Systems

Stochastic algorithms

Francesco Corona

Department of Computer Science
Federal University of Ceará, Fortaleza

Systems

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2018.1

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Control, optimisation and validation
Fault diagnosis
Classification
By typology
Representation
Input-output representation
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System properties
Dynamical v Instantaneous
Linear v Nonlinear
Stationary v non-stationary
Proper v improper
With v without delay

Systems

We study how to mathematically model a broad variety of systems

Our scope is to analyse their dynamical behaviour

- ⇝ We want to operate them appropriately
- ⇝ The design of control devices
- ⇝ Under external stimuli

The methodological approach shall be formal and system independent

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General concepts

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General topics

There is a wide spectrum of problems around systems theory

- ⇝ System modelling, identification and analysis
- System control, optimisation and verification
- System diagnosis

Modelling, identification and analysis

General concepts

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Modelling

To study a system, the availability of a **mathematical model** is crucial

⇝ A quantitative description of the behaviour of the system

The model is often constructed on the knowledge of the component devices

- A knowledge of the laws the system obeys to must be available

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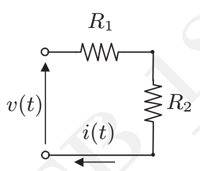
Modelling (cont.)

Example

Consider the electric circuit consisting of two serially arranged resistors

The current flow $i(t)$ through the system depends on tension $v(t)$



- $R_1 = 1\Omega$
- $R_2 = 3\Omega$

Both resistors can be assumed to follow Ohm's law,

$$v(t) = (R_1 + R_2)i(t) = 4i(t)$$

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Identification

At times, we only have an incomplete knowledge on the system's devices

- The model must be constructed from observations
- (Observations of the system behaviour, that is)

Case A) We have a knowledge on the type/number of component devices

- Not all of their parameters are known
- System observations are available

⇝ **Parametric identification**

⇝ **White-box identification**

Case B) We have no knowledge on the components and their parameters

- Observations of the system are available

⇝ **Black-box identification**

Identification (cont.)

Example

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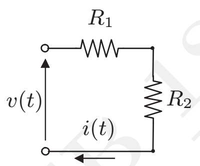

- $R_1 = ?$
- $R_2 = ?$

Both resistors can be assumed to follow Ohm's laws,

$$v(t) = (R_1 + R_2)i(t) = Ri(t)$$

R is an unknown parameter, it can be identified from data

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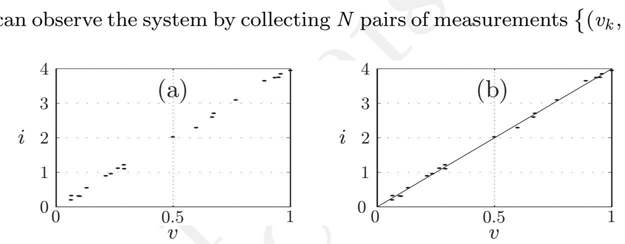
Identification (cont.)

We can observe the system by collecting N pairs of measurements $\{(v_k, i_k)\}_{k=1}^N$



Often, such points will not be perfectly aligned along a line with slope R

- **Disturbances** alter the behaviour of the system
- **Measurement errors** are always present

We choose R corresponding to the line that best approximates the data

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Analysis

Systems analysis is about forecasting the future behaviour of a system

- Based on the external stimuli it is subjected to

The availability of a mathematical model of the system is fundamental

- Needed to approach the problem in a quantitative manner

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Analysis (cont.)

Example

The marine ecosystem is described thru time evolution of its fauna and flora

- Birth-growth-dead processes

The behaviour of the system is influenced by many factors

- Climate conditions, availability of food, ...
- Human predators, pollutants, ...
- ..., and so on

They *recently* spoke of reducing CO_2 emissions by injecting it into the sea

- CO_2 dissolves in sea water

The lack of a valid model limits our understanding about the system

- We do not know the response of the ecosystem

Control, optimisation and validation

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Control

The objective of **control** is to impose a desired behaviour on a system

- We need to define what we mean by desired behaviour
- The **specifications** that such behaviour must satisfy

We need to design a device for implementing this task, the **controller**

- ⇝ Drive the system evolution towards the desired behaviour
- ⇝ The scope of a controller is to stimulate the system

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Control (cont.)

Example

Consider a conventional network for the distribution of drinking water

- Water pressure must be kept constant throughout the net

We can have measurements of pressure at various network locations

- Locations have some nominal (target) pressure value

Specs suggest instantaneous variations within $\pm 10\%$ of nominal value

We can identify two stimuli that act on the system and modify its behaviour

- The flow-rate of water withdrawn from the network
- The pressures imposed by the network pumps

We cannot control water withdrawals, they are understood as disturbances

Pump pressures can be manipulated to meet specifications

- This manipulation is done by the controller

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Optimisation

Achieve a certain system's behaviour, while optimising a performance index

- **Optimisation** can be understood as a special case of control

We impose a desired behaviour, while optimising a **performance index**

- The index measures the quality of the behaviour of the system
- (Economic or operational terms)

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Optimisation (cont.)

Example

Consider a conventional suspension system in a car

- ⇝ Passenger's comfort
- ⇝ Good handling

Modern cars have suspensions based on 'semi-active' technology

- A controller (dynamically) changes the dumping factor
- Optimises (a compromise between) the two criteria

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Validation

Suppose that a mathematical model of a system under study is available

- Suppose that a set of desired properties can be formally expressed

Validation allows to see whether the model satisfies the properties

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Validation (cont.)

Example

Consider a conventional elevator

The system is controlled to guarantee that it responds correctly to requests

- The controller is an abstract machine
- Programmable logic controller (PLC)

Formal verification can be used to guarantee the correct functioning

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Fault diagnosis

General concepts

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Fault diagnosis

Systems deviate from nominal behaviour because of occurrence of faults

- ⇝ We need to detect the presence of an anomaly
- ⇝ We need to identify the typology of fault
- ⇝ We need to devise a corrective action

Fault diagnosis

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Fault diagnosis (cont.)

Example

The human body is a complex system subjected to many potential faults

- We conventionally call them diseases

Consider the presence of fever, or another anomalous conditions

- Symptoms reveal the presence of a disease

A doctor, once identified the pathology, prescribes a therapy

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Classification

The diversity of systems leads to a number of methodological approaches

- Each model/system pertains a particular class

Conventional methodological approaches and model classification

By typology

- Time-evolving systems
- Discrete-event systems
- Hybrid systems

By representation

- Input-output models
- State-space models

Systems by typology

Classification

Time-evolving systems

Time-evolving systems

The behaviour is described with functions, or signals

- The independent variable is time, t

$\rightsquigarrow$ **Continuous time-evolving systems**

- The time variable varies continuously

$\rightsquigarrow$ **Discrete time-evolving systems**

- The time variable takes discrete values

The signal can take values in a discrete set, **digital time-evolving systems**

Time-evolving systems

Systems by typology

Time-evolving systems (cont.)

The evolution of such systems is completely based on the passage of time

The functions of system behaviour satisfy differential/difference equations

- A specification of the relation between functions and their derivatives
- (The relation is instantaneous)

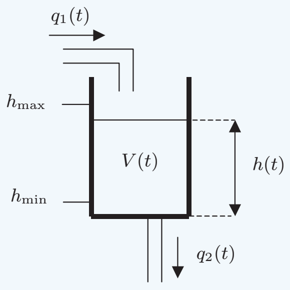
Time-evolving systems(cont.)

Example

Continuous time-evolving systems

Consider a tank whose volume of liquid $V(t)$ [m^3] varies in time

- Variation is due to input and output flow rates
- (By two externally operated pumps)



Tank cannot be emptied or filled

$$\rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}V(t) = q_1(t) - q_2(t)$$

- Output flow $q_2(t) \geq 0$ [m^3s^{-1}]
- Input flow $q_1(t) \geq 0$ [m^3s^{-1}]

The differential equation is a relation between continuous-time functions

$$V(t), q_1(t), q_2(t)$$

Time-evolving systems(cont.)

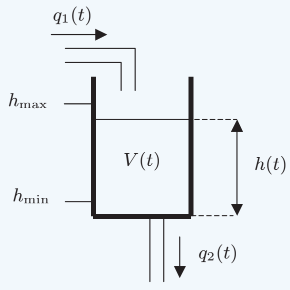
Example

Discrete time-evolving systems

Consider a tank whose volume of liquid $V(t)$ [m^3] varies in time

- Measurements are not continuously available
- Acquisitions only every Δt units of time

We are interested in the system behaviour at $0, \Delta t, 2\Delta t, \dots, k\Delta t, \dots$



We consider discrete-time variables

- $V(k) = V(k\Delta t)$
- $q_1(k) = q_1(k\Delta t)$
- $q_2(k) = q_2(k\Delta t)$

Time-evolving systems(cont.)

We approximate the derivative with the difference quotient

$$\frac{\mathrm{d}}{\mathrm{d}}V(t) \approx \frac{\Delta V}{\Delta t} = \frac{V(k+1) - V(k)}{\Delta t} = q_1(k) - q_2(k)$$

Multiplying both sides of $\dfrac{\Delta V}{\Delta t} = q_1(k) - q_2(k)$ by Δt yields

$$\rightsquigarrow \quad V(k+1) - V(k) = q_1(k)\Delta t - q_2(k)\Delta t$$

The difference equation is a relation between discrete-time variables

$$V(k), q_1(k), q_2(k)$$

Discrete-event systems

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Discrete-event systems

Discrete-event systems

Dynamic systems whose states take logical or symbolic values (not numeric)

The behaviour is characterised by the occurrence of instantaneous events
⇝ [At irregular (perhaps unknown) times]

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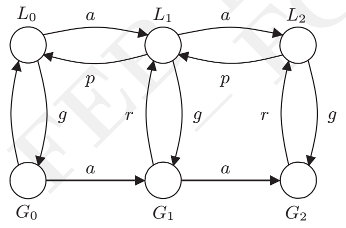
Discrete-event systems (cont.)

Example

Discrete-event systems

Consider a depot where parts are awaiting to be processed by some machine

- The number of parts awaiting to be processed cannot be larger than 2
- The machine can be either healthy (working) or faulty (stopped)



The state of the system

- Number of awaiting parts
- Status of the machine

The events of the system

- Changes in state

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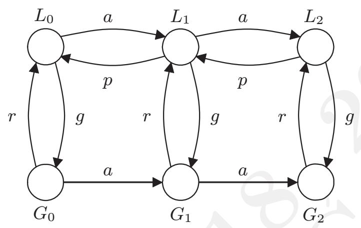
Discrete-event systems (cont.)



Six possible states

- L_0, L_1 and L_2
- G_0, G_1 and G_2

- L_0, the machine is working and the depot is empty
- L_1, the machine is working and there is one part in the depot
- L_2, the machine is working and there are two parts in the depot

- G_0, the machine is not working and the depot is empty
- G_1, the machine is not working and there is one part in the depot
- G_2, the machine is not working and there are two parts in the depot

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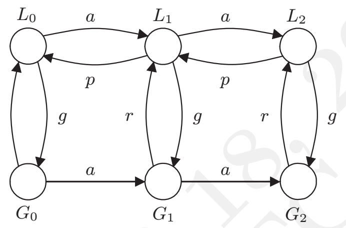
Discrete-event systems (cont.)

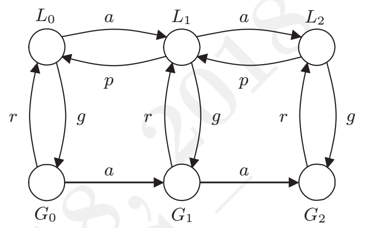


Four possible events

- a and p
- g and r

- a, a new part arrives to the depot
- p, the machines takes one part from the depot
- g, the machine gets faulty
- r, the machine gets fixed

Discrete-event systems (cont.)



Event a can only occur when the depot does not have two parts

$$\begin{cases} L_i \to L_{i+1} \\ G_i \to G_{i+1} \end{cases}$$

Event p can only occur when the deport is not empty

$$\begin{cases} L_i \to L_{i-1} \end{cases}$$

Event g and r determine the switches $L_i \to G_i$ and $G_i \to L_i$, respectively

Hybrid systems

Systems by typology

Hybrid systems

A model that combines time-evolving dynamics and discrete-event dynamics

$\rightsquigarrow$ Thus, the most general class of dynamical systems

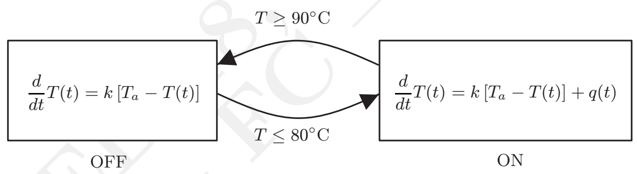
Hybrid systems (cont.)

Example

Hybrid systems

Consider a modern sauna, a cabin where the temperature is regulated

- A thermostat controls a stove used as heat generator
- Keep the temperature between 80°C and 90°C



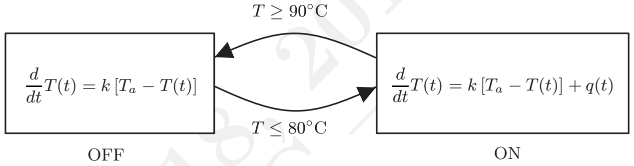
The thermostat can be represented using a discrete-event model

- Switch the heater ON
- Switch the heater OFF

The cabin can be represented using a time-evolving model

- Temperature $T(t)$

Hybrid systems(cont.)



When the thermostat is OFF

$T(t)$ in the cabin decreases, heat exchanged with the outside $[T_a < T(t)]$

$$\frac{d}{dt}T(t) = k[T_a - T(t)], \quad \text{with } k > 0$$

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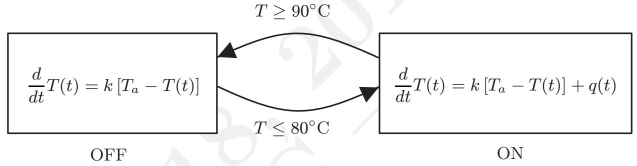

When the thermostat is ON

$T(t)$ in the cabin increases, heat generated by the stove $q(t)$

$$\frac{d}{dt}T(t) = k[T_a - T(t)] + q(t)$$

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Hybrid systems(cont.)

The state of the system is $x = (l, T)$

- A logical variable $l \in \{\mathrm{ON}, \mathrm{OFF}\}$, representing the discrete state
- A real function $T \in \mathbb{R}$, representing the continuous state

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Representation

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Representation

Fundamental step to use formal techniques to study time-evolving systems
$\rightsquigarrow$ We describe the system behaviour in terms of variables, or functions

There are two possible such model/system descriptions

- **Input-output** (**IO**) representation
- **State-space** (**SS**) representation

We define the mathematical elements and properties of these models

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Input-output representation

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Input-output representation

The quantities in the input-output (IO) representation of a system

Causes

- Quantities that are generated outside the system
- Their evolution influences the behaviour of the system
- They are not influenced by the system's behaviour

Effects

- Quantities whose behaviour is influenced by the causes
- Their evolution is influenced by nature of the system

By convention

$$\begin{cases} \text{Causes} & \longrightarrow & \text{Inputs} \\ \text{Effects} & \longrightarrow & \text{Outputs} \end{cases}$$

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Input-output representation (cont.)

The system S can be seen as an operator or a computational unit

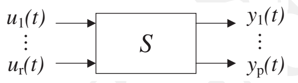
- It assigns a specific evolution to the output variables
- One for each possible evolution of the input variables

Input-output representation (cont.)

A system can have more than one (r) input and more than one (p) output

- Both inputs and outputs are assumed to be measurable/observable

A graphical IO system representation



r **inputs** (in $\mathcal{R}^r$)

$$\mathbf{u}(t) = \big[u_1(t) \cdots u_r(t)\big]^T$$

p **outputs** (in $\mathcal{R}^p$)

$$\mathbf{y}(t) = \big[y_1(t) \cdots y_p(t)\big]^T$$

Manipulable inputs

- They can be used for control

Non-manipulable inputs

- They are called disturbances

Input-output representation (cont.)

Example

A car (IO representation)

Consider a car

Let its position and speed be the output variables

- They are both measurable

As input variables, we can consider wheel and gas position

- They are both measurable and manipulable

By acting on the input variables, we influence the output behaviour

- The changes depend on the specific system (car)
- (More precisely, by its dynamics)

Wind speed could be considered as an additional input variable

- It may be measurable but it is not manipulable

$r = 3$ inputs and $p = 2$ outputs (A MIMO system)

Input-output representation (cont.)

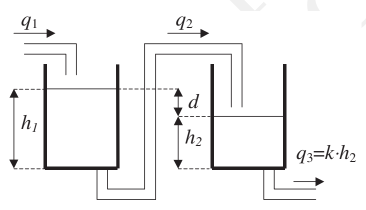
Example

Two tanks (IO representation)

Consider a system consisting of two cylindric tanks, both of base B [m^2]

- Output flow-rate from tank 1 is the input flow-rate to tank 2

⇝ q_2



First tank

- Input flow-rate q_1 [m^{3}s^{-1}]
- Output flow-rate q_2 [m^{3}s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Input flow-rate q_2 [m^{3}s^{-1}]
- Output flow-rate q_3 [m^{3}s^{-1}]
- h_2 is the liquid level [m]

Input-output representation (cont.)

Suppose that flow-rates q_1 and q_2 can be set to some desired value (pumps)

Also suppose that q_3 depends linearly on the liquid level in the tank

- $q_3 = kh_2$, with k [m^{2}s^{-1}]
- k appropriate constant

Inputs, q_1 and q_2

⇝ Measurable and manipulable

⇝ They influence the liquid levels in the tanks

Output, $d = h_1 - h_2$

⇝ Measurable, but not manipulable

⇝ It is influenced indirectly only through inputs

State-space representation

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State-space representation

For a given behaviour of the inputs, S defines the behaviour of the outputs

- ⇝ The output at time t is not only dependent on the inputs at time t
- ⇝ It also depends on the past behaviour of the system

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State-space description(cont.)

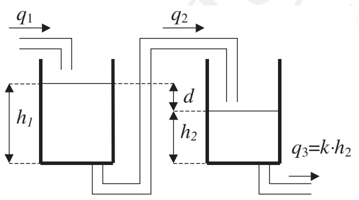
Example

Two tanks (SS representation)

Consider a system consisting of two cylindric tanks, both of base B [m^2]

- Let $d_0 = h_{1,0} - h_{2,0}$ be the output variable at time t_0
- ($h_{1,0}$ and $h_{2,0}$ are the liquid levels at time t_0)

Suppose that all input variables (q_1 and q_2) are zero at time t_0



- $q_{1,0} = 0$
- $q_{2,0} = 0$

Output $d(t)$ at any time $t > t_0$ depends on input values $q_1(t)$ and $q_2(t)$

- Over the entire interval $[t_o, t]$

■

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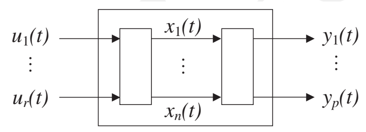
State-space representation (cont.)

We can take this into account by introducing an intermediate variable

A variable that *exists* between inputs and outputs

- The **state** variable of the system

The state condenses information about past and present of the system



r **inputs** (in $\mathcal{R}^r$)

$$\mathbf{u}(t) = [u_1(t) \cdots u_r(t)]^T$$

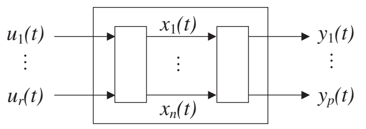
n **states** (in $\mathcal{R}^n$)

$$\mathbf{x}(t) = [x_1(t) \cdots x_n(t)]^T$$

p **outputs** (in $\mathcal{R}^p$)

$$\mathbf{y}(t) = [y_1(t) \cdots y_p(t)]^T$$

State-space representation (cont.)



The state vector $\mathbf{x}(t) = [x_1(t) \cdots x_n(t)]^T$ has n components

- n is the order of the system

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State-space representation (cont.)

Definition

State variable

*The **state** of a system at time t_0 is a variable that contains the necessary information to univocally determine the behaviour of output $\mathbf{y}(t)$ for $t \geq t_0$*

- *Given the behaviour of input $\mathbf{u}(t)$ for $t \geq t_0$ and the state itself at t_0*

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State-space description (cont.)

In general, it is possible to select different physical entities as state variables

- The state variable is neither univocally defined, nor it is determined
- It is anything that can be seen as an *internal cause* of evolution
- (In general)

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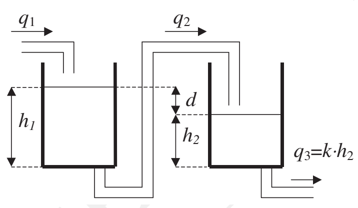
State-space representation (cont.)

Example

Two tanks (SS representation)

Consider a system consisting of two cylindric tanks, both of base B [m^2]



First tank

- Input flow-rate q_1 [m^{3}s^{-1}]
- Output flow-rate q_2 [m^{3}s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Input flow-rate q_2 [m^{3}s^{-1}]
- Output flow-rate q_3 [m^{3}s^{-1}]
- h_2 is the liquid level [m]

- Let $d_0 = h_{1,0} - h_{2,0}$ be the output variable at time t_0
- Let $h_{1,0}$ and $h_{2,0}$ be the liquid levels at time t_0

As state variable, select the volume of liquid in the tanks, $V_1(t)$ and $V_2(t)$

State-space representation (cont.)

We shall see that weare able to evaluate the output $d(t)$ for $t > t_0$

- ⇝ Need to know the initial state, $V_{1,0}$ and $V_{2,0}$, at t_0
- ⇝ Need to know the input, $q_1(t)$ and $q_2(t)$, in $[t_0, t]$

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State-space representation (cont.)

Common to choose as state those variables that characterise system energy

- For a spring with elastic constant k, the potential energy at time t is $E_k(t) = 1/2kz^2(t)$ with $z(t)$ the spring deformation with respect to an equilibrium position. $z(t)$ can be used as state variable
- For a cylindric tank of base B and liquid level $h(t)$, the potential energy at time t is $E_p(t) = 1/2\rho g V^2(t)/B$, with ρ the density of the liquid and $V(t) = Bh(t)$. $V(t)$ or equivalently $h(t)$ can be used as state variable
- For a mass m moving with speed $v(t)$ on a plane, the kinetic energy at time t is $E_m(t) = 1/2mv^2(t)$. $v(t)$ can be used as state of the system

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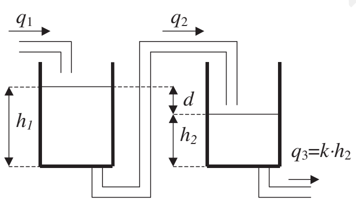
State-space representation (cont.)

Example

Two tanks (SS representation, reloaded)

Consider any of the tanks in the two cylindric tanks system, base B [m^2]



First tank

- Input flow-rate q_1 [m^{3}s^{-1}]
- Output flow-rate q_2 [m^{3}s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Input flow-rate q_2 [m^{3}s^{-1}]
- Output flow-rate q_3 [m^{3}s^{-1}]
- h_2 is the liquid level [m]

Each of the tanks can store a certain amount of potential energy

- The amount depends on the liquid volume (level)

The entire (two-tank) system has thus order 2

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State-space description(cont.)

Consider a system in which there is energy stored, its state is not zero

- The system can evolve even in the absence of external inputs

The state can be understood as a possible (internal) cause of evolution

Mathematical model

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Mathematical model

System analysis studies the relations between the system inputs and outputs (IO) or, alternatively, between the system inputs, states and outputs (SS)

We are given certain input functions

⇝ Interest in understanding how states and outputs evolve in time

We need a model to describe quantitatively the system behaviour

- The relations between inputs (states) and outputs

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Mathematical model (cont.)

Input-output model

The relationship between the system output $\mathbf{y}(t) \in \mathcal{R}^p$ and its derivatives, the system input $\mathbf{u}(t) \in \mathcal{R}^r$ and its derivatives (a **differential equation**)

State-space model

It describes how the evolution of the system state $\dot{\mathbf{x}}(t) \in \mathbb{R}^n$ depends on the state $\mathbf{x}(t) \in \mathcal{R}^n$ itself and on the output $\mathbf{u}(t) \in \mathcal{R}^r$ (the **state equation**)

It describes how the system output $\mathbf{y}(t) \in \mathcal{R}^p$ depends on system state $\mathbf{x}(t) \in \mathcal{R}^n$ and on system input $\mathbf{u}(t) \in \mathcal{R}^r$ (the **output transformation**)

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Input-output model

Mathematical model

Input-output model

The IO model of a SISO system is given as a differential equation

$$h\Big[\underbrace{y(t), \dot{y}(t), \ldots, y^{(n)}(t)}_{\text{output}}, \underbrace{u(t), \dot{u}(t), \ldots, u^{(m)}(t)}_{\text{input}}, \underbrace{t}_{\text{time}}\Big] = 0$$

- $\dot{y}(t) = \dfrac{d}{dt}y(t)$, and $\ldots, y^{(n)}(t) = \dfrac{d^n}{dt^n}y(t)$
- $\dot{u}(t) = \dfrac{d}{dt}u(t)$, and $\ldots, u^{(m)}(t) = \dfrac{d^m}{dt^m}u(t)$

h is a multi-parametric function that depends on the system

- n is the maximum order of derivation of the output
- m is the maximum order of derivation of the input

The **order of the system** (**model**) is n

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Input-output model (cont.)

Example

Consider a system model given as the differential equation

$$2\underbrace{\dot{y}(t)y(t)}_{\text{output}} + 2\sqrt{\underbrace{t}_{\text{time}}}\,\underbrace{u(t)\ddot{u}(t)}_{\text{input}} = 0$$

We have,

- Output order of derivation, $n = 1$
- Input order of derivation, $m = 2$

Function h links y and $\dot{y}$, and u and $\ddot{u}$, and t is the independent variable

The relationship *explicitly* depends on the independent variable (time)

$$\rightsquigarrow \quad 2\sqrt{t}$$

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Input-output model (cont.)

The IO model of a MIMO system with p outputs and r inputs

$$\begin{cases} h_1\Big[\underbrace{y_1(t), \dot{y}_1(t), \ldots, y_1^{(n_1)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_1,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \ldots, u_r^{(m_1,r)}(t)}_{\text{input r}}, t\Big] \\ \quad = 0 \\ h_2\Big[\underbrace{y_2(t), \dot{y}_2(t), \ldots, y_2^{(n_2)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_2,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \ldots, u_r^{(m_2,r)}(t)}_{\text{input r}}, t\Big] \\ \quad = 0 \\ \quad \vdots \\ h_p\Big[\underbrace{y_p(t), \dot{y}_p(t), \ldots, y_p^{(n_p)}(t)}_{\text{output p}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_p,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \ldots, u_r^{(m_p,r)}(t)}_{\text{input r}}, t\Big] \\ \quad = 0 \end{cases}$$

h_i $(i = 1, \ldots, p)$ are multi-parametric functions depending on the system

- n_i, max order of derivation of the i-th component of output $y_i(t)$
- m_i, max order of derivation of the i-th component of input $u_i(t)$

A total of p differential equations

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State-space model

Mathematical model

State-space model

The SS model of a SISO system is not a differential equation of order n

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State-space model (cont.)

State equation

$$\begin{cases} \dot{x}_1(t) = f_1\big[x_1(t), \dots, x_n(t), u(t), t\big] \\ \dot{x}_2(t) = f_2\big[x_1(t), \dots, x_n(t), u(t), t\big] \\ \quad \vdots \\ \dot{x}_n(t) = f_n\big[x_n(t), \dots, x_n(t), u(t), t\big] \end{cases}$$

It links the derivative of each state with the other states and the input

Output transformation

$$y(t) = g\big[x_1(t), \dots, x_n(t), u(t), t\big]$$

It further links the output with each state variable and the input

f_i with $i = 1, \dots, n$ and g are multi-parametric functions

- They depend on (are) the dynamics of the system

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State-space model (cont.)

Let $\dot{\mathbf{x}}(t)$ be the vector whose components are the derivatives of the state

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}x_1(t) \\ \vdots \\ \dfrac{\mathrm{d}}{\mathrm{d}t}x_n(t) \end{bmatrix}$$

State-space model

$$\rightsquigarrow \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), u(t), t\big] \\ y(t) = g\big[\mathbf{x}(t), u(t), t\big] \end{cases}$$

$\mathbf{f}$ is a vectorial function whose i-th component is f_i, with $i = 1, \dots, n$

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State-space model (cont.)

The SS model of a MIMO system with r inputs and p outputs

State equation

$$\begin{cases} \dot{x}_1(t) = f_1\big[x_1(t), \dots, x_n(t), u_1(t), \dots, u_r(t), t\big] \\ \dot{x}_2(t) = f_2\big[x_1(t), \dots, x_n(t), u_1(t), \dots, u_r(t), t\big] \\ \quad \vdots \\ \dot{x}_n(t) = f_n\big[x_n(t), \dots, x_n(t), u_1(t), \dots, u_r(t), t\big] \end{cases}$$

Output transformation

$$\begin{cases} y_1(t) = g_1\big[x_1(t), \dots, x_n(t), u_1(t), \dots, u_r(t), t\big] \\ y_2(t) = g_2\big[x_1(t), \dots, x_n(t), u_1(t), \dots, u_r(t), t\big] \\ \quad \vdots \\ y_p(t) = g_p\big[x_1(t), \dots, x_n(t), u_1(t), \dots, u_r(t), t\big] \end{cases}$$

Multi-parametric functions depending on the system

- f_i with $i = 1, \dots, n$
- g_i with $i = 1, \dots, p$

State-space model (cont.)

State-space model

$$\rightsquigarrow \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \\ \mathbf{y}(t) = \mathbf{g}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \end{cases}$$

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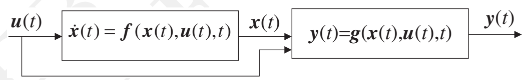
State-space model (cont.)

The state equation is a set of n first-order differential equations

- Regardless of the fact that the system is SISO or MIMO

The output transformation is a scalar or vectorial algebraic equation

- Depending on the number p of output variables



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State-space model (cont.)

The state-space representation of a system is central in our methods

- A consistent framework for analysing systems
- Any arbitrary degree of complexity

Conversion of a scalar n-th order ordinary differential equation

$\rightsquigarrow$ n first-order ordinary differential equations

$\rightsquigarrow$ A first-order vector equation, dimension n

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State-space model (cont.)

Example

Consider the third-order scalar equation

$$\dddot{x}(t) + c_3\ddot{x}(t) + c_2\dot{x}(t) + c_1 x(t) = bu(t)$$

We define,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix}$$

Thus,

$$\rightsquigarrow \quad \dot{x}_3(t) + c_3 x_3(t) + c_2 x_2(t) + c_1 x_1(t) = bu(t)$$

This equation can be first integrated to get $x_3(t)$ $[\rightsquigarrow \ddot{x}(t)]$

- Two more integrations to get $x_2(t)$ $[\rightsquigarrow \dot{x}(t)]$ and x_1 $[\rightsquigarrow x(t)]$

State-space model (cont.)

The three scalar differential equations that must be solved

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ x_3(t) \\ [-c_3 x_3(t) - c_2 x_2(t) - c_1 x_1(t) + bu(t)] \end{bmatrix}$$
$$= \begin{bmatrix} f_1[\mathbf{x}(t), u(t), t] \\ f_2[\mathbf{x}(t), u(t), t] \\ f_3[\mathbf{x}(t), u(t), t] \end{bmatrix}$$

A single vector (state) equation

State-space model (cont.)

Example

Consider two interconnected second-order systems

$$\ddot{y} - a_2(\dot{z} - \dot{y}) - a_1(z - y)^2 = b_1 u_1 + b_2 u_2$$
$$\ddot{z} - c_2 \dot{z}^2 - c_1(y + z) = d u_1^2$$

The state vector could be

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ a_2(x_4 - x_2) + a_1(x_3 - x_1)^2 = b_1 u_1 + b_2 u_2 \\ x_4 \\ c_2 x_4^2 + c_1(x_3 + x_1) + d u_1^2 \end{bmatrix}$$

State-space model (cont.)

The state vector is not necessarily unique

$$\mathbf{x} = \begin{bmatrix} y \\ \dot{y} \\ (z - y) \\ (\dot{z} - \dot{y}) \end{bmatrix}$$

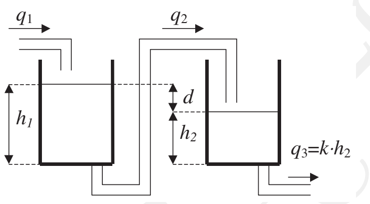
The dynamic equation would get adjusted, accordingly

Examples

Systems

Examples

Example



First tank

- Input flow-rate q_1 [$\mathrm{m^3s^{-1}}$]
- Output flow-rate q_2 [$\mathrm{m^3s^{-1}}$]
- h_1 is the liquid level [m]

Second tank

- Input flow-rate q_2 [$\mathrm{m^3s^{-1}}$]
- Output flow-rate q_3 [$\mathrm{m^3s^{-1}}$]
- h_2 is the liquid level [m]

- $u_i = q_i$ with $i = 1, 2$, the input variables
- $y = d$, the output variable
- $x_1 = V_1$ and $x_2 = V_2$, the state variables

We are interested in the IO and the SS representations of the system

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Examples (cont.)

For an incompressible fluid, by mass conservation

$$\rightsquigarrow \begin{cases} \dfrac{\mathrm{d}\,V_1(t)}{\mathrm{d}t} &= q_1(t) - q_2(t) \\ \dfrac{\mathrm{d}\,V_2(t)}{\mathrm{d}t} &= q_2(t) - q_3(t) = q_2(t) - kh_2(t) \end{cases} \quad (1)$$

We have $h_1 = V_1/B$ and $h_2 = V_2/B$, thus

$$\rightsquigarrow \begin{cases} \dot{h}_1(t) &= \dfrac{1}{B}q_1(t) - \dfrac{1}{B}q_2(t) \\ \dot{h}_2(t) &= \dfrac{1}{B}q_2(t) - \dfrac{1}{B}q_3(t) = \dfrac{1}{B}q_2(t) - \dfrac{k}{B}h_2(t) \end{cases}$$

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Examples (cont.)

Moreover, we have $y(t) = d(t) = h_1(t) - h_2(t)$

Thus,

$$\dot{y}(t) = \dot{d}(t) = \dot{h}_1(t) - \dot{h}_2(t) = \underbrace{\left[\frac{1}{B}q_1(t) - \frac{1}{B}q_2(t)\right]}_{\dot{h}_1(t)} - \underbrace{\left[\frac{1}{B}q_2(t) - \frac{k}{B}h_2(t)\right]}_{\dot{h}_2(t)}$$

$$= \frac{1}{B}q_1(t) - \frac{2}{B}q_2(t) + \frac{k}{B}h_2(t)$$
$$= \frac{1}{B}u_1(t) - \frac{2}{B}u_2(t) + \frac{k}{B}[h_1(t) - y(t)]$$

We used $u_1(t) = q_1(t)$ and $u_2(t) = q_2(t)$

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Examples (cont.)

Furthermore, by taking the second derivative

$$\ddot{y}(t) = \frac{1}{B}\dot{u}_1(t) - \frac{2}{B}\dot{u}_2(t) + \frac{k}{B}\dot{h}_1(t) - \frac{k}{B}\dot{y}(t)$$
$$= \frac{1}{B}\dot{u}_1(t) - \frac{2}{B}\dot{u}_2(t) + \underbrace{\frac{k}{B^2}u_1(t) - \frac{k}{B^2}u_2(t)}_{\frac{k}{B}\dot{h}_1(t)} - \frac{k}{B}\dot{y}(t)$$

The IO system representation is a ordinary differential equation

$$\rightsquigarrow \quad \ddot{y}(t) + \frac{k}{B}\dot{y}(t) = \frac{1}{B}\dot{u}_1(t) - \frac{2}{B}\dot{u}_2(t) + \frac{k}{B^2}u_1(t) - \frac{k}{B}u_2(t)$$

Examples (cont.)

$$\ddot{y}(t) + \frac{k}{B}\dot{y}(t) = \frac{1}{B}\dot{u}_1(t) - \frac{2}{B}\dot{u}_2(t) + \frac{k}{B^2}u_1(t) - \frac{k}{B}u_2(t)$$

The obtained model is in the general IO form

$$\begin{cases}
h_1\Big[\underbrace{y_1(t), \dot{y}_1(t), \ldots, y_1^{(n_1)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_1,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \ldots, u_r^{(m_1,r)}(t)}_{\text{input r}}, t\Big] \\
\quad = 0 \\
h_2\Big[\underbrace{y_2(t), \dot{y}_2(t), \ldots, y_2^{(n_2)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_2,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \ldots, u_r^{(m_2,r)}(t)}_{\text{input r}}, t\Big] \\
\quad = 0 \\
\quad \vdots \\
h_p\Big[\underbrace{y_p(t), \dot{y}_p(t), \ldots, y_p^{(n_p)}(t)}_{\text{output p}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m_p,1)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \ldots, u_r^{(m_p,r)}(t)}_{\text{input r}}, t\Big] \\
\quad = 0
\end{cases}$$

- $p = 1$
- $n_1 = 2$
- $r = 2$
- $m_1 = m_2 = 1$

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Examples (cont.)

The SS system representation is derived from mass conservation

$$\rightsquigarrow \begin{cases}
\dfrac{\mathrm{d}\,V_1(t)}{\mathrm{d}t} & = q_1(t) - q_2(t) \\
\dfrac{\mathrm{d}\,V_2(t)}{\mathrm{d}t} & = q_2(t) - q_3(t) = q_2(t) - kh_2(t)
\end{cases}$$

The state equation is obtained by setting $h_2 = x_2/B$

The output transformation,

$$y(t) = \frac{1}{B}x_1(t) - \frac{1}{B}x_2(t)$$

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Examples (cont.)

The resulting SS representation of the system

$$\rightsquigarrow \begin{cases}
\dot{x}_1(t) & = u_1(t) - u_2(t) \\
\dot{x}_2(t) & = -\dfrac{k}{B}x_2(t) + u_2(t) \\
\dot{y}(t) & = \dfrac{1}{B}x_1(t) - \dfrac{1}{B}x_2(t)
\end{cases}$$

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Examples (cont.)

The model is in the general SS form

State equation

$$\begin{cases}
\dot{x}_1(t) = f_1\big[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t\big] \\
\dot{x}_2(t) = f_2\big[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t\big] \\
\quad \vdots \\
\dot{x}_n(t) = f_n\big[x_n(t), \ldots, x_n(t), u(t), t\big]
\end{cases}$$

Output transformation

$$\begin{cases}
y_1(t) = g_1\big[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t\big] \\
y_2(t) = g_2\big[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t\big] \\
\quad \vdots \\
y_p(t) = g_p\big[x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t\big]
\end{cases}$$

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Examples (cont.)

In general, the choice of the states is not unique

- We could have chosen the levels as states
- $x_1 = h_1$ and $x_2 = h_2$

$$\rightsquigarrow \begin{cases} \dot{x}_1(t) & = -Bu_1(t) - Bu_2(t) \\ \dot{x}_2(t) & = -kx_2(t) + Bu_2(t) \\ y(t) & = x_1(t) - x_2(t) \end{cases}$$

■

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System properties

We discuss a set of fundamental properties of time-evolving models

- Proper v Improper
- Linear v Non-linear
- With v Without delay
- Dynamical v Instantaneous
- Stationary v Non-stationary
- *Lumped v Distributed parameters*

Yet another way of classifying dynamical systems/models

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System properties

Properties

Dynamical v Instantaneous

Definition

Instantaneity

A system is said to be instantaneous if the value of the output $\mathbf{y}(t) \in \mathcal{R}^p$ at time t only depends on the value of the input $\mathbf{u}(t) \in \mathcal{R}^r$ at time t

A system is said to be dynamical, otherwise

Dynamical v Instantaneous (cont.)

Proposition

IO representation - SISO

A necessary and sufficient condition for a SISO system to be instantaneous is that the IO relationship is expressed by an equation in the form

$$h\big[y(t), \cancel{\dot{y}(t)}, \ldots, \cancel{y^{(n)}(t)}, u(t), \cancel{\dot{u}(t)}, \ldots, \cancel{u^{(m)}(t)}, t\big] = h\big[y(t), u(t), t\big] = 0$$

If a SISO system is instantaneous, the IO relation is an algebraic equation
- ⇝ The order of the derivatives of y and u is zero ($n = m = 0$)

Dynamical v Instantaneous (cont.)

This is necessary but not sufficient for the system to be instantaneous

Consider as IO representation of a SISO system a differential equation
- ⇝ Then, the system is certainly dynamical

Dynamical v Instantaneous systems (cont.)

Example

Counter-intuition

Consider as IO representation of a SISO system the algebraic equation

$$y(t) = u(t - T), \quad \text{with } T \in \mathcal{R}^+$$

Such a system is not instantaneous, it is dynamical
- ⇝ Finite time delay system

The output $y(t)$ at time t does not depend on the input $u(t)$ at time t
- It depends on the input value $u(t - T)$, at a preceding moment

Dynamical v Instantaneous (cont.)

Proposition

IO representation - MIMO

A necessary and sufficient condition for a MIMO system with r inputs and p outputs to be instantaneous is that the IO representation is expressed by a system of equations in the form

$$\begin{cases} h_1\big[y_1(t), u_1(t), u_2(t), \dots, u_r(t), t\big] = 0 \\ h_2\big[y_2(t), u_1(t), u_2(t), \dots, u_r(t), t\big] = 0 \\ \quad\vdots \\ h_p\big[y_p(t), u_1(t), u_2(t), \dots, u_r(t), t\big] = 0 \end{cases}$$

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Dynamical v Instantaneous (cont.)

If a MIMO system is instantaneous, then the following conditions are true

- $\rightsquigarrow$ The order of the derivatives of y_i is $n_i = 0$, for all $i = 1, \dots, p$
- $\rightsquigarrow$ The order of the derivatives of u_i is $m_{j,i} = 0$, for all $j = 1, \dots, p$ and $i = 1, \dots, r$

The IO relation can be expressed as a system of p algebraic equations

If any of the IO relations is a differential equation, the system is dynamical

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Dynamical v Instantaneous

Proposition

SS representation

Consider a model of a system expressed in SS form

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \\ \mathbf{y}(t) = \mathbf{g}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \end{cases}$$

A necessary and sufficient condition for a system to be instantaneous is that the SS model is zero order (i.e., there exists no state vector)

$$\rightsquigarrow \quad \mathbf{y}(t) = \mathbf{g}\big[\mathbf{u}(t), t\big]$$

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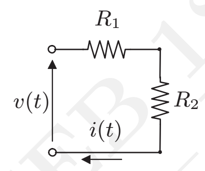
Dynamical v Instantaneous

Example

Consider the two serially arranged resistors, the system is instantaneous

The IO representation corresponds to the SS output transformation



$$v(t) = (R_1 + R_2)i(t) = Ri(t)$$

$$\rightsquigarrow \quad y(t) = \frac{1}{R}u(t)$$

The order of the system is zero (no device to store energy)

Linear v Nonlinear

Properties

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Linear v Nonlinear

Definition

Linearity

A system is said to be linear if it obeys the superposition principle

A system is said nonlinear, otherwise

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Linear v Non-linear (cont.)

Superposition principle

Consider some system

- Let the system response to cause c_1 be equal to effect e_1
- Let the system response to cause c_2 be equal to effect e_2

The system response to cause $(\alpha c_1 + \beta c_2)$ equals effect $(\alpha e_1 + \beta e_2)$

- (whatever the constants α and β)

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Linear v Non-linear (cont.)

Proposition

IO representation - SISO

A necessary and sufficient condition for a SISO system to be linear is that the IO representation is expressed by a linear differential equation

$$\begin{aligned} a_0(t)y(t) + a_1(t)\dot{y}(t) + \cdots + a_n(t)y^{(n)}(t) \\ = b_0(t)u(t) + b_1(t)\dot{u}(t) + \cdots + b_m(t)u^{(m)}(t) \end{aligned}$$

The coefficients of the IO representation are, in general, time dependent

Linear v Non-linear (cont.)

Linear differential equation

Consider the differential equation

$$h\big[y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t), t\big] = 0$$

The equation is linear if and only if the function h is a linear combination of the output and its derivatives, and of the input and its derivatives

$$\begin{aligned}\alpha_0(t)y(t) + \alpha_1(t)\dot{y}(t) + \cdots + \alpha_n(t)y^{(n)}(t) \\ + \beta_0(t)u(t) + \beta_1(t)\dot{u}(t) + \cdots + \beta_m(t)u^{(m)}(t) = 0\end{aligned}$$

$\rightsquigarrow$ A zero-sum weighted sum of inputs, outputs, and derivatives

Linear v non-linear (cont.)

Consider a MIMO system in IO representation

Each function h_i, $i = 1, \dots, p$, must be a linear combination of the i-th component of the output and its n_i derivatives, and the input and its derivatives

The condition is necessary and sufficient

Linear v non-linear (cont.)

Proposition

SS representation

A necessary and sufficient condition for a system to be linear is that state equation and output transformation in the SS model are linear equations

$$\begin{cases}\dot{x}_1(t) = a_{1,1}(t)x_1(t) + \cdots + a_{1,n}(t)x_n(t) + b_{1,1}u_1(t) + \cdots + b_{1,r}(t)u_r(t) \\ \quad\vdots \\ \dot{x}_n(t) = a_{n,1}(t)x_1(t) + \cdots + a_{n,n}(t)x_n(t) + b_{n,1}u_1(t) + \cdots + b_{n,r}(t)u_r(t) \\ y_1(t) = c_{1,1}(t)x_1(t) + \cdots + c_{1,n}(t)x_n(t) + d_{1,1}u_1(t) + \cdots + d_{1,r}(t)u_r(t) \\ \quad\vdots \\ y_p(t) = c_{p,1}(t)x_1(t) + \cdots + c_{p,n}(t)x_n(t) + d_{p,1}u_1(t) + \cdots + d_{p,r}(t)u_r(t)\end{cases}$$

Linear v non-linear (cont.)

$$\rightsquigarrow \begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{cases}$$

- $\rightsquigarrow$ $\mathbf{A}(t) = \big\{a_{i,j}(t)\big\} \in \mathcal{R}^{n \times n}$
- $\rightsquigarrow$ $\mathbf{B}(t) = \big\{b_{i,j}(t)\big\} \in \mathcal{R}^{n \times r}$
- $\rightsquigarrow$ $\mathbf{C}(t) = \big\{c_{i,j}(t)\big\} \in \mathcal{R}^{p \times n}$
- $\rightsquigarrow$ $\mathbf{D}(t) = \big\{d_{i,j}(t)\big\} \in \mathcal{R}^{p \times r}$

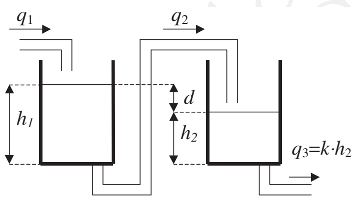
Coefficient matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are, in general, time dependent ∎

Linear v non-linear (cont.)

Example

Consider a system consisting of two cylindric tanks, both of base B [m^2]

- Output flow-rate from tank 1 is the input flow-rate to tank 2, q_2



First tank

- Input flow-rate q_1 [m^{3}s^{-1}]
- Output flow-rate q_2 [m^{3}s^{-1}]
- h_1 is the liquid level [m]

Second tank

- Input flow-rate q_2 [m^{3}s^{-1}]
- Output flow-rate q_3 [m^{3}s^{-1}]
- h_2 is the liquid level [m]

Linear v non-linear (cont.)

The SS representation of the system

$$\rightsquigarrow \begin{cases} \rightsquigarrow \begin{cases} \dfrac{\mathrm{d}\,V_1(t)}{\mathrm{d}t} = q_1(t) - q_2(t) \\ \dfrac{\mathrm{d}\,V_2(t)}{\mathrm{d}t} = q_2(t) - q_3(t) = q_2(t) - kh_2(t) \end{cases} \\ \rightsquigarrow \begin{cases} y(t) = \dfrac{1}{B}x_1(t) - \dfrac{1}{B}x_2(t) \end{cases} \end{cases}$$

Examples (cont.)

We obtained,

$$\rightsquigarrow \begin{cases} \dot{x}_1(t) = u_1(t) - u_2(t) \\ \dot{x}_2(t) = -\dfrac{k}{B}x_2(t) + u_2(t) \\ \dot{y}(t) = \dfrac{1}{B}x_1(t) - \dfrac{1}{B}x_2(t) \end{cases}$$

We have,

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{k}{B} \end{bmatrix}, \quad \mathbf{B}(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C}(t) = \begin{bmatrix} \dfrac{1}{B} & -\dfrac{1}{B} \end{bmatrix}, \quad \mathbf{D}(t) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Linear v non-linear (cont.)

Example

Counter-intuition

Consider the system described by the IO model

$$y(t) = u(t) + 1$$

The system violates the superposition principle

- It is thus nonlinear

Consider two constant inputs

- $u_1(t) = 1$
- $u_2(t) = 2$

We can calculate the outputs

- $y_1(t) = u_1(t) + 1 = 2$
- $y_2(t) = u_2(t) + 2 = 3$

Linear v non-linear (cont.)

Consider the combined input

$$u_3(t) = u_1(t) + u_2(t) = 3$$

The resulting output

$$y_3(t) = u_3(t) + 1 = 4$$

We thus have,

$$y_3(t) = 4 \neq y_1(t) + y_2(t) = 5$$

The IO representation is a nonlinear algebraic equation

- Blame the +1 on the RHS for nonlinearity

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Linear v non-linear (cont.)

Example

Counter-intuition

Consider the system described by the IO model

$$\dot{y}(t) + y(t) = \sqrt{t-1}u(t)$$

The system is linear

Consider the IO representation for a linear SISO system

$$a_0(t)y(t) + a_1(t)\dot{y}(t) + \cdots + \cancel{a_n(t)y^{(n)}}(t) = b_0(t)u(t) + \cancel{b_1(t)\dot{u}(t)} + \cdots + \cancel{b_m(t)u^{(m)}}(t)$$

- $\rightsquigarrow$ $a_0(t) = 1$
- $\rightsquigarrow$ $a_1(t) = 1$
- $\rightsquigarrow$ $b_0(t) = \sqrt{t-1}$

System $\dot{y}(t) + y(t) = \sqrt{t-1}u(t)$ is thus linear

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Stationary and non-stationary systems

Definition

Stationarity

A system is said to be stationary (or time invariant), if it obeys the translation principle

A system is said to be non-stationary (or time varying), otherwise

Stationary and non-stationary systems (cont.)

Translation principle

Consider some system

- Let the system response to a cause $c_1(t)$ be equal to an effect $e_1(t)$

System response to cause $c_2(t) = c_1(t-T)$ equals effect $e_2(t) = e(t-T)$

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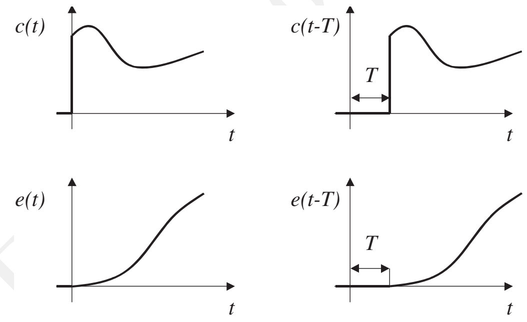
Stationary and non-stationary systems (cont.)

Let the same cause be applied to system S at 2 different times

- ⇝ $t = 0$
- ⇝ $t = T$



The resulting effect is analogous

- Shifted by time interval T

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Stationary and non-stationary systems (cont.)

In nature no system is stationary

Yet, there exists a wide range of variations that can be neglected

- Over large time intervals

Over such intervals, the systems can be considered as stationary

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Stationary and non-stationary systems (cont.)

Proposition

IO representation

A necessary and sufficient condition for a system to be stationary

⇝ *The IO representation must not explicitly depend on time*

Consider the SISO system

$$h\big[y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t), t\big] = 0$$

Then, the stationary model becomes

$$\rightsquigarrow \quad h\big[y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t)\big] = 0$$

Stationary and non-stationary systems (cont.)

Consider a linear SISO system

$$a_0(t)y(t) + a_1(t)\dot{y}(t) + \cdots + a_n(t)y^{(n)}(t) = b_0(t)u(t) + b_1(t)\dot{u}(t) + \cdots + b_m(t)u^{(m)}(t)$$

The model becomes a linear differential equation

$$\rightsquigarrow \quad a_0 y(t) + a_1 \dot{y}(t) + \cdots + a_n y^{(n)}(t) = b_0 u(t) + b_1 \dot{u}(t) + \cdots + b_m u^{(m)}(t)$$

The coefficients are constant

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Stationary and non-stationary systems (cont.)

Theorem

SS representation

A necessary and sufficient condition for a system to be stationary

- $\rightsquigarrow$ *The SS representation must not explicitly depend on time*
- *(Both state equation and output transformation)*

Consider the system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \\ \mathbf{y}(t) = \mathbf{g}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \end{cases}$$

Then, the stationary model becomes

$$\rightsquigarrow \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), \mathbf{u}(t)\big] \\ \mathbf{y}(t) = \mathbf{g}\big[\mathbf{x}(t), \mathbf{u}(t)\big] \end{cases}$$

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Stationary and non-stationary systems (cont.)

Consider a linear system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{cases}$$

The model becomes

$$\rightsquigarrow \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

The (elements of the) coefficient matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are constant

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Stationary and non-stationary systems (cont.)

Example

Consider the instantaneous and linear system

$$y(t) = tu(t)$$

The system is non-stationary

We can show this by using the translation principle

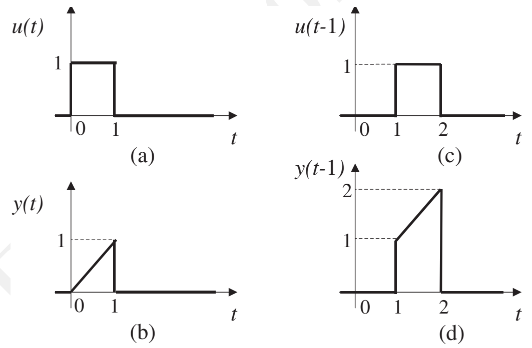
Consider the input

$$u(t) = \begin{cases} 1, & \text{if } t \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

If the same input is applied with a delay, the output is not simply shifted

Stationary and non-stationary systems (cont.)

The same input is applied with one (1) time-unit delay



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Proper and improper systems

Definition

Appropriateness

A system is said to be proper, if it obeys the causality principle

The system is said to be improper, otherwise

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Proper and improper systems (cont.)

Causality principle

The effect does not precede the generating cause

In nature, all systems are (obviously?) proper

- Only the model can be improper

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Proper v improper (cont.)

Proposition

IO representation - SISO

A necessary and sufficient condition for a SISO system to be proper

$\rightsquigarrow$ *The order of derivation of the output (n) is equal or larger than the order of derivation of the input (m)*

$$h\big[y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t), t\big] = 0, \text{ with } n \geq m$$

A system where $n > m$ is said to be **strictly proper**

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Proper and improper systems (cont.)

The result can be extended to MIMO systems

$$\begin{cases} h_1\big[\underbrace{y_1(t), \dot{y}_1(t), \dots, y_1^{(n_1)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{1,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_r(t), \dots, u_r^{(m_{1,r})}(t)}_{\text{input r}}, t\big] = 0 \\ h_2\big[\underbrace{y_2(t), \dot{y}_2(t), \dots, y_2^{(n_2)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{2,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_r(t), \dots, u_r^{(m_{2,r})}(t)}_{\text{input r}}, t\big] = 0 \\ \vdots \\ h_p\big[\underbrace{y_p(t), \dot{y}_p(t), \dots, y_p^{(n_p)}(t)}_{\text{output p}}, \underbrace{u_1(t), \dot{u}_1(t), \dots, u_1^{(m_{p,1})}(t)}_{\text{input 1}}, \dots, \underbrace{u_r(t), \dots, u_r^{(m_{p,r})}(t)}_{\text{input r}}, t\big] = 0 \end{cases}$$

None of the equations must include derivatives of the input variables whose order is larger than the derivation order of corresponding output variables

$$n_i \geq \max_{j=1,\dots,r} m_{i,j}, \quad \text{for all } i = 1, \dots, p$$

A system is strictly proper if the inequality is strictly true, for all $i = 1, \dots, p$

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Proper and improper systems (cont.)

Proposition

SS representation

Consider a system described by a SS model

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \\ \mathbf{y}(t) = \mathbf{g}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \end{cases}$$

Such a system/model is always proper

A strictly proper system has an output transformation independent on $\mathbf{u}(t)$

$$\rightsquigarrow \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}\big[\mathbf{x}(t), \mathbf{u}(t), t\big] \\ \mathbf{y}(t) = \mathbf{g}\big[\mathbf{x}(t), t\big] \end{cases}$$

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Proper and improper systems (cont.)

The SS model of a linear, stationary and strictly proper system

$$\rightsquigarrow \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

With v without delay

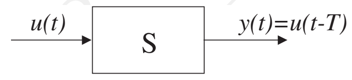
Properties

Systems with and without delay

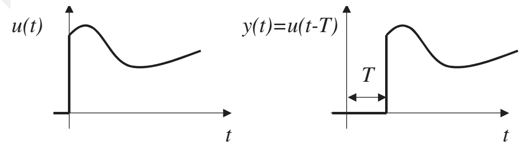
Definition

Finite time delay

A finite delay is a system whose output $y(t)$ at time t is equal to the input $u(t-T)$ at time $t-T$



$\rightsquigarrow$ $T \in (0, +\infty)$ is called the **time delay**



Systems with and without delay (cont.)

Consider the algebraic equation describing a finite delay element

$$y(t) = u(t-T), \text{ with } T \in \mathcal{R}^+$$

- Such a system is not instantaneous
- The system is dynamic

The output at time t depends on the previous values of the input

---

Systems with and without delay (cont.)

Proposition

IO and SS representation

A necessary and sufficient condition for a system to be without a time delay

$\rightsquigarrow$ *All the signals in the model (IO or SS) must share the same argument*

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Systems with and without delay (cont.)

Example

Consider a system described by the IO model

$$4\dot{y}(t) + 2y(t) = u(t - T)$$

The system has delay elements

- There are signals that are dependent on t
- There are signals that are dependent on $t - T$

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Systems with and without delay (cont.)

Example

Consider a system described by the SS model

$$\begin{cases} \dot{x}(t) &= x(t - T) + u(t) \\ y(t) &= 7x(t) \end{cases}$$

The system/model has delay elements

- There are signals that are dependent on t
- There are signals that are dependent on $t - T$