CHEM-E7190/2021: Exercise VII - Observability, Controllability

1. The dynamic equations of the stirred tank system in state-space form are the following:

$$\dot{x}(t) = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 1 & 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u(t)$$

   (a) Calculate the controllability gramian of the system.
   (b) Calculate the controllability matrix and the controllability of the system.
   (c) Calculate the observability gramian of the system.
   (d) Calculate the observability matrix and the observability of the system.

2. Given $x(0) = 0$ and any $\bar{x}$, compute $u(t)$ such that $x(\bar{t}) = \bar{x}$ for some $\bar{t} > 0$

Solution

We know that

$$\bar{x} = x(\bar{t}) = \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B u(\tau) d\tau$$

If we limit our search to controls u of the form

$$u(t) = B^T e^{A^T(\bar{t}-t)} \bar{z}$$

we have

$$\begin{aligned} \bar{x} &= \int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B B^T e^{A^T(\bar{t}-\tau)} \bar{z} d\tau \\ &= \left(\int_0^{\bar{t}} e^{A(\bar{t}-\tau)} B B^T e^{A^T(\bar{t}-\tau)} d\tau \right) \bar{z}, \xi = \bar{t} - \tau \\ &= \left(\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right) \bar{z} \end{aligned}$$

and

$$\bar{x} = \left(\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right)^{-1} \bar{x}$$

$$u(t) = B^T e^{A^T(\bar{t}-t)} \left(\int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi \right)^{-1} \bar{x}$$

The symmetric matrix

$$X(t) := \int_0^{\bar{t}} e^{A\xi} B B^T e^{A^T\xi} d\xi$$

is the Controllability Gramian.