Aalto University

Ordinary differential equations

Process Automation (CHEM-E7140)
2019-2020

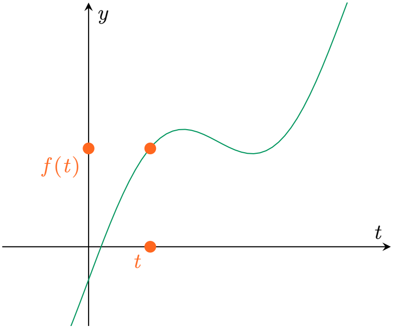
Francesco Corona

Chemical and Metallurgical Engineering
School of Chemical Engineering

A brief review of calculus

Functions and their derivatives

A **function** $y = f(t)$ encodes the relation between two quantities, variables, y and t



Consider the **rate of change** of quantity y corresponding to a change in t

- It is the ratio between the differential change in y and the corresponding differential change in variable t

We conventionally call the **ratio of differential changes** the derivative of function f

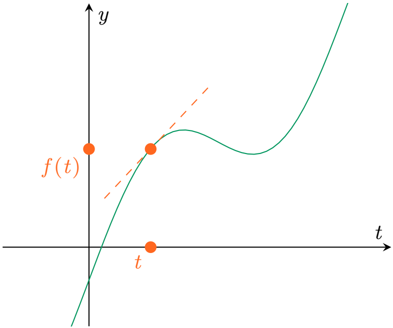
Functions and their derivatives (cont.)

The **derivative** of a function $f(t)$ is the rate of change of the function, it is a number

- $\rightsquigarrow$ The derivative is defined with respect to the independent variable (here, t)
- $\rightsquigarrow$ It can be computed at any point t of the domain of the function

---

We are given some function $f(t)$, we are interested in its derivative at some point t



Derivative of function f with respect to t

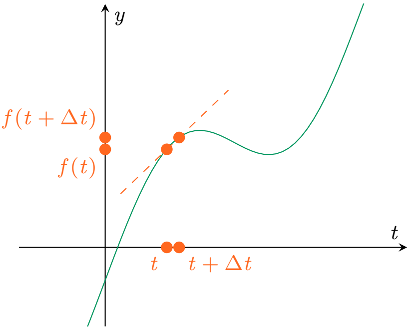
$$\rightsquigarrow \quad \frac{\mathrm{d}f(t)}{\mathrm{d}t}$$

The rate of change is understood as the slope of the tangent line to the function,

- ... at that specific point t

Functions and their derivatives (cont.)

The value of the derivative can be approximated by using small changes in t and $f(t)$



Consider the small change $t \to t + \Delta t$ and the associated $f(t) \to f(t + \Delta t)$

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} \approx \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

The tangent line will be approximated

- By the secant line to function
- Its slope is the approximation

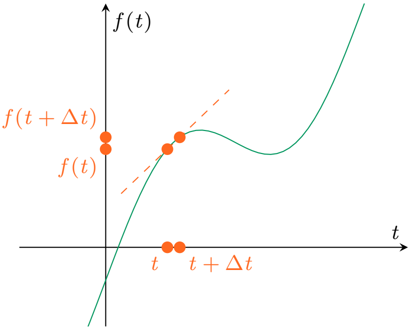
Remember the equation of a line $y = mx + c$ through two points (x_1, y_1) and (x_2, y_2)

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1 = \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{\Delta y / \Delta x} x + \frac{y_2 - y_1}{x_2 - x_1} x_1 + y_1$$

Functions and their derivatives (cont.)

We can improve the quality of the approximation, by letting Δt become smaller

- As $\Delta t \to 0$, the approximation will converge to the true derivative



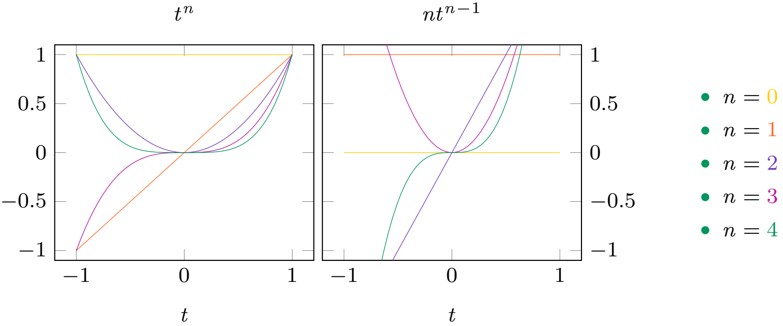
$$\rightsquigarrow \quad \frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Functions and their derivatives (cont.)

Example

Power law

Consider the function $f(t) = t^n$ (the power law) and its derivative $\mathrm{d}f(t)/\mathrm{d}t = nt^{n-1}$



- The derivative is commonly known, but we can derive it
- By using the approximation/definition of derivative

Functions and their derivatives (cont.)

$$f(t) = t^n$$

By definition of derivative, we have

$$\begin{aligned}
\frac{\mathrm{d}f(t)}{\mathrm{d}t} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} &= \frac{1}{\Delta t}\Big[\underbrace{(t+\Delta t)^n}_{\text{Powers of a binomial}} - t^n\Big] \\
&= \frac{1}{\Delta t}\Big[\underbrace{t^n + nt^{n-1}(\Delta t) + \frac{n(n-1)}{2}t^{n-2}(\Delta t)^2 + \cdots}_{\text{Binomial theorem}} - t^n\Big] \\
&= \frac{1}{\Delta t}\Big[\cancel{t^n} + nt^{n-1}(\Delta t) + \frac{n(n-1)}{2}t^{n-2}(\Delta t)^2 + \cdots - \cancel{t^n}\Big] \\
&= \frac{1}{\Delta t}\Big[nt^{n-1}(\Delta t) + \frac{n(n-1)}{2}t^{n-2}(\Delta t)^2 + \underbrace{\mathcal{O}((\Delta t)^3)}_{\text{HO terms}}\Big] \\
&= nt^{n-1} + \frac{n(n-1)}{2}t^{n-2}(\Delta t) + \mathcal{O}((\Delta t)^2) \\
&= nt^{n-1} + \mathcal{O}(\Delta t) \\
&\approx nt^{n-1}
\end{aligned}$$

Functions and their derivatives (cont.)

The first order derivative $\mathrm{d}f(t)/\mathrm{d}t$ is the ratio of two distinct quantities $\mathrm{d}f(t)$ and $\mathrm{d}t$

⇝ The ratio can be manipulated by algebraic procedures

Thus, we can have

$$\mathrm{d}t\frac{\mathrm{d}f}{\mathrm{d}t} = \mathrm{d}f$$

And, we can have

$$\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{\mathrm{d}f}{\mathrm{d}z}$$

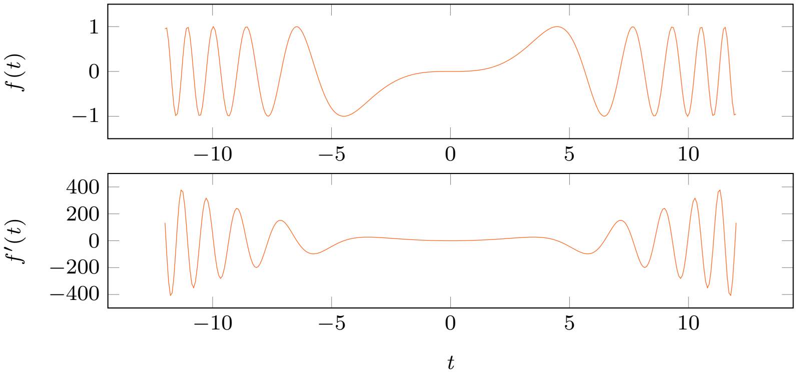
As an application, we get the **chain law** of derivation

$$\frac{\mathrm{d}f(g(t))}{\mathrm{d}t} = \underbrace{\frac{\mathrm{d}f(g(t))}{\mathrm{d}g(t)}}_{f'(g(t))}\underbrace{\frac{\mathrm{d}g(t)}{\mathrm{d}t}}_{g'(t)}$$

Functions and their derivatives (cont.)

Example

Consider the function $f(t) = \sin(t^3)$, compute its first derivative with respect to t

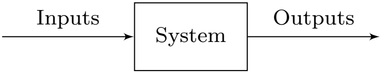


Ordinary differential equations

Introduction to ODEs

Ordinary differential equations (ODEs) are probably our most useful modelling tool

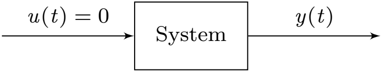
⇝ (Together with probability)



First some motivating and yet simple examples of ordinary differential equations

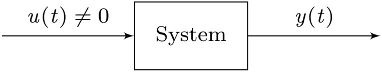
Systems for which the input is identically null over time

- Non-zero initial conditions
- **Force-free response**
- $y(t)$, when $u(t) = 0$



Systems for which the input is not identically null over time

- Zero initial conditions
- **Forced response**
- $y(t)$, when $u(t) \neq 0$



Introduction to ODEs

Example

Consider the problem of modelling the number of bacteria in some bacterial colony

- We assume that each bacterium in the colony gives rise to new individuals
- We can also assume that we know the birth rate, let us denote it by $\lambda > 0$

We assume that, on average, each bacterium will produce λ offsprings per unit time

$\rightsquigarrow$ The size y of the colony varies in time proportionally to its size

$\rightsquigarrow$ (The larger the population, the larger the rate of change)

$$\underbrace{\frac{\mathrm{d}y(t)}{\mathrm{d}t}}_{\dot{y}(t)} = \lambda y(t) \qquad \text{(This identity is an ODE)}$$

We are interested in knowing the size of the population over time, $y(t)$

- Function $y(t)$ is the solution to the ODE

Introduction to ODEs (cont.)

The solution to the ODE is a (class of) function(s) $y(t)$ that satisfies the given identity

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \dot{y}(t) = \lambda y(t)$$

There are many techniques that can be used to solve the ordinary differential equation

- Separate variables and integrate

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \lambda y(t) \quad \rightsquigarrow \quad \int_{y_0}^{y} \frac{1}{y(t)}\mathrm{d}y = \int_{t_0}^{t} \lambda \mathrm{d}t$$

---

1. Move all terms in y to one side
2. Move all terms in t to the other side
3. Integrate both sides over appropriate intervals
4. The intervals are set in terms of initial conditions $(t_0, y_0 = y(t = t_0))$

Introduction to ODEs (cont.)

We have,

$$
\begin{aligned}
&\rightsquigarrow \quad \int_{y_0}^{y} \frac{1}{y(t)} \mathrm{d}y = \int_{t_0}^{t} \lambda \mathrm{d}t \\
&\rightsquigarrow \quad \ln \big[y(t)\big]_{y_0}^{y} = \lambda [t]_{t_0}^{t} \\
&\rightsquigarrow \quad \ln \big[y(t)\big] - \ln \big(y_0\big) = \lambda t - \lambda t_0 \\
&\rightsquigarrow \quad \ln \big[y(t)\big] = \lambda t - \underbrace{\lambda t_0 + \ln \big(y_0\big)}_{\text{constant}} \\
&\rightsquigarrow \quad y(t) = \underbrace{e^{(\lambda t + \text{constant})} = e^{\lambda t} e^{\text{constant}}}_{e^{(\alpha+\beta)} = e^{\alpha} e^{\beta}} = e^{\lambda t} \cdot \text{constant}
\end{aligned}
$$

The bacteria population $y(t)$ evolves in time as an exponential function (grows, $\lambda > 0$)

- The exponential grow is weighted by some constant
- To find the constant, we use initial conditions

Introduction to ODEs (cont.)

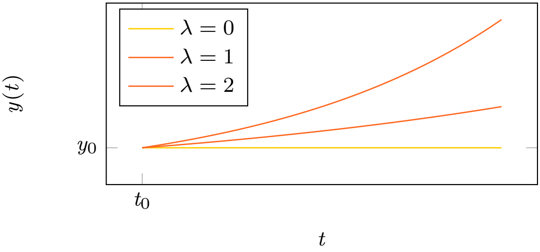
$$y(t) = e^{\lambda t} \cdot \text{constant}$$

Suppose that at time $t = 0$, the population size is known to be $y(t = 0) = y_0 = y(0)$

$$y_0 = \underbrace{e^{\lambda \cdot 0}}_{1} \cdot \text{constant} \quad \rightsquigarrow \quad \text{constant} = y_0$$

That is, the solution to the ODE is given by $y(t) = e^{\lambda t} y_0$

Starting from some initial bacterial population size y_0 at time t_0, the system evolution



$\rightsquigarrow$ $y(t) = e^{\lambda t} y_0$

$\rightsquigarrow$ $y_0 = 10$

Introduction to ODEs (cont.)

```
y0 = 10;                                              % Set initial condition
lambda = 2;                                           % Set model parameter

tMin = 00; tMax = 01; deltaT=0.1;                     % Define time range
tRange = tMin:deltaT:tMax;                            % Min, max, delta

y_clf = @(t) exp(lambda*t)*y0;                        % Set analytical solution

[timeR,y_num] = ode45(@(t,y) lambda*y,tRange,y0);     % Compute numerical the
                                                      % solution using ODE45

figure(1);                                            % Plotting stuff
hold on                                               %
                                                      %
fplot(y_clf,[tMin,tMax],'k');                         % Analytical
plot(timeR,y_num,'.-k');                              % Numerical
                                                      %
stairs(timeR,y_num,'--r');                            % Numerical
hold off                                              %
                                                      %
xlabel('Time','FontSize',24)                          %
ylabel('N. of bacteria','FontSize',24)                %
                                                      %
xlim([tMin,tMax]);                                    %
ylim([0,max(y_num)]);                                 %
                                                      % Could set a legend, ...
```

Introduction to ODEs (cont.)

Example

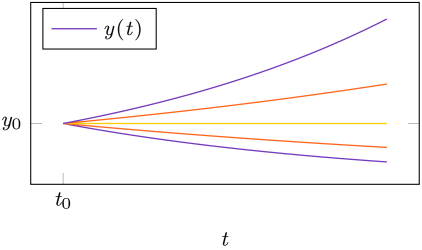
Reconsider the problem of modelling the number of bacteria in some bacterial colony

- We assume that bacteria procreate, at rate λ_1, and die, at rate λ_2

$$\dot{y}(t) = \lambda_1 y(t) - \lambda_2 y(t) = (\lambda_1 - \lambda_2) y(t) = \lambda y(t) \qquad (\lambda = \lambda_1 - \lambda_2)$$

The resulting ODE has not changed, we already know the solution $y(t) = (e^{\lambda t}) y_0$

Suppose that at time $t = 0$, the population size is known to be $y(t = 0) = y_0 = y(0)$



We cannot anylonger discriminate between the effect of birth λ_1 and the effect of death λ_2 anylonger

$\rightsquigarrow$ $\lambda = \{-2, -1, 0, 1, 2\}$

$\rightsquigarrow$ $y_0 = 10$

Solution by Taylor expansion

Ordinary differential equations

Introduction to ODEs (cont.)

Consider the ODE $\dot{y}(t) = \lambda y(t)$ and suppose that we approximate the solution $y(t)$

- Suppose we can express function $y(t)$ by its Taylor series expansion

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

$\rightsquigarrow$ This is a parametric representation of function $y(t)$

$\rightsquigarrow$ The parameters $\{c_0, c_1, c_2, c_3, \dots\}$ are constants

We are interested in determining the actual solution $y(t)$, from this approximation

- That is, we must determine the parameters
- (Fix the coefficients of the expansion)

In general, the Taylor series expansion of some function $f(x)$ around some point x_0

$$f(x) = f(x_0) + \frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x_0} \frac{x - x_0}{1!} + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\Big|_{x_0} \frac{(x - x_0)^2}{2!} + \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}\Big|_{x_0} \frac{(x - x_0)^3}{3!} + \cdots$$

Solution by Taylor series expansion

Consider the ODE $\dot{y}(t) = \lambda y(t)$, we could compute its solution by variable separation

- $\rightsquigarrow$ We considered some value of λ and some initial condition $y(t=0) = y(0)$
- $\rightsquigarrow$ Then, we calculated the closed-form solution $y(t) = e^{\lambda t} y(0)$

By expressing the solution $y(t)$ in terms of its Taylor series expansion, we have

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \mathcal{O}(t^5)$$

Given an expression of $y(t)$, we could also calculate its first derivative $\dot{y}(t)$

$$\rightsquigarrow \quad \dot{y}(t) = 0 + c_1 + 2c_2 t + 3t^2 + 4c_4 t^3 + \mathcal{O}(t^4)$$

Solution by Taylor series expansion (cont.)

We substitute $\dot{y}(t)$ and $y(t)$ into the given ordinary differential equation, $\dot{y}(t) = \lambda y(t)$

$$\underbrace{0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \mathcal{O}(t^4)}_{\dot{y}(t)} = \underbrace{\lambda c_0 + \lambda c_1 t + \lambda c_2 t^2 + \lambda c_3 t^3 + \mathcal{O}(t^4)}_{\lambda y(t)}$$

The identity is satisfied when the coefficients of the powers of t of both sides match

---

- $\rightsquigarrow$ (t^0) $1c_1 = \lambda c_0$
- $\rightsquigarrow$ (t^1) $2c_2 = \lambda c_1$
- $\rightsquigarrow$ (t^2) $3c_3 = \lambda c_2$
- $\rightsquigarrow$ (t^3) $4c_4 = \lambda c_3$
- $\rightsquigarrow$ $\cdots$

If we knew c_0, we could calculate c_1, given c_1 we could calculate c_2, from c_2 we could calculate c_2, given c_2 we could calculate c_3, ...

- $\rightsquigarrow$ c_0 can be determined

By the initial condition, we know that at $t = 0$ we have $y_0 = c_0 + \cancel{c_1 t} + \cancel{c_2 t^2} + \cancel{c_3 t^3} + \cdots$

Solution by Taylor series expansion (cont.)

- $c_0 = y_0$
- $c_1 = \lambda c_0 = \lambda y_0$
- $c_2 = \dfrac{1}{2}\lambda c_1 = \dfrac{1}{2}\lambda^2 y_0$
- $c_3 = \dfrac{1}{3}\lambda c_2 = \dfrac{1}{3!}\lambda^3 y_0$
- $c_4 = \dfrac{1}{4}\lambda c_3 = \dfrac{1}{4!}\lambda^4 y_0$
- $\cdots$

By substituting in the Taylor series expansion of $y(t)$, we obtain

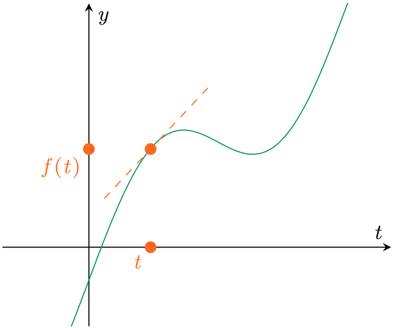
$$\begin{aligned}
y(t) &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots \\
&= y_0 + \lambda y_0 t + \frac{1}{2}\lambda^2 y_0 t^2 + \frac{1}{3!}\lambda^3 y_0 t^3 + \frac{1}{4!}\lambda^4 y_0 t^4 + \mathcal{O}(t^5) \\
&= \underbrace{\left[1 + \lambda t + \frac{1}{2}\lambda^2 t^2 + \frac{1}{3!}\lambda^3 t^3 + \frac{1}{4!}\lambda^4 t^4 + \mathcal{O}(t^5)\right]}_{\text{Exponential function } e^{\lambda t}} y_0 = e^{\lambda t} y_0
\end{aligned}$$

Solution by Taylor series expansion (cont.)

Taylor series expansion

Any smooth function $f(t+\Delta t)$ can be expanded as a Taylor series at some point t

$$f(t \pm \Delta t) = f(t) + \frac{\mathrm{d}f(t)}{\mathrm{d}t}\Delta t + \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\frac{(\Delta t)^2}{2!} + \frac{\mathrm{d}^3 f(t)}{\mathrm{d}t^3}\frac{(\Delta t)^3}{3!} + \cdots + \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\frac{(\Delta t)^n}{n!} + \cdots$$



If we know the function, its first derivative, its second order derivative, ... at point t, we can approximate f near that point

⇝ The more derivatives we add, the more accurate the approximation

⇝ Also, $f(t) = f(t_0) + \frac{\mathrm{d}f(t_0)}{\mathrm{d}t}(t-t_0) + \frac{\mathrm{d}^2 f(t_0)}{\mathrm{d}t^2}\frac{(t-t_0)^2}{2!} + \frac{\mathrm{d}^3 f(t_0)}{\mathrm{d}t^3}\frac{(t-t_0)^3}{3!} + \cdots$

Solution by Taylor series expansion (cont.)

Example

Consider functions $f(t) = \sin(t)$ and $f(t) = \cos(t)$, compute the Taylor expansions

- Expand them about point $t_0 = 0$ (MacLaurin series expansions)

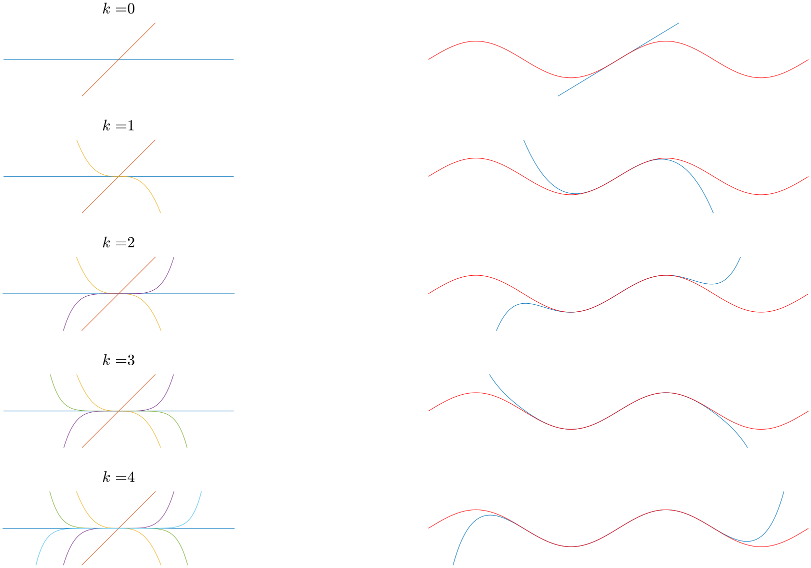
In general, we can write the expansion

$$f(t) = f(t_0) + \frac{\mathrm{d}f(t_0)}{\mathrm{d}t}(t - t_0) + \frac{\mathrm{d}^2 f(t_0)}{\mathrm{d}t^2}(t - t_0)^2 + \frac{\mathrm{d}^3 f(t_0)}{\mathrm{d}t^3}(t - t_0)^3 + \cdots$$

For the sine function, we have

$$\begin{aligned}
\sin(t) &= \sin(0) + \cos(0)t - \frac{1}{2!}\sin(0)t^2 - \frac{1}{3!}\cos(0)t^3 + \frac{1}{4!}\sin(0)t^4 - \frac{1}{5!}\cos(0)t^5 + \cdots \\
&= \cancel{\sin(0)} + \cos(0)t - \cancel{\frac{1}{2!}\sin(0)t^2} - \frac{1}{3!}\cos(0)t^3 + \cancel{\frac{1}{4!}\sin(0)t^4} - \frac{1}{5!}\cos(0)t^5 + \cdots \\
&= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} t^{2k+1}
\end{aligned}$$

Solution by Taylor series expansion (cont.)



Solution by Taylor series expansion (cont.)

$$f(t) = f(t_0) + \frac{\mathrm{d}f(t_0)}{\mathrm{d}t}(t - t_0) + \frac{\mathrm{d}^2 f(t_0)}{\mathrm{d}t^2}(t - t_0)^2 + \frac{\mathrm{d}^3 f(t_0)}{\mathrm{d}t^3}(t - t_0)^3 + \cdots$$

For the cosine function, we have

$$\begin{aligned}
\cos(t) &= \cos(0) - \cancel{\sin(0)t} - \frac{1}{2!}\cos(0)t^2 + \cancel{\frac{1}{3!}\sin(0)t^3} + \frac{1}{4!}\cos(0)t^4 - \cancel{\frac{1}{5!}\sin(0)t^5} + \cdots \\
&= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k}
\end{aligned}$$

Solution by Taylor series expansion (cont.)

$$\cos(t) = 1 + 0t - \frac{t^2}{2!} + 0t^3 + \frac{t^4}{4!} + 0t^5 - \frac{t^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

```
tRange = -2*pi:0.01:+2*pi;                           # Define the t-range

Fcos = @(t) cos(t);                                  # Define functional variable
                                                     # Fcos of t

Ccos_1 = [0 1];                                      # Set coefficients of a 1st
    Tcos_1 = polyval(Ccos_1,xRange);                 # order expansion, evaluation

Ccos_3 = [-1/factorial(2) 0 1];                      # Set coefficients of a 2nd
    Tcos_3 = polyval(Ccos_3,xRange);                 # order expansion, evaluation

Ccos_5 = [1/factorial(4) 0 -1/factorial(2) 0 1];     # Set coefficients of a 3rd
    Tcos_5 = polyval(Ccos_5,xRange);                 # order expansion, evaluation

figure(1); hold on                                   # Some plotting

fplot(Fcos);                                         # Plots function Fcos

plot(xRange, Tcos_1);                                # Plots 1st approximation
plot(xRange, Tcos_3);                                # Plots 2nd approximation
plot(xRange, Tcos_5);                                # Plots 3rd approximation

hold off
```

Solution by Taylor series expansion (cont.)

We can consider the Taylor series expansion of the exponential function e^t (important)

$$\rightsquigarrow \quad e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

We can consider the Taylor series expansion of the function $e^{\lambda t}$ (this is also important)

- By replacing t with λt, we obtain

$$\rightsquigarrow \quad e^{\lambda t} = 1 + (\lambda t) + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!} + \cdots$$

We may want to write the Taylor series expansion of function e^{it}, with $i = \sqrt{(-1)}$

$$\begin{aligned}
\rightsquigarrow \quad e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \\
&= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} + \cdots \\
&= \underbrace{1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots}_{\cos(t)} + i\Big(\underbrace{t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots}_{\sin(t)}\Big) \\
&= \cos(t) + i\sin(t)
\end{aligned}$$

Again, by replacing t with it

$\rightsquigarrow$ (Euler's formula)

Second- and higher-order systems

Ordinary differential equation

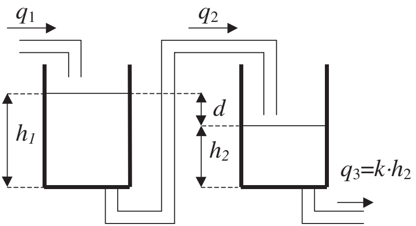
Input-output representation (cont.)

Example

Two tanks (IO representation)

Consider a system consisting of two cylindric liquid tanks, same cross section B [m^2]

- A main inflow to tank 1, a main outflow from tank 2
- The outflow from tank 1 is the inflow to tank 2



First liquid tank

- Inflow, rate q_1 [$m^3 s^{-1}$]
- Outflow, rate q_2 [$m^3 s^{-1}$]
- h_1 is the liquid level [m]

Second liquid tank

- Inflow, rate q_2 [$m^3 s^{-1}$]
- Outflow, rate q_3 [$m^3 s^{-1}$]
- h_2 is the liquid level [m]

Second- and higher-order systems (cont.)

Suppose that flow-rates q_1 and q_2 can be set to some desired value (pumps)

Also, suppose that q_3 depends linearly on the liquid level in the tank, h_2

- $q_3 = k \cdot h_2$ $[\mathrm{m}^3\mathrm{s}^{-1}]$, with k $[\mathrm{m}^2\mathrm{s}^{-1}]$ some appropriate constant

$$\xrightarrow{u(t) = [q_1(t), q_2(t)]'} \boxed{\text{Two tanks}} \xrightarrow{y(t) = d(t)}$$

Inputs, q_1 and q_2, both measurable and manipulable

$\rightsquigarrow$ They influence the liquid levels in the tanks

Output, $d = h_1 - h_2$, measurable, not manipulable

$\rightsquigarrow$ It is influenced by the inputs

Second- and higher-order systems (cont.)

For an incompressible fluid, by mass conservation

$$\begin{cases} \dfrac{\mathrm{d}\,V_1(t)}{\mathrm{d}t} = q_1(t) - q_2(t) \\ \dfrac{\mathrm{d}\,V_2(t)}{\mathrm{d}t} = q_2(t) - q_3(t) \end{cases} \quad \rightsquigarrow \quad \begin{cases} \dot{h}_1(t) = \dfrac{1}{B}q_1(t) - \dfrac{1}{B}q_2(t) \\ \dot{h}_2(t) = \dfrac{1}{B}q_2(t) - \dfrac{k}{B}h_2(t) \end{cases}$$

By taking the first derivative of $y(t) = h_1(t) - h_2(t)$ and rearranging, we obtained

$$\dot{y}(t) = \dot{h}_1(t) - \dot{h}_2(t) = \frac{1}{B}u_1(t) - \frac{2}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)]$$

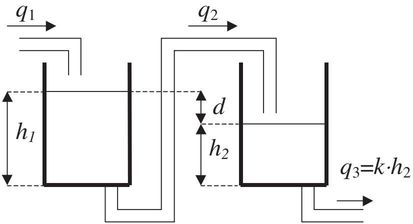
By taking the second derivative of $y(t)$ and rearranging, we obtained

$$\begin{aligned} \ddot{y}(t) &= \frac{1}{B}\dot{u}_1(t) - \frac{2}{B}\dot{u}_2(t) + \frac{k}{B}\dot{h}_1(t) - \frac{k}{B}\dot{y}(t) \\ &= \frac{1}{B}\dot{u}_1(t) - \frac{2}{B}\dot{u}_2(t) + \underbrace{\frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t)}_{\frac{k}{B}\dot{h}_1(t)} - \frac{k}{B}\dot{y}(t) \end{aligned}$$

Second- and higher-order systems (cont.)

Rearranging terms, the IO system's representation is an ordinary differential equation

$$\rightsquigarrow \quad \ddot{y}(t) + \frac{k}{B}\dot{y}(t) - \frac{1}{B}\dot{u}_1(t) + \frac{2}{B}\dot{u}_2(t) - \frac{k}{B^2}u_1(t) + \frac{k}{B}u_2(t) = 0$$



Suppose that the inputs are zero

- $\rightsquigarrow$ $u_1(t) = q_1(t) = 0$
- $\rightsquigarrow$ $u_2(t) = q_2(t) = 0$

Also their derivatives are zero

$$\ddot{y}(t) + \frac{k}{B}\dot{y}(t) = +\cancel{\frac{1}{B}\dot{u}_1(t)} - \cancel{\frac{2}{B}\dot{u}_2(t)} + \cancel{\frac{k}{B^2}u_1(t)} - \cancel{\frac{k}{B}u_2(t)} \quad \rightsquigarrow \quad \ddot{y}(t) + \frac{k}{B}\dot{y}(t) = 0$$

What's $y(t)$, for some $y(0)$ and $\dot{y}(0)$?

Homogeneous equation and modes (cont.)

Homogeneous equation

Consider the ordinary differential equation of a IO model (linear and time invariant)

$$\alpha_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + \alpha_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \alpha_0 y(t) = \beta_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + \cdots + \beta_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + \beta_0 u(t)$$

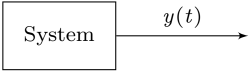
Let the RHS of be zero, define the **homogenous equation** associated to the model

$$\rightsquigarrow \quad a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{dt} + a_0 y(t) = 0$$

The solution $y(t)$ to the homogeneous equation can be defined as the system response (the output) for an input $u(t)$ that is null for $t \geq t_0$ and for given initial conditions

Input- or **force-free response**

- We may denote it as $h(t)$



Second- and higher-order systems (cont.)

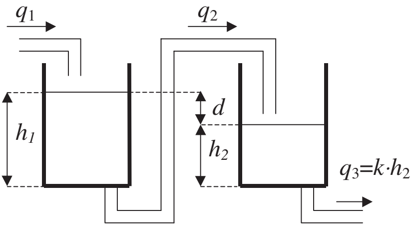
Example

$$\ddot{y}(t) + \frac{k}{B}\dot{y}(t) = 0$$

Interest is in $y(t)$ for some given initial conditions $y(0)$ and $\dot{y}(0)$ (assuming $u(t) = 0$)

- We want to use the Taylor expansion of the solution $y(t)$
- For simplicity, let $k/B = 1$

⇝ We solve $\ddot{y}(t) + \dot{y}(t) = 0$



The differential equation is second-order

⇝ Initial *position* $y(t = 0) = y(0)$

⇝ Initial *velocity* $\dot{y}(t = 0) = \dot{y}(0)$

Second- and higher-order systems (cont.)

$$\ddot{y}(t) + \dot{y}(t) = 0$$

- We assume that $y(t)$ can be expressed using a Taylor expansion

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots$$

- We can compute the first derivative of the solution $y(t)$, $\dot{y}(t)$

$$\dot{y}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \cdots$$

- We compute the second derivative of the solution $y(t)$, $\ddot{y}(t)$

$$\ddot{y}(t) = 2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \cdots$$

$\cdots$, we substitute function and derivatives into the system ODEs

Second- and higher-order systems (cont.)

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots$$

After considering initial conditions $y(t=0) = y(0)$ and $\dot{y}(t=0) = \dot{y}(0)$, we have

$$y(t=0) = c_0 + \cancel{c_1 t} + \cancel{c_2 t^2} + \cancel{c_3 t^3} + \cancel{c_4 t^4} + \cdots = y(0)$$

$$\rightsquigarrow \quad c_0 = y(0)$$

$$\dot{y}(t=0) = c_1 + \cancel{2c_2 t} + \cancel{3c_3 t^2} + \cancel{4c_4 t^3} + \cancel{5c_5 t^4} + \cdots = \dot{y}(0)$$

$$\rightsquigarrow \quad c_1 = \dot{y}(0)$$

- Then, from the ordinary differential equation $\ddot{y}(t) + \dot{y}(t) = 0$ we have

$$\underbrace{2c_2 + 2 \cdot 3c_3 t + 3 \cdot 4c_4 t^2 + 4 \cdot 5c_5 t^3 + \cdots}_{\ddot{y}(t)}$$

$$= \underbrace{-(c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \cdots)}_{\dot{y}(t)}$$

Second- and higher-order systems (cont.)

$$\underbrace{2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \cdots}_{\ddot{y}(t)}$$

$$= \underbrace{-c_1 - 2c_2 t - 3c_3 t^2 - 4c_4 t^3 - 5c_5 t^4 - \cdots)}_{\dot{y}(t)}$$

By equating the coefficients to satisfy the identity and rearranging, we obtain

- $\rightsquigarrow$ $2c_2 = -c_1$
- $\rightsquigarrow$ $2\cdot 3c_3 = -2c_2$
- $\rightsquigarrow$ $3\cdot 4c_4 = -3c_3$
- $\rightsquigarrow$ $4\cdot 5c_5 = -4c_4$
- $\rightsquigarrow$ $\cdots$

- $c_0 = y(0)$
- $c_1 = \dot{y}(0)$
- $c_2 = -\frac{1}{2!}\dot{y}(0)$
- $c_3 = +\frac{1}{3!}\dot{y}(0)$
- $c_4 = -\frac{1}{4!}\dot{y}(0)$
- $c_5 = +\frac{1}{5!}\dot{y}(0)$
- $\cdots$

Second- and higher-order systems (cont.)

$$y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots$$

Substituting the coefficients in the assumed (Taylor's) solution form, we obtain

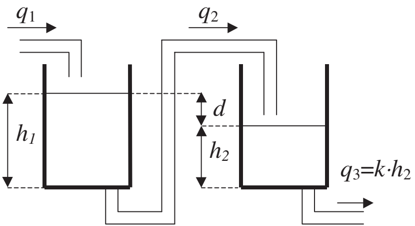
$$\begin{aligned}
y(t) &= y(0) + \dot{y}(0)t - \frac{1}{2!}\dot{y}(0)t^2 + \frac{1}{3!}\dot{y}(0)t^3 - \frac{1}{4!}\dot{y}(0)t^4 + \frac{1}{5!}\dot{y}(0)t^5 - \cdots \\
&= y(0) - \dot{y}(0)\Big(\underbrace{-t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \frac{1}{4!}t^4 + \frac{1}{5!}t^5 - \cdots}_{-1+e^{-t}}\Big) \\
&= \underbrace{y(0) + \dot{y}(0)}_{k_1}\underbrace{-\dot{y}(0)}_{k_2}\, e^{-t} \\
&= k_1 + k_2 e^{-t}
\end{aligned}$$

With k_1 and k_2 some constants depending on the initial conditions

- $k_1 = y(0) + \dot{y}(0)$
- $k_2 = -\dot{y}(0)$

Second- and higher-order systems (cont.)

Example



$$\ddot{y}(t) + \dot{y}(t) = 0$$

For simplicity, we let $\frac{2k}{B} = 1$ and obtained the system evolution by solving the ODE

$$y(t) = y(0) + \dot{y}(0) - \dot{y}(0)e^{-t}$$

```
y0   = ?;                                   % Initial position, set me!
yd0  = ?;                                   % Initial velocity, set me!

tMin = 0;                                   % Initial time is zero,
tMax = ?;                                   % Final time, set me!
tRnage = [tMin, tMax]                       % Define the time interval

yt = @(t) y0 + yd0 - yd0*exp(-t)            % Define the solution function

fplot(yt,tRange)                            % Plot solution over time
```

Second- and higher-order systems (cont.)

Higher-order systems

Consider the general linear time-invariant system and homogeneous (with no inputs)

$$\alpha_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + \alpha_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + \alpha_2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \alpha_1 \frac{\mathrm{d}y}{\mathrm{d}t} + \alpha_0 y = 0$$

Or, equivalently

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \ddot{y} + \alpha_0 y = 0$$

We consider an alternative to assuming that the solution is written as Taylor expansion

---

Instead of using Taylor expansions, we assume that the solution is given by $y(t) = e^{\lambda t}$

- (Which is not very different, in practice)[1]

[1] From $y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$ to $y(t) = e^{\lambda t} = 1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots$.

Second- and higher-order systems (cont.)

If we set the solution to be $y(t) = e^{\lambda t}$, then we can easily compute its derivatives

- $\rightsquigarrow$ $\dot{y}(t) = \lambda e^{\lambda t}$
- $\rightsquigarrow$ $\ddot{y}(t) = \lambda^2 e^{\lambda t}$
- $\rightsquigarrow$ $\cdots$
- $\rightsquigarrow$ $y^{(n)}(t) = \lambda^n e^{\lambda t}$

These functions can be substituted into homogeneous linear time-invariant ODEs

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \dot{y} + \alpha_0 y = 0$$

By substituting the assumed solution and derivatives into the differential equation

$$\rightsquigarrow \quad \left[\alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0\right] e^{\lambda t} = 0$$

The identity is verified for all n values of λ that solve the characteristic equation

$$\underbrace{\underbrace{\alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0}_{\textbf{Characteristic polynomial}} = 0}_{\textbf{Characteristic equation}}$$

Second- and higher-order systems (cont.)

$$\alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0$$

The characteristic equation has n solutions, or roots, collected in set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

- They can be real and/or complex (and associated complex-conjugate) numbers
- They can be positive and/or negative, distinct and repeated (multiplicity)

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \dot{y} + \alpha_0 y = 0$$

For distinct (real and complex) roots, the ODE solution has the simple form

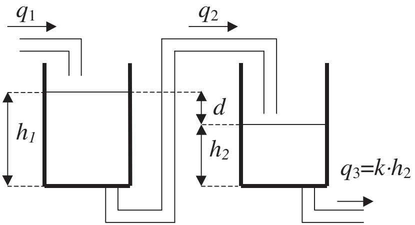
$$\rightsquigarrow \quad y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}$$

The solution is a sum of exponential functions, each weighted by coefficients

- The coefficients are determined from the n initial conditions

Second- and higher-order systems (cont.)

Example



$$\ddot{y}(t) + \frac{k}{B}\dot{y}(t) = 0$$

For simplicity, let $\dfrac{2k}{B} = 1$ and obtained the system evolution from $\ddot{y}(t) + \dot{y}(t) = 0$

$$y(t) = y(0) + \dot{y}(0) - \dot{y}(0)e^{-t}$$

Start by assuming a solution $y(t) = e^{\lambda t}$ and computing its derivatives $\dot{y}(t)$ and $\ddot{y}(t)$

- Substitute then in the original system ODE
- Compute the characteristic equation
- Solve the characteristic equation

Second- and higher-order systems (cont.)

Definition

Characteristic polynomial

Consider the homogeneous part of a linear time-invariant differential equation

$$\alpha_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + \alpha_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = 0$$

The **characteristic polynomial** is a n-order polynomial in the variable λ whose coefficients correspond to the coefficients $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ of the homogeneous equation

$$\rightsquigarrow \quad P(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0 = \sum_{i=0}^{n} \alpha_i \lambda^i$$

Any polynomial of order n with real coefficients has n real or complex-conjugate roots

- The roots are solutions of the **characteristic equation**

$$\rightsquigarrow \quad P(\lambda) = \sum_{i=0}^{n} \alpha_i \lambda^i = 0$$

Second- and higher-order systems (cont.)

In general, there are $r \leq n$ **distinct roots** p_i, each with **multiplicity** ν_i

$$\rightsquigarrow \quad \overbrace{\underbrace{p_1 \quad \cdots \quad p_1}_{\nu_1} \quad \underbrace{p_2 \quad \cdots \quad p_2}_{\nu_2} \quad \cdots \quad \underbrace{p_r \quad \cdots \quad p_r}_{\nu_r}}^{n}$$

- $\rightsquigarrow$ If $i \neq j$, then $p_i \neq p_j$
- $\rightsquigarrow$ $\sum_{i=1}^{r} \nu_i = n$

Consider the particular case in which all roots have multiplicity equal one

$$\rightsquigarrow \quad \overbrace{p_1 \quad p_2 \quad \cdots \quad p_{n-1} \quad p_n}^{n}$$

- $\rightsquigarrow$ If $i \neq j$, then $p_i \neq p_j$
- $\rightsquigarrow$ $\nu_i = 1$, for every i

Second- and higher-order systems (cont.)

Definition

Modes

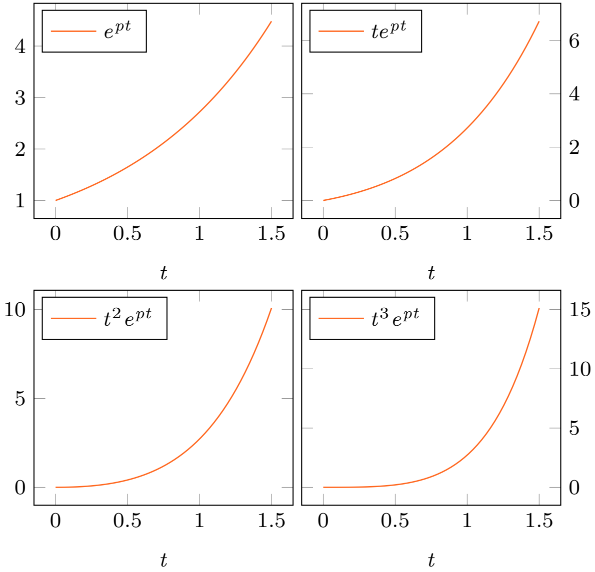
Let p be a root with multiplicity ν of the characteristic polynomial

The **modes** associated to that root are the ν functions of time

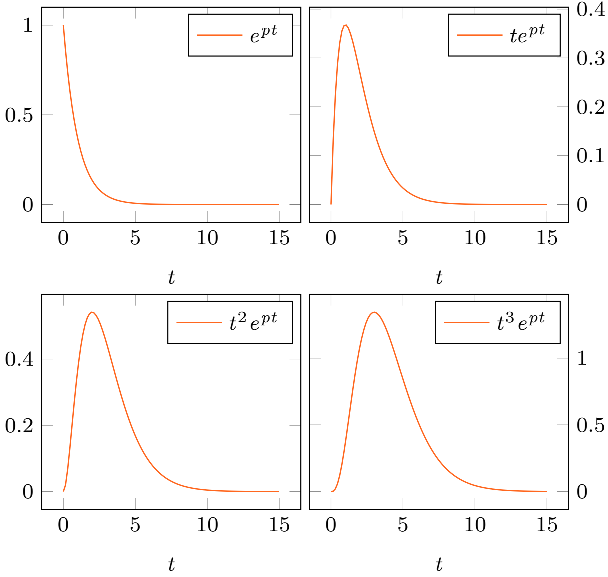
$$\rightsquigarrow \quad e^{pt}, te^{pt}, t^2 e^{pt}, \cdots, t^{\nu-1} e^{pt}$$

A system with a n-order characteristic polynomial has n modes

Let $p = 1$ and $\nu = 4$



Let $p = -1$ and $\nu = 4$



Second- and higher-order systems (cont.)

Modes are functions of time, their linear combinations are a family of functions of time

- The family is parameterised by the coefficients of the combination

Definition

Linear combinations of modes

A linear combination of the n modes is a function $h(t)$ is a weighted sum of the modes

- Each node is weighted by some coefficient

Each root p_i with multiplicity ν_i is associated to a combination of ν_i terms

$$A_{i,0}e^{p_i t} + A_{i,1}te^{p_i t} + \cdots + A_{i,\nu_i-1}t^{\nu_i-1}e^{p_i t} = \underbrace{\sum_{k=0}^{\nu_i-1} A_{i,k}t^k e^{p_i t}}_{\text{root } p_i}$$

There is a total of r distinct roots, $i = 1, \ldots, r$

Second- and higher-order systems (cont.)

$$A_{i,0}e^{p_it} + A_{i,1}te^{p_it} + \cdots + A_{i,\nu_i-1}t^{\nu_i-1}e^{p_it} = \underbrace{\sum_{k=0}^{\nu_i-1} A_{i,k}t^k e^{p_it}}_{\text{root } p_i}$$

As there are r distinct roots, $i = 1, \dots, r$, the complete linear combination of modes

$$h(t) = \underbrace{\sum_{k=0}^{\nu_1-1} A_{1,k}t^k e^{p_1t}}_{\text{root } p_1} + \underbrace{\sum_{k=0}^{\nu_2-1} A_{2,k}t^k e^{p_2t}}_{\text{root } p_2} + \cdots + \underbrace{\sum_{k=0}^{\nu_r-1} A_{r,k}t^k e^{p_rt}}_{\text{root } p_r}$$

$$\rightsquigarrow = \sum_{i=1}^{r}\sum_{k=0}^{\nu_i-1} A_{i,k}t^k e^{p_it}$$

Consider the case in which all roots (n) have multiplicity equal to one

$$\rightsquigarrow \quad h(t) = A_1e^{p_1t} + A_2e^{p_2t} + \cdots + A_ne^{p_nt} = \sum_{i=1}^{n} A_ie^{p_it}$$

(We have omitted the second subscript of coefficients A)

Second- and higher-order systems (cont.)

Example

Consider the following homogenous differential equation

$$3\frac{\mathrm{d}^4 y(t)}{\mathrm{d}t^4} + 21\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 45\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 39\frac{\mathrm{d} y(t)}{\mathrm{d}t} + 12y(t) = 0$$

The associated characteristic polynomial

$$P(\lambda) = 3\lambda^4 + 21\lambda^3 + 45\lambda^2 + 39\lambda + 12 = 3(\lambda+1)^3(\lambda+4)$$

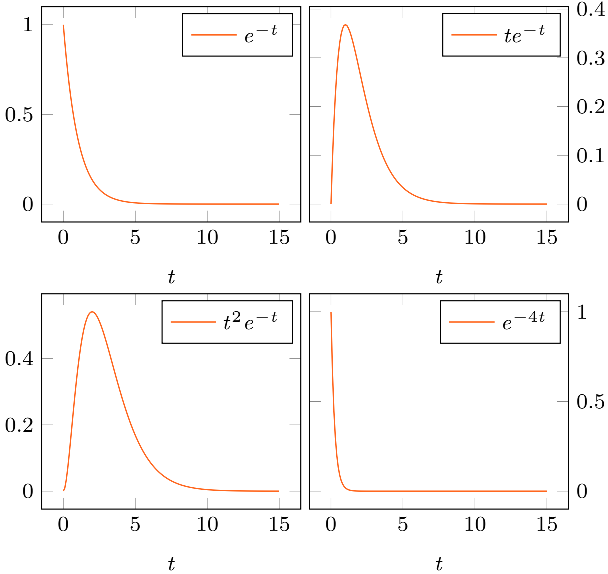
The characteristic equation has four roots, the system has four modes

$$p_1 = -1, \quad (\nu_1 = 3) \quad \rightsquigarrow \quad \begin{cases} e^{-t} \\ te^{-t} \\ t^2 e^{-t} \end{cases}$$

$$p_2 = -4, \quad (\nu_2 = 1) \quad \rightsquigarrow \quad \begin{cases} e^{-4t} \end{cases}$$

The family of functions $h(t)$ is given as a linear combination of the modes

$$h(t) = \underbrace{A_{1,0}e^{-t} + A_{1,1}te^{-t} + A_{1,2}t^2e^{-t}}_{\text{root } p_1} + \underbrace{A_2 e^{-4t}}_{\text{root } p_2}$$


e^{-t}
te^{-t}
t^2e^{-t}
e^{-4t}
0
0.5
1
0.1
0.2
0.3
0.4
0.2
0.4
5
10
15
t

Second- and higher-order systems (cont.)

The modes from the characteristic polynomial, the mixing coefficients are parameters

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

The coefficients determine the force-free evolution, from every possible initial condition

Theorem

Solution of the homogeneous equation

Consider the homogeneous equation

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

A real function $h(t)$ is the solution of a homogeneous linear time-invariant differential equation if and only if it can be written as a linear combination of its modes

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

Second- and higher-order systems (cont.)

Complex and conjugate roots

A characteristic polynomial $P(s)$ with complex roots will have complex signal modes

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} \quad \text{(Yet, their combination must be a real function)}$$

Let $P(s)$ be a characteristic polynomial with roots $p_i = \alpha_i + j\omega_i$ of multiplicity ν_i

- Let $p_i' = \alpha_i - j\omega_i$ with multiplicity $\nu_i' = \nu_i$ be the conjugate complex root

The contribution of each (p_i, p_i') to the linear combination can be re-written

$$\sum_{k=0}^{\nu_i - 1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \quad \text{(Coefficients } M_{i,k} \text{ and } \phi_{i,k}\text{)}$$

Or, equivalently

$$\sum_{k=0}^{\nu_i - 1} \left[B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \right] \quad \text{(Coefficients } B_{i,k} \text{ and } C_{i,k}\text{)}$$

Second- and higher-order systems (cont.)

The solution equations

$$h(t) = \sum_{i=1}^{R}\sum_{k=0}^{\nu_i-1} A_{i,k}\,t^k\,e^{p_i t} + \sum_{i=R+1}^{R+S}\sum_{k=0}^{\nu_i-1} M_{i,k}\,t^k\,e^{\alpha_i t}\cos(\omega_i t + \phi_{i,k})$$

$$\left(\rightsquigarrow \sum_{i=1}^{R} A_i\,e^{p_i t} + \sum_{i=R+1}^{R+S} M_i\,e^{\alpha_i t}\cos(\omega_i t + \phi_i)\right)$$

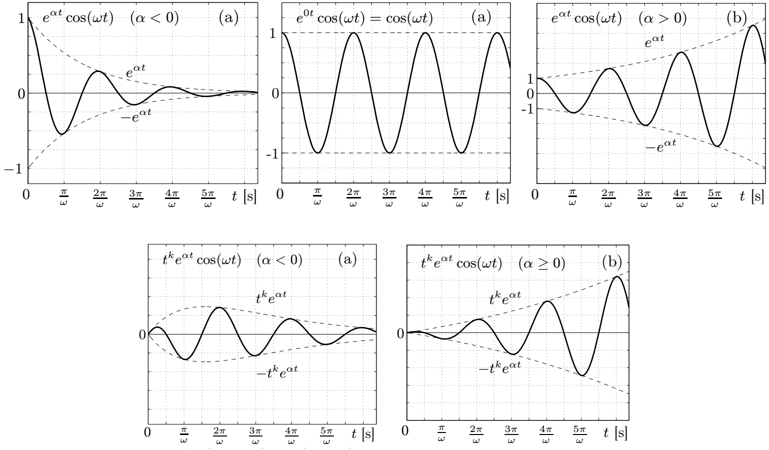
The solution equations

$$h(t) = \sum_{i=1}^{R}\sum_{k=0}^{\nu_i-1} A_{i,k}\,t^k\,e^{p_i t} + \sum_{i=R+1}^{R+S}\sum_{k=0}^{\nu_i-1} \left[B_{i,k}\,t^k\,e^{\alpha_i t}\cos(\omega_i t) + C_{i,k}\,t^k\,e^{\alpha_i t}\sin(\omega_i t)\right]$$

$$\left(\rightsquigarrow \sum_{i=1}^{R} A_i\,e^{p_i t} + \sum_{i=R+1}^{R+S} \left[B_i\,e^{\alpha_i t}\cos(\omega_i t) + C_i\,e^{\alpha_i t}\sin(\omega_i t)\right]\right)$$

They provide the parametric structure of the linear combination and are all equivalent

Second- and higher-order systems (cont.)



Second- and higher-order systems (cont.)

Consider the general linear time-invariant n-order system and homogeneous (no inputs)

$$\alpha_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + \alpha_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + \alpha_2 \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \alpha_1 \frac{\mathrm{d} y}{\mathrm{d}t} + \alpha_0 y = 0$$

Or, equivalently

$$\alpha_n y^{(n)} + \alpha_{n-1} y^{(n-1)} + \cdots + \alpha_1 \ddot{y} + \alpha_0 y = 0$$

We can convert the n-order equation into a set of n first order equations, and solve it

---

As a preprocessing step, we start by dividing all the coefficients by α_n

$$y^{(n)}(t) + \underbrace{a_{n-1}}_{\alpha_{n-1}/\alpha_n} y^{(n-1)}(t) + \cdots + \underbrace{a_2}_{\alpha_2/\alpha_n} \ddot{y}(t) + \underbrace{a_1}_{\alpha_1/\alpha_n} \dot{y}(t) + \underbrace{a_0 y}_{\alpha_0/\alpha_n} = 0$$

Second- and higher-order systems (cont.)

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y = 0$$

Firstly, we introduce a set of n new variables $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]'$

$$\begin{aligned}
x_1(t) &= y(t) \\
x_2(t) &= \dot{y}(t) \\
x_3(t) &= \ddot{y}(t) \\
\cdots &= \cdots \\
x_{n-1}(t) &= y^{(n-2)}(t) \\
x_n(t) &= y^{(n-1)}(t)
\end{aligned}$$

Then, we introduce their first-order derivatives $\dot{x}(t) = [\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t)]'$

$$\begin{aligned}
\dot{x}_1(t) &= \dot{y}(t) = x_2(t) \\
\dot{x}_2(t) &= \ddot{y}(t) = x_3(t) \\
\dot{x}_3(t) &= \dddot{y}(t) = x_4(t) \\
\cdots &= \cdots \\
\dot{x}_{n-1}(t) &= y^{(n-1)}(t) = x_n(t) \\
\dot{x}_n(t) &= y^{(n)}(t) = -a_{n-1}x_n(t) - a_{n-1}x_{n-1}(t) - \cdots - a_2 x_3(t) - a_1 x_2(t) - a_0 x_1(t)
\end{aligned}$$

Second- and higher-order systems (cont.)

That is,

$$
\begin{aligned}
\dot{x}_1 &= 0x_1 + 1x_2 + 0x_3 + 0x_4 + \cdots + 0x_n \\
\dot{x}_2 &= 0x_1 + 0x_2 + 1x_3 + 0x_4 + \cdots + 0x_n \\
\dot{x}_3 &= 0x_1 + 0x_2 + 0x_3 + 1x_4 + \cdots + 0x_n \\
\cdots &= \cdots \\
\dot{x}_{n-1} &= 0x_1 + 0x_2 + 0x_3 + 0x_4 + \cdots + 1x_n \\
\dot{x}_n &= -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 - \cdots - a_{n-1} x_n
\end{aligned}
$$

We can write a $\dot{x}(t)$ as a matrix-vector multiplication $Ax(t)$, a system of equations

$$
\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}}_{\dot{x}(t)}
=
\underbrace{\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1}
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{x(t)}
$$

Second- and higher-order systems (cont.)

Example

Consider a linear and time-invariant homogeneous system $\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = 0$

- The system in IO representation is a third-order ODE

We are interested in formulating the system as a matrix system of 3 first-order ODEs

We introduce 3 variables $x = [x_1, x_2, x_3]'$ and compute their derivatives $\dot{x} = [\dot{x_1}, \dot{x_2}, \dot{x_3}]'$

$$
\begin{aligned}
x_1(t) &= y(t) & \dot{x_1}(t) &= \dot{y}(t) = x_2(t) \\
x_2(t) &= \dot{y}(t) & \dot{x_2}(t) &= \ddot{y}(t) = x_3(t) \\
x_3(t) &= \ddot{y}(t) & \dot{x_3}(t) &= \dddot{y}(t) = -a_2 x_3(t) - a_1 x_2(t) - a_0 x_1(t)
\end{aligned}
$$

In matrix form, we can write

$$
\rightsquigarrow \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \rightsquigarrow \quad \dot{x}(t) = Ax(t)
$$

■

Second- and higher-order systems (cont.)

Example

Consider the following linear and time-invariant homogeneous system

$$3\frac{\mathrm{d}^4 y(t)}{\mathrm{d}t^4} + 21\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 45\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 39\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 12y(t) = 0$$

The system in IO representation is a forth-order ODE

$\rightsquigarrow$ A system of 4 first-order ODEs

We first divide by the leading coefficient ($a_4 = 3$)

$$\frac{\mathrm{d}^4 y(t)}{\mathrm{d}t^4} + \underbrace{7}_{a_3}\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + \underbrace{15}_{a_2}\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \underbrace{13}_{a_1}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \underbrace{4}_{a_0} y(t) = 0$$

By using the general expression derived earlier,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_0 & -a_1 & -a_2 & -a_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -4 & -13 & -15 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$