LINEAR QUADRATIC CONTROL

$\dot{x} = Ax + Bu$

$u = -Kx$

- STABILITY
- CONTROLLABILITY TESTS
  - CONTROLLABILITY MATRIX
  - POPOV, BELEVITCH, HAUTUS
  - CONTROLLABILITY GRAMIAN
- EIGENVALUE PLACEMENT (Arbitrary)

IS THERE AN OPTIMAL WAY OF PLACING THE EIGENVALUES OF THE CONTROLLED SYSTEM? (Equivalently, what's the best K?)

→ THE LINEAR QUADRATIC REGULATOR ANSWERS THIS QUESTION

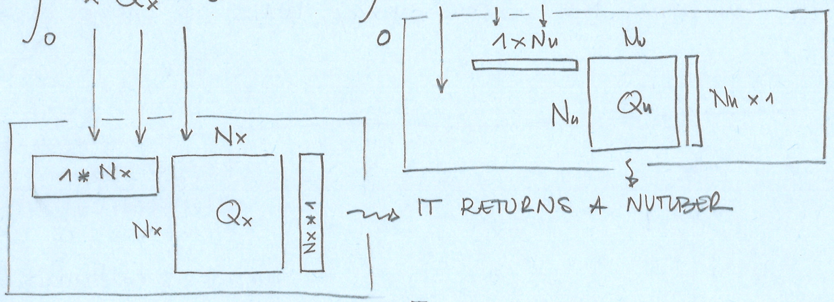
→ LINEAR FOR LINEAR SYSTEMS

→ REGULATOR FOR BRINGING THE SYSTEM BACK TO SS

→ QUADRATIC IS THE COST (OR OBJECTIVE) FUNCTION THAT WE OPTIMIZE

> IT IS MEANT TO MEASURE THE QUALITY OF DIFFERENT CONTROL SOLUTIONS (matrices K)
> • WE WANT TO CHOOSE K THAT IS OPTIMAL

$$J = \int_0^\infty x^T Q_x x \, dt + \int_0^\infty u^T Q_u u \, dt$$

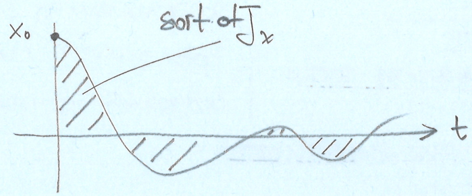


→ IT RETURNS A NUMBER

IT IS SHORTHAND FOR $(x - x_{ss})^T Q_x (x - x_{ss})$, with $x_{ss} = 0$

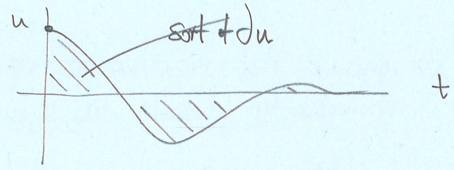
IT IS SHORTHAND FOR $(u - u_{ss})^T Q_u (u - u_{ss})$, with $u_{ss} = 0$

BECAUSE THE MODEL IS IN PERTURBATION VARS



It is a measure of the state energy

+



It is a measure of the input energy = 'Control energy'

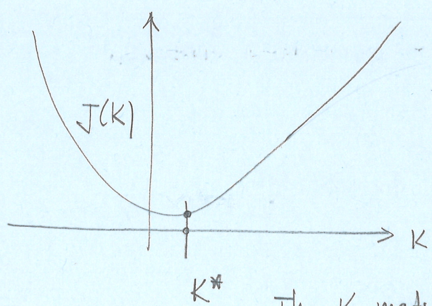
WE CALL Q_x AND Q_u, THE 'WEIGHTING MATRICES'

→ THEY ARE POSITIVE-DEFINITE

IN LQR, WE ARE INTERESTED IN MINIMISING BOTH 'ENERGIES'

BUT,

1. MINIMISING THE STATE ENERGY, REQUIRES LARGE CONTROLS
2. MINIMISING THE CONTROL ENERGY, LEADS TO LARGE OUTPUTS



PICTURE THE OPTIMIZATION TASK

Imagine a continuum of matrices K

The K matrix that minimises the cost

$$J = \int_0^{\infty} \underbrace{x^T Q_x x \, dt}_{J_x} + \int_0^{\infty} \underbrace{u^T Q_u u \, dt}_{J_u}$$

$$(x' - 0)^T \quad (x' - 0) \qquad\qquad (u' - 0)^T \quad (u' - 0)$$

$$Q_x = \begin{bmatrix} \otimes & & & \\ & \otimes & & \\ & & \ddots & \\ & & & \otimes \end{bmatrix} \Big\} N_x$$

$$\underbrace{\qquad\qquad}_{N_x}$$

ENTRIES ALONG THE DIAGONAL QUANTIFY THE SEVERITY OF BEING FAR FROM ZERO (FROM STEADY STATE)

THE USER DEFINES IT, WITH SOME RESTRICTIONS

- ELEMENT MUST BE POSITIVE
- MATRIX MUST BE SYMMETRIC
- (TYPICALLY IT IS CHOSEN TO BE A DIAGONAL MATRIX

The entries are understood as penalties, the larger the penalty.

$$Q_u = \begin{bmatrix} \otimes & & & \\ & \otimes & & \\ & & \ddots & \\ & & & \otimes \end{bmatrix} \Big\} N_u$$

$$\underbrace{\qquad\qquad}_{N_u}$$

ENTRIES ALONG THE DIAGONAL QUANTIFY THE SEVERITY OF USING CONTROL 'ENERGY' FAR FROM ZERO (FROM STEADY STATE)

$\gg$ lqr (A, B, Q_x, Q_u) (help lqr)

$\rightarrow$ WILL RETURN THE MATRIX K THAT MAKES THE OBJECTIVE J AS SMALL AS POSSIBLE

$\rightarrow$ $(A - BK)$ IS THE OPTIMAL CLOSED LOOP A MATRIX

$\rightarrow$ EIGENVALUES

$\rightarrow$ EIGENVECTORS

$$Q_x = \begin{bmatrix} q_x^{11} & 0 \\ 0 & q_x^{22} \end{bmatrix} \qquad \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \underbrace{\begin{bmatrix} q_x^{11} & 0 \\ 0 & q_x^{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}$$

$$x_1(t) q_x^{11} x_1(t) + x_2(t) q_x^{22} x_2(t) \leftarrow \begin{bmatrix} q_x^{11} x_1(t) \\ q_x^{22} x_2(t) \end{bmatrix} = \begin{bmatrix} q_x^{11} x_1(t) + 0 x_2(t) \\ 0 x_1(t) + q_x^{22} x_2(t) \end{bmatrix}$$

$$= x_1^2(t) q_x^{11} + x_2^2(t) q_x^{22}$$

IF CHOSEN <u>LARGE</u> IT MEANS WE WILL HAVE A LARGE COST FOR NOT HAVING $x_1(t) = 0$ $(x_2(t) = 0)$

HOW ABOUT Q_u?

$$J = \int_0^\infty \Big[\underbrace{x^T(t)\, Q_x\, x(t)}_{\substack{\geq 0 \\ \text{for all } x}} + \underbrace{u^T(t)\, Q_u\, u(t)}_{\substack{> 0 \\ \text{for all } u}} \Big] dt$$

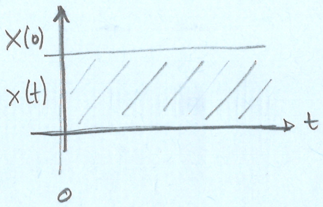
WE ARE INTERESTED IN THE MINIMISATION OF J

- $\underset{u \in \mathbb{R}^{N_u}}{\text{minimise}}\ J$

  such that $\dot{x} = Ax + Bu$

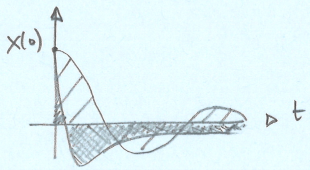
WE ARE AFTER SOME CONTROL LAW THAT MAKES THIS COST FUNCTION MINIMAL SUBJECT TO THE DYNAMICS

Suppose that the system is at $x(0) \neq 0$ (say, 1-STATE VARIABLE)

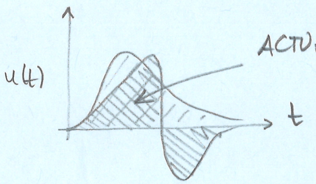


IF WE DO NOTHING, $\longrightarrow$ the integral increases indefinitely ($\to \infty$)

WE WANT TO BRING IT x BACK TO THE ORIGIN



$\rightarrow$ WE CAN ONLY DO THIS BY CHANGING THE CONTROLS



ACTUATE, THEN RELEASE SO THAT IT ALSO GETS BACK TO ITS ZERO

Q_x AND Q_u SET THE COMPROMISE THAT IS ACCEPTABLE BETWEEN PRIORITIES

- QUICK BACK TO $x=0$ (Q_x LARGER THAN Q_u)
- NOT USE TOO MUCH CONTROL (Q_u LARGER THAN Q_x)

FULL-STATE FEEDBACK / LQR

TO CHANGE THE EIGENVALUES OF THE CLOSED-LOOP A-MATRIX, A_{FB}

$$\dot{x} = \underbrace{(A - BK)}_{A_{FB}} x$$

- WE NEED (A, B) ARE CONTROLLABLE!

WE CAN CHANGE THE DYNAMICS (THE EIGENVALUES) OF THE CLOSED LOOP SYSTEM

- WE CAN PLACE THE EIGENVALUES <u>ANYWHERE</u> IN THE COMPLEX PLANE

HOW DO WE PICK (DESIGN) AN APPROPRIATE MATRIX K?

<u>Example</u>

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix} ; \quad B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- check stability
- controllability
- Find eigenvalues (and evector)
- check responses
- Place evalues of closed loop sys.

>> eig(A)?

$$\mathcal{C} = [B \;\; AB] = \left[\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right] \rightsquigarrow \text{rank}(\mathcal{C}) = 2 \quad \text{(CONTROLLABLE!)}$$

>> place

>> lqr

$$Q_x = \begin{bmatrix} q_x^{11} & 0 \\ 0 & q_x^{22} \end{bmatrix} ; \quad Q_u = \begin{bmatrix} q_u^{11} & 0 \\ 0 & q_u^{22} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} 0.5811 \\ 1 \end{bmatrix}$$

- Check stability
- Controllability
- Find eigendecomposition
- check responses
- Place eigenvalues of the closed loop system

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>> eig(A)
>> ctrb(A,B)
>> rank(ctrb(A,B))
>> place(A,B,...)      ?
>> lqr(A,B,Qx,Qy)      ?
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$Qx = ?$

$Qu = ?$

Example

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 4 \end{bmatrix}; \quad B = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

?