Consider a n-order homogeneous ordinary differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \text{———} + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0$$

Alternatively, we write

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \text{———} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0$$

Let $y(t) = e^{\lambda t}$ be the solution, then differentiate

$$\begin{cases} \dot{y}(t) = \lambda e^{\lambda t} \\ \ddot{y}(t) = \lambda^2 e^{\lambda t} \\ \vdots \\ y^{(n)}(t) = \lambda^n e^{\lambda t} \end{cases}$$

Substituting in the ODE, we get

$$e^{\lambda t} \underbrace{\underbrace{\left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \text{———} + a_2 \lambda^2 + a_1 \lambda + a_0\right)}_{\text{CHARACTERISTIC POLYNOMIAL}} = 0}_{\text{CHARACTERISTIC EQUATION}}$$

The equation is verified when $(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \text{———} + a_2 \lambda^2 + a_1 \lambda + a_0)$ equals zero, we thus need to find its roots

THIS IS IN GENERAL SATISFIED BY n VALUES OF λ

$$\underbrace{\{\lambda_1, \lambda_2, \ldots, \lambda_n\}}_{\text{IT RELATES TO THE SPECTRUM OF SOME MATRIX}}$$

- POSITIVE AND NEGATIVE
- REAL OR COMPLEX NUMBER (+ COMPL - CONJ)
- SINGLE OR REPEATED

The solution to the differential equation can be written as

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \text{———} + C_n e^{\lambda_n t}$$

The coefficients must be determined by using n initial cond.

A WEIGHTED COMBINATION OF EXPONENTIAL FUNCTIONS

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n)}(0) \end{bmatrix} \xrightarrow{\text{TO GET}} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix}$$

ALTERNATIVE APPROACH

1. START FROM GENERAL n- ORDER DIFFERENTIAL EQUATION

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \text{———} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y^{(0)} = 0$$

2. DIVIDE BY THE COEFFICIENT OF THE HIGHEST ORDER DERIVAT.
$\rightsquigarrow$ (HERE a_n)

3. WE OBTAIN

$$y^{(n)} + \alpha_{n-1} y^{(n-1)} + \alpha_{n-2} y^{(n-2)} + \text{———} + \alpha_2 y^{(2)} + \alpha_1 y^{(1)} + \alpha_0 y^{(0)} = 0 \qquad (*)$$

WITH $\alpha_{n-1} = \dfrac{a_{n-1}}{a_n}$; $\alpha_{n-2} = \dfrac{a_{n-2}}{a_n}$; ———

4. WE NOW INTRODUCE A SET OF DUMMY VARIABLES

$$\begin{cases} x_1 = y \\ x_2 = y^{(1)} \\ x_3 = y^{(2)} \\ \vdots \\ x_n = y^{(n-1)} \\ x_{n+1} = y^{(n)} \end{cases}$$

WE CAN THEN COMPUTE THEIR FIRST DERIVATIVES

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = \underbrace{-\left[\alpha_{n-1} x_n + \alpha_{n-2} x_{n-1} + \dots + \alpha_2 x_3 + \alpha_1 x_2 + \alpha_0 x_1\right]}_{\text{FROM THE ODE } (*)}$$

5. WRITE IT IN MATRIX FORM

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & \alpha_1 & \alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = 0$$

$$\boxed{\begin{array}{c}\text{GENERAL FORM} \\ \dot{x} = Ax\end{array}}$$

Example $\ddot{y} + 3\dot{y} + 2y = 0$ w/ $\begin{cases} y(0) = 2 \\ \dot{y}(0) = -3 \end{cases}$

Let $y(t) = e^{\lambda t}$ be the assumed solution

Then $\dot{y}(t) = \lambda e^{\lambda t}$

$\ddot{y}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0 \rightsquigarrow e^{\lambda t}\left[\lambda^2 + 3\lambda + 2\right] = 0$

$\lambda^2 + 3\lambda + 2 = 0$ when $\boxed{\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -2 \end{matrix}}$

The general solution is $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$= C_1 e^{-t} + C_2 e^{-2t}$

$y(0) = 2 \rightsquigarrow C_1 + C_2 = 2.$

$\dot{y}(t) = -C_1 e^{-t} - 2C_2 e^{-2t} \rightsquigarrow \dot{y}(0) = -C_1 - 2C_2 = -3$

$\left| \begin{matrix} C_1 = 1 \\ C_2 = 1 \end{matrix} \right. \rightsquigarrow y(t) = e^{-t} + e^{-2t}$

STABLE SOLUTION (both exponentials decay)

THE OTHER APPROACH $y^{(2)} + 3y^{(1)} + 2y^{(0)} = 0$

1) WE INTRODUCE DUMMY VARIABLES

$\begin{cases} x_1 = y \\ x_2 = y^{(1)} \end{cases} \xrightarrow{\text{derivatives}} \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -3x_2 - 2x_1 \end{cases} \xrightarrow{\text{matrix form}}$

2) WE GET

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

CHECK

$\det(A - \lambda I) = 0$

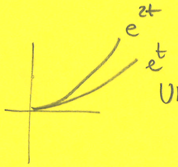
or eig(A)

Example $y^{(2)} - 3y^{(1)} + 2y = 0$ w/ $\begin{cases} y(0) = 2 \\ \dot{y}(0) = 3 \end{cases}$

↑ CHANGED THIS ONLY

$(\lambda^2 - 3\lambda + 2) \longrightarrow (\lambda - 2)(\lambda - 1) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \end{cases}$ $\quad C_1 e^{t} + C_2 e^{2t} = y(t)$



UNSTABLE

Example $y^{(2)} + y^{(1)} - 2y = 0$ w/ $\begin{cases} y(0) = 3 \\ \dot{y}(0) = 0 \end{cases}$

$(\lambda^2 + \lambda - 2) = 0 \longrightarrow (\lambda + 2)(\lambda - 1) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -2 \end{cases}$ $\quad C_1 e^{t} + C_2 e^{-2t}$

UNSTABLE

THERE IS A CLOSE CONNECTION BETWEEN THE ROOT OF THE CHARACTERISTIC POLYNOMIAL AND THE EIGEN VALUES OF THE STATE MATRIX

$\det(A - \lambda I) \equiv$ characteristic polynomial

Easily checked for a 2×2 system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leadsto A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & a_1 & -a_2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\right) = -\lambda\left[\lambda^2 + a_2\lambda + a_1\right] - 1\left[a_0\right] + 0$$

$$= \underbrace{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0}_{\text{CHARACTERISTIC EQUATION}}$$

WE ARE INTERESTED IN SOLVING THE GENERAL SYSTEM $A x = \dot{x}$

CASE 1 Uncoupled dynamics — MATRIX A is DIAGONAL

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

WE'VE SOLVED THOSE ALREADY

THIS MEANS THAT

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 & \rightsquigarrow \quad x_1(t) = e^{\lambda_1 t} x_1(0) \\ \dot{x}_2 = \lambda_2 x_2 & \rightsquigarrow \quad x_2(t) = e^{\lambda_2 t} x_2(0) \\ \vdots \\ \dot{x}_n = \lambda_n x_n & \rightsquigarrow \quad x_n(t) = e^{\lambda_n t} x_n(0) \end{cases}$$

EQUIVALENTLY, IN MATRIX FORM, THE SOLUTION

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

$$x(t) = \underbrace{e^{At}} x(0)$$

THE MATRIX EXPONENTIAL OF STATE MATRIX A
(Easy because A is diagonal)

$\longrightarrow$ THE STATE TRANSITION MATRIX

WE HAVE A GENERAL $\dot{x} = Ax$, WE ARE INTERESTED IN A CHANGE OF COORDINATES SUCH THAT $x = Tz$ THAT DIAGONALIZES THE ODE

$\dot{z} = Dz$ WITH D DIAGONAL ‖ TARGET REPRESENTATION

WE START FROM $x = Tz \rightsquigarrow \dot{x} = T\dot{z} = Ax$

THAT IS, WE HAVE $T\dot{z} = Ax$ WE CAN LEFT MULTIPLY BY T^{-1}

$$\underbrace{T^{-1}T}_{I}\dot{z} = T^{-1}Ax \rightsquigarrow \dot{z} = T^{-1}Ax = \underbrace{T^{-1}AT}z$$

WE WANT THIS TO BE DIAGONAL, D

WE HAVE $D = T^{-1}AT$

WE CAN LEFT MULTIPLY BY T, TO GET $TD = \underbrace{TT^{-1}}_{I}AT$

SO THAT $AT = TD$ THIS IS THE EIGEN EQUATION IN MATRIX FORM

$$Av = \lambda v$$

$$A\begin{bmatrix} | \\ v_i \\ | \end{bmatrix} = \lambda_i \begin{bmatrix} | \\ v_i \\ | \end{bmatrix}$$

THE SCALAR CASE

$$A\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} T$$

$\{\lambda_i\}$ $\{v_i\}$ EIGENVALUES AND EIGENVECTORS

FROM $D = T^{-1}AT$, WE CAN LEFT MULTIPLY BY T AND THEN WE CAN RIGHT MULTIPLY BY T^{-1}

$$TD = \cancel{T}\overset{I}{\cancel{T^{-1}}}AT$$

$$\rightsquigarrow TD = AT$$

$$TDT^{-1} = A\overset{=1}{\cancel{TT^{-1}}}$$

$$\rightsquigarrow A = TDT^{-1}$$

THEN $A^2 = (TD\underbrace{T^{-1})(T}_{I}DT^{-1}) = TD^2T^{-1}$

$$A^3 = (TD\underbrace{T^{-1})(T}_{I}D\underbrace{T^{-1})(T}_{I}DT^{-1})$$

$$= TD^3T^{-1}$$

$$A^n = TD^nT^{-1}$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \text{———}$$

$$= T\left[1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \text{——}\right]T^{-1}$$

$$= Te^{Dt}T^{-1}$$

THE SOLUTION THEN BECOMES $y(t) = Te^{Dt}T^{-1}y(0)$