Signals and distributions
Stochastic algorithms

Francesco Corona

Department of Computer Science
Federal University of Ceará, Fortaleza

Canonical signals
Signals and distributions

Canonical signals

We describe some **signals** or **functions** in the real variable t, time

$$f : \mathcal{R} \to \mathcal{C}$$

In our studies, such functions are often discontinuous

- We introduce a generalisation of function
- The **distribution**

Unit step
Canonical signals

Unit step

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

The function is continuous over the domain, except for the origin
- Discontinuity, size 1

Unit step (cont.)

We can use the unit step to define new functions

Consider some function $f(t) : \mathcal{R} \to \mathcal{R}$, we have

$$f(t)\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

Values of $f(t)$ for $t < 0$ have been set to zero

Graphically,

Ramps

Canonical signals

Ramps

$$\delta_{-2}(t) = \int_{-\infty}^{t} \delta_{-1}(\tau)\mathrm{d}\tau$$

$$= \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

$$= t\delta_{-1}(t)$$

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Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals
Derivatives of a
discontinuous
function
Convolution
integrals
Convolution with
canonical signals
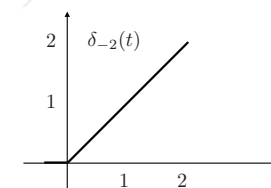
Ramps (cont.)

Definition

Ramp functions

The family of ramp functions $\delta_{-k}(t)$ can be be recursively defined for $k > 2$

$$\delta_{-k}(t) = \underbrace{\int_{-\infty}^{t} \cdots \int_{-\infty}^{t}}_{k-1 \ times} \delta_{-1}(\tau)d\tau = \begin{cases} 0, & t < 0 \\ \dfrac{t^{k-1}}{(k-1)!}, & t \geq 0 \end{cases}$$

$$= \frac{t^{k-1}}{(k-1)!}\delta_{-1}(t)$$

Signals and
distributions

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Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals
Derivatives of a
discontinuous
function
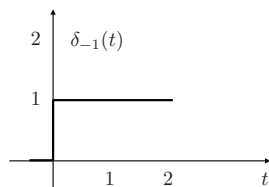Convolution
integrals
Convolution with
canonical signals

Ramps (cont.)

$\delta_{-1}(t)$ (a) $\delta_{-2}(t) = t\delta_{-1}(t)$ (b) $\delta_{-3}(t) = \dfrac{t^2}{2}\delta_{-1}(t)$ (c)

- **Quadratic ramp**, $k = 3$

$$\rightsquigarrow \quad \delta_{-3}(t) = \frac{t^2}{2!}\delta_{-1}(t)$$

- **Cubic ramp**, $k = 4$

$$\rightsquigarrow \quad \delta_{-4}(t) = \frac{t^3}{3!}\delta_{-1}(t)$$

Signals and
distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals
Derivatives of a
discontinuous
function
Convolution
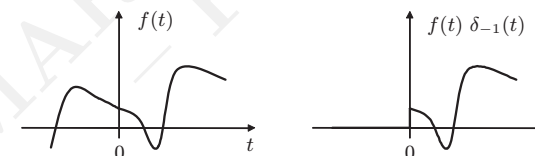integrals
Convolution with
canonical signals

Ramps (cont.)

Definition

Exponential ramp

*A generalisation of the ramp function is the **exponential ramp**, or **cisoid***

$$\frac{t^k}{k!}e^{at}\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t^k}{(k)!}e^{at}, & t \geq 0 \end{cases}$$

It is defined in terms of two parameters
$\rightsquigarrow k \in \mathcal{N}$
$\rightsquigarrow a \in \mathcal{C}$

Signals and
distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals
Derivatives of a
discontinuous
function
Convolution
integrals
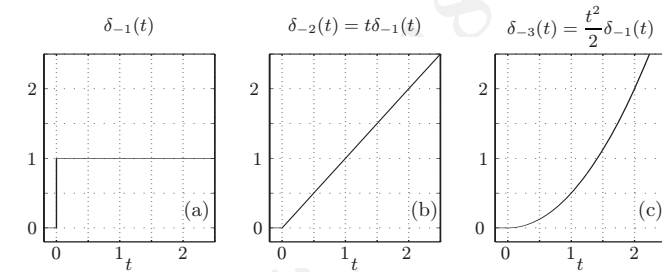Convolution with
canonical signals

Ramps (cont.)

$$\frac{t^k}{k!}e^{at}\delta_{-1}(t)$$

Particular cases that can be generated from the exponential ramp

$\rightsquigarrow a = 0$ and $k = 1, 2, \cdots$, the family of ramp functions
$\rightsquigarrow a = 0$ and $k = 0$, the unit ramp

$\rightsquigarrow k = 0$ and $a \in \mathcal{R}$, exponential function e^{at}
$\rightsquigarrow k = 0$ and $a = j\omega \in \mathcal{I}$, a linear combinations of exponential ramps can be used to describe sinusoidal functions

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Ramps (cont.)

Linear combinations of ramps can be used for polynomial functions

$$c_2 t^2 + c_1 t + c_0$$

Impulse

Canonical signals

Impulse

We can extend the family of canonical signals

We consider the derivatives of the unit step (so far, we integrated it)

- The results of classical calculus cannot be used for the purpose
- The derivative of a discontinuous function is not defined

Impulse (cont.)

We need to generalise the concept of function

$\rightsquigarrow$ The distribution

Slide 1

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Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Impulse (cont.)

Let $\varepsilon > 0$ be some positive scalar

Define the function $\delta_{-1,\varepsilon}(t)$

$$\delta_{-1,\varepsilon}(t) = \begin{cases} 0, & t < 0 \\ t/\varepsilon, & t \in [0,\varepsilon) \\ 1, & t \geq \varepsilon \end{cases}$$

This function is understood as a continuous approximation of the unit step

$$\leadsto \lim_{\varepsilon \to 0} \delta_{-1,\varepsilon}(t) = \delta_{-1}(t)$$

Slide 2

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Impulse (cont.)

Definition

Finite impulse

Function $\delta_{-1,\varepsilon}(t)$ is continuous, it thus possesses a derivative

$$\delta_\varepsilon(t) = \frac{d}{dt}\delta_{-1,\varepsilon} = \begin{cases} 1/\varepsilon, & t \in [0,\varepsilon) \\ 0, & elsewhere \end{cases}$$

*Function $\delta_\varepsilon(t)$ is denoted as **finite impulse** of base ε*

Function $\delta_\varepsilon(t)$ is a rectangle with base ε and with height $1/\varepsilon$

• Area equal to 1, whatever the value of ε

Slide 3

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Impulse (cont.)

We can define the derivative of the unit step

Slide 4

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Impulse (cont.)

Definition

Unit impulse or Dirac function

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t) = \frac{d}{dt}\lim_{\varepsilon \to 0}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon \to 0}\frac{d}{dt}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon \to 0}\delta_\varepsilon(t)$$

Such a definition is not formally correct in the sense of the classical calculus

• It is valid only if we accept the generalisation of a function
• The impulse $\delta(t)$ is not a function, it is a distribution

Slide 1

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Impulse (cont.)

Slide 2

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals
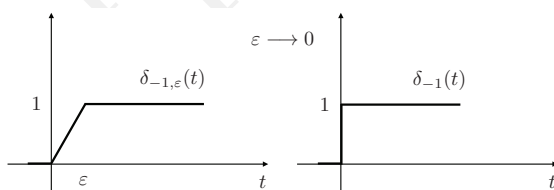
Convolution with canonical signals

Impulse (cont.)

The following properties hold

⤳ $\delta(t)$ is equal to zero everywhere, except the origin

$$\delta(t) = 0, \quad \text{if } t \neq 0$$

⤳ $\delta(t)$ at the origin is equal to infinity

$$\delta(t) = \infty, \quad \text{if } t = 0$$

⤳ The area under $\delta(t)$ is equal to 1

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

Slide 3

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Impulse (cont.)

Theorem

Let $f(t)$ be some continuous function in $t = 0$

- *The product of $f(t)$ and the impulse $\delta(t)$*

$$\rightsquigarrow \quad f(t)\delta(t) = f(0)\delta(t)$$

Let $f(t)$ be some continuous function in $t = T$

- *The product of $f(t)$ and $\delta(t - T)$*

$$\rightsquigarrow \quad f(t)\delta(t - T) = f(T)\delta(t - T)$$

Proof

We have that $\delta(t) = 0$, for $t \neq 0$

The values taken by $f(t)$ for $t \neq 0$ are not significant (impulse is zero)

Slide 4

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Derivative of the impulse

Canonical signals

Slide 1

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
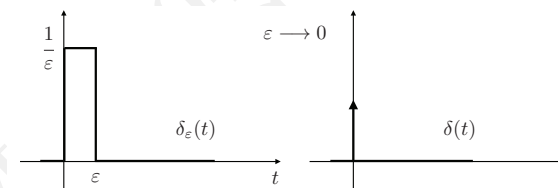function

Convolution
integrals

Convolution with
canonical signals

Derivative of the impulse

We use the limit reasoning to define higher-order derivatives of the impulse

Slide 2

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Derivative of the impulse (cont.)

Consider the function $\hat{\delta}_\varepsilon(t)$

$$\hat{\delta}_\varepsilon(t) = \begin{cases} 4t/\varepsilon^2, & t \in [0, \varepsilon/2) \\ 4/\varepsilon - 4t/\varepsilon^2, & t \in [\varepsilon/2, \varepsilon) \\ 0, & \text{elsewhere} \end{cases}$$

The impulse can be re-defined

$$\rightsquigarrow \quad \delta(t) = \lim_{\varepsilon \to 0} \hat{\delta}_\varepsilon(t)$$

Slide 3

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Derivative of the impulse (cont.)

We define the first-order derivative of the impulse

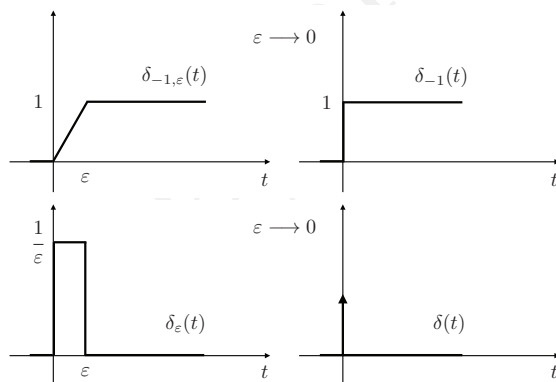$$\rightsquigarrow \quad \delta_1(t) = \frac{d}{dt}\delta(t) = \frac{d}{dt}\lim_{\varepsilon \to 0}\hat{\delta}_\varepsilon(t) = \lim_{\varepsilon \to 0}\frac{d}{dt}\delta_{1,\varepsilon}(t)$$

Slide 4

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Derivative of the impulse (cont.)

Slide 1

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

Derivative of the impulse (cont.)

The higher-order $(k > 1)$ derivatives of the impulse

$$\rightsquigarrow \quad \delta_k(t) = \frac{d^k}{dt^k}\delta(t) = \frac{d}{dt}\delta_{k-1}(t)$$

Slide 2

Signals and distributions

UFC/DC
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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
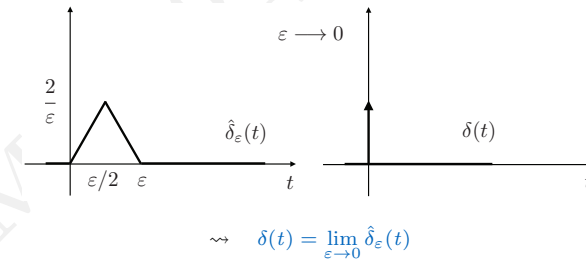Convolution integrals
Convolution with canonical signals

The family of canonical signals

Canonical signals

Slide 3

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals

For $k \in \mathcal{Z}$, we can define a family of canonical signals, $\delta_k(t)$

$\rightsquigarrow$ $\delta_0(t) = \delta(t)$, the impulse $(k = 0)$
$\rightsquigarrow$ $k < 0$, the integrals of the impulse
$\rightsquigarrow$ $k > 0$, the derivatives of the impulse

Such signals are linearly independent

Slide 4

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
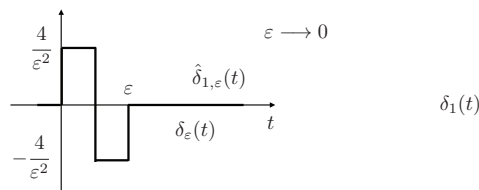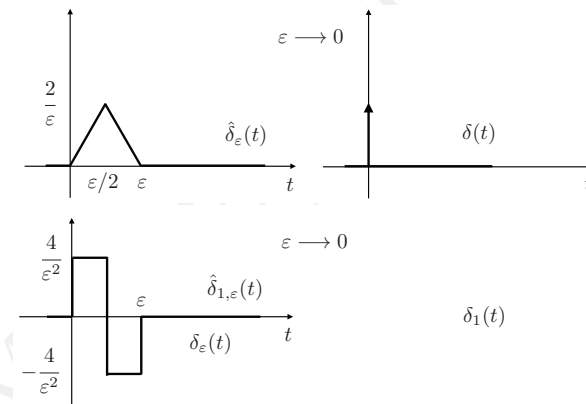The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Definition

Linear dependence of scalar functions

Consider a set of scalar real functions $f_1(t), f_2(t), \ldots, f_n(t)$

$$f_i(t) : \mathcal{R} \rightarrow \mathcal{R}$$

*Such functions are said to be **linearly dependent** over the interval $[t_1, t_2]$, if and only if there exist a set of real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ that are not all equal to zero and such that*

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \cdots + \alpha_n f_n(t) = 0, \quad \forall t \in [t_1, t_2]$$

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Example

Consider two functions $f_1(t)$ and $f_2(t)$

$$f_1(t) = t, \quad t \in (-\infty, \infty)$$

$$f_2(t) = |t| = \begin{cases} -t, & t \in (-\infty, 0) \\ t, & t \in (0, \infty) \end{cases}$$

The two functions are linearly dependent on each interval $[t_1, t_2]$ with $t_2 \leq 0$

- Let $\alpha_1 = \alpha_2 \neq 0$, we have $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$, for every $t \in [t_1, t_2]$

The two functions are linearly dependent on each interval $[t_1, t_2]$ with $t_1 \geq 0$

- Let $\alpha_1 = \alpha_2 \neq 0$, we have $\alpha_1 f_1(t) + \alpha_2 f_2(t) = 0$, for every $t \in [t_1, t_2]$

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

The two functions are linearly independent on $[t_1, t_2]$, $t_1 < 0$ and $t_2 > 0$

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Remark

Two or more functions can be linearly dependent in an interval

- Yet, they can be linearly independent in a larger interval

Conversely, linear independence in a given interval implies linear independence in any larger interval of which the initial interval is a subset

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals
Derivatives of a discontinuous function
Convolution integrals
Convolution with canonical signals

The family of canonical signals (cont.)

Consider the function

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \delta_k(t)$$

Suppose that such a function is identically null over $[a, b]$, with $a \neq b$

$\rightsquigarrow$ Then, $a_k = 0$ for all $k \in \mathcal{Z}$

Derivatives of a discontinuous function

Signals and distributions

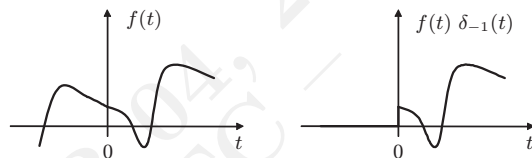## Derivatives of a discontinuous function

We can formally calculate the derivative of discontinuous functions

Discontinuous functions are common in systems analysis

- Zero for $t < 0$ and continuous for $t \geq 0$
- Discontinuity in the origin

Derivatives of a discontinuous function (cont.)

Let $f(t)$ be a continuous function

We are interested in calculating the derivative of function $f(t)\delta_{-1}(t)$

- If $f(0) \neq 0$, then $f(t)\delta_{-1}(t)$ has a discontinuity in $t = 0$

Derivatives of a discontinuous function (cont.)

The first-order derivative

$$\frac{d}{dt}f(t)\delta_{-1}(t) = \left[\frac{d}{dt}f(t)\right]\delta_{-1}(t) + f(t)\left[\underbrace{\frac{d}{dt}\delta_{-1}(t)}_{\delta_{(t)}}\right]$$

$$= \dot{f}(t)\delta_{-1}(t) + f(0)\delta(t)$$

It is the first-order derivative of the original function multiplied by $\delta_{-1}(t)$

- plus the impulse at the origin multiplied by $f(0)$

Derivatives of a discontinuous function (cont.)

The second-order derivative

$$\frac{d^2}{dt^2}f(t)\delta_{-1}(t) = \left[\frac{d}{dt}\dot{f}(t)\right]\delta_{-1}(t) + f(t)\left[\frac{d}{dt}\delta_{-1}(t)\right] + f(0)\left[\frac{d}{dt}\delta(t)\right]$$
$$= \ddot{f}(t)\delta_{-1}(t) + \dot{f}(0)\delta(t) + f(0)\delta_1(t)$$

It is the second-order derivative of the original function multiplied by δ_{-1}
- plus the impulse at the origin multiplied by $\dot{f}(0)$
- plus $\delta_1(t)$ times $f(0)$

Derivatives of a discontinuous function (cont.)

Higher-order derivatives are calculated analogously

$$\frac{d^k}{dt^k}f(t)\delta_{-1}(t) = f^{(k)}\delta_1(t) + f^{(k-1)}(0)\delta(t) + \cdots + f(0)\delta_{k-1}(t)$$
$$= f^{(k)}(t)\delta_{-1}(t) + \sum_{i=0}^{k-1} f^{(i)}(0)\delta_{k-1-i}(t)$$

We used $\delta_0(t) = \delta(t)$

Derivatives of a discontinuous function (cont.)

Example

Consider the function
$$f(t) = \cos{(t)}\delta_{-1}(1)$$

We are interested in its derivatives

The first-order derivative,
$$\frac{d}{dt}\cos{t}\delta_{-1}(t) = \left[\frac{d}{dt}\cos{(t)}\right] + \sin{(0)}\delta_t + \cos{(0)}\delta_1(t)$$
$$= -\sin{(t)}\delta_{-1}(t) + \delta_1(t)$$

The second-order derivative,
$$\frac{d^2}{dt^2}\cos{(t)}\delta_{-1}(t) = \left[\frac{d^2}{dt^2}\cos{(t)}\right]\delta_{-1}(t) - \sin{(0)}\delta(t) + \cos{(0)}\delta_1(t)$$
$$= -\cos{(t)}\delta_{-1}(t) + \delta_{-1}(t)$$

Derivatives of a discontinuous function (cont.)

Example

Consider the cisoid function
$$f(t) = te^{(at)}\delta_{-1}(t)$$

We are interested in its derivatives

The first-order derivative,
$$\frac{d}{dt}te^{(at)}\delta_{-1}(t) = e^{(at)}\delta_{-1}(t) + ate^{(at)} + [te^{(at)}]_{t=0}\delta(t)$$
$$= (1 - at)e^{(at)}\delta_{-1}(t)$$

The second-order derivative,
$$\frac{d^2}{dt^2} = ae^{(at)}\delta_{-1}(t) + a(1 + at)e^{(at)}\delta_{-1}(t) + [(1 + at)e^{(at)}]_{t=0}\delta(t)$$
$$= (2a + a^2t)e^{(at)}\delta_{-1}(t) + \delta(t)$$

Slide 1

Convolution integrals

Signals and distributions

Slide 2

Definition

Convolution

Consider the two functions

$$f, g : \mathcal{R} \to \mathcal{C}$$

*The **convolution** of f with g is a function $h : \mathcal{R} \to \mathcal{C}$ in the real variable t,*

$$h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau$$

Function $h(t)$ is built by using the operator **convolution integral**, $\star$

Slide 3

Example

Consider the two functions

$$f(\tau) = \begin{cases} 1, & \text{if } \tau \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$g(\tau) = \begin{cases} \tau, & \text{if } \tau \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

We want to calculate

Slide 4

To calculate

$$g(\tau - t) = \begin{cases} \tau - t, & \text{if } \tau \in [t, t + 1] \\ 0, & \text{otherwise} \end{cases}$$

we need to shift $g(\tau)$ by a quantity t

- $t > 0$, to the right
- $t < 0$, to the left

Signals and distributions

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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

To calculate

$$g(t - \tau) = \begin{cases} t - \tau, & \text{if } \tau \in [t - 1, t] \\ 0, & \text{otherwise} \end{cases}$$

we need to flip $g(\tau)$ around $\tau = t$

(d)

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

To calculate $f(\tau)g(t - \tau)$

(e)   (f)

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

(g)

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

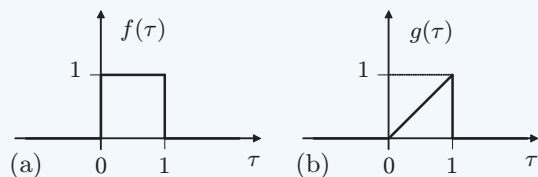Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

Theorem

The convolution operator is commutative

$$f \star g(t) = g \star f(t)$$

Proof

Let $\rho = t - \tau$, then write

$$f \star g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t - \tau)\mathrm{d}\tau = \int_{-\infty}^{+\infty} f(t - \rho)g(\rho)\mathrm{d}\rho = g \star f(t)$$

Signals and distributions

UFC/DC
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2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

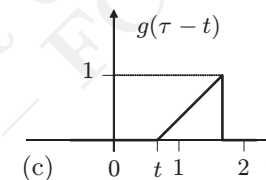Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

Definition

Consider the two functions $f, g : \mathbb{R} \to \mathbb{C}$

Let their derivatives be

$$\dot{f}(t) = \frac{d}{dt}f(t)$$

$$\dot{g}(t) = \frac{d}{dt}g(t)$$

Let their integrals be

$$\mathcal{F}(t) = \int_{-\infty}^{t} f(\tau)\,d\tau$$

$$\mathcal{G}(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$$

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

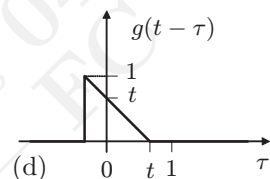Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

The following statements are true

(1) The derivative of the convolution between two functions is given by the convolution of one function with the derivative of the other function

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$$

(2) The integral of the convolution between two functions is given by the convolution of one function with the integral of the other function

$$\int_{-\infty}^{t} f \star g(\tau)\mathrm{d}\tau = f \star \mathcal{G}(t) = \mathcal{F} \star g(t)$$

(3) The integral of a convolution between two function does not change if one of the two operands is derived and the other one is integrated

$$f \star g(t) = \mathcal{F} \star \dot{g}(t) = \dot{f} \star \mathcal{G}(t)$$

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
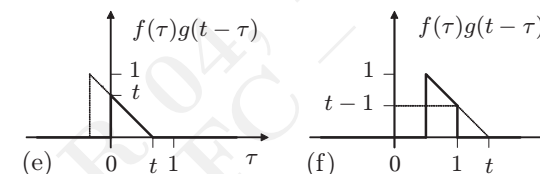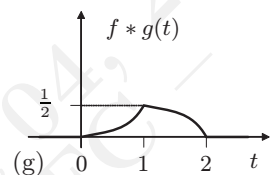The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

Proof

To demonstrate (1), observe that we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star g(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{+\infty} f(\tau)\frac{\mathrm{d}}{\mathrm{d}t}g(t-\tau)\mathrm{d}\tau$$

$$= \int_{-\infty}^{+\infty} f(\tau)\dot{g}(t-\tau)\mathrm{d}\tau = f \star \dot{g}(t)$$

Because of the commutative property $f \star g(t) = g \star f(t)$, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star g(t) = \frac{\mathrm{d}}{\mathrm{d}t}g \star f(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}f(t-\tau)g(\tau)\mathrm{d}\tau$$

$$= \int_{-\infty}^{+\infty} \dot{f}(t-\tau)g(\tau)\mathrm{d}\tau = g \star \dot{f}(t) = \dot{f} \star g(t)$$

Signals and distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the impulse
The family of canonical signals

Derivatives of a discontinuous function

Convolution integrals

Convolution with canonical signals

Convolution integrals (cont.)

To demonstrate (2) where the three functions are identical, we use (1)

Observe that all three functions when evaluated for $t = -\infty$ are null

- Whereas their derivatives are equal, for all values of t

This is because of the definition of integral

$$\frac{d}{dt}\int_{-\infty}^{0} f * g(\tau)d\tau = f * g(t)$$

And, because

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star \mathcal{G}(t) = f \star \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}\right](t) = f \star g(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \star g(t) = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\right](t) = f \star g(t)$$

Signals and
distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Convolution integrals (cont.)

To demonstrate (3), we use (1) again

$\mathcal{F} \star \dot{g}(t)$ is obtained from (1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} \star g(t) = \mathcal{F} \star \left[\frac{\mathrm{d}}{\mathrm{d}t}g\right](t) = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\right] \star g(t) \quad \rightsquigarrow \quad \mathcal{F} \star \dot{g}(t) = f \star g(t)$$

$\dot{f} \star \mathcal{G}(t)$ is obtained by differentiating $f \star \mathcal{G}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}f \star \mathcal{G}(t) = f \star \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}\right](t) = \left[\frac{\mathrm{d}}{\mathrm{d}t}f\right] \star \mathcal{G}(t) \quad \rightsquigarrow \quad f \star g(t) = \dot{f} \star \mathcal{G}(t)$$

■

Signals and
distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Convolution with canonical signals

Signals and distributions

Signals and
distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Convolution with canonical signals

Theorem

Convolution with the impulse

Consider a function $f : \mathcal{R} \rightarrow \mathcal{R}$ continuous in t

We have,
$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t - \tau)d\tau$$

For any interval (t_a, t_b) containing t, we have
$$f(t) = \int_{t_a}^{t_b} f(\tau)\delta(t - \tau)d\tau$$

Signals and
distributions

UFC/DC
SA (CK0191)
2018.1

Canonical signals
Unit step
Ramps
Impulse
Derivative of the
impulse
The family of
canonical signals

Derivatives of a
discontinuous
function

Convolution
integrals

Convolution with
canonical signals

Convolution with canonical signals (cont.)

Proof

Observe that $\delta(t - \tau) = \delta(\tau - t)$ is an impulse centred in $\tau = t$

Thus,

$$\int_{-\infty}^{+\infty} f(\tau)\delta(t - \tau)d\tau = \int_{-\infty}^{+\infty} \underbrace{f(t)\delta(t - \tau)}_{f(t)\delta(t-T)=f(T)\delta(t-T)} \mathrm{d}\tau$$

$$= f(t) \underbrace{\int_{-\infty}^{+\infty} \delta(t - \tau)\mathrm{d}\tau}_{\int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t=\int_{0^-}^{0^+} \delta(t)dt=1} = f(t)$$

■

Convolution with canonical signals (cont.)

Theorem

Consider a function $f : \mathcal{R} \to \mathcal{R}$ continuous with k continuous derivatives

We have,

$$\frac{d^k}{dt^k} f(t) = \int_{-\infty}^{+\infty} f(\tau) \delta_k(t - \tau) d\tau$$

Proof

Observe that $f(t) = f \star \delta(t)$

By repeatedly differentiating and using that $\dfrac{\mathrm{d}}{\mathrm{d}t} f \star g(t) = f \star \dot{g}(t) = \dot{f} \star g(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta(t) = f \star \left[\frac{\mathrm{d}}{\mathrm{d}t} \delta \right](t) = f \star \delta_1(t)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta_1(t) = f \star \delta_2(t)$$

$$\cdots = \cdots$$

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f \star \delta_{k-1}(t) = f \star \delta_k(t)$$