



Aalto University

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Sylvester's formula

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Modal matrix

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Linear time-invariant processes: Dynamics

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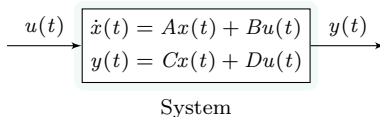
Representation and analysis

LTI systems - Dynamics

Representation and analysis

Consider a linear and time-invariant system of order N_x , in **state-space** representation

$$\begin{aligned} \rightsquigarrow \text{Let } N_x \text{ be the number of outputs} \\ \rightsquigarrow \text{Let } N_u \text{ be the number of inputs} \end{aligned} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$



A ($N_x \times N_x$), B ($N_x \times N_u$), C ($N_y \times N_x$) and D ($N_y \times N_u$) are the system matrices

$\rightsquigarrow x(t)$ is the **state vector**

- (N_x components)

$\rightsquigarrow \dot{x}(t)$ is the derivative of the state vector

- (N_x components)

$\rightsquigarrow u(t)$ is the **input vector**

- (N_u components)

$\rightsquigarrow y(t)$ is the **output vector**

- (N_y components)

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The analysis problem: Determine the behaviour of state $x(t)$ and output $y(t)$ for $t \geq t_0$

- We are given the input function $u(t)$, for $t \geq t_0$
- We are given the initial state $x(t_0)$

The solution to the analysis, for $t \geq t_0$, an initial state $x(t_0)$ and an input $u(t \geq t_0)$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0) + C \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau}_{Cx(t)} + Du(t)$$

The solution is known as the **Lagrange formula**

- Based on the **state transition matrix**

$$\rightsquigarrow e^{At}$$

Force-free and forced evolution

Note that we can write the state solution $x(t)$, for $t \geq t_0$, as the sum of two terms

$$\begin{aligned} x(t) &= \underbrace{e^{A(t-t_0)}x(t_0)}_{x_u(t)} + \underbrace{\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau}_{x_f(t)} \\ &= x_u(t) + x_f(t) \end{aligned}$$

↪ The **force-free evolution** of the state, $x_u(t)$

↪ The **forced evolution** of the state, $x_f(t)$

The **force-free evolution** of the state, from the initial condition $x(t_0)$

↪ $e^{A(t-t_0)}$ determines the transition from $x(t_0)$ to $x(t)$

↪ In the absence of contribution from the input

The **forced evolution** of the state, from the contribution of input $u(t)$

↪ In the absence of an initial condition $x(t_0)$

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Consider a square $(N_x \times N_x)$ matrix A , the exponential e^A is square $(N_x \times N_x)$ matrix

$$\rightsquigarrow e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$
$$= \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

The **state transition matrix** is the matrix exponential e^{At} of the matrix At

\rightsquigarrow It is a matrix whose elements are functions of time

\rightsquigarrow We discuss its meaning and how to compute it

The state transition matrix (cont.)

The exponential function

Let z be some scalar, by definition its exponential is a scalar

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \end{aligned}$$

The series always converges

The matrix exponential

Let A be a $(n \times n)$ matrix, by definition its exponential is a $(n \times n)$ matrix

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \end{aligned}$$

The series always converges

The state transition matrix (cont.)

The product of several matrices

The product of matrix A and B is only possible when the matrixes are compatible

- Number of columns of A must equal the number of rows of B

The same applies to the product of several matrixes

$$\underbrace{M}_{(m \times n)} = \underbrace{A_1}_{(m \times m_1)} \underbrace{A_2}_{(m_1 \times m_2)} \cdots \underbrace{A_{k-1}}_{(m_{k-2} \times m_{k-1})} \underbrace{A_k}_{(m_{k-1} \times n)}$$

Powers of a matrix

Let A be an order- n square matrix, we want to define the k -th power of matrix A

The k -th power of matrix A is the n -order matrix A^k

$$A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$$

Some special cases,

$$\leadsto A^{k=0} = I$$

$$\leadsto A^{k=1} = A$$

The state transition matrix (cont.)

Definition

The state transition matrix

Consider a linear and time-invariant state-space model with $(N_x \times N_x)$ state matrix A

$$\begin{array}{ccc} u(t) & \xrightarrow{\quad} & \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array}} \xrightarrow{\quad} y(t) \end{array} \quad \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}$$

System

The **state transition matrix** of this system is given by the $(N_x \times N_x)$ matrix e^{At}

$$\begin{aligned} e^{At} &= \underbrace{\frac{A^0 t^0}{0!}}_I + \underbrace{\frac{A^1 t^1}{1!}}_{At} + \underbrace{\frac{A^2 t^2}{2!}}_{(A^2 t^2)/2!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \end{aligned}$$

The state transition matrix is well defined for any square matrix A

- (The series always converges)

The state transition matrix (cont.)

It is not convenient to determine the state transition matrix starting from its definition

↪ One exception is when A is (block-)diagonal

The matrix exponential of block-diagonal matrixes

Consider any block-diagonal matrix A , we have

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_q \end{bmatrix} \rightsquigarrow e^A = \begin{bmatrix} e^{A_1} & 0 & \cdots & 0 \\ 0 & e^{A_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{A_q} \end{bmatrix}$$

The matrix exponential of diagonal matrixes (as special case)

For any diagonal ($n \times n$) matrix A , we have

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \rightsquigarrow e^A = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}$$

The state transition matrix (cont.)

Example

Consider a linear and time-invariant state-space model with (2×2) diagonal matrix A

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

We are interested in the corresponding state transition matrix

We have,

$$e^{At} = \begin{bmatrix} e^{(-1)t} & 0 \\ 0 & e^{(-2)t} \end{bmatrix}$$

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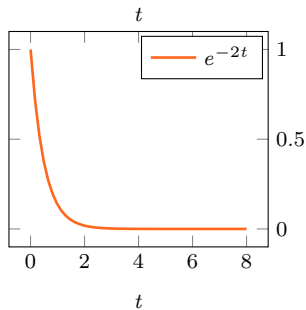
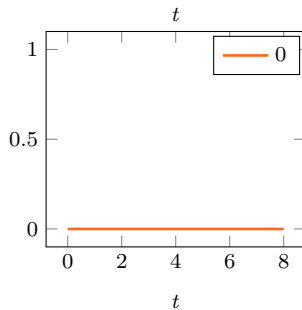
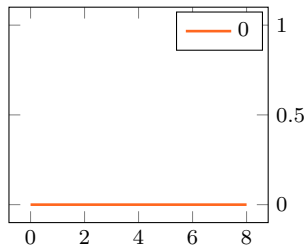
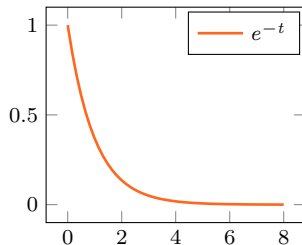
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We state without proof some fundamental results about a state transition matrix e^{At}
 \rightsquigarrow They are needed to derive Lagrange formula

Proposition

Derivative of the state transition matrix

Consider the state transition matrix e^{At} , we have,

$$\begin{aligned}\frac{d}{dt}e^{At} &= Ae^{At} \\ &= e^{At}A\end{aligned}$$

By using the derivative property, we have that A commutes with e^{At}
 \rightsquigarrow (This result is important)

Properties (cont.)

Proposition

Composition of two state transition matrices

Consider the two state transition matrices e^{At} and $e^{A\tau}$, we have

$$e^{At}e^{A\tau} = e^{A(t+\tau)}$$

Proposition

Inverse of the state transition matrix

Let e^{At} be a state transition matrix, its inverse $(e^{At})^{-1}$ is matrix e^{-At}

$$\begin{aligned}e^{At}e^{-At} &= e^{-At}e^{At} \\ &= I\end{aligned}$$

A state transition matrix e^{At} is always invertible (non-singular)

- Even if A were singular

Properties (cont.)

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Matrix inverse

Consider a square matrix A of order n

We define the **inverse** of A the square matrix of order n , A^{-1}

$$A^{-1}A = AA^{-1} = I$$

The inverse of matrix A exists if and only if A is non-singular

- When the inverse exists, it is also unique

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The state transition matrix

Sylvester's expansion

We determine the analytical expression of the state transition matrix e^{At}

- The procedure is known as **Sylvester expansion**
- (Does not require computing the infinite series)
- There are also other procedures (later)

Proposition

The Sylvester's expansion

Let A be a $(n \times n)$ matrix and let the corresponding state transition matrix be e^{At}

We have,

$$\begin{aligned} e^{At} &= \beta_0(t)I + \beta_1(t)A + \beta_2(t)A^2 + \cdots + \beta_{n-1}(t)A^{n-1} \\ &= \sum_{i=0}^{n-1} \beta_i(t)A^i \end{aligned}$$

The coefficients β_i of the expansion are appropriate functions of time

- ↪ They can be determined by solving a set of linear equations
- ↪ There is a finite number (n) of them

Sylvester's expansion (cont.)

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We show how to determine the coefficients when A has eigenvalues of multiplicity one

We will not consider the other cases, because rather involved and tedious to derive

- ↪ Matrix A has complex eigenvalues (with multiplicity larger one)
- ↪ Matrix A has complex eigenvalues (with multiplicity one)
- ↪ Eigenvalues of A have multiplicity larger than one

Sylvester's expansion (cont.)

Eigenvalues with multiplicity one

Let matrix A have distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

$$\begin{aligned} e^{At} &= \sum_{i=0}^{n-1} \beta_i(t) A^i \\ &= \beta_0(t)I + \beta_1(t)A + \beta_2(t)A^2 + \dots + \beta_{n-1}(t)A^{n-1} \end{aligned}$$

The n unknown functions $\beta_i(t)$ are those that solve the system

$$\rightsquigarrow \begin{cases} 1\beta_0(t) + \lambda_1\beta_1(t) + \lambda_1^2\beta_2(t) + \dots + \lambda_1^{n-1}\beta_{n-1}(t) = e^{\lambda_1 t} \\ 1\beta_0(t) + \lambda_2\beta_1(t) + \lambda_2^2\beta_2(t) + \dots + \lambda_2^{n-1}\beta_{n-1}(t) = e^{\lambda_2 t} \\ \dots \\ 1\beta_0(t) + \lambda_n\beta_1(t) + \lambda_n^2\beta_2(t) + \dots + \lambda_n^{n-1}\beta_{n-1}(t) = e^{\lambda_n t} \end{cases}$$

Sylvester's expansion (cont.)

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Or, equivalently, in matrix form

$$V\beta = \eta$$

- The vector of unknowns

$$\rightsquigarrow \beta = [\beta_0(t) \quad \beta_1(t) \quad \cdots \quad \beta_{n-1}(t)]^T$$

- The coefficients matrix¹

$$\rightsquigarrow V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

- The known vector

$$\rightsquigarrow \eta = [e^{\lambda_1 t} \quad e^{\lambda_2 t} \quad \cdots \quad e^{\lambda_n t}]^T$$

¹A matrix in this form is known to be a Vandermonde matrix.

Sylvester's expansion (cont.)

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$$\eta = [e^{\lambda_1 t} \quad e^{\lambda_2 t} \quad \dots \quad e^{\lambda_n t}]^T$$

The components of vector η are special functions of time, $e^{\lambda t}$

↪ Functions $e^{\lambda t}$ are the **modes** of matrix A

↪ Mode $e^{\lambda t}$ associates with eigenvalue λ

Each element of e^{At} is a linear combination of such modes

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Example

Consider a (2×2) matrix A , we want to determine the state transition matrix e^{At}

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

Matrix A is triangular, the eigenvalues correspond to the diagonal elements

Matrix A has 2 distinct eigenvalues

$$\rightsquigarrow \lambda_1 = -1$$

$$\rightsquigarrow \lambda_2 = -2$$

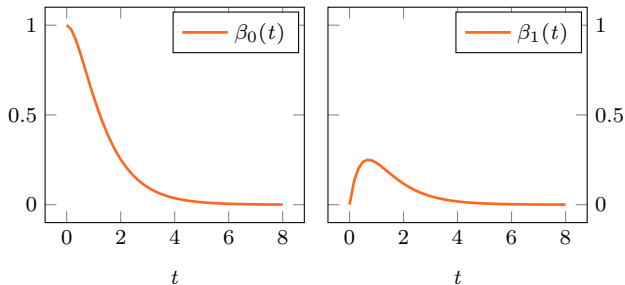
To determine e^{At} , we write the system

$$\begin{cases} 1\beta_0(t) + \lambda_1\beta_1(t) = e^{\lambda_1 t} \\ 1\beta_0(t) + \lambda_2\beta_1(t) = e^{\lambda_2 t} \end{cases} \rightsquigarrow \begin{cases} \beta_0(t) + (-1)\beta_1(t) = e^{(-1)t} \\ \beta_0(t) + (-2)\beta_1(t) = e^{(-2)t} \end{cases}$$

Sylvester's expansion (cont.)

By simple manipulation, we get

$$\rightsquigarrow \begin{cases} \beta_0(t) = 2e^{-t} - e^{-2t} \\ \beta_1(t) = e^{-t} - e^{-2t} \end{cases}$$



Thus,

$$\begin{aligned} e^{At} &= \beta_0(t)I_2 + \beta_1(t)A \\ &= (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

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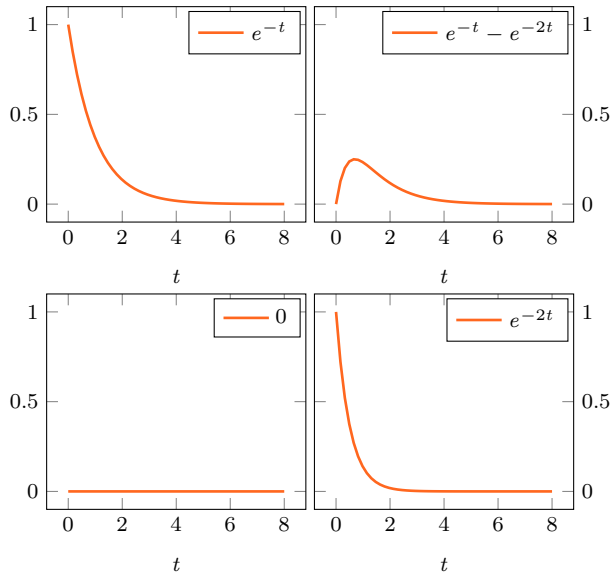
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Each element of e^{At} is a linear combination of the two system modes, e^{-t} and e^{-2t}

Sylvester's expansion (cont.)

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Eigenvalues and eigenvectors

Let $\lambda \in \mathcal{R}$ be some scalar and let $v \neq 0$ be some $(n \times 1)$ column vector

Consider a square matrix A of order n , suppose that the identity holds

$$Av = \lambda v$$

The scalar λ is called an **eigenvalue** of A

Vector v is the associated **eigenvector**

Consider a square matrix A of order n whose elements are real numbers

Matrix A has n (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

- They can be real numbers or conjugate-complex pairs
- If $\lambda_i \neq \lambda_j$ for $i \neq j$, A has multiplicity one

Sylvester's expansion (cont.)

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Eigenvalues of triangular and diagonal matrices

Let matrix $A = \{a_{i,j}\}$ be a triangular or a diagonal matrix

- The eigenvalues of A are the n diagonal elements $\{a_{i,i}\}$

Sylvester's expansion (cont.)

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Characteristic polynomial

The **characteristic polynomial** of a square matrix A of order n

- The n -order polynomial in the variable s

$$P(s) = \det(sI - A)$$

Computing eigenvalues and eigenvectors

The eigenvalues of matrix A of order n solve its characteristic polynomial

\rightsquigarrow The roots of the equation $P(s) = \det(sI - A) = 0$

Let λ be an eigenvalue of matrix A

Each eigenvector v associated to it is a non-trivial solution to the system

$$(\lambda I - A)v = 0$$

Sylvester's expansion (cont.)

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Systems of linear equations

Consider a system of n linear equations in n unknowns $Ax = b$

$\leadsto A$ is a $(n \times n)$ matrix of **coefficients**

$\leadsto b$ is a $(n \times 1)$ vector of **known terms**

$\leadsto x$ is a $(n \times 1)$ vector of **unknowns**

If A is non-singular, the system admits one and only one solution

If matrix A is singular, let $M = [A|b]$ be a $[n \times (n + 1)]$ matrix

- If $\text{rank}(A) = \text{rank}(M)$, system has infinite solutions
- If $\text{rank}(A) < \text{rank}(M)$, system has no solutions

Sylvester's expansion (cont.)

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Matrix rank

The **rank** of a $(m \times n)$ matrix A is equal to the number of columns (or rows) of the matrix that are linearly independent, $\text{rank}(A)$

Matrix kernel or null space

Consider a $(m \times n)$ matrix A , we define its **null space** or **kernel**

$$\ker(A) = \{x \in \mathcal{R}^n | Ax = 0\}$$

It is the set of all vectors $x \in \mathcal{R}^n$ that left-multiplied by A produce the null vector

The set is a vector space, its dimension is called the **nullity** of matrix A , $\text{null}(A)$

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LTI systems - Dynamics

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We can now prove the solution to the analysis problem for MIMO systems

- Lagrange formula

Theorem

Lagrange formula

Consider the state-space representation of a time-invariant linear system of order n

$$\begin{array}{ccc}
 u(t) & \xrightarrow{\quad} & \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array}} \xrightarrow{\quad} y(t) \\
 & & \text{System}
 \end{array}
 \qquad
 \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

The solution for $t \geq t_0$, for an initial state $x(t_0)$ and an input $u(t \geq t_0)$

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$y(t) = Ce^{A(t-t_0)} x(t_0) + C \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

Lagrange formula (cont.)

Proof

By left-multiplying the state equation $\dot{x}(t) = Ax(t) + Bu(t)$ by e^{-At} , we get

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Bu(t)$$

The resulting state equation can be rewritten,

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Bu(t)$$

Then, by using the result on the derivative of the state transition matrix²,

$$\begin{aligned}\frac{d}{dt}\left[e^{-At}x(t)\right] &= e^{-At}\dot{x}(t) - e^{-At}Ax(t) \\ &= e^{-At}Bu(t)\end{aligned}$$

²Derivative of the state transition matrix

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}\left[\frac{d}{dt}x(t)\right] + \left[\frac{d}{dt}e^{-At}\right]x(t) = e^{-At}\dot{x}(t) - e^{-At}Ax(t).$$

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$$\frac{d}{dt} \left[e^{-At} x(t) \right] = e^{-At} B u(t)$$

By integrating between t_0 and t , we obtain

$$\left[e^{-A\tau} x(\tau) \right]_{t_0}^t = \int_{t_0}^t e^{-A\tau} B u(\tau) d\tau$$

That is,

$$e^{-At} x(t) - e^{-At_0} x(t_0) = \int_{t_0}^t e^{-A\tau} B u(\tau) d\tau$$

Thus,

$$e^{-At} x(t) = e^{-At_0} x(t_0) + \int_{t_0}^t e^{-A\tau} B u(\tau) d\tau$$

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$$e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^t e^{-A\tau}Bu(\tau)d\tau$$

The first Lagrange formula is obtained by multiplying both sides by e^{At}

$$\rightsquigarrow x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

The second formula is obtained by substituting $x(t)$ in the output equation

$$y(t) = Cx(t) + Du(t)$$
$$\rightsquigarrow C \underbrace{\left[e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau \right]}_{x(t)} + Du(t)$$



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$$x(t) = \underbrace{e^{A(t-t_0)} x(t_0)}_{x_u(t)} + \underbrace{\int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau}_{x_f(t)}$$

We can write the state solution (for $t \geq t_0$) as the sum of two terms

$$\rightsquigarrow x(t) = x_u(t) + x_f(t)$$

\rightsquigarrow The **force-free evolution** of the state, $x_u(t)$

\rightsquigarrow The **forced evolution** of the state, $x_f(t)$

Force-free and forced evolution (cont.)

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$$x(t) = \underbrace{e^{A(t-t_0)} x(t_0)}_{x_u(t)} + \underbrace{\int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau}_{x_f(t)}$$

The **force-free evolution** of the state, from the initial condition $x(t_0)$

$$\rightsquigarrow x_l(t) = e^{A(t-t_0)} x(t_0)$$

$\rightsquigarrow e^{A(t-t_0)}$ indicates the transition from $x(t_0)$ to $x(t)$

\rightsquigarrow In the absence of contribution from the input

The **forced evolution** of the state

$$\rightsquigarrow x_f(t) = \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau$$

\rightsquigarrow The contribution of $u(\tau)$ to state $x(t)$

\rightsquigarrow Through a weighting function, $e^{A(t-\tau)} B$

Force-free and forced evolution (cont.)

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$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)}_{y_u(t)} + \underbrace{C \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)}_{y_f(t)}$$

We can write the output solution (for $t \geq t_0$) as the sum of two terms

$$\rightsquigarrow y(t) = y_l(t) + y_f(t)$$

\rightsquigarrow The **force-free evolution** of the output, $y_u(t)$

\rightsquigarrow The **forced evolution** of the output, $y_f(t)$

Free and forced evolution (cont.)

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$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)}_{y_u(t)} + \underbrace{C \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)}_{y_f(t)}$$

The **force-free evolution** of the output, from initial condition $y(t_0) = Cx(t_0)$

$$\rightsquigarrow \quad \begin{aligned} y_u(t) &= Ce^{A(t-t_0)}x(t_0) \\ &= Cx_u(t) \end{aligned}$$

The **forced-evolution** of the output

$$\rightsquigarrow \quad \begin{aligned} y_f(t) &= C \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \\ &= Cx_f(t) + Du(t) \end{aligned}$$

Free and forced evolution (cont.)

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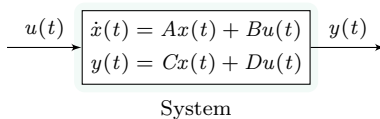
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Note that for $t_0 = 0$, we have

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x(0) + C \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

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Example

Consider a linear time-invariant system with the state-space representation,

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

We want to determine the state and the output evolution for $t \geq 0$

- We consider the input signal $u(t) = 2\delta_{-1}(t)$, a 2-step
- We consider the initial state $x(0) = (3, 4)^T$

The state transition matrix for this state-space representation,

$$e^{At} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

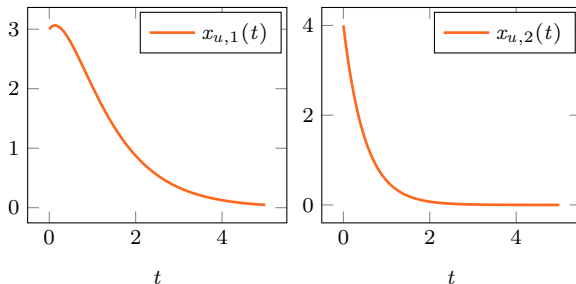
We computed it earlier

Free and forced evolution (cont.)

The force-free evolution of the state, for $t \geq 0$

$$\begin{aligned}
 \rightsquigarrow \quad x_u(t) &= e^{At} x(0) \\
 &= \underbrace{\begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}}_{e^{At}} \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{x(0)} \\
 &= \begin{bmatrix} (7e^{-t} - 4e^{-2t}) \\ 4e^{-2t} \end{bmatrix}
 \end{aligned}$$

That is,

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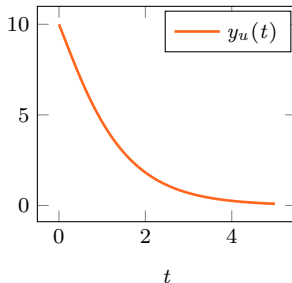
Transition matrix

Free and forced evolution (cont.)

The force-free evolution of the output, for $t \geq 0$

$$\begin{aligned}
 \rightsquigarrow \quad y_u(t) &= Cx_u(t) \\
 &= \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} 7e^{-t} - 4e^{-2t} \\ 4e^{-2t} \end{bmatrix}}_{x_u(t)} \\
 &= 14e^{-t} - 4e^{-2t}
 \end{aligned}$$

That is,



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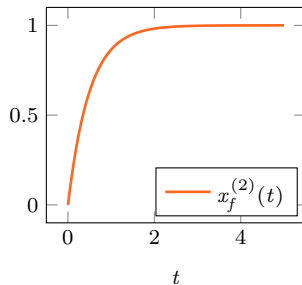
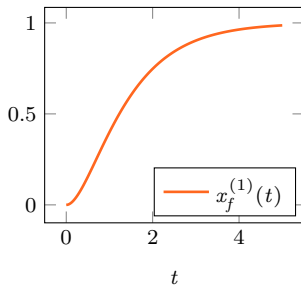
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Free and forced evolution (cont.)

The forced evolution of the state, for $t \geq 0$

$$\begin{aligned}
 \leadsto \quad x_f(t) &= \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\
 &= \int_0^t \underbrace{\begin{bmatrix} e^{t-\tau} & (e^{t-\tau} - e^{-2(t-\tau)}) \\ 0 & e^{-2(t-\tau)} \end{bmatrix}}_{e^{A(t-\tau)}} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B \underbrace{2}_{u(\tau)} d\tau \\
 &= 2 \int_0^t \begin{bmatrix} (e^{t-\tau} - e^{-2(t-\tau)}) \\ e^{-2(t-\tau)} \end{bmatrix} d\tau = 2 \begin{bmatrix} \int_0^t (e^{t-\tau} - e^{-2(t-\tau)}) d\tau \\ \int_0^t e^{-2(t-\tau)} d\tau \end{bmatrix} \\
 &= 2 \begin{bmatrix} (1 - e^{-t}) - 1/2(1 - e^{-2t}) \\ 1/2(1 - e^{-2t}) \end{bmatrix} = \begin{bmatrix} (1 - 2e^{-t} + e^{-2t}) \\ (1 - e^{-2t}) \end{bmatrix}
 \end{aligned}$$

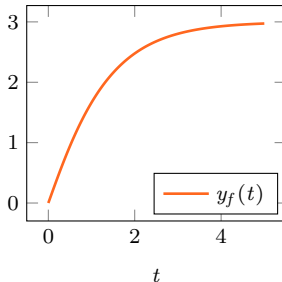


Free and forced evolution (cont.)

Since $D = 0$, the forced evolution of the output for $t \geq 0$

$$\begin{aligned}
 \rightsquigarrow y_f(t) &= Cx_f(t) \\
 &= \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_C \underbrace{\begin{bmatrix} (1 - 2e^{-t} + e^{-2t}) \\ (1 - e^{-2t}) \end{bmatrix}}_{x_f(t)} \\
 &= 3 - 4e^{-t} + e^{-2t}
 \end{aligned}$$

That is,

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The form of the state space representation depends on the choice of state variables

- The choice is not unique, even when we are coming from a physical model

There is an infinite number of different representations of the same system

- The representations are related by a **similarity transformation**
- These transformations allow flexibility in the analysis
- (We can change to easier system representations)

The state matrix can be transformed to have a **canonical form**

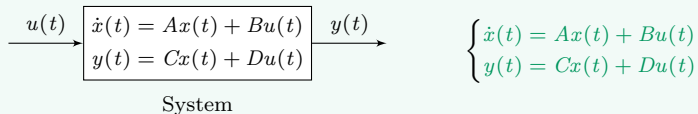
↪ **Diagonal form, Jordan form, ...**

Similarity transformation (cont.)

Definition

Similarity transformation

Consider the state-space representation of a linear time-invariant system of order N_x



- $x(t)$ and $\dot{x}(t)$, state vector and its derivative (N_x components)
- $u(t)$, input vector (N_u components)
- $y(t)$, output vector (N_y components)

Let vector $z(t)$ be related to $x(t)$ by some linear transformation P , $x(t) = Pz(t)$

P can be any ($N_x \times N_x$) non-singular matrix (its inverse always exists)

- Because of non-singularity, we have $z(t) = P^{-1}x(t)$

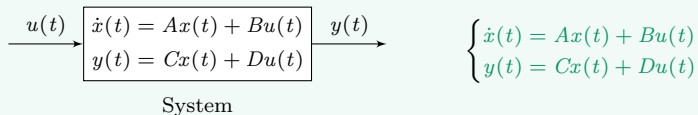
The transformation/matrix P is called a **similarity transformation/matrix**

Similarity transformation (cont.)

Proposition

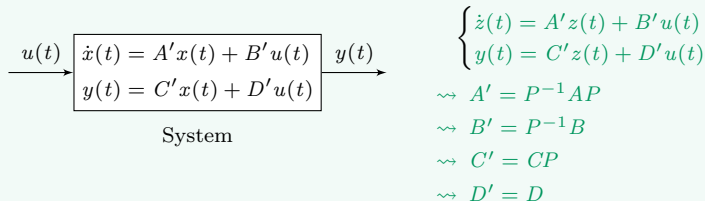
Similar representations

Consider the state-space representation of a linear time-invariant system of order N_x



Let P be some similarity transformation matrix such that $x(t) = Pz(t)$

Vector $z(t) = P^{-1}x(t)$ satisfies the new state-space representation

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Similarity transformation (cont.)

Proof

By taking the time-derivative of the state vector $x(t) = Pz(t)$, we have

$$\rightsquigarrow \dot{x}(t) = P\dot{z}(t)$$

By substituting $x(t)$ and $\dot{x}(t)$ into the state-space representation,

$$\rightsquigarrow \begin{cases} \underbrace{P\dot{z}(t)}_{\dot{x}(t)} = A \underbrace{Pz(t)}_{x(t)} + Bu(t) \\ y(t) = C \underbrace{Pz(t)}_{x(t)} + Du(t) \end{cases}$$

Pre-multiply the state equation by P^{-1} , to complete the proof

$$\begin{aligned} P^{-1} \underbrace{P\dot{z}(t)}_{\dot{x}(t)} &= P^{-1} A \underbrace{Pz(t)}_{x(t)} + P^{-1} Bu(t) \\ P^{-1} y(t) &= P^{-1} C \underbrace{Pz(t)}_{x(t)} + P^{-1} Du(t) \end{aligned}$$

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$$P^{-1}P\dot{z}(t) = P^{-1}APz(t) + P^{-1}Bu(t)$$

$$P^{-1}y(t) = P^{-1}CPz(t) + P^{-1}Du(t)$$

For the state equation, we have

$$\underbrace{P^{-1}P}_I \dot{z}(t) = \underbrace{P^{-1}AP}_{A'} z(t) + \underbrace{P^{-1}B}_{B'} u(t)$$

$$\dot{z}(t) = A'z(t) + B'u(t)$$

For the measurements, we have

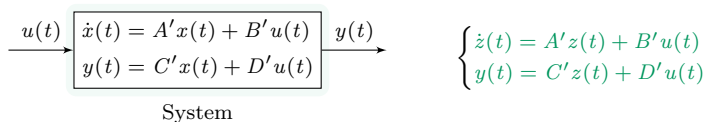
$$\underbrace{PP^{-1}}_I y(t) = \underbrace{PP^{-1}}_I CPz(t) + \underbrace{PP^{-1}}_I Du(t)$$

$$\underbrace{CP}_{C'} z(t) + \underbrace{D}_{D'} u(t)$$

$$y(t) = C'z(t) + D'u(t)$$



Similarity transformation (cont.)



We obtained a different state-space representation of the same dynamical system

- Input $u(t)$ and output $y(t)$ are left unchanged (problem data)
- We defined some new (transformed) state variables, $z(t)$

There is an infinite number of non-singular matrixes P that could be used

\rightsquigarrow Thus, there is also an infinite number of equivalent representations

$$\rightsquigarrow A' = P^{-1}AP$$

$$\rightsquigarrow B' = P^{-1}B$$

$$\rightsquigarrow C' = CP$$

$$\rightsquigarrow D' = D$$

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Example

Consider a linear time-invariant system with state-space representation $\{A, B, C, D\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}}^A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^B u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_C \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1.5 \\ 0 \end{bmatrix}}_D u(t) \end{cases}$$

Consider the similarity transformation of the state using some matrix P

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_P \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

What is the $\{A', B', C', D'\}$ state-space representation for state $z(t)$

Similarity transformation (cont.)

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We are given the similarity transformation P ,

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We compute its inverse,

$$P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since $z(t) = P^{-1}x(t)$, we have

$$\begin{aligned} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} x_2(t) \\ x_1(t) - x_2(t) \end{bmatrix} \end{aligned}$$

The second component of z is the difference between first and second component of x

↪ The first component of z simply equals the second component of x

Similarity transformation (cont.)

We conclude by calculating the resulting state-space representation

$$\begin{aligned}A' &= P^{-1}AP \\&= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\&= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix} \\B' &= P^{-1}B \\&= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\C' &= CP \\&= \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \\D' &= D \\&= \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}\end{aligned}$$



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Similarity transformation (cont.)

Proposition

Similarity and state transition matrix

Consider the state matrix $A' = P^{-1}AP$ from some similarity transformation P

The corresponding state transition matrix,

$$e^{A't} = P^{-1}e^{At}P$$

Proof

Note that

$$(A')^k = \underbrace{(P^{-1}AP) \cdot (P^{-1}AP) \cdots (P^{-1}AP)}_{k \text{ times}} = P^{-1} \underbrace{AA \cdots A}_{k \text{ times}} P = P^{-1}A^kP$$

Thus, by definition

$$e^{A't} = \sum_{k=0}^{\infty} \frac{(A')^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(P^{-1}A^kP)t^k}{k!} = P^{-1} \left(\sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \right) P = P^{-1}e^{At}P$$



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Similarity transformation (cont.)

We show how two similar state-space representations describe the same IO relation

Proposition

Invariance of the IO relationship under similarity

Consider two similar state-space representations of a linear time-invariant system

$$\rightsquigarrow \{A, B, C, D\} \text{ and } \{A', B', C', D'\}$$

$$\rightsquigarrow P \text{ is the transformation matrix}$$

Suppose that the system be subjected to some known input

$$\rightsquigarrow u(t)$$

The two representations produce the same forced response

$$\rightsquigarrow y_f(t)$$

Similarity transformation (cont.)

Proof

Consider the Lagrange formula

The forced response of the second representation due to input $u(t)$

$$\begin{aligned}
 y_f(t) &= C' \underbrace{\int_{t_0}^t e^{A'(t-\tau)} B' u(\tau) d\tau}_{z_f(t)} + Du(t) \\
 &= \underbrace{CP}_{C'} \int_{t_0}^t \underbrace{P^{-1} e^{A(t-\tau)} P}_{e^{A'(t-\tau)}} \underbrace{P^{-1} B}_{B'} u(\tau) d\tau + Du(t) \\
 &= C \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau + Du(t)
 \end{aligned}$$

This response corresponds to the one of the original representation

$$y_f(t) = C \int_{t_0}^t e^{A(t-\tau)} B u(\tau) d\tau + Du(t)$$

Similarity transformation (cont.)

Proposition

Invariance of the eigenvalues under similarity transformations

Matrix A and $P^{-1}AP$ have the same characteristic polynomial

Proof

The characteristic polynomial of matrix A'

$$\begin{aligned}
 \det(\lambda I - A') &= \det(\lambda I - \underbrace{P^{-1}AP}_{A'}) \\
 &= \det(\lambda \underbrace{P^{-1}P}_I - P^{-1}AP) \\
 &= \det[P^{-1}(\lambda I - A)P] \\
 &= \det(P^{-1}) \det(\lambda I - A) \det(P) \\
 &= \det(\lambda I - A)
 \end{aligned}$$

The last equality is obtained from $\det(P^{-1})\det(P) = 1$

A and A' share the same characteristic polynomial

\rightsquigarrow Thus, also the eigenvalues are the same

Similarity transformation (cont.)

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Two similar representations have the same modes, the modes characterise the dynamics

The modes are therefore independent of the representation

~> This is important

Similarity transformation (cont.)

Example

Consider two similar state-space representations of a linear time-invariant system

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$A' = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$$

The similarity transformation matrix

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We are interested in the eigenvalues and modes of the system

Matrix A and A' have two eigenvectors

- $\lambda_1 = -1$
- $\lambda_2 = -2$

The system modes are e^{-t} and e^{-2t}

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We consider a special similarity transformation P , we seek for a diagonal matrix A'

↪ A state-space representation with a diagonal state matrix

↪ **Diagonal canonical form**

↪ $\Lambda = A' = P^{-1}AP$

Consider the linear time-invariant system with a single input (and, say, single output)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

The evolution of the i -th component of the state vector

$$\rightsquigarrow \dot{x}_i(t) = \lambda_i x_i(t) + b_i u(t)$$

State derivatives are not related to other components

Diagonalisation (cont.)

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We can understand a system with diagonal matrix A as a collection of sub-systems

- ↪ Each sub-system is described by a single state component
- ↪ Each state component evolves independently
- ↪ The representation is **decoupled**
- ↪ N_x first-order subsystems

The characteristic polynomial of the system for the i -th component

$$\rightsquigarrow P_i(s) = (s - \lambda_i)$$

This subsystem has mode $e^{-\lambda_i t}$

We show how to determine a similarity transformation that leads to a diagonal form

- This can be understood as a somehow special similarity transformation

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Definition

Modal matrix

Consider a system in state-space representation with $(N_x \times N_x)$ matrix A

- Let v_1, v_2, \dots, v_n be a set of all the eigenvectors of matrix A
- Suppose that they correspond to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$

Suppose that eigenvectors in this set are also linearly independent

We define the **modal matrix** of A as the $(N_x \times N_x)$ matrix V

$$V = [v_1 | v_2 | \dots | v_n]$$

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If a matrix A has N_x distinct eigenvalues λ , then the modal matrix A always exists

- Its N_x eigenvectors v are linearly independent

Distinct eigenvalues

Let A be a n -order matrix whose n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct

Then, there is a set of n linearly independent eigenvectors

- Vectors v_1, v_2, \dots, v_n form a basis for \mathcal{R}^n

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Example

Consider a state-space representation of a linear time-invariant system with matrix A

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

We are interested in the modal matrix V of A

The eigenvalues and eigenvectors of A

$$\rightsquigarrow \lambda_1 = 1 \text{ and } v_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

$$\rightsquigarrow \lambda_2 = 5 \text{ and } v_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$$

The modal matrix V ,

$$V = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

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$$V = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

It is important to note that the eigenvectors are determined up to a scaling constant

- (Plus, the ordering of the eigenvalues is arbitrary)
- There can be more than one modal matrix

These two modal matrices of matrix A are equivalent

$$V' = [2v_1 | 3v_2] = \begin{bmatrix} 2 & 3 \\ -2 & 9 \end{bmatrix}$$

$$V'' = [v_2 | v_1] = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$



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Consider a matrix A with some eigenvalues λ that have multiplicity ν larger than one

- The modal matrix V exists if and only if to each eigenvalue λ_i with multiplicity ν_i is possible to associate ν_i linearly independent eigenvectors $\{v_{i,1}, v_{i,2}, \dots, v_{i,\nu_i}\}$

This is not always possible

But, ...

If a matrix admits a modal matrix, then it can be diagonalised

- (This is what matters to us)

Example

Consider a state space representation of a linear time-invariant system with matrix A

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Its only eigenvalue $\lambda = 2$ has multiplicity $\nu = 2$

Its eigenvectors are obtained by solving the system $[\lambda I - A]v = 0$

$$[2I - A]v = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{cases} 0 = 0 \\ 0 = 0 \end{cases}$$

We can choose any two linearly independent eigenvectors for λ

- As the equation is satisfied for any value of a and b

A modal matrix with the eigenvectors from the canonical basis

$$\rightsquigarrow V = [v_1 | v_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

Consider a state space representation of a linear time-invariant system with matrix A

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Its only eigenvalue $\lambda = 2$ has multiplicity $\nu = 2$

Its eigenvectors are obtained by solving the system $[\lambda I - A]v = 0$

$$[2I - A]v = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{cases} -b = 0 \\ 0 = 0 \end{cases}$$

As $b = 0$, we can choose only one linearly independent eigenvector for λ

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Matrix A does not admit a modal matrix

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Proposition

Diagonalisation

Consider a state-space representation of a linear time-invariant system with matrix A

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues and $V = [v_1 | v_2 | \dots | v_n]$ one of its modal matrices

Let Λ be the state matrix transformed according to $\Lambda = V^{-1}AV$

$\leadsto \Lambda$ is diagonal



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Example

Consider a linear time-invariant system with matrixes $\{A, B, C, D\}$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \end{cases}$$

We are interested in a diagonal representation by similarity

We can compute the eigenvalues and eigenvectors of A

- $\lambda_1 = -1$ and $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $\lambda_2 = -2$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Diagonalisation (cont.)

Then, we can determine a modal matrix and its inverse

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

From the similarity transformation expressions, we get

$$\begin{aligned} A' &= V^{-1}AV = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \Lambda \\ B' &= V^{-1}B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ C' &= CV = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \\ D' &= D = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} \end{aligned}$$



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We show a procedure alternative to Sylvester's formula for the state transition matrix

- We assume a linear time-invariant state-space system representation
- We assume that the state matrix A can be diagonalised

Transition matrix by diagonalisation (cont.)

Proposition

State transition matrix by diagonalisation

Consider a $(n \times n)$ state matrix A and let $\lambda_1, \lambda_2, \dots, \lambda_n$ be its eigenvalues

Suppose that A admits the modal matrix V

We have for the state transition matrix

$$e^{At} = V e^{\Lambda t} V^{-1} = V \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} V^{-1}$$

Because we have a diagonal state matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

State transition matrix by diagonalisation (cont.)

Proof

We have shown that the identity holds (see similarity and state transition matrices³)

$$e^{\Lambda t} = V^{-1} e^{At} V$$

To complete the proof, multiply both sides by V on the left and by V^{-1} on the right 

³Given $A' = P^{-1}AP$, we have $e^{A't} = P^{-1}e^{At}P$.

State transition matrix by diagonalisation (cont.)

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$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u(t) \end{cases}$$

We are interested in computing the state transition matrix e^{At}

State transition matrix by diagonalisation (cont.)

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We have already computed the modal matrix of A and its inverse, V and V^{-1}

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Thus, we have

$$\begin{aligned} e^{At} &= V \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{-t} \\ 0 & -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-t} & (e^{-t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

This is the same result we determined by using the Sylvester expansion

