Matrix Book

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Chapter 1

Sets, Logic, Numbers, Relations, Orderings, Graphs, and Functions

2i

In this chapter we review basic terminology and results concerning sets, logic, numbers, relations, orderings, graphs, and functions. This material is used throughout the book.

3

1.1 Sets

[4]A *set* $\{x, y, \ldots\}$ is a collection of elements. A set can include either a finite or infinite number of elements. The set $\mathcal{X}$ is *finite* if it has a finite number of elements; otherwise, $\mathcal{X}$ is *infinite*. [5]The set $\mathcal{X}$ is *countably infinite* if $\mathcal{X}$ is infinite and its elements are in one-to-one correspondence with the positive integers. The set $\mathcal{X}$ is *countable* if it is either finite or countably infinite.

[6]Let $\mathcal{X}$ be a set. Then,

$$x \in \mathcal{X} \tag{1.1}$$

means that x is an *element* of $\mathcal{X}$. If w is not an element of $\mathcal{X}$, then we write

$$w \notin \mathcal{X}. \tag{1.2}$$

[7]No set can be an element of itself. Therefore, there does not exist a set that includes every set. The set with no elements, denoted by $\varnothing$, is the *empty set*. If $\mathcal{X} \neq \varnothing$, then $\mathcal{X}$ is *nonempty*.

[8]Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. The *intersection* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of common elements of $\mathcal{X}$ and $\mathcal{Y}$, which is given by

$$\mathcal{X} \cap \mathcal{Y} \triangleq \{x\colon\ x \in \mathcal{X} \text{ and } x \in \mathcal{Y}\} = \{x \in \mathcal{X}\colon\ x \in \mathcal{Y}\} = \{x \in \mathcal{Y}\colon\ x \in \mathcal{X}\} = \mathcal{Y} \cap \mathcal{X}, \tag{1.3}$$

[9]The *union* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of elements in either $\mathcal{X}$ or $\mathcal{Y}$, which is the set

$$\mathcal{X} \cup \mathcal{Y} \triangleq \{x\colon\ x \in \mathcal{X} \text{ or } x \in \mathcal{Y}\} = \mathcal{Y} \cup \mathcal{X}. \tag{1.4}$$

[10]The *complement* of $\mathcal{X}$ *relative* to $\mathcal{Y}$ is

$$\mathcal{Y} \backslash \mathcal{X} \triangleq \{x \in \mathcal{Y}\colon\ x \notin \mathcal{X}\}. \tag{1.5}$$

[11]If $\mathcal{Y}$ is specified, then the *complement* of $\mathcal{X}$ is

$$\mathcal{X}^\sim \triangleq \mathcal{Y} \backslash \mathcal{X}. \tag{1.6}$$

[12]The *symmetric difference* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of elements that are in either $\mathcal{X}$ or $\mathcal{Y}$ but not both, which is given by

$$\mathcal{X} \ominus \mathcal{Y} \triangleq (\mathcal{X} \cup \mathcal{Y}) \backslash (\mathcal{X} \cap \mathcal{Y}). \tag{1.7}$$

[13]If $x \in \mathcal{X}$ implies that $x \in \mathcal{Y}$, then $\mathcal{X}$ is a *subset* of $\mathcal{Y}$ (equivalently, $\mathcal{Y}$ *contains* $\mathcal{X}$), which is written as

$$\mathcal{X} \subseteq \mathcal{Y}. \tag{1.8}$$

[14]Equivalently,

$$\mathcal{Y} \supseteq \mathcal{X}. \tag{1.9}$$

[15]Note that $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $\mathcal{X} \backslash \mathcal{Y} = \varnothing$. [16]Furthermore, $\mathcal{X} = \mathcal{Y}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} \neq \mathcal{Y}$, then $\mathcal{X}$ is a *proper subset* of $\mathcal{Y}$ and we write $\mathcal{X} \subset \mathcal{Y}$. [17]The sets $\mathcal{X}$ and $\mathcal{Y}$ are *disjoint* if $\mathcal{X} \cap \mathcal{Y} = \varnothing$. [18]A *partition* of $\mathcal{X}$ is a set of pairwise-disjoint and nonempty subsets of $\mathcal{X}$ whose union is equal to $\mathcal{X}$.

[19]The symbols $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of nonnegative integers, positive integers, integers, rational numbers, and real numbers, respectively.

[20]A set cannot have repeated elements. Therefore, $\{x, x\} = \{x\}$. [21]A *multiset* is a finite collection of elements that allows for repetition. The multiset consisting of two copies of x is written as $\{x, x\}_{\rm ms}$. [22i]For example, the roots of the polynomial $p(x) = (x - 1)^2$ are the elements of the multiset $\{1, 1\}_{\rm ms}$, while the prime factors of 72 are the elements of the multiset $\{2, 2, 2, 3, 3\}_{\rm ms}$.

[23]The operations "$\cap$," "$\cup$," "$\backslash$," "$\ominus$," and "$\times$" and the relations "$\subset$" and "$\subseteq$" extend to multisets. [24]For example,

$$\{x, x\}_{\rm ms} \cup \{x\}_{\rm ms} = \{x, x, x\}_{\rm ms}. \tag{1.10}$$

[25]By ignoring repetitions, a multiset can be converted to a set, while a set can be viewed as a multiset with distinct elements.

[26]The *Cartesian product* $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ of sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is the set consisting of *tuples* of the form $(x_1, \ldots, x_n)$, where, for all $i \in \{1, \ldots, n\}$, $x_i \in \mathcal{X}_i$. A tuple with n components is an *n-tuple*. [27]The components of a tuple are ordered but need not be distinct. Therefore, a tuple can be viewed as an ordered multiset. [28]We thus write

$$(x_1, \ldots, x_n) \in \bigtimes_{i=1}^n \mathcal{X}_i \triangleq \mathcal{X}_1 \times \cdots \times \mathcal{X}_n. \tag{1.11}$$

$\mathcal{X}^n$ denotes $\bigtimes_{i=1}^n \mathcal{X}$.
[29]

Definition 1.1.1. A *sequence* $(x_i)_{i=1}^{\infty} = (x_1, x_2, \ldots)$ is a tuple with a countably infinite number of components. Now, let $i_1 < i_2 < \cdots$. Then, $(x_{i_j})_{j=1}^{\infty}$ is a *subsequence* of $(x_i)_{i=1}^{\infty}$.

[30i] Let $\mathcal{X}$ be a set, and let $X \triangleq (x_i)_{i=1}^{\infty}$ be a sequence whose components are elements of $\mathcal{X}$; that is, $\{x_1, x_2, \ldots\} \subseteq \mathcal{X}$. For convenience, we write either $X \subseteq \mathcal{X}$ or $X \subset \mathcal{X}$, where X is viewed as a set and the multiplicity of the components of the sequence is ignored. [31] For sequences $X, Y \subset \mathbb{F}^n$, define $X + Y \triangleq (x_i + y_i)_{i=1}^{\infty}$ and $X \odot Y \triangleq (x_i \odot y_i)_{i=1}^{\infty}$, where "$\odot$" denotes component-wise multiplication. In the case $n = 1$, we define $XY \triangleq (x_i y_i)_{i=1}^{\infty}$.

[32i]

[33]

1.2 Logic

[34] Every *statement* is either true or false, and no statement is both true and false. [35][36] A *proof* is a collection of statements that verify that a statement is true. [37] A *conjecture* is a statement that is believed to be true but whose proof is not known.

[38] Let A and B be statements. [39] The *not* of A is the statement (not A), the *and* of A and B is the statement (A and B), and the *or* of A and B is the statement (A or B). [40] The statement (A or B) does not contradict the statement (A and B); hence, the word "or" is inclusive. [41] The *exclusive or* of A and B is the statement (A xor B), which is [(A and not B) or (B and not A)]. [42] Equivalently, (A xor B) is the statement [(A or B) and not(A and B)], that is, A or B, but not both. [43] Note that (A and B) = (B and A), (A or B) = (B or A), and (A xor B) = (B xor A).

[44i] Let A, B, and C be statements. Then, the statements (A and B or C) and (A or B and C) are ambiguous. For clarity, we thus write, for example, [A and (B or C)] and [A or (B and C)]. In words, we write "A and either B or C" and "A or both B and C," respectively, where "either" and "both" signify parentheses. [45i] Furthermore,

$$(A \text{ and } B) \text{ or } C = (A \text{ and } C) \text{ or } (B \text{ and } C), \tag{1.12}$$
$$(A \text{ or } B) \text{ and } C = (A \text{ or } C) \text{ and } (B \text{ or } C). \tag{1.13}$$

[46] Let A be a statement. To analyze statements involving logic operators, define $\mathrm{truth}(A) = 1$ if A is true, and $\mathrm{truth}(A) = 0$ if A is false. Then,

$$\mathrm{truth}(\text{not } A) = \mathrm{truth}(A) + 1, \tag{1.14}$$

[47] where $0 + 0 = 0$, $1 + 0 = 0 + 1 = 1$, and $1 + 1 = 0$. [48i] Therefore, A is true if and only if (not A) is false, while A is false if and only if (not A) is true. [49i] Note that

$$\begin{aligned}\mathrm{truth}[\text{not}(\text{not } A)] &= \mathrm{truth}(\text{not } A) + 1\\ &= [\mathrm{truth}(A) + 1] + 1\\ &= \mathrm{truth}(A).\end{aligned}$$

[50] Furthermore, note that $\mathrm{truth}(A) + \mathrm{truth}(A) = 0$ and $\mathrm{truth}(A)\,\mathrm{truth}(A) = \mathrm{truth}(A)$.

[51] Let A and B be statements. Then,

$$\mathrm{truth}(A \text{ and } B) = \mathrm{truth}(A)\,\mathrm{truth}(B), \tag{1.15}$$
$$\mathrm{truth}(A \text{ or } B) = \mathrm{truth}(A)\,\mathrm{truth}(B) + \mathrm{truth}(A) + \mathrm{truth}(B), \tag{1.16}$$

$$\text{truth}(A \text{ xor } B) = \text{truth}(A) + \text{truth}(B). \tag{1.17}$$

[52]Hence,

$$\text{truth}(A \text{ and } B) = \min\{\text{truth}(A), \text{truth}(B)\}, \tag{1.18}$$

$$\text{truth}(A \text{ or } B) = \max\{\text{truth}(A), \text{truth}(B)\}. \tag{1.19}$$

[53][54i]Consequently, truth(A and B) = truth(B and A), truth(A or B) = truth(B or A), and truth(A xor B) = truth(B xor A). [55]Furthermore, truth(A and A) = truth(A or A) = truth(A), and truth(A xor A) = 0.

[56]Let A and B be statements. The *implication* $(A \Longrightarrow B)$ is the statement [(not A) or B]. [57]Therefore,

$$\text{truth}(A \Longrightarrow B) = \text{truth}(A)\,\text{truth}(B) + \text{truth}(A) + 1. \tag{1.20}$$

[58]The implication $(A \Longrightarrow B)$ is read as either "if A, then B," "if A holds, then B holds," or "A implies B." [59]The statement A is the *hypothesis*, while the statement B is the *conclusion*. [60]If $(A \Longrightarrow B)$, then A is a *sufficient condition* for B, and B is a *necessary condition* for A. [61]It follows from (1.20) that, if A and B are true, then $(A \Longrightarrow B)$ is true; if A is true and B is false, then $(A \Longrightarrow B)$ is false; and, if A is false, then $(A \Longrightarrow B)$ is true whether or not B is true. [62i]For example, both implications $[(2+2=5) \Longrightarrow (3+3=6)]$ and $[(2+2=5) \Longrightarrow (3+3=8)]$ are true. [63]Finally, note that $[(A \Longrightarrow B)$ and $A] = A$ and B.

[64i]

[65]A *predicate* is a statement that depends on a variable. Let $\mathcal{X}$ be a set, let $x \in \mathcal{X}$, and let $A(x)$ be a predicate. [66]There are two ways to use a predicate to create a statement. An *existential statement* has the form

$$\text{there exists } x \in \mathcal{X} \text{ such that } A(x) \text{ holds}, \tag{1.21}$$

whereas a *universal statement* has the form

$$\text{for all } x \in \mathcal{X}, A(x) \text{ holds}. \tag{1.22}$$

[67][68]Note that

$$\text{truth}[\text{there exists } x \in \mathcal{X} \text{ such that } A(x) \text{ holds}] = \max_{x\in\mathcal{X}} \text{truth}[A(x)], \tag{1.23}$$

$$\text{truth}[\text{for all } x \in \mathcal{X}, A(x) \text{ holds}] = \min_{x\in\mathcal{X}} \text{truth}[A(x)]. \tag{1.24}$$

[69]An *argument* is an implication whose hypothesis and conclusion are predicates that depend on the same variable. In particular, letting x denote a variable, and letting $A(x)$ and $B(x)$ be predicates, the implication $[A(x) \Longrightarrow B(x)]$ is an argument. [70]For example, for each real number x, the implication $[(x=1) \Longrightarrow (x+1=2)]$ is an argument. [71]Note that the variable x links the hypothesis and the conclusion, thereby making this implication useful for the purpose of *inference*. In particular, for all real numbers x, $\text{truth}[(x=1) \Longrightarrow (x+1=2)] = 1$. The statements (for all $x, [A(x) \Longrightarrow B(x)]$ holds) and (there exists x such that $[A(x) \Longrightarrow B(x)]$ holds) are inferences.

[72]Let A and B be statements. The *bidirectional implication* $(A \Longleftrightarrow B)$ is the statement $[(A \Longrightarrow B)$ and $(A \Longleftarrow B)]$, where $(A \Longleftarrow B)$ means $(B \Longrightarrow A)$. If $(A \Longleftrightarrow B)$, then A and B are *equivalent*. [73]Furthermore,

$$\text{truth}(A \Longleftrightarrow B) = \text{truth}(A) + \text{truth}(B) + 1. \tag{1.25}$$

[74i]Therefore, A and B are equivalent if and only if either both A and B are true or both A and B are false. [75i]

[76]Let A and B be statements, and assume that $(A \iff B)$. Then, A holds *if and only if* B holds. The implication $A \implies B$ (the "only if" part) is *necessity*, while $B \implies A$ (the "if" part) is *sufficiency*.

[77]Let A and B be statements. The *converse* of $(A \implies B)$ is $(B \implies A)$. [78]Note that

$$\begin{aligned}(A \implies B) &\iff [(\text{not } A) \text{ or } B]\\ &\iff [(\text{not } A) \text{ or not}(\text{not } B)]\\ &\iff [\text{not}(\text{not } B) \text{ or not } A]\\ &\iff (\text{not } B \implies \text{not } A).\end{aligned}$$

[79]Therefore, the statement $(A \implies B)$ is equivalent to its *contrapositive* $[(\text{not } B) \implies (\text{not } A)]$.

[80]Let A, B, A', and B' be statements, and assume that $(A' \implies A \implies B \implies B')$. Then, $(A' \implies B')$ is a *corollary* of $(A \implies B)$.

[81]Let A, B, and A' be statements, and assume that $A \implies B$. Then, $(A \implies B)$ is a *strengthening* of $[(A \text{ and } A') \implies B]$. If, in addition, $(A \implies A')$, then the statement $[(A \text{ and } A') \implies B]$ has a *redundant assumption*.

[82]An *interpretation* is a feasible assignment of true or false to all statements that comprise a statement. [83]For example, there are four interpretations of the statement $(A \text{ and } B)$, depending on whether A is assigned to be true or false and B is assigned to be true or false. Likewise, $[(x = 1) \text{ and } (x = 2)]$ has three interpretations, which depend on the value of x.

[84]Let $A_1, A_2, \ldots$ be statements, and let B be a statement that depends on $A_1, A_2, \ldots$ Then, B is a *tautology* if B is true whether or not $A_1, A_2, \ldots$ are true. [85]For example, let B denote the statement $(A \text{ or not } A)$. Then,

$$\text{truth}(A \text{ or not } A) = 1, \tag{1.26}$$

and thus the statement $(A \text{ or not } A)$ is true whether or not A is true. Hence, $(A \text{ or not } A)$ is a tautology. Likewise, $(A \implies A)$ is a tautology. Furthermore, since

$$\text{truth}[(A \text{ and } B) \implies A] = \text{truth}(A)^2\,\text{truth}(B) + \text{truth}(A)\,\text{truth}(B) + 1 = 1, \tag{1.27}$$

it follows that $[(A \text{ and } B) \implies A]$ is a tautology. [86]Likewise, $\text{truth}([A \text{ and not } A] \implies B) = 1$, and thus $([A \text{ and not } A] \implies B)$ is a tautology. [87i]

[88]Let $A_1, A_2, \ldots$ be statements, and let B be a statement that depends on $A_1, A_2, \ldots$ Then, B is a *contradiction* if B is false whether or not $A_1, A_2, \ldots$ are true. [89]For example, let B denote the statement $(A \text{ and not } A)$. Then,

$$\text{truth}(A \text{ and not } A) = 0, \tag{1.28}$$

and thus the statement $(A \text{ and not } A)$ is false whether or not A is true. Hence, $(A \text{ and not } A)$ is a contradiction. [90i]

[91]Let A and B be statements. If the implication $(A \implies B)$ is neither a tautology nor a contradiction, then $\text{truth}(A \implies B)$ depends on the truth of the statements that comprise A and B. [92]For example, $\text{truth}(A \implies \text{not } A) = \text{truth}(A) + 1$, and thus the statement $(A \implies \text{not } A)$ is true if and only if A is false, and false if and only if A is true. Hence, $(A \implies \text{not } A)$ is neither a tautology nor a contradiction. [93]A statement that is neither a tautology nor a

contradiction is a *contingency*. [94]For example, the implication $[A \implies (A \text{ and } B)]$ is a contingency. Likewise, for each real number x, $\text{truth}[(x=1) \implies (x=2)] = \text{truth}(x \neq 1)$, and thus the statement $[(x=1) \implies (x=2)]$ is a contingency.

[95]An argument that is a contingency is a *theorem*, *proposition*, *corollary*, or *lemma*. [96]A theorem is a significant result; a proposition is a theorem of less significance. The primary role of a lemma is to support the proof of a theorem or a proposition. A *corollary* is a consequence of a theorem or a proposition. A *fact* is either a theorem, proposition, lemma, or corollary.

[97]In order to visualize logic operations on predicates, it is helpful to replace statements with sets and logic operations by set operations; the truth of a statement can then be visualized in terms of Venn diagrams. [98]To do this, let $\mathcal{X}$ be a set, for all $x \in \mathcal{X}$, let $A(x)$ and $B(x)$ be predicates, and define $\mathcal{A} \triangleq \{x \in \mathcal{X} \colon \text{truth}[A(x)] = 1\}$ and $\mathcal{B} \triangleq \{x \in \mathcal{X} \colon \text{truth}[B(x)] = 1\}$. Then, the logic operations "and," "or," "xor," and "not" are equivalent to "$\cap$," "$\cup$," "$\ominus$," and "$\sim$," respectively. [99]For example, $\{x \in \mathcal{X} \colon \text{truth}[(\text{not } A(x)) \text{ and } B(x)] = 1\} = \mathcal{A}^\sim \cap \mathcal{B}$. Furthermore, since $[A(x) \implies B(x)]$ is equivalent to $[(\text{not } A(x)) \text{ or } B(x)]$, it follows that $\{x \in \mathcal{X} \colon \text{truth}[A(x) \implies B(x)] = 1\} = \mathcal{A}^\sim \cup \mathcal{B}$. Similarly, since $[A(x) \iff B(x)]$ is equivalent to $[(A(x) \text{ or not } B(x)) \text{ and } ([\text{not } A(x)] \text{ or } B(x))]$, it follows that $\{x \in \mathcal{X} \colon A(x) \iff B(x)\} = (\mathcal{A} \cup \mathcal{B}^\sim) \cap (\mathcal{A}^\sim \cup \mathcal{B}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cup \mathcal{B})^\sim$.

[100]Now, define $\mathcal{X}$, $A(x)$, $B(x)$, $\mathcal{A}$, and $\mathcal{B}$ as in the previous paragraph, and assume that, for all $x \in \mathcal{X}$, $A(x) \implies B(x)$. Therefore, $\mathcal{A}^\sim \cup \mathcal{B} = \{x \in \mathcal{X} \colon \text{truth}[(\text{not } A(x)) \text{ or } B(x)] = 1\} = \mathcal{X}$, and thus $\mathcal{A} \backslash \mathcal{B} = (\mathcal{A}^\sim \cup \mathcal{B})^\sim = \{x \in \mathcal{X} \colon \text{truth}[(\text{not } A(x)) \text{ or } B(x)] = 0\} = \varnothing$. Consequently, $\mathcal{A} \subseteq \mathcal{B}$. [101][102]This means that the logic operator "$\implies$" is represented by "$\subseteq$." [103] For example, for all $x \in \mathcal{X}$, let $C(x)$ be a predicate, and define $\mathcal{C} \triangleq \{x \in \mathcal{X} \colon \text{truth}[C(x)] = 1\}$. Then, for all $x \in \mathcal{X}$, $\text{truth}[(A(x) \text{ and } B(x)) \implies C(x)] = 1$ if and only if $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{C}$. [104]Likewise, for all $x \in \mathcal{X}$,

$$\text{truth}([A(x) \text{ and } (B(x) \text{ or } C(x))] \iff [(A(x) \text{ and } B(x)) \text{ or } (A(x) \text{ and } C(x))]) = 1 \quad (1.29)$$

if and only if

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C}). \quad (1.30)$$

Note that (1.30) represents a tautology. [105i]

[106]

1.3 Relations and Orderings