Expectations

Probability and distributions

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Expectation

Probability and distributions

Expectation of a random variable

Definition

Expectation

Let X be a random variable

If X is a continuous random variable with PDF $f(x)$ and

$$\int_{-\infty}^{\infty} |x| f(x) dx < \infty,$$

*then the **expectation** of X is*

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

If X is a discrete random variable with PMF $p(x)$ and

$$\sum_x |x| p(x) < \infty,$$

*then the **expectation** of X is*

$$E(X) = \sum_x x p(x)$$

Expectation of a random variable (cont.)

The expectation is also called the **mathematical expectation** of X

- The **expected value** of X
- The **mean** of X
- ...

When the mean specification is used, we indicate $E(X)$ by μ

$$\rightsquigarrow \quad \mu = E(X)$$

Expectation of a random variable (cont.)

Example

Expectation of a constant

Consider a constant 'random' variable K, all of its mass is at constant k

- A discrete 'random' variable with PMF $p_K(k) = 1$, for $k = k$

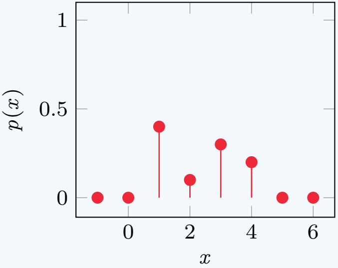
Because $|k|$ is finite, by definition

$$E(K) = \sum_k kp(k) = \underbrace{k}_{K=k} \underbrace{p(k)}_{p(K=k)} = k$$

Expectation of a random variable (cont.)

Example

Let the random variable X of the discrete type have PMF of form



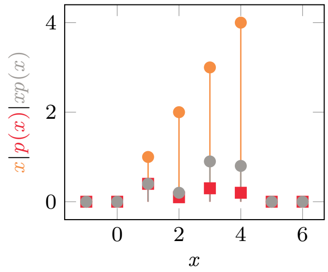
x	1	2	3	4
$p(x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

$p(x) = 0$ elsewhere

$$\begin{aligned} E(X) &= \sum_x x p(x) \\ &= \underbrace{1}_{x=1} \underbrace{(4/10)}_{p(X=1)} + \underbrace{2}_{x=2} \underbrace{(1/10)}_{p(X=2)} + \underbrace{3}_{x=3} \underbrace{(3/10)}_{p(X=3)} + \underbrace{4}_{x=4} \underbrace{(2/10)}_{p(X=4)} \\ &= 2.3 \end{aligned}$$

Expectation of a random variable (cont.)

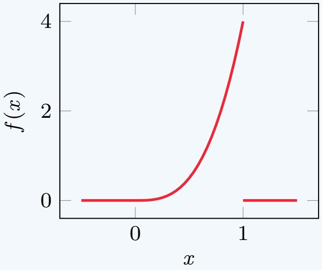
$$E(X) = \sum_x xp(x) = 1(4/10) + 2(1/10) + 3(3/10) + 4(2/10) = 2.3$$



Expectation of a random variable (cont.)

Example

Let X have the PDF

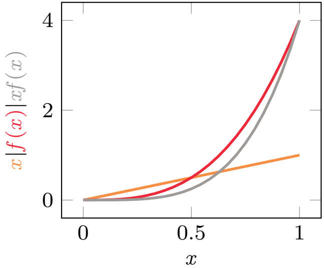


$$f(x) = \begin{cases} 4x^3, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$E(X) = \int_0^1 x(4x^3)\mathrm{d}x = \int_0^1 4x^4 \mathrm{d}x = 4\Big[\frac{x^5}{5}\Big]_0^1 = 4/5$$

Expectation of a random variable (cont.)

$$E(X) = \int_0^1 x(4x^3)\mathrm{d}x = \int_0^1 4x^4\mathrm{d}x = 4\Big[\frac{x^5}{5}\Big]_0^1 = 4/5$$



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Expectation of a random variable (cont.)

Let us consider a function $Y = g(X)$ of a random variable X

- Y is a RV, we could get its expectation
- We could use its distribution

We show that we can get the expectation of Y from the distribution of X

Expectation of a random variable (cont.)

Theorem

Let X be a random variable and let $Y = g(X)$ for some function g

Suppose X is continuous with PDF $f_X(x)$

- *If $\int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty$, then the expectation of Y exists and it is*

$$E(Y) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{1}$$

Suppose X is discrete with PMF $p_X(x)$

- *If $\sum_{x \in \mathcal{S}_X} |g(x)| p_X(x) < \infty$, then the expectation of Y exists and it is*

$$E(Y) = \sum_{x \in \mathcal{S}_X} g(x) p_X(x) \tag{2}$$

$\mathcal{S}_X$ indicates the support of X

Expectation of a random variable (cont.)

We show that the expectation operator E is a linear operator

Expectation of a random variable (cont.)

Theorem

Let $g_1(X)$ and $g_2(X)$ be some functions of a random variable X

Suppose that the expectations of $g_1(X)$ and $g_2(X)$ exist

Then, for any constants k_1 and k_2, expectation of $k_1 g_1(X) + k_2 g_2(X)$ exists

$$E\big[k_1 g_1(X) + k_2 g_2(X)\big] = k_1 E\big[g_1(X)\big] + k_2 E\big[g_2(X)\big] \tag{3}$$

Proof

For the continuous case, existence follows from

1. Existence
2. Triangle inequality
3. Linearity of the integral

Expectation of a random variable (cont.)

$$\begin{aligned}\int_{-\infty}^{\infty} \big|k_1 g_1(X) + k_2 g_2(X)\big| f_X(x) \mathrm{d}x \\ \leq |k_1| \int_{-\infty}^{\infty} \big|g_1(X)\big| f_X(x) \mathrm{d}x + |k_2| \int_{-\infty}^{\infty} \big|g_2(X)\big| f_X(x) \mathrm{d}x \\ < \infty \end{aligned} \quad (4)$$

Result in Equation (3) follows from linearity (additivity) of the integral

- The proof for the discrete case follows from the linearity of sums

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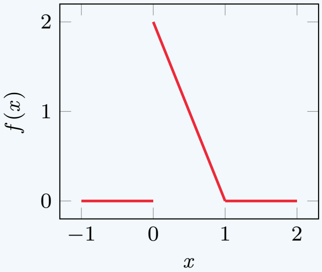
Expectation of a random variable (cont.)

Example

Let X have the PDF



$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

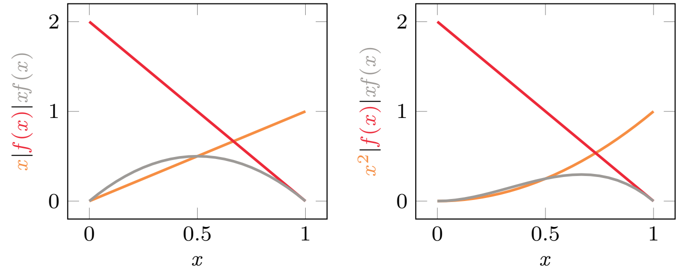
$$E(X) = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x = \int_0^1 (x)\big[2(x-1)\big]\mathrm{d}x = 1/3$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x = \int_0^1 (x^2)\big[2(x-1)\big]\mathrm{d}x = 1/6$$

And,

$$E(6X + 3X^2) = 6\underbrace{(1/3)}_{E(X)} + 3\underbrace{(1/6)}_{E(X^2)} = 5/2$$

Expectation of a random variable (cont.)



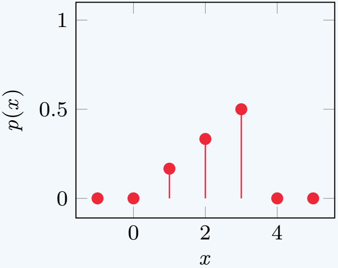
$$E(X) = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x = \int_0^1 (x)\big[2(x-1)\big] \mathrm{d}x = 1/3$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x = \int_0^1 (x^2)\big[2(x-1)\big] \mathrm{d}x = 1/6$$

Expectation of a random variable (cont.)

Example

Let X have the PDF



$$p(x) = \begin{cases} x/6, & x = 1,2,3 \\ 0, & \text{elsewhere} \end{cases}$$

$$E(X^3) = \sum_x x^3 p(x) = \sum_{x=1}^{3} (x^3) x/6$$
$$= 1/6 + 16/6 + 81/6 = 98/6$$

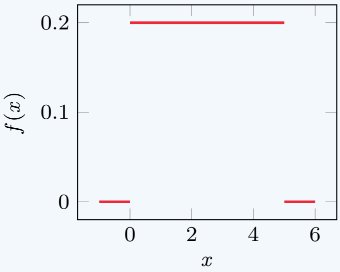
Expectation of a random variable (cont.)

Example

Consider a line segment of length 5 and split it into two parts, at random

Let X be the length of the left part

- Assume that X has the PDF



$$f(x) = \begin{cases} 1/5, & 0 < x < 5 \\ 0, & \text{elsewhere} \end{cases}$$

Expectation of a random variable (cont.)

The expected value of the length X

$$E(X) = \int_0^5 x(1/5)\mathrm{d}x = 5/2$$

The expected value of the length $(5 - X)$

$$E\big[(5 - X)\big] = \int_0^5 (5 - x)(1/5)\mathrm{d}x = 5/2$$

Expectation of a random variable (cont.)

The expected value of the product of the two lengths $E\big[X(5-X)\big]$

$$E\big[X(5-X)\big] = \int_0^5 \big[x(5-x)\big](1/5)\mathrm{d}x = 25/6 \neq (5/2)(5/2)$$

The expected value of a product is not the product of the expected values

Some common expectations

Probability and distributions

Some common expectations

Certain expectations, when they exist, have some special symbols and names

Some common expectations

Let X be a random variable of the discrete type with PMF $p(x)$

$$E(X) = \sum_x xp(x)$$

If the support of X is the set $\{a_1, a_2, a_3, \ldots\}$, then

$$E(X) = a_1 p(a_1) + a_2 p(a_2) + a_3 p(a_3) + \cdots$$

This sum of products can be understood as a weighted average

- It is the weighted sum of terms a_1, a_2, a_3, $\cdots$
- The weight associated with each a_i is $p(a_i)$

Some common expectations

We usually call $E(X)$ the **arithmetic mean** of the values of X

- The mean value of the distribution
- The **mean value** of X
- ...

Definition

Mean

Let X be a random variable and assume the expectation of X exists

*The **mean value** μ of X is defined to be $\mu = E(X)$*

The mean is the first moment (about 0) of a RV

Some common expectations (cont.)

Let X be a discrete RV with support $\{a_1, a_2, a_3, \cdots\}$ and with PMF $p(x)$

$$
\begin{aligned}
E\big[(X-\mu)^2\big] &= \sum_x (x-\mu)^2 p(x) \\
&= (a_1-\mu)^2 p(a_1) + (a_2-\mu)^2 p(a_2) + (a_3-\mu)^2 p(a_3) + \cdots
\end{aligned}
$$

This sum of products can be understood as the weighted average

- It is the weighted sum of the squares of deviation terms
- Deviations of $a_1, a_2, a_3, \ldots$ from their mean value μ
- The weight associated with each $(a_i - \mu)^2$ is $p(a_i)$

Some common expectations (cont.)

This is an important expectation for all kinds of random variables

- It is the second moment of the RV

We usually call $E\big[(X-\mu)^2\big]$, the **variance** of X

Definition

Variance

Let X be a random variable with finite mean μ

Assume that $E[(X-\mu)^2]$ is finite

*The **variance** of X is defined to be $Var(X) = E\big[(X-\mu)^2\big]$*

Some common expectations (cont.)

It is worthwhile to note the identity

$$\text{Var}(X) = E\big[(X-\mu)^2\big] = E\big[(X^2 - 2\mu X + \mu^2)\big]$$

Since the operator E is linear,

$$\begin{aligned}\text{Var}(X) &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2\end{aligned}$$

We usually call $\sigma = \sqrt{\text{Var}(X)}$ the **standard deviation** of X

Some common expectations (cont.)

Number σ is sometimes interpreted as a measure of dispersion

- The spread of the points relative to their mean value μ

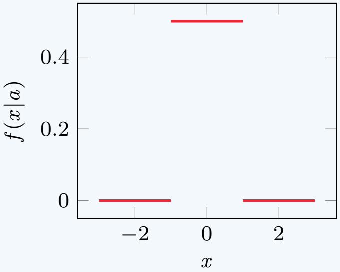
Remark

If the space contains a single point k for which $P(k) > 0$, then

$$p(k) = 1, \quad \mu = k, \quad \sigma = 0$$

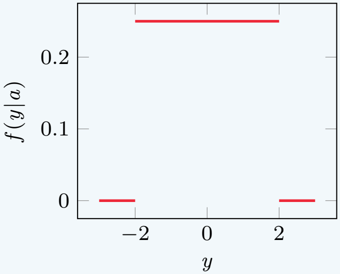
Some common expectations (cont.)

Remark



$$f(x) = \begin{cases} (2a)^{-1}, & -a < x < a \\ 0, & \text{elsewhere} \end{cases}$$

$$\sigma_X = a/\sqrt{3}$$



$$f(y) = \begin{cases} (4a)^{-1}, & -2a < y < 2a \\ 0, & \text{elsewhere} \end{cases}$$

$$\sigma_Y = 2a/\sqrt{3}$$

Some common expectations (cont.)

The standard deviation of Y is twice that of X

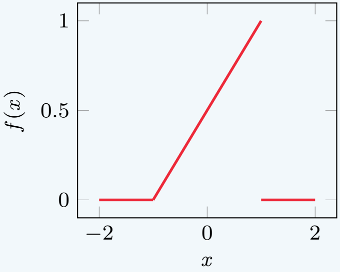
The probability of RV Y is spread out twice as much as is that of X

- Relative to the mean $\mu = 0$

Some common expectations (cont.)

Example

Let X have the PDF



$$f(x) = \begin{cases} \dfrac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

The mean value of X

$$\mu = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x = \int_{-1}^{1} x \frac{x+1}{2} \mathrm{d}x = 1/3$$

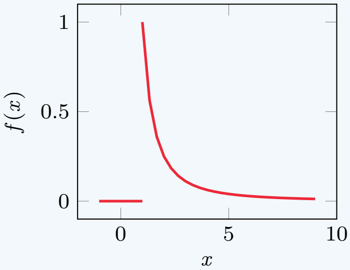
The variance of X

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x - \mu^2 = \int_{-1}^{1} x^2 \frac{x+1}{2} \mathrm{d}x - (1/3)^2 = 2/9$$

Some common expectations (cont.)

Example

Let X have the PDF



$$f(x) = \begin{cases} 1/x^2, & 1 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

The mean value of X does not exist, because the quantity

$$\int_1^\infty |x| \frac{1}{x^2} \mathrm{d}x = \lim_{b \uparrow \infty} \int_1^b \frac{1}{x} \mathrm{d}x = \lim_{b \uparrow \infty} \left[\log(b) - \log(1) \right]$$

does not exist

Some common expectations (cont.)

Definition

Moment generating function

Let X be a random variable

Assume that the expectation of $e^{(tX)}$ exists for $t \in (-h, h)$, for some $h > 0$

*The **moment generating function** (**MGF**) of X is the function*

$$M(t) = E\Big[\exp(tX)\Big], \quad \textit{for } -h < t < h$$

All is needed is that the MGF exists in an open neighbourhood around 0

- Such interval includes an interval $(-h, h)$, for some $h > 0$
- If we set $t = 0$, we have $M(0) = 1$

For the MGF to exist, it must exist in an open interval about 0

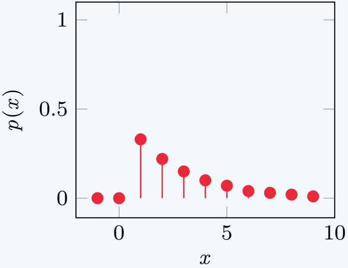
Some common expectations (cont.)

Example

Suppose we have a fair spinner with numbers 1, 2 and 3

- Let X be the number of spins until the first 3

Assuming the spins are independent, the PMF of X is



$$p(x) = \frac{1}{3}(2/3)^{x-1}$$

for $x = 1, 2, 3, \dots$

Some common expectations (cont.)

Using the geometric series[1], the MGF of X is

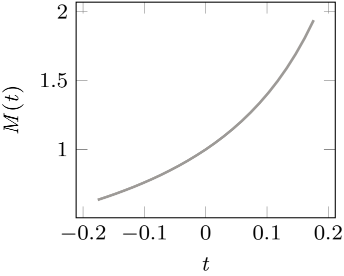
$$M(t) = \sum_{x=1}^{\infty} e^{(tx)} \big[\frac{1}{3}(2/3)^{x-1}\big] = \sum_{x=1}^{\infty} \frac{1}{3} e^{t} (e^{t})^{x-1} (2/3)^{x-1}$$
$$= \frac{1}{3} e^{(t)} \sum_{x=1}^{\infty} \big[e^{(t)}(2/3)\big]^{x-1} = \frac{1}{3} e^{(t)} \frac{1}{\big[1 - e^{(t)}(2/3)\big]}$$

provided that $e^{(t)}(2/3) < 1$, or $t < \log(3/2)$

[1] If $x \in [0, 1)$, then $\sum_{n=0}^{\infty} x^n = 1/(1-x)$. The series diverges for $x \geq 1$

Some common expectations (cont.)

The last interval is an open interval of 0, MGF exists as determined above



Some common expectations (cont.)

Remark

When several RVs are around, useful to indicate the MGF M of X as M_X

Some common expectations (cont.)

Let X and Y be two random variables equipped with their respective MGFs

If X and Y have the same distribution ($F_X(z) = F_Y(z)$, for all z), then

- $M_X(t) = M_Y(t)$, in a neighbourhood of 0
- The converse is also true

$\rightsquigarrow$ The MGF uniquely identify the distribution

Some common expectations (cont.)

Theorem

Let X and Y be random variables with moment generating functions M_X and M_Y, respectively, each existing in some open intervals about 0

Then,

- $F_X(z) = F_Y(z)$ *for all* $z \in \mathcal{R}$, *iff* $M_X(t) = M_Y(t)$,
- *for all neighbourhoods* $t \in (-h, h)$, *for some* $h > 0$

This theorem is important, it is desirable to make the statement persuasive

Some common expectations (cont.)

The random variable X is of the discrete type

Let the MGF of a random variable X be

$$M(t) = \frac{1}{10}e^{t} + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t}, \quad \forall t \in \mathcal{R}$$

Let $p(x)$ be the PMF of X with support $\mathcal{S}_X = \{a_1, a_2, a_3, \dots\}$

Then, because

$$M(t) = \sum_{x} e^{tx} p(x)$$

we have

$$\frac{1}{10}e^{t} + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t} = p(a_1)e^{a_1 t} + p(a_2)e^{a_2 t} + \cdots$$

Some common expectations (cont.)

$$\frac{1}{10}e^{t} + \frac{2}{10}e^{2t} + \frac{3}{10}e^{3t} + \frac{4}{10}e^{4t} = p(a_1)e^{a_1 t} + p(a_2)e^{a_2 t} + \cdots$$

This is an identity that holds for all real values of t

It appears that the RHS should consist of four terms only

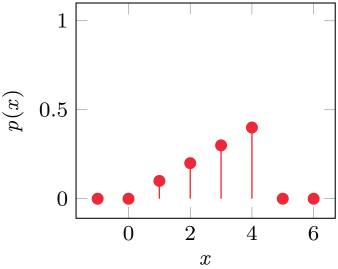
- Each term in the RHS should be equal to one term in the LHS

Hence, we can take

- ⇝ $a_1 = 1,\ p(a_1) = 1/10$
- ⇝ $a_2 = 2,\ p(a_2) = 2/10$
- ⇝ $a_3 = 3,\ p(a_3) = 3/10$
- ⇝ $a_4 = 4,\ p(a_4) = 4/10$

Some common expectations (cont.)

Or, more simply, the PMF of X is



$$p(x) = \begin{cases} x/10, & x = 1, 2, 3, 4 \\ 0, & \text{elsewhere} \end{cases}$$

Some common expectations (cont.)

The random variable X is of the continuous type

Let the MGF of random variable X be

$$M(t) = \frac{1}{1-t}, \quad t < 1$$

Let $f_X(x)$ be the PDF of X over $(-\infty, +\infty)$

Then, because

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \mathrm{d}x$$

we have

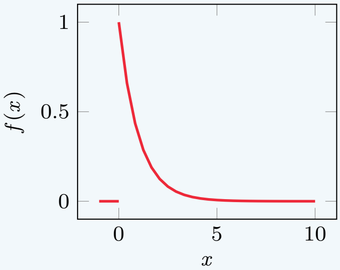
$$\frac{1}{1-t} = \int_{-\infty}^{\infty} e^{tx} f(x) \mathrm{d}x, \quad t < 1$$

It is not obvious how $f(x)$ can be found

Some common expectations (cont.)

Example

It is easy to see that a distribution with PDF



$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

has the MGF $M(t) = (1-t)^{-1}$, for $t < 1$

The random variable X has a distribution with this PDF

- In accordance with the uniqueness statement

Some common expectations (cont.)

A distribution equipped with MGF $M(t)$ is completely determined by it

$\rightsquigarrow$ We can get some properties of the distro straight out of $M(t)$

Existence of $M(t)$ for $t \in (-h, h)$ implies existence its derivatives

- Of all orders, at $t = 0$

We can also swap the order of differentiation and integration

- (summation in the discrete case)

Some common expectations (cont.)

The random variable X is of the continuous type

$$M^{(1)}(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} e^{tx} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} e^{tx} p(x)\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} x e^{tx} f(x)\mathrm{d}x$$

The random variable X is of the discrete type

$$M^{(1)}(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} = \sum_{x} x e^{tx} p(x)$$

In either case, setting $t = 0$ yields

$$M^{(1)}(0) = E(X) = \mu$$

Some common expectations (cont.)

The second derivative,

$$M^{(2)}(t) = \int_{-\infty}^{\infty} x^2 e^{tx} f(x) \mathrm{d}x$$

or

$$M^{(2)}(t) = \sum_{x} x^2 e^{tx} p(x)$$

In either case, setting $t = 0$ yields

$$M^{(2)}(0) = E(X^2)$$

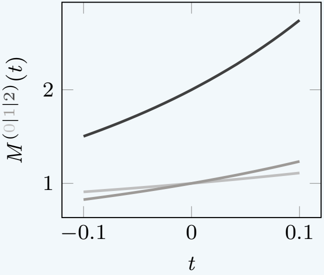
So that

$$\sigma^2 = E(X^2) - \mu^2 = M^{(2)}(0) - \left[M^{(1)}(0)\right]^2$$

Some common expectations (cont.)

Example

Let $M(t) = (1-t)^{-1}$ for $t < 1$, then we have



$$M^{(1)}(t) = (1-t)^{-2}$$
$$M^{(2)}(t) = 2(1-t)^{-3}$$

Hence,

$$\mu = M^{(1)}(0) = 1$$
$$\sigma^2 = M^{(2)}(0) - \mu^2 = 2 - 1 = 1$$

Some common expectations (cont.)

To compute the first two moments, we could have used the PDF

$$\mu = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x$$
$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) \mathrm{d}x - \mu^2$$

Some common expectations (cont.)

In general, if m is a positive integer and if $M^{(m)}(t)$ indicates the m-th derivative of $M(t)$, we have, by repeated differentiation with respect to t,

$$M^{(m)}(0) = E(X^m)$$

Integrals and sums as

$$E(X^m) = \int_{-\infty}^{\infty} x^m f(x) \mathrm{d}x$$
$$\text{or} = \int_x x^m p(x)$$

are called **moments**

$M(t)$ generates them, and it is called the **moment generating function**

- $E(X^m)$ is the **m-th moment of the distribution**

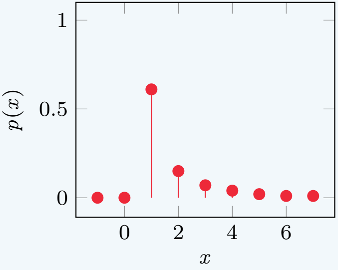
Some common expectations (cont.)

Example

The convergence of the series below is a known result[2]

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \pi^2/6$$

We use it to define the PMF of a random variable X of the discrete type



$$p(x) = \begin{cases} \dfrac{6}{\pi^2 x^2} & x = 1, 2, 3, \ldots \\ 0, & \text{elsewhere} \end{cases}$$

2 ⇝ Evaluating $\xi(2)$

Some common expectations (cont.)

The MGF of this distribution, if it exists, is given by

$$M(t) = E(e^{tX}) = \sum_x e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

To check that the series diverges for $t > 0$, the ratio test[3] can be used

- ⇝ No positive number h such that $M(t)$ exists for $t \in (-h, h)$

The distribution with the PMF $p(x)$ above does not admit a MGF

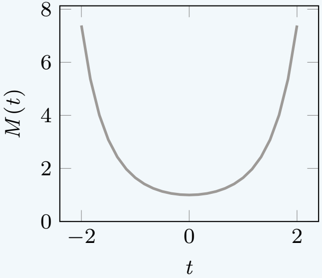
■

[3] $\sum_{n=1}^{\infty} a_n$ and let $L = \lim_{n \uparrow \infty} |a_{n+1}/a_n|$. If $L < 1$, the series converges absolutely, if $L > 1$ it diverges, and if $L = 1$ or the limit fails to exist the test is inconclusive

Some common expectations (cont.)

Example

Let X have the MGF



$$M(t) = e^{t^2/2}$$
for $t \in (-\infty, \infty)$

We can differentiate $M(t)$ any number of times to get the moments of X

Function $M(t)$ can represented as the Maclaurin's series

$$\begin{aligned} e^{t^2/2} &= 1 + \frac{1}{1!}\Big(\frac{t^2}{2}\Big) + \frac{1}{2!}\Big(\frac{t^2}{2}\Big)^2 + \cdots + \frac{1}{k!}\Big(\frac{t^2}{2}\Big)^k + \cdots \\ &= 1 + \frac{1}{2!}t^2 + \frac{(3)(1)}{4!}t^4 + \cdots + \frac{(2k-1)\cdots(3)(1)}{(2k)!}t^{2k} + \cdots \end{aligned}$$

Some common expectations (cont.)

Remark

In general, the Maclaurin's series for $M(t)$

$$
\begin{aligned}
M(t) &= M(0) + \frac{M^{(1)}(0)}{1!}t + \frac{M^{(2)}(0)}{2!}t^2 + \cdots + \frac{M^{(m)}(0)}{m!}t^m + \cdots \\
&= 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^m}{m!}E(X^m) + \cdots
\end{aligned}
$$

The coefficient of $(\frac{t^m}{m!})$ in this expansion of $M(t)$ corresponds to $E(X^m)$

Some common expectations (cont.)

For the considered $M(t)$, we have

$$E(X^{2k}) = (2k-1)(2k-3)\cdots(3)(1) = \frac{2k!}{2^k k!}, \quad k = 1, 2, 3\dots \tag{5}$$
$$E(X^{2k-1}) = 0, \quad k = 1, 2, 3, \dots$$

■

Remark

There exist distribution that do not have the MGFs

Some common expectations (cont.)

Let i indicate the imaginary unit and let t be an arbitrary real number

- We define $\varphi(t) = E(e^{itX})$

This expectation exists for every distribution

$\rightsquigarrow$ **Characteristic function**

To see why $\varphi(t)$ exists for all real t, in the continuous case

$$|\varphi(t)| = \Big| \int_{-\infty}^{\infty} e^{itx} f(x) \mathrm{d}x \Big| \leq \int_{-\infty}^{\infty} |e^{itx} f(x)| \mathrm{d}x$$

Since $f(x)$ is non-negative, and because

$$\big| e^{itx} \big| = \big| \cos(tx) + i \sin(tx) \big| = \sqrt{\cos^2(tx) + \sin^2 tx} = 1,$$

we have that $|f(x)| = f(x)$

Some common expectations (cont.)

Thus,

$$\left|\varphi(t)\right| \leq \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$$

Accordingly, the integral for $\varphi(t)$ exists for all real values of t

Remark

Every distribution of probability has its unique characteristic function

Every characteristic function its its unique distribution of probability

There are similarities between $\varphi(t)/M(t)$ and Fourier/Laplace transforms

Some common expectations (cont.)

Characteristic functions do more than moment generating functions

- $\varphi(t)$ can be used to generate the moments of $F_X(x)$
- (when they exist)

Let X have a distribution with characteristic function $\varphi(t)$

If $E(X)$ and $E(X^2)$ exist they are given respectively by

- $iE(X) = \varphi^{(1)}(0)$
- $i^2 E(X^2) = \varphi^{(2)}(0)$