On Maxitive Image Processing

Olivier Strauss, Kevin Loquin, Florentin Kucharczak

Abstract Digital image processing has become the most common form of image processing. Many transformations can be achieved by very simple and versatile algorithms such as contrast enhancing, restoration, color correction, etc. However, a wide branch of image processing algorithms makes an extensive use of spatial transformations that are only defined in the analog domain such as rotation, translation, zoom, anamorphosis, homography, distortion, derivation, etc. Designing a digital image processing algorithm that mimics a spatial transformation is usually achieved by using the so-called *kernel based approach*. This approach involves two kernels to ensure the continuous to discrete interplay: the sampling kernel and the reconstruction kernel, whose choice is highly arbitrarily made. The maxitive kernel based approach can be seen as an extension of the conventional kernel based approach that reduces the impact of such an arbitrary choice. It consists in replacing at least one of the kernels by a normalized fuzzy subset of the image plane. When considering digital image spatial transformation, this replacement leads to compute the convex set of all the images that would have been obtained by using a (continuous convex set) of conventional kernels. Using this set induces a kind of robustness that can reduce the risk of false interpretation. Medical imaging for example would be a kind of applications that could benefit of such an approach.

Olivier Strauss
LIRMM – Université de Montpellier, Montpellier, France, e-mail: olivier.strauss@lirmm.fr

Kevin Loquin
DMS Imaging, Montpellier, France, e-mail: kloquin@dms-imaging.com

Florentin Kucharczak
LIRMM – Université de Montpellier, Montpellier, France, e-mail: florentin.kucharczak@lirmm.fr

1 Introduction

In digital image processing, fuzzy subsets have been used from its very first introduction for representing image information at different levels (see e.g. [2] for a nice overview on fuzzy set based image processing). Most research work has rather considered high level image processing [16]. The maxitive approach we present in this paper belongs to the class of low-level processing where a membership function is used to represent a local neighborhood around each point of the image plane.

Digital image processing refers to the set of algorithms used to transform, filter, enhance, modify, analyze, distort, fuse, etc., digital images. Many of these algorithms are designed to mimic an underlying physical operation defined in the continuous illumination domain and formerly achieved via optical or electronic filters or through manipulations including painting, cutting, moving or pasting of image patches. Spatial domain operations like derivation [23], morphing, filtering, geometric and perspective transformations [12], super-resolution [17], etc. are usually derived by using a kernel based approach [25]. Among all spatial domain operations, we are interested here by geometric transformations like affine and projective transformations or, more generally, diffeomorphisms or homeomorphisms. In kernel-based approaches, the choice of a particular kernel shape or spread is usually prompted more by practical aspects than by any theoretical purpose. Unfortunately, this choice can highly impact the output of the obtained discrete operator. Figure 2 illustrates this fact by highlighting the noise introduced by a digital approximation of a continuous rigid transformation operation. Two different interpolation methods have been used to rotate a detail of Figure 1 (see Figures 2.a,b). The obtained images are not identical as illustrated by enhancing their absolute difference (Figure 2.c).

This dependance is not a real problem when the considered operations are dedicated to artistic modifications of an image. Photographs have their own rule to choose among the three main interpolation methods (nearest neighbor, bilinear, bicubic), while some dedicated softwares have developed their own interpolation (or more generally reconstruction) method. It is more problematic when the information carried on by an image is quantitative, e.g. in medical applications where quantization is expected.

For example to study a lung tumor growth, the patient is subjected to hybrid PET-CT scan, where the CT[1] gives the anatomical structure information and PET[2] gives quantitative information about the tumor metabolism. After image acquisition, it is necessary to bring the image of either of one the modality w.r.t to other (PET w.r.t CT or vice versa). This operation involves geometrical transformations. If details in the images are comparable to the image resolution, the choice of registration algorithm can be critical.

[1] Computed Tomography scan is a computational method for reconstructing cross-sectional images from sets of X-ray measurements taken from different angles.

[2] Positron-Emission Tomography is a nuclear medicine functional imaging technique used to observe metabolic processes in the body.

A more careful approach would be to compute not a single transformed image but the set of all images that can be obtained by using different kernels. In medical diagnosis, this can lead to confirm a diagnosis (if all the images lead to the same diagnosis) or highlight the need for a complementary medical investigation (if different images lead to different diagnosis). This is what the maxitive approach proposes.

Maxitive image processing takes advantage of the obvious analogy between probability density functions (pdf) and positive kernels to extend the signal processing theory to the case where the modeling is imprecisely known [13]. Within this technique, the possibilistic interpretation [5] of fuzzy subsets is used to define *maxitive kernels* that can be seen as convex sets of conventional positive kernels. These convex sets aim at representing scant knowledge on the appropriate kernel to be used in a given application. Maxitive-based signal processing extensions lead to interval-valued signal that includes the set of all signals that would have been obtained by the corresponding conventional method using a positive kernel that belongs to the core of the maxitive kernel [18]. This approach has been extended into two dimensions and involved in image processing applications (see e.g. [14, 6, 20, 10]).

In this article, we propose a formalization of the maxitive approach for extending geometrical transformations in digital image processing, i.e. we show how this technique can be used to compute the convex set of all images that would have been obtained by considering a convex set of possible kernels when applying geometrical transformations to digital images.

Fig. 1 A digitalized version of an illustration of Ivan Bilibin for Fairytale of the Tsar Saltan (1905).

After this introduction, Section 2 provides some notations and some necessary background knowledge. In Section 3 we present the kernel-based method to design discrete operators mimicking continuous geometrical transformations. We then propose the maxitive approach as a simple extension of the former method. We propose to use this extension to design a rigid transformation. We then conclude this article.

2 Preliminary Considerations, Definitions and Notations

2.1 Notations

Let $\mathbb{R}$ be the real line and $\mathbb{IR}$ be the set of all intervals of $\mathbb{R}$. Let Ω be the image plane, i.e. a box of $\mathbb{R}^2$: $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1, \Omega_2 \subseteq \mathbb{R}$. Let $\mathscr{P}(\Omega)$ be the set of all Lebesgue measurable subsets of Ω. With N being a positive integer, we define Θ_N by $\Theta_N = \{1, \ldots, N\} \subset \mathbb{N}$. Let $\mathscr{P}(\Theta_N) = 2^{\Theta_N}$ be the power set of Θ_N.

2.2 Digital Images

A continuous (real) image is usually obtained by projecting on a plane, via an optical device, the light reflected by the objects placed in front of the camera. This projection is often referred to as *the illumination function*.

A digital image is a numeric representation of a continuous image. It is composed of a finite set of digital values called pixels, each value being associated to the measurement of the illumination function at a specific location on the continuous image plane.

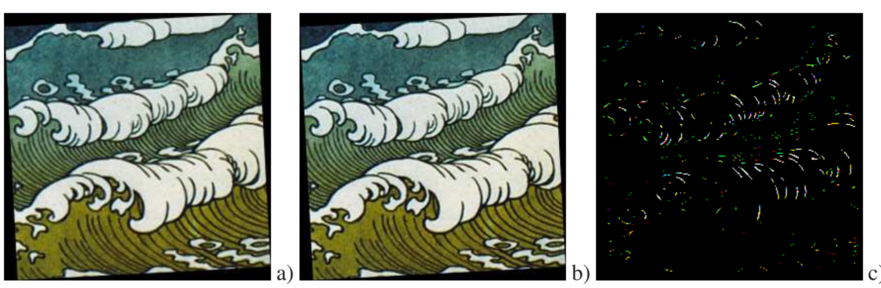

Fig. 2 Rotation of 3° of a detail of Figure 1 by using bilinear interpolation (a) bicubic interpolation (b) and an enhanced view of the difference between the two rotated images (c)

2.3 Capacities and Expectations

A capacity is a confidence measure that is more general than a probability measure [8]. It can be defined on both continuous and discrete domains. A capacity defined on a continuous reference set is called a *continuous capacity*, while a capacity defined on a discrete reference set is called a *discrete capacity*. Let Φ be either Ω or Θ_N.

Definition 1. *A (continuous or discrete) capacity ν is a set function $\nu : \mathscr{P}(\Phi) \to [0,1]$ such that $\nu(\varnothing) = 0$, $\nu(\Phi) = 1$ and $\forall A, B \in \mathscr{P}(\Phi)$, $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$.*

Given a capacity ν, its conjugate ν^c is defined as: $\nu^c(A) = 1 - \nu(A^c)$ for any subset $A \in \mathscr{P}(\Phi)$, with A^c being the complementary set of A in Φ. A capacity ν such that for all A, B in $\mathscr{P}(\Phi)$, $\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B)$ is said to be concave. Here we only consider this kind of capacity. The core of a concave capacity ν, denoted $\mathscr{M}(\nu)$, is the set of probabilities P on $\mathscr{P}(\Phi)$ such that $\nu(A) \geq P(A)$ for all subsets $A \in \mathscr{P}(\Phi)$. A probability measure is a capacity that equals its conjugate. Thus, if ν is a probability, $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$ [4].

The concept of expected value associated with a probability measure has been extended to concave capacities by means of a Choquet integral (see e.g. [19]). Let ν be a concave capacity defined on Φ and let $f : \Phi \to \mathbb{R}$ be a L_1 bounded function.

The (imprecise) expectation of f w.r.t. ν is the real interval $\overline{\underline{\mathbb{E}}}_\nu(f)$ defined by:

$$\overline{\underline{\mathbb{E}}}_\nu(f) = \left[\underline{\mathbb{E}}_\nu(f), \overline{\mathbb{E}}_\nu(f)\right] = \left[\check{\mathbb{C}}_{\nu^c}(f), \check{\mathbb{C}}_\nu(f)\right],$$

where $\check{\mathbb{C}}$ denotes the asymmetric Choquet integral (see e.g. [22]). One of the important properties of this extension is that $\overline{\underline{\mathbb{E}}}_\nu(f)$ is an interval that contains all of the $\mathbb{E}_P(f)$ with $P \in \mathscr{M}(\nu)$. Conversely, any value of this interval corresponds to an expected value of the form $\mathbb{E}_P(f)$ with $P \in \mathscr{M}(\nu)$ [4].

Remark 1. *Upper expectations and concave capacities coincide when considering the characteristic function. Let ν be a concave capacity, $\forall A \in \mathscr{P}(\Phi)$, $\nu(A) = \overline{\mathbb{E}}_\nu(\chi_A)$, with χ_A being the characteristic function of A.*

2.4 Summative and Maxitive Kernels

In discrete image processing, the role of a kernel is to define a weighted neighborhood of spatial locations in the image plane, with those weights being used in an aggregation process. Among different kernels, summative kernels play an important role since they define normalized positive weighted neighborhoods. They are intensively used to establish discrete operators defined in the continuous domain [13]. Let $N \in \mathbb{N}$ and Φ be either Ω or Θ_N.

A **summative kernel** is a positive function $\kappa : \Phi \longrightarrow \mathbb{R}^+$ complying with the summative property, i.e. $\int_\Omega \kappa(x)dx = 1$ (if $\Phi = \Omega$) or $\sum_{n \in \Theta_N} \kappa_n = 1$ (if $\Phi = \Theta_N$). Such a function defines a probability measure P_κ on Φ by: $\forall A \in \mathscr{P}(\Phi)$, $P_\kappa(A) = \int_A \kappa(x)\, dx$

(if $\Phi = \Omega$) or $P_\kappa(A) = \sum_{n\in A} \kappa_n$ (if $\Phi = \Theta_N$). $\mathscr{K}(\Phi)$ is the set of all summative kernels defined on Φ.

A **maxitive kernel** [13] is a function $\pi : \Phi \longrightarrow [0,1]$ complying with the maxitive property, i.e. $\sup_{x\in\Phi} \pi(x) = 1$. Such a function defines a concave capacity on Φ called a possibility measure Π_π by: $\Pi_\pi(A) = \sup_{x\in A} \pi(x)$. A maxitive kernel π defines a convex subset of $\mathscr{K}(\Phi)$ as follows:
$\mathscr{M}(\pi) = \{\kappa \in \mathscr{K}(\Phi) | \, \forall A \in \mathscr{P}(\Phi),\, P_\kappa(A) \leq \Pi_\pi(A)\}$ called its *core*.

2.5 Crisp and Fuzzy Partitions

In image processing, partitioning is mandatory to define the relation between the continuous domain, where the illumination function is defined, and the discrete domain, where the measured illumination is depicted. Traditional image processing is based on crisp partition, while more advanced image processing has been based on fuzzy partition (see e.g. [6, 15]).

An image partition of Ω is a set of N subsets $\{C_n\}_{n\in\Theta_N}$ such that (i) $\forall(n,m) \in \Theta_N$, $C_n \cap C_m \neq \varnothing \Longleftrightarrow n = m$, (ii) $\forall\omega \in \Omega$, $\exists n \in \Theta_N$ such that $\omega \in C_n$.

An image partition is said to be uniform if it can be generated by a simple generic subset E: let χ_E be the characteristic function of E, $\forall n \in \Theta_N$, $\exists\omega_n \in \Omega$ such that $\forall\omega \in \Omega$, $\chi_{C_n}(\omega) = \chi_E(\omega - \omega_n)$.

A fuzzy image partition of Ω is a set of N fuzzy subsets $\{C_n\}_{n\in\Theta_N}$ such that $\forall\omega \in \Omega$: (i) $\sum_{n=1}^{N} \mu_{C_n}(\omega) = 1$ and (ii) μ_{C_n} is continuous, with μ_{C_n} being the membership function of C_n ($n \in \Theta_N$). A fuzzy partition is said to be uniform if it can be generated by a simple generic fuzzy subset E: let $\{\omega_n\}_{n\in\Theta_N}$ be a set of N regularly spaced locations of Ω, then C_n is generated by E i.e. $\forall\omega \in \Omega$, $\mu_{C_n}(\omega) = \mu_E(\omega_n - \omega)$. A fuzzy partition is said to be normalized if $\forall n \in \Theta_N$, $\exists\omega \in \Omega$ such that $\mu_{C_n}(\omega) = 1$ [21]. Usually, in image processing, partitions are uniform to comply with the geometry of the image sensors.

Fuzzy partitions are instrumental for performing reconstructions. Let $\{F_n\}_{n\in\Theta_N}$ be a discrete function, a reconstructed continuous function $\hat{F} : \Omega \to \mathbb{R}$ can be defined by $\forall\omega \in \Omega$, $\hat{F}(\omega) = \sum_{n\in\Theta_N} F_n \mu_{C_n}(\omega)$. The following definition will allow us to extend this instrumentality to link the continuous space Ω to the discrete space Θ_N.

Definition 2. *Let $A \subseteq \Theta_N$. We define Υ_A as being the membership function of $\bigcup_{n\in A} C_n$, where the union is defined by the Łukasiewicz T-conorm: $\forall\omega \in \Omega$,*
$\Upsilon_A(\omega) = min(1, \sum_{n\in A} \mu_{C_n}(\omega)) = \sum_{n\in A} \mu_{C_n}(\omega)$ due to the fact that $\sum_{n=1}^{N} \mu_{C_n}(\omega) = 1$.

Remark 2. *Note that a uniform crisp partition is a special case of fuzzy partition where the generic fuzzy subset E is a crisp subset.*

3 From Continuous to Digital Image Processing

3.1 Continuous Image / Digital Image

In the continuous domain, an image can be seen as a measurable physical illumination phenomenon, i.e. the projection, via an optical device, of the real-world light information in a particular direction. It is generally modeled by a bounded positive integrable function $\mathscr{I}$ defined on $\mathbb{R}^2$. More precisely $\mathscr{I}$ is a $L_1(\mathbb{R}^2)^+$ function defined on a compact subset Ω (e.g. a closed rectangle) of $\mathbb{R}^2$. This function is usually extended throughout the continuous domain $\mathbb{R}^2$ by assigning an arbitrary value (usually 0) to Ω^c (the complementary set of Ω in $\mathbb{R}^2$).

There are some optical systems that allow to perform image processing in the continuous domain. Yet, nowadays image processing is mainly performed on computer or smartphones, i.e. on image stored in computer memory as a discrete quantities. From a signal processing point of view, a sampled image can be considered as being obtained by measuring the continuous illumination function $\mathscr{I}$ defined on $\mathbb{R}^2$ projected by an optical device on a matrix of sensors called the retina (see Figure 3). The sensors are usually regularly spaced along each axis at a limited number N of locations, called the *sampling locations*. Those measurement values or locations are usually referred to as *pixel values or locations*. Let $\Theta_N = \{1,\dots,N\} \subset \mathbb{N}$ be the set of indices of the sampling locations and $\{\omega_n\}_{n\in\Theta_N}$ be the set of sampling locations also referred to as the *sampling grid*.

Ideally, each measure I_n can be modeled by an integral of the illumination function $\mathscr{I}$ in a crisp neighborhood around ω_n. Let $\phi^{\omega_n} \subset \mathbb{R}^2$ be this neighborhood, then the relation between I and $\mathscr{I}$ can be expressed by: $\forall n \in \Theta_N,\ I_n = \int_{\phi^{\omega_n}} \mathscr{I}(\omega)d\omega$. When $\chi_{\phi^{\omega_n}}$ is the characteristic function of the subset ϕ^{ω_n}, it can be rewritten as:

$$\forall n \in \Theta_N,\ I_n = \int_{\mathbb{R}^2} \mathscr{I}(\omega)\chi_{\phi^{\omega_n}}(\omega)d\omega. \tag{1}$$

Finally, the measured pixel values are quantized to obtain the digital image. What has to be kept in mind is that what we have at hand is not the image but discrete measures of it.

There are many operations for which accounting for the underlying continuous nature of the image is mandatory: derivation, morphing, filtering, geometric and perspective transformations, etc. For those operations, the aim is to define a discrete operator that can mimic the equivalent operation in the continuous domain. The idea is illustrated in Figure 4 when considering a rotation of a detail of the image depicted in Figure 1 around the optical axis. Let us consider the input discrete image as being obtained by sampling a continuous image. The image we would like to obtain by using the discrete rotation operator is the image that would have been obtained by rotating and then sampling the original continuous image. Such an operation is not possible due to the loss of information induced by the sampling. It has to be approximated in a way that preserves at best the original (discrete) information. For example, a particularly desirable property would be the reversibility of a digital operation. However, continuous based discrete operations always lead to information loss [3]. Therefore, the original information cannot be reconstructed from the processed image.

Different methods have been proposed in the relevant literature to counteract this information loss (see e.g. [9] for a lossless rotation that uses permutations of pixels values). The method we propose does not have this target but aims at quantifying the information loss induced by a geometrical transformation.

3.2 Kernel-based Image Processing

Kernel-based image processing, as illustrated in Figure 4, consists in defining discrete operations on digital images that are analog to operations defined on continuous images in the continuous domain. Kernels are used for defining weighted neighborhoods of a location in the image plane, aiming at reconstructing a continuous image from a discrete image, or sampling a continuous image to built a discrete image.

3.2.1 Kernel-based Image Sampling and Reconstruction

Let $\{\omega_n\}_{n\in\Theta_N}$ be the N sampling locations. In Section 3.1, sampling a continuous image has been very straightforwardly modeled by integrating the illumination function $\mathscr{I}$ in a crisp neighborhood around each sampling location ω_n (see Equation 1). This modeling supposes that the impulse response of the sensor is uniform. To account for a known non-uniform impulse response of the sensor, a more general

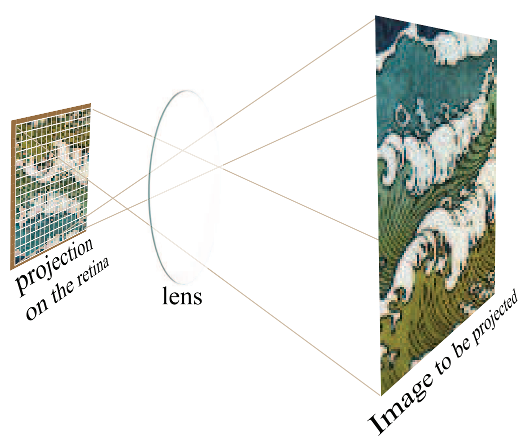

Fig. 3 Measure of a continuous image.

link between the sampled pixel values I and the continuous illumination $\mathscr{I}$ can be obtained by replacing the neighborhood function χ_ϕ by a sampling kernel κ [6]:

$$\forall n \in \Theta_N,\ I_n = \int_{\mathbb{R}^2} \mathscr{I}(\omega)\kappa(\omega - \omega_n)d\omega = \int_{\mathbb{R}^2} \mathscr{I}(\omega)\kappa^n(\omega)d\omega, \qquad (2)$$

with κ^n being the kernel κ translated in ω_n.

A certain consistency has to be kept between the continuous and the discrete domain. The range of the real measure of the illumination value being generally unknown, we can suppose without loss of generality that the correspondance between digital gray-scale and real value is linear, i.e. quantifying a real illumination value amounts to replacing this value by its nearest integer. That comes down to assuming the original illumination measurement scale to be the real valued counterpart of the digital available grayscale (e.g. if the digital grayscale is $\{0,\dots 255\}$ then the real illumination scale is $[0,255]$). The consistency of both continuous and discrete images can be expressed in that way: "if $\mathscr{I}$ is a constant image such that $\forall\omega \in \Omega$, $\mathscr{I}(\omega) = a$, then $\forall n \in \Theta_N$, $I_n = a$". Considering Equation 2, $a = \int_{\mathbb{R}^2} a\kappa(\omega - \omega_n)d\omega$, i.e. $\int_{\mathbb{R}^2} \kappa(\omega)d\omega = 1$, thus κ is a summative kernel.

Reconstruction can be thought of as the converse procedure of sampling. However, since sampling induces information loss, the recomposed image usually cannot be seen as a perfect reconstruction of the original continuous illumination $\mathscr{I}$, but rather as an estimate $\hat{\mathscr{I}}$ of the continuous function $\mathscr{I}$. This estimate is obtained by a finite weighted sum of the pixel values I_n $(n \in \Theta_N)$:

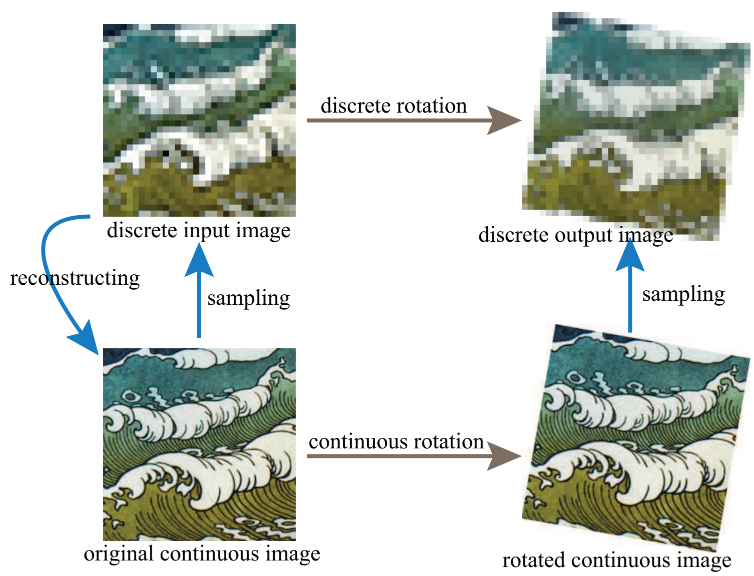

Fig. 4 How to go from continuous to discrete image processing?

$$\hat{\mathscr{I}}(\omega) = \sum_{n\in\Theta_N} I_n \eta(\omega_n - \omega) = \sum_{n\in\Theta_N} I_n \eta_n^{\omega}, \tag{3}$$

η being a continuous reconstruction kernel and η_n^{ω} being the discrete kernel induced by sampling η translated in ω on the sampling grid.

The same consistency between continuous and discrete domain evoked above implies that $\forall \omega \in \Omega$, $\sum_{n\in\Theta_N} \eta(\omega - \omega_n) = \sum_{n\in\Theta_N} \eta_n^{\omega} = 1$: sampling a reconstruction kernel translated at any location $\omega \in \Omega$ leads to a discrete summative kernel. Moreover, if the sampling is uniform, then η is even [24].

3.2.2 From Summative to Maxitive Kernel-based Image Processing

Let φ be a geometric transformation (i.e. a mapping form $\mathbb{R}^2$ to $\mathbb{R}^2$) transforming a continuous image $\mathscr{I}$ into another continuous image $\mathscr{I}'$ such that $\forall \omega \in \Omega$, $\mathscr{I}'(\omega) = \mathscr{I}(\varphi(\omega))$. Let $\{\omega_n\}_{n\in\Theta_N}$ be the sampling locations and $\{I_n\}_{n\in\Theta_N}$ be the pixel values of the original discrete image – supposedly obtained by sampling $\mathscr{I}$.

Kernel based image processing consists in deriving a discrete operation that is equivalent to sampling $\mathscr{I}'$ on the sampling grid.

Following Equation 2, sampling the continuous image $\mathscr{I}'$ leads to the sampled image I' such that:

$$\forall k \in \Theta_N, I'_k = \int_{\mathbb{R}^2} \mathscr{I}'(\omega)\kappa(\omega - \omega_k)d\omega, \tag{4}$$

κ being the sampling kernel. Considering $\mathscr{I}'(\omega) = \mathscr{I}(\varphi(\omega))$, Equation 4 becomes:

$$\forall k \in \Theta_N, I'_k = \int_{\mathbb{R}^2} \mathscr{I}(\varphi(\omega))\kappa(\omega - \omega_k)d\omega. \tag{5}$$

Now, let η be a reconstruction kernel, i.e. $\forall \omega \in \Omega$, $\mathscr{I}(\omega) = \sum_{n\in\Theta_N} I_n \eta(\omega_n - \omega)$, then $\forall k \in \Theta_N$:

$$\begin{aligned} I'_k &= \int_{\mathbb{R}^2} \sum_{n\in\Theta_N} I_n \eta(\varphi(\omega_n - \omega))\kappa(\omega - \omega_k)d\omega \\ &= \sum_{n\in\Theta_N} I_n \int_{\mathbb{R}^2} \eta(\varphi(\omega_n - \omega))\kappa(\omega - \omega_k)d\omega = \sum_{n\in\Theta_N} I_n \rho_n^k, \end{aligned} \tag{6}$$

with $\rho_n^k = \int_{\mathbb{R}^2} \eta(\varphi(\omega_n - \omega))\kappa(\omega - \omega_k)d\omega$.

Thus, estimating the discrete values of I' based on I comes down to defining, for each location k, a positive discrete kernel ρ^k by sampling the continuous kernel κ^k, defined by $\forall \omega \in \Omega$, $\kappa^k(\omega) = \kappa(\omega - \omega_k)$, with the continuous kernel η^{φ} defined by: $\forall \omega \in \mathbb{R}$, $\eta^{\varphi}(\omega) = \eta(\varphi(\omega))$. The obtained weights define discrete summative kernels (for each k) if $\sum_{n\in\Theta_N} \rho_n^k = 1$ which is not guaranteed for every transformation φ.

This is induced by the fact that φ is not an equiareal mapping. Within this approach, a way to cope with this problem is to normalize ρ by replacing ρ_n^k by $\frac{\rho_n^k}{\sum_{i\in\Theta_N}\rho_i^k}$.

Since we consider a uniform partition, η is a positive even kernel. Let C_n be the (possibly non normalized) fuzzy subset whose membership function is defined by: $\forall\omega\in\Omega$, $\mu_{C_n}(\omega)=\eta(\omega-\omega_n)=\eta(\omega_n-\omega)$. By construction $\forall\omega\in\Omega$, $\sum_{n\in\Theta_N}\mu_{C_n}(\omega)=1$. Therefore the subsets $\{C_n\}_{n\in\Theta_N}$ form a fuzzy partition. Now, let C_n^φ $(n\in\Theta_N)$ the fuzzy subsets defined by $\forall\omega\in\Omega$, $\mu_{C_n^\varphi}(\omega)=\eta(\varphi(\omega-\omega_n))=\eta(\varphi(\omega_n-\omega))$. Since φ is a mapping, the property $\forall\omega\in\Omega^\varphi$, $\sum_{n\in\Theta_N}\mu_{C_n^\varphi}(\omega)=1$ is kept (Ω^φ being defined by: $\Omega^\varphi=\{\omega|\exists u\in\Omega,\omega=\varphi(u)\}$).

Now, let Q_κ^k be the measure defined by: $\forall A\subseteq\Theta_N$, $Q_\kappa^k(A)=\sum_{n\in A}\rho_n^k$, the value I'_k can be seen as being the estimate of I w.r.t. Q_κ^k:

$$I'_k=\mathbb{E}_{Q_\kappa^k}(I). \tag{7}$$

The measure Q_κ^k can also be seen as an estimate:

$$\begin{aligned}\forall A\subseteq\Theta_N,\quad Q_\kappa^k(A)&=\sum_{n\in A}\int_{\mathbb{R}^2}\eta^\varphi(\omega_n-\omega)\kappa(\omega-\omega_k)d\omega\\&=\int_{\mathbb{R}^2}\kappa^k(\omega)\big(\sum_{n\in A}\mu_{C_n^\varphi}(\omega)\big)d\omega\\&=\int_{\mathbb{R}^2}\kappa^k(\omega)\Upsilon_A^\varphi(\omega)d\omega,\end{aligned} \tag{8}$$

Υ_A^φ being defined as in Definition 2: $\forall\omega\in\mathbb{R}$, $\Upsilon_A^\varphi(\omega)=\sum_{n\in A}\mu_{C_n^\varphi}(\omega)$. Thus Equation 8 becomes:

$$\forall A\subseteq\Theta_N, Q_\kappa^k(A)=\mathbb{E}_{P_{\kappa^k}}(\Upsilon_A), \tag{9}$$

P_{κ^k} being the set measure whose density is defined by the kernel κ^k.

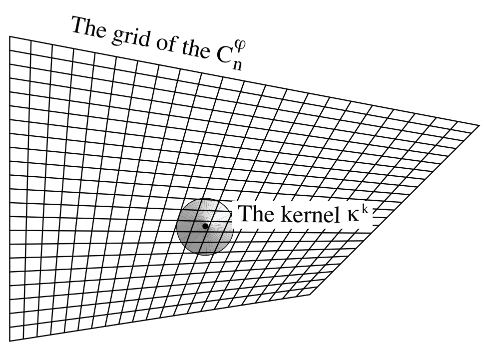

Fig. 5 Computing the ρ_n^k by sampling the kernel κ^k on the transformed interpolation partition when φ is a perspective transformation and $\eta=\eta^\square$.

As an example, let φ be a perspective transformation, then $\varphi(\omega) = \frac{A.\omega+\alpha}{B.\omega+\beta}$, where A and B are a 2×2 matrix and α and β translation vectors. Then, $\forall\omega\in\Omega$, $\mu_{C_n^\varphi}(\omega) = \eta(\varphi(\omega - \omega_n)) = \eta(\frac{A.\omega+\alpha-A.\omega_n}{B.\omega+\beta-B.\omega_n})$. This is illustrated in Figure 5 by considering the interpolation kernel being the nearest-neighbor kernel $\eta^\square$ defined by: $\forall\omega\in\mathbb{R}^2$, $\eta^\square(\omega)=1$, if $\omega\in\Xi$ and 0 else, with $\Xi =]-0.5,0.5]\times]-0.5,0.5]$.

3.2.3 Maxitive Kernel-based Image Processing

The maxitive extension of the kernel-based image processing simply consists in replacing the summative sampling kernel by a maxitive sampling kernel. This replacement aims at representing imprecise knowledge on the sampling kernel, either because the original sampling kernel is unknown, either because the kernel that would lead to the least distortion in the transformed image is unknown.

Let π be a maxitive kernel. Replacing κ by π in Equation 2 leads to an interval-valued discrete image $\underline{\overline{I}}$ such that $\forall n\in\Theta_N, \underline{\overline{I}}_n = \underline{\overline{\mathbb{E}}}_{\Pi_{\pi^k}}(\mathscr{I})$, Π_{π^k} being the possibility measure induced by π^k, the maxitive kernel π translated in ω_k. This interval valued discrete image represents the convex set of all the discrete images that could have been obtained by sampling $\mathscr{I}$ with a kernel $\kappa\in\mathscr{M}(\pi)$ (see [13]). Following [7], Expression 9 can be extended to define the capacity ν^k, for each location ω_k:

$$\forall A\subseteq\Theta_N, \nu_\pi^k(A) = \overline{\mathbb{E}}_{\Pi_{\pi^k}}(\Upsilon_A^\varphi). \tag{10}$$

By construction, ν_π^k is a concave capacity such that $\forall\kappa\in\mathscr{M}(\pi)$ and $\forall A\subseteq\Theta_N$, $Q_\kappa^k(A)\leq\nu_\pi^k(A)$.

The final step of this extension consists of estimating the interval valued image $\underline{\overline{I}}'$ by replacing, in Equation 7, the discrete additive set measure Q_κ^k by the discrete non additive set measure ν_π^k. This leads to:

$$\forall k\in\Theta_N, \underline{\overline{I_k}}' = \underline{\overline{\mathbb{E}}}_{\nu_\pi^k}(I). \tag{11}$$

The estimation operator $\underline{\overline{\mathbb{E}}}$ propagates imprecise knowledge of the sampling kernel to the interval-valued transformed image.

3.3 Example: Rigid Transformations

As it will be seen here, the computation can be very easy. But at first we should answer the question: “which maxitive kernel has to be chosen?”. As in image processing, mostly separable kernels or radial kernels are considered, a very interesting kernel would be a one that dominates any sampling kernel whose support is bounded. This problem has been addressed in [6] by considering separable kernels and separable estimations. As mentioned by the authors, such an approach is not suitable for transformations like rotations. A nice answer to this problem is proposed in [1] that gives different $2D$ maxitive kernels depending on which summative kernels have to be considered. For example, the continuous maxitive kernel $\breve{\pi}(\omega) = \max(0, 1 - \|\omega\|^2)$ dominates every bell-shaped radial continuous summative kernel whose support is included in $[-1, 1]$. This is very convenient because it can represent the fact that the support of the sampling kernel is at most the distance between two pixels but all what is known about the shape is that it is bell-shaped and radial. Note that the Dirac impulse is included in $\mathscr{M}(\breve{\pi})$, ensuring a kind of guaranteed preservation of the information carried on by the digital original image.

Thus now let us choose $\breve{\pi}$ to be the sampling maxitive kernel and $\eta^{\square}$ to be the interpolation kernel. Then the partition $\{C_n\}$ $(n \in \Theta_N)$ is a crisp partition, each C_n being a box centered in ω_n. Then Equation 10 leads to:

$$\forall A \subseteq \Theta_N, \nu_\pi^k(A) = \overline{\mathbb{E}}_{\Pi_{\pi^k}}(\Upsilon_A) = \sup_{\omega \in \bigcup_{n\in A} C_n^\varphi} \breve{\pi}(\omega - \omega_k).$$

In that case, since the C_n^φ form a crisp partition, ν_π^k is a possibility measure associated to the discrete possibility distribution γ^k defined by:

$$\forall n \in \Theta_N, \gamma_n^k = \sup_{\omega \in C_n^\varphi} \breve{\pi}(\omega - \omega_k).$$

φ being a rigid transform, it is defined by: $\forall \omega \in \mathbb{R}^2$, $\varphi(\omega) = R.\omega + \tau$, where R is a 2×2 orthonormal matrix and $\tau \in \mathbb{R}^2$ is a translation vector.

C_n^φ can be computed very easily by using geometric considerations. If the partition is not crisp, then ν_π^k is not a possibility measure but can be computed analytically (see [7] for an example of such a computation).

To illustrate this approach, we propose to take another look at the experiment illustrated in Figure 2. The same detail of Figure 1 has been rotated by $3°$ using the approach proposed in Section 3.2.3 by considering a crisp partition of the original image and the maxitive kernel $\breve{\pi}(\omega)$ described above. The discrete capacity ν_n^k is a possibility measure defined by the distribution computed by mean of Equation **??**. By constructions, the images of Figures 2.a and 2.b are included in the interval-valued image represented by its lower bound 6.a and its upper bound 6.b. Figure 1.c represents the imprecision of the interval valued image. It can also be noticed that the most important imprecision is concentrated near the edges of the drawing, i.e. the regions where two different interpolations yield to two different values.

4 Conclusion and Discussion

In many applications, images are subject to geometric distortions introduced by perspective, misalignment, optical aberrations, movement of the imaging sensor etc. which need to be corrected for further interpretation. Digital geometrical transformations are instrumental for reversing these distortions, aligning different images in a common frame or simply enable image contain analyse or interpretation. Many of those transformations aim a mimicking a physical operation defined in the continuous illumination domain. Kernel-based approach is a convenient way for designing digital transformations that may ensure a certain preservation of digital image topological and illumination properties. However, this method relies on modeling the interplay between continuous and discrete domain via two arbitrarily chosen kernels that model the sampling (to go from continuous to discrete domain) and the reconstruction (to go from discrete to continuous domain) operations. The arbitrariness of this choice can have severe consequences in applications where details have to be preserved that are comparable with image resolution.

Maxitive image processing can be an interesting solution to preserve the information carried on by the original image. It allows to compute de (convex) set of all images that would have been obtained by considering a (convex) set of kernels that could have been appropriate to compute this transformation. This computation can be achieved with a very low increase of the computation complexity (see e.g. [6]).

This article is focussed on modeling imprecise knowledge on the sampling kernel. Imprecise knowledge on the approximation kernel has to be taken into account in another way. Note however that, due to the interchangeable role of kernels in Expression 6, this modeling in fact addresses imprecise knowledge on the convolution of both reconstruction and sampling kernels (see e.g. [7]). This needs to be further investigated. Other problems have to be investigated including reversibility – how both illumination and topological information preserved when an image is subject to a transformation and then its inverse transformation? – and selection, in the convex set of obtained images, of a representative image to be presented to the expert (see e.g. [11]).

References

1. Akrout, H., Crouzet, J.F., Strauss, O.: Multidimensional possibility/probability domination for extending maxitive kernel based signal processing. Fuzzy Sets and Systems To appear

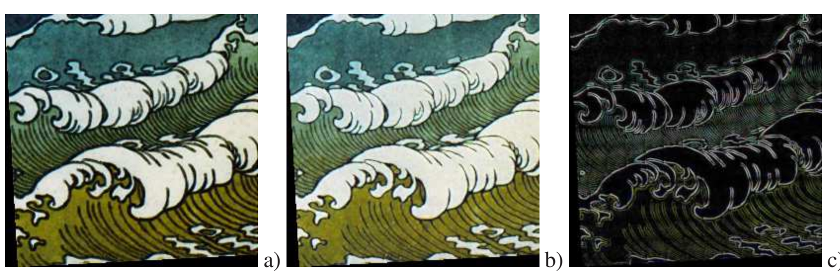

Fig. 6 Imprecise rotation by 3° of the same detail of Figure 1. The interval-valued image is represented by its lower bound (a) and its upper bound (b). (c) is an enhanced view of the difference between the upper and the lower image, i.e. the imprecision of the interval-valued images.

2. Bloch, I.: Fuzzy sets for image processing and understanding. Fuzzy Sets and Systems **281**, 280–291 (2015). DOI https://doi.org/10.1016/j.fss.2015.06.017. URL http://www.sciencedirect.com/science/article/pii/S0165011415003061. Special Issue Celebrating the 50th Anniversary of Fuzzy Sets
3. Danielsson, P.E., Hammerin, M.: High-accuracy rotation of images. CVGIP: Graph. Models Image Process. **54**(4), 340–344 (1992)
4. Denneberg, D.: Non-Additive Measure and Integral. Kluwer Academic Publishers (1994)
5. Dubois, D.: Possibility theory and statistical reasoning. Computational Statistics and Data Analysis **51**(1), 47 – 69 (2006). The Fuzzy Approach to Statistical Analysis
6. Graba, F., Comby, F., Strauss, O.: Non-additive imprecise image super-resolution in a semi-blind context. IEEE Transactions on Image Processing **26**(3), 1379–1392 (2017). DOI 10.1109/TIP.2016.2621414
7. Graba, F., Strauss, O.: An interval-valued inversion of the non-additive interval-valued f-transform: Use for upsampling a signal. Fuzzy Sets and Systems **288**, 26–45 (2016)
8. Grabisch, M., Marichal, J.L., Mesiar, R., Pap, E.: Aggregation Functions. Cambridge University Press (2009)
9. Komatsu, K., Sezaki, K.: Lossless rotation transformations with periodic structure. In: IEEE International Conference on Image Processing 2005, vol. 2, pp. II–273 (2005)
10. Kucharczak, F., Loquin, K., Buvat, I., Strauss, O., Mariano-Goulart, D.: Interval-based reconstruction for uncertainty quantification in pet. Physics in Medicine & Biology **63**(3), 035,014 (2018)
11. Kucharczak, F., Mory, C., Strauss, O., Comby, F., Mariano-Goulart, D.: Regularized selection: a new paradigm for inverse based regularized image reconstruction techniques. In: Proc. IEEE International Conference on Image Processing. Beijin, China (2017)
12. Lehmann, T., Gonner, C., Spitzer, K.: Survey: interpolation methods in medical image processing. IEEE Transactions on Medical Imaging **18**(11), 1049–1075 (1999)
13. Loquin, K., Strauss, O.: On the granularity of summative kernels. Fuzzy Sets and Systems **159**(15), 1952–1972 (2008)
14. Loquin, K., Strauss, O.: Linear filtering and mathematical morphology on an image: a bridge. In: Proc. IEEE International Conference on Image Processing, pp. 3965–3968. Le Caire, Egypt (2009)
15. Nie, Y., Barner, K.E.: The fuzzy transformation and its applications in image processing. IEEE Transactions on Image Processing **15**(4), 910–927 (2006). DOI 10.1109/TIP.2005.863111
16. Omhover, J.., Detyniecki, M., Rifqi, M., Bouchon-Meunier, B.: Ranking invariance between fuzzy similarity measures applied to image retrieval. In: 2004 IEEE International Conference on Fuzzy Systems (IEEE Cat. No.04CH37542), vol. 3, pp. 1367–1372 vol.3 (2004)
17. Park, S.C., Park, M.K., Kang, M.G.: Super-resolution image reconstruction: a technical overview. Signal Processing Magazine, IEEE **20**(3), 21–36 (2003)
18. Rico, A., Strauss, O.: Imprecise expectations for imprecise linear filtering. International Journal of Approximate Reasoning **51**(8), 933–947 (2010)
19. Rico, A., Strauss, O.: Imprecise expectations for imprecise linear filtering. International Journal of Approximate Reasoning **51**(8), 933–947 (2010)
20. Rico, A., Strauss, O., Mariano-Goulart, D.: Choquet integrals as projection operators for quantified tomographic reconstruction. Fuzzy Sets and Systems **160**(2), 198–211 (2009)
21. Ruspini, E.: New experimental results in fuzzy. Information Sciences **6**, 273284 (1973)
22. Schmeidler, D.: Integral representation without additivity. Processing of the American Mathematical Society **2** (1986)
23. Shen, J., Castan, S.: Towards the unification of band-limited derivative operators for edge detection. Signal Processing **31**(2), 103–119 (1993)
24. Strauss, O.: Non-additive interval-valued f-transform. Fuzzy sets and systems pp. 1–24 (2015)
25. Unser, M.: Splines: A perfect fit for signal and image processing. IEEE Signal Process. Mag. **16**, 22–38 (1999)