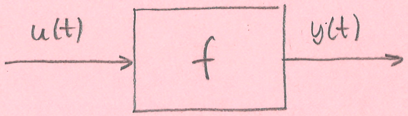
INTRO/REFRESHER DYNAMICAL SYSTEMS



INPUT SIGNAL $u(t) \in \mathbb{R}^{N_u}$

OUTPUT SIGNAL $y(t) \in \mathbb{R}^{N_y}$

DYNAMICS + MEASUREMENT
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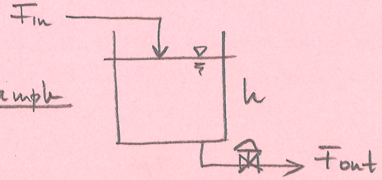
The representation can be regarded as MAPPING IN TIME

$u : t \longrightarrow u(t)$

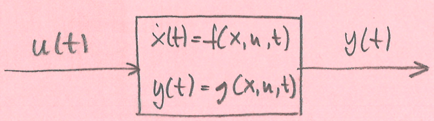
$y : t \longrightarrow y(t)$

$u \longmapsto y$ as $f\{u\}$

Example



If the system dynamics are given as ordinary differential equations and the measurement process is given as an algebraic equation,



$x \in \mathbb{R}^{N_x}$ INTERNAL STATE VARIABLE

$\dot{x}(t) = f(x(t), u(t), t)$ DYNAMICS OR STATE EQUATION (determine the time evolution of the state v.)

$y(t) = g(x(t), u(t), t)$ MEASUREMENT OR OUTPUT EQUATION (map the state to the output)

Ob

SINGLE STATE VARIABLE

$\dot{x} = f(x) = dx/dt$, WE APPROXIMATE THE NONLINEARITY USING A TAYLOR SERIES

$\rightarrow$ FIXED POINT x_s NEGLECT H.O.T.

$$\rightarrow f(x) = f(x_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s}(x - x_s) + \cancel{\frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s}(x-x_s)^2 + \cdots}$$

$$\rightarrow f(x) \cong f(x_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s}(x-x_s) \quad \text{with } f(x_s) = \left.\frac{dx}{dt}\right|_{x_s} = 0, \text{ AS WE ARE AT STEADY STATE}$$

$$\rightarrow f(x) \cong \left.\frac{\partial f}{\partial x}\right|_{x_s}(x - x_s) \cong \frac{dx}{dt} \quad \text{AND ALSO} \quad \frac{dx}{dt} = \frac{d(x-x_s)}{dt}$$

LET $x' = x - x_s \rightsquigarrow \dfrac{dx'}{dt} = \left.\dfrac{\partial f}{\partial x}\right|_{x_s} x'$ (PERTURBATION VARIABLES)

SINGLE STATE VARIABLE + SINGLE INPUT VARIABLE

$\rightarrow \dot{x}(t) = f(x,u)$, FIXED POINT (x_s, u_s)

$$\rightarrow f(x,u) = f(x_s,u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x-x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u-u_s) + \text{H.O.T.}$$

with $f(x_s,u_s) = \left.\dfrac{dx}{dt}\right|_{ss}$

$$\rightarrow f(x,u) \cong \left.\frac{\partial f}{\partial x}\right|_{x_s u_s}(x-x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s u_s}(u-u_s) \cong \frac{dx}{dt} = \frac{d(x-x_s)}{dt} \quad = 0$$

LET $x' = x - x_s$, $u' = u - u_s \rightsquigarrow dx'/dt = \left.\dfrac{\partial f}{\partial x}\right|_{x_s u_s} x' + \left.\dfrac{\partial f}{\partial u}\right|_{x_s u_s} u'$

MULTIPLE (2) STATE VARIABLES + MULTIPLE (1) INPUT VARIABLES

$x(t) = f(x,u)$ with $x = (x_1, x_2)$, $\dot{x}_1(t) = f_1(x_1, x_2, u)$, $\dot{x}_2(t) = f_2(x_1, x_2, u)$, FIXED POINT (x_s, u_s)

$$\rightarrow f_1(x_1,x_2,u) \cong \underbrace{\cancel{f(x_1^{ss}, x_2^{ss}, u^{ss})}}_{=0} + \left.\frac{\partial f_1}{\partial x_1}\right|_{ss}(x_1 - x_1^{ss}) + \left.\frac{\partial f_1}{\partial x_2}\right|_{ss}(x_2 - x_2^{ss}) + \left.\frac{\partial f_1}{\partial u}\right|_{ss}(u - u^{ss})$$

$$f_2(x_1,x_2,u) \cong \cancel{f_2(x_1^{ss}, x_2^{ss}, u_{ss})} + \left.\frac{\partial f_2}{\partial x_1}\right|_{ss}(x_1 - x_1^{ss}) + \left.\frac{\partial f_2}{\partial x_2}\right|_{ss}(x_2 - x_2^{ss}) + \left.\frac{\partial f_2}{\partial u}\right|_{ss}(u - u_{ss})$$

$\rightsquigarrow$

LINEAR-TIME INVARIANT SYSTEMS

A dynamical system f is said to be linear if the following holds

- SUPERPOSITION $f(u_1 + u_2) = f(u_1) + f(u_2)$

- AMPLIFICATION $f(cu) = c f(u)$

A dynamical system is said to be time-invariant if the dynamics f are not dependent on time : $f(x, u, \cancel{t})$

The general linear and time-invariant system :

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \qquad \begin{cases} A \in \mathbb{R}^{N_x \times N_x} \\ B \in \mathbb{R}^{N_x \times N_u} \\ C \in \mathbb{R}^{N_y \times N_x} \\ D \in \mathbb{R}^{N_y \times N_u} \end{cases}$$

LINEARISATION : The idea is to approximate the behaviour of a generally nonlinear system around a reference or steady-state point

→ LINEARIZATION OF THE ODE

$$\begin{cases} \dot{x}(t) = f(x, u) \\ y(t) = g(x, u) \end{cases} \xrightarrow{\text{Linearization}} \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Perturbation vars

→ APPROXIMATION VALID AROUND A FIXED POINT

- STEADY-STATE
- EQUILIBRIUM

FOR THE LINEAR TIME-INVARIANT CASE →

→ LET $\phi(t, t_0) = e^{A(t-t_0)}$

SOLUTION of THE STATE-SPACE ODE

Consider the LTI STATE EQUATION $\dot{x}(t) = Ax(t) + Bu(t)$ with initial conditions $x(t_0) = x_0$

HOMOGENEOUS SOLUTION (NO CONTROLS)

$$x(t) = \underbrace{e^{A(t-t_0)}} x_0 \quad \longleftarrow \text{ SOLVES } \quad \dot{x}(t) = Ax(t), \quad x(0) = x_0$$

We used the MATRIX EXPONENTIAL FUNCTION

$$\underbrace{e^{A(t-t_0)}} = \sum_{\nu=0}^{\infty} \frac{A^{\nu}(t-t_0)^{\nu}}{\nu!}$$

The derivative

$$\frac{d}{dt} e^{A(t-t_0)} = \frac{d}{dt} \sum_{\nu=0}^{\infty} \frac{A^{\nu}(t-t_0)^{\nu}}{\nu!} = \sum_{\nu=0}^{\infty} \frac{A^{\nu}\nu(t-t_0)^{\nu-1}}{\nu!}$$

$$= A \sum_{\nu=0}^{\infty} \frac{A^{\nu-1}(t-t_0)^{\nu-1}}{(\nu-1)!} = \underbrace{A e^{A(t-t_0)}}$$

Thus, by computing the derivative of the solution, we have

$$\dot{x}(t) = A \underbrace{e^{A(t-t_0)} x_0}_{x(t)} = Ax(t) \qquad \square$$

$$\phi(t,t_0) = e^{A(t-t_0)}$$

GENERAL SOLUTION (WITH CONTROLS)

$$x(t) = \underbrace{\phi(t,t_0)x_0}_{\text{HOMOGENEOUS SOLUTION}} + \underbrace{\int_{t_0}^{t} \phi(t,\tau) Bu(\tau)\, d\tau}_{\text{CONVOLUTION INTEGRAL OF THE INPUT}} \qquad \phi(t,\tau) = e^{A(t-\tau)}$$

IS THIS REALLY THE SOLUTION?

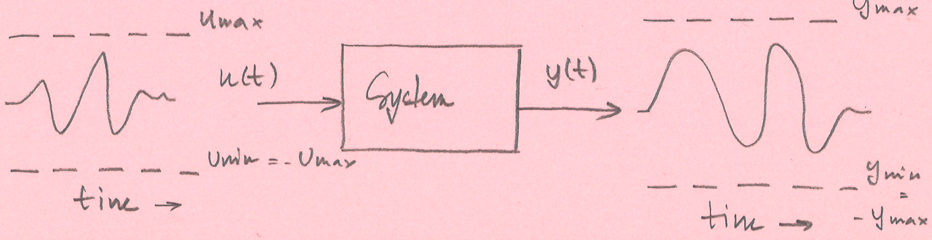
$$\dot{x}(t) = A\phi(t,t_0)x_0 + \underbrace{\phi(t,t)}_{=I} Bu(t) + \int_{t_0}^{t} \underbrace{\frac{d}{dt}\phi(t,\tau)}_{A\phi(t,\tau)} Bu(\tau)\, d\tau$$

$$= \underbrace{A\left(\phi(t,t_0)x_0 + \int_{t_0}^{t} Bu(\tau)\, d\tau\right)}_{Ax(t)} + Bu(t)$$

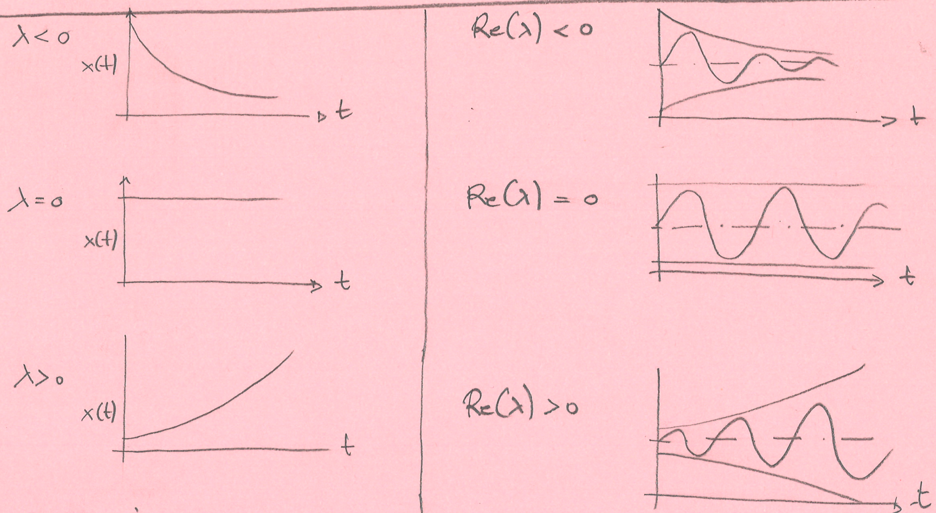
DYNAMICS AND STABILITY

BIBO (Bounded Input, Bounded Output) STABILITY

- A system is said to have BIBO stability if every bounded input results in a bounded output



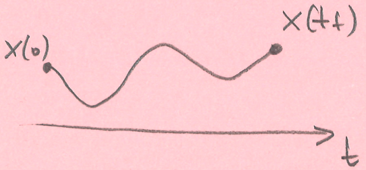
$$\boxed{\text{A LTI INVARIANT SYSTEM IS BIBO STABLE IF } Re(\lambda_i) < 0 \text{ FOR ALL } \lambda_i \in \sigma(A) \text{ (all the eigenvalues of A)}}$$



REAL EIGENVALUES

CONTROLLABILITY

CONTROLLABILITY (GENERAL CASE) – A system is controllable if it can be steered from any initial state $x(0)$ to any final state $x(t_f)$ in a finite time (t_f) by an appropriate choice of the input $u(t)$, $0 \leq t \leq t_f$



For any $x(0)$ and for any $x(t_f)$

→ FIND $u(t)$

→ AN OPTIMAL ONE, POSSIBLY

FOR LINEAR AND TIME-INVARIANT SYSTEMS THERE EXIST A NUMBER OF CRITERIA FOR CHECKING CONTROLLABILITY

→ CONTROLLABILITY MATRIX

$$\mathbb{C} = \left[B \;\; AB \;\; AB^2 \;\; \cdots \;\; AB^{N_x - 1} \right]$$

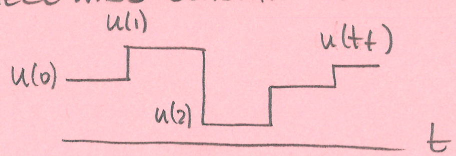
(A,B) CONTROLLABLE IF $\mathbb{C}$ IS FULL RANK
$\det(\mathbb{C}) \neq 0$

→ POPOV – BELEVITCH – HAUTUS TEST

$$\operatorname{rank} \left[\lambda I - A \mid B \right] = N_x \,, \quad \forall \lambda \in \sigma(A)$$

THE SPECTRUM OF MATRIX A

CONTROLS, IN PRACTICE, CAN ONLY BE DISCRETE, FOR EXAMPLE PIECEWISE CONSTANT FUNCTIONS



OBSERVABILITY

OBSERVABILITY (GENERAL CASE) – A system is observable if it is possible to uniquely determine its initial state from a sequence of measurements over a finite time interval

FOR LINEAR AND TIME-INVARIANT SYSTEMS THERE EXIST A NUMBER OF CRITERIA FOR CHECKING OBSERVABILITY

→ OBSERVABILITY MATRIX

$$\mathbb{O} = \left[C^T \;\; A^T C^T \;\; (A^T)^2 C^T \;\; \cdots \;\; (A^T)^{N_x - 1} C^T \right]^T$$

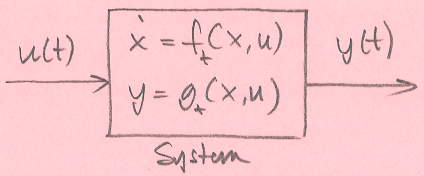
(A,C) OBSERVABLE IF $\mathbb{O}$ IS FULL-RANK $\det(\mathbb{O}) \neq 0$

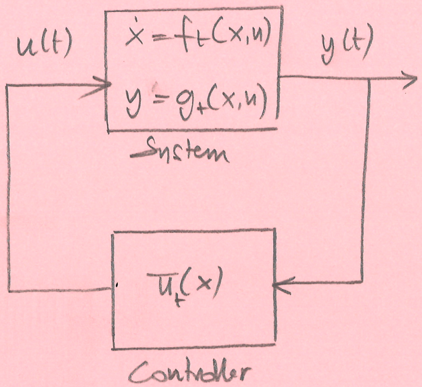
→ POPOV – BELEVITCH TEST

$$\text{rank} \left[\lambda I - A^T \mid C^T \right]^T = N_x \, , \quad \forall \lambda \in \sigma(A)$$

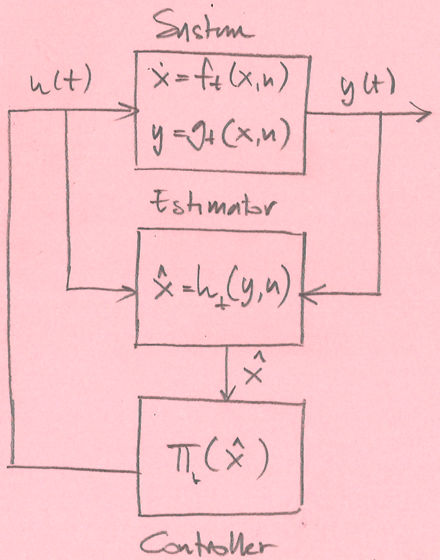
THE SPECTRUM OF MATRIX A

STATE FEEDBACK CONTROL



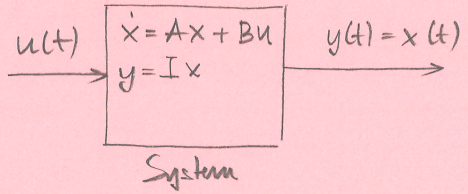


- WE CAN MEASURE THE STATE VARIABLES, $x(t)$
- WE DEFINE CONTROL ACTIONS $u(t)$



- WE CANNOT MEASURE THE STATE VARIABLES $x(t)$
- WE ESTIMATE THEM, $\hat{x}(t)$
- THEN, WE DEFINE CONTROL ACTIONS, $u(t)$

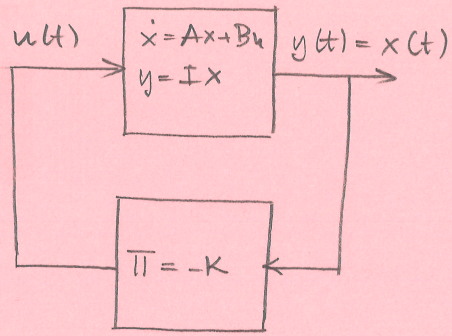
LINEAR QUADRATIC REGULATOR (LQR)



– LINEAR AND TIME – INVARIANT DYNAMICS

⇝ STABLE OR UNSTABLE
⇝ CONTROLLABLE

⇝ OPTIMAL PLACEMENT OF THE EIGENVALUES OF THE CONTROLLED SYSTEM



– LINEAR AND TIME INVARIANT CONTROLLER

– $\dot{x} = (A - BK)\, x$

OPTIMAL IN THE FOLLOWING SENSE

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

$\left.\begin{matrix} Q \\ \\ R \end{matrix}\right\}$ WEIGHTING MATRIXES

– x IS SHORT HAND FOR $x - x_{ss}$, x_{ss} STEADY STATE

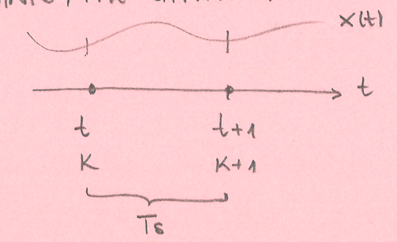
– u IS SHORT HAND FOR $u - u_{ss}$, u_{ss} STEADY STATE

Dynamical Systems 9

DISCRETE-TIME SYSTEMS, LINEAR CASE

- THIS TYPE OF MODELS ARE OFTEN CREATED BY DISCRETISING CONTINUOUS-TIME MODELS
- WE FOCUS ON LINEAR-TIME-INVARIANT MODELS

THE CONTINUOUS-TIME SYSTEM IS DISCRETISED AT EQUIDISTANT TIME POINTS, THE SAMPLING TIME IS T_S

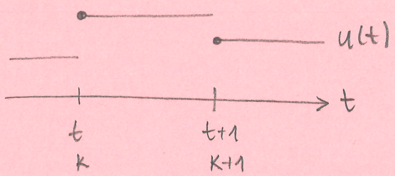


T_S IS THE TIME BETWEEN TWO CONSECUTIVE SAMPLING TIMES

EXAMPLE: THE STATE VECTOR $x(t)$ AT THE K-TH SAMPLING TIME IS THE VALUE OF THE VECTOR AT TIME $t = KT_S$

$\rightsquigarrow x_K = x(KT_S)$

THE INPUT VECTOR IS ASSUMED CONSTANT BETWEEN SAMPLING TIMES



EXAMPLE: THE INPUT VECTOR $u(t)$ IS ASSUMED PIECEWISE CONSTANT

(it could have been piecewise linear, piecewise quadratic, ...)

CONSIDER THE GENERAL LTI SYSTEM → STATE-SPACE CONTINUOUS-TIME

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$
$$y(t) = C x(t) + D_c u(t)$$

('c' denotes continuous-time)

THE DISCRETE-TIME REPRESENTATION IS OBTAINED FROM THE CONTINUOUS-TIME ONE, BY MEANS OF SIMULATION (INTEGRATION)

FOR LTI SYSTEMS WE CAN DERIVE AN ANALYTICAL EXPRESSION FOR THE DISCRETE-TIME FORM OF THE MODEL UNDER THE PIECEWISE CONSTANT ASSUMPTION FOR THE INPUT

⇝ We do not need numerical integration

WE OBTAIN, THE TIME INVARIANT REPRESENTATION

$$\begin{cases} x_{k+1} = A x_k + B u_k \\ y_k = C x_k + D u_k \end{cases}$$

, for all $k = 0, \_\_\_, K-1$

A, B, C, D constant with k

CONSIDER TIME $t_0 = 0$ AND $t = T$, WE CAN WRITE

$$x_1 = x(T_s) = e^{A_c(T_s - 0)} x(0) + \int_0^{T_s} e^{A_c(T_s - t)} B_c u(\tau) d\tau$$

$$= e^{A_c T_s} x(0) + \int_0^{T_s} e^{A_c t} dt \left(B_c \underbrace{u(0)} \right) = A x_0 + B u_0$$

CHANGE OF VARIABLE
$t = T_s - \tau$

IT IS PIECEWISE CONSTANT IN $[0, T_s]$ AND THUS CAN BE MOVED OUTSIDE THE INTEGRAL

THE DISCRETE-TIME MATRICES WRT THE CONTINUOUS-TIME ONES

$$\begin{cases} A = e^{A_c T_s} \\ B = \int_0^{T_s} e^{A_c t}\, dt \; (B_c) \\ C = C_c \\ D = D_c \end{cases}$$

$\left.\begin{matrix} C = C_c \\ D = D_c \end{matrix}\right\}$ UNCHANGED, THE MAPPING FROM STATE VARS TO MEASUREMENTS IS STATIC

THE HOMOGENEOUS RESPONSE (only A is needed)

$$x_1 = A x_0$$
$$x_2 = A x_1 = A(A x_0) = A^2 x_0$$
$$\vdots = \vdots$$
$$x_k = A x_{k-1} = A(A^{k-1} x_0) = A^k x_0$$

FORCED RESPONSE

The forced response with arbitrary input can be computed by induction

$$\begin{aligned} x_{k+1} &= A x_{k+1} + B u_{k+1} \\ &= A(A x_k + B u_k) + B u_{k+1} \\ &= A^2 x_k + A B u_k + B u_{k+1} \end{aligned}$$

⇝ It generalizes to $x_k = A^k x_0 + \sum_{m=0}^{k-1} A^{k-m-1} B u_m \quad (k \geq 0)$

⇝ FOR THE OUTPUT, WE HAVE

$$y_k = C A^k x_0 + \sum_{m=0}^{k-1} C A^{k-m-1} B u_m + D u_k$$

STABILITY IN DISCRETE TIME

The eigenvalues of the state matrix A are defined as

$$\lambda_i v_i = A v_i \quad , \text{for } v_i \neq 0$$

We can write $(\lambda_i I - A) v_i = 0$, which implies that

$$\det(\lambda_i I - A) = 0 \qquad (\text{characteristic polynomial})$$

Let A be $n \times n$, then the characteristic polynomial is

$$\lambda^n + a_{n-1}\lambda^{n-1} + \text{—} + a_1\lambda + a_0 = 0$$

By factorization, we get

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_{N-1}) = 0 \quad , \text{for } \lambda_i \in \mathbb{C}$$

We define $V = [v_1 \; v_2 \; \text{—} \; v_N]$

Then, we define

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} \end{bmatrix}$$

Then, we also define

$$\Lambda^k = (\Lambda)^k$$

And, finally $A^k = (V \Lambda V^{-1})^k = V \Lambda^k V^{-1}$