OPTIMAL CONTROL IN DISCRETE TIME

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- DISCRETE OPTIMAL CONTROL
  - Formulation
  - A toy example
  - Sparsity structure

- DYNAMIC PROGRAMMING
  - DP in discrete state space
  - Linear Quadratic case
  - Infinite horizon problem
  - Linear Quadratic Regulator
  - (Robust and Stochastic DP)
  - (Properties of the DP operator)
  - Gradient of the value function
  - Discrete-time Minimum Principle
  - Iterative Dynamic Programming
  - Differential DP

NUMERICAL SIMULATION

CONTINUOUS TIME DYNAMICS CAN BE DISCRETISED BY USING NUMERICAL SIMULATION TOOLS, THEY CAN BE APPLIED TO SYSTEMS REPRESENTED IN TERMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH DEFINED INITIAL CONDITIONS

⇝ INITIAL VALUE PROBLEMS

(THE EXISTENCE AND UNIQUENESS OF A SOLUTION TO AN INITIAL VALUE PROBLEM IS GUARANTEED BY THE PICARD-LINDELÖF TH.

NUMERICAL INTEGRATION METHODS ARE USED TO APPROXIMATELY SOLVE A WELL POSED INITIAL VALUE PROBLEM

→ THERE ARE DIFFERENT TYPES OF TECHNIQUES

→ TWO MAIN CATEGORIES

- → SINGLE STEP
- → MULTIPLE STEP

→ TWO MAIN 'OTHER' CATEGORIES

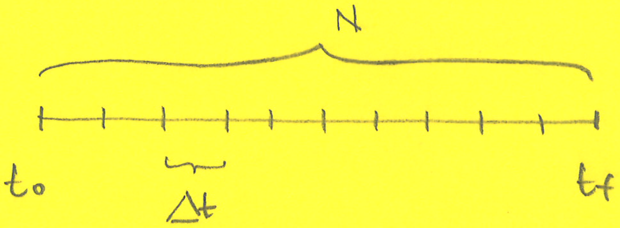
- → EXPLICIT
- → IMPLICIT

WE OVERVIEW SOME NOTIONS IN NUMERICAL INTEGRATION OVER ARBITRARY TIME INTERVALS

t_0 ——— t_f

* WE START WITH AN EXPLICIT ONE STEP METHOD

THE FIRST STEP IN A NUMERICAL INTEGRATION / SIMULATION METHOD IS THE <u>DISCRETISATION OF THE STATE TRAJECTORIES</u> OVER A <u>DISCRETISED TIME GRID</u> OF THE INTEGRATION INTERVAL $[t_0, t_f]$



→ WE CAN CONSIDER A UNIFORM TIME-GRID, WITH FIXED SIZE

→ N (SUB) INTERVALS

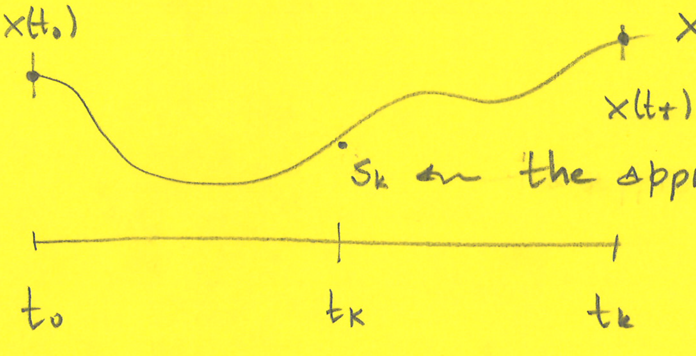
→ SIZE Δt

→ $\Delta t = \dfrac{(t_f - t_f)}{N}$

THE <u>DISCRETISATION TIME GRID</u> IS $t_k \triangleq t_0 + k\,\Delta t \quad (k = 0, 1, \ldots, N)$

THE SOLUTION OF THE INITIAL VALUE PROBLEM IS THEN APPROXIMATED ON THE TIME GRID, AT THE POINTS t_k

⇝ AT POINT t_k, THE SOLUTION IS APPROXIMATED BY A VALUE S_k $(\approx x(t_k))$



← the true, continuous-time solution to the IVP

← the approximated, discrete-time solution at t_k

DIFFERENT TECHNIQUES COMPUTE S_k $(k = 0, 1, \ldots, N)$ IN DIFFERENT WAYS BUT THEY ARE ALL SUCH THAT AS $N \to \infty$ THEN $S_k \to x(t_k)$

⇝ THEY MUST CONVERGE TO THE TRUE SOLUTION AS THE NUMBER OF SUBINTERVAL GROWS

$$\| S_k - x(t_k) \| \xrightarrow{\Delta t \to 0} 0$$

THE EXPLICIT EULER'S METHOD

→ SET $S_0 = X_0$

→ COMPUTE $S_{k+1} = S_k + \Delta t \underbrace{f(S_k, t_k)}$ $(k = 0, 1, \ldots, N-1)$

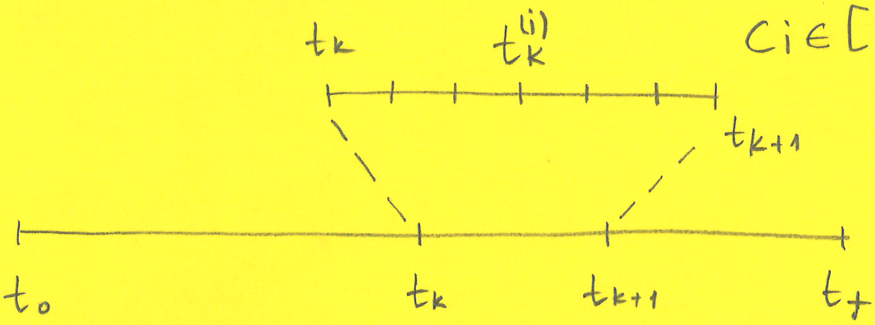
↓

EACH APPROXIMATION IS BASED ON ONE FUNCTION EVALUATION (FIRST-ORDER METHOD)

TECHNIQUES THAT DEPEND ON A LARGER NUMBER OF EVALUATIONS OF f AT EACH STEP ARE CALLED HIGHER-ORDER METHODS

⇝ INTERMEDIATE STATE VALUES $S_k^{(i)}$ ARE COMPUTED _WITHIN_ EACH INTERVAL $[t_k, t_{k+1}]$ AT POINTS $t_k^{(i)} = t_k + c_i(\Delta t)$ WITH $i = 1, 2, \ldots, m$

$c_i \in [0, 1]$



⇝ $S_k^{(1)} = S_k$

$$S_k^{(2)} = S_k + \Delta t\, a_{21} f(S_k^{(1)}, t_k^{(1)})$$

$$S_k^{(3)} = S_k + \Delta t \left(a_{31} f(S_k^{(1)}, t_k^{(1)}) + a_{32} f(S_k^{(2)}, t_k^{(2)})\right)$$

$$\vdots$$

$$S_k^{(i)} = S_k + \Delta t \sum_{j=1}^{i-1} a_{ij} f(S_k^{(j)}, t_k^{(j)})$$

$$\vdots$$

$$S_k^{(m)} = S_k + \Delta t \sum_{j=1}^{m-1} a_{mj} f(S_k^{(j)}, t_k^{(j)})$$

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THE APPROXIMATION S_{k+1} IS THEN CONSTRUCTED AS

$$S_{k+1} = S_k + \Delta t \sum_{j=1}^{m} b_j f\left(S_k^{(j)}, t_k^{(j)}\right)$$

WITH COEFFICIENTS b_j EXPRESSED IN TERMS OF THE BUTCHER TABLE

$$\begin{array}{c|cccc}
c_1 & & & & \\
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
\vdots & \vdots & \vdots & & \\
c_m & a_{m1} & a_{m2} & \dots & a_{m\,m-1} \\
\hline
 & b_1 & b_2 & & b_m
\end{array}$$

THE EXPLICIT EULERS METHOD USES $m=1$, $c_1 = 0$ AND $b_1 = 1$

THE ORDER-4 RUNGE-KUTTA METHOD USES

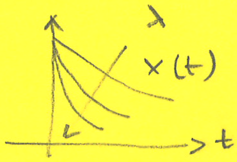
$$\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}$$

(RK 4)

EXPLICIT INTEGRATORS USED ON STABLE SYSTEMS CAN OVERSHOOT THE TRUE TRAJECTORY OF THE ODE $\rightsquigarrow$ INACCURATE SOLUTIONS

* IT CAN BE OBSERVED IN FIRST-ORDER SYSTEMS LIKE

$\dot{x} = -\lambda x$ (the solution is $x(t) = x_0 e^{-\lambda t}$)



* FOR $\lambda \gg 1$ THE ODE HAS A FAST AND STABLE DECAY

IF WE WOULD USE THE EXPLICIT EULER'S METHOD TO SIMULATE THE SIMULATE SYSTEM AND APPROXIMATE ITS SOLUTION, THEN

$$S_{k+1} = S_k - \Delta t (\lambda S_k) \qquad (S_0 = X_0)$$

THIS QUANTITY IS VERY DIFFERENT FROM $x(t_{k+1})$

$\rightsquigarrow$ (the approximation becomes even unstable for $\Delta t > 2/\lambda$)

THIS COULD BE ADDRESSED BY REDUCING THE SIZE OF THE SUBINTERVALS

* STILL NOT GREAT BECAUSE OF THE LONG SIMULATION TIME REQUIRED BUT UNNECESSARY FOR SLOW MODES

This can be in general understood as a fast mode for a single variable system

The coexistence of slow and fast modes is often referred to as STIFFNESS of the ODE