Chapter 1

Sets, Logic, Numbers, Relations, Orderings, Graphs, and Functions

In this chapter we review basic terminology and results concerning sets, logic, numbers, relations, orderings, graphs, and functions. This material is used throughout the book.

1.1 Sets

A *set* $\{x, y, \ldots\}$ is a collection of elements. A set can include either a finite or infinite number of elements. The set $\mathcal{X}$ is *finite* if it has a finite number of elements; otherwise, $\mathcal{X}$ is *infinite*. The set $\mathcal{X}$ is *countably infinite* if $\mathcal{X}$ is infinite and its elements are in one-to-one correspondence with the positive integers. The set $\mathcal{X}$ is *countable* if it is either finite or countably infinite.

Let $\mathcal{X}$ be a set. Then,

$$x \in \mathcal{X} \tag{1.1}$$

means that x is an *element* of $\mathcal{X}$. If w is not an element of $\mathcal{X}$, then we write

$$w \notin \mathcal{X}. \tag{1.2}$$

No set can be an element of itself. Therefore, there does not exist a set that includes every set. The set with no elements, denoted by $\varnothing$, is the *empty set*. If $\mathcal{X} \neq \varnothing$, then $\mathcal{X}$ is *nonempty*.

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. The *intersection* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of common elements of $\mathcal{X}$ and $\mathcal{Y}$, which is given by

$$\mathcal{X} \cap \mathcal{Y} \triangleq \{x\colon\ x \in \mathcal{X} \text{ and } x \in \mathcal{Y}\} = \{x \in \mathcal{X}\colon\ x \in \mathcal{Y}\} = \{x \in \mathcal{Y}\colon\ x \in \mathcal{X}\} = \mathcal{Y} \cap \mathcal{X}, \tag{1.3}$$

The *union* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of elements in either $\mathcal{X}$ or $\mathcal{Y}$, which is the set

$$\mathcal{X} \cup \mathcal{Y} \triangleq \{x\colon\ x \in \mathcal{X} \text{ or } x \in \mathcal{Y}\} = \mathcal{Y} \cup \mathcal{X}. \tag{1.4}$$

The *complement* of $\mathcal{X}$ *relative* to $\mathcal{Y}$ is

$$\mathcal{Y}\backslash\mathcal{X} \triangleq \{x \in \mathcal{Y}\colon\ x \notin \mathcal{X}\}. \tag{1.5}$$

If $\mathcal{Y}$ is specified, then the *complement* of $\mathcal{X}$ is

$$\mathcal{X}^{\sim} \triangleq \mathcal{Y}\backslash\mathcal{X}. \tag{1.6}$$

The *symmetric difference* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of elements that are in either $\mathcal{X}$ or $\mathcal{Y}$ but not both, which is given by

$$\mathcal{X} \ominus \mathcal{Y} \triangleq (\mathcal{X} \cup \mathcal{Y})\backslash(\mathcal{X} \cap \mathcal{Y}). \tag{1.7}$$

If $x \in \mathcal{X}$ implies that $x \in \mathcal{Y}$, then $\mathcal{X}$ is a *subset* of $\mathcal{Y}$ (equivalently, $\mathcal{Y}$ *contains* $\mathcal{X}$), which is written as

$$\mathcal{X} \subseteq \mathcal{Y}. \tag{1.8}$$

Equivalently,

$$\mathcal{Y} \supseteq \mathcal{X}. \tag{1.9}$$

Note that $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $\mathcal{X}\backslash\mathcal{Y} = \varnothing$. Furthermore, $\mathcal{X} = \mathcal{Y}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} \neq \mathcal{Y}$, then $\mathcal{X}$ is a *proper subset* of $\mathcal{Y}$ and we write $\mathcal{X} \subset \mathcal{Y}$. The sets $\mathcal{X}$ and $\mathcal{Y}$ are *disjoint* if $\mathcal{X} \cap \mathcal{Y} = \varnothing$. A *partition* of $\mathcal{X}$ is a set of pairwise-disjoint and nonempty subsets of $\mathcal{X}$ whose union is equal to $\mathcal{X}$.

The symbols $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of nonnegative integers, positive integers, integers, rational numbers, and real numbers, respectively.

A set cannot have repeated elements. Therefore, $\{x, x\} = \{x\}$. A *multiset* is a finite collection of elements that allows for repetition. The multiset consisting of two copies of x is written as $\{x, x\}_{\mathrm{ms}}$. For example, the roots of the polynomial $p(x) = (x-1)^2$ are the elements of the multiset $\{1, 1\}_{\mathrm{ms}}$, while the prime factors of 72 are the elements of the multiset $\{2, 2, 2, 3, 3\}_{\mathrm{ms}}$.

The operations "$\cap$," "$\cup$," "$\backslash$," "$\ominus$," and "$\times$" and the relations "$\subset$" and "$\subseteq$" extend to multisets. For example,

$$\{x, x\}_{\mathrm{ms}} \cup \{x\}_{\mathrm{ms}} = \{x, x, x\}_{\mathrm{ms}}. \tag{1.10}$$

By ignoring repetitions, a multiset can be converted to a set, while a set can be viewed as a multiset with distinct elements.

The *Cartesian product* $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ of sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is the set consisting of *tuples* of the form $(x_1, \ldots, x_n)$, where, for all $i \in \{1, \ldots, n\}$, $x_i \in \mathcal{X}_i$. A tuple with n components is an *n-tuple*. The components of a tuple are ordered but need not be distinct. Therefore, a tuple can be viewed as an ordered multiset. We thus write

$$(x_1, \ldots, x_n) \in \textstyle\bigtimes_{i=1}^{n}\mathcal{X}_i \triangleq \mathcal{X}_1 \times \cdots \times \mathcal{X}_n. \tag{1.11}$$

$\mathcal{X}^n$ denotes $\bigtimes_{i=1}^{n} \mathcal{X}$.

Definition 1.1.1. A *sequence* $(x_i)_{i=1}^{\infty} = (x_1, x_2, \ldots)$ is a tuple with a countably infinite number of components. Now, let $i_1 < i_2 < \cdots$. Then, $(x_{i_j})_{j=1}^{\infty}$ is a *subsequence* of $(x_i)_{i=1}^{\infty}$.

Let $\mathcal{X}$ be a set, and let $X \triangleq (x_i)_{i=1}^{\infty}$ be a sequence whose components are elements of $\mathcal{X}$; that is, $\{x_1, x_2, \ldots\} \subseteq \mathcal{X}$. For convenience, we write either $X \subseteq \mathcal{X}$ or $X \subset \mathcal{X}$, where X is viewed as a set and the multiplicity of the components of the sequence is ignored. For sequences $X, Y \subset \mathbb{F}^n$, define $X + Y \triangleq (x_i + y_i)_{i=1}^{\infty}$ and $X \odot Y \triangleq (x_i \odot y_i)_{i=1}^{\infty}$, where "$\odot$" denotes component-wise multiplication. In the case $n = 1$, we define $XY \triangleq (x_i y_i)_{i=1}^{\infty}$.