deficient}$$

Example

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\quad C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

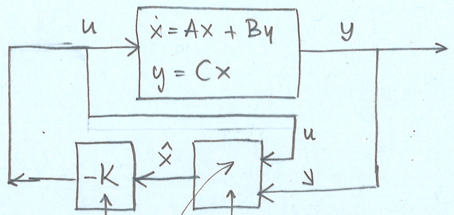
- STABILITY
- CONTROLLABILITY
- TUNE K
- OBSERVABILITY

IF THE SYSTEM IS OBSERVABLE, IT IS POSSIBLE TO ESTIMATE $x(t)$ FROM MEASUREMENT $y(t)$

⇝ IF WE HAVE AN ESTIMATE OF $x(t)$, WE CAN USE IT IN A FEEDBACK CONTROL SCHEME

FULL-STATE

1. WE NEED TO VERIFY WHETHER A SYSTEM IS OBSERVABLE
2. WE NEED TO DESIGN A DEVICE THAT ESTIMATES THE STATE



SENSOR (OBSERVATION) BASED FEEDBACK CONTROL

CONTROLLER (EPLACEMENT LQR, ...)

STATE ESTIMATOR (IT IS ITSELF A DYNAMICAL SYSTEM FOR THE ESTIMATE $\hat{x}$ OF THE FULL STATE VECTOR $\hat{x} \in \mathbb{R}^{N_x}$

WE WILL USE AN OPTIMAL TECHNIQUE FOR THIS KNOWN AS THE KALMAN FILTER

But, first, we need to make sure that the system is OBSERVABLE

(WE APPLIED AN IDENTICAL REASONING WHEN WE CHECKED (TEST) FOR THE CONTROLLABILITY)

⇝ OBSERVABILITY TEST(S)

⇝ LIMITED MEASUREMENT INFORMATION

OBSERVABILITY (A, C)

→ OBSERVABILITY MATRIX $\mathcal{O}$ MUST BE FULL RANK

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_x-1} \end{bmatrix}$$

→ $\mathcal{C} = [B \; AB \; A^2B \; A^3B \; \cdots \; A^{N_x-1}B]$

THE OBSERVABILITY MATRIX

- THE SYSTEM, THE PAIR (A, C), IS OBSERVABLE IF THE COLUMN SPACE OF $\mathcal{O}$ IS FULLRANK
  $\mathrm{rank}(\mathcal{O}) = N_x$
- POPOV, BELEVITCH, HAUTUS
  $\mathrm{rank}\begin{bmatrix}(\lambda I - A) & C\end{bmatrix} = N_x$ for all $\lambda_i \in \sigma(A)$
- OBSERVABILITY GRAMIAN IS NONSINGULAR
  $W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$
  For all $t > 0$ $\det(W_o(t)) \neq 0$

IN MATLAB >> obsv(A,C) BUILD THE OBSERVABILITY MATRIX

~ // ~ // ~

$\mathcal{C} = [B \; AB \; A^2B \; \cdots \; A^{N_x-1}B]$

$\mathcal{O}^T = [C^T \; A^TC^T \; (A^T)^2C^T \cdots (A^T)^{N_x-1}C^T]$

$$\mathcal{O} = \begin{bmatrix} C \\ AC \\ A^2C \\ \vdots \\ A^{N_x-1}C \end{bmatrix}$$

OBSERVABILITY IS THUS EQUIVALENT TO THE CONTROLLABILITY OF SOME 'DUAL SYSTEM' $\dot{x} = A^T x + C^T u$

~ // ~ // ~

Example

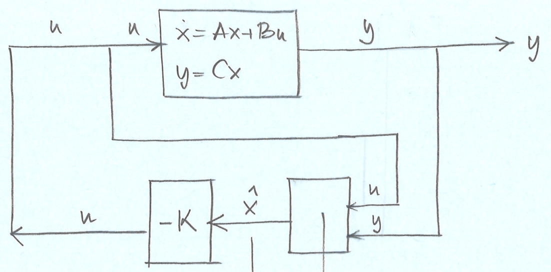
$$A = \begin{bmatrix} 1.8 & 0.6 & -0.2 \\ 0.8 & 1.6 & -0.2 \\ -0.4 & -0.8 & 2.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad C = [0 \; 1 \; 2]$$

— STUDY ME!

Example

$$A = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \; 1]$$

— STUDY ME!

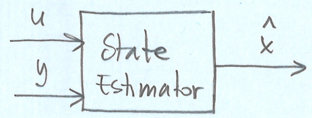


State estimator — LUENBERGER OBSERVER / KALMAN FILTER

State estimate

THE KALMAN FILTER IS AN OPTIMAL (FULL) STATE ESTIMATOR

→ REQUIRES OBSERVABILITY



IT IS IMPORTANT TO UNDERSTAND THE STATE ESTIMATOR AS A DYNAMICAL SYSTEM ITSELF

(it evolves alongside the process)

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K_o(y-\hat{y})$$

$$\hat{y} = C\hat{x}$$

It performs a comparison between the actual observation y and what the estimator predicts ($\hat{y}$)

- WHENEVER THERE IS A DIFFERENCE, WE STIMULATE THE ESTIMATOR

→ TO CORRECT THE ESTIMATE $\hat{x}$ towards → x so that y and $\hat{y}$ match

* INPUTS TO THIS SYSTEM:
  - → CONTROLS, $u(t)$
  - → MEASUREMENTS y

* OUTPUTS TO THIS SYSTEM
  - → ESTIMATED MEASUREMENTS $\hat{y}$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$ THE SYSTEM

$$\begin{cases} \dfrac{d\hat{x}}{dt} = A\hat{x} + Bu + K_0(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$ THE STATE ESTIMATOR

By substituting $\hat{y}$, we get $\dfrac{d\hat{x}}{dt} = A\hat{x} + Bu + K_0 y - K_0 C\hat{x}$

$$= \underbrace{(A - K_0C)}\hat{x} + \underbrace{Bu + K_0 y}$$

THE 'STATE' MATRIX FOR THE STATE ESTIMATES (THE DYNAMICS OF IT)

Also reminds of the full-state feedback controller

"WE CAN TUNE IT BY CHOOSING ITS EIGENVALUES"

$$\underbrace{\begin{bmatrix} B & K_0 \end{bmatrix}}\begin{bmatrix} u \\ y \end{bmatrix} \qquad \begin{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N_u} \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N_y} \end{bmatrix} \end{bmatrix}$$

$(N_x \times N_u)$

$(N_x \times N_y)$

$[B \;\; K_0]$ IS UNDERSTOOD AS THE 'INPUT' MATRIX FOR STATE ESTIMATION

→ WE CAN COMPUTE THE STABILITY

→ (WE CAN COMPUTE THE CONTROLLABILITY)

→ (WE CAN COMPUTE THE OBSERVABILITY)

WE CAN TUNE K_0 BY PLACING THE EIGENVALUES OF $(A - K_0C)$

SUPPOSE THAT $e = x - \hat{x}$

→ THIS IS THE ERROR THAT OUR ESTIMATOR (FULL STATE) MAKES AT ANY t

→ WE ARE INTERESTED IN MINIMISING IT

WE CONSIDER THE DYNAMICS OF THIS ERROR $\dfrac{de}{dt}$

→ WE WANT THAT $de/dt \to 0$

→ THAT THE ESTIMATOR IS STABLE

} ASYMPTOTICALLY, AS $t \to \infty$

$$\frac{de}{dt} = \frac{dx}{dt} - \frac{d\hat{x}}{dt}$$

$$\boxed{Ax + Bu} \qquad \boxed{A\hat{x} + Bu + K_o y - K_o \hat{y}}$$

$$= (Ax + \cancel{Bu}) - A\hat{x} - \cancel{Bu} + K_o y - K_o \hat{y}$$

$$= Ax - A\hat{x} + K_o y + K_o \hat{y}$$

$$= A(x - \hat{x}) + K_o(y - \hat{y}) \qquad \text{with} \quad y = Cx, \quad \hat{y} = C\hat{x}$$

$$= A(x - \hat{x}) + K_o(C\hat{x} - Cx)$$

$$= A(x - \hat{x}) - K_o C(x - \hat{x})$$

$$= (A - K_o C)(x - \hat{x})$$

$$= \underbrace{(A - K_o C)}\, e$$

THE LUENBERGER (STATE) OBSERVER

the dynamics of the error

→ stable / unstable

→ stabilisable using K_o (place its eivalues) or LQR :)

OBSERVABILITY MUST BE VERIFIED

FULL STATE FEED BACK CONTROL WITH A STATE OBSERVER

1.
$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

with $\begin{cases} x \in \mathbb{R}^{Nx} \\ u \in \mathbb{R}^{Nu} \\ y \in \mathbb{R}^{Ny} \end{cases}$ $\begin{cases} A \in \mathbb{R}^{Nx \times Nx} \\ B \in \mathbb{R}^{Nx \times Nu} \\ C \in \mathbb{R}^{Ny \times Nx} \end{cases}$

2.
$$u(t) = -Kx(t)$$

$\{ K \in \mathbb{R}^{Nu \times Nx}$

3.
$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K_o\left[y(t) - \hat{y}(t)\right], \quad \text{with } \hat{y} = C\hat{x} \\ &= A\hat{x}(t) + Bu(t) + K_o y(t) - K_o \hat{y}(t) \\ &= (A - K_o C)\hat{x}(t) + Bu(t) + K_o y(t) \end{aligned}$$

THE RESULTING CLOSED LOOP HAS ORDER $2Nx$ AND ITS EIGENVALUES ARE GIVEN BY $\{\sigma(A-BK) \cup \sigma(A-K_oC)\}$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ K_oC & A-BK-K_oC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

LET $P = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = P^{-1}$

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\hat{z}}(t) \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-K_oC \end{bmatrix} \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}$$

FULL STATE FEEDBACK CONTROL WITH A KALMAN FILTER