Input-output representation

Stochastic algorithms

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Input-output representation
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Representation and analysis
Homogeneous equation and modes
Force-free evolution
Modes
Aperiodic
Pseudo-periodic
Impulse response
Forced evolution

Input-output models

We concentrate on single-input single-output (SISO) systems

- Input-output (IO) representation
- Linear and stationary systems

Linear ordinary differential equations w/ constant coefficients

- Direct integration of the ODEs in time

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Input-output models (cont.)

The analysis consists of determining the output signal for a given model

$\rightsquigarrow$ **Force-free** and **forced evolution**

$\rightsquigarrow$ Decomposition by linearity

We study the homogeneous equation associated to the model equation

$\rightsquigarrow$ A definition of the **system modes**

$\rightsquigarrow$ They characterise this evolution

The force-free evolution is given by a linear combination of modes

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Input-output models (cont.)

We study the forced response of the system to the unit impulse

- It is a **canonical regime**

$\rightsquigarrow$ Full characterisation

The forced evolution to any input is given as a convolution

- The input and the response to the unit impulse
- The **Duhamel integral**

Representation and analysis

Input-output representation

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Representation and analysis

Consider a SISO system represented by a linear, time-invariant IO model

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + \cdots + b_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t) \quad (1)$$

The **independent variable**

- ⇝ Time, $t \in \mathcal{R}$

The **dependent variables**

- ⇝ The input, $u(t) : \mathcal{R} \rightarrow \mathcal{R}$
- ⇝ The output, $y(t) : \mathcal{R} \rightarrow \mathcal{R}$

The **parameters**

- ⇝ $a_i \in \mathcal{R}$, with $i = 0, \dots, n$
- ⇝ $b_i \in \mathcal{R}$, with $i = 0, \dots, m$

The order of the system is the highest order of derivation of the output

- We suppose that the system is proper ($n \geq m$)

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Representation and analysis (cont.)

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + \cdots + b_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t)$$

The problem

The fundamental problem of analysis for an IO model representation

- ⇝ Calculate the solution of the differential equation $y(t)$
- ⇝ From a given initial time t_0 ($t \geq t_0$)

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Representation and analysis (cont.)

This corresponds to determine the evolution of output $y(t)$, for $t \geq t_0$

Initial conditions

$$\begin{aligned} y(t)\Big|_{t=t_0} &= y_0 \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t}\Big|_{t=t_0} &= y_0' \\ \cdots &= \cdots \\ \frac{\mathrm{d}^{n-1}y(t)}{\mathrm{d}t^{n-1}}\Big|_{t=t_0} &= y_0^{(n-1)} \end{aligned} \quad (2)$$

The values of the output and its derivatives at the initial time t_0

Input signal

$$u(t), \quad \text{for } t \geq t_0 \quad (3)$$

The value of the input at the initial time t_0

Representation and analysis (cont.)

We overview standard solution methods of ordinary differential equations

$\rightsquigarrow$ And, some less standard methods will be introduced

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Representation and analysis (cont.)

The past of the system for $t \in (-\infty, t_0]$ is summarised by the state $\mathbf{x}(t_0)$

- The initial state is not given/available in the IO representation
- We have initial conditions for the output and its derivatives

$\rightsquigarrow$ The information is equivalent

Initial state and initial conditions are univocally related[1]

[1] This is strictly true only for observable systems.

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Representation and analysis (cont.)

Initial state and initial conditions

If the initial state of the system is null, then all initial conditions are null

$$\mathbf{x}(t_0) = \mathbf{0} \quad \rightsquigarrow \quad y_0 = y_0' = \cdots = y_0^{(n-1)} = 0$$

If the initial state is not null, then not all initial conditions are null

$$\mathbf{x}(t_0) \neq \mathbf{0} \quad \rightsquigarrow \quad (\exists i \in \{0, 1, \ldots, n-1\}) \quad y_0^{(i)} \neq 0$$

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Representation and analysis (cont.)

The solution (in terms of force-free and forced evolution)

We will consider the evolution of the **output** of a system

- We assumed that this is an **effect**

We assumed that the effect is due to two types of **causes**

$\rightsquigarrow$ **Internal causes** in the system, the **initial state**

$\rightsquigarrow$ **External causes** to the system, the **input**

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Representation and analysis (cont.)

Consider a linear system (superposition principle)

The effect is due to the simultaneous existence of both causes

The response can be determined as the sum of effects

- Each cause is acting alone

$$\rightsquigarrow \quad y(t) = y_u(t) + y_f(t), \quad \text{for } t \geq t_0$$

$y_u(t)$ is called the **force-free response**

- Contribution to the output that is only due to **initial state** at $t = t_0$

$y_f(t)$ is called the **forced response**

- Contribution to the output that is only due to **input** for any $t \geq t_0$

Representation and analysis (cont.)

Force-free and forced response

$$y(t) = y_u(t) + y_f(t), \quad \text{for } t \geq t_0$$

$y_u(t) \quad \rightsquigarrow \quad$ can be defined as the system response for an input $u(t)$ that is identically null for $t \geq t_0$ and for given initial conditions

$y_f(t) \quad \rightsquigarrow \quad$ can be defined as the system response for a given input $u(t)$ for $t \geq t_0$ and for initial conditions that are identically null

Representation and analysis (cont.)

We want to study the two terms separately and show how they are calculated

- The analysis is restricted to stationary models

We introduce a simplification that will not disrupt generality

- We will assume that the initial time is $t_0 = 0$

If $t_0 \neq 0$, solve for $\tau = (t - t_0)$ to get $y(\tau)$ for $\tau \geq 0$

Homogeneous equation and modes

Input-output representation

Homogeneous equation and modes

Consider SISO system represented by a linear, time-invariant IO model

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + \cdots + b_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t)$$

We study a simplified form of this differential equation

- The **homogeneous equation** (RHS is null)

$$\rightsquigarrow \quad a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = 0$$

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Homogeneous equation and modes (cont.)

This form allow us to introduce the fundamental concept of **system mode**

System modes are functions that characterises the system evolution

$\rightsquigarrow$ The number of modes equals the system's order

Linear combinations of modes solve the homogeneous equations

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Homogeneous equation and modes (cont.)

Definition

Homogeneous equation

Consider the differential equation of a IO model

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

Suppose that we let the RHS of the IO representation be zero

*Define the **homogenous equation** associated to it*

$$\rightsquigarrow \quad a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

$\rightsquigarrow$ $t \in \mathcal{R}$

$\rightsquigarrow$ $y : \mathcal{R} \rightarrow \mathcal{R}$

$\rightsquigarrow$ $a_i \in \mathcal{R}$, *with* $i = 0, \ldots, n$

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Homogeneous equation and modes (cont.)

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{dt} + a_0 y(t) = 0$$

The homogeneous equation is a simplified form of the differential equation

It is possible to associate a polynomial to any homogenous equation

$\rightsquigarrow$ **Characteristic polynomial**

Homogeneous equation and modes (cont.)

Definition

Characteristic polynomial

Consider the homogeneous differential equation

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

The ***characteristic polynomial*** *of a homogenous differential equation is a n-order polynomial in the variable s whose coefficients correspond to the coefficients $\{a_0, a_1, \ldots, a_n\}$ of the homogeneous equation*

$$\rightsquigarrow \quad P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = \sum_{i=0}^{n} a_i s^i \tag{4}$$

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Homogeneous equation and modes (cont.)

Consider any polynomial of order n with real coefficients

- It has n real or complex-conjugate roots

The roots are solutions of the **characteristic equation**

$$\rightsquigarrow \quad P(s) = \sum_{i=0}^{n} a_i s^i = 0$$

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Homogeneous equation and modes (cont.)

In general, there are $r \leq n$ **distinct roots** p_i, each with **multiplicity** ν_i

$$\rightsquigarrow \quad \overbrace{\underbrace{p_1 \;\cdots\; p_1}_{\nu_1} \; \underbrace{p_2 \;\cdots\; p_2}_{\nu_2} \;\cdots\; \underbrace{p_r \;\cdots\; p_r}_{\nu_r}}^{n}$$

- $\rightsquigarrow$ If $i \neq j$, then $p_i \neq p_j$
- $\rightsquigarrow$ $\sum_{i=1}^{r} \nu_i = n$

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Homogeneous equation and modes (cont.)

Consider the particular case in which all roots have multiplicity equal one

$$\rightsquigarrow \quad \overbrace{p_1 \quad p_2 \quad \cdots \quad p_{n-1} \quad p_n}^{n}$$

- $\rightsquigarrow$ If $i \neq j$, then $p_i \neq p_j$
- $\rightsquigarrow$ $\nu_i = 1$, for every i

Homogeneous equation and modes (cont.)

Definition

Modes

Let p be a root with multiplicity ν of the characteristic polynomial

The modes associated to that root are the ν functions of time

$$\rightsquigarrow \quad e^{pt}, te^{pt}, t^2e^{pt}, \cdots, t^{\nu-1}e^{pt}$$

A system with a n-order characteristic polynomial has n modes

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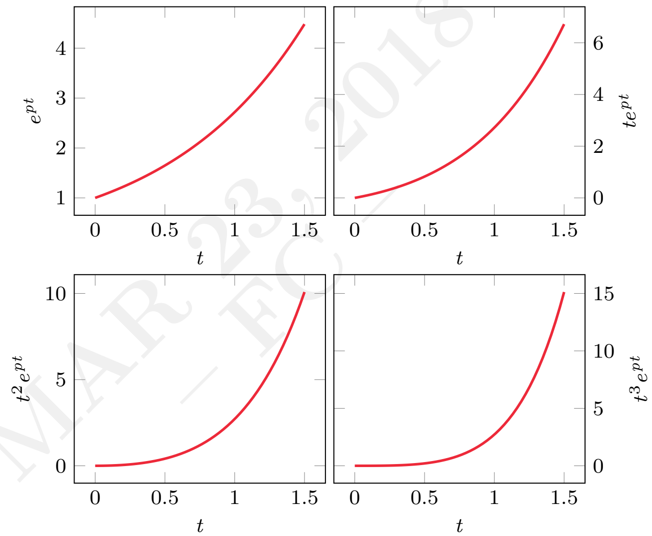
Let $p = 1$



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Homogeneous equation and modes (cont.)

Example

Consider the following homogenous differential equation

$$3\frac{\mathrm{d}^4y(t)}{\mathrm{d}t^4} + 21\frac{\mathrm{d}^3y(t)}{\mathrm{d}t^3} + 45\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 39\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 12y(t) = 0$$

The associated characteristic polynomial

$$P(s) = 3s^4 + 21s^3 + 45s^2 + 39s + 12 = 3(s+1)^3(s+4)$$

Its roots

$$\rightsquigarrow \quad \begin{cases} p_1 = -1, & \text{multiplicity } \nu_1 = 3 \\ p_2 = -4, & \text{multiplicity } \nu_2 = 1 \end{cases}$$

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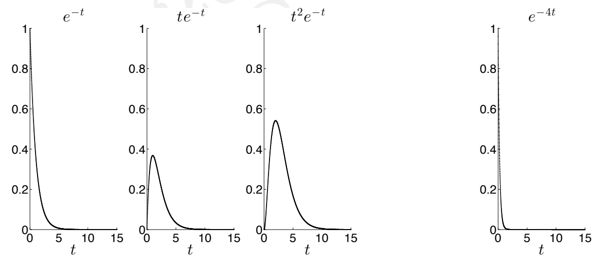
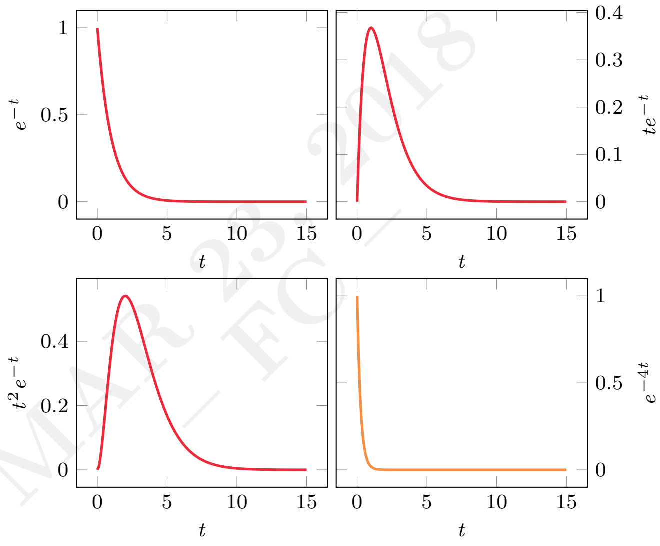
Homogeneous equation and modes (cont.)

Such a system has four modes

$$p_1 = -1, \quad (\nu_1 = 3) \quad \rightsquigarrow \begin{cases} e^{-t} \\ te^{-t} \\ t^2e^{-t} \end{cases}$$

$$p_2 = -4, \quad (\nu_2 = 1) \quad \rightsquigarrow \left\{ e^{-4t} \right.$$





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Homogeneous equation and modes (cont.)

As modes are functions, their linear combinations are a family of functions

- The family is parameterised by the coefficients of the combination

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Homogeneous equation and modes (cont.)

Definition

Linear combinations of modes

A linear combination of the n modes of a system is a function $h(t)$

- *A sum of the modes, each weighted by some coefficient*

Each root p_i with multiplicity ν_i is associated to a combination of ν_i terms

$$\rightsquigarrow \quad A_{i,0}e^{p_it} + A_{i,1}te^{p_it} + \cdots + A_{i,\nu_i-1}t^{\nu_i-1}e^{p_it} = \underbrace{\sum_{k=0}^{\nu_i-1} A_{i,k}t^ke^{p_it}}_{\text{root } p_i} \quad (5)$$

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Homogeneous equation and modes (cont.)

$$A_{i,0}e^{p_it} + A_{i,1}te^{p_it} + \cdots + A_{i,\nu_i-1}t^{\nu_i-1}e^{p_it} = \underbrace{\sum_{k=0}^{\nu_i-1} A_{i,k}t^ke^{p_it}}_{\text{root } p_i}$$

There is a total of r distinct roots

The complete linear combination of modes

$$h(t) = \underbrace{\sum_{k=0}^{\nu_1-1} A_{1,k}t^ke^{p_1t}}_{\text{root } p_1} + \underbrace{\sum_{k=0}^{\nu_2-1} A_{2,k}t^ke^{p_2t}}_{\text{root } p_2} + \cdots + \underbrace{\sum_{k=0}^{\nu_r-1} A_{r,k}t^ke^{p_rt}}_{\text{root } p_r} \quad (6)$$

$$\rightsquigarrow = \sum_{i=1}^{r}\sum_{k=0}^{\nu_i-1} A_{i,k}t^ke^{p_it}$$

Homogeneous equation and modes (cont.)

Consider the case in which all roots (n) have multiplicity equal to one

$$\rightsquigarrow \quad h(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t} = \sum_{i=1}^{n} A_i e^{p_i t}$$

(We have omitted the second subscript of coefficients A)

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Homogeneous equation and modes (cont.)

Example

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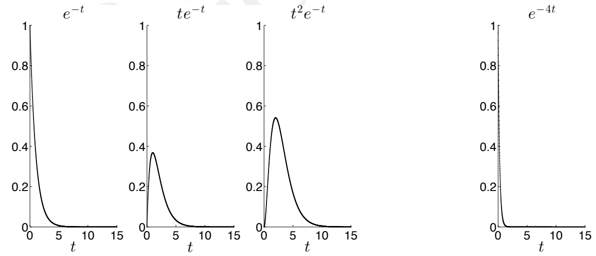
- Two roots $p_1 = -1$ ($\nu_1 = 3$) and $p_2 = -4$ ($\nu_2 = 1$)
- Four modes e^{-t}, te^{-t}, $t^2 e^{-t}$ and e^{-4t}



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Homogeneous equation and modes (cont.)

A linear combination of the modes

$$h(t) = \underbrace{A_{1,0} e^{-t} + A_{1,1} te^{-t} + A_{1,2} t^2 e^{-t}}_{\text{root } p_1} + \underbrace{A_2 e^{-4t}}_{\text{root } p_2}$$

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Homogeneous equation and modes (cont.)

The modes are known through the characteristic polynomial

The coefficients of their linear combination are parameters

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

The equation is a parametric form of a family of functions

The actual coefficients determine the force-free evolution

$\rightsquigarrow$ From every possible initial condition

Homogeneous equation and modes (cont.)

Theorem

Solution of the homogeneous equation

Consider the homogeneous equation

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = 0$$

A real function $h(t)$ is a solution of the homogeneous equation if and only if it is a linear combination of its modes

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

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Homogeneous equation and modes (cont.)

Proof

We demonstrate only the necessary condition

Consider the case in which all n roots have multiplicity equal to one

$$h(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t} = \sum_{i=1}^{n} A_i e^{p_i t}$$

For the k-th order derivative of function $h(t)$, we have

$$\begin{aligned} \rightsquigarrow \quad \frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t) &= \frac{\mathrm{d}^k}{\mathrm{d}^k}\Big(A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t} \Big) \\ &= p_1^k A_1 e^{p_1 t} + p_2^k A_2 e^{p_2 t} + \cdots + p_n^k A_n e^{p_n t} \\ &= \sum_{i=1}^{n} p_i^k A_i e^{p_i t}, \quad \text{for } k = 0, 1, \dots, n \end{aligned}$$

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$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t) = \sum_{i=1}^{n} p_i^k A_i e^{p_i t}, \quad \text{for } k = 0, 1, \dots, n$$

We substitute the k-th order derivatives of $h(t)$ in the homogeneous equation

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = \sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t) = 0$$

We have,

$$\begin{aligned} \rightsquigarrow \quad \sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t) &= \sum_{k=0}^{n} a_k \sum_{i=1}^{n} p_i^k A_i e^{p_i t} \\ &= \sum_{k=0}^{n} \sum_{i=1}^{n} a_k p_i^k A_i e^{p_i t} = \sum_{i=1}^{n} A_i e^{p_i t} \Big(\sum_{k=0}^{n} a_k p_i^k \Big) \end{aligned}$$

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Homogeneous equation and modes (cont.)

$$\sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t) = \sum_{i=1}^{n} A_i e^{p_i t} \Big(\sum_{k=0}^{n} a_k p_i^k \Big) = 0$$

For all values of $i = 1, \cdots, n$, the term between parenthesis is equal to zero

$\rightsquigarrow$ As p_i is a root of the characteristic polynomial

$$\sum_{k=0}^{n} a_k p_i^k = a_n p_i^n + \cdots + a_1 p_i + a_0 = P(s)\Big|_{s=p_i} = 0$$

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Homogeneous equation and modes (cont.)

Complex and conjugate roots

Consider as characteristic polynomial $P(s)$ whose roots are complex

The modes in $h(t)$ are complex signals

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

Let $P(s)$ be a polynomial with real coefficients and complex roots

- Let $p_i = \alpha_i + j\omega_i$ with multiplicity ν_i be a complex root

For each $p_i = \alpha_i + j\omega_i$ there is a conjugate complex root $p_i' = \alpha_i - j\omega_i$

- Multiplicity $\nu_i' = \nu_i$

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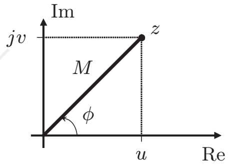
Homogeneous equation and modes (cont.)

Complex numbers (Cartesian representation)

Consider the set of complex numbers $\mathcal{C} = \{u + jv \,|\, u, v \in \mathcal{R}\}$ $(j = \sqrt{-1})$

A **complex number**

$$\begin{aligned} z &= \text{Re}(z) + \text{Im}(z) \\ &= u + jv \end{aligned}$$



It consists of two parts

- **Real part**, $\text{Re}(z) = u$
- **Imaginary part**, $\text{Im}(z) = v$

The **complex conjugate** of z

$$z' = \text{Re}(z) - j\text{Im}(z)$$

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Homogeneous equation and modes (cont.)

The complex exponential function

Consider an imaginary number $z = 0 + j\phi$

We have,

$$\rightsquigarrow \quad e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

The exponential of an imaginary number is a complex number

- Real part, $\cos(\phi)$
- Imaginary part, $\sin(\phi)$

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Homogeneous equation and modes (cont.)

Proof

Let $z \in \mathcal{C}$ be any scalar

We have (by definition),

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Let $z = j\phi$, for this particular case

$$\begin{aligned} \rightsquigarrow \quad e^{j\phi} &= 1 + j\phi - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + \cdots \\ &= \Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!}\Big] + j\Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!}\Big] \end{aligned}$$

Homogeneous equation and modes (cont.)

$$e^{j\phi} = \underbrace{\Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!}\Big]}_{\cos(\phi)} + j\Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!}\Big]$$

The first sum is the McLaurin expansion of the cosine function

$$\begin{aligned}\cos(\phi) &= \sum_{k=0}^{\infty} \frac{\phi^k}{k!}\Big[\frac{\mathrm{d}^k \cos(x)}{\mathrm{d}x^k}\Big]_{x=0} \\ &= \cos(0) - \sin(0)\phi - \cos 0 \frac{\phi^2}{2!} + \sin(0)\frac{\phi^3}{3!} + \cdots \\ &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots = \Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!}\Big]\end{aligned}$$

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Homogeneous equation and modes (cont.)

$$e^{j\phi} = \Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k}}{(2k)!}\Big] + j\underbrace{\Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!}\Big]}_{\sin(\phi)}$$

The second sum is the McLaurin expansion of the sine function

$$\begin{aligned}\sin(\phi) &= \sum_{k=0}^{\infty} \frac{\phi^k}{k!}\Big[\frac{\mathrm{d}^k \sin(x)}{\mathrm{d}x^k}\Big]_{x=0} \\ &= \sin(0) - \cos(0)\phi - \sin 0 \frac{\phi^2}{2!} + \cos(0)\frac{\phi^3}{3!} + \cdots \\ &= \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \cdots = \Big[\sum_{k=0}^{\infty} (-1)^k \frac{\phi^{2k+1}}{(2k+1)!}\Big]\end{aligned}$$

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Homogeneous equation and modes (cont.)

A pair of roots (p_i, p_i') is associated to a linear combination of $2\nu_i$ modes

$$\rightsquigarrow \quad \underbrace{(A_{i,0}\, e^{p_i t} + A'_{i,0}\, e^{p'_i t})}_{k=0} + \cdots + \underbrace{t^{\nu_i - 1}(A_{i,\nu_i-1}\, e^{p_i t} + A'_{i,\nu-1}\, e^{p'_i t})}_{k=\nu_i-1} \qquad (7)$$

(Pairs of terms for $k = 0, \dots, \nu_i - 1$ have been grouped up)

$$\underbrace{(A_{i,0}\, e^{p_i t} + A'_{i,0}\, e^{p'_i t})}_{k=0} + \cdots + \underbrace{t^{\nu_i - 1}(A_{i,\nu_i-1}\, e^{p_i t} + A'_{i,\nu-1}\, e^{p'_i t})}_{k=\nu_i-1}$$

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Homogeneous equation and modes (cont.)

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k}\, t^k\, e^{p_i t}$$

Function $h(t)$ is a real function (must take real values for all values of t)

$$\underbrace{(A_{i,0}\, e^{p_i t} + A'_{i,0}\, e^{p'_i t})}_{k=0} + \cdots + \underbrace{t^{\nu_i - 1}(A_{i,\nu_i-1}\, e^{p_i t} + A'_{i,\nu-1}\, e^{p'_i t})}_{k=\nu_i-1}$$

Coefficients $A_{i,k}$ and $A'_{i,k}$ need be complex and conjugated

- For all $k = 0, \dots, \nu_i - 1$

Then, $A_{i,k}\, e^{p_i t}$ and $A'_{i,k}\, e^{p'_i t}$ are complex and conjugated

- Their sum will be a real number (as desired)
- For all values of t

Homogeneous equation and modes (cont.)

Consider a characteristic polynomial $P(s)$ that has complex roots

It is possible to derive a *proper* parameterisation of $h(t)$

$\rightsquigarrow$ (That is, one that only contains real terms)

Homogeneous equation and modes (cont.)

Proposition

Consider the contribution of $(p_i, p_i') = \alpha_i \pm j\omega_i$ *a pair of conjugate complex roots with multiplicity* ν_i *to the linear combination of the* $(2\nu_i)$ *modes*

$$\underbrace{(A_{i,0}\, e^{p_i t} + A'_{i,0}\, e^{p'_i t})}_{k=0} + \cdots + \underbrace{t^{\nu_i - 1}(A_{i,\nu_i-1}\, e^{p_i t} + A'_{i,\nu-1}\, e^{p'_i t})}_{k=\nu_i-1}$$

This sum of terms can be re-written

$$\rightsquigarrow \sum_{k=0}^{\nu_i-1} M_{i,k}\, t^k\, e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \qquad (8)$$

The $2\nu_i$ *complex coefficients,* $A_{i,k}$ *and* $A'_{i,k}$*, are replaced by* $2\nu_i$ *real ones*

$\rightsquigarrow$ $M_{i,k}$

$\rightsquigarrow$ $\phi_{i,k}$

Homogeneous equation and modes (cont.)

Proof

Consider the term $(Ae^{pt} + A'e^{p't})$ in which $(p, p') = \alpha \pm j\omega$

Write the coefficients A and A' in polar form

$$A = |A|e^{j\phi}$$
$$A' = |A|e^{-j\phi}$$

$\rightsquigarrow$ $|A|$ denotes the magnitude of coefficient A

$\rightsquigarrow$ $\phi = \arg(A)$ is the phase of coefficient A

We have,

$$\begin{aligned}
Ae^{pt} + A'e^{p't} &= |A|e^{j\phi}e^{(\alpha+j\omega)t} + |A|e^{-j\phi}e^{(\alpha-j\omega)t} \\
&= |A|e^{\alpha t}\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right] \\
&= 2|A|e^{\alpha t}\cos(\omega t + \phi) \text{ [Euler's formula]} \\
&= \underbrace{M}_{M=2|A|\geq 0}\; e^{\alpha t}\cos(\omega t + \phi)
\end{aligned}$$

■

Homogeneous equation and modes (cont.)

The linear combination of two modes $(At^k e^{pt} + A't^k e^{p't})$

$$\rightsquigarrow \quad M\, t^k\, e^{\alpha t} \cos(\omega t + \phi)$$

The term is denoted **pseudo-periodic mode**

Homogeneous equation and modes (cont.)

Complex numbers (Polar representation)

Consider the set of complex numbers $\mathcal{C} = \{u + jv \big| u, v \in \mathcal{R}\}$ $(j = \sqrt{-1})$

The **complex number** $z = \text{Re}(z) + \text{Im}(z) = u + jv$

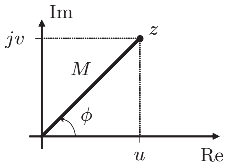
We can define

Module

- $M = |z| = \sqrt{u^2 + v^2}$

Phase

- $\phi = \arg(z) = \arctan(v/u)$



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Homogeneous equation and modes (cont.)

The inverse formulæ hold

- $\rightsquigarrow$ $u = M\cos(\phi)$
- $\rightsquigarrow$ $v = M\sin(\phi)$

We have,

$$z = u + jv$$
$$= M\cos(\phi) + jM\sin(\phi) = M\big[\cos(\phi) + j\sin(\phi)\big]$$
$$\rightsquigarrow \quad = Me^{j\phi}$$

The polar representation of a complex number

$$z = Me^{j\phi} = |z|e^{j\phi} = |z|e^{j\arg(z)}$$

The complex conjugate

$$\rightsquigarrow \quad z' = |z|e^{-\arg(z)}$$

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Homogeneous equation and modes (cont.)

Euler's formula

Relationships to write a periodic function as sum of exponential functions

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$
$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

Proof

$$\frac{e^{j\phi} + e^{-j\phi}}{2} = \frac{[\cos(\phi) + j\sin(\phi)] + [\cos(-\phi) + j\sin(-\phi)]}{2}$$
$$= \frac{[\cos(\phi) + j\sin(\phi)] + [\cos(\phi) - j\sin(\phi)]}{2} = \frac{2\cos(\phi)}{2} = \cos(\phi)$$

$$\frac{e^{j\phi} - e^{-j\phi}}{2} = \frac{[\cos(\phi) + j\sin(\phi)] - [\cos(-\phi) + j\sin(-\phi)]}{2}$$
$$= \frac{[\cos(\phi) + j\sin(\phi)] - [\cos(\phi) - j\sin(\phi)]}{2} = \frac{2j\sin(\phi)}{2} = \sin(\phi)$$

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Homogeneous equation and modes (cont.)

We can define an alternative structure of the linear combination of modes

- The structure will be equivalent to the form in A

$$\sum_{i=1}^{r}\sum_{k=0}^{\nu_i-1} A_{i,k}\, t^k e^{p_i t}$$

Pairs of conjugate complex roots are expressed using the form in M and ϕ

$$\sum_{k=0}^{\nu_i-1} M_{i,k}\, t^k e^{\alpha_i t}\cos(\omega_i t + \phi_{i,k})$$

Homogeneous equation and modes (cont.)

Let R be the number of distinct real roots p_i

- Multiplicity ν_i $(i = 1, \dots, R)$

$$\rightsquigarrow \quad p_1, p_2, \dots, p_i, \dots, p_R$$

Let S be the number of pairs of distinct complex conjugate roots (p_i, p_i')

- Multiplicity ν_i $(i = R+1, \dots, R+S)$

$$\rightsquigarrow \quad (p_{R+1}, p'_{R+1}), (p_{R+2}, p'_{R+2}), \dots, (p_i, p'_i), \dots, (p_{R+S}, p'_{R+S})$$

Clearly, the total number of roots

$$\rightsquigarrow \quad n = \sum_{i=1}^{R} \nu_i + 2 \sum_{i=R+1}^{R+S} \nu_i$$

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Homogeneous equation and modes (cont.)

$$n = \sum_{i=1}^{R} \nu_i + 2 \sum_{i=R+1}^{R+S} \nu_i$$

We consider a particular representation of the linear combination of modes

We distinguish modes associated with real and conjugate complex roots

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k}) \qquad (9)$$

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Homogeneous equation and modes (cont.)

Consider the case in which all roots have multiplicity equal to one

$$n = R + 2S$$

We have,

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} A_i e^{p_i t} + \sum_{i=R+1}^{R+S} M_i e^{\alpha_i t} \cos(\omega_i t + \phi_i) \qquad (10)$$

(We have omitted the second subscript of the coefficients A, M and ϕ)

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Homogeneous equation and modes (cont.)

Example

Consider a system with homogeneous differential equation

$$\frac{d^3 y(t)}{dt^3} + 2\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} = 0$$

The characteristic polynomial without constant term

$$P(s) = s^3 + 2s^2 + 5s = s(s^2 + 2s + 5)$$

Its roots, from $P(s) = 0$

$$\rightsquigarrow \begin{cases} p_1 = 0, & (\nu_1 = 1) \\ p_2 = \alpha_2 + j\omega_2 = -1 + j2, & (\nu_2 = 1) \\ p_2' = \alpha_2 - j\omega_2 = -1 - j2, & (\nu_2' = 1) \end{cases}$$

Homogeneous equation and modes (cont.)

We can write a linear combination of the modes

$$h(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)}_{\text{root } (p_2, p_2')}$$

$$= A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

■

Homogeneous equation and modes (cont.)

We can define yet another structure of the linear combination of modes

Homogeneous equation and modes (cont.)

Proposition

Consider the contribution of $(p_i, p_i') = \alpha_i \pm j\omega_i$ a pair of conjugate complex roots with multiplicity ν_i to the linear combination of the $(2\nu_i)$ modes

$$\underbrace{(A_{i,0} e^{p_i t} + A_{i,0}' e^{p_i' t})}_{k=0} + \cdots + \underbrace{t^{\nu_i - 1}(A_{i,\nu_i - 1} e^{p_i t} + A_{i,\nu - 1}' e^{p_i' t})}_{k=\nu_i - 1}$$

This sum of terms can be re-written

$$\rightsquigarrow \sum_{k=0}^{\nu_i - 1} \Big[B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \Big] \qquad (11)$$

The $2\nu_i$ complex coefficients, $A_{i,k}$ and $A_{i,k}'$, are replaced by $2\nu_i$ real ones

- $\rightsquigarrow$ $B_{i,k}$
- $\rightsquigarrow$ $C_{i,k}$

Homogeneous equation and modes (cont.)

Proof

Consider the term $(Ae^{pt} + A' e^{p' t})$ in which $(p_i, p_i') = \alpha + j\omega$

Write the coefficients A and A' in cartesian form

$$A = u + jv$$
$$A' = u - jv$$

We have,

$$\begin{aligned} Ae^{pt} + A' e^{p' t} &= (u + jv) e^{\alpha t} \big[\cos(\omega t) + j \sin(\omega t)\big] \\ &+ (u - jv) e^{\alpha t} \big[\cos(\omega t) - j \sin(\omega t)\big] \\ &= 2u e^{\alpha t} \cos(\omega t) - 2v e^{\alpha t} \sin(\omega t) \\ &= \underbrace{B}_{B=2u} e^{\alpha t} \cos(\omega t) + \underbrace{C}_{C=-2v} e^{\alpha t} \sin(\omega t) \end{aligned}$$

■

Homogeneous equation and modes (cont.)

We distinguish modes associated with real and conjugate complex roots

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} \Big[B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \Big] \quad (12)$$

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Homogeneous equation and modes (cont.)

Consider the case in which all roots have multiplicity equal to one

$$\rightsquigarrow \quad n = R + 2S$$

We have,

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} A_i e^{p_i t} + \sum_{i=R+1}^{R+S} \Big[B_i e^{\alpha_i t} \cos(\omega_i t) + C_i e^{\alpha_i t} \sin(\omega_i t) \Big] \quad (13)$$

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Homogeneous equation and modes (cont.)

The equations

$$h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

$$\Big(\rightsquigarrow \sum_{i=1}^{R} A_i e^{p_i t} + \sum_{i=R+1}^{R+S} M_i e^{\alpha_i t} \cos(\omega_i t + \phi_i) \Big)$$

The equations

$$h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} \Big[B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \Big]$$

$$\Big(\rightsquigarrow \sum_{i=1}^{R} A_i e^{p_i t} + \sum_{i=R+1}^{R+S} \Big[B_i e^{\alpha_i t} \cos(\omega_i t) + C_i e^{\alpha_i t} \sin(\omega_i t) \Big] \Big)$$

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Homogeneous equation and modes (cont.)

They provide the parametric structure of the linear combination

$\rightsquigarrow$ They are all equivalent

Homogeneous equation and modes (cont.)

Example

Consider a system with homogeneous differential equation

$$\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0$$

Characteristic polynomial $P(s)$ w/o constant term and the roots of $P(s) = 0$

$$P(s) = s^3 + 2s^2 + 5s = s(s^2 + 2s + 5)$$

$$\rightsquigarrow \begin{cases} p_1 = 0, & (\nu_1 = 1) \\ p_2 = \alpha_2 + j\omega_2 = -1 + j2, & (\nu_2 = 1) \\ p_2' = \alpha_2 - j\omega_2 = -1 - j2, & (\nu_2' = 1) \end{cases}$$

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Homogeneous equation and modes (cont.)

This problem can be solved in two equivalent ways

$$\rightsquigarrow \quad h(t) = \underbrace{A_1}_{\text{root } p_1} + \underbrace{B_2 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)}_{\text{root } (p_2, p_2')}$$

$$\rightsquigarrow \quad h(t) = \underbrace{A_1}_{\text{root } p_1} + \underbrace{M_2 e^{-t}\cos(2t + \phi_2)}_{\text{root } (p_2, p_2')}$$

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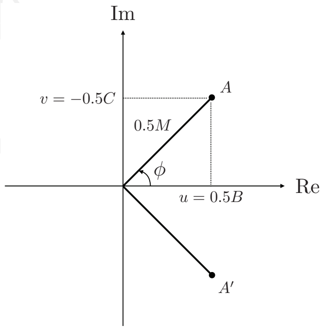
Homogeneous equation and modes (cont.)

The two coefficients A and A' in the complex plane

$$A = (M/2)e^{+j\omega} = B/2 - jC/2$$
$$A' = (M/2)e^{-j\omega} = B/2 + jC/2$$

$$M = 2|A| = \sqrt{B^2 + C^2}$$
$$\phi = \arg(A) = \arctan(-C/B)$$

$$B = +M\cos\phi = +2u$$
$$C = -M\sin\phi = -2v$$



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Force-free evolution

The **force-free response** is a particular contribution to the output

It is due to the fact that the system is NOT initially at rest

- (This is the cause due to the non-zero state at t_0)

$$y(t) = \underbrace{y_u(t)}_{\text{force-free response}} + y_f(t), \quad \text{for } t \geq t_0$$

We study how to characterise it

Force-free evolution (cont.)

Proposition

Free-force response

Consider a SISO system represented by a linear, time-invariant IO model

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

*The **free-force response** $y_u(t)$ is a linear combination of the modes*

Proof

Let the input $u(t)$ be always zero for $t \geq 0$

- Then, also its derivatives are zero

$$\rightsquigarrow \quad a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = 0$$

Force-free evolution (cont.)

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = 0$$

The force-free response $y_u(t)$ for $t \geq 0$ is equal to the solution of the associated homogeneous differential equation, for some given initial conditions

$$\begin{cases} y_0 = y(t)\Big|_{t=t_0} \\ y_0' = \dfrac{\mathrm{d}y(t)}{\mathrm{d}t}\Big|_{t=t_0} \\ \cdots = \cdots \\ y_0^{(n-1)} = \dfrac{\mathrm{d}^{n-1}y(t)}{\mathrm{d}t^{n-1}}\Big|_{t=t_0} \end{cases}$$

$h(t)$ solves the homogeneous equation iff it is linear combination of modes

$\rightsquigarrow$ Thus, $y_u(t)$ can be expressed as a linear combination of the modes

- (The n coefficients are still unknown)

■

Force-free evolution (cont.)

The coefficients of the force-free response depend on the initial conditions

$\rightsquigarrow$ So, does its evolution

The force-free response $y_u(t)$ is a particular linear combination of the modes

- The n coefficients are determined from initial conditions

$$\begin{cases} y_0 = y(t)\Big|_{t=t_0} \\ y_0' = \dfrac{\mathrm{d}y(t)}{\mathrm{d}t}\Big|_{t=t_0} \\ \cdots = \cdots \\ y_0^{(n-1)} = \dfrac{\mathrm{d}^{n-1}y(t)}{\mathrm{d}t^{n-1}}\Big|_{t=t_0} \end{cases}$$

Force-free evolution (cont.)

Example

Consider a system with homogeneous differential equation

$$\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt} + 21\frac{dy(t)}{dt} + 18y(t) = 0$$

We are interested in the force-free response $y_u(t)$, for $t \geq 0$

- The initial conditions

$$y_0 = 2$$
$$y'_0 = 1$$
$$y''_0 = -20$$

Force-free evolution (cont.)

The characteristic polynomial

$$P(s) = s^3 + 8s^2 + 21s + 18 = (s+2)(s+3)^2$$

Its roots from $P(s) = 0$ are all real

$$\begin{cases} p_1 = -2, & \text{multiplicity } \nu_1 = 1 \\ p_2 = -3, & \text{multiplicity } \nu_2 = 2 \end{cases}$$

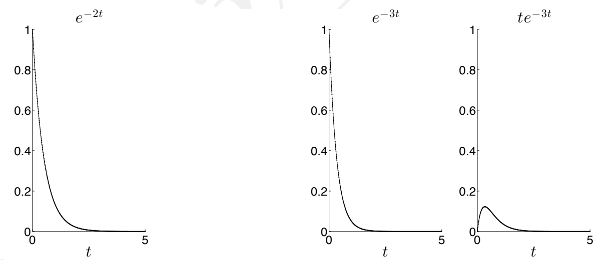
The force-free response

$$\rightsquigarrow \quad y_u(t) = \underbrace{A_1 e^{-2t}}_{\text{root } p_1} + \underbrace{A_{2,0} e^{-3t} + A_{2,1} te^{-3t}}_{\text{root } p_2}$$

Force-free evolution (cont.)

$$y_u(t) = \underbrace{A_1 e^{-2t}}_{\text{root } p_1} + \underbrace{A_{2,0} e^{-3t} + A_{2,1} te^{-3t}}_{\text{root } p_2}$$

The three modes to be combined to get the force-free response



Force-free evolution (cont.)

The force-free response

$$y_u(t) = A_1 e^{-2t} + A_{2,0} e^{-3t} + A_{2,1} te^{-3t}$$

Its first- and second-order derivatives

$$\frac{dy_u(t)}{dt} = -2A_1 e^{-2t} - 3A_{2,0} e^{-3t} + A_{2,1}(e^{-3t} - 3te^{-3t})$$
$$\frac{d^2y_u(t)}{dt^2} = 4A_1 e^{-2t} + 9A_{2,0} e^{-3t} + A_{2,1}(-6e^{-3t} + 9te^{-3t})$$

Force-free evolution (cont.)

We substitute the initial conditions

$$\begin{aligned} y_u(t)\Big|_{t=0} &= A_1 + A_{2,0} = 2 \\ \frac{\mathrm{d}y_u(t)}{\mathrm{d}t}\Big|_{t=0} &= -2A_1 - 3A_{2,0} + A_{2,1} = 1 \\ \frac{\mathrm{d}^2 y_u(t)}{\mathrm{d}t^2}\Big|_{t=0} &= 4A_1 + 9A_{2,0} - 6A_{2,1} = -20 \end{aligned}$$

Force-free evolution (cont.)

$$\begin{aligned} y_u(t)\Big|_{t=0} &= A_1 + A_{2,0} = 2 \\ \frac{\mathrm{d}y_u(t)}{\mathrm{d}t}\Big|_{t=0} &= -2A_1 - 3A_{2,0} + A_{2,1} = 1 \\ \frac{\mathrm{d}^2 y_u(t)}{\mathrm{d}t^2}\Big|_{t=0} &= 4A_1 + 9A_{2,0} - 6A_{2,1} = -20 \end{aligned}$$

We have,

$$\mathbf{Ax} = \mathbf{b} \quad \rightsquigarrow \begin{cases} \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & 1 \\ 4 & 9 & -6 \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \\ \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -20 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ \mathbf{x} = \begin{bmatrix} A_1 \\ A_{2,0} \\ A_{2,1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Force-free evolution (cont.)

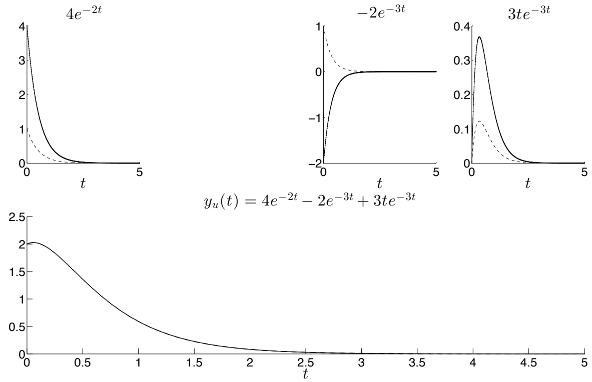
The solutions of the linear system of equations

- $A_1 = x_1 = 4$
- $A_{2,0} = x_2 = -2$
- $A_{2,1} = x_3 = 3$

Force-free evolution (cont.)

We can write the complete expression of the force-free evolution $y_u(t)$

$$y_u(t) = A_1 e^{p_1 t} + A_{2,0} e^{p_{2,} t} + A_{2,1} t e^{p_2 t}$$
$$= 4e^{-2t} - 2e^{-3t} + 3te^{-3t}$$



Force-free evolution (cont.)

Complex conjugate roots

Consider a characteristic polynomial $P(s)$ with conjugate complex roots

$$(p_i, p_i') = \alpha_i \pm j\omega_i$$

We want to determine an expression for force-free evolution

- We need to use a(ny) linear combination of the modes

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} \Big[B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \Big]$$

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Force-free evolution (cont.)

Example

Consider a system with homogeneous differential equation

$$\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 0$$

We are interested in the force-free response $y_u(t)$, for $t \geq 0$

- The initial condition

$$y_0 = 3$$
$$y_0' = 2$$
$$y_0'' = 1$$

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Force-free evolution (cont.)

The characteristic polynomial

$$P(s) = s^3 + 2s^2 + 5s = s(s^2 + 2s + 5)$$

Its roots from $P(s) = 0$

$$\begin{cases} p_1 = -0, & \text{multiplicity } \nu_1 = 1 \\ p_2 = -\alpha_2 + j\omega = -1 + j2, & \text{multiplicity } \nu_2 = 1 \\ p_2' = -\alpha_2 - j\omega = -1 - j2, & \text{multiplicity } \nu_2' = 1 \end{cases}$$

- $R = 1$ distinct real roots
- $S = 1$ distinct pair of complex conjugate roots

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Force-free evolution (cont.)

We first consider a parameterisation in the form

$$h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

We get the force-free response

$$y_u(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)}_{\text{root } (p_2, p_2')} = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

Force-free evolution (cont.)

The force-free response and its derivatives of order 1 and order 2

$$y_u(t) = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$
$$\frac{\mathrm{d}y_u(t)}{\mathrm{d}t} = -M_2 e^{-t} \cos(2t + \phi_2) - 2M_2 e^{-t} \sin(2t + \phi_2)$$
$$\frac{\mathrm{d}^2 y_u(t)}{\mathrm{d}t^2} = -2M_2 e^{-t} \cos(2t + \phi_2) + 4M_2 e^{-t} \sin(2t + \phi_2)$$

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Force-free evolution (cont.)

We substitute the initial conditions

$$y_u(t)\Big|_{t=0} = A_1 + M_2 \cos(\phi_2) = 3$$
$$\frac{\mathrm{d}y_u(t)}{\mathrm{d}t}\Big|_{t=0} = -M_2 \cos(\phi_2) - 2M_2 \sin(\phi_2) = 2$$
$$\frac{\mathrm{d}^2 y_u(t)}{\mathrm{d}t^2}\Big|_{t=0} = -3M_2 \cos(\phi_2) + 4M_2 \sin(\phi_2) = 1$$

The system of equations is non-linear in the unknowns

- $\rightsquigarrow$ M_2
- $\rightsquigarrow$ ϕ_2
- $\rightsquigarrow$ (A_1)

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Force-free evolution (cont.)

$$y_u(t)\Big|_{t=0} = A_1 + M_2 \cos(\phi_2) = 3$$
$$\frac{\mathrm{d}y_u(t)}{\mathrm{d}t}\Big|_{t=0} = -M_2 \cos(\phi_2) - 2M_2 \sin(\phi_2) = 2$$
$$\frac{\mathrm{d}^2 y_u(t)}{\mathrm{d}t^2}\Big|_{t=0} = -3M_2 \cos(\phi_2) + 4M_2 \sin(\phi_2) = 1$$

The system of equations is linear in the unknowns

- $\rightsquigarrow$ $x = M_2 \cos(\phi_2)$
- $\rightsquigarrow$ $y = M_2 \sin(\phi_2)$

For consistency, we let $z = A_1$

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Force-free evolution (cont.)

$$y_u(t)\Big|_{t=0} = \underbrace{A_1}_{z} + \underbrace{M_2 \cos(\phi_2)}_{x} = 3$$
$$\frac{dy_u(t)}{dt}\Big|_{t=0} = -\underbrace{M_2 \cos(\phi_2)}_{x} - 2\underbrace{M_2 \sin(\phi_2)}_{y} = 2$$
$$\frac{d^2 y_u(t)}{dt^2}\Big|_{t=0} = -3\underbrace{M_2 \cos(\phi_2)}_{x} + 4\underbrace{M_2 \sin(\phi_2)}_{y} = 1$$

The resulting system of linear equation

$$\rightsquigarrow \quad \begin{cases} z + x = 3 \\ -x - 2y = 2 \\ -3x + 4y = 1 \end{cases}$$

Force-free evolution (cont.)

The solution

- $z = 4 = A_1$
- $x = -1 = M_2 \cos(\phi_2)$
- $y = -0.5 = M_2 \sin(\phi_2)$

Thus, we get

$$\rightsquigarrow \quad \begin{cases} A_1 = 4 \\ \\ M_2 = \sqrt{x^2 + y^2} = \sqrt{1^2 + 0.5^2} = 1.12 \\ \phi_2 = \arctan(y/x) = \arctan(-0.50/-1) = -2.68 \text{ [rad]} \end{cases}$$

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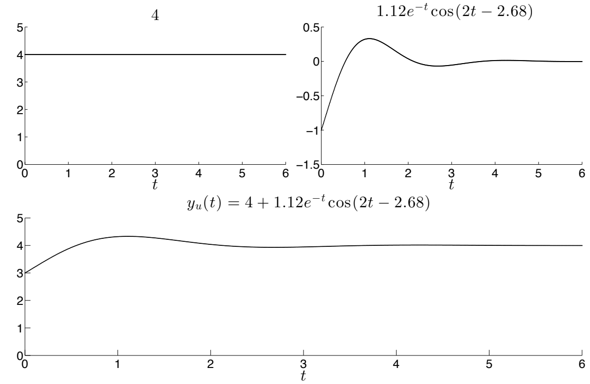
Force-free evolution (cont.)

The force-free response for $t \geq 0$

$$y_u(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{M_2 e^{\alpha_2 t} \cos(\omega_2 t + \phi_2)}_{\text{root } (p_2, p_2')} = A_1 + M_2 e^{-t} \cos(2t + \phi_2)$$

$$\rightsquigarrow \quad = 4 + 1.12 e^{-t} \cos(2t - 2.68)$$



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Force-free evolution (cont.)

We now consider a parameterisation in the form

$$h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} \left[B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t) \right]$$

We get the force-free response

$$y_u(t) = \underbrace{A_1 e^{p_1 t}}_{\text{root } p_1} + \underbrace{B_2 e^{\alpha_2 t} \cos(\omega_2 t) + C_2 e^{\alpha_2 t} \sin(\omega_2 t)}_{\text{root } (p_2, p_2')}$$

$$= A_1 + B_2 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t)$$

$$\rightsquigarrow \quad = 4 - e^{-t} \cos(2t) + 0.5 e^{-t} \sin(2t)$$

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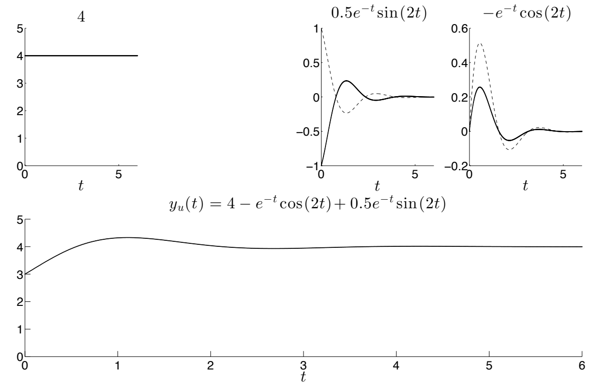
Force-free evolution (cont.)



Force-free evolution (cont.)

We can compare the different forms of the solution

$$A = (M/2)e^{+j\omega} = B/2 - jC/2$$
$$A' = (M/2)e^{-j\omega} = B/2 + jC/2$$

$$M = 2|A| = \sqrt{B^2 + C^2}$$
$$\phi = \arg(A) = \arctan(-C/B)$$

$$B = +M\cos\phi = +2u$$
$$C = -M\sin\phi = -2v$$

We get,

$$M_2 = \sqrt{B_2^2 + C_2^2}$$
$$\phi_2 = \arctan(-C_2/B_2)$$

$$M_2 = +M_2\cos(\phi_2)$$
$$C_2 = -M_2\sin(\phi_2)$$

Force-free evolution (cont.)

Initial time not equal zero

How to calculate the force-free response from an initial time $t \neq 0$

Force-free evolution (cont.)

Example

Consider a system with homogeneous differential equation

$$\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 8\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t} + 21\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 18y(t) = 0$$

We are interested in the force-free response, for $t \geq t_0 \neq 0$

- The initial condition

$$y(t)\Big|_{t=t_0} = y_0 = 2$$
$$\frac{\mathrm{d}y(t)}{\mathrm{d}t}\Big|_{t=t_0} = y_0' = 1$$
$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}\Big|_{t=t_0} = y_0'' = -20$$

Classification of modes

Input-output representation

Classification of modes

Modes fully characterise the dynamics of a system

- It is important to study their form
- It is important to classify them

We provide an intuitive classification

⇝ **Aperiodic modes**
⇝ **Pseudo-periodic modes**

Aperiodic modes have no oscillatory behaviour, pseudo-periodic ones do

Classification of modes (cont.)

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = \sum_{i=0}^{n} a_i s^i = 0$$

Aperiodic modes

$$t^k e^{\alpha t}, \quad \text{for } k = 0, \dots, \nu - 1$$

Associate to real roots $p = \alpha \in \mathcal{R}$ (multiplicity ν)

Pseudo-periodic modes

$$\begin{cases} t^k e^{\alpha t} \cos(\omega t) \\ t^k e^{\alpha t} \sin(\omega t) \end{cases}, \quad \text{for } k = 0, \dots, \nu - 1$$

$$t^k e^{\alpha t} \cos(\omega t + \phi_k), \quad \text{for } k = 0, \dots, \nu - 1$$

Associate to conjugate complex roots $(p, p') = \alpha \pm j\omega \in \mathcal{C}$ (multiplicity ν)

Aperiodic modes

Modes

Aperiodic modes

These are the modes associated to real roots $p = \alpha \in \mathcal{R}$, multiplicity ν

$$t^k e^{\alpha t}, \quad k = 0, \dots, \nu - 1$$

The fundamental parameter of the generic aperiodic mode is $\alpha \neq 0$

$$\rightsquigarrow \quad \tau = -1/\alpha, \quad (\alpha = p \neq 0)$$

The exponent t/τ in $t^k e^{\alpha t} = t^k e^{-t/\tau}$ is dimensionless

- Parameter τ has the units of time

⇝ **Time-constant**

The time constant is not defined for $\alpha = 0$

Aperiodic modes (cont.)

Roots with multiplicity one

Let real root α have multiplicity $\nu = 1$, there is only one associated mode

$$\rightsquigarrow \quad e^{\alpha t}$$

This mode (a simple exponential) is aperiodic

Stable or **convergent**, if $\alpha < 0$

$\rightsquigarrow$ As t increases, the mode $e^{\alpha t}$ tends to 0 asymptotically

Stability limit or **constant**, if $\alpha = 0$

$\rightsquigarrow$ The mode is equal to $e^{0t} = 1$, for any $t \geq 0$

Unstable or **divergent**, if $\alpha > 0$

$\rightsquigarrow$ As t increases, the mode $e^{\alpha t}$ tends to ∞ asymptotically

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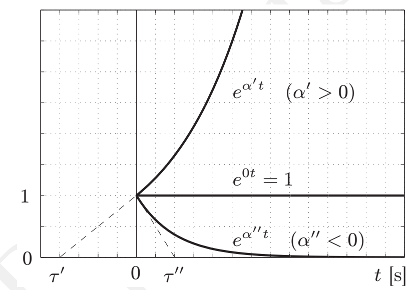
Aperiodic (cont.)



Unstable $(\alpha > 0)$

$\rightsquigarrow$ The time-constant takes negative values $\tau < 0$

Stable $(\alpha < 0)$

$\rightsquigarrow$ The time-constant takes positive values $\tau > 0$

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Aperiodic (cont.)

τ is geometrically understood as the (below) tangent to mode at $t = 0$

The value of the tangent where it intersects the abscissa

$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} e^{\alpha t}\Big|_{t=0} = \alpha e^{\alpha t}\Big|_{t=0} = \alpha$$

The line tangent to $e^{\alpha t}$ in $t = 0$ is $f(t) = at + b$ with slope $a = \alpha$

- The intercept (at $t = 0$) is $b = f(0) = 1$
- $f(t) = \alpha t + 1 = 0$ when $t = -1/\alpha = \tau$

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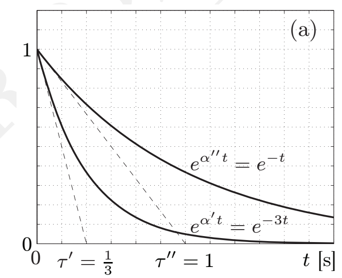
Aperiodic (cont.)

τ is also the time after which the mode has lost $\approx 63\%$ of its initial value

t	0	τ	2τ	3τ	4τ	5τ
$e^{\alpha t} = e^{-t/\tau}$	1	0.37	0.14	0.05	0.02	0.01

The smaller the time-constant $\tau = -1/\alpha$, the faster a (stable) mode vanishes



Aperiodic (cont.)

Roots with multiplicity larger than one

Let real root α have multiplicity $\nu > 1$, there are ν associated modes

$$\rightsquigarrow \quad e^{\alpha t}, te^{\alpha t}, t^2 e^{\alpha t}, \dots, t^k e^{\alpha t}, \dots, t^{\nu-1} e^{\alpha t}$$

We consider only modes in the form $t^k e^{\alpha t}$, with $k > 0$

Stable, if $\alpha < 0$ and $k \geq 1$

$\rightsquigarrow$ As t increases, the mode $t^k e^{\alpha t}$ tends to 0 asymptotically

Unstable, if $\alpha \geq 0$ and $k \geq 1$

$\rightsquigarrow$ As t increases, the mode $t^k e^{\alpha t}$ tends to ∞ asymptotically

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Aperiodic (cont.)

Case with $\alpha < 0$ and $k > 0$ ($k \geq 1$)

Consider the case in which $\alpha < 0$ and $k > 0$

$$\rightsquigarrow \quad t^k e^{\alpha t}$$

If we study the asymptotic behaviour of the mode, we get

$$\rightsquigarrow \quad \lim_{t\to\infty} t^k e^{\alpha t} = \lim_{t\to\infty} \frac{t^k}{e^{-\alpha t}} = \infty/\infty$$

The undetermined form is solved by differentiating k times (de l'Hospital)

$$\rightsquigarrow \quad \lim_{t\to\infty} t^k e^{\alpha t} = \lim_{t\to\infty} \frac{t^k}{e^{-\alpha t}} = \lim_{t\to\infty} \frac{k!}{(-\alpha)^k e^{-\alpha t}} = 0 \quad \text{(stable)}$$

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Aperiodic (cont.)

Case with $\alpha \geq 0$ and $k \geq 1$

Consider the case in which $\alpha \geq 0$ and $k \geq 1$

$$t^k e^{\alpha t}$$

If the root is null ($\alpha = 0$), the mode is t^k and it grows

- $\rightsquigarrow$ $k = 1$, a line
- $\rightsquigarrow$ $k = 2$, a parabola
- $\rightsquigarrow$ $k = 3$, a cubic

For a positive root ($\alpha > 0$), the mode grows faster

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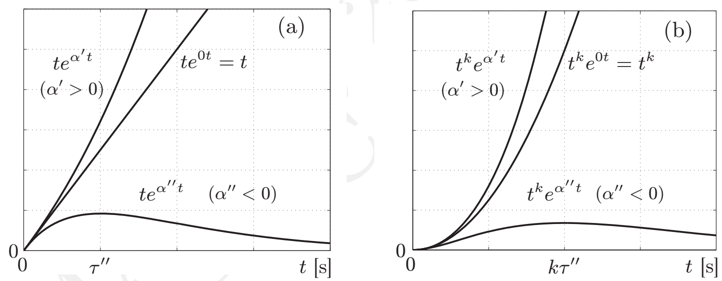
Aperiodic (cont.)

(a) For $k = 1$, the tangent to the mode has unit slope in $t = 0$

(b) For $k > 1$, the tangent to the mode has zero slope in $t = 0$



Stable modes have one maximum at $t = k\tau$

$$\rightsquigarrow \quad \tau = -1/\alpha$$

Aperiodic (cont.)

Consider a stable (decreasing) mode of the form $t^k e^{\alpha t}$ and $k \geq 1$

Still, the smaller the time-constant $\tau = -1/\alpha$, the faster a mode vanishes

- Different geometrical interpretation compared to the case $k = 0$

⇝ $t = k\tau$ is the value of t where the mode has its single maximum

To appreciate this fact, we can differentiate the mode

$$\frac{\mathrm{d}}{\mathrm{d}t} t^k e^{\alpha t} = k t^{k-1} e^{\alpha t} + \alpha t^k e^{\alpha t} = t^{k-1} e^{\alpha t}(k + \alpha t)$$

The derivative is zero for $t > 0$ and $a < 0$ at $t = -k/\alpha = k\tau$

- Curve $e^k e^{\alpha t}$ for $\alpha < 0$ has a maximum at $t = k\tau$

Pseudo-periodic modes

Modes

Pseudo-periodic modes

These are the modes associated to conjugate complex roots $(p, p') = \alpha \pm j\omega$

Pseudo-periodic modes can take various forms

- We restrict our presentation to one type

$$t^k e^{\alpha t} \cos{(\omega t)}, \quad \text{with } k = 0, \dots, \nu - 1$$

The other cases (phased) are not considered

Pseudo-periodic modes (cont.)

$$t^k e^{\alpha t} \cos{(\omega t)}, \quad (k = 0, \dots, \nu - 1)$$

The parameters that characterise the generic pseudo-periodic mode

⇝ **Time-constant**

$$\tau = -\frac{1}{\alpha}, \quad \alpha \neq 0$$

⇝ **Natural pulsation**

$$\omega_n = \sqrt{\alpha^2 + \omega^2}$$

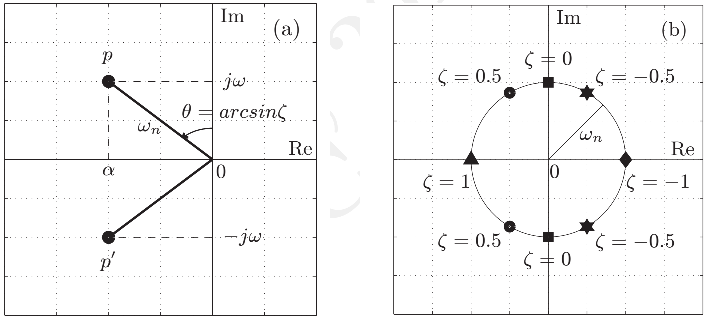
⇝ **Dumping coefficient**

$$\zeta = -\frac{\alpha}{\omega_n} = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

Pseudo-periodic modes (cont.)

Natural pulsation

We can represent the pair of roots $(p, p') = \alpha \pm j\omega$ on the complex plane



Suppose that $p = \alpha + j\omega$ is a pole on the positive imaginary half-plane

- ω_n is the module of the vector that connects pole p (p') and origin

$$\rightsquigarrow \quad \omega_n = \sqrt{\alpha^2 + \omega^2}$$

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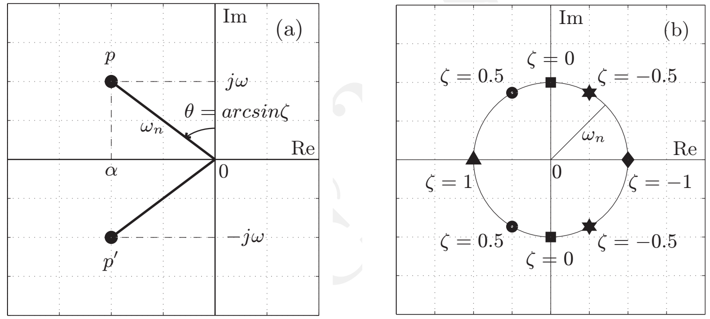
Pseudo-periodic modes (cont.)

Dumping coefficient



ζ is the sine of the angle θ between the vector connecting p with the origin and the positive imaginary half-axis (counterclock-wise = positive)

$$\rightsquigarrow \quad \zeta = -\frac{\alpha}{\omega_n} = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

- Negative α, $\rightsquigarrow$ positive θ
- Null α, $\rightsquigarrow$ null θ
- Positive α, $\rightsquigarrow$ negative θ

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Pseudo-periodic modes (cont.)

Roots with multiplicity one

Consider a pair of conjugate complex roots $(p, p') = \alpha \pm j\omega$ with $\nu = 1$

The corresponding pseudo-periodic mode

$$\rightsquigarrow \quad e^{\alpha t}\cos(\omega t)$$

Such a mode has an oscillatory behaviour

- This is due to the cosine factor

The mode envelops $\cos(\omega t)$ with functions $-e^{\alpha t}$ and $e^{\alpha t}$

$$\rightsquigarrow \quad e^{\alpha t}\cos(\omega t) = \begin{cases} -e^{\alpha t}, & t = (2h+1)\dfrac{\pi}{\omega}, \quad h \in \mathcal{N} \\ e^{\alpha t}, & t = 2h\dfrac{\pi}{\omega}, \quad h \in \mathcal{N} \end{cases}$$

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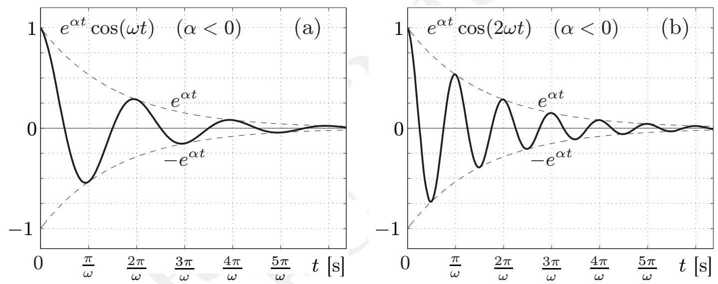
Pseudo-periodic modes (cont.)



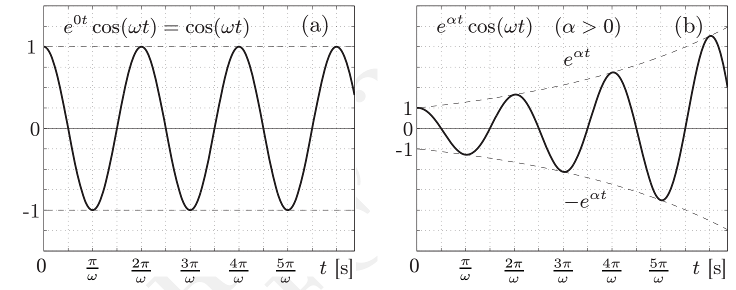
Stable $(\alpha < 0)$

$\rightsquigarrow$ As t increases the envelops tend to 0 asymptotically

Case (a) has a larger damping factor than case (b)

- Time constant is equal (same α)

Pseudo-periodic modes (cont.)



Stability limit $(\alpha = 0)$

⇝ The mode becomes equal to $\cos(\omega t)$ and it is periodic

⇝ As functions of t, the envelops are constant ± 1 curves

Unstable $(\alpha > 0)$

⇝ As t increases the envelops tend to $\pm\infty$ asymptotically

Pseudo-periodic modes (cont.)

The time constant

Again, the time constant indicates the velocity of the mode (envelops)

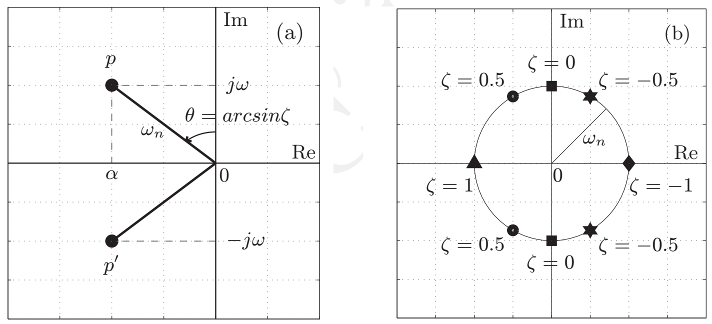
- (As in the aperiodic case)

Pseudo-periodic modes (cont.)

The dumping factor ζ

The dumping factor is a real number in the interval $[-1, 1]$

⇝ As it is equal to $\sin(\theta)$

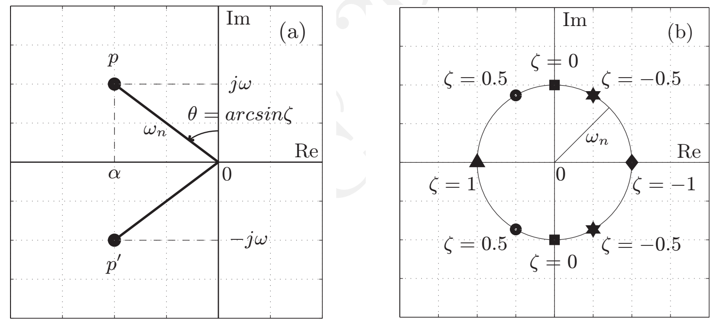


$$\zeta = -\frac{\alpha}{\omega_n} = -\frac{\alpha}{\sqrt{\alpha^2 + \omega^2}}$$

Pseudo-periodic modes (cont.)

We study pairs of roots with the same natural pulsation (same ω_n)

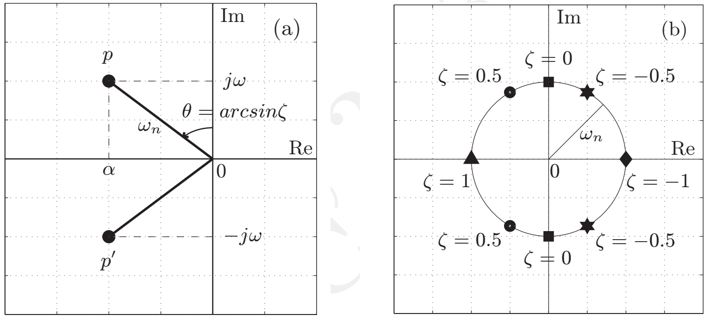
- Dumping coefficient is changed (different α and ω)



These roots lie on the complex plane

- Along a circle, radius ω_n

Pseudo-periodic modes (cont.)



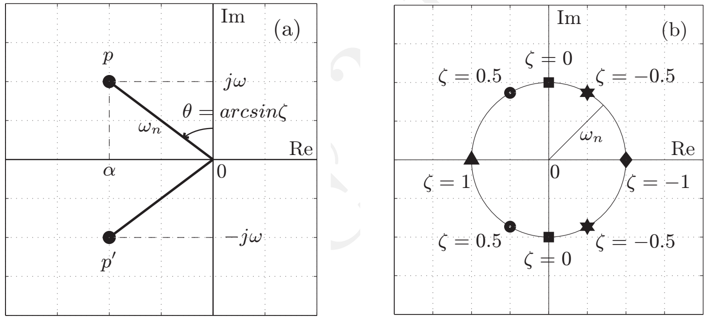
$\zeta = +1$, if $\alpha = -\omega_n < 0$ and $\omega = 0$

- Two complex roots coinciding with a negative real root, multiplicity 2
- The associated modes are $e^{-\omega_n t}$ and $te^{-\omega_n t}$

$\zeta = -1$, if $\alpha = -\omega_n > 0$ and $\omega = 0$

- Two complex roots coinciding with a positive real root, multiplicity 2
- The associated modes are $e^{\omega_n t}$ and $te^{\omega_n t}$

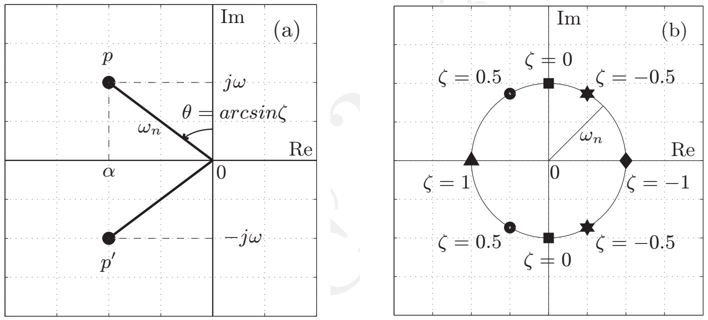
Pseudo-periodic modes (cont.)



$\zeta = 0$, if $\alpha = 0$ and $\omega = \omega_n$

- Two conjugate imaginary roots
- The associated mode is at the stability limit

Pseudo-periodic modes (cont.)



$\zeta \in (0, 1)$, if $\alpha < 0$ and $\omega > 0$

- The two complex roots have a negative real part
- The associated mode is stable

$\zeta \in (-1, 0)$, if $\alpha > 0$ and $\omega > 0$

- The two complex roots have a positive real part
- The associated mode is unstable

Pseudo-periodic modes (cont.)

Roots with multiplicity larger than one

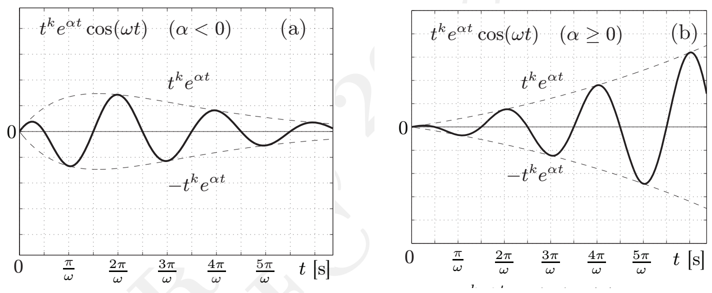
Consider a pair of conjugate complex roots $(p, p') = \alpha \pm j\omega$ with $\nu > 1$

The corresponding pseudo-periodic modes

$$e^{\alpha t}\cos(\omega t), te^{\alpha t}\cos(\omega t), t^2 e^{\alpha t}\cos(\omega t), \ldots, t^k e^{\alpha t}\cos(\omega t), \ldots, t^{\nu-1} e^{\alpha t}\cos(\omega t)$$

We consider the modes in the form $t^k e^{\alpha t}\cos(\omega t)$ with $k > 0$

Pseudo-periodic modes (cont.)



Stable ($\alpha < 0$ and $k > 0$)

⇝ As t increases the mode tends to 0 asymptotically

Unstable ($\alpha \geq 0$ and $k \geq 1$)

⇝ As t increases the mode tends to ∞ asymptotically

The mode envelops $\cos(\omega t)$ with functions $\pm t^k e^{\alpha t}$

Pseudo-periodic modes (cont.)

Example

3.16

Impulse response

Input-output representation

Impulse response

We will study the general forced response of a system due to arbitrary inputs

We start by studying a particular forced response

⇝ **Impulse response**

Impulse response (cont.)

Definition

Impulse response

*The **impulse response** $w(t)$ is the forced evolution of a system subjected to an input $u(t) = \delta(t)$ applied at time $t = 0$*

The impulse response is an important function, as it is a **canonical regime**

What do we get from its knowledge?

- $\rightsquigarrow$ The forced evolution of the system under any input
- $\rightsquigarrow$ The force-free evolution for any initial condition

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Impulse response (cont.)

Unit impulse and unit step

The **unit impulse** $\delta(t)$ is the *derivative* of the unit step $\delta_{-1}(t)$

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t)$$

The **unit step** is the Heaviside function

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

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Structure of the impulse response

Proposition

Structure of the impulse response

Consider a linear, stationary and proper SISO system

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

For $t < 0$, the impulse response $w(t)$ is null

$$\rightsquigarrow \quad w(t) = 0$$

For $t \geq 0$, the impulse response $w(t)$ can be parameterised as linear combination $h(t)$ of the n modes of the system and, possibly, an impulsive term

$$\rightsquigarrow \quad w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

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Structure of the impulse response (cont.)

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

Let ν_i be the multiplicity of root p_i of the characteristic polynomial

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

The impulsive term is present iff the system is not strictly proper

$$A_0 = \begin{cases} b_n/a_n, & m = n \\ 0, & m < n \end{cases}$$

Structure of the impulse response (cont.)

Proof

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

In a causal/proper ($n \geq m$) system, the effect cannot precede the cause

When subjected to impulse $\delta(t)$ at $t = 0$, the response is null for $t < 0$

- This is imposed by $\delta_{-1}(t)$ in $w(t)$

Moreover, an impulsive input $u(t) = \delta(t)$ is (definition) null for $t > 0$

The system is assumed initially at rest in $t = 0^-$

- At time $t = 0^+$, it is in a non-null initial state

Because of the action due to the impulsive input

After $t = 0^+$ the input is null

The evolution is a particular force-free response

- Unknown coefficients $A_{i,k}$ to be determined
- This is given by $h(t)$ in $w(t)$

■

Structure of the impulse response (cont.)

Example

Consider an instantaneous system with the model

$$3y(t) = 2u(t)$$

We are interested in the force-free response to a unit impulse

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + \cdots + b_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t)$$

The system has $m = n = 0$ (non strictly proper)

- $a_n = 3$
- $b_n = 2$

Structure of the impulse response (cont.)

$$3y(t) = 2u(t)$$

Model is an algebraic equation, characteristic polynomial has order $n = 0$

$\rightsquigarrow$ A system represented by this model does not have any mode

The impulsive response for an input $u(t) = \delta(t)$

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t) \rightsquigarrow (2/3)\delta(t)$$

$\rightsquigarrow$ $h(t) = 0$

$\rightsquigarrow$ $A_0 = b_n/a_n$

■

Structure of the impulse response (cont.)

Consider a characteristic polynomial of the system

- R distinct real roots
- S distinct pairs of conjugate complex roots

We can re-write

$$h(t) = \sum_{i=1}^{r} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t}$$

We can use one of the forms where the pseudo-periodic modes are explicit

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} \left(B_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t) + C_{i,k} t^k e^{\alpha_i t} \sin(\omega_i t)\right)$$

$$\rightsquigarrow \quad h(t) = \sum_{i=1}^{R} \sum_{k=0}^{\nu_i - 1} A_{i,k} t^k e^{p_i t} + \sum_{i=R+1}^{R+S} \sum_{k=0}^{\nu_i - 1} M_{i,k} t^k e^{\alpha_i t} \cos(\omega_i t + \phi_{i,k})$$

Structure of the impulse response (cont.)

Unknown coefficients in the expression of $h(t)$ in the impulse response $w(t)$

- We used the symbols A, M, ϕ, B and C
- As in force-free responses

In the force-free case, coefficients can take an infinity of arbitrary values

- They depend on the initial conditions

In the impulse case, coefficients depend univocally only on the system

We study a technique/algorithm to find their value

Calculation of the impulse response

Computing the impulse response

A complicated technique to calculate the impulse response in time-domain

The algorithm is based on the knowledge of the impulse response $w(t)$

- We know that $w(t)$ has a known parameterisation

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

As such, $w(t)$ must satisfy the model

$$\sum_{i=0}^{n} a_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} y(t) = \sum_{i=0}^{m} b_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} u(t)$$

$\rightsquigarrow$ For a given impulse input $u(t) = \delta(t)$

$\rightsquigarrow$ For any value of t

Calculation of the impulse response (cont.)

$$w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$$

In the parameterisation of $w(t)$ there are $(n+1)$ unknown coefficients

- The n coefficients associated to the modes
- The coefficient A_0 of the impulsive term

Calculation of the impulse response (cont.)

The impulse response $w(t)$ must satisfy the model for all t, $t = 0$ included

- All of the nasty things happen here

$\rightsquigarrow$ Discontinuities or impulsive terms

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + \cdots + b_1 \frac{\mathrm{d}u(t)}{dt} + b_0 u(t)$$

Calculation of the impulse response (cont.)

We calculate derivatives[2] of $w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$, up to order n

$$
\begin{aligned}
w(t) &= h(t)\delta_{-1}(t) + A_0\delta(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}w(t) &= \dot{h}(t)\delta_{-1}(t) + h(0)\delta(t) + A_0\delta_1(t) \\
\cdots &= \cdots \\
\frac{\mathrm{d}^n}{\mathrm{d}t^n}w(t) &= h^n(t)\delta_{-1}(t) + h^{(n-1)}(0)\delta(t) + h^{(n-2)}(0)\delta(t) + \cdots + A_0\delta_n(t)
\end{aligned}
$$

[2]In the sense of distributions,

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(t)\delta_{-1}(t) = f^{(k)}(t)\delta_{-1}(t) + \sum_{i=0}^{k-1} f^{(i)}(0)\delta_{k-1-i}(t)$$

and

$$\delta_k(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\delta_{k-1}(t), \text{ with } k > 1.$$

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Calculation of the impulse response (cont.)

Moreover, we have,

$$
\begin{aligned}
u(t) &= \delta(t) \\
\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= \delta_1(t) \\
\cdots &= \cdots \\
\frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} &= \delta_m(t)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\rightsquigarrow \quad a_n\frac{\mathrm{d}^n w(t)}{\mathrm{d}t^n} + \cdots + a_1\frac{\mathrm{d}w(t)}{\mathrm{d}t} + a_0 w(t) \\
= b_m\delta_m(t) + \cdots + b_1\delta_1(t) + b_0\delta(t)
\end{aligned}
$$

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Calculation of the impulse response (cont.)

We can now substitute for the expressions of $w(t)$ and its derivatives

- We solve after imposing equality between the coefficients
- Those that multiply the terms $\delta(t), \delta_1(t), \dots, \delta_m(t)$

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Calculation of the impulse response (cont.)

A set of $n+1$ equations in $n+1$ unknowns coefficients of $w(t)$

- A_0, $\{A_i\}$ and $\{M_i\}$ and $\{\phi_i\}$ (or, $\{B_i\}$ and $\{C_i\}$)

$$
\begin{aligned}
b_0 &= a_0 A_0 + a_1 h(0) + \cdots + a_{n-1}h^{(n-2)}(0) + a_n h^{(n-1)}(0) \\
b_1 &= a_1 A_0 + a_2 h(0) + \cdots + a_n h^{(n-2)}(0) \\
\cdots &= \cdots \\
b_{n-1} &= a_{n-1}A_0 + a_n h(0) \\
b_n &= a_n A_0
\end{aligned}
$$

The unknown coefficients A_0, $\{A_i\}$ and $\{M_i\}$ and $\{\phi_i\}$ (or, $\{B_i\}$ and $\{C_i\}$)

- They appear also in the expression of $h(0), \dot{h}(0), \cdots, h^{(n-1)}(0)$

$\rightsquigarrow$ The coefficients a_i and b_i with $i = 1, \dots, n$ are given by the model
$\rightsquigarrow$ If we have $n < m$, we can set $b_{m+1} = b_{m+2} = \cdots = b_n = 0$
$\rightsquigarrow$ Terms that multiply $\delta_{-1}(t)$ cancel out (missing from RHS)

Calculation of the impulse response (cont.)

$$
\begin{aligned}
b_0 &= a_0 A_0 + a_1 h(0) + \cdots + a_{n-1} h^{(n-2)}(0) + a_n h^{(n-1)}(0) \\
b_1 &= a_1 A_0 + a_2 h(0) + \cdots + a_n h^{(n-2)}(0) \\
\cdots &= \cdots \\
b_{n-1} &= a_{n-1} A_0 + a_n h(0) \\
b_n &= a_n A_0
\end{aligned}
$$

From $b_n = a_n A_0$,

- If $m = n$, then $a_n A_0 = b_n \neq 0$ and $A_0 = b_n / a_n \neq 0$
- If $m < n$, then $a_n A_0 = b_n = 0$ and $A_0 = 0$

It thus is possible to simplify the calculation

- Determine a priori the term A_0
- Treat it as constant

(Last equation of the system becomes an identity)

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Calculation of the impulse response (cont.)

Algorithm

1. Determine the characteristic polynomial $P(s)$ of the homogeneous equation associated to the IO model and calculate its roots
2. Determine the n modes of the model
3. Write $w(t) = A_0\delta(t) + h(t)\delta_{-1}(t)$ using a parameterisation of $h(t)$
4. Calculate the derivatives of $h(t)$, up to the $(n-1)$-th order
5. Write the system of n equations in the n unknown coefficients of $h(t)$

$$
\begin{aligned}
b_0 - a_0 A_0 &= a_1 h(0) + a_2 \dot{h}(0) + \cdots + a_{n-1} h^{(n-2)}(0) + a_n h^{(n-1)}(0) \\
b_1 - a_1 A_0 &= a_2 h(0) + a_3 \dot{h}(0) + \cdots + a_n h^{(n-2)}(0) \\
\cdots &= \cdots \\
b_{n-2} - a_{n-2} A_0 &= a_{n-1} h(0) + a_n \dot{h}(0) \\
b_{n-1} - a_{n-1} A_0 &= a_n h(0)
\end{aligned}
$$

6. Solve for the n unknown coefficients A_i of $w(t)$

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Calculation of the impulse response (cont.)

Example

Consider a system described by the IO model

$$2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 4y(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t} + 3u(t)$$

We are interested in computing the the impulse response

The characteristic polynomial

$$P(s) = 2s^2 + 6s + 4$$

From $P(s) = 0$, two real roots, both with multiplicity one

- ⇝ $p_1 = -1$, mode $e^{-p_1 t}$
- ⇝ $p_2 = -2$, mode $e^{-p_2 t}$

A strictly proper model, $m = 1 < n = 2$

- ⇝ $w(t)$ w/o the impulsive term
- ⇝ Thus, $A_0 = 0$

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Calculation of the impulse response (cont.)

The structure of the impulse response and its derivative

$$w(t) = \underbrace{\Big(A_1 e^{-t} + A_2 e^{-2t}\Big)}_{h(t)} \delta_{-1}(t)$$

$$\frac{dw(t)}{dt} = \underbrace{\Big(-A_1 e^{-t} - 2A_2 e^{-2t}\Big)}_{\dot{h}(t)} \delta_{-1}(t) + \underbrace{\Big(A_1 + A_2\Big)}_{h(0)} \delta(t)$$

$$\frac{d^2 w(t)}{dt^2} = \underbrace{\Big(A_1 e^{-t} + 4A_2 e^{-2t}\Big)}_{\ddot{h}(t)} \delta_{-1}(t) + \underbrace{\Big(-A_1 - 2A_2\Big)}_{\dot{h}(0)} \delta(t) + \underbrace{\Big(A_1 + A_2\Big)}_{h(0)} \delta_1(t)$$

Calculation of the impulse response (cont.)

By substituting $w(t)$ and its derivatives in the model and setting $u(t) = \delta(t)$

$$\underbrace{4(A_1 e^{-t} + A_2 e^{-2t})\delta_{-1}(t)}_{a_0 w(t)}$$
$$+ \underbrace{6(-A_1 e^{-t} - 2A_2 e^{-2t})\delta_{-1}(t) + 6(A_1 + A_2)\delta(t)}_{a_1 \frac{\mathrm{d}}{\mathrm{d}t} w(t)}$$
$$+ \underbrace{2(A_1 e^{-t} + 4A_2 e^{-2t})\delta_{-1}(t) + 2(-A_1 - 2A_2)\delta(t) + 2(A_1 + A_2)\delta_1(t)}_{a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} w(t)}$$
$$= \underbrace{3\delta(t) + \delta_1(t)}_{b_0 \delta(t) + b_1 \delta_1(t)}$$

The coefficients multiplying $\delta_{-1}(t)$ will cancel each other out, always

Calculation of the impulse response (cont.)

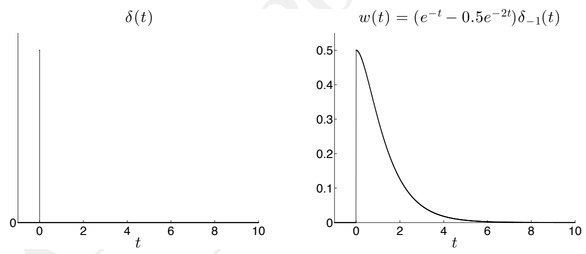
Since $m < n$ and thus $A_0 = 0$, we can write a system of two equations

$$\begin{cases} \big[a_1 h(0) + a_2 \dot{h}(0)\big]\delta(t) = b_0 \delta(t) \\ a_2 h(0)\delta_{-1}(t) = b_1 \delta_{-1}(t) \end{cases}$$
$$\rightsquigarrow \begin{cases} 4A_1 + 2A_2 = 3 \\ 2A_1 + 2A_2 = 1 \end{cases} \rightsquigarrow \begin{cases} A_1 = 1 \\ A_2 = -0.5 \end{cases}$$

The resulting impulse response

$$w(t) = (e^{-t} - 0.5e^{-2t})\delta_{-1}(t)$$

Calculation of the impulse response (cont.)



□

Calculation of the impulse response (cont.)

Example

Calculate the impulse response for the system described by the IO model

$$\frac{\mathrm{d}^3 y(t)}{\mathrm{d}t^3} + 2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 5\frac{\mathrm{d}y(t)}{\mathrm{d}t} = 4\frac{\mathrm{d}u(t)}{\mathrm{d}t} + u(t)$$

The characteristic polynomial

$$P(s) = s^3 + 2s^2 + 5s$$

From $P(s) = 0$, the roots

- A real root $p_1 = \alpha_1 = 0$ with multiplicity one ν_1
- A pair of conjugate complex roots $p_2 = \alpha_2 \pm j\omega_2 = -1 \pm j2$ also with multiplicity one $\nu_2 = \nu_2' = 1$

Calculation of the impulse response (cont.)

A strictly proper model, $m = 1 < n = 3$

- $w(t)$ without the impulsive term

$$\begin{aligned}\rightsquigarrow \quad w(t) &= h(t)\delta_{-1}(t) = \big[A_1 e^{p_1 t} + M_2 e^{\alpha_2 t} \cos{(\omega_2 t + \phi_2)}\big]\delta_{-1}(t)\\ &= \big[A_1 + M_2 e^{-t} \cos{(2t + \phi_2)}\big]\delta_{-1}(t)\end{aligned}$$

Calculation of the impulse response (cont.)

By differentiating $h(t)$ two times,

$$\begin{aligned}h(t) &= A_1 + M_2 e^{-t} \cos{(2t + \phi_2)}\\ \dot{h}(t) &= -M_2 e^{-t} \cos{(2t + \phi_2)} - 2M_2 e^{-t} \sin{(2t + \phi_2)}\\ \ddot{h}(t) &= -3M_2 e^{-t} \cos{(2t + \phi_2)} + 4M_2 e^{-t} \sin{(2t + \phi_2)}\end{aligned}$$

We have the system of equations

$$\begin{cases} a_1 h(0) + a_2 \dot{h}(0) + a_3 \ddot{h}(0) = b_0 \\ a_2 h(0) + a_3 \dot{h}(0) = b_1 \\ a_3 h(0) = b_2 \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} 5A_1 = 1 \\ 2A_1 + 5M_2 \cos{(\phi_2)} - 2M_2 \sin{(\phi_2)} = 4 \\ A_1 + M_2 \cos{(\phi_2)} = 0 \end{cases}$$

Calculation of the impulse response (cont.)

Let $u_2 = M_2 \cos{(\phi_2)}$ and $v_2 = M_2 \sin{(\phi_2)}$

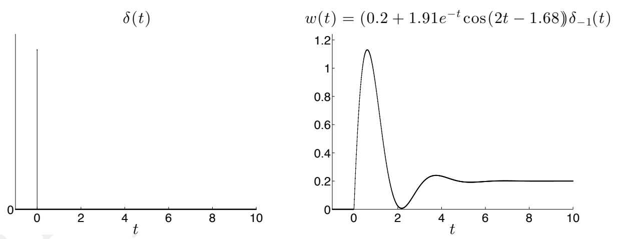
$$\begin{cases} 5A_1 = 1 \\ 2A_1 + u_2 - 2v_2 = 4 \\ A_1 + u_2 = 0 \end{cases} \rightsquigarrow \begin{cases} A_1 = +0.2 \\ u_2 = -0.2 \\ v_2 = -1.9 \end{cases}$$

$$\rightsquigarrow \quad \begin{cases} M_2 = \sqrt{u^2 + v^2} = 1.91 \\ \phi_2 = \arctan{(u/v)} = \arctan{(-1.9/-0.2)} = -1.68 \text{ [rad]} \end{cases}$$

Calculation of the impulse response (cont.)

The impulse response

$$\rightsquigarrow \quad w(t) = \big[0.2 + 1.91 e^{-t} \cos{(2t - 1.68)}\big]\delta_{-1}(t)$$



Forced evolution and Duhamel's integral

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Forced evolution (cont.)

We show a fundamental result in the analysis of linear IO models

- The **Duhamel's integral**

The forced evolution $y_f(t)$ of a linear time-invariant system subjected to input $u(t)$ is determined by its convolution with the impulse response $w(t)$

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Forced evolution (cont.)

Convolution

Consider two functions $f, g : \mathcal{R} \rightarrow \mathcal{C}$

The **convolution** of f with g is function $h : \mathcal{R} \rightarrow \mathcal{C}$ in the real variable t

$$\rightsquigarrow \quad h(t) = f \star g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau$$

Function $h(t)$ is constructed using the operator **convolution integral** $\star$

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Forced evolution (cont.)

We start by assuming that the system is at some remote time $t = -\infty$

- We assume that no cause has ever acted on it before

$\rightsquigarrow$ The system is therefore assumed to be at rest

At such a remote time, the system is subjected to an input $u(t)$

- We assume that the input $u(t)$ is known in $(-\infty, t]$

$\rightsquigarrow$ This is needed to determine the output at time t

Forced evolution (cont.)

Proposition

Duhamel's integral

Consider a system at rest at $t = -\infty$, for every value of $t \in \mathcal{R}$

We have,

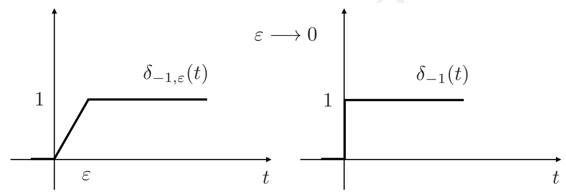
$$\rightsquigarrow \quad \underbrace{y(t) = \int_{-\infty}^{t} u(\tau)w(t-\tau)d\tau}_{\textit{Duhamel's integral}}$$

Proof

Let $w_\varepsilon(t)$ be the forced response of the system due to a finite impulse $\delta_\varepsilon(t)$

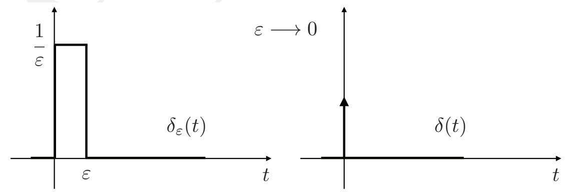
$$\delta_\varepsilon(t) = \frac{d}{dt}\delta_{-1,\varepsilon} = \begin{cases} 1/\varepsilon, & t \in [0, \varepsilon) \\ 0, & \text{elsewhere} \end{cases}$$

Forced evolution (cont.)



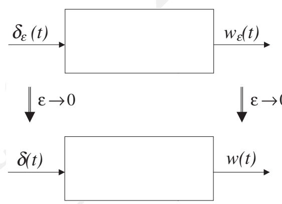
From the definition of the derivative of the unit step $\delta_{-1}(t)$, we have

$$\delta(t) = \frac{d}{dt}\delta_{-1}(t) = \frac{d}{dt}\lim_{\varepsilon\to 0}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon\to 0}\frac{d}{dt}\delta_{-1,\varepsilon}(t) = \lim_{\varepsilon\to 0}\delta_\varepsilon(t)$$



Forced evolution (cont.)

Because $\delta(t) = \lim_{\varepsilon\to 0}\delta_\varepsilon(t)$, it is intuitive to see that $w(t) = \lim_{\varepsilon\to 0} w_\varepsilon(t)$

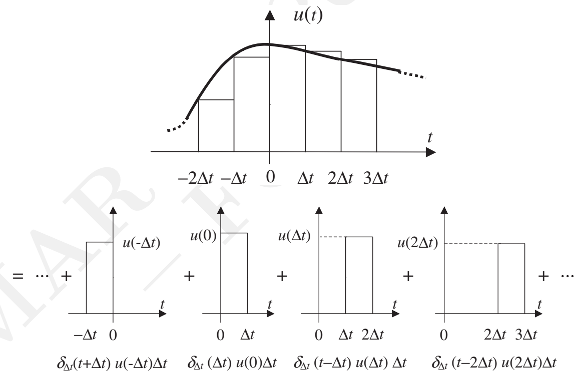


Forced evolution (cont.)

We are interested in approximating the function $u(t)$

We approximate $u(t)$ with a series of rectangles

- Each rectangle is a finite impulse

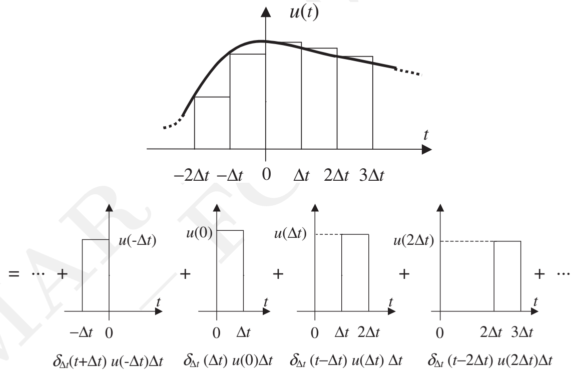


Δt denotes the sampling time (the base of the rectangles)

Forced evolution (cont.)

Each rectangle is a assumed to be a finite impulse, $\delta_{\Delta t}(t - k\Delta t)$

- Subscript Δt is the base of the rectangle (was ε)
- Argument $(t - k\Delta t)$ right-shifts it by $k\Delta t$



Each finite impulse is multiplied by the scaling factor $u(k\Delta t)\Delta t$

- The area of a rectangle with base Δt and height $u(k\Delta t)$

Forced evolution (cont.)

The approximation gets better as Δt gets smaller

Thus, we define

$$\rightsquigarrow \quad u_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} u(k\Delta t)\delta_{\Delta t}(t - k\Delta t)\Delta t$$

We have,

$$\rightsquigarrow \quad u(t) = \lim_{\Delta t \to 0} u_{\Delta t}(t)$$

Forced evolution (cont.)

The system is assumed to be linear (the superposition principle)

We approximate the total system output due to such an input

- A sum of the outputs due to the component inputs

$$\rightsquigarrow \quad y_{\Delta t}(t) = \sum_{k=-\infty}^{\infty} u(k\Delta t) w_{\Delta t}(t - k\Delta t)\Delta t$$

Again, the approximation gets better as Δt gets smaller

We have,

$$\rightsquigarrow \quad y(t) = \lim_{\Delta t \to 0} y_{\Delta t}(t) = \lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} u(k\Delta t) w_{\Delta t}(t - k\Delta t)\Delta t$$
$$= \int_{-\infty}^{\infty} u(\tau) w(t - \tau) d\tau$$

As we let $k\Delta t = \tau$ and $\Delta t = d\tau$, τ is now a real variable

Forced evolution (cont.)

The system is assumed to be proper (causes first, then effects)

- $w(t - \tau)$ is zero when $(t - \tau) < 0$ $(\tau \geq t)$

We have,

$$\rightsquigarrow \quad y(t) = \int_{-\infty}^{\infty} u(\tau) w(t - \tau) d\tau = \underbrace{\int_{-\infty}^{t} u(\tau) w(t - \tau) d\tau}_{\text{Duhamel's integral}}$$

Forced evolution (cont.)

The Duhamel's integral is a convolution integral

$$y(t) = \int_{-\infty}^{\infty} u(\tau)w(t-\tau)d\tau = \underbrace{\int_{-\infty}^{t} u(\tau)w(t-\tau)d\tau}_{\text{Duhamel's integral}}$$

The upper-extreme is set to be t instead of $+\infty$ only for convenience

- The convolution of $u(\tau)$ and $w(\tau)$ is zero for $\tau \geq t$

Because of the commutativity of convolution integrals, we write

$$y(t) = u * w(t) = w * u(t) = \int_{-\infty}^{+\infty} u(t-\tau)w(\tau)d\tau = \int_{0}^{+\infty} u(t-\tau)w(\tau)d\tau$$

Moreover, for $\tau < 0$ we have

$$w(\tau) = 0$$

Forced evolution (cont.)

$$y(t) = \int_{0}^{+\infty} u(t-\tau)w(\tau)d\tau$$

Consider the contributions to $y(t)$ at time t

They are due to the value of the input $u(t-\tau)$ τ times earlier

- Weighted by the impulse response $w(\tau)$

Consider a system whose modes are all stable

- The impulse response $w(t)$ tends to zero
- It is virtually zero for $\tau > \bar{\tau}$
- $\bar{\tau}$ depends on system time-constant

The system loses memory of the input after a time $\bar{\tau}$ from application

Forced evolution (cont.)

Decomposition in forced and force-free response

Consider some initial time $t = t_0$

We decompose the Duhamel's integral

$$y(t) = \underbrace{\int_{-\infty}^{t_0} u(\tau)w(t-\tau)\mathrm{d}\tau}_{y_u(t)} + \underbrace{\int_{t_0}^{t} u(\tau)w(t-\tau)\mathrm{d}\tau}_{y_f(t)}, \quad \text{for } t \geq t_0$$

The first term $y_u(t)$ is the contribution to the output signal at time t due to the values taken by the input before the initial time t_0

- $\rightsquigarrow$ At time t_0, the system is a non null state
- $\rightsquigarrow$ (Non-zero initial conditions)
- $\rightsquigarrow$ Force-free evolution

The second term $y_f(t)$ is the contribution to the output signal at time t due to the value taken by the input after the initial time t_0

- $\rightsquigarrow$ Forced evolution

Forced evolution (cont.)

Forced response by convolution

Consider some initial time t_0

The forced evolution

$$\rightsquigarrow \quad y_f(t) = \int_{t_0}^{t} u(\tau)w(t-\tau)\mathrm{d}\tau = \int_{0}^{t-t_0} u(t-\tau)w(\tau)\mathrm{d}\tau$$

The second formula is derived from the first one

- Change variable, $\rho = t - \tau$

$$\rightsquigarrow \int_{t_0}^{t} u(\tau)w(t-\tau)\mathrm{d}\tau = \int_{t-t_0}^{0} u(t-\rho)w(\rho)(-\mathrm{d}\rho) = \int_{0}^{t-t_0} u(t-\rho)w(\rho)\mathrm{d}\rho$$

Forced evolution (cont.)

Let $t_0 = 0$, we have the expression

$$\rightsquigarrow \quad y_f(t) = \int_{t_0=0}^{t} u(\tau)w(t-\tau)d\tau = \int_{t_0=0}^{t} u(t-\tau)w(\tau)d\tau$$

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Forced evolution (cont.)

Example

Consider the system represented by the IO model

$$2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 4y(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t} + 3u(t)$$

We are interested in the forced evolution ($t \geq 0$) due to input $u(t) = 4\delta_{-1}(t)$

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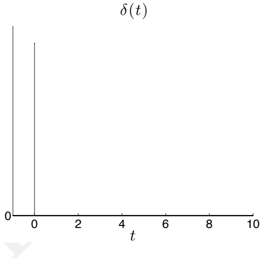
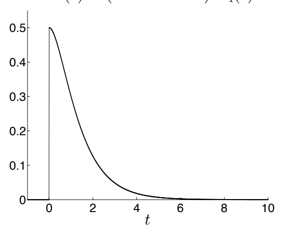
Forced evolution (cont.)

The impulse response of the system

$$\rightsquigarrow \quad w(t) = (e^{-t} - 0.5e^{-2t})\delta_{-1}(t)$$





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Forced evolution (cont.)

The forced response will be zero for $t < 0$

For $t \geq 0$, we have

$$y_f(t) = \int_0^t u(\tau)w(t-\tau)\mathrm{d}\tau, \quad u(\tau) = 4, \text{ for } \tau \in [0, t]$$

$$\begin{aligned}
\rightsquigarrow \quad y_f(t) &= \int_0^t u(\tau)w(t-\tau)\mathrm{d}\tau = \int_0^t 4[e^{-(t-\tau)} - 0.5e^{-2(t-\tau)}]\mathrm{d}\tau \\
&= 4e^{-t}\int_0^t e^{\tau}\mathrm{d}\tau - 2e^{-2t}\int_0^t e^{2\tau}\mathrm{d}\tau \\
&= 4e^{-t}(e^{-t} - 1) - 2e^{-2t}(0.5e^{2t} - 0.5) \\
&= 3 - 4e^{-t} + e^{-2t}
\end{aligned}$$

Forced evolution (cont.)

The forced response will be zero for $t < 0$

For $t \geq 0$, we have

$$y_f(t) = \int_0^t u(t-\tau)w(\tau)\mathrm{d}\tau, \quad u(\tau) = 4 \text{ for } \tau \in [0, t]$$

$$\rightsquigarrow \quad y_f(t) = \int_0^t 4\big[e^{-\tau} - 0.5e^{-2\tau}\big]\mathrm{d}\tau = 4\int_0^t e^{-\tau}\mathrm{d}\tau - 2\int_0^t e^{-2\tau}\mathrm{d}\tau$$
$$= 4(e^{-t} - 1) - 2(0.5e^{-2t} - 0.5) = 3 - 4e^{-t} + e^{-2t}$$

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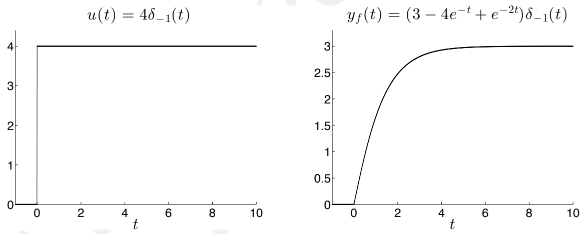

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Forced evolution (cont.)

Example

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$$2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 4y(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t} + 3u(t)$$

We are interested in the forced evolution ($t \geq 0$) due to input $u(t)$

$$u(t) = \begin{cases} 2, & \text{if } t \in [1, 4) \\ 0, & \text{otherwise} \end{cases}$$

This input can be understood as the sum of two functions

- A step with size $+2$, at $t = 1$
- A step with size -2, at $t = 4$

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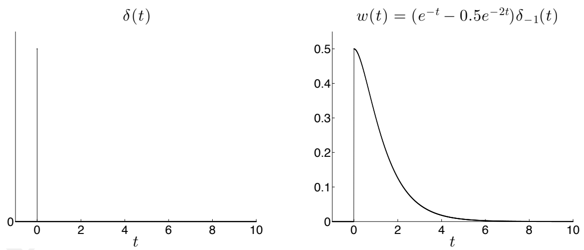
Forced evolution (cont.)

The impulse response of the system

$$\rightsquigarrow \quad w(t) = (e^{-t} - 0.5e^{-2t})\delta_{-1}(t)$$



Forced evolution (cont.)

Using the Duhamels integral, we can calculate the forced response

$$y_f(t) = \int_{-\infty}^{t} u(\tau) w(t-\tau) \mathrm{d}\tau$$
$$= \begin{cases} 0, & t \in (-\infty, 1) \\ 2\int_1^t w(t-\tau)\mathrm{d}\tau, & t \in [1, 4) \\ 2\int_1^4 w(t-\tau)\mathrm{d}\tau, & t \in [4, +\infty) \end{cases}$$

Forced evolution (cont.)

The change of variable $\rho = t - \tau$

For $1 \leq t < 4$

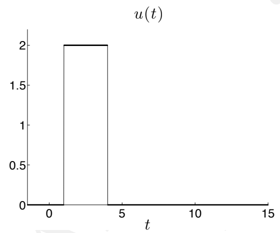
$$\rightsquigarrow \quad \int_0^t w(t-\tau)\mathrm{d}\tau = \int_0^{t-1} w(\rho)\mathrm{d}\rho = \int_0^{t-1} (e^{-\rho} - 0.5e^{-2\rho})\mathrm{d}\rho$$
$$= 0.75 - e^{-(t-1)} + 0.25e^{-2(t-1)}$$
$$= 0.75 - 2.72e^{-t} + 1.85e^{-2t}$$

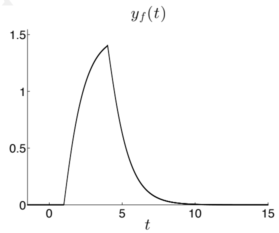
For $t \geq 4$

$$\rightsquigarrow \quad \int_1^4 w(t-\tau)\mathrm{d}\tau = \int_{t-4}^{t-1} w(\rho)\mathrm{d}\rho = \int_{t-4}^{t-1} (e^{-\rho} - 0.5e^{-2\rho})\mathrm{d}\rho$$
$$= -e^{-(t-1)} + 0.25e^{-2(t-1)} + e^{-(t-4)} - 0.25e^{-2(t-4)}$$
$$= 51.9e^{-t} + 743e^{-2t}$$

Forced evolution (cont.)

$$\rightsquigarrow \quad y_f(t) = \begin{cases} 0, & t \in (-\infty, 1] \\ 1.5 - 5.44e^{-t} + 3.69e^{-2t}, & t \in [1, 4) \\ 104e^{-t} + 1487e^{-2t}, & t \in [4, +\infty) \end{cases}$$





The input signal $u(t)$ is active only in the interval $t \in [1, 4]$

- The response is not null for $t \geq 4$
- At $t = 4$ there is a non-null state

From $t = 4$, the evolution is force-free