Matrix Book

Dennis Bernstein

February 2021

1

Chapter 1

Sets, Logic, Numbers, Relations, Orderings, Graphs, and Functions

[2i]

In this chapter we review basic terminology and results concerning sets, logic, numbers, relations, orderings, graphs, and functions. This material is used throughout the book.

[3]

1.1 Sets

[4]A *set* $\{x, y, \ldots\}$ is a collection of elements. A set can include either a finite or infinite number of elements. The set $\mathcal{X}$ is *finite* if it has a finite number of elements; otherwise, $\mathcal{X}$ is *infinite*. [5]The set $\mathcal{X}$ is *countably infinite* if $\mathcal{X}$ is infinite and its elements are in one-to-one correspondence with the positive integers. The set $\mathcal{X}$ is *countable* if it is either finite or countably infinite.

[6]Let $\mathcal{X}$ be a set. Then,

$$x \in \mathcal{X} \tag{1.1}$$

means that x is an *element* of $\mathcal{X}$. If w is not an element of $\mathcal{X}$, then we write

$$w \notin \mathcal{X}. \tag{1.2}$$

[7]No set can be an element of itself. Therefore, there does not exist a set that includes every set. The set with no elements, denoted by $\varnothing$, is the *empty set*. If $\mathcal{X} \neq \varnothing$, then $\mathcal{X}$ is *nonempty*.

[8]Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. The *intersection* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of common elements of $\mathcal{X}$ and $\mathcal{Y}$, which is given by

$$\mathcal{X} \cap \mathcal{Y} \triangleq \{x\colon\ x \in \mathcal{X} \text{ and } x \in \mathcal{Y}\} = \{x \in \mathcal{X}\colon\ x \in \mathcal{Y}\} = \{x \in \mathcal{Y}\colon\ x \in \mathcal{X}\} = \mathcal{Y} \cap \mathcal{X}, \tag{1.3}$$

[9]The *union* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of elements in either $\mathcal{X}$ or $\mathcal{Y}$, which is the set

$$\mathcal{X} \cup \mathcal{Y} \triangleq \{x\colon\ x \in \mathcal{X} \text{ or } x \in \mathcal{Y}\} = \mathcal{Y} \cup \mathcal{X}. \tag{1.4}$$

[10]The *complement* of $\mathcal{X}$ *relative* to $\mathcal{Y}$ is

$$\mathcal{Y}\backslash\mathcal{X} \triangleq \{x \in \mathcal{Y}\colon\ x \notin \mathcal{X}\}. \tag{1.5}$$

[11]If $\mathcal{Y}$ is specified, then the *complement* of $\mathcal{X}$ is

$$\mathcal{X}^{\sim} \triangleq \mathcal{Y}\backslash\mathcal{X}. \tag{1.6}$$

[12]The *symmetric difference* of $\mathcal{X}$ and $\mathcal{Y}$ is the set of elements that are in either $\mathcal{X}$ or $\mathcal{Y}$ but not both, which is given by

$$\mathcal{X} \ominus \mathcal{Y} \triangleq (\mathcal{X} \cup \mathcal{Y})\backslash(\mathcal{X} \cap \mathcal{Y}). \tag{1.7}$$

[13]If $x \in \mathcal{X}$ implies that $x \in \mathcal{Y}$, then $\mathcal{X}$ is a *subset* of $\mathcal{Y}$ (equivalently, $\mathcal{Y}$ *contains* $\mathcal{X}$), which is written as

$$\mathcal{X} \subseteq \mathcal{Y}. \tag{1.8}$$

[14]Equivalently,

$$\mathcal{Y} \supseteq \mathcal{X}. \tag{1.9}$$

[15]Note that $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $\mathcal{X}\backslash\mathcal{Y} = \varnothing$. [16]Furthermore, $\mathcal{X} = \mathcal{Y}$ if and only if $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{X} \neq \mathcal{Y}$, then $\mathcal{X}$ is a *proper subset* of $\mathcal{Y}$ and we write $\mathcal{X} \subset \mathcal{Y}$. [17]The sets $\mathcal{X}$ and $\mathcal{Y}$ are *disjoint* if $\mathcal{X} \cap \mathcal{Y} = \varnothing$. [18]A *partition* of $\mathcal{X}$ is a set of pairwise-disjoint and nonempty subsets of $\mathcal{X}$ whose union is equal to $\mathcal{X}$.

[19]The symbols $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of nonnegative integers, positive integers, integers, rational numbers, and real numbers, respectively.

[20]A set cannot have repeated elements. Therefore, $\{x, x\} = \{x\}$. [21]A *multiset* is a finite collection of elements that allows for repetition. The multiset consisting of two copies of x is written as $\{x, x\}_{\mathrm{ms}}$. [22]For example, the roots of the polynomial $p(x) = (x-1)^2$ are the elements of the multiset $\{1, 1\}_{\mathrm{ms}}$, while the prime factors of 72 are the elements of the multiset $\{2, 2, 2, 3, 3\}_{\mathrm{ms}}$.

[23]The operations "$\cap$," "$\cup$," "$\backslash$," "$\ominus$," and "$\times$" and the relations "$\subset$" and "$\subseteq$" extend to multisets. [24]For example,

$$\{x, x\}_{\mathrm{ms}} \cup \{x\}_{\mathrm{ms}} = \{x, x, x\}_{\mathrm{ms}}. \tag{1.10}$$

[25]By ignoring repetitions, a multiset can be converted to a set, while a set can be viewed as a multiset with distinct elements.

[26]The *Cartesian product* $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ of sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is the set consisting of *tuples* of the form $(x_1, \ldots, x_n)$, where, for all $i \in \{1, \ldots, n\}$, $x_i \in \mathcal{X}_i$. A tuple with n components is an *n-tuple*. [27]The components of a tuple are ordered but need not be distinct. Therefore, a tuple can be viewed as an ordered multiset. [28]We thus write

$$(x_1, \ldots, x_n) \in \bigtimes_{i=1}^{n} \mathcal{X}_i \triangleq \mathcal{X}_1 \times \cdots \times \mathcal{X}_n. \tag{1.11}$$

$\mathcal{X}^n$ denotes $\bigtimes_{i=1}^{n} \mathcal{X}$.

[29]

Definition 1.1.1. A *sequence* $(x_i)_{i=1}^{\infty} = (x_1, x_2, \ldots)$ is a tuple with a countably infinite number of components. Now, let $i_1 < i_2 < \cdots$. Then, $(x_{i_j})_{j=1}^{\infty}$ is a *subsequence* of $(x_i)_{i=1}^{\infty}$.

[30]Let $\mathcal{X}$ be a set, and let $X \triangleq (x_i)_{i=1}^{\infty}$ be a sequence whose components are elements of $\mathcal{X}$; that is, $\{x_1, x_2, \ldots\} \subseteq \mathcal{X}$. For convenience, we write either $X \subseteq \mathcal{X}$ or $X \subset \mathcal{X}$, where X is viewed as a set and the multiplicity of the components of the sequence is ignored. [31]For sequences $X, Y \subset \mathbb{F}^n$, define $X + Y \triangleq (x_i + y_i)_{i=1}^{\infty}$ and $X \odot Y \triangleq (x_i \odot y_i)_{i=1}^{\infty}$, where "$\odot$" denotes component-wise multiplication. In the case $n = 1$, we define $XY \triangleq (x_i y_i)_{i=1}^{\infty}$.

[32]

[33]

1.2 Logic

[34]Every *statement* is either true or false, and no statement is both true and false. [35][36]A *proof* is a collection of statements that verify that a statement is true. [37]A *conjecture* is a statement that is believed to be true but whose proof is not known.

[38]Let A and B be statements. [39]The *not* of A is the statement (not A), the *and* of A and B is the statement (A and B), and the *or* of A and B is the statement (A or B). [40] The statement (A or B) does not contradict the statement (A and B); hence, the word "or" is inclusive. [41]The *exclusive or* of A and B is the statement (A xor B), which is [(A and not B) or (B and not A)]. [42]Equivalently, (A xor B) is the statement [(A or B) and not(A and B)], that is, A or B, but not both. [43]Note that (A and B) = (B and A), (A or B) = (B or A), and (A xor B) = (B xor A).

Let A, B, and C be statements. Then, the statements (A and B or C) and (A or B and C) are ambiguous. For clarity, we thus write, for example, [A and (B or C)] and [A or (B and C)]. In words, we write "A and either B or C" and "A or both B and C," respectively, where "either" and "both" signify parentheses. Furthermore,

$$(A \text{ and } B) \text{ or } C = (A \text{ and } C) \text{ or } (B \text{ and } C), \tag{1.12}$$
$$(A \text{ or } B) \text{ and } C = (A \text{ or } C) \text{ and } (B \text{ or } C). \tag{1.13}$$

Let A be a statement. To analyze statements involving logic operators, define $\text{truth}(A) = 1$ if A is true, and $\text{truth}(A) = 0$ if A is false. Then,

$$\text{truth}(\text{not } A) = \text{truth}(A) + 1, \tag{1.14}$$

where $0+0 = 0$, $1+0 = 0+1 = 1$, and $1+1 = 0$. Therefore, A is true if and only if (not A) is false, while A is false if and only if (not A) is true. Note that

$$\begin{aligned} \text{truth}[\text{not}(\text{not } A)] &= \text{truth}(\text{not } A) + 1 \\ &= [\text{truth}(A) + 1] + 1 \\ &= \text{truth}(A). \end{aligned}$$

Furthermore, note that $\text{truth}(A) + \text{truth}(A) = 0$ and $\text{truth}(A)\,\text{truth}(A) = \text{truth}(A)$.

Let A and B be statements. Then,

$$\text{truth}(A \text{ and } B) = \text{truth}(A)\,\text{truth}(B), \tag{1.15}$$
$$\text{truth}(A \text{ or } B) = \text{truth}(A)\,\text{truth}(B) + \text{truth}(A) + \text{truth}(B), \tag{1.16}$$

$$\text{truth}(A \text{ xor } B) = \text{truth}(A) + \text{truth}(B). \tag{1.17}$$

Hence,

$$\text{truth}(A \text{ and } B) = \min\{\text{truth}(A), \text{truth}(B)\}, \tag{1.18}$$

$$\text{truth}(A \text{ or } B) = \max\{\text{truth}(A), \text{truth}(B)\}. \tag{1.19}$$

Consequently, $\text{truth}(A \text{ and } B) = \text{truth}(B \text{ and } A)$, $\text{truth}(A \text{ or } B) = \text{truth}(B \text{ or } A)$, and $\text{truth}(A \text{ xor } B) = \text{truth}(B \text{ xor } A)$. Furthermore, $\text{truth}(A \text{ and } A) = \text{truth}(A \text{ or } A) = \text{truth}(A)$, and $\text{truth}(A \text{ xor } A) = 0$.

Let A and B be statements. The *implication* $(A \Longrightarrow B)$ is the statement $[(\text{not } A) \text{ or } B]$. Therefore,

$$\text{truth}(A \Longrightarrow B) = \text{truth}(A)\,\text{truth}(B) + \text{truth}(A) + 1. \tag{1.20}$$

The implication $(A \Longrightarrow B)$ is read as either "if A, then B," "if A holds, then B holds," or "A implies B." The statement A is the *hypothesis*, while the statement B is the *conclusion*. If $(A \Longrightarrow B)$, then A is a *sufficient condition* for B, and B is a *necessary condition* for A. It follows from (1.20) that, if A and B are true, then $(A \Longrightarrow B)$ is true; if A is true and B is false, then $(A \Longrightarrow B)$ is false; and, if A is false, then $(A \Longrightarrow B)$ is true whether or not B is true. For example, both implications $[(2 + 2 = 5) \Longrightarrow (3 + 3 = 6)]$ and $[(2+2=5) \Longrightarrow (3+3=8)]$ are true. Finally, note that $[(A \Longrightarrow B) \text{ and } A] = A \text{ and } B$.

A *predicate* is a statement that depends on a variable. Let $\mathcal{X}$ be a set, let $x \in \mathcal{X}$, and let $A(x)$ be a predicate. There are two ways to use a predicate to create a statement. An *existential statement* has the form

$$\text{there exists } x \in \mathcal{X} \text{ such that } A(x) \text{ holds}, \tag{1.21}$$

whereas a *universal statement* has the form

$$\text{for all } x \in \mathcal{X}, A(x) \text{ holds}. \tag{1.22}$$

Note that

$$\text{truth}[\text{there exists } x \in \mathcal{X} \text{ such that } A(x) \text{ holds}] = \max_{x \in \mathcal{X}} \text{truth}[A(x)], \tag{1.23}$$

$$\text{truth}[\text{for all } x \in \mathcal{X}, A(x) \text{ holds}] = \min_{x \in \mathcal{X}} \text{truth}[A(x)]. \tag{1.24}$$

An *argument* is an implication whose hypothesis and conclusion are predicates that depend on the same variable. In particular, letting x denote a variable, and letting $A(x)$ and $B(x)$ be predicates, the implication $[A(x) \Longrightarrow B(x)]$ is an argument. For example, for each real number x, the implication $[(x = 1) \Longrightarrow (x + 1 = 2)]$ is an argument. Note that the variable x links the hypothesis and the conclusion, thereby making this implication useful for the purpose of *inference*. In particular, for all real numbers x, $\text{truth}[(x = 1) \Longrightarrow (x + 1 = 2)] = 1$. The statements (for all $x, [A(x) \Longrightarrow B(x)]$ holds) and (there exists x such that $[A(x) \Longrightarrow B(x)]$ holds) are inferences.

Let A and B be statements. The *bidirectional implication* $(A \Longleftrightarrow B)$ is the statement $[(A \Longrightarrow B) \text{ and } (A \Longleftarrow B)]$, where $(A \Longleftarrow B)$ means $(B \Longrightarrow A)$. If $(A \Longleftrightarrow B)$, then A and B are *equivalent*. Furthermore,

$$\text{truth}(A \Longleftrightarrow B) = \text{truth}(A) + \text{truth}(B) + 1. \tag{1.25}$$

Therefore, A and B are equivalent if and only if either both A and B are true or both A and B are false.

Let A and B be statements, and assume that $(A \Longleftrightarrow B)$. Then, A holds *if and only if* B holds. The implication $A \Longrightarrow B$ (the "only if" part) is *necessity*, while $B \Longrightarrow A$ (the "if" part) is *sufficiency*.

Let A and B be statements. The *converse* of $(A \Longrightarrow B)$ is $(B \Longrightarrow A)$. Note that

$$\begin{aligned}(A \Longrightarrow B) &\Longleftrightarrow [(\text{not } A) \text{ or } B]\\ &\Longleftrightarrow [(\text{not } A) \text{ or not}(\text{not } B)]\\ &\Longleftrightarrow [\text{not}(\text{not } B) \text{ or not } A]\\ &\Longleftrightarrow (\text{not } B \Longrightarrow \text{not } A).\end{aligned}$$

Therefore, the statement $(A \Longrightarrow B)$ is equivalent to its *contrapositive* $[(\text{not } B) \Longrightarrow (\text{not } A)]$.

Let A, B, A', and B' be statements, and assume that $(A' \Longrightarrow A \Longrightarrow B \Longrightarrow B')$. Then, $(A' \Longrightarrow B')$ is a *corollary* of $(A \Longrightarrow B)$.

Let A, B, and A' be statements, and assume that $A \Longrightarrow B$. Then, $(A \Longrightarrow B)$ is a *strengthening* of $[(A \text{ and } A') \Longrightarrow B]$. If, in addition, $(A \Longrightarrow A')$, then the statement $[(A \text{ and } A') \Longrightarrow B]$ has a *redundant assumption*.

An *interpretation* is a feasible assignment of true or false to all statements that comprise a statement. For example, there are four interpretations of the statement $(A \text{ and } B)$, depending on whether A is assigned to be true or false and B is assigned to be true or false. Likewise, $[(x = 1) \text{ and } (x = 2)]$ has three interpretations, which depend on the value of x.

Let $A_1, A_2, \ldots$ be statements, and let B be a statement that depends on $A_1, A_2, \ldots$ Then, B is a *tautology* if B is true whether or not $A_1, A_2, \ldots$ are true. For example, let B denote the statement $(A \text{ or not } A)$. Then,

$$\text{truth}(A \text{ or not } A) = 1, \tag{1.26}$$

and thus the statement $(A \text{ or not } A)$ is true whether or not A is true. Hence, $(A \text{ or not } A)$ is a tautology. Likewise, $(A \Longrightarrow A)$ is a tautology. Furthermore, since

$$\text{truth}[(A \text{ and } B) \Longrightarrow A] = \text{truth}(A)^2\, \text{truth}(B) + \text{truth}(A)\, \text{truth}(B) + 1 = 1, \tag{1.27}$$

it follows that $[(A \text{ and } B) \Longrightarrow A]$ is a tautology. Likewise, $\text{truth}([A \text{ and not } A] \Longrightarrow B) = 1$, and thus $([A \text{ and not } A] \Longrightarrow B)$ is a tautology.

Let $A_1, A_2, \ldots$ be statements, and let B be a statement that depends on $A_1, A_2, \ldots$ Then, B is a *contradiction* if B is false whether or not $A_1, A_2, \ldots$ are true. For example, let B denote the statement $(A \text{ and not } A)$. Then,

$$\text{truth}(A \text{ and not } A) = 0, \tag{1.28}$$

and thus the statement $(A \text{ and not } A)$ is false whether or not A is true. Hence, $(A \text{ and not } A)$ is a contradiction.

Let A and B be statements. If the implication $(A \Longrightarrow B)$ is neither a tautology nor a contradiction, then $\text{truth}(A \Longrightarrow B)$ depends on the truth of the statements that comprise A and B. For example, $\text{truth}(A \Longrightarrow \text{not } A) = \text{truth}(A) + 1$, and thus the statement $(A \Longrightarrow \text{not } A)$ is true if and only if A is false, and false if and only if A is true. Hence, $(A \Longrightarrow \text{not } A)$ is neither a tautology nor a contradiction. A statement that is neither a tautology nor a contradiction is a *contingency*. For example, the implication $[A \Longrightarrow (A \text{ and } B)]$ is a contingency. Likewise, for each real number x, $\text{truth}[(x=1) \Longrightarrow (x=2)] = \text{truth}(x \neq 1)$, and thus the statement $[(x=1) \Longrightarrow (x=2)]$ is a contingency.

An argument that is a contingency is a *theorem*, *proposition*, *corollary*, or *lemma*. A theorem is a significant result; a proposition is a theorem of less significance. The primary role of a lemma is to support the proof of a theorem or a proposition. A *corollary* is a consequence of a theorem or a proposition. A *fact* is either a theorem, proposition, lemma, or corollary.

In order to visualize logic operations on predicates, it is helpful to replace statements with sets and logic operations by set operations; the truth of a statement can then be visualized in terms of Venn diagrams. To do this, let $\mathcal{X}$ be a set, for all $x \in \mathcal{X}$, let $A(x)$ and $B(x)$ be predicates, and define $\mathcal{A} \triangleq \{x \in \mathcal{X}\colon \text{truth}[A(x)] = 1\}$ and $\mathcal{B} \triangleq \{x \in \mathcal{X}\colon \text{truth}[B(x)] = 1\}$. Then, the logic operations "and," "or," "xor," and "not" are equivalent to "$\cap$," "$\cup$," "$\ominus$," and "$\sim$," respectively. For example, $\{x \in \mathcal{X}\colon \text{truth}[(\text{not } A(x)) \text{ and } B(x)] = 1\} = \mathcal{A}^\sim \cap \mathcal{B}$. Furthermore, since $[A(x) \Longrightarrow B(x)]$ is equivalent to $[(\text{not } A(x)) \text{ or } B(x)]$, it follows that $\{x \in \mathcal{X}\colon \text{truth}[A(x) \Longrightarrow B(x)] = 1\} = \mathcal{A}^\sim \cup \mathcal{B}$. Similarly, since $[A(x) \Longleftrightarrow B(x)]$ is equivalent to $[(A(x) \text{ or not } B(x)) \text{ and } ([\text{not } A(x)] \text{ or } B(x))]$, it follows that $\{x \in \mathcal{X}\colon A(x) \Longleftrightarrow B(x)\} = (\mathcal{A} \cup \mathcal{B}^\sim) \cap (\mathcal{A}^\sim \cup \mathcal{B}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cup \mathcal{B})^\sim$.

Now, define $\mathcal{X}$, $A(x)$, $B(x)$, $\mathcal{A}$, and $\mathcal{B}$ as in the previous paragraph, and assume that, for all $x \in \mathcal{X}$, $A(x) \Longrightarrow B(x)$. Therefore, $\mathcal{A}^\sim \cup \mathcal{B} = \{x \in \mathcal{X}\colon \text{truth}[(\text{not } A(x)) \text{ or } B(x)] = 1\} = \mathcal{X}$, and thus $\mathcal{A} \backslash \mathcal{B} = (\mathcal{A}^\sim \cup \mathcal{B})^\sim = \{x \in \mathcal{X}\colon \text{truth}[(\text{not } A(x)) \text{ or } B(x)] = 0\} = \varnothing$. Consequently, $\mathcal{A} \subseteq \mathcal{B}$. This means that the logic operator "$\Longrightarrow$" is represented by "$\subseteq$." For example, for all $x \in \mathcal{X}$, let $C(x)$ be a predicate, and define $\mathcal{C} \triangleq \{x \in \mathcal{X}\colon \text{truth}[C(x)] = 1\}$. Then, for all $x \in \mathcal{X}$, $\text{truth}[(A(x) \text{ and } B(x)) \Longrightarrow C(x)] = 1$ if and only if $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{C}$. Likewise, for all $x \in \mathcal{X}$,

$$\text{truth}([A(x) \text{ and } (B(x) \text{ or } C(x))] \Longleftrightarrow [(A(x) \text{ and } B(x)) \text{ or } (A(x) \text{ and } C(x))]) = 1 \tag{1.29}$$

if and only if

$$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C}). \tag{1.30}$$

Note that (1.30) represents a tautology.

1.3 Relations and Orderings

Let $\mathcal{X}$, $\mathcal{X}_1$, and $\mathcal{X}_2$ be sets. A *relation* $\mathcal{R}$ on $(\mathcal{X}_1, \mathcal{X}_2)$ is a subset of $\mathcal{X}_1 \times \mathcal{X}_2$. A *relation* $\mathcal{R}$ on $\mathcal{X}$ is a subset of $\mathcal{X} \times \mathcal{X}$. Likewise, a *multirelation* $\mathcal{R}$ on $(\mathcal{X}_1, \mathcal{X}_2)$ is a multisubset of $\mathcal{X}_1 \times \mathcal{X}_2$, while a *multirelation* $\mathcal{R}$ on $\mathcal{X}$ is a multisubset of $\mathcal{X} \times \mathcal{X}$.

Let $\mathcal{X}$ be a set, and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be relations on $\mathcal{X}$. Then, the sets $\mathcal{R}_1 \cap \mathcal{R}_2$, $\mathcal{R}_1 \backslash \mathcal{R}_2$, and $\mathcal{R}_1 \cup \mathcal{R}_2$ are relations on $\mathcal{X}$. Furthermore, if $\mathcal{R}$ is a relation on $\mathcal{X}$ and $\mathcal{X}_0 \subseteq \mathcal{X}$, then we define the *restricted relation* $\mathcal{R}|_{\mathcal{X}_0} \triangleq \mathcal{R} \cap (\mathcal{X}_0 \times \mathcal{X}_0)$, which is a relation on $\mathcal{X}_0$.

Definition 1.3.1. Let $\mathcal{R}$ be a relation on the set $\mathcal{X}$. Then, the following terminology is defined:

i) $\mathcal{R}$ is *reflexive* if, for all $x \in \mathcal{X}$, it follows that $(x, x) \in \mathcal{R}$.

ii) $\mathcal{R}$ is *symmetric* if, for all $(x_1, x_2) \in \mathcal{R}$, it follows that $(x_2, x_1) \in \mathcal{R}$.

iii) $\mathcal{R}$ is *transitive* if, for all $(x_1, x_2) \in \mathcal{R}$ and $(x_2, x_3) \in \mathcal{R}$, it follows that $(x_1, x_3) \in \mathcal{R}$.

iv) $\mathcal{R}$ is an *equivalence relation* if $\mathcal{R}$ is reflexive, symmetric, and transitive.

Proposition 1.3.2. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be relations on the set $\mathcal{X}$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are (reflexive, symmetric) relations, then so are $\mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_1 \cup \mathcal{R}_2$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are (transitive, equivalence) relations, then so is $\mathcal{R}_1 \cap \mathcal{R}_2$.

Definition 1.3.3. Let $\mathcal{R}$ be a relation on the set $\mathcal{X}$. Then, the following terminology is defined:

i) The *complement* $\mathcal{R}^{\sim}$ of $\mathcal{R}$ is the relation $\mathcal{R}^{\sim} \triangleq (\mathcal{X} \times \mathcal{X})\backslash\mathcal{R}$.

ii) The *support* $\operatorname{supp}(\mathcal{R})$ of $\mathcal{R}$ is the smallest subset $\mathcal{X}_0$ of $\mathcal{X}$ such that $\mathcal{R}$ is a relation on $\mathcal{X}_0$.

iii) The *reversal* $\operatorname{rev}(\mathcal{R})$ of $\mathcal{R}$ is the relation $\operatorname{rev}(\mathcal{R}) \triangleq \{(y,x)\colon (x,y) \in \mathcal{R}\}$.

iv) The *shortcut* $\operatorname{shortcut}(\mathcal{R})$ of $\mathcal{R}$ is the relation $\operatorname{shortcut}(\mathcal{R}) \triangleq \{(x,y) \in \mathcal{X}\times\mathcal{X}\colon$ x and y are distinct and there exist $k \geq 1$ and $x_1,\ldots,x_k \in \mathcal{X}$ such that $(x,x_1),(x_1,x_2),\ldots,(x_k,y) \in \mathcal{R}\}$.

v) The *reflexive hull* $\operatorname{ref}(\mathcal{R})$ of $\mathcal{R}$ is the smallest reflexive relation on $\mathcal{X}$ that contains $\mathcal{R}$.

vi) The *symmetric hull* $\operatorname{sym}(\mathcal{R})$ of $\mathcal{R}$ is the smallest symmetric relation on $\mathcal{X}$ that contains $\mathcal{R}$.

vii) The *transitive hull* $\operatorname{trans}(\mathcal{R})$ of $\mathcal{R}$ is the smallest transitive relation on $\mathcal{X}$ that contains $\mathcal{R}$.

viii) The *equivalence hull* $\operatorname{equiv}(\mathcal{R})$ of $\mathcal{R}$ is the smallest equivalence relation on $\mathcal{X}$ that contains $\mathcal{R}$.

Proposition 1.3.4. Let $\mathcal{R}$ be a relation on the set $\mathcal{X}$. Then, the following statements hold:

i) $\operatorname{ref}(\mathcal{R}) = \mathcal{R} \cup \{(x,x)\colon x \in \mathcal{X}\}$.

ii) $\operatorname{sym}(\mathcal{R}) = \mathcal{R} \cup \operatorname{rev}(\mathcal{R})$.

iii) $\operatorname{trans}(\mathcal{R}) = \mathcal{R} \cup \operatorname{shortcut}(\mathcal{R})$.

iv) If $\mathcal{R}$ is symmetric, then $\operatorname{trans}(\mathcal{R}) = \operatorname{sym}(\operatorname{trans}(\mathcal{R}))$.

v) $\operatorname{equiv}(\mathcal{R}) = \operatorname{trans}(\operatorname{sym}(\operatorname{ref}(\mathcal{R})))$.

Furthermore, the following statements hold:

vi) $\mathcal{R}$ is reflexive if and only if $\mathcal{R} = \operatorname{ref}(\mathcal{R})$.

vii) The following statements are equivalent:

a) $\mathcal{R}$ is symmetric.

b) $\mathcal{R} = \operatorname{sym}(\mathcal{R})$.

c) $\mathcal{R} = \operatorname{rev}(\mathcal{R})$.

viii) $\mathcal{R}$ is transitive if and only if $\mathcal{R} = \operatorname{trans}(\mathcal{R})$.

ix) $\mathcal{R}$ is an equivalence relation if and only if $\mathcal{R} = \operatorname{equiv}(\mathcal{R})$.

For an equivalence relation $\mathcal{R}$ on the set $\mathcal{X}$, $(x_1,x_2) \in \mathcal{R}$ is denoted by $x_1 \equiv x_2$. If $\mathcal{R}$ is an equivalence relation and $x \in \mathcal{X}$, then the subset $\mathcal{E}_x \triangleq \{y \in \mathcal{X}\colon\; y \equiv x\}$ of $\mathcal{X}$ is the *equivalence class of x induced by* $\mathcal{R}$.

Theorem 1.3.5. Let $\mathcal{R}$ be an equivalence relation on a set $\mathcal{X}$. Then, the set $\{\mathcal{E}_x\colon x \in \mathcal{X}\}$ of equivalence classes induced by $\mathcal{R}$ is a partition of $\mathcal{X}$.

Proof. Since $\mathcal{X} = \bigcup_{x\in\mathcal{X}} \mathcal{E}_x$, it suffices to show that, if $x,y \in \mathcal{X}$, then either $\mathcal{E}_x = \mathcal{E}_y$ or $\mathcal{E}_x \cap \mathcal{E}_y = \varnothing$. Hence, let $x,y \in \mathcal{X}$, and suppose that $\mathcal{E}_x$ and $\mathcal{E}_y$ are not disjoint so that there exists $z \in \mathcal{E}_x \cap \mathcal{E}_y$. Thus, $(x,z) \in \mathcal{R}$ and $(z,y) \in \mathcal{R}$. Now, let $w \in \mathcal{E}_x$. Then, $(w,x) \in \mathcal{R}$, $(x,z) \in \mathcal{R}$, and $(z,y) \in \mathcal{R}$ imply that $(w,y) \in \mathcal{R}$. Hence, $w \in \mathcal{E}_y$, which implies that $\mathcal{E}_x \subseteq \mathcal{E}_y$. By a similar argument, $\mathcal{E}_y \subseteq \mathcal{E}_x$. Consequently, $\mathcal{E}_x = \mathcal{E}_y$. $\square$

The following result, which is the converse of Theorem 1.3.5, shows that a partition of a set $\mathcal{X}$ defines an equivalence relation on $\mathcal{X}$.

Theorem 1.3.6. Let $\mathcal{X}$ be a set, let $\mathcal{P}$ be a partition of $\mathcal{X}$, and define the relation $\mathcal{R}$ on $\mathcal{X}$ by $(x,y) \in \mathcal{R}$ if and only if x and y belong to the same element of $\mathcal{P}$. Then, $\mathcal{R}$ is an equivalence relation on $\mathcal{X}$.

Theorem 1.3.5 shows that every equivalence relation induces a partition, while Theorem 1.3.6 shows that every partition induces an equivalence relation.

Definition 1.3.7. Let $\mathcal{X}$ be a set, let $\mathcal{P}$ be a partition of $\mathcal{X}$, and let $X_0 \subseteq \mathcal{X}$. Then, X_0 is a *representative subset* of $\mathcal{X}$ *relative to* $\mathcal{P}$ if, for all $X \in \mathcal{P}$, exactly one element of X_0 is an element of X.

Definition 1.3.8. Let $\mathcal{R}$ be a relation on the set $\mathcal{X}$. Then, the following terminology is defined:

i) $\mathcal{R}$ is *antisymmetric* if $(x_1,x_2) \in \mathcal{R}$ and $(x_2,x_1) \in \mathcal{R}$ imply that $x_1 = x_2$.

ii) $\mathcal{R}$ is a *partial ordering* if $\mathcal{R}$ is reflexive, antisymmetric, and transitive.

iii) $(\mathcal{X},\mathcal{R})$ is a *partially ordered set* if $\mathcal{R}$ is a partial ordering.

Let $(\mathcal{X},\mathcal{R})$ be a partially ordered set. Then, $(x_1,x_2) \in \mathcal{R}$ is denoted by $x_1 \preceq x_2$. If $x_1 \preceq x_2$ and $x_2 \preceq x_1$, then, since $\mathcal{R}$ is antisymmetric, it follows that $x_1 = x_2$. Furthermore, if $x_1 \preceq x_2$ and $x_2 \preceq x_3$, then, since $\mathcal{R}$ is transitive, it follows that $x_1 \preceq x_3$.

Definition 1.3.9. Let $(\mathcal{X},\mathcal{R})$ be a partially ordered set. Then, the following terminology is defined:

i) Let $\mathcal{S} \subseteq \mathcal{X}$. Then, $y \in \mathcal{X}$ is a *lower bound* for $\mathcal{S}$ if, for all $x \in \mathcal{S}$, it follows that $y \preceq x$.

ii) Let $\mathcal{S} \subseteq \mathcal{X}$. Then, $y \in \mathcal{X}$ is an *upper bound* for $\mathcal{S}$ if, for all $x \in \mathcal{S}$, it follows that $x \preceq y$.

The following result shows that every partially ordered set has at most one lower bound that is "greatest" and at most one upper bound that is "least."

Lemma 1.3.10. Let $(\mathcal{X},\mathcal{R})$ be a partially ordered set, and let $\mathcal{S} \subseteq \mathcal{X}$. Then, there exists at most one lower bound $y \in \mathcal{X}$ for $\mathcal{S}$ such that every lower bound $x \in \mathcal{X}$ for $\mathcal{S}$ satisfies $x \preceq y$. Furthermore, there exists at most one upper bound $y \in \mathcal{X}$ for $\mathcal{S}$ such that every upper bound $x \in \mathcal{X}$ for $\mathcal{S}$ satisfies $y \preceq x$.

Proof. For $i = 1,2$, let $y_i \in \mathcal{X}$ be such that y_i is a lower bound for $\mathcal{S}$ and, for all $x \in \mathcal{X}$, $x \preceq y_i$. Therefore, $y_1 \le y_2$ and $y_2 \le y_1$. Since "$\preceq$" is antisymmetric, it follows that $y_1 = y_2$. $\square$

Definition 1.3.11. Let $(\mathcal{X},\mathcal{R})$ be a partially ordered set. Then, the following terminology is defined:

i) Let $\mathcal{S} \subseteq \mathcal{X}$. Then, $y \in \mathcal{X}$ is the *greatest lower bound* for $\mathcal{S}$ if y is a lower bound for $\mathcal{S}$ and every lower bound $x \in \mathcal{X}$ for $\mathcal{S}$ satisfies $x \preceq y$. In this case, we write $y = \mathrm{glb}(\mathcal{S})$.

ii) Let $\mathcal{S} \subseteq \mathcal{X}$. Then, $y \in \mathcal{X}$ is the *least upper bound* for $\mathcal{S}$ if y is an upper bound for $\mathcal{S}$ and every upper bound $x \in \mathcal{X}$ for $\mathcal{S}$ satisfies $y \preceq x$. In this case, we write $y = \mathrm{lub}(\mathcal{S})$.

iii) $(\mathcal{X},\preceq)$ is a *lattice* if, for all distinct $x,y \in \mathcal{X}$, the set $\{x,y\}$ has a least upper bound and a greatest lower bound.

iv) $(\mathcal{X},\preceq)$ is a *complete lattice* on $\mathcal{X}$ if every subset $\mathcal{S}$ of $\mathcal{X}$ has a least upper bound and a greatest lower bound.

Example 1.3.12. Consider the partially ordered set $(\mathbb{P}, \preceq)$, where $m \preceq n$ indicates that n is an integer multiple of m. For example, $3 \preceq 21$, but it is not true that $2 \preceq 3$. Next, note that the greatest lower bound of a subset $\mathcal{S}$ of $\mathbb{P}$ is the greatest common divisor of the elements of $\mathcal{S}$. For example, $\operatorname{glb}\{9, 21\} = 3$. Likewise, the least upper bound of a subset $\mathcal{S}$ of $\mathbb{P}$ is the least common multiple of the elements of $\mathcal{S}$. For example, $\operatorname{lub}\{2, 3, 4\} = 12$. Therefore, $(\mathbb{P}, \preceq)$ is a lattice. Next, note that 1 is a lower bound for every subset of $\mathbb{P}$. Since every subset of $\mathbb{P}$ has a smallest element in the usual ordering, it follows that every subset of $\mathbb{P}$ has a greatest lower bound. In particular, $\operatorname{glb}(\mathbb{P}) = 1$. However, no subset of $\mathbb{P}$ that has an infinite number of elements has an upper bound. Therefore, $(\mathbb{P}, \preceq)$ is not a complete lattice. Now, consider $(\mathbb{N}, \preceq)$. Note that 1 is a lower bound for every subset of $\mathbb{N}$. Since every subset of $\mathbb{N}$ has a smallest element in the usual ordering, it follows that every subset of $\mathbb{N}$ has a greatest lower bound. In particular, $\operatorname{glb}(\mathbb{N}) = 1$. Furthermore, for all $m \in \mathbb{N}$, $0 = 0 \cdot m$, and thus 0 is an upper bound for every subset of $\mathbb{N}$. In particular, since 0 is the unique upper bound of $\mathbb{N}$, it follows that 0 is the least upper bound of $\mathbb{N}$. Hence, $(\mathbb{N}, \preceq)$ is a complete lattice. $\diamond$

Proposition 1.3.13. Let $(\mathcal{X}, \preceq)$ be a lattice, and let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{X}$. Then,

$$\operatorname{glb}(\mathcal{S}_1 \cup \mathcal{S}_2) = \operatorname{glb}[\mathcal{S}_1 \cup \{\operatorname{glb}(\mathcal{S}_2)\}], \quad \operatorname{lub}(\mathcal{S}_1 \cup \mathcal{S}_2) = \operatorname{lub}[\mathcal{S}_1 \cup \{\operatorname{lub}(\mathcal{S}_2)\}]. \tag{1.31}$$

Definition 1.3.14. Let $(\mathcal{X}, \mathcal{R})$ be a partially ordered set. Then, $\mathcal{R}$ is a *total ordering* on $\mathcal{X}$ if, for all $x, y \in \mathcal{X}$, either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$.

Let $\mathcal{S} \subseteq \mathbb{R}$. Then, it is traditional to write $\inf \mathcal{S}$ and $\sup \mathcal{S}$ for $\operatorname{glb}(\mathcal{S})$ and $\operatorname{lub}(\mathcal{S})$, respectively, where "inf" and "sup" denote infimum and supremum, respectively. If $\mathcal{S} = \varnothing$, then we define $\inf \varnothing \triangleq \infty$ and $\sup \varnothing \triangleq -\infty$. Finally, if $\mathcal{S}$ has no lower bound, then we write $\inf \mathcal{S} = -\infty$, whereas, if $\mathcal{S}$ has no upper bound, then we write $\sup \mathcal{S} = \infty$.

The following result uses the fact that "$\subseteq$" is a partial ordering on every collection of sets.

Proposition 1.3.15. Let $\mathcal{S}$ be a collection of sets. Then,

$$\operatorname{glb}(\mathcal{S}) = \bigcap_{S \in \mathcal{S}} S, \quad \operatorname{lub}(\mathcal{S}) = \bigcup_{S \in \mathcal{S}} S. \tag{1.32}$$

Hence, for all $S \in \mathcal{S}$,

$$\operatorname{glb}(\mathcal{S}) \subseteq S \subseteq \operatorname{lub}(\mathcal{S}). \tag{1.33}$$

Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets. Then, by viewing $\mathcal{S}$ as the collection of sets $\{S_1, S_2, \ldots\}$, it follows that

$$\operatorname{glb}(\mathcal{S}) = \bigcap_{i=1}^{\infty} S_i, \quad \operatorname{lub}(\mathcal{S}) = \bigcup_{i=1}^{\infty} S_i. \tag{1.34}$$

Hence, for all $i \geq 1$,

$$\operatorname{glb}(\mathcal{S}) \subseteq S_i \subseteq \operatorname{lub}(\mathcal{S}). \tag{1.35}$$

Note that $\operatorname{glb}(\mathcal{S})$ and $\operatorname{lub}(\mathcal{S})$ are independent of the ordering of the sequence $\mathcal{S}$.

Proposition 1.3.16. Let $\mathcal{S}$ be a collection of sets, let A be a set, let $\mathcal{S}_0 \triangleq \{S \in \mathcal{S} \colon A \subseteq S\}$, and assume that $\mathcal{S}_0 \neq \varnothing$. Then, $A \subseteq \operatorname{glb}(\mathcal{S}_0)$. If, in addition, $\operatorname{glb}(\mathcal{S}_0) \in \mathcal{S}_0$, then $\operatorname{glb}(\mathcal{S}_0)$ is the smallest element of $\mathcal{S}$ that contains A in the sense that, if $S \in \mathcal{S}$ and $A \subseteq S$, then $\operatorname{glb}(\mathcal{S}_0) \subseteq S$.

Proposition 1.3.17. Let $\mathcal{S}$ be a collection of sets, let A be a set, and let $\mathcal{S}_0 \triangleq \{S \in \mathcal{S} \colon S \subseteq A\}$. Then, $\operatorname{lub}(\mathcal{S}_0) \subseteq A$. If, in addition, $\operatorname{lub}(\mathcal{S}_0) \in \mathcal{S}_0$, then $\operatorname{lub}(\mathcal{S}_0)$ is the largest element of $\mathcal{S}$ that is contained in A in the sense that, if $S \in \mathcal{S}$ and $S \subseteq A$, then $S \subseteq \operatorname{lub}(\mathcal{S}_0)$.

Definition 1.3.18. Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets. Then, the *essential greatest lower bound* of $\mathcal{S}$ is defined by

$$\operatorname{essglb}(\mathcal{S}) \triangleq \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} S_i, \tag{1.36}$$

and the *essential least upper bound* of $\mathcal{S}$ is defined by

$$\operatorname{esslub}(\mathcal{S}) \triangleq \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} S_i. \tag{1.37}$$

Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets. Then, the set $\operatorname{essglb}(\mathcal{S})$ consists of all elements of $\cup_{i=1}^{\infty} S_i$ that belong to all but finitely many of the sets in $\mathcal{S}$. Furthermore, the set $\operatorname{esslub}(\mathcal{S})$ consists of all elements of $\cup_{i=1}^{\infty} S_i$ that belong to infinitely many of the sets in $\mathcal{S}$. Therefore, $\operatorname{essglb}(\mathcal{S})$ and $\operatorname{esslub}(\mathcal{S})$ are independent of the ordering of the sequence $\mathcal{S}$, and

$$\operatorname{glb}(\mathcal{S}) \subseteq \operatorname{essglb}(\mathcal{S}) \subseteq \operatorname{esslub}(\mathcal{S}) \subseteq \operatorname{lub}(\mathcal{S}). \tag{1.38}$$

Note that $\operatorname{lub}(\mathcal{S}) \backslash \operatorname{esslub}(\mathcal{S})$ is the set of elements of $\cup_{i=1}^{\infty} S_i$ that belong to at most finitely many of the sets in $\mathcal{S}$.

Example 1.3.19. Consider the sequence of sets given by

$$(\{1,4\}, \{1,2\}, \{1,2,3\}, \{1,2\}, \{1,2,3\}, \{1,2\}, \{1,2,3\}, \ldots).$$

Then, (1.38) becomes $\{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \{1,2,3,4\}$. $\diamond$

Definition 1.3.20. Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets, and assume that $\operatorname{essglb}(\mathcal{S}) = \operatorname{esslub}(\mathcal{S})$. Then, the *essential limit* of $\mathcal{S}$ is defined by

$$\operatorname{esslim}(\mathcal{S}) \triangleq \operatorname{essglb}(\mathcal{S}) = \operatorname{esslub}(\mathcal{S}). \tag{1.39}$$

Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets. Then, $\mathcal{S}$ is *nonincreasing* if, for all $i \in \mathbb{P}$, $S_{i+1} \subseteq S_i$. Furthermore, $\mathcal{S}$ is *nondecreasing* if, for all $i \in \mathbb{P}$, $S_i \subseteq S_{i+1}$.

Proposition 1.3.21. Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets. If $\mathcal{S}$ is nonincreasing, then

$$\operatorname{esslim}(\mathcal{S}) = \operatorname{glb}(\mathcal{S}) = \operatorname{essglb}(\mathcal{S}) = \operatorname{esslub}(\mathcal{S}). \tag{1.40}$$

Furthermore, if $\mathcal{S}$ is nondecreasing, then

$$\operatorname{esslim}(\mathcal{S}) = \operatorname{essglb}(\mathcal{S}) = \operatorname{esslub}(\mathcal{S}) = \operatorname{lub}(\mathcal{S}). \tag{1.41}$$

Example 1.3.22. Consider the nonincreasing sequence of sets

$$(\mathbb{N}, \mathbb{N}\backslash\{1\}, \mathbb{N}\backslash\{1,2\}, \mathbb{N}\backslash\{1,2,3\}, \ldots).$$

Then, (1.38) becomes $\{0\} = \{0\} = \{0\} \subseteq \mathbb{N}$. Now, consider the nondecreasing sequence of subsets of $\mathbb{R}$ given by

$$(\{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \ldots).$$

Then, (1.38) becomes $\{1\} \subseteq \mathbb{P} = \mathbb{P} = \mathbb{P}$, where $\mathbb{P}$ is the set of positive integers. $\diamond$

Let $\mathcal{S} \triangleq (S_i)_{i=1}^{\infty}$ be a sequence of sets. Then, the sequence $\hat{\mathcal{S}} \triangleq (\cap_{j=1}^{k}[\cup_{i=j}^{\infty} S_i])_{k=1}^{\infty} = (\cup_{i=k}^{\infty} S_i)_{k=1}^{\infty} = (\hat{S}_k)_{i=1}^{\infty}$ is nonincreasing. Hence,

$$\operatorname{esslub}(\mathcal{S}) = \operatorname{esslim}(\hat{\mathcal{S}}) = \operatorname{glb}(\hat{\mathcal{S}}) = \operatorname{essglb}(\hat{\mathcal{S}}) = \operatorname{esslub}(\hat{\mathcal{S}}). \tag{1.42}$$

Furthermore, the sequence $\tilde{\mathcal{S}} \triangleq (\cup_{j=1}^{k}[\cap_{i=j}^{\infty} S_i])_{k=1}^{\infty} = (\cap_{i=k}^{\infty} S_i)_{k=1}^{\infty} = (\tilde{S}_k)_{i=1}^{\infty}$ is nondecreasing. Hence,

$$\operatorname{essglb}(\mathcal{S}) = \operatorname{esslim}(\tilde{\mathcal{S}}) = \operatorname{essglb}(\tilde{\mathcal{S}}) = \operatorname{esslub}(\tilde{\mathcal{S}}) = \operatorname{lub}(\tilde{\mathcal{S}}). \tag{1.43}$$

1.4 Directed and Symmetric Graphs

Let $\mathcal{X}$ be a finite, nonempty set, and let $\mathcal{R}$ be a multirelation on $\mathcal{X}$. Then, the pair $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ is a *directed multigraph*. The elements of $\mathcal{X}$ are the *nodes* of $\mathcal{G}$, while the elements of $\mathcal{R}$ are the *directed edges* of $\mathcal{G}$. If $\mathcal{R}$ is a relation on $\mathcal{X}$, then $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ is a *directed graph*. We focus on directed graphs, which have distinct (that is, nonrepeated) directed edges.

The directed graph $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ can be visualized as a set of points in the plane representing the nodes in $\mathcal{X}$ connected by the directed edges in $\mathcal{R}$. Specifically, the directed edge $(x, y) \in \mathcal{R}$ from x to y can be visualized as a directed line segment or curve connecting node x to node y. The direction of a directed edge can be denoted by an arrowhead. A directed edge of the form (x, x) is a *self-directed edge*.

If the relation $\mathcal{R}$ is symmetric, then $\mathcal{G}$ is a *symmetric graph*. In this case, it is convenient to represent the pair of directed edges (x, y) and (y, x) in $\mathcal{R}$ by a single *edge* $\{x, y\}$, which is a subset of $\mathcal{X}$. For the self-directed edge (x, x), the corresponding edge is the single-element *self-edge* $\{x\}$. To illustrate these notions, consider a directed graph that represents a city with streets (directed edges) connecting intersections (nodes). Each directed edge represents a one-way street, while the presence of the one-way street (x, y) and its *reverse* (y, x) represents a two-way street. A symmetric relation is a street plan consisting entirely of two-way streets (that is, edges) and thus no one-way streets (directed edges), whereas an antisymmetric relation is a street plan consisting entirely of one-way streets (directed edges) and thus no two-way streets (edges).

Definition 1.4.1. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. Then, the following terminology is defined:

i) If $x, y \in \mathcal{X}$ are distinct and $(x, y) \in \mathcal{R}$, then y is the *head* of (x, y) and x is the *tail* of (x, y).

ii) If $x, y \in \mathcal{X}$ are distinct and $(x, y) \in \mathcal{R}$, then x is a *parent* of y, and y is a *child* of x.

iii) If $x, y \in \mathcal{X}$ are distinct and either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$, then x and y are *adjacent*.

iv) If $x \in \mathcal{X}$ has no parent, then x is a *root*.

v) If $x \in \mathcal{X}$ has no child, then x is a *leaf*.

Definition 1.4.2. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. Then, the following terminology is defined:

i) The *reversal* of $\mathcal{G}$ is the graph $\operatorname{rev}(\mathcal{G}) \triangleq (\mathcal{X}, \operatorname{rev}(\mathcal{R}))$.

ii) The *complement* of $\mathcal{G}$ is the graph $\mathcal{G}^{\sim} \triangleq (\mathcal{X}, \mathcal{R}^{\sim})$.

iii) The *reflexive hull* of $\mathcal{G}$ is the graph $\operatorname{ref}(\mathcal{G}) \triangleq (\mathcal{X}, \operatorname{ref}(\mathcal{R}))$.

iv) The *symmetric hull* of $\mathcal{G}$ is the graph $\operatorname{sym}(\mathcal{G}) \triangleq (\mathcal{X}, \operatorname{sym}(\mathcal{R}))$.

v) The *transitive hull* of $\mathcal{G}$ is the graph $\mathrm{trans}(\mathcal{G}) \triangleq (\mathcal{X}, \mathrm{trans}(\mathcal{R}))$.

vi) The *equivalence hull* of $\mathcal{G}$ is the graph $\mathrm{equiv}(\mathcal{G}) \triangleq (\mathcal{X}, \mathrm{equiv}(\mathcal{R}))$.

vii) $\mathcal{G}$ is *reflexive* if $\mathcal{R}$ is reflexive.

viii) $\mathcal{G}$ is *transitive* if $\mathcal{R}$ is transitive.

ix) $\mathcal{G}$ is an *equivalence graph* if $\mathcal{R}$ is an equivalence relation.

x) $\mathcal{G}$ is *antisymmetric* if $\mathcal{R}$ is antisymmetric.

xi) $\mathcal{G}$ is *partially ordered* if $\mathcal{R}$ is a partial ordering on $\mathcal{X}$.

xii) $\mathcal{G}$ is *totally ordered* if $\mathcal{R}$ is a total ordering on $\mathcal{X}$.

xiii) $\mathcal{G}$ is a *tournament* if $\mathcal{G}$ is antisymmetric and $\mathrm{sym}(\mathcal{R}) = \mathcal{X} \times \mathcal{X} \backslash \{(x, x)\colon x \in \mathcal{X}\}$.

Definition 1.4.3. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. Then, the following terminology is defined:

i) The directed graph $\mathcal{G}' = (\mathcal{X}', \mathcal{R}')$ is a *directed subgraph* of $\mathcal{G}$ if $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{R}' \subseteq \mathcal{R}$.

ii) The directed subgraph $\mathcal{G}' = (\mathcal{X}', \mathcal{R}')$ of $\mathcal{G}$ is a *spanning directed subgraph* of $\mathcal{G}$ if $\mathrm{supp}(\mathcal{R}) = \mathrm{supp}(\mathcal{R}')$.

iii) If $\mathcal{X}_0 \subseteq \mathcal{X}$, then $\mathcal{G}|_{\mathcal{X}_0} \triangleq (\mathcal{X}_0, \mathcal{R}|_{\mathcal{X}_0})$.

iv) If $\mathcal{G}' = (\mathcal{X}', \mathcal{R}')$ is a directed graph, then $\mathcal{G} \cup \mathcal{G}' \triangleq (\mathcal{X} \cup \mathcal{X}', \mathcal{R} \cup \mathcal{R}')$ and $\mathcal{G} \cap \mathcal{G}' \triangleq (\mathcal{X} \cap \mathcal{X}', \mathcal{R} \cap \mathcal{R}')$.

v) For $x, y \in \mathcal{X}$, a *directed walk* in $\mathcal{G}$ from x to y is an n-tuple of directed edges of $\mathcal{G}$ of the form $((x, y)) \in \mathcal{R}$ for $n = 1$ and $((x, x_1), (x_1, x_2), \ldots, (x_{n-1}, y)) \in \mathcal{R}^n$ for all $n \geq 2$. The *length* of the directed walk is n. The nodes $x, x_1, \ldots, x_{n-1}, y$ are the *nodes* of the walk. Furthermore, if $n \geq 2$, then the nodes $x_1, \ldots, x_{n-1}$ are the *intermediate nodes* of the walk.

vi) For $x, y \in \mathcal{X}$, a *directed trail* in $\mathcal{G}$ from x to y is a directed walk in $\mathcal{G}$ from x to y whose directed edges are distinct.

vii) For $x, y \in \mathcal{X}$, a *directed path* in $\mathcal{G}$ from x to y is a directed trail in $\mathcal{G}$ from x to y whose intermediate nodes are distinct and do not include x and y.

viii) For $x \in \mathcal{X}$, a *directed cycle* in $\mathcal{G}$ at x is a directed path in $\mathcal{G}$ from x to x whose length is at least 2.

ix) $\mathcal{G}$ is *directionally acyclic* if $\mathcal{G}$ has no directed cycles.

x) If $\mathcal{G}$ has at least one directed cycle, then the *directed period* of $\mathcal{G}$ is the greatest common divisor of the lengths of the directed cycles of $\mathcal{G}$.

xi) $\mathcal{G}$ is *directionally aperiodic* if it has at least one directed cycle and the greatest common divisor of the lengths of the directed cycles in $\mathcal{G}$ is 1.

xii) A *directed Hamiltonian path* is a directed path whose nodes include all of the nodes of $\mathcal{X}$.

xiii) A *directed Hamiltonian cycle* is a directed cycle whose nodes include every node in $\mathcal{X}$.

xiv) $\mathcal{G}$ is a *directed tree* if $\mathcal{G}$ has exactly one root x and, for all $y \in \mathcal{X}$ such that $y \neq x$, y has exactly one parent.

xv) $\mathcal{G}$ is a *directed forest* if $\mathcal{G}$ is a union of disjoint directed trees.

xvi) $\mathcal{G}$ is a *directed chain* if $\mathcal{G}$ is a tree and has exactly one leaf.

xvii) $\mathcal{G}$ is *directionally connected* if, for all distinct $x, y \in \mathcal{X}$, there exist directed walks in $\mathcal{G}$ from x to y and from y to x.

xviii) $\mathcal{G}$ is *bipartite* if there exist nonempty, disjoint sets $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ and $\mathcal{R} \cap (\mathcal{X}_1 \times \mathcal{X}_1) = \mathcal{R} \cap (\mathcal{X}_2 \times \mathcal{X}_2) = \varnothing$.

xix) The *indegree* of $x \in \mathcal{X}$ is $\operatorname{indeg}(x) \triangleq \operatorname{card}\{y \in \mathcal{X}\colon y \text{ is a parent of } x\}$.

xx) The *outdegree* of $x \in \mathcal{X}$ is $\operatorname{outdeg}(x) \triangleq \operatorname{card}\{y \in \mathcal{X}\colon y \text{ is a child of } x\}$.

xxi) Let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are nonempty and disjoint, and assume that $\mathcal{X} = \operatorname{supp}(\mathcal{G})$. Then, $(\mathcal{X}_1, \mathcal{X}_2)$ is a *directed cut* of $\mathcal{G}$ if, for all $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, there does not exist a directed walk from x_1 to x_2.

A self-directed edge is a directed path; however, a self-directed edge is not a directed cycle.

A directed Hamiltonian cycle is both a directed Hamiltonian path and a directed cycle, both of which are directed paths.

Definition 1.4.4. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a symmetric graph. Then, the following terminology is defined:

i) For $x, y \in \mathcal{X}$, a *walk* in $\mathcal{G}$ connecting x and y is an n-tuple of edges of $\mathcal{G}$ of the form $(\{x, y\}) \in \mathcal{E}$ for $n = 1$ and $(\{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, y\}) \in \mathcal{E}^n$ for $n \ge 2$. The *length* of the walk is n. The nodes $x, x_1, \ldots, x_{n-1}, y$ are the *nodes* of the walk. Furthermore, if $n \ge 2$, then the nodes $x_1, \ldots, x_{n-1}$ are the *intermediate nodes* of the walk.

ii) For $x, y \in \mathcal{X}$, a *trail* in $\mathcal{G}$ connecting x and y is a walk in $\mathcal{G}$ connecting x to y whose edges are distinct.

iii) For $x, y \in \mathcal{X}$, a *path* in $\mathcal{G}$ connecting x and y is a trail in $\mathcal{G}$ connecting x and y whose intermediate nodes are distinct and do not include x and y.

iv) For $x \in \mathcal{X}$, a *cycle* in $\mathcal{G}$ at x is a path in $\mathcal{G}$ connecting x and x whose length is at least 3.

v) $\mathcal{G}$ is *acyclic* if $\mathcal{G}$ has no cycles.

vi) If $\mathcal{G}$ has at least one cycle, then the *period* of $\mathcal{G}$ is the greatest common divisor of the lengths of the cycles of $\mathcal{G}$.

vii) $\mathcal{G}$ is *aperiodic* if the period of $\mathcal{G}$ is 1.

viii) A *Hamiltonian path* is a path whose nodes include every node in $\mathcal{X}$.

ix) $\mathcal{G}$ is *Hamiltonian* if $\mathcal{G}$ has a *Hamiltonian cycle* $\mathcal{P}$, which is a cycle such that every node in $\mathcal{X}$ is a node of $\mathcal{P}$.

x) $\mathcal{G}$ is a *tree* if there exists a directed tree $\mathcal{G}' = (\mathcal{X}, \mathcal{R}')$ such that $\mathcal{G} = \operatorname{sym}(\mathcal{G}')$.

xi) $\mathcal{G}$ is a *forest* if $\mathcal{G}$ is a union of disjoint trees.

xii) $\mathcal{G}$ is a *chain* if there exists a directed chain $\mathcal{G}' = (\mathcal{X}, \mathcal{R}')$ such that $\mathcal{G} = \operatorname{sym}(\mathcal{G}')$.

xiii) $\mathcal{G}$ is *connected* if, for all distinct $x, y \in \mathcal{X}$, there exists a walk in $\mathcal{G}$ connecting x and y.

xiv) $\mathcal{G}$ is *bipartite* if there exist nonempty, disjoint sets $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ and $\{\{x, y\} \in \mathcal{R}\colon x \in \mathcal{X}_1 \text{ and } y \in \mathcal{X}_2\} = \varnothing$.

xv) The *degree* of $x \in \mathcal{X}$ is $\deg(x) \triangleq \operatorname{indeg}(x) = \operatorname{outdeg}(x)$.

A self-edge is a path; however, a self-edge is not a cycle.

A Hamiltonian cycle is both a Hamiltonian path and a cycle, both of which are paths.

Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph, and let $w\colon \mathcal{X} \times \mathcal{X} \mapsto [0, \infty)$, where $w(x, y) > 0$ if $(x, y) \in \mathcal{R}$ and $w(x, y) = 0$ if $(x, y) \notin \mathcal{R}$. For each directed edge $(x, y) \in \mathcal{R}$, $w(x, y)$ is the *weight* associated with the directed edge (x, y), and the triple $\mathcal{G} = (\mathcal{X}, \mathcal{R}, w)$ is a *weighted* *directed graph*. The graph $\mathcal{G}' = (\mathcal{X}', \mathcal{R}', w')$ is a *weighted directed subgraph* of $\mathcal{G}$ if $\mathcal{X}' \subseteq \mathcal{X}$, $\mathcal{R}'$ is a relation on $\mathcal{X}'$, $\mathcal{R}' \subseteq \mathcal{R}$, and w' is the restriction of w to $\mathcal{R}'$. Finally, if $\mathcal{G}$ is symmetric, then w is *symmetric* if, for all $(x, y) \in \mathcal{R}$, $w(x, y) = w(y, x)$. In this case, w is defined on each edge $\{x, y\}$ of $\mathcal{G}$.

1.5 Numbers

Let x and y be real numbers. Then, x *divides* y if there exists an integer n such that $y = nx$, In this case, we write $x|y$. For example, $6|12$, $3|-9$, $\pi|-2\pi$, $3|0$, and $0|0$. The notation $x \nmid y$ means that x does not divide y.

Let $n_1, \ldots, n_k$ be integers, not all of which are zero. Then, the *greatest common divisor* of the set $\{n_1, \ldots, n_k\}$ is the positive integer defined by

$$\gcd\{n_1, \ldots, n_k\} \triangleq \max\{i \in \mathbb{P}\colon i \text{ divides } n_1, \ldots, n_k\}.$$

For example, $\gcd\{5, 10\} = 5$, and $\gcd\{0, 2\} = 2$. The set $\{n_1, \ldots, n_k\}$ is *coprime* if $\gcd\{n_1, \ldots, n_k\} = 1$. For example, $\gcd\{-3, -7\} = 1$, and thus $\{-3, -7\}$ is coprime.

Let $n_1, \ldots, n_k$ be nonzero integers. Then, the *least common multiple* of the set $\{n_1, \ldots, n_k\}$ is the positive integer defined by

$$\operatorname{lcm}\{n_1, \ldots, n_k\} \triangleq \min\{i \in \mathbb{P}\colon n_1, \ldots, n_k \text{ divide } i\}.$$

For example, $\operatorname{lcm}\{-3, -7\} = 21$, and $\operatorname{lcm}\{-2, 3\} = 6$.

Let m be a nonzero integer, and let n be an integer. Then, $m|n$ if and only if $\gcd\{m, n\} = |m|$.

Let n be an integer, and let k be a positive integer. Furthermore, let l be an integer, and let $r \in [0, k-1]$ be an integer satisfying $n = kl + r$. Then, we write

$$r = \operatorname{rem}_k(n). \tag{1.44}$$

where r is the *remainder* after dividing n by k. For example, $\operatorname{rem}_3(-11) = 1$ and $\operatorname{rem}_3(11) = 2$. Furthermore, $k|n$ if and only if $\operatorname{rem}_k(n) = 0$.

Proposition 1.5.1. Let m and n be integers, and let k be a positive integer. Then,

$$\operatorname{rem}_k(n-m) = \operatorname{rem}_k[\operatorname{rem}_k(n) - \operatorname{rem}_k(m)]. \tag{1.45}$$

Furthermore, $k|n-m$ if and only if $\operatorname{rem}_k(n) = \operatorname{rem}_k(m)$.

Definition 1.5.2. Let n and m be integers, and let k be a positive integer. Then, n and m are *congruent modulo* k if k divides $n - m$. In this case, we write

$$n \overset{k}{\equiv} m. \tag{1.46}$$

Proposition 1.5.1 implies that $n \overset{k}{\equiv} m$ if and only if the remainders of n and m after dividing by k differ by a multiple of k. For example, $-1 \overset{3}{\equiv} 2 \overset{3}{\equiv} 8 \overset{3}{\equiv} 26 \overset{3}{\equiv} 29$.

Let n be an integer. Then, n is *even* if 2 divides n, whereas n is *odd* if 2 does not divide n. Now, assume that $n \geq 2$. Then, n is *prime* if, for all integers m such that $2 \leq m < n$, m does not divide n. Note that 2 is prime, but 1 is not prime. Letting p_n denote the nth prime, it follows that

$$(p_i)_{i=1}^{25} = (2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97).$$

The nth *harmonic number* is denoted by

$$H_n \triangleq \sum_{i=1}^{n} \frac{1}{i}. \tag{1.47}$$

Then,

$$(H_i)_{i=0}^{12} = \left(0, 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \frac{7381}{2520}, \frac{83711}{27720}, \frac{86021}{27720}\right).$$

For all $\alpha \in \mathbb{R}$, the nth *generalized harmonic number of order* α is denoted by

$$H_{n,\alpha} \triangleq \sum_{i=1}^{n} \frac{1}{i^\alpha}. \tag{1.48}$$

Define $H_0 \triangleq H_{0,\alpha} \triangleq 0$. Then,

$$(H_{i,2})_{i=0}^{10} = \left(0, 1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}, \frac{5369}{3600}, \frac{266681}{176400}, \frac{1077749}{705600}, \frac{9778141}{6350400}, \frac{1968329}{1270080}\right).$$

The symbol $\mathbb{C}$ denotes the set of complex numbers. The elements of $\mathbb{R}$ and $\mathbb{C}$ are *scalars*. Define

$$\jmath \triangleq \sqrt{-1}. \tag{1.49}$$

Let $z \in \mathbb{C}$. Then, $z = x + y\jmath$, where $x, y \in \mathbb{R}$. Define the *complex conjugate* $\overline{z}$ of z by

$$\overline{z} \triangleq x - y\jmath \tag{1.50}$$

and the real part $\operatorname{Re} z$ of z and the imaginary part $\operatorname{Im} z$ of z by

$$\operatorname{Re} z \triangleq \tfrac{1}{2}(z + \overline{z}) = x, \quad \operatorname{Im} z \triangleq \tfrac{1}{2\jmath}(z - \overline{z}) = \tfrac{1}{2}(\overline{z} - z)\jmath = y. \tag{1.51}$$

Furthermore, the *absolute value* $|z|$ of z is defined by

$$|z| \triangleq \sqrt{x^2 + y^2}. \tag{1.52}$$

Finally, the *argument* $\arg z \in (-\pi, \pi]$ of z is defined by

$$\arg z \triangleq \begin{cases} 0, & y = x = 0, \\ \operatorname{atan} \frac{y}{x}, & x > 0, \\ -\frac{\pi}{2}, & y < 0, x = 0, \\ \frac{\pi}{2}, & y > 0, x = 0, \\ -\pi + \operatorname{atan} \frac{y}{x}, & y < 0, x < 0, \\ \pi + \operatorname{atan} \frac{y}{x}, & y \geq 0, x < 0, \end{cases} \tag{1.53}$$

where atan: $\mathbb{R} \mapsto (-\frac{\pi}{2}, \frac{\pi}{2})$.

Let z be a complex number. Then,

$$z = |z| e^{(\arg z)\jmath}. \tag{1.54}$$

z is a nonnegative number if and only if $\arg z = 0$, and z is a negative number if and only if $\arg z = -\pi$. If z is not a nonnegative number, then $\arg z \in (-\pi, 0) \cup (0, \pi]$ is the angle from the positive real axis to the line segment connecting z to the origin in the complex plane, where clockwise angles are negative and confined to the set $(-\pi, 0)$, and counterclockwise angles are positive and confined to the set $(0, \pi]$. Furthermore, if z is nonzero, then

$$\arg \frac{1}{z} = \begin{cases} -\arg z, & \arg z \in (-\pi, \pi), \\ \pi, & \arg z = \pi. \end{cases} \tag{1.55}$$

Let z_1 and z_2 be nonzero complex numbers. Then, there exists $k \in \{-1, 0, 1\}$ such that

$$\arg z_1 z_2 = \arg z_1 + \arg z_2 + 2k\pi. \tag{1.56}$$

Hence, $2\pi | \arg z_1 z_2 - \arg z_1 - \arg z_2$. For example,

$$\arg\,(-1)(-1) = \arg 1 = 0 = \pi + \pi - 2\pi = \arg -1 + \arg -1 - 2\pi,$$

$$\arg\,(1)(-1) = \arg -1 = \pi = 0 + \pi = \arg 1 + \arg -1,$$

$$\arg\,(-\jmath)(-\jmath) = \arg -1 = \pi = -\pi/2 - \pi/2 + 2\pi = \arg -\jmath + \arg -\jmath + 2\pi.$$

The *closed left half plane* (CLHP), *open left half plane* (OLHP), *closed right half plane* (CRHP), and *open right half plane* (ORHP) are the subsets of $\mathbb{C}$ defined by

$$\mathrm{OLHP} \triangleq \{x \in \mathbb{C}\colon\ \operatorname{Re} x < 0\}, \quad \mathrm{ORHP} \triangleq \{x \in \mathbb{C}\colon\ \operatorname{Re} x > 0\}, \tag{1.57}$$

$$\mathrm{CLHP} \triangleq \{x \in \mathbb{C}\colon\ \operatorname{Re} x \le 0\}, \quad \mathrm{CRHP} \triangleq \{x \in \mathbb{C}\colon\ \operatorname{Re} x \ge 0\}. \tag{1.58}$$

The imaginary numbers are represented by IA . Note that 0 is a real number, an imaginary number, and a complex number.

Next, we define the *open inside unit disk* (OIUD) and the *closed inside unit disk* (CIUD) by

$$\mathrm{OIUD} \triangleq \{x \in \mathbb{C}\colon\ |x| < 1\}, \quad \mathrm{CIUD} \triangleq \{x \in \mathbb{C}\colon\ |x| \le 1\}. \tag{1.59}$$

The complements of the open inside unit disk and the closed inside unit disk are given, respectively, by the *closed outside unit disk* (COUD) and the *open outside unit disk*, which are defined by

$$\mathrm{COUD} \triangleq \{x \in \mathbb{C}\colon\ |x| \ge 1\}, \quad \mathrm{OOUD} \triangleq \{x \in \mathbb{C}\colon\ |x| > 1\}. \tag{1.60}$$

The unit circle in $\mathbb{C}$ is denoted by UC .

Since $\mathbb{R}$ is a proper subset of $\mathbb{C}$, we state many results for $\mathbb{C}$. In other cases, we treat $\mathbb{R}$ and $\mathbb{C}$ separately. To do this efficiently, we use the symbol $\mathbb{F}$ to consistently denote either $\mathbb{R}$ or $\mathbb{C}$.

Let $n \in \mathbb{N}$. Then,

$$n! \triangleq \begin{cases} n(n-1)\cdots(2)(1), & n \ge 1, \\ 1, & n = 0. \end{cases} \tag{1.61}$$

Then,

$$(i!)_{i=0}^{12} = (1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, 479001600).$$

Let $z \in \mathbb{C}$ and $k \in \mathbb{Z}$. Then,

$$\binom{z}{k} \triangleq \begin{cases} \dfrac{z(z-1)\cdots(z-k+1)}{k!}, & k > 0, \\ 1, & k = 0, \\ 0, & k < 0. \end{cases} \tag{1.62}$$

In particular, if $n, k \in \mathbb{N}$, then

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{(n-k)!k!}, & n \geq k \geq 0, \\ 0, & k > n \geq 0. \end{cases} \tag{1.63}$$

Hence,

$$\binom{n}{n} = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{1.64}$$

For example,

$$\binom{-1}{-1} = 0, \quad \binom{-1}{1} = -1, \quad \binom{1}{-1} = 0, \quad \binom{-1}{0} = 1, \quad \binom{0}{0} = 1,$$

$$\binom{-1}{3} = -1, \quad \binom{-\frac{1}{2}}{3} = \frac{-5}{16}, \quad \binom{0}{3} = 0, \quad \binom{\frac{1}{2}}{3} = \frac{1}{16}, \quad \binom{1}{3} = 0.$$

Note that, for all $n \geq k \geq 1$, $\binom{n}{k}$ is the number of k-element subsets of $\{1, \ldots, n\}$.

Let $z, w \in \mathbb{C}$, and assume that $z \notin -\mathbb{P}$, $w \notin -\mathbb{P}$, and $z - w \notin -\mathbb{P}$. Then,

$$\binom{z}{w} \triangleq \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}. \tag{1.65}$$

For $k_1, \ldots, k_l \in \mathbb{N}$, where $\sum_{i=1}^{l} k_i = n$, we define the *multinomial coefficient*

$$\binom{n}{k_1, \ldots, k_l} \triangleq \frac{n!}{k_1! \cdots k_l!}. \tag{1.66}$$

Note that, if $1 \leq m \leq n$, then

$$\binom{n}{m} = \binom{n}{m, n-m}.$$

For $z \in \mathbb{C}$ and $k \in \mathbb{N}$, we define the *falling factorial*

$$z^{\underline{k}} \triangleq \begin{cases} z(z-1)\cdots(z-k+1), & k \geq 0, \\ 1, & k = 0. \end{cases} \tag{1.67}$$

In particular, if $n \in \mathbb{N}$, then $n^{\underline{n}} = n!$. Hence, if $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, then

$$\binom{z}{k} \triangleq \begin{cases} \dfrac{z^{\underline{k}}}{k!}, & k \geq 0, \\ 0, & k < 0. \end{cases} \tag{1.68}$$

Furthermore, for all $z \in \mathbb{C}$ and $k \in \mathbb{N}$, we define the *rising factorial*

$$z^{\overline{k}} \triangleq \begin{cases} z(z+1)\cdots(z+k-1), & k \geq 1, \\ 1, & k = 0. \end{cases} \tag{1.69}$$

In particular, if $n \in \mathbb{N}$, then $1^{\overline{n}} = n!$. Finally, if $z \in \mathbb{C}$ and $k \in \mathbb{N}$, then

$$z^{\underline{k}} = (z-k+1)^{\overline{k}}, \quad z^{\overline{k}} = (z+k-1)^{\underline{k}}, \quad z^{\underline{k}} = (-1)^k(-z)^{\overline{k}}. \tag{1.70}$$

The *double factorial* is defined by

$$n!! \triangleq \begin{cases} n(n-2)(n-4)\cdots(2) = 2^{n/2}(n/2)!, & n \text{ even}, \\ n(n-2)(n-4)\cdots(3)(1) = \dfrac{(n+1)!}{2^{(n+1)/2}[\frac{1}{2}(n+1)]!}, & n \text{ odd}. \end{cases} \tag{1.71}$$

By convention, $(-1)!! = 0!! = 1$. Finally, if $n \geq 1$, then $(2n)!!(2n-1)!! = (2n)!$ and $(2n+1)!!(2n)!! = (2n+1)!$.

1.6 Functions and Their Inverses

Let $\mathcal{X}$ and $\mathcal{Y}$ be nonempty sets. Then, a *function* f that maps $\mathcal{X}$ into $\mathcal{Y}$ is a rule $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$ that assigns a unique element $f(x)$ (the *image* of x) of $\mathcal{Y}$ to each element x of $\mathcal{X}$. Equivalently, a function $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$ can be viewed as a subset $\mathcal{F}$ of $\mathcal{X} \times \mathcal{Y}$ such that, for each $x \in \mathcal{X}$, there exists a unique $y \in \mathcal{Y}$ such that $(x, y) \in \mathcal{F}$. In this case,

$$\mathcal{F} = \operatorname{Graph}(f) \triangleq \{(x, f(x))\colon\ x \in \mathcal{X}\}. \tag{1.72}$$

The set $\mathcal{X}$ is the *domain* of f, while the set $\mathcal{Y}$ is the *codomain* of f. For $\mathcal{X}_1 \subseteq \mathcal{X}$, it is convenient to define

$$f(\mathcal{X}_1) \triangleq \{f(x)\colon\ x \in \mathcal{X}_1\}. \tag{1.73}$$

The *range* of f is the set $\mathcal{R}(f) \triangleq f(\mathcal{X})$. The function f is *one-to-one* if, for all $x_1, x_2 \in \mathcal{X}$ such that $f(x_1) = f(x_2)$, it follows that $x_1 = x_2$. The function f is *onto* if $\mathcal{R}(f) = \mathcal{Y}$. The function $I_{\mathcal{X}}\colon\ \mathcal{X} \mapsto \mathcal{X}$ defined by $I_{\mathcal{X}}(x) \triangleq x$ for all $x \in \mathcal{X}$ is the *identity mapping* on $\mathcal{X}$. Finally, if $\mathcal{S} \subseteq \mathcal{X}$, $f_{\mathcal{S}}\colon \mathcal{S} \mapsto \mathcal{Y}$, and, for all $x \in \hat{\mathcal{X}}$, $f_{\mathcal{S}}(x) = f(x)$, then $f_{\mathcal{S}}$ is the *restriction* of f to $\mathcal{S}$.

Note that the subset $\mathcal{F}$ of $\mathcal{X} \times \mathcal{Y}$ can be viewed as a relation on $(\mathcal{X}, \mathcal{Y})$. Consequently, a function can be viewed as a special case of a relation.

Let $\mathcal{X}$ be a set, and let $\hat{\mathcal{X}}$ be a partition of $\mathcal{X}$. Furthermore, let $f\colon \hat{\mathcal{X}} \mapsto \mathcal{X}$, where, for all $\mathcal{S} \in \hat{\mathcal{X}}$, it follows that $f(\mathcal{S}) \in \mathcal{S}$. Then, f is a *canonical mapping*, and $f(\mathcal{S})$ is a *canonical form*. That is, for each element $\mathcal{S} \subseteq \mathcal{X}$ in the partition $\hat{\mathcal{X}}$ of $\mathcal{X}$, the function f assigns an element of $\mathcal{S}$ to the set $\mathcal{S}$. For example, let $\mathcal{S} \triangleq \{1,2,3,4\}$, $\hat{\mathcal{X}} \triangleq \{\{1,3\},\{2,4\}\}$, $f(\{1,3\}) = 1$, and $f(\{2,4\}) = 2$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. If $f\colon \mathcal{X} \mapsto \mathcal{Y}$ is one-to-one and onto, then $\mathcal{X}$ and $\mathcal{Y}$ have the same *cardinality*, which is written as $\operatorname{card}(\mathcal{X}) = \operatorname{card}(\mathcal{Y})$. Consequently, if $\mathcal{X}$ is finite, then $\operatorname{card}(\mathcal{X})$ is the number of elements of $\mathcal{X}$. If $f\colon \mathcal{X} \mapsto \mathcal{Y}$ is one-to-one, then $\operatorname{card}(\mathcal{X}) \leq \operatorname{card}(\mathcal{Y})$. If every function $f\colon \mathcal{X} \mapsto \mathcal{Y}$ that is one-to-one is not onto, then $\operatorname{card}(\mathcal{X}) < \operatorname{card}(\mathcal{Y})$. If

$\mathrm{card}(\mathcal{X}) = \mathrm{card}(\mathbb{P})$, then $\mathcal{X}$ is *countable*. Note that $\mathrm{card}(\mathbb{N}) = \mathrm{card}(\mathbb{P}) = \mathrm{card}(\mathbb{Z}) = \mathrm{card}(\mathbb{Q}) < \mathrm{card}([0,1]) = \mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathbb{R}^2)$.

Let $\mathcal{X}$ be a finite multiset. Then, $\mathrm{card}(\mathcal{X})$ is the number of elements in $\mathcal{X}$. Cardinality is not defined for infinite multisets.

Let $\mathcal{X}$ be a set, and let $f\colon\ \mathcal{X} \mapsto \mathcal{X}$. Then, f is a function on $\mathcal{X}$. The element $x \in \mathcal{X}$ is a *fixed point* of f if $f(x) = x$.

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be sets, let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$, and let $g\colon\ f(\mathcal{X}) \mapsto \mathcal{Z}$. Then, the *composition* of g and f is the function $g \circ f\colon\ \mathcal{X} \mapsto \mathcal{Z}$ defined by $(g \circ f)(x) \triangleq g[f(x)]$. The following result shows that function composition is associative.

Proposition 1.6.1. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, and $\mathcal{W}$ be sets, and let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$, $g\colon\ \mathcal{Y} \mapsto \mathcal{Z}$, $h\colon\ \mathcal{Z} \mapsto \mathcal{W}$. Then,

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{1.74}$$

Hence, we write $h \circ g \circ f$ for $h \circ (g \circ f)$ and $(h \circ g) \circ f$.

Proposition 1.6.2. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be sets, and let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$ and $g\colon\ \mathcal{Y} \mapsto \mathcal{Z}$. Then, the following statements hold:

i) If $g \circ f$ is onto, then g is onto.

ii) If $g \circ f$ is one-to-one, then f is one-to-one.

Proof. To prove *i*), note that $\mathcal{Z} = g(f(\mathcal{X})) \subseteq g(\mathcal{Y}) \subseteq \mathcal{Z}$. Hence, $g(\mathcal{Y}) = \mathcal{Z}$. To prove *ii*), suppose that f is not one-to-one. Then, there exist distinct $x_1, x_2 \in \mathcal{X}$ such that $f(x_1) = f(x_2)$. Therefore, $g(f(x_1)) = g(f(x_2))$, and thus $g \circ f$ is not one-to-one. □

Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. Then, f is *left invertible* if there exists a function $f^{\mathsf{L}}\colon\ \mathcal{Y} \mapsto \mathcal{X}$ (a *left inverse* of f) such that $f^{\mathsf{L}} \circ f = I_{\mathcal{X}}$, whereas f is *right invertible* if there exists a function $f^{\mathsf{R}}\colon\ \mathcal{Y} \mapsto \mathcal{X}$ (a *right inverse* of f) such that $f \circ f^{\mathsf{R}} = I_{\mathcal{Y}}$. In addition, the function $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$ is *invertible* if there exists a function $f^{\mathsf{Inv}}\colon\ \mathcal{Y} \mapsto \mathcal{X}$ (the *inverse* of f) such that $f^{\mathsf{Inv}} \circ f = I_{\mathcal{X}}$ and $f \circ f^{\mathsf{Inv}} = I_{\mathcal{Y}}$; that is, f^{Inv} is both a left inverse of f and a right inverse of f.

Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$, and let $\tilde{\mathcal{X}}$ denote the set of subsets of $\mathcal{X}$. Then, for all $y \in \mathcal{Y}$, the *set-valued inverse* $f^{\mathsf{inv}}\colon \mathcal{Y} \mapsto \tilde{\mathcal{X}}$ is defined by $f^{\mathsf{inv}}(y) \triangleq \{x \in \mathcal{X}\colon f(x) = y\}$. If f is one-to-one, then, for all $y \in \mathcal{R}(f)$, the set $f^{\mathsf{inv}}(y)$ has a single element, and thus $f^{\mathsf{inv}}\colon \mathcal{R}(f) \mapsto \mathcal{X}$ is a function. If f is invertible, then, for all $y \in \mathcal{Y}$, $f^{\mathsf{inv}}(y) = \{f^{\mathsf{Inv}}(y)\}$. The *inverse image* $f^{\mathsf{inv}}(\mathcal{S})$ of $\mathcal{S} \subseteq \mathcal{Y}$ is the set

$$f^{\mathsf{inv}}(\mathcal{S}) \triangleq \bigcup_{y \in \mathcal{S}} f^{\mathsf{inv}}(y) = \{x \in \mathcal{X}\colon\ f(x) \in \mathcal{S}\}. \tag{1.75}$$

Note that $f^{\mathsf{inv}}(\mathcal{S})$ is defined whether or not f is invertible. In fact, $f^{\mathsf{inv}}(\mathcal{Y}) = f^{\mathsf{inv}}[f(\mathcal{X})] = \mathcal{X}$ and $f[f^{\mathsf{inv}}(\mathcal{Y})] = f(\mathcal{X})$.

Proposition 1.6.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets, let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$, and let $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$. Then, the following statements are equivalent:

i) f is a left inverse of g.

ii) g is a right inverse of f.

Proposition 1.6.4. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets, let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$, and assume that f is invertible. Then, f has a unique inverse. Now, let $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$. Then, the following statements are equivalent:

i) g is the inverse of f.

ii) f is the inverse of g.

Theorem 1.6.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets, and let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. Then, the following statements hold:

i) f is left invertible if and only if f is one-to-one.

ii) f is right invertible if and only if f is onto.

Furthermore, the following statements are equivalent:

iii) f is invertible.

iv) f has a unique inverse.

v) f is one-to-one and onto.

vi) f is left invertible and right invertible.

vii) f has a unique right inverse.

viii) f has a one-to-one left inverse.

ix) f has an onto right inverse.

If, in addition, $\operatorname{card}(\mathcal{X}) \ge 2$, then the following statement is equivalent to *iii)*–*ix)*:

x) f has a unique left inverse.

Proof. To prove *i)*, suppose that f is left invertible with left inverse $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$. Furthermore, suppose that $x_1, x_2 \in \mathcal{X}$ satisfy $f(x_1) = f(x_2)$. Then, $x_1 = g[f(x_1)] = g[f(x_2)] = x_2$, which shows that f is one-to-one. Conversely, suppose that f is one-to-one so that, for all $y \in \mathcal{R}(f)$, there exists a unique $x \in \mathcal{X}$ such that $f(x) = y$. Hence, define the function $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$ by $g(y) \triangleq x$ for all $y = f(x) \in \mathcal{R}(f)$ and by $g(y)$ arbitrary for all $y \in \mathcal{Y}\backslash\mathcal{R}(f)$. Consequently, $g[f(x)] = x$ for all $x \in \mathcal{X}$, which shows that g is a left inverse of f.

To prove *ii)*, suppose that f is right invertible with right inverse $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$. Then, for all $y \in \mathcal{Y}$, it follows that $f[g(y)] = y$, which shows that f is onto. Conversely, suppose that f is onto so that, for all $y \in \mathcal{Y}$, there exists at least one $x \in \mathcal{X}$ such that $f(x) = y$. Selecting one such x arbitrarily, define $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$ by $g(y) \triangleq x$. Consequently, $f[g(y)] = y$ for all $y \in \mathcal{Y}$, which shows that g is a right inverse of f. $\square$

Let $f\colon \mathcal{X} \mapsto \mathcal{Y}$, and assume that f is one-to-one. Then, the function $\hat{f}\colon \mathcal{X} \mapsto \mathcal{R}(f)$ defined by $\hat{f}(x) \triangleq f(x)$ is one-to-one and onto and thus invertible. For convenience, we write $f^{\mathsf{Inv}}\colon \mathcal{R}(f) \mapsto \mathcal{X}$.

The sine and cosine functions $\sin\colon \mathbb{R} \mapsto [-1,1]$ and $\cos\colon \mathbb{R} \mapsto [-1,1]$ can be defined in an elementary way in terms of ratios of sides of triangles. The additional trigonometric functions $\tan\colon \mathbb{R}\backslash\pi(\frac{1}{2}+\mathbb{Z}) \mapsto \mathbb{R}$, $\csc\colon \mathbb{R}\backslash\pi\mathbb{Z} \mapsto \mathbb{R}$, $\sec\colon \mathbb{R}\backslash\pi(\frac{1}{2}+\mathbb{Z}) \mapsto \mathbb{R}$, and $\cot\colon \mathbb{R}\backslash\pi\mathbb{Z} \mapsto \mathbb{R}$ are defined by

$$\tan x \triangleq \frac{\sin x}{\cos x}, \quad \csc x \triangleq \frac{1}{\sin x}, \quad \sec x \triangleq \frac{1}{\cos x}, \quad \cot x \triangleq \frac{\cos x}{\sin x}. \tag{1.76}$$

The exponential function $\exp\colon \mathbb{R} \mapsto (0,\infty)$ is defined by

$$\exp(x) \triangleq e^x, \tag{1.77}$$

where $e \triangleq \lim_{x\to\infty}(1+1/x)^x \approx 2.71828\ldots$. The exponential function can be extended to complex arguments as follows. For all $x \in \mathbb{R}$, the power series for "exp" is given by

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}. \tag{1.78}$$

Hence, for all $y \in \mathbb{R}$, we define

$$\exp(y\jmath) = e^{y\jmath} \triangleq \sum_{i=0}^{\infty} \frac{(y\jmath)^i}{i!} = \sum_{i=0}^{\infty}(-1)^i \frac{y^{2i}}{(2i)!} + \sum_{i=0}^{\infty}(-1)^{2i+1} \frac{y^{2i+1}}{(2i+1)!}\jmath = \cos y + (\sin y)\jmath. \tag{1.79}$$

Thus, for all $y \in \mathbb{R}$,

$$\sin y = \frac{1}{2\jmath}(e^{y\jmath} - e^{-y\jmath}), \quad \cos y = \tfrac{1}{2}(e^{y\jmath} + e^{-y\jmath}). \tag{1.80}$$

Now, let $z = x + y\jmath$, where $x, y \in \mathbb{R}$. Then, $\exp\colon \mathbb{C} \mapsto \mathbb{C}\backslash\{0\}$ is defined by

$$\exp(z) = \exp(x + y\jmath) \triangleq e^{x+y\jmath} = e^x e^{y\jmath} = e^x[\cos x + (\sin x)\jmath]. \tag{1.81}$$

In particular, $e^{\pi\jmath} = -1$.

The six trigonometric functions can now be extended to complex arguments. In particular, by replacing $y \in \mathbb{R}$ in (1.80) by $z \in \mathbb{C}$, we define $\sin\colon \mathbb{C} \mapsto \mathbb{C}$ and $\cos\colon \mathbb{C} \mapsto \mathbb{C}$ by

$$\sin z \triangleq \frac{1}{2\jmath}(e^{z\jmath} - e^{-z\jmath}), \quad \cos z \triangleq \tfrac{1}{2}(e^{z\jmath} + e^{-z\jmath}). \tag{1.82}$$

Hence,

$$e^{z\jmath} = \cos z + (\sin z)\jmath, \quad e^{-z\jmath} = \cos z - (\sin z)\jmath. \tag{1.83}$$

Likewise, $\tan\colon \mathbb{C}\backslash\pi(\frac{1}{2}+\mathbb{Z}) \mapsto \mathbb{R}$, $\csc\colon \mathbb{C}\backslash\pi\mathbb{Z} \mapsto \mathbb{R}$, $\sec\colon \mathbb{C}\backslash\pi(\frac{1}{2}+\mathbb{Z}) \mapsto \mathbb{R}$, and $\cot\colon \mathbb{C}\backslash\pi\mathbb{Z} \mapsto \mathbb{R}$ are defined by

$$\tan z \triangleq \frac{\sin z}{\cos z}, \quad \csc z \triangleq \frac{1}{\sin z}, \quad \sec z \triangleq \frac{1}{\cos z}, \quad \cot z \triangleq \frac{\cos z}{\sin z}. \tag{1.84}$$

Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. If f is not one-to-one, then f is not invertible. This is the case, for example, for a periodic function such as $\sin\colon \mathbb{R} \mapsto [-1, 1]$, respectively. In particular, $\sin^{\mathsf{inv}}(1) = \{(4k+1)\pi/2\colon k \in \mathbb{Z}\}$. However, it is convenient to define a *principal inverse* asin of sin by choosing an element of the set $\sin^{\mathsf{inv}}(y)$ for each $y \in [-1, 1]$. Although this choice can be made arbitrarily, it is traditional to define

$$\text{asin: } [-1, 1] \mapsto [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]. \tag{1.85}$$

Similarly,

$$\text{acos: } [-1, 1] \mapsto [0, \pi], \quad \text{atan: } \mathbb{R} \mapsto (-\tfrac{\pi}{2}, \tfrac{\pi}{2}), \tag{1.86}$$

$$\text{acsc: } (-\infty, -1] \cup [1, \infty) \mapsto [-\tfrac{\pi}{2}, 0) \cup (0, \tfrac{\pi}{2}], \tag{1.87}$$

$$\text{asec: } (-\infty, -1] \cup [1, \infty) \mapsto [0, \tfrac{\pi}{2}) \cup (\tfrac{\pi}{2}, \pi], \tag{1.88}$$

$$\text{acot: } \mathbb{R} \mapsto (-\tfrac{\pi}{2}, 0) \cup (0, \tfrac{\pi}{2}]. \tag{1.89}$$

An analogous situation arises for the exponential function $f(z) = e^z$, which is not one-to-one and thus requires a principal inverse in the form of a logarithm defined on $\mathbb{C}\backslash\{0\}$. Let w be a nonzero complex number, and, for all $i \in \mathbb{Z}$, define

$$z_i \triangleq \log|w| + (\arg w + 2i\pi)\jmath. \tag{1.90}$$

Then, for all $i \in \mathbb{Z}$,

$$e^{z_i} = |w|e^{(\arg w)\jmath}e^{2i\pi\jmath} = |w|e^{(\arg w)\jmath} = w. \tag{1.91}$$

Consequently, $f^{\mathsf{inv}}(w) = \{z_i \colon i \in \mathbb{Z}\}$. For example, $f^{\mathsf{inv}}(1) = \{2i\pi\jmath \colon i \in \mathbb{Z}\}$, and $f^{\mathsf{inv}}(-1) = \{(2i+1)\pi\jmath \colon i \in \mathbb{Z}\}$. The *principal logarithm* $\log w$ of w is defined by choosing z_0, which yields

$$\log w \triangleq z_0 = \log|w| + (\arg w)\jmath. \tag{1.92}$$

Therefore,

$$\log\colon \mathbb{C}\backslash\{0\} \mapsto \{z\colon \operatorname{Re} z \neq 0 \text{ and } -\pi < \operatorname{Im} z \leq \pi\}. \tag{1.93}$$

Hence,

$$\operatorname{Re}\log w = \log|w|, \quad \operatorname{Im}\log w = \arg w. \tag{1.94}$$

Let w_1 and w_2 be nonzero complex numbers. Then, with $f\colon \mathbb{C} \mapsto \mathbb{C}\backslash\{0\}$ given by (1.81),

$$f^{\mathsf{inv}}(w_1w_2) = f^{\mathsf{inv}}(w_1) + f^{\mathsf{inv}}(w_2). \tag{1.95}$$

However,

$$\log w_1w_2 = \log w_1 + \log w_2 \tag{1.96}$$

if and only if

$$\arg w_1w_2 = \arg w_1 + \arg w_2. \tag{1.97}$$

For example,

$$\arg\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\jmath\right)^2 = \arg\jmath = \frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{4} = \arg\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\jmath\right) + \arg\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\jmath\right),$$

and thus

$$\log\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\jmath\right)^2 = \log\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\jmath\right) + \log\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\jmath\right).$$

However,

$$\arg(-1)^2 = \arg 1 = 0 \neq 2\pi = \pi + \pi = \arg(-1) + \arg(-1),$$

and thus

$$\log(-1)^2 = \log 1 = 0 \neq 2\pi\jmath = \pi\jmath + \pi\jmath = \log(-1) + \log(-1).$$

Therefore, there exist nonzero complex numbers w_1 and w_2 such that the principal logarithm does not satisfy (1.96).

Let w be a nonzero complex number. Then,

$$w = e^{\log w}. \tag{1.98}$$

Now, let z be a complex number. Then,

$$\log e^z = z - \left(\operatorname{round}\frac{\operatorname{Im} z}{2\pi}\right)2\pi\jmath, \tag{1.99}$$

where, for all $x \in \mathbb{R}$, round(x) denotes the closest integer to x except in the case where $2x$ is an integer, in which case round$(x) = \lfloor x \rfloor$. Therefore, $\log e^z = z$ if and only if $\operatorname{Im} z \in (-\pi, \pi]$.

An analogous situation arises for nth roots. Consider $f\colon \mathbb{R} \mapsto [0, \infty)$ defined by $f(x) = x^2$. Then, for all $y \in [0, \infty)$, it follows that $f^{\mathsf{inv}}(y) = \{-\sqrt{y}, \sqrt{y}\}$, where $\sqrt{y}$ represents the nonnegative square root of $y \geq 0$. For complex-valued extensions, let $n \geq 1$, and define $f\colon \mathbb{C} \mapsto \mathbb{C}$ by $f(z) = z^n$. Let w be a nonzero complex number. If z satisfies $z^n = w$, then $\log z^n = \log w = \log |w| + (\arg w)\jmath$, where "log" is the principal log. Furthermore, z satisfies $z^n = w$ if and only if there exists an integer i such that $n \log z = \log |w| + (\arg w + 2i\pi)\jmath$. Therefore, for all $i \in \mathbb{Z}$, define

$$z_i \triangleq e^{\frac{1}{n}[\log |w| + (\arg w + 2i\pi)\jmath]}, \tag{1.100}$$

which satisfies

$$z_i^n = w. \tag{1.101}$$

Note that, for all $i \in \mathbb{Z}$, $z_{n+i} = z_i$. Therefore, for all $i \in \{0, \ldots, n-1\}$, define the n distinct numbers

$$z_i \triangleq \sqrt[n]{|w|}e^{\frac{\arg w}{n}\jmath}e^{\frac{2i\pi}{n}\jmath}, \tag{1.102}$$

where $\sqrt[n]{|w|}$ is the nonnegative nth root of $|w|$. Consequently, $f^{\mathsf{inv}}(w) = \{z_0, \ldots, z_{n-1}\}$. The *principal nth root $w^{1/n}$ of* w is defined by choosing z_0, which yields

$$w^{1/n} \triangleq z_0 = \sqrt[n]{|w|}e^{\frac{\arg w}{n}\jmath}. \tag{1.103}$$

In particular, if w is a positive number, then $w^{1/n} = \sqrt[n]{w}$, which is the positive nth root of w. However, for an odd integer n and a negative number a, a notational conflict arises between the principal nth root of a and the negative nth root of a. For example, $(-1)^{1/3} = e^{(\pi/3)\jmath}$, whereas, for all odd integers n, it is traditional to interpret $\sqrt[n]{-1}$ as -1. In other words, for all $a < 0$ and odd $n \geq 1$, $\sqrt[n]{a} \triangleq -\sqrt[n]{|a|}$, and thus

$$a^{1/n} = \sqrt[n]{|a|}e^{(\pi/n)\jmath} = \sqrt[n]{a}e^{[(1/n-1)\pi]\jmath}. \tag{1.104}$$

Let z and α be complex numbers, and assume that z is not zero. As an extension of the functions $f(z) = z^n$ and $f(z) = z^{1/n}$, define

$$z^\alpha \triangleq e^{\alpha \log z}, \tag{1.105}$$

where $\log z$ is the principal logarithm of z. For example,

$$\frac{1}{\jmath^{2\jmath}} = e^{-2\jmath \log \jmath} = e^{-2\jmath(\pi/2)\jmath} = e^{\pi}.$$

Next, let z_1 and z_2 be complex numbers, and let α be a real number. Then, $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$. Now, let α be a complex number. Then, $\alpha^{z_1}\alpha^{z_2} = \alpha^{z_1+z_2}$. However, $(z_1 z_2)^\alpha$ and $z_1^\alpha z_2^\alpha$ are not necessarily equal. For example, $(-1)^\jmath(-1)^\jmath = e^{-\pi}e^{-\pi} = e^{-2\pi} \neq 1 = 1^\jmath = [(-1)(-1)]^\jmath$. However,

$$(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha e^{2n\pi\alpha\jmath}, \tag{1.106}$$

where

$$n = \begin{cases} 1, & -2\pi < \arg z_1 + \arg z_2 \le -\pi, \\ 0, & -\pi < \arg z_1 + \arg z_2 \le \pi, \\ -1, & \pi < \arg z_1 + \arg z_2 \le 2\pi. \end{cases} \tag{1.107}$$

Finally,

$$(\alpha^{z_1})^{z_2} = \alpha^{z_1 z_2} e^{2n\pi z_2 \jmath}, \tag{1.108}$$

where

$$n = \left\lfloor \frac{1}{2} - \frac{(\operatorname{Im} z_1)\log|\alpha| + (\operatorname{Re} z_1)\arg\alpha}{2\pi} \right\rfloor. \tag{1.109}$$

For example, setting $\alpha = -1$, $z_1 = -1$, and $z_2 = \frac{1}{2}$ yields $n = 1$, and thus $\jmath = (-1)^{1/2} = [(-1)^{-1}]^{1/2} = (-1)^{-1/2} e^{n\pi\jmath} = (1/\jmath)(-1) = \jmath$. Furthermore,

$$(e^{z_1})^{z_2} = e^{z_1 z_2} e^{2n\pi z_2 \jmath}, \tag{1.110}$$

where $n = \left\lfloor \frac{1}{2} - \frac{\operatorname{Im} z_1}{2\pi} \right\rfloor$. See [pennisi, pp. 108–114] and [ponnusamy, pp. 91, 114–119].

Finally, let z, α, and β be complex numbers. Then, $(z^\alpha)^\beta$, $(z^\beta)^\alpha$, and $z^{\alpha\beta}$ may be different as can be seen from the example $z = \frac{1}{2}\jmath$, $\alpha = 2 - \jmath$, and $\beta = -3 - \jmath$, where $(z^\alpha)^\beta \approx 0.03 + 0.04\jmath$, $(z^\beta)^\alpha \approx 9104 + 10961\jmath$, and $z^{\alpha\beta} \approx 17 + 20\jmath$. A similar situation can occur in the case where z, α, and β are real. For example, if $z = -1$, $\alpha = 1/2$, and $\beta = 2$, then $(z^\alpha)^\beta = z^{\alpha\beta} = -1 \ne 1 = (z^\beta)^\alpha$. As a final example, let $z = e$, $\alpha = 2\pi i\jmath$, where $i \ge 1$, and $\beta = \pi$. Then, $(z^\beta)^\alpha = (e^\pi)^{2\pi i\jmath} = e^{2\pi i\jmath \log e^\pi} = e^{2\pi^2 i\jmath} = z^{\alpha\beta} = \cos 2\pi^2 i + \jmath \sin 2\pi^2 i$ and $(z^\alpha)^\beta = (e^{2\pi i\jmath})^\pi = 1^\pi = e^{\pi \log 1} = e^{\pi 0} = 1$. Since, for all $i \ge 1$, $\cos 2\pi^2 i + \jmath \sin 2\pi^2 i \ne 1$, it follows that $(z^\beta)^\alpha = z^{\alpha\beta} \ne (z^\alpha)^\beta$. See [nahinit, pp. 166, 167].

Definition 1.6.6. Let $\mathcal{I} \subset \mathbb{R}$ be a finite or infinite interval, and let $f\colon \mathcal{I} \mapsto \mathbb{R}$. Then, f is *convex* if, for all $\alpha \in [0,1]$ and $x, y \in \mathcal{I}$,

$$f[\alpha x + (1-\alpha)y] \le \alpha f(x) + (1-\alpha) f(y). \tag{1.111}$$

Furthermore, f is *strictly convex* if, for all $\alpha \in (0,1)$ and distinct $x, y \in \mathcal{I}$,

$$f[\alpha x + (1-\alpha)y] < \alpha f(x) + (1-\alpha) f(y). \tag{1.112}$$

Finally, f is (*concave, strictly convex*) if $-f$ is (convex, strictly convex).

A more general definition of a convex function is given by Definition **??**.

Let $\mathcal{X}$ be a set, and let $\sigma\colon \mathcal{X} \times \cdots \times \mathcal{X} \mapsto \mathcal{X} \times \cdots \times \mathcal{X}$, where each Cartesian product has n factors. Then, σ is a *permutation* if, for all $(x_1, \ldots, x_n) \in \mathcal{X} \times \cdots \times \mathcal{X}$, the tuples $(x_1, \ldots, x_n)$ and $\sigma[(x_1, \ldots, x_n)]$ have the same components with the same multiplicity but possibly in a different order. For convenience, we write $(\sigma(x_1), \ldots, \sigma(x_n))$ for $\sigma[(x_1, \ldots, x_n)]$. In particular, we write $(\sigma(1), \ldots, \sigma(n))$ for $\sigma[(1, \ldots, n)]$. The permutation σ is a *transposition* if $(\sigma(x_1), \ldots, \sigma(x_n))$ and $(x_1, \ldots, x_n)$ differ by exactly two distinct interchanged components. Finally, let $\operatorname{sign}(\sigma)$ denote -1 raised to the smallest number of transpositions needed to transform $(\sigma(1), \ldots, \sigma(n))$ to $(1, \ldots, n)$. Note that, if σ_1 and σ_2 are permutations of $(1, \ldots, n)$, then $\operatorname{sign}(\sigma_1 \circ \sigma_2) = \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2)$.

1.7 Facts on Logic

Fact 1.7.1. Let A and B be statements. Then, the following statements hold:

i) $[A \text{ and } (A \Longrightarrow B)] \Longrightarrow B$.

ii) $\text{not}(A \text{ and } B) \Longleftrightarrow [(\text{not } A) \text{ or not } B]$.

iii) $\text{not}(A \text{ or } B) \Longleftrightarrow [(\text{not } A) \text{ and not } B]$.

iv) $(A \text{ or } B) \Longleftrightarrow [(\text{not } A) \Longrightarrow B] \Longleftrightarrow [(A \text{ and } B) \text{ xor } (A \text{ xor } B)]$.

v) $(A \Longrightarrow B) \Longleftrightarrow [(\text{not } A) \text{ or } B] \Longleftrightarrow \text{not}(A \text{ and not } B)] \Longleftrightarrow [(A \text{ and } B) \text{ xor not } A]$.

vi) $\text{not}(A \text{ and } B) \Longleftrightarrow (A \Longrightarrow \text{not } B) \Longleftrightarrow (B \Longrightarrow \text{not } A)$.

vii) $[A \text{ and not } B] \Longleftrightarrow [\text{not}(A \Longrightarrow B)]$.

Remark: Each statement is a tautology. **Remark:** *ii)* and *iii)* are *De Morgan's laws.* See [blochbook, p. 24]. See Fact 1.8.1.

Fact 1.7.2. Let A and B be statements. Then, the following statements are equivalent:

i) $A \Longleftrightarrow B$.

ii) $(A \text{ or not } B) \text{ and not}(A \text{ and not } B)$.

iii) $(A \text{ or not } B) \text{ and } [(\text{not } A) \text{ or } B]$.

iv) $(A \text{ and } B) \text{ or } [(\text{not } A) \text{ and not } B]$.

v) $\text{not}(A \text{ xor } B)$.

Remark: The equivalence of each pair of statements is a tautology.

Fact 1.7.3. Let A, B, and C be statements. Then,

$$[(A \Longrightarrow B) \text{ and } (B \Longrightarrow C)] \Longrightarrow (A \Longrightarrow C).$$

Fact 1.7.4. Let A, B, and C be statements. Then, the following statements are equivalent:

i) $A \Longrightarrow (B \text{ or } C)$.

ii) $[A \text{ and } (\text{not } B)] \Longrightarrow C$.

Remark: The statement that *i)* and *ii)* are equivalent is a tautology.

Fact 1.7.5. Let A, B, and C be statements. Then, the following statements are equivalent:

i) $(A \text{ and } B) \Longrightarrow C$.

ii) $[B \text{ and } (\text{not } C)] \Longrightarrow (\text{not } A)$.

iii) $[A \text{ and } (\text{not } C)] \Longrightarrow (\text{not } B)$.

Source: To prove *i)* $\Longrightarrow$ *ii)*, note that $[(A \text{ and } B) \text{ or } (\text{not } B)] \Longrightarrow [C \text{ or } (\text{not } B)]$, that is, $[A \text{ or } (\text{not } B)] \Longrightarrow [C \text{ or } (\text{not } B)]$, and thus $A \Longrightarrow [C \text{ or } (\text{not } B)]$. Hence, $[B \text{ and } (\text{not } C)] \Longrightarrow (\text{not } A)$. Conversely, to prove *ii)* $\Longrightarrow$ *i)*, note that $[(B \text{ and } (\text{not } C)) \text{ or } (\text{not } B)] \Longrightarrow [(\text{not } A) \text{ or}$
$(\text{not } B)]$, that is, $[(\text{not } C) \text{ or } (\text{not } B)] \Longrightarrow [(\text{not } A) \text{ or } (\text{not } B)]$, and thus $(\text{not } C) \Longrightarrow [(\text{not } A) \text{ or}$
$(\text{not } B)]$. Hence, $(A \text{ and } B) \Longrightarrow C$.

Fact 1.7.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets, and let Z be a statement that depends on elements of $\mathcal{X}$ and $\mathcal{Y}$. Then, the following statements are equivalent:

i) Not[for all $x \in \mathcal{X}$, Z holds].

ii) There exists $x \in \mathcal{X}$ such that Z does not hold.

Furthermore, the following statements are equivalent:

iii) Not[there exists $y \in \mathcal{Y}$ such that Z holds].

iv) For all $y \in \mathcal{Y}$, Z does not hold.

Finally, the following statements are equivalent:

v) Not[for all $x \in \mathcal{X}$, there exists $y \in \mathcal{Y}$ such that Z holds].

vi) There exists $x \in \mathcal{X}$ such that, for all $y \in \mathcal{Y}$, Z does not hold.

1.8 Facts on Sets

Fact 1.8.1. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of a set $\mathcal{X}$. Then, the following statements hold:

i) $\mathcal{A} \cap \mathcal{A} = \mathcal{A} \cup \mathcal{A} = \mathcal{A}$.

ii) $\mathcal{A} \backslash \mathcal{B} = \mathcal{A} \cap \mathcal{B}^\sim$.

iii) $(\mathcal{A} \cup \mathcal{B})^\sim = \mathcal{A}^\sim \cap \mathcal{B}^\sim$.

iv) $(\mathcal{A} \cap \mathcal{B})^\sim = \mathcal{A}^\sim \cup \mathcal{B}^\sim$.

v) $(\mathcal{A} \backslash \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$.

vi) $\mathcal{A} \backslash (\mathcal{A} \cap \mathcal{B}) = \mathcal{A} \cap \mathcal{B}^\sim$.

vii) $\mathcal{A} \cap (\mathcal{A}^\sim \cup \mathcal{B}) = \mathcal{A} \cap \mathcal{B}$.

viii) $(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{B}^\sim) = \mathcal{A}$.

ix) $[\mathcal{A} \backslash (\mathcal{A} \cap \mathcal{B})] \cup \mathcal{B} = \mathcal{A} \cup \mathcal{B}$.

x) $(\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A}^\sim \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{B}^\sim) = \mathcal{A} \cap \mathcal{B}$.

xi) $(\mathcal{A}^\sim \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{B}^\sim) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A}^\sim \cap \mathcal{B}^\sim) = [(\mathcal{A} \cup \mathcal{B}) \backslash (\mathcal{A} \cap \mathcal{B})]^\sim = [(\mathcal{A} \cap \mathcal{B}^\sim) \cup (\mathcal{A}^\sim \cap \mathcal{B})]^\sim$.

Remark: *iii*) and *iv*) are De Morgan's laws. See Fact 1.7.1.

Fact 1.8.2. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be subsets of a set $\mathcal{X}$. Then, the following statements hold:

i) $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$.

ii) $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$.

iii) $(\mathcal{A} \backslash \mathcal{B}) \backslash \mathcal{C} = \mathcal{A} \backslash (\mathcal{B} \cup \mathcal{C})$.

iv) $(\mathcal{A} \cap \mathcal{B}) \backslash \mathcal{C} = (\mathcal{A} \backslash \mathcal{C}) \cap (\mathcal{B} \backslash \mathcal{C})$.

v) $(\mathcal{A} \cap \mathcal{B}) \backslash (\mathcal{C} \cap \mathcal{B}) = (\mathcal{A} \backslash \mathcal{C}) \cap \mathcal{B}$.

vi) $(\mathcal{A} \cup \mathcal{B}) \backslash \mathcal{C} = (\mathcal{A} \backslash \mathcal{C}) \cup (\mathcal{B} \backslash \mathcal{C}) = [\mathcal{A} \backslash (\mathcal{B} \cup \mathcal{C})] \cup (\mathcal{B} \backslash \mathcal{C})$.

vii) $(\mathcal{A} \cup \mathcal{B}) \backslash (\mathcal{C} \cap \mathcal{B}) = (\mathcal{A} \backslash \mathcal{B}) \cup (\mathcal{B} \backslash \mathcal{C})$.

viii) $\mathcal{A} \backslash (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \backslash \mathcal{B}) \cap \mathcal{A} \backslash \mathcal{B})$.

ix) $\mathcal{A} \backslash (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \backslash \mathcal{B}) \cup \mathcal{A} \backslash \mathcal{B})$.

Fact 1.8.3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be subsets of a set $\mathcal{X}$. Then, the following statements hold:

i) $\mathcal{A} \ominus \varnothing = \varnothing \ominus \mathcal{A} = \mathcal{A}$, $\mathcal{A} \ominus \mathcal{A} = \varnothing$.

ii) $\mathcal{A} \ominus \mathcal{B} = \mathcal{B} \ominus \mathcal{A}$.

iii) $\mathcal{A} \ominus \mathcal{B} = (\mathcal{A} \cap \mathcal{B}^\sim) \cup (\mathcal{B} \cap \mathcal{A}^\sim) = (\mathcal{A} \backslash \mathcal{B}) \cup (\mathcal{B} \backslash \mathcal{A}) = (\mathcal{A} \cup \mathcal{B}) \backslash (\mathcal{A} \cap \mathcal{B})$.

iv) $\mathcal{A} \ominus \mathcal{B} = \{x \in \mathcal{X}\colon (x \in \mathcal{A}) \text{ xor } (x \in \mathcal{B})\}$.

v) $\mathcal{A} \ominus \mathcal{B} = \varnothing$ if and only if $\mathcal{A} = \mathcal{B}$.

vi) $\mathcal{A} \ominus (\mathcal{B} \ominus \mathcal{C}) = (\mathcal{A} \ominus \mathcal{B}) \ominus \mathcal{C}$.

vii) $(\mathcal{A} \ominus \mathcal{B}) \ominus (\mathcal{B} \ominus \mathcal{C}) = \mathcal{A} \ominus \mathcal{C}$.

viii) $\mathcal{A} \cap (\mathcal{B} \ominus \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \ominus (\mathcal{A} \cap \mathcal{C})$.

If, in addition, $\mathcal{A}$ and $\mathcal{B}$ are finite, then

$$\operatorname{card}(\mathcal{A} \ominus \mathcal{B}) = \operatorname{card}(\mathcal{A}) + \operatorname{card}(\mathcal{B}) - 2\operatorname{card}(\mathcal{A} \cap \mathcal{B}).$$

Fact 1.8.4. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be finite sets. Then,

$$\operatorname{card}(\mathcal{A} \times \mathcal{B}) = \operatorname{card}(\mathcal{A}) \operatorname{card}(\mathcal{B}),$$

$$\operatorname{card}(\mathcal{A} \cup \mathcal{B}) = \operatorname{card}(\mathcal{A}) + \operatorname{card}(\mathcal{B}) - \operatorname{card}(\mathcal{A} \cap \mathcal{B}),$$

$$\begin{aligned}\operatorname{card}(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) = {} & \operatorname{card}(\mathcal{A}) + \operatorname{card}(\mathcal{B}) + \operatorname{card}(\mathcal{C}) - \operatorname{card}(\mathcal{A} \cap \mathcal{B}) - \operatorname{card}(\mathcal{A} \cap \mathcal{C}) - \operatorname{card}(\mathcal{B} \cap \mathcal{C}) \\ & + \operatorname{card}(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}).\end{aligned}$$

Remark: The second and third equalities are versions of the *inclusion-exclusion principle*. See [benjaminquinn, p. 82], [herman2, p. 67], and [rpstanley, pp. 64–67]. **Remark:** The inclusion-exclusion principle holds for multisets $\mathcal{A}$ and $\mathcal{B}$ with "$\mathcal{A} \cup \mathcal{B}$" defined as the smallest multiset that contains both $\mathcal{A}$ and $\mathcal{B}$. For example, $\operatorname{card}(\{1,1,2,2\}) = \operatorname{card}(\{1,1,2\} \cup \{1,2,2\}) = \operatorname{card}(\{1,1,2\}) + \operatorname{card}(\{1,2,2\}) - \operatorname{card}(\{1,2\})$; that is, $4 = 3+3-2$. See [wildbergermultisets,].

Fact 1.8.5. Define $\mathcal{A} \triangleq \{x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n\}_{\mathrm{ms}}$, where, for all $i \in \{1, \ldots, n\}$, k_i is the number of repetitions of x_i. Then, the number of multisubsets of $\mathcal{A}$ is $\prod_{i=1}^n (k_i + 1)$. **Source:** [singhsingh,].

Fact 1.8.6. Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$. Then, the following statements hold:

i) $\sup(-\mathcal{A}) = -\inf A$.

ii) $\inf(-\mathcal{A}) = -\sup A$.

iii) $\sup(\mathcal{A} + \mathcal{B}) = \sup \mathcal{A} + \sup \mathcal{B}$.

iv) $\sup(\mathcal{A} - \mathcal{B}) = \sup \mathcal{A} - \inf \mathcal{B}$.

v) $\inf(\mathcal{A} + \mathcal{B}) = \inf \mathcal{A} + \inf \mathcal{B}$.

vi) $\inf(\mathcal{A} - \mathcal{B}) = \inf \mathcal{A} - \sup \mathcal{B}$.

vii) $\sup(\mathcal{A} \cup \mathcal{B}) = \max\{\sup \mathcal{A}, \sup \mathcal{B}\}$.

viii) $\inf(\mathcal{A} \cup \mathcal{B}) = \min\{\inf \mathcal{A}, \inf \mathcal{B}\}$.

ix) If $0 \notin \mathcal{A}$, then

$$\sup\left\{\frac{1}{x} \colon x \in \mathcal{A}\right\} = \max\left\{\frac{1}{\inf[\mathcal{A} \cap (-\infty, 0)]}, \frac{1}{\inf[\mathcal{A} \cap (0, \infty)]}\right\}.$$

x) $\sup\{xy \colon x \in \mathcal{A}, y \in \mathcal{B}\} = \max\{(\inf \mathcal{A}) \inf \mathcal{B}, (\inf \mathcal{A}) \sup \mathcal{B}, (\sup \mathcal{A}) \inf \mathcal{B}, (\sup \mathcal{A}) \sup \mathcal{B}\}$.

Source: [kaczor1, p. 3].

Fact 1.8.7. Let $S_1, \ldots, S_m$ be finite sets, and let $n \triangleq \sum_{i=1}^m \operatorname{card}(S_i)$. Then,

$$\left\lceil \frac{n}{m} \right\rceil \le \max_{i \in \{1, \ldots, m\}} \operatorname{card}(S_i).$$

In particular, if $m < n$, then there exists $i \in \{1, \ldots, m\}$ such that $\operatorname{card}(S_i) \ge 2$. **Remark:** This is the *pigeonhole principle*.

Fact 1.8.8. Let $S_1, \ldots, S_m$ be sets, assume that, for all $i \in \{1, \ldots, m\}$, $\operatorname{card}(S_i) = n$, and assume that, for all distinct $i, j \in \{1, \ldots, m\}$, $\operatorname{card}(S_i \cap S_j) \le k$. Then,

$$\frac{n^2 m}{n + (m-1)k} \le \operatorname{card}\left(\cup_{i=1}^m S_i\right).$$

Source: [jukna, p. 23].

Fact 1.8.9. Let X be a set, let $n \triangleq \operatorname{card}(X)$, let $S_1, \ldots, S_m \subseteq X$, and assume that, for all distinct $i, j \in \{1, \ldots, m\}$, $S_i \backslash S_j$ and $S_j \backslash S_i$ are nonempty. Then, $m \le \binom{n}{\lfloor n/2 \rfloor}$. **Source:** [matousek, p. 57]. **Remark:** This is a *Sperner lemma.*

Fact 1.8.10. Let X be a set, let $n \triangleq \operatorname{card}(X)$, let $S_1, \ldots, S_m \subseteq X$, let $k \le n/2$, assume that, for all $i \in \{1, \ldots, m\}$, $\operatorname{card}(S_i) = k$, and, for all distinct $i, j \in \{1, \ldots, m\}$, $S_i \cap S_j$ is nonempty. Then, $m \le \binom{n-1}{k-1}$. **Source:** [matousek, p. 57]. **Remark:** This is the *Erdös-Ko-Rado theorem.*

Fact 1.8.11. Let X be a set, let $n \triangleq \operatorname{card}(X)$, let $S_1, \ldots, S_m \subseteq X$, assume that, for all $i \in \{1, \ldots, m\}$, $\operatorname{card}(S_i)$ is odd, and, for all distinct $i, j \in \{1, \ldots, m\}$, $\operatorname{card}(S_i \cap S_j)$ is even. Then, $m \le n$. **Source:** [matousek, p. 57]. **Remark:** This is the *oddtown theorem.*

Fact 1.8.12. Let X be a set, let $n \triangleq \operatorname{card}(X)$, let $S_1, \ldots, S_m \subseteq X$, let $p \ge 2$ be prime, and assume that, for all $i \in \{1, \ldots, m\}$, $\operatorname{card}(S_i) = 2p - 1$, and, for all distinct $i, j \in \{1, \ldots, m\}$, $\operatorname{card}(S_i \cap S_j) \ne p-1$. Then, $m \le \sum_{i=1}^{p-1} \binom{n}{i}$. **Source:** [matousek, p. 58]. **Remark:** Excluding intersections of cardinality $p - 1$ restricts the number of possible subsets of X.

Fact 1.8.13. Let X be a set, let $S_1, \ldots, S_m, T_1, \ldots, T_m \subseteq X$, let $k \ge 1$ and $l \ge 1$, and assume that, for all $i \in \{1, \ldots, m\}$, $\operatorname{card}(S_i) = k$, $\operatorname{card}(T_i) = l$, and $S_i \cap T_i = \varnothing$, and, for all $i, j \in \{1, \ldots, m\}$ such that $i < j$, $S_i \cap T_j \ne \varnothing$. Then, $m \le \binom{k+l}{l}$. **Source:** [matousek, pp. 171–173].

Fact 1.8.14. Let S be a set, and let $\mathcal{S}$ denote the set of all subsets of S. Then, "$\subset$" and "$\subseteq$" are transitive relations on $\mathcal{S}$, and "$\subseteq$" is a partial ordering on $\mathcal{S}$.

Fact 1.8.15. Define the relation $\mathcal{R}$ on $\mathbb{R} \times \mathbb{R}$ by

$$\mathcal{R} \triangleq \{((x_1, y_1), (x_2, y_2)) \in (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})\colon x_1 \le x_2 \text{ and } y_1 \le y_2\}.$$

Then, $\mathcal{R}$ is a partial ordering.

Fact 1.8.16. Define the relation $\mathcal{L}$ on $\mathbb{R} \times \mathbb{R}$ by

$$\mathcal{L} \triangleq \{((x_1, y_1), (x_2, y_2)) \in (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})\colon x_1 \le x_2 \text{ and, if } x_1 = x_2, \text{ then } y_1 \le y_2\}.$$

Then, $\mathcal{L}$ is a total ordering on $\mathbb{R} \times \mathbb{R}$.

Remark: Denoting this total ordering by "$\preceq$," note that $(1, 4) \preceq (2, 3)$ and $(1, 4) \preceq (1, 5)$. **Remark:** This ordering is the *lexicographic ordering* or *dictionary ordering*, where "book" $\preceq$ "box". Note that the ordering of words in a dictionary is reflexive, antisymmetric, and transitive, and that every pair of words can be ordered. **Related:** Fact **??**.

Fact 1.8.17. Let $n \ge 1$ and $x_1, \ldots, x_{n^2+1} \in \mathbb{R}$. Then, at least one of the following statements holds:

i) There exist $1 \le i_1 \le \cdots \le i_{n+1} \le n^2 + 1$ such that $x_{i_1} \le \cdots \le x_{i_{n+1}}$.

ii) There exist $1 \le i_1 \le \cdots \le i_{n+1} \le n^2 + 1$ such that $x_{i_1} \ge \cdots \ge x_{i_{n+1}}$.

Source: [radu, p. 53] and [steeleerdos,]. **Remark:** This is the *Erdös-Szekeres theorem.*

1.9 Facts on Graphs

Fact 1.9.1. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. Then, the following statements hold:

i) $\mathcal{R}$ is the graph of a function on $\mathcal{X}$ if and only if every node in $\mathcal{X}$ has exactly one child.

Furthermore, the following statements are equivalent:

ii) $\mathcal{R}$ is the graph of a one-to-one function on $\mathcal{X}$.

iii) $\mathcal{R}$ is the graph of an onto function on $\mathcal{X}$.

iv) $\mathcal{R}$ is the graph of a one-to-one and onto function on $\mathcal{X}$.

v) Every node in $\mathcal{X}$ has exactly one child and not more than one parent.

vi) Every node in $\mathcal{X}$ has exactly one child and at least one parent.

vii) Every node in $\mathcal{X}$ has exactly one child and exactly one parent.

Related: Fact 1.10.1.

Fact 1.9.2. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph, and assume that $\mathcal{R}$ is the graph of a function $f\colon\ \mathcal{X} \mapsto \mathcal{X}$. Then, either f is the identity function or $\mathcal{G}$ has a directed cycle.

Fact 1.9.3. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph, and assume that $\mathcal{G}$ has a directed Hamiltonian cycle. Then, $\mathcal{G}$ has no roots and no leaves.

Fact 1.9.4. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. Then, $\mathcal{G}$ has either a root or a directed cycle.

Fact 1.9.5. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. If $\mathcal{G}$ is a directed tree, then it is not transitive.

Fact 1.9.6. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph, and assume that $\mathcal{G}$ is directionally acyclic. Furthermore, for all $x, y \in \mathcal{X}$, let "$x \preceq y$" denote the existence of directional path from x to y. Then, "$\preceq$" is a partial ordering on $\mathcal{X}$. **Remark:** This result provides the foundation for the *Hasse diagram*, which illustrates the structure of a partially ordered set. See [schroderbook,trotter,].

Fact 1.9.7. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a directed graph. If $\mathcal{G}$ is a directed forest, then $\mathcal{G}$ is directionally acyclic.

Fact 1.9.8. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a symmetric graph, and let $n = \operatorname{card}(\mathcal{X})$. Then, the following statements are equivalent:

i) $\mathcal{G}$ is a forest.

ii) $\mathcal{G}$ is acyclic.

iii) No pair of nodes in $\mathcal{X}$ is connected by more than one path.

Furthermore, the following statements are equivalent:

iv) $\mathcal{G}$ is a tree.

v) $\mathcal{G}$ is a connected forest.

vi) $\mathcal{G}$ is connected and has no cycles.

vii) $\mathcal{G}$ is connected and has $n-1$ edges.

viii) $\mathcal{G}$ has no cycles and has $n-1$ edges.

ix) Every pair of nodes in $\mathcal{X}$ is connected by exactly one path.

Fact 1.9.9. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a tournament. Then, $\mathcal{G}$ has a directed Hamiltonian path. If, in addition, $\mathcal{G}$ is directionally connected, then $\mathcal{G}$ has a directed Hamiltonian cycle. **Remark:** The second statement is *Camion's theorem.* See [bangjensen, p. 16]. **Remark:** The directed edges in a tournament distinguish winners and losers in a contest where every player (that is, node) encounters every other player exactly once.

Fact 1.9.10. Let $\mathcal{G} = (\mathcal{X}, \mathcal{R})$ be a symmetric graph without self-edges, where $\mathcal{X} \subset \mathbb{R}^2$, assume that $v \triangleq \operatorname{card}(\mathcal{X}) \geq 3$, assume that $\mathcal{G}$ is connected, and assume that the edges in $\mathcal{R}$ can be represented by line segments that lie in the same plane and that pairwise either are disjoint or intersect at a node. Furthermore, let e denote the number of edges of $\mathcal{G}$, and let f denote the number of disjoint regions in $\mathbb{R}^2$ whose boundaries are the edges of $\mathcal{G}$. Then,

$$f + v - e = 2, \quad \frac{3}{2}f \leq e \leq 3v - 6, \quad f \leq 2v - 4.$$

If, in addition, $\mathcal{G}$ has no triangles, then $e \leq 2v - 4$. **Source:** [pearls, pp. 162–166] and [trudeau, pp. 97–116]. **Remark:** The equality gives the *Euler characteristic* for a planar graph. A related result for the surfaces of a convex polyhedron is given by Fact **??**. See [richeson,].

1.10 Facts on Functions

Fact 1.10.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite sets, and let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. Then, the following statements hold:

i) If $\operatorname{card}(\mathcal{X}) < \operatorname{card}(\mathcal{Y})$, then f is not onto.

ii) If $\operatorname{card}(\mathcal{Y}) < \operatorname{card}(\mathcal{X})$, then f is not one-to-one.

iii) If f is one-to-one and onto, then $\operatorname{card}(\mathcal{X}) = \operatorname{card}(\mathcal{Y})$.

Now, assume that $\operatorname{card}(\mathcal{X}) = \operatorname{card}(\mathcal{Y})$. Then, the following statements are equivalent:

iv) f is one-to-one.

v) f is onto.

vi) $\operatorname{card}[f(\mathcal{X})] = \operatorname{card}(\mathcal{X})$.

Related: Fact 1.9.1.

Fact 1.10.2. Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$ be invertible. Then, f^{Inv} is invertible, and $(f^{\mathsf{Inv}})^{\mathsf{Inv}} = f$.

Fact 1.10.3. Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. Then, for all $A, B \subseteq \mathcal{X}$, the following statements hold:

i) $A \subseteq f^{\mathsf{inv}}[f(A)] \subseteq \mathcal{X}$.

ii) $f^{\mathsf{inv}}[f(\mathcal{X})] = \mathcal{X} = f^{\mathsf{inv}}(\mathcal{Y})$.

iii) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

iv) $f(A \cap B) \subseteq f(A) \cap f(B)$.

v) $f(A \cup B) = f(A) \cup f(B)$.

vi) $f(A) \backslash f(B) \subseteq f(A \backslash B)$.

Furthermore, the following statements are equivalent:

vii) f is one-to-one.

viii) For all $A \subseteq \mathcal{X}$, $f^{\mathsf{inv}}[f(A)] = A$.

ix) For all $A, B \subseteq \mathcal{X}$, $f(A \cap B) = f(A) \cap f(B)$.

x) For all disjoint $A, B \subseteq \mathcal{X}$, $f(A)$ and $f(B)$ are disjoint.

xi) For all $A, B \subseteq \mathcal{X}$, $f(A) \backslash f(B) = f(A \backslash B)$.

Source: [apostolbook, pp. 44, 45] and [carothers, p. 64]. **Remark:** To show that equality does not necessarily hold in *iv)*, let $f(x) = x^2$, $A = [-2, 1]$, and $B = [-1, 2]$. Then, $f(A \cap B) = [0, 1] \subset [0, 4] = f(A) \cap f(B)$. **Related:** Fact **??**.

Fact 1.10.4. Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. Then, for all $A, B \subseteq \mathcal{Y}$, the following statements hold:

i) $f[f^{\mathsf{inv}}(A)] = A \cap f(\mathcal{X}) \subseteq A$.

ii) $f[f^{\mathsf{inv}}(\mathcal{Y})] = f(\mathcal{X})$.

iii) If $A \subseteq B$, then $f^{\mathsf{inv}}(A) \subseteq f^{\mathsf{inv}}(B)$.

iv) $f^{\mathsf{inv}}(A \cap B) = f^{\mathsf{inv}}(A) \cap f^{\mathsf{inv}}(B)$.

v) $f^{\mathsf{inv}}(A \cup B) = f^{\mathsf{inv}}(A) \cup f^{\mathsf{inv}}(B)$.

vi) $f^{\mathsf{inv}}(A)\backslash f^{\mathsf{inv}}(B) = f^{\mathsf{inv}}(A\backslash B)$.

In addition, the following statements are equivalent:

vii) f is onto.

viii) For all $A \subseteq \mathcal{Y}$, $f[f^{\mathsf{inv}}(A)] = A$.

Source: [apostolbook, pp. 44, 45] and [carothers, p. 64]. **Related:** Fact **??**.

Fact 1.10.5. Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$. Then, the following statements hold:

i) If f is invertible, then, for all $y \in \mathcal{Y}$, $f^{\mathsf{inv}}(y) = \{f^{\mathsf{Inv}}(y)\}$.

ii) Assume that f is left invertible, and define $\hat{f}\colon \mathcal{X} \mapsto \mathcal{R}(f)$, where, for all $x \in \mathcal{X}$, $\hat{f}(x) \triangleq f(x)$. Then, $\hat{f}$ is invertible, and, for all $y \in \mathcal{R}(f)$, $f^{\mathsf{inv}}(y) = \{\hat{f}^{\mathrm{Inv}}(y)\}$.

iii) If f is left invertible and f^{L} is a left inverse of f, then, for all $y \in \mathcal{R}(f)$, $f^{\mathsf{inv}}(y) = \{f^{\mathsf{L}}(y)\}$.

iv) If f is right invertible and f^{R} is a right inverse of f, then, for all $y \in \mathcal{Y}$, $f^{\mathsf{R}}(y) \in f^{\mathsf{inv}}(y)$.

Related: Fact **??**.

Fact 1.10.6. Let $g\colon\ \mathcal{X} \mapsto \mathcal{Y}$ and $f\colon\ \mathcal{Y} \mapsto \mathcal{Z}$. Then, the following statements hold:

i) If $A \subseteq \mathcal{Z}$, then $(f \circ g)^{\mathsf{inv}}(A) = g^{\mathsf{inv}}[f^{\mathsf{inv}}(A)]$.

ii) $f \circ g$ is one-to-one if and only if g is one-to-one and the restriction $\hat{f}\colon g(\mathcal{X}) \mapsto \mathcal{Z}$ of f is one-to-one. If these conditions hold and g^{L} and $\hat{f}^{\mathsf{L}}$ are left inverses of g and $\hat{f}$, respectively, then $g^{\mathsf{L}} \circ \hat{f}^{\mathsf{L}}$ is a left inverse of $f \circ g$.

iii) $f \circ g$ is onto if and only if the restriction $\hat{f}\colon g(\mathcal{X}) \mapsto \mathcal{Z}$ of f is onto. Let $\hat{g}\colon \mathcal{X} \mapsto g(\mathcal{X})$, where, for all $x \in \mathcal{X}$, $\hat{g}(x) = g(x)$. If these conditions hold and $\hat{g}^{\mathsf{R}}$ and $\hat{f}^{\mathsf{R}}$ are right inverses of $\hat{g}$ and $\hat{f}$, respectively, then $\hat{g}^{\mathsf{R}} \circ \hat{f}^{\mathsf{R}}$ is a right inverse of $f \circ g$.

iv) $f \circ g$ is invertible if and only if g is one-to-one and the restriction $\hat{f}\colon g(\mathcal{X}) \mapsto \mathcal{Z}$ of f is one-to-one and onto. If these conditions hold, g^{L} is a left inverse of g, and $\hat{f}^{\mathsf{Inv}}$ is the inverse of $\hat{f}$, then $(f \circ g)^{\mathsf{Inv}} = g^{\mathsf{L}} \circ \hat{f}^{\mathsf{Inv}}$.

Remark: A matrix version of this result is given by Fact **??** and Fact **??**.

Fact 1.10.7. Let $f\colon\ \mathcal{X} \mapsto \mathcal{Y}$, let $g\colon\ \mathcal{Y} \mapsto \mathcal{X}$, and assume that f and g are one-to-one. Then, there exists $h\colon\ \mathcal{X} \mapsto \mathcal{Y}$ such that h is one-to-one and onto. **Source:** [duren, pp. 311, 312] and [moschovakis, pp. 16, 17]. **Remark:** This is the *Schroeder-Bernstein theorem.*

Fact 1.10.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets, let $f\colon \mathcal{X} \mapsto \mathcal{Y}$, and, for $i \in \{1, 2\}$, let $g_i\colon \mathcal{R}(f) \mapsto \mathbb{F}^n$ and $\alpha_i \in \mathbb{F}$. Then, $(\alpha_1 g_1 + \alpha_2 g_2) \circ f = \alpha_1(g_1 \circ f) + \alpha_2(g_2 \circ f)$. **Remark:** The composition operator $\mathcal{C}(g, f) \triangleq g \circ f$ is linear in its first argument.

1.11 Facts on Integers

Fact 1.11.1. Let $n, m \ge 0$ and $k, l \ge 2$. Then, $\prod_{i=1}^{k}(n+i) \ne m^l$. **Source:** [erdosselfridge,

]. **Remark:** A product of consecutive integers cannot be a power of an integer.

Fact 1.11.2. Let n be an integer. Then, $n(n+1)(n+2)(n+3)+1 = (n^2+3n+1)^2$. Hence, $n(n+1)(n+2)(n+3)+1$ is a square. **Example:** $5(6)(7)(8)+1 = 41^2$. **Related:** Fact **??**.

Fact 1.11.3. Let x be a real number, and assume that $x+\frac{1}{2}$ is not an integer. Then, the integer closest to x is $\lfloor x+\frac{1}{2}\rfloor$.

Fact 1.11.4. Let w, x, y, and z be real numbers, and let n and m be integers. Then, the following statements hold:

i) If $w|x$ and $y|z$, then $wy|xz$.
ii) If $x|y$ and $x|z$, then $x^2|yz$.
iii) If $x|y$, then $x|ny$.
iv) If $x|y$ and $y|z$, then $x|z$.
v) If $x|y$ and $x|z$, then $x|my+nz$.

Fact 1.11.5. Let n and m be integers, at least one of which is nonzero. Then, the following statements hold:

i) Assume that m is positive. Then, there exist unique integers q and $r \in [0, m-1]$ such that $n = qm + r$. In particular, $q = \lfloor n/m \rfloor$ and $r = \mathrm{rem}_m(n) = n - qm = n - m\lfloor n/m \rfloor \in [0, m-1]$.
ii) If m is positive, then $\lceil n/m \rceil = \lfloor (n+m-1)/m \rfloor$.
iii) If $n|m$, then $\gcd\{n,m\} = |n|$.
iv) If k is prime and $k|mn$, then either $k|m$ or $k|n$.
v) $\gcd\{n/\gcd\{n,m\}, m/\gcd\{n,m\}\} = 1$.
vi) If both n and m are prime and $m \neq n$, then n and m are coprime.
vii) If $n > 0$ and $m > 0$, then $1 \le \gcd\{n,m\} \le \min\{n, m, |n-m|\}$.
viii) $(\mathrm{lcm}\{n,m\})\gcd\{n,m\} = |nm|$.
ix) n and m are coprime if and only if $\mathrm{lcm}\{n,m\} = |nm|$.
x) There exist integers k, l such that $\gcd\{n,m\} = kn + lm$.

Now, assume that n and m are coprime, and let k be an integer. Then, the following statements hold:

xi) $\gcd\{n-m, n+m, nm\} = 1$.
xii) $\gcd\{n^k - m^k, n^k + m^k\} \le 2$.
xiii) $\gcd\{(n-m)^k, (n+m)^k\} \le 2^k$.
xiv) $\gcd\{n^2 - nm + m^2, n+m\} \le 3$.
xv) $\gcd\{nk, m\} = \gcd\{k, m\}$.

Finally, let $n_1, \ldots, n_k$ and $m_1, \ldots, m_l$ be integers. Then, the following statement holds:

xvi) $\gcd\{n_1m_1, n_1m_2, \ldots, n_km_l\} = (\gcd\{n_1, \ldots, n_k\})\gcd\{m_1, \ldots, m_l\}$.

Source: [sav, p. 12]. *x)*–*xiv)* are given in [larson, pp. 86, 89, 105]; *xv)* is given in [grimaldi, p. 123]. **Example:** $\gcd\{221, 754\} = 13 = -17(221) + 5(754)$. See [larson, pp. 86, 87]. **Remark:** The first set in *xvi)* contains kl products. **Remark:** *x)* is the *GCD identity*. See [andersonfeil, p. 17].

Fact 1.11.6. Let $l, m, n \ge 1$. Then, the following statements hold:

i) $\gcd\{l,m,n\} = \gcd\{\gcd\{l,m\}, \gcd\{m,n\}, \gcd\{n,l\}\}$.
ii) $lmn = (\gcd\{lm, mn, nl\})\,\mathrm{lcm}\{l,m,n\}$.

iii) $\gcd\{l, \mathrm{lcm}\{m, n\}\} = \mathrm{lcm}\{\gcd\{l, m\}, \gcd\{l, n\}\}$.

iv) $\mathrm{lcm}\{l, \gcd\{m, n\}\} = \gcd\{\mathrm{lcm}\{l, m\}, \mathrm{lcm}\{l, n\}\}$.

v) $\gcd\{\mathrm{lcm}\{l, m\}, \mathrm{lcm}\{m, n\}, \mathrm{lcm}\{n, l\}\} = \mathrm{lcm}\{\gcd\{l, m\}, \gcd\{m, n\}, \gcd\{n, l\}\}$.

vi) $lmn \gcd\{l, m, n\} = (\mathrm{lcm}\{l, m, n\})(\gcd\{l, m\})(\gcd\{m, n\}) \gcd\{n, l\}$.

vii) $\gcd\{l, m\} = \gcd\{l + m, \mathrm{lcm}\{l, m\}\}$.

viii)
$$\frac{(\gcd\{l, m, n\})^2}{\gcd\{l, m\} \gcd\{m, n\} \gcd\{n, l\}} = \frac{(\mathrm{lcm}\{l, m, n\})^2}{\mathrm{lcm}\{l, m\} \mathrm{lcm}\{m, n\} \mathrm{lcm}\{n, l\}}.$$

Source: [larson, p. 105]. *i)* is given in [five, pp. 25, 144]; *viii)* is given in [gelca, p. 310].

Fact 1.11.7. Let $n \ge 1$. Then, $\gcd\{n^2+1, (n+1)^2+1\} \in \{1, 5\}$. Furthermore, $\gcd\{n^2 + 1, (n + 1)^2 + 1\} = 5$ if and only if $n \overset{5}{\equiv} 2$. **Source:** [five, pp. 31, 165].

Fact 1.11.8. Let $k_1, \ldots, k_n$ be positive integers, and assume that $k_1 < \cdots < k_n$. Then,
$$\sum_{i=1}^{n-1} \frac{1}{\mathrm{lcm}\{k_i, k_{i+1}\}} \le 1 - \frac{1}{2^{n-1}}.$$

Source: [sav, p. 12].

Fact 1.11.9. Let m and n be integers. Then, the following statements are equivalent:

i) Either both m and n are even or both m and n are odd.

ii) $n \overset{2}{\equiv} m$.

Furthermore, the following statements are equivalent:

iii) $m|n$.

iv) $n \overset{|m|}{\equiv} 0$.

v) $n \overset{|m|}{\equiv} m$.

Fact 1.11.10. Let $k \ge 1$, and let m, n, p, q be integers. Then, the following statements hold:

i) If $n = m$, then $n \overset{k}{\equiv} m$.

ii) $n \overset{k}{\equiv} n$.

Furthermore, the following statements are equivalent:

iii) $k|(n - m)$.

iv) $n \overset{k}{\equiv} m$.

v) $m \overset{k}{\equiv} n$.

vi) $-n \overset{k}{\equiv} -m$.

vii) $n - m \overset{k}{\equiv} 0$.

Furthermore, the following statement holds:

viii) If $n \overset{k}{\equiv} m$ and $m \overset{k}{\equiv} p$, then $n \overset{k}{\equiv} p$.

Next, if $p \overset{k}{\equiv} q$ and $n \overset{k}{\equiv} m$, then the following statements hold:

ix) $n + p \overset{k}{\equiv} m + q$.

x) $n - p \overset{k}{\equiv} m - q$.

xi) $np \overset{k}{\equiv} mq$.

Finally, the following statements hold:

xii) If $n \overset{k}{\equiv} m$, and p is a positive integer, then $pn \overset{k}{\equiv} pm$.

xiii) If $n \overset{k}{\equiv} m$, and p is a positive integer, then $n^p \overset{k}{\equiv} m^p$.

xiv) If $pn \overset{k}{\equiv} pm$, then $n \overset{k/\gcd\{k,p\}}{\equiv} m$.

xv) If $pn \overset{k}{\equiv} pm$ and $\gcd\{k,p\} = 1$, then $n \overset{k}{\equiv} m$.

xvi) $k!|\prod_{i=0}^{k-1}(n+i)$. For example, $11(12)(13) = 6(286)$ and $(22)(23)\cdots(28) = 5040(1184040)$.

xvii) If $n \overset{k}{\equiv} n_0$ and $m \overset{k}{\equiv} m_0$, then $nm \overset{k}{\equiv} \mathrm{rem}_k(n_0m_0)$.

Source: *xiv)* is given in [underwood, pp. 30, 31]. **Remark:** "$\overset{k}{\equiv}$" is an equivalence relation on $\mathbb{Z}$, which partitions $\mathbb{Z}$ into *residue classes*.

Fact 1.11.11. Let $n \geq 1$, and let m be the sum of the decimal digits of n. Then, the following statements hold:

i) $3|n$ if and only if $3|m$.

ii) $n \overset{9}{\equiv} m$.

Source: [underwood, pp. 31, 32].

Fact 1.11.12. Let n be a positive integer. Then, the following statements hold:

i) $n^2 \overset{3}{\equiv} 0$ if and only if $n \overset{3}{\equiv} 0$.

ii) $n^2 \overset{3}{\equiv} 1$ if and only if either $n \overset{3}{\equiv} 1$ or $n \overset{3}{\equiv} 2$.

Source: [nelsenpww,]. **Example:** $3 \overset{3}{\equiv} 6 \overset{3}{\equiv} 9 \overset{3}{\equiv} 12 \overset{3}{\equiv} 15 \overset{3}{\equiv} 0$, $9 \overset{3}{\equiv} 36 \overset{3}{\equiv} 81 \overset{3}{\equiv} 144 \overset{3}{\equiv} 225 \overset{3}{\equiv} 0$, $1 \overset{3}{\equiv} 4 \overset{3}{\equiv} 7 \overset{3}{\equiv} 10 \overset{3}{\equiv} 13 \overset{3}{\equiv} 1$, $2 \overset{3}{\equiv} 5 \overset{3}{\equiv} 8 \overset{3}{\equiv} 11 \overset{3}{\equiv} 14 \overset{3}{\equiv} 2$, and $1 \overset{3}{\equiv} 4 \overset{3}{\equiv} 16 \overset{3}{\equiv} 25 \overset{3}{\equiv} 49 \overset{3}{\equiv} 64 \overset{3}{\equiv} 100 \overset{3}{\equiv} 121 \overset{3}{\equiv} 169 \overset{3}{\equiv} 196 \overset{3}{\equiv} 1$.

Fact 1.11.13. Let $k, l, m, n \geq 1$. Then, the following statements hold:

i) If $m \leq n$ is prime, then m does not divide $n! + 1$. Hence, there exists a prime $k \in [n+1, n!+1]$ such that $k|n! + 1$.

ii) None of the integers $n! + 2, n! + 3, \ldots, n! + n$ are prime.

iii) Assume that $n \geq 2$ is not prime, and let k be the smallest prime such that $k|n$. Then, $k \leq \sqrt{n}$. If, in addition, $\sqrt[3]{n} < k$, then n/k is prime.

iv) If n is prime, then $(2^{n-1} - 1)/n$ is an integer.

v) If $n \geq 3$ is odd, then $n^2 \overset{8}{\equiv} 1$.

vi) If n is prime and $n \geq 5$, then either $n \overset{6}{\equiv} 1$ or $n \overset{6}{\equiv} 5$.

vii) If $n \overset{8}{\equiv} 7$, then n is not the sum of three squares of integers.

viii) If $n \overset{9}{\equiv} 4$, then n is not the sum of three cubes of integers.

ix) The last digit of n^2 is neither 2, 3, 7, nor 8.

x) Neither 3 nor 5 divides $(n+1)^3 - n^3$.

xi) If $n \geq 2$, then $n^4 + 4^n$ is not prime.

xii) $3|n(n^2 - 3n + 8)$, $6|n^3 + 5n$, $8|(n-1)(n^3 - 5n^2 + 18n - 8)$.

xiii) $9|4^n + 15n - 1$, $30|n^5 - n$, $120|n^5 - 5n^3 + 4n$.

xiv) 121 does not divide $n^2 + 3n + 5$.

xv) $3^{n+1} | 2^{3^n} + 1$.

xvi) 2^n does not divide $n!$.

xvii) If $m \le n$, then $m! | n^{\underline{m}}$.

xviii) $\gcd\{2^m - 1, 2^n - 1\} = 2^{\gcd\{m,n\}} - 1$. Hence, $n|m$ if and only if $2^n - 1 | 2^m - 1$.

xix) If n and 6 are coprime, then $24 | n^2 - 1$.

xx) If n is even, then $n^2 - 1 | 2^{n!} - 1$.

xxi) If $6 | k + l + m$, then $6 | k^3 + l^3 + m^3$.

xxii) If $n \ge 4$ and $m \ge 4$ are prime, then $24 | n^2 - m^2$.

xxiii) If n is not prime, then $2^n - 1$ is not prime. Furthermore, if $n \in \{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279\}$, then $2^n - 1$ is prime.

xxiv) If $n \ge 1$ and $2^n + 1$ is prime, then there exists $k \ge 0$ such that $n = 2^k$. If $k \in \{0, 1, 2, 3, 4\}$ then $2^{2^k} + 1$ is prime. If $5 \le k \le 32$, then $2^{2^k} + 1$ is not prime.

xxv) If $n \ge 3$ is odd, then $3 \nmid 2^n - 1$.

xxvi) If $n \ge 4$ is even, then $3 \nmid 2^n + 1$.

xxvii) If $4^n + 2^n + 1$ is prime, then there exists a positive integer m such that $n = 3^m$.

xxviii) If $n \ge 5$ is prime, then there exists a positive integer m such that $n = \sqrt{24m + 1}$.

xxix) $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$ is a positive integer.

xxx) If $n \ge 12$, then $\sqrt{n^2 - 19n + 89}$ is not an integer.

xxxi) If $n = k^2 + l^2 + m^2$, then there exist positive integers p, q, r such that $n^2 = p^2 + q^2 + r^2$.

xxxii) There exist infinitely many multisets of integers $\{x, y, z, p, q, r\}_{\rm ms}$ such that $x^2 + y^2 + z^2 = p^3 + q^3 + r^3$.

xxxiii) If m and n are coprime, then $\{m + in \colon i \ge 1\}$ contains an infinite number of primes.

xxxiv) If n is prime, then $(k + l)^n \overset{n}{\equiv} k^n + l^n$.

xxxv) If $kl = mn$, then $k + l + m + n$ is not prime.

xxxvi) $2(k^4 - l^4) \ne m^2$.

xxxvii) If k, l, m, n are nonnegative, $\{k, l\} \ne \{m, n\}$, and $k^2 + l^2 = m^2 + n^2$, then $k^2 + l^2$ is not prime.

xxxviii) If m and n are prime and $m < n$, then $mn | \binom{n+m}{m} - \binom{n}{m} - 1$.

xxxix) If $n \ge 3$ is prime, then $([(n - 1)/2]!)^2 \overset{n}{\equiv} (-1)^{(n+1)/2}$.

xl) $10 | 1 + 8^n - 3^n - 6^n$.

xli) $5 | 1 + 2^n + 3^n + 4^n$ if and only if $n/4$ is not an integer.

xlii) If $k \ge 3$ and $l \ge 5$ are consecutive primes, then $k + l$ is the product of at least three primes.

xliii) $\sqrt{n^4 + 2n^3 + 2n^2 + 2n + 1}$ is not an integer.

Source: [AMR, pp. 595–598], [engel, pp. 118, 131–137, 208]. *iii)* is given in [underwood, pp. 13, 19]; *viii)* and *ix)* are given in [underwood, pp. 31, 33]; *xi)* is given in [engel, p. 120]; *xx)* is given in [gelca, p. 266]; *xxix)* is given in [larson, p. 64]; *xxxiii)* is given in [steinfourier, chapter 8]; *xxxiv)* is given in [mollnf, p. 68]; *xxxv)–xxxvii)* are given in [AMR, pp. 595–599]; *xlviii)* is given in [AMR2, pp. 51, 294, 295]; *xxxix)* is given in [aebi2,]; *xl)–xliii)* are given in [five, pp. 7, 11, 32, 36, 72, 73, 82, 167, 178]. **Remark:** *vi)* implies that, if n is prime and $n \ge 5$, then there exists a positive integer k such that either $n = 6k - 1$ or $n = 6k + 1$.

For example, $23 \overset{6}{\equiv} 5$ and $31 \overset{6}{\equiv} 1$. For $k = 20$, neither $n = 6k-1 = 119 = 7(17)$ nor $n = 6k+1 = 121 = 11^2$ is prime. **Remark:** *i*) and *xxxiii*) imply that there are an infinite number of primes. **Remark:** *xxxiv*) is *Dirichlet's theorem.* **Remark:** The prime numbers 2^n-1 listed in *xxiii*) are *Mersenne primes.* It is unknown whether or not there exist infinitely many Mersenne primes. **Remark:** The prime numbers 2^n+1 listed in *xxiv*), namely, 3, 5, 17, 257, 65537, are *Fermat primes.* These are the only known Fermat primes. **Example:** In *xxxvii*), $1^2+7^2 = 5^2+5^2 = 50$, $1^2+8^2 = 4^2+7^2 = 65$, and $0^2+10^2 = 6^2+8^2 = 100$ are not prime. **Example:** In *xlviii*), $2(3)|\binom{5}{2}-\binom{3}{2}-1$; that is, $6|6$; $3(5)|\binom{8}{3}-\binom{5}{3}-1$, that is, $15|45$; and $11(13)|\binom{24}{11}-\binom{13}{11}-1$, that is, $143|2496065$.

Fact 1.11.14. Let $n \ge 1$. Then, there exist n consecutive positive integers whose sum of squares is prime if and only if $n \in \{2,3,6\}$. **Source:** [AMR3, pp. 74, 75]. **Example:** $1^2+2^2 = 5$, $2^2+3^2+4^2 = 29$, and $2^2+3^2+4^2+5^2+6^2+7^2 = 139$.

Fact 1.11.15. Let $n \ge 2$ be prime, and let $k \ge 1$. Then, $n|k^n-k$. Equivalently, $k^n \overset{n}{\equiv} k$. **Source:** [benjaminquinn, p. 115], [dragovic,], [engel, p. 119]. **Remark:** This is *Fermat's little theorem.* **Remark:** An equivalent statement is the following: Let n be prime, let k be a positive integer, and assume that n and k are coprime. Then, $k^{n-1} \overset{n}{\equiv} 1$. See [underwood, p. 42]. **Example:** $4^7-4 = 7(2340)$ and $13^3-13 = 3(728)$. **Remark:** $341|2^{341}-2$, but $341 = 11(31)$ is not prime. See [engel, p. 120].

Fact 1.11.16. Let $n \ge 2$ be prime, and let k and l be positive integers. Then, $(k+l)^n \overset{n}{\equiv} k^n+l^n$. **Source:** [apagodufd,].

Fact 1.11.17. Let $n \ge 2$. Then, n is prime if and only if $\sum_{i=1}^{n-1} i^{n-1} \overset{n}{\equiv} n-1$. **Remark:** Necessity follows from Fermat's little theorem given by Fact 1.11.15. Sufficiency is a conjecture. **Example:** $1^6+2^6+3^6+4^6+5^6+6^6 = 67171 = 7(9595)+6 \overset{7}{\equiv} 6$.

Fact 1.11.18. Let n be prime, and let $k \ge 1$. Then,

$$\sum_{i=1}^{n} i^k \overset{p}{\equiv} \begin{cases} -1, & n-1|k, \\ 0, & n-1\nmid k. \end{cases}$$

Source: [macmillansondow,]. **Example:** Let $n = 3$ and $k = 2$. Then, $1^2+2^2+3^2 = 14 \overset{3}{\equiv} -1$.

Fact 1.11.19. Let $n \ge 5$ be prime. Then,

$$\sum_{i=0}^{n-1} \binom{2i}{i} \overset{n}{\equiv} \begin{cases} 1, & n \overset{3}{\equiv} 1, \\ -1, & n \overset{3}{\equiv} 2. \end{cases}$$

Source: [apagodufd,].

Fact 1.11.20. Let $n \ge 2$. Then, n is prime if and only if $n|(n-1)!+1$. **Remark:** This is *Wilson's theorem.* **Remark:** $n|(n-1)!+1$ is equivalent to $(n-1)! \overset{n}{\equiv} -1$. **Example:** $4!+1 = 5(5)$ and $12!+1 = 13(36846277)$.

Fact 1.11.21. Let $n \ge 3$. Then, n is prime if and only if $\prod_{i=1}^{n-1}(2^i-1) \overset{2^n-1}{\equiv} n$. **Remark:** This is *Vantieghem's theorem.* **Example:** $4!+1 = 5(5)$. **Source:** [gaitanas2,].

Fact 1.11.22. Let $p \ge 2$ be prime and let $1 \le n \le p$. Then, $p|(n-1)!(p-n)!+(-1)^{n+1}$. **Source:** [mollnf, p. 67]. **Remark:** This is an extension of Wilson's theorem given by Fact 1.11.20. **Example:** $4!6!+1 = 11(1571)$ and $13!9!-1 = 23(98246143821913)$.

Fact 1.11.23. Let $m, n \ge 1$. Then, $(m^2-n^2)^2+(2mn)^2 = (m^2+n^2)^2$. **Remark:** This result characterizes all *Pythagorean triples* within an integer multiple. **Example:** If $m = 2$

and $n=1$, then $3^2+4^2=5^2$; if $m=3$ and $n=2$, then $5^2+12^2=13^2$; if $m=4$ and $n=1$, then $8^2+15^2=17^2$; if $m=4$ and $n=3$, then $7^2+24^2=25^2$.

Fact 1.11.24. Let $n\ge 1$. Then, there exist $k\ge 1$ and $\delta_1,\ldots,\delta_k\in\{-1,1\}$ such that $n=\sum_{i=1}^k\delta_i i^2$. **Source:** [five, pp. 33, 171] and [gelca, p. 9]. **Example:** $7=1-4-9+16+25-36$, $12=-1+4+9$, and $18=1-4-9+16+25-36-49-64+81-100+121$.

Fact 1.11.25. Let n be a positive integer. Then, the number of 4-tuples of integers (j,k,l,m) such that $j^2+k^2+l^2+m^2=n$ is equal to 8 times the sum of the distinct divisors of n that are not divisible by 4. **Source:** Fact **??** and [aez,zprice,]. **Remark:** This is *Jacobi's four-square theorem.* **Example:** The distinct divisors of 4 that are not divisible by 4 are 1 and 2. Accordingly, the number of ways of writing 4 as a sum of squares of the components of a 4-tuple of integers is 24. Two of these are $0^2+0^2+0^2+2^2$ and $1^2+(-1)^2+1^2+1^2$.

Fact 1.11.26. Let $n\ge 0$. Then, the following statements hold:

i) There exist nonnegative integers $m_1,\ldots,m_4$ such that $n=\sum_{i=1}^4 m_i^2$.

ii) There exist nonnegative integers $m_1,\ldots,m_9$ such that $n=\sum_{i=1}^9 m_i^3$.

iii) There exist nonnegative integers $m_1,\ldots,m_{19}$ such that $n=\sum_{i=1}^{19} m_i^4$.

iv) There exist nonnegative integers $m_1,\ldots,m_{37}$ such that $n=\sum_{i=1}^{37} m_i^5$.

v) There exist nonnegative integers $m_1,\ldots,m_{73}$ such that $n=\sum_{i=1}^{73} m_i^6$.

Source: [gruber, pp. 372, 373]. **Remark:** These are solutions of *Waring's problem.* The first result is *Lagrange's four-square theorem.* For example, $3=0^2+1^2+1^2+1^2$ and $310=1^2+2^2+4^2+17^2$.

Fact 1.11.27. Let $n\ge 0$. Then, the following statements hold:

i) There exist nonnegative integers $m_1,\ldots,m_4$ such that $n=m_1^2+m_2^2+m_3^2+m_4^2$.

ii) There exist nonnegative integers $m_1,\ldots,m_4$ such that $n=m_1^2+m_2^2+2m_3^2+2m_4^2$.

iii) There exist nonnegative integers $m_1,\ldots,m_4$ such that $n=m_1^2+2m_2^2+4m_3^2+14m_4^2$.

iv) $\operatorname{rem}(n,4)\ne 3$ if and only if there exist nonnegative integers $m_1,\ldots,m_4$ such that $n=m_1^2+m_2^2+4m_3^2+4m_4^2$.

v) If $n\ge 2$, then there exist nonnegative integers $m_1,\ldots,m_4$ such that $n=2m_1^2+3m_2^2+4m_3^2+5m_4^2$.

vi) Let k_1,k_2,k_3,k_4 be positive integers, and assume that, for all $k\in\{1,2,3,5,6,7,10,14,15\}$, there exist nonnegative integers $m_1,\ldots,m_4$ such that $k=k_1m_1^2+k_2m_2^2+k_3m_3^2+k_4m_4^2$. Then, there exist nonnegative integers $m_1,\ldots,m_4$ such that $n=k_1m_1^2+k_2m_2^2+k_3m_3^2+k_4m_4^2$.

Remark: *i)*–*iii)* are universal positive integer-matrix quaternary quadratic forms. There are 54 such forms. See [lure, pp. 123–125] and [williamsfour,]. **Related:** Fact **??**.

Fact 1.11.28. Let i,j,k,l be odd positive integers. Then, there exist even nonnegative integers q,r,s,t such that $q^2+r^2+s^2+t^2=i^2+j^2+k^2+l^2$. If, in addition, i,j,k,l are distinct, then so are q,r,s,t. **Example:** $1^2+3^2+5^2+7^2=0^2+2^2+4^2+8^2$. **Source:** [nelsensquares,]. **Related:** Fact **??**.

Fact 1.11.29. Let $n\ge 1$, let $d_1,\ldots,d_l$ be the distinct positive divisors of n, and, for all $i\in\{1,\ldots,l\}$, let a_i denote the number of distinct positive divisors of d_i. Then,

$$\sum_{i=1}^l a_i^3=\left(\sum_{i=1}^l a_i\right)^2.$$

Source: [sav, p. 64]. **Remark:** This is *Liouville's theorem.* **Related:** Fact 1.12.1. **Example:** Let $n = 8$ so that $d_1 = 1$, $d_2 = 2$, $d_3 = 4$, $d_4 = 8$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_4 = 4$. Then, $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$. Let $n = 15$ so that $d_1 = 1$, $d_2 = 3$, $d_3 = 5$, $d_4 = 15$, $a_1 = 1$, $a_2 = 2$, $a_3 = 2$, and $a_4 = 4$. Then, $1^3 + 2^3 + 2^3 + 4^3 = (1 + 2 + 2 + 4)^2$.

Fact 1.11.30. The following statements hold:

i) $1^2 + 7^2 = 5^2 + 5^2 = 50$, $1^2 + 8^2 = 4^2 + 7^2 = 65$, $2^2 + 9^2 = 6^2 + 7^2 = 85$, $2^2 + 11^2 = 5^2 + 10^2 = 125$.

ii) $5^2 + 14^2 = 10^2 + 11^2 = 221$, $4^2 + 19^2 = 11^2 + 16^2 = 377$, $7^2 + 24^2 = 15^2 + 20^2 = 25^2 = 625$.

iii) $1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2 = 325$, $20^2 + 107^2 = 43^2 + 100^2 = 68^2 + 85^2 = 11849$.

iv) $15^2 + 70^2 = 30^2 + 65^2 = 34^2 + 63^2 = 47^2 + 54^2 = 5125$, $10^2 + 11^2 + 12^2 = 13^2 + 14^2 = 365$.

v) $25^2 + 60^2 = 33^2 + 56^2 = 16^2 + 63^2 = 39^2 + 52^2 = 65^2 = 4225$, $1 + 3 + 3^2 + 3^3 + 3^4 = 11^2$.

vi) $7^2 + 74^2 = 14^2 + 73^2 = 22^2 + 71^2 = 25^2 + 70^2 = 41^2 + 62^2 = 50^2 + 55^2 = 5525$.

vii) $5^2 + 17^2 + 18^2 = 9^2 + 14^2 + 19^2 = 638$, $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2 = 2030$.

viii) $36^2 + 37^2 + 38^2 + 39^3 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2 = 7230$.

ix) $55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2 = 19855$.

x) $297^2 = (88 + 209)^2 = 88209$, $7777^2 = (6048 + 1729)^2 = 60481729$.

xi) $3^3 + 4^3 + 5^3 = 6^3 = 216$, $58^3 + 59^3 + 69^3 = 90^3 = 729000$, $1^3 + 12^3 = 9^3 + 10^3 = 1729$.

xii) $10^3 + 27^3 = 19^3 + 24^3 = 20683$, $4^3 + 48^3 = 36^3 + 40^3 = 110656$, $1 + 18 + 18^2 = 7^3$.

xiii) $167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3 = 87539319$, $11^3 + 12^3 + 13^3 + 14^3 = 20^3 = 8000$.

xiv) $31^3 + 33^3 + 35^3 + 37^3 + 39^3 + 41^3 = 66^3 = 287496$, $2^4 + 2^4 + 3^4 + 4^4 + 4^4 = 5^4$.

xv) $59^4 + 158^4 = 133^4 + 134^4 = 635318657$, $30^4 + 120^4 + 272^4 + 315^4 = 353^4 = 15527402881$.

xvi) $240^4 + 340^4 + 430^4 + 599^4 = 651^4 = 179607287601$.

xvii) $27^5 + 84^5 + 110^5 + 133^5 = 144^5 = 61917364224$, $1 + 7 + 7^2 + 7^3 = 20^2$.

xviii) $1^6 - 2^6 + 3^6 = 3(6 + 6^3) = 2^2 + 3^2 + 5^2 + 7^2 + 11^2 + 13^2 + 17^2 = 666$.

xix) $95800^4 + 217519^4 + 414560^4 = 422481^4 = 31858749840007945920321$.

xx) $3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6 = 160426514$, $13^2 + 7^3 = 2^9$, $2^7 + 17^3 = 71^2$.

xxi) $10^7 + 14^7 + 123^7 + 149^7 = 15^7 + 90^7 + 129^7 + 146^7 = 2056364173794800$.

xxii) $81^8 + 539^8 + 966^8 = 158^8 + 310^8 + 481^8 + 725^8 + 954^8 = 765381793634649192581218$.

xxiii) $42^9 + 99^9 + 179^9 + 475^9 + 542^9 + 574^9 + 625^9 + 668^9 + 822^9 + 851^9 = 917^9 = 458483827502199203411828597$.

xxiv) $62^{10} + 115^{10} + 172^{10} + 245^{10} + 295^{10} + 533^{10} + 689^{10} + 927^{10} + 1011^{10} + 1234^{10} + 1603^{10} + 1684^{10} = 1772^{10} = 303518810756415395921574821458201$.

xxv) For all $i \in \{1, 2, 3\}$, $1^i + 21^i + 36^i + 56^i = 2^i + 18^i + 39^i + 55^i$.

xxvi) For all $i \in \{1, 3, 9\}$, $1^i + 13^i + 13^i + 14^i + 18^i + 23^i = 5^i + 9^i + 10^i + 15^i + 21^i + 22^i$.

xxvii) For all $i \in \{-1, 1\}$, $4^i + 10^i + 12^i = 5^i + 6^i + 15^i$, $6^i + 14^i + 14^i = 7^i + 9^i + 18^i$, and $3^i + 40^i = 4^i + 15^i + 24^i = 5^i + 8^i + 30^i$.

xxviii) For all $i \in \{-2, -1, 1, 2\}$, $(-230)^i + (-92)^i + 23^i + 46^i = (-220)^i + (-110)^i + 22^i + 55^i$.

xxix) For all $i \in \{1, 2, 6\}$, $83^i + 211^i + (-300)^i = (-124)^i + (-185)^i + 303^i$, and $43^i +$

$371^i + (-372)^i = 140^i + 307^i + (-405)^i$.

xxx) For all $i \in \{1,3,5\}$, $(-51)^i + (-33)^i + (-24)^i + 7^i + 13^i + 38^i + 50^i = (-134)^i + (-75)^i + (-66)^i + 8^i + 47^i + 87^i + 133^i = 0$.

xxxi) For all $i \in \{1,2,3,9\}$, $(-621)^i + 51^i + 253^i + 412^i + 600^i = (-624)^i + 187^i + 100^i + 429^i + 603^i$.

xxxii) For all $i \in \{1,3,5,7\}$, $(-98)^i + (-82)^i + (-58)^i + (-34)^i + 13^i + 16^i + 69^i + 75^i + 99^i = (-169)^i + (-161)^i + (-119)^i + (-63)^i + 8^i + 50^i + 132^i + 148^i + 174^i = 0$.

xxxiii) For all $i \in \{1,2,3,4,5\}$, $(-461)^i + (-233)^i + (-199)^i + 465^i + 237^i + 203^i = (-435)^i + (-343)^i + 1^i + 3^i + 347^i + 439^i$.

xxxiv) $13! = 112296^2 - 79896^2 = 6227020800$.

Source: [bremner9,castellanos2,choudhrylike,ekl7,harper,Mclaughlin,piezas,] and [posamentier, pp. 48, 49]. **Remark:** *xvii)* and *xix)* are counterexamples to Euler's conjecture, which states that, for all $n \geq 4$, the nth power of a positive integer cannot be decomposed into the sum of $n-1$ or fewer nth powers of integers. Euler's conjecture is true in the case $n = 3$; that is, the cube of a positive integer cannot be the sum of the cubes of two positive integers. This case is given by Fact 1.11.39.

Fact 1.11.31. Let i, j, k, l be positive integers. Then, there exist positive integers m, n, r, s such that $\{m, n\} \neq \{r, s\}$ and

$$(i^2 + j^2)(k^2 + l^2) = m^2 + n^2 = r^2 + s^2.$$

In particular, $m = |ik - jl|$, $n = jk + il$, $r = ik + jl$, and $s = |il - jk|$. **Source:** Fact **??** and [nahinit, pp. 25, 26]. **Example:** $(2^2 + 3^2)(4^2 + 5^2) = 533 = 7^2 + 22^2 = 23^2 + 2^2$.

Fact 1.11.32. Let $k, m, n \geq 1$, assume that $k > m + n$, let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be integers, and assume that $\sum_{i=1}^m x_i^k = \sum_{i=1}^n y_i^k$. Then, $m = n$ and $x^\downarrow = y^\downarrow$. **Remark:** This is *Euler's extended conjecture*. See [Ekl,LPS,].

Fact 1.11.33. Let $n \geq 0$. Then, there exist $k, l \geq 0$ such that $n = k^2 + l^2$ if and only if n does not have a prime factor of the form $4k + 3$ raised to an odd exponent. **Source:** [aigner, Chapter 4] and [sierpinski, p. 378]. **Remark:** $29 = 2^2 + 5^2$, but neither 27, 71, nor 243 is the sum of two squares.

Fact 1.11.34. Let $n \geq 0$. Then, there exist $k, l, m \geq 0$ such that $n = k^2 + l^2 + m^2$ if and only if there do not exist $i, j \geq 0$ such that $n = 4^i(8j + 7)$. Hence, if $k, l \geq 1$, $k \stackrel{8}{\equiv} 3$, and $l \stackrel{8}{\equiv} 5$, then $kl \stackrel{8}{\equiv} 7$, and thus kl is not the sum of three squares. **Source:** [grosswald, p. 38] and [lure, p. 59]. **Remark:** $14 = 1^2 + 2^2 + 3^2$, but $15 = 4^0(8 \cdot 1 + 7)$ is not the sum of three squares.

Fact 1.11.35. Let $n \geq 0$. Then, there exist positive integers k, l, m such that $k < l < m$ and $n = k^2 + l^2 - m^2$. **Source:** [sav, pp. 56, 57]. **Example:** $0 = 3^2 + 4^2 - 5^2$, $1 = 4^2 + 7^2 - 8^2$, and $2 = 5^2 + 11^2 - 12^2$.

Fact 1.11.36. Let $l, m, n \geq 1$. Then, there exist integers j, k such that $j^2 + k^2 = (l^2 + m^2)^n$. **Source:** [larson, p. 115]. **Example:** $(2^2 + 3^2)^3 = 2197 = 9^2 + 46^2$.

Fact 1.11.37. Let $n \geq 1$. Then, the following statements are equivalent:

i) There exist $k, l \geq 1$ such that $n = k^3 + l^3$.

ii) There exists a divisor m of n such that $\sqrt[3]{n} \leq m \leq 2^{2/3}\sqrt[3]{n}$, $3|m^2 - n/m$, and $\sqrt{\frac{4n}{3m} - \frac{m^2}{3}}$ is an integer.

Furthermore, the following statements are equivalent:

iii) There exist $k, l \geq 1$ such that $n = k^3 - l^3$.

iv) There exists a divisor m of n such that $1 \le m \le \sqrt[3]{n}$, $3|m^2 - \frac{n}{m}$, and $\sqrt{\frac{4n}{3m} - \frac{m^2}{3}}$ is an integer.

Source: [broughan,]. **Example:** $91 = 3^3 + 4^3$ and $m = 7$.

Fact 1.11.38. Let $n \ge 2$. Then, H_n is not an integer. **Source:** [larson, p. 105].

Fact 1.11.39. Let $k, l, m \ge 1$ and $n \ge 3$. Then, $k^n + l^n \ne m^n$. **Remark:** This is *Fermat's last theorem.* **Credit:** A. Wiles.

Fact 1.11.40. Let $n \ge 2$ be prime, and assume that $n \overset{4}{\equiv} 1$. Then, there exist $k, l \ge 1$ such that $n = k^2 + l^2$. **Source:** [AAR, p. 41] and [zagier90,]. **Credit:** P. de Fermat. **Example:** $29 = 4 + 25$ and $89 = 25 + 64$.

Fact 1.11.41. Let $k, l, m, n \ge 2$, and assume that $k^l - m^n = 1$. Then, $k = 3$, $l = 2$, $m = 2$, and $n = 3$. **Remark:** This is *Catalan's conjecture.* **Credit:** P. Mihăilescu.

Fact 1.11.42. Let $n \ge 1$. Then, there exists a prime $m \in (n, 2n]$. If, in addition, $n \ge 2898242$, then there exists a prime $m \in (n, n + n/(111 \log^2 n)]$. **Source:** [aigner, Chapter 2] and [axler,trudgian,]. **Remark:** The first statement is *Bertrand's postulate.*

Fact 1.11.43. Let $n \ge 20$, and, for all $i \ge 1$, let p_i denote the ith prime. Then,

$$n(\log n + \log\log n - \tfrac{3}{2}) < p_n < n(\log n + \log\log n - \tfrac{1}{2}).$$

Source: [havil, p. 183] and [rosser,].

Fact 1.11.44. Let $n \ge 1$, and, for all $i \ge 1$, let p_i denote the ith prime. Then, $\prod p_i \le 4^n$, where the product is taken over all i such that $p_i \le n$. **Source:** [mollnf, p. 90].

Fact 1.11.45. For all $i \ge 1$, let p_i denote the ith prime. Then, for all $k \ge 4$, $p_{k+1}^2 < \prod_{i=1}^k p_i$. **Remark:** This is *Bonse's inequality.* **Remark:** $121 < 210$ and $169 < 2310$.

Fact 1.11.46. Let $n \ge 4$ be even. Then, there exist primes k and l such that $n = k + l$. **Remark:** This is the *Goldbach conjecture.* **Example:** $44 = 13 + 31$ and $100 = 17 + 83$. **Remark:** An incomplete proof is given in [bruckmanGC,].

Fact 1.11.47. Let $n \ge 1$, and let d_n denote the sum of all positive integers (not counting multiplicity) that divide n. Then, $d_n \le H_n + e^{H_n} \log H_n$. **Remark:** This result is equivalent to the *Riemann hypothesis.* See [borweinchoi, p. 48] and [lagarias,]. Equivalent statements are given by Fact **??**. **Remark:** Let $r_n \triangleq d_n/(H_n + e^{H_n} \log H_n)$. Then, $r_{12} \approx .98864$, $r_{120} \approx .98344$, $r_{360} \approx .97111$, and $r_{2520} \approx .97831$.

Fact 1.11.48. Let $n \ge 1$, let $\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\} = \{1, \dots, 2n\}$, and assume that $a_1 < \cdots < a_n$ and $b_n < \cdots < b_1$. Then, $\sum_{i=1}^n |a_i - b_i| = n^2$. **Source:** [sav, p. 66].

Fact 1.11.49. If $n \ge 1$, then there exist finitely many multisets $\{k_1, \dots, k_n\}_{\rm ms}$ of positive integers such that $\sum_{i=1}^n \frac{1}{k_i} = 1$. Now, define $S_1 \triangleq 2$ and, for all $n \ge 2$, define $S_n \triangleq 1 + \prod_{i=1}^{n-1} S_i$. In particular, $(S_i)_{i=1}^6 = (2, 3, 7, 43, 1807, 3263443)$. If $n \ge 2$ and the positive integers $k_1, \dots, k_n$ satisfy $\sum_{i=1}^n \frac{1}{k_i} = 1$, then $\max\{k_1, \dots, k_n\} \le S_{n-1} - 1$. **Source:** [rose, p. 288] and [sandowmac,].

Fact 1.11.50. Let $n \ge 1$. Then,

$$\frac{4}{4n+1} = \frac{1}{n} - \frac{1}{n(4n+1)}, \quad \frac{4}{4n-1} = \frac{1}{n} + \frac{1}{n(4n-1)}.$$

If n is odd, then

$$\frac{4}{n} = \frac{2}{n-1} + \frac{2}{n+1} - \frac{4}{n(n^2-1)}.$$

If $n \stackrel{3}{\equiv} 2$, then

$$\frac{4}{n} = \frac{1}{n} + \frac{3}{n+1} + \frac{3}{n(n+1)}.$$

Source: [wikiEfr,]. **Remark:** These equalities concern *Egyptian fractions* and are associated with the Erdös-Straus conjecture. See Fact 1.11.51.

Fact 1.11.51. Let $n \geq 2$. Then, there exist $k, l, m \geq 1$ such that $4/n = 1/k + 1/l + 1/m$. **Example:** $4/5 = 1/2 + 1/4 + 1/20 = 1/2 + 1/5 + 1/10$. **Remark:** This is the *Erdös-Straus conjecture*. **Related:** Fact 1.11.50.

Fact 1.11.52. Let $n \geq 1$. Then, $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$. **Source:** [five, pp. 19, 119].

1.12 Facts on Finite Sums

Fact 1.12.1. Let $n, k \geq 1$. Then,

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1} \sum_{i=0}^{k} B_i \binom{k+1}{i} (n+1)^{k+1-i} = \frac{1}{k+1} \left[\left(\sum_{i=0}^{k+1} B_{k+1-i} \binom{k+1}{i} (n+1)^i \right) - B_{k+1} \right].$$

In particular,

$$\sum_{i=1}^{n} i = \binom{n+1}{2} = \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n^2 + \tfrac{1}{2}n,$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{4}\binom{2n+2}{3} = \binom{n+1}{2} + 2\binom{n+1}{3} = \tfrac{1}{6}n(n+1)(2n+1) = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n,$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \binom{n+1}{2}^2 = \tfrac{1}{4}n^2(n+1)^2 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2,$$

$$\sum_{i=1}^{n} i^4 = \tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) = \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n,$$

$$\sum_{i=1}^{n} i^5 = \tfrac{1}{12}n^2(n+1)^2(2n^2+2n-1) = \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2,$$

$$\sum_{i=1}^{n} i^6 = \tfrac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1) = \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n,$$

$$\sum_{i=1}^{n} i^7 = \tfrac{1}{24}n^2(n+1)^2(3n^4+6n^3-4n+2),$$

$$\sum_{i=1}^{n} i^8 = \tfrac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3),$$

$$\sum_{i=1}^{n} i(i+1) = \tfrac{1}{3}n(n+1)(n+2), \quad \sum_{i=1}^{n} i(i+1)(i+2) = \tfrac{1}{4}n(n+1)(n+2)(n+3),$$

$$\sum_{i=1}^{n} i(i+1)^2 = \tfrac{1}{12}n(n+1)(n+2)(3n+5), \quad \sum_{i=1}^{n} i(i+1)^3 = \tfrac{1}{60}n(n+1)(n+2)(12n^2+39n+29),$$

$$\sum_{i=0}^{n-1}(2i+1) = n^2, \quad \sum_{i=0}^{n-1}(2i+1)^2 = \tfrac{1}{3}n(4n^2-1), \quad \sum_{i=0}^{n-1}(2i+1)^3 = n^2(2n^2-1),$$

$$\sum_{i=0}^{n-1}(2i+1)^4 = \tfrac{1}{15}n(48n^4-40n^2+7), \quad \sum_{i=0}^{n-1}(2i+1)^5 = \tfrac{1}{3}n^2(16n^4-20n^2+7),$$

$$\sum_{i=0}^{n-1}(2i+1)^6 = \tfrac{1}{21}n(4n^2-1)(48n^4-72n^2+31), \quad \sum_{i=0}^{n-1}(2i+1)^7 = \tfrac{1}{3}n^2(48n^6-112n^4+98n^2-31).$$

Now, let $k \ge 1$ and $n \ge 1$, and define $p_k(n) \triangleq \sum_{i=1}^n i^k$. Then, the following statements hold:

i) $p_k(n)$ is a polynomial whose degree is $k+1$ and whose leading coefficient is $1/(k+1)$.

ii) The coefficient of n in $p_k(n)$ is $(-1)^k B_k$.

iii) $p_k(1) = 1$.

iv) For all $z \in \mathbb{C}$, $p_k'(z) = kp_{k-1}(z) + (-1)^k B_k$.

v) p_2 divides p_{2k}, and p_3 divides p_{2k+1}.

vi) $p_1(n)$ divides $p_{2k+1}(n)$.

vii) $\sum_{i=1}^k \binom{k+1}{i} p_i(n) = (n+1)^{k+1} - n - 1$.

Source: The first equality is the *Bernoulli formula*, where B_i is the ith Bernoulli number. See Fact **??**. See [comtet, pp. 153–155], [GR, pp. 2, 3], [GKP, pp. 283, 284], and [torabi,]. *i)*–*iv)* are given in [wubernoulli,]; *v)* is given in [comtet, p. 155]; *vi)* is given in [macmillansondowdiv,]; and *vii)* is given in [schaum, p. 135]. **Remark:** *v)* is a statement about polynomials, whereas *vi)* is a statement about integers. **Remark:** A matrix approach to sums of powers of integers is given in [dubeau,]. The expressions involving binomial coefficients are given in [benjamincubes,] and [benjaminquinn, pp. 109–112]. See also [mackiw,]. **Related:** Fact 1.11.29, Fact 1.12.2, Fact **??**, and [herman, p. 11].

Fact 1.12.2. Let $n \ge 1$, let $k \ge 0$, and define $\sigma_k \triangleq \sum_{i=1}^n i^k$. Then,

$$\sigma_1 = \tfrac{1}{2}(n+\tfrac{1}{2})^2 - \tfrac{1}{8}, \quad \sigma_2 = \tfrac{1}{3}(n+\tfrac{1}{2})^3 - \tfrac{1}{12}(n+\tfrac{1}{2}), \quad 2\sigma_1^4 = \sigma_5 + \sigma_7,$$
$$\sigma_3 = \sigma_1^2, \quad \sigma_4 = (\tfrac{6}{5}\sigma_1 - \tfrac{1}{5})\sigma_2, \quad \sigma_5 = \tfrac{4}{3}\sigma_1^3 - \tfrac{1}{3}\sigma_1^2, \quad \sigma_6 = (\tfrac{12}{7}\sigma_1^2 - \tfrac{6}{7}\sigma_1 + \tfrac{1}{7})\sigma_2,$$
$$\sigma_7 = 2\sigma_1^4 - \tfrac{4}{3}\sigma_1^3 + \tfrac{1}{3}\sigma_1^2, \quad \sigma_1^3 = \tfrac{1}{4}\sigma_3 + \tfrac{3}{4}\sigma_5, \quad \sigma_1^5 = \tfrac{1}{16}\sigma_5 + \tfrac{5}{8}\sigma_7 + \tfrac{5}{16}\sigma_9,$$
$$8\sigma_1^3 + \sigma_1^2 - 9\sigma_2^2 = 0, \quad 81\sigma_2^4 - 18\sigma_2^2\sigma_3 + \sigma_3^2 - 64\sigma_3^3 = 0, \quad 16\sigma_3^3 - \sigma_3^2 - 6\sigma_3\sigma_5 - 9\sigma_5^2 = 0.$$

Furthermore,

$$\sum_{i=0}^{k}\binom{k+1}{i}\sigma_i = (n+1)^{k+1} - 1.$$

Next, define the polynomial

$$F_k(s) \triangleq \frac{1}{k+1}\sum_{i=0}^{k} B_i \binom{k+1}{i}(s+1)^{k+1-i}.$$

Then,

$F_3(s) = s^2,\ F_4(s) = \frac{6}{5}s - \frac{1}{5},\ F_5(s) = \frac{4}{3}s^3 - \frac{1}{3}s^2,\ F_6(s) = \frac{12}{7}s^2 - \frac{6}{7}s + \frac{1}{7},\ F_7(s) = 2s^4 - \frac{4}{3}s^3 + \frac{1}{3}s^2.$

If $k \ge 3$ is odd, then $\sigma_k = F_k(\sigma_1)$ and $\deg F_k = \frac{1}{2}(k+1)$. If $k \ge 2$ is even, then $\sigma_k = \sigma_2 F_k(\sigma_1)$ and $\deg F_k = \frac{1}{2}(k-2)$. **Source:** [beardon,]. **Remark:** F_k is a *Faulhaber polynomial.* Generating functions are given in [beardon,]. **Remark:** B_i is the ith Bernoulli number. See Fact **??**. **Related:** Fact 1.12.1.

Fact 1.12.3. For all $n \ge 0$, define the nth *triangular number* by $T_n \triangleq \frac{1}{2}n(n+1)$. Then, the following statements hold:

i) $(T_i)_{i=0}^{20} = (0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210)$.

ii) If $n \ge 1$, then $T_n = \sum_{i=1}^n i = \binom{n+1}{2} = \frac{1}{2}n(n+1) = \sqrt{\sum_{i=1}^n i^3}$.

iii) $2T_nT_{n+1} = T_{n^2+2n}$, $T_nT_{n+2} = 2T_{(n^2+3n)/2}$, and $T_n \overset{2}{\equiv} \lfloor (n+1)/2 \rfloor$.

iv) If $n \ge 1$, then $8T_n + 1 = (2n+1)^2$ and $T_n + T_{n+1} = (n+1)^2$.

v) If $n \ge 2$, then $8T_n + 1 = T_{n+1} + 6T_n + T_{n-1}$.

vi) If $n \ge 1$, then $T_n^2 = \sum_{i=1}^n i^3$, $T_{2n^2-1} = \sum_{i=1}^n (2i-1)^3$, and $\sum_{i=1}^{2n-1} (-1)^{i+1} T_i = n^2$.

vii) If $n \ge 1$, then $9T_n + 1 = T_{3n+1}$, $T_{(n+1)^2} = T_n^2 + T_{n+1}^2 = T_{n^2} + T_{n+1}^2 - T_{n-1}^2$, and $\sum_{i=0}^n 9^i = T_{(3^{n+1}-1)/2}$.

viii) If $n \ge 1$, then $\sum_{i=1}^n T_i = \frac{n}{3}T_{n+1} = \frac{1}{3}(n+2)T_n = \frac{1}{6}n(n+1)(n+2)$ and $2T_n^2 = T_{n^2} + n^3$.

ix) If $n \ge 2$, then $T_n^2 = T_n + T_{n-1}T_{n+1}$.

x) If $n \ge 2$, then none of $\sqrt[3]{T_n}$, $\sqrt[4]{T_n}$, $\sqrt[5]{T_n}$ are integers.

xi) If $n \ge 1$, then the last digit of T_n is not an element of $\{2, 4, 7, 9\}$.

xii) If $n \ge 1$ and T_n is prime, then $n = 2$ and $T_n = 3$.

xiii) If $n, m \ge 1$, then $T_{m+n} = T_m + T_n + mn$, $T_{mn} = T_mT_n + T_{m-1}T_{n-1}$, $T_{mn-1} = T_{m-1}T_n + T_mT_{n-1}$.

xiv) For all $n \ge 1$, let τ_n be the nth positive integer such that $T_{\tau_n - 1}$ is square. Then, $(\tau_i)_{i=1}^{12} = (1, 2, 9, 50, 289, 1682, 9801, 57122, 332929, 1940450, 11309769, 65918162)$.

xv) There are infinitely many square triangular numbers.

xvi) For all $n \ge 1$, $\tau_{n+2} = 6\tau_{n+1} - \tau_n - 2$.

xvii) For all $n \ge 1$, $\tau_n = \frac{1}{2}[(3-2\sqrt{2})^{n-1} + 1]^2/(3-2\sqrt{2})^{n-1}$.

xviii) For all $n \ge 1$, $\sqrt{\tau_n} - \sqrt{\tau_n - 1} = (\sqrt{2}-1)^{n-1}$.

xix) For all $n \ge 2$,

$$T_{\tau_n - 1} = \left[\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 2^i \binom{n-1}{2i}\right]^2 \left[\sum_{i=0}^{\lfloor (n-2)/2 \rfloor} 2^i \binom{n-1}{2i+1}\right]^2.$$

In particular, $(T_{\tau_i - 1})_{i=1}^9 = (0, 1, 36, 1225, 41616, 1413721, 48024900, 1631432881)$.

xx) For all $n \ge 1$, $T_{\tau_{n+2}} = (6\sqrt{T_{\tau_{n+1}}} - \sqrt{T_{\tau_n}})^2$.

xxi) $(\sqrt{T_{\tau_i - 1}})_{i=1}^{12} = (0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214, 46611179)$.

xxii) For all $n \ge 1$, $T_{\tau_{n+2}-1} = 34T_{\tau_{n+1}-1} - T_{\tau_n - 1} + 2$.

xxiii) For all $n \ge 1$, $T_{\tau_n - 1} = \frac{1}{32}[(1+\sqrt{2})^{2n-2} - (1-\sqrt{2})^{2n-2}]^2$.

xxiv) n is a triangular number and a Fibonacci number if and only if $n \in \{1, 3, 21, 55\}$.

xxv) Every nonnegative integer is the sum of three triangular numbers.

xxvi) Every triangular number except T_1 and T_3 is the sum of three positive triangular numbers.

xxvii) $T_{132}^2 + T_{143}^2 = T_{164}^2$.

xxviii) If $n \ge 3$, then $\prod_{i=1}^n T_i < T_{n!}$.

xxix) If $n \ge 0$, then $\sum_{i=0}^n \binom{n}{i} T_i = 2^{n-2}(T_{n+1} - 1)$.

xxx) If $n \ge 1$, then $T_{n^2+n-1} + T_{n^2+3n+1} = (n+1)^4$.

Source: [fasctrinumb,pandey,trottertri,]. *xiii)* and *xxviii)* are given in [asiru,]; *xviii)* is given in [ibstedt,]; *xix)* is given in [AMR2, pp. 55, 312, 313]; *xx)* is given in [keedwell,]; *xxv)* is given in [grosswald, p. 25] and [sunpent,]; *xxix)* is given in [gonzalezgci,]. *xxx)* is given in [marion,]. **Remark:** T_n is given by $P_2(n)$ in Fact 1.12.6. See Fact 1.20.1. **Related:** Fact **??**.

Fact 1.12.4. For all $n \in \mathbb{N}$, define the nth *pentagonal number* by $P_n \triangleq \frac{1}{2}n(3n-1)$. Then, the following statements hold:

i) $(P_i)_{i=0}^{18} = (0, 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330, 376, 425, 477)$.

ii) If $n \ge 1$, then $\frac{1}{n}\sum_{i=1}^n P_i = T_n$ and $3P_n = T_{3n-1}$.

iii) If n is a pentagonal number, then $\frac{1}{6}(\sqrt{24n+1} + 1) = n$.

iv) Let $n \ge 1$. Then, the following statements are equivalent:

a) n is a pentagonal number.

b) $24n + 1$ is a square, and $\sqrt{24n+1} \overset{6}{\equiv} 5$.

c) $\frac{1}{6}(\sqrt{24n+1} + 1)$ is an integer.

v) Every nonnegative integer is the sum of five pentagonal numbers.

vi) If $n \ge 1$ and $n \notin \{9, 21, 31, 43, 55, 89\}$, then n is the sum of four pentagonal numbers.

Finally, for all $n \ge 1$, define the nth *dual pentagonal number* by $P_n' \triangleq \frac{1}{2}n(3n+1)$. Then, the following statements hold:

vii) For all $n \ge 1$, $P_n < P_n' < P_{n+1}$.

viii) $(P_i')_{i=1}^{18} = (2, 7, 15, 26, 40, 57, 77, 100, 126, 155, 187, 222, 260, 301, 345, 392, 442, 495)$.

Source: [sunpent,wolframpent,]. **Remark:** For all $n \ge 1$, $P_n' = P_{-n}$. **Remark:** See [fuchs, pp. 45–47]. **Related:** Fact **??**.

Fact 1.12.5. For all $n \ge 0$, define the nth *generalized pentagonal number* by

$$g_n \triangleq \begin{cases} \frac{1}{8}(n+1)(3n+1), & n \text{ odd}, \\ \frac{1}{8}n(3n+2), & n \text{ even}. \end{cases}$$

Then, the following statements hold:

i) $(g_i)_{i=0}^{21} = (0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, 92, 100, 117, 126, 145, 155, 176)$.

ii) For all $n \ge 1$, $g_{2n-1} = \frac{1}{2}n(3n-1)$ and $g_{2n} = \frac{1}{2}n(3n+1)$.

iii) $(g_i)_{i=-\infty}^{\infty} = (\frac{1}{2}i(3i-1))_{i=-\infty}^{\infty}$.

iv) For all $n \ge 0$,

$$g_n \triangleq \begin{cases} P_{(n+1)/2}, & n \text{ odd}, \\ P_{-n/2} = P_{n/2}', & n \text{ even}. \end{cases}$$

v) $(g_0, g_1, g_2, g_3, g_4, \ldots) = (P_0, P_1, P_1', P_2, P_2', \ldots)$, where P_n is the nth pentagonal number and P_n' is the nth dual pentagonal number.

vi) Every nonnegative integer is the sum of three generalized pentagonal numbers.
Source: [wikipentagonal,guy,sunpent,]. **Related:** Fact 1.12.4 and Fact **??**.

Fact 1.12.6. Let $n, k \geq 1$, and define $P_k(n) \triangleq \operatorname{card}\{(i_1, \ldots, i_k) : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$. Then,

$$P_k(n) = \binom{n+k-1}{k}.$$

In particular,

$$P_2(n) = \sum_{i=1}^{n} i = \binom{n+1}{2} = \frac{n(n+1)}{2}, \quad P_2(n) = \binom{n+2}{3} = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \frac{n(n+1)(n+2)}{6},$$

$$P_3(n) = \binom{n+3}{4} = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{l=1}^{j} l = \frac{n(n+1)(n+2)(n+3)}{24}.$$

Remark: $P_2(n)$, $P_3(n)$, and $P_4(n)$ are the *triangular*, *tetrahedral*, and *pentatopic* numbers.
Remark: $P_k(n)$ is the number of k-element multisubsets of $\{1, \ldots, n\}$; that is, $P_k(n) = \binom{n}{k}_{\mathrm{r}}$. See Fact 1.16.16. **Related:** Fact 1.12.3.

Fact 1.12.7. Let $n \geq 0$ and $k \geq 3$, and define the (n,k) *polygonal number* $p_k(n) \triangleq (k-2)\binom{n}{2} + n$. Then, the following statements hold:

i) $p_k(n) = \frac{1}{2}n[(k-2)n + 4 - k]$.
ii) $p_3(n) = \frac{1}{2}n(n+1)$ is the nth triangular number.
iii) $p_4(n) = n^2$.
iv) $p_5(n) = \frac{1}{2}n(3n-1)$ is the nth pentagonal number.
v) $p_k(n) = \left.\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{x[(k-3)x+1]}{n!(1-x)^3}\right|_{x=0}$.
vi) Let $m \geq 0$. Then, there exist nonnegative integers $n_1, \ldots, n_k$ such that $m = \sum_{i=1}^{k} p_k(n_i)$.

Source: [andrewseureka,guy,sunpent,wolfpolygonal,]. **Credit:** The last statement is due to A. L. Cauchy.

Fact 1.12.8. Let $n \geq 1$. Then,

$$\sum_{i,j=1}^{n} |i-j| = \frac{1}{3}n(n^2-1), \quad \sum_{i,j=1}^{n} (i-j)^2 = \frac{1}{6}n^2(n^2-1).$$

Now, let $k \geq 1$, and define $\sigma_k \triangleq \sum_{i=1}^{n} i^k$. Then,

$$\sum_{i,j=1}^{n} |i^k - j^k| = 4\sigma_{k+1} - 2(n+1)\sigma_k, \quad \sum_{i,j=1}^{n} |(i-j)(i^k - j^k)| = 2n\sigma_{k+1} - n(n+1)\sigma_k.$$

Source: [benzceAS,]. **Related:** Fact **??**.

Fact 1.12.9. Let $n \geq 1$. Then,

$$\exp\sum_{i,j=1}^{n}\left|\log\frac{i}{j}\right| = \frac{\prod_{i=1}^{n} i^{4i}}{(n!)^{2n+2}}, \quad \exp\sum_{i,j=1}^{n}\left|(i-j)\log\frac{i}{j}\right| = \left(\frac{\prod_{i=1}^{n} i^{2i}}{(n!)^{n+1}}\right)^n.$$

Source: [benzceAS,]. **Related:** Fact **??**.

Fact 1.12.10. Let $1 \le k \le n$, let $r \in \mathbb{R}$, and define

$$S_{k,r} \triangleq \sum \prod_{j=1}^{k} i_j^r,$$

where the sum is taken over all k-tuples $(i_1, \ldots, i_k)$ such that $1 \le i_1 < \cdots < i_k \le n$. Then,

$$S_{1,1} = \tfrac{1}{2}n(n+1), \quad S_{2,1} = \frac{1}{24}n(n^2-1)(3n+2), \quad S_{3,1} = \frac{1}{48}(n-2)(n-1)n^2(n+1)^2,$$

$$S_{1,2} = \frac{1}{6}n(n+1)(2n+1), \quad S_{2,2} = \frac{1}{432}n(n+1)(2n+1)(10n^3-3n^2-13n+6).$$

Furthermore, for all $r \in \mathbb{R}$,

$$S_{3,r} = \frac{1}{6}(2S_{1,3r} - 3S_{1,r}S_{1,2r} + S_{1,r}^2).$$

Source: [benczeAOI,].

Fact 1.12.11. Let $k \ge 1$ and $n \ge 1$. If $n^2 \le k \le (n+1)^2 - 1$, then $\lfloor \sqrt{k} \rfloor = n$. If $n^3 \le k \le (n+1)^3 - 1$, then $\lfloor \sqrt[3]{k} \rfloor = n$. Now, assume that $n \ge 2$. Then,

$$\sum_{i=1}^{n^2-1} \lfloor \sqrt{i} \rfloor = \frac{1}{6}n(n-1)(4n+1), \quad \sum_{i=1}^{n^3-1} \lfloor \sqrt[3]{i} \rfloor = \frac{1}{4}(n-1)n^2(3n+1).$$

Source: [five, pp. 39, 187].

Fact 1.12.12. Let $n \ge 1$. Then,

$$\sum_{i=1}^{n} \left\lfloor \frac{i}{2} \right\rfloor = \frac{n^2}{4} + \frac{(-1)^n - 1}{8}, \quad \sum_{i=1}^{\lfloor n/2 \rfloor} \left\lfloor \frac{i}{2} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+2}{4} \right\rfloor.$$

Source: [beauregard,].

Fact 1.12.13. Let $n \ge 3$ and $m \ge 1$, assume that n is prime, and assume that $n \nmid m$. Then,

$$\sum_{i=1}^{n-1} \left\lfloor \frac{im}{n} \right\rfloor = \tfrac{1}{2}(n-1)(m-1), \quad \sum_{i=1}^{n-1} \left\lfloor \frac{(-1)^i i^2 m}{n} \right\rfloor = \tfrac{1}{2}(n-1)(m-1),$$

$$\sum_{i=1}^{n-1} \left\lfloor \frac{i^3 m}{n} \right\rfloor = \tfrac{1}{4}(n-1)(n^2 m - nm - 2), \quad \sum_{i=1}^{n-1} \left\lfloor \frac{(-1)^i i^4 m}{n} \right\rfloor = \tfrac{1}{2}(n-1)[m(n^2-n-1)-1].$$

In particular,

$$\sum_{i=1}^{n-1} \left\lfloor \frac{i}{n} \right\rfloor = 0, \quad \sum_{i=1}^{n-1} \left\lfloor \frac{(-1)^i i^2}{n} \right\rfloor = 0, \quad \sum_{i=1}^{n-1} \left\lfloor \frac{i^3}{n} \right\rfloor = \tfrac{1}{4}(n-2)(n^2-1), \quad \sum_{i=1}^{n-1} \left\lfloor \frac{(-1)^i i^4}{n} \right\rfloor = \tfrac{1}{2}(n-2)(n^2-1).$$

If n is odd, then

$$\sum_{i=1}^{n-1} (-1)^i \left\lfloor \frac{i^2}{n} \right\rfloor = \tfrac{1}{2}(n-1).$$

If n is even and $\frac{n}{2} \overset{4}{\equiv} 1$, then

$$\sum_{i=1}^{n-1}(-1)^i \left\lfloor \frac{i^2}{n} \right\rfloor = 1 - \frac{n}{2}.$$

Finally, if n is an odd prime, then

$$\sum_{i=1}^{n-1} \left\lfloor \frac{i^n}{n^2} \right\rfloor = \frac{1}{n^2}\left(\sum_{i=1}^{n-1} i^n\right) - \tfrac{1}{2}(n-1).$$

Source: [AMR, pp. 428–432] and [koshycurious,].

Fact 1.12.14. Let $1 \le m \le n$. Then,

$$\sum \prod_{j=1}^{m} i_j = \frac{n}{(2m-1)!} \prod_{i=1}^{m-1}(n^2 - i^2),$$

where the sum is taken over all m-tuples $(i_1, \ldots, i_m)$ of positive integers such that $\sum_{j=1}^{m} i_j = n$. In particular,

$$\sum ij = \frac{1}{6}n(n^2-1),$$

where the sum is taken over all ordered pairs (i,j) of positive integers such that $i + j = n$. **Source:** [comtet, pp. 33, 85].

Fact 1.12.15. Let $1 \le m < n$. Then,

$$\sum_{i=m+1}^{n} i \prod_{j=1}^{m}(i^2 - j^2) = \frac{(n+m+1)!}{2(m+1)(n-m-1)!}.$$

Source: [AMR2, pp. 31, 188].

Fact 1.12.16. Let $1 \le k \le n$. Then,

$$\sum \operatorname{card}(\cap_{i=1}^{k} \mathcal{S}_i) = 2^{k(n-1)}n,$$

where the sum is taken over all k-tuples $(\mathcal{S}_1, \ldots, \mathcal{S}_k)$ of subsets of $\{1, \ldots, n\}$. In particular,

$$\sum \operatorname{card}(\mathcal{S}_1 \cap \mathcal{S}_2) = 4^{n-1}n,$$

where the sum is taken over all ordered pairs $(\mathcal{S}_1, \mathcal{S}_2)$ of subsets of $\{1, \ldots, n\}$. **Source:** [comtet, pp. 33, 34].

Fact 1.12.17. Let $n \ge 2$. Then,

$$\operatorname{card}(\{(i,j) \colon i, j \in \{1, \ldots, n\} \text{ and } i < j\}) = \binom{n}{2}.$$

Fact 1.12.18. Let $n \ge 1$. Then,

$$\sum_{i=1}^{n}(2i-1) = n^2, \quad \sum_{i=1}^{2n} i = \sum_{i=1}^{n}(4i-1) = (2n+1)n, \quad \sum_{i=1}^{2n-1} i = \sum_{i=1}^{n}(4i-3) = (2n-1)n.$$

Fact 1.12.19. Let $m, n \geq 1$. Then,

$$\sum_{i=1}^{n}(mi-1) = \tfrac{1}{2}mn(n+1) - n, \quad \sum_{i=1}^{n}(mi-1)^2 = \tfrac{1}{6}m^2n(n+1)(2n+1) - mn(n+1) + n,$$

$$\sum_{i=1}^{n}(mi-1)^3 = \tfrac{1}{4}m^3n^2(n+1)^2 - \tfrac{1}{2}m^2n(n+1)(2n+1) + \tfrac{3}{2}mn(n+1) - n.$$

In particular,

$$\sum_{i=1}^{n}(2i-1) = n^2, \quad \sum_{i=1}^{n}(3i-1) = \tfrac{3}{2}n^2 + \tfrac{1}{2}n, \quad \sum_{i=1}^{n}(2i-1)^2 = \tfrac{4}{3}n^3 - \tfrac{1}{3}n,$$

$$\sum_{i=1}^{n}(3i-1)^2 = 3n^3 - \tfrac{3}{2}n^2 - \tfrac{1}{2}n, \quad \sum_{i=1}^{n}(2i-1)^3 = 2n^4 - n^2.$$

Source: [GR, pp. 2, 3] and [jeffrey, p. 37].

Fact 1.12.20. Let $m \geq n \geq 1$. Then,

$$\sum_{i,j=1}^{m,n} \min\{i,j\} = \frac{1}{6}n(n+1)(3m-n+1), \quad \sum_{i,j=1}^{m,n} \max\{i,j\} = \frac{1}{6}n(n^2-1) + \frac{1}{2}mn(m+1).$$

Source: [comtet, p. 168].

Fact 1.12.21. Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} 2^i i = 2^{n+1}(n-1) + 2, \quad \sum_{i=1}^{n} 2^i i^2 = 2^{n+1}(n^2-2n+3) - 6,$$

$$\sum_{i=1}^{n} 2^i i^3 = 2^{n+1}(n^3 - 3n^2 + 9n - 13) + 26.$$

If $n \geq 2$, then

$$\sum_{i=1}^{n-1} 2^{i-1}(n-i) = 2^n - n - 1.$$

Source: [pwz, pp. 95, 97].

Fact 1.12.22. Let $n \geq 1$, let x be a complex number, and assume that $x \neq 1$. Then,

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}, \quad \sum_{i=1}^{n} x^i = \frac{x-x^{n+1}}{1-x}, \quad \sum_{i=0}^{n-1}(n-i)x^i = \frac{x^{n+1}-(n+1)x+n}{(x-1)^2},$$

$$\sum_{i=1}^{n} ix^i = \frac{[nx^{n+1}-(n+1)x^n+1]x}{(x-1)^2} = \frac{(nx^n - \sum_{i=0}^{n-1} x^i)x}{x-1},$$

$$\sum_{i=1}^{n} i^2x^i = \frac{([n(x-1)-1]^2 + x)x^{n+1} - x^2 - x}{(x-1)^3}$$

$$= \frac{[n^2(x-1)^2 - 2n(x-1) + x + 1]x^{n+1} - x^2 - x}{(x-1)^3}$$
$$= \frac{[n^2x^{n+1} - (n^2+2n-1)x^n + 2\sum_{i=1}^{n-1} x^i + 1]x}{(x-1)^2}$$
$$= \frac{[n^2x^n - \sum_{i=0}^{n-1}(2i+1)x^i]x}{x-1}.$$

In particular,

$$\sum_{i=1}^{n} \frac{i}{2^i} = \frac{2^{n+1}-n-2}{2^n}, \quad \sum_{i=1}^{n} \frac{i^2}{2^i} = 6 - \frac{n^2+4n+6}{2^n}.$$

Source: [five, pp. 22, 132], [larson, pp. 54, 55], [man93,], and [pwz, pp. 95, 97]. **Related:** Fact **??**.

Fact 1.12.23. Let $n \ge 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i+1}i = \begin{cases} -\frac{1}{2}n, & n \text{ even}, \\ \frac{1}{2}(n+1), & n \text{ odd}, \end{cases} \qquad \sum_{i=1}^{n}(-1)^{i+1}i^2 = \begin{cases} -\frac{1}{2}n(n+1), & n \text{ even}, \\ \frac{1}{2}n(n+1), & n \text{ odd}. \end{cases}$$

Now, let $m \ge 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i+1}(mi-1) = \tfrac{1}{4}(-1)^n[2-m(2n+1)] + \tfrac{1}{4}(m-2),$$

$$\sum_{i=1}^{n}(-1)^{i+1}(mi-1)^2 = \tfrac{1}{2}(-1)^{n+1}[m^2n(n+1) - m(2n+1) + 1] + \tfrac{1}{2}(1-m).$$

Source: [demaio2,] and [GR, pp. 2, 3].

Fact 1.12.24. Let $n \ge 2$. Then,

$$\sum_{i=n+1}^{2n-1} \frac{1}{i^2} = 4\sum_{i=1}^{n-1}(-1)^{n-1-i}\frac{\left(\frac{i}{n^2-i^2}\right)^2}{\binom{2n}{n-i}}.$$

Source: [almkvist,]. **Example:** $\frac{1}{16} + \frac{1}{25} = \frac{8}{75} - \frac{1}{240} = \frac{41}{400}$.

Fact 1.12.25. Let $n, m, k \ge 1$. Then,

$$\sum_{i=1}^{n} \frac{1}{[m+k(i-1)](m+ki)} = \frac{n}{m(kn+m)}, \quad \sum_{i=0}^{n} \frac{1}{(ki+m)(ki+m+k)} = \frac{n+1}{m(kn+m+k)}.$$

In particular,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}, \quad \sum_{i=1}^{n} \frac{1}{4i^2-1} = \frac{n}{2n+1}, \quad \sum_{i=1}^{n} \frac{1}{(i+1)(i+2)} = \frac{n}{2n+4},$$

$$\sum_{i=1}^{n} \frac{1}{(i+2)(i+3)} = \frac{n}{3n+9}, \quad \sum_{i=1}^{n} \frac{1}{(3i+1)(3i-2)} = \frac{n}{3n+1}, \quad \sum_{i=1}^{n} \frac{1}{(5i+2)(5i-3)} = \frac{n}{10n+4}.$$

Source: [GR, p. 3]. **Related:** Fact **??**.

Fact 1.12.26. Let $n, k \geq 1$. Then,

$$\sum_{i=0}^{n} \frac{1}{(i+k)(i+k+1)} = \frac{n+1}{k(n+k+1)},$$

$$\sum_{i=0}^{n} \frac{1}{(i+k)(i+k+2)} = \frac{(n+1)[(2k+1)n+2(k+1)^2]}{2k(k+1)(n+k+1)(n+k+2)}.$$

Related: Fact **??**.

Fact 1.12.27. Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} \frac{1}{i(2i+1)} = 2 - 2\sum_{i=1}^{2n+1} (-1)^{i+1}\frac{1}{i}, \quad \sum_{i=1}^{n} \frac{i}{4i^4+1} = \frac{1}{4} - \frac{1}{4(2n^2+2n+1)},$$

$$\sum_{i=1}^{n} \frac{2i^2-1}{4i^4+1} = \frac{1}{2} - \frac{2n+1}{2(2n^2+2n+1)}, \quad \sum_{i=1}^{n} \frac{i}{i^4+4} = \frac{3}{8} - \frac{2n^2+2n+3}{4(n^2+1)(n^2+2n+2)},$$

$$\sum_{i=1}^{n} \frac{i}{\prod_{j=0}^{i}(2j+1)} = \frac{1}{2} - \frac{1}{2}\frac{1}{\prod_{i=0}^{n}(2j+1)}, \quad \sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n^2+3n}{4(n+1)(n+2)},$$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(2i+1)} = 1 + 4\sum_{i=3}^{2n+1} (-1)^{i}\frac{1}{i} + \frac{1}{n+1}, \quad \sum_{i=1}^{n} \frac{2^i(i^2-2i-1)}{i^2(i+1)^2} = \frac{2^{n+1}}{(n+1)^2} - 2,$$

$$\sum_{i=1}^{n} \frac{3i^2+3i+1}{i^3(i+1)^3} = \frac{-1}{(n+1)^3} + 1, \quad \sum_{i=1}^{n} \frac{6i+3}{4i^4+8i^3+8i^2+4i+3} = \frac{n^2+2n}{2n^2+4n+3},$$

$$\sum_{i=1}^{n} \frac{i^2+3i+3}{i^4+2i^3-3i^2-4i+2} = -\frac{2n^2+5n}{n^2+2n-1}, \quad \sum_{i=1}^{n} \frac{4^i i^2}{(i+1)(i+2)} = \frac{2}{3} + \frac{4^{n+1}(n-1)}{3(n+2)},$$

$$\sum_{i=1}^{n} \frac{2^i(i^3-3i^2-3i-1)}{i^3(i+1)^3} = \frac{2^n}{n^3}, \quad \sum_{i=0}^{n} \frac{i^3+6i^2+11i+5}{(i+3)!} = \frac{5}{2} - \frac{n^2+6n+10}{(n+3)!}.$$

If $n \geq 2$, then

$$\sum_{i=2}^{n} \frac{1}{i^2-1} = \frac{3n^2-n-2}{4n^2+4n} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}.$$

If $n \geq 3$, then

$$\sum_{i=1,\, i\neq 2}^{n} \frac{1}{i^2-4} = \frac{3}{16} - \frac{1}{4(n-1)} - \frac{1}{4n} - \frac{1}{4(n+1)} - \frac{1}{4(n+2)} = \frac{3}{16} - \frac{2n^3+3n^2-n-1}{2(n-1)n(n+1)(n+2)}.$$

Source: [GR, pp. 2, 3]. The first equality is given in [bonar, p. 119]. The second equality is given in [MOC, p. 41]. The third equality is given in [bonar, p. 235]. The fourth equality is given in [bonar, p. 122]. The fifth equality is given in [larson, p. 171]. The sixth equality is given in [bonar, p. 118]. The penultimate equality is given in [GR, p. 3]. **Related:** Fact **??** and Fact **??**.

Fact 1.12.28. Let $n \ge 1$. Then,

$$\sum_{i=1}^n \frac{1}{i^2(i+1)^2} = 2\sum_{i=1}^n \frac{1}{i^2} - 3 + \frac{1}{(n+1)^2} + \frac{2}{n+1},$$

$$\sum_{i=1}^n \frac{(-1)^{i+1}}{i^2(i+1)^2} = 3 - 4\sum_{i=1}^n \frac{(-1)^{i+1}}{i} + \frac{(-1)^{n+1}}{(n+1)^2} + \frac{(-1)^{n+1}2}{n+1}.$$

Source: [AMR2, pp. 34, 203–205]. **Related:** Fact **??**.

Fact 1.12.29. Let $n \ge 1$. Then,

$$\sum_{i=1}^n \frac{\prod_{j=1}^{n-1}(4i^4 + j^4)}{i^2 \prod_{j=1, j\ne i}^n (i^4 - j^4)} = \frac{1}{2n^2}\binom{2n}{n}.$$

Source: [almkvist,]. **Remark:** For $n = 1$, both products are set to 1.

Fact 1.12.30. Let $n \ge 1$. Then,

$$\sqrt{1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}} = 1 + \frac{1}{n} - \frac{1}{n+1},$$

$$\sum_{i=1}^n \sqrt{1 + \frac{1}{i^2} + \frac{1}{(i+1)^2}} = \sum_{i=1}^n \frac{i^2+i+1}{i(i+1)} = \frac{n(n+2)}{n+1}, \quad \sum_{i=1}^n \frac{i^2+i-1}{i(i+1)} = \frac{n^2}{n+1}.$$

Source: [benczemayhem,].

Fact 1.12.31. Let $n \ge 1$. Then,

$$\sum_{i=1}^n \frac{1}{\sqrt{i} + \sqrt{i+1}} = \sqrt{n+1} - 1.$$

Source: [gelca, p. 121].

Fact 1.12.32. Let $n \ge 1$. Then,

$$\sum_{i=1}^n \frac{1}{\sqrt{1 + (1+1/i)^2} + \sqrt{1 + (1-1/i)^2}} = \frac{1}{4}(\sqrt{(n+1)^2 + n^2} - 1).$$

Source: [AMR, pp. 3, 70, 71].

Fact 1.12.33. Let $n \ge 1$. Then,

$$\sum_{i=1}^n \frac{1}{(\sqrt{i} + \sqrt{i+1})(\sqrt[4]{i} + \sqrt[4]{i+1})} = \sqrt[4]{n+1} - 1.$$

Source: [AMR, pp. 4, 73].

Fact 1.12.34. Let $n \ge 1$. Then,

$$\sum_{1\le i\le n+1} \frac{1}{i} + \sum_{1\le i<j\le n+1} \frac{1}{ij} + \cdots + \sum \frac{1}{\prod_{j=1}^n i_j} = n + 1 - \frac{1}{(n+1)!},$$

where the last sum is taken over all n-tuples $(i_1,\ldots,i_n)$ such that $1\le i_1<\cdots<i_n\le n+1$. Furthermore,

$$\sum_{i=1}^{n}\sum\frac{1}{\prod_{j=1}^{i}k_j}=n,$$

where the last sum is taken over all i-tuples $(k_1,\ldots,k_i)$ such that $1\le k_1<\cdots<k_i\le n$. Now, let $n\ge 2$. Then,

$$\sum_{i=1}^{n-1}(-1)^{i+1}\sum\frac{1}{\prod_{j=1}^{i}k_j}=\frac{n-1}{n},\quad \sum_{i=1}^{n-1}(-1)^{i+1}\sum\frac{1}{\prod_{j=1}^{i}k_j^2}=\frac{n-1}{2n},$$

$$\sum_{i=1}^{n-1}(-1)^{i+1}\sum\frac{2^i}{\prod_{j=1}^{i}(k_j^3+1)}=\frac{(n-1)(n+2)}{3n(n+1)},$$

where the second sum in each equality is taken over all i-tuples $(k_1,\ldots,k_i)$ such that $2\le k_1<\cdots<k_i\le n$. **Source:** [benczeabout,diaztelescopic,].

Fact 1.12.35. Let $n\ge 1$. Then,

$$\frac{2}{3}n^{3/2}<\left(\frac{2n}{3}+\frac{1}{8}-\frac{1}{8\sqrt{n+1}}\right)\sqrt{n+1}<\sum_{i=1}^{n}\sqrt{i}<\left(\frac{2n}{3}+\frac{1}{6}-\frac{1}{6\sqrt{n+1}}\right)\sqrt{n+1}<\frac{2}{3}n^{3/2}+\frac{1}{2}\sqrt{n}.$$

Source: [mercaAM,]. **Remark:** It is conjectured in [mercaAM,] that

$$\left\lfloor\frac{1}{n}\sum_{i=1}^{n}\sqrt{i}\right\rfloor=\left\lfloor\left(\frac{2}{3}+\frac{1}{6n}\right)\sqrt{n+1}\right\rfloor.$$

Fact 1.12.36. Let $n\ge 1$. Then,

$$\sum_{i=1}^{n-1}i^2<\frac{n^3}{3}<\sum_{i=1}^{n}i^2,\quad \sum_{i=1}^{n-1}i^3<\frac{n^4}{4}<\sum_{i=1}^{n}i^3.$$

Fact 1.12.37. Let $n\ge 1$ and $p\ge 1$. Then,

$$n\left(\frac{n+1}{2}\right)^p\le\sum_{i=1}^{n}i^p.$$

Source: [cvet, p. 103].

Fact 1.12.38. Let $n\ge 1$. Then,

$$\frac{2}{3}<\sum_{i=n}^{2n}\frac{1}{i},\quad 1<\sum_{i=n+1}^{3n+1}\frac{1}{i},\quad \frac{1}{2}<\sum_{i=3n+1}^{5n+1}\frac{1}{i}<\frac{2}{3}.$$

Source: [kaczor1, p. 9].

Fact 1.12.39. Let $n>m\ge 1$. Then, $\sum_{i=m}^{n}\frac{1}{i}$ is not an integer. **Source:** [havil, p. 24].

Fact 1.12.40. Let $n\ge 1$ and $p\in(0,\infty)$. Then,

$$\sum_{i=1}^{n}\left(\frac{1}{i}\right)^{1/p}<\frac{p}{p-1}n^{1-1/p},\quad \sum_{i=1}^{n}\frac{1}{\sqrt{i}}<2\sqrt{n}.$$

Source: [larson, p. 63] and [radu, p. 282].

Fact 1.12.41. Let $n \geq 1$. Then,

$$\frac{n}{\sqrt{n^2+n}} < \sum_{i=1}^{n} \frac{1}{\sqrt{n^2+i}} < \frac{n}{\sqrt{n^2+1}}.$$

Source: [larson, p. 278].

Fact 1.12.42. Let $n \geq 1$ and $r > 0$. Then,

$$\frac{n}{n+1} \leq \left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{1/r} \leq \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \leq \sqrt{\frac{n}{n+1}}.$$

Source: [ABMP,bennettMI,qiineq,]. **Remark:** The first and second inequality are *Alzer's inequality* and *Martins's inequality*, respectively. **Related:** Fact 1.13.13.

Fact 1.12.43. Let $n \geq 1$, let p be a real number, and define the sequences

$$\mathcal{S}_1(p) \triangleq \left(\frac{\sum_{i=1}^{n} i^p}{n(n!)^{p/n}}\right)_{n=1}^{\infty}, \quad \mathcal{S}_2(p) \triangleq \left(\frac{n(n+1)^p}{\sum_{i=1}^{n} i^p}\right)_{n=1}^{\infty}.$$

Then, the following statements hold:

i) If $p > 0$, then $\mathcal{S}_1(p)$ is increasing.

ii) If $p \in (0,1)$, then $\mathcal{S}_2(p)$ is increasing.

iii) If $p \in (-\infty,0) \cup (1,\infty)$, then $\mathcal{S}_2(p)$ is decreasing.

Source: [bennettMI,].

Fact 1.12.44. Let $n \geq 1$. Then,

$$\sum_{i=1}^{2n} (-1)^{i+1}\frac{1}{i} = H_{2n} - H_n.$$

Source: [gelca, p. 10] and [sav, pp. 21, 22]. **Remark:** This is *Catalan's identity.* **Example:** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{1}{3} + \frac{1}{4}$.

Fact 1.12.45. Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} H_i = (n+1)(H_{n+1}-1) = (n+1)H_n - n, \quad \sum_{i=1}^{n} iH_i = \frac{1}{4}n(n+1)(2H_{n+1}-1),$$

$$\sum_{i=1}^{n} i^2H_i = \frac{1}{36}n(n+1)[6(2n+1)H_{n+1} - 4n - 5],$$

$$\sum_{i=1}^{n} i^3H_i = \frac{1}{48}n(n+1)[12n(n+1)H_{n+1} - 3n^2 - 7n - 2], \quad \sum_{i=1}^{n} \frac{H_i}{i} = \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{1}{ij} = \frac{1}{2}(H_n^2 + H_{n,2}).$$

If, in addition, $k \geq 0$, then

$$(H_{n+k} - H_k)^2 + H_{n+k,2} - H_{k,2} = \left(\sum_{i=1}^{n} \frac{1}{i+k}\right)^2 + \sum_{i=1}^{n} \frac{1}{(i+k)^2} = \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{2}{(i+k)(j+k)}.$$

Furthermore,

$$\sum_{i=1}^{n} (-1)^i H_i = \begin{cases} \frac{1}{2}H_{n/2}, & n \text{ even}, \\ \frac{1}{2}H_{(n+1)/2} - H_{n+1}, & n \text{ odd}. \end{cases}$$

Source: [batailleharm,], [benjaminquinn, p. 91], [GKP, pp. 279, 280], and [sofosm,].

Fact 1.12.46. Let $n \ge 1$. Then,

$$\log n + \frac{1}{2n} + \frac{1}{2} < H_n < \log n + \frac{1}{n} + \frac{29}{50}.$$

Now, let $n \ge 2$. Then,

$$\log n + \frac{1}{n} < H_n < \log n + 1, \quad \frac{1}{n} < H_n - \log n < 1,$$

$$n(\sqrt[n]{n+1} - 1) < H_n \le n - \frac{n-1}{\sqrt[n-1]{n}} < n - \frac{n}{\sqrt[n]{n+1}} + \frac{n}{n+1} < 1 + n\left(1 - \frac{1}{\sqrt[n]{n}}\right).$$

Source: [havil, p. 47], [herman, pp. 158, 161], [kaczor1, p. 9], [larson, p. 250], and [radu, p. 10]. In the last string, the ordering of the third and fifth terms follows from the arithmetic-mean–geometric mean inequality. **Remark:** The second inequality in the last string is strict for all $n \ge 3$. **Related:** Fact ??.

Fact 1.12.47. Let $n \ge 2$. Then,

$$\tfrac{1}{2}(\lfloor \log_2 n \rfloor + 1) < H_n \le \lfloor \log_2 n \rfloor + 1.$$

Equivalently,

$$\frac{1}{2}\left(\left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1\right) < H_n \le \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1.$$

Source: [GKP, p. 276].

Fact 1.12.48. Let $n \ge 1$. Then,

$$\sum_{i=1}^n \frac{1}{iH_i^2} < 1.85, \quad \sum_{i=1}^n \frac{1}{iH_i^3} < 1.34.$$

Source: [cvet, p. 183].

Fact 1.12.49. Let $n \ge 1$. Then,

$$\frac{n(3n+5)}{2(n+1)^2} \le H_{n,2} \le 2 - \frac{1}{n}.$$

Source: [benczeassi,].

Fact 1.12.50. Let $n \ge 5$ be prime. Then, the following statements hold:

i) Let $H_{n-1} = N_{n-1}/D_{n-1}$, where N_{n-1} and D_{n-1} are positive coprime integers. Then, $n^2 | N_{n-1}$.

ii) Let $H_{n-1,2} = N_{n-1,2}/D_{n-1,2}$, where $N_{n-1,2}$ and $D_{n-1,2}$ are positive coprime integers. Then, $n | N_{n-1,2}$.

Source: [wikiwolst,] and [mollnf, pp. 29, 305]. **Example:** $H_{18} = 14274301/4084080 = (19)^2 39541/$
4084080 and $H_{10,2} = 1968329/1270080 = (11)178939/1270080$.

Fact 1.12.51. Le $n \ge 1$. Then,

$$\exp \frac{2n}{2n+1} \le \exp \frac{en^{n+1}}{(n+1)^{n+1}} \le \left(1 + \frac{1}{n}\right)^n \le \exp \sqrt{\frac{n}{n+1}},$$

$$\exp \frac{2}{n+2} \le \exp \frac{e}{(n+1)^{(n+1)/n}} \le (n+1)^{1/n} \le \exp \frac{1}{\sqrt{n+1}},$$

$$\sum_{i=1}^n \log^2\left(1+\frac{1}{i}\right) \le \frac{n}{n+1}, \quad \prod_{i=1}^n \log(i+1) \le \sqrt{\frac{n!}{n+1}}, \quad (n+1)! \le \exp\sum_{i=1}^n \frac{i}{\sqrt{i+1}}.$$

Source: [benczesnie,].

1.13 Facts on Factorials

Fact 1.13.1. Let n and m be positive integers such that $m < n$. Then, $n^{\underline{m}}$ is the number of m-tuples whose components are distinct elements of $\{1, \ldots, n\}$. **Remark:** $n^{\underline{m}}$ is the number of permutations of m distinct elements chosen from a set of n elements.

Fact 1.13.2. Let $n \ge 3$, and assume that n is prime. Then,

$$(n-1)!! \overset{n}{\equiv} \prod_{i=1}^{n-1} i! \overset{n}{\equiv} (-1)^{(n-1)/2} \prod_{i=1}^{n-1} i^i.$$

Source: [aebi2,].

Fact 1.13.3. Let $n \ge 1$. Then,

$$\sum_{i=1}^n i(i!) = (n+1)! - 1, \quad \sum_{i=1}^n (i^2+1)i! = n(n+1)!,$$

$$\sum_{i=0}^n \frac{1}{i!} = \frac{\lfloor en! \rfloor}{n!}, \quad \sum_{i=1}^n \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}, \quad \sum_{i=1}^n \frac{i^2+i-1}{(i+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!},$$

$$\sum_{i=1}^n \frac{i^2+3i+1}{(i+2)!} = \frac{3}{2} - \frac{n+3}{(n+2)!}, \quad \sum_{i=1}^n \frac{(4i+1)i!}{(2i+1)!} = 1 - \frac{n!}{(2n+1)!},$$

$$\sum_{i=1}^n \frac{i^3+6i^2+11i+5}{(i+3)!} = \frac{5}{3} - \frac{n^2+6n+10}{(n+3)!} = \frac{5}{3} - \frac{1}{(n+1)!} - \frac{1}{(n+2)!} - \frac{1}{(n+3)!}.$$

Source: The second equality is given in [gelca, p. 123]. The third equality is given in [cameron, pp. 33, 34] and [hassani,]. The fourth equality is given in [sav, pp. 4, 5]. The seventh equality is given in [pwz, p. 78]. The last inequality is given in [kaczor1, pp. 28, 165].

Fact 1.13.4. Let $n \ge 1$ and $k \ge 1$. Then,

$$\sum_{i=1}^n i!(i^2+ki+1) = (n+1)!(n+k) - k.$$

In particular,

$$\sum_{i=1}^n i!(i^2+i+1) = (n+1)!(n+1) - 1, \quad \sum_{i=1}^n i!(i+1)^2 = (n+1)!(n+2) - 2.$$

Source: [MOC, p. 39] and [benjaminquinn, p. 92].

Fact 1.13.5. Let $n \ge 2$ and $k \in \{1, \ldots, n\}$. Then,

$$\prod_{i=1,\, i\ne k}^n (k-i) = (-1)^{n-k}(n-k)!(k-1)!.$$

Source: [korus,].

Fact 1.13.6. Let $n \geq 1$. Then,

$$\sum_{i=0}^{n}(1+i-\sqrt{i})\sqrt{i!} = (n+1)\sqrt{n!}.$$

Source: [mollnf, p. 8].

Fact 1.13.7. Let $n \geq 2$. Then,

$$\sum_{i=1}^{n}\frac{1}{(i-1)!}\sum_{j=0}^{n-i}(-1)^j\frac{1}{j!} = 1, \quad \sum_{i=1}^{n}\frac{i}{(i-1)!}\sum_{j=0}^{n-i}(-1)^j\frac{1}{j!} = 2.$$

Source: [gelca, p. 313].

Fact 1.13.8. Let $n \geq 2$. Then,

$$\sum_{i=1}^{n}(-1)^i\frac{1}{i!}\sum\frac{1}{\prod_{j=1}^{i}k_j} = \sum_{i=1}^{n}(-1)^i\frac{1}{i}\sum\frac{1}{\prod_{j=1}^{i}k_j!} = 0,$$

where the second and fourth sums are taken over all i-tuples $(k_1,\ldots,k_i)$ of positive integers such that $\sum_{j=1}^{i}k_j = n$. **Source:** [frumosu,].

Fact 1.13.9. Let $n \geq 1$. Then,

$$(n+1)! \leq \sqrt[n]{\prod_{i=1}^{n}(2i)!}.$$

Source: [larson, p. 63].

Fact 1.13.10. Let $n \geq 2$. Then, $2^{H_n} \leq \sqrt[n]{n!} < \frac{1}{2}(n+1) < e^{H_n}$. **Source:** [AMR3, pp. 172–174].

Fact 1.13.11. Let $n \geq 1$. Then,

$$\sqrt[n]{n!} \leq \prod p^{1/(p-1)},$$

where the product is taken over all primes p that divide n. **Source:** [larson, p. 169]. **Remark:** This implies there are infinitely many primes.

Fact 1.13.12. If $n \geq 2$, then

$$(n-1)! < \frac{n^n}{e^{n-1}} < n! \leq \frac{(n+1)^n}{2^n}, \quad n! \leq \left[\frac{(n+1)(2n+1)}{6}\right]^{n/2}.$$

If $n \geq 3$, then

$$n! < 2^{n(n-1)/2}.$$

If $n \geq 6$, then

$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$

Source: [five, pp. 13, 89, 90], [experimentation, p. 210], [herman, p. 137], and [radu, p. 346].

Fact 1.13.13. Let $n \ge 1$. Then,

$$\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}, \quad \frac{n}{\sqrt[n]{n!}} < \frac{n+1}{\sqrt[n+1]{(n+1)!}}, \quad \frac{n+1}{\sqrt[n]{n!}} < \frac{n+2}{\sqrt[n+1]{(n+1)!}}.$$

Source: [bennettMI,]. **Remark:** The first inequality is the *Minc-Sathre inequality*. The second inequality is given by Fact 1.12.42.

Fact 1.13.14. Let $n \ge 1$. Then,

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n+\frac{1}{6}\right)} < n! \le \left(\frac{n}{e}\right)^n \sqrt{2\pi\left(n+\frac{e^2}{2\pi}-1\right)},$$

$$n^{n/2} \le n! \le \left(\frac{n+1}{2}\right)^n, \qquad \sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{1/(12n)},$$

$$\frac{\sqrt{2\pi e}}{e^{(3-\sqrt{3})/3}}\left(\frac{n+(3-\sqrt{3})/3}{e}\right)^{n+1/2} < n! < \frac{\sqrt{2\pi e}}{e^{(3+\sqrt{3})/3}}\left(\frac{n+(3+\sqrt{3})/3}{e}\right)^{n+1/2}.$$

Now, let $n \ge 3$. Then, $n^{n/2} < n!$ and

$$2\left(\frac{n}{e}\right)^n < e\left(\frac{n}{e}\right)^n < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n < \frac{n+\frac{13}{12}}{n+1}\sqrt{2n\pi}\left(\frac{n}{e}\right)^n$$
$$< n! < \frac{n-\frac{23}{12}}{n-2}\sqrt{2n\pi}\left(\frac{n}{e}\right)^n < \sqrt{\frac{n}{n-1}}\sqrt{2n\pi}\left(\frac{n}{e}\right)^n < \left(\frac{n+1}{2}\right)^n < e\left(\frac{n}{2}\right)^n.$$

Therefore,

$$\frac{2}{\sqrt[n]{e}} < \frac{2n}{n+1} < \sqrt[2n]{\frac{n-1}{n}}\frac{e}{\sqrt[2n]{2n\pi}} < \sqrt[n]{\frac{n-2}{n-\frac{23}{12}}}\frac{e}{\sqrt[2n]{2n\pi}}$$
$$< \frac{n}{\sqrt[n]{n!}} < \sqrt[n]{\frac{n+1}{n+\frac{13}{12}}}\frac{e}{\sqrt[2n]{2n\pi}} < \frac{e}{\sqrt[2n]{2n\pi}} < \frac{e}{\sqrt[n]{e}} < \frac{e}{\sqrt[n]{2}}.$$

Finally,

$$\sqrt[2n]{2n\pi}\frac{n}{\sqrt[n]{n!}} < \sqrt[n]{\frac{n+\frac{13}{12}}{n+1}}\sqrt[2n]{2n\pi}\frac{n}{\sqrt[n]{n!}} < e < \sqrt[n]{\frac{n-\frac{23}{12}}{n-2}}\sqrt[2n]{2n\pi}\frac{n}{\sqrt[n]{n!}} < \sqrt[2n]{\frac{n}{n-1}}\sqrt[2n]{2n\pi}\frac{n}{\sqrt[n]{n!}}.$$

Now, let $6 \le n \le 9$. Then,

$$n^{n/2} \le 2\left(\frac{n}{e}\right)^n < e\left(\frac{n}{e}\right)^n < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$

Finally, let $n \ge 10$. Then,

$$n^{n/2} < \left(\frac{n}{3}\right)^n < 2\left(\frac{n}{e}\right)^n < e\left(\frac{n}{e}\right)^n < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$

Source: [BBplausible, p. 197], [batirfac,], [gelca, p. 10], [hirschhornrefine,], [kaczor1, p. 10], [sandordebnath,], and [taorandom, pp. 35–37]. **Remark:** $\sqrt{2n\pi}\left(\frac{n}{e}\right)^n < n!$ is Stirling's formula. See Fact **??**. **Remark:** $\sqrt{2\pi} < e < 2\sqrt{\pi}$ and $e \approx (\pi^4+\pi^5)^{1/6}$. See [castellanos1,]. **Remark:** $0.16666 \approx 1/6 < e^2/(2\pi) - 1 \approx 0.17600$. **Related:** Fact **??**.

Fact 1.13.15. Let $n \geq 1$. Then,

$$\sum_{i=0}^{n}(i+1)i!! = (n+1)!! + (n+2)!! - 2, \quad \sum_{i=0}^{n}(-1)^i(i+1)i!! = (-1)^n[(n+2)!! - (n+1)!!],$$

$$\sum_{i=0}^{n}\frac{i}{(i+1)!!} = 2 - \frac{1}{(n+1)!!} - \frac{1}{n!!}, \quad \sum_{i=0}^{n}\frac{1}{(2i)!!(2n-2i)!!} = \frac{1}{n!},$$

$$\sum_{i=0}^{n}\binom{n}{i}(2i-1)!!(2n-2i-1)!! = (2n)!!, \quad \sum_{i=0}^{n}\frac{(2i-1)!!(2n-2i-1)!!}{(2i)!!(2n-2i)!!} = 1,$$

$$\prod_{i=1}^{n}\frac{2i}{2i-1} = \frac{\sqrt{\pi}\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} = \frac{(2n)!!}{(2n-1)!!} = \frac{[(2n)!!]^2}{(2n)!} = \frac{4^n}{\binom{2n}{n}}.$$

If, in addition, $n \geq 5$, then

$$\sqrt{2n} < \sqrt{2n+1} < \sqrt{3n+1} < \sqrt{\frac{2n(2n+1)\pi}{4n+1}} < \sqrt{(n+\tfrac{1}{4})\pi} < \frac{(2n)!!}{(2n-1)!!}$$
$$< \frac{\sqrt{\pi}(2n+1)}{\sqrt{4n+3}} < \sqrt{\frac{(4n+3)(2n+1)\pi}{8n+8}} < \sqrt{(n+\tfrac{4}{\pi}-1)\pi} < \sqrt{(n+\tfrac{1}{2})\pi} < 2\sqrt{n}.$$

Source: [castellanos1,chenqi,jameson,], [koshcat, pp. 49, 52], and [sav, p. 51]. **Remark:** This result yields the Wallis product given by Fact **??**. **Related:** Fact **??** and Fact **??**.

Fact 1.13.16. Let $n, m \geq 1$. Then, $m!n!(m+n)!|(2m)!(2n)!$. **Source:** [comtet, p. 79].

Fact 1.13.17. Let $n, m \geq 1$. Then, $(n!)^m|(mn)!$. Now, define $m \triangleq \max\{k^l\colon k$ is prime, $l \geq$ 1,
and $k^l \leq n\}$. Then, $(n!)^{m+1}|(mn)!$. **Source:** [morris3,].

1.14 Facts on Finite Products

Fact 1.14.1. If $n \geq 1$, then

$$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right) = n+1, \quad \prod_{i=1}^{n}\left(1+\frac{1}{i^2}\right) = \frac{\sinh(\pi)\Gamma(n+1-\jmath)\Gamma(n+1+\jmath)}{\pi(n!)^2},$$

$$\prod_{i=1}^{n}\left(1+\frac{1}{i^3}\right) = \frac{(n+1)\cosh(\sqrt{3}\pi/2)\Gamma(n+1/2-\sqrt{3}\jmath/2)\Gamma(n+1/2+\sqrt{3}\jmath/2)}{\pi(n!)^2},$$

$$\prod_{i=1}^{n}\frac{1}{4i^2-1} = \frac{(n!)(n+1)!2^{2n+1}}{(2n)!(2n+2)!}.$$

If $n \geq 2$, then

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right) = \frac{1}{n}, \quad \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}, \quad \prod_{i=2}^{n}\left(1-\frac{1}{i^4}\right) = \frac{(n+1)\prod_{i=1}^{n}(1+i^2)}{4n^3[(n-1)!]^2},$$

$$\prod_{i=2}^{n}\frac{i^2}{i^2-1} = \frac{2n}{n+1}, \quad \prod_{i=2}^{n}\frac{i^3-1}{i^3+1} = \frac{2(n^2+n+1)}{3n(n+1)}.$$

Source: [MOC, p. 39]. **Related:** Fact **??**.

1.15 Facts on Numbers

Fact 1.15.1. Let $n \geq 1$, let $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ be positive rational numbers, and assume that $q_1^{p_1}$ is an irrational number. Then, $\sum_{i=1}^n q_i^{p_i}$ is an irrational number. **Source:** [havilirrational, p. 129] and [patruno,]. **Related:** Fact ??.

Fact 1.15.2. Let a be a nonzero rational number. Then, e^a is irrational. **Source:** [aigner, Chapter 7]. **Remark:** If $x \in (0, \infty)$ is transcendental, then, for all $n \geq 1$, x^n is irrational.

Fact 1.15.3. There exist positive irrational numbers a and b such that a^b is rational. **Source:** Note that $\sqrt{2}$ is irrational, and define $\alpha \triangleq \sqrt{2}^{\sqrt{2}}$. Then, $\alpha^{\sqrt{2}} = 2$. Suppose that α is irrational. Then, the result holds with $a = \alpha$ and $b = \sqrt{2}$. Alternatively, suppose that α is rational. Then, the result holds with $a = b = \sqrt{2}$. **Remark:** This proof does not depend on knowing whether or not $\sqrt{2}^{\sqrt{2}}$ is irrational. In fact, $\sqrt{2}^{\sqrt{2}}$ and e^π are irrational.

Fact 1.15.4.

$$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}, \quad \sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}, \quad \sqrt{3\sqrt{2}-4} = \sqrt[4]{2}(\sqrt{2}-1),$$

$$\sqrt{19-4\sqrt{21}} = 2\sqrt{3}-\sqrt{7}, \quad \sqrt{21-4\sqrt{17}} = \sqrt{17}-2, \quad \sqrt{25-4\sqrt{21}} = \sqrt{21}-2,$$

$$\sqrt[3]{2+\sqrt{5}} = \tfrac{1}{2}(1+\sqrt{5}), \quad \sqrt[3]{\sqrt{5}-2} = \tfrac{1}{2}(\sqrt{5}-1),$$

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{\sqrt{5}-2} = \sqrt{5}, \quad \sqrt[3]{2+\sqrt{5}} - \sqrt[3]{2-\sqrt{5}} = 1,$$

$$\sqrt[3]{2+\tfrac{10\sqrt{3}}{9}} + \sqrt[3]{2-\tfrac{10\sqrt{3}}{9}} = 2, \quad \sqrt[3]{16+12\sqrt[3]{7}+9\sqrt[3]{49}} = 2\sqrt[3]{3\sqrt[3]{7}-5} + \sqrt[3]{2+3\sqrt[3]{49}},$$

$$(19+17\sqrt[3]{2}+22\sqrt[3]{4})(5\sqrt[3]{2}-\sqrt[3]{4}-3) = 129, \quad \sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{1/9} - \sqrt[3]{2/9} + \sqrt[3]{4/9},$$

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = \tfrac{1}{3}(\sqrt[3]{2}+\sqrt[3]{20}-\sqrt[3]{25}), \quad \sqrt{\sqrt[3]{28}-\sqrt[3]{27}} = \tfrac{1}{3}(\sqrt[3]{98}-\sqrt[3]{28}-1),$$

$$\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} = \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}, \quad \sqrt[5]{1+\sqrt[5]{2}+\sqrt[5]{8}}\,\sqrt[10]{5} = \sqrt{1+\sqrt[5]{4}},$$

$$\sqrt[3]{\sqrt[5]{32/5}-\sqrt[5]{27/5}} = \sqrt[5]{1/25}+\sqrt[5]{3/25}-\sqrt[5]{9/25}, \quad \sqrt[6]{7\sqrt[3]{20}-19} = \sqrt[3]{5/3}-\sqrt[3]{2/3}.$$

Source: [berele,hirschhornlost,landauhow,sofogen,witula2,]. **Remark:** $\sqrt[3]{2-\sqrt{5}} = \frac{1}{2}(1-\sqrt{5})$.

Fact 1.15.5. Let $n \geq 1$. Then,

$$\sqrt[3]{3n-1+n\sqrt{8n-3}} + \sqrt[3]{3n-1-n\sqrt{8n-3}} = 1.$$

In particular,

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = \sqrt[3]{5+2\sqrt{13}} + \sqrt[3]{5-2\sqrt{13}} = \sqrt[3]{8+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{21}} = 1.$$

Source: [sofogen,]. **Remark:** For $a < 0$, $\sqrt[3]{a} = -\sqrt[3]{|a|}$.

Fact 1.15.6. Let $n \geq 1$, and define $\alpha \triangleq \frac{1}{2}(\sqrt{5}+1)$. Then,

$$\underbrace{\sqrt{-2+\sqrt{2+\cdots+\sqrt{2+\sqrt{5}}}}}_{n+1 \text{ square roots}} = \alpha^{2^{-n}} - \alpha^{-2^{-n}}, \quad \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{5}}}}}_{n+1 \text{ square roots}} = \alpha^{2^{-n}} + \alpha^{-2^{-n}}.$$

In particular,

$$\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} = \sqrt{\sqrt{5}-2}, \quad \sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} = \sqrt{2+\sqrt{5}}.$$

Source: [nyblomrad,].

Fact 1.15.7. Define

$$\pi_{-12} \triangleq \frac{223}{71}, \quad \pi_{-11} \triangleq \sqrt[3]{31}, \quad \pi_{-10} \triangleq \frac{1}{3}\sqrt{120-18\sqrt{3}}, \quad \pi_{-9} \triangleq \frac{7^7}{4^9}, \quad \pi_{-8} \triangleq \frac{52163}{16604},$$

$$\pi_{-7} \triangleq \frac{20^3+47^3}{30^3} - 1, \quad \pi_{-6} \triangleq \frac{689}{396\log\frac{689}{396}}, \quad \pi_{-5} \triangleq \frac{66\sqrt{2}}{33\sqrt{29}-148}, \quad \pi_{-4} \triangleq \sqrt[4]{\frac{2143}{22}},$$

$$\pi_{-3} \triangleq \frac{3\log 5280}{\sqrt{67}}, \quad \pi_{-2} \triangleq \frac{3\log(640320)}{\sqrt{163}}, \quad \pi_{-1} \triangleq \frac{\log(640320^3+743)}{\sqrt{163}},$$

$$\pi_1 \triangleq \frac{\log(640320^3+744)}{\sqrt{163}}, \quad \pi_2 \triangleq \sqrt[4]{\frac{35444733}{363875}}, \quad \pi_3 \triangleq \frac{63(17+15\sqrt{5})}{25(7+15\sqrt{5})}, \quad \pi_4 \triangleq \frac{104348}{33215},$$

$$\pi_5 \triangleq \sqrt[5]{\frac{77729}{254}}, \quad \pi_6 \triangleq \frac{99^2}{2206\sqrt{2}}, \quad \pi_7 \triangleq \frac{99}{80}\left(\frac{7}{7-3\sqrt{2}}\right), \quad \pi_8 \triangleq \frac{355}{113}, \quad \pi_9 \triangleq \log_5 157,$$

$$\pi_{10} \triangleq \frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right), \quad \pi_{11} \triangleq \sqrt{7+\sqrt{6+\sqrt{5}}}, \quad \pi_{12} \triangleq \frac{9}{5}+\frac{3}{\sqrt{5}}, \quad \pi_{13} \triangleq \frac{19\sqrt{7}}{16},$$

$$\pi_{14} \triangleq \frac{22}{7}, \quad \pi_{15} \triangleq \sqrt{2}+\sqrt{3}.$$

Then, $\pi_{-12} < \cdots < \pi_{-1} < \pi < \pi_1 < \cdots < \pi_{15}$. **Source:** [BBplausible,castellanos1,fuks,], [hardyetal, pp. xxxiv, 34, 35], [havil, p. 96], and [parkps,].

Fact 1.15.8. Let x be a nonzero rational number. Then, $\tan x$ is an irrational number. **Source:** [havilirrational, pp. 104–107]. **Remark:** Since $\tan \pi/4 = 1$, it follows that π is an irrational number.

Fact 1.15.9. Let $x < 0$ and $p \in \mathbb{R}$. Then, the following statements hold:

i) x^p is real if and only if p is an integer.

ii) x^p is positive if and only if p is an even integer.

Remark: $(-1)^{1/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}\jmath$, and $\sqrt[3]{-1} = -1$.

Fact 1.15.10. Let $x \in [-11/4, -5/4] \cup [5/4, 11/4]$. Then,

$$[e^{(x-1/4)\pi\jmath}]^{x+1/4} + [e^{(x+1/4)\pi\jmath}]^{x-1/4} = 0.$$

1.16 Facts on Binomial Coefficients

Fact 1.16.1. Let n and k be positive integers. Then, the following statements hold:

i) $\binom{n}{k}$ is an integer.

ii) $\binom{2n}{n}$ is an even integer.

iii) $\frac{1}{n+1}\binom{2n}{n}$, $\frac{3}{n}\binom{2n}{n-3}$, $\frac{3}{n}\binom{3n}{n+1}$, and $\frac{4}{(3n+1)(3n+2)}\binom{3n+2}{n}$ are integers.

iv) If $n \ge 3$, then $\frac{(3n)!}{n!(n+1)!(n+2)!}$ is an integer.

v) $\frac{(nk)!}{n!(k!)^n}$, $\frac{(2n)!(2k)!}{n!k!(n+k)!}$, $\frac{\gcd\{n,k\}(n+k-1)!}{n!k!}$, and $\frac{[\binom{2n}{n}\binom{2k}{k}]^2}{\binom{n+k}{k}}$ are integers.

vi) If $1 \le k \le n$, then $\frac{\gcd\{n,k\}}{n}\binom{n}{k}$ and $\frac{\gcd\{n+1,k\}}{n-k+1}\binom{n}{k}$ are integers. If, in addition, $\gcd\{n,k\} = 1$, then $n|\binom{n}{k}$.

vii) Let $n \ge 2$ be prime, and assume that $k < n < 2k$. Then, $n|(2k)!$ and $n|\binom{2k}{k}$. In addition, $\frac{(k)!}{n}$ is not an integer.

viii) If n is prime, then $\binom{2n}{n} \overset{n}{\equiv} 2$.

ix) Let $p \ge 2$ be prime, and assume that $\max\{k, n-k\} < p \le n$. Then, $p|\binom{n}{k}$. In particular, if $k \le p-1$, then $p|\binom{p}{k}$.

x) If $\gcd\{n,k\} = \gcd\{n-1,k\} = 1$, then $\frac{1}{2}n(n-1)|\binom{n}{k}$.

xi) If $2k \le n$, then $\binom{n}{k}$ has a prime factor $p_1 \ge k+1$ and a prime factor $p_2 \le \max\{n/k, n/2\}$.

xii) If k is prime, then $\binom{n}{k} \overset{k}{\equiv} \lfloor \frac{n}{k} \rfloor$.

xiii) If n is prime and $k \le n-2$, then $\binom{n-1}{k} \overset{n}{\equiv} (-1)^k$.

xiv) If n is prime and $k \le n-3$, then $\binom{n-2}{k} \overset{n}{\equiv} (-1)^k(k+1)$.

xv) If n is prime and $k \le n-4$, then $\binom{n-3}{k} \overset{n}{\equiv} (-1)^k\binom{k+2}{2}$.

xvi) Assume that $n \ge 3$. Then, n is prime if and only if, for all $i \in \{1, \dots, n-1\}$, $\binom{n-1}{i} \overset{n}{\equiv} (-1)^i$.

xvii) If $n \ne 3$ is prime, then $n^2|\sum_{i=1}^{n^2-1}\binom{2i}{i}$.

xviii) If $n \ge 5$ is prime and $k \ge l \ge 1$, then $\binom{kn-1}{n-1} \overset{n^3}{\equiv} 1$, $\binom{kn}{ln} \overset{n^3}{\equiv} \binom{k}{l}$, and $\binom{kn}{n} \overset{n^3}{\equiv} k$.

xix) There exist integers $0 \le m_1 < \cdots < m_k$ such that $n = \sum_{i=1}^k \binom{m_i}{i}$.

xx) If $n \ge 5$ is prime, then $\binom{n^2}{n} \overset{n^5}{\equiv} n$.

xxi) If $n \ge 5$ is prime and $k, l, m \ge 1$, then $\binom{ln^k}{mn^k} \overset{n^{3k}}{\equiv} \binom{ln^{k-1}}{mn^{k-1}}$.

xxii) $2(2n+1)\binom{2n}{n} \Big| \binom{6n}{3n}\binom{3n}{n}$, $\binom{2k}{k} \Big| \binom{4n+2k+2}{2n+k+1}\binom{2n+k+1}{2k}\binom{2n-k+1}{n}$, $\binom{2k}{k} \Big| C_{n+k}(2n+1)\binom{2n}{n}\binom{n+k+1}{2k}$.

Source: [koshcat, pp. 9–11, 15, 18, 21, 23, 25, 44, 45, 65]. *i)* is given in [gelca, p. 296]; *xii)*–*xv)* are given in [comtet, pp. 78, 79]; *xvi)* is given in [AMR2, pp. 21, 142]; *xvii)* is given in [callanp,]; *xviii)* is given in [siong,]; *xix)* is given in [comtet, p. 75]; *xx)* and *xxi)* are given in [fuchs, pp. 37–39]; *xxii)* is given in [zwsun,]. **Example:** To illustrate *xix)*, let $k = 3$ and note that $1 = \binom{0}{1} + \binom{1}{2} + \binom{3}{3}$, $2 = \binom{0}{1} + \binom{2}{2} + \binom{3}{3}$, $3 = \binom{1}{1} + \binom{2}{2} + \binom{3}{3}$, $4 = \binom{0}{1} + \binom{1}{2} + \binom{4}{3}$, $5 = \binom{0}{1} + \binom{2}{2} + \binom{4}{3}$, and $6 = \binom{1}{1} + \binom{2}{2} + \binom{4}{3}$.

Fact 1.16.2. Let $n \ge 1$. Then, the following statements hold:

i) $2^{n-1} \le \operatorname{lcm}\{1,2,\ldots,n\} \le 3^n$.

ii) $\sqrt{n}2^{n-2} \le n\binom{n-1}{\lfloor (n-1)/2\rfloor} \le \operatorname{lcm}\{1,2,\ldots,n\} \le 3^n$.

iii) As $n \to \infty$, $\log \operatorname{lcm}\{1,2,\ldots,n\} \sim n$.

iv) $\operatorname{lcm}\left\{\binom{n}{0},\binom{n}{1},\ldots,\binom{n}{n}\right\} = \dfrac{\operatorname{lcm}\{1,2,\ldots,n+1\}}{n+1}$.

Source: [farhi,]. **Remark:** If $n \ge 4$, then $2^{n-1} \le \sqrt{n}2^{n-2}$.

Fact 1.16.3. Let $n \ge 5$, and assume that n is prime. Then,

$$4^{n-1} \overset{n^3}{\equiv} \pm\binom{n-1}{\frac{1}{2}(n-1)}.$$

Source: [aebi,]. **Remark:** For each $n \ge 5$, the congruence holds for either "+" or "−."
Credit: F. Morley. **Example:** $256 = 4^4 \overset{125}{\equiv} 6 = \binom{4}{2}$ and $4096 = 4^6 \overset{343}{\equiv} -20 = -\binom{6}{3}$.

Fact 1.16.4. Let $n \ge 1$ and $k \ge 1$. Then,

$$n^k = \sum_{i=1}^{k} \alpha_{i,k}\binom{n}{i},$$

where, for all $i \in \{1,\ldots,k\}$,

$$\alpha_{i,k} = \sum_{j=0}^{i} (-1)^{i-j}\binom{i}{j}j^k.$$

In particular,

$$n = \binom{n}{1}, \quad n^2 = 2\binom{n}{2}+\binom{n}{1}, \quad n^3 = 6\binom{n}{3}+6\binom{n}{2}+\binom{n}{1}, \quad n^4 = 24\binom{n}{4}+36\binom{n}{3}+14\binom{n}{2}+\binom{n}{1}.$$

Source: [hirschhornbin,].

Fact 1.16.5. Let $n \ge 1$. Then,

$$\binom{n}{2} = \tfrac{1}{2}(n^2-n), \quad \binom{n+1}{2} = \binom{n}{2} + n = \tfrac{1}{2}(n^2+n), \quad \binom{n+1}{2} + \binom{n}{2} = n^2,$$

$$\binom{2n+2}{n+1} = \frac{4n+2}{n+1}\binom{2n}{n}, \quad \binom{\binom{n}{2}}{2} = 3\binom{n+1}{4},$$

$$\sum_{i=1}^{n} \tfrac{1}{2}i(i+1) = \sum_{i=1}^{n}\binom{i+1}{2} = \binom{n+2}{3} = \tfrac{1}{6}n(n+1)(n+2),$$

$$\sum_{i=1}^{n}(2i-1)^2 = \binom{2n+1}{3} = \tfrac{1}{3}n(2n-1)(2n+1),$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{4}\binom{2n+2}{3} = 2\binom{n+1}{3} + \binom{n+1}{2} = \tfrac{1}{6}n(n+1)(2n+1),$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \binom{n+1}{2}^2 = \tfrac{1}{4}n^2(n+1)^2 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2,$$

$$\sum_{i=1}^{n} i^4 = \tfrac{1}{20}(3n^2+3n-1)\binom{2n+2}{3} = \tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1),$$

$$\sum_{i=1}^{n} i^5 = \binom{n+1}{2} + 30\binom{n+2}{4} + 120\binom{n+3}{6}.$$

Source: [five, pp. 17, 110] and [beardon,demaio,].

Fact 1.16.6. Let $n \geq 0$. Then,

$$\prod_{i=0}^{n} \binom{n}{i} = \frac{(n!)^{n+1}}{\prod_{i=0}^{n}(i!)^2}.$$

Source: [hirschhornabo,].

Fact 1.16.7. Let $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n$ be complex numbers. Then, for all $k \in \{0, 1, \ldots, n\}$,

$$y_k = \sum_{i=0}^{k} \binom{k}{i} x_i$$

if and only if, for all $k \in \{0, 1, \ldots, n\}$,

$$x_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} y_i.$$

Furthermore, for all $k \in \{0, 1, \ldots, n\}$,

$$y_k = \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} x_i$$

if and only if, for all $k \in \{0, 1, \ldots, n\}$,

$$x_k = \sum_{i=0}^{k} (-1)^{i+1} \binom{k}{i} y_i.$$

Source: Each equality is a *binomial transform.* See [boyadzhiev3,].

Fact 1.16.8. The following statements hold:

i) Let $n \geq 0$. Then,

$$\binom{0}{n} = \text{truth}(n=0), \quad \binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = n + \text{truth}(n=0).$$

ii) Let $k, n \geq 0$. Then,

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n+k}{n} = \binom{n+k}{k}, \quad (n-k)\binom{n}{k} = n\binom{n-1}{k}, \quad \binom{n}{k} = \frac{n+1-k}{n+1}\binom{n+1}{k},$$

$$k\binom{n}{k} = n\binom{n-1}{k-1} = (n+1-k)\binom{n}{k-1}, \quad k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}, \quad \binom{kn}{k} = n\binom{kn-1}{k-1},$$

$$\binom{n+k}{2} = \binom{n}{2} + \binom{k}{2} + nk, \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = (-1)^k\binom{k-n-1}{k}.$$

iii) Let $n \ge 1$. Then,

$$\binom{2n}{n} = 2\binom{2n-1}{n} = \frac{n+1}{2n+1}\binom{2n+1}{n} = \frac{n+1}{2(2n+1)}\binom{2n+2}{n+1} = \frac{(2n)!}{(n!)^2},$$

$$\binom{2n}{n} = (n+1)\binom{2n+1}{n+1} - 2(n+1)\binom{2n}{n+1} = \sum_{i=0}^{n}\frac{(n!)^2}{(i!)^2[(n-i)!]^2}, \quad \binom{2n}{n}^2 = \sum_{i=0}^{n}\frac{(2n)!}{(i!)^2[(n-i)!]^2}.$$

iv) Let $0 \le k \le n$. Then,

$$\binom{-n}{k} = \frac{-n(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k\binom{n+k-1}{k}.$$

For example, $\binom{-1}{n} = (-1)^n$.

v) Let $k \ge 0$ and $n \ge 1$. Then,

$$\binom{n+k}{k+1} = \frac{1}{k!}\sum_{i=0}^{n-1}\frac{(k+i)!}{i!}.$$

vi) Let $n \ge k \ge 0$, and let m be prime. Then,

$$\binom{mn}{mk} \overset{m}{\equiv} \binom{n}{k}, \quad \binom{mn}{mk} \overset{m^2}{\equiv} \binom{n}{k}.$$

If, in addition, $m \ge 5$, then

$$\binom{mn}{mk} \overset{m^3}{\equiv} \binom{n}{k}.$$

vii) Let m be prime, and let n, k, q, r be nonnegative integers such that $q < m$ and $r < m$. Then,

$$\binom{mn+q}{mk+r} \overset{m}{\equiv} \binom{n}{k}\binom{q}{r}.$$

viii) Let $0 \le k \le n$. Then,

$$\binom{n-\frac{1}{2}}{k} = \frac{\binom{2n}{n}\binom{n}{k}}{4^k\binom{2n-2k}{n-k}} = \frac{(2n-1)!!}{(2n-2k-1)!!(2k)!!},$$

$$\binom{n+\frac{1}{2}}{n-k} = \frac{2n+1}{2k+1}\frac{\binom{2n}{n}\binom{n}{k}}{4^{n-k}\binom{2k}{k}} = \frac{(2n+1)!!}{(2n-2k)!!(2k+1)!!}.$$

ix) Let $1 \le k \le n$. Then,

$$\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1},$$

$$\frac{n+2}{n+1}\binom{n+1}{k}\binom{n+1}{k+1} = \binom{n}{k-1}\binom{n}{k} + \binom{n}{k}\binom{n}{k+1} + 2\binom{n}{k}^2,$$

$$\frac{n}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}\binom{k+1}{2} = \frac{n}{n+1}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n-k+2}{2} = \binom{n}{k-1}\binom{n}{k}\binom{n+1}{2}.$$

x) Let $1 \le k \le n$. Then,

$$\binom{n}{k}^3 + \binom{n}{k+1}^3 + 3\binom{n}{k}\binom{n}{k+1}\binom{n+1}{k+1} = \binom{n+1}{k+1}^3,$$

$$\frac{1}{\binom{n+1}{k}^3} + \frac{1}{\binom{n+1}{k+1}^3} + \frac{3(n+2)}{(n+1)\binom{n}{k}\binom{n+1}{k}\binom{n+1}{k+1}} = \frac{(n+2)^3}{(n+1)^3\binom{n}{k}^3}.$$

Source: [benczeAOI,], [benjaminquinn, pp. 123, 124], [chuzhangqi,], [fuchs, p. 31], [goulddouble,], [GKP, p. 174], [herman, p. 10], and [koshcat, pp. 5, 43]. *viii*) is given in [grimaldi, p. 274]. **Example:** $252 = \binom{10}{5} \overset{125}{\equiv} \binom{2}{1} = 2$.

Fact 1.16.9. The following statements hold:

i) Let $x \in \mathbb{C}$. Then,

$$\binom{x}{0} = 1, \quad \binom{x}{1} = x.$$

ii) Let $k \ge 1$ and $x \in \mathbb{C}$. Then,

$$\binom{x}{k} = \frac{x}{k}\binom{x-1}{k-1} = \frac{x+1-k}{k}\binom{x}{k-1}.$$

iii) Let $k \ge 2$ and $x \in \mathbb{C}$. Then,

$$\binom{x}{k} = \frac{x(x-1)}{k(k-1)}\binom{x-2}{k-2}.$$

iv) Let $k \ge 0$ and $x \in \mathbb{C}$. Then,

$$(x-k)\binom{x}{k} = x\binom{x-1}{k}.$$

v) Let $0 \le k \le n$ and $x \in \mathbb{C}$. Then,

$$\binom{x}{k} = \binom{x-1}{k} + \binom{x-1}{k-1}.$$

vi) Let $k \ge 0$ and $x \in \mathbb{C}$. Then,

$$\binom{x}{k} = (-1)^k\binom{k-x-1}{k}.$$

vii) Let $n, k \ge 0$ and $x \in \mathbb{C}$. Then,

$$\binom{n}{k}\binom{x+n}{n} = \binom{x+n}{n-k}\binom{x+k}{k}.$$

Fact 1.16.10. The following statements hold:

i) Let $x, y, z \in \mathbb{C}$ and $n \ge 0$. Then,

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x(x-iz)^{i-1}(y+iz)^{n-i}.$$

In particular,

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i} = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i.$$

ii) Let $0 \le k \le m \le n$. Then,

$$\sum_{i=m}^{n} \binom{i}{k} = \binom{n+1}{k+1} - \binom{m}{k+1}.$$

In particular,

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}.$$

iii) Let $n \ge k \ge 0$ and $m \ge 0$. Then,

$$\sum_{i=k}^{n} \binom{m+i}{i} = \binom{n+m+1}{n} - \binom{k+m}{k-1}.$$

In particular,

$$\sum_{i=0}^{n} \binom{m+i}{i} = \sum_{i=0}^{n} \binom{m+i}{m} = \binom{n+m+1}{n} = \binom{n+m+1}{m+1},$$

$$\sum_{i=0}^{n} \binom{n+i}{i} = \sum_{i=0}^{n} \binom{n+i}{n} = \binom{2n+1}{n}.$$

iv) Let $n, m \ge 1$. Then,

$$\sum_{i=1}^{m} \binom{n+m-i}{n} = \binom{n+m}{n+1}.$$

v) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n} \frac{1}{2^i}\binom{n+i}{i} = 2^n, \quad \sum_{i=0}^{n} \frac{i}{2^i}\binom{n+i}{i} = \frac{n+1}{2^{n+1}}\left[2^{2n+1} - \binom{2n+2}{n+1}\right].$$

vi) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Let $n \ge 3$ be odd. Then,

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i} = 2^{n-1} - 1.$$

Let $n \ge 2$ be even. Then,

$$\sum_{i=1}^{n/2} \binom{n}{i} = \frac{1}{2}\binom{n}{n/2} + 2^{n-1} - 1.$$

vii) Let $n, k \ge 1$. Then,

$$\sum_{i=0}^{\lfloor n/k \rfloor} \binom{n}{ki} = \frac{2^n}{k} \sum_{i=1}^{k} \cos^n \frac{i\pi}{k} \cos \frac{ni\pi}{k}.$$

In particular,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} = 2^{n-1}.$$

viii) Let $n \geq 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(2i-x)\binom{x}{i} = (n-x)\binom{x}{n} = -(n+1)\binom{x}{n+1}.$$

ix) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} i\binom{n}{i} = 2^{n-1}n.$$

x) Let $n, k \geq 1$, and define

$$S_{n,k} \triangleq \sum_{i=1}^{n} i^k\binom{n}{i}.$$

Then,

$$S_{n,k+1} = n(S_{n,k} - S_{n-1,k}).$$

xi) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} i^2\binom{n}{i} = 2^{n-2}n(n+1), \quad \sum_{i=1}^{n} i^3\binom{n}{i} = 2^{n-3}n^2(n+3),$$

$$\sum_{i=1}^{n} i^4\binom{n}{i} = 2^{n-4}n(n+1)(n^2+5n-2).$$

xii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i} = \frac{2^{n+1}-1}{n+1}.$$

xiii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \frac{(i+1)!}{(n+1)^{i+1}}\binom{n}{i} = 1.$$

xiv) Let $n \geq 0$. Then,

$$\sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \frac{1}{2i}\binom{n+1}{2i} = \sum_{i=0}^{n} \frac{2^i-1}{i+1}.$$

xv) Let $n \geq 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n} \frac{x^{i+1}}{i+1}\binom{n}{i} = \frac{(x+1)^{n+1}-1}{n+1}.$$

xvi) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n}\binom{2n+1}{i} = \sum_{i=0}^{n}\binom{2n+1}{2i} = \sum_{i=0}^{2n}\binom{2n}{i} = 4^n.$$

xvii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n}\binom{2n}{n-i} = \sum_{i=0}^{n}\binom{2n}{i} = \frac{1}{2}\left[4^n + \binom{2n}{n}\right], \quad \sum_{i=0}^{\lfloor n/2 \rfloor}\binom{2n}{n-2i} = \frac{1}{2}\binom{2n}{n} + 4^{n-1},$$

$$\sum_{i=0}^{\lfloor n/3\rfloor}\binom{2n}{n-3i}=\frac{1}{2}\binom{2n}{n}+\frac{1}{3}(2^{2n-1}+1),\quad \sum_{i=0}^{\lfloor n/4\rfloor}\binom{2n}{n-4i}=\frac{1}{2}\binom{2n}{n}+2^{2n-3}+2^{n-2},$$

$$\sum_{i=0}^{\lfloor n/5\rfloor}\binom{2n}{n-5i}=\frac{1}{2}\binom{2n}{n}+\frac{1}{5(2^n)}[2^{3n-1}+(3+\sqrt{5})^n+(3-\sqrt{5})^n],$$

$$\sum_{i=0}^{\lfloor n/6\rfloor}\binom{2n}{n-6i}=\frac{1}{2}\binom{2n}{n}+\frac{1}{6}(2^{2n-1}+3^n+1),\quad \sum_{i=1}^{n}\binom{2n-1}{n-i}=4^{n-1}.$$

If $n\ge 2$, then

$$\sum_{i=1}^{\lfloor n/3\rfloor}\binom{2n-3}{n-3i}=\frac{1}{3}(4^{n-2}-1).$$

If $n\ge 3$, then

$$\sum_{i=1}^{\lfloor n/5\rfloor}\binom{2n-5}{n-5i}=\frac{1}{5}4^{n-3}-\frac{1}{5(2^{2n-5})}[(\sqrt{5}+1)^{2n-5}-(\sqrt{5}-1)^{2n-5}].$$

xviii) Let $n\ge 0$. Then,

$$\sum_{i=0}^{n}i\binom{2n}{i}=\tfrac{1}{2}4^n n.$$

xix) Let $n>1$. Then,

$$\sum_{i=0}^{n}i^2\binom{2n}{i}=4^{n-1}n(2n+1)-\tfrac{1}{2}n^2\binom{2n}{n}.$$

xx) Let $n>1$. Then,

$$\sum_{i=0}^{n-1}(n-i)^2\binom{2n}{i}=4^{n-1}n.$$

xxi) Let $n\ge 0$. Then,

$$\sum_{i=0}^{n}i\binom{2n+1}{2i}=(2n+1)2^{2n-2}.$$

xxii) Let $n\ge 0$. Then,

$$\sum_{i=0}^{n}i^2\binom{2n+1}{2i}=(n+1)(2n+1)2^{2n-3}.$$

xxiii) Let $n\ge 1$ and $k\ge 1$. Then,

$$\sum_{i=0}^{n}k^i\binom{n}{i}=(k+1)^n,\quad \sum_{i=0}^{n}k^i i\binom{n}{i}=kn(k+1)^{n-1},$$

$$\sum_{i=0}^{n}k^i i^2\binom{n}{i}=kn(kn+1)(k+1)^{n-2},\quad \sum_{i=0}^{n}k^i i^3\binom{n}{i}=kn(k^2n^2+3kn+1-k)(k+1)^{n-3},$$

$$\sum_{i=0}^{n} k^i i^4 \binom{n}{i} = kn[k^3n^3 + 6k^2n^2 + (7k - 4k^2)n + (k-2)^2 - 3](k+1)^{n-4}.$$

In particular,

$$\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n, \quad \sum_{i=0}^{n} 2^i i \binom{n}{i} = 2n3^{n-1}, \quad \sum_{i=0}^{n} 2^i i^2 \binom{n}{i} = 2n(2n+1)3^{n-2},$$

$$\sum_{i=0}^{n} 2^i i^3 \binom{n}{i} = 2n(4n^2+6n-1)3^{n-3}, \quad \sum_{i=0}^{n} 2^i i^4 \binom{n}{i} = 2n(8n^3+24n^2-2n-3)3^{n-4},$$

$$\sum_{i=0}^{n} 3^i \binom{n}{i} = 4^n, \quad \sum_{i=0}^{n} 3^i i \binom{n}{i} = 3n4^{n-1}, \quad \sum_{i=0}^{n} 3^i i^2 \binom{n}{i} = 3n(3n+1)4^{n-2},$$

$$\sum_{i=0}^{n} 3^i i^3 \binom{n}{i} = 3n(9n^2+9n-2)4^{n-3}, \quad \sum_{i=0}^{n} 3^i i^4 \binom{n}{i} = 3n(27n^3+54n^2-15n-2)4^{n-4},$$

$$\sum_{i=0}^{n} 4^i \binom{n}{i} = 5^n, \quad \sum_{i=0}^{n} 4^i i \binom{n}{i} = 4n5^{n-1}, \quad \sum_{i=0}^{n} 4^i i^2 \binom{n}{i} = 4n(4n+1)5^{n-2},$$

$$\sum_{i=0}^{n} 4^i i^3 \binom{n}{i} = 4n(16n^2+12n-3)5^{n-3}, \quad \sum_{i=0}^{n} 4^i i^4 \binom{n}{i} = 4n(64^3+96n^2-36n+1)5^{n-4}.$$

xxiv) Let $n \ge 1$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} 4^i \binom{n}{2i} = \tfrac{1}{2}[3^n + (-1)^n], \qquad \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 5^{2i+1} \binom{n}{2i+1} = \tfrac{1}{2}[6^n - (-4)^n].$$

xxv) Let $n \ge 1$, and let F_n denote the nth Fibonacci number. Then,

$$\sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \binom{n-i}{i-1} = F_k, \quad \sum_{i=0}^{\lfloor n/3 \rfloor} 2^{n-3i} \binom{n-i}{2i} = F_{2n} + 1.$$

xxvi) Let $n \ge 1$, and let L_n denote the nth Lucas number. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} = L_n, \quad \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{2^i n}{n-i} \binom{n-i}{i} = 2^n + (-1)^n,$$

$$\sum_{i=0}^{\lfloor n/2 \rfloor} 2^i \binom{n-i}{i} = \frac{1}{3}[2^{n+1} + (-1)^n], \quad \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{2^i i}{n-i} \binom{n-i}{i} = \frac{1}{3}[2^n + (-1)^n 2].$$

xxvii) Let $n \ge 1$, and let F_n and L_n denote the nth Fibonacci number and nth Lucas number, respectively. Then,

$$\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} i \binom{n-i-1}{i} = \frac{1}{10}[(5n-4)F_n - nL_n],$$

$$\sum_{i=1}^{\lfloor (n-1)/2 \rfloor} i^2 \binom{n-i-1}{i} = \frac{1}{50}[(15n^2 - 20n + 4)F_n - (5n^2 - 6n)L_n].$$

xxviii) Let $n \geq 1$, and let P_n denote the nth Pell number. Then,

$$\sum_{i=0}^{\lfloor n/4 \rfloor} 2^{n+1-4i} \binom{n-2i}{2i} = P_{n+1} + n + 1.$$

xxix) Let $n \geq k \geq 0$. Then,

$$\sum_{i=k}^{n} \binom{n}{i} = \sum_{i=0}^{n-k} 2^i \binom{n-i-1}{k-1}.$$

xxx) Let $n \geq 1$. Then,

$$S_n \triangleq \sum_{i=0}^{n} \binom{n}{i}^{-1} = \frac{n+1}{2^n} \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{2i+1} \binom{n+1}{2i+1} = \frac{n+1}{2^n} \sum_{i=0}^{n} \frac{2^i}{i+1},$$

$$\sum_{i=0}^{n} i \binom{n}{i}^{-1} = \frac{n}{2} S_n = \frac{n(n+1)}{2^{n+1}} \sum_{i=0}^{n} \frac{2^i}{i+1}, \quad \sum_{i=0}^{n} i^2 \binom{n}{i}^{-1} = \frac{(n+1)(n-2)}{4} S_n + \frac{(n+1)^2}{2},$$

$$\sum_{i=0}^{n} i^3 \binom{n}{i}^{-1} = \frac{n(n^2-3n-6)}{8} S_n + \frac{3n(n+1)^2}{4},$$

$$\sum_{i=0}^{n} i^4 \binom{n}{i}^{-1} = \frac{(n+1)(n^3-7n^2-2n+16)}{16} S_n + \frac{(7n-8)(n+1)^3}{8}.$$

Furthermore, for all $n \geq 2$,

$$S_n = \frac{n+1}{2n} S_{n-1} + 1.$$

xxxi) Let $n \geq 1$ and $z \in \mathbb{C}$, and assume that $\operatorname{Re} z \neq -1$. Then,

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} z^i = (n+1) \left(\frac{z}{z+1} \right)^{n+1} \sum_{i=1}^{n+1} \frac{1}{i} \frac{(1+z^i)(1+z)^{i-1}}{z^i}.$$

In particular,

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} = \frac{n+1}{2^{n+1}} \sum_{i=1}^{n+1} \frac{2^i}{i} = \frac{n+1}{2^n} \sum_{i=0}^{n} \frac{2^i}{i+1}.$$

xxxii) Let $n \geq 2$ and $m \geq 0$. Then,

$$\sum_{i=n}^{n+m} \binom{i}{n}^{-1} = \frac{n}{n-1} \left[1 - \binom{n+m}{n-1}^{-1} \right].$$

xxxiii) Let $n \geq 0$ and $x, y \in \mathbb{C}$, assume that $x + y \neq 0$, and define

$$S_n \triangleq \sum_{i=0}^{n} \binom{n}{i}^{-1} x^i y^{n-i}.$$

Then,

$$S_n = x^n + (n+1) \left(\frac{xy}{x+y} \right)^n \sum_{i=0}^{n-1} \frac{[(i+1)y^{i+2} + yx^{i+1}](x+y)^i}{(xy)^{i+1}(i+1)(i+2)}.$$

Furthermore, for all $n \geq 2$,

$$S_n = \frac{(n+1)xy}{n(x+y)} S_{n-1} + \frac{x^{n+1} + y^{n+1}}{x+y}.$$

xxxiv) Let $n \geq 0$, and define

$$S_n \triangleq \sum_{i=0}^{n} \binom{n}{i}^{-2}.$$

Then,

$$S_n = 1 + \frac{(n+1)^2(n+1)!}{4^n(\frac{1}{2})^{\overline{n+2}}} \sum_{i=0}^{n-1} \frac{4^i(\frac{1}{2})^{\overline{i+2}}(3i^2 + 12i^2 + 18i + 10)}{(i+1)^2(i+2)^3(i+1)!}.$$

Furthermore, for all $n \geq 2$,

$$S_n = \frac{(n+1)^3}{2n^2(2n+3)} S_{n-1} + \frac{3n+3}{2n+3}.$$

xxxv) Let $n \geq 1$ and $k \geq 1$. Then,

$$\sum_{i=0}^{n} \binom{n}{i}^{-k} = (n+1)^k \sum_{i=0}^{n} \left[\sum_{j=0}^{i} (-1)^i \frac{1}{n+1+j-i} \binom{i}{j} \right]^k.$$

In particular,

$$\sum_{i=0}^{n} \binom{n}{i}^{-1} = (n+1) \sum_{i=0}^{n} \frac{1}{2^i(n+1-i)} = (n+1) \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^j}{n+1+j-i} \binom{i}{j},$$

$$\sum_{i=0}^{n} \binom{n}{i}^{-2} = (n+1)^2 \sum_{i=0}^{n} \frac{2}{n+1-i} \sum_{j=0}^{i} \frac{(-1)^j}{n+2+j} \binom{i}{j} = (n+1)^2 \sum_{i=0}^{n} \left[\sum_{j=0}^{i} \frac{(-1)^j}{n+1+j-i} \binom{i}{j} \right]^2.$$

Furthermore,

$$\lim_{n \to \infty} \sum_{i=0}^{n} \binom{n}{i}^{-k} = 2.$$

xxxvi) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} \binom{2n}{2i}^{-1} = \frac{2(2n+1)}{2^{2n+2}} \sum_{i=0}^{2n+1} \frac{2^i}{i+1}.$$

xxxvii) Let $n, m \geq 0$ and $1 \leq k \leq m$. Then,

$$\sum_{i=0}^{n} \frac{\binom{n}{i}}{\binom{n+m}{k+i}} = \frac{n+m+1}{(m+1)\binom{m}{k}}.$$

xxxviii) Let $n \geq 1$. Then,

$$\sum_{i=n}^{n^2-n+1} \frac{\binom{(n-1)^2}{i-n}}{i\binom{n^2}{i}} = \frac{1}{n\binom{2n-1}{n}}.$$

xxxix) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} \frac{1}{i^2\binom{2i}{i}} = \frac{2}{3}H_{n,2} - \frac{1}{3}\sum_{i,j=1}^{n} \frac{(i-1)!(j-1)!}{(i+j)!}.$$

xl) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \frac{4^i}{\binom{2i}{i}} = \frac{4^{n+1}(2n+1)}{3\binom{2n+2}{n+1}} + \frac{1}{3}.$$

xli) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} \frac{4^i}{i\binom{2i}{i}} = \frac{2(4^n)}{\binom{2n}{n}} - 2.$$

xlii) Let $n \geq 2$. Then,

$$\sum_{i=1}^{n-1} \frac{4^i}{i(2i+1)\binom{2i}{i}} = 2 - \frac{4^n}{n\binom{2n}{n}}.$$

xliii) Let $n \geq 3$. Then,

$$\sum_{i=2}^{n-1} \frac{4^i}{i(i-1)\binom{2i}{i}} = 4 - \frac{4^n(2n-1)}{n(n-1)\binom{2n}{n}}.$$

xliv) Let $n \geq 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} x^{n-2i} = \tfrac{1}{2}(x+1)^n + \tfrac{1}{2}(x-1)^n,$$

$$2x\sum_{i=0}^{n} \binom{n}{i} x^{2\lfloor i/2 \rfloor} = (1+x)^{n+1} - (1-x)^{n+1}.$$

xlv) Let $n \geq 0$ and $x \in \mathbb{C}$. Then,

$$x\sum_{i=0}^{n} \binom{n+i}{2i} \left(\frac{x^2-1}{4}\right)^{n-i} = [\tfrac{1}{2}(x+1)]^{2n+1} + [\tfrac{1}{2}(x-1)]^{2n+1}.$$

xlvi) Let $n \geq 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n} \frac{1}{i}\binom{n+i-1}{2i-1}(x-1)^{2i}x^{n-i} = \frac{1}{n}(x^n-1)^2.$$

xlvii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} H_i\binom{n}{i} = 2^n\left(H_n - \sum_{i=1}^{n} \frac{1}{2^i i}\right).$$

xlviii) Let $n \geq k \geq 1$. Then,

$$\sum_{i=k}^{n} H_i\binom{i}{k} = \binom{n+1}{k+1}\left(H_{n+1} - \frac{1}{k+1}\right).$$

xlix) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} H_i \binom{n}{i}^2 = (2H_n - H_{2n}) \binom{2n}{n}.$$

l) Let $n \geq k \geq 1$. Then,

$$\sum_{i=k}^{n} \frac{1}{n+1-i} \binom{i}{k} = \binom{n+1}{k} (H_{n+1} - H_k).$$

li) Let $n \geq k \geq 1$. Then,

$$\sum_{i=0}^{k-1} \sum_{j=k}^{n} (-1)^{i+j-1} \frac{1}{j-i} \binom{n}{i} \binom{n}{j} = \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i),$$

$$\sum_{i=1}^{n} i \binom{n}{i}^2 (H_i - H_{n-i}) = \sum_{i=1}^{n} (2i - n) \binom{n}{i}^2 H_i = \binom{2n-1}{n}.$$

lii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} \frac{i}{4^i} \binom{2i}{i} = \frac{n(n+1)(n+2)}{6(4^n)} C_{n+1} = \frac{(2n+2)!}{6(4^n)(n+1)!(n-1)!}.$$

liii) Let $n \geq 0$ and $m \geq 1$. Then,

$$\sum_{i=0}^{n} \binom{n+1}{i} \sum_{j=1}^{m} j^i = (m+1)^{n+1} - 1.$$

liv) Let p be prime and $n \geq 1$. Then,

$$\sum \binom{n}{i} \stackrel{p}{\equiv} 0,$$

where the sum is taken over all $i \in \{1, \ldots, n-1\}$ such that $(p-1)|i$. Equivalently,

$$\sum_{i=1}^{\lfloor \frac{n-1}{p-1} \rfloor} \binom{n}{(p-1)i} \stackrel{p}{\equiv} 0.$$

lv) Let $n \geq 2$. Then, $\sum_{i=1}^{n-1} \binom{n}{i}$ is even.

lvi) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n-1} i^{n-i} (n-i)^i \binom{n}{i} = (-1)^n n! \left(1 + \sum_{i=1}^{n-2} (-1)^i \frac{n^i}{i!} \right).$$

lvii) Let p be prime and $n \in [p, 2p-2]$. Then, $p | \binom{n}{p-1}$.

lviii) Let $n \geq 0$ and $x, y \in \mathbb{C}$. Then,

$$(x+y)^{\underline{n}} = \sum_{i=0}^{n} \binom{n}{i} x^{\underline{n-i}} y^{\underline{i}}.$$

lix) Let $n, m \geq 0$ and $x \in \mathbb{C}$. Then,

$$(1-x)^{n+1}\sum_{i=0}^{m}\binom{n+i}{i}x^i + x^{m+1}\sum_{i=0}^{n}\binom{m+i}{i}(1-x)^i = 1.$$

lx) Let $n \geq 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}\binom{n}{i}^2 x^i = \sum_{i=1}^{n}\binom{n}{i}\binom{2n-i}{n}(x-1)^i.$$

lxi) Let $x_0, x_1, \ldots, x_n \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n} x_i\binom{n}{i} = \sum_{i=0}^{n-1}(x_i + x_{i+1})\binom{n-1}{i}.$$

lxii) Let $n \geq 1$. Then,

$$\sum_{i=1,\, i\,\mathrm{odd}}^{n}\binom{n}{i} = \sum_{i=0,\, i\,\mathrm{even}}^{n}\binom{n}{i} = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i} = 2^{n-1}.$$

lxiii) Let $n \geq 1$ and let $z \in \mathbb{C}$, where $|z| < 1/4$. Then,

$$\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}z^i = \frac{1}{\sqrt{1+4z}}\left[\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1} - \left(\frac{1+\sqrt{1-4z}}{2}\right)^{n+1}\right],$$

$$\sum_{i=0}^{\lfloor n/2\rfloor}\frac{n}{n-i}\binom{n-i}{i}z^i = \left(\frac{1+\sqrt{1+4z}}{2}\right)^{n} + \left(\frac{1+\sqrt{1+4z}}{2}\right)^{n}.$$

lxiv) Let $n \geq 1$ and let $z \in \mathbb{C}$, where $|z| > 2$. Then,

$$\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n-i}{i}z^{n-2i} = \frac{1}{\sqrt{z^2-4}}\left[\left(\frac{z+\sqrt{z^2-4}}{2}\right)^{n+1} - \left(\frac{z-\sqrt{z^2-4}}{2}\right)^{n+1}\right].$$

lxv) Let $n \geq 2$. Then,

$$\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{i}(n-2i) = n\binom{n-1}{\lfloor n/2\rfloor}.$$

lxvi) Let $n \geq 0$, and assume that n is even. Then,

$$\sum_{i=0}^{2n}\binom{2n}{i}|n-i| = n\binom{2n}{n}.$$

lxvii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}\frac{i}{i+2}\binom{n}{i} = 2^n - \frac{2(2^{n+1}n+1)}{(n+1)(n+2)}, \quad \sum_{i=1}^{n}\frac{i}{i+3}\binom{n}{i} = 2^n - \frac{6(2^n n^2 + 2^n n + 2^{n+1} - 1)}{(n+1)(n+2)(n+3)},$$

$$\sum_{i=1}^{n}\frac{i}{i+4}\binom{n}{i} = 2^n - \frac{8(2^n n^3 + 3(2^n)n^2 + 2^{n+3}n + 3)}{(n+1)(n+2)(n+3)(n+4)}.$$

lxviii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} 2^{n-i}\binom{n+i}{2i} = \frac{1}{3}(2^{2n+1}+1).$$

lxix) Let $n \geq 0$ and $k \geq 0$, and define

$$S_k(n) \triangleq \sum_{i=0}^{n} i^k \binom{n+i}{i}.$$

Then,

$$S_0(n) = \binom{2n+1}{n}, \quad S_1(n) = \frac{n(n+1)}{n+2}\binom{2n+1}{n}, \quad S_2(n) = \frac{n(n+1)^3}{(n+2)(n+3)}\binom{2n+1}{n},$$

$$S_3(n) = \frac{n^2(n+1)^2(n^2+4n+5)}{(n+2)(n+3)(n+4)}\binom{2n+1}{n}, \; S_4(n) = \frac{n(n+1)^3(n^4+7n^3+17n^2+9n-4)}{(n+2)(n+3)(n+4)(n+5)}\binom{2n+1}{n},$$

$$S_{k+1}(n) = (n+1)\left[S_k(n+1) - S_k(n) - (n+1)^k\binom{2n+2}{n+1}\right].$$

Furthermore, as $n \to \infty$, $S_k(n) \sim 2n^k\binom{2n}{n}$.

lxx) Let $n \geq 2$. Then,

$$\sum_{i=1}^{n-1} \frac{in^{n-i}}{i+1}\binom{n}{i} = \frac{n(n^n-1)}{n+1}.$$

Source: [benjaminquinn, pp. 64–68, 78], [gould,], [greene, pp. 1, 2], and [herman, pp. 2–10]. *ii)* and *iii)* are given in [plazaPT,]; *iv)* is given in [pandc, p. 159]; *v)* is given in [engel, pp. 96, 97, 207], [GKP, p. 167], and [larcombemlip,]; *vi)* follows from *i)* with $x = y = 1$; *vii)* is given in [benjaminquinn, pp. 65, 81]; *viii)* is given in [pwz, pp. 95, 97]; *ix)* is given in [benjaminquinn, p. 66]; *x)* is given in [gonzalezgci,]; *xi)* is given in [engel, pp. 95, 96], [gonzalezgci,], and [herman2, p. 62]; *xii)* is given in [jeffrey, pp. 35, 36]; *xiii)* is given in [comtet, p. 173]; *xiv)* is given in [SuryWZ,]; *xv)* is given in [GR, p. 5]; *xvi)* is given in [herman, p. 3], [IIbook, p. 16], and [chuzhangqi,]; *xvii)* is given in [GR, p. 12] and [mercacos,]; *xviii)* and *xix)* are given in [GR, p. 12]; *xx)* is given in [chuzhangqi,]; *xxi)* and *xxii)* are given in [IIbook, p. 16]; *xxiii)* is given in [bona, p. 77] and [herman2, p. 55]; *xxiv)* is given in [bona, p. 78]; *xxv)* is given in [bruckmanagain,], [engel, p. 205], and [gelca, p. 301]; *xxvi)* is given in [kisielewicz,koshyflp,]; *xxvii)* is given in [gauthierbinom,]; *xxviii)* is given in [bruckmanpell,]; *xxix)* is given in [comtet, p. 72]; *xxx)* is given in [experimentation, p. 55] and [pippenger,rockett,sprugnoli1,SuryWZ,]; *xxxi)* is given in [comtet, p. 294]; *xxxii)* is given in [bonar, pp. 136, 137]; *xxxiii)* is given in [apagodu,]; *xxxiv)* is given in [apagodu,]; *xxxv)* is given in [apagodu,mansour,]; *xxxvi)* is given in [mansour,]; *xxxvii)* is given in [trif,]; *xxxviii)* is given in [stanford,]; *xxxix)* is given in [wolfharm,]; *xl)*–*xliii)* are given in [sprugnolicentral,]; *xliv)* is given in [benjaminquinn, p. 113]; *xlv)* and *xlvi)* are given in [wilf, p. 174]; *xlvii)* is given in [boyadzhiev3,]; *xlviii)* is given in [benjaminquinn, p. 92] and [GKP, pp. 279, 280]; *xlviv)* is given in [chuharm,]; *l)* is given in [benjaminquinn, p. 92]; *li)* is given in [sondow2005,]; *lii)* is given in [koshcat, p. 57]; *liii)*–*lv)* and *lvii)* are given in [macmillansondow,]; *lvi)* is given in [melzak, p. 179]; *lviii)* is given in [gelca, p. 298] and [gonzalezgci,]; *lix)* is given in [koornwinder,]; *lx)* is given in [comtet, p. 168]; *lxi)* and *lxii)* are given in [gonzalezgci,]; *lxiii)* is given in [GKP,]; *lxiv)* is given in [wilf, pp. 131, 132]; *lxv)* and *lxvi)* are given in [tuenter,]; *lxvii)* is given in [lopezaguayo,]; *lxviii)* is given in [wilf, p. 133]; *lxix*) is given in [larcombesps,parislarcombe,]; *lxx*) is given in [benczebinom,]. **Remark:** *i*) is the *Abel identity*; the special case $z = 0$ is the *binomial identity*. See [comtet, pp. 128–130] and [guojen,]. An extension is given in [abelleib,]. *liii*) is *Pascal's identity*. See [macmillansondow,].

Fact 1.16.11. The following statements hold:

i) Let $0 \le k \le n$. Then,

$$\sum_{i=0}^{k}(-1)^i\binom{n}{i} = (-1)^k\binom{n-1}{k}.$$

ii) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0.$$

iii) Let $n \ge 1$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{2n}{i} = (-1)^n\frac{1}{2}\binom{2n}{n}, \quad \sum_{i=0}^{n}(-1)^i\binom{2n}{n-i} = \frac{1}{2}\binom{2n}{n},$$

$$\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{2n}{n-2i} = \frac{1}{2}\binom{2n}{n} + 2^{n-1}, \quad \sum_{i=0}^{\lfloor n/3\rfloor}(-1)^i\binom{2n}{n-3i} = \frac{1}{2}\binom{2n}{n} + 3^{n-1},$$

$$\sum_{i=0}^{\lfloor n/4\rfloor}(-1)^i\binom{2n}{n-4i} = \frac{1}{2}\binom{2n}{n} + \frac{1}{4}[(2+\sqrt{2})^n + (2-\sqrt{2})^n],$$

$$\sum_{i=0}^{\lfloor n/5\rfloor}(-1)^i\binom{2n}{n-5i} = \frac{1}{2}\binom{2n}{n} + \frac{1}{5(2^n)}[(5+\sqrt{5})^n + (5-\sqrt{5})^n].$$

iv) Let $n \ge 2$. Then,

$$\sum_{i=1}^{n}(-1)^i i\binom{n}{i} = 0.$$

v) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i 2^i\binom{n}{i} = (-1)^n.$$

vi) Let $n \ge 1$ and $1 \le k \le n-1$. Then,

$$\sum_{i=1}^{n}(-1)^i i^{2k}\binom{2n}{n+i} = 0.$$

Furthermore,

$$\sum_{i=1}^{n}(-1)^i\binom{2n}{n+i} = -\frac{1}{2}\binom{2n}{n}, \quad \sum_{i=1}^{n}(-1)^i i^{2n}\binom{2n}{n+i} = (-1)^n\tfrac{1}{2}(2n)!.$$

vii) Let $n > k \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^i i^k \binom{n}{i} = 0.$$

In particular, if $n \geq 3$, then

$$\sum_{i=1}^{n}(-1)^i i^2 \binom{n}{i} = 0.$$

viii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^i i^n \binom{n}{i} = (-1)^n n!.$$

ix) Let $n \geq 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i (x+i)^n \binom{n}{i} = (-1)^n n!.$$

x) Let $n > k \geq 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i (x+i)^k \binom{n}{i} = 0.$$

xi) Let $n \geq 0$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{2i} = 2^{n/2} \cos \frac{n\pi}{4}, \quad \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i \binom{n}{3i} = (2)3^{n/2-1} \cos \frac{n\pi}{6}.$$

xii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \binom{n}{2i+1} = 2^{n/2} \sin \frac{n\pi}{4}.$$

xiii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} = \begin{cases} -1, & n \overset{6}{\equiv} 3 \text{ or } n \overset{6}{\equiv} 4, \\ 0, & n \overset{6}{\equiv} 2 \text{ or } n \overset{6}{\equiv} 5, \\ 1, & n \overset{6}{\equiv} 0 \text{ or } n \overset{6}{\equiv} 1. \end{cases}$$

xiv) Let $n \geq 0$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i 2^{n-2i} \binom{n-i}{i} = n+1, \quad \sum_{i=0}^{\lfloor n/3 \rfloor} (-1)^i 2^{n-3i} \binom{n-2i}{2i} = F_{n+3} - 1.$$

xv) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n}(-1)^i 4^i \binom{n+i}{2i} = (-1)^n (2n+1).$$

xvi) Let $n \geq 0$. Then,

$$\sum_{i=0}^{4n+1}(-1)^{i(i+1)/2}\binom{4n+1}{i} = 0, \quad \sum_{i=0}^{4n+3}(-1)^{i(i+1)/2}\binom{4n+3}{i} = (-4)^{n+1},$$

$$\sum_{i=0}^{2n+2}(-1)^{i(i+1)/2}\binom{2n+2}{i} = (-1)^{(n+1)(n+2)/2}2^{n+1}.$$

xvii) Let $n, k \geq 0$. Then,

$$\sum_{i=0}^{n}(-1)^i(n-i)^k\binom{n}{i} = \begin{cases} 0, & k < n, \\ n!, & k = n, \\ \frac{1}{2}n(n+1)!, & k = n+1. \end{cases}$$

xviii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor}(-1)^i\frac{n}{n-i}\binom{n-i}{i} = \begin{cases} -2, & n \overset{6}{\equiv} 3, \\ -1, & n \overset{6}{\equiv} 2 \text{ or } n \overset{6}{\equiv} 4, \\ 1, & n \overset{6}{\equiv} 1 \text{ or } n \overset{6}{\equiv} 5, \\ 2, & n \overset{6}{\equiv} 0. \end{cases}$$

xix) Let $k > n \geq 0$. Then,

$$\sum_{i=0}^{n}(-1)^{n-i}\frac{1}{k-i}\binom{n}{i} = \frac{1}{(k-n)\binom{k}{n}}.$$

xx) Let $n \geq 1$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor}(-1)^i 2^{n-2i}\frac{n}{n-i}\binom{n-i}{i} = 2.$$

xxi) Let $n \geq 1$ and $x, y \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\frac{1}{i}(x+iy)^n = x^n H_n + nx^{n-1}y.$$

xxii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\frac{1}{i}(n-i)^n = n^n(H_n - 1).$$

xxiii) Let $n \geq 1$ and $1 \leq k \leq n$. Then,

$$\sum_{i=1}^{n}(-1)^{i+1}\frac{1}{i^k}\binom{n}{i} = \sum\prod_{j=1}^{k}\frac{1}{i_j},$$

where the sum is taken over all k-tuples $(i_1, \ldots, i_k)$ of positive integers such that $1 \leq i_1 \leq \cdots \leq i_k \leq n$. In particular,

$$\sum_{i=1}^{n}(-1)^{i+1}\frac{1}{i}\binom{n}{i} = H_n.$$

xxiv) Let $n \ge 0$, $0 \le k \le n$, and $x \in \mathbb{C}$, and assume that $-x \notin \{0, \ldots, n\}$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{i^k}{i+x}\binom{n}{i} = (-1)^k \frac{n!x^k}{\overline{x^{n+1}}}, \ \sum_{i=0}^{n}(-1)^i \frac{1}{i+x}\binom{n}{i} = \frac{n!}{\overline{x^{n+1}}} = \frac{1}{x}\prod_{i=1}^{n}\frac{i}{x+i} = \frac{1}{x\binom{x+n}{n}} = \frac{1}{(n+1)\binom{x+n}{n+1}},$$

$$\sum_{i=0}^{n}(-1)^i \frac{x}{i+x}\binom{n}{i} = \frac{1}{\binom{n+x}{n}}, \quad \sum_{i=0}^{n}(-1)^i \binom{n}{i}\left(\frac{x}{x+i}\right)^2 = \left(\prod_{i=1}^{n}\frac{i}{x+i}\right)\left(1+\sum_{i=1}^{n}\frac{x}{x+i}\right).$$

If $n \ge 2$, then

$$\sum_{i=1}^{n}(-1)^{n-i+1}\frac{i^{n-2}}{x+i}\binom{n-1}{i-1} = \frac{(n-1)!x^{n-2}}{\prod_{i=1}^{n}(x+i)}.$$

xxv) Let $n \ge 1$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{1}{i+1}\binom{n}{i} = \frac{1}{n+1}, \quad \sum_{i=1}^{n}(-1)^{i+1}\frac{1}{i+1}\binom{n}{i} = \frac{n}{n+1}.$$

xxvi) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{1}{2i+1}\binom{n}{i} = \frac{4^n}{(2n+1)\binom{2n}{n}} = \frac{(2n)!!}{(2n+1)!!} = \frac{4^n(n!)^2}{(2n+1)!}.$$

xxvii) Let $n, m \ge 1$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{1}{m+i+1}\binom{n}{i} = \frac{1}{(n+m+1)\binom{n+m}{m}} = \frac{n!m!}{(n+m+1)!}.$$

xxviii) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^{n+i}\frac{1}{2n-2i+1}\binom{2n+1}{i} = \frac{2^{4n}}{(n+1)\binom{2n+1}{n}}.$$

xxix) Let $n \ge 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i}\binom{n}{i} = H_n, \quad \sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i+1}\binom{n}{i} = \frac{n}{n+1}, \quad \sum_{i=1}^{n}(-1)^{i-1}\frac{H_i}{i+1}\binom{n}{i} = \frac{H_n}{n+1},$$

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i+2}\binom{n}{i} = \frac{n(n+3)}{2(n+1)(n+2)}, \quad \sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i+3}\binom{n}{i} = \frac{n(n^2+6n+11)}{3(n+1)(n+2)(n+3)}.$$

xxx) Let $n \ge 0$ and $m \ge 1$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{1}{(i+m)^2}\binom{n}{i} = \frac{(H_{n+m} - H_{m-1})}{m\binom{n+m}{n}}.$$

In particular,

$$\sum_{i=0}^{n}(-1)^i \frac{1}{(i+1)^2}\binom{n}{i} = \frac{H_{n+1}}{n+1}, \quad \sum_{i=0}^{n}(-1)^i \frac{1}{(i+2)^2}\binom{n}{i} = \frac{H_{n+2}-1}{(n+1)(n+2)},$$

$$\sum_{i=0}^{n}(-1)^i\frac{1}{(i+3)^2}\binom{n}{i} = \frac{2H_{n+3}-3}{(n+1)(n+2)(n+3)}.$$

Furthermore,

$$\sum_{i=0}^{n}(-1)^i\frac{1}{(i+m)^3}\binom{n}{i} = \frac{(H_{n+m}-H_{m-1})^2+H_{n+m,2}-H_{m-1,2}}{2m\binom{n+m}{n}}, \quad \sum_{i=0}^{n}(-1)^i\frac{1}{(i+m)^4}\binom{n}{i}$$
$$= \frac{(H_{n+m}-H_{m-1})^3+3(H_{n+m}-H_{m-1})(H_{n+m,2}-H_{m-1,2})+2(H_{n+m,3}-H_{m-1,3})}{6m\binom{n+m}{n}}.$$

xxxi) Let $n \ge 1$, $k \in \{0,1,\ldots,n\}$, and $x,y \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(ix+y)^k = \begin{cases} 0, & 0 \le k \le n-1, \\ n!(-x)^n, & k=n. \end{cases}$$

xxxii) Let $n \ge 1$ and $k \in \{1,\ldots,n\}$, and assume that n and k are odd. Then,

$$\sum_{i=0}^{(n-1)/2}(-1)^i\binom{n}{i}(n-2i)^k = \begin{cases} 0, & k<n, \\ 2^{n-1}n!, & k=n. \end{cases}$$

xxxiii) Let $n \ge 2$ and $k \in \{2,\ldots,n\}$, and assume that n and k are even. Then,

$$\sum_{i=0}^{(n-2)/2}(-1)^i\binom{n}{i}(n-2i)^k = \begin{cases} 0, & k<n, \\ 2^{n-1}n!, & k=n. \end{cases}$$

xxxiv) Let $k,n \ge 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i(x-ki)^n\binom{n}{i} = k^n n!.$$

xxxv) Let $n \ge 1$, let $p \in \mathbb{F}[s]$, assume that $\deg p \le n$, and let $\alpha \in \mathbb{C}$ be the coefficient of s^n in $p(s)$. Then,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}p(i) = \alpha(-1)^n n!.$$

If, in particular, $\deg p < n$, then,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}p(i) = 0.$$

xxxvi) Let $n \ge 1$ and $k \in \{0,1,\ldots,n\}$. Then,

$$\sum_{i=0}^{n}(-1)^i i^k\binom{n}{i} = \begin{cases} 0, & k<n, \\ (-1)^n n!, & k=n. \end{cases}$$

xxxvii) Let $n,k \ge 1$, and assume that n is prime. Then,

$$\sum_{i=1}^{n-1}(-1)^i i^k\binom{n-1}{i} \overset{n}{\equiv} \begin{cases} -1, & n-1|k, \\ 0, & \text{otherwise.} \end{cases}$$

xxxviii) Let $n \geq 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n} H_i \binom{n}{i} x^{n-i} = H_n(x+1)^n - \sum_{i=1}^{n} \frac{1}{i} x^i (x+1)^{n-i}.$$

In particular,

$$\sum_{i=1}^{n} (-1)^{i+1} H_i \binom{n}{i} = \frac{1}{n}, \quad \sum_{i=1}^{n} H_i \binom{n}{i} = 2^n H_n - \sum_{i=1}^{n} \frac{2^{n-i}}{i},$$

$$\sum_{i=1}^{n} (-1)^i 2^{n-i} H_i \binom{n}{i} = H_n - \sum_{i=1}^{n} \frac{2^i}{i}, \quad \sum_{i=1}^{n} (-1)^{n-i} 2^i H_i \binom{n}{i} = H_n - \sum_{i=1}^{n} (-1)^i \frac{1}{i}.$$

xxxix) Let $n \geq 1$ and $1 \leq k \leq \lfloor n/2 \rfloor$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} C_{n-k-1} = 0.$$

xl) Let $n \geq 1$ and $k \in \{0, 1, \ldots, n\}$. Then,

$$\sum_{i=k}^{n} (-1)^i \binom{n}{i} \binom{i}{k} = \begin{cases} 0, & k < n, \\ (-1)^n, & k = n. \end{cases}$$

xli) Let $n \geq 1$ and $k \in \{0, 1, \ldots, n\}$. Then,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k = \begin{cases} 0, & k < n, \\ n!, & k = n. \end{cases}$$

xlii) Let $n, m \geq 1$ and $k \in \{0, 1, \ldots, n-1\}$. Then,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (mi)^k = 0.$$

xliii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} (-1)^i \frac{1}{\binom{n}{i}} = \begin{cases} 0, & n \text{ odd}, \\ \dfrac{2n+2}{n+2}, & n \text{ even}, \end{cases} \quad \sum_{i=0}^{2n} (-1)^i \frac{1}{\binom{2n}{i}} = \frac{2n+1}{n+1}.$$

xliv) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} (-1)^{i+1} \frac{1}{\binom{2n}{i}} = \frac{1}{2(n+1)} + \frac{(-1)^{i+1}}{2\binom{2n}{n}}, \quad \sum_{i=1}^{2n-1} (-1)^{i+1} \frac{i}{\binom{2n}{i}} = \frac{n}{n+1}.$$

xlv) Let $n \geq 0$. Then,

$$\sum_{i=0}^{2n} (-1)^i \frac{\binom{2n}{i}}{\binom{4n}{2i}} = \frac{4n+1}{2n+1}, \quad \sum_{i=0}^{2n} (-1)^i \frac{\binom{4n}{2i}}{\binom{2n}{i}} = \frac{1}{1-2n}, \quad \sum_{i=0}^{n} (-1)^i \frac{4^i \binom{n}{i}}{\binom{2i}{i}} = \frac{1}{1-2n}.$$

xlvi) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^i\frac{4^i\binom{n}{i}}{i\binom{2i}{i}} = H_n - 2H_{2n}, \quad \sum_{i=1}^{n}(-1)^i\frac{4^i\binom{n-1}{i-1}}{i^2\binom{2i}{i}} = \frac{1}{n}(H_n - 2H_{2n}).$$

xlvii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(H_i - 2H_{2i}) = \frac{4^n}{n\binom{2n}{n}}.$$

xlviii) Let $n \geq 1$ and $k \geq 2$. Then,

$$\sum_{i=0}^{n}(-1)^i\frac{\binom{kn}{i}}{\binom{2kn}{2i}} = \begin{cases} 0, & kn \text{ odd}, \\ \dfrac{2kn+1}{2(kn+1)}, & kn \text{ even}, \end{cases} \quad \sum_{i=0}^{n}(-1)^i\frac{i\binom{kn}{i}}{\binom{2kn}{2i}} = \begin{cases} -\dfrac{(kn+1)(2kn+1)}{2(kn+2)}, & kn \text{ odd}, \\ \dfrac{kn(2kn+1)}{2(kn+1)}, & kn \text{ even}. \end{cases}$$

xlix) Let $n, k \geq 1$. Then,

$$\sum_{i=0}^{n}(-1)^i\frac{1}{\binom{n+k}{i+k}} = \frac{n+k+1}{n+k+2}\left[\binom{n+k+1}{k}^{-1} + (-1)^n\right],$$

$$\sum_{i=0}^{n}(-1)^i\frac{i}{\binom{n+k}{i+k}} = (-1)^n(n+1)\frac{n+k+1}{n+k+2} - \frac{n+k+1}{n+k+3}\left[\binom{n+k+2}{k+1}^{-1} + (-1)^n\right].$$

l) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n}(-1)^i\sum_{j=1}^{n}\frac{1}{i+j}\binom{n}{i} = \frac{1}{n}\left[1 - \binom{2n}{n}^{-1}\right].$$

li) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{1}{(i+k)^2} = \frac{(k-1)!}{(n+1)^{\overline{k}}}(H_{n+k} - H_{k-1}), \quad \frac{(k-1)!}{(n+1)^{\overline{k}}} = \sum_{i=1}^{k}(-1)^{i-1}\frac{1}{n+i}\binom{k-1}{i-1}.$$

In particular,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{1}{(i+1)^2} = \frac{H_{n+1}}{n+1}, \quad \sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{1}{(i+2)^2} = \frac{H_{n+2}-1}{(n+1)(n+2)},$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{1}{(i+3)^2} = \frac{2H_{n+3}-3}{(n+1)(n+2)(n+3)}.$$

lii) Let $n, k \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{H_i}{\binom{k+i}{i}} = \frac{k}{n+k}(H_{k-1} - H_{n+k-1}), \quad \sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{iH_i}{\binom{k+i}{i}} = \frac{nk(H_{n+k-2} - H_{k-1} - 1)}{(n+k)(n+k-1)},$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{i^2H_i}{\binom{k+i}{i}} = \frac{nk[2n-k+(k-n)(H_{n+k-3} - H_{k-1})}{(n+k)(n+k-1)(n+k-2)}.$$

liii) Let $n, k \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{H_{k+i}-H_k}{\binom{k+i}{i}} = \frac{kH_{n+k}}{n+k}, \quad \sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{i(H_{k+i}-H_i)}{\binom{k+i}{i}} = \frac{nk(1-H_{n+k})}{(n+k)(n+k-1)},$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{i(H_{k+i}-H_k)}{\binom{k+i}{i}} = \frac{n(n^2-n-k^2)}{(n+k)^2(n+k-1)^2},$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{i^2(H_{k+i}-H_i)}{\binom{k+i}{i}} = \frac{nk(1+k-2n)+nk(n-k)H_{n+k}}{(n+k)(n+k-1)(n+k-2)}.$$

liv) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{\binom{n}{i}}{i^2\binom{n+i}{n}} = \sum_{i=1}^{n}(-1)^{i-1}\frac{\binom{2n}{n+i}}{i^2\binom{2n}{n}} = \tfrac{1}{2}H_{n,2}.$$

lv) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i}\binom{n}{i}\sum_{j=1}^{i}\frac{H_j}{j} = 3\sum_{i=1}^{n}(-1)^{i-1}\frac{(n!)^2\binom{n}{i}}{i^3\prod_{j=1}^{n}(i^2+ij+j^2)} = H_{n,3}.$$

lvi) Let $n \geq 1$ and $k \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{\binom{n}{i}}{i^k\prod_{j=1}^{n}\left(\sum_{l=0}^{k-1}(j/i)^l\right)} = \tfrac{1}{k}H_{n,k}.$$

lvii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\frac{1}{1-2i}\binom{n}{i} = \frac{4^n}{\binom{2n}{n}}.$$

lviii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{i}{(2i-1)^2}\binom{n}{i} = \frac{4^n}{2\binom{2n}{n}}\sum_{i=1}^{n}\frac{1}{2i-1}.$$

lix) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{n+i}{(2i-1)^3}\binom{2n}{n+i} = \frac{16^n}{4\binom{2n}{n}}\sum_{i=1}^{n}\frac{1}{(2i-1)^2}.$$

lx) Let $n \geq 1$ and $z \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}(1-z^i) = (1-z)^n.$$

lxi) Let $m \geq 0$, $k \geq 1$, and $n \geq km+1$. Then,

$$\sum_{i=1}^{n-m}(-1)^{i+1}\binom{n}{i}\binom{n-i}{m}^k = \binom{n}{m}^k.$$

lxii) Let $n \ge 1$. If $0 \le k \le n-1$, then

$$\sum_{i=k+1}^{n} (-1)^{k+i-1} \frac{1}{i-k}\binom{n}{i} = \binom{n}{k}(H_n - H_k).$$

If $1 \le k \le n$, then

$$\sum_{i=0}^{k-1}(-1)^{k+i-1}\frac{1}{k-i}\binom{n}{i} = \binom{n}{k}(H_n - H_{n-k}), \quad \sum_{i=0}^{k-1}\sum_{j=k}^{n}(-1)^{i+j-1}\frac{1}{j-i}\binom{n}{i}\binom{n}{j} = \sum_{i=0}^{k-1}\binom{n}{i}^2(H_{n-i} - H_i).$$

lxiii) Let $n \ge 1$. Then,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}(H_i^2 - 3H_{i,2}) = \frac{2}{n}(H_n + H_{n-1}),$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}(H_i^3 - 9H_iH_{i,2} + 14H_{i,3}) = -\frac{3}{n}[(H_n + H_{n-1})^2 + H_{n,2} - H_{n-1,2}].$$

Source: *i)* and *ii)* are given in [gelca, p. 295]; *iii)* is given in [koshcat, p. 84] and [mercacos,]; *iv)*, *vii)*–*x)* are given in [jeffrey, pp. 35, 36] and [GR, pp. 4, 5]; *v)* is given in [pwz, p. 133]; *vi)* is given in [koshcat, pp. 40–42]; the first equality *xi)* and *xii)* are given in [herman, p. 75]; the second equality in *xi)* is given in [wilf, pp. 54, 55]; *xiii)*, *xvii)*, and *xviii)* are given in [benjaminquinn, pp. 85, 86, 89, 90]; *xiv)* is given in [bruckmansnake,] and [comtet, p. 168]; *xv)* is given in [larson, pp. 183–185]; *xvi)* is given in [plazafalcon,]; *xix)* is given in [IIbook, pp. 48–52]; *xx)* follows from Fact **??**; *xxi)* and *xxii)* are given in [GKP, pp. 280–282]; *xxiii)* is given in [boyadzhiev3,dilcher,], [GR, p. 5], and [larson, pp. 160, 161]; *xxiv)* is given in [wenghangjat,furduilim,petersonbin,] and [pwz, p. 31]; *xxvi)* is given in [IIbook, p. 166]; *xxvii)* is given in [spivey16,SuryWZ,]; *xxviii)* is given in [clarkewallis,]; *xxix)* is given in [srivastavachoi, p. 254]; *xxx)* is given in [chuharm,larcombetasb,] and [larson, p. 163]; *xxxi)*–*xxxiii)* are given in [katsuurabinom,]; *xxxiv)* is given in [equalsums,]; *xxxv)* is given in [fengming,gonzalezgci,katsuurabinom,]; *xxxvi)* is given in [fengming,]; *xxxvii)* is given in [apostolnumbertheory, p. 275]; *xxxviii)* is given in [boyadzhiev2,munarini,]; *xxxix)* is given in [koshcat, p. 131]; *xl)*–*xliii)* are given in [fengming,]; *xliv)* is given in [sprugnoli2,]; *xlv)* is given in [sprugnolicentral,trif,]; *xlvi)* and *xlvii)* are given in [sprugnolicentral,]; *l)* is given in [SuryWZ,]; *xlix)* is given in [SuryWZ,trif,]; *l)* is given in [koshcat, p. 38]; *li)*–*liii)* are given in [chuharm,]; *liv)* is given in [korus,]; *lv)* is given in [guoqibang,korus,]; *lvi)* is given in [korus,]; *lvii)* is given in [sprugnolicentral,]; *lviii)* and *lix)* are given in [korus,]; *lx)* is given in [barrerojl,]; *lxi)* is given in [abelpow,]; *lxii)* is given in [sondowciec,] and [srivastavachoi, p. 254]; *lxiii)* is given in [wangjia,]. **Related:** Fact **??**.

Fact 1.16.12. The following statements hold:

i) Let $n \ge 0$ and $l \ge 1$, and define $\omega \triangleq e^{(2\pi/l)\jmath}$. Then,

$$\sum_{i=0}^{\lfloor n/l \rfloor}\binom{n}{li} = \frac{1}{l}\sum_{i=0}^{l-1}(1+\omega^i)^n.$$

ii) Let $n \ge 0$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor}\binom{n}{2i} = 2^{n-1}, \quad \sum_{i=0}^{\lfloor n/3 \rfloor}\binom{n}{3i} = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right), \quad \sum_{i=0}^{\lfloor n/4 \rfloor}\binom{n}{4i} = \frac{1}{2}\left(2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}\right).$$

iii) Let $k, l, n \geq 0$, assume that $k < l$, and define $\omega \triangleq e^{(2\pi/l)\jmath}$. Then,

$$\sum_{i=0}^{\lfloor (n-k)/l \rfloor} \binom{n}{li+k} = \frac{1}{l}\sum_{i=0}^{l-1} \omega^{-ik}(1+\omega^i)^n.$$

iv) Let $n \geq 0$. Then,

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} = 2^{n-1}, \qquad \sum_{i=0}^{\lfloor (n-1)/3 \rfloor} \binom{n}{3i+1} = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right),$$

$$\sum_{i=0}^{\lfloor (n-2)/3 \rfloor} \binom{n}{3i+2} = \frac{1}{3}\left(2^n + 2\cos\frac{(n-4)\pi}{3}\right), \qquad \sum_{i=0}^{\lfloor (n-1)/4 \rfloor} \binom{n}{4i+1} = \frac{1}{2}\left(2^{n-1} + 2^{n/2}\sin\frac{n\pi}{4}\right),$$

$$\sum_{i=0}^{\lfloor (n-2)/4 \rfloor} \binom{n}{4i+2} = \frac{1}{2}\left(2^{n-1} - 2^{n/2}\cos\frac{n\pi}{4}\right), \qquad \sum_{i=0}^{\lfloor (n-3)/4 \rfloor} \binom{n}{4i+3} = \frac{1}{2}\left(2^{n-1} - 2^{n/2}\sin\frac{n\pi}{4}\right).$$

v) Let $n \geq 1$. Then,

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \frac{2^i n}{n-i}\binom{n-i}{2i} = 2^{n-1} + \cos\frac{n\pi}{2}.$$

vi) Let $n, k \geq 1$. Then,

$$\sum_{\substack{0 \leq i \leq n \\ k \mid 2i-n}} \binom{n}{i} = \frac{2^n}{k}\sum_{i=0}^{k-1} \cos^n \frac{2i\pi}{k}.$$

vii) Let $n, k \geq 1$, and assume that k divides n. Then,

$$\sum_{i=0}^{n} \binom{n}{ki} = \frac{2^n}{k}\sum_{i=0}^{k-1} (-1)^{ni/k}\cos^n \frac{i\pi}{k}.$$

Source: [benjaminchen,], [jeffrey, p. 35], [krebs,], and [pwz, p. 104]. **Remark:** If $l = 1$, then $\omega = 1$ and *i)* yields *v)* of Fact 1.16.10. If $l = 2$, then $\omega = -1$ and *i)* yields the first equality in *ii)*. If $l = 3$, then $\omega = \frac{1}{2}(-1+\sqrt{3}\jmath)$ and *i)* yields the second equality in *ii)*. For example,

$$\binom{8}{0} + \binom{8}{3} + \binom{8}{6} = 1 + 56 + 28 = \tfrac{1}{3}(2^8 - 1) = 85.$$

If $l = 4$, then $\omega = \jmath$, and *i)* yields the third equality in *ii)*. For example,

$$\binom{13}{0} + \binom{13}{4} + \binom{13}{8} + \binom{13}{12} = 1 + 715 + 1287 + 13 = \tfrac{1}{2}(2^{12} - 2^6) = 2016.$$

Remark: The second equality in *ii)* can be written as

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{3i} = \tfrac{1}{3}(2^n + m),$$

where $m = 2, 1, -1, -2, -1, 1$ correspond to $n \overset{6}{\equiv} 0, 1, 2, 3, 4, 5$, respectively. Likewise,

$$\sum_{i=0}^{\lfloor n/4 \rfloor} \binom{n}{4i} = \tfrac{1}{4}(2^n + m2^{\lceil n/2 \rceil}),$$

where $m = 3, 1, 0, -1, -2, -1, 0, 1$ correspond to $n \overset{8}{\equiv} 0, 1, 2, 3, 4, 5, 6, 7$, respectively. See [benjaminscott,].

Fact 1.16.13. The following statements hold:

i) Let $k, m \geq 0$ and $x \in \mathbb{C}$. Then,

$$\binom{x}{k}\binom{k}{m} = \binom{x}{m}\binom{x-m}{k-m}.$$

ii) Let $k, m \geq 0$ and $x \in \mathbb{C}$. Then,

$$\binom{x}{k}\binom{x-k}{m} = \binom{x}{m}\binom{x-m}{k}.$$

iii) Let $k, m \geq 0$ and $x \in \mathbb{C}$. Then,

$$\binom{k}{m}\binom{x-m}{k} = \binom{x-m}{k-m}\binom{x-k}{m}.$$

iv) Let $k, l, m, n \geq 0$. Then,

$$\sum_{i=l-m}^{n-k} \binom{n}{k+i}\binom{m}{l-i} = \binom{n+m}{k+l}, \quad \sum_{i=\max\{-k,-l\}}^{\min\{n-k,m-l\}} \binom{n}{k+i}\binom{m}{l+i} = \binom{n+m}{n-k+l}.$$

In particular,

$$\sum_{i=1}^{n} \binom{n}{i}\binom{n}{i-1} = \binom{2n}{n+1}.$$

v) Let $m, n \geq 0$ and $0 \leq k \leq m$. Then,

$$\sum_{i=0}^{\min\{n,k\}} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}.$$

vi) Let $k, m, n \geq 0$. If $k \leq m$, then

$$\sum_{i=0}^{\min\{n,k\}} i\binom{n}{i}\binom{m}{k-i} = n\binom{n+m-1}{k-1}.$$

If $k \leq m-1$ and $n \leq m$, then

$$\sum_{i=\max\{1,n+k-m\}}^{\max\{n,k\}} i\binom{n}{i}\binom{m-n}{k-i} = \frac{kn}{m-k}\binom{m-1}{k}.$$

vii) Let $n, k \geq 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n} \binom{n}{i} \binom{x}{k+i} = \binom{n+x}{n+k}.$$

viii) Let $n, m \geq 0$. Then,

$$\sum_{i=0}^{\min\{n,m\}} \binom{n}{i} \binom{m}{i} = \binom{n+m}{n}.$$

In particular,

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}, \quad \sum_{i=0}^{n} \binom{n}{i} \binom{2n}{i} = \binom{3n}{n}.$$

ix) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \binom{n}{i} \binom{2n}{n-i} = \binom{3n}{n}.$$

x) Let $n \geq 0$ and $x, y \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n} \binom{x}{i} \binom{y}{n-i} = \binom{x+y}{n}.$$

xi) Let $l, m \geq 0$, and assume that $n \geq k \geq 0$. Then,

$$\sum_{i=0}^{l} \binom{l-i}{m} \binom{k+i}{n} = \binom{l+k+1}{m+n+1}.$$

xii) Let $0 \leq k \leq n/2$. Then,

$$\sum_{i=k}^{n-k} \binom{i}{k} \binom{n-i}{k} = \binom{n+1}{2k+1}.$$

xiii) Let $n, m, k \geq 0$. Then,

$$\sum_{i=0}^{k} \binom{n+k-i}{n} \binom{m+i}{m} = \binom{n+m+k+1}{k}.$$

xiv) Let $0 \leq k \leq n$. Then,

$$\sum_{i=0}^{n-k} \binom{n}{i} \binom{n}{i+k} = \sum_{i=k}^{n} \binom{n}{i} \binom{n}{i-k} = \binom{2n}{n+k} = \frac{(2n)!}{(n-k)!(n+k)!}.$$

xv) Let $n \geq 0$ and $1 \leq l \leq k$. Then,

$$\sum_{i=l}^{n+l-k} \binom{i-1}{l-1} \binom{n-i}{k-l} = \binom{n}{k}.$$

xvi) Let $0 \le k \le n$. Then,
$$\sum_{i=0}^{k} \binom{n}{i}\binom{n-i}{k-i} = 2^k \binom{n}{k}.$$

xvii) Let $0 \le k \le n$. Then,
$$\sum_{i=k}^{n} \binom{n}{i}\binom{i}{k} = 2^{n-k} \binom{n}{k}.$$

xviii) Let $n \ge 1$ and $0 \le k \le \lfloor n/2 \rfloor$. Then,
$$\sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n}{2i}\binom{i}{k} = \frac{2^{n-2k-1}n}{n-k}\binom{n}{k}.$$

xix) Let $0 \le k \le n-1$. Then,
$$\sum_{i=\lceil k/2 \rceil}^{\lfloor n/2 \rfloor} \binom{n}{2i}\binom{2i}{k} = 2^{n-k-1}\binom{n}{k}.$$

xx) Let $n, k \ge 1$. If $k \le n$, then
$$\sum_{i=0}^{k} 2^{2i}\binom{n}{k-i}\binom{n-k+i}{2i} = \binom{2n}{2k}.$$

If $2k+1 \le n$, then
$$\sum_{i=0}^{k} 2^{2i+1}\binom{n}{k-i}\binom{n-k+i}{2i+1} = \binom{2n}{2k+1}.$$

xxi) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,
$$\sum_{i=0}^{\lfloor x/2 \rfloor} 2^{2i+1}\binom{x-2i}{n-i}\binom{x+1}{2i+1} = \binom{2x+2}{2n+1}.$$

xxii) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,
$$\sum_{i=1}^{n} i\binom{n}{i}\binom{x}{i} = n\binom{n+x-1}{n}.$$

xxiii) Let $n \ge 1$. Then,
$$\sum_{i=1}^{n} i\binom{n}{i}^2 = \frac{n}{2}\binom{2n}{n} = n\binom{2n-1}{n-1} = (2n-1)\binom{2n-2}{n-1}.$$

xxiv) Let $n \ge 1$. Then,
$$\sum_{i=1}^{n} i^2\binom{n}{i}^2 = \frac{n^3}{2n-1}\binom{2n-1}{n-1}.$$

xxv) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \frac{1}{4^{2i}(i+1)}\binom{2i}{i}^2 = \frac{(2n+1)^2}{4^{2n}(n+1)}\binom{2n}{n}^2.$$

xxvi) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \frac{(4i-1)}{4^{2i}(2i-1)^2}\binom{2i}{i}^2 = \frac{(4n-1)}{4^{2n}(4n-1)}\binom{2n}{n}^2.$$

xxvii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} \frac{\binom{n}{i}^2}{(2i+1)\binom{2n}{2i}} = \frac{2^{4n}(n!)^4}{(2n+1)!(2n)!}.$$

xxviii) Let $m, n \geq 1$. Then,

$$\sum_{i=0}^{\min\{m,n\}} 4^i \frac{\binom{m}{i}\binom{n}{i}}{\binom{2i}{i}} = \frac{\binom{2m+2n}{2m}}{\binom{m+n}{m}}.$$

In particular,

$$\sum_{i=0}^{n} 4^i \frac{\binom{n}{i}^2}{\binom{2i}{i}} = \frac{\binom{4n}{2n}}{\binom{2n}{n}}.$$

xxix) Let $n, k \geq 1$. Then,

$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2n-2i}{n-i} = \sum_{i=0}^{n-m} \binom{2i+k}{i}\binom{2n-2i-k}{n-i} = 4^n, \quad \sum_{i=1}^{n} i^2\binom{2i}{i}\binom{2n-2i}{n-i} = \frac{4^{n-1}}{2}n(3n+1),$$

$$\sum_{i=0}^{n} \binom{4i}{2i}\binom{4n-4i}{2n-2i} = 2^{4n-1} + 2^{2n-1}\binom{2n}{n}.$$

xxx) Let $n, k \geq 1$. Then,

$$\sum_{i=1}^{n} i\binom{n}{i}\binom{k}{k-i} = n\binom{n+k-1}{k-1},$$

$$\sum_{i=1}^{\min\{n,k\}} \binom{2n}{n-i}\binom{2k}{k-i} = \frac{1}{2}\left[\binom{2n+2k}{n+k} - \binom{2n}{n}\binom{2k}{k}\right],$$

$$\sum_{i=1}^{\min\{n,k\}} i\binom{2n}{n-i}\binom{2k}{k-i} = \frac{nk}{2(n+k)}\binom{2n}{n}\binom{2k}{k},$$

$$\sum_{i=0}^{\min\{n,k\}} (2i+1)\binom{2n+1}{n-i}\binom{2k+1}{k-i} = \frac{(2n+1)(2k+1)}{n+k+1}\binom{2n}{n}\binom{2k}{k}.$$

In particular,

$$\sum_{i=1}^{n} \binom{2n}{n-i}^2 = \frac{1}{2}\left[\binom{4n}{2n} - \binom{2n}{n}^2\right].$$

xxxi) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \frac{1}{2i+1} \binom{2i}{i} \binom{2n-2i}{n-i} = \frac{16^n}{(2n+1)\binom{2n}{n}},$$

$$\sum_{i=0}^{n} \frac{1}{(2i+1)(2n-2i+1)} \binom{2i}{i} \binom{2n-2i}{n-i} = \frac{16^n}{(n+1)(2n+1)\binom{2n}{n}}.$$

xxxii) Let $n \geq 1$ and $0 \leq k \leq n$. Then,

$$\sum_{i=k}^{n} \frac{1}{i+1} \binom{2i}{i} \binom{2n-2i}{n-i} = \frac{n-k+1}{2(n+1)} \binom{2k}{k} \binom{2n+2-2k}{n+1-k}.$$

In particular,

$$\sum_{i=0}^{n} \frac{1}{i+1} \binom{2i}{i} \binom{2n-2i}{n-i} = \binom{2n+1}{n}.$$

xxxiii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} 2^{n-2i} \binom{2i}{i} \binom{n}{2i} = \sum_{i=0}^{\lfloor n/2 \rfloor} 2^{n-2i} \binom{n}{i} \binom{n-i}{i} = \binom{2n}{n}.$$

xxxiv) Let $n \geq 1$. Then,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{4^i} \binom{2i}{i} \binom{n}{2i} = \frac{1}{2^{n-1}} \binom{2n-1}{n-1}.$$

xxxv) Let $n \geq 0$ and $x, y, z \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n} \frac{x}{x+iz} \binom{x+iz}{i} \binom{y-iz}{n-i} = \binom{x+y}{n}.$$

xxxvi) $n \geq k \geq 0$. Then,

$$\sum_{i=0}^{n-1} \frac{1}{i+1} \binom{2i}{i} \binom{n+k-2i-1}{n-i-1} = \binom{n+k}{n-1}.$$

xxxvii) $n \geq 1$. Then,

$$\sum_{i=1}^{n} \frac{n+1}{i(n-i+1)} \binom{2i-2}{i-1} \binom{2n-2i}{n-i} = \binom{2n}{n}.$$

xxxviii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n-1} \sum_{j=i+1}^{n+1} \binom{n}{i} \binom{n+1}{j} = 4^n - 1.$$

xxxix) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} 2^{n-i} \binom{n}{i} \binom{i}{\lfloor i/2 \rfloor} = \binom{2n+1}{n}.$$

xl) Let $n \geq 1$ and $1 \leq k \leq (n-1)/2$. Then,

$$\sum_{i=k}^{(n-1)/2} \binom{n}{2i+1}\binom{i}{k} = 2^{n-2k-1}\binom{n-k-1}{k}.$$

xli) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} 2^i \binom{2n-2i}{n-i}\binom{n+i}{n} = \frac{2^n}{n!}\prod_{i=1}^{n}(4i-1).$$

xlii) Let $n \geq 1$ and $k \geq 2$. Then,

$$\sum_{i=0}^{n} \frac{1}{kn-ki+1}\binom{kn-ki+1}{i}\binom{ki}{n-i} = \frac{1}{kn+1}\binom{kn+1}{n}.$$

In particular,

$$\sum_{i=0}^{n} \frac{1}{2n-2i+1}\binom{2n-2i+1}{i}\binom{2i}{n-i} = C_n.$$

xliii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} \binom{n}{i}^2 \binom{3n+i}{2n} = \binom{3n}{n}^2.$$

xliv) Let $n \geq k \geq 1$. Then,

$$\sum_{i=0}^{k} \binom{k}{i}^2 \binom{n+2k-i}{2k} = \binom{n+k}{n}^2, \quad \sum_{i=0}^{n} \binom{n}{i}^2 \binom{3n-i}{2n} = \binom{2n}{n}^2.$$

xlv) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j}^3 = \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{i}.$$

xlvi) Let $n \geq 1$ and $0 \leq k \leq n-1$. Then,

$$\sum_{i=0}^{k} \binom{2n}{2i+1}\binom{n-i-1}{k-i} = 2^{2k+1}\binom{n+k}{2k+1}, \quad \sum_{i=0}^{k} \binom{2n-1}{2i}\binom{n-i-1}{k-i} = 4^k\binom{n+k-1}{2k},$$

$$\sum_{i=0}^{k} \binom{2n}{2i}\binom{n-i}{k-i} = \frac{n4^k}{2k}\binom{n+k-1}{2k-1}, \quad \sum_{i=0}^{k} \binom{m+i}{i}\binom{n-i}{k-i} = \binom{n+m+1}{k},$$

$$\sum_{i=0}^{k} \binom{2n-1}{2i+1}\binom{n-i-1}{k-i} = \frac{(2n-1)4^k}{2k+1}\binom{n+k-1}{2k}.$$

xlvii) Let $n \geq 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i}\binom{n+i}{i}(1+x)^i = \sum_{i=0}^{n} \binom{n}{i}\binom{n+i}{i}x^i,$$

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n+i}{i}(x-\tfrac{1}{2})^i = (-1)^n \sum_{i=0}^{n} \binom{n}{i}\binom{n+i}{i}(-x-\tfrac{1}{2})^i.$$

xlviii) Let $k, m, n \geq 0$. Then,

$$\sum_{i=0}^{m-k+l} \binom{m-k+l}{i} \binom{n+k-l}{n-i} \binom{k+i}{m+n} = \binom{k}{m} \binom{l}{n}.$$

xlix) Let $n \geq 0$. Then,

$$\sum \binom{i+j}{i} \binom{j+k}{j} \binom{k+i}{k} = \sum_{i=0}^{n} \binom{2i}{i}, \quad \sum \binom{2i}{i} \binom{2j}{j} \binom{2k}{k} = (2n+1)\binom{2n}{n},$$

where both sums are taken over all 3-tuples (i, j, k) of nonnegative integers such that $i + j + k = n$.

l) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \binom{n}{i} \binom{i}{\lfloor i/2 \rfloor} \binom{n-i}{\lfloor (n-i)/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor} \binom{n+1}{\lfloor (n+1)/2 \rfloor}.$$

li) Let $n \geq 0$, and define

$$S(n) \triangleq \sum_{i=0}^{n} \binom{n}{i}^3 = \sum_{i=0}^{n} \binom{n}{i}^2 \binom{2i}{n}.$$

Then, for all $n \geq 2$,

$$S(n) = \frac{7n^2 - 7n + 2}{n^2} S(n-1) + \frac{8(n-1)^2}{n^2} S(n-2).$$

In particular, $S(0) = 1$, $S(1) = 2$, $S(2) = 10$, $S(3) = 56$, $S(4) = 346$, and $S(5) = 2252$.

lii) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n} \left[\sum_{j=0}^{i} \binom{n}{j} \right]^3 = n2^{3n-1} - 2^{3n} - 3n2^{n-2} \binom{2n}{n}.$$

liii) Let $n \geq 0$, and define

$$S(n) \triangleq \sum_{i=0}^{n} \binom{n}{i}^2 \binom{n+i}{i}^2.$$

Then,

$$S(n) = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} S(n-1) - \frac{(n-1)^3}{n^3} S(n-2).$$

In particular, $S(0) = 1$, $S(1) = 5$, $S(2) = 73$, $S(3) = 1445$, $S(4) = 33001$, and $S(5) = 819005$.

liv) Let $n \geq 1$. Then,

$$S(n) \triangleq \sum_{i=0}^{n} \binom{n}{i}^4 = \sum_{i,j=0}^{n} (-1)^{n+i+j} \binom{n}{i} \binom{n}{j} \binom{n+i}{i} \binom{n+j}{j} \binom{2n-i-j}{n}.$$

Furthermore, for all $n \ge 2$,

$$S(n) = \frac{2(2n-1)(3n^2-3n+1)}{n^3}S(n-1) + \frac{(4n-3)(4n-4)(4n-5)}{n^3}S(n-2).$$

In particular, $S(0) = 1$, $S(1) = 2$, $S(2) = 18$, $S(3) = 164$, $S(4) = 1810$, and $S(5) = 21252$.

lv) Let $n \ge k \ge 0$ and $z \in \mathbb{C}$.

$$\sum_{i=0}^{k} \binom{z}{i}\binom{-z}{n-i} = \frac{n-k}{n}\binom{z-1}{k}\binom{-z}{n-k}.$$

In particular,

$$\sum_{i=0}^{n} \binom{z}{i}\binom{-z}{n-i} = 0.$$

lvi) Let $n > k \ge 0$ and $z \in \mathbb{C}$.

$$\sum_{i=0}^{n} \binom{z}{i}\binom{1-z}{n-i} = \frac{(n-1)(1-z)-k}{n(n-1)}\binom{z-1}{k}\binom{-z}{n-k-1}.$$

lvii) Let $n \ge 1$. Then,

$$\sum_{i=0}^{n} \binom{n}{i}^2\binom{n+i}{i}^2 = \sum_{i=0}^{n}\sum_{j=0}^{i} \binom{n}{i}\binom{n+i}{i}\binom{i}{j}^3.$$

lviii) Let $n \ge 1$ and $z \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n} \binom{n+i}{2i}\binom{2i}{i}\binom{2i}{i+1}z^{i-1}(z+1)^{i+1} = n(n+1)\sum_{i=0}^{n}\left[\binom{n+i}{2i}C_i z^i\right]^2.$$

lix) Let $n \ge 1$. Then,

$$\sum_{i=1}^{n} H_i\binom{n}{i}^2 = H_n\binom{2n}{n} - \sum_{i=1}^{n}\frac{1}{i}\binom{2n-i}{n-i}, \quad \sum_{i=1}^{n} H_i\binom{n}{i}\binom{2n}{i} = H_n\binom{3n}{n} - \sum_{i=1}^{n}\frac{1}{i}\binom{3n-i}{n-i}.$$

lx) Let $n \ge 0$ and $z_1, \ldots, z_m \in \mathbb{C}$. Then,

$$\sum \prod_{j=1}^{m} \binom{i_j + z_j}{i_j} = \binom{n+m-1+\sum_{i=1}^{m} z_i}{n},$$

where the sum is taken over all multisets $\{i_1, \ldots, i_m\}_{\mathrm{ms}}$ of nonnegative integers such that $\sum_{j=1}^{m} i_j = n$.

lxi) Let $n \ge 1$ and $k \ge 1$. Then,

$$\sum (-1)^{\sum_{j=1}^{n} i_j}\binom{\sum_{j=1}^{n} i_j}{i_1, \ldots, i_n}\prod_{j=1}^{n}\binom{n}{j}^{i_j} = \binom{n+k-1}{k},$$

where the sum is over all n-tuples $(i_1, \ldots, i_n)$ of nonnegative integers such that $\sum_{j=1}^{n} j i_j = n$.

lxii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2.$$

lxiii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{n-i-j} = \sum_{i=0}^{n}\binom{2i}{i}.$$

lxiv) Let $m, n \geq 1$. Then,

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{i+j}{i}\binom{m-i+j}{j}\binom{n-j+i}{i}\binom{m+n-i-j}{m-i} = \frac{(m+n+1)!}{m!n!}\sum_{i=0}^{\min\{m,n\}}\frac{1}{2i+1}\binom{m}{i}\binom{n}{i}.$$

lxv) Let $n \geq 0$. Then,

$$\sum_{i=0}^{n}\frac{1}{4^i}\binom{2i}{i}\binom{2n-i}{n} = \sum_{i=0}^{n}\frac{1}{4^i}\binom{2i}{i}\binom{2n+1}{2i} = \frac{1}{4^n}\binom{4n+1}{2n}.$$

lxvi) Let $n \geq 1$. Then,

$$\sum\prod_{j=1}^{m}\binom{2i_j}{i_j} = \frac{4^n\Gamma(n+m/2)}{\Gamma(m/2)},$$

where the sum is taken over all m-tuples $(i_1,\ldots,i_m)$ of nonnegative integers such that $\sum_{j=1}^{m} i_j = n$.

lxvii) Let $n \geq 0$ and $z_1,\ldots,z_m \in \mathbb{C}$. Then,

$$\sum\prod_{j=1}^{m}\binom{z_j}{i_j} = \binom{\sum_{i=1}^{m} z_i}{n},$$

where the sum is taken over all m-tuples $(i_1,\ldots,i_m)$ of nonnegative integers such that $\sum_{j=1}^{m} i_j = n$.

lxviii) Let $n, m \geq 1$. Then,

$$\sum_{i,j=-n}^{n}\binom{2n}{n+i}\binom{2n}{n+j}|i^2-j^2| = 2n^2\binom{2n}{n}^2.$$

lxix) Let $n \geq 1$. Then,

$$\sum_{i=1}^{2n}4^i\binom{\frac{1}{2}}{i}\binom{-\frac{1}{2}}{i}\binom{-2i}{2n-i} = \frac{4n+1}{16^n}\binom{2n}{n}^2.$$

lxx) Let $n \geq 0$, $0 \leq k \leq 2n+1$, and $x \in \mathbb{C}$, and assume that $-x \notin \{0,1,\ldots,n\}$. Then,

$$\sum_{i=0}^{n}\binom{n}{i}^2\left[\frac{(-i)^k}{(x+i)^2}+\frac{(-i)^{k-1}}{x+i}[k-2i(H_i-H_{n-i})]\right] = \left(\frac{n!}{x^{\overline{n+1}}}\right)^2 x^k,$$

$$\sum_{i=0}^{n}\binom{n}{i}^2\left[\frac{1}{(x+i)^2}+\frac{2}{x+i}(H_i-H_{n-i})\right]=\left(\frac{n!}{x^{\overline{n+1}}}\right)^2,$$

$$\sum_{i=0}^{n}\binom{n}{i}^2\binom{n+i}{i}\left[\frac{1}{(x+i)^2}+\frac{1}{x+i}(3H_i-2H_{n-i}-H_{n+i})\right]=\frac{n!(1-x)^{\overline{n}}}{(x^{\overline{n+1}})^2},$$

$$\sum_{i=0}^{n}\binom{n}{i}^2\binom{n+i}{i}^2\left[\frac{1}{(x+i)^2}+\frac{2}{x+i}(2H_i-H_{n-i}-H_{n+i})\right]=\left(\frac{(1-x)^{\overline{n}}}{x^{\overline{n+1}}}\right)^2,$$

$$\sum_{i=0}^{n}\binom{n}{i}^2(H_i-H_{n-i})=0,\quad \sum_{i=0}^{n}\binom{n}{i}^2\binom{n+i}{i}(3H_i-2H_{n-i}-H_{n+i})=0,$$

$$\sum_{i=0}^{n}\binom{n}{i}^2\binom{n+i}{i}^2(2H_i-H_{n-i}-H_{n+i})=0,\quad \sum_{i=0}^{n}(2i-n)\binom{n}{i}^2(H_i-H_{n-i})=\binom{2n}{n}.$$

lxxi) Let $n\ge 0$ and $x\in\mathbb{C}$, and assume that $-x\notin\{0,1,\ldots,n\}$. Then,

$$\frac{1}{x}+\sum_{i=1}^{n}\binom{n}{i}^2\binom{n+i}{i}^2\left[\frac{-i}{(x+i)^2}+\frac{1+2iH_{n+i}+2iH_{n-i}-4H_i}{x+i}\right]=x\left[\frac{(1-x)^{\overline{n}}}{x^{\overline{n+1}}}\right]^2,$$

$$1+\sum_{i=1}^{n}\binom{n}{i}^2\binom{n+i}{i}^2\left[\frac{i^2}{(x+i)^2}-\frac{2i^2}{x+i}\left(\frac{1}{i}+H_{n+i}+H_{n-i}-2H_i\right)\right]=\left[\frac{(1-x)^{\overline{n}}}{(1+x)^{\overline{n}}}\right]^2.$$

Furthermore,

$$\sum_{i=1}^{n}\binom{n}{i}^2\binom{n+i}{i}^2(1+2iH_{n+i}+2iH_{n-i}-4H_i)=0,$$

$$\sum_{i=1}^{n}i^2\binom{n}{i}^2\binom{n+i}{i}^2\left(\frac{1}{i}+H_{n+i}+H_{n-i}-2H_i\right)=n(n+1).$$

lxxii) Let $n\ge 1$. Then,

$$\sum_{i=1}^{n}\binom{n}{i}^2[2H_i+(n-2i)(2H_i^2+H_{i,2})]=-\frac{1}{n},$$

$$\sum_{i=0}^{n}\binom{n}{i}^3[1+3(n-2i)H_i]=(-1)^n,\quad \sum_{i=1}^{n}\binom{n}{i}^3[2H_i+(n-2i)(3H_i^2+H_{i,2})]=(-1)^n2H_n,$$

$$\sum_{i=1}^{n}\binom{n}{i}^4[1+4(n-2i)H_i]=(-1)^n\binom{2n}{n},\quad \sum_{i=0}^{n}\binom{n}{i}\binom{2n}{i}\binom{2n}{n+i}[1+(n-2i)(2H_i+H_{n+i}]=(-1)^n.$$

lxxiii) Let $n\ge 0$ and $0\le k\le 2n+1$. Then,

$$\sum_{i=0}^{n}i^{k-1}\binom{n}{i}^2[k-2i(H_i-H_{n-i})]=\begin{cases}0, & 0\le k\le 2n,\\ (n!)^2, & k=2n+1.\end{cases}$$

lxxiv) Let $n\ge 1$. Then,

$$\sum_{i=0}^{n}\frac{1}{(2i+1)(2n-2i-1)}\binom{2i}{i}\binom{2(n-i-1)}{n-i-1}=\frac{16^n}{8n^2\binom{2n}{n}}.$$

lxxv) Let $n, m, k_1, \ldots, k_m$ be positive integers, and define $k \triangleq \sum_{i=1}^m k_i$. Then,

$$\sum \prod_{j=1}^{m} \binom{i_j + k_j - 1}{i_j} = \binom{n+k-1}{n},$$

where the sum is taken over all $\binom{n+m-1}{n}$ m-tuples $(i_1, \ldots, i_m)$ of nonnegative integers such that $\sum_{j=1}^m i_j = n$.

Source: *i)*, *iii)*, *iv)*, *xi)*, and *lxvii)* are given in [GKP, pp. 167, 169, 171, 172]; *ii)* is given in [bona, p. 75]; *v)*, *xii)*, *xv)*, *xvii)*, *xix)*, *xxiii)*, and *xxiv)* are given in [benjaminquinn, pp. 64–68, 78]; *vi)* is given in [gonzalezgci,nihous,]; *vii)* is given in [greene, p. 2] and [pwz, p. 31]; *viii)* is given in [benjaminquinn, p. 78] and [pwz, p. 130]; *ix)* is given in [wilf, p. 138]; *x)* is given in [vignat,]; *xiii)* is given in [cossali,]; *xiv)* is given in [herman, p. 9]; *xvi)* is given in [herman2, p. 62]; *xviii)* is given in [shattuckls,]; *xx)* is given in [schottpaper,]; *xxi)* is given in [pwz, p. 31]; *xxii)* is given in [pwz, p. 138]; *xxiv)* follows from [larson, p. 163]; *xxv)* and *xxvi)* are given in [pwz, pp. 95, 96]; *xxvii)* is given in [poghosyan,]; *xxviii)* is given in [chamberlandLAA13,]; *xxix)* is given in [bruckmanps,changxu,gauthiereval,vignat,] and [GKP, p. 187]; *xxx)* is given in [benjaminquinn, p. 79] and [chamberlandLAA13,popescu,]; *xxxi)* is given in [koshcat, p. 84] and [sprugnolicentral,]; *xxxii)* is given in [koshcat, p. 140]; *xxxiii)* is given in [amdeberhan,], [benjaminquinn, p. 78], and [koshcat, p. 97]; *xxxiv)* is given in [pwz, p. 113]; *xxxv)* is given in [chuabel,], [GKP, p. 201], [guojen,], and [pwz, p. 142]; *xxxvi)* and *xxxvii)* are given in [herman2, p. 66]; *xxxviii)* is given in [larson, pp. 161, 162]; *xxxix)*–*xl)* are given in [gelca, pp. 300, 303]; *xli)* is given in [IIbook, p. 156]; *xlii)* is given in [callanflex,]; *xliv)* is given in [comtet, p. 173]; *xlv)* is given in [callanJIS,]; *xlvi)* is given in [hirschhorn,rzad,] and [koshcat, p. 96]; *xlvii)* is given in [gouldcur,]; *xlix)* is given in [ghalayini,] and [pwz, p. 22]; *l)* is given in [gelca, p. 304]; *li)* is given in [comtet, p. 90], [mollnf, p. 171], and [strehl,]; *lii)* is given in [calkin,] *liii)* is given in [AAR, pp. 399, 400] and [mollnf, p. 171]; *liv)* is given in [comtet, p. 90], [mollnf, p. 171], and [pwz, p. 33]; *lv)* and *lvi)* are given in [comtet, p. 169]; *lvii)* is given in [strehl,]; *lviii)* is given in [sunacta,]; *lix)* is given in [munarini,]; *lx)* is given in [abelleib,]; *lxi)* is given in [mercatrudi,]; *lxii)*–*lxiv)* are given in [chentelescope,]; *lxv)* is given in [amdeberhanpbi,]; *lxvi)*–*lxvii)* are given in [vignat,]; *lxviii)* is given in [tuenter,]; *lxix)* is given in [gesselthesum,]; *lxx)* and *lxxi)* are given in [wenghangpfd,wenghangjat,]; *lxxii)* is given in [wangjia,]; *lxxiii)* is given in [wenghangjat,]; *lxxiv)* is given in [chenpseia,]; *lxxv)* is given in [kataria,]. **Remark:** *v)* is *Vandermonde's convolution*. *xxxv)* is *Rothe's identity*; see [chuabel,guojen,]. $S(n)$ in *l)* is the nth *Franel number*. See Fact **??**.

Fact 1.16.14. The following statements hold:

i) Let $n, m \ge 0$. Then,

$$\sum_{i=0}^{\min\{n,m\}} (-1)^i \binom{n+m}{n+i}\binom{m+n}{m+i} = \binom{n+m}{n} = \binom{m+n}{m}.$$

ii) Let $n, m \ge 0$. Then,

$$\sum_{i=-m}^{m} (-1)^i \binom{2n}{n-i}\binom{2m}{m-i} = \sum_{i=-n}^{n} (-1)^i \binom{2n}{n-i}\binom{2m}{m-i} = \frac{\binom{2n}{n}\binom{2m}{m}}{\binom{n+m}{n}} = \frac{\binom{2n}{n}\binom{2m}{m}}{\binom{n+m}{m}} = \frac{(2n)!(2m)!}{n!m!(n+m)!}.$$

iii) Let $0 \le n \le k \le m$. Then,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{m+i}{k} = (-1)^n \binom{m}{k-n}.$$

In particular,

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{m+i}{k} = (-1)^k.$$

iv) Let $0 \le k \le n$. Then,

$$\sum_{i=k}^{n}(-1)^i\binom{n}{i}\binom{i}{k} = \begin{cases} 0, & k < n, \\ (-1)^n, & k = n. \end{cases}$$

v) Let $n, m, k \ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{m+i}{k} = \begin{cases} 0, & k < n, \\ (-1)^n, & k = n, \\ (-1)^n \dfrac{m^{\underline{k-n}}}{(k-n)!}, & k > n. \end{cases}$$

vi) Let $n, m, k, l \ge 0$. Then,

$$\sum_{i=\max\{-m,n-k\}}^{l-m}(-1)^i\binom{l}{m+i}\binom{k+i}{n} = (-1)^{l+m}\binom{k-m}{n-l}.$$

vii) Let $n, m, k, l \ge 0$. Then,

$$\sum_{i=n}^{\min\{l-m,k+n\}}(-1)^i\binom{l-i}{m}\binom{k}{i-n} = (-1)^{l+m}\binom{k-m-1}{l-m-n}.$$

viii) Let $n \ge k \ge 1$. Then,

$$\sum_{i=0}^{k}(-1)^{i+1}\binom{n}{i}\binom{n-i}{k-i} = 0.$$

ix) Let $n \ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\frac{1}{2^i}\binom{2i}{i}\binom{n}{i} = \begin{cases} 0, & n \text{ even}, \\ \dfrac{1}{2^n}\dbinom{n}{n/2}, & n \text{ odd}. \end{cases}$$

x) Let $n \ge 0$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i+x}{i} = (-1)^n\binom{x}{n}.$$

xi) Let $n \ge 1$.

$$\sum_{i=0}^{n}(-1)^i\frac{1}{i+1}\binom{2i}{i}\binom{n+i}{2i} = 0.$$

xii) Let $n \ge k \ge 1$. Then,

$$\sum_{i=0}^{n-k}(-1)^i\frac{1}{i+1}\binom{2i}{i}\binom{n+i}{k+2i} = \binom{n-1}{k-1}.$$

xiii) Let $n,k\ge 0$. Then,

$$\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\binom{2k}{k-n+i}=(-1)^n\frac{\binom{2n}{n}\binom{2k}{k}}{\binom{n+k}{n}}.$$

In particular,

$$\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}^2=(-1)^n\binom{2n}{n}.$$

xiv) Let $n\ge 0$. Then,

$$\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n}{i}\binom{2n-2i}{n}=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n-i}{i}\binom{2n-2i}{n-i}=2^n,$$

$$\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n+1}{i}\binom{2n-2i}{n}=n+1,\quad \sum_{i=0}^{n}(-1)^i\binom{2i}{i}\binom{2n-2i}{n-i}=\begin{cases}2^n\binom{n}{n/2}, & n \text{ even},\\ 0, & n \text{ odd}.\end{cases}$$

xv) Let $n,k\ge 1$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{(n-i)k}{n+1}=nk^{n-1}\binom{k}{2}.$$

xvi) Let $n\ge 0$. Then,

$$\sum_{i=0}^{2n+1}(-1)^i\binom{2n+1}{i}^2=0.$$

xvii) Let $n,k\ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n+k+1}{k+i+l}\binom{k+i}{k}=1.$$

xviii) Let $n,m\ge 0$. Then,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{2n-i}{m-i}=\binom{n}{m}.$$

xix) Let $n\ge 1$ and $1\le m\le 2n$. Then,

$$\sum_{i=0}^{\min\{n,2n-m\}}(-4)^i\binom{n}{i}\binom{2n-2i}{m-i}=(-1)^m\binom{2n}{m}.$$

xx) Let $n\ge 1$ and $1\le m\le n$. Then,

$$\sum_{i=m}^{n}(-1)^i4^{n-i}\binom{n}{i}\binom{2i}{i-m}=\binom{2n}{n-m}.$$

In particular,

$$\sum_{i=0}^{n}(-1)^i4^{n-i}\binom{n}{i}\binom{2i}{i}=\binom{2n}{n}.$$

xxi) Let $n, k \geq 0$. Then,

$$\sum_{i=0}^{n} (-1)^{i+k} 2^{n-2i} \binom{n-i}{i} \binom{i}{k} = \binom{n+1}{2k+1}.$$

xxii) Let $n \geq 0$ and $k \geq n+1$. Then,

$$\sum_{i=0}^{n} (-1)^{i} \frac{1}{i+1} \binom{k}{i} \binom{k-1-i}{n-i} = \frac{1}{k+1} \left[\binom{k}{n+1} + (-1)^n \right].$$

xxiii) Let $n \geq k \geq 0$. Then,

$$\sum_{j=k}^{n} \sum_{i=0}^{n-j} (-1)^{i} \binom{i+j}{j} \binom{j}{k} = 1.$$

In particular,

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} (-1)^{i} \binom{i+j}{j} = \sum_{j=1}^{n} \sum_{i=0}^{n-j} (-1)^{i} j \binom{i+j}{j} = 1.$$

xxiv) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} (-1)^{i} \frac{4^i \binom{n}{i}}{\binom{2i}{i}} = \frac{1}{1-2n}.$$

xxv) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} (-1)^{i} \frac{\binom{n}{i}^2}{\binom{2n}{i}} = \frac{1}{\binom{2n}{n}}.$$

xxvi) Let $n \geq 0$ and $x \in \mathbb{C}$, and, if $n \geq 1$, assume that $-x \notin \{1, \ldots, n\}$. Then,

$$\sum_{i=0}^{n} (-1)^{i} \frac{\binom{n}{i}}{\binom{x+i}{i}} = \frac{x}{x+n}.$$

xxvii) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} (-1)^{i} \frac{\binom{n}{i}}{\binom{n+i}{i}} = \sum_{i=1}^{n} (-1)^{i+1} \frac{\binom{n}{i}}{\binom{n+i}{i}} = \frac{1}{2}.$$

xxviii) Let $n \geq 1$. Then,

$$(-1)^n \sum_{i=0}^{2n} (-1)^{i} \binom{2n}{i}^3 = \sum_{i=-n}^{n} (-1)^{i} \binom{2n}{n+i}^3 = \frac{(3n)!}{(n!)^3}.$$

xxix) Let $n \geq 0$. Then,

$$\sum_{i=0}^{2n} (-1)^{i} \binom{2n}{i} \binom{2i}{i} \binom{4n-2i}{2n-i} = \binom{2n}{n}^2.$$

xxx) Let $k, m, n \geq 0$. Then,

$$\sum_{i=-k}^{k} (-1)^i \binom{2k}{k+i}\binom{2m}{m+i}\binom{2n}{n+i} = \frac{(k+m+n)!(2k)!(2m)!(2n)!}{(k+m)!(m+n)!(n+k)!k!m!n!}.$$

xxxi) Let $k, l, m, n, p \geq 0$. Then,

$$\sum_{i,j=0}^{\min\{k,n\}} (-1)^{i+j} \binom{i+j}{j+l}\binom{k}{i}\binom{n}{j}\binom{p+n-i-j}{m-i} = (-1)^l \binom{n+k}{n+l}\binom{p-k}{m-n-l}.$$

xxxii) Let $n \geq 1$ and $x, y \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i \binom{x+i}{i}\binom{y+n}{n-i} = \binom{y-x+n-1}{n}, \quad \sum_{i=0}^{n}(-1)^i \frac{\binom{n}{i}\binom{x+i}{i}}{\binom{y+i}{i}} = \frac{(y-x)^{\overline{n}}}{(y+1)^{\overline{n}}}.$$

xxxiii) Let $n, k \geq 1$ and $x \in \mathbb{C}$, where $x \notin \{-n, \ldots, -1, 0\}$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{\binom{n}{i}}{\binom{x+i}{i}} \sum \prod_{j=1}^{k} \frac{1}{x+i_j} = \frac{x}{(x+n)^{k+1}},$$

where the second sum is taken over all k-tuples of integers $(i_1, \ldots, i_k)$ such that $0 \leq i_1 \leq \cdots \leq i_k \leq i$. In particular,

$$\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\sum\prod_{j=1}^{k}\frac{1}{i_j} = \frac{1}{n^k}, \quad \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\sum_{j=1}^{i}\sum_{l=1}^{j}\frac{1}{jl} = \frac{1}{n^2},$$

$$\sum_{i=0}^{n}(-1)^i \frac{\binom{n}{i}}{\binom{x+i}{i}} \sum_{j=1}^{i}\sum_{l=1}^{j} \frac{1}{(x+j)(x+l)} = \frac{x}{(x+n)^3},$$

$$\sum_{i=1}^{n}(-1)^{i+1} \frac{\binom{n}{i}}{\binom{x+i}{i}} \sum_{j=1}^{i}\sum_{l=1}^{j} \frac{1}{(x+j)(x+l)} = \frac{n}{(x+n)^3},$$

$$\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\left(\sum_{j=1}^{i}\frac{1}{j^3} + \sum_{1\leq j<k\leq i}\frac{1}{jk(j+k)} + \sum_{1\leq j<k<l\leq i}\frac{1}{jkl}\right) = \frac{1}{n^3}.$$

xxxiv) For all $n \geq 1$ and $k \geq 0$,

$$\sum_{i=1}^{n}(-1)^{i+1}\frac{1}{i^2}\binom{n}{i} = \frac{1}{2}(H_n^2 + H_{n,2}) = \sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{ij},$$

$$\sum_{i=0}^{n}(-1)^i \frac{\binom{n}{i}}{\binom{i+k}{k}} \sum_{j=1}^{i} \frac{1}{j+k} = -\frac{n}{(n+k)^2}, \quad \sum_{i=0}^{n}(-1)^i H_i \binom{n}{i} = -\frac{1}{n},$$

$$\sum_{i=1}^{n}(-1)^i \frac{\binom{n}{i}}{\binom{i+k}{k}} \sum_{j=1}^{i}\sum_{l=1}^{j} \frac{1}{(j+k)(l+k)} = -\frac{n}{(n+k)^3},$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\sum_{j=1}^{i}(-1)^{j+1}\frac{1}{j^2}\binom{i}{j} = -\frac{1}{2}\sum_{i=1}^{n}(-1)^i(H_i^2+H_{i,2})\binom{n}{i} = \frac{1}{n^2},$$

$$\sum_{i=1}^{n}(-1)^i\frac{\binom{n}{i}}{\binom{i+k}{k}}\left[\left(\sum_{j=1}^{i}\frac{1}{j+k}\right)^3 + 3\left(\sum_{j=1}^{i}\frac{1}{j+k}\right)\sum_{j=1}^{i}\frac{1}{(j+k)^2} + 2\sum_{j=1}^{i}\frac{1}{(j+k)^3}\right] = -\frac{6n}{(n+k)^4},$$

$$\sum_{i=1}^{n}(-1)^i(H_i^3+3H_iH_{i,2}+2H_{i,3})\binom{n}{i} = -\frac{6}{n^3},$$

$$\sum_{i=1}^{n}(-1)^i(H_i^4+6H_i^2H_{i,2}+3H_{i,2}^2+8H_iH_{i,3}+6H_{i,4})\binom{n}{i} = -\frac{24}{n^4},$$

$$\sum_{i=1}^{n}(-1)^iH_i^3\binom{n}{i} = \frac{1}{2n}\left(5H_{n-1,2}+\frac{4}{n}H_{n-1}-H_{n-1}^2-\frac{2}{n^2}\right),$$

$$\sum_{i=1}^{n}(-1)^iH_iH_{i,2}\binom{n}{i} = \frac{1}{2n}\left(H_{n-1}^2-H_{n-1,2}-\frac{2}{n^2}\right),$$

$$\sum_{i=1}^{n}(-1)^iH_{i,3}\binom{n}{i} = -\frac{1}{2n}\left(H_{n-1,2}+H_{n-1}^2+\frac{2}{n}H_{n-1}+\frac{2}{n^2}\right).$$

If $n+k\ge 2$, then

$$\sum_{i=1}^{n}(-1)^i i\frac{\binom{n}{i}}{\binom{i+k}{k}}\sum_{j=1}^{i}\frac{1}{j+k} = \frac{n(n^2-n-k^2)}{(n+k)^2(n+k-1)^2},$$

$$\sum_{i=1}^{n}(-1)^i i\frac{\binom{n}{i}}{\binom{i+k}{k}}\left[\left(\sum_{j=1}^{i}\frac{1}{j+k}\right)^2+\sum_{j=1}^{i}\frac{1}{(j+k)^2}\right] = \frac{2n[2n^3+3n^2(k-1)-n(3k-1)-k^3]}{(n+k)^3(n+k-1)^3}.$$

If $n\ge 2$, then

$$\sum_{i=1}^{n}(-1)^i iH_i\binom{n}{i} = \frac{1}{n-1},\quad \sum_{i=1}^{n}(-1)^i i(H_i^2+H_{i,2})\binom{n}{i} = \frac{2(2n-1)}{n(n-1)^2},$$

$$\sum_{i=1}^{n}(-1)^i i(H_i^3+3H_iH_{i,2}+2H_{i,3})\binom{n}{i} = \frac{6(3n^2-3n+1)}{n^2(n-1)^3}.$$

xxxv) Let $n\ge 0$, $m\in\mathbb{Z}$, and $x\in\mathbb{C}$. Then,

$$\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\binom{x+i}{2n+m}\binom{x+2n-i}{2n+m} = (-1)^n\binom{2n}{n}\binom{x+n}{2n+m}\frac{\binom{x+n}{n+m}}{\binom{x+n}{n}}.$$

In particular,

$$\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\binom{x+i}{2n}\binom{x+2n-i}{2n} = (-1)^n\binom{x}{n}\binom{x+n}{n},$$

$$\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\binom{x+i}{2n+1}\binom{x+2n-i}{2n+1} = (-1)^n\frac{x}{n+1}\binom{2n}{n}\binom{x+n}{2n+1}.$$

xxxvi) Let $n, m, k \geq 0$. Then,

$$\sum_{i=0}^{\min\{n,m,k\}} (-1)^i \binom{n+m}{n+i}\binom{m+k}{m+i}\binom{k+n}{k+i} = \frac{(n+m+k)!}{(n!)(m!)(k!)}.$$

xxxvii) Let $n \geq 0$ and $x \in \mathbb{C}$, and assume that $-x \notin \{0, \ldots, n\}$. Then,

$$\sum_{i=0}^{n}(-1)^i \frac{1}{x+i}\binom{n}{i}\binom{n+i}{i} = \frac{(1-x)^{\overline{n}}}{x^{\overline{n+1}}},$$

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i}^3 \left(\frac{1}{(x+i)^3} + \frac{3}{(x+i)^2}(H_i - H_{n-i}) + \frac{3}{2(x+i)}[3(H_i - H_{n-i})^2 + H_{i,2} + H_{n-i,2}]\right) = \left(\frac{n!}{x^{\overline{n+1}}}\right)^3.$$

Furthermore,

$$\sum_{i=0}^{n}(-1)^i[3(H_i - H_{n-i})^2 + H_{i,2} + H_{n-i,2}]\binom{n}{i}^3 = 0.$$

xxxviii) Let $n \geq 0$ and $x \in \mathbb{C}$, and assume that $-x \notin \{0, \pm 1 \ldots, \pm(n+1)\}$. Then,

$$\sum_{i=1}^{n}(-1)^i \frac{1}{(n+1)(x-i)}\frac{\binom{2n}{n+i}}{\binom{n+i}{n+1}} + \sum_{i=0}^{n}\left[\frac{1}{(x+i)^2} + \frac{H_i + H_{n+i} - 2H_{n-i}}{x+i}\right]\binom{n}{i}\binom{2n}{n+i} = \frac{n!(2n)!}{(x^{\overline{n+1}})^2(1-x)^{\overline{n}}}.$$

Furthermore,

$$\sum_{i=1}^{n}(-1)^i \frac{1}{n+1}\frac{\binom{2n}{n+i}}{\binom{n+i}{n+1}} = \sum_{i=0}^{n}(2H_{n-i} - H_i - H_{n+i})\binom{n}{i}\binom{2n}{n+i}.$$

xxxix) Let $n \geq 1$ and $x, y \in \mathbb{C}$. Then,

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i}\frac{\binom{y+i}{i}}{\binom{x+i}{i}} = \frac{\binom{x-y+n-1}{n}}{\binom{x+n}{n}}.$$

xl) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}H_i = (-1)^n 2H_n, \quad \sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}(H_i^2 + H_{i,2}) = (-1)^n 4H_n^2,$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}(H_i^3 + 3H_iH_{i,2} + 2H_{i,3}) = (-1)^n 4(2H_n^3 + H_{n,3}),$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}(H_i^4 + 6H_i^2H_{i,2} + 8H_iH_{i,3} + 3H_{i,2}^2 + 6H_{i,4}) = (-1)^n 16H_n(H_n^3 + 2H_{n,3}),$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{2n+i}{i}H_i = (-1)^n\binom{2n}{n}H_{2n},$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{2n+i}{i}(H_i^2 + H_{i,2}) = (-1)^n\binom{2n}{n}(2H_{n,2} + H_{2n}^2 - H_{2n,2}),$$

$$\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{2n+i}{i}(H_i^3 + 3H_iH_{i,2} + 2H_{i,3}) = (-1)^n\binom{2n}{n}(H_{2n}^3 + 2H_{2n,3} + 3H_{2n}(2H_{n,2} - H_{2n,2})).$$

Source: *i*), *vi*), *vii*), *xxxi*), and *xxxvi*) are given in [GKP, pp. 167, 169, 171, 172, 187]; *ii*) is given in [larcombeszily,]; *iii*) is given in [greene, p. 2]; *iv*) is given in [benjaminquinn, p. 84]; *v*) is a corrected version of a result given in [henrici, p. 26]; *viii*) is given in [herman2, p. 62]; *ix*) is given in [pwz, pp. 63, 114]; *x*) is given in [pwz, p. 133]; *xi*) is given in [pwz, p. 115]; *xii*) is given in [AAR, p. 108]; *xiii*) is given in [GR, p. 5] and [menikhes,]; *xiv*) is given in [AMR, pp. 45, 247, 248] and [spivey14,]; *xv*) is given in [coxthieu,]; *xvi*) is given in [GR, p. 5]; *xvii*) is given in [IIbook, p. 65]; *xviii*) is given in [henrici, p. 26]; *xviv*) is given in [koshcat, p. 72]; *xx*) is given in [melhammore,]; *xxi*) is given in [bataillebinom,]; *xxii*) is given in [AAR, pp. 107, 108]; *xxiii*) is given in [janjic,]; *xxiv*), *xxvi*), *xxviii*), and *xxix*) are given in [pwz, pp. 22, 32, 44–47]; *xxv*) is given in [koshcat, p. 85]; *xxvii*) is given in [latulippe,]; *xxx*) is given in [AAR, pp. 108, 109]; *xxxii*) is given in [chuharm,]; *xxxiii*) is given in [chuharm,barrerojl,diazbarreroRS,srivastavaCS,srivastavaOSI,]; *xxxiv*) is given in [wenghangpfd,sofosm,wangjia,]; *xxxv*) is given in [gouldvos,]; *xxxvii*) and *xxxviii*) are given in [wenghangpfd,wenghangjat,]; *xxxix*) and *xl*) are given in [wangjia,]. **Remark:** If $n > k$, then both terms in *iv*) are zero. *ii*) is *Dixon's identity.* See [pwz, p. 43]. *xxviii*) is a special case. **Remark:** *xxxv*) is *Vosmansky's identity.* **Remark:** Additional equalities for products of binomial coefficients are given in [riordan, pp. 141–146].

Fact 1.16.15. The following statements hold:

i) If $n \ge 2$, then

$$2^n < \frac{4^n}{n+1} < \frac{4^n}{2\sqrt{n}} < \binom{2n}{n} < \frac{4^n}{\sqrt{(n+1/4)\pi}} < \left\{ \begin{array}{c} \dfrac{4^n}{\sqrt{n\pi}} \\ \dfrac{4^n}{\sqrt{3n+1}} \end{array} \right\} < \frac{4^n}{\sqrt{2n+1}} < \frac{4^n}{\sqrt{2n}} < \frac{4^n \log 3}{\log(2n+3)} < 4^n.$$

ii) If $n \ge 3$, then

$$2^n < \frac{4^n}{n+1} < \frac{4^n}{2\sqrt{n}} < \frac{4^n}{\sqrt{n\pi}}\left(1 - \frac{1}{4n}\right) < \binom{2n}{n} < \frac{4^n}{\sqrt{(n+1/4)\pi}} < \frac{4^n}{\sqrt{n\pi}}.$$

iii) If $n \ge 1$, then

$$\binom{2n+1}{n} < 4^n.$$

If, in addition, $n \ge 4$, then

$$\binom{2n+2}{n+1} < 4^n.$$

iv) If $n \ge 11$, then

$$\frac{2^{2n+1}}{n+1} < \binom{2n}{n}.$$

v) If $n \ge 1$, then

$$\prod_{\substack{n+1 \le i \le 2n, \\ i \text{ prime}}} i \le \binom{2n}{n}.$$

vi) If $n \ge 2$ and $1 \le k \le n-1$, then

$$\binom{n}{k-1}\binom{n}{k+1} \le \binom{n}{k}^2.$$

vii) If $n \geq 1$ and $0 \leq k \leq n-1$, then

$$\binom{n-1}{k}\binom{n+1}{k} \leq \binom{n}{k}^2.$$

viii) If $1 \leq k \leq n$, then

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \min\left\{\frac{n^k}{k!}, 2^n\right\}.$$

ix) If $1 \leq k < n/2$, then

$$\binom{n}{k} \leq \binom{n}{k+1}.$$

x) If $0 \leq k \leq n$, then

$$(n+1)^k \binom{n}{k} \leq n^k \binom{n+1}{k}.$$

xi) If $1 \leq k \leq n-1$, then

$$\sum_{i=1}^{k} i(i+1)\binom{2n}{k-i} < \frac{2^{2n-2}k(k+1)}{n}.$$

xii) If $1 \leq k < n$, then

$$n^k \leq (k+1)^{k-1}\binom{n}{k} \leq k^{k/2}(k+1)^{(k-1)/2}\binom{n}{k}.$$

xiii) If $n \geq 2$, then

$$\prod_{i=0}^{n} \binom{n}{i} \leq \left(\frac{2^n-2}{n-1}\right)^{n-1}.$$

xiv) If $n \geq 1$, then

$$\sum_{i=1}^{n} \sqrt{\binom{n}{i}} \leq \sqrt{n(2^n-1)}.$$

xv) If $n \geq 2$, and $1 \leq k \leq n-1$, then

$$\frac{1}{n+1} < \binom{n}{k}\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}.$$

xvi) If $n \geq 2$, $k \geq 0$, and $2k+1 < n$, then

$$\sum_{i=0}^{k} \binom{n}{i} < \frac{2^n(k+1)}{n+1}.$$

xvii) If $n \geq 2$, $k \geq 1$, and $n < 2k+1$, then

$$\frac{2^n(k+1)}{n+1} < \sum_{i=0}^{k} \binom{n}{i}.$$

xviii) If $n \ge 1$, then

$$\left|\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{2n}{i}\right| \le (2\sqrt{2})^n.$$

xix) If $n \ge 1$, then

$$\left|\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{3n}{i}\right| \le 4^n.$$

xx) If $n \ge 1$, then

$$\sum_{i=1}^{n}\frac{1}{i!}\binom{n-1}{i-1} < \frac{2^n}{n}.$$

xxi) If $n \ge 1$, then

$$\frac{4}{n+1} \le \sum_{i=0}^{n}\frac{\binom{n}{i}^2}{\binom{2n-1}{i}^2}.$$

Source: *i*) is given in [bruckmanGC,], [experimentation, p. 210], [herman, p. 137], and [koshcat, pp. 45–50]; *ii*) is given in [hirschhornAMM15,]; *iii*) is given in [grimaldi, p. 156]; *iv*) is given in [thuan3602,]; *v*) is given in [bak, p. 287]; *vi*) and *vii*) are given in [bona, pp. 76, 80]; *viii* and *ix*) is given in [zeilberger52,]; *x*) is given in [herman, p. 111]; *xi*) is given in [chuzhangqi,]; *xii*) is given in [elsnerBLMS,]; *xiii*) is given in [kaczor1, p. 14] and [larson, p. 253]; *xiv*) is given in [kaczor1, p. 14]; *xv*)–*xvii*) are given in [lopezmarengo,]; *xviii*) and *xix*) are given in [bak, pp. 157, 160]; *xx*) is given in [thuan3602,]; *xxi*) is given in [diaztelescopic,]. **Remark:** If $k = n \ge 1$, then $\sum_{i=0}^{k}\binom{n}{i} = \frac{2^n(k+1)}{n+1} = 2^n$, while, if $n \ge 1$ is odd and $2k+1 = n$, then $\sum_{i=0}^{k}\binom{n}{i} = \frac{2^n(k+1)}{n+1} = 4^n$. See Fact 1.16.10.

Fact 1.16.16. Let $n \ge k \ge 1$, and let $\binom{n}{k}_{\mathrm{r}}$ denote the number of k-element multisubsets of $\{1,\dots,n\}$. Then,

$$\binom{n}{k}_{\mathrm{r}} = \binom{n+k-1}{k}.$$

Furthermore, for all $z \in \mathbb{C}$,

$$\binom{z}{k}_{\mathrm{r}} = \binom{z+k-1}{k} = \frac{z^{\underline{k}}}{k!}, \quad \binom{z}{k}_{\mathrm{r}} = \binom{z}{k-1}_{\mathrm{r}} + \binom{z-1}{k}_{\mathrm{r}},$$

$$\binom{z}{k}_{\mathrm{r}} = \frac{z}{k}\binom{z+1}{k-1}_{\mathrm{r}} = \frac{z+k-1}{k}\binom{z}{k-1}_{\mathrm{r}}, \quad \binom{z+1}{k}_{\mathrm{r}} = \sum_{i=0}^{k}\binom{z}{i}_{\mathrm{r}}.$$

Source: [comtet, pp. 15–17]. **Remark:** $\binom{n}{k}_{\mathrm{r}}$ is the *binomial coefficient with repetition.* **Related:** Fact 1.12.6 and Fact **??**.

Fact 1.16.17. Let $n \ge 1$. Then,

$$n^n = \sum\binom{n}{k_1,\dots,k_{n-1}} = \sum\frac{n!}{k_1!\cdots k_{n-1}!},$$

where the sum is taken over all $n-1$-tuples $(k_1,\dots,k_{n-1})$ such that $0 \le k_1 \le 1$, $0 \le k_1+k_2 \le 2$, $\dots$, $0 \le k_1+\cdots+k_{n-1} \le n-1$. **Source:** [benjaminjuhnke,]. **Example:** $3^3 = \binom{3}{0,0} + \binom{3}{0,1} + \binom{3}{0,2} + \binom{3}{1,0} + \binom{3}{1,1} = 6+6+3+6+6 = 27$.

Fact 1.16.18. Let $n, m \geq 1$, and assume that $m \leq n$. Then,

$$\binom{2n}{m} = \sum_{i=0}^{\lfloor m/2 \rfloor} 2^{m-2i} \binom{n}{i, m-2i, n-m+i}.$$

Source: [sullivan,].

1.17 Facts on Fibonacci, Lucas, and Pell Numbers

Fact 1.17.1. Define $F_1 \triangleq F_2 \triangleq 1$ and, for all $k \in \mathbb{Z}$, define F_k by $F_{k+2} = F_{k+1} + F_k$. Then,

$$(F_i)_{i=-5}^{18} = (5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584).$$

Furthermore, for all $k, l \in \mathbb{Z}$, the following statements hold:

i) If $k \geq 2$, then F_k is the number of tuples each of whose components is either 1 or 2 and the sum of whose components is $n - 1$.

ii) If $k \geq 3$, then F_k is the number of subsets of $\{1, \ldots, k-2\}$ that do not contain a pair of consecutive integers.

iii) $F_{-k} = (-1)^{k+1} F_k$.

iv) If $3|k$, then F_k is even.

v) If $4|k$, then $3|F_k$; if $5|k$, then $5|F_k$; if $6|k$, then $8|F_k$.

vi) If $k \geq 3$ is prime, then $k|F_{2k} - F_k$.

vii) $\gcd\{F_k, F_l\} = F_{\gcd\{k,l\}}$.

viii) If $k|l$, then $F_k|F_l$. Hence, $F_k|F_{lk}$.

ix) $\gcd\{F_k, F_{k+1}\} = \gcd\{F_k, F_{k+2}\} = 1$.

x) If $k \geq 4$, then $F_k + 1$ is not prime.

xi) If $n \geq 1$, then there exists a unique set $\{i_1, \ldots, i_m\}$ of integers $2 \leq i_1 < \cdots < i_m$ such that, for all $j \in \{1, \ldots, m-1\}$, $i_j + 1 < i_{j+1}$ and such that $n = \sum_{j=1}^m F_{i_j}$.

xii) For all $k \in \mathbb{Z}$,

$$F_{2k+2} = 2F_{2k} + F_{2k-1} = F_{k+1}^2 + 2F_kF_{k+1} = F_{k+2}^2 - F_k^2 = F_kF_{k+1} + F_{k+1}F_{k+2},$$
$$F_{2k+3} = F_kF_{k+2} + F_{k+1}F_{k+3} = F_{k+2}F_{k+3} - F_kF_{k+1}, \quad F_kF_{k+2} = F_{k+1}^2 + (-1)^{k+1},$$
$$F_{k+5}F_{k+2} = F_{2k+5} + F_{k+3}F_k, \quad F_{k+4} + F_k = 3F_{k+2}, \quad F_{3k+3} = F_{k+2}^3 + F_{k+1}^3 - F_k^3,$$
$$F_{3k+6} = 4F_{3k+3} + F_{3k}, \quad F_k^2 = F_{k+3}F_{k-3} + 4(-1)^{3-k}, \quad F_{2k+1} = F_{k+1}^2 + F_k^2,$$
$$F_k^2 + F_{k+3}^2 = 2(F_{k+2}^2 + F_{k+1}^2), \quad F_{2k}^2 + 1 = F_{2k-1}F_{2k+1}, \quad F_{2k}^2 = F_{2k-2}F_{2k+2} + 1,$$
$$F_{2k+1}^2 = F_{2k}F_{2k+2} + 1, \quad F_{k+2}^2 + F_k^2 = F_{k+1}^2 + F_kF_{k+3} = 3F_{k+1}^2 + 2(-1)^{k+1},$$
$$F_{2k+3}^2 + 1 = F_{2k+1}F_{2k+5}, \quad F_{k+1}F_{k+2} - F_kF_{k+3} = (-1)^k,$$
$$F_{k+2}^2 - F_{k+1}^2 = F_kF_{k+3}, \quad F_{k+1}^2 - F_kF_{k+1} - F_k^2 = (-1)^k, \quad 2F_{2k+1} = 5F_k^2 + F_{2k} + 2(-1)^k,$$
$$F_{2k+2} = F_kF_{k+2} + F_{k+2}^2 + (-1)^k, \quad F_k^2 + F_{k+4}^2 = F_{k+1}^2 + 4F_{k+2}^2 + F_{k+3}^2,$$
$$F_{2k+1}^2 = (2F_kF_{k+1})^2 + (F_{k+1}^2 - F_k^2)^2, \quad F_{2k+3}^2 = (F_kF_{k+3})^2 + (2F_{k+1}F_{k+2})^2,$$
$$F_{k+3}^2 = 2F_{k+2}^2 + 2F_{k+1}^2 - F_k^2 = 4F_{k+2}F_{k+1} + F_k^2, \quad F_{k+4}^2 = (F_{k+2} + F_{k+3})^2 = (2F_k + 3F_{k+1})^2,$$

$$5F_k^3 = F_{3k} + 3(-1)^{k+1}F_k, \quad 25F_k^5 = F_{5k} + 5(-1)^{k+1}F_{3k} + 10F_k,$$
$$F_{k+1}^3 = F_{k+3}F_k^2 + (-1)^kF_{k+2}, \quad F_{k+4}^3 = 3F_{k+3}^3 + 6F_{k+2}^3 - 3F_{k+1}^3 - F_k^3,$$
$$F_{k+2}^3 = F_{3k+3} - F_{k+1}^3 + F_k^3 = F_{k+1}^3 + F_k^3 + 3F_kF_{k+1}F_{k+2} = F_kF_{k+3}^2 + (-1)^kF_{k+1},$$
$$F_kF_{k+1}F_{k+2} = F_{k+1}^3 + (-1)^{k+1}F_{k+1}, \quad F_{k+1}F_{k+2}F_{k+6} = F_{k+3}^3 + (-1)^kF_k,$$
$$F_kF_{k+4}F_{k+5} = F_{k+3}^3 + (-1)^{k+1}F_{k+6}, \quad F_{3k+9} = F_{k+2}F_{k+5}^2 + F_{k+1}F_{k+4}^2 - F_kF_{k+3}^2,$$
$$F_{k+2}^4 = F_kF_{k+1}F_{k+3}F_{k+4} + 1, \quad (F_k^2 + F_{k+1}^2 + F_{k+2}^2)^2 = 2(F_k^4 + F_{k+1}^4 + F_{k+2}^4),$$
$$2F_{4k+6} = F_{k+3}^4 + 2F_{k+2}^4 - 2F_{k+1}^4 - F_k^4, \quad F_kF_{k+4}^3 = F_{k+3}^4 + (-1)^{k+1}(F_{k+2}F_{k+6} + 2F_{k+3}^2),$$
$$6F_{4k+4} = F_{k+3}^4 + 3F_{k+2}^4 - 3F_k^4 - F_{k-1}^4, \quad F_{k+5}^4 = 5F_{k+4}^4 + 15F_{k+3}^4 - 15F_{k+2}^4 - 5F_{k+1}^4 + F_k^4,$$
$$6F_{2k+3}^2 = F_k^4 + 4F_{k+1}^4 + 4F_{k+2}^4 + F_{k+3}^4, \quad 10F_{2k+3}^3 = F_k^6 + 8F_{k+1}^6 + 8F_{k+2}^6 + F_{k+3}^6,$$
$$6F_{2k+4}^2 = F_k^4 - 4F_{k+1}^4 - 10F_{k+2}^4 - F_{k+3}^4 + F_{k+4}^4,$$
$$56F_{2k+6}^2 = F_k^4 - 6F_{k+2}^4 - 6F_{k+4}^4 + F_{k+6}^4 - 20,$$
$$216F_{2k+8}^2 = -F_k^4 + 81F_{k+2}^4 - 520F_{k+4}^4 + 81F_{k+6}^4 - F_{k+8}^4,$$
$$1224F_{2k+9}^2 = F_k^4 + 19F_{k+3}^4 + 19F_{k+6}^4 + F_{k+9}^4 + 480,$$
$$20304F_{2k+12}^2 = F_k^4 - 46F_{k+4}^4 - 46F_{k+8}^4 + F_{k+12}^4 - 8100,$$
$$3264F_{2k+12}^2 = F_k^4 - 256F_{k+3}^4 - 4930F_{k+6}^4 - 256F_{k+9}^4 + F_{k+12}^4,$$
$$(F_kF_{k+1})^2 + (F_kF_{k+2})^2 + (F_{k+1}F_{k+2})^2 = (F_k^2 + F_{k+1}F_{k+2})^2, \quad F_{k+1}^2F_{k+3}^2 - F_k^2F_{k+4}^2 = 4(-1)^kF_{k+2}^2,$$
$$F_{k+1}^5 = F_{k+1} + F_k^3F_{k+2}F_{k+3} + (-1)^kF_kF_{k+2}F_{k+3}, \quad 6F_{5k+10} = F_{k+4}^5 + 3F_{k+3}^5 - 6F_{k+2}^5 - 3F_{k+1}^5 + F_k^5,$$
$$(F_k^2 + F_{k+1}^2)(F_{k+2}^2 + F_{k+3}^2) = F_{2k+3}^2 + 1, \quad (F_k^2 + F_{k+2}^2)(F_{k+4}^2 + F_{k+6}^2) = F_{2k+6}^2 + [2F_{k+3}^2 - 5(-1)^k]^2,$$
$$\sum_{i=0}^{5} F_{k+i} = 4F_{k+4}, \quad \sum_{i=0}^{9} F_{k+i} = 11F_{k+6}.$$

xiii) For all $k \ge 1$,
$$\operatorname{atan}\frac{1}{F_{2k+1}} + \operatorname{atan}\frac{1}{F_{2k+2}} = \operatorname{atan}\frac{1}{F_{2k}}.$$

xiv) For all $k \ge 0$,
$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k-i}{i} = (-1)^k \sum_{j=0}^{k} \sum_{i=0}^{\lfloor (k-j)/2 \rfloor} (-2)^j \binom{k-i}{i}\binom{k-2i}{j} = F_{k+1},$$
$$\sum_{i=0}^{\lfloor k/2 \rfloor} 5^i \binom{k+1}{2i+1} = 2^kF_{k+1}, \quad \sum_{i=0}^{k} \binom{k+i}{2i} = F_{2k+1}, \quad \sum_{i=0}^{k} \binom{k+i+1}{2i+1} = F_{2k+2}.$$

xv) For all $k \ge 1$,
$$\sum_{i=1}^{k} F_i = F_{k+2} - 1, \quad \sum_{i=1}^{k} F_{2i-1} = F_{2k}, \quad \sum_{i=1}^{k} F_{2i} = F_{2k+1} - 1, \quad \sum_{i=1}^{k} F_{4i-2} = F_{2k}^2,$$
$$\sum_{i=1}^{k} F_{4i-1} = F_{2k}F_{2k+1}, \quad \sum_{i=1}^{k} (-1)^{i+1}F_{i+1} = (-1)^{k-1}F_k, \quad \sum_{i=1}^{k} iF_i = kF_{k+2} - F_{k+3} + 2,$$
$$5\sum_{i=1}^{k} F_{2i-1}^2 = F_{4k} + 2k, \quad \sum_{i=1}^{k} F_i^2 = F_kF_{k+1}, \quad \sum_{i=1}^{2k-1} F_iF_{i+1} = F_{2k}^2, \quad \sum_{i=1}^{2k} F_iF_{i+1} = F_{2k+1}^2 - 1,$$

$$\sum_{i=1}^{k} F_iF_{i+1} = F_{k+1}^2 - \tfrac{1}{2}[1+(-1)^k], \quad \sum_{i=1}^{k} F_iF_{3i} = F_kF_{k+1}F_{2k+1}, \quad \sum_{i=1}^{k-1} F_iF_{k-i} = \frac{1}{5}[2kF_{k+1} - (k+1)F_k,$$

$$\sum_{i=1}^{k} F_i^2F_{i+1} = \frac{1}{2}F_kF_{k+1}F_{k+2}, \quad \sum_{i=1}^{k}(F_iF_{i+1})^3 = \frac{1}{4}(F_kF_{k+1}F_{k+2})^2, \quad \sum_{i=1}^{k}\frac{1}{F_{2^i}} = 2 - \frac{F_{2^k-1}}{F_{2^k}},$$

$$\sum_{i=1}^{k}\binom{k}{i}F_i = F_{2k}, \quad \sum_{i=0}^{k}\binom{k}{i}F_{i+1} = F_{2k+1}, \quad \sum_{i=1}^{k}\binom{k+1}{i+1}F_i = F_{2k+1} - 1,$$

$$\sum_{i=1}^{k}2^i\binom{k}{i}F_i = \sum_{i=1}^{k}2^{k-i}\binom{k}{i}F_{k-i} = F_{3k}, \quad \sum_{i=0}^{k}2^i\binom{k}{i}F_{i+1} = F_{3k+1}, \quad \sum_{i=0}^{k}2^i\binom{k}{i}F_{i+2} = F_{3k+2},$$

$$\sum_{i=1}^{2k}\frac{1}{2^i}\binom{2k}{i}F_{2i} = \left(\frac{5}{4}\right)^k F_{2k}, \quad \sum_{i=0}^{2k+1}\frac{1}{2^i}\binom{2k+1}{i}F_{2i} = \frac{1}{2}\left(\frac{5}{4}\right)^k L_{2k+1},$$

$$\sum_{i=1}^{k}2^i\binom{k}{i}F_{3i} = F_{6k}, \quad \sum_{i=1}^{2k}\frac{1}{3^i}\binom{2k}{i}F_{3i} = \left(\frac{20}{9}\right)^k F_{2k}, \quad \sum_{i=0}^{2k+1}\frac{1}{3^i}\binom{2k+1}{i}F_{3i} = \frac{2}{3}\left(\frac{20}{9}\right)^k L_{2k+1},$$

$$\sum_{i=1}^{2k}2^{2k-i}\binom{2k}{i}F_i = 5^kF_{2k}, \quad \sum_{i=1}^{2k+1}2^{2k+1-i}\binom{2k}{i}F_i = 5^kL_{2k+1}, \quad \sum_{i=1}^{k}\left(\frac{3}{2}\right)^i\binom{k}{i}F_i = \frac{1}{2^k}F_{4k},$$

$$\sum_{i=1}^{k}\binom{2k}{2i}F_{4i} = \tfrac{1}{2}(5^k+1)F_{2k}, \quad \sum_{i=1}^{k}\binom{2k+1}{2i}F_{4i} = \tfrac{1}{2}(5^kL_{2k+1} - F_{2k+1}),$$

$$\sum_{i=0}^{k-1}\binom{2k}{2i+1}F_{4i+2} = \tfrac{1}{2}(5^k-1)F_{2k}, \quad \sum_{i=0}^{k}\binom{2k+1}{2i+1}F_{4i+2} = \tfrac{1}{2}(5^{k+1/2}L_{2k+1} + F_{2k+1}),$$

$$\sum_{i=1}^{k}4^i\binom{k}{i}F_i = F_{3k}, \quad \sum_{i=1}^{2k}(-1)^{2k-i}4^i\binom{2k}{i}F_{2i} = 5^kF_{6k}, \quad \sum_{i=1}^{2k+1}(-1)^{2k+1-i}4^i\binom{2k+1}{i}F_{2i} = 5^kL_{6k+3},$$

$$\sum_{i=1}^{2k}\binom{2k}{i}F_{2i} = 5^kF_{2k}, \quad \sum_{i=0}^{k}\binom{2k}{2i}F_{2i} = \frac{2}{5}\sum_{i=0}^{4}(\sin\tfrac{2i\pi}{5})(\sin\tfrac{4i\pi}{5})(1+2\cos\tfrac{i\pi}{5})^k = \tfrac{1}{2}(F_{2k} - F_k),$$

$$\sum_{i=0}^{k}\binom{2k}{2i}F_{2i-1} = \tfrac{1}{2}(F_{2k-1} + F_{k+1}), \quad \sum_{i=0}^{k}\binom{2k+1}{2i+1}F_{2i+1} = \sum_{i=0}^{k}F_{2i-1}F_{k-i} = \tfrac{1}{2}(F_{2k} + F_k),$$

$$\sum_{i=0}^{k}\binom{2k+1}{2i+1}F_{2i} = \sum_{i=0}^{k-1}F_{2i}F_{k-i-1} = \tfrac{1}{2}(F_{2k-1} - F_{k+1}),$$

$$\sum_{i=0}^{k}\binom{k}{i}F_{3i} = 2^kF_{2k}, \quad \sum_{i=0}^{k}\binom{2k}{2i}F_{4i} = \tfrac{1}{2}F_{2k}(5^k+1), \quad \sum_{i=0}^{k}\binom{2k+1}{2i}F_{4i} = \tfrac{1}{2}(5^kL_k - F_k),$$

$$\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}F_i = \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}F_{2i} = F_k, \quad \sum_{i=1}^{k}(-1)^{k-i}2^i\binom{k}{i}F_{2i} = F_{3k},$$

$$\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}F_{3i} = 2^kF_k, \quad \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}F_{6i} = 4^kF_{3k},$$

$$\sum_{i=1}^{2k}(-1)^i 2^{i-1}\binom{2k}{i}F_i = 0, \quad \sum_{i=1}^{2k+1}\binom{2k+1}{i}F_i^2 = 5^k F_{2k+1},$$

$$\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i \frac{k}{k-i}\binom{k-i}{i}F_{2k-3i} = F_k, \quad \sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i \frac{k}{k-i}\binom{k-i}{i}L_{2k-3i} = L_k + 2,$$

$$\sum_{i=1}^{k}\binom{2k}{2i}F_{6i} = 2^{2k-1}(F_{4k}+F_{2k}), \quad \sum_{i=1}^{k}\binom{2k+1}{2i}F_{6i} = 4^k(F_{4k+2}-F_{2k+1}),$$

$$\sum_{i=0}^{k}\binom{2k}{2i+1}F_{6i+3} = 2^{2k-1}(F_{4k}-F_{2k}), \quad \sum_{i=0}^{k}\binom{2k+1}{2i+1}F_{6i+3} = 4^k(F_{4k+2}+F_{2k+1}),$$

$$2F_{k+2} + 2\sum_{1\le i<j\le k}F_iF_j = F_{2k+1} + F_kF_{k+1} + 1, \quad F_k^5 + F_{k+1}^5 + \frac{5}{7}\left(\frac{F_{k+2}^7 - F_{k+1}^7 - F_k^7}{F_{k+2}^2 - F_kF_{k+1}}\right) = F_{k+2}^5,$$

$$\sum_{j=1}^{k}\sum_{i=1}^{j}\sum_{h=1}^{i}\sum_{g=1}^{h}F_g^4 = \frac{1}{100}[4F_{k+2}^4 + (k+2)^4 - 5(k+2)^2].$$

xvi) For all $k \ge 2$,

$$\sum_{i=1}^{k}F_{4i-6} = F_{2k-3}F_{2k-1} - 1, \quad \sum_{i=1}^{k}F_{4i-7} = F_{2k-3}F_{2k-2}.$$

xvii) For all $n, k \ge 1$,

$$5^n F_k^{2n+1} = \sum_{i=0}^{n}(-1)^{i(k+1)}\binom{2n+1}{i}F_{[2(n-i)+1]k},$$

$$5^n F_k^{2n} = \sum_{i=0}^{n-1}(-1)^{i(k+1)}\binom{2n}{i}L_{2(n-i)k} + (-1)^{n(k+1)}\binom{2n}{n}.$$

xviii) For all $n \ge 4$,

$$\left\lfloor\left(\sum_{i=n}^{2n}\frac{1}{F_i}\right)^{-1}\right\rfloor = F_{n-2}.$$

xix) For all $n \ge 1$ and $k \ge 3$,

$$\left\lfloor\left(\sum_{i=2n}^{2kn}\frac{1}{F_i}\right)^{-1}\right\rfloor = F_{2n-2}, \quad \left\lfloor\left(\sum_{i=2n+1}^{k(2n+1)}\frac{1}{F_i}\right)^{-1}\right\rfloor = F_{2n-1} - 1.$$

xx) For all $n \ge 1$ and $k \ge 2$,

$$\left\lfloor\left(\sum_{i=2n}^{2kn}\frac{1}{F_i^2}\right)^{-1}\right\rfloor = F_{2n}F_{2n-1} - 1, \quad \left\lfloor\left(\sum_{i=2n-1}^{k(2n-1)}\frac{1}{F_i^2}\right)^{-1}\right\rfloor = F_{2n-1}F_{2n-2}.$$

xxi) For all $k, l \in \mathbb{Z}$,

$$F_k = F_{l+1}F_{k-l} + F_lF_{k-l-1}, \quad F_{k+l+1} = F_{k+1}F_{l+1} + F_kF_l,$$

$$(-1)^kF_{k-l} = F_{l+1}F_k - F_lF_{k+1}, \quad F_{k+l+2} = F_{k+1}F_l + F_{k+2}F_{l+1} = F_{k+2}F_{l+2} - F_kF_l,$$

$$F_{k+l+1}^2 = F_{k+1}^2F_{l+1}^2 + \tfrac{1}{3}F_kF_l(F_{k+3}F_{l+3} + 2F_{k-1}F_{l-1}),$$

$$F_{k+l+1}^2 + F_{k-l}^2 = F_{2k+1}F_{2l+1}, \quad F_k^2 + F_{k+2l+1}^2 = F_{2l+1}F_{2k+2l+1}^2,$$

$$F_{k+2l}^2 - F_k^2 = F_{2l}F_{2k+2l}, \quad F_k^2 = F_{k-l}F_{k+l} + (-1)^{k+l}F_l^2,$$

$$F_{k+l+1}F_{k+l-1} = F_{k+l}^2 + (F_{k+1}F_{k-1} - F_k)^2(F_{l+1}F_{l-1} - F_l^2),$$

$$F_{k+l+2} = F_kF_{l+1} + F_{k+1}F_l + F_{k+1}F_{l+1}, \quad F_{k+2}F_{l+2} = F_kF_l + F_kF_{l+1} + F_{k+1}F_l + F_{k+1}F_{l+1}.$$

xxii) For all $k, l \in \mathbb{Z}$,

$$\sum_{i=1}^{2k} 2^{2k-i}\binom{2k}{i}F_{i+l} = 5^kF_{2k+l}, \quad \sum_{i=1}^{2k+1} 2^{2k+1-i}\binom{2k}{i}F_{i+l} = 5^kL_{2k+1+l}.$$

xxiii) For all $k \ge 1$ and $l \ge 1$,

$$\sum_{i=0}^{k}\binom{k}{i}F_{3i+l} = 2^kF_{2k+l}.$$

xxiv) For all $k \ge 1$ and $l \ge 2$,

$$\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\frac{F_{il}}{F_{l-1}^i} = \left(\frac{F_l}{F_{l-1}}\right)^kF_k.$$

xxv) For all $k \ge 1$ and $l \ge 3$,

$$\sum_{i=1}^{k}\binom{k}{i}\left(\frac{F_l}{F_{l-1}}\right)^iF_i = \frac{F_{kl}}{F_{l-1}^k}.$$

xxvi) For all $k \ge 1$ and $l \in \{0, \ldots, k\}$, $F_{2k+1-l}^2 + F_l^2 = F_{2k+1}F_{2k-2l+1}$.

xxvii) For all $k \in \mathbb{Z}$ and $l \in \{0, 1, 2, 3\}$, $F_{3k+1}F_{k+l+1}^3 + F_{3k+2}F_{k+l}^3 = F_{l-2k-1}^3 + F_{3k+1}F_{3k+2}F_{3l}$.

xxviii) For all $k \in \mathbb{Z}$ and $l \ge 1$,

$$\sum_{i=1}^{l}F_{k+i} = F_{k+l+2} - F_{k+2}.$$

xxix) For all $k \ge 0$ and $l \in \mathbb{Z}$,

$$\sum_{i=0}^{k}\binom{k}{i}F_{i+l} = F_{2k+l}.$$

xxx) For all $k, l, m \in \mathbb{Z}$,

$$F_{k+l}F_{k+m} = F_kF_{k+l+m} + (-1)^kF_lF_m,$$

$$F_{k+l+m+3} + F_kF_lF_m = F_{k+2}F_{l+2}F_{m+2} + F_{k+1}F_{l+1}F_{m+1}.$$

xxxi) For all $n \geq 1$,

$$\prod_{i=1}^{n}\left(1+4\sin^2\frac{2i\pi}{n}\right)=[1+F_n-2F_{n+1}+(-1)^n]^2,$$

$$F_{2n+1}=\prod_{i=1}^{n}\left(5-4\sin^2\frac{2i\pi}{2n+1}\right),\quad F_{4n+2}=\prod_{i=1}^{n}\left(5+4\sin^2\frac{i\pi}{2n+1}\right).$$

xxxii) For all $n \geq 1$ and $z \in \mathbb{C}$,

$$\prod_{i=0}^{n-1}\left[3+2\cos\left(\frac{2i\pi}{n}-z\right)\right]=5F_n^2+4(-1)^n\sin^2\frac{nz}{2}.$$

xxxiii) For all $n \geq 2$,

$$F_n=\frac{2^{n-1}}{n}\sqrt{\prod_{i=1}^{n-1}\left[1-\left(\cos\frac{i\pi}{n}\right)\cos\frac{3i\pi}{n}\right]}.$$

xxxiv) For all $n \geq 4$,

$$F_n=\prod_{i=1}^{\lfloor (n-1)/2\rfloor}\left(1+4\sin^2\frac{i\pi}{n}\right)=\prod_{i=1}^{\lfloor (n-1)/2\rfloor}\left(1+4\cos^2\frac{i\pi}{n}\right)=\prod_{i=1}^{\lfloor (n-1)/2\rfloor}\left(3+2\cos\frac{2i\pi}{n}\right).$$

xxxv) For all $k \geq 1$,

$$\sum_{0\le j\le i\le k}\binom{k}{i-j}\binom{k-i}{j}=F_{2k-1},\quad \sum_{i,j=1}^{k}\binom{k-i}{j-1}\binom{k-j}{i-1}=F_{2k},$$

$$\sum_{i,j=0}^{k}\binom{k+i}{2j}\binom{k+j}{2i}=F_{4k-1},\quad \sum_{i,j=0}^{k}\binom{k+i}{2j-1}\binom{k+j}{2i}=F_{4k},\quad \sum_{i,j=0}^{k}\binom{k+i}{2j+1}\binom{k+j}{2i+1}=F_{4k-3}.$$

xxxvi) Let $n \geq 0$. Then, there exists $k \geq 0$ such that $n = F_k$ if and only if either $\sqrt{5n^2-4}$ or $\sqrt{5n^2+4}$ is an integer.

xxxvii) Let $n \geq 1$, and define $\mathcal{A}_n \triangleq \{(i_1,\ldots,i_k) \in \times_{i=1}^{k}\{1,2\}\colon k \geq 1 \text{ and } \sum_{j=1}^{k} i_j = n\}$. Then, $\operatorname{card}(\mathcal{A}_n) = F_{n+1}$.

xxxviii) For all $n \geq 0$, $9F_n^2 \leq F_{n+3}^2$.

xxxix) For all $n \notin \{-2,-1\}$,

$$\left(1+\frac{F_n}{F_{n+1}}-\frac{F_n}{F_{n+2}}\right)^2=1+\left(\frac{F_n}{F_{n+1}}\right)^2+\left(\frac{F_n}{F_{n+2}}\right)^2.$$

Source: [AMR, pp. 63, 330, 331], [AMR3, pp. 186, 187, 239–241], [benczeassi,], [benjamin-quinn, pp. 10–12, 70, 78, 125, 126, 144], [bibak,chamberlandusing,chamberlandtrig,chamberlandLAA13,chenchenfib,clarkeqi,d], [engel, p. 206], [fairgrieve,], [fuchs, p. 63], [garnier,], [gelca, p. 297], [grimaldi, pp. 10, 11, 12, 38, 39, 56, 57, 61, 109, 115, 116, 117, 121], [griffithsrv,griffithsext,griffiths,griffithsfrom,griffiths2,kasturiwale,keskin,kilic,], [koshy, pp. 6–8, 239–241, 362, 363], [koshcat, pp. 78, 79], [langlang,], [larson, pp. 72, 175], [mahonado,melhamfam,melhammore,melhamsome,melhamsimson,], [mollnf, pp. 110–113], [ohtsukasc,ohtsukahigher,ohtsukaH766,ollerton,simons,terrana,], [vajda, pp. 37, 70–72, 182, 183], [wangwen,welukar,werman,]. **Remark:** F_n is the nth *Fibonacci number.* **Remark:** Concerning *ii*), $\binom{n-k-1}{k}$ is the number of k-element subsets of $\{1,\dots,n-2\}$ that do not contain a pair of consecutive integers. See [AMR3, p. 187]. **Remark:** *xi*) is *Zeckendorf's theorem.* **Related:** Fact **??**. The generating function is given by Fact **??**.

Fact 1.17.2. Define $L_1 \triangleq 1$, $L_2 \triangleq 3$ and, for all $k \in \mathbb{Z}$, define L_k by $L_{k+2} = L_{k+1} + L_k$. Then,

$(L_i)_{i=-5}^{16} = (-11, 7, -4, 3, -1, 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207).$

Then, for all $k \in \mathbb{Z}$, the following statements hold:

i) $L_{-k} = (-1)^k L_k$.

ii) L_{3k} is even.

iii) If k is prime, then $L_k \overset{k}{\equiv} 1$.

iv) 5 divides $L_{k+1} - 3L_k$, $3^{k-1} - L_k$, $kL_{k+1} + 2F_k$, and $T_kL_{k+1} + (k+1)F_k$.

v) 10 divides $4kL_{k+1} - 2F_k$.

vi) $\gcd\{L_k, L_{k+1}\} = \gcd\{L_k, L_{k+2}\} = 1$.

For all $k \in \mathbb{Z}$,

$$L_k^2 = L_{2k} + 2(-1)^k, \quad L_{k+1}^2 = L_kL_{k+2} + 5(-1)^{k+1}, \quad L_{2k}^2 = L_{4k} + 2,$$

$$L_{k+1}^2 - L_{k+1}L_k - L_k^2 = 5(-1)^{k+1}, \quad L_k^2 + L_{k+1}^2 = L_{2k} + L_{2k+2},$$

$$L_{3k} = L_k[L_{2k} + (-1)^{k+1}], \quad (L_k^2 + L_{k+1}^2 + L_{k+2}^2)^2 = 2(L_k^4 + L_{k+1}^4 + L_{k+2}^4),$$

$$L_{k+3}^2 = 2L_{k+2}^2 + 2L_{k+1}^2 - L_k^2, \quad L_{k+4}^3 = 3L_{k+3}^3 + 6L_{k+2}^3 - 3L_{k+1}^3 - L_k^3,$$

$$L_{k+2}^3 + L_{k+1}^3 - L_k^3 = 5L_{3k+3}, \quad L_{k+5}^4 = 5L_{k+4}^4 + 15L_{k+3}^4 - 15L_{k+2}^4 - 5L_{k+1}^4 + L_k^4,$$

$$2L_{k+1} = L_k + 5F_k, \quad 2L_{k+2} = 3L_k + 5F_k, \quad L_{k+1} = F_k + F_{k+2}, \quad L_{k+2} = 3F_{k+1} + F_k,$$

$$5F_{k+1} = L_k + L_{k+2}, \quad 2F_{k+1} = F_k + L_k, \quad 2F_{k+2} = 3F_k + L_k, \quad F_{k+4} = F_k + L_{k+2}, \quad F_{2k+4} = F_{2k} + L_{2k+2},$$

$$2L_k = L_k^2 + 5F_k^2, \quad L_{2k} = 5F_k^2 + 2(-1)^k, \quad 5F_{2k+1} = L_k^2 + L_{k+1}^2, \quad F_{2k+1} + F_kL_k = F_{k+1}L_{k+1},$$

$$F_{2k} = F_kL_k, \quad F_{k+1}L_k = F_{2k+1} + (-1)^k, \quad 2L_{2k} = L_k^2 + 5F_k^2, \quad L_k^2 = 5F_k^2 + 4(-1)^k,$$

$$L_{k+3}^2 = F_kF_{k+4} + F_{k+4}F_{k+5} + F_{k+2}^2, \quad 5F_{2k+3}F_{2k-3} = L_{4k} + 18, \quad F_k \overset{2}{\equiv} L_k,$$

$$L_{k+1}^2 + F_{k+1}^2 = 2F_{k+2}^2 + 2F_k^2, \quad 25F_{k+1}^2 + L_{k+1}^2 = 2L_{k+2}^2 + 2L_k^2,$$

$$L_{2k+7} = F_{k+4}^2 + L_{k+3}^2 - F_kF_{k+4} + F_{2k} + (-1)^k,$$

$$F_k^4 + L_{k+2}^4 + F_{k+4}^4 = 9(F_{k+1}^4 + F_{k+2}^4 + F_{k+3}^4), \quad L_k^4 + L_{k+4}^4 + 625F_{k+2}^4 = 9(L_{k+1}^4 + L_{k+2}^4 + L_{k+3}^4),$$

$$2F_k^2L_{k+6}^2 = -17F_{k+1}^4 + 57F_{k+2}^4 + 402F_{k+3}^4 + 113F_{k+4}^4 - 25F_{k+5}^4,$$

$$50L_k^2F_{k+6}^2 = -17L_{k+1}^4 + 57L_{k+2}^4 + 402L_{k+3}^4 + 113L_{k+4}^4 - 25L_{k+5}^4,$$

$$F_k^3 + F_{k+1}^3 + 3F_kF_{k+1}F_{k+2} = F_{k+2}^3, \quad L_k^3 + L_{k+1}^3 + 3L_kL_{k+1}L_{k+2} = L_{k+2}^3,$$

$$F_{k+1} = \frac{1}{2}\left[F_k + (-1)^{\min\{k,0\}}\sqrt{5F_k^2 + 4(-1)^k}\right], \quad L_{k+1} = \frac{1}{2}\left[L_k + (-1)^{\min\{k+1,0\}}\sqrt{5(L_k^2 - 4(-1)^k)}\right].$$

For all $k \ge 1$,

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k}{k-i}\binom{k-i}{i} = L_k,$$

$$\sum_{i=0}^{k} L_i = L_{k+2} - 1, \quad \sum_{i=1}^{k} L_{2i-1} = L_{2i} - 2, \quad \sum_{i=0}^{k} L_{2i} = L_{2k+1} + 1, \quad \sum_{i=0}^{k} L_i^2 = L_k L_{k+1} + 2,$$

$$\sum_{i=1}^{k} L_{2i-1}^2 = F_{4k} - 2k, \quad \sum_{i=1}^{k} i L_i = k L_{k+2} - L_{k+3} + 4, \quad \sum_{i=0}^{k} 2^i L_i = 2^{n+1} F_{k+1},$$

$$\sum_{i=0}^{k} 3^i L_i + \sum_{i=0}^{k+1} 3^{i-1} F_i = 3^{k+1} F_{k+1}, \quad \sum_{i=0}^{k} \binom{k}{i} L_i = L_{2k}, \quad \sum_{i=0}^{2k} (-1)^i 2^{i-1} \binom{2k}{i} L_i = 5^i,$$

$$\sum_{i=0}^{k} \binom{k}{i} L_i L_{k-i} = 2^k L_k + 2, \quad 5 \sum_{i=0}^{k} \binom{k}{i} F_i F_{k-i} = 2^k L_k - 2, \quad \sum_{i=0}^{k} \binom{k}{i} F_i L_{k-i} = 2^k F_k,$$

$$\sum_{i=0}^{k} (-1)^i L_{k-2i} = 2F_{k+1}, \quad \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} L_i = (-1)^k L_k, \quad \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} L_{2i} = L_k,$$

$$\sum_{i=0}^{k-1} (-1)^i \binom{2k}{i} L_{2k-2i} = 1 + (-1)^{n-1} \binom{2k}{k}, \quad \sum_{i=0}^{k} (-1)^i \binom{2k+1}{i} L_{2k+1-2i} = 1,$$

$$\sum_{i=0}^{2k} \binom{2k}{i} L_{2i} = \sum_{i=0}^{2k} \binom{2k}{i} L_i^2 = 5^k L_{2k}, \quad \sum_{i=0}^{2k+1} \binom{2k+1}{i} L_{2i} = \sum_{i=0}^{2k+1} \binom{2k+1}{i} L_i^2 = 5^{k+1} F_{2k+1},$$

$$\sum_{i=0}^{2k+1} \binom{2k+1}{i} F_{2i} = 5^k L_{2k+1}, \quad \sum_{i=0}^{2k} \binom{2k}{i} F_i^2 = 5^{k-1} L_{2k}, \quad \sum_{i=0}^{k} F_i L_{k-i} = (k+1) F_k, \quad \sum_{i=0}^{\lfloor k/2 \rfloor} 5^i \binom{k}{2i} = 2^{k-1} L_k,$$

$$L_{2k} = 2 + \prod_{i=0}^{k-1} \left(1 + 4\sin^2 \frac{i\pi}{k}\right) = 2 \prod_{i=1}^{k} \left(\frac{1}{4} + \frac{5}{4} \tan^2 \frac{(2i-1)\pi}{4k}\right), \quad L_{2k+1} = \prod_{i=1}^{k} \left(\frac{1}{4} + \frac{5}{4} \tan^2 \frac{2i\pi}{2k+1}\right),$$

$$\sum_{i=1}^{n} \frac{\tan^2 \frac{2i\pi}{2k+1}}{1 + 5\tan^2 \frac{2i\pi}{2k+1}} = \frac{(2k+1)F_{2k}}{4L_{2k+1}}, \quad \sum_{i=1}^{n} \frac{\tan^2 \frac{(2i-1)\pi}{4k}}{1 + 5\tan^2 \frac{(2i-1)\pi}{4k}} = \frac{kF_{2k-1}}{2L_{2k}},$$

$$1 + 10 \sum_{i=1}^{\lfloor (n-1)/2 \rfloor} \frac{\cos^2 \frac{i\pi}{k}}{3 + 2\cos \frac{2i\pi}{k}} = \frac{kL_{k-1}}{2F_k}.$$

For all $k, l \in \mathbb{Z}$,

$$L_{k+l} + (-1)^l L_{k-l} = L_k L_l, \quad L_{k+l} = 5F_k F_l + (-1)^l L_{k-l}, \quad L_{2k} L_{2l} = L_{k+l}^2 + 5F_{k-l}^2,$$

$$5F_k F_l = L_{k+l} + (-1)^{l+1} L_{k-l}, \quad 5F_k^2 = L_{2k} + 2(-1)^{k+1},$$

$$2L_{k+l} = L_k L_l + 5F_k F_l, \quad 2(-1)^l L_{k+l} = L_k L_l - 5F_k F_l, \quad 2(-1)^l L_{k-l} = L_k L_l - 5F_k F_l,$$

$$2F_{k+l} = F_k L_l + F_l L_k, \quad F_{k+l} + (-1)^l F_{k-l} = F_k L_l, \quad F_{k+l} + (-1)^l F_{k-l} = F_k L_l,$$

$$F_k L_l = L_k F_l + 2(-1)^l F_{k-l}, \quad L_{k+l+1} = F_{k+1} L_{l+1} + F_k L_l, \quad L_{k+l+1}^2 + L_{k-l}^2 = 5F_{2k+1} F_{2l+1},$$

$$F_k^4 + [(-1)^{l+1} L_{2l} + 1](F_{k+l}^4 + F_{k+2l}^4) + F_{k+3l}^4 = F_l L_{2l} F_{3l} F_{2k+3l}^2 + 10(-1)^l F_{l-1} F_l^4 F_{l+1},$$

$$(-1)^{l+1} F_k^6 + (L_{4l}+1)[F_{k+l}^6 + (-1)^{l+1} F_{k+2l}^6] + F_{k+3l}^6 = F_l F_{3l} F_{5l} F_{2k+3l}^3 + 15(-1)^l F_{l-1} F_l^4 F_{l+1} F_{3l} F_{2k+3l},$$

$$(-1)^{l+1} F_l F_k^4 + (-1)^l L_l^3 F_{2l} (F_{k+l}^4 + F_{k+3l}^4) - [L_{4l} + 2(-1)^l L_{2l} + 4] F_{3l} F_{k+2l}^4 + (-1)^{l+1} F_l F_{k+4l}^4 = 3F_{2l}^2 F_{3l} F_{2k+4l}^2.$$

For all $k, l, m \in \mathbb{Z}$,

$$L_{m+k}L_{m+l-1} = L_{m-1}L_{m+l+k} + (-1)^{m-1}F_{k+1}(L_l - 2L_{l+1}).$$

If p is prime, then

$$\sum_{i=0}^{\lfloor (p-1)/2 \rfloor} (-1)^i \binom{2i}{i} \overset{p}{\equiv} F_p, \quad \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} 5^i \binom{p}{2i} \overset{p}{\equiv} L_p, \quad \sum_{i=0}^{\lfloor (p-1)/2 \rfloor} (-1)^i (i+1)^{p-2} \binom{2i}{i} \overset{p}{\equiv} L_{p-1}.$$

For all $n \geq 1$,

$$\sum_{i=1}^{n} F_{n-i}H_i = F_{2n}H_n - \sum_{i=1}^{n} \frac{F_{2n-i}}{i}, \quad \sum_{i=1}^{n} 2^{n-i}F_{n-i}H_i = F_{3n}H_n - \sum_{i=1}^{n} \frac{2^i F_{3n-2i}}{i},$$

$$\sum_{i=1}^{n} L_{n-i}H_i = L_{2n}H_n - \sum_{i=1}^{n} \frac{L_{2n-i}}{i}, \quad \sum_{i=1}^{n} 2^{n-i}L_{n-i}H_i = L_{3n}H_n - \sum_{i=1}^{n} \frac{2^i L_{3n-2i}}{i},$$

$$\sum_{i=1}^{n} \binom{n+i+1}{n-i} H_i = \sum_{i=1}^{n} \frac{1}{i}(L_{2i}-2)F_{2n-2i+2}, \quad \sum_{i=1}^{n} \binom{n+i}{n-i} \frac{2n+1}{2i+1} H_i = \sum_{i=1}^{n} \frac{1}{i}(L_{2i}-2)L_{2n-2i+1},$$

$$\sum_{i=1}^{2n} F_{2i}^3 = \frac{1}{4}F_{2n-1}F_{2n}^2L_{2n+1}^2L_{2n+2}, \quad \sum_{i=1}^{2n+1} F_{2i}^3 = \frac{1}{4}L_{2n}L_{2n+1}^2F_{2n+2}^2F_{2n+3},$$

$$\sum_{i=1}^{n} F_{2i}^3 = \frac{1}{4}(F_{2n+1}-1)^2(F_{2n+1}+2), \quad \prod_{i=0}^{n-1}[1+e^{(2\pi i/n)\jmath}-e^{(4\pi i/n)\jmath}] = (-1)^{n+1}L_n+(-1)^n+1,$$

$$\operatorname{atan}\frac{2}{L_{2n-1}} = 2\operatorname{atan}\frac{1}{L_{2n}} + \operatorname{atan}\frac{2}{L_{2n+1}},$$

$$\operatorname{atan}\frac{2}{L_{2n-1}} = \operatorname{atan}\frac{1}{F_{2n}} + \operatorname{atan}\frac{1}{L_{2n}}, \quad \operatorname{atan}\frac{2}{L_{2n+1}} = \operatorname{atan}\frac{1}{F_{2n}} - \operatorname{atan}\frac{1}{L_{2n}},$$

$$\operatorname{atan}\frac{1}{F_{2n-1}} = \operatorname{atan}\frac{1}{L_{2n-2}} + \operatorname{atan}\frac{1}{L_{2n}}, \quad \operatorname{atan}\frac{2}{\sqrt{5}F_{2n}} = \operatorname{atan}\frac{\sqrt{5}}{L_{2n+1}} + \operatorname{atan}\frac{1}{\sqrt{5}F_{2n+1}},$$

$$\operatorname{atan}\frac{F_{2n}}{F_{2n+1}} = \sum_{i=1}^{2n}\operatorname{atan}\frac{1}{L_{2i}} = \operatorname{atan}1 - \frac{1}{2}\operatorname{atan}\frac{1}{2} - \frac{1}{2}\operatorname{atan}\frac{2}{L_{4n+1}},$$

$$\operatorname{atan}\frac{F_{2n-1}}{F_{2n}} = \operatorname{atan}2 - \sum_{i=1}^{2n-1}\operatorname{atan}\frac{1}{L_{2i}} = \operatorname{atan}1 - \frac{1}{2}\operatorname{atan}\frac{1}{2} + \frac{1}{2}\operatorname{atan}\frac{2}{L_{4n-1}},$$

$$[5F_k^2 + (-1)^k 4]^{2/3} L_k^{2/3} = 5^{1/3}[L_k^2 + (-1)^{k+1}4]^{2/3}F_k^{2/3} + (-1)^k 4,$$

$$\sum_{i=1}^{n} \frac{2}{F_{i+3}} \leq \log F_{n+2}, \quad \sum_{i=1}^{n} \frac{2}{L_{i+3}} \leq \log L_{n+2}.$$

For all $n, k \geq 1$,

$$L_k^{2n+1} = \sum_{i=0}^{n}(-1)^{ik}\binom{2n+1}{i}L_{[2(n-i)+1]k}, \quad L_k^{2n} = \sum_{i=0}^{n-1}(-1)^{ik}\binom{2n}{i}L_{2(n-i)k} + (-1)^{nk}\binom{2n}{n}.$$

For all $k \geq 1$ and $n \geq 1$,

$$[\tfrac{1}{2}(L_k \pm \sqrt{5}F_k)]^n = \tfrac{1}{2}(L_{nk} \pm \sqrt{5}F_{nk}), \quad F_{2^k n} = F_n \prod_{i=1}^{k} L_{2^{k-i}n}, \quad \sum_{i=0}^{k} (-1)^{in} \frac{1}{L_{(i+1)n}L_{in}} = \frac{F_{(k+1)n}}{2F_n L_{(k+1)n}}.$$

Source: [adegoke,], [AMR, pp. 36, 195], [AMR3, p. 187], [benczeAOI,benczesnie,], [benjaminquinn, pp. 10–12, 78, 125, 126, 144], [bibak,chamberlandtrig,chenchenfib,deshpande5,griffithsfrom,], [grimaldi, pp. 97–99, 108, 110], [hindin,keskin,], [koshy, pp. 6–8, 239–241, 362, 363], [koshcat, pp. 78, 79], [koshyflp,lewisb,marquesnew,melhamfam,melhamcertain,], [mollnf, p. 112], [munarini,ohtsukaoom,suryAMM14,terrana,], [vajda, pp. 70–72, 182, 183], and [voll,]. **Remark:** L_n is the nth *Lucas number*. **Remark:** The generating function is given by Fact **??**. **Remark:** F_n and L_n are analogous to the sine and cosine functions, respectively. See [lewisb,].

Fact 1.17.3. Define $P_1 \triangleq 1$, $P_2 \triangleq 2$ and, for all $k \in \mathbb{Z}$, define P_k by $P_{k+2} = 2P_{k+1} + P_k$. Then,

$$(P_i)_{i=-5}^{14} = (29, -12, 5, -2, 1, 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782).$$

For all $k \in \mathbb{Z}$,

$$P_k = \frac{\sqrt{2}}{4}[(1+\sqrt{2})^k - (1-\sqrt{2})^k], \quad P_{k+1}P_{k-1} = P_k^2 + (-1)^k,$$
$$P_{2k+1} = P_k^2 + P_{k+1}^2, \quad 5P_{6k+3} = P_{3k}^2 + P_{3k+3}^2,$$
$$P_{k+3}^2 = 5P_{k+2}^2 + 5P_{k+1}^2 - P_k^2, \quad P_{k+4}^3 = 12P_{k+3}^3 + 30P_{k+2}^3 - 12P_{k+1}^3 - P_k^3,$$
$$P_{k+5}^4 = 29P_{k+4}^4 + 174P_{k+3}^4 - 174P_{k+2}^4 - 29P_{k+1}^4 + P_k^4,$$
$$(2P_kP_{k+1})^2 + (P_{k+1}^2 - P_k^2)^2 = (P_{k+1}^2 + P_k^2)^2.$$

For all $k \geq 1$,

$$P_k = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} 2^i \binom{k}{2i+1}, \quad P_{k+1} = \sum_{0 \le i \le j \le k} \binom{k-i}{j}\binom{j}{i},$$

$$\sum_{i=1}^{4k+1} P_i = \left[\sum_{i=1}^{n} 2^i \binom{2n+1}{2i}\right]^2 = (P_{2k} + P_{2k+1})^2, \quad \sum_{i=0}^{\lfloor (k-1)/4 \rfloor} \frac{1}{16^i}\binom{k-1-2i}{2i} = \frac{1}{2^k}(P_k + k),$$

$$2^{2-k} \sum_{i=0}^{\lfloor (k-3)/4 \rfloor} \binom{4k-2}{2k-8i-5} = 2^{k-2}(4^{k-1}+1) - P_{2k-1}, \quad \sum_{i=0}^{n} 5^i 2^{n-i} \binom{n}{i} P_i = P_{3n}.$$

Source: [chenchenfib,diazego,gauthierpell,kilic,mahonsum,seiffertpell,]. **Remark:** P_n is the nth *Pell number*.

1.18 Facts on Arrangement, Derangement, and Catalan Numbers

Fact 1.18.1. For all $n \geq 1$, let a_n denote the nth *arrangement number*, which is the number of k-tuples whose components are distinct elements of $\{1, \ldots, n\}$ and where $0 \leq k \leq n$. Define $a_0 = 1$. Then,

$(a_i)_{i=0}^{12} = (1, 2, 5, 16, 65, 326, 1957, 13700, 109601, 986410, 108505112, 1302061345, 16926797486)$.

For all $n \geq 1$,

$$a_n = \sum_{i=0}^{n} n^{\underline{i}} = n! \sum_{i=0}^{n} \frac{1}{i!} = \lfloor n!e \rfloor, \quad a_{n+1} = na_{n-1} + 1.$$

Remark: a_n is the nth *arrangement number*. See [comtet, p. 75]. **Remark:** The five arrangements of $\{1, 2\}$ are $\varnothing$, (1), (2), $(1, 2)$, and $(2, 1)$. **Remark:** The generating function is given by Fact **??**. **Related:** Fact **??**.

Fact 1.18.2. For all $n \geq 1$, let d_n denote the number of permutations of $(1, \ldots, n)$ that leave no component unchanged. Define $d_0 = 1$. Then,

$(d_i)_{i=0}^{13} = (1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932)$.

For all $n \geq 1$,

$$d_n = n! \sum_{i=0}^{n} (-1)^i \frac{1}{i!} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{n!}{e} + \frac{1}{n} \right\rfloor = \int_0^\infty e^{-x} (x-1)^n \, \mathrm{d}x,$$

and, for all $n \geq 1$,

$$d_{n+1} = n(d_n + d_{n-1}) = (n+1)d_n + (-1)^{n+1}, \quad n! = \sum_{i=0}^{n} \binom{n}{i} d_{n-i}, \quad d_n \overset{n}{\equiv} (-1)^n.$$

Finally,

$$\lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}.$$

Source: $d_n \overset{n}{\equiv} (-1)^n$ is given in [aebi2,]. **Remark:** d_n is the nth *derangement number*. See [cameron, pp. 57, 58] and [hassani,]. **Remark:** The permutation $(1, 2, 3, 4, 5) \mapsto (3, 1, 2, 4, 5)$ is not a derangement, but $(1, 2, 3, 4, 5) \mapsto (3, 1, 2, 5, 4)$ is a derangement. Each derangement is represented by a permutation matrix whose diagonal entries are zero. **Remark:** The generating function is given by Fact **??**. **Related:** Fact 1.18.3.

Fact 1.18.3. For all $n_1, \ldots, n_k \geq 1$, let $D_{n_1,\ldots,n_k}$ denote the number of permutations of $(1, \ldots, 1, 2,$
$\ldots, 2, \ldots, n, \ldots, n)$ that leave no component unchanged, where i appears n_i times. Then,

$$D_{n_1,\ldots,n_k} = (-1)^{\sum_{i=1}^{k} n_i} \int_0^\infty e^{-x} \prod_{i=1}^{n} L_{n_i}(x) \, \mathrm{d}x.$$

Source: [evengillis,]. **Remark:** $D_{n_1,\ldots,n_k}$ is a *generalized derangement number*, where the nth derangement number is $d_n = D_{1,\ldots,1}$. See Fact 1.18.2. L_n is the nth Laguerre polynomial. See Fact **??**.

Fact 1.18.4. Let $n \geq 0$, and let C_n denote the number of ways that n factors can be grouped for multiplication. Then,

$(C_i)_{i=0}^{15} = (1,1,2,5,14,42,132,429,1430,4862,16796,58786,208012,742900,2674440,9694845).$

For all $n \geq 1$,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = 2\binom{2n}{n} - \binom{2n+1}{n} = 4\binom{2n-1}{n} - \binom{2n+1}{n} = \binom{2n+1}{n+1} - 2\binom{2n}{n+1},$$

$$C_n = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n}\binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n},$$

$$C_n = \frac{2^n(2n-1)!!}{n!} = \prod_{i=2}^{n} \frac{n+i}{i} = \frac{4^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+2)},$$

$$C_n = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n}{i}^2 = \frac{1}{n}\sum_{i=1}^{n}\binom{n}{i}\binom{n}{i-1} = \sum_{i=0}^{\lfloor n/2\rfloor}\left[\binom{n}{i} - \binom{n}{i-1}\right]^2 = \sum_{i=0}^{\lfloor n/2\rfloor}\left[\frac{n+1-2i}{n+1}\binom{n+1}{i}\right]^2,$$

$$C_{n+1} = \frac{4n+2}{n+2}C_n = 2C_n + \frac{2}{n}\binom{2n}{n-2} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}2^{n-2i}C_i, \quad \sum_{i=0}^{n}\binom{2n-2i}{n-i}C_i = \binom{2n+1}{n},$$

$$C_{n+1} = \sum_{i=0}^{n} C_iC_{n-i} = \frac{n+3}{2n}\sum_{i=1}^{n} C_iC_{n+1-i} = \frac{1}{4^{n+1}}\sum_{i=0}^{n+1} C_{2i}C_{2n+2-2i},$$

$$\sum_{i=1}^{n} iC_iC_{n-i} = \frac{n}{2}C_{n+1}, \quad \sum_{i=1}^{n} i^2C_iC_{n-i} = \frac{n^2+2n+2}{2}C_{n+1} - 4^n,$$

$$\sum_{i=1}^{n} i^3C_iC_{n-i} = \frac{n}{2}[(n^2+3n+3)C_{n+1} - 3(4^n)], \quad C_{2n+1} = \sum_{i=1}^{n+1}\left[\frac{2i}{n+1+i}\binom{2n+1}{n+1-i}\right]^2,$$

$$\sum_{i=0}^{n-1}\frac{i+1}{2i+1}C_iC_{n-i+1} = \frac{1}{2(2n+1)}\left[(n+1)C_n + \frac{2^{4n-1}}{n(n+1)C_n}\right].$$

For all $n \geq 2$,

$$\sum_{i=1}^{\lfloor n/2\rfloor}(-1)^i\binom{n-i}{i}C_{n-1-i} = 0.$$

Furthermore, C_n is odd if and only if there exists $k \geq 1$ such that $n = 2^k - 1$. In addition, C_n is prime if and only if $n = 3$. Finally,

$$\lim_{n\to\infty}\frac{C_{n+1}}{C_n} = 4.$$

Source: [wikicatalan,chamberlandLAA13,cofman,gauthiercat,gauthierconvol,], [gelca, p. 299], [koshcat, pp. 112, 123, 127, 129, 329, 330], [larcombeutil,larcombegessel,], [mollnf, pp. 184, 186], and [penson,]. **Remark:** C_n is the nth *Catalan number*. See Fact **??** and Fact **??**. **Remark:** The generating function is given by Fact **??**. **Remark:** Additional interpretations of the Catalan numbers are given in [stanleycat,].

1.19 Facts on Cycle, Subset, Eulerian, Bell, and Ordered Bell Numbers

Fact 1.19.1. For $n \ge k \ge 1$, let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the number of permutations of $(1,\ldots,n)$ that have exactly k cycles. Furthermore, define $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \triangleq 1$, and, for all $k \ge 1$, define $\begin{bmatrix} k \\ 0 \end{bmatrix} \triangleq 0$. Then, the following statements hold:

i) Let $n \ge 1$. Then,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!, \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1},$$

$$\begin{bmatrix} n \\ 3 \end{bmatrix} = \frac{(n-1)!}{2}(H_{n-1}^2 - H_{n-1,2}), \quad \begin{bmatrix} n \\ 4 \end{bmatrix} = \frac{(n-1)!}{3!}(H_{n-1}^3 - 3H_{n-1}H_{n-1,2} + 2H_{n-1,3}).$$

ii) Let $n \ge 0$. Then, $\begin{bmatrix} n \\ n \end{bmatrix} = 1$ and $\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} = n!$.

iii) Let $n \ge 1$. Then, $\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}$.

iv) Let $n \ge 2$. Then, $\begin{bmatrix} n \\ n-2 \end{bmatrix} = 2\binom{n}{3} + 3\binom{n}{4} = \frac{3n-1}{4}\binom{n}{3}$.

v) Let $n \ge 3$. Then, $\begin{bmatrix} n \\ n-3 \end{bmatrix} = 6\binom{n}{4} + 20\binom{n}{5} + 15\binom{n}{6} = \binom{n}{2}\binom{n}{4}$.

vi) Let $n \ge 4$. Then, $\begin{bmatrix} n \\ n-4 \end{bmatrix} = \frac{1}{48}(15n^3 - 30n^2 + 5n + 2)$.

vii) Let $n > k \ge 1$. Then, $\begin{bmatrix} n \\ n-k \end{bmatrix} = (-1)^k \binom{n-1}{k} \frac{\mathrm{d}^k}{\mathrm{d}z^k} \frac{z^n}{(e^z-1)^n}\bigg|_{z=0}$.

viii) Let $n \ge k \ge 1$. Then,

$$\begin{bmatrix} n \\ n-k \end{bmatrix} = (-1)^k \frac{1}{(n-k-1)!} \sum (-1)^\kappa \frac{(n+\kappa-1)!}{\prod_{j=1}^k i_j![(j+1)!]^{i_j}},$$

where $\kappa \triangleq \sum_{j=1}^k i_j$ and the sum is taken over all k-tuples $(i_1,\ldots,i_k)$ of nonnegative integers such that $\sum_{j=1}^k j i_j = k$.

ix) Let $n \ge k \ge 1$. Then,

$$\sum \frac{1}{\prod_{j=1}^k i_j} = \frac{k!}{n!}\begin{bmatrix} n \\ k \end{bmatrix},$$

where the sum is taken over all k-tuples $(i_1,\ldots,i_k)$ of positive integers such that $\sum_{j=1}^k i_j = n$.

Source: [benjaminquinn, pp. 93–96], [GKP, pp. 257–267], and [zwillinger, p. 139]. *viii)* is given in [malenfant,], and *ix)* is given in [comtet, p. 172]. **Remark:** The permutation $(1,2,3,4,5) \mapsto (3,1,2,5,4)$ has two cycles, while the permutation $(1,2,3,4,5) \mapsto (3,1,2,4,5)$ has three cycles. Each cycle is represented by a diagonally located block in the canonical form of a permutation matrix given by Fact **??**. **Remark:** $\begin{bmatrix} n \\ k \end{bmatrix}$ is a *cycle* *number*, which is related to the *Stirling number of the first kind* $s(n,k) = (-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}$. See [benjaminquinn, pp. 103–107]. **Remark:** *vii*) relates the cycle number $\begin{bmatrix} n \\ n-k \end{bmatrix}$ to the coefficients of the power series for $z^n/(e^z-1)^n$. See [malenfant,]. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^n}{(e^z-1)^n}\bigg|_{z=0} = -\frac{n}{2}, \quad \frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{z^n}{(e^z-1)^n}\bigg|_{z=0} = \frac{1}{12}n(3n-1), \quad \frac{\mathrm{d}^3}{\mathrm{d}z^3}\frac{z^n}{(e^z-1)^n}\bigg|_{z=0} = -\frac{1}{8}n^2(n-1),$$

$$\frac{\mathrm{d}^4}{\mathrm{d}z^4}\frac{z^n}{(e^z-1)^n}\bigg|_{z=0} = \frac{1}{240}n(15n^3-30n^2+5n+2), \quad \frac{\mathrm{d}^5}{\mathrm{d}z^5}\frac{z^n}{(e^z-1)^n}\bigg|_{z=0} = -\frac{1}{96}n^2(3n^3-10n^2+5n+2),$$

$$\frac{\mathrm{d}^6}{\mathrm{d}z^6}\frac{z^n}{(e^z-1)^n}\bigg|_{z=0} = \frac{1}{4032}n(63n^5-315n^4+315n^3+91n^2-42n-16).$$

For the case $k=3$, note that, for all $n \ge 4$, $8\binom{n}{2}\binom{n}{4} = n^2(n-1)\binom{n-1}{3}$. **Example:** $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 11$, $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \binom{4}{2} = \frac{3!}{2}(H_3^2 - H_{3,2}) = 6$, $\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 50$, $\begin{bmatrix} 5 \\ 3 \end{bmatrix} = 35$, $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = 10$, $\begin{bmatrix} 6 \\ 2 \end{bmatrix} = 274$, $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = 225$, $\begin{bmatrix} 6 \\ 4 \end{bmatrix} = 85$, $\begin{bmatrix} 6 \\ 5 \end{bmatrix} = 15$. To illustrate *ix*), note that

$$\frac{1}{3\cdot 1} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 2} = \frac{2!}{4!}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{11}{12}, \quad \frac{1}{1\cdot 1\cdot 2} + \frac{1}{1\cdot 2\cdot 1} + \frac{1}{2\cdot 1\cdot 1} = \frac{3!}{4!}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{3}{2}.$$

Related: Fact **??** and Fact **??**.

Fact 1.19.2. The following statements hold:

i) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n}\begin{bmatrix} n \\ i \end{bmatrix} x^i = x^{\overline{n}}.$$

In particular,

$$\sum_{i=1}^{n} 2^i \begin{bmatrix} n \\ i \end{bmatrix} = (n+1)!.$$

ii) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n}(-1)^{n-i}\begin{bmatrix} n \\ i \end{bmatrix} x^i = x^{\underline{n}}.$$

iii) Let $n \ge 2$. Then,

$$\sum_{i=0}^{n}(-1)^i\begin{bmatrix} n \\ i \end{bmatrix} = 0.$$

iv) Let $n \ge k \ge 1$. Then,

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

v) Let $k, n \ge 0$. Then,

$$\sum_{i=k}^{n}\begin{bmatrix} n \\ i \end{bmatrix}\binom{i}{k} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}.$$

vi) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} i \begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} = n!H_n.$$

vii) Let $n, k \geq 0$. Then,

$$\sum_{i=0}^{k} (n+i) \begin{bmatrix} n+i \\ i \end{bmatrix} = \begin{bmatrix} n+k+1 \\ k \end{bmatrix}.$$

viii) Let $n, k \geq 0$. Then,

$$\sum_{i=k}^{n} \frac{n!}{i!} \begin{bmatrix} i \\ k \end{bmatrix} = \sum_{i=k}^{n} n^{\underline{n-i}} \begin{bmatrix} i \\ k \end{bmatrix} = \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}.$$

ix) Let $n, k, l \geq 0$. Then,

$$\sum_{i=0}^{n} \binom{n}{i} \begin{bmatrix} i \\ l \end{bmatrix} \begin{bmatrix} n-i \\ k \end{bmatrix} = \begin{bmatrix} n \\ l+k \end{bmatrix} \binom{l+k}{l}.$$

x) Let $n \geq k \geq 0$. Then,

$$\sum_{i=k}^{n} (-1)^{k-i} \binom{i}{k} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

xi) Let $n \geq 1$. Then,

$$\sum_{i=0}^{n} (-1)^{i} \binom{2n}{i} \begin{bmatrix} 2n-i \\ n-i \end{bmatrix} = \prod_{i=1}^{n} (2i-1).$$

Source: [benjaminquinn, pp. 93–96], [kauers,], and [zwillinger, p. 139]. **Example:**

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} z - \begin{bmatrix} 3 \\ 2 \end{bmatrix} z^2 + \begin{bmatrix} 3 \\ 3 \end{bmatrix} z^3 = z - 3z^2 + z^3 = z(z-1)(z-2) = z^{\underline{3}}.$$

Fact 1.19.3. For $n \geq k \geq 1$, let $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denote the number of partitions of a set of n elements into k subsets. Furthermore, define $\left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} \triangleq 1$ and, for all $n, k \geq 1$, define $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} \triangleq 0$ and $\left\{ \begin{matrix} 0 \\ k \end{matrix} \right\} \triangleq 0$. Then, the following statements hold:

i) Let $n \geq k \geq 1$. Then,

$$\begin{aligned} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} &= \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n = \frac{1}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n = \sum_{i=1}^{k} (-1)^{k-i} \frac{i^{n-1}}{(i-1)!(k-i)!} \\ &= \frac{n!}{k!} \sum \frac{1}{i_1! \cdots i_k!} = \sum \frac{n!}{(1!)^{i_1} i_1! (2!)^{i_2} i_2! \cdots (n!)^{i_n} i_n!}, \end{aligned}$$

where the penultimate sum is taken over all k-tuples $(i_1, \ldots, i_k)$ of positive integers whose sum is n, and the last sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of nonnegative integers whose sum is k and satisfy $\sum_{j=1}^{n} j i_j = n$. In particular, if $n \geq 1$, then

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1, \quad \left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1, \quad \left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2},$$

$$\left\{ {n \atop n-2} \right\} = \binom{n}{3} + 3\binom{n}{4} = \frac{1}{4}\binom{n}{3}(3n-5), \quad \left\{ {n \atop n-3} \right\} = \binom{n}{4} + 10\binom{n}{5} + 15\binom{n}{6} = \frac{1}{2}\binom{n}{4}(n^2-5n+6).$$

ii) Let $n \ge k \ge 0$. Then,

$$\left\{ {n \atop k} \right\} \le \left[{n \atop k} \right], \quad k^{n-k} \le \left\{ {n \atop k} \right\} \le \binom{n-1}{k-1} k^{n-k}.$$

iii) Let $n \ge k \ge 1$. Then,

$$\left\{ {n \atop n-k} \right\} = \sum \frac{n!}{(n-k-\kappa)! \prod_{j=1}^{k} i_j! [(j+1)!]^{i_j}},$$

where $\kappa \triangleq \sum_{j=1}^{k} i_j$ and the sum is taken over all k-tuples $(i_1, \ldots, i_k)$ of nonnegative integers such that $\sum_{j=1}^{k} j i_j = k$.

iv) Let $n, k, m \ge 1$. Then,

$$\sum x_{i_j}^{n} = k \sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} \binom{k+m-1}{m-i},$$

where the sum is over all k-tuples $(i_1, \ldots, i_k)$ of nonnegative integers such that $\sum_{j=1}^{k} i_j = m$.

v) Let $n \ge 2$. Then,

$$\sum_{i=1}^{n} (-1)^i (n-1)! \left\{ {n \atop i} \right\} = 0.$$

Source: [benjaminquinn, p. 103] and [zwillinger, p. 140]. *i)* is given in [aldrovandi, p. 159], [frumosu,], and [johnsoncurious,] (see (2.1)); *ii)* is given in [GKP, p. 260] and [comtet, p. 292]; *iii)* is given in [malenfant,]; *iv)* is given in [comtet, pp. 172, 173]; *v)* is given in [murty,]. **Remark:** $\left\{ {n \atop k} \right\}$ is a *subset number*, which is also called a *Stirling number of the second kind* denoted by $S(n,k)$. The curly braces are reminiscent of set notation. See [benjaminquinn, pp. 103–107], [GKP, pp. 257–267], and [knuthnotes,]. **Example:** $\left\{ {3 \atop 2} \right\} = 3$, $\left\{ {4 \atop 2} \right\} = 7$, $\left\{ {5 \atop 2} \right\} = 15$, $\left\{ {6 \atop 2} \right\} = 31$, $\left\{ {4 \atop 3} \right\} = 6$, $\left\{ {5 \atop 3} \right\} = 25$, $\left\{ {6 \atop 3} \right\} = 90$, $\left\{ {6 \atop 4} \right\} = 65$.
Related: Fact **??**.

Fact 1.19.4. The following statements hold:

i) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n} \left\{ {n \atop i} \right\} x^{\underline{i}} = x^n.$$

ii) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,

$$\sum_{i=1}^{n} (-1)^{n-i} \left\{ {n \atop i} \right\} x^{\overline{i}} = x^n.$$

iii) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,

$$x^{\overline{n}} = \sum_{i=1}^{n} \left[{n \atop i} \right] x^i = \sum_{i=1}^{n} \sum_{j=i}^{n} \left[{n \atop j} \right] \left\{ {j \atop i} \right\} x^{\underline{i}} = \sum_{i=1}^{n} \binom{n}{i} \frac{(n-1)!}{(i-1)!} x^{\underline{i}}.$$

iv) Let $n \geq 1$ and $x \in \mathbb{C}$. Then,

$$x^{\underline{n}} = \sum_{i=1}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} x^i = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{n-j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{Bmatrix} i \\ j \end{Bmatrix} x^{\bar{j}}.$$

v) Let $k \geq 1$ and $n \geq k+1$. Then,

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}.$$

vi) Let $n, k \geq 0$. Then,

$$\sum_{i=1}^{\min\{k,n\}} i! \begin{Bmatrix} k \\ i \end{Bmatrix} \binom{n}{i} = n^k.$$

vii) Let $n, k \geq 0$. Then,

$$\sum_{i=1}^{\min\{k,n\}} i! \begin{Bmatrix} k \\ i \end{Bmatrix} \binom{n+1}{i+1} = \sum_{i=1}^{n} i^k.$$

viii) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} (-1)^i (i-1)! \begin{Bmatrix} n \\ i \end{Bmatrix} = 0.$$

ix) Let $n \geq k \geq 1$. Then,

$$\sum_{i=1}^{k} (-1)^{k-i} i^n \binom{k}{i} = k! \begin{Bmatrix} n \\ k \end{Bmatrix}.$$

x) Let $n, k \geq 0$. Then,

$$\sum_{i=k}^{n} \binom{n}{i} \begin{Bmatrix} i \\ k \end{Bmatrix} = \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}.$$

xi) Let $n, k \geq 0$. Then,

$$\sum_{i=1}^{k} i \begin{Bmatrix} n+i \\ i \end{Bmatrix} = \begin{Bmatrix} n+k+1 \\ k \end{Bmatrix}.$$

xii) Let $n, k \geq 0$. Then,

$$\sum_{i=0}^{n} \begin{Bmatrix} i \\ k \end{Bmatrix} (k+1)^{n-i} = \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}.$$

xiii) Let $n \geq k \geq 0$. Then,

$$\sum_{i=k}^{n} (-1)^{n-i} \binom{n}{i} \begin{Bmatrix} i+1 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix}.$$

xiv) Let $n, k \geq 0$. Then,

$$\sum_{i=1}^{k} (-1)^i \begin{Bmatrix} k \\ i \end{Bmatrix} i^n = (-1)^k k! \begin{Bmatrix} n \\ k \end{Bmatrix}.$$

xv) Let $n \geq 0$. Then,

$$\sum_{i=1}^{n} (-1)^i \frac{i!}{i+1} \begin{Bmatrix} n \\ i \end{Bmatrix} = B_n.$$

xvi) Let $n \geq k \geq 1$. Then,
$$\sum_{i=k}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{Bmatrix} i \\ k \end{Bmatrix} = \binom{n}{k} \frac{(n-1)!}{(k-1)!}.$$

xvii) Let $n \geq k \geq 1$. Then,
$$\sum_{i=k}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix} \begin{Bmatrix} i \\ k \end{Bmatrix} = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

xviii) Let $n \geq k \geq 1$. Then,
$$\sum_{i=k}^{n} (-1)^{n-i} \begin{Bmatrix} n \\ i \end{Bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

xix) Let $n, k \geq 0$. Then,
$$\sum_{i=k}^{n} (-1)^{k-i} \begin{Bmatrix} n+1 \\ i+1 \end{Bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} = \binom{n}{k}.$$

xx) Let $n \geq k \geq 0$. Then,
$$\sum_{i=k}^{n} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} \begin{Bmatrix} i \\ k \end{Bmatrix} = \frac{n!}{m!}.$$

xxi) Let $n \geq k \geq 0$. Then,
$$\sum_{i=k}^{n} (-1)^{k-i} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} \begin{Bmatrix} i \\ k \end{Bmatrix} = n^{\underline{n-k}}.$$

xxii) Let $n, k \geq 0$, and assume that $k \leq n+1$. Then,
$$\sum_{i=0}^{n} (-1)^{i+1-k} \frac{1}{i+1} \begin{bmatrix} i+1 \\ k \end{bmatrix} \begin{Bmatrix} n \\ i \end{Bmatrix} = \frac{1}{n+1} \binom{n+1}{k} B_{n+1-k}.$$

xxiii) Let $n, k \geq 0$. Then,
$$\sum_{i=1}^{k} (-1)^{i+1} i \begin{bmatrix} n+1 \\ n+1-i \end{bmatrix} \begin{Bmatrix} n+k-i \\ n \end{Bmatrix} = \sum_{i=1}^{n} i^k.$$

xxiv) Let $n, k, l \geq 0$. Then,
$$\sum_{i=0}^{n} \binom{n}{i} \begin{Bmatrix} i \\ l \end{Bmatrix} \begin{Bmatrix} n-i \\ k \end{Bmatrix} = \binom{l+k}{l} \begin{Bmatrix} n \\ l+k \end{Bmatrix}.$$

xxv) Let $n \geq k \geq 0$. Then,
$$\sum_{i=0}^{n} \binom{k-n}{k+i} \binom{k+n}{n+i} \begin{bmatrix} k+i \\ i \end{bmatrix} = \begin{Bmatrix} n \\ n-k \end{Bmatrix}.$$

xxvi) Let $n \geq k \geq 0$. Then,
$$\sum_{i=0}^{n} \binom{k-n}{k+i} \binom{k+n}{n+i} \begin{Bmatrix} k+i \\ i \end{Bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

xxvii) Let $n \ge k \ge 0$. Then,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{i=0}^{n-k} (-1)^i \binom{n-1+i}{n-k+i} \binom{2n-k}{n-k-i} \begin{Bmatrix} n-k-i \\ i \end{Bmatrix},$$

where $\kappa \triangleq \sum_{j=1}^{k} i_j$ and the sum is taken over all k-tuples $(i_1, \ldots, i_k)$ such that, for all $j \in \{1, \ldots, k\}$, $0 \le i_j \le k$ and such that $\sum_{j=1}^{k} j i_j = k$.

Source: [benjaminquinn, pp. 103, 106, 107], [boyadzhiev,daboullah,], [engel, p. 95], [frumosu,gonzalezgci,], [GKP, pp. 264, 265, 289], and [merceafaul,riordan,]. **Remark:** In *xv*), B_n is the nth Bernoulli number. See Fact **??**. **Remark:** The coefficient $L(n,k) \triangleq \binom{n}{k}\binom{n-1}{k-1}(n-k)! = \binom{n}{k}\frac{(n-1)!}{(k-1)!}$ in *iii*) is a *Lah number*. See [daboullah,]. **Example:**

$$1^2 + 2^2 + 3^2 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \binom{4}{2} 1! + \begin{Bmatrix} 2 \\ 2 \end{Bmatrix} \binom{4}{3} 2! = 1(6)(1) + 1(4)(2) = 14,$$

$$\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} z + \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} z(z-1) + \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} z(z-1)(z-2) = z + 3z(z-1) + z(z-1)(z-2) = z^3.$$

Fact 1.19.5. Let $n \ge 1$, let $0 \le k \le n-1$, and let $\left\langle {n \atop k} \right\rangle$ denote the number of permutations of $(1, \ldots, n)$ in which exactly k components are larger than the previous component. Then, the following statements hold:

i) Let $n \ge 1$ and $0 \le k \le n-1$. Then,

$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k+1-i)^n,$$

$$\sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle = n!, \quad \sum_{i=0}^{n-1} (-1)^i \left\langle {n \atop i} \right\rangle = \frac{2^{n+1}(2^{n+1}-1)B_{n+1}}{n+1}.$$

ii) Let $n \ge 1$. Then,

$$\left\langle {n \atop 0} \right\rangle = \left\langle {n \atop n-1} \right\rangle = 1, \quad \left\langle {n \atop 1} \right\rangle = 2^n - n - 1, \quad \left\langle {n \atop 2} \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}, \quad \left\langle {n \atop n} \right\rangle \triangleq 0,$$

$$\left\langle {2 \atop 1} \right\rangle = 1, \quad \left\langle {3 \atop 1} \right\rangle = 4, \quad \left\langle {4 \atop 1} \right\rangle = \left\langle {4 \atop 2} \right\rangle = 11, \quad \left\langle {5 \atop 1} \right\rangle = \left\langle {5 \atop 3} \right\rangle = 26, \quad \left\langle {5 \atop 2} \right\rangle = 66.$$

iii) Let $n \ge 1$ and $0 \le k \le n-1$. Then,

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle, \quad \left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle,$$

where $\left\langle {0 \atop 0} \right\rangle \triangleq 1$ and $\left\langle {0 \atop k} \right\rangle \triangleq 0$.

iv) Let $n \ge 1$ and $x \in \mathbb{C}$. Then,

$$x^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{x+i}{n}.$$

In particular,

$$x^2 = \binom{x}{2}+\binom{x+1}{2},\ x^3 = \binom{x}{3}+4\binom{x+1}{3}+\binom{x+2}{3},\ x^4 = \binom{x}{4}+11\binom{x+1}{4}+11\binom{x+2}{4}+\binom{x+3}{4}.$$

v) Let $k \ge 1$ and $1 \le n \le k$, Then,

$$k^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+i}{n} = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i-1}{n}.$$

vi) If $k, n \ge 1$, then

$$\sum_{i=1}^{k} i^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+i+1}{n+1}.$$

vii) If $k, n \ge 1$, then

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=n-k}^{n} \left\{ {n \atop i} \right\} \binom{i}{n-k}, \quad \left\langle {n \atop k} \right\rangle = \sum_{i=1}^{n-k} (-1)^{n-k-i} i! \left\{ {n \atop i} \right\} \binom{n-i}{k}.$$

viii) If $n \ge 2$, then

$$\sum_{i=0}^{n-1} (-1)^i \frac{\left\langle {n \atop i} \right\rangle}{\binom{n-1}{i}} = 0, \quad \sum_{i=0}^{n-1} (-1)^i \frac{\left\langle {n \atop i} \right\rangle}{\binom{n}{i}} = (n+1)B_n.$$

Source: [GKP, pp. 267–269], [knuth3, pp. 35–39], and [petersen2,]. **Remark:** $\left\langle {n \atop m} \right\rangle$ is an *Eulerian number*. An alternative definition is used in [belbachir,]. **Remark:** *iv)* is *Worpitzky's identity*. **Related:** Fact ??.

Fact 1.19.6. Let $\mathcal{B}_n$ denote the number of partitions of $\{1, \ldots, n\}$, and define $\mathcal{B}_0 \triangleq 1$. Then, the following statements hold:

i) Let $n \ge 1$. Then,

$$\mathcal{B}_n = \sum_{i=1}^{n} \left\{ {n \atop i} \right\} = \sum_{i=0}^{n} \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n = 1 + \left\lfloor \frac{1}{e} \sum_{i=1}^{2n} \frac{i^n}{i!} \right\rfloor.$$

ii) $(\mathcal{B}_i)_{i=0}^{13} = (1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437)$.

iii) If p is prime, then $\mathcal{B}_{n+p} \overset{p}{\equiv} \mathcal{B}_n + \mathcal{B}_{n+1}$.

iv) If $n, m \ge 1$, then

$$\mathcal{B}_{n+m} = \sum_{i=0}^{n} \sum_{j=1}^{m} \left\{ {m \atop j} \right\} \binom{n}{i} j^{n-i} \mathcal{B}_i, \quad \mathcal{B}_{n+1} = \sum_{i=0}^{n} \binom{n}{i} \mathcal{B}_i.$$

v) Let $k \ge n \ge 1$. Then,

$$\mathcal{B}_n = \sum_{i=1}^{k} \frac{i^n}{i!} \sum_{j=0}^{k-i} (-1)^j \frac{1}{j!} = \sum_{i=1}^{n} \frac{i^n}{i!} \sum_{j=0}^{n-i} (-1)^j \frac{1}{j!}.$$

Remark: $\mathcal{B}_n$ is the nth *Bell number*. See [aldrovandi, p. 160], [bressoud, p. 623], and [spivey,]. **Related:** Fact **??**.

Fact 1.19.7. For all $n \geq 1$, let $\mathcal{O}_n$ denote the number of possible orderings of the multiset $\{i_1, \ldots, i_n\}_{\mathrm{ms}}$ of real numbers, and define $\mathcal{O}_0 \triangleq 1$. Then, the following statements hold:

i) Let $n \geq 1$. Then,

$$\mathcal{O}_n = \sum_{i=1}^{n} i! \left\{ {n \atop i} \right\}.$$

ii) $(\mathcal{O}_i)_{i=0}^{11} = (1, 1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573)$.

iii) Let $n \geq 1$. Then,

$$\mathcal{O}_n = \sum_{i=0}^{n-1} \binom{n}{i} \mathcal{O}_i = (-1)^{n-1} + 2\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \mathcal{O}_i = \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n = \sum_{i=0}^{\infty} \frac{i^n}{2^{i+1}} = \sum_{i=0}^{n-1} 2^i \left\langle {n \atop i} \right\rangle.$$

iv) Let $n \geq 1$. Then,

$$\sum_{i=1}^{n} \mathcal{O}_{i-1}\mathcal{O}_{n-i} \binom{n}{i} = \frac{n}{2}\mathcal{O}_{n-1} + \frac{1}{2}\sum_{i=1}^{n} i! H_i \left\{ {n \atop i} \right\},$$

$$\sum_{i=1}^{n} \mathcal{O}_{i-1}\mathcal{O}_{n-i} \binom{n}{i} = \frac{n+1}{2}\mathcal{O}_{n-1} - \frac{1}{4}\delta_{n,1} + \frac{1}{4}\sum_{i=1}^{n} i! H_i \left\{ {n+1 \atop i+1} \right\}.$$

Source: [dil,dilgf,munarini,]. **Remark:** $\mathcal{O}_n$ is the nth *ordered Bell number*. **Remark:** For $n = 3$, the 13 possible orderings of $\{x, y, z\}_{\mathrm{ms}}$ are $x = y = z$, $x < y = z$, $y = z < x$, $y < x = z$, $x = z < y$, $z < x = y$, $x = y < z$, $x < y < z$, $x < z < y$, $y < x < z$, $y < z < x$, $z < x < y$, $z < y < x$. **Remark:** The generating function is given by Fact **??**.

1.20 Facts on Partition Numbers, the Totient Function, and Divisor Sums

Fact 1.20.1. For all $n \geq 1$, let p_n denote the number of partitions of the n-element multiset $\{1, \ldots, 1\}_{\mathrm{ms}}$. Equivalently, for all $n \geq 1$, let p_n denote the number of ways of representing n as a sum of one or more positive integers. Define $p_0 \triangleq 1$. Furthermore, for all $n, k \geq 1$, let $p_{n,k}$ denote the number of ways of representing n as a sum of k positive integers, and, for all $n, k, l \geq 1$, let $p_{n,k,l}$ denote the number of ways of representing n as a sum of k positive integers the largest of which is l. Then, the following statements hold:

i) For all $n \geq 1$, $p_n = \operatorname{card}\{(k_1, \ldots, k_n) \in \mathbb{N}^n \colon k_1 \leq \cdots \leq k_n \text{ and } \sum_{i=1}^{n} k_i = n\}$.

ii) For all $n \geq 1$, $p_n = \operatorname{card}\{(k_1, \ldots, k_n) \in \mathbb{N}^n \colon \sum_{i=1}^{n} i k_i = n\}$.

iii) $(p_i)_{i=0}^{20} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627)$.

iv) For all $n \in \{1, 2\}$, $p_n = \mathcal{B}_n$. For all $n \geq 3$, $p_n < \mathcal{B}_n$.

v) For all $n \in \{1, 2, 3, 4\}$, $p_n = F_{n+1}$. For all $n > 5$, $p_n < F_{n+1}$.

vi) For all $n, k \geq 1$, $p_{n,k}$ is the number of ways of representing n as a sum of positive integers, the largest of which is k.

vii) Let $n \geq 1$. Then, $\sum_{i=1}^{n} p_{n,i} = p_n$.

viii) Let $n \geq 1$. Then,

$$p_{n,1} = p_{n,n-1} = p_{n,n} = 1, \quad p_{n,2} = \tfrac{1}{4}[2n + 3 + (-1)^n], \quad p_{n,3} = \left\lfloor \tfrac{1}{12}(n+3)^2 + \tfrac{1}{2} \right\rfloor,$$

$$p_{n,4} = \left\lfloor \tfrac{1}{144}(n+5)(n^2 + n + 22 + 18\lfloor n/2 \rfloor) + \tfrac{1}{2} \right\rfloor,$$

$$p_{n,5} = \left\lfloor \tfrac{1}{2880}(n+8)(n^3 + 22n^2 + 44n + 248 + 180\lfloor n/2 \rfloor) + \tfrac{1}{2} \right\rfloor,$$

$$\sum_{i=1}^{2} p_{n,i} = 1 + \left\lfloor \frac{n}{2} \right\rfloor, \quad \sum_{i=1}^{3} p_{n,i} = 1 + \left\lfloor \frac{n^2 + 6n}{12} \right\rfloor.$$

ix) Let $k \geq 1$. Then, as $n \to \infty$,

$$\sum_{i=1}^{k} p_{n,i} \sim \frac{n^{k-1}}{k!(k-1)!}.$$

x) Let $n \geq k \geq 1$. Then,

$$\sum_{i=1}^{n} p_{n,i,k} = \sum_{i=1}^{n} p_{n,k,i} = p_{n,k}.$$

In particular,

$$\sum_{i=1}^{n} p_{n,i,1} = \sum_{i=1}^{n} p_{n,1,i} = \sum_{i=1}^{n} p_{n,i,n-1} = \sum_{i=1}^{n} p_{n,n-1,i} = \sum_{i=1}^{n} p_{n,i,n} = \sum_{i=1}^{n} p_{n,n,i} = 1.$$

xi) Let $n \geq k \geq 1$. Then, the number of ways of representing n as a sum of k or fewer positive integers is

$$P_{n,k} \triangleq \sum_{i,j=1}^{n,k} p_{n,i,j} = \sum_{i,j=1}^{n,k} p_{n,j,i} = \sum_{i=1}^{k} p_{n,i}.$$

Furthermore,

$$P_{n,k+1} = P_{n-k-1,k+1} + P_{n,k}.$$

xii) Let $n, m \geq 1$. Then, the number of ways of representing all of the positive integers less than or equal to mn as the sum of n or fewer positive integers, the largest of which is less than or equal to m, is

$$\sum_{i,j,k=1}^{mn,m,n} p_{i,j,k} = \binom{n+m}{m} - 1.$$

xiii) For all $n \geq 1$, $5|p_{5n+4}$, $7|p_{7n+5}$, $11|p_{11n+6}$, and $13|p_{17303n+237}$.

xiv) For each prime m, there exists $n \geq 1$ such that $m|p_n$.

xv) For all $n \geq 1$,

$$p_n = \frac{1}{n} \sum_{i=1}^{n} s_i p_{n-i},$$

where s_i is the sum of the divisors of i.

xvi) For all $n \geq 1$,

$$p_n = \sum (-1)^{\lfloor (i-1)/2 \rfloor} p_{n-g_i} = \sum (-1)^i (p_{n-P_i} + p_{n-P_i'}) = \sum_{i=1}^{n-1} e_i p_{n-i},$$

where the first sum is taken over all $i \geq 1$ such that $g_i \leq n$, the second sum is taken over all $i \geq 1$ such that $P_i \leq n$ and $P_i' \leq n$, and, for all $k \geq 0$,

$$e_k \triangleq \begin{cases} 1, & k = 0, \\ (-1)^i, & k \in \{g_{2i-1}, g_{2i}\} = \{\frac{1}{2}i(3i-1), \frac{1}{2}i(3i+1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

xvii) For all $n \geq 2$, let $(n_\mathrm{e}, n_\mathrm{o})$ denote the number of partitions of $\{1, \ldots, n\}$ into an (even, odd) number of subsets. Then, $n_\mathrm{e} - n_\mathrm{o} = e_n$.

xviii) Let $n \geq 1$, define $\mathcal{P}_n \triangleq \{(k_1, \ldots, k_n) \in \mathbb{N}^n \colon k_1 \leq \cdots \leq k_n \text{ and } \sum_{i=1}^n k_i = n\}$, and let $i \in \{1, \ldots, n\}$. Then,

$$\sum \sum_{j=1}^n \operatorname{truth}(k_j = i) = \sum \sum_{j=1}^n \operatorname{card} \left\{ j \in \{1, \ldots, n\} \colon \sum_{l=1}^n \operatorname{truth}(k_l = j) \geq i \right\},$$

where the first sum on both sides is taken over all $(k_1, \ldots, k_n) \in \mathcal{P}_n$. (Example: The number 1 appears 12 times over all 7 elements of $\mathcal{P}_5$, while the number of times that an integer appears 1 or more times in each element of $\mathcal{P}_n$ summed over all elements of $\mathcal{P}_5$ is 12. Likewise, the number 2 appears 4 times over all the 7 elements of $\mathcal{P}_5$, while the number of times that an integer appears 2 or more times in each element of $\mathcal{P}_n$ summed over all 7 elements of $\mathcal{P}_5$ is 4.)

Remark: p_n is the nth *partition number*. See [andrewspart, p. 70], [AAR, pp. 553–576], [andrewseriksson, pp. 58–61, 125], [benjaminquinn, p. 79], [bressoud, pp. 624–627], [comtet, Chapter II], [hirschhorncong,], [OHC,], [zwillinger, p. 138]. *xvii*) is given in [santosdasilva,]. **Remark:** To illustrate $p_5 = 7$, note that $5 = 1+1+1+1+1 = 2+1+1+1 = 2+2+1 = 3+1+1 = 4+1 = 3+2$. **Remark:** P_i is the ith pentagonal number, P_i' is the i dual pentagonal number, and g_i is the ith generalized pentagonal number. See Fact 1.12.5 and Fact **??**. For example, $15 = p_7 = p_6 + p_5 - p_2 - p_0 = 11 + 7 - 2 - 1 = 15$. **Remark:** *xii*) implies that the number of ways of representing positive integers less than or equal to mn as the sum of less than or equal to n positive integers, the largest of which is less than or equal to m, is equal to the number of ways of representing positive integers less than or equal to mn as the sum of less than or equal to m positive integers, the largest of which is less than or equal to n. **Remark:** $(e_i)_{i=0}^6 = (1, -1, -1, 0, 0, 1, 0)$. **Remark:** The generating function for $(p_i)_{i=1}^\infty$ is given by Fact **??**. **Remark:** *xviii*) is *Elder's theorem*. The case $i = 1$ is *Stanley's theorem*. See [gilbert,]. **Remark:** For all $n \geq 1$, the number of tuples of positive integers whose components sum to n is 2^{n-1}. **Credit:** *xiii*) is due to S. A. Ramanujan and A. O. L. Atkin. **Related:** Fact **??**, Fact **??**, and Fact **??**.

Fact 1.20.2. For all $n \geq 1$, let s_n denote the sum of the distinct positive divisors of n, and define $(p_i)_{i=0}^\infty$ and $(e_i)_{i=1}^\infty$ as in Fact 1.20.1. Then, the following statements hold:

i) If n is prime and $k \geq 1$, then $s_{n^k} = \frac{n^{k+1}-1}{n-1}$.

ii) If $n \geq 1$ and $m \geq 1$ are coprime, then $s_{nm} = s_n s_m$.

iii) Let $n \geq 2$, and let $m_1, \ldots, m_l$ be distinct primes and $k_1, \ldots, k_l$ be positive integers such that $n = \prod_{i=1}^l m_i^{k_i}$. Then, $s_n = \prod_{i=1}^l s_{m_i^{k_i}}$.

iv) Let $n \geq 1$. Then, $s_n = -ne_n - \sum_{i=0}^{n-1} e_{n-i}s_i = -ne_n - \sum_{i=1}^{n} e_n s_{n-i}$, where $s_0 \triangleq n$.

v) Let $n \geq 1$. Then, $s_n = -\sum_{i=1}^{n} ie_i p_{n-i}$.

Source: [OHC,]. **Remark:** The generating function for $(s_i)_{i=1}^{\infty}$ is given by Fact **??**. **Remark:** To illustrate *iv)*, note that $12 = s_6 = -6e_6 - e_6s_0 - e_5s_1 - e_4s_2 - e_3s_3 - e_2s_4 - e_1s_5 = -6(0) - 0(12) - (1)1 - 0(3) - 0(4) - (-1)(7) - (-1)6 = 12$. **Remark:** To illustrate *v)*, note that $12 = s_6 = -[e_1p_5 + 2e_2p_4 + 3e_3p_3 + 4e_4p_2 + 5e_5p_1 + 6e_6p_0] = -[1(-1)7 + 2(-1)5 + 3(0)3 + 4(0)2 + 5(1)1 + 6(0)1] = 12$.

Fact 1.20.3. Let $n \geq 1$, let $\tau(n)$ be the number of positive divisors of n, and let $\sigma(n)$ be the sum of the positive divisors of n, where $\sqrt{n}$ is counted and summed twice if $\sqrt{n}$ is an integer. Then, $\sqrt{n} \leq \sigma(n)/\tau(n)$. **Source:** [kaczor1, p. 17]. **Example:** $\sqrt{4} \leq (1+2+2+4)/4 = 9/4$ and $\sqrt{20} \leq (1+2+4+5+10+20)/6 = 7$. **Related:** Fact **??**.

Fact 1.20.4. Let $k \geq 1$, and let $\phi(k) \triangleq \operatorname{card}\{i \in \{1,\ldots,k\}\colon \gcd\{k,i\} = 1\}$. Then, the following statements hold:

i) $(\phi(i))_{i=1}^{28} = (1,1,2,2,4,2,6,4,6,4,10,4,12,6,8,8,16,6,18,8,12,10,22,8,20,12,18,12)$.

ii) Let $n \geq 2$ be prime. Then, $\phi(n) = n-1$.

iii) Let m be an integer, let $n \geq 1$, and assume that m and n are coprime. Then, $m^{\phi(n)} \overset{n}{\equiv} 1$.

iv) Let $n \geq 1$, let $n_1,\ldots,n_l$ be distinct primes, let $i_1,\ldots,i_l \geq 1$, and assume that $n = \prod_{j=1}^{l} n_j^{i_j}$. Then,

$$\phi(n) = n\prod_{i=1}^{l}\left(1 - \frac{1}{n_i}\right).$$

If, in addition, $i_1 = \cdots = i_l = 1$, then $\phi(n) = \prod_{i=1}^{l}(n_i - 1)$.

v) Let $n \geq 1$ and $m \geq 1$, and assume that n and m are coprime. Then, $\phi(nm) = \phi(n)\phi(m)$.

vi) Let $n \geq 1$. Then, $\sum \phi(i) = n$, where the sum is taken over all $i \geq 1$ that divide n.

vii) If $n \geq 2$, then $\sqrt{n/2} \leq \phi(n)$. If, in addition, $n \geq 3$ and $n \neq 6$, then $\sqrt{n} \leq \phi(n)$.

viii) Let c_n denote the nth positive number such that n and $\phi(n)$ are coprime. Then, $(c_i)_{i=1}^{27} = (1,2,3,5,7,11,13,15,17,19,23,29,31,33,35,37,41,43,47,51,53,59,61,65,67,69,71)$.

ix) Let $n \geq 3$. Then, n and $\phi(n)$ are coprime if and only if n has distinct prime factors and, for all distinct prime factors k and l of n such that $k < l$, $k \nmid l-1$.

Source: [benjaminquinn, pp. 116, 117]. *vii)* is given in [kendallosburn,]. **Remark:** ϕ is the *totient function*. See [apostolnumbertheory, pp. 25–28]. **Remark:** *iii)* is *Euler's theorem*. See [larson, p. 148]. **Example:** For *iii)*, note that, for $m = 4$ and $n = 3$, $4^2 - 1 = 3 \cdot 5$. Furthermore, for $m = 7$ and $n = 5$, $7^4 - 1 = 5 \cdot 480$. **Example:** For *iv)*, note that $\phi(23) = 22 = 23(1 - 1/23)$, $6 = \phi(9) = 9(1 - 1/3)$, and $40 = \phi(55) = 55(1 - 1/5)(1 - 1/11)$. **Example:** For *v)*, note that $\phi(35) = 24 = 4 \cdot 6 = \phi(5)\phi(7)$ and $\phi(68) = 32 = 2 \cdot 16 = \phi(4)\phi(17)$. **Example:** For *vi)*, note that $20 = \phi(1) + \phi(2) + \phi(4) + \phi(5) + \phi(10) + \phi(20) = 1+1+2+4+4+8$ and $23 = \phi(1) + \phi(23) = 1 + 22$. **Remark:** c_n is the nth *cyclic number*. See Fact **??**. **Remark:** The first ten cyclic numbers that are not prime are $1, 15, 33, 35, 51, 65, 69, 77, 85, 87$. **Related:** Fact **??**.

1.21 Facts on Convex Functions

Fact 1.21.1. let $a, b \in \mathbb{R}$, assume that $a < b$, and let $f\colon (a,b) \mapsto \mathbb{R}$. Then, the following statements are equivalent:

i) f is convex.

ii) For all $x_1, x, x_2 \in (a,b)$ such that $x_1 < x < x_2$,

$$\frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}.$$

iii) For all $x_1, x, x_2 \in (a,b)$ such that $x_1 < x < x_2$,

$$\frac{f(x) - f(x_1)}{x - x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_2) - f(x)}{x_2 - x}.$$

iv) For all $x_1, y_1, x_2, y_2 \in (a,b)$ such that $x_1 \ne y_1$, $x_2 \ne y_2$, $x_1 \le x_2$, and $y_1 \le y_2$,

$$\frac{f(x_1) - f(y_1)}{x_1 - y_1} \le \frac{f(x_2) - f(y_2)}{x_2 - y_2}.$$

Furthermore, the following statements are equivalent:

v) f is strictly convex.

vi) For all $x_1, x, x_2 \in (a,b)$ such that $x_1 < x < x_2$,

$$\frac{f(x) - f(x_1)}{x - x_1} < \frac{f(x_2) - f(x)}{x_2 - x}.$$

vii) For all $x_1, y_1, x_2, y_2 \in (a,b)$ such that $x_1 \ne y_1$, $x_2 \ne y_2$, $x_1 < x_2$, and $y_1 < y_2$,

$$\frac{f(x_1) - f(y_1)}{x_1 - y_1} < \frac{f(x_2) - f(y_2)}{x_2 - y_2}.$$

Source: [gruber, p. 4] and [kadelburg,].

Fact 1.21.2. Let a and b be nonnegative numbers such that $a < b$, let $f\colon [a,b] \mapsto \mathbb{R}$, assume that f is convex, and let $x_1, \ldots, x_n$ be positive numbers such that $\sum_{i=1}^n x_i \le b - a$. Then,

$$\sum_{i=1}^n f(a + x_i) \le f\left(a + \sum_{i=1}^n x_i\right) + (n-1)f(a).$$

Source: [kadelburg,].

Fact 1.21.3. Let a, b, c be nonnegative numbers such that $0 \le a \le b \le c$, let $f\colon [0,c] \mapsto \mathbb{R}$, and assume that f is convex. Then,

$$f(c - b + a) + f(b) \le f(c) + f(a).$$

Source: Fact 1.21.2 and [kadelburg,].

Fact 1.21.4. Let $\mathcal{I}$ be a finite or infinite interval, and let $f\colon \mathcal{I} \mapsto \mathbb{R}$. Then, in each case below, f is convex:

i) $\mathcal{I} = (0, \infty)$, $f(x) = -\log x$.

ii) $\mathcal{I} = (0, \infty)$, $f(x) = x \log x$.

iii) $\mathcal{I} = (0,\infty)$, $f(x) = x^p$, where $p < 0$.

iv) $\mathcal{I} = [0,\infty)$, $f(x) = -x^p$, where $p \in (0,1)$.

v) $\mathcal{I} = [0,\infty)$, $f(x) = x^p$, where $p \in (1,\infty)$.

vi) $\mathcal{I} = [0,\infty)$, $f(x) = (1+x^p)^{1/p}$, where $p \in (1,\infty)$.

vii) $\mathcal{I} = \mathbb{R}$, $f(x) = \frac{a^x - b^x}{c^x - d^x}$, where $0 < d < c < b < a$ and $f(0) \triangleq (\log a/b)/\log c/d$.

viii) $\mathcal{I} = \mathbb{R}$, $f(x) = \log \frac{a^x - b^x}{c^x - d^x}$, $0 < d < c < b < a$, $ad \ge bc$, and $f(0) \triangleq \log[(\log a/b)/(\log c/d)]$.

ix) $\mathcal{I} = \mathbb{R}$, $f(x) = \log \frac{c^x - d^x}{a^x - b^x}$, $0 < d < c < b < a$, $ad < bc$, and $f(0) \triangleq \log[(\log c/d)/(\log a/b)]$.

x) $\mathcal{I} = (0,\infty)$, $f(x) = \log \Gamma(x)\Gamma(1/x)$.

Source: *vii*) and *viii*) are given in [experimentation, p. 39]; *x*) is given in [jamesonmia,].

Fact 1.21.5. Let $\mathcal{I} \subseteq (0,\infty)$ be a finite or infinite interval, let $f\colon \mathcal{I} \mapsto \mathbb{R}$, and define $g\colon \mathcal{I} \mapsto \mathbb{R}$ by $g(x) = xf(1/x)$. Then, f is (convex, strictly convex) if and only if g is (convex, strictly convex). **Source:** [niculescupersson, p. 13].

Fact 1.21.6. Let $f\colon \mathbb{R} \mapsto \mathbb{R}$, assume that f is convex, and assume that there exists $\alpha \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$, $f(x) \le \alpha$. Then, f is constant. **Source:** [niculescupersson, p. 35].

Fact 1.21.7. Let $\mathcal{I} \subseteq \mathbb{R}$ be a finite or infinite interval, let $f\colon \mathcal{I} \mapsto \mathbb{R}$, and assume that f is continuous. Then, the following statements are equivalent:

i) f is convex.

ii) For all $n \in \mathbb{P}$, $x_1,\ldots,x_n \in \mathcal{I}$, and $\alpha_1,\ldots,\alpha_n \in [0,1]$ such that $\sum_{i=1}^n \alpha_i = 1$, it follows that

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \le \sum_{i=1}^n \alpha_i f(x_i).$$

Remark: This is *Jensen's inequality.* **Remark:** Setting $f(x) = x^p$ yields Fact **??**, whereas setting $f(x) = \log x$ for all $x \in (0,\infty)$ yields the arithmetic-mean–geometric-mean inequality given by Fact **??**. **Related:** Fact **??**.

Fact 1.21.8. Let $[a,b] \subset \mathbb{R}$, let $f\colon [a,b] \mapsto \mathbb{R}$ be convex, and let $x,y \in [a,b]$. Then,

$$\tfrac{1}{2}[f(x)+f(y)] - f[\tfrac{1}{2}(x+y)] \le \tfrac{1}{2}[f(a)+f(b)] - f[\tfrac{1}{2}(a+b)].$$

Remark: This is *Niculescu's inequality.* See [bagdasar, p. 13].

Fact 1.21.9. Let $\mathcal{I} \subseteq \mathbb{R}$ be a finite or infinite interval, let $f\colon \mathcal{I} \mapsto \mathbb{R}$. Then, the following statements are equivalent:

i) f is convex.

ii) f is continuous, and, for all $x,y,z \in \mathcal{I}$,

$$\tfrac{2}{3}(f[\tfrac{1}{2}(x+y)] + f[\tfrac{1}{2}(y+z)] + f[\tfrac{1}{2}(z+x)]) \le \tfrac{1}{3}[f(x)+f(y)+f(z)] + f[\tfrac{1}{3}(x+y+z)].$$

Remark: This is *Popoviciu's inequality.* See [niculescupersson, p. 12]. **Remark:** For a scalar argument and $f(x) = |x|$, this result implies Hlawka's inequality given by Fact **??**. See Fact **??** and [nicpop,]. **Problem:** Extend this result so that it yields Hlawka's inequality for vector arguments.

Fact 1.21.10. Let $[a,b] \subset \mathbb{R}$, let $f\colon [a,b] \mapsto \mathbb{R}$, and assume that f is convex. Then,

$$f[\tfrac{1}{2}(a+b)] \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \tfrac{1}{2}[f(a)+f(b)].$$

Source: [niculescupersson, pp. 50–53], [sandorhadamard1,sandorhadamard2,]. **Remark:** This is the *Hermite-Hadamard inequality.*

Fact 1.21.11. Let $[a, b] \subset \mathbb{R}$, let $f\colon [a, b] \mapsto \mathbb{R}$, assume that f is concave, and let p and q be positive numbers such that $p < q$. Then,

$$\left(\frac{q+1}{b-a}\int_a^b [f(x)]^q \,\mathrm{d}x\right)^{1/q} \le \left(\frac{p+1}{b-a}\int_a^b [f(x)]^p \,\mathrm{d}x\right)^{1/p}.$$

Source: [PPT, p. 216].

1.22 Notes

Some of the preliminary material in this chapter can be found in [naylor,]. A related treatment of mathematical preliminaries is given in [robbin,]. An extensive introduction to logic and mathematical fundamentals is given in [blochbook,]. In [blochbook,], the notation "$A \to B$" denotes an implication, which is called a *disjunction*, while "$A \Longrightarrow B$" denotes a tautology.

The "truth" operator is represented by square brackets in [GKP,]. See [knuthnotes,].

Multisets are discussed in [blizard,grattan,singhsingh,wildbergermultisets,].

Partially ordered sets are considered in [schroderbook,trotter,]. Lattices are discussed in [blochbook,]. For a pair of elements x, y, $\mathrm{glb}(\{x, y\})$ is alternatively written as $x \wedge y$, where "$\wedge$" is the *meet* operator. Similarly, $\mathrm{lub}(\{x, y\})$ is alternatively written as $x \vee y$, where "$\vee$" is the *join* operator.

A directed graph is also called a *digraph*. A directionally connected graph is traditionally called *strongly connected* [westgraph, p. 56].

Alternative terminology for "one-to-one" and "onto" is *injective* and *surjective*, respectively, while a function that is injective and surjective is *bijective*.

Subtle aspects of compositions of complex functions are discussed in [boasmoc,].