Calculus 1

November 21, 2025

Contents

An Introduction to Differentiable Functions

An introduction to what it means for a function to be differentiable.

Your precalculus classes were about functions, exponential, logarithmic, rational, trigonometric ... , all from a global perspective. Differential calculus is about these same functions, but from a *local* perspective. In one sense, the main idea of this class is to describe how *small* changes to the input of a function change the output. After all, *differential* has the same root as *difference* and *calculus* the same root as *calculate*. So *differential calculus* really means to calculate differences. Put more simply, this course is about subtraction.

One way to understand how small changes in the input of a function affect thes output is to zom in close to a point $(a, f(a))$ on the graph of a function $y = f(x)$. Do this and you'll most likely notice that the graph looks more and more like the graph of a linear function (ie. like a non-vertical line). If so, we say the function $y = f(x)$ is *differentiable* at $x = a$. And the *derivative* of the function, evaluated at $x = a$, written as

$$\left.\frac{dy}{dx}\right|_{x=a},$$

is just the slope of that line or the rate of change of the linear function.

When the independent variable is time, we can interpret the derivative as an instantaneous rate of change.

Example 1. The graph of the function

$$T = f(m)\,,\ 0 \leq m \leq 60,$$

expressing the temperature (in °C) of a cup of coffee in terms of the number of minutes past noon is shown below.

Desmos link: `https://www.desmos.com/calculator/fuftb4mq0k`

151: Cooling Coffee

(a) Zoom in close enough to the point P above to make the graph look like a straight line. Then approximate the slope of this line. Include units.

(b) Use the resut of part (a) to interpret the value and the meaning of the derivative

$$\left.\frac{dT}{dm}\right|_{T=40}.$$

Learning outcomes:
Author(s):

(c) Repeat parts (a) and (b) for other temperatures by draggind the slider T_0 in Line 2 of the worksheet above. Then see if you can find a relationship between the temperature T and the rate dT/dm at which the temperature of the coffee is changing.

You should have found that

$$\left.\frac{dT}{dm}\right|_{T=40} = -2^{\circ}\mathrm{C/min}.$$

This means that at the moment the temperature of the coffee is 40°C, the temperature is decreasing at the rate of 2°C/min. As long as we do not think too carefully about what exactly we mean by a moment in time, the meaning of the derivative is pretty clear.

But while this interpretation of the derivative as an instantaneous rate of change is often useful, it is fairly limited in scope. Perhaps the biggest problem is that the idea of slope does not extend to higher dimensions.

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$, like the function

$$(u, v) = f(x, y) = (0.2(x^2 - y^2), 0.4xy),$$

maps the two-dimensional plane to itself. You can get a feel for the gobal action of this function by dragging the slider u in Line 2 from $u = 0$ to $u = 1$. This transforms the checkerboard of horizontal and vertical lines into two families of intersecting parabolas and gives a global perspective of the function.

Desmos link: `https://www.desmos.com/calculator/h4sslgjjyx`

151: Complex Squaring Function

By focusing on the shaded square you can get a feeling for the local action of the function near point P. Here f maps this square to a square-like region with a different size and orientation. The image becomes more square-like by shrinking the side length s of the original square (do this by dradding Slider s in Line 4 toward $s = 0.1$.

The derivative of this function at the point P describes how the function acts near P. It can be described by just two numbers. One is a scaling (stretching) factor k that measures the ratio of the sides lengths of the original very small square and its square-like image,

$$k = \lim_{s\to 0} \frac{\text{side length of square-like image}}{s}.$$

The other number is an angle. It is the angle through which we would rotate the orginal very small square to make its sides parallel to its square-like image.

Question 1 (a) *Use the animation above to approximate the scaling factor k.*

(b) *Use the animation above to approximate the rotation angle θ.*

Fortuntately for us, because our class is about functions $f : \mathbb{R} \to \mathbb{R}$ that map the real line (or a subset thereof) to itself, rotations do not come into play. So the local behavior of such functions, at least where they are differentiable, can be described by just one number. This number, the derivative, measures the local stretching (or scaling) factor.

Stretching Factors

Interpreting the derivative as a local stretching/scaling factor.

1 Rubber Band Calculus

Think of a function and the first thing that might come to mind is its graph. But we can also think of a function's domain as an elastic band and the function as acting on that band by stretching or compressing it.

For example, imagine a thin elastic band of length 4 meters running along the horizontal L-axis from $L = 0$ meters to $L = 4$ meters. Now hold the left end fixed at $L = 0$ and stretch the band by moving right end an additional four meters to the right . Then the function

$$H = g(L) = 2L\,,\ 0 \le L \le 4,$$

describes this stretching action. It takes as an input the distance (in meters) of a point on the band from the origin ($L = 0$) and returns as an output the distance between the origin and the corresponding point on the stretched band. The exploration below shows this stretching action.

Exploration 2 *Drag the slider k in Line 2 below to illustrate the stretching action.*

Desmos link: `https://www.desmos.com/calculator/qejivz36ui`

151: Rubber Band 1

The *global stretching factor* for a linear function like the one above,

$$H = g(L) = 2L\,,\ 0 \le L \le 4,$$

is the slope of its graph. We can calculate this factor as an average stretching factor (ie. an average rate of change) between the points $L = a$ meters and $L = b$ meters from the origin. For the function $H = f(L) = 2L$, the stretching

Learning outcomes:
Author(s):

factor is

$$\frac{\Delta H}{\Delta L} = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{2(b-a)}{b-a}$$
$$= 2,$$

This tells us that any two points on the stretched band are twice as far apart as they were on the unstretched band.

Question 3 (a) *What are the units of the stretching factor?*

Free Response:

(b) *Find an expression for the inverse function*

$$L = f^{-1}(H).$$

Include a domain.

(c) *Interpret the inverse function as a deformation of a thin elastic band. What is the global stretching factor for this function?*

Free Response:

2 Non-Linear Stretching Functions

Example 2. Here's an example

$$H = f(L) = 10 - \sqrt{100 - L^2}\,,\ 0 \leq L \leq 10,$$

of a non-linear stretching function (where L and H are measured in meters as before). Like most functions in this class, it acts like a linear function near most points in its domain.

To stretch the elastic band in the demonstration below, drag the slider u in Line 2 from $u = 0$ to $u = 1$. Then zoom in close enough to the point $H = f(2)$ in the stretched band (highlighted in black) to make the stretching function look linear.

Desmos link: `https://www.desmos.com/calculator/nyd60dbezj`

151: Rubber Band Ladder 5

(a) Use the close-up view of the stretched band to approximate the stretching factor at the input $L = 2$.

(b) Drag the slider m in Line 4 to $m = 95$ and repeat part (a) to approximate the stretching factor at the input $L = 9.5$ meters.

(c) Parts of the elastic band get stretched, others compressed. Identify these.

We can also approximate the stretching factors at different inputs by zooming in near enough to the graph of the function $H = f(L)$ (shown below) to make the graph look like a line.

Desmos link: `https://www.desmos.com/calculator/jfwvbbrts1`

151: Rubber Band Ladder 6

(a) Zoom in close enough to the point $P(2, f(2))$ to make the graph above look like a straight line. Then click on the graph to get the coordinates of P and a point close to P. Use these to approximate the stretching factor at $L = 2$.

(b) Zoom back out and use the graph of the function $H = f(L)$ to determine the stretching factors at $L = 0$ and $L = 10$.

We can also approximate the stretching factors numerically using the expression

$$H = f(L) = 10 - \sqrt{100 - L^2}\,,\ 0 \le L \le 10,$$

for the stretching function.

For example, at the input $L = 2$, we'll to first compute the *average stretching factor* over the interval between lengths $L = 2$ and $L = v$. This average factor is

$$\begin{aligned} k(v) &= \frac{\Delta H}{\Delta L} \\ &= \frac{f(v) - f(2)}{v - 2} \\ &= \frac{10 - \sqrt{100 - v^2} - (10 - \sqrt{96})}{v - 2} \\ &= \frac{\sqrt{100 - v^2} - \sqrt{96}}{v - 2}\,,\ v \neq 2. \end{aligned}$$

(a) The idea to approximate the stretching factor at $L = 2$ is to see if these average stretching factors appear to approach some number as $v \to 2$ (ie. as v approaches 2). To do this we'll use the Table in Line 3 of the worksheet above. But first you'll need to input the correct expression for the average stretching function in Line 2. Then use the table to approximate the stretching factor at the input $L = 2$.

(b) Repeat part (a) to approximate the stretching factor at the input $L = 9.5$ and compare this with your earlier estimate.

(c) Repeat part (a) to approximate the stretching factor at the right end where $L = 10$. Anything different here?

(d) Use the geometry of the graph of the function $H = f(L)$ to compute the *exact* stretching factors at $L = 2, 9.5, 10$.

Example 3. Here's another example

$$H = f(L) = 4 - 4\cos(L/2)\,,\ 0 \leq L \leq 2\pi.$$

of a non-linear stretching function. We still need to talk about a *local* stretching factor, one that varies from point to point.

Exploration 4 *Drag the slider u in Line 2 below from $u = 0$ to $u = 1$ to illustrate the stretching action.*

Desmos link: `https://www.desmos.com/calculator/hqvyhormhf`

151: Rubber Band Cosine

(a) Use the worksheet above to approximate the local stretching factor at the point $L = \pi \sim 3.1$ meters from the origin on the unstretched band. Explain your reasoning.

(b) Drag slider m in Line 4 above to approximate the input(s) at which the local stretching factor is equal to 1. (ie. where the stretched band is neither in tension or in compression). Explain your reasoning.

(c) Use the graph of the stretching function below to approximate the local stretching factor at the input $L = \pi$. Start by dragging the slider u in Line 2 below from $u = 0$ to $u = 1$ to illustrate the stretching action and change the graph from $H = L$ to $H = f(L)$. Then zoom in close to the point $P(\pi, 4)$.

Desmos link: `https://www.desmos.com/calculator/aczfty35qj`

151: Rubber Band Cosine 3

(d) Finally, approximate the local stretching factor at $L = \pi$ numerically by finding entering an expression for the average stretching factor $k(v)$ between inputs $L = 2$ and $L = v$ in Line 3 of the worksheet above. Then use the table in Line 4.

3 A Hanging Slinky

Example 4. A hanging slinky stretched under its own weight gives a example of a stretching function.

Desmos link: https://www.desmos.com/calculator/zqjjgael5j

151: Slinky Photo

We can model the stretching of an ideal spring stretched under its own weight with the function

$$H = f(L) = L + \frac{g\rho}{2kL_0}L^2\,,\ 0 \leq L \leq L_0,$$

where

- L_0 is the length (in meters) of the relaxed (unstretched) spring,
- g is the magnitude of the gravitational acceleration (measured in meters/sec/sec),
- ρ is the linear density (in kg/meter) of the spring,
- k (measured in Netwons/meter) is the spring constant,
- L is the distance of a point on the relaxed spring from the spring's bottom end, and
- $H = f(L)$ is the distance from the corresponding point on the stretched spring to the spring's bottom end.

Free Response: Use the above information to check that the above expression for $H = f(L)$ has the correct units. You will need to know that Newtons (a measure of force) have units $\mathrm{kg} \cdot \mathrm{m/sec}^2$.

Drag the slider g in Line 5 of the worksheet below from $g = 0$ to $g = g_0 = 0.5\mathrm{m/sec}^2$ to turn on the gravitational field and stretch the slinky.

Desmos link: https://www.desmos.com/calculator/vjjibjkdrz

151: Slinky 3

We'll now work with the stretching function in the workhseet above,

$$H = f(L) = L + \frac{1}{4}L^2\,,\ 0 \leq L \leq 4.$$

(a) Use the pictures of the springs at the left of the graph to estimate the local stretching factor

$$\left.\frac{dH}{dL}\right|_{L=3}$$

at the point in the relaxed spring $L = 3$ meters from its bottom end.

(b) Find an expression for the average stretching factor between inputs $L = 3$ and $L = v$ meters. This factor is

$$\begin{aligned} a = m(v) &= \frac{\Delta H}{\Delta L} \\ &= \frac{f(v) - f(3)}{v - 3} \\ &= \frac{v^2 + 4v - 21}{4(v - 3)} \end{aligned}$$

(c) Enter this expression in Line 1 of the worksheet below. Then

(i) Use the table in Line 2 to guess the exact value of the local stretching factor at $L = 3$. Compare this with your estimate.

(ii) Drag the slider v in Line 11 and use the output in Line 12 ($m(v)$) to guess the exact value of the stretching factor at $L = 3$.

(iii) Describe how the line PQ is related to the average stretching factor $a = m(v)$.

Desmos link: `https://www.desmos.com/calculator/zsbupxubm6`

151: Slinky 4

(d) Now we'll compute the local stretching factor at $L = 3$ algebraically. This factor is the derivative

$$\left.\frac{dH}{dL}\right|_{L=3}$$

of H with respect to L evaluated at $L = 3$. It is equal to the limit, as v approaches 3, of the average stretching factor $m(v)$ between lengths $L = 3$ and $L = v$. Here's the computation with some algebra left for you near the end:

$$\begin{aligned}
\left.\frac{dH}{dL}\right|_{L=3} &= \lim_{\Delta L \to 0} \frac{\Delta H}{\Delta L} \\
&= \lim_{v \to 3} \frac{f(v) - f(3)}{v - 3} \\
&= \lim_{v \to 3} \frac{v^2 + 4v - 21}{4(v-3)} \\
&= \lim_{v \to 3} \frac{(v+7)(v-3)}{4(v-3)} \\
&= \lim_{v \to 3} \frac{v+7}{4} \\
&= \frac{3+7}{4} \\
&= 2.5.
\end{aligned}$$

(e) So our conclusion is that the local stretching factor of the stretching function

$$H = f(L) = L + \frac{1}{4}L^2\,,\ 0 \le L \le 4,$$

at $L = 3$ is equal to 2.5 meters/meter. This is a dimensionless scaling factor. It means that a small interval of length $\Delta L \sim 0$ meters around the point $L = 3$ on the relaxed spring gets stretched to a small interval approximately 2.5 times as long. Symbolically, we can write this as

$$\Delta H \sim 2.5 \Delta L,$$

where

$$\Delta H = f(L) - f(3)$$

and

$$\Delta L = L - 3 \sim 0.$$

(f) What about the inverse function $L = f^{-1}(H)$ that relaxes the stretched band? What is the local scaling factor

$$\left.\frac{dL}{dH}\right|_{L=3}$$

of the inverse at $L = 3$? Explain your reasoning.

(g) Use the algebra of limits to find the local stretching factor

$$\left.\frac{dH}{dL}\right|_{L=b}$$

of this same function at the point $L = b$ meters from the bottom end of the relaxed spring by evaluating the limit (as $v \to b$) of the average stretching factor between $L = v$ and $L = b$. Then check your result by substituting $b = 3$. Check also with the above worksheets at a few other values of b.

4 Gas Consumption

Most of our applications will not be about springs or elastic bands. Nevertheless, we can still think of the derivative as a local stretching factor. But iit might be better to talk about a local *scaling factor* instead. The derivative scales (ie. multiplies) a small change in the input to give an approximate corresponding change in the output. Here's an example.

Example 5. The function

$$G = f(s) = \frac{1}{2000}(s-100)^2\,,\ 0 \le s \le 80,$$

expresses the number of gallons of gas in a car in terms of the trip odometer reading in miles. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/wdnmaszvgb`

151: Gas Consumption A

(a) What are the units of the factor 1/2000 in the function f? How do you know?

(b) Find the average rate (in gal/mile) at which the car burned gas during the trip. Then find the average gas mileage (in miles/gal) over the entire trip.

(c) Use the graph above to approximate the rate (in gal/mile) at which the car burns gas when the odometer reads 60 miles. Then approximate the gas mileage (in miles/gal) at this odometer reading.

(d) Enter the correct expression in Line 5 of the worksheet for the average rate at which the car burns gas between odometer readings $s = v$ and $s = 60$ miles. Then use the table to guess the exact rate at which the car burns gas at the odometer reading $s = 60$ miles.

(e) Use the algebra of limits to compute the exact rate (in gal/mile) at which the car burns gas when the odometer reads 60 miles.

(f) Approximate the change

$$\Delta G = f(v) - f(60)$$

in the number of gallons of gas in the tank from odometer reading $s = 60$ miles to odometer reading $s = v$ miles in terms of the change

$$\Delta v = v - 60$$

in the odometer reading. Assume $\Delta v \sim 0$.

(g) Use the algebra of limits as above to compute the exact rate (in gal/mile) at which the car burns gas at the odometer reading $s = u$ miles.

5 A Falling Slinky

Falling Slinky

6 Atmospheric Pressure

Atmospheric Pressure

Average Rates of Change

Graphing average rate of change functions.

With time as the independent variable, the derivative is most naturally interpreted as an instantaneous rate of change. Here's an example.

Question 5 *Due to a late-season frost in eastern Washington, the price of Cosmic Crisp apples is rising precipitously.*

The function

$$N = f(t)\,,\, 0 \le t \le 8,$$

expresses the number of pounds of apples you can buy with \$36 *in terms of the number of hours past 9am on April 23, 2005.*

The graph of the function $N = f(t)$ is shown below.

Desmos link: `https://www.desmos.com/calculator/njanpkrqex`

151: Apples Drought

(a) *Let $b \in [0, 8]$ be a constant. Interpret the meaning of the function*

$$m(v) = \frac{f(v) - f(b)}{v - b}$$

in the context of this scenario. Include units. State the domain of the function as well.

(b) *Use the graph of the function $N = f(t)$ above and the slider v on Line 4 to to sketch a rough graph of the function $r = m(v)$ when $b = 5$. Label the axes with the appropriate variable names and units.*

(c) *Activate the folder (avg. rate of change function) in Line 11 to check your graph of the function $r = m(v)$.*

(d) *Click on the appopriate points of the curve $r = m(v)$ to find the average rate of change (with respect to time) in the number of pounds of apples you can buy with* \$36 *between*

(i) *11:00 am and 2:00 pm*

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Author(s):

(ii) *1:00 pm and 2:00 pm*
Be sure to include units. Note that because of the different scale on the vertical axis for the function $r = m(t)$, you will need to divide the r-coordinates by 10.

(e) *Zoom in close enough near the missing point on the curve $r = m(v)$ to approximate the rate of change at 2pm in the number of pounds of apples you can buy with \$36 at 2:00pm.*

(f) *Open the Table in Line 1 by clicking the* right arrow *at the left of the line. Use it to guess the rate of change (with respect to time) in the number of pounds of apples you can buy with \$36 at 2:00pm.*

(g) *Now suppose between 9am and 5pm the price of apples increases at a constant rate. Suppose also that the price is \$2/lb at 9am and \$4/lb at 2pm.*

(i) *Find an expression for the function $N = f(t)$.*
Hint: *First find a function*

$$P = g(t), 0 \le t \le 8,$$

that expresses the price (in dollars/pound) in terms of the number of hours past 9am.

(ii) *Find an expression for the function $r = m(v)$. Include the appropriate domain.*

Explanation. Working with the function

$$\begin{aligned} N &= f(t) \\ &= \frac{36}{2 + \frac{2}{5}t} \\ &= \frac{90}{5+t}, \ 0 \le t \le 8, \end{aligned}$$

we have

$$\begin{aligned} m(v) &= \frac{f(v) - f(5)}{v-5} \\ &= \frac{1}{v-5}\left(\frac{90}{5+v} - 9\right) \\ &= \frac{1}{v-5}\left(\frac{45-9v}{v+5}\right) \\ &= \frac{-9}{v+5} \text{ if } v \ne 5. \end{aligned}$$

(iii) *Use your expression for $m(v)$ and the algebra of limits to compute the rate of change (with respect to time) at 2pm in the number of pounds of apples you can buy with \$36.*

Explanation. The question is asking us to compute the derivative

$$\left.\frac{dN}{dt}\right|_{t=5} = \lim_{v\to 5} \frac{f(v) - f(5)}{v - 5}.$$

Using the previous part, we get

$$\begin{aligned}\left.\frac{dN}{dt}\right|_{t=5} &= \lim_{v\to 5} \frac{f(v) - f(5)}{v - 5} \\ &= \lim_{v\to 5} \frac{-9}{v+5} \\ &= \frac{-9}{5+5} \\ &= -0.9.\end{aligned}$$

Conclusion: At 2pm we can buy 9 pounds of apples with \$36. And at 2pm the number of pounds of apples we can buy with \$36 is decreasing at the rate of 0.9 lb/hour.

(iv) *Compare at 2pm the relative rate of change in the number of pounds of apples you can buy with* \$36 *and the relative rate of change in the price.*

Explanation. To compute the *relative* rate of change in the price at 2pm, we divide the absolute rate of change, $0.40(\$/\text{lb})/hr$, by the price $P = f(5) = \$4/\text{lb}$ at that time. This tells us the price is increasing at the relative rate of

$$\frac{1}{P} \cdot \left.\frac{dP}{dt}\right|_{t=5} = \frac{0.40(\$/\text{lb})/\text{hr}}{4(\$/\text{lb})} = 10\%/\text{hr}$$

at 2pm.

At the same time, the number of pounds of apples we can buy with \$36 is changing at the relative rate of

$$\frac{1}{N} \cdot \left.\frac{dN}{dt}\right|_{t=5} = \frac{-0.9 \text{ lb/hr}}{9 \text{ lb}} = -10\%/\text{hr}.$$

So at 2pm the number of pounds of apples we can buy with \$36 is decreasing at the relative rate of 10%/hr.

Free Response: *Notice anything?*

Question 6 *This question is a continuation of Question 1. We'll work with the same function*

$$N = f(t) = \frac{36}{2 + \frac{2}{5}t}\,,\; 0 \le t \le 8$$

and the expression

$$\frac{\Delta N}{\Delta t} = \frac{f(v) - f(b)}{v - b}$$

for the average rate of change of N (the number of pounds of apples we can buy with \$36) with respect to time over the time interval between $t = b$ and $t = v$ hours past 9am.

(a) *Use the algebra of limits and the average rate of change function above to compute the derivative*

$$\left.\frac{dN}{dt}\right|_{t=b}.$$

(b) *Use the result of part (a) to compare at 9am the relative rates of change in the price and the number of pounds of apples we can buy with \$36.*

(c) *Express the derivative from part (a) in terms of N and*

$$P = 2 + \frac{2}{5}t.$$

Use this equation to relate the relative rates of change in the price and the number of pounds of apples we can buy with \$36 at any instant.

Question 7 *We can often get more insight into a problem by taking away the numbers. Here we'll generalize the first two questions by supposing the (differentiable) function*

$$P = g(t), 0 \leq t \leq 8,$$

expresses the price (in dollars/pound) of Cosmic Crisp apples in terms of the number of hours past 9am. And we'll let

$$N = f(t), 0 \leq t \leq 8,$$

be the function that expresses the number of pounds of apples we can buy with \$100 dollars in terms of he number of hours past 9am.

(a) *Find an expression for the function $N = f(t)$ in terms of the function $g(t)$.*

(b) *Use your expression from part (a) to find an expression (in terms of the function g) for the average rate of change*

$$\frac{\Delta N}{\Delta t} = \frac{f(v) - f(b)}{v - b}.$$

(c) *Use your expression from part (b) to find an expression for the derivative*

$$\left.\frac{dN}{dt}\right|_{t=b}$$

in terms of the function $P = g(t)$ and its derivative dP/dt.

(d) *Express the derivative from part (c) in terms of N, P, and dP/dt.*

(e) *Use your equation from part (c) to compare the relative rates of change in the price and the number of pounds of apples we can buy with \$100 at any instant.*

Question 8 *The function*

$$G = f(s) = \frac{1}{2000}(s - 100)^2 \, , \; 0 \leq s \leq 80,$$

expresses the number of gallons of gas in a car in terms of the trip odometer reading in miles. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/igubjr4nw1`

151: Gas Consumption B

(a) *What are the units of the factor $1/2000$ in the function f? How do you know?*

(b) *Find the average gas mileage (in mile/gal) between odometer readings $s = 20$ and $s = 60$ miles.*

(c) *Zoom in on the appropriate point in the graph above to approximate the gas mileage (in miles/gal) at the instant the odometer reads 20 miles.*

(d) *Find an expression for the average gas mileage between odometer readings $s = b$ and $s = v$ miles. Simplify this expression.*

(e) *Use the previous part to find an expression for the gas mileage at the instant the odometer reads b miles.*

(f) *What is the gas mileage at the instant the odometer reads $s = 20$ miles.*

(g) *At what odometer reading is the car getting 20 miles/gal?*

Question 9 *The function*

$$g = f(v) \, , \; 10 \leq v \leq 55,$$

expresses the rate (in gal/hr) at which a car burns gas in terms of its speed (in miles/hour).

The graph of the function $g = f(v)$ is shown below.

Desmos link: `https://www.desmos.com/calculator/3ubcc1x8kh`

151: Rate of Gas Consumption

(a) *Let $b \in [10, 55]$ be a constant. Interpret the meaning of the function*

$$m(w) = \frac{f(w) - f(b)}{w - b}$$

in the context of this scenario. Include units. State the domain of the function as well.

(b) *Use the graph of the function $g = f(v)$ above and the slider w on Line 4 to to sketch a rough graph of the function $r = m(w)$ when $b = 20$. Label the axes with the appropriate variable names and units.*

(c) *Activate the folder (avg. rate of change function) in Line 12 to check your graph of the function $r = m(v)$.*

(d) *Click on the appopriate points of the curve $r = m(v)$ to find the average rate of change (with respect to speed) in the fuel consumption rate between speeds of*

   (i) *15 miles/hour and 20 miles/hour*

   (ii) *20 miles/hour and 30 miles/hour*
   Be sure to include units. Note that because of the different scale on the vertical axis for the function $r = m(v)$, you will need to divide the r-coordinates by 10.

(e) *Zoom in close enough near the missing point on the curve $r = m(v)$ to approximate the rate of change (with respect to speed) in the rate of fuel consumption at a speed of* 20 *miles/hr.*

(f) *Open the Table in Line 1 by clicking the* right arrow *at the left of the line. Use it to guess the rate of change (with respect to speed) in the rate of fuel consumption at a speed of* 20 *miles/hr.*

(g) *Now suppose between speeds of* 10 *miles/hour and* 55 *miles/hour that the car's gas mileage is a linear function of its speed. Suppose also that the car gets* 25 *miles/gal at at a speed of* 20 *miles/hour and* 35 *miles/gal at a speed of* 40 *miles/hour.*

   (i) *Find an expression $G = k(v)$ for the function that expresses the car's gas mileage (in miles/gal) in terms of its speed (in miles/hr).*

   (ii) *Find an expression for the function $g = f(v)$ that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour). Note: This is not a linear function.*

(iii) *Find an expression for the function $r = m(v)$. Include the appropriate domain.*

(iv) *Use your expression for $m(v)$ and the algebra of limits to compute the rate of change (with respect to speed) in the rate of fuel consumption at a speed of 20 miles/hr.*

(v) *Use the language of small changes to interpret the meaning of the derivative*

$$\left.\frac{dg}{dv}\right|_{v=20}.$$

(vi) *Simplify the units of the derivative above. What do these units suggest about its meaning? Is this correct? Why or why not?*

Question 10 *The function*

$$h = f(t)\,,\ 0 \leq t \leq 2.2,$$

expresses the height of a balloon (in hundreds of feet) in terms of the number of minutes past noon.

The graph of the function $h = f(t)$ is shown below.

Desmos link: `https://www.desmos.com/calculator/yd4xm6x6ub`

151: Balloon

(a) *Let $b \in [0, 2.2]$ be a constant. Interpret the meaning of the function*

$$m(v) = \frac{f(v) - f(b)}{v - b}$$

in the context of this scenario. Include units. State the domain of the function as well.

(b) *Use the graph of the function $h = f(t)$ above and the slider u to to sketch a rough graph of the function $r = m(v)$ when $b = 2$. Label the axes with the appropriate variable names and units.*

(c) *Activate the folder (avg. rate of change function) in Line 11 to check your graph of the function $r = m(v)$.*

(d) *Click on the appopriate points of the curve $r = m(v)$ to find the balloon's average rate of ascent between*

(i) *12:00pm and 12:02pm*

(ii) *12:01pm and 12:02pm*

(e) *Zoom in close enough near the missing point on the curve $r = m(v)$ to approximate the balloon's rate of ascent at 12:02pm.*

(f) *Open the Table in Line 1 by clicking the* right arrow *at the left of the line. Use it to guess the instantaneous rate of ascent at 12:02pm.*

(g) *Now suppose*

$$f(t) = -2t^3 + 7t^2 - 8t + 8\,,\ 0 \le t \le 2.2.$$

(i) *Find a simplified expression for the function $r = m(v)$. Include the appropriate domain.*

(ii) *Use your simplified expression to compute the balloon's rate of ascent at 12:02pm.*

Question 11 (a) *Drag the slider u from $u = 0$ to $u = 1$ in Line 2 below. Then use the graph/animation of the stretching function*

$$H = f(L) = 0.1 + 0.5L + 0.1(L-1)^3\,,\ 0 \le L \le 3,$$

to approximate the local stretching factor at $L = 2$. Include units.

Desmos link: `https://www.desmos.com/calculator/boubpczsne`

151: Rubber Band 12

(b) *Use the graph of the same stretching function $H = f(L)$ show again below to graph (by hand) the function*

$$r = m(v)$$

that gives the average stretching factor of the portion of the band between lengths $L = v$ and *$L = 2$ meters. Label the axes with the appropriate variable names and units. Then activate the average rate of change folder in Line 11 to see how you did.*

Desmos link: `https://www.desmos.com/calculator/vss7ofbwii`

151: Rubber Band 12

(c) *Use the graph of the average rate of change function and the Table in Line 1 to guess the local stretching factor at $L = 2$.*

(d) *Use the algebra of limits to compute the local stretching factor at $L = 2$.*

The Rythm of a Function

Listening to the beat of a function.

Have you ever sat by the train tracks listening to the trains roll by? If so, and you're near a joint in a rather old track, you'll hear a regular beat of the train as it passes by. The video below (especially recommended for engineers), explains how modern technonlogy gets around these types of joints.

Expansion Joints

Exploration 12 *But we can use this idea, the old-fashioned clickity-clack rythm of a train, to listen to the beat of a function.*

We'll imagine shrinking our train to a point that moves upward along a straight track (the s-axis in the demonstration below). And we'll start listening to our train at time $t = 0$ seconds as it passes the point $s = 0$. For the demonstration below, the joints are spaced just 0.1 meters apart.

Desmos link: `https://www.desmos.com/calculator/qjsfuvhsv9`

151: Sound Squaring Function 3

Follow the directions in the exploration above and then do the following.

(a) *Describe how the train's speed varies.*

(b) *Sketch a rough graph of the function $v = f(t)$, $0 \leq t \leq 1$, that expresses the train's speed (in meters/sec) as fuction of the number of seconds since the train passed the point $y = 0$. Label the axes with the correct variable names and units.*

(c) *Use your graph from part (b) to sketch a graph of the function $s = g(t)$, that expresses the train's distance (in meters) from the point $s = 0$ as a function of time (measured in seconds as in part (b)).*

(d) *Finally, use your graphs from part (b) and (c) to sketch a graph of the function $v = h(s)$, that expresses the train's speed (in meters/sec) in terms of its distance (in meters) from $s = 0$.*

Desmos link: `https://www.desmos.com/calculator/xaajoakuug`

Learning outcomes:
Author(s):

151: Sound Squaring Function 5B

Exploration 13 *Desmos link:* `https://www.desmos.com/calculator/xyfteumpre`

151: Sound Function Cube

Exploration 14 *Desmos link:* `https://www.desmos.com/calculator/uyym1ow6zq`

151: Sound Function Exp2

Exploration 15 *Can you hear an inflection point?*

Desmos link: `https://www.desmos.com/calculator/oojiur3a93`

151: Sound Function Oscillating

Exploration 16 *Desmos link:* `https://www.desmos.com/calculator/xwj3qnoqia`

151: Sound Function Root

Differentiable Functions

An introduction to what it means for a function to be differentiable.

1 Differentiability

Zoom in closer and closer near a point $(a, f(a))$ on the graph of a common function f and you'll most likely notice that the graph looks more and more like the graph of a linear function (ie. like a non-vertical line). If so, we say the function $y = f(x)$ is *differentiable* at $x = a$. And the *derivative* of the function, evaluated at $x = a$, written as

$$\left.\frac{dy}{dx}\right|_{x=a},$$

is just the slope of that line or the rate of change of the linear function.

When the independent variable is time, we can interpret the derivative as an instantaneous rate of change.

Example 6. The graph of the function

$$T = f(m)\,,\ 0 \leq m \leq 60,$$

expressing the temperature (in °C) of a cup of coffee in terms of the number of minutes past noon is shown below.

Desmos link: `https://www.desmos.com/calculator/fuftb4mq0k`

151: Cooling Coffee

(a) Zoom in close enough to the point P above to make the graph look like a straight line. Then approximate the slope of this line. Include units.

(b) Use the resut of part (a) to interpret the value and the meaning of the derivative

$$\left.\frac{dT}{dm}\right|_{T=40}.$$

(c) Repeat parts (a) and (b) for other temperatures by draggind the slider T_0 in Line 2 of the worksheet above. Then see if you can find a relationship between the temperature T and the rate dT/dm at which the temperature of the coffee is changing.

Learning outcomes:
Author(s):

We'll get back to applications a bit later. The next few examples focus on the geometric meaning of differentiability.

Exploration 17 *Use the graph of the function $y = f(x)$ below to answer the following questions.*

Desmos link: `https://www.desmos.com/calculator/ojdj4r3r9v`

151: Diff0

(a) *Zoom in toward the point A to determine if the function $y = f(x)$ is differentiable at $x = 1$. If so, approximate the derivative*

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{d}{dx}\left(f(x)\right)\Big|_{x=1}.$$

The derivative is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3.$$

(b) *Zoom in toward the point B to determine if the function $y = f(x)$ is differentiable at $x = 0$. If so, approximate the derivative*

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{d}{dx}\left(f(x)\right)\Big|_{x=0}.$$

The derivative is

$$\left.\frac{dy}{dx}\right|_{x=0} = -2.$$

Exploration 18 *Use the graph of the function*

$$y = f(x) = |x|$$

below to evaluate the following derivatives if possible. Explain your reasoning.

Desmos link: `https://www.desmos.com/calculator/us2fruzbra`

151: Diff1

(a)

$$\left.\frac{d(|x|)}{dx}\right|_{x=2} = 1$$

(b)
$$\left.\frac{d(|x|)}{dx}\right|_{x=0} =$$

(c)
$$\left.\frac{d(|x|)}{dx}\right|_{x=-3} = -1$$

Exploration 19 *Use the graph of the function $y = f(x)$ below to determine if the function is differentiable at $x = 0$. Explain your reasoning.*

Desmos link: `https://www.desmos.com/calculator/ov8qt938ot`

151: Diff2

Exploration 20 (a) *Use the graph of the function $y = f(x)$ below to determine if the function is differentiable at $x = 0.2$. If so, approximate the derivative*
$$\left.\frac{dy}{dx}\right|_{x=0.2} = \left.\frac{d}{dx}\left(f(x)\right)\right|_{x=0.2}.$$
Explain your reasoning.

(b) *Use the graph of the function $y = f(x)$ below to determine if the function is differentiable at $x = 0$. If so, approximate the derivative*
$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{d}{dx}\left(f(x)\right)\right|_{x=0}.$$
Explain your reasoning.

Desmos link: `https://www.desmos.com/calculator/tvwtbx9hco`

151: Not Differentiable

Exploration 21 *Use the graph of $y = f(x)$ below to determine whether each of the following derivatives are negative, positive, or zero. Explain your reasoning.*

(a)
$$\left.\frac{d}{dx}\left(f(x)\right)\right|_{x=0.5}$$

(b)
$$\frac{d}{dx}\left(f(x)\right)\Big|_{x=1}$$

(c)
$$\frac{d}{dx}\left(f(x)\right)\Big|_{x=1.5}$$

(d)
$$\frac{d}{dx}\left(f(x)\right)\Big|_{x=2.29}$$

Desmos link: `https://www.desmos.com/calculator/ks2yui6ofs`

151:Diff 7

Question 22 (a) *Summarize your understanding of the main ideas of this section.*

(b) *What questions do you have about this section?*

Free Response:

2 Using Algebra to Compute Derivatives

Exploration 23 *The aim of this problem is to use numerical methods to approximate and algebra to evaluate the derivative*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1}.$$

(a) *First use the graph of the function $y = f(x) = x^3$ below to approximate or guess the value of the above derivative by zooming in on the appropriate point.*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1} = 3.$$

Desmos link: `https://www.desmos.com/calculator/rto22qzlvm`

151: Cubing Function

(b) *The idea to compute the derivative algebraically is this: Fix the point $P(1,1)$ on the graph of $y = f(x) = x^3$. Then choose a variable point Q on the graph, different from P, with coordinates (v, v^3). When Q is sufficiently close to P, the line PQ approximates the curve $y = x^3$ near $P(1,1)$ and the slope of this line approximates the derivative above.*

(i) *Our first step is to find an expression for the slope of line PQ as a function of v. The slope is*

$$\begin{aligned} m(v) &= \frac{\Delta y}{\Delta v} \\ &= \frac{f(v) - f(1)}{v - 1} \\ &= \frac{v^3 - 1}{v - 1} \end{aligned}$$

all assuming $v \neq 1$.

(ii) *Now we'll use the slope (or average rate of change) function $m(x)$ to create a table of slopes for the lines PQ. Reveal the contents of the Table folder in Line 1 of the worksheet above by clicking the Right Arrow just to the left of "Table".*

i. *Add a few entries to the table to get* better *approximations to the above derivative.*

ii. *The slopes $m(v)$ should appear should appear to approach some number as v approaches* 1*. What is that number?*

iii. *This suggests that*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1} = \lim_{v \to 1} \frac{v^3 - 1}{v - 1} = 3.$$

(iii) *Another approach to approximating the derivative is to graph the slope function*

$$m(v) = \frac{v^3 - 1}{v - 1},\ v \neq 1.$$

Activate the Average Rate of Change folder on Line 11 by clicking the camera icon to the left of the line to see the graph of this function. Activate also the Table folder in Line 1.

i. *Drag the slider v on Line 4. Describe the relationship between the graph of the slope function and the line PQ.*

ii. *There is a hole in the graph of the function $y = m(x)$. Where is it? Why is it there?*

(iv) *To verify the numerical and graphical evidence that*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1} = \lim_{v\to 1}\frac{v^3-1}{v-1} = 3,$$

we'll use algebra to evaluate the above limit.
The idea is to factor the numerator v^3-1. *Since*

$$\left(v^3-1\right)\Big|_{v=1} = 1^3-1 = 0,$$

we know that $v-1$ *is a factor of* v^3-1. *To simplify the quotient*

$$\frac{v^3-1}{v-1}\,,\ v\neq 1,$$

we could use long division or factor the difference of two cubes:

$$a^3-b^3 = (a-b)(a^2+ab+b^2).$$

Either way, the result is that

$$v^3-1 = (v-1)(v^2+v+1).$$

So

$$\frac{v^3-1}{v-1} = v^2+v+1\,,\ v\neq 1.$$

Putting this all together, we get

$$\begin{aligned}\frac{d}{dx}\left(x^3\right)\Big|_{x=1} &= \lim_{v\to 1}\frac{v^3-1}{v-1}\\ &= \lim_{v\to 1}\frac{(v-1)(v^2+v+1)}{v-1}\\ &= \lim_{v\to 1}(v^2+v+1)\\ &= 1^2+1+1\\ &= 3.\end{aligned}$$

Exploration 24 *Repeat all parts of Exploration 2 for the following derivatives.*

(a)

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=2} = 12.$$

(b)

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=a} = 3a^2.$$

Use the demonstration below to check your work for part (b) by dragging the sliders v (Line 4) and a (Line 16). Then activate the Derivative Function folder on Line 17.

Desmos link: `https://www.desmos.com/calculator/ewy7jqij6s`

151: Cubing Function 2

Keep following the method of Exploration 2 to evaluate the following derivatives.

(c)

$$\frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=1} = -1.$$

(d)

$$\frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=a} = -1/a^2.$$

(e)

$$\frac{d}{dx}\left(\frac{1}{x^2}\right)\Big|_{x=3} = -2/27.$$

(f)

$$\frac{d}{dx}\left(\frac{1}{x^2}\right)\Big|_{x=a} = -2/a^3.$$

(g)

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\Big|_{x=1} = -\frac{1}{2}.$$

(h)

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\Big|_{x=a} = -\frac{2a}{(1+a^2)^2}.$$

3 Applications

Question 25 *The function*

$$h = f(t)\,,\; 0 \leq t \leq 2.2,$$

expresses the height of a balloon (in hundreds of feet) in terms of the number of minutes past noon.

The graph of the function $h = f(t)$ is shown below.

Desmos link: `https://www.desmos.com/calculator/yd4xm6x6ub`

151: Balloon

(a) *Find an expression for the function $r = m(v)$ that gives the balloon's average rate of ascent (measured in hundreds of feet per minute) between time v minutes past noon and 12:02 pm. Include also the function's domain.*

(b) *How is the average rate of ascent function in part (a) related to the line PQ in the demonstration above?*

(c) *Use the graph of the function $h = f(t)$ above to sketch a rough graph of the function $r = m(v)$.*

(d) *Activate the folder (avg. rate of change function) in Line 11 to check your graph of the function $r = m(v)$.*

(e) *Open the Table in Line 1 by clicking the* right arrow *at the left of the line.*

  (i) *What is the balloon's average rate of ascent between 12:02:00 pm and 12:02:06 pm?*

  (ii) *Use the graph of the function $h = f(t)$ above to estimate the balloon's rate of ascent at 12:02pm by zooming in on the appropriate point.*

  (iii) *What does the table suggest about the balloon's rate of ascent at 12:02pm? Explain.*

(f) *Now suppose*

$$f(t) = -2t^3 + 7t^2 - 8t + 8.$$

  (i) *Find a simplified expression for the function $r = m(v)$. Include the appropriate domain.*

  (ii) *Use your simplified expression to compute the balloon's rate of ascent at 12:02pm.*

Differentiable Functions, Part 2

Using limits to compute the derivative of a function at a general input.

1 Using Limits to Compute Derivatives

Exploration 26 *In part (b) of Exploration 7 from the previous chapter where we computed*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1}.$$

Now we'll use limits to evaluate the derivative

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=a}.$$

In other words, we'll compute the derivative of the function $f(x) = x^3$ at a general input $x = a$. Think about zooming in on the graph of $y = x^3$ sufficiently close to the point (a, a^3) so that the graph looks like a straight line. We'll compute the slope of that line.

The algebra here is nearly identical to what we did earlier. You should compare the two computations.

Desmos link: `https://www.desmos.com/calculator/8eiffwbgt5`

151: Cubing Function 3

The idea to compute the derivative algebraically is this: Fix the point $P(a, a^3)$ on the graph of $y = f(x) = x^3$. Then choose a variable point Q on the graph, different from P, with coordinates (v, v^3). When Q is sufficiently close to P, the line PQ approximates the curve $y = x^3$ near $P(a, a)$ and the slope of this line is above derivative.

(a) *Our first step is to find an expression for the slope of line PQ as a function*

Learning outcomes:
Author(s):

of v. The slope is

$$\begin{aligned} m(v) &= \frac{\Delta y}{\Delta v} \\ &= \frac{f(v) - f(a)}{v - a} \\ &= \frac{v^3 - a^3}{v - a} \end{aligned}$$

all assuming $v \neq a$.

(b) *The next step is to factor the numerator $v^3 - a^2$. Since*

$$(v^3 - a^3)\Big|_{v=a} = a^3 - a^3 = 0,$$

we know that $v - a$ is a factor of $v^3 - a^3$. To simplify the quotient

$$\frac{v^3 - a^3}{v - a}\,,\ v \neq a,$$

we could use long division or factor the difference of two cubes:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2).$$

Either way, the result is that

$$v^3 - a^3 = (v - a)(v^2 + va + a^2).$$

So

$$\frac{v^3 - a^3}{v - a} = v^2 + va + a^2\,,\ v \neq a.$$

Putting this all together, we get

$$\begin{aligned} \frac{d}{dx}\left(x^3\right)\Big|_{x=a} &= \lim_{v \to a} \frac{v^3 - a^3}{v - a} \\ &= \lim_{v \to a} \frac{(v - a)(v^2 + va + a^2)}{v - a} \\ &= \lim_{v \to a} (v^2 + va + a^2) \\ &= a^2 + a^2 + a^2 \\ &= 3a^2. \end{aligned}$$

(c) *More simply put, we just write the derivative*

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

as a function of x.

(d) *We can check our result using the desmos worksheet above. Do this as follows:*

(i) *Input the correct expression for the derivative $d(x^3)/dx$ on Line 17.*

(ii) *For a differentiable function $f(x)$ and values of h near zero, we can approximate the derivative $d(f(x))/dx$ as the slope*

$$\frac{d}{dx}(f(x)) = \frac{f(x+h) - f(x-h)}{2h}$$

of the line through the points $(x-h, f(x-h))$ and $(x+h, f(x+h))$. Now activate the folder Approximation to the derivaitve function *on Line 19. Then drag the slider h on Line 21 to get a better approximation to the derivative.*

Free Response: *Describe what happens to the above approximation as $h \to 0$.*

Exercise 27 (a) *Follow the method of Exploration 1 for the function $f(x) = x^4$ to compute the derivative*

$$\frac{d}{dx}(x^4)\bigg|_{x=a}.$$

(b) *Modify the desmos worksheet below for the function $f(x) = x^4$.*

151: Cubing Function 3

Then to check your work, do part (d) of Exploration 1 for $f(x) = x^4$. Describe also what happens to the approximation in part (d) as $h \to 0$. Include a screenshot to help with your description.

Exercise 28 *Repeat all parts of Exercise for the function $f(x) = 1/x^4$.*

Exercise 29 *Repeat all parts of Exercise for the function $f(x) = 1 + x^2$.*

Exercise 30 *Repeat all parts of Exercise for the function $f(x) = 1/(1+x^2)$.*

Practice with Limits and Derivatives

Practice with limits and derivatives.

1 More Practice with Rational Functions

Question 31 *The function*

$$W = f(r) = \frac{2000}{r^2}, r \geq 4,$$

expresses the weight of an astronaut (measured in pounds) in terms of her distance from the center of the earth (measured in thousands of miles).

(a) *Find an expression for the average rate of change in the astronaut's weight with respect to her distance from the earth's center between distances $r = b$ and $r = c$ thousands of miles from the center. Assume $b, c \geq 4$ and that $b \neq c$.*

(b) *Use your* expression from part (a) directly and the definition of the derivative *to find an expression (fully simplified) for the derivative*

$$\left.\frac{dW}{dr}\right|_{r=b}.$$

(c) *Use part (b) to evaluate the derivative*

$$\left.\frac{dW}{dr}\right|_{r=2}.$$

(d) *What are the units of the derivative above?*

(e) *Explain the meaning of the derivative in part (c) using the language of small changes.*

Question 32 *The function*

$$g = f(v) = \frac{1}{2}v + 10\,,\ 10 \leq v \leq 55,$$

expresses the gas mileage (in miles/gal) of a car in terms of its speed.

Learning outcomes:
Author(s):

(a) *At what rate (in gal/hr) does the car burn gas at a speed of 40 miles/hour?*

(b) *Find a function*

$$r = g(v)\,,\ 10 \leq v \leq 55,$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed.

Answer: The function is

$$r = g(v) = \frac{v}{\frac{1}{2}v + 15}\,,\ 10 \leq v \leq 55.$$

(c) *Find an expression for the average rate of change in the rate at which the car burns gas with respect to its speed between speeds of $v = w$ and $v = b$ miles/hour. Then use this expression and the algebra of limits to find an expression for the derivative*

$$\left.\frac{dr}{dv}\right|_{v=b}.$$

(d) *Evaluate the derivative*

$$\left.\frac{dr}{dv}\right|_{v=40}.$$

(e) *What are the units of the derivative above. Explain its meaning in terms of small changes.*

2 The Squaring Function and Its Inverse

Question 33 *The function*

$$A = f(s) = s^2\,,\ s \geq 0,$$

expresses the area of a square (measured in square feet) in terms of its side length (in feet).

(a) *Find an expression for the average rate of change in the area of a square with respect to its side length between side lengths $s = b$ and $s = w$ feet.*

(b) *Use your expression from part (a) and the algebra of limits to find an expression for the derivative*

$$\left.\frac{dA}{ds}\right|_{s=b}.$$

(c) *Use the result of part (b) to evaluate the derivative*

$$\left.\frac{dA}{ds}\right|_{s=5}.$$

(d) *What are the units of the derivative in part (c).*

(e) *Explain the meaning of the derivative (c) in terms of small changes.*

Explanation. You should have found that

$$\left.\frac{dA}{ds}\right|_{s=5} = 10 \text{ ft}^2/\text{ft}.$$

This means that if the change

$$\Delta s = s - 5 \sim 0,$$

in the side length is small, and

$$\Delta A = f(s) - f(5) = f(s) - 25$$

is the resulting change in the area, the derivative is a good approximation to the average rate of change. That is,

$$\frac{\Delta A}{\Delta s} \sim \left.\frac{dA}{ds}\right|_{s=5} = 10 \text{ ft}^2/\text{ft},$$

and

$$\Delta A \sim 10\Delta s.$$

For example, if

$$\Delta s = 0.1,$$

then

$$\Delta A \sim (10 \text{ ft})(0.1 \text{ ft}) = 1 \text{ ft}^2.$$

Compare this approximation with the actual change

$$\Delta A = f(5.1) - f(5) = 1.01 \text{ ft}^2$$

in the area. Pretty good.

Question 34 *The function*

$$s = g(A) = \sqrt{A}\,,\ A \geq 0,$$

expresses the side length (in feet) of a square in terms of its area (in square feet).

(a) *Use the algebra of limits to find an expression for the derivative*

$$\left.\frac{ds}{dA}\right|_{A=a}.$$

(b) *Evaluate the derivative*

$$\left.\frac{ds}{dA}\right|_{A=25}.$$

What are its units? Explain its meaning in terms of small changes.

Explanation. The standard approach goes something like this.

$$\begin{aligned}\left.\frac{ds}{dA}\right|_{A=a} &= \lim_{w\to a}\frac{g(w)-g(a)}{w-a}\\ &= \lim_{w\to a}\frac{\sqrt{w}-\sqrt{a}}{w-a}\\ &= \lim_{w\to a}\frac{\sqrt{w}-\sqrt{a}}{(\sqrt{w}-\sqrt{a})(\sqrt{w}+\sqrt{a})}\\ &= \lim_{w\to a}\frac{1}{\sqrt{w}+\sqrt{a}}\\ &= \frac{1}{2\sqrt{a}}.\end{aligned}$$

This approach obscures a key point. Namely, that the function g is the *inverse* of the function f in Question 1. We can exploit this by instead rewriting the equation

$$s = \sqrt{A} = g(A)$$

as

$$A = s^2 = f(s).$$

Now we'll find an expression for the derivative

$$\left.\frac{ds}{dA}\right|_{s=b}$$

in terms of the side length $s = b$. This gives

$$\begin{aligned}\left.\frac{ds}{dA}\right|_{s=b} &= \lim_{w\to b}\frac{w-b}{f(w)-f(b)}\\ &= \lim_{w\to b}\frac{w-b}{w^2-b^2)}\\ &= \lim_{w\to b}\frac{1}{w+b)}\\ &= \frac{1}{2b}.\end{aligned}$$

Letting $a = b^2$ be the area, we have $b = \sqrt{a}$, and

$$\left.\frac{ds}{dA}\right|_{A=a} = \frac{1}{2\sqrt{A}}.$$

This makes it clear that the derivative of the inverse of a function is the *reciprocal* of the function's derivative.

(c) *Use the result of part (b) to evaluate the derivative*

$$\left.\frac{dA}{ds}\right|_{s=5}.$$

(d) *What are the units of the derivative in part (c).*

(e) *Explain the meaning of the derivative (c) in terms of small changes.*

3 The Cubing Function and its Inverse

Question 35 *Repeat the two questions of the previous section with the function*

$$V = f(s) = s^3\,,\ s \geq 0,$$

that expresses the volume of a cube in terms of its side length.

4 A Square Root Function

Question 36 *A rock dropped from a height of* 100 *feet falls to the surface of the planet Krypton without air resistance.*

The function

$$v = f(h) = 6\sqrt{100 - h}\,,\ 0 \leq h \leq 100,$$

expresses the rock's speed (in ft/sec) in terms of its height (in feet).

(a) *Find an expression for the average rate of change in the rock's speed with respect to its height between heights $h = v$ and $h = b$ feet.*

(b) *Use your* expression from part (a) directly and the definition of the derivative *to find an expression (fully simplified) for the derivative*

$$\left.\frac{dv}{dh}\right|_{h=b}.$$

(c) *Use part (b) to evaluate the derivative*

$$\left.\frac{dv}{dh}\right|_{h=36}.$$

(d) *What are the units of the derivative above?*

(e) *Explain the meaning of the derivative in part (c) using the language of small changes.*

5 More on the Squaring Function

Check back later.

Free Fall

Rocks in free fall.

1 Speed vs. Velocity

In colloquial English, speed and velocity are typically used interchangeably. A baseball announcer might say, for example, that a pitcher has good velocity, when he really means the pitcher is throwing hard (fast).

But we need to be precise and for us *velocity* is a vector. It is the rate of change of position (also a vector) with respect to time. As such, we can talk about *average velocity* over a time interval and *instantaneous velocity* at a moment in time. For example, at some intstant we might be driving due south at a speed of 50 miles/hour and this would be a description of our instantaneous velocity. Or over a three-hour period we might have driven 120 miles and ended up at a point due south of where we started. Then we could describe our average velocity over this three-hour period as a displacement (ie. change in position) due south at an average speed of 40 miles/hour.

In colloquial English *acceleration* is also used incorrectly. An advertisment might claim a car can accelerate from rest to a speed of 60 miles/hour in 4 seconds. But acceleration, being the rate of change of veloicty with respect to time, is also a vector. So instead, we might say that a car is speeding up at a constant rate of 22 (ft/sec)/sec. This implies, for example, that every second the speed of the car increases by 22 ft/sec. It would *not* be correct to say that the car's acceleration is 22 (ft/sec)/sec. Acceleration, like velocity, needs a direction.

It also obscures meaning if we say that the speed of a car is increasing at the rate of 22 ft/sec^2. What exactly do we mean by a square second. Better to say what we really mean, that the speed is increasing at the rate of 22 (ft/sec)/sec.

Velocity plays only a very small part in our class, and this activity is all about speed. Velocity is defined as a rate of change (of position with respect to time), but what about speed? To think of speed this way we need a *cumulative distance function*. The odometer of your car is one such function. It records the total number of miles the car has been driven since its birth. The trip odometer also keeps track of cumulative distance. In either case, the average rate of change of the odometer reading with respect to time measures average speed (distance travelled / time) over an interval of time. The limit of these average rates of

Learning outcomes:
Author(s):

change, as the time interval shrinks to zero, measures the (instantaneous) speed at any moment. This is the reading on the speedometer.

For this class, motion is generally restricted to a line, and the cumulative distance function measures distance along this line from some select point. Later on, you'll learn about *arclength*, distance meaured along a curve. Typically the parameters s is used for this distance function. Don't confuse this with speed. In this activity we drop a rock and measure the cumulative distance function $s = f(t)$ straight down, from the rock's initial position.

It's a consequence of Newton's second law and his law of univeral graviation, that in the absence of air restistance the speed of a rock dropped near the surface of a planet increases at a constant rate. On the earth, this rate is about 32 (ft/sec)/sec (or about 9.8 (m/s)/s). On the moon, the rate is about one-sixth these values. On Mars, the *magnitude* of the gravitational acceleration is approximately 3.73 (m/s)/s.

2 A Rock in Freefall

Suppose we drop a rock from rest near the surface of a planet without an atmosphere (or very near the surface of the earth). Our focus here will be in addressing two questions. Don't try to answer these now. Wait until the end of the activity.

Question 37 (a) *How does the speed at which the rock hits the surface compare with its average speed during the time it takes to hit the surface?*

(b) *Through approximately what fraction of its initial height does the rock fall during the last* $1/20$ *of the time it takes to hit the surface? Or during the last similarly* small *fraction of the falling time?*

(c) *What is the exact fraction?*

Question 38 *Play the slider* u *in Line 2 of the desmos worksheet below to watch two motions. One shows a rock falling to the surface of a planet when it is dropped from rest near the surface. The other is a constant-speed motion.*

Desmos link: `https://www.desmos.com/calculator/dmlrxahkld`

151: Free Fall 2

(a) *Which motion is which?*

(b) *What does the animation suggest about the speed of the rock as it hits the surface compared with its average speed during the time it takes to hit the ground?*

(c) *What does the animation suggest about the fraction of its initial height the rock falls through during the last* $1/20$ *of the time it takes to hit the surface? Give a quick estimate.*

(d) *Now suppose the rock is dropped from a height of* H *meters and that it takes* T *seconds to hit the surface. Then, as we'll verify another day, the function expressing the rock's distance from its starting point (in meters) in terms of the number of seconds since it was dropped is of the form*

$$s = f(t) = kt^2$$

for some positive constant k.

(i) *What are the units of the constant* k*? How do you know?*

(ii) *Express the constant* k *in terms of* H *and* T.

(iii) *Find a fully-simplified expression for the average speed of the rock over the time interval* $b \leq t \leq w$. *Assume* $b < w$.

(iv) *Use the algebra of limits to find an expression for the (instantaneous) speed of the rock* b *seconds after being dropped. Check that your expression has the correct units.*

(v) *Use the result of part (iv) to express in terms of* H *and* T *the speed of the rock (in metes/sec) as it hits the surface. Check that your expression has the correct units.* Hint: *Evaluate the derivative above at the appropriate time* $t = b$.

(vi) *Compare the speed of the rock as it hits the ground with its average speed during the time it falls.*

(vii) *Use the result of part (vi) to approximate the fraction of its initial height through which the rock fall during the last* $1/20$ *of the time it takes to hit the surface*

(viii) *Through what* exact *fraction of its initial height the rock fall through during the last* $1/20$ *of the time it takes to hit the surface? Compare this with your earlier estimate.*

3 Takeaways

In the last problem you showed that the derivative of the function

$$y = f(x) = kx^2$$

is

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) = 2kx.$$

But in some contexts, like for the falling rock, it is more insightful to write to write this derivative as

$$\begin{aligned}\frac{dy}{dx} &= 2kx \\ &= 2\left(\frac{2kx^2}{x}\right), \; x \neq 0, \\ &= 2\left(\frac{y}{x}\right), \; x \neq 0.\end{aligned}$$

Geometrically, this says that the slope ($2ka$) of the tangent line to the parabola

$$y = f(x) = kx^2, k \neq 0,$$

at the point $P(a, ka^2)$ is *twice* the slope (ka) of the line OP through the origin and P.

Desmos link: `https://www.desmos.com/calculator/aamjtxiqgi`

151: Squaring Function

In the context of a rock dropped from rest, this means that the rock's speed at any time t is twice its average speed during the time interval from the moment of release to time t.

This way of writing the derivative has other implications as well.

For example, in the context of the function

$$A = s^2, s > 0,$$

that expresses the area of a square of its side length, we have for $\Delta s \sim 0$,

$$\frac{\Delta A}{\Delta s} \sim \frac{dA}{ds} = \frac{2A}{s}.$$

So that for $\Delta s \sim 0$,

$$\frac{\Delta A}{A} \sim 2\left(\frac{\Delta s}{s}\right).$$

In other words, the relative change in the area of a square (or the relative error in computing the area) is approximately twice the relative change in its side length (or twice the relative error in meausring its side length), at least for *small* changes in (or errors in measuring) the side length.

We don't really need calculus to understand this. A picture does just as well.

Desmos link: `https://www.desmos.com/calculator/h2fm6mm8ua`

151: Square Error

Question 39 *Explain exactly what the picutre suggests and how it does so.*

Free Response:

For more on this idea, see the chapter about *Elasticity*.

The Weighted Chain

Lifting a suspended weight.

Question 40 *Attach a small weight to the center of a twenty-inch string, hold the ends of the string together and then pull your hands apart. Play the slide a in Line 2 of the animation below to watch the motion.*

Desmos link: `https://www.desmos.com/calculator/jmqscra2if`

Desmos activity available at 151: Weighted Chain 22

What we're most interested in here is how moving your hands a bit farther apart changes the height of the weight. You might, for example, move each hand 1 cm outward. What happens to the weight? Perhaps it moves upward about half as far, only 0.5 cm. Or perhaps about twice as far. The ratio of these two small distances depends upon the distance between your hands. And while we're not literally stretching an elastic band here, the effect is the same. We can talk about the local scaling factor, *the one that converts a small outward movement of your hands to an approximate change in height. This local scaling factor, in some configurations less than one, in others greater than one, is the* derivative. *Our aim here is to see how this factor depends on the distance between your hands, or really half that distance.*

(a) *Play the animation. Then sketch by hand a graph of the function*

$$h = f(s)\,,\; 0 \leq s \leq 10$$

that expresses the distance (in inches) of the weight from its starting point in term of half the distance (in inches) between your hands (ie. the distance MR above). Activate the folder in Line 15 to see how you did.

(b) *Use the graph of the height function $h = f(s)$ to sketch by hand a rough graph of its derivative $r = dh/ds = f'(s)$. Be sure to label the axes with the appropriate variable names and units. Keep in mind the following when drawing your graph:*

(i) *For what values of s is the scaling factor less than one?*

(ii) *For what value(s) of s is the scaling factor equal to one?*

(iii) *Greater than one?*

(iv) *Equal to zero?*

Learning outcomes:
Author(s):

Then activate the folder in Line 20 to see how you did.

(c) *Find an expression for the function $h = f(s)$.*

(d) *Use the geometry of the curve $h = f(s)$ to evaluate the local scaling factor*

$$\left.\frac{dh}{ds}\right|_{s=b}.$$

(e) *Now we'll go about computing this scaling factor algebraically:*

(i) *Find an expression for the average rate of change of h with respect to s between $s = b$ and $s = w$.*

(ii) *Use the algebra of limits to find an expression for the derivative*

$$\left.\frac{dh}{ds}\right|_{s=b}.$$

(f) *Drag the slider a in Line 2 to $a = 9.5$ and use the animation to approximate the derivative*

$$\left.\frac{dh}{ds}\right|_{s=9.5}$$

(g) *Compute the exact value of the above derivative. Include units. Compare this with your approximation.*

(h) *Explain the meaning of the above derivative in terms of small changes.*

Question 41 *The last question had nothing to do with time. But suppose now, for example, we move our hands outward at a constant speed of 4 inches/sec. Then the speed of the weight would depend on the half-distnace between our hands. How?*

We'll work more generally, and suppose we have a function

$$s = g(t)\,,\; 0 \le t \le T_0,$$

that expresses the distance MR (ie. half the distance between our hands), measured in inches, in terms of the number of seconds since we began moving our hands.

Desmos link: `https://www.desmos.com/calculator/x0e7h7sobg`

Desmos activity available at 151: Weighted Chain 23

Purchasing Power

How a small change in price affects the quantity of an item you can buy.

1 Purchasing Power

Question 42 *This problem investigates how a small change in the price of apples would affect the number of pounds we can buy with* \$10.

(a) *We'll start by finding a function*

$$n = f(p)\,,\ p > 0,$$

that expresses the number of pounds of apples we can buy with \$10 *in terms of the price (measured in dollars/pound).*

The function is

$$n = f(p) = 10/p,\ p > 0.$$

(b) *Would you expect the derivative dn/dp to be postive, negative, or zero? Explain.*

(c) *Use limits to find an expression for the derivative*

$$\left.\frac{dn}{dp}\right|_{p=a}.$$

The derivative is (click the Hint tab above for help)

Hint:

$$\begin{aligned}\left.\frac{dn}{dp}\right|_{p=a} &= \lim_{p\to a}\frac{1}{p-a}\left(f(p)-f(a)\right)\\ &= \lim_{p\to a}\frac{1}{p-a}\left(\frac{10}{p}-\frac{10}{a}\right)\\ &= \lim_{p\to a}\frac{10}{p-a}\left(\frac{a-p}{pa}\right)\\ &= \lim_{p\to a}\frac{-10}{pa}\\ &= \frac{-10}{a^2}.\end{aligned}$$

Learning outcomes:
Author(s):

(d) *Evaluate the derivative*

$$\left.\frac{dn}{dp}\right|_{p=2}.$$

(i) *What are the units of the above derivative?*

(ii) *What do you get by "simplifying" the units? Explain how simpifying the units of this derivative gives you insight into its meaning.*

(e) *Interpret the meaning of the above derivative. Click the arrow to the lower right for help.*

For a way to interpret the meaning of the derivative

$$\left.\frac{dn}{dp}\right|_{p=2}.$$

we can ask ourselves what happens to the number of pounds of apples we can buy with \$10 *if the price changes by a small amount from* \$2/lb*. For this, let* Δp *be a small change in price from* \$2/lb *(measured in dollars/lb) and let* Δn *be the corresponding change in the number of pounds we can buy with* \$10*.*

For $\Delta p \sim 0$*, we have*

$$-2.5 = \left.\frac{dn}{dp}\right|_{p=2} \sim \frac{\Delta n}{\Delta p}$$

and (type "Delta" for Δ*)*

$$\Delta n \sim -2.5\Delta p.$$

This tells us that if the price increases from \$2/lb *to say* \$2.10/lb*, then*

$$\Delta n \sim \left(-2.5\frac{lb}{\$/lb}\right)\left(0.10\frac{\$}{lb}\right) = -0.25\ lbs.$$

So if the price increases from \$2/lb *to say* \$2.10/lb*, we can buy about* 0.25 *fewer pounds of apples with ten dollars.*

(f) *Next we'll use the worksheet below to visualize the approximate change*

$$\begin{aligned}\Delta n &\sim \left(\left.\frac{dn}{dp}\right|_{p=2}\right)\Delta p\\ &= \left(-2.5\frac{lb}{\$/lb}\right)\left(0.10\frac{\$}{lb}\right)\\ &= -0.25\ lbs.\end{aligned}$$

in the number of pounds of apples we can buy with \$10 *as the price increases from* \$2/lb *to* \$2.10/lb*.*

Desmos link: `https://www.desmos.com/calculator/qw7wislq0c`

151: Apples

(i) *To get started, find an equation (in point-slope form) of the tangent line to the curve $n = f(p)$ at the point P with coordinates $(2, 5)$. The tangent line has a slope equal to the derivative*

$$\left.\frac{dn}{dp}\right|_{p=2}.$$

So its equation is

$$n = 5 + -2.5(p - 2).$$

(ii) *Enter your equation of the tangent line on Line 17 of the desmos worksheet.*

(iii) *Activate the folders* tangent line *and* linear approximation *on Lines 18 and 22.*

(iv) *Explain why the difference in the n-coordinates of points P and R' (ie. n-coordinate of R' minus n-coordinate of P) is equal to our approximation of Δn above when Q as coordinates $(2.10, f(2.10))$.*

(v) *Drag Slider v on Line 2 to make Q approach P. What so you think happens to the ratio of Δn to our approximation of Δn as $v \to 2$?*

(g) *We can get a better understanding of the changes*

$$\Delta p = 2.1 - 2 = 0.10 \text{ dollars/pound}$$

and

$$\Delta n = f(2.1) - f(2) \sim 0.25 \text{ pounds}$$

by thinking about relative instead of absolute change.

An increase in price from \$2/lb to \$2.10/lb is a relative change of

$$\frac{\Delta p}{p} = \frac{\$0.1/lb}{\$2/lb} = 0.05 = 5\%.$$

This causes a relative change in the number of pounds we can buy of approximately

$$\frac{\Delta n}{n} \sim \frac{-0.25 lbs}{5 lbs} = -0.05 = -5\%.$$

(h) *We can get a the same result relating the two relative changes of the previous question by working in general.*

Suppose we increase the price of apples by $Q\%$, where $Q \sim 0$. What can we say about the relative change in the number of pounds of apples we can buy with \$10?

Well, if the change in price Δp is near zero and Δn is the corresponding change in the number of pounds we can buy, then

$$\frac{\Delta n}{\Delta p} \sim \frac{dn}{dp} = -10/p^2.$$

So

$$\Delta n \sim \left(\frac{-10}{p^2}\right) \Delta p.$$

Dividing both sides by $n = 10/p$ tells us that

$$\frac{\Delta n}{n} \sim \left(\frac{-10}{p^2}\right)\left(\frac{\Delta p}{\frac{10}{p}}\right) = -\frac{\Delta p}{p} = -Q\%.$$

So if we increase the price by $Q\% \sim 0$, then the number of apples we can buy decreases by approximately that same $Q\%$.

(i) *We can think about this relationship between the relative changes geometrically. To do this, remove your equation of the tangent line on Line 17 and activate the folder* tangent line *in the worksheet above. Then drag slider a on Line 4 to move point P. What do you notice about the ratio $PB : PA$ of the distances PA and PB as P moves? More on this later.*

Question 43 *Due to a printing error, the graph of the function*

$$n = f(p)\,,\, p > 0,$$

expressing the number of pounds of an item we can buy with \$100 in terms of the price (measured in \$/lb) is not shown below. All we can see is a point A on the graph.

Desmos link: `https://www.desmos.com/calculator/y5nqs8fkvj`

151: Printers Error 1

The problem is to draw the tangent line to the graph at A without *sketching the graph or doing any kind of computation. Click the arrow below for a hint.*

The key idea is to relate the slope of the tangent line at A to the slope of the line OA through the origin and A.

We know that

$$n = f(p) = \frac{100}{p}\,,\, p > 0,$$

and

$$\frac{dn}{dp} = -100/p^2.$$

Now write this derivative in terms of both n and p,

$$\frac{dn}{dp} = -\frac{100}{p^2} = -\frac{n}{p}.$$

Now relate this to the slope of line OA and draw the tangent line at A.

2 Weight in Space

Question 44 *The weight of an object is the gravitational force that the earth exerts on the object's mass and varies with the object's height above the surface. The function*

$$W = f(h) = \frac{k}{(h+4)^2}\ , \ h \geq 0,$$

expresses the weight (in pounds) of an object in terms of its height (in thousands of miles) above the surface of the earth. We'll suppose the object weighs 200 pounds on the surface.

Desmos link: `https://www.desmos.com/calculator/zklsucctjp`

151: Weight in Space

(a) *Find the value of the constant k. What are its units?*

(b) *Find the average rate of change of W with respect to h over the interval between heights of h and v thousands of miles.*

(c) *Use part (b) to find an expression for the derivative dW/dh.*

(d) *Use the graph of the function $W = f(h)$ above to approximate the derivative*

$$\left.\frac{dW}{dh}\right|_{h=4}.$$

Include units

(e) *Evaluate the derivative*

$$\left.\frac{dW}{dh}\right|_{h=4}$$

using your expression for dW/dh and compare this with your estimate.

(f) *What are the units of the derivative above? Interpret it meaning.*

(i) *At approximately what altitude does your weight decrease by 4 pounds when your altitude increases by 200 miles? Use your expression for the derivative dW/dh to help.*

(ii) *Compute the exact change in your weight over the interval you found in part (i).*

3 A Rock in Free Fall

Question 45 *A rock is dropped from a height of 100 meters on the planet Krypton.*

The function

$$v = f(h)\,,\; 0 \le h \le 100,$$

expresses the speed (in meters/sec) of the rock in terms of its height (in meters) above the surface.

Desmos link: `https://www.desmos.com/calculator/hcqb5nxc18`

151: Free Fall Speed and Height

(a) *Would you expect the derivative*

$$\left.\frac{dv}{dh}\right|_{h=36}$$

to be postive or negative? Why?

(b) *Suppose*

$$v = f(h) = 4\sqrt{100-h}\,,\; 0 \le h \le 100$$

and find an expression for the average rate of change of the rock's speed (measured in meters/sec) with respect to its height (in meters) between heights of v meters and a meters.

(c) *Use your expression from part (b) to find an expression for the derivative*

$$dv/dh\Big|_{h=a}.$$

(d) *Use the graph above to approximate the derivative*

$$dv/dh\Big|_{h=64}.$$

Include units.

(e) *Evaluate the derivative*

$$dv/dh\Big|_{h=64}$$

and compare the exact value with your estimate.

(f) *What are the units of the derivative above? What is its meaning?*

(g) *"Simplify the units of the derivative". What insight does this give you into its meaning?*

(h) *Use part (e) to approximate the rocks's speed at a height of* 61 *feet. Compare your estimate with the actual speed.*

(i) *Use part (c) to approximate the interval over which the rock falls two meters as its speed increases by* 0.5m/s.

4 Distance to the Horizon

Question 46 *The function*

$$s = f(h) = 1.22\sqrt{h}\,,\ 0 \leq h \leq 10{,}000$$

expresses the distance to the horizon (measured in miles) in terms of your alitude (measured in feet).

(a) *Find an expression for the average rate of change in the distance to the horizon with respect to altitude between altitudes h feet and v feet.*

(b) *Use your expression from part (a) to find an expression for the derivative ds/dh.*

(c) *Evaluate the derivative*

$$\frac{ds}{dh}\Big|_{h=25}.$$

(d) *What are the units of the derivative above? Interpret its meaning.*

(e) *Use the result of part (c) to approximate the distance to the horizon at an altitude of* 24 *feet. Then compare this approximation with the actual distance.*

(f) *Approximate the relative change in the distance to the horizon in terms of a small relative change in altitude.*

(g) *Use your result from part (b) to approximate the altitude at which moving* 10 *feet higher increase the distance to the horizon by* 0.5 *miles.*

5 Intensity of Sound

Question 47 *The* intensity *of sound is measured in Watts per square meter and is a function of the distance from the source.*

Suppose for a jet taking off, this function is

$$I = f(r) = \frac{100}{r^2}\,,\ r \geq 10,$$

where r is the distance (in meters) from the jet.

(a) *Find an expression for the average rate of change of the sound intensity (in Watts/m^2) with respect to the distance from the source (in meters) between distances r and a meters.*

(b) *Use your expression from part (a) to find an expression for the derivative*

$$dI/dr\Big|_{r=a}$$

(c) *Evaluate the derivative*

$$dI/dr\Big|_{r=20}.$$

(d) *What are the units of the derivative above? What is its meaning?*

(e) *Approximate the relative change in the sound intensity in terms of a small relative change in the distance to the source.*

Practice Quiz 1

First practice quiz, Weeks 1-2

Directions:

(a) Show all work.

(b) Give brief explanations for each problem. Include these explanations in the flow of the solution.

(c) Show all units in all computations.

1 Part 1

Question 48 *Explain what it means intuitively for a function to be differentiable at some input in terms of the graph of that function.*

Question 49 *The function*

$$s = f(h) = 1.22\sqrt{h}\,,\ 0 \leq h \leq 10,000,$$

expresses the distance to the horizon (measured in miles) in terms of your altitude (measured in feet).

(a) *Find an expression for the average rate of change in the distance to the horizon with respect to altitude between altitudes b feet and v feet.*

(b) *Use your expression from part (a) to find an expression for the derivative*

$$\left.\frac{ds}{dh}\right|_{h=b}.$$

(c) *Evaluate the derivative*

$$\left.\frac{ds}{dh}\right|_{h=25}.$$

(d) *What are the units of the derivative above? Interpret its meaning.*

Learning outcomes:
Author(s):

(e) *Use the result of part (c) to approximate the distance to the horizon at an altitude of 24 feet. Then compare this approximation with the actual distance.*

(f) *Approximate the relative change in the distance to the horizon in terms of a small relative change in altitude.*

(g) *Use your result from part (b) to approximate the altitude at which moving 10 feet higher increase the distance to the horizon by 0.5 miles.*

Question 50 *The* intensity *of sound is measured in Watts/(square meter) and is a function*

$$I = f(r) = \frac{100}{r^2}\ ,\ r \geq 0.5,$$

of the distance from the source.

Suppose r is measured in meters and that the intensity of the sound emitted by a vacuum cleaner is 10^{-4}watts/m^2 at a distance of 0.5 meters.

(a) *Compute the value of the constant k. What are its units?*

(b) *Find an expression for the average rate of change of the sound intensity (in Watts/m^2) with respect to the distance from the source (in meters) between distances b and r meters.*

(c) *Use your expression from part (b) to find an expression for the derivative*

$$\left.\frac{dI}{dr}\right|_{r=b}.$$

(d) *Evaluate the derivative*

$$\left.\frac{dI}{dr}\right|_{r=2}.$$

(e) *What are the units of the derivative above? What is its meaning?*

(f) *Approximate the relative change in the sound intensity in terms of a small relative change in the distance to the source.*

Question 51 *Between speeds of 70 miles/hr and 84 miles/hr, the gas mileage of a car (in miles/gal) is a one-to-one function $G = f(v)$ of its speed (in miles/hour). The car gets 10 miles/gal at a speed of 80 miles/hour.*

(a) *Which of the following is more likely to be true? Explain your reasoning.*

(i) $\left.\frac{dG}{dv}\right|_{v=80} = 0.25$ *or*

(ii) $\left.\frac{dG}{dv}\right|_{v=80} = -0.25$

(b) *What are the units of the correct derivative above? Explain its meaning.*

(c) *Assuming the correct choice in part (b), evaluate the derivative*

$$\left.\frac{dv}{dG}\right|_{G=10}.$$

(d) *Simplify the units of the derivative in part (c). What does this tell you about its meaning?*

(e) *At what rate (in gal/hour) does the car burn gas at a speed of* 80 *miles/hour?*

Question 52 *Between speeds of* 55 *miles/hour and* 70 *miles/hour the gas milesage of a car is a linear function of its speed. The car gets* 40 *miles/gal at a speed of* 55 *miles/hour and* 30 *miles/gal at a speed of* 70 *miles/hour.*

(a) *Find a function*

$$r = f(v)\,,\ 55 \le v \le 70,$$

that express the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour). Note: *This function is* not *linear*

(b) *Find an expression for the average rate of change of r with respect to v between speeds of b miles/hour and w miles/hour.*

(c) *Use your expression from part (b) to find an expression for the derivative*

$$\left.\frac{dr}{dv}\right|_{v=b}.$$

(d) *Evaluate the derivative*

$$\left.\frac{dr}{dv}\right|_{v=60}.$$

(e) *What are the units of the derivative above? Interpret its meaning.*

Introduction to Motion

An introduction to motion

Exploration 53 *Desmos link:* `https://www.desmos.com/calculator/bk1z9cwbhb`

151: Sound Squaring Function

Learning outcomes:
Author(s):

Introduction

Introduction to Differential Calculus

This course is listed as Calculus I in the catalogue, but it really should be called Differential Calculus. *Differential* has the same root as *difference* and *calculus* the same root as *calculate*. So this class is really about calculating differences, or more simply put it's about subtracting. But it's about subtracting in the context of functions.

Pick a specific input to specific function and you'll likely find that the changes in the function's output are approximately proportional to *small* changes in the input. If so, we say that the function is *differentiable* at that input and we call the proportionality constant the *derivative*.

For example, let's look at the behavior of the function

$$A = f(s) = s^2, s \geq 0,$$

near the input $s = 5$. To emphasize the importance of units, let's define the input s to be the side length of a square measured in feet and the output $f(s)$ to be the area of that square, measured in square feet. The problem before us is to describe a simple relationship between a small change in the side length

$$\Delta s = s - 5$$

of the square and the change

$$\Delta A = f(s) - f(5)$$

in its area.

Question 54 *(a) We'll first take a numerical approach and compute some small changes and their ratios. Fill in the missing entries in the table below.*

s (ft)	$A = s^2$ (ft^2)	$\Delta s = s - 5$ (ft)	$\Delta A = s^2 - 25$ (ft^2)	$\Delta A/\Delta s$ (ft^2/ft)
4.9	24.01	-0.1	-0.99	9.9
4.99	24.9001	-0.01	-0.0999	9.99
5	*25*	0	0	–
5.01	25.1001	0.01	0.1001	10.01
5.1	26.01	0.1	1.01	10.1

Learning outcomes:
Author(s):

(b) The data in the table above suggests an approximate proportional relationship between ΔA and Δs. We can guess the constant of proportionality from the fifth column. As $s \to 5$ (as s approaches 5), it looks like the ratio $\Delta A/\Delta s$ approaches some number, the constant of proportionality.

i) What is that number? 10

ii) What are its units?

Free Response:

So for $\Delta s \sim 0$, we suspect that

$$\Delta A \sim 10\Delta s.$$

(c) The constant of proportionality is called the derivative, in this case of the function $A = s^2$, at the input $s = 5$. We write this as

$$\left.\frac{dA}{ds}\right|_{s=5} = 10.$$

(d) We could have taken an algebraic approach to determe this constant of proportionality instead. The idea is to first simplfiy the quotient $\Delta A/\Delta s$ as

$$\begin{aligned}\frac{\Delta A}{\Delta s} &= \frac{s^2 - 25}{s - 5}\\ &= \frac{(s+5)((s-5)}{s-5}\\ &= s + 5 \text{ if } s \neq 5.\end{aligned}$$

So, for example, if $s = 4.99$, then

$$\frac{\Delta A}{\Delta s} = 4.99 + 5 = 9.99$$

as shown in the last column of the second row of the above table.

The advantage of this algebraic approach is that we can now compute the proportionality constant as a limit:

$$\begin{aligned}\left.\frac{dA}{ds}\right|_{s=5} &= \lim_{s\to 5}\frac{\Delta A}{\Delta s}\\ &= \lim_{s\to 5}(s+5)\\ &= 5+5\\ &= 10.\end{aligned}$$

(d) We can also use the graph of the function $A = f(s) = s^2$ to intrepret the ratios

$$\frac{\Delta A}{\Delta s} = \frac{f(s) - f(5)}{s - 5}$$

geometrically. Move the slider s in the demonstration below and describe

(i) how the line through the points P and Q is related to the ratio $\Delta A/\Delta s$ show on Line 2,

(ii) what happens to the line PQ as $s \to 5$, and

(iii) what happens to the line PQ when $s = 5$.

Free Response:

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/vz9ud5txva`

Continuing with the above demonstration,

(i) Open the Code folder in Line 3 and turn off the line PQ in Line 7.

(ii) Write an equation for the line through the point P with slope equal to the proportionality constant in the line below and on Line 8 in the desmos worksheet:

$$A = L(s) = 25 + 10(s - 5).$$

(iii) Zoom in close enough to the point P to make the graph of the function $A = f(s)$ look like a line. How do the graph of the function and the graph of the line $A = L(s)$ compare in this close-up view?

Free Response:

(e) **Summary:**

- *If we change the side of a square from a length of 5 feet to a length of $s \sim 5$ feet, then the area of the square changes by approximately*

  $$\Delta A = s^2 - 25 \sim 10\Delta s = 10(s - 5)$$

  square feet. The proportionality constant 10 has units $ft^2/ft = ft$.

- *Zoom in close enough to the graph of the function $A = f(s) = s^2$ near the point $P(5, 25)$ and the graph looks like a line with slope equal to the proportionality constant.*

- *We can compute the proportionality constant as the limit*

$$\left.\frac{dA}{ds}\right|_{s=5} = \lim_{s\to 5} \frac{f(s)-f(5)}{s-5}.$$

- *Suppose for example, we wanted to approximate the side length s of a square with area 25.06 ft^2. Then*

$$\Delta A = s^2 - 25 = 25.06 - 25 = 0.06.$$

And since

$$\Delta A \sim 10\Delta s = 10(s-5),$$

$$0.06 \sim 10\Delta s.$$

So

$$\Delta s \sim 0.006$$

and a square with area 25.06 ft^2 has an approximate side length (measured in feet) of

$$s = 5 + \Delta s \sim 5.006.$$

Question 55 *On a clear day with an unobstructed view (like you might have at the beach or in a hot air balloon), the distance to the horizon is limited by the curvature of the earth as illustrated in the demonstration below.*

In fact, as long as you are not too high above the surface of the earth, the function

$$s = f(h) = 1.22\sqrt{h}, 0 \le h \le 20,000,$$

gives a good approximation to the distance to the horizon (the length of the red arc AT below, measured in miles) in terms of your height above the ground (the distance AP below, measured in feet).

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

Desmos activity available at

151:Distance to Horizon 1

Our aim is to approximate the change in the distance to the horizon (in miles) in terms of a small change in height (in feet) from a height of 25 feet.

(i) To start, what are the units of the constant 1.22 above? Explain how you know.

Free Response:

(ii) Go through a similar analysis as in parts (a)-(e) of Example 1, to approximate the change $\Delta s = s - f(25)$ in the distance to the horizon in terms of the change $\Delta h = h - 25$ in your height above the ground. Start by completing the column headings (with units) and the missing entries in the table below.

h (ft)	$s = 1.22\sqrt{h}$ (miles)	$\Delta h = h - 25$ (ft)	$\Delta s = f(h) - f(25)$ (miles)	$\Delta s/\Delta h$ (units?)
4.9^2				
4.99^2				
25		0	0	–
5.01^2				
5.1^2				

Question 56 *This question is similar to the last, but suppose instead we are looking down on the earth from the space station or a rocket. Then the approximation to the distance to the horizon from the previous problem will* not *work.*

So our first step is to find a function

$$s = f(h), h \geq 0,$$

that expresses the distance to the horizon (still measured in miles) in terms of our height above the earth's surface, now measured in miles instead of feet. *We'll suppose the earth to be a perfect sphere of radius* 3960 *miles. The distance to the horizon is the arclength AT below, meaured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance AP.*

Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

(a) Find an expression for the above function.

Hint: *Use right triangle ΔOTP to find an expression for the radian measure of angle $\angle POT$. Then use this angle to find an expression for the arclength AT.*

Here are more details.

(i) Enter the two side lengths, measured in feet, in right triangle ΔOPT below.

$$OT = 3960$$

and

$$OP = h + 3960.$$

(ii) Let θ be the radian measure of $\angle TOP$. Write an equaion with a trigonometric function of θ that relates the two lengths in part (i). Use the Math Editor tab to enter the trig function and the angle θ.

$$\cos\theta = \frac{3960}{h + 3960}.$$

(iii) Now solve the equation from part (ii) for θ in terms of h. Then use what you know about measuring arclength along a circle to find an expression for the function f. Use the Math Editor tab to help.

$$s = f(h) = 3960 \arccos(\frac{3960}{h + 3960}).$$

(b) Now suppose we are 165 miles above the surface of the earth and we wish to approximate how a small change in our altitude changes the distance to the horizon.

To do this, fill in the missing entries in the table below.

h *(miles)*	$s = f(h)$ *(miles)*	$\Delta h = h - 165$ *(miles)*	$\Delta s = f(h) - f(165)$ *(miles)*	$\Delta s/\Delta h$ *(units?)*
162				
163				
164				
165				
166				
167				
168				

(c) Do the data above suggest that the quotients $\Delta s/\Delta h$ approach some number as h approaches 165? If so, use the data to approximate that number. If not, explain why not.

(d) Make your own table similar to the one above to get a better approximation, correct to the nearest thousandth, to

$$\lim_{h\to 165} \frac{f(h) - f(165)}{h - 165}.$$

(e) Use your result from part (d), rounded to the nearest thousandth, to approximate Δs in terms of Δh and enter your result below.

$$\Delta s \sim 3.291\Delta h, \text{ for } \Delta h \sim 0.$$

(f) Explain the meaning of the proportionality constant in parts (d) and (e). Be sure to include units in your explanation.

Question 57 *The function*

$$h = f(v)\,,\ 80 \leq v \leq 120,$$

expresses the height of a helicopter (measured in feet) in terms of its speed (measured in ft/sec).

Suppose

$$\left.\frac{dh}{dv}\right|_{v=100} = -20.$$

(a) *What are the units of the above derivative?*

(b) *Explain the meaning of the derivative using the language of small changes.*

Small Changes

We explore how small changes to the input of a function change the output.

The main idea of differential calculus is to approximate the change in the output of a function in terms of a small change in the input. For some functions, called *differentiable*, the change in the output is approximately proportional to the (small) change in the input. The proportionality factor is called the derivative. In this chapter we explore this idea.

1 Odometer Readings

Example 7. The graph of the function

$$s = f(t), 0 \leq t \leq 2,$$

that expresses the trip odometer reading (measured in miles) on your car in terms of the number of hours past noon during a two-hour trip is shown below.

Access Desmos interactives through the online version of this text at

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Desmos link: https://www.desmos.com/calculator/iiw69lr1bc

Desmos activity available at

151: Odometer

Our goal is to approximate the car's speed at 12:30pm in three ways:

(1) geometrically, using the above graph as is.

(2) geometrically, by zooming in on the above graph.

(3) arithmetically, using the specific expression for the function f.

(a) Start by using the graph above to describe how the speed of the car varies over the two-hour period. Explain your reasoning. Then play the Slider u in Line 2 and use the animation of the motion to check if your description was accurate. Explain.

Learning outcomes:
Author(s):

(b) Set the slider in Line 6 to $n = 20$. Use *only* the graph (with $u = 2$) to create a table with five columns showing the values of t, s (approximate this), $\Delta t = t - 0.5$, $\Delta s = f(t) - f(0.5)$, and $\Delta s/\Delta t$. Include units in the heading of each column. The table should include five rows, with $t = 0.3, 0.5, \ldots 0.7$.

t (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.3				
0.4				
0.5				
0.6				
0.7				

(c) Explain the meaning of the fifth column in the table of part (b). What do the entries in this column suggest about the speed of the car at 12:30pm?

(d) Now we'll use the fact that

$$s = f(t) = 48t^2 - 16t^3, 0 \le t \le 2,$$

to construct another table like the one in part (b). Do this by finding expressions for

$$\Delta s = f(t) - f(0.5)$$

and

$$r = g(t) = \frac{\Delta s}{\Delta t} = \frac{f(t) - f(0.5)}{t - 0.5},$$

both in terms of t (and *not* Δt). Use these functions to fill in the missing entries in the table below.

t (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.49				
0.499				
0.4999				
0.5				
0.5001				
0.501				
0.51				

(e) Does your table from part (d) suggest that the ratios $\Delta s/\Delta t$ approach some number as $t \to 0.5$? If so, what would be your guess for the exact value of this number? What are its units? What is its meaning?

(f) Activate the folder "Graph of average speed" on Line 9.

(i) Use the graph to check some your entries in the fifth column of your table from part (d). Explain.

(ii) How is the line PQ, through the fixed point $P(0.5, 10)$ and the variable point $Q(t, s)$ on the graph of the function $r = g(t)$ related to the ratio $\Delta s/\Delta t$?

(iii) Describe what happens to the line PQ as point Q approaches point P.

(g) For a quicker way to approximate car's speed at 12:30pm, zoom in sufficiently close to point P in the graph above to make the graph of $s = f(t)$ look like a line. Use the coordinates of point P and a second point in the window far away from P to estimate the car's speed at 12:30pm. Explain your method.

(h) Summarize your conclusions by comparing your three estimates for the car's speed at 12:15pm. Which estimate do you think is most accurate? Least accurate?

Example 8. This is a continuation of the previous example where we'll algebraically compute the exact speed of the car at 12:30pm, using the odometer function

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2.$$

The idea is to first find an algebraic expression for the car's average speed between time t and time $t = 0.5$ hours past noon. Then we'll evaluate the limit of this average speed as $t \to 0.5$ to find the (instantaneous) speed at 12:30pm.

Question 58 *First we'll find the average speed between time t and time $t =$ 0.5.*

(a) Explain in general how to compute a car's average speed over some time interval. What do you need to know? What is the computation? Make up your own specific example.

(b) Now for our particular odometer function above, the average speed $v_{\text{avg}}(t)$, measured in miles/hour, between time t and time $t = 0.5$ is

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{48t^2 - 16t^3 - 10}{t - 0.5} \\ &= \frac{(2t-1)(-8t^2 + 20t + 10)}{t - 0.5} \\ &= -16t^2 + 40t + 20 \text{ if } t \neq 0.5. \end{aligned}$$

The key step in the computation above is in the third line. How did we know $2t - 1$ was a factor of

$$f(t) - f(0.5) = 48t^2 - 16t^3 - 10?$$

The reason is that $t = 0.5$ is a root of the polynomial $f(t) - f(0.5)$ and therefore $t - 0.5$ is a factor. And so

$$2(t - 0.5) = 2t - 1$$

is also a factor. Then we can use long division to find the quotient.

(c) Show the steps in the long division.

(d) The final step in computing the car's speed v (in miles/hour) at 12:30pm is to evaluate the limit of these average speeds as $t \to 0.5$. We get

$$\begin{aligned} v &= \lim_{t \to 0.5} (-16t^2 + 40t + 20) \\ &= 36. \end{aligned}$$

(e) Here's another way to simplify the average speed in part (b). Fill in the missing steps.

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{(48t^2 - 16t^3) - (48(0.5)^2 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{(48t^2 - 48(0.5)^2) - (16t^3 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{48(t^2 - (0.5)^2) - 16(t^3 - (0.5)^3)}{t - 0.5} \\ &= \frac{48(t - 0.5)(t + 0.5)) - 16(t - 0.5)(t^2 + 0.5t + 0.25)}{t - 0.5} \\ &= 48(t + 0.5)) - 16(t^2 + 0.5t + 0.25) \ \textit{if}\ t \neq 0.5. \end{aligned}$$

(f) Use the above expression for the average speed function to compute the (instantaneous) speed of the car at 12:30pm by evaluating the approriate limit.

(g) Sketch by hand *a graph of the average speed function $y = v_{avg}(t)$ over the appropriate domain. Be sure also to state this function's domain.*

2 A Projectile

Example 9. Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/l4fknr0hpl`

Desmos activity available at

151: Projectile

3 The Falling Ladder, Part 1

Example 10. The top end of a ten-foot ladder leans against a vertical wall and the bottom end rests on the horizontal floor. We analyze how a small change in the distance between the wall and the bottom of the ladder affects the height of the ladder's top above the floor.

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/dvyuifyyg4`

Desmos activity available at

151: Ladder 1B

Question 59 *(a) The slider s in Line 1 of the demonstration above controls the distance between the wall and the bottom of the ladder, measured in feet. Use the slider s to describe* qualitatively *how a small change in s changes the height h (also measured in feet) of the ladder's top end above the floor.*

(i) Do the small changes have the same or opposite signs?

(ii) At what positions of the ladder does a small change in s result in a comparatively large change in h?

(b) Now let's focus on the particular position of the ladder where the bottom end is $s = 8$ feet from the wall. For this, turn on the "one position" folder in Line 3.

(i) Drag the slider s close to $s = 8$ and use the cordinates of the endpoints of the ladder to construct a table with five columns showing the values of s, h, $\Delta s = s - 8$, $\Delta h = f(s) - f(8)$, and $\Delta h/\Delta s$. Include units in the heading of each column. The table should include seven rows, with $s = 7.7, 7.8, \ldots 8.2, 8.3$. Here $h = f(s)$ is the function described in part (ii) below.

(ii) Find a function

$$h = f(s), 0 \leq s \leq 10,$$

that expresses the height of the ladder's top end above the ground (in feet) in terms of the distance of its bottom end from the wall (in feet).

(iii) Use your function f to construct another table, like the one in part (i), with $s = 7.9, 7.99, 7.999, 8, 8.0001, 8.01, 8.1$. Do this by finding expressions for

$$\Delta h = f(s) - f(8)$$

and

$$r = g(s) = \frac{\Delta h}{\Delta s} = \frac{f(s) - f(8)}{s - 8},$$

both in terms of s (and not *Δs).*

(iv) Does your table from part (iii) suggest that the ratios $\Delta h/\Delta s$ approach some number as $s \to 8$? If so, what would is your guess for the exact value of this limit? What are its units? What is its meaning?

(v) Activate the folder "graph of function" on Line 8. How is the line PQ, through the fixed point $P(8, 6)$ and the variable point $Q(s, h)$ on the graph of the function $h = f(s)$ related to the ratios $\Delta h/\Delta s$?

(vi) Change the bounds for s in Line 1 to run between $s = 7.9$ and $s = 8.1$. Then activate the folder "graph: average rate of change function" on Line 18. Move the slider s and use the graph of the function $r = g(s)$ to check your computations in part (iii).

(vii) Use the result of part (iv) to write an approximation for the change in height

$$\Delta h = h - 8$$

in terms of the change

$$\Delta s = s - 8.$$

(viii) The graph of the function $h = f(s)$ suggests another, geometric way to find the proportionality constant (of part (vii)) that relates Δh to Δs. Explain how.

4 The Falling Ladder, Part 2

Example 11. A tree leans precariously with its trunk making an angle of $\phi = \pi/6$ radians with the ground. One end of a ten-foot ladder leans against the trunk, the other rests on the horizontal ground. We analyze how a small change in the distance between the bottom of the ladder and the base of the trunk changes the distance between the top of the ladder and the base of the trunk.

Access Geogebra interactives through the online version of this text at

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Geogebra link: `https://www.geogebra.org/m/qmke5y7x`

We'll let t be the distance between the top of the ladder and the base of the trunk (measured in feet) and s the distance between the bottom of the ladder and the base of the trunk (also measured in feet).

The slider θ in the demonstration above controls the angle that the ladder makes with the ground, but this angle does not come into play in our problem.

(a) Use the slider θ to describe qualitatively how a small change in s (the length of segment GC) changes t (the length of segment GB):

(i) For what positions of the ladder do these small changes have the same signs? Opposite signs?

(ii) For what positions of the ladder does a small change in s result in a comparatively large change in t?

(b) Now let's focus on the particular position of the ladder when the bottom end C is 16 feet from the trunk's base and the top end D is about 8 feet $f$rom the base *as illustrated above*. Our first step is to find a function

$$t = f(s)$$

that expresses t in terms of s for values of s and t near $s = 16$ and $t = 8$ respectively.

To do this, first use the law of cosines to write an equation relating s and t. Then *complete the square* to solve this equation for t in terms of s to find the function $t = f(s)$. Keep in mind that when $s = 16$, we must have $t \sim 8$.

Hint: *Here is an outline of the steps to check your work:*

(i) Use the law of cosines to relate s and t.

$$s^2 + t^2 - \sqrt{3}st = 100.$$

(ii) Solve the above equation for t in terms of s as follows:

First rewrite the equation with the two terms with t as a factor on the left side and the other two terms on the right.

$$t^2 - \sqrt{3}st = 100 - s^2.$$

Then complete the square by adding the same perfect square to each side.

$$t^2 - \sqrt{3}st + \left(\frac{\sqrt{3}s}{2}\right)^2 = 100 - s^2 + \left(\frac{\sqrt{3}s}{2}\right)^2.$$

Then factor the LHS and simplify the RHS.

$$\left(t - \frac{\sqrt{3}s}{2}\right)^2 = 100 - \frac{s^2}{4}.$$

Next, solve for t in terms of s.

$$t = \frac{\sqrt{3}}{2}s \pm \sqrt{100 - \frac{s^2}{4}}$$

Finally, make the correct choice of $\pm$ to solve for t in terms of s, when s is near 16 and t near 8.

$$t = \frac{\sqrt{3}}{2}s - \sqrt{100 - \frac{s^2}{4}}.$$

(c) Use your function from part (b) to find an expressions for

$$\Delta t = f(s) - f(16)$$

and for the function

$$r = g(s) = \frac{\Delta t}{\Delta s} = \frac{f(s) - f(16)}{s - 16}.$$

Explain what the output of the function g measures. What are its units?

(d) Use the results of part (c) to construct a table with five columns showing the values of s, t, $\Delta s = s - 16$, $\Delta t = f(s) - f(16)$, and $\Delta t/\Delta s$. Include units in the heading of each column. The table should include seven rows, with $s = 15.9, 15.99, 15.999, 16, 16.001, 16.01, 16.1$.

(e) Does your table from part (d) suggest that the ratio $\Delta t/\Delta s$ approaches some number as $s \to 16$? If so, approximate the value of this number. What are its units?

(f) Check the box "GraphofRelation" in the demonstration above and explain how the line EF is related to part (d).

(g) Use the result of part (e) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of s near 16. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 16.4 feet from the trunk's base. Compare your approximation with the exact distance.

5 Riding a Ferris Wheel

Suppose you ride a ferris wheel

Limits

Limits in context.

1 Limits and Tangent Lines

Example 12. Let
$$g(x) = \frac{x^2 - 9}{3x - 9}.$$

(a) Evaluate each of the following expressions.

(i) $g(7)$

(ii) $\lim_{x \to 7} g(x)$

(iii) $g(3)$

(iv) $\lim_{x \to 3} g(x)$

(b) Simplify and then graph the function $g(x)$.

(c) Interpret the expressions in part (a) geometrically by considering the graph of the function $f(x) = x^2/3$ as in the demonstration below.

Desmos link: `https://www.desmos.com/calculator/u0uvuchnrk`

Desmos activity available at 151: Parabola Basic

2 Limits and Gas Mielage

Example 13. The function

$$G = f(s) = \frac{2}{5} + \frac{1}{5000}(40(s+2)^2 - (s+2)^3),\ 0 \leq s \leq 23,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/xzkknfpkw3`

Learning outcomes:
Author(s):

Desmos activity available at 151: Gas as a Function of Distance

(a) Use the graph of the function f shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Find your average gas mileage (in miles/gallon) over the interval $s \in [8, 18]$.

(c) Use the graph to approximate your gas mileage at the moment you are 18 miles from home. Do this by zooming in on the appropriate point.

(d) Use the algebra of limits to determine your exact gas mileage at the moment you are 18 miles from home.

(e) Use the result of part (d) to approximate the change in the volume of gas

$$\Delta G = f(s) - f(18)$$

in terms of the change

$$\Delta s = s - 18$$

in your distance from home for values of s near 18 miles. What are the units of the proportionality constant?

(f) Use the result of part (e) to approximate your distance from home when there are 1.9 gallons of gas in your tank.

3 Limits, Gas Mileage and Speed

Example 14. Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and that the car gets 38 miles/gallon at a speed of 40 miles/hour.

(a) Find an expression for the function

$$G = f(v)\,,\; 30 \leq v \leq 70,$$

that gives the gas mileage (in miles/gal) in terms of the speed (in miles/hour).

(b) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the car's speed at 60 miles/hour gives an approximately proportional change in its gas mileage.

(c) Use the results of part (b) to approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find the exact value of the proportionality constant.

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta G = g - f(60)$$

in gas mileage in terms of a small change

$$\Delta v = v - 60$$

in the car's speed.

(g) Use part (f) to approximate the speed at which the car gets 36 miles/gallon.

(h) Would you expect your approximation in part (g) to be greater or less than the actual speed? Explain your reasoning with a graph.

(i) Simplify the units of the proportionality constant. What might these units suggest about a way to interpret the constant?

(j) At what rate (in gal/hr) does the car burn gas at a speed of 60 miles/hour?

(k) How is the rate in part (j) related to the proportionality constant?

4 Limits, Speed and Altitude

Question 60 *A rock dropped from a height of* 100 *feet falls to the surface of Planet Krypton without air resistance.*

(a) By considering only the physical situation and without *doing any computations, sketch a graph of the function*

$$v = g(h)\,, 0 \le h \le 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet).

(b) Use the results from part (a) to choose a reasonable expression for the function g from the list below.

Multiple Choice:

(a) $g(t) = 100 - 9t^2$, $0 \le t \le 10/3$

(b) $g(h) = 100 - 9h^2$, $0 \le h \le 100$

(c) $g(h) = 0.005(100 - h)^2$, $0 \le h \le 100$

(d) $g(h) = 6\sqrt{100 - h}$, $0 \le h \le 100$ ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 *feet gives an approximately proportional change in its speed.*

(d) Use the results of part (c) to approximate the proportionality constant. What are its units?

(e) Use the algebra of limits to find the exact value of the proportionality constant.

(f) Explain the meaning of the proportionality constant.

(g) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(h) Use part (g) to approximate the rock's speed at a height of 63 *feet.*

(i) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(j) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

5 Limits and Gas Mileage

6 Limits and Purchasing Power

The Derivative

Computing derivatives with limits.

Example 15. Suppose

$$y = f(x) = x^2$$

and let's use limits to evaluate

$$f'(3) = \frac{dy}{dx}\bigg|_{x=3} = \frac{d(x^2)}{dx}\bigg|_{x=3}.$$

We have

$$\begin{aligned} f'(3) = \frac{d(x^2)}{dx}\bigg|_{x=3} &= \lim_{v \to 3} \frac{f(v) - f(3)}{v - 3} \\ &= \lim_{v \to 3} \frac{v^2 - 9}{v - 3} \\ &= \lim_{v \to 3} \frac{(v-3)(v+3)}{v - 3} \\ &= \lim_{v \to 3} (v + 3) \\ &= (3 + 3) \\ &= 6. \end{aligned}$$

Next let's do almost the same thing and compute

$$f'(x) = \frac{dy}{dx}$$

for the function

$$y = f(x) = x^2$$

by replacing 3 in the above computation with x.

Learning outcomes:
Author(s):

We get

$$\begin{aligned}\frac{dy}{dx} = \frac{d(x^2)}{dx} &= \lim_{v\to x} \frac{f(v)-f(x)}{v-x} \\ &= \lim_{v\to x} \frac{v^2-x^2}{v-x} \\ &= \lim_{v\to x} \frac{(v-x)(v+x)}{v-x} \\ &= \lim_{v\to x} (v+x) \\ &= (x+x) \\ &= 2x.\end{aligned}$$

Just as a check, when $x = 3$,

$$\left.\frac{dy}{dx}\right|_{x=3} = (2x)\Big|_{x=3} = 2(3) = 6.$$

Question 61 *Use the result of the previous example to solve each of the following problems. Do* not *use limits.*

(a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(-4, 16)$.

(b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).

(c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.

(d) Let $\mathcal{L}$ be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where $\mathcal{L}$ intersects the y-axis.

(e) Repeat parts (a)-(d) above for the tangent line to the parabola $y = x^2$ at the general point (a, a^2). What do you notice?

Question 62 *(a) Use the method of Example 1 for the function*

$$y = g(x) = 1/x^2$$

to compute

$$g'(3) = \left.\frac{dy}{dx}\right|_{x=3} = \left.\frac{d(1/x^2)}{dx}\right|_{x=3}$$

and

$$g'(x) = \frac{dy}{dx}.$$

(b) Use the result of part (a) to find an equation of the tangent line to the curve $y = 1/x^2$ at the point $(3, 1/9)$.

Question 63 *(a) Use numerical methods to estimate the slope of the tangent line to the curve*

$$y = f(x) = x^3$$

at the point $(2, 8)$. Include enough data to suggest a progression toward a limit.

(b) Use the algebra of limits to find the exact slope of the tangent line in part (a).

(c) Use algebra to find the coordinates of the all pointw where the tangent line in part (a) intersects the curve $y = x^3$.

(d) Suppose you measure the edge length of a cube to be 2cm and then use this measurement to compute the volume of the cube. Use the result of part (b) to approximate your error in computing the volume in terms of your error in measuring the edge length. Assume the latter error is small.

Then compare your exact error in computing the volume with your approximation for some specific edge length near 2cm. You should start this problem defining a function with meaningful variable names (do not use x and y).

Question 64 *(a) Find a function*

$$s = g(V), V \geq 0$$

that expresses the edge length (measured in cm) of a cube in terms of its volume (measured in cubic centimeters).

(b) Use the algebra of limits to evaluate the derivative

$$g'(V_0) = \left.\frac{ds}{dV}\right|_{V=V_0}.$$

(c) What are the units of the derivative in part (b)? Explain how you know.

(d) Suppose you submerge the cube in water and measure its volume to be 8 cm^3. You then use this measurement to compute the edge length of the cube. Use the result of part (b) to approximate your error in computing the edge length in terms of your error in measuring the volume. Assume the latter error is small.

Start this problem by defining the errors Δs and ΔV in terms of the actual edge length s of the cube.

(e) Use the result of part (d) to approximate your error in computing the edge length of the cube if the cube's actual volume is 7.7cm^3.

(f) Explain how this question is related to Question 4.

Question 65 *Suppose that between speeds of* 60 *miles/hr and* 72 *miles/hr, the gas mileage of a car is a linear function of its speed. Suppose also that the car gets* 36 *miles/gallon at a speed of* 60 *miles/hour and* 32 *miles/gallon at a speed of* 72 *miles/hour.*

(a) Find a function

$$r = f(v), 60 \leq v \leq 72,$$

that expresses the rate (measured in gal/hr) at which the car burns gas in terms of its speed (measured in miles/hour). Explain your reasoning. This function is not *linear.*

Hint:

$$r = f(v) = \frac{3v}{168 - v}, 60 \leq v \leq 72.$$

(b) Use numerical methods to estimate the value of the derivative

$$f'(63) = \left.\frac{dr}{dv}\right|_{v=63}.$$

Make a table that shows enough data to suggest a progression toward a limit. Include units in all column headings.

(c) Use the algebra of limits to find an expression for the derivative

$$f'(v) = \frac{dr}{dv}.$$

Then use this expression to find the exact value of the derivative in part (b).

(d) What are the units of the derivative in part (b)? Explain its meaning.

(e) Use the result of part (c) to approximate the change

$$\Delta r = f(v) - f(63)$$

in the rate at which your car burns gas in terms of the change

$$\Delta v = v - 63$$

of the car's speed. Assume $\Delta v \sim 0$.

Question 66 *Suppose that between speeds of* 30 *miles/hour and* 70 *miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that* *the car gets a maximum of* 42 *miles/gal at a speed of* 50 *miles/hour and* 34 *miles/gallon at a speed of* 30 *miles/hour.*

(a) Find a function

$$r = h(v)\,,\ 30 \le v \le 70,$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour).

Hint: *(i) At what rate does the car burn gas at a speed of* 50 *miles/hour?* 25/21 *gal/mile*

(ii) Find a function that expresses the gas mileage G (measured in miles/gallon) in terms of the speed (measured in miles/hr).

$$G = 42 - 0.02(v-50)^2\,,\ 30 \le v \le 70.$$

(ii) The rate (in gal/hr) at which the car burns gas as a function of its speed (in miles/hr) is

$$r = h(v) = \frac{v}{42 - 0.02(v-50)^2}\,,\ 30 \le v \le 70.$$

(b) Use the algebra of limits to evaluate the derivative

$$h'(40) = \left.\frac{dr}{dv}\right|_{v=40}.$$

(c) What are the units of the above derivative? How do you know?

(d) Express the meaning of the derivative in the context of small changes.

Question 67 *At 10:00am on April 18, the wholesale price of Cosmic Crisp apples is* \$2.00*/lb and is decreasing at the rate of* \$0.10/*lb*/*hour*.

Use the algebra of limits to determine the rate (in pounds/hour) at which the number of pounds of apples a store can purchase with \$1000 *is changing at this time.*

Start this question by defining a function that expresses the number of pounds of apples the store can buy with \$1000 *in terms of the price (in* \$*/lb). Choose meaningful variable names (not x and y). Do* not *assume the price is decreasing at a constant rate.*

1 Limits, Speed and Altitude

Question 68 *A rock dropped from a height of* 100 *feet falls to the surface of Planet Krypton without air resistance.*

(a) By considering only the physical situation and without *doing any computations, sketch a graph of the function*

$$v = g(h)\,, 0 \leq h \leq 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet). Explain your reasoning.

(b) Use the results from part (a) to choose a reasonable expression for the function g from the list below.

Multiple Choice:

(a) $g(t) = 100 - 9t^2$, $0 \leq t \leq 10/3$

(b) $g(h) = 100 - 9h^2$, $0 \leq h \leq 100$

(c) $g(h) = 0.005(100 - h)^2$, $0 \leq h \leq 100$

(d) $g(h) = 6\sqrt{100 - h}$, $0 \leq h \leq 100$ ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 *feet gives an approximately proportional change in its speed. Then approximate the proportionality constant. What are its units?*

(d) Use the algebra of limits to find an expression for the derivative

$$g'(h) = \frac{dv}{dh}.$$

Then use this expression to find the exact value of the proportionality constant in part (c).

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(g) Use part (g) to approximate the rock's speed at a height of 63 *feet.*

(h) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(i) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

Derivatives of Polynomials

Working with polynomials and their derivatives.

1 Differentiating Polynomials

Question 69 *Let $f(x) = x^5$.*

(a) *Use the algebra of limits to find an expression for the derivative*

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(x^5\right).$$

(b) *Use the result of part (a) to find an expression for the derivative*

$$\frac{d}{dx}(f^{-1}(x)) = \frac{d}{dx}\left(x^{1/5}\right).$$

Use set-builder notation to state the domains of $f^{-1}(x)$ and its derivative.

(c) *Use the result of part (a) and the algebra of limits to find an expression for the derivative*

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{d}{dx}\left(x^{-5}\right).$$

(d) *Use the results of parts (a)-(c) to find equations of the tangent lines to the three curves $y = f(x)$, $y = f^{-1}(x)$, and $y = 1/f(x)$ at the point $(1, 1)$. Graph the curves and their tangent line in Desmos to check your work.*

Question 70 *(a) Use the results of Question 1 to make a conjecture about the derivative*

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^n)$$

of the function $f(x) = x^n$.

(b) What do you get for the derivative when $n = 0$? When $n = 1$? Are these results correct? Explain.

Learning outcomes:
Author(s):

Question 71 *Due to a printing error, the graph of the function $y = f(x) = 234x^5$ is missing below. All we see is a point A on the graph of the function.*

Desmos link: `https://www.desmos.com/calculator/pjyqjtdcxm`

151: Printers Error 2

Without *sketching a graph, drag the slider m to draw the tangent line to the curve $y = 234x^5$. Explain your reasoning.*

Question 72 *(a) Suppose one giraffe is always twice as tall as another. What can you say about their growth rates at any instant?*

(b) Suppose one giraffe is always two feet taller than another. What can you say about their growth rates at any instant?

(c) What do parts (a) and (b) suggest about how to compute the derivatives

$$\frac{d}{dx}\left(f(x) + b\right),$$

$$\frac{d}{dx}\left(af(x)\right),$$

and

$$\frac{d}{dx}\left(af(x) + b\right)$$

for constants $a, b \in \mathbb{R}$?

(d) Make up your own scenario that suggests how to compute the derivative

$$\frac{d}{dx}\left(f(x) + g(x)\right)$$

of the sum of two functions.

Question 73 (a) *Find an equation of the tangent line to the curve $y = 36/x$ at the point $P(6, 6)$. Then explain how you could have solved this problem without using calculus.*

(b) *Find an equation of the tangent line to the curve*

$$y = -x^3 + 4x^2 - 3x + 1$$

at the point $(2, 3)$. Graph the curve and its tangent line on Desmos.

(c) *The function*

$$s = f(h) = 1.22\sqrt{h}\,,\ 0 \le h \le 30000,$$

expresses the distance to the horizon (in miles) in terms of your altitude (in feet).

(i) *Evaluate $f(25)$ and interpret its meaning.*

(ii) *Evaluate the derivative*

$$\left.\frac{ds}{dh}\right|_{h=25}.$$

Include units.

(iii) *Use the language of small changes to interpret the meaning of the derivative above.*

2 Stock Price

Question 74 *The function*

$$P = f(t) = 0.1t^3 - t^2 + t + 20\,,\ 0 \le t \le 8,$$

expresses the price (in dollars/share) of a stock in terms of the number of hours past 9am. The graph of the function is shown below.

Desmos link: `https://www.desmos.com/calculator/gpqicgxvk3`

151: Stock Price

(a) *Is the price increasing or decreasing at 10am? Use the graph to estimate this rate as follows:*

(i) *First drag the slider u (another name for t) on Line 5 to the appropriate value.*

(ii) *Then drag the slider m (the slope of the orange line) so that the orange line looks parallel to the (red) tangent line to the curve $P = f(t)$.*

(b) *Use the function to compute the exact rate at which the price is changing at 10am. Include units.*

(c) *At what relative rate is the price changing at 10am? Include units.*

(d) *At what rate is the number of shares you can buy with* \$1000 *changing at 10am? Do* not *use the quotient rule or the chain rule (we have not learned these yet). Use limits instead, but work in general (ie. not with this specific price function). See the Explanation below for help.*

(e) *Use the sliders m and b to control the line $P = b + mt$ and approximate when the price is decreasing at the rate of (\$1.8/share)/hr. Then use calculus and algebra to compute the exact time(s).*

(f) *Use calculus and technology to approximate the time(s) when the price is decreasing at the relative rate of 10%/hour.*

(g) *Use the graph to approximate the minimum and maximum prices of the stock during the eight-hour period. Then use calculus and algebra to compute these exact prices.* Hint: *What is the value of the derivative dP/dt when the price is a maximum/minimum?*

(h) *Use the slider u (another name for the variable t) to approximate the time(s) when the price is decreasing at the fastest rate. Then use calculus to compute the exact time(s).*

Hint: *It helps to graph the fuction*

$$dP/dt = 0.03t^2 - 2t + 1\,,\; 0 \le t \le 8,$$

Then think about how you would use calculus *to compute the exact time when the price is decreasing at the fastest rate. This is similar to the previous question.*

(i) *Use the graph above to approximate the end of the time interval beginning at 9am over which the price decreases at the greatest average rate. Do this by activating the folder* secant line *on Line 6 (click the circle on the left) and dragging the slider u.*

(j) *The use calculus and algebra to find the end of the time interval beginning at 9am over which the price decreases at the greatest average rate.*

Hint: *Think about the function*

$$m(t) = \frac{f(t) - f(0)}{t - 0}\,,\; 0 < t \le 8,$$

that expresses the average rate of change of the price (with respect to time) over the time interval between 9am and time t hours past 9am. Then use this function and the ideas of parts (g) and (h).

Explanation. (d) Here's the solution to part (d).

- Start by finding a function

$$S = g(t)\,,\; 0 \le t \le 8,$$

that expresses the number of shares we can buy with $\$1,000$ in terms of the number of hours past 9am.

This function is

$$S = g(t) = \frac{1000}{f(t)}\ ,\ 0 \leq t \leq 8$$

- Next we'll use limits to find an expression for the derivative

$$\left.\frac{dS}{dt}\right|_{t=a}$$

as follows.

$$\begin{aligned}
\left.\frac{dS}{dt}\right|_{t=a} &= \lim_{v \to a} \frac{g(v) - g(a)}{v - a} \\
&= \lim_{v \to a} \left(\frac{1000}{v - a}\right)\left(\frac{1}{f(v)} - \frac{1}{f(a)}\right) \\
&= \lim_{v \to a} \left(\frac{1000}{v - a}\right)\left(\frac{f(a) - f(v)}{f(v)f(a)}\right) \\
&= \lim_{v \to a} \left(\frac{-1000}{f(v)f(a)}\right)\left(\frac{f(v) - f(a)}{v - a}\right) \\
&= \lim_{v \to a} \frac{-1000}{f(v)f(a)} \cdot \lim_{v \to a} \frac{f(v) - f(a)}{v - a}
\end{aligned}$$

Let's pause here for a moment. The last equality above follows from the fact that both limits in the product exist. The first limit is easy enough to evaluate. The second limit exists because *it is equal to the derivative*

$$\left.\frac{dP}{dt}\right|_{t=a}.$$

That's the key point. Now we know that

$$\begin{aligned}
\left.\frac{dS}{dt}\right|_{t=a} &= \lim_{v \to a} \frac{-1000}{f(v)f(a)} \cdot \lim_{v \to a} \frac{f(v) - f(a)}{v - a} \\
&= \left(\frac{-1000}{(f(a))^2}\right)\left(\left.\frac{dP}{dt}\right|_{t=a}\right)
\end{aligned}$$

This is enough to compute the rate at which the number of shares we can buy with $1000 is changing at 10am.

But we can get a better understanding of this by writing the above expression in a more meaningful way. Remember that $f(a) = P$ is the price at time t. So

$$\begin{aligned}\left.\frac{dS}{dt}\right|_{t=a} &= -\left(\frac{1000}{(f(a))^2}\right)\left(\left.\frac{dP}{dt}\right|_{t=a}\right)\\ &= -\left(\frac{1000}{P^2}\right)\left(\left.\frac{dP}{dt}\right|_{t=a}\right)\\ &= -\left(\frac{1000}{P}\right)\left(\frac{1}{P}\cdot\left.\frac{dP}{dt}\right|_{t=a}\right)\\ &= -S\left(\frac{1}{P}\cdot\left.\frac{dP}{dt}\right|_{t=a}\right),\end{aligned}$$

where $S = 1000/P$ is the number of shares we can buy with $\$1,000$ at a price of P dollars/share.

Now remember that the product

$$\frac{1}{P}\cdot\left.\frac{dP}{dt}\right|_{t=a}$$

is the *relative* rate of change in the price (with respect to time) at time $t = a$ hours past 9am.

From part (c) we know at 10am this relative rate is equal to

$$\frac{1}{P}\cdot\left.\frac{dP}{dt}\right|_{t=1} = -\frac{0.7}{20.1}\text{hr}^{-1} \sim -3.4826\%/\text{hour}$$

So at 10am we can buy

$$S = \frac{\$1000}{\$20.1/\text{share}} \sim 47.7512 \text{ shares}$$

with $\$1,000$. And the number of shares we can buy with is increasing at the rate of approximately

$$(47.7512 \text{ shares})\,(-3.4826\%/\text{hour}) \sim 1.733 \text{ shares/hour}.$$

at 10am.

Thinking about Parabolas

Question 75 *(a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(-3, 9)$.*

(b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).

(c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.

(d) Let $\mathcal{L}$ be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where $\mathcal{L}$ intersects the y-axis.

(e) Repeat parts (a)-(d) above for the tangent line to the parabola $y = x^2$ at the general point (b, b^2). What do you notice? Enter your work in the Desmos activity below.

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/qe7mgnu5sv`

Question 76 *Let*

$$y = f(x) = ax^2$$

where $a \in \mathbb{R}$ is a constant, and the variables x, y are measured in meters.

(a) What are the units of the constant a? How do you know?

(b) Answer part (e) of the previous question for this function. Modify the desmos activity in the previous question to check your work.

Question 77 *In the absense of air resistance, a rock released from rest near the surface of the earth falls $s = 16t^2$ feet during the first t seconds of its fall.*

Compare the speed of the rock when it hits the ground with the average speed of the rock during the entire time interval of its fall.

Question 78 *(a) The demonstration below shows two normal lines to a parabola and their point of intersection P. What do you think happens to P as point B approaches A? Answer this question* without *dragging the slider b.*

(b) Now drag the slider b near $a = 2$ and observe what happens to point P. Were you correct?

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/ybaivhc2tl`

Access this activity online at 151: Normals to Parabola

The parabola has equation $y = x^2/4$, point A has coordinates $(2, 1)$, and point B has coordinates $(b, b^2/4)$.

(c) Find an equation of the normal lines to the parabola at A and B.

(d) Use algebra to find an equation of the point P where the normal lines intersect.

(e) The point P approaches some point Q as B appraoches A. Use the algebra of limits to find the coordinates of Q.

(f) Find an equation of the circle centered at Q through A. Do this by first using vector algebra to find the coordinates of the center of the circle.

(g) Repeat parts (c)-(g), replacing the point $A(2, 1)$ with the point $A(a, a^2)$.

Applications

Question 79 *Play the slider u in the animation below to watch the motion of a balloon as it leaves behind track's at equal time intervals. Ignore the balloon's rightward jogs. Their purpose is to prevent the tracks from overlapping.*

Desmos link: `https://www.desmos.com/calculator/h91txxjcmi`

151: Balloon Motion 7

(a) Use the animation to sketch by hand the graph of the function

$$h = f(t)\,,\ -1 \le t \le 1.8,$$

that expresses the balloon's height (in thousands of feet) in terms of the number of hours past noon.

(b) Use the animation or your graph from part (a) to sketch by hand the graph of the function

$$r = g(t)\,,\ -1 \le t \le 1.8,$$

that expresses the balloon's rate of ascent (in thousands of feet/hour in terms of the number of hours past noon.

(c) Use your graph(s) or the animation to approximate when the balloon descends at the fastest rate.

Question 80 *The function*

$$h = f(t) = 5 - 2t - \frac{t^2}{4} + t^3\,,\ -1 \leq t \leq 1.8,$$

expresses the height (in thousands of feet) of a balloon in terms of the number of hours past noon.

(a) Find the balloon's average rate of ascent between 11:00am and 11:30am.

(b) Is the balloon rising or falling at 1:00pm? At what rate? Use the graph of the function $h = f(t)$ below to approximate the rate. Then compute the exact rate.

(c) When is the balloon descending at the rate of 1000 ft/hour? Use the sliders m anb b below to approximate the time(s). Then compute the exact time(s).

(d) Use the graph below to approximate when the balloon is descending at the fastest rate. Approximate this rate from the graph. Then compute the exact time and rate.

(e) Use the graph below to approximate when the balloon is at its lowest point. Then compute the exact time.

(f) Use the graph below to approximate when the balloon is at its highest point between 11am and 1:36pm. Then compute the exact time.

(g) Use algebra to find all half-hour time intevals during which the balloon descends at an average rate of 500 ft/hour.

Desmos link: `https://www.desmos.com/calculator/bvtukd0vlc`

Access this activity online at 151: Height of Balloon

Question 81 *The function*

$$T = f(m) = \frac{m^3}{3} - 3m^2 + m + 35\,,\ 0 \leq m \leq 8,$$

expresses the temperature (in Celsius degrees) of a beaker of water in terms of the number of minutes past noon. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/jrzsv06pwt`

Desmos activity available at 151: Beaker Temperature

(a) *Use the sliders above to appoximate the time(s) when the temperature is decreasing at the rate of* $5^\circ C/min$.

(b) *Use calculus and algebra to find the exact times in part (a). Do* not *rely on the quadratic formula. Complete the square instead.*

(c) *When is the temperature a minimum? Use the graph to approximate the time. Then use calculus and algebra to find the exact time.*

Question 82 *The function*

$$P = f(t) = 8 + 4t - t^2\,,\ 0 \le t \le 5,$$

expresses the price (in dollars/share) of a stock in terms of the number of hours past 9am. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/jrwe6t0x41`

Desmos activity available at 151: Stock Price Short

(a) *Is the price increasing or decreasing at 10am? At what relative rate?*

(b) *Use the slider above to appoximate the time(s) when the price is decreasing at the relative rate of 40%/hour.*

(c) *Use calculus and algebra to find the exact time(s) in part (b). Do* not *rely on the quadratic formula. Complete the square instead.*

Question 83 *The function*

$$G = f(s) = \frac{11}{5} + \frac{1}{5000}\left(s^3 - 50s^2 + 300s\right)\,,\ 3 \le s \le 28,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/cphmgnrtm7`

Desmos activity available at 151: Gas as a Function of Distance 2c

(a) Use the graph of the function f shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Zoom in on the graph to approximate your gas mileage at the moment you are 20 miles from home. Show a screenshot to help explain how you got your approximation. Then compute the exact gas mileage.

(c) Use the sliders m and b_1 in the graph to approximate your distance from home at the moment your car gets 30 miles/gallon. Show a screenshot to help explain how you got your approximation. Then compute the exact distances.

(d) Use the sliders m and b_1 in the graph to approximate an interval beginning or ending when you are 20 miles from home over which your average gas mileage is equal to you gas mileage at the moment you are 20 miles from home. Show a screenshot to help explain how you got your approximation. Then compute the exact interval.

(e) Sensors on your car measure both the (instantaneous) gas mileage and the number of gallons of gas in your tank at each instant. A computer then uses these measurements to estimate the number of additional miles you can drive before running out of gas. Use this idea to find a function

$$m = g(s)\,,\ 3 \leq s \leq 28,$$

that expresses the number of miles you can drive before running out of gas (assuming your gas mileage remains constant for the remainder of your trip) in terms of your distance from home. Explain your reasoning.

Question 84 *(a) Make up your own* quadratic *function*

$$v = f(G) = aG^2 + bG + c,$$

with a,b, and c all not equal to zero, that expresses the speed of a car (measured in miles/hour) in terms of its gas mileage (measured in miles/gallon). Be sure to include a domain. Explain why you think your function is reasonable.

(b) Compute the derivative dv/dG and evaluate it at a spefic gas mileage. Include units.

(c) Evaluate the derivative dv/dG at a specific gas mileage and its meaning. Include units in your explanation.

(d) Use your function from part (a) to find a function

$$r = h(G)$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its gas mileage (in miles/gal). Explain your logic thoroughly.

(e) Evaluate $h(G)$ at the same gas mileage, say G_0, you used in part (c). Compare the units of $h(G_0)$ and the derivative $dv/dG\Big|_{G=G_0}$. Are these two numbers related? What does this tell you about simplifying the units of a derivative?

(f) Use the ideas of this chapter (ie. derivatives of polynomials, and nothing beyond) to find an expression for the derivative dr/dG .

(g) What are the units of the derivative dr/dG?

(h) Evalute the derivative dr/dG at a specific gas mileage and explain its meaning. Include units in your explanation.

(i) Make up and answer your own question about the derivative dr/dG at a specific gas mileage.

A Few More Problems

Question 85 *The function*

$$h = f(t) = kt^3, t \geq 0,$$

expresses the height of a balloon (in thousands of feet) in terms of the number of hours past noon. Here $k > 0$ is a positive constant.

(a) What are the units of the constant k? Explain how you know.

(b) Find the balloon's rate of ascent (measured in thousands of ft/hr) at time $t = 3$ hours past noon. Then find its average rate of ascent (measured in thousands of ft/hr) between noon and 3pm. How are these rates related? Note that both will be expressed in terms of k.

(c) Find a function

$$r = g(a)$$

that expresses the balloon's rate of ascent (measured in thousands of ft/hr) at time $t = u$ hours past noon in terms of its average rate of ascent (measured in thousands of ft/hr) over the time interval $t \in [0, u]$. Assume $u > 0$.

(d) Interpret your result from part (b) geometrically on the graph of the function f. Follow the directions on Lines 4, 6, and 8 in the demonstration below to help explain your interpretation.

(e) Which of the following expresses the balloon's rate of ascent at time t hours past noon in terms of t and $h = f(t)$?

Multiple Choice:

(a) $\dfrac{h}{t}$

(b) $\dfrac{h}{3t}$

(c) $\dfrac{3h}{t}$ ✓

Desmos link: `https://www.desmos.com/calculator/0byjcy77yw`

Desmos activity available at 151: Subtangent 1

Question 86 *The function*

$$v = f(G)$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon) for speeds between 55 *miles/hour and* 70 *miles/hour.*

Suppose $f(30) = 60$.

(a) Which of the following is more likely to be true?

$$\frac{dv}{dG}\bigg|_{G=30} = 4$$

or

$$\frac{dv}{dG}\bigg|_{G=30} = -4?$$

Explain your reasoning.

(b) What are the units of the above derivative? Do not simplify the units and do not write "per" in place of "/".

(c) Explain the meaning of the derivative in part (a). It is not *enough to say "the rate of change of something with respect to something else." Remember this class is all about small changes and your explanation should be about an approximate relationship between small changes in this setting.*

(d) Simplify the units of the derivative. What does this suggest about its meaning?

(e) At what rate does the car burn gas (in gal/hour) at a speed of 60 *miles/hour?*

(f) What does this problem suggest about simplifying the units of a derivative?

Question 87 *The function*

$$v = f(G), 20 \leq G \leq 40,$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon). Ue the graph of the function f *to find approximate answers to the following questions. Change the position and slope of line* AB *by dragging either the line or the points* A *or* B*. Change the position of the tangent line by dragging the slider* G*.*

(a) Label the axes with the appropriate variable names and units.

(b) At what speed does the car burn gas as the fastest rate?

Hint: *One of questions (b), (c) is related to the tangent lines to the curve, the other is related to the lines through the origin and the points of the curve.*

(c) At what speed does increasing the speed by 0.1 *miles/hour result in the greatest change in the gas mileage? Approximate that change.*

Geogebra link: `https://www.geogebra.org/m/vjdf6x6z`

Geogebra activity available at 151: Gas Mileage

3 Homework Solution

Desmos link: `https://www.desmos.com/calculator/jku58rp2ve`

Gas Consumption

Gas consumption and derivatives.

Question 88 (a) *Use calculus to find an expression for the x-coordinate of the vertex of the parabola*

$$y = ax^2 + bx + c\,,\ a \neq 0.$$

(b) *Find an equation of the parabola's axis of symmetry.*

Question 89 *The function*

$$\begin{aligned} v &= f(G) \\ &= \frac{1}{20}G^2 - 2G + 40\,,\ 20 \leq G \leq 40, \end{aligned}$$

graphed below expresses the speed (in miles/hr) of a car in terms of its gas mileage (in miles/gal).

Desmos link: `https://www.desmos.com/calculator/lkxc2lxrvz`

151: Gas Mileage 34

(a) *Use the graph to determine the rate (in gal/hr) at which the car burns gas at a speed of 30 miles/hour.*

(b) *Drag the slider u in Line 2 to approximate the speeds between 20 miles/hour and 40 miles/hour at which the car burns gas at the maximum and minimum rates (measured in gal/hr). Explain your reasoning.*

(c) *Use the slider u to sketch by hand a rough graph of the function $r = h(G)$ that expresses the rate (in gal/hr) at which the car burns gas in terms of its gas mileage (in miles/gal).*

(d) *Use calculus to determine the exact speeds in part (b).*

(e) *Explain how you could justify your answers to part (d)* without *relying on the graph of the function $v = f(G)$.*

Learning outcomes:
Author(s):

Question 90 *The function*

$$\begin{aligned} G &= f(v) \\ &= -\frac{v^2}{20} + 5v - 90\,,\ 35 \le v \le 70, \end{aligned}$$

graphed below expresses the gas mileage (in miles/gal) of a car in terms of its speed (in miles/hour).

Desmos link: `https://www.desmos.com/calculator/fapdhcqptl`

151: Burning Gas

(a) *Use the graph to determine the rate (in gal/hr) at which the car burns gas at a speed of 50 miles/hour.*

(b) *Drag the slider v in Line 1 to approximate the speeds between 35 miles/hour and 70 miles/hour at which the car burns gas at the maximum and minimum rates (measured in gal/hr). Explain your reasoning.*

(c) *Use calculus to determine the exact speeds in part (b). Find a way that avoids using the quotient rule.*

Question 91 *The function*

$$G = f(s) = \frac{11}{5} + \frac{1}{5000}\left(s^3 - 50s^2 + 300s\right),\ 3 \le s \le 28,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/pb8v4t3cxg`

Desmos activity available at 151: Gas as a Function of Distance 33

(a) *Use the graph above to determine if you are driving toward or away from home. Explain your reasoning.*

(b) *Sensors on your car measure both the (instantaneous) gas mileage and the number of gallons of gas in your tank at each instant. A computer then uses these measurements to estimate the number of additional miles you can drive before running out of gas.*

(i) *Use the graph and the slider s_0 above to approximate the reading for the number of additional miles you can drive when you are* 10 *miles from home and* 20 *miles from home. Explain your reasoning.*

(ii) *Find a function*

$$m = g(s)\,,\ 3 \leq s \leq 28,$$

that expresses the number of miles you can drive before running out of gas (assuming your gas mileage remains constant for the remainder of your trip) in terms of your distance from home. Explain your reasoning.

(iii) *Use your function g to find the reading for the number of additional miles you can drive when you are* 10 *miles from home. When you are* 20 *miles from home. Compare these with your estimates above.*

The Quotient and Product Rules

Quotient and product rules.

Here are examples of how to use the Leibniz notation correctly in computing the derivative of a product and quotient of functions.

Example 16. Find expressions for the derivatives

(a)
$$\frac{d}{dx}\big((5x^3-2)(4-6x)\big)$$

(b)
$$\frac{d}{dt}\left(\frac{5t^3-2}{4-6t}\right)$$

Explanation. (a)
$$\begin{aligned}\frac{d}{dx}\big((5x^3-2)(4-6x)\big) &= (4-6x)\frac{d}{dx}\big(5x^3-2\big)+(5x^3-2)\frac{d}{dx}(4-6x)\\ &= (4-6x)15x^2+(5x^3-2)(-6).\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dt}\left(\frac{5t^3-2}{4-6t}\right) &= \frac{1}{(4-6t)^2}\left((4-6t)\frac{d}{dt}\big(5t^3-2\big)-(5t^3-2)\frac{d}{dt}(4-6t)\right)\\ &= \frac{(4-6t)15t^2-(5t^3-2)(-6)}{(4-6t)^2}.\end{aligned}$$

1 Relative Rates of Change

Suppose $P=f(t)$ and $Q=g(t)$ are differentiable functions of t and $g(t)\leq 0$. Then the quotient and product rules are better written in the forms
$$\frac{d}{dt}\left(\frac{P}{Q}\right)=\frac{P}{Q}\left(\frac{1}{P}\cdot\frac{dP}{dt}-\frac{1}{Q}\cdot\frac{dQ}{dt}\right) \tag{1}$$
and
$$\frac{d}{dt}(PQ)=PQ\left(\frac{1}{P}\cdot\frac{dP}{dt}+\frac{1}{Q}\cdot\frac{dQ}{dt}\right).$$

Learning outcomes:
Author(s):

Question 92 (a) *Verify the statements above.*

(b) *What do they say about the relative rate of change in the quotient of two functions? In their product?*

2 Exercises

Exercise 93 *Between speeds of 45 miles/hr and 75 miles/hr, the function*

$$G = f(v)\,,\ 45 \leq v \leq 75,$$

expresses the gas mileage (in miles/gal) of a car in terms of its speed (in miles/hour).

Suppose $f(50) = 25$ and

$$\left.\frac{dG}{dv}\right|_{v=50} = 0.8.$$

(a) *What are the units of the derivative above? What is the meaning? Explain in terms of small changes.*

(b) *What are the simplified units of the derivative above? What insight do these units give you about the derivative's meaning?*

(c) *Approximate the gas mileage at a speed of 48 miles/hour.*

(d) *Let*

$$r = g(v)\,,\ 45 \leq v \leq 75,$$

be the function that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hr).

(i) *Use Equation (1) above to evaluate the derivative*

$$\left.\frac{dr}{dv}\right|_{v=50}.$$

Include units for all numbers.

(ii) *What are the units of the derivative above? What is the meaning? Explain in terms of small changes.*

(iii) *What are the simplified units of the derivative above? What insight do these units give you about the derivative's meaning?*

(iv) *Approximate the rate (in gal/hr) at which the car burns gas at a speed of 48 miles/hr.*

Exercise 94 *At 9:00am on February 23, 2023, the price of oil was decreasing at the relative rate of 2%/hour. At what relative rate was the number of gallons of oil you could buy with $100,000 changing at that time? Use calculus to justify your assertion.*

Exercise 95 *The function*

$$P = f(t) = 5 - 3t + t^2 \,,\, 0 \le t \le 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

(a) Use the graphs of the function $P = f(t)$ and the function $r = f'(t)/f(t)$ to estimate when the stock price is increasing at the greatest relative rate.

(b) Use algebra to find the exact time when the stock price is increasing at the greatest relative rate.

Hint: *What is the value of the derivative dr/dt at this time? But start by finding an expression for the instantaneous relative rate of change in the stock price.*

Desmos link: `https://www.desmos.com/calculator/xuupp3srqv`

Desmos activity available at 151: Stock Price 4

Exercise 96 *An h-foot tall giraffe walks directly toward a spotlight on the ground as the light casts its shadow on a vertical wall as illustrated below. The wall is b feet from the light.*

Desmos link: `https://www.desmos.com/calculator/2eiyjjpu9n`

Desmos activity available at 151: Spotlight

Suppose at a certain instant the giraffe is w feet from the spotlight and is walking at a speed of v ft/sec.

(a) *Is the length of the giraffe's shadow increasing or decreasing at this instant?*

(b) *At what rate?*

(c) *Check that your answer in part (b) has the correct units.*

(d) *Assume v is constant and describe what happens to the rate in part (b) as the giraffe approaches the spotlight. How is your conclusion revealed in the animation?*

Exercise 97 (a) *On January 1, 2024 the national debt of a country was decreasing at the rate of 3%/yr and the population was increasing at the rate of 2%/yr. Was the per-captita (ie. per person) share of the national debt increasing or decreasing at this time? At what relative rate?*

(b) *During the year 2024 the national debt of a country decreased 3% and the population increased 2%. Did the per-captita share of the national debt increase or decrease during the year? By what percent?*

(c) *Compare the two questions above and their answers.*

Test 1

Test 1.

(a) (6 points) The function

$$W = f(r) = \frac{2000}{r^2}, r \geq 4,$$

expresses the weight of an astronaut (measured in pounds) in terms of her distance from the center of the earth (measured in thousands of miles).

(i) (2 points) Find an expression for the average rate of change in the astronaut's weight with respect to her distance from the earth's center between distances $r = b$ and $r = c$ thousands of miles from the center. Assume $b, c \geq 4$ and that $b \neq c$.

(ii) (4 points) Use your *expression from part (a) directly and the definition of the derivative* to find an expression (fully simplified) for the derivative

$$\left.\frac{dW}{dr}\right|_{r=b}.$$

In particular, do *not* use the power rule. Show all work in its mathematically correct form. Write vertically, one equal sign per line. No need to explain algebra.

Explanation. (i) The average rate of change is

$$\begin{aligned}\frac{\Delta W}{\Delta h} &= \frac{f(c) - f(b)}{c - b} \\ &= \left(\frac{1}{c-b}\right)\left(\frac{2000}{c^2} - \frac{2000}{b^2}\right)\end{aligned}$$

(ii) (Some algebra omitted, left for you to fill in).

$$\begin{aligned}\left.\frac{dW}{dr}\right|_{r=b} &= \lim_{c \to b}\left(\frac{1}{c-b}\right)\left(\frac{2000}{c^2} - \frac{2000}{b^2}\right) \\ &= \lim_{c \to b} \frac{-2000(c+b)}{c^2 b^2} \\ &= -\frac{2000(b+b)}{b^2 b^2} \\ &= -\frac{4000}{b^3}.\end{aligned}$$

Learning outcomes:
Author(s):

(b) (5 points) At 3pm, you can buy 5 pounds of cod with \$100. And at 3pm the number of pounds of cod you can buy with \$100 is decreasing at the rate of 0.4 lbs/hour.

At what rate (with respect to time) is the price of cod changing at 3pm? Explain your reasoning and be sure to end with a concluding sentence.

Explanation. Let $P = f(t)$ be the function that expresses the price of cod (in dollars/pound) in terms of the number of hours since 3pm.

Let $Q = g(t)$ be the function that expresses the number of pounds of cod you can buy with \$100 in terms of the number of hours since 3pm.

Then

$$P = \frac{100}{Q}$$

and the quotient rule gives

$$\begin{aligned}\frac{dP}{dt} &= \frac{1}{Q^2}\left(Q\frac{d}{dt}(100) - 100\frac{dQ}{dt}\right)\\ &= -\frac{100}{Q^2}\cdot\frac{dQ}{dt}\end{aligned}$$

Since at 3pm the number of pounds we can buy with \$100 is decreasing at the rate of 0.4 lbs/hour, we know that

$$\left.\frac{dQ}{dt}\right|_{t=0} = -0.4 \text{ lb/hr}.$$

And since we can buy $Q = g(0) = 5$ pounds with \$100 at 3pm,

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{t=0} &= -\left(\frac{100}{Q^2}\cdot\frac{dQ}{dt}\right)\Big|_{t=0}\\ &= \left(-\frac{100 \text{ dollars}}{25 \text{ lb}^2}\right)\left(-0.4\frac{\text{lb}}{\text{hr}}\right)\\ &= (1.6 \text{ dollars/lb})/\text{hr}.\end{aligned}$$

So at 3pm the price of cod is increasing at the rate of (1.6 dollars/lb)/hr.

(c) (5 points) Between speeds of 60 miles/hour and 80 miles/hour, the function

$$v = f(G)\,,\ 8 \le G \le 20,$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon).

Suppose $f(12) = 72$.

(i) (1 point) Which would be more likely, that

$$\left.\frac{dv}{dG}\right|_{G=12} = 4 \qquad \text{or that} \qquad \left.\frac{dv}{dG}\right|_{G=12} = -4?$$

Explain your reasoning.

(ii) (1 point) What are the units of the correct derivative above?

(iii) (2 points) Explain the meaning of the correct derivative above using the language of *small changes*.

(iv) (1 point) Simplify the units of the correct derivative above. What do these simplified units tell you about the derivative's meaning?

Explanation. (i) At the high speed of 72 miles/hour it is likely that a small increase in the speed will decrease the gas mileage. So we would expect the derivative to be negative, so

$$\left.\frac{dv}{dG}\right|_{G=12} = -4.$$

(ii) Since v has units miles/hour and G has units miles/gal, the derivative dv/dG has units

$$\frac{\text{miles/hr}}{\text{miles/gal}}.$$

(iii) We know that

$$\left.\frac{dv}{dG}\right|_{G=12} = -\frac{0.4}{0.1}\frac{\text{miles/hr}}{\text{miles/gal}} \sim \frac{\Delta v}{\Delta G}$$

if $\Delta G \sim 0$. So increasing the speed by 0.4 mile/hour (from 72 miles/hr to 72.4 miles/hr) would decrease the gas mileage by about 0.1 miles/gal (from 12 miles/gal to about 11.9 miles/gal).

(iv) The units of the simplified derivative are

$$\frac{\text{miles/hr}}{\text{miles/gal}} = \frac{\text{gal}}{\text{hr}}.$$

Although this suggests the derivative is related to the rate at which the car burns gas, this is *not* true. Our interpretation of the derivative had nothing to do with this rate and in fact the car burns gas at the rate of 6 gal/hour at a speed of 72 miles/hour. So the simplified units are misleading and tell us *nothing* about the derivative's meaning.

(d) (5 points) The function

$$G = f(s) = \frac{3s+25}{s+5}\,,\ 0 \le s \le 10,$$

expresses the number of gallons of gas in a car in terms of the trip odometer reading (measured in miles).

Sensors on the car measure both the number of gallons of gas in the tank and the current (instantaneous) gas mileage (measured in miles/gallon).

A computer then calculates the number of miles you have left to drive assuming the car's gas mileage remains constant for the remainder of your trip (and equal to the current gas mileage). This number is displayed on the dashboard.

Determine the exact trip odometer reading when the dashboard reading indicates that you have 13 miles left to drive. Explain your reasoning and be sure to end with a concluding sentence.

Explanation. I will not solve this problem, but will give you a hint instead by solving a closely related problem.

Suppose, for example, that we wish to find the dashboard reading (for the number of miles left to drive) at the odometer reading $s = 5$ miles.

First, we would find the number of gallons in the tank at this odometer reading. At this time, the tank has

$$f(5) = \frac{3(5) + 25}{5 + 5} = 4$$

gallons of gas.

Now at $s = 5$, the derivative dG/ds is (left for you to verify)

$$\left.\frac{dG}{ds}\right|_{s=5} = -0.10\frac{\text{gal}}{\text{mile}}.$$

This means that five miles into our trip the car is burning gas at the rate of 0.10 gallons/mile. So if this rate were to remain constant for the remainder of the trip, we would be able to drive an additional

$$\frac{4 \text{ gallons}}{0.10\frac{\text{gal}}{\text{mile}}} = 40 \text{ miles}$$

before running out of gas.

Derivatives of Exponential Functions

Working with exponential functions and their derivatives.

Relative Changes and Relative Rates of Change

Relative changes and relative errors are often more meaningful than absolute changes and errors. For example, I might measure the distance from Shoreline's Central Market to the Richmond beach library to be 5 km with an error of at most 0.2 km, while NASA might measure the between the earth and the moon on the first day of spring to be $384,400$ km with an error of at most 100 km. The relative error in my measurement is at most

$$\frac{0.2 \text{ km}}{5 \text{ km}} = 0.04 = 4\%,$$

while the relative error in NASA's measurement is at most

$$\frac{100 \text{ km}}{384,400 \text{ km}} \sim 0.00026 = 0.026\%.$$

Relatively speaking, NASA's measurement was about 150 times more accutate than mine.

Question 98 *At 10:00am the preices of Stock A and Stock B are both increasing at the rate of (\$2/share)/hour. At 10:00am Stock A sells for \$50/share and Stock B for \$10/share. Compare the relative rates at which the share prices are changing at 10:00am.*

Question 99 *The function*

$$P = f(t) = 5 - 3t + t^2 \,,\, 0 \leq t \leq 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

(a) At what relative rate is the price of the stock changing at 10am?

(b) When is the share price increasing at a rate of 60%/hr?

(c) During what time interval is the price of the stock increasing?

Learning outcomes:
Author(s):

(d) During what time interval is the relative rate of change in the price of the stock increasing?

Desmos link: `https://www.desmos.com/calculator/hhkveu6lxp`

Desmos activity available at 151: Stock Price

Question 100 *The function*

$$P = f(t)\,,\ 1 \leq t \leq 3.6,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am. Use the graph below to approximate the answers to the following questions without *putting a scale on the P axis.*

(a) At what relative rate is the price changing at 11am? At 12:30pm?

(b) When is the stock price increasing at its maximum relative rate? At its minimum relative rate? Approximate these rates.

Desmos link: `https://www.desmos.com/calculator/jyebaj5jif`

Desmos activity available at 151: Stock Price 2

Question 101 *The function*

$$W = f(t) = 200 + 4t + 2t^2, 0 \leq t \leq 12,$$

expresses the weight (in pounds) of a baby elephant in terms of its age (in months).

(a) Find the average rate at which the elephant gained weight between ages 4 and 10 months.

(b) Find the relative average rate at which the elephant gained weight between ages 4 and 10 months.

(c) Find the relative instantaneous rate at which the elephant is gaining weight at age 4 month.

(d) Find the relative instantaneous rate at which the elephant is gaining weight at age 10 months.

(e) Use the graph below to interpret your answers to parts (b)-(d) geometrically.

Desmos link: `https://www.desmos.com/calculator/2xj6xy7ggo`

Desmos activity available at 151: Elephant

Exponential Growth

Question 102 *(a) What does it mean for a population to grow exponentially?*

(b) Is it possible for a population to increase by 20% every year and not grow exponentially?

Question 103 *Suppose between noon and 10pm a colony of bacteria grows exponentially. The population is 200,000 at noon and 242,200 at 1pm.*

(a) Describe how the population grows. Keeping Question 2(b) in mind, is your description sufficient?

(b) How might we find a complete description of the exponential growth?

(c) Determine the relative average growth rate between noon and 12:30pm. Between 1pm and 1:30pm. Over any half-hour time period. Use the slider u in the graph below to intrepret these rates geometrically.

(d) Approximate the instantaneous relative growth rates in the population at noon, at 1pm, and at 2pm. Modify the definition of $v = u + 1/2$ in the demonstration below and interpret these rates geometrically.

(e) Use limits to write an expression that gives the instantaneous growth rate at time t hours past noon. What can you conclude?

(f) Try to answer part (b) again.

Desmos link: `https://www.desmos.com/calculator/wvpsotdhby`

Desmos activity available at 151: Exp Growth 1

Question 104 *The function*

$$P = f(t) = 3(2)^t, -2 \leq t \leq 4,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) Describe how the population grows. Is your description sufficient?

The population *doubles* every *hour*.

(b) Find an expression for the relative growth rate *between time $t = u$ hours past noon and time $t = u + h$ hours past noon. Measure the rate relative to the population at time $t = u$. Is this question asking about an average or an instantaneous relative growth rate?*

The relative growth rate is

$$\frac{1}{P}\left(\frac{\Delta P}{\Delta t}\right) = \frac{1}{f(u)}\left(\frac{f(u+h)-f(u)}{h}\right)$$

$$= \frac{1}{3(2)^u}\left(\frac{3(2)^{u+h}-3(2)^u}{h}\right)$$

$$= \frac{1}{3(2)^u}\left(\frac{3(2)^u(2^h-1}{h}\right)$$

$$= \frac{2^h-1}{h}$$

(c) What are the units of the relative average growth rate in part (b)?

(d) Input your function from part (b) on Line 5 in the worksheet below.

(e) What do you notice about the distance between points R and S as you drag the slider u below. How is this distance related to the relative average growth rate in part (b)?

(f) Use limits to write an expression for the relative growth rate at time $t = u$ hours past noon. Simplify this expression as much as possible. What can you conclude about how the relative instantaneous growth rate varies with time?

The relative growth rate is

$$\lim_{h\to 0}\frac{2^h-1}{h}.$$

(g) Interpret your expression from part (f) as the derivative of a specific function evaluated at a specific input. What does this tell you about the relative growth rate of this particular population?

The relative instantaneous growth rate of the population is equl to the derivative of the function

$$f(x) = 2^x$$

evaluated at $x = 0$.

(h) Use part (f) to numerically approximate the relative (instantaneous) growth rate of the population. Show a table that suggests a progression toward a limit.

(i) Use a similar method to approximate the relative instantaneous growth rate of the population

$$P = f(t) = 5(3)^t.$$

Desmos link: `https://www.desmos.com/calculator/omjbec2hpu`

Desmos activity available at 151: Exponential Growth 1

Question 105 *Parts (h) and (i) of the previous question suggest that there is a number e between 2 and 3 that makes the relative growth rate of the function*

$$P = f(t) = P_0 e^t, -3 \le t \le 5$$

equal to 100%/hr, where we assume here that t is measured in hours.

(a) What is the one-hour growth factor for this population?

(b) Describe what happens to the population every hour.

(c) At what relative rate is the population increasing at 1:00pm?

(d) Suppose at 1:00pm the population is $500,000$. Approximate the population at 1:03pm and compare your approximation to the actual population at that time.

Exponential Functions with Bases other than e

Question 106 *The function*

$$P = g(t) = P_0 e^{t/2}, -6 \le t \le 10$$

expresses the population (Colony B) of bacteria in terms of the number of hours past noon.

(a) Describe a transformation that takes the graph of the population function

$$P = f(t) = P_0 e^t, -3 \le t \le 5$$

for Colony A (where t is also the number of hours past noon) to the graph of $P = g(t)$.

(b) Suppose that the population of Colony A is $400,000$ at 4:00pm.

(i) When is the population of Colony B equal to $400,000$?

(ii) What are the growth rates of the two populations when they each have respective populations of 400,000 bacteria?

(iii) What are the relative growth rates of the two populations when they each have respective populations of 400,000 bacteria?

Question 107 *Here's another way to think about differentiating the function*

$$P = g(t) = P_0 e^{t/2}, -6 \leq t \leq 10$$

that expresses the population of a colony of bacteria in terms of the number of hours past noon.

We'll let $u = t/2$ be the number of two-hour periods since noon.

(a) Express the population in terms of u.

(b) Use what you know about the exponential function base e to express the growth rate of the population in terms of u.

(c) Use part (b) to find the growth rate of the population at 6pm. Pay careful attention to units.

(d) Use the idea of part (c) to express the growth rate

$$\frac{dP}{dt} = g'(t)$$

in terms of t.

(e) Suppose instead that the population grows exponentially and doubles every hour. Find the relative instantaneous growth rate of the population.

Question 108 *Between 11am and 8pm, a population of bacteria grows exponentially. The population is 4 million at noon and 5 million at 1pm.*

(a) What is the one-hour growth factor? 1.25

(b) Describe how the population grows.

The population increases by 25% *every hour.*

(c) Use your description from part (b) to find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon. Do not use e in your function. Define meaningful variables and include a domain.

The function

$$P = f(t) = 4(5/4)^t, \ -1 \leq t \leq 8$$

expresses the population (in millions of bacteria) in terms of the number of hours past noon.

(d) Use the fact that $k = e^{\ln k}$ for $k > 0$ to express your function from part (c) using an exponential function with base e.

$$P = f(t) = 4e^{t \ln(1.25)}, \ -1 \leq t \leq 8$$

(e) Use u-substitution and the chain rule (show all steps) along with the fact that

$$\frac{d}{du}(e^u) = e^u$$

to find the relative instantaneous growth rate of the population. Include units in your conclusion.

The relative instantaneous growth rate of the population is

$$\frac{1}{P}\left(\frac{dP}{dt}\right) = \ln(5/4) \sim 22.314\%/hour.$$

(f) At what rate is the population growing when there are 10 million bacteria? Do this the easy way. Note that the question is not *asking for a relative rate.*

(g) Find the population when the population is increasing at the rate of 3 million bacteria/hour. Do this the easy way.

Question 109 *Between 11am and 8pm, a population of bacteria decreases exponentially. The population is 5 million at noon and 4 million at 4pm.*

(a) What is the four-hour growth factor? (This is the number you multiply the current popluation by to get the population four hours later). 0.8

(b) What is the one-hour growth factor? $(4/5)^{1/4}$

(c) Describe how the population decreases.

The population decreases by 20% *every* 4 *hours.*

(d) Use your description from part (c) to find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon. Do *not use e in your function. Start by define meaningful variables. Include a domain with your function.*

The function

$$P = f(t) = 5(4/5)^{t/4}\,,\ -1 \le t \le 8$$

expresses the population (in millions of bacteria) in terms of the number of hours past noon.

(e) Express your function from part (c) using an exponential function with base e. Then use u-substitution and the chain rule to find the relative instantaneous growth rate of the population. Show all steps in using the chain rule.

The population function is

$$P = f(t) = 5e^{0.25\ln(0.8)t}\,,\ -1 \le t \le 8.$$

The relative intstantaneous growth rate is

$$\frac{1}{P}\left(\frac{dP}{dt}\right) = 0.25\ln(0.8) \sim -5.58\%/hour.$$

(f) At what rate is the population decreasing when there are 2 million bacteria? Do this the easy way.

(g) Find the population when the population is decreasing at the rate of 300,000 bacteria/hour. Do this the easy way.

(h) Use the result of part (f) to approximate the population 4 minutes after there are 2 million bacteria.

Relative Rates Again

Question 110 *The function*

$$P = f(t), 0 \leq t \leq 2,$$

expresses the balance (in dollars) in an account in terms of the number of years since the start of 2022. Suppose

$$\frac{1}{P}\frac{dP}{dt}\bigg|_{P=5000} = 0.08.$$

(a) What are the units of the above derivative? How do you know?

(b) Interpret the meaning of the above derivative.

(c) Approximate the balance in the account four days after the account has $5,000. *Explain your reasoning.*

Question 111 *The function*

$$q = f(p) = 0.5(p-18)^2, \ 6 \leq p \leq 15,$$

expresses the average number of burgers/day sold at Five Guys of Edmonds in terms of the price (in \$/burger).

(a) At what relative rate does the quantity sold (q) change with respect to the price (p) at a price of \$10/burger?

(b) What are the units of the above relative rate of change?

(c) Explain the meaning of the relative rate of change in part (a).

(d) Use the graph of the function $q = f(p)$ and the slider u in the desmos activity below to interpret the relative rate of change in part (a) geometrically. Explain your reasoning.

(e) Use the result of part (a) to approximate the relative change in the average number of burgers sold per day if the Five Guys increases the price from \$10/burger to \$10.25/burger. Explain your reasoning.

(f) Use the result of part (a) to approximate the relative change $\Delta a\%$ in the average number of burgers sold per day in terms of a small relative change $\Delta r\%$ in the price from \$10/burger. Explain your reasoning.

Desmos link: `https://www.desmos.com/calculator/ylgk03oaza`

Desmos activity available at 151: Burgers 1

Question 112 *The function*

$$P = f(t), -2 \le t \le 5,$$

expresses the population of a colony (call it Colony A) of bacteria in terms of the number of hours past noon.

The function

$$P = g(t) = f(t/2), -4 \le t \le 10,$$

expresses the population of Colony B in terms of the number of hours past noon.

The populations do not *necessarily grow exponentially.*

(a) Compare the populations at noon.

(b) Suppose Colony A as 50,000 bacteria at 3:00pm. When does Colony B have 50,000 bacteria? Explain.

(c) Suppose the population of Colony A takes three hours to grow from 20,000 to 50,000. How long does it take the population of Colony B to grow from 20,000 to 50,000?

(d) Suppose the population of Colony A is increasing at the rate of 10,000 bac/hr at 3pm. What is the growth rate of Colony B at 6pm? Explain.

(e) What is the relative growth rate of Colony A at 3pm? What is the relative growth rate of Colony B at 6pm? Explain.

Question 113 *The function*

$$P = f(u) = 10e^u, -2 \le u \le 5$$

expresses the population (in millions of bacteria) of a colony of bacteria in terms of the number of hours since noon.

(a) What are the units of the input to the exponential function in the above expression for P?

(b) At what rate is the population growing when there are 30 million bacteria?

(c) At what rate is the population growing when there are P million bacteria? Do the units of your answer make sense?

(d) What is the relative instantaneous growth rate of the population?

(e) Find a function

$$P = g(t)$$

that expresses the population (in millions of bacteria) of the colony of bacteria in terms of the number of minutes *since noon. Include a domain.*

(f) Use common sense to evaluate the derivative

$$\left.\frac{dP}{dt}\right|_{P=30}.$$

Explain your reasoning.

(g) Use the idea of part (f) to find an expression for the derivative

$$\frac{dP}{dt} = g'(t)$$

at time t minutes past noon.

Question 114 *The function*

$$P = P_0 e^{kt}, -4 \leq t \leq 5,$$

expresses the population of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of the constant P_0?

(b) What are the units of the constant k?

(c) Use the ideas of the previous question to find an expression for the growth rate of the population at time t hours past noon. Include units in your answer.

(d) Find an expression for the relative growth rate at time t hours past noon. Include units in your answer.

Question 115 *One of the two functions graphed below is an exponential function. Which one? How do you know?*

The Derivative as a Magnification Factor

Exploration 116 *It sometimes helps to think of the derivative as a magnification factor that maps a small interval around an input to a function to a corresponding interval around the output.*

(a) Use this idea for the function $y = f(x)$ graphed below to approximate the derivatives

$$\frac{d}{dx}(f(x))\Big|_{x=4} \text{ and } \frac{dy}{dx}\Big|_{x=5}.$$

Desmos link: `https://www.desmos.com/calculator/la4f5ots3r`

Desmos activity available at 151: Magnification Factor 1

Differentiating the Exponential Function e^x

Exploration 117 *(a) Use the graph of the function $y = f(x)$ below to approximate the derivatives*

$$\frac{dy}{dx}\Big|_{y=k} \text{ for } k = 1, 2, \ldots, 6.$$

Note the above derivatives are evaluated at the outputs *of the function f.*

(b) What do you notice?

Desmos link: `https://www.desmos.com/calculator/k08dphtuca`

Desmos activity available at 151: Magnification Factor 2

Question 118 *The function*

$$P = 400e^t, 0 \le t \le 2$$

expresses the population (in thousands) of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of the factor 400?

(b) What are the units of the exponent in the factor e^t? Be careful.

(c) Find the (instantaneous) growth rate of the population when there are 1,200,000 bacteria.

(d) Find the relative (instantaneous) growth rate when there are 1,200,000 bacteria.

(e) Find the relative (instantaneous) growth rate at any time.

(f) Approximate the population 30 seconds after there are 1,200,000 bacteria.

Question 119 *Let $k > 0$ be a constant and let*

$$f(x) = ke^x.$$

For $a \in \mathbb{R}$ let point P with coordinates $(a, f(a))$ be a curve on the curve $y = f(x)$. Let Q be the point where the tangent line to the curve intersects the x-axis and let R be the point with coordinates $(a, 0)$

(a) Find the length of segment $\overline{QR}$.

(b) How is part (a) related to part (e) of the previous question?

1 Exercises

Exercise 120 *Find an expression for the derivative*

$$\frac{d}{dt}\left(e^{\cos t}\right).$$

Explanation. We need to use the chain rule because we are differentiating the composition of functions. So let

$$y = e^{\cos t}$$

and

$$u = \cos t.$$

Then

$$y = e^u$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du} \cdot \frac{du}{dt} \\ &= \left(\frac{d}{du}\left(e^u\right)\right)\left(\frac{d}{dt}(\cos t)\right) \\ &= e^u\left(-\sin t\right) \\ &= (-\sin t)e^{\cos t}.\end{aligned}$$

Exercise 121 *Find expressions for each of the following derivatives. Follow the previous exercise exactly in using the chain rule.*

(a) $\dfrac{d}{dt}\left(1000e^{\frac{t}{5}}\right)$

(b) $\dfrac{d}{dt}\left(\dfrac{10}{3+5e^{t}}\right)$

(c) $\dfrac{d}{dx}\left(\sin(e^{4x})\right)$

(d) $\dfrac{d}{dt}\left(1000(2^{t})\right)$

Hint: *Write* 2^t *as*

$$2^t = \left(e^{\ln 2}\right)^t = e^{t\ln 2}$$

and use the chain rule.

The Chain Rule

An introduction to the chain rule.

1 Introduction

The chain rule tells us how to differentiate the composition of functions. Computationally, it is the *most* important part of this course. Conceptually, it is the most intuitive. In fact, even though you might not realize it, you already know the chain rule. And you've known it for a long time, well before you ever heard of a derivative.

Here's an example. Suppose you're ascending in a hot air balloon at the constant rate of 200 ft/min. Suppose also that the air temperature decreases at the constant rate of 0.003°C/ft. The question is at what rate, with respect to time, is the temperature of your surrounding environment changing.

It should be intuitive that we need only multiply these rates. So the temperature is changing at the constant rate of

$$\left(-0.003\frac{^\circ\mathrm{C}}{\mathrm{ft}}\right)\left(200\frac{\mathrm{ft}}{\mathrm{min}}\right) = -0.6\frac{^\circ\mathrm{C}}{\mathrm{min}}.$$

Here we have two linear functions, one

$$h = g(t)$$

that expresses our altitude (in feet) as a function of time (say in minutes past 1pm), the other

$$C = f(h)$$

that expresses the temperature (in Celsius degrees) in terms of altitude (in feet). And the composition

$$C = f(g(t))$$

expresses the temperature in terms of time. Since the height and temperature functions h and f have respective constant rates of change $\Delta h/\Delta t$ and $\Delta C/\Delta h$, the compostion has constant rate of change

$$\frac{\Delta C}{\Delta t} = \frac{\Delta C}{\Delta h}\cdot\frac{\Delta h}{\Delta t}.$$

Learning outcomes:
Author(s):

And this is what the chain rule says, that the derivative of the composition of functions is the product of their derivatives.

Well, not quite. We need to be more precise because things are more subtle when the rates of change are not constant. Suppose for example, we know only that at precisely 2pm our balloon is rising at the rate of 200 ft/min. Suppose also at 2pm we are 2500 feet above the ground and that at this altitude the temperature decreases at the rate of 0.003°C/ft. What can we say about the rate, with respect to time, at which the temperature of our surrounding environment is changing at 2pm?

Well, there's nothing else we can really do expcept multiply the two rates and end up with almost the same conclusion, that at 2pm the temperature of our surrounding environment is decreasing at the rate of 0.6°C/min. But this reasoning rests on the subtle assumption that the temperature ($C = f(h)$) and altitude ($h = g(t)$) functions are *differentiable* at their respective inputs $h = 2500$ and $t = 60$. What this means intuitively is that these functions act sufficiently like linear functions near their respective inputs to make the above computation remain valid.

It all boils down to what we mean by these rates of change. The idea that at 2pm our balloon is ascending at the rate of 200 ft/min seems intuitive, but is it? What do we really mean by a moment in time? Instead of addressing this question directly, we can think of the ascent rate as a *scaling factor* that converts a small interval of time

$$\Delta t = t - 60 \sim 0$$

to an approximation of the actual change in height

$$\Delta h = f(t) - f(6) = f(t) - 2500$$

during that interval. We might say, for example, that our altitude increases by about 200 feet from 2:00pm to 2:01pm. But because our rate of ascent might change considerably during time interval, this approximation might not be very accurate. It would be better to say that in the 1/100th of a minute following 2pm, our altitude increases by about two feet.

More generally, we could approximate the change in height (in feet) over the time interval Δt minutes beginning at 2pm by the product

$$\Delta h \sim \left(\frac{dh}{dt}\bigg|_{t=60} \right) \Delta t = 200\Delta t.$$

If we assume the altitude function is *differentiable* at 2pm (which would seem almost certain) then this is a good approximation for small time intervals $\Delta t \sim 0$ in the sense that the error

$$\Delta h - 200\Delta t$$

in the approximation approaches zero *faster* than Δt. That is, we should expect that

$$\lim_{\Delta t \to 0} \frac{\Delta h - 200\Delta t}{\Delta t} = 0.$$

And this is equivalent to saying that the altitude function is differentiable at 2pm and that

$$\left.\frac{dh}{dt}\right|_{t=60} = \lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t} = 200 \text{ ft/min}.$$

Similarly, the rate of change of temperature with respect to altitude,

$$\left.\frac{dC}{dh}\right|_{h=2500} = -0.003^{\circ}\text{C/ft},$$

is best understood in terms of small changes. Over a small change in altitude of $\Delta h = h - 2500$ feet, the change in temperature

$$\Delta C = g(h) - g(2500)$$

is well-approximated by the product

$$\Delta C \sim \left(\left.\frac{dC}{dh}\right|_{h=2500}\right) \Delta h = -0.003\Delta h.$$

By this we mean that

$$\left.\frac{dC}{dh}\right|_{h=2500} = \lim_{\Delta h \to 0} \frac{\Delta C}{\Delta h} = -0.003^{\circ}\text{C/ft}.$$

With these assumptions we can compute the rate of change in temperature with respect to time at 2pm as

$$\begin{aligned}
\left.\frac{dC}{dt}\right|_{t=60} &= \lim_{t \to 0} \frac{\Delta C}{\Delta t} \\
&= \lim_{t \to 0} \frac{\Delta C}{\Delta h} \cdot \frac{\Delta h}{\Delta t} \\
&= \lim_{h \to 0} \frac{\Delta C}{\Delta h} \cdot \lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t} \\
&= \left.\frac{dC}{dh}\right|_{h=2500} \cdot \left.\frac{dh}{dt}\right|_{t=60} \\
&= \left(-0.003 \frac{^{\circ}\text{C}}{\text{ft}}\right) \left(200 \frac{\text{ft}}{\text{min}}\right) \\
&= -0.6 \frac{^{\circ}\text{C}}{\text{min}}.
\end{aligned}$$

2 Distance to the Horizon

Here's an example of a question more typical of what you see in this class. It looks more complicated, but it is really just like the last example.

Example 17. The function

$$h = g(t) = -4t^2 + 16t + 100\,,\ -2 \le t \le 5$$

expresses the altitude (in feet) of a balloon in terms of the number of seconds past 2pm.

The function

$$s = f(h) = 1.22\sqrt{h}\,,\ 0 \le h \le 30000,$$

expresses the distance to the horizon (in miles) in terms of altitude (in feet).

Our problem is to determine the rate at which the distance to the horizon (limited by the curvature of the earth) is changing (with respect to time) for the occupants of the ballon at 2pm.

Explanation. To determine this rate, just as in the previous question, we need only multiply two rates of change:

(a) the balloon's rate of ascent at 2pm and

(b) the rate of change in the distance to the horizon with respect to altitude, evaluated at the height of the balloon at 2pm.

These rates are both derivatives. The rate of ascent at 2pm is the derivative

$$\begin{aligned} \left.\frac{dh}{dt}\right|_{t=0} &= \frac{d}{dt}\left(-4t^2 + 16t + 100\right)\Big|_{t=0} \\ &= (-8t + 16)\Big|_{t=0} \\ &= 16 \text{ ft/sec.} \end{aligned}$$

For the second rate of change, in the distance to the horizon with repsect to altitude, we first compute the derivative

$$\frac{ds}{dh} = \frac{d}{dh}\left(1.22\sqrt{h}\right) = \frac{0.61}{\sqrt{h}} \text{ miles/ft}$$

and then evaluate it at the altitude

$$h = f(0) = 100 \text{ feet.}$$

We get

$$\left.\frac{ds}{dh}\right|_{h=100} = \left.\frac{0.61}{\sqrt{h}}\right|_{h=100} = 0.061 \text{ miles/ft.}$$

Then the rate of change in the distance to the horizon for the occupants of the balloon at 2pm is

$$\left.\frac{ds}{dt}\right|_{t=0} = \left(\left.\frac{ds}{dh}\right|_{h=g(0)}\right)\left(\left.\frac{dh}{dt}\right|_{t=0}\right)$$

$$= \left(0.061\frac{\text{miles}}{\text{ft}}\right)\left(16\frac{\text{ft}}{\text{sec}}\right)$$

$$= 0.976 \text{ miles/sec.}$$

Note: What we really did here was to evaulate the derivative of the composition

$$s = f(g(h)) = 1.22\sqrt{-4t^2 + 16t + 100},$$

exressing the distance to the horizon in terms of time, at time $t = 0$.

3 Visualizing the Chain Rule

The image of stretching an elastic band gives us a way to visualize the chain rule.

Suppose we have two stretching functions

$$H = g(L)\,,\ 0 \le L \le 10$$

and

$$s = f(H),$$

where the inputs and outputs are all measured in meters.

We wish to approximate the local stretching factor of the composition

$$s = f(g(L))\,,\ 0 \le L \le 10$$

at the points $L = 9.5$ meters from the origin on the relaxed band.

Let's think first about what this means. Imagine a thin elastic band running along the x-axis, here between $L = 0$ and $L = 10$ meters. First we stretch the band with the function g. This actions sends the points L meters from the origin on the relaxed band to the points $H = g(L)$ meters on the stretched band.

(a) Watch this first stretching action by sliding the Slider u in Line 2 of the worksheet below from $u = 0$ to $u = 1$.

Desmos link: `https://www.desmos.com/calculator/8whlpa61hn`

Worksheet available at 151: Rubber Band Composition

Nex we stretch the stretched band with function f. This action takes the points $H = g(L)$ meters from the origin on the stretched band to the points $s = f(g(L))$ meters from the origin.

(b) Watch the second stretch by sliding the Slider u_2 on Line 4 from $u_2 = 0$ to $u_2 = 1$.

(c) Our goal now is to approximate the local stretching factor

$$\frac{ds}{dL}\bigg|_{L=9.5}$$

of the compostion $f \circ g$ at the points $L = 9.5$ meters from the origin in the relaxed band.

Here are two approaches:

(i) Use the animation to approximate the local stretching factors

$$\frac{dH}{dL}\bigg|_{L=9.5} \quad \text{and} \quad \frac{ds}{dH}\bigg|_{H=g(9.5)}.$$

Then use these factors to approximate the local stretching factor of the composition $f \circ g$ at the points $L = 9.5$ meters from the origin in the relaxed band. Explain your reasoning in detail. Include screen shots to help with your explanation. Include units in all computations.

Free Response:

(ii) Use the animation to approximate the local stretching factor

$$\frac{ds}{dL}\bigg|_{L=9.5}$$

directly. Explain your reasoning in detail. Include screen shots to help with your explanation. Include units in all computations.

Free Response:

(iii) Compare the results of the two methods.

Free Response:

4 Computing with the Chain Rule

The chain rule tells us how to differentiate the composition of functions. It say that the derivative of the composition is the product of the derivatives.

Here is a more formal statement.

Theorem 1. (The Chain Rule) If $y = f(u)$ is a differentiable function of u and $u = g(t)$ is a differentiable function of t, then the composition

$$y = f(g(t))$$

is a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}.$$

Or paying closer attention to the inputs,

$$\left.\frac{dy}{dt}\right|_{t=t_0} = \left.\frac{dy}{du}\right|_{u=g(t_0)} \cdot \left.\frac{du}{dt}\right|_{t=t_0}.$$

We'll go through some examples to get an understanding of how to use the chain rule and also why it works.

Examples

Example 18. Find an equation of the tangent line to the curve

$$y = f(x) = \left(2x^3 + 1\right)^2$$

at the point $(1, 9)$.

Explanation. Let

$$y = \left(2x^3 + 1\right)^2$$

and

$$u = 2x^3 + 1.$$

Then

$$y = u^2$$

and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}\left(u^2\right) \cdot \frac{d}{dx}\left(2x^3 + 1\right) \\ &= 2u(6x^2) \\ &= 2(2x^3 + 1)(6x^2). \end{aligned}$$

Then the slope of the tangent line to the curve $y = (2x^3 + 1)^2$ at the point $(1, 9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 2(3)(6) = 36,$$

and an equation of the tangent line is

$$y - 9 = 36(x - 1).$$

We can get the same result without the chain rule, by rewriting the original function as

$$y = f(x) = (2x^3 + 1)^2 = 4x^6 + 4x^3 + 1.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(4x^6 + 4x^3 + 1\right)\\ &= 4\frac{d}{dx}\left(x^6\right) + 4\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}(1)\\ &= 24x^5 + 12x^2.\end{aligned}$$

Then the slope of the tangent line to the curve $y = (2x^3+1)^2$ at the point $(1, 9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 14 + 12 = 36,$$

as before.

Example 19. Find an equation of the tangent line to the curve

$$x^2 + y^2 = 25$$

at the point $P(4, -3)$.

Explanation. Solve the above equation for y in terms of x to get

$$y = \pm\sqrt{25 - x^2}.$$

While this equation does not define y as a function of x, near the point $(4, -3)$, the equation

$$y = -\sqrt{25 - x^2}$$

does define y as a function of x.

Now to find the derivative of the function

$$y = -\sqrt{25 - x^2},$$

let

$$u = 25 - x^2.$$

Then

$$y = \sqrt{u} = -u^{1/2}$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}\left(-u^{1/2}\right) \cdot \frac{d}{dx}\left(25 - x^2\right) \\ &= -\frac{1}{2\sqrt{u}}(-2x) \\ &= \frac{x}{\sqrt{25 - x^2}}.\end{aligned}$$

So the slope of the tangent line to the curve

$$x^2 + y^2 = 25$$

at the point $(P(4, -3)$ is

$$\left.\frac{dy}{dx}\right|_{x=4} = \left(\frac{x}{\sqrt{25 - x^2}}\right)\Big|_{x=4} = \frac{4}{3}$$

and an equation of the tangent line is

$$y + 3 = \frac{4}{3}(x - 4).$$

To find the slope of the tangent line without appealing to the chain rule, note that the curve

$$x^2 + y^2 = 25$$

is a circle centered at the origin. So the tangent line at $P(4, -3)$ is perpendicular to the radius $\overline{OP}$ from the origin to P. Since $\overline{OP}$ has slope $m_1 = -3/4$, the tangent line at $P(4, -3)$ has slope

$$m_2 = -1/m_1 = 4/3.$$

Question 122 *(a) Describe a transformation that takes the circle*

$$x^2 + y^2 = 25$$

to the ellipse

$$4x^2 + y^2 = 25. \tag{2}$$

(b) Find the image (call it Q) of the point $P(4, -3)$ under the transformation in part (a).

(c) Use the result of Example 2 to find an equation of the tangent line to the ellipse (2) at Q.

(d) Use the chain rule to first find the slope of the tangent line to the ellipse (2) at Q. Then find an equation of the tangent line.

Question 123 *The function*

$$P = f(t) = 22 + \frac{t}{2} - \frac{3t^2}{4}, 0 \le t \le 3,$$

expresses the price of a stock (in dollars/share) in terms of the number of hours past 9am.

(a) Is the price of the stock increasing or decreasing at 11am? At what rate? At what relative rate?

(b) Is the number of shares you can buy with $1000 *increasing or decreasing at 11am? Approximate the rate numerically.*

(c) Compute the exact rate in part (b). Compute also the relative rate of change (with respect to time) in the number of shares you can buy with $1000 *at 11am.*

Start your solution to this problem by defining a new function with a new function name and a new depedent variable.

Question 124 *At 10am the price of a stock is increasing at the relative rate of p%/hr. Is the number of shares you can buy with* $1000 *increasing or decreasing at 10am? At what relative rate?*

Answer this question in two ways:

(i) Going back to limits and using the definition of the derivative.

(ii) Using the chain rule.

Either way, start your solution with defintions as in the previous problem.

Question 125 *A tree leans precariously with its trunk making an angle of* $\phi = \pi/3$ *radians with the ground. One end of a 14-foot ladder leans against the trunk, the other rests on the horizontal ground.*

Let t *be the distance between the top of the ladder and the base of the trunk (measured in feet) and* s *the distance between the bottom of the ladder and the base of the trunk (also measured in feet).*

Geogebra link: `https://www.geogebra.org/m/ctf2bcqz`

Geogebra activity available at 151: Ladder and Tree Part 2

(a) Use the law of cosines to write an equation relating t *and* s.

(b) Use the result of part (a) to find the two possible values of t *when* $s = 16$.

For the remainder of this problem we'll focus on positions of the ladder when $s \sim 16$ *and* $t \sim 10$.

(c) For these positions, complete the square to find a function

$$t = f(s)$$

that expresses the distance (in feet) from the top of the ladder to the base of the trunk in terms of the distance (in feet) from the bottom of the ladder to the trunk's base.

(d) Drag the slider θ in the worksheet above to approximate the value of the derivative

$$\left.\frac{dt}{ds}\right|_{s=16}.$$

Include units and explain your reasoning.

(e) Use your function from part (c) to find an expression for the derivative dt/ds. Show every step *in computing the derivative as in Examples 1 and 2 above.*

(f) Evaluate the derivative

$$\left.\frac{dt}{ds}\right|_{s=16}$$

and compare the exact value with your estimate from part (d).

(g) What are the units of the derivative in part (f)? Explain the meaning of the derivative, not *by giving a standard response about the rate of change of some quantity with respect to another, but by relating small changes.*

(h) Use your response to part (f) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of s near $s = 16$. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 15.6 feet from the base of the trunk. Compare your approximation with the exact distance.

(i) Let $\theta = m\angle GBA$ be the radian measure of the angle the ladder makes with the trunk.

part (i) Find an expression for the derivative $d\theta/ds$.

part (ii) Evaluate the derivative

$$\left.\frac{d\theta}{ds}\right|_{s=16}$$

and interpret its meaning in terms of small changes. Be sure to explain the significance of the derivative's sign as well.

Example 20. (a) Find the slope of the tangent line to the curve

$$y = f(\theta) = 6\sin\theta$$

at the point with coordinates $P(2\pi/3, 3\sqrt{3})$. We do not need the chain rule for this.

(b) Describe a transformation that takes the graph of $y = f(\theta)$ to the graph of the function

$$y = g(\theta) = 6\sin(\theta/2).$$

(c) Find the image of the point $P(2\pi/3, 3\sqrt{3})$ under the above transformation.

(d) Use the results of parts (a) - (c) and the demonstration below to guess the slope of the tangent line to the curve $y = g(\theta)$ at the point $Q(4\pi, 3\sqrt{3})$.

Desmos link: `https://www.desmos.com/calculator/mqjxpsqyo5`

Desmos activity available at 151: Chain Rule 1

(e) Use the chain rule to confirm your guess from part (d).

Example 21. Find an equation of the tangent line to the curve

$$y = f(t) = 4\sin^2\left(\frac{\pi}{6}t\right)$$

at the point $(2, 3)$.

Explanation. Let

$$y = 4\sin^2\left(\frac{\pi}{6}t\right) = 4\left(\sin\left(\frac{\pi}{6}t\right)\right)^2$$

and

$$u = \sin\left(\frac{\pi}{6}t\right).$$

Then

$$y = 4u^2$$

and

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{du}\cdot\frac{du}{dt} \\
&= \frac{d}{du}\left(4u^2\right)\cdot\frac{d}{dt}\left(\sin\left(\frac{\pi}{6}t\right)\right) \\
&= 8u\cos\left(\frac{\pi}{6}t\right)\frac{d}{dt}\left(\frac{\pi}{6}t\right) \\
&= 8\sin\left(\frac{\pi}{6}t\right)\cos\left(\frac{\pi}{6}t\right)\frac{\pi}{6}.
\end{aligned}$$

Question 126 *(a) The third equality above uses the chain rule again. Fill in the missing details of this computation by first letting*

$$z = \sin\left(\frac{\pi}{6}y\right)$$

and making an explicit substitution using the variable v in place of u.

(b) Complete the solution by finding an equation of the tangent line to the curve at the point $(2, 3)$.

Question 127 *Use the chain rule to compute each of the following derivatives. Show* all *steps as in Examples 1, 2, and 8 above.*

(a) $\dfrac{d}{dx}\left(\dfrac{12}{1+x^2}\right)$

(b) $\dfrac{d}{dt}\left(12(3)^{t/5}\right)$

(b) $\dfrac{d}{dt}\left(e^{3\sin(4t)}\right)$

(b) $\dfrac{d}{d\theta}\left(12 + 5\cos(\theta/4)\right)$

Example 22. The function

$$P = f(t) = 10e^{\frac{1}{4}t}\,,\ -4 \le t \le 12,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of 1/4 in the function above? How do you know?

(b) Use the chain rule to find an expression for the growth rate of the population at time t hours past noon. What are the units of the growth rate?

(c) Express the growth rate from part (b) in terms of the population $P = f(t)$ at time t hours past noon.

(d) What is the relative instantaneous growth rate of the population? Include units.

(e) Find the relative average growth rate of the population over a one-hour period.

(f) Describe what happens to the population every hour.

Example 23. (a) Use the chain rule to find the relative instantaneous growth rate of the population function

$$P = 10(2^t)\,,\ -2 \le t \le 5,$$

where P is measured in millions of bacteria and t is the number of hours past noon.

(b) Describe what happens to the population every hour.

(c) Estimate the population 2 minutes after there are 30 million bacteria.

Question 128 *This problem suggests a way to think about the chain rule geometrically.*

You ride a ferris wheel for one revolution and get off. The function

$$h = f(\theta) = , \ 0 \leq \theta \leq 2\pi,$$

expresses your height (in feet) in terms of the wheel's angle of rotation (measured in radians from the moment you boarded).

The function

$$\theta = g(t) \, , \ 0 \leq t \leq 44,$$

expresses the rotation angle of the wheel in terms of the number of seconds since you boarded.

Our goal is to use the graphs of the function f and g below (take the times t to be postive, not negative as shown) to approximate your rate of ascent $t = 16$ seconds after boarding. We'll do this in two different ways.

(a) The first way involves a few steps and uses the graphs of both f and g.

(i) Approximate the value of the derivative

$$\left.\frac{dh}{d\theta}\right|_{\theta=g(16)}$$

and interpet its meaning. Include units.

(ii) Approximate the value of the derivative

$$\left.\frac{d\theta}{dt}\right|_{t=16}$$

and interpet its meaning. Include units.

(iii) Use the results of (i) and (ii) and common sense (do not appeal to the chain rule directly) to approximate the value of the derivative

$$\left.\frac{dh}{dt}\right|_{t=16} = h'(16),$$

where $h(t) = f(g(t))$. Explain your logic and include all units in your computation. Interpret the meaning of this derivative.

(b) The second way has just one step and that's to use the graph of $h = f(\theta)$ to approximate your rate of ascent at time $t = 16$ using the fact that the horizontal blue lines are drawn at intervals of $\Delta t = 2$ seconds. Be sure to explain your logic.

Desmos link: `https://www.desmos.com/calculator/fkgfpsowe8`

Desmos activity available at 151: Ferris Wheel 2

Question 129 *This is a continuation of the previous question.*

Approximate the time between times $t = 5$ and $t = 22$ seconds when you are ascending at the slowest rate. Approximate this rate of ascent.

Question 130 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) Use the geometry of the ferris wheel (see the picture below) to find a function

$$h = f(\theta)\,,\ 0 \le \theta \le 2\pi,$$

that expresses your height (in feet) above the ground in terms of the rotation angle of the wheel, measured in radians. Use the cosine *function,* not *the sine function.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

(b) Suppose you move at the constant speed of 10 ft/sec as you ride the ferris wheel.

(i) Use the result of part (a) to find a function $h = g(t)$ that expresses your height (in feet) in terms of the number of seconds since you boarded. Include the appropriate domain.

(ii) Find a function $r = h(t)$ that expresses your rate of ascent (in ft/sec) in terms of the number of seconds since you boarded. Include the appropriate domain. Use vectors to interpret this rate geometrically, in terms of the angle your velocity makes with the horizontal and your speed.

(iii) Write an equation that relates your height h (in feet) and your rate of ascent r (in ft/sec) at any instant. Graph the relation by hand.

(iv) Are you going up or down the second time you are 100 feet above the ground? At what rate?

(v) What is your height when you are descending at the rate of 4 ft/sec? Find all possibilities.

Question 131 *The function*

$$\theta = f(t), t \ge 0,$$

expresses the radian measure of a ferris wheel's rotation angle in terms of the number of seconds since you boarded. The wheel has radius r feet and its center is b feet above the ground.

(a) Find a function

$$h = g(t), t \geq 0,$$

that expresses your height (in feet) in terms of the number of seconds since you boarded.

(b) Find an expression for your rate of ascent (in ft/sec) at time t seconds after you boarded. Assume g is a differentiable function of t.

(c) Interpret your rate of ascent (part (b)) in terms of your speed at time t and the angle your velocity vector makes with the horizontal. Assume here that $\theta = f(t)$ is an increasing function of t.

Exercises

Exercise 132 *Use the facts that*

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

to compute each of the following derivatives. Show all steps in using the chain rule as in Examples 1 and 2 above.

(a) $\dfrac{d}{d\theta}(\sec\theta)$

(b) $\dfrac{d}{d\theta}(\csc\theta)$

(c) $\dfrac{d}{dt}\left(\sqrt{\sin^2 t + (4-\cos t)^2}\right)$

Hint: *Simplify the function first.*

(d) $\dfrac{d}{d\theta}\left(\sec^2\theta\right)$

(e) $\dfrac{d}{dt}\left(\dfrac{1}{5+3\cos(t/2)}\right)$

Explanation. (d) Let

$$y = \sec^2\theta = (\sec\theta)^2$$

and let

$$u = \sec\theta.$$

Then

$$y = u^2$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= \frac{d}{du}\left(u^2\right)\cdot\frac{d}{d\theta}\left(\sec\theta\right)\\ &= 2u(\sec\theta\tan\theta)\\ &= 2(\sec\theta)(\sec\theta\tan\theta).\end{aligned}$$

(e) Let

$$y = \frac{1}{5+3\cos(t/2)}$$

and let

$$u = 5 + 3\cos(t/2).$$

Then

$$y = u^{-1}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du}\cdot\frac{du}{dt}\\ &= \frac{d}{du}\left(u^{-1}\right)\cdot\frac{d}{dt}\left(5+3\cos(t/2)\right)\\ &= -\frac{1}{u^2}\left(-3\sin\left(\frac{t}{2}\right)\frac{d}{dt}\left(\frac{t}{2}\right)\right)\\ &= \frac{3\sin\left(\frac{t}{2}\right)}{2(5+3\cos(t/2))^2}.\end{aligned}$$

Exercise 133 *You jog around a circular track of radius r feet at the constant speed of speed of v ft/sec. A flagpole lies b feet due east of the track's center.*

(a) Use the animation below (and nothing else) to sketch by hand a graph of the function

$$s = f(\theta)\,, \theta \geq 0,$$

that expresses your distance (in feet) to the flagpole in terms of your angle $\theta = \angle FOJ$ of rotation about the track's center, measured in radians from the time you start. Assume you start at the point A on the track nearest the flagpole. Be sure to include variable names, units, and scales on your axes. Explain your reasoning. For this particular graph assume that $r = 40$ and $b = 96$ (be sure to adjust the sliders in the worksheet to have these values).

Desmos link: `https://www.desmos.com/calculator/4pndurvhdd`

Worksheet available at 151: Jogger

(b) Using only your graph of the function f from part (a), sketch by hand a graph of the derivative

$$y = f'(\theta) = ds/d\theta$$

when $r = 40$ and $b = 96$. Be sure to include variable names, units, and scales on your axes. Explain your reasoning.

(c) Using only the animation (stop the motion), approximate the value of the derivative

$$\left.\frac{ds}{d\theta}\right|_{\theta=3\pi/2}$$

when $r = 40$ and $b = 96$. Include units. Explain your reasoning.

(d) Use trigonometry to find an expression for the function $s = f(\theta)$ in terms of the parameters r and b (ie. do not *assume $r = 40$ and $b = 96$).*

(e) Find an expression for the derivative $ds/d\theta$. Show every step *in using the chain rule as in Examples 1 and 2. Do* not *assume $r = 40$ and $b = 96$*

(f) Use your result from part (e) to compute the exact value of the derivative

$$\left.\frac{ds}{d\theta}\right|_{\theta=3\pi/2}$$

when $r = 40$ and $b = 96$. Compare this with your estimate from part (c).

(g) What are the units of the derivative in part (f)? Explain the meaning of the derivative, not *by giving a standard response about the rate of change of some quantity with respect to another, but by relating small changes.*

(h) Use your response to part (f) with $r = 40$ and $b = 96$ to write an approximation for the change

$$\Delta s = f(\theta) - f(3\pi/2)$$

in terms of the change

$$\Delta\theta = \theta - 3\pi/2$$

for values of θ near $\theta = 3\pi/2$.

(i) Find a function

$$s = g(t)\,, t \geq 0,$$

that expresses your distance (in feet) to the flagpole in terms of the number of seconds since you started jogging. Assume you start at the point on the track nearest the flagpole.

(j) Find a function that expresses the rate of change (with respect to time) in your distance to the flagpole in terms of the number of seconds since you began jogging.

(k) Express the rate of change in part (b) in terms of your speed and the angle between the following two vectors:

- *the vector that gives your position relative to the flagpole*
- *the vector that points in the direction of your motion*

Exercise 134 *High tide of 10.83 feet at Edmonds at 12:28am, May 15, low tide at 6:58am of ?? feet, lowest tide of 1.64 feet at 5:32pm.*

Desmos link: `https://www.desmos.com/calculator/zta9tkzzmx`

Worksheet available at 151: Edmonds Tides

Exercise 135 *A pendulum of length L feet is $L+5$ oscillates sinuosoidally between angles $-\theta_0$ and θ_0 with period $2\pi\sqrt{L/g}$ seconds, where g is a constant. The angles $\pm\theta_0$ are measured in radians from the downward vertical.*

(a) Find a function

$$h = f(\theta)\,, -\theta_0 \leq \theta \leq \theta_0,$$

that expresses the height of the pendulum above its stagble equilibrium position (ie. its lowest point) in terms of the angle of rotation. Use the cosine function.

(b) Assume now that the pendulum is released from rest from the displacement angle $\theta = \theta_0$ at time $t = 0$ seconds. Find a function

$$\theta = g(t)\,, t \geq 0,$$

that expresses the displacement angle (in radians) in terms of the number of seconds since the pendulum was released. Use the cosine function.

(c) Find a function

$$h = w(t)\,, t \geq 0$$

that expresses the height (in feet) of the pendulum above equilibrium in terms of the number of seconds since the pendulum was released.

(d) Find an expression for the derivative dh/dt and interpret its meaning.

Exercise 136 *The function*

$$h = f(s) = 3 + 2\cos s\,,\ 0 \leq s \leq 3,$$

expresses the altitude (in thousand of feet) along a mountain trail in terms of the distance (measured in miles) from the summit. The distance is measured along the trail.

The function

$$s = g(t) = t + \frac{t^2}{4}\,,\ 0 \le t \le 2,$$

expresses your distance from the summit (in miles) in terms of the number of hours past noon.

The goal of this problem is to compute the rate (with respect to time) at which your altitude is changing at 1pm.

Desmos link: `https://www.desmos.com/calculator/twfsqmmgyb`

Worksheet available at 151: Mountain Road

(a) *Find an expression for a function $h = k(t)$ that expresses your altitude (in thousands of feet) in terms of the number of hours past noon. Include a domain.*

(b) *Evaluate the derivative*

$$\left.\frac{ds}{dt}\right|_{t=1}$$

and interpret its meaning.

(c) *Evaluate the derivative*

$$\left.\frac{dh}{ds}\right|_{s=g(1)}$$

and interpret its meaning.

(d) *Use the results of (b) and (c) to compute the rate (with respect to time) at which your altitude is changing at 1pm.*

(e) *Express the rate of part (d) as a derivative.*

(f) *Express the derivative*

$$\left.\frac{dh}{dt}\right|_{t=1}$$

in terms of the derivatives

$$\left.\frac{ds}{dt}\right|_{t=1}$$

and

$$\left.\frac{dh}{ds}\right|_{s=g(1)}.$$

Exercise 137 *A weight is attached to the midpoint of a 10-foot rope.*

You hold the ends of the rope 10 feet apart at the same height. You then move the ends directly toward each other, each at the constant speed of 2 ft/sec.

Desmos link: `https://www.desmos.com/calculator/grostrmdln`

Worksheet available at *151: Weighted Rope*

(a) *Find a function*

$$v = f(t)\,,\; 0 \le t \le 2.5,$$

that expresses the speed of the weight (in feet/sec) in terms of the number of seconds since you began moving your hands.

(b) *Describe how the speed of the weight varies.*

(c) *Comments?*

The Chain Rule and Polynomials

Problems with polynomials and the chain rule.

1 Practice Questions

Question 138 *The function*

$$h = f(s)\,,\ 0 \le s \le 3,$$

expresses the altitude (in thousands of feet) of a trail in terms of the distance (in miles) from the trailhead.

The function

$$s = g(t)\,,\ 0 \le t \le 4,$$

expresses your distance from the trailhead (in miles) as you hike along the trail in terms of the number of hours past noon.

The functions f and g are graphed below.

Desmos link: `https://www.desmos.com/calculator/sm5rxml9zi`
sm5rxml9zi

Worksheet available at 151: Hiking Trail

(a) *Use the graphs above to estimate the following. Include units and write each as a derivative using the Leibniz notation.*

  (i) *your elevation at 2:30pm.*
  (ii) *your hiking speed at 2:30pm.*
  (iii) *the steepness of your part of the trail at 2:30pm. Interpret the meaning of this in terms of small changes.*
  (iv) *the rate (in feet/hour) at which you are gaining or losing altitude at 2:30pm.*

(b) *Now suppose that*

$$h = f(s) = -\frac{s^3}{3} + s^2 + s + 2\,,\ 0 \le s \le 3$$

and

$$s = g(t) = 0.1t^2 + 0.35t\,,\ 0 \le t \le 4.$$

Learning outcomes:
Author(s):

(i) *Find the exact values of each of the three derivatives in part (a). Use the desmos worksheet to save time by entering f'(?) and g'(?) in a blank line, where you need to determine the inputs to these derivatives.*

Question 139 *Due to a late-season frost in eastern Washington, the price of Cosmic Crisp apples is rising precipitously. The function*

$$P = f(t)\,,\; 0 \le t \le 8,$$

expresses the pirce (in dollars/pound) in terms of the number of hours past 9am on April 25, 2025.

(a) *Find an expression for the function*

$$N = g(P)$$

for the number of pounds of apples you can buy with \$36 *in terms of the price (in dollars/pound).*

(b) *Evaluate the derivative*

$$\left.\frac{dN}{dP}\right|_{P=4}.$$

Include units.

(c) *Interpret the meaning of the above derivative in terms of small changes.*

(d) *Find an expression for the derivative dN/dt in terms of P and the derivative dP/dt.*

(e) *Suppose*

$$P = f(t) = 2 + \frac{2}{5}t\,,\; 0 \le t \le 8,$$

and determine the rate of change (with respect to time) at 2pm in then number of pounds of apples you can buy with \$36.

(f) *Suppose (for an arbitrary differentiable price function) that at some instant the price is decreasing at the relative rate of* 10%*/hour. Is the number of pounds of apples you can buy with* \$36 *increasing or decreasing at this instant? At what relative rate?*

Question 140 *The bottom end of a ten-foot ladder slides across a horizontal floor as its top end slides along a vertical wall.*

(a) *Find a function*

$$h = f(s)\,,\ 0 \le s \le 10,$$

that expresses the height of the top end above the floor (in feet) in terms of the distance of the bottom end from the wall (in feet).

(b) *Use the demonstration below to estimate the derivative*

$$\left.\frac{dh}{ds}\right|_{s=8}.$$

Include units. Then use calculus to find the exact value.

Desmos link: `https://www.desmos.com/calculator/jqwsnh33o1`

Worksheet available at 151: Ladder and Tree 23

(c) *Interpret the meaning of the above derivative in terms of small changes.*

(d) *Suppose when the ladder's bottom end is eight feet from the wall, it is sliding away from the wall at a speed of 5 ft/sec. Is the top end moving up or down the wall at this instant? At what speed?*

(e) *Now suppose the function*

$$s = g(t) = 10 - \frac{2}{5}(t-5)^2\,,\ 0 \le t \le 10,$$

expresses the distance of the ladder's bottom end from the wall (in feet) in terms of the number of seconds past noon.

(i) *Play the slider in Line 2 in the animation below to watch the motion. Then sketch by hand a rough graph of the composition*

$$h = f(g(t))\,,\ 0 \le t \le 10.$$

Label the axes with the appropriate variable names and units. Then activate the folder in Line 43 to see how you did.

Desmos link:
`https://www.desmos.com/calculator/duq8yqxozk`

Worksheet available at 151: Ladder and Tree 25

(ii) *Use the graph of $h = f(g(t))$ to sketch a graph of the function*

$$r = \frac{dh}{dt}.$$

Label the axes with the appropriate variable names and units. Then activate the folder in Line 47 to see how you did. Explain the meaning of this function.

(iii) *What comments/questions do you have about the graph of $r = dh/dt$?*

(iv) *At time $t = 7$ seconds past noon, is the top end of the ladder moving up or down? At what speed? Use calculus to find the exact speed.*

Question 141 *The function*

$$g = f(v)\,,\ 20 \le v \le 50,$$

expresses the gas mileage (in miles/gal) of a car in terms of its speed (in miles/hour).

The function

$$G = h(w)\,,\ 32 \le w \le 90,$$

expresses the gas mileage (in km/liter) of the same car in terms of its speed (in km/hour).

Assume there are exactly 1.6 km in one mile and exactly 4 liters in one gallon.

Suppose also that

$$\left.\frac{dg}{dv}\right|_{v=40} = 0.3.$$

(a) *What are the units of the derivative above? Explain its meaning in terms of small changes.*

(b) *Express the function h in terms of the function f.*

(c) *Use common sense to evaluate the derivative*

$$\left.\frac{dG}{dw}\right|_{w=64}.$$

(d) *Use common sense to express the derivative dG/dw in terms of dg/dv. Then use the chain rule and check if you get the same result.*

Question 142 *The function*

$$W = f(h) = \frac{2400}{(h+4)^2}\,,\ h \ge 0,$$

expresses the weight (in pounds) of an astronaut in terms of her height (in thousands of miles above the surface of the earth).

(a) *What are the units of the constant 2400? How do you know?*

(b) *Suppose when her spaceship is 100 miles above the earth's surface, the astronaut is moving directly toward the earth at a speed of 15,000 miles/hour. At what rate (with respect to time) is her weight changing at this instant. Compute this rate as the product of two other rates. Include units for these rates and explain their meanings in terms of small changes.*

Question 143 *Suppose an oil spill in the Gulf of Mexico retains the shape of a perfect circle as it expands. At some moment the circle's radius is increasing at the relative rate of 5%/hour. At what relative rate is its area changing at this moment?*

2 Discussion Questions

Question 144 *The function*

$$P = f(t)\,,\ 0 \le t \le 7,$$

expresses the price of a stock (in dollars/share) in terms of the number of hours past 9am. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/0yv49behlw`

Worksheet available at 151: Stock Chain Rule

(a) *Is the number of shares you can buy with $200 increasing or decreasing at 3pm? At what rate? Drag points P and Q in the graph to help approximate the rate. Explain your reasoning.*

(b) *Suppose*

$$P = f(t) = -\frac{1}{2}t^2 + 4t + 4\,,\ 0 \le t \le 7.$$

(i) *Find the exact rate (with respect to time) at which the number of shares you can buy with $200 is changing at 3pm.*

(ii) *Find the relative rates of change (with respect to time) at 3pm in the stock price and the number of shares you can buy with $200. Include units. Compare these rates.*

Question 145 *A rock is dropped from rest near the surface of some planet and allowed to fall until it hits the surface. The function*

$$v = f(h)\,,\ 0 \le h \le 50,$$

expresses the speed of the rock (in meters/sec) in terms of its height above the surface (in meters).

(a) *Would you expect the derivative*

$$\left.\frac{dv}{dh}\right|_{h=18}$$

to be positive or negative? Explain your reasoning.

(b) *Suppose*

$$v = g(h) = \frac{\sqrt{100-2h}}{3}\,,\ 0 \le h \le 50.$$

(i) *Use the graph of the function $y = \sqrt{x}$ and the sliders u and v below to help approximate the derivative in part (a).*

Desmos link:
`https://www.desmos.com/calculator/ilb4wptgxe`

Worksheet available at 151: Stock Chain Rule

(ii) *Use the expression for the function $v = g(h)$ to find the exact value of the derivative in part (a).*

(iii) *Interpret the meaning of the derivative in part (a) in terms of small changes.*

(iv) *Use part (iii) to approximate the rock's speed at a height of 17.7 meters.*

Question 145.1 *Go back to Question 3, part (e) of this chapter and determine all possible angles the ladder makes with the ground when its top end is moving twice as fast as its bottom end.*

The Doppler Effect: Stretching Time

The Doppler effect.

The Doppler effect, in its simplest form, describes how the pitch of a sound emitted from a moving source changes frequency as heard by a stationary observer. It's difficult for me to think about frequency and pitch, so we'll follow Hermann Bondi (*Relativity and Common Sense*, 1962, pp. 41-49) and imagine instead a moving source that emits pings at frequent, regularly spaced intervals, say at the rate of f pings/second. Then we'll compute the frequency (measured in 1/sec) at which these pings are heard by a stationary observor.

Relativity and Common Sense

Example 24. In this the simplest case, we'll suppose that the source is heading directly toward the observor at the constant speed of v meters/sec. We'll take the speed of sound to be w meters/sec. What determines the frequency heard by the observor is the ratio $\lambda = v/w$ and we'll assume $\lambda < 1$.

We'll also assume the source emits pings at the rate of f_s pings/second. Our problem is to express the frequency f_o of the pings heard by the observor in terms of λ and f_s.

Drag the slider t_1 below to start the motion of the source P. The observor is at the origin O. We assume the source passes harmlessly through the observor and then begins to move directly away. The pings are emitted at the frequency of 1/sec, on the second (the blinking light at A indicates when the pings are emitted). You should be able to approximate the frequency of the pings heard by the observor in the two cases, when the source is moving directly toward the observor and when it is moving directly away.

What are these frequencies?

Free Response:

Desmos link: `https://www.desmos.com/calculator/h2rw1sr8li`

Doppler Stationary Observer 1

We don't need calculus to analyze this problem. Nevertheless, we'll take an approach that will work in the next variation when the source does not move either directly toward or away from the observor.

Suppose the source collides with the observor at noon. The idea is to find a function $T = f(t)$ that takes as an input the time (measured in the number of

Learning outcomes:
Author(s):

seconds past noon, possibly negative) when a ping was emitted and returns as an ouput the time (measured in the same manner) that the observor hears that ping.

(a) Find an expression for the function

$$T = f(t)\,,\ t \in \mathbb{R}.$$

Work in general with speeds v m/sec for the source and w m/sec for the speed of sound. Note the important point that the speed of sound is constant (at a given air temperature) and independent of the motion of the source.

(b) Compute the derivative dT/dt. What are its units? There are two cases to consider, when $t < 0$ and when $t > 0$. Evaluate the derivative (two cases) for $\lambda = 1/2, 2/3, 9/10$.

(c) Interpret the meanings of these derivatives. Then write an equation expressing the frequence f_0 of the pings heard by the observor in terms of $\lambda = v/w$ and the frequency f_s of the pings emitted by the source. There are two cases to consider as above.

(d) Check your results with the demonstration above where $\lambda = 1/2$.

Example 25. This is a continuation of the previous example, where we now suppose the source (suppose it's an airplane) flies at a constant altitude h km and passes directly over the observor at noon.

In the demonstration below, the pings are emitted once/second and the light at A flashes each second. The light at B flashes each time the obersvor (still at the origin O) hears a ping.

Desmos link: `https://www.desmos.com/calculator/ow5li8h6o6`

Doppler Stationary Observer 2

(a) Use the demonstration to approximate the frequency of the pings recieved by the observor when the plane is

  (i) 10 km west (to the left) of O
  (ii) directly above O
  (iii) 10 km east of O

(b) Now find a function

$$T = g(t)\,,\ t \in \mathbb{R}.$$

as in the first example. This function takes as an input the time (measured in the number of seconds since noon) when a ping was emitted and returns as an output the time (measured the same way) when the observor hears this ping. Work in general, with v km/sec the speed of the source, w km/hour the speed of sound, and h (in km) the altitude of the plane.

The problem here should be to relate the frequencies at time $t = -0.5$ when $v = 1$, $w = 2$, and $h = 4$. The result should be that

$$f_1 = 0.7f_2.$$

Doppler Effect with Sound:

Desmos link: `https://www.desmos.com/calculator/gxzmjpgkrr`

Doppler Effect Stationary Observer 2 With Sound

On Trails

Hiking bugs.

1 Trails Desribed with Functions

In this section we model a trail with a curve $h = f(x)$, where x and h are measured in meters. We take the h-coordinate to measure the height (in meters) above sea-level and suppose the positive x-axis points due east.

Question 146 *A beetle crawls eastward along the trail $h = f(x)$ at a constant speed of v_0 meters/sec.*

At what rate is the beetle gaining altitude if

(a) $f(x) = x$?

(b) $f(x) = 2x$?

(c) $f(x) = mx$?

(d) *Find the limit of the beetle's rate of ascent as $m \to \infty$ as it crawls on the trail $h = mx$ at the constant speed of v_0 meters/sec.*

Question 147 (a) *A beetle crawls along the parabola*

$$h = 9 - \frac{1}{4}(x-6)^2 \ , \ 0 \leq x \leq 12,$$

(coordinates in meters) in the eastward direction at a constant speed of v_0 m/sec. At what rate is it ascending when it is 10 meters above the ground for the second time?

(b) *A beetle crawls along the curve*

$$h = 4 - 3\cos\left(\frac{x}{2}\right) \ , \ 0 \leq x \leq 12,$$

(coordinates in meters) in the eastward direction at a constant speed of v_0 m/sec. At what rate is it ascending when it is 5 meters above the ground for the second time?

Learning outcomes:
Author(s):

2 Parametrically Defined Trails

Derivatives of Trigonometric Functions

Working with trigonometric functions and their derivatives.

1 Introduction

Question 148 *You ride a ferris wheel of radius* 50 *feet at a constant speed of* 5 *ft/sec. The wheel's center is* 60 *feet above the ground.*

Desmos link: https://www.desmos.com/calculator/rlgxg2wbms

151: Vector Ferris Wheel

(a) *Use precalculus and common sense to find your rate of ascent when you are*

  (i) 60 *feet above the ground for the first time*
  (ii) 60 *feet above the ground for the second time*
  (iii) 110 *feet above the ground*
  (iv) 80 *feet above the ground for the first time*
  (v) 80 *feet above the ground for the second time*

(b) *Express your rate of ascent in terms of the angle θ your position vector relative to the wheel's center makes with the rightward horizontal. Assume θ is measured in the same sense as the rotation of the wheel.*

(c) *Express your rate of ascent in terms of the angle ϕ your position vector relative to the wheel's center makes with the upward vertical. Assume ϕ is measured in the same sense as the rotation of the wheel.*

2 Velocity

Velocity is the rate of change of position with respect to time. Since position is a vector, so is velocity. The instantaneous velocity vector is the derivative

Learning outcomes:
Author(s):

(with respect to time) of the position vector. It (the velocity vector) points in the direction of motion and is therefore tangent to the path. The length of the velocity vector is the speed.

Play the slider u (time) below to watch a projectile motion in a uniform gravitational field.

Desmos link: `https://www.desmos.com/calculator/vdtsb2fc8j`

151: Projectile Motion

(a) When is the object speeding up? Slowing down? How can you tell from looking at its velocity vector?

(b) Suppose the position vector (measured in meters) at time t seconds past launch is

$$\langle x, y\rangle = \langle (v_0 \cos\phi)t, (v_0 \sin\phi)t - \frac{1}{2}gt^2\rangle\,,\ 0 \le t \le \frac{2v_0 \sin\phi}{g}.$$

Here v_0 is the initial speed (in m/sec), ϕ is the launch angle relative to the horizontal, and g is the magnitude of the gravitational acceleration (in $\mathrm{m/sec^2}$).

Find an expression for the velocity vector.

3 Derivatives of the Sine and Cosine Functions

We just saw we can differentiate the position vector of a motion to find the velocity vector, at least if we know how to differentiate the component functions. There is one case where we can reverse this process. The next example shows how and gives a way to find the derivatives of the sine and cosine functions.

Example 26. The idea is to look at the motion with position vector (in meters)

$$\overrightarrow{OP} = \langle 1\cos(1t), 1\sin(1t)\rangle,$$

expressed in terms of the number of seconds past noon. Play the slider u (another name for t) in the worksheet below to watch the motion.

Desmos link: `https://www.desmos.com/calculator/ov0azsjucq`

151: Circular Motion

(a) What are the units of the blue 1 and the red 1 in the parameterization.

(b) What is the speed of the motion?

(c) Use the geometry of the circle to find the components of the velocity vector.

(d) What does this tell us about the derivatives

$$\frac{d}{dt}(\cos t) \qquad \text{and} \qquad \frac{d}{dt}(\sin t)\,.$$

(e) Use the same idea to compute the derivatives

$$\frac{d}{dt}(\cos(3t)) \qquad \text{and} \qquad \frac{d}{dt}(\sin(3t))?$$

Then use the chain rule to compute the derivatives again.

4 Riding a Ferris Wheel

Question 149 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) *Use the figure below to find a function*

$$h = f(\theta)\,,\ 0 \le \theta \le 2\pi,$$

that expresses your height above the ground in terms of the rotation angle of the wheel (measured in radians since the time you boarded). Use the cosine *function,* not *the sine function.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

(b) *The function*

$$\theta = a(t)\,,\ 0 \le t \le 120,$$

expresses the rotation angle (in radians, measured since you boarded the wheel) in terms of the number of seconds since you got on. Use the graph of this function below to approximate the your rate of ascent at time $t = 70$ seconds.

Desmos link: `https://www.desmos.com/calculator/kijv21gots`

151: Ferris Wheel Angle

(c) *Now suppose*

$$\theta = a(t) = \pi\left(1 - \cos\left(\frac{\pi}{120}t\right)\right)\,,\ 0 \le t \le 120,$$

and compute your exact rate of ascent at time $t = 70$ seconds. Compare this with your estimate.

5 Simple Harmonic Motion

Question 150 *Suppose for this problem that the earth is a ball with uniform density of radius 4000 miles. Now imagine drilling a straight tunnel through the earth from the North Pole to the South Pole.*

A subway car dropped from rest at the north pole falling through the tunnel would then oscillate in simple harmonic motion between the poles and return to the north pole every 84 minutes. This means we can think of the car as being dragged along by a point moving around the earth at constant speed as illustrated below.

Desmos link: `https://www.desmos.com/calculator/ij8dqowgza`

Desmos activity available at 142: Simple Harmonic Motion

(a) *Find a function*

$$s = f(t)\,,\ t \geq 0,$$

that expresses the distance of the rock (measured in thousands of miles) form the South Pole in terms of the number of minutes since the rock was released. Assume the rock was dropped at noon on July 1, 2085.

(b) *Use your function and calculus to determine the car's speed as it passes the center of the earth. Then find the speed without calculus.*

(c) *Express the average speed of the car during the time for one oscillation in terms of its maximum speed.*

(d) *Express the car's speed when it is 2/3 of the way from the earth's center to the South Pole in terms of its maximum speed.*

Question 151 *Assume for this question that each month has 30 days and that the number of hours of daylight/day in Seattle is a sinusoidal function of time. Assume also that on June 21, Seattle gets a maximum of 16 hours of daylight/day and that on December 21, Seattle gets a minimu of 8 hours of daylight/day.*

(a) *Find a function*

$$H = f(t)\,,\ 0 \leq t \leq 12,$$

that expresses the number of daylight hours/day in Seattle in terms of the number of months since June 21. Use the cosine function. Start by sketching a graph. Explain your reasoning.

(b) *Use your function to determine the number of hours of daylight/day that Seattle gets on August 21.*

(c) *Use calculus to approximate about how many more minutes of daylight/-day we get tomorrow than today.*

(d) *Use your function to determine the day(s) of the year when Seattle gets 14 hours of daylight/day.*

6 Two Optimization Problems

Question 152 *The function*

$$h = f(t) = 1200 + 500\cos t + 300\sin t\,,\ 0 \le t \le 10,$$

expresses the altitude (in feet) of a balloon in terms of the number of hours past noon.

Desmos link: `https://www.desmos.com/calculator/f1ldi6yrek`

151: Trig 1

(a) *Use calculus to find the minimum and maximum heights of the balloon between noon and 6pm. Do not use a calculator.*

(b) *Find the balloon's maximum rate of ascent. No calculator.*

Question 153 *Use calculus and some trigonometry (but not just trigonometry) to find the minimum and maximum values of the function*

$$f(\theta) = 3\cos\theta - 5\sin\theta\,,\ 0 \le \theta \le \pi.$$

Do not use a calculator.

7 Exercises

Question 154 *The function*

$$h = f(t) = 800 - 200\tan t\,,\ -1.3 \leq t \leq 1.3,$$

expresses the altitude (in feet) of a balloon in terms of the number of hours past noon.

Desmos link: `https://www.desmos.com/calculator/re6nqofgs0`

151: Trig 2

Find exact answers to the following questions without using a calculator.

(a) *When is the balloon descending at the rate of 150 ft/hour?*

(b) *How high is the balloon when it is descending at the rate of* 500 *ft/hour?*

(c) *Express the balloon's rate of ascent (in feet/hour) in terms of its altitude.*

Question 155 *The function*

$$h = f(t) = 11 - 7\cos t\,,\ 0 \leq t \leq 6,$$

expresses the depth of the water (in feet) at the Edmonds Pier in terms of the number of two-hour periods *past noon.*

Desmos link: `https://www.desmos.com/calculator/4bh7kimi7f`

151: Trig 3

Find exact answers to the following questions without using a calculator.

(a) *Is the depth of the water increasing or decreasing at 2pm? At what rate?*

(b) *At what rate is the depth of the water changing when the water is* 6 *feet deep?*

Question 156 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) *Use the figure below to find a function*

$$h = f(\theta)\,,\ 0 \leq \theta \leq 2\pi,$$

that expresses your height above the ground in terms of the rotation angle of the wheel (measured in radians since the time you boarded). Use the cosine *function,* not *the sine function.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

(b) *The wheel stops when you are* 100 *feet above the ground for the* second *time. It then starts again and turns through a small angle of* $\Delta\theta$ *radians before stopping again.*

Desmos link: `https://www.desmos.com/calculator/gi8obqmnav`

151: Ferris Wheel 34

(i) *Zoom in on point* P *above to approximate the change in your height (in feet) if* $\Delta\theta = 0.1$.

(ii) *Use a derivative to approximate the change in your height if* $\Delta\theta = 0.1$.

(iii) *Use a deriviative to approximate the change* Δh *in your height (measured in feet) as the wheel turns through the small angle* $\Delta\theta$ *radians.*

8 Visualizing Derivatives

Question 157 *You ride a ferris wheel for one revolution and get off. Let*

$$h = f(s)\,,0 \leq s \leq ??,$$

be the function that expresses your height above the ground (measured in feet) in terms of your distance traveled, measured (in feet) along your path from your starting point.

(a) Choose a radius for the ferris wheel that you think is reasonable and fill in the missing upper bound for the domain of f *above.*

(b) Use the demonstration below to sketch by hand a graph of the function

$$r = \frac{dh}{ds} = f'(s).$$

Do not *make any computations. Just use the demonstration below. Here are the key points to keep in mind:*

- *Approximate the derivative dh/ds by the ratio $\Delta h/\Delta s$.*
- *The length of the red arclength (when the ferris wheel is on the way up) is the input s.*
- *The length of the purple segment is the output h.*
- *The lengths of the orange arc and orange segments are Δs.*
- *The (signed) length of the green segment is Δh.*

Explain your reasoning thoroughly. Be sure to include at least the following points:

- *The units of the input and output to the derivative*
- *Scales on the vertical and horizontal axes*
- *A discussion of how a small change in the input to the function f changes the output at various positions along your ride.*
- *A discussion of where a small change in the input to f gives the greatest positive change in the output and a consideration of the ratios of these changes*
- *A discussion of where a small change in the input to f gives the negative change in the output with the greatest magnitude and a consideration of the ratios of these changes*
- *A discussion of where a small change in the input to f barely changes the output and a consideration of the ratios of these changes*

(c) How would your graph of the derivative dh/ds change if you doubled the radius of the ferris wheel? Sketch the new graph.

Desmos link: `https://www.desmos.com/calculator/pxsmo04nmg`

9 The Derivative of the Sine Function

Question 158 *Desmos link:* `https://www.desmos.com/calculator/jcmcyrpndw`

Desmos activity available at *151: Derivative of Sine*

Question 159 *(a) Describe a transformation that takes the graph of the function*

$$y = f(\theta) = \sin\theta$$

to the graph of the function

$$y = g(\theta) = \cos\theta.$$

(b) Does that same transformation take the graph of the derivative of the sine function to the graph of the derivative of the cosine function? Do not use any particular facts about these functions to answer this question. Instead, give a general answer that would apply to all pairs of functions similarly related.

(c) Use your answer to part (b) and the fact that

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

to find an expression for the derivative

$$\frac{d}{d\theta}(\cos\theta)$$

of the cosine function.

10 Applications 1

Question 160 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) Find a function

$$h = f(\theta)\,,\ 0 \le \theta \le 2\pi,$$

that expresses your height above the ground in terms of the rotation angle of the wheel, measured in radians. Use the cosine *function,* not *the sine function.*

(b) The wheel stops when you are 100 *feet above the ground and on the way up. It then starts again and turns through a small angle of $\Delta\theta$ radians before stopping again. Use the appropriate linear approximation to estimate the change Δh in your height (measured in feet) as the wheel turned through the angle θ.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

11 Transformations of the Sine Function

Question 161 *(a) Describe a transformation that takes the graph of the function*

$$y = f(\theta) = \sin\theta$$

to the graph of the function

$$y = g(\theta) = \sin(2\theta).$$

(b) How does that same transformation affect the slope of a line?

(c) Use your answer to part (b) to find an expression for the derivative

$$g'(\theta) = \frac{d}{d\theta}\left(\sin(2\theta)\right).$$

12 Applications 2

Question 162 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You travel at a constant speed of 5 ft/sec as you ride the ferris wheel.*

(a) Find a function

$$h = f(t)\,, t \geq 0$$

that expresses your height above the ground in terms of the number of seconds since you got on. Use the cosine *function,* not *the sine function.*

(b) Are you ascending or descending the second time you are 90 feet above the ground? At what rate? Use the methods of this class, not vectors, to answer this question.

(c) Find your height when you are descending at the rate of 4.8 feet/sec. Give all possibilities. Do not use a calculator except to do arithmetic.

13 MATH 142

Question 163 *The graph below shows the x-coordinate function of a beetle moving around a circle at a constant speed.*

Desmos link: `https://www.desmos.com/calculator/qi6c9xbhnw`

142: Edmonds Pier 2

Use the graph to answer the following questions. Be sure to include units.

(a) *Find the x-coordinate of the circle's center.*

(b) *Find the radius of the circle.*

(c) *Find the period of the motion. This the time it takes the beetle to make one revolution about the center of its circular path.*

(d) *Find a time when the beetle's x-coordinate is a maximum.*

(e) *Use (a)-(d) to find an expression $x = f(t)$ for the function that expresses the x-coordinate of the beetle (measured in feet) in terms of the number of minutes past noon. Include the domain.*

(f) *Check your expression from part (e) by substituting the two times given in the graph.*

Circular Interpolation

Circular Interpolation

1 Three Problems

The following three questions might look dramatically different, yet they are essentially identical.

Question 164 *A mass at the end of a spring oscillates in simple harmonic motion on a horizontal surface as shown below. The mass is released from rest* 2 *meters from its equilibrium position and it completes one oscillation every* 5 *seconds.*

142: Simple Harmonic motion.

Desmos link: `https://www.desmos.com/calculator/l0bp34oisg`

(a) *Which way is the mass moving* 1.5 *seconds after it is released? What is its speed at this time?*

(b) *Is the mass moving left or right the second time it is* 1.6 *meters from its equilibrium position? At what speed?*

Question 165 *The center of a ferris wheel with radius* 80 *feet is* 90 *feet above the ground. The wheel rotates at a constant rate, completing one revolution every* 100 *seconds.*

(a) *Are you ascending or descending* 30 *seconds after boarding. At what rate?*

(b) *Are you ascending or descending the second time you are* 42 *feet above the ground? At what rate?*

Learning outcomes:
Author(s):

Question 166 *Over the course of a 24-hour period, from midnight October 29 to midnight October 30, the depth of the water at the Edmonds Pier is a sinusoidal function of time. Suppose also that a high tide of 21 feet occurs at 2:00am and the following low tide of 5 feet occurs at 8:00am.*

(a) *At what rate is the depth of the water changing at 5:36am?*

(b) *At what rate is the depth of the water is changing the* second *time the water is* 8.2 *feet deep?*

What is usually called *sinusoidal modeling* is kind of a misnomer. First, it's usually easier to model osciallatory behavior with a cosine function. Second, and more importantly, the name *sinusuoidal modeling* fails to convey the main idea of replacing linear interpolation with circular interpolation.

Here's an example.

Example 27. Suppose that over the course of a 24-hour period, from midnight October 29 to midnight October 30, the depth of the water at the Edmonds Pier is a sinusoidal function of time. Suppose also that a high tide of 21 feet occurs at 2:00am and the following low tide of 5 feet occurs at 8:00am.

Our question is to determine the rate at which the depth of the water is changing the *second* time the water is 10 feet deep. One approach is to find a function that expresses the depth in terms of time and then evaluate the derivative at the appropriate time. This solution is shown below, split into two parts.

But there is an easier way. We'll look at that after the long way.

Explanation. We first find a function

$$h = f(t), 0 \leq t \leq 24,$$

that expresses the depth of the water (in feet) in terms of the number of hours past midnight, October 29.

Note that to say "sinusoidal function" means that the graph of f is generated by uniform circular motion. But the graphs of the sine and cosine functions are both generated this way, so it is ok to express $f(t)$ in terms of the cosine function, and we will do just that.

Here are the steps.

(a) First we'll use the information above to sketch by hand a graph of the function f. Label the axes with the appropriate variable names and units. Label the coordinates of two key points on the graph.

Desmos activity available at:

142: Edmonds Pier.

Desmos link: `https://www.desmos.com/calculator/x2kocpkcfm`

(b) Next we'll compute the mean depth of the water over the 24 hours and the maximum deviation of the depth from this mean. Including units in our computation, the mean depth is

$$h_{avg} = 0.5(21\text{ ft } + 5\text{ ft }) = 13\text{ ft.}$$

And the maximum deviation from the mean is

$$21\text{ ft } - 13\text{ ft } = 8\text{ ft.}$$

Activate the Amplitude folder on Line 5 of the above demonstration to draw the horizontal line showing the average depth. Note this line is labeled with its equation. There is also a vertical line that shows the the maximum deviation from the mean.

(c) Next we'll use the graph to find the period of oscillation. Since high tide occurs at 2:00am and low tide at 8:00am, the period (the time between succesive) high (or low) tides is

$$2(8\text{ hours } - 2\text{ hours }) = 12\text{ hours}$$

Activate the Period folder on Line 11 to show the period on the graph.

(d) Now we'll compute the rotation rate of a uniform circular motion that generates the sinusoidal variation in the depth of the water. This rotation rate is

$$\omega = \frac{2\pi\text{ radians}}{12\text{ hours}} = \frac{\pi}{6}\text{ radians/hr.}$$

Activate the Uniform Circular Motion folder on Line 18 to show the period on the graph.

(e) Next we'll use parts (a)-(d) to find an expression for the function

$$h = f(t), 0 \le t \le 24,$$

that gives the depth of the water (in feet) at time t hours past midnight. Use the cosine function. Keeping in mind that high tide occrs at 2:00am, our function is

$$h = f(t) = 13 + 8\cos\left(\frac{\pi}{6}(t-2)\right), 0 \le t \le 24.$$

(f) To check that our function is correct, we'll use the given information that the depth of the water is 21 feet at 2:00am and 5 feet at 8:00am.

Substituting $t = 2$ gives the depth at 2:00am (in feet) as

$$\begin{aligned} f(2) &= 13 + 8\cos\left(\frac{\pi}{6}(2-2)\right) \\ &= 13 + 8\cos(0) \\ &= 13 + 8 \\ &= 21. \end{aligned}$$

Substituting $t = 2$ gives the depth at 2:00am (in feet) as

$$\begin{aligned} f(8) &= 13 + 8\cos\left(\frac{\pi}{6}(8-2)\right) \\ &= 13 + 8\cos(\pi) \\ &= 13 + 8(-1) \\ &= 5. \end{aligned}$$

These check out.

(g) Finally, we'll use our function to estimate the depth (in feet) of the water at 5:30am, October 29 to be

$$\begin{aligned} f(5.5) &= 13 + 8\cos\left(\frac{\pi}{6}(5.5-2)\right) \\ &= 13 + 8\cos(7\pi/12) \\ &\sim 10.92 \end{aligned}$$

(h) Enter the coordinates of the appropriate point in Line 27 of the Desmos Activity above to check that the depth at 5:30am is reasonable.

Example 28. This is a continuation of the previous example, and we'll start with the function

$$h = f(t) = 13 + 8\cos\left(\frac{\pi}{6}(t-2)\right), 0 \le t \le 24,$$

that expresses the depth of the water (in feet) at the Edmonds Pier in terms of the number of hours past midnight.

Our question now is to find the rate at which the depth of the water is changing the *second* time the water is 10 feet deep.

One approach would be to first determine when the water is 10 feet deep for the second time. But this is not necessary. We can determine the rate without finding the time. Here's how.

(a) We'll first take time out of the picture by letting

$$\theta = \frac{\pi}{6}(t-2)$$

Then we'll let t_0 be the second time when the water is 10 feet deep and define the angle θ_0 as

$$\theta_0 = \frac{\pi}{6}(t_0 - 2)$$

(b) Our aim is to evaluate the derivative

$$\begin{aligned} \left.\frac{dh}{dt}\right|_{t=t_0} &= -\frac{4\pi}{3}\sin\left(\frac{\pi}{6}(t_0-2)\right) \\ &= -\frac{4\pi}{3}\sin(\theta_0), \end{aligned}$$

where I've left the computation of the derivative dh/dt to you.

Note the key point. To evaluate this derivative we need *not* know the value of θ_0, but only the value of $\sin\theta_0$.

(c) To determine the value of $\sin\theta_0$, we go back to the height function, which we now write in the form (since $\theta = \pi(t-2)/6$)

$$h = g(\theta) = 13 + 8\cos\theta.$$

When the water is 10 feet deep for the second time,

$$10 = 13 + 8\cos\theta_0$$

and

$$\cos\theta_0 = -3/8.$$

(d) Now since

$$\cos^2\theta_0 + \sin^2\theta_0 = 1,$$

we find that

$$\sin\theta_0 = \pm\frac{\sqrt{55}}{8}.$$

(e) To choose the correct sign, we go back to the graph.

Desmos activity available at:

142: Edmonds Pier 2.

Desmos link: `https://www.desmos.com/calculator/fs9xy1lf0e`

Here we can see that at the point Q (with coordinates $(t_0, f(t_0))$,

$$8 < t_0 < 14.$$

This tells us that

$$\theta_0 = \frac{\pi}{6}(t_0 - 2)$$

is between

$$\frac{\pi}{6}(8-2) < \theta_0 < \frac{\pi}{6}(14-2)$$

or that

$$\pi < \theta_0 < 2\pi.$$

In other words, the angle θ_0 is in the third or fourth quadrant. This means that $\sin\theta_0 < 0$ so

$$\sin\theta_0 = -\frac{\sqrt{55}}{8}.$$

(f) Finally, we evaluate the derivative

$$\begin{aligned} \left.\frac{dh}{dt}\right|_{t=t_0} &= -\frac{4\pi}{3}\sin(\theta_0) \\ &= \left(-\frac{4\pi}{3}\right)\left(-\frac{\sqrt{55}}{8}\right) \\ &= \frac{\pi\sqrt{55}}{6} \end{aligned}$$

So when the water is 10 feet deep for the second time, the depth is increasing at the rate of

$$\frac{\pi\sqrt{55}}{6}\ \text{ft/hr} \sim 3.88\ \text{ft/hr}.$$

2 A Quicker Approach: Circular Interpolation

Here's another approach to the same question.

Question 167 *Suppose that over the course of a 24-hour period, from midnight October 29 to midnight October 30, the depth of the water at the Edmonds Pier is a sinusoidal function of time. Suppose also that a high tide of 21 feet occurs at 2:00am and the following low tide of 5 feet occurs at 8:00am. We still wish to determine exact rate at which the water's depth is changing the second teim the water is* 10 *feet deep.*

The key idea is that sinusoidal variation is generated by uniform circular motion. So we can replace our problem about tides by one about a ferris wheel.

(a) *Drag point B below to the appropriate position for the water to be* 10 *feet deep. Then use the circle to help approximate the rate at which the water's depth is changing at the above time.*

(b) *Use a little trigonometry to find the exact rate.*

Explain your reasoning and include screenshots to help with your explanations.

Free Response:

Desmos link: `https://www.desmos.com/calculator/0wxwmkzvky`

142: Circular Interpolation Tides

Review of Sine and Cosine

Some problems.

1 Derivatives

Question 168 *Follow the method of Example 2 in the chapter* The Chain Rule *of these notes* exactly *to find expressions for each of the following derivatives.*

(a) $\dfrac{d}{d\theta}\left(\sec^2\theta\right)$

(b) $\sqrt{12+3\sin\theta}$

(c) $\dfrac{d}{d\theta}\left(\cos^2\theta+\sin^2\theta\right)$

Question 169 *Divide each side of the identity*

$$\cos^2\theta+\sin^2\theta=1$$

by $\cos^2\theta$ *to get*

$$1+\tan^2\theta=\sec^2\theta.$$

Now use the result of Question 1(a) and the chain rule to find an expression for the derivative

$$\frac{d}{d\theta}\left(\tan\theta\right).$$

2 Transformations

Question 170 (a) *The point with coordinates* (a,b) *lies on the graph of the function* $y=f(x)$. *What are the coordinates of the corresponding point on the graph of the function*

$$y=g(x)=f(x/2)?$$

Learning outcomes:
Author(s):

(b) *Express the derivative*

$$\frac{d}{dx}\left(g(x)\right)\Big|_{x=2a} = \frac{d}{dx}\left(f(x/2)\right)\Big|_{x=2a}$$

in terms of the derivative

$$\frac{d}{dx}\left(f(x)\right)\Big|_{x=a}.$$

(c) *Compute the slopes of the tangent lines to the curves* $y = \sin\theta$ *and* $y = \sin(\theta/2)$ *at the origin. Drag the slider* k *in the worksheet below to visualize the transformation.*

151: Tranform Sine.

Desmos link: `https://www.desmos.com/calculator/2tvlkyi1sj`

(d) *Which of the following curves share the same tangent line at the origin with the curve* $y = \sin\theta$*?*

Select All Correct Answers:

(i) $y = 2\sin\theta$

(ii) $y = 2\sin(2\theta)$

(iii) $y = 2\sin(\theta/2)$ ✓

(iv) $y = \frac{1}{3}\sin(3\theta)$ ✓

(v) $y = \frac{1}{3}\sin(\theta/3)$

3 Simple Harmonic Motion

Question 171 *A relaxed spring has a length of* 3 *meters. A mass is attached to the right end of the relaxed spring and then pulled an additional* 2 *meters to the right. The mass is then released from rest and oscillates without friction on a horizontal surface about its equilibirium postion (*$x = 0$ *meters) with a period of* 5 *seconds as shown below.*

A consequence of Hooke's Law (that the force of the spring acting on the mass points toward equilibrium and has magnitude proportional to the distance of the mass from equilibrium) is that the mass oscillates in simple harmonic motion about its equilibrium position. This means that the motion of the mass is driven by uniform circular motion around the circle of radius 5 *meters below.*

151: Simple Harmonic motion 23.

Desmos link: https://www.desmos.com/calculator/noxooak1au

(a) *Find a function*
$$s = f(t)\,,\ t \geq 0,$$
that expresses the displacement of the mass from equilibrium in terms of the number of seconds since the mass was released.

(b) *Find a function*
$$v = g(t)\,,\ t \geq 0,$$
that gives the horizontal component of the velocity of the mass (in meters/sec) in terms of the number of seconds since release. Take this component to be negative when the mass is moving to the left, positive when moving to the right.

(c) *Use your function from part (b) to find the maximum speed of the mass.*

(d) *Find an equation relating s and v. Use it to answer the following questions.*

   (i) *Where is the mass when it is moving at $2/3$ of its maximum speed?*

   (ii) *At what fraction of its maximum speed is the mass traveling when it is $3/4$ of the way from its equilibrium position to one if its extreme positions?*

4 Ferris Wheels

Question 172 *A ferris wheel rotates at a constant rate, making one revolution every 90 seconds. The wheel has a radius of 50 feet and its center is 60 feet above the ground.*

You ride the wheel for one revolution and jump off.

(a) *What is your speed as you ride the wheel?*

(b) *Find an equation relating your height h above the ground (in feet) and your rate of ascent r (in ft/sec) at any instant. Start by finding an expression for the height as a function of time.*

(c) *Use your equation from part (b) to determine your rate of ascent when you are 20 feet above the ground for the second time.*

Question 173 *The graph below shows your height (in feet) as a function of the number of seconds since you boarded a ferris wheel. The wheel does not change its sense of rotation, but its rate of rotation might vary during the time you make your first revolution.*

151: Ferris Wheel 99.

Desmos link: `https://www.desmos.com/calculator/4y7ljahkwn`

(a) *Use the graph above to approximate your rate of ascent when you are* 150 *feet above the ground for the first time.*

(b) *Use part (a) to approximate the wheel's rotation rate when you are* 150 *feet above the ground for the first time.*

Hint: *Start by expressing your height in terms of the wheel's angle of rotation since you boarded.*

5 Flagpole and Track

Question 174 *Let r and a be constants, with $a > r > 0$.*

You run around a circular track of radius r feet, passing point A at time $t = 0$ seconds past noon (see the animation below where $r = 50$). A flagpole is located at point F, a feet from the track's center and $a - r$ feet from A as shown below (where $a = 80$).

151: Track and Flagpole.

Desmos link: `https://www.desmos.com/calculator/xymelsnabl`

Answer the following questions for general parameters a and r. Do not *use the specific values in the worksheet.*

(a) *Find a function*

$$s = f(\theta)\,,\ \theta \geq 0,$$

that expresses your distance to the flagpole (in feet) in terms of the angle $\theta = \angle AOP$.

(b) *Find an expression for derivative $ds/d\theta$.*

(c) *With* $r = 50$ *and* $a = 80$*, evaluate the derivative*

$$\left.\frac{ds}{d\theta}\right|_{\theta=\pi/2}.$$

Include units. Then interpret the meaning of this derivative in terms of small changes.

(d) *Now le'ts return to working with the general parameters* a *and* r*. Suppose you run around the track at the constant speed of* v *ft/sec.*

(i) *Find an expression for the derivative* ds/dt*, where* t *is the number of seconds past noon.*

(ii) *Express the derivative* ds/dt *in terms of* v *and the angle* $\phi = \angle OPF$ *above as you run around the "top" half of the track.*

(iii) *Use the result of part (ii) to describe your position on the track when your distance from the flagpole is increasing at the greatest rate. What is this maximum rate?*

(iv) *How would your answer to part (iii) change if* $0 < a < r$ *(ie. if the flagpole were inside the track)?*

Exponential Functions, Part 2

More on exponential functions and their derivatives.

QR Code

Desmos link: `https://www.desmos.com/calculator/emp4pukecm`

1 Introduction

Consider two models for the population of a bacteria colony. The first,

$$P = g(t) = 3 + \frac{1}{2}t\,,\ 0 \le t \le 6,$$

is linear and expresses the population (in millions of bacteria) in terms of the number of hours past noon. The growth rate is easy to describe. The population increases at the contant rate of 0.5 million bacteria/hour. We know this from precalculus. We could also use calculus to compute the growth rate as

$$\frac{dP}{dt} = \frac{d}{dt}\left(3 + \frac{1}{2}t\right) = \frac{1}{2}\,\frac{\text{bac}}{\text{hr}}.$$

We could avoid thinking about an instantaneous growth rate by saying that the average growth rate over *any* time interval is 0.5 million bacteria/hour. This statement implies that the (instantaneous) growth rate is constant.

But saying that the population increases by 0.5 million bacteria every hour would not completely describe how the population grows. It does *not* imply a constant growth rate because it does not tell us anything about what happens to the popluation over, for example, a half-hour period. The function graphed below shows an example of a non-linear function that increases by 0.5 million over *every* one-hour interval. Drag the slider u in Line 1 to convince yourself of this.

Describe what you see in the animation and how this suggests the function increases by 0.5 million over *every* one-hour interval.

Free Response:

151: Not Constant Rate

Learning outcomes:
Author(s):

Desmos link: `https://www.desmos.com/calculator/ehtzdzckmy`

The second model,

$$P = f(t) = 3(2^t)\,,\ 0 \le t \le 6,$$

is exponential. Like the first it expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon. How can we desribe how this population grows?

We could say that the population doubles every hour. Equivalently, we could say that the population increases by 100% every hour. But these descriptions would be incomplete. They don't tell us, nor do they imply, anything about what happens to the population over a half-hour period for example. The function $P = h(t)$ graphed below shows an example of a non-exponential function that doubles every hour. Drag the slider u in Line 1 and keep your eye on Line 2 (where $h = 1$, and $P = f_7(t) = h(t)$) to convince yourself of this.

Question 175 *Describe what you see in the animation and how this suggests the function doubles over* every *one-hour interval.*

151: Not Constant Rate 3

Desmos link: `https://www.desmos.com/calculator/zapwlt6qju`

We might try describing what happens to the population over a shorter time interval, say Δt hours. Over the time interval from time t hours past noon to time $t + \Delta t$ hours past noon the population increases by

$$\begin{aligned}\Delta P &= f(t + \Delta t) - f(t)\\ &= 3(2^{t+\Delta t} - 2^t)\\ &= 3(2^t \cdot 2^{\Delta t} - 2^t)\\ &= 3(2^t)(2^{\Delta t} - 1)\end{aligned}$$

millions of bacteria. To get the relative change over this time interval, we divide this absolute change by the population at the start of the time interval. This gives the relative rate of change as

$$\begin{aligned}\frac{\Delta P}{P} &= \frac{f(t + \Delta t) - f(t)}{f(t)}\\ &= \frac{3(2^t)(2^{\Delta t} - 1)}{3(2^t)}\\ &= 2^{\Delta t} - 1.\end{aligned}$$

Question 176 *What are the units of this rate of change?*

For example, the relative rate of change in the population over a one hour period is

$$2^1 - 1 = 1 = 100\%,$$

and over a half-hour period is

$$\sqrt{2} - 1 \sim 0.414 \sim 41.4\%.$$

Question 177 *Explain why this rate is less than 50%.*

Exploration 178 *The worksheet below shows the graph of the function $P = 3(2^t)$.*

151: Exp Growth Subtangent

Desmos link: `https://www.desmos.com/calculator/zu0f4jnh4a`

(a) *Experiment with the worksheet by first dragging the slider u (another name for t) in Line 2. Watch how Lines 6, 8, 10 change, as well as how the length of segment AB changes. Describe what you see.*

Notes:

(i) *Line 6*

$$\frac{f(u+h)}{f(u)}$$

This is the growth factor over the inteval $t = u$ to $t = u + h$ hours.

(ii) *Line 8:*

$$\frac{1}{f(u)} \cdot \frac{f(u+h) - f(u)}{h}$$

This is the relative average rate of change over the inteval $t = u$ to $t = u + h$ hours.

(iii) *Line 10:*

$$f(u) \cdot \frac{h}{f(u+h) - f(u)}$$

This is the reciprocal of Line 8.

(b) *Now Drag $h = \Delta t$ in Line 4 to $h = 0.5$ and repeat part (a). Describe what you see.*

(c) *Drag h closer to zero and describe how Lines 6, 8, and 10 change. Include units for these three expressions.*

(d) *What are the units of Line 10? What is its meaning?*

(e) *Summarize your observations.*

2 Relative Growth Rates

We still have not answered the question about how to describe exponential growth *without* a formula. What we know so far is that over time intervals of fixed duration an exponential function increaseses by a fixed percentage. And Exploration 4 might have suggested that the relative instantaneous growth rate

$$\frac{1}{P} \cdot \frac{dP}{dt}$$

is constant for an exponential function. We can show this is true for our function $f(t) = 3(2^t)$ using our prior work where we found that

$$\frac{\Delta P}{P} = 2^{\Delta t} - 1.$$

So the relative instantaneous growth rate of the function $f(t) = 3(2^t)$ is

$$\begin{aligned} \frac{1}{P} \cdot \frac{dP}{dt} &= \lim_{\Delta t \to 0} \frac{1}{P} \cdot \frac{\Delta P}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot \frac{\Delta P}{P} \\ &= \lim_{\Delta t \to 0} \frac{2^{\Delta t} - 1}{\Delta t}. \end{aligned}$$

Now we have no way to evaluate this limit algebraically (at least as of yet), but we can recognize it as the derivative

$$\frac{d}{dt}\left(2^t\right)\Big|_{t=0} = \lim_{\Delta t \to 0} \frac{2^{\Delta t} - 2^0}{\Delta t}.$$

From Exploration 4 we know the approximate value of this limit, and hence the relative rate of change of our population function $P = f(t) = 3(2^t)$. It is

$$\frac{1}{P} \cdot \frac{dP}{dt} \sim 0.693 = 69.3\%/\text{h}r.$$

So our population increases at the constant relative rate of approximately 69.3%/hr. The exact relative rate turns out to be

$$\frac{1}{P} \cdot \frac{dP}{dt} = \ln 2/\text{h}r.$$

But to understand why, we first need to come to terms with the number $e \sim 2.71828$.

3 The Number e

For our purposes, the best definition definition of the number e is this: It is the number for which the exponential function (t in hours)

$$P = g(t) = e^t$$

increases at the constant relative rate of 100%/hr. Put another way, the number e is the one-hour growth factor for a population that increases at the relative rate of 100%/hr. But to make sense of this, we need to understand what it means to say that at any instant a population is increasing at the relative rate of 100%/hr.

It does *not* mean that the population increases 100% every hour (the function $f(t) = P_0(2^t)$ does that). Rather, like every derivative the meaning is best understood in terms of small changes.

For example, in 1/100 of an hour the population would increase by about

$$\left(\frac{1}{100}\right)(100\%) = 1\%.$$

So the 1/100-hour growth factor is about 1.01. And the one-hour growth factor is about

$$\left(1 + \frac{1}{100}\right)^{100} \sim 2.705.$$

This tells us that $e \sim 2.705$ and that every hour a population increasing at the relative instantaneous rate of 100%/hr increases by about 170.5%.

We can get a better approximation to e by taking a smaller time interval. For example, a better approximation is

$$e \sim \left(1 + \frac{1}{1000}\right)^{1000} \sim 2.717.$$

And as you might have guessed,

$$e = \lim_{n \to infty} \left(1 + \frac{1}{n}\right)^n \sim 2.718.$$

4 An Example

When we learned about derivatives of inverse functions, we saw that it was helpful to express the derivative of a one-to-one function in terms of its *output*. For the function

$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \le \theta \le \pi/2,$$

for example,

$$\frac{dy}{d\theta} = \cos\theta = \sqrt{1-y^2}.$$

When working with exponential functions, or simple transformations of these, it is almost always best to do the same. Here's an example.

Example 29. The function

$$P = 3e^{\frac{1}{5}t}\,,\ 0 \le t \le 20,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

Find the population when it is increasing at the rate of 2 million bacteria/hour.

Explanation. There's a hard way and an easy way to solve this problem. The hard way would be to first find the time when the population is increasing at the given rate. Then we could use this time to find the population.

The easy way is to ignore time altogether and express the growth rate of the population in terms of the population.

Before we do this, let's think first about the units of the factor $1/5$ in the exponent. Since any exponential function takes only dimensionless inputs and t is measured in hours, the units of $1/5$ are

Question 179 ***Multiple Choice:***

(a) *hours*

(b) *dimensionless*

(c) *millions of bacteria*

(d) *hours*$^{-1}$ ✓

The units of $1/5$ give a clue to its meaning. It is in fact the relative growth rate of the population. So at any instant this population is increasing at the relative rate of

Question 180 20%*/hour* .

Question 181 (a) *Use the chain rule to compute the absolute growth rate*

$$\frac{dP}{dt} = \frac{d}{dt}\left(3e^{\frac{1}{5}t}\right).$$

Make the explicity u-substitution. Do not skip any steps.

(b) *Use part (a) to verify that the relative growth rate*

$$\frac{1}{P} \cdot \frac{dP}{dt}$$

is 20%*/hour.*

You should have found the relative growth rate to be

$$\frac{1}{P} \cdot \frac{dP}{dt} = \frac{1}{5}\ \mathrm{hr}^{-1},$$

so that the absolute growth rate (in millions of bacteria/hour) is

$$\frac{dP}{dt} = \frac{1}{5}P.$$

Question 182 *Use this to find the population when it is increasing at the rate of* 2 *million bacteria/hour.*

5 Exercises

Question 183 *The function*

$$P = f(t) = 50e^{\frac{t}{10}}\ ,\ 0 \leq t \leq 20,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past midnight. Graph shown below.

Desmos link: `https://www.desmos.com/calculator/kffjsoepmo`

151: Exp Function 5

(a) *Use the graph of the function $P = f(t)$ above to approximate the growth rate when there are 200 million bacteria.*

(b) *Find a function $r = g(P)$ that expresses the growth rate (measured in millions of bacteria/hour) in terms of the popuation (measured in millions). Include a domain.*

(c) *Find the* exact *growth rate when there are 200 million bacteria. No calculator.*

(d) *What is the population when it is increasing at the rate of 8 million bacteria/hour?*

(e) *Use the ideas of this question to describe how the population grows.* Hint: *Think about the relative instantaneous growth rate.*

Question 184 *Between noon and 6pm, a population of bacteria grows exponenially, increasing by 20% every hour. There are 5 million bacteria at noon.*

(a) *Find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon.*

(b) *Find a function that expresses the growth rate (in millions of bacteria/hour) in terms of the population (in millions).*

(c) *Find the population when it is increasing at the rate of 2 million bacteria/hour.*

(d) *What is the relative growth rate of the population?*

(e) *Consider a second population that grow exponentially, increasing by 20% every four hours.*

   (i) *Use common sense to find its relative growth rate.*

   (ii) *Use calculus to verify your answer from part (i) with a computation.*

Question 185 *The function*

$$N = f(t) = \frac{20}{1 + 5e^{-t/5}}\ ,\ 0 \leq t \leq 30,$$

models the spread of a virus throughout a population. It takes as an input the number of months since January 1st and returns as an output the number of infected individuals (measured in millions). Graph shown below.

Desmos link: `https://www.desmos.com/calculator/ikis451wxu`

151: Logistic Model

(a) *Use the sliders in the worksheet above to approximate the number of infected individuals when the virus is spreading at the rate of* $800,000$ *people/month.*

(b) *Find a function*

$$r = g(N)$$

that expresses the infection rate (in millions of people/month) in terms of the population.

(c) *Use calculus and algebra to determine the number of infected individuals when the virus is spreading at the rate of* $750,000$ *people/month.*

(d) *Use the sliders in the worksheet above to approximate the number of infected individuals when the virus is spreading at its fastest rate. Also approximate this rate.*

(e) *Use calculus and algebra to determine the number of infected individuals when the virus is spreading at its fastest rate. Then find this exact rate.*

Question 186 *Use the chain rule and your knowledge of the derivative* $d(e^x)/dx$ *to compute each of the following derivatives.*

(a)

$$\frac{d}{dt}\left(100 \cdot 2^t\right)$$

(b)

$$\frac{d}{dt}\left(100 \cdot 2^{t/5}\right)$$

Explanation. (a) The key is to realize that

$$2 = e^{\ln 2}.$$

So we can write the function to differentiate as

$$P = 100 \cdot 2^t = 100\left(e^{\ln 2}\right)^t = 100e^{t\ln 2}.$$

Now we can make a u-substitution and use the chain rule. We let

$$u = t\ln 2.$$

Then

$$P = 100e^{t \ln 2} = 100e^u$$

and

$$\begin{aligned}
\frac{dP}{dt} &= \frac{dP}{du} \cdot \frac{du}{dt} \\
&= \frac{d}{du}\left(100e^u\right) \cdot \frac{d}{dt}\left(t \ln 2\right) \\
&= 100e^u (\ln 2) \\
&= P \ln 2 \\
&= (100 \cdot 2^t) \ln 2 \\
&= (100 \ln 2)(2^t)
\end{aligned}$$

Question 187 *A colony of bacteria has a population of* 20 *million at noon and a population of* 25 *million at 2pm. The population grows exponentially between noon and midnight.*

(a) *Describe precisely how the population grows.*

(b) *Use your description from part (a) to find a function*

$$P = f(t)\,,\ 0 \le t \le 12,$$

that expresses the population (in millions) in terms of the number of hours past noon. Do not *use* e.

(c) *Find the relative instantaneous growth rate of the population.*

(d) *Find a function* $r = g(P)$ *that expresses the (absolute) growth rate in terms of the population.*

(e) *At what rate is the population changing when there are* 30 *million bacteria?*

Explanation. (a) The population increases by

$$\frac{25 \text{ million} - 20 \text{ million}}{20 \text{ million}} = 25\%$$

every two hours.

Better yet, at least for determining the function, is to say that the two-hour growth factor is

$$\frac{25 \text{ million}}{20 \text{ million}} = 1.25.$$

This means that every two hours the population gets multiplied by 1.25.

(b) Since the two-hour growth factor is 1.25, the one-hour growth factor is $1.25^{1/2}$. So

$$P = 20(1.25^{1/2})^t = 20(1.25)^{\frac{t}{2}}\,,\ 0 \le t \le 12.$$

(c) The key to computing the derivative dP/dt is to write the population function using base e. Since

$$1.25 = e^{\ln 1.25},$$

$$\begin{aligned} P &= 20(1.25)^{\frac{t}{2}} \\ &= 20\left(e^{\ln 1.25}\right)^{\frac{t}{2}} \\ &= 20e^{t(\ln 1.25)/2} \end{aligned}$$

So the growth rate (in millions of bacteria/hour) is

$$\begin{aligned} \frac{dP}{dt} &= \frac{d}{dt}\left(20(1.25)^{\frac{t}{2}}\right) \\ &= 20\frac{d}{dt}\left(e^{t(\ln 1.25)/2}\right) \\ &= \left(20e^{t(\ln 1.25)/2}\right)\frac{d}{dt}\left(t(\ln 1.25)/2\right) \\ &= P(\ln 1.25)/2. \end{aligned}$$

And the relative growth rate (units are hr^{-1}) is

$$\frac{1}{P}\cdot\frac{dP}{dt} = \frac{1}{2}\ln 1.25.$$

(d) From part (c), the function

$$g(P) = \frac{dP}{dt} = P(\ln 1.25)/2$$

expresses the growth rate (in millions of bacteria/hour) in terms of the population.

(e) When there are 30 million bacteria, the population is increasing at the rate of

$$g(30) = (30 \text{ million bacteria})\left(\frac{1}{2}\ln 1.25/\mathrm{hr}\right) = 15\ln 1.25 \text{ million bacteria/hr}.$$

Question 188 *Between ground level and an altitude of* 10 *km, atmospheric pressure on the planet Krypton is an exponential function of altitude. The pressure is 100 kPa at ground level and 80 kPa at an altitude of* 0.75 *km.*

Desmos link: `https://www.desmos.com/calculator/j5h8kaj8xs`

151: Atomspheric Pressure

(a) *Describe exactly how the pressure decreases.*

(b) *Use your description from part (a) to find a function*

$$P = f(h)\,,\; 0 \leq h \leq 10,$$

that expresses the atmospheric pressure (in kPa) in terms of the altitude (in kilometers). Do not *use the number e.*

(c) *Input your expression for f in Line 2 of the worksheet above.*

(d) *Evaluate the derivative*

$$\left.\frac{dP}{dh}\right|_{h=5}$$

and interpret its meaning.

(e) *Find a function $r = g(P)$ that expresses the derivative dP/dh in terms of P.*

(f) *Use calculus to approximate the altitude at which the pressure is 0.2 kPa less at an altitude 0.1 km higher.*

6 Conceptual Questions

Question 189 *A population of bacteria grows exponentially. If the growth rate is 4 million bacteria/hour when the population is 20 million, determine the growth rate when the population is 30 million.*

Question 190 *A cup of coffee at a temperature of $90^\circ C$ is brought into a room held at a constant temperature of $20^\circ C$. Suppose that the difference in temperature between the coffee and the room decreases exponentially. Suppose also that the temperature of the coffee initially decreases at the rate of $5^\circ C$/min. What is the temperature of the coffee when its temperature is decreasing at the rate of $2^\circ C$/min?*

Exponential Functions, Short Version

More on exponential functions and their derivatives.

1 Introduction

To claim a population that increases by 0.5 million bacteria every hour grows at a constant rate would be a falsehood. The reason is the statement does not imply anything about what the population does over a shorter period. The graph below shows a example of a non-linear function that increases by 0.5 million over *every* one-hour interval. Drag the slider u in Line 1 to convince yourself of this.

151: Not Constant Rate

Desmos link: `https://www.desmos.com/calculator/ehtzdzckmy`

We would make a similar error in asserting a population that doubles every hour grows exponentially. The function $P = h(t)$ graphed below shows an example of a non-exponential function that doubles every hour. Drag the slider u in Line 1 and keep your eye on Line 2 (where $h = 1$, and $P = f_7(t) = h(t)$) to convince yourself of this.

151: Not Constant Rate 3

Desmos link: `https://www.desmos.com/calculator/zapwlt6qju`

We can describe linear growth by saying a population grows at a constant rate of 0.5 million bacteria/hour. Or we could avoid the idea of an instantaneous growth rate by saying that the population increases at the average rate of 0.5 million bacteria/hour over *every* time interval.

But to correct the second error and describe exponential growth, we are almost forced into thinking about the instantaneous growth rate. What characterizes exponential growth is that this rate is proportional to the population. So for example, when the population is eight times the current population, its growth rate would be eight times the current growth rate.

To understand why in a specific example, consider the graph of the function

$$P = f(t) = P_0 \cdot 2^t$$

Learning outcomes:
Author(s):

shown below.

151: Exp Growth Translate Dilate

Desmos link: `https://www.desmos.com/calculator/dmdpvei3d6`

The key point is that because

$$f(t-3) = 2^{t-3} = (2^t)(2^{-3}) = \frac{1}{8} f(t),$$

$$8f(t-3) = f(t).$$

Geometrically, this means that translating the graph of f three hours to the right (Drage the slider u in Line 2 above from $u = 0$ to $u = 3$) and then multiplying the outputs by 8 (drag the slider v in Line 4 from $v = 1$ to $v = 8$) sends the graph of $P = f(t)$ to itself and the point $A(0, f(0) = A(0, P_0)$ to the point $A''(3, f(3)) = A''(3, 8P_0)$. The translation leaves the slope of the tangent line at $(0, P_0)$ unchanged and the dilation multiplies the slope by 8. So

$$\left.\frac{dP}{dt}\right|_{t=3} = 8\left(\left.\frac{dP}{dt}\right|_{t=0}\right).$$

The same kind of reasoning shows that for any for any exponential function, the growth rate is proportional to the population. Here's a more computational proof of this fact for the function $f(t) = P_0 2^t$.

Over the time interval from time t hours past noon to time $t + \Delta t$ hours past noon the population increases by

$$\begin{aligned} \Delta P &= f(t + \Delta t) - f(t) \\ &= 3(2^{t+\Delta t} - 2^t) \\ &= 3(2^t \cdot 2^{\Delta t} - 2^t) \\ &= 3(2^t)(2^{\Delta t} - 1) \\ &= P(2^{\Delta t} - 1). \end{aligned}$$

millions of bacteria. Here $P = f(t)$ is the population at the start of the time interval.

So over the period from t hours past noon to $t + \Delta t$ hours past noon the average growth rate is

$$\frac{\Delta P}{\Delta t} = P\left(\frac{2^{\Delta t} - 1}{\Delta t}\right),$$

measured in millions of bacteria/hour.

Then at time t hours past noon, the instantaneous growth rate (in millions of bacteria/hour) is

$$\begin{aligned}\frac{dP}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} \\ &= \lim_{\Delta t \to 0} P \left(\frac{2^{\Delta t} - 1}{\Delta t} \right) \\ &= P \lim_{\Delta t \to 0} \frac{2^{\Delta t} - 1}{\Delta t}.\end{aligned}$$

This shows what we wanted, that at any instant the growth rate is proportional to the poluation. Expressed differently, this says that the relative growth rate (measured in hr^{-1}) is constant and equal to

$$\frac{1}{P} \cdot \frac{dP}{dt} = \lim_{\Delta t \to 0} \frac{2^{\Delta t} - 1}{\Delta t}.$$

The best we can do here is to approximate the above limit numerically, to get the relative growth rate as

$$\frac{1}{P} \cdot \frac{dP}{dt} \sim 0.6931 \text{ hr}^{-1} \sim 69.31\%/\text{hr}.$$

Key Point: Over a one-hour period, the population doubles. So over a one-hour period the population increases at a relative average rate of 100%/hour. This is greater than the relative instantaneous growth rate of 69.31%/hr because of compounding.

2 Relative Growth Rates

We still have not answered the question about how to describe exponential growth *without* a formula. What we know so far is that over time intervals of fixed duration an exponential function increaseses by a fixed percentage. And Exploration 4 might have suggested that the relative instantaneous growth rate

$$\frac{1}{P} \cdot \frac{dP}{dt}$$

is constant for an exponential function. We can show this is true for our function $f(t) = 3(2^t)$ using our prior work where we found that

$$\frac{\Delta P}{P} = 2^{\Delta t} - 1.$$

So the relative instantaneous growth rate of the function $f(t) = 3(2^t)$ is

$$\begin{aligned} \frac{1}{P} \cdot \frac{dP}{dt} &= \lim_{\Delta t \to 0} \frac{1}{P} \cdot \frac{\Delta P}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot \frac{\Delta P}{P} \\ &= \lim_{\Delta t \to 0} \frac{2^{\Delta t} - 1}{\Delta t}. \end{aligned}$$

Now we have no way to evaluate this limit algebraically (at least as of yet), but we can recognize it as the derivative

$$\frac{d}{dt}\left(2^t\right)\Big|_{t=0} = \lim_{\Delta t \to 0} \frac{2^{\Delta t} - 2^0}{\Delta t}.$$

From Exploration 4 we know the approximate value of this limit, and hence the relative rate of change of our population function $P = f(t) = 3(2^t)$. It is

$$\frac{1}{P} \cdot \frac{dP}{dt} \sim 0.693 = 69.3\%/\mathrm{h}r.$$

So our population increases at the constant relative rate of approximately 69.3%/hr. The exact relative rate turns out to be

$$\frac{1}{P} \cdot \frac{dP}{dt} = \ln 2/\mathrm{h}r.$$

But to understand why, we first need to come to terms with the number $e \sim 2.71828$.

3 The Number e

For our purposes, the best definition definition of the number e is this: It is the number for which the exponential function (t in hours)

$$P = g(t) = e^t$$

increases at the constant relative rate of 100%/hr. Put another way, the number e is the one-hour growth factor for a population that increases at the relative rate of 100%/hr. But to make sense of this, we need to understand what it means to say that at any instant a population is increasing at the relative rate of 100%/hr.

It does *not* mean that the population increases 100% every hour (the function $f(t) = P_0(2^t)$ does that). Rather, like every derivative the meaning is best understood in terms of small changes.

For example, in 1/100 of an hour the population would increase by about

$$\left(\frac{1}{100}\right)(100\%) = 1\%.$$

So the 1/100-hour growth factor is about 1.01. And the one-hour growth factor is about

$$\left(1+\frac{1}{100}\right)^{100} \sim 2.705.$$

This tells us that $e \sim 2.705$ and that every hour a population increasing at the relative instantaneous rate of 100%/hr increases by about 170.5%.

We can get a better approximation to e by taking a smaller time interval. For example, a better approximation is

$$e \sim \left(1+\frac{1}{1000}\right)^{1000} \sim 2.717.$$

And as you might have guessed,

$$e = \lim_{n \to infty} \left(1+\frac{1}{n}\right)^{n} \sim 2.718.$$

4 An Example

When we learned about derivatives of inverse functions, we saw that it was helpful to express the derivative of a one-to-one function in terms of its *output*. For the function

$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \le \theta \le \pi/2,$$

for example,

$$\frac{dy}{d\theta} = \cos\theta = \sqrt{1-y^2}.$$

When working with exponential functions, or simple transformations of these, it is almost always best to do the same. Here's an example.

Example 30. The function

$$P = 3e^{\frac{1}{5}t}\,,\ 0 \le t \le 20,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

Find the population when it is increasing at the rate of 2 million bacteria/hour.

Explanation. There's a hard way and an easy way to solve this problem. The hard way would be to first find the time when the population is increasing at the given rate. Then we could use this time to find the population.

The easy way is to ignore time altogether and express the growth rate of the population in terms of the population.

Before we do this, let's think first about the units of the factor $1/5$ in the exponent. Since any exponential function takes only dimensionless inputs and t is measured in hours, the units of $1/5$ are

Question 191 ***Multiple Choice:***

(a) *hours*

(b) *dimensionless*

(c) *millions of bacteria*

(d) *hours*$^{-1}$ ✓

The units of $1/5$ give a clue to its meaning. It is in fact the relative growth rate of the population. So at any instant this population is increasing at the relative rate of

Question 192 20%*/hour* .

Question 193 (a) *Use the chain rule to compute the absolute growth rate*

$$\frac{dP}{dt} = \frac{d}{dt}\left(3e^{\frac{1}{5}t}\right).$$

Make the explicity u-substitution. Do not skip any steps.

(b) *Use part (a) to verify that the relative growth rate*

$$\frac{1}{P}\cdot\frac{dP}{dt}$$

is 20%*/hour.*

You should have found the relative growth rate to be

$$\frac{1}{P}\cdot\frac{dP}{dt} = \frac{1}{5}\ \mathrm{hr}^{-1},$$

so that the absolute growth rate (in millions of bacteria/hour) is

$$\frac{dP}{dt} = \frac{1}{5}P.$$

Question 194 *Use this to find the population when it is increasing at the rate of* 2 *million bacteria/hour.*

5 Exercises

Question 195 *The function*

$$P = f(t) = 50e^{\frac{t}{10}}, \ 0 \le t \le 20,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past midnight. Graph shown below.

Desmos link: `https://www.desmos.com/calculator/kffjsoepmo`

151: Exp Function 5

(a) *Use the graph of the function $P = f(t)$ above to approximate the growth rate when there are* 200 *million bacteria.*

(b) *Find a function $r = g(P)$ that expresses the growth rate (measured in millions of bacteria/hour) in terms of the popuation (measured in millions). Include a domain.*

(c) *Find the* exact *growth rate when there are* 200 *million bacteria. No calculator.*

(d) *What is the population when it is increasing at the rate of* 8 *million bacteria/hour?*

(e) *Use the ideas of this question to describe how the population grows.* Hint: *Think about the relative instantaneous growth rate.*

Question 196 *Between noon and 6pm, a population of bacteria grows exponenially, increasing by* 20% *every hour. There are* 5 *million bacteria at noon.*

(a) *Find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon.*

(b) *Find a function that expresses the growth rate (in millions of bacteria/hour) in terms of the population (in millions).*

(c) *Find the population when it is increasing at the rate of* 2 *million bacteria/hour.*

(d) *What is the relative growth rate of the population?*

(e) *Consider a second population that grow exponentially, increasing by* 20% *every four hours.*

   (i) *Use common sense to find its relative growth rate.*

   (ii) *Use calculus to verify your answer from part (i) with a computation.*

Question 197 *The function*

$$N = f(t) = \frac{20}{1 + 5e^{-t/5}}\,,\ 0 \le t \le 30,$$

models the spread of a virus throughout a population. It takes as an input the number of months since January 1st and returns as an output the number of infected individuals (measured in millions). Graph shown below.

Desmos link: `https://www.desmos.com/calculator/ikis451wxu`

151: Logistic Model

(a) *Use the sliders in the worksheet above to approximate the number of infected individuals when the virus is spreading at the rate of* $800,000$ *people/month.*

(b) *Find a function*

$$r = g(N)$$

that expresses the infection rate (in millions of people/month) in terms of the population.

(c) *Use calculus and algebra to determine the number of infected individuals when the virus is spreading at the rate of* $750,000$ *people/month.*

(d) *Use the sliders in the worksheet above to approximate the number of infected individuals when the virus is spreading at its fastest rate. Also approximate this rate.*

(e) *Use calculus and algebra to determine the number of infected individuals when the virus is spreading at its fastest rate. Then find this exact rate.*

Question 198 *Use the chain rule and your knowledge of the derivative $d(e^x)/dx$ to compute each of the following derivatives.*

(a)
$$\frac{d}{dt}\left(100 \cdot 2^t\right)$$

(b)
$$\frac{d}{dt}\left(100 \cdot 2^{t/5}\right)$$

Explanation. (a) The key is to realize that

$$2 = e^{\ln 2}.$$

So we can write the function to differentiate as

$$P = 100 \cdot 2^t = 100\left(e^{\ln 2}\right)^t = 100e^{t \ln 2}.$$

Now we can make a u-substitution and use the chain rule. We let

$$u = t \ln 2.$$

Then
$$P = 100e^{t \ln 2} = 100e^u$$

and
$$\begin{aligned}
\frac{dP}{dt} &= \frac{dP}{du} \cdot \frac{du}{dt} \\
&= \frac{d}{du}\left(100e^u\right) \cdot \frac{d}{dt}\left(t \ln 2\right) \\
&= 100e^u(\ln 2) \\
&= P \ln 2 \\
&= (100 \cdot 2^t) \ln 2 \\
&= (100 \ln 2)(2^t)
\end{aligned}$$

Question 199 *A colony of bacteria has a population of 20 million at noon and a population of 25 million at 2pm. The population grows exponentially between noon and midnight.*

(a) *Describe precisely how the population grows.*

(b) *Use your description from part (a) to find a function*

$$P = f(t)\,,\ 0 \le t \le 12,$$

that expresses the population (in millions) in terms of the number of hours past noon. Do not *use* e.

(c) *Find the relative instantaneous growth rate of the population.*

(d) *Find a function* $r = g(P)$ *that expresses the (absolute) growth rate in terms of the population.*

(e) *At what rate is the population changing when there are* 30 *million bacteria?*

Explanation. (a) The population increases by

$$\frac{25 \text{ million} - 20 \text{ million}}{20 \text{ million}} = 25\%$$

every two hours.

Better yet, at least for determining the function, is to say that the two-hour growth factor is

$$\frac{25 \text{ million}}{20 \text{ million}} = 1.25.$$

This means that every two hours the population gets multiplied by 1.25.

(b) Since the two-hour growth factor is 1.25, the one-hour growth factor is $1.25^{1/2}$. So

$$P = 20(1.25^{1/2})^t = 20(1.25)^{\frac{t}{2}}\,,\ 0 \le t \le 12.$$

(c) The key to computing the derivative dP/dt is to write the population function using base e. Since

$$1.25 = e^{\ln 1.25},$$

$$\begin{aligned} P &= 20(1.25)^{\frac{t}{2}} \\ &= 20\left(e^{\ln 1.25}\right)^{\frac{t}{2}} \\ &= 20e^{t(\ln 1.25)/2} \end{aligned}$$

So the growth rate (in millions of bacteria/hour) is

$$\begin{aligned} \frac{dP}{dt} &= \frac{d}{dt}\left(20(1.25)^{\frac{t}{2}}\right) \\ &= 20\frac{d}{dt}\left(e^{t(\ln 1.25)/2}\right) \\ &= \left(20e^{t(\ln 1.25)/2}\right)\frac{d}{dt}\left(t(\ln 1.25)/2\right) \\ &= P(\ln 1.25)/2. \end{aligned}$$

And the relative growth rate (units are hr^{-1}) is

$$\frac{1}{P}\cdot\frac{dP}{dt}=\frac{1}{2}\ln 1.25.$$

(d) From part (c), the function

$$g(P)=\frac{dP}{dt}=P(\ln 1.25)/2$$

expresses the growth rate (in millions of bacteria/hour) in terms of the population.

(e) When there are 30 million bacteria, the population is increasing at the rate of

$$g(30)=(30\text{ million bacteria})\left(\frac{1}{2}\ln 1.25/\mathrm{hr}\right)=15\ln 1.25\text{ million bacteria/hr}.$$

Question 200 *Between ground level and an altitude of* 10 *km, atmospheric pressure on the planet Krypton is an exponential function of altitude. The pressure is 100 kPa at ground level and 80 kPa at an altitude of* 0.75 *km.*

Desmos link: `https://www.desmos.com/calculator/j5h8kaj8xs`

151: Atomspheric Pressure

(a) *Describe exactly how the pressure decreases.*

(b) *Use your description from part (a) to find a function*

$$P=f(h)\,,\ 0\le h\le 10,$$

that expresses the atmospheric pressure (in kPa) in terms of the altitude (in kilometers). Do not *use the number e.*

(c) *Input your expression for f in Line 2 of the worksheet above.*

(d) *Evaluate the derivative*

$$\left.\frac{dP}{dh}\right|_{h=5}$$

and interpret its meaning.

(e) *Find a function $r=g(P)$ that expresses the derivative dP/dh in terms of P.*

(f) *Use calculus to approximate the altitude at which the pressure is* 0.2 *kPa less at an altitude* 0.1 *km higher.*

6 Conceptual Questions

Question 201 *A population of bacteria grows exponentially. If the growth rate is* 4 *million bacteria/hour when the population is* 20 *million, determine the growth rate when the population is* 30 *million.*

Question 202 *A cup of coffee at a temperature of* $90^\circ C$ *is brought into a room held at a constant temperature of* $20^\circ C$*. Suppose that the difference in temperature between the coffee and the room decreases exponentially. Suppose also that the temperature of the coffee initially decreases at the rate of* $5^\circ C/min$*. What is the temperature of the coffee when its temperature is decreasing at the rate of* $2^\circ C/min$*?*

Practice Quiz 2

Practice quiz, Weeks 4-6

Directions:

(a) Show all work.

(b) Give brief explanations for each problem. Include these explanations in the flow of the solution.

(c) Show all units in all computations.

(d) No calculators.

(e) Show each step when using the chain, product, and quotient rules and use the Leibniz notation when doing so. Here is an example.

Example 31. Find an expression for the derivative

$$\frac{d}{d\theta}\left(4 + 5\cos^3(2\theta)\right).$$

Explanation. With $y = 4 + 5\cos^3(2\theta)$, the derivative $dy/d\theta$ is

$$\begin{aligned}
\frac{dy}{d\theta} &= \frac{d}{d\theta}\left(4 + 5\cos^3(2\theta)\right) \\
&= \frac{d}{d\theta}(4) + 5\frac{d}{d\theta}\left((\cos(2\theta))^3\right) \\
&= 5(3)(\cos(2\theta))^2\frac{d}{d\theta}(\cos(2\theta)) \\
&= 15(\cos(2\theta))^2(-\sin(2\theta))\frac{d}{d\theta}(2\theta) \\
&= -30(\cos(2\theta))^2\sin(2\theta).
\end{aligned}$$

Question 203 *Find expressions for each of the following derivatives.*

(a)

$$\frac{d}{dt}\left(4 - t^4 e^{-5t}\right)$$

(b)

$$\frac{d}{d\theta}\left(\frac{12}{4 + 5\tan(4\theta)}\right)$$

Learning outcomes:
Author(s):

(c)
$$\frac{d}{dw}\left(\sqrt{40-5w^2}\right)$$

(d)
$$\frac{d}{dt}\left(2^{t/8}\right)$$

Question 204 (a) *Find an expression for the derivative*

$$\frac{d}{dx}\left(\sin(\arcsin x)\right).$$

Do this directly using the chain rule. Do not *simplify the function first.*

(b) *Find an expression for the same derivative by first simplifying the function* $y = \sin(\arcsin x)$ *and then using the chain rule.*

(c) *Find an expression for the same derivative by first simplifying the function* $y = \sin(\arcsin x)$ *and then using geometry.*

Question 205 (a) *Find an equation of the tangent line to the curve*

$$y = 4 - t^4 e^{-5t}$$

at the point on the curve with t*-coordinate* $t = 0$.

(b) *Find an equation of the tangent line to the curve*

$$y = \frac{12}{4 + 5\tan(4\theta)}$$

at the point on the curve with θ*-coordinate* $\theta = 0$.

Question 206 *Assume for this question that each month has 30 days and that the number of hours of daylight/day in Seattle is a sinusoidal function of time. Assume also that on June 21, Seattle gets a maximum of 16 hours of daylight/day and that on December 21, Seattle gets a minimu of 8 hours of daylight/day.*

(a) *Find a function*

$$H = f(t),\ 0 \le t \le 12,$$

that expresses the number of daylight hours/day in Seattle in terms of the number of months since June 21. Use the cosine function. Start by sketching a graph. Explain your reasoning.

(b) *Evaluate the derivative*

$$\left.\frac{dH}{dt}\right|_{t=2}.$$

(c) *What are the units of the derivative above?*

(d) *Explain the meaning of the above derivative.*

Question 207 *The function*

$$T = f(m) = 20 + 70e^{-m/20}\,,\ 0 \leq m \leq 60,$$

expresses the temperature (in Celsius degress) of a cup of coffee in terms of the number of minutes past noon.

(a) *At what rate is the temperature changing at 12:40pm?*

(b) *Does the temperature change at a constant relative rate? Justify your assertion.*

(c) *Find a function $r = g(T)$ that expresses the rate of change in the temperature (measured in $^\circ C$/min) in terms of the temperature (measured in Celsius degrees).*

(d) *Find the temperature when it is decreasing at the rate of $3^\circ C$/min.*

(e) *Find a function $s = h(T)$ that expresses the relative rate of change in the temperature in terms of the temperature (measured in Celsius degrees).*

(f) *Find the temperature when it is decreasing at the rate of 3%/min.*

Question 208 *Assume for this problem that over the course of a 24-hour period beginning at midnight of July 21, the temperature in Shoreline is a sinusoidal function of time. Assume also that the temperature reaches its minimum of $50^\circ F$ at 5am and its maximum of 80° at 5pm.*

At what rate (with respect to time) is the temperature changing when it is $68^\circ F$ for the second time? Start by finding a function that expresses the temperature in terms of time. Be sure to define your variables.

Question 209 *The function*

$$h = f(t) = 3 + 5\tan\left(\frac{t-9}{12}\right)\,,\ 0 \leq t \leq 12,$$

expresses the height (in thousands of feet) of a balloon in terms of the number of hours past noon.

Determine the possible height(s) of the balloon when it is ascending at the rate 500 ft/hour.

Question 210 *The function*

$$G = f(s) = 5e^{-s/20}\,,\ 0 \leq s \leq 40,$$

expresses the number of gallons of gas in a truck in terms of the trip odometer reading (measured in miles).

Find a function

$$m = h(s)\,,\ 0 \leq s \leq 40,$$

that takes expresses the dashboard reading for the number of miles left to drive before running out of gas in terms of the trip odometer reading (in miles). The dashboard reading assumes the truck burns gas at the current rate (ie. the rate at odometer reading s miles) for the remainder of the trip.

Derivatives of Inverse Functions, Short Version

Derivatives of inverse functions.

1 The Main Idea

The main idea here is simple. That the derivative of the *inverse* of a function is the *reciprocal* of the function's derivative. Or expressed more succinctly, *the derivative of the inverse is the reciprocal of the derivative.*

We don't really need to know much about calculus to understand why.

Question 211 *Take for example, the function*

$$G = f(s)\,,\ 0 \leq s \leq 20,$$

that expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route and the graph of the function is shown below.

Desmos link: `https://www.desmos.com/calculator/oltntpzth9`

151: Inverse Function 1

(a) *Use the slider s above to approximate (with units) the derivative*

$$\left.\frac{dG}{ds}\right|_{s=2}.$$

Explain the derivative's meaning.

(b) *Use the slider s above to approximate (with units) the derivative*

$$\left.\frac{ds}{dG}\right|_{G=f(2)}.$$

Explain the derivative's meaning.

(c) *Write an equation that expresses the relationship between the derivatives in parts (a) and (b). Explain why this relationship holds.*

Learning outcomes:
Author(s):

Question 212 *This question is a continuation of Question 1. Now we are given an expression*

$$G = f(s) = \frac{s^2}{2000} + \frac{23s}{1000} + \frac{1}{10}\,,\ 0 \leq s \leq 20,$$

for the function f.

(a) *Use calculus to find an expression for the derivative dG/ds and to compute the exact value of the derivative*

$$\left.\frac{dG}{ds}\right|_{s=2}.$$

(b) *Use the idea of Question 1 to find an expression for the derivative ds/dG and the exact value of the derivative*

$$\left.\frac{ds}{dG}\right|_{G=f(2)}.$$

(c) *For a more algebraic way to find an expression for the derivative ds/dG, we'll differentiate both sides of the equation*

$$G = \frac{s^2}{2000} + \frac{23s}{1000} + \frac{1}{10}$$

with respect to G. Keeping in mind that this equation, for values of s between 0 and 20, defines s implicitly as a function of G (why?). Then using the chain rule tells us that

$$\frac{dG}{dG} = \left(\frac{s+23}{1000}\right)\left(\frac{ds}{dG}\right).$$

And solving for the derivative ds/dG gives

$$\frac{ds}{dG} = \frac{1000}{s+23}.$$

2 Derivatives of Inverse Power Functions

Let's use what we just learned to verify something we already know, but in a more meaningful context.

Example 32. The function

$$A = f(s) = s^2\,,\; s \geq 0,$$

expresses the area of a square (measured in square feet) in terms of its side length (in feet).

Since

$$\frac{dA}{ds} = 2s,$$

we know that the derivative of the inverse function

$$s = f^{-1}(A) = \sqrt{A}\,,\; A \geq 0,$$

expressing the side length of a square in terms of its area is

$$\begin{aligned} \frac{ds}{dA} &= \frac{1}{dA/ds} \\ &= \frac{1}{2s} \\ &= \frac{1}{2\sqrt{A}}. \end{aligned}$$

We could verify this using the power rule:

$$\begin{aligned} \frac{ds}{dA} &= \frac{d}{dA}\left(\sqrt{A}\right) \\ &= \frac{1}{2\sqrt{A}}. \end{aligned}$$

To make this more meaningful, let's look at the derivative

$$\begin{aligned} \left.\frac{dA}{ds}\right|_{s=5} &= \left.(2s)\right|_{s=5} \\ &= 10\frac{\text{ft}^2}{\text{ft}} \\ &= 10 \text{ ft}. \end{aligned}$$

Although it's almost never a good idea to simplify the units of the derivative, it is here. The derivative is 10 feet and is equal to *half* the perimeter of our square with side length $s = 5$ feet. To get an idea why this is so, we'll interpret the derivative in terms of small changes.

Suppose we make a small change

$$\Delta s = s - 5 \sim 0$$

in the side length of the square. What can we say about the change

$$\Delta A = f(s) - f(5) = A - 25$$

in its area?

Well since $\Delta s \sim 0$,

$$\frac{\Delta A}{\Delta s} \sim \left.\frac{dA}{ds}\right|_{s=5} = 10,$$
$$\Delta A \sim \left(\left.\frac{dA}{ds}\right|_{s=5}\right) \Delta s = 10\Delta s.$$

This tells us that *to approximate the change in the area we multiply the change in the side length by* 10 *feet*.

(a) The picture below suggests why this is a good approximation. Explain how.

(b) Do some algebra and compare the actual change

$$\Delta A = f(5+s) - f(5) = (5+\Delta s)^2 - 25$$

in the area with the approximate change. Reconcile the difference with the picture.

Free Response:

Desmos link: `https://www.desmos.com/calculator/h2fm6mm8ua`

151: Square Error

We can interpret the derivative

$$\left.\frac{ds}{dA}\right|_{A=25} = \frac{1}{10}\text{ft}^{-1}$$

in much the same way by rewriting the above approximation as

$$\Delta s \sim \left(\left.\frac{ds}{dA}\right|_{A=25}\right) \Delta A = \left(\frac{1}{10}\right) \Delta A.$$

This tells us *to approximate the change in the side length we divide the change in the area by* 10 *feet*.

This interpretation of the derivative is useful in approximating square roots. For example, the side length of a square with an area of 25.2 ft^2 is

$$\begin{aligned}\sqrt{25.2\ \text{ft}^2} &= 5\ \text{ft} + \Delta s\ \text{ft}\\ &\sim 5\ \text{ft} + \left(\frac{1}{10\ \text{ft}}\right)(0.2\ \text{ft}^2)\\ &= 5.02\ \text{ft},\end{aligned}$$

giving us a pretty good approximation to

$$\sqrt{25.2} \sim 5.01996.$$

Exercise 213 *Carry out a similar analysis with the function*

$$V = s^3 \, , \, s \geq 0,$$

expressing the volume of a cube (in m^3) in terms of its edge length (in meters). Try to include a picture as well.

3 Derivative of the Arc Sine Function

Our goal here is to find an expression for the derivative

$$\frac{d\theta}{dy} = \frac{d}{dy}\,(\arcsin y)$$

of the inverse sine function

$$\theta = \arcsin y = \sin^{-1}(y)\,,\, -1 \leq y \leq 1.$$

The first step is to recognize that the inverse sine function is *not* the inverse of the sine function. Because the sine function is not one-to-one, its inverse is not a function.

Rather, the inverse sine function $\theta = \arcsin y$ is the inverse of the function

$$y = f(\theta) = \sin\theta\,,\, -\pi/2 \leq \theta \leq \pi.$$

Example 33. Before computing the derivative of the inverse sine function in general, we'll look at a specific example and evaluate the derivative

$$\frac{d}{dy}\,(\arcsin y)\Big|_{y=4/5}.$$

Desmos link: `https://www.desmos.com/calculator/alp2mnxqhc`

151: Inverse Sine Function

The idea to computing the above derivative is to first find the slope of the tangent line to the curve $y = \sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$, at the point P on the graph above. Since $d(\sin\theta)/d\theta = \cos\theta$, the slope is the derivative

$$\begin{aligned}\frac{d}{d\theta}\,(\sin\theta)\Big|_{\sin\theta=4/5} &= \frac{d}{d\theta}\,(\sin\theta)\Big|_{\theta=\arcsin(4/5)}\\ &= \cos\theta\Big|_{\theta=\arcsin(4/5)}\\ &= \cos(\arcsin(4/5)).\end{aligned}$$

To evaluate this composition, we need to find the cosine of the angle

$$\theta_0 = \arcsin(4/5)$$

between $-\pi/2$ and $\pi/2$ whose sine is $4/5$. But since

$$\cos^2\theta_0 + \sin^2\theta_0 = 1,$$

$$\begin{aligned}\cos\theta_0 &= \pm\sqrt{1-\sin^2\theta_0}\\ &= \pm\sqrt{1-\left(\frac{4}{5}\right)^2}\\ &= \pm 3/5.\end{aligned}$$

To choose the correct sign, we use the fact that $-\pi/2 \le \theta_0 \le \pi/2$. This tells us $\cos\theta_0 \ge 0$, so $\cos\theta_0 = 3/5$.

We now know that for the function

$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \le \theta \le \pi/2,$$

$$\left.\frac{dy}{d\theta}\right|_{y=4/5} = 3/5.$$

So for the inverse function

$$\theta = \arcsin y,$$

$$\begin{aligned}\left.\frac{d\theta}{dy}\right|_{y=4/5} &= \left(\left.\frac{dy}{d\theta}\right|_{y=4/5}\right)^{-1}\\ &= \frac{1}{\sqrt{1-(4/5)^2}}\\ &= 5/3.\end{aligned}$$

Next we'll compute the derivative of the arcsine function for a general input.

Example 34. The inverse sine function

$$\theta = \arcsin(y) = \sin^{-1}(y)\ -1 \le y \le 1,$$

gives the angle between $-\pi/2$ and $\pi/2$ whose sine is equal to y. The key to expressing its derivative

$$\frac{d\theta}{dy} = \frac{d}{dy}\left(\arcsin y\right)$$

in terms of y, is to express the derivative of its inverse

$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \le \theta \le \pi/2,$$

in terms of $y = \sin\theta$.

Now

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta) = \cos\theta.$$

To express this derivative in terms of $y = \sin\theta$, we use the fact that

$$\cos^2\theta + \sin^2\theta = 1.$$

So

$$\frac{dy}{d\theta} = \cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - y^2}.$$

To choose between $\pm$, we must remember that by the definition of the arcsine function (see equation (1)),

$$\pi/2 \leq \theta = \arcsin y \leq \pi/2.$$

This tells us that $\cos\theta \geq 0$ and so $\cos\theta = \sqrt{1 - y^2}$. So the derivative of the function

$$y = \sin\theta\,,\ -\pi/2 \leq \theta \leq \pi/2,$$

is

$$\frac{dy}{d\theta} = \cos\theta = \sqrt{1 - y^2}.$$

And the derivative of its inverse

$$\theta = \arcsin y$$

is

$$\begin{aligned} \frac{d}{dy}(\arcsin y) &= \frac{d\theta}{dy} \\ &= \left(\frac{dy}{d\theta}\right)^{-1} \\ &= \frac{1}{\sqrt{1 - y^2}}. \end{aligned}$$

4 Derivative of the Arc Cosine Function

The inverse of the function

$$x = \cos\theta\,,\ 0 \leq \theta \leq \pi,$$

is the arccosine function

$$\theta = \arccos x, -1 \leq x \leq 1.$$

It returns the angle between 0 and π whose cosine is x.

Exercise 214 (a) *Follow the method of Example 5 above to evaluate the derivative*

$$\frac{d}{dx}(\arccos x)\Big|_{x=-3/4}$$

Desmos link: `https://www.desmos.com/calculator/hgxqjixv4n`

151: Inverse Cosine Function

(b) *Follow the method of Example 6 above to find an expression for the derivative*

$$\frac{d}{dx}(\arccos x) .$$

5 Exercises

Exercise 215 *Express the derivative of each of the following functions in terms of its* output.

(a)
$$x = \cos\theta\,,\ 0 \le \theta \le \pi$$

(b)
$$x = \cos\theta\,,\ \pi \le \theta \le 2\pi$$

(c)
$$y = \tan\theta\,,\ -\pi/2 < \theta < \pi/2$$

(d)
$$P = e^{t/5}\,,\ t \in \mathbb{R}$$

(e)
$$y = \frac{e^x - e^{-x}}{2}\,,\ x \in \mathbb{R}$$

(f)
$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,,\ x \in \mathbb{R}$$

(g)
$$y = \frac{e^x + e^{-x}}{2}\,,\ x \le 0$$

(h)
$$G = \frac{s^2}{2000} + \frac{23s}{1000} + \frac{1}{10}\,,\ 0 \le s \le 20,$$

Exercise 216 *Use the results of Exercise 5 to express the derivative of the inverse of each of the functions in that exercise in terms of its* independent *variable.*

Exercise 217

- *Find simplified expressions for each of the following derivatives.*
- *Do* not *simplify a function before taking its derivative.*
- *Show all steps in using the chain rule. Use the Leibniz notation.*
- *Use Desmos to graph each function (before taking the derivative).*
- *Explain how each graph is related to the derivative.*

(a)
$$\frac{d}{dx}\left(\sin(\arcsin x)\right)$$

(b)
$$\frac{d}{d\theta}\left(\arcsin(\sin\theta)\right)$$

(c)
$$\frac{d}{dx}\left(\arctan x + \arctan(1/x)\right)$$

(d)
$$\frac{d}{dx}\left(\arccos x + \arccos\left(\sqrt{1-x^2}\right)\right)$$

(e)
$$\frac{d}{dx}\left(\arcsin x + \arcsin\left(\sqrt{1-x^2}\right)\right)$$

Derivatives of Inverse Functions

Using the chain rule to compute the derivative of the inverse of a function.

Introduction

We saw at the beginning of the class the relationship between the derivative of a function and the derivative its inverse. Recall the two problems:

- You measure the edge length of a cube to be 2 cm and use this measurement to compute its volume. How is the error in computing the volume related to the error (assumed to be small) in your measurement of the edge length?
- You submerge a cube in water and measure its volume to be 8 cm^3. You then use this measurement to compute the edge length of the cube. How is the error in computing the edge length related to the error (assumed to be small) in your measurement of the volume?

The key to the first problem was to differentiate the function

$$V = f(s) = s^3\,, s > 0$$

that expresses the volume (in cubic cm) of the cube in terms of its edge length (in cm). We used limits and found that

$$\left.\frac{dV}{ds}\right|_{s=2} = 12.$$

Question 218 *(a) What are the units of the above derivative? How do you know?*

(b) Interpret the meaning of the above derivative in terms of the geometry of the cube.

Question 219 *From here, we let*

$$\Delta s = s - 2$$

Learning outcomes:
Author(s):

be the error in our measurement (measured in cm) and

$$\Delta V = s^3 - 8$$

the error in the computed volume (measured in cubic cm). Then for small errors $\Delta s \sim 0$,

$$\frac{\Delta V}{\Delta s} \sim \left.\frac{dV}{ds}\right|_{s=2} = 12,$$

and so

$$\Delta V \sim 12\Delta s.$$

For the second problem we used limits to differentiate the function

$$s = f^{-1}(V)\,,\; V > 0,$$

and found that

$$\left.\frac{ds}{dV}\right|_{V=8} = 1/12.$$

Question 219.1 *What are the units of the above derivative? How do you know?*

Then for small errors ΔV *in our measurement of the volume,*

$$\Delta s \sim \frac{1}{12}\Delta V$$

But all this work took us back to the same approximate relation between the errors ΔV *and* Δs *that we had already found in solving the first problem.*

Expressed another way, having found that

$$\left.\frac{dV}{ds}\right|_{s=2} = 12,$$

we should have known immediately (without any computations) that

$$\left.\frac{ds}{dV}\right|_{V=8} = 1/12.$$

For the same reason, this same relationship between the derivative of any function and the derivative of its inverse holds. That is, for any differentiable function $y = f(x)$, *the derivative* dx/dy *of the inverse function* $x = f^{-1}(y)$ *is*

$$\begin{aligned}\frac{d}{dy}\left(f^{-1}(y)\right) &= \frac{dx}{dy}\\ &= \frac{1}{dy/dx}.\end{aligned}$$

Well almost. We need to take some care in evaluating the above derivatives at the correct inputs.

Supposing that $b = f(a)$, a more precise statement of the relation between the derivative of a function $y = f(x)$ and the derivative dx/dy of its inverse is

$$\begin{aligned} \frac{d}{dy}\left(f^{-1}(y)\right)\Big|_{y=b} &= \frac{dx}{dy}\Big|_{y=b} \\ &= \frac{1}{\frac{dy}{dx}\Big|_{x=a}}. \end{aligned}$$

That is, the derivative of the inverse (of a function) is the reciprocal of the derivative (of that function).

Well, not quite. The above relationship is true if

$$\frac{dy}{dx}\Big|_{x=a} \neq 0.$$

This condition also guarantees that the function $y = f(x)$ is one-to-one in a sufficiently small neighborhood near $x = a$ and is therefore invertible in that neighborhood.

Here are some examples.

The Derivative of the Natural Log Function

Example 35. (a) Use the graph of the function $y = f(x)$ below to estimate the value of the derivative

$$\frac{dy}{dx}\Big|_{x=1.1} = \frac{dy}{dx}\Big|_{y=3} \sim 3$$

(b) Use the result of part (a) and the graph below to estimate the value of the derivative

$$\frac{dx}{dy}\Big|_{x=1.1} = \frac{dx}{dy}\Big|_{y=3} \sim 1/3.$$

Desmos link: `https://www.desmos.com/calculator/nnshzdh6jp`

(c) Set $u = 0.6931$ and $n = 90$ in the demonstration above and estimate the value of the derivative

$$\frac{dy}{dx}\Big|_{x=0.6931} = \frac{dy}{dx}\Big|_{y=2} \sim 2$$

(d) Use the result of part (c) and the graph above to estimate the value of the derivative

$$\left.\frac{dx}{dy}\right|_{x=0.6931} = \left.\frac{dx}{dy}\right|_{y=2} \sim 0.5$$

Question 220 *Parts (a) and (c) and perhaps a few more derivatives suggest that*

$$f(x) = e^x.$$

(e) Use the results of parts (b) and (d) to evaluate the derivatives

$$\left.\frac{dx}{dy}\right|_{y=3} = \frac{d}{dy}(\ln y)\Big|_{y=3} = 1/3,$$

$$\left.\frac{dx}{dy}\right|_{y=2} = \frac{d}{dy}(\ln y)\Big|_{y=2} = 1/2,$$

and more generally

$$\left.\frac{dx}{dy}\right|_{y=a} = \frac{d}{dy}(\ln y)\Big|_{y=a} = 1/a.$$

Question 221 *Our conclusion is that*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Geogebra activity available at 151: Magnification Factor 3

Question 222 *Here's an equivalent, but more computational way to show that*

$$\frac{d}{dx}(\ln x) = 1/x.$$

The key is to recognize that the chain rule tells us that if $u = g(x)$ is a differentiable function of x, then

$$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}.$$

Now to compute the derivative above, we know that since the functions $f(x) = \ln x$ and $g(x) = e^x$ are inverses of one another,

$$e^{\ln x} = x.$$

Then differentiate both sides of this equation with respect to x to get

$$\frac{d}{dx}\left(e^{\ln x}\right) = \frac{dx}{dx}.$$

And by the chain rule we can rewrite this equation as

$$\left(e^{\ln x}\right)\frac{d}{dx}(\ln x) = 1.$$

And since $e^{\ln x} = x$,

$$\frac{d}{dx}(\ln x) = 1/x.$$

Question 223 *Find an equation of the tangent line to the curve $y = \ln x$ at the point $(4, \ln 4)$.*

$$y - \ln 4 = \frac{1}{4}(x - 4).$$

Question 224 *(a) Use the chain rule to compute the derivative*

$$\frac{d}{dx}(\ln(4x)).$$

(b) Compute the same derivative without *using the chain rule.*

Explanation. Let

$$y = \ln(4x)$$

and

$$u = 4x.$$

Then

$$y = \ln u$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{d}{du}(\ln u)\cdot\frac{d}{dx}(4x)\\ &= \frac{4}{u}\\ &= \frac{1}{x}.\end{aligned}$$

(b) We can compute the derivative

$$\frac{d}{dx}\left(\ln(4x)\right)$$

wihout the chain rule as follows:

$$\begin{aligned}\frac{d}{dx}\left(\ln(4x)\right) &= \frac{d}{dx}\left(\ln 4 + \ln x\right) \\ &= \frac{d}{dx}\left(\ln 4\right) + \frac{d}{dx}\left(\ln x\right) \\ &= 0 + \frac{1}{x}.\end{aligned}$$

Question 225 *(a) Compute the derivative*

$$\frac{d}{dx}\left(\ln(x^2)\right)$$

both with and without the chain rule. Follow the steps exactly *as in the previous example when using the chain rule.*

(b) What are the domains of the functions $f(x) = \ln x$ and $g(x) = \ln(x^2)$ and their derivatives.

(c) Graph the function $g(x) = \ln(x^2)$ and its derivative on the same coordinate system by hand.

Question 226 *(a) Use the chain rule to compute the derivative*

$$\frac{d}{dx}\left(\ln|x|\right).$$

Do this by noting that

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

and using the chain rule to compute

$$\frac{d}{dx}\left(-x\right).$$

(b) A reflection about the y-axis takes the graph of a function $y = f(x)$ to the graph of a function $y = g(x)$. Describe the transformation that takes the graph of $y = f'(x)$ to the graph of $y = g'(x)$. Explain your reasoning.

Question 227 *(a) Find a function*

$$T = f(k),\ k \geq 1,$$

that expresses the time (in years) it takes an investment to grow by a factor of k. Assume the investment grows at a constant relative instantaneous rate of i%/yr. So, for example if $i = 5$, $f(3)$ would be the time it takes an investment to triple at an relative instantaneous growth rate of 5%/yr.

Hint: *Let B_0 be the initial investment. Then the value of the investment T years later is*

$$kB_0 = B_0 e^{\frac{i}{100}T}.$$

Solve this equation for T to find an expression for the function f. You should get that

$$T = f(k) = (100 \ln k)/i$$

(b) Suppose $i = 5$ and evaluate the derivative

$$\left.\frac{dT}{dk}\right|_{k=3}.$$

(c) What are the units of the above derivative? Explain its meaning in terms of small changes.

I'll leave the units up to you. But the meaning is that it takes an investment (increasing at the relative instantaneous rate of 5%/yr) about 0.067 years longer to increase by 201% than it does to triple.

(d) Use the result of parts (b) and (c) to estimate how much longer it would take an investment growing at an relative instantaneous rate of 5%/yr to increase by 210% than it would take to triple. Just use simple arithmetic (ie. multiplication), no calculator.

Question 228 *The function*

$$G = f(v) = 40 - 0.08\left(\frac{v}{2} - 25\right)^2 \quad , 25 \leq v \leq 65,$$

expresses the gas mileage of a car (in miles/gallon) in terms of its speed (in miles/hr).

(a) Explain the meaning of the derivative

$$\frac{d}{dv}\left(\ln(f(v))\right).$$

Include units in your explanation. Also, what are the units of 25 in the above expression? How do you know?

(b) Evaluate the above derivative at $v = 30$ and explain its meaning in terms of small changes.

Question 229 *The function*

$$v = g(h)\,,\; 50 \leq h \leq 200,$$

expresses the speed (in ft/sec) of a hawk in terms of its altitude (in feet) during a portion of its flight.

Suppose that $f(150) = 80$ and

$$\left.\frac{dv}{dh}\right|_{h=150} = -2.$$

(a) What are the units of the above derivative? Explain the meaning of the derivative in the context of small changes in this particular scenario. Be specific.

I'll leave the units up to you, but if we assume the hawk is descending, then its speed would increase by about 2 ft/sec as it falls from 150 to 149 feet.

(b) Evaluate the derivative

$$\left.\frac{d}{dh}\left(\ln(g(h))\right)\right|_{h=150}.$$

Show all work and explain your reasoning.

The key here is to use the chain rule to evaluate the derivative. So with

$$v = g(h)$$

and

$$w = \ln(g(h)),$$

we have

$$w = \ln v.$$

Then by the chain rule,

$$\begin{aligned}\frac{dw}{dh} &= \frac{dw}{dv}\cdot\frac{dv}{dh} \\ &= \left(\frac{1}{v}\right)\left(\frac{dv}{dh}\right),\end{aligned}$$

and continue from here by evaluating the above expression at $h = 150$.

(c) What are the units of the derivative in part (b)? Explain the meaning of the derivative in the context of small changes in this particular scenario. Be specific by answering this question much like part (a) above (appropriately modified).

The Inverse Sine Function

Let
$$\theta = g(y) = \arcsin y = \sin^{-1}(y).$$

(a) What is the domain of the function g?

(b) What is its range?

(c) Explain the meaning of $\arcsin y$.

(d) True or false: The inverse sine function is the inverse of the sine function.

(e) What function is the inverse of the inverse sine function?

Question 230 *The graph of the function*

$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \leq \theta \leq \pi/2,$$

is shown below.

Desmos link: `https://www.desmos.com/calculator/lxwoeir1pt`

Worksheet available at 151: Arc Sine

(a) Use the graph to estimate the derivative

$$\frac{d}{dy}(\arcsin y)\Big|_{y=0.8}.$$

Explain your reasoning.

(b) Use the graph and the slider u to estimate the derivative

$$\frac{d}{dy}(\arcsin y)\Big|_{y=0.8}.$$

(c) Express the derivative of the function

$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \leq \theta \leq \pi/2,$$

in terms of y.

(d) Use part (c) and the ideas in parts (a),(b) to find an expression for the derivative

$$\frac{d}{dy}(\arcsin y)\,.$$

(e) Check your answer to part (d) by evaluating the derivative at $y = 0.6, 0.8$.

(f) What is the domain of the derivative in part (d)?

Question 231 *The top of a 25-foot long ladder slides down a vertical wall at the constant rate of 4 ft/sec.*

(a) Find a function

$$\theta = f(t)\,,\ 0 \leq t \leq 6.25,$$

that expresses the angle the ladder makes with the ground (measured in radians) in terms of the number of seconds since the ladder was in the vertical position.

(b) Find the rotation rate of the ladder when the top of the ladder is

(i) 15 *feet above the ground.*

(ii) 24 *feet above the ground.*

(iii) 24.9 *feet above the ground.*

(c) Solve this problem again by working directly with the sine function, not the arcsine function.

Desmos link: `https://www.desmos.com/calculator/5c4lssovbi`

Worksheet available at 151: Ladder and ArcSine

Question 232 *The bottom end of a 25-foot long ladder slides across a horizontal floor at the constant rate of 4 ft/sec as the top end slides down a vertical wall.*

(a) Find a function

$$\theta = g(u)\,,\ 0 \leq u \leq 25,$$

that expresses the angle the ladder makes with the wall (measured in radians) in terms of the distance (in feet) between the wall and the bottom end of the ladder.

Hint: *Let A be the bottom end of the ladder, B the top end, and point O the point on the wall closest to A (as in the demonstration below). Use right triangle trigonometry in ΔAOB.*

(b) Use the slider u in the animation below to approximate each of the following derivatives. Include units. Note that the tick marks on the radian protractor are spaced at intervals of 0.1 *radians.*

(i) $\left.\dfrac{d\theta}{du}\right|_{u=1}$

(ii) $\left.\dfrac{d\theta}{du}\right|_{u=15}$

(iii) $\left.\dfrac{d\theta}{du}\right|_{u=24.9}$

(c) What do the above derivatives tell you?

(d) Find an expression for the derivative $d\theta/du$. Use your expression to evaluate the three derivatives in part (b) and compare these with your estimates.

(e) Find a function

$$\theta = f(t)\,,\ 0 \le t \le 6.25,$$

that expresses the angle the ladder makes with the ground (measured in radians) in terms of the number of seconds since the ladder was in the vertical position.

Hint: *Express the distance u (in feet) between the bottom end of the ladder and the wall in terms of t. Then substitute this expression for u in your function from part (a).*

(f) Find an expression for the derivative

$$r = \omega(t) = d\theta/dt.$$

Interpret its meaning. Include units.

(g) Find the rotation rate of the ladder when the bottom end of the ladder is

(i) 1 foot from the wall.

(ii) 15 feet from the wall.

(iii) 24.9 feet above the wall.

(h) Evaluate the limit

$$\lim_{t\to 6.25} \omega(t)$$

and interpret its meaning.

Desmos link: `https://www.desmos.com/calculator/egolipj5qg`

Worksheet available at 151: Ladder and ArcSine 2

Exercises

Question 233 *(a) Simplfiy the derivative*

$$\frac{d}{d\theta}\left(\arcsin(\sin\theta)\right)$$

(b) Use the result of part (a) to graph the function

$$y = \arcsin(\sin\theta).$$

Explain your reasoning.

Question 234 *Find the measure of the acute angle that the tangent line to the curve*

$$y = f(\theta) = \ln|\sec\theta|$$

at the point $(\pi/7, f(\pi/7))$ makes with the x-axis Do not *use a calculator.*

Question 235 *The function*

$$q = f(p) = 0.2\,(2p - 40)^2 \;,\; 5 \le p \le 12,$$

expresses the average number of burgers sold per day at Five Guys in Edmonds in terms of the price (in dollars/burger).

(a) Evaluate the derivative

$$\frac{d}{dp}\left(\ln(f(p))\right)\Big|_{p=7.5}$$

(b) What are the units of the derivative above? Explain its meaning.

Question 236 *The bottom end of a 25-foot ladder lies 24 feet from the base of a vertical wall. Use the appropriate linear approximation to estimate the angle through which the ladder rotates when the bottom end is pulled an additional 0.1 feet away from the wall along a horizontal floor.*

Solve this problem twice, first using an inverse trig function and again without an inverse trig function.

Start by defining your variables, with units.

Use a calculator if need be, but only *for arithmetic and not to evaluate any trigonometric functions.*

Compare your estimate with the actual angle of rotation.

Question 237 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a 14-foot ladder leans against the trunk, the other rests on the horizontal ground. The bottom end of the ladder is pulled away from the trunk at the constant speed of 4 ft/sec. At what rate is the ladder rotating when the bottom and tops ends are respectively 16 and 10 feet from the base of the trunk?*

Hint: *Use the law of sines.*

Desmos link: `https://www.desmos.com/calculator/rpms2jqfpm`

Desmos activity available at 151: Tree and Ladder

Question 238 *(a) The animation below shows water draining from a tank. Play the animation and sketch by hand a graph of the function $V = f(t)$ that expresses the depth of the water as a function of time. Explain your reasoning. Label the axes with units and the appropriate variable names.*

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/pdghky6tie`

(b) Torricelli's law says that the rate, say in cm^3/sec, at which water drains out of a small hole in the bottom of a tank is proportional to the square root of the depth of the water. So if $V = f(h)$ is a function that expresses the volume (in cm^3) of water in the tank in terms of the depth (in feet) and $h = g(t)$ is a function that expresses the depth of the water (in feet) in terms of number of seconds past noon, then

$$\frac{dV}{dt} = -k\sqrt{h}$$

for some positive constant k.

For a cylindrical tank of radius r cm,

$$V = f(h) = \pi r^2 h$$

and

$$\frac{dV}{dh} = \pi r^2.$$

So by the chain rule

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt}\\ &= \pi r^2 \frac{dh}{dt}.\end{aligned}$$

So for the cylindrical tank we can write Torricelli's law as

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} = -k\sqrt{h}.$$

Or equivalently as

$$\frac{dh}{dt} = -\frac{k}{\pi r^2}\sqrt{h} = -k_2\sqrt{h},$$

where

$$k_2 = \frac{k}{\pi r^2}$$

is a positive constant.

(c) Which of the following functions might express the depth of water in a cylindrical tank (in terms of time) as the water drains out of a small hole in the bottom of the tank? Justify your reasoning.

(i) $h = g(t) = 2(5-t)^3, 0 \le t \le 5$

(ii) $h = g(t) = 2(5-t)^2, 0 \le t \le 5$

Solution:

The key idea is to express the derivative dh/dt *as a function of the depth* h *of the water.*

(i) If $h = 2(5-t)^3$*, then*

$$\begin{aligned}\frac{dh}{dt} &= \frac{d}{dt}\left(2(5-t)^3\right) \\ &= 2(3)(5-t)^2 \cdot \frac{d}{dt}(5-t) \\ &= -6(5-t)^2.\end{aligned}$$

Now to express dh/dt *in terms of* h*, solve the equation*

$$h = 2(5-t)^3$$

for t *to get*

$$t = 5 - \left(\frac{h}{2}\right)^{1/3}.$$

Then, substitute this expression for t *into the derivative* dh/dt*:*

$$\begin{aligned}\frac{dh}{dt} &= -6(5-t)^2 \\ &= -6\left(\frac{h}{2}\right)^{2/3} \\ &= -\left(\frac{6}{2^{2/3}}\right)h^{2/3}.\end{aligned}$$

This tells us that the rate of change in the depth of the water is not *proportional to the square root of the water's depth as Torricelli's law requires. So the function*

$$h = 2(5-t)^3, 0 \le t \le 5$$

is not a possible depth function for water draining from a cylindrical tank.

Free Response: *Give a similar analysis for the depth function of part (ii).*

Question 239 *This is a continuation of the previous problem.*

Now we pour water into a cylindrical tank at a constant rate, while at the same time water leaks out through a small hole in the bottom of the tank. We'll suppose that the tank starts with some initial volume of water.

Free Response: *(a) What do you think happens to the water level in the tank initially?*

(b) What do you think happens to the water level in the long run?

To model this situtation, we need to modify Torricelli's law. For this, let's suppose that we pour water into the cylindrical tank (of radius r) at the constant rate of k_3 cm^3/sec. Then with the same notation as before,

$$\frac{dV}{dt} = k_3 - k\sqrt{h}.$$

But since

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt}\\ &= \pi r^2 \cdot \frac{dh}{dt},\end{aligned}$$

the above modification of Torricelli's law becomes

$$\begin{aligned}\frac{dh}{dt} &= \frac{k_3}{\pi r^2} - \frac{k}{\pi r^2}\sqrt{h}\\ &= k_1 - k_2\sqrt{h},\end{aligned}$$

where

$$k_1 = \frac{k_3}{\pi r^2}$$

and

$$k_2 = \frac{k}{\pi r^2}$$

are positive constants.

Free Response: *(a) What are the units of k_1 and k_2? How do you know?*

(b) Explain the meaning of k_1.

The equation

$$\frac{dh}{dt} = k_1 - k_2\sqrt{h}$$

expresses the rate of change in the water's depth as a function of the depth. It is called a differential equation and you will learn a little about how to solve equations like this next quarter.

For this particular differential equation, it is not possible to express the depth of the water explicitly as a function of time. But assuming that the depth of the water is h_0 cm at time $t = 0$, it turns out that the function

$$t = g^{-1}(h) = -\frac{2}{k_2}\left(\sqrt{h} - \sqrt{h_0} + \frac{k_1}{k_2}\ln\left|\frac{k_1 - k_2\sqrt{h}}{k_1 - k_2\sqrt{h_0}}\right|\right),\ t \geq 0,$$

expresses the time (in seconds) in terms of the depth of the water (in cm). We can check that this is indeed correct as follows:

(a) Show algebraically that the depth of the water at time $t = 0$ is $h = h_0$.

(b) Use the above expression for $t = g^{-1}(h)$ to compute and then simplify the derivative dt/dh.

(c) Use the result of part (b) to show that

$$\frac{dh}{dt} = k_1 - k_2\sqrt{h}.$$

Desmos link: `https://www.desmos.com/calculator/c78kv7wifv`

Worksheet available at 151: Draining Cylinder 2

(d) Experiment with the sliders above and summarize your observations about how the graph of the function $h = g(t)$ changes depending on the initial depth of the water and the constants k_1 and k_2.

(e) Express the equilibrium depth in terms of k_1 and k_2. Check that your expression has the correct units. The equilibrium depth is the depth at which the water level remains constant. It is also the depth which the water level approaches (independent of the initial depth).

(f) What happens to the equilibrium depth when k_1 increases (and k_2) is held constant? When k_2 changes and k_1 is held constant?

The Quotient Rule

An introduction to the quotient rule.

Discussion Questions

Question 240 *Consider the following two questions.*

(a) *During the course of one year, the national debt of a country increases by 20% and its population increases by 5%. Find the relative change in the per-capita (ie. per-person) share of the national debt over the one-year period.*

(b) *At some instant the national debt of a country is increasing at the rate of 20%/yr and its population is increasing at the rate of 5%/yr. At what relative rate is the per-capita share of the national debt changing at this instant?*

(a) *Are these questions identical?*

(b) *Answer the first question.*

(c) *Use numerical methods to approximate the answer to the second question.* Hint: *Approximate the relative changes in the national debt and population over a small time interval. Then use these to approximate the relative change and the relative average rate of change in the per-capita share of the debt over this time interval. Use the worksheet below to help.*

Desmos link: `https://www.desmos.com/calculator/rk4aocmccz`

Desmos activity available at 151: Relative Quotient Rule 1

(d) *For a more algebraic approach to the second question, let's suppose that at the moment in question, say time $t = 0$ measured in years, the national debt is D_0 dollars, the population is P_0, and the per-capita share is $C_0 = D_0/P_0$. We'll also suppose at this instant that the debt is increasing at the relative rate of $100p$%/yr and the population at the relative rate of $100q$%/yr. We wish to determine the relative rate at which the per-capita share of the debt is changing at this time. We'll do this using limits.*

Learning outcomes:
Author(s):

To start let $D = f(t)$ *and* $P = g(t)$ *be the debt (in dollars) and population functions expressed in terms of the number of years since the moment in question. Also, let*

$$C = \frac{D}{P} = \frac{f(t)}{g(t)}$$

be the per-capita share of the debt.

Now the relative average rate of change in the per-capita share over the time interval $t \in [0, \Delta t]$ *is*

$$\begin{aligned} \frac{1}{C_0} \cdot \frac{\Delta C}{\Delta t} &= \frac{1}{\Delta t} \cdot \frac{P_0}{D_0} \cdot \Delta C \\ &= \frac{1}{\Delta t} \cdot \frac{P_0}{D_0} \left(\frac{f(\Delta t)}{g(\Delta t)} - \frac{f(0)}{g(0)} \right). \end{aligned}$$

The key now is to approximate the debt

$$D = f(\Delta t)$$

and the population

$$P = g(\Delta t)$$

at time Δt*. For this we need to use the respective relative rates of change* $100p\%$*/yr and* $100q\%$*/yr in the nationl debt and population at time* $t = 0$.

At time $t = \Delta t$ *hours , the national debt is approximately*

$$D = f(\Delta t) \sim D_0(1 + p\Delta t)$$

and the population approximately

$$P = g(\Delta t) \sim P_0(1 + q\Delta t).$$

So the change in the per-capital share of the debt over this time interval is approximately

$$\begin{aligned} \Delta C &= \frac{D}{P} - \frac{D_0}{P_0} \\ &\sim \frac{D_0(1 + p\Delta t)}{P_0(1 + q\Delta t)} - \frac{D_0}{P_0} \\ &= \frac{D_0}{P_0} \left(\frac{(p - q)\Delta t}{1 + q\Delta t} \right) \end{aligned}$$

and the relative change approximately

$$\begin{aligned} \frac{\Delta C}{C_0} &= \frac{P_0}{D_0} \cdot \Delta C \\ &= \frac{(p - q)\Delta t}{1 + q\Delta t}. \end{aligned}$$

So over the time interval $t \in [0, \Delta t]$, the relative average rate of change in the per-capita share of the debt is

$$\frac{1}{C_0} \cdot \frac{\Delta C}{\Delta t} = \frac{1}{\Delta t} \cdot \frac{\Delta C}{C_0} = \frac{p - q}{1 + q\Delta t}.$$

Finally, the relative instantaneous rate of change in the per-capita share of the debt at time $t = 0$ is

$$\frac{1}{C_0} \cdot \left.\frac{dC}{dt}\right|_{t=0} = \lim_{\Delta t \to 0} \frac{p - q}{1 + q\Delta t} = p - q.$$

(e) *For yet another way to approach the second question, assume the debt and population functions increase at constant relative rates. Then find expressions for these functions and use these to answer the question.*

(f) *What is your guess about the relative (instantaneous) rate of change in the quotient of two functions? How is it related to the relative rates of change of the functions?*

Question 241 *At some instant the per-capita share of a country's national debt is increasing at the rate of 4%/yr and its population is increasing the rate of 6%/yr.*

(a) *Find exponential functions $P = f(t)$ and $C = g(t)$ that model the population and per-capita share of the debt, each expressed in terms of the number of years since the moment described above.*

(b) *Use your expressions from part (a) to find a function $D = h(t)$ that expresses the national debt in terms of the same variable t. Simplify the expression for $D = h(t)$.*

(c) *Use your simplified expression for the function $D = h(t)$ to find the relative rate at which the national debt is changing at the instant in question.*

(d) *Repeat parts (a)-(c) using linear functions for the population and per-capita share of the debt instead.*

Question 242 *Use your guess for the relative rate of change in the quotient of two functions to find an expression for the derivative of the quotient of two functions.*

Question 243 (a) *At 9am on May 29, the balance in an account is increasing at the relative rate of 8%/yr. At the same time, the price of a stock in increasing at the rate of 5%/yr. Is the number of shares you can buy with the balance in the account increasing or decreasing at this time? At what relative rate? No computations. Just explain what you think.*

(b) *How would your answer to part (a) change if instead the stock price were decreasing at the rate of 5%/yr?*

Question 244 *Let $B = f(t)$ and $P = g(t)$ be functions that respectively express the balance (in dollars) in an account and the price (in dollars/share) of a stock in terms of the number of years past 9am on May 29. Let $S = h(t)$ be the number of shares of the stock you can buy with the balance in the account at time t years past 9am on May 29.*

(a) *Interpret the meanings of the following derivatives. Include units.*

(i) $\frac{d}{dt}(\ln(f(t))) = \frac{d}{dt}(\ln B))$

(ii) $\frac{d}{dt}(\ln(g(t))) = \frac{d}{dt}(\ln P))$

(iii) $\frac{d}{dt}(\ln(h(t))) = \frac{d}{dt}(\ln S))$

item Express the derivative

$$\frac{d}{dt}(\ln S)) = \frac{d}{dt}\left(\ln\left(\frac{B}{P}\right)\right)$$

in terms of the derivatives

$$\frac{d}{dt}(\ln B))$$

and

$$\frac{d}{dt}(\ln P)).$$

The Relative Quotient Rule

Had we never learned about relative changes and relative rates of change, the following questions would have almost forced these ideas upon us.

Question 245 *(a) Suppose over a six-month perioid, the national debt of a small country increases from \$4 billion to \$4.16 billion while the population increases from 20 million to 21 million. Does the per-capita (ie. per-person) share of the national debt increase or decrease during this period?*

(b) At noon on July 1, the national debt of a small country with a population of 20 million is \$4 billion.At that same instant the population is increasing at the rate of 1 million people/yr while the national debt is increasing at the rate of \$0.16 billion/yr. Is the per-capita share of the national debt increasing or decreasing at this instant? At what rate?

Explanation. (a) The question is really to determine which is greater,

$$c_1 = \frac{4 \times 10^9}{2 \times 10^7} \frac{\text{dollars}}{\text{person}}$$

or

$$c_2 = \frac{4.16 \times 10^9}{2.1 \times 10^7} \frac{\text{dollars}}{\text{person}} = \frac{4 \times 10^9 \times 1.04}{2 \times 10^7 \times 1.05} \frac{\text{dollars}}{\text{person}} = \left(\frac{1.04}{1.05}\right) c_1.$$

And because the total debt increased by 4% while the population increased by 5%, the per-capita share of the national debt decreased during the six-month period.

But the per-capita share of the debt did *not* decrease by 1%, but rather by about (to the nearest thousandth of a percent) 0.952%.

(b) For the instantaneous rate, let

$$Q = f(t)$$

be the total national debt in billions of dollars at time t years past noon on July 1 and

$$P = g(t)$$

the population in millions at time t. Also, let

$$C = h(t) = \frac{f(t)}{g(t)}$$

be the per-capita share of the debt in thousands of dollars/person at time t years past noon on July 1. We wish to find the value of the derivative

$$\left.\frac{dC}{dt}\right|_{t=0}.$$

Part (a) suggests we express the relative instantaneous rate of change

$$\frac{1}{C} \cdot \left.\frac{dC}{dt}\right|_{t=0}$$

in terms of the relative rates

$$\frac{1}{Q} \cdot \left.\frac{dQ}{dt}\right|_{t=0} = 0.04$$

and

$$\frac{1}{P} \cdot \frac{dP}{dt}\bigg|_{t=0} = 0.05$$

Perhaps the easiest way to do this, although not entirely complete, is to assume that the population and national debt each grow exponentially. In that case, the relative growth rates of each are constant and

$$Q = f(t) = 4e^{0.04t}$$

and

$$P = g(t) = 20e^{0.05t}.$$

This tells us that

$$C = \frac{f(t)}{g(t)} = 0.2e^{-0.01t}$$

and

$$\frac{1}{C} \cdot \frac{dC}{dt}\bigg|_{t=0} = -0.01$$

So at noon on July 1, we suspect that the per-capita share of the debt is decreasing at the relative rate of 1%/yr and at the absolute rate of \$2/person/yr.

To be sure this is true even if the relative rates of change in the population and total debt are *not* constant, we can be more general and compute the relative instantaneous rate of change in the per-capita share of the national debt as

$$\begin{aligned}
\frac{1}{C} \cdot \frac{dC}{dt} &= \frac{d}{dt}\left(\ln\left(\frac{Q}{P}\right)\right) \\
&= \frac{d}{dt}(\ln Q - \ln P) \\
&= \frac{d}{dt}(\ln Q) - \frac{d}{dt}(\ln P) \\
&= \frac{1}{Q} \cdot \frac{dQ}{dt} - \frac{1}{P} \cdot \frac{dP}{dt}.
\end{aligned}$$

Theorem 2. (a) (The Relative Quotient Rule) If $Q = f(t)$ and $P = g(t)$ are differentiable functions of t, then if $g(t) \neq 0$,

$$C = \frac{f(t)}{g(t)} = \frac{Q}{P}$$

is a differentiable function of t and

$$\frac{1}{C} \cdot \frac{dC}{dt} = \frac{1}{Q} \cdot \frac{dQ}{dt} - \frac{1}{P} \cdot \frac{dP}{dt}.$$

The relative rate of change in a quotient of two functions is equal to the difference in the relative rates of change in the functions.

(b) (The Quotient Rule) With the same hypotheses (and obtained by multiplying both side of the previous equation by $C = Q/P$),

$$\frac{d}{dt}\left(\frac{Q}{P}\right) = \frac{1}{P} \cdot \frac{dQ}{dt} - \frac{Q}{P^2} \cdot \frac{dP}{dt}.$$

The Tangent Fuction

Question 246 *Compute the derivative*

$$\frac{d}{d\theta}(\tan\theta)$$

Explanation. We compute this derivative from scratch by letting

$$y = \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Then

$$\ln|y| = \ln|\sin\theta| - \ln|\cos\theta|$$

and

$$\frac{d}{d\theta}(\ln|y|) = \frac{d}{d\theta}(\ln|\sin\theta| - \ln|\cos\theta|).$$

So

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{d\theta} &= \frac{1}{\sin\theta} \cdot \frac{d}{d\theta}(\sin\theta) - \frac{1}{\cos\theta} \cdot \frac{d}{d\theta}(\cos\theta) \\ &= \cot\theta + \tan\theta. \end{aligned}$$

Then mulitplying both sides by $y = \tan\theta$ gives

$$\frac{d}{d\theta}(\tan\theta) = 1 + \tan^2\theta.$$

Question 247 *(a) Use the graph of the function*

$$y = f(\theta) = \tan(2\theta)$$

below to estimate the y-coordinates of all points on the curve where the tangent lines are parallel to the lines

(i) $6x - y = 12$.

(ii) $6x + y = 12$.

(b) Find the exact y-coordinates without *using a calculator.*

Desmos link: `https://www.desmos.com/calculator/obz6ghw3ej`

Desmos activity available at 151: Tangent Graph

Question 248 *You stand 50 feet from the base of a tree and measure the angle of elevation to the top of the tree with an error of at most $\pm 2°$. You then compute the height of the tree above eye level to be 100 feet.*

Desmos link: `https://www.desmos.com/calculator/yjyghsoeog`

Desmos activity available at 151: Angle of Elevation 1

(a) Use the demonstration above to appoimate your error in computing the tree's height.

(b) Use the appropriate linear approximation to estimate your error in computing the tree's height. Compare this with your estimate from part (a). Do not use a calculator. Do this as follows:

- *Find a function*
  $$h = f(\theta)\,,\ 0 < \theta < \pi/2,$$
  that expresses the computed height of the tree above eye level (measured in feet) in terms of the measured angle of elevation (in radians). Draw a picture to help with your explanation.

  This function is
  $$h = f(\theta) = 50\tan\theta\,,\ 0 < \theta < \pi/2.$$

- *Next find an expression for the derivative $dh/d\theta$ and evaluate the derivative*
  $$\left.\frac{dh}{d\theta}\right|_{h=100}.$$

- *Now we'll take the exact height of the tree above eye level to be 100 feet and let*
  $$\Delta h = h - 100$$
  be the error in the computed height and
  $$\Delta\theta = \theta - \arctan(2)$$
  be the error in the measured angle.

  Then if $\Delta\theta \sim 0$,
  $$\left.\frac{dh}{d\theta}\right|_{h=100} \sim \frac{\Delta h}{\Delta\theta}$$

and so

$$\Delta h \sim 250\,(\Delta\theta)\,.$$

- *I'll let you continue from here.*

The Inverse Tangent Function

Question 249 *(a) Explain the meaning of the function*

$$\theta = f(y) = \arctan y.$$

In particular, what does it take as an input and what does it return as an output? Include the domain and range of the function as part of your explanation.

(b) Express the derivative

$$\frac{d}{d\theta}(\tan\theta) = 1 + \tan^2\theta$$

of the function

$$y = \tan\theta$$

in terms of y.

(c) Use part (b) to find an expression for the derivative

$$\frac{d\theta}{dy} = \frac{d}{dy}(\arctan y) = \frac{d}{dy}\left(\tan^{-1} y\right).$$

(d) Evaluate the derivatives

$$\left.\frac{d}{d\theta}(\tan\theta)\right|_{\theta=\pi/4}$$

and

$$\left.\frac{d}{dx}(\arctan x)\right|_{x=1}.$$

Comments?

Question 250 *You stand 60 feet from the base of a tree and measure the angle of elevation to the top of an 85-foot tall tree. You then move 4 feet closer to the tree and measure the same angle. Assume your eyes are five feet above the ground.*

Desmos link: `https://www.desmos.com/calculator/qqgvq3noah`

Desmos activity available at 151: Tree 3

(a) Drag the slider u in the demonstration above to estimate the chnage in the angle of elevation after you move 4 feet closer to the tree. Note that the angles of elevation are $\angle TEB$ and $\angle TFB$ and that $\angle ETF$ measures their difference. Consecutive tick marks on the protractor are spaced at intervals of 0.01 radians.

(b) Use derivatives to approximate the change in the angle of elevation and compare your approximation with your estimate in part (a) and with the actual change. Use a calculator with only addition, subtradction, multiplication, and division.

Go about this by first finding a function

$$\theta = f(s), s > 0,$$

that expresses the angle of elevation to the top of the tree (measured in radians) in terms of your distance from the tree (measured in feet). Use the arctangent function in your expression. Do not *use the inverse cotangent function.*

The function is

$$\theta = f(s) = \arctan(80/s), s > 0.$$

Then continue in a manner similar to Question 4.

Two Motions

Question 251 *(a) Play the slider s in the demonstration below to show the motion of a beetle crawling along the y-axis as it leaves behind tracks spaced at equal time intervals.*

(b) Use the animation to sketch (by hand) a graph of the function

$$s = f(t)$$

that expresses the position (in this case the y-coordinate) of the beetle as a function of time. Label the axes with the appropriate variable names and units. Then activate the folder in Line 2 to see how you did.

(c) Use the animation to sketch (by hand) a graph of the function

$$v = g(t)$$

that expresses the beetle's velocity (in this case the rate of change, with respect to time, in the beetle's y-coordinate). Label the axes with the appropriate variable names and units. Then activate the folder in Line 7 to see how you did.

Desmos link: `https://www.desmos.com/calculator/srotstrdzm`

Desmos activity available at 151: Tangent Motion

Question 252 *Repeat parts (a)-(c) of the previous question for the motion below.*

Desmos link: `https://www.desmos.com/calculator/h6vq21lfql`

Desmos activity available at 151: ArcTangent Motion

Exercises

Question 253 *(a) Use the website below to compute an accurate estimate of the current rate (in dollars/year) at which the U.S. national debt is changing. Explain your method.*

National Debt Clock

(b) Use the website below to compute an accurate estimate (in people/yr) at which the U.S. population is currently increasing.

Popluation Clock

(c) Use your estimates above and the current national debt and U.S. population to compute

(i) the current rate of change in the per-capita share of the national debt.

(ii) the current relative rate of change in the per-captia share of the national debt.

Include units in every number in each step of your computations.

Question 254 *The function*

$$C = f(t) = At^4 e^{-kt}\,,\, t \geq 0,$$

expresses the concentration of a drug (measured in mg/L) in the bloodstream in terms of the number of hours since the drug was injected. Here A and k are postive constants.

Desmos link: `https://www.desmos.com/calculator/lvsxu3wa9a`

Desmos activity available at 151: Drug Concentration

(a) What are the units of the constant A? How do you know?

(b) What are the units of k? How do you know?

(c) Use the graph of $C = f(t)$ above to sketch by hand a graph of the derivative $r = dC/dt$. Be sure to label the axes with the appropriate variable names and units. Do not *use technology. Explain your reasoning.*

(d) Use calculus and algebra to find an expression in terms of k for the time when the concentration is a maximum. Work in general. Do not *use a specific value of k. Check your work by following the directions, Lines 4 and 5, in the desmos demonstration. Label also the coordinates of the corresponding point on your hand-drawn graph of the derivative.*

(e) Find expressions (in terms of k) for the times when the concentration is increasing and decreasing at the maximum rates. Check that your expressions have the correct units. Work in general. Do not *use a specific value of k. Check your work by following the directions, Lines 6-9, in the desmos demonstration. Label also the coordinates of the corresponding points on your hand-drawn graph of the derivative.*

(f) Find an expression (in terms of k) for the relative rate at which the concentration is changing t hours after the injection. Check that your expression has the correct units. Work in general. Do not *use a specific value of k.*

(g) Suppose the concentration is at its maximum five hours after injection and determine when the concetration is increasing at the rate of 50%/hr. Determine also when the concentration is decreasing at the rate 50%/hr.

(h) Observations?

Question 255 *The function*

$$P = f(t) = 5 - 3t + t^2 \,,\ 0 \le t \le 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

(a) Use the graphs of the function $P = f(t)$ and the function $r = f'(t)/f(t)$ to estimate when the stock price is increasing at the greatest relative rate.

(b) Use algebra to find the exact time when the stock price is increasing at the greatest relative rate.

Hint: *What is the value of the derivative dr/dt at this time? But start by finding an expression for the instantaneous relative rate of change in the stock price.*

Desmos link: `https://www.desmos.com/calculator/y78fnyy7s3`

Desmos activity available at 151: Stock Price 4

Question 256 *You jog once around a circular track of radius r meters at the constant speed of speed of v m/sec. A flagpole lies b meters due east of the track's center.*

(a) Find a function

$$s = f(t)\,, 0 \leq t \leq 2\pi r/v,$$

that expresses your distance (in meters) to the flagpole in terms of the time (measured in seconds) since you started running. Assume you start at the point A on the track nearest the flagpole. Explain your reasoning. Work with the general parameters r, v, and b, not *with any specific values for these parameters.*

(b) Find an expression fro the time when your distance to the flagpole is increasing at the greatest rate. Try to give a geometric interpretation of your position at this time.

Desmos link: `https://www.desmos.com/calculator/bxofhvfbfs`

Demonstration available at Math 151: Jogger 3

Moving Averages and their Rates of Change

Moving averages and the quotient rule.

Question 257 (a) *Your average for the first four exams in a class is* 90%. *How would a score of* 100% *on the fifth exam* change *your average?*

(b) *Your average for the first nine exams in a class is* 90%. *How would a score of* 100% *on the tenth exam change your average?*

(c) *Over the first four hours of a ten-hour car trip, your average speed was* 90 *km/hour. During the fifth hour you drove at a constant speed of* 100 *km/hour. Compare your average speeds over the first four hours and over the first five hours of your trip.*

(d) *Over the first nine hour of a ten-hour car trip, your average speed was* 90 *km/hour. During the tenth hour you drove at a constant speed of* 100 *km/hour. Compare your average speeds over the first nine hours and over the entire ten hours of your trip.*

(e) *A computer display keeps track of your average speed since the start of your trip. Suppose four hours into a ten-hour car trip, your trip odometer reading (set to zero at the start of your trip) reads* 360*km and your speedometer reads* 100 *km/hour. Find the rate of change, with respect to time (measured in hours), in your average speed as shown on the display at this instant.*

(f) *A computer display keeps track of your average speed since the start of your trip. Suppose nine hours into a ten-hour car trip, your trip odometer reading (set to zero at the start of your trip) reads* 810*km and your speedometer reads* 100 *km/hour. Find the rate of change, with respect to time (measured in hours), in your average speed as shown on the display at this instant.*

Learning outcomes:
Author(s):

Implicit Differentiation

An introduction to implicit differentiation.

Discussion Questions

Question 258 *Which of the following equations define y implicty as a function of x in a sufficiently small neighborhood of the given point? Supplement your reasoning with a graph of each relation.*

(a) $x^2 + y^2 = 25$ *near the point* $(4, -3)$

(b) $x^2 + y^2 = 25$ *near the point* $(0, -5)$

(c) $x^2 + y^2 = 25$ *near the point* $(-5, 0)$

(d) $x^2 - xy + y^2 = 1$ *near the point* $(1, 1)$.

(e) $x^2 - xy + y^2 = 3$ *near the point* $(1, 2)$.

(f) $x^2 - xy + y^2 = 3$ *near the point* $(2, 1)$.

Introduction to Implicit Differentiation

Question 259 *Let $p = f(t)$ be a differentiable function of t. Find expressions for each of the following derivatives, first supposing that*

$$p = f(t) = t^3 + 1,$$

and then more generally, not assuming any particular expression for the function f.

(a) $\dfrac{d}{dp}\left(p^5\right)$

(b) $\dfrac{d}{dt}\left(p^5\right)$

(c) $\dfrac{d}{dt}\left(e^{2p}\right)$

(d) $\dfrac{d}{dt}\left(t^4 \sin(p)\right)$

Learning outcomes:
Author(s):

Question 260 *(a) Find equations of the tangent lines to the ellipse*

$$x^2 - xy + 2y^2 = 9$$

at the points where the ellipse intersects the x-axis.

(b) Find equations of all horizontal tangent lines to the ellipse.

(c) Find equations of all vertical tangent lines.

Desmos link: `https://www.desmos.com/calculator/ox3wvfitqk`

Desmos activity available at 151: Implicit Ellipse

Question 261 *(a) Find an equation of the tangent line to the curve*

$$4xe^y - 3y\cos(2xy) = 0$$

at the origin.

(b) Enter your equation in Line 5 of the worksheet below to check your answer.

Desmos link: `https://www.desmos.com/calculator/1qprxg226m`

Desmos activity available at 151: Implicit 1

Astroids

Question 262 *(a) The coordinate axes cut out a segment from the tangent line to the curve*

$$x^{2/3} + y^{2/3} = 5$$

at the point $P(8,1)$. Find the length of that segment. Do this by using implicit differentiation to help find an equation of the tangent line at P and go from there. But first show that P lies on the curve.

(b) As a bug crawls around the above curve and passes the point $P(8,1)$, it is moving away from the x-axis at the rate of 4 cm/sec. Is the bug moving toward or away from the y-axis at this time? At what rate?

(c) Find the slope of the tangent line in part (a) without using implicit differentiation.

Question 263 *(a) Use implicit differentiation to show that segments cut by the coordinate axes from the tangent lines to the astroid*

$$x^{2/3} + y^{2/3} = a^{2/3}$$

all have the same length. Here $a > 0$ is a constant.

(b) Prove the same result by using trigonometric functions to parameterize the astroid.

Desmos link: `https://www.desmos.com/calculator/vrythrvjuc`

Desmos activity available at 151: Astroid

Ellipses and Related Curves

Question 264 *(a) Use implicit differentiation to find an equation of the tangent line to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point P with coordinates (x_0, y_0). Here a, b are positive constants.

(b) Input your equation from part (a) on Line 3 of the desmos worksheet below. Drag point P to check your equation is correct.

(c) Solve part (c) again without *calculus by considering a composition of transformations that takes the circle*

$$x^2 + y^2 = 1$$

to the ellipse in part (a).

Desmos link: `https://www.desmos.com/calculator/t1e9v7ncpv`

Desmos activity available at 151: Tangents to Ellipse

Question 265 *An ellipse through the point $P(0, b)$ has focal points F_1 at the origin and F_2 at the point $A(a, 0)$.*

(a) Use the definition of an ellipse as the set of points whose distances to the foci have a constant sum to find an equation of the ellipse.

(b) Use implicit differentiation to find the slope of the tangent line to the ellipse at P.

(c) Find an equation of the line normal to the ellipse at Q.

(d) Find the coordinates of the point Q where the normal line intersects the x-axis.

(e) Express the ratio $F_1Q : QF_2$ in terms of a and b. Interpret the ratio geometrically.

The ratio is

$$\frac{F_1Q}{QF_2} = \frac{b}{\sqrt{a^2+b^2}}.$$

(f) Check your work in the demonstration below.

Desmos link: `https://www.desmos.com/calculator/xqywotxxf9`

Desmos activity available at 151:Normal to an Ellipse

Question 266 *Let F_1 and F_2 be respectively the origin and the point with coordinates $(a, 0)$. The curve $\mathcal{C}$ passes through the point $P(0, b)$. The curve is defined by the property that the sum of the distance from a point Q of $\mathcal{C}$ to F_1 and k times its distance to F_2 is a constant.*

(a) Find an equation of the curve $\mathcal{C}$.

(b) Find an equation of the tangent line to $\mathcal{C}$ at P.

(c) Find an equation of the line normal to $\mathcal{C}$ at P.

(d) Find the coordinates of the point Q where the normal line intersects the x-axis.

(e) Express the ratio $F_1Q : QF_2$ in terms of a and b. Interpret the ratio geometrically.

The ratio is

$$\frac{F_1Q}{QF_2} = \frac{bk}{\sqrt{a^2+b^2}}.$$

(f) Check your work in the demonstration below.

Desmos link: `https://www.desmos.com/calculator/uz7w5uh0vl`

Desmos activity available at 151: Generalized Ellipse

The Sliding Ladder

Question 267 *You walk along a straight path, moving either directly toward or directly away from a tree at the end of the path.*

Let $s = f(t)$ be a function that expresses your distance to the tree in terms of the number of minutes past noon and suppose that

$$\left.\frac{ds}{dt}\right|_{t=5} = -240.$$

(a) Interpret the meaning of the above derivative.

(b) What is your speed at 12:05pm?

(c) Find an expression for your speed at time t minutes past noon.

Question 268 *A tree leans precariously with its trunk making an angle of ϕ radians with the ground. One end of a ladder with length L feet leans against the trunk, the other rests on the horizontal ground. You slide the bottom end of the ladder along the ground at random.*

(a) Let s and t be the respective distances (measured in feet) from the base of the trunk to the bottom and top ends of the ladder. Find an equation that relates these distances. Use the parameters L and ϕ, not *their particular values in the demonstration below.*

(b) Let u be time measured in seconds and suppose that s and t are (unknown) functions of u. Express the derivative dt/du in terms of the derivative ds/du. Interpret the meanings of the these derivatives.

(c) Switch s and t in your equation from part (b). What do you notice? Explain why.

(d) Use part (b) to express the speed of the top end of the ladder in terms of the speed of its bottom end.

(e) Suppose $\phi = \pi/3$ and that at some instant the bottom end of the ladder is three times as far from the trunk's base as the top end. What can you say about the speeds of the ends of the ladder at this instant?

(f) Still supposing $\phi = \pi/3$, find all possible angles the ladder makes with the ground when the top end of the ladder is moving four times as fast as the bottom end. Characterize each angle in one of two ways:

(i) One end of the ladder is moving toward the trunk's base and the other end away from the base.

(ii) Both ends of the ladder are either moving toward or way from the trunk's base.

Give exact angles. Then use a calculator to approximate their radian measures to the nearest hundredth.

(g) Use the animation below to check that your answers to parts (e) and (f) are reasonable. The graph is of the relation

Desmos link: `https://www.desmos.com/calculator/58jqjo0inl`

Desmos activity available at 151: Tree and Ladder

Question 269 *A tree leans precariously with its trunk making an angle of ϕ radians with the ground. One end of a ladder with length L feet leans against the trunk, the other rests on the horizontal ground. You slide the bottom end of the ladder along the ground at random.*

Suppose at some instant the ladder makes an angle of β_0 radians with the trunk and that the bottom end is s_0 feet from the trunk's base. Suppose also that the ladder is rotating at the rate of ω_0 rad/sec at this same moment.

(a) Find an expression for rate of change in the distance from the bottom end of the ladder to the base of the trunk at this time.

(b) Find an expression for rate of change in the distance from the top end of the ladder to the base of the trunk at this time.

(c) Find expressions for the speeds of the ends of the ladder at this time.

Hint: *Use the law of sines in ΔBOT in the demonstration from the previous question.*

Question 270 *The function*

$$h = f(\delta) = \frac{24}{\pi} \arccos(-\tan\phi \tan\delta)\,, \; -\pi/2 + \phi < \delta < \pi/2 - \phi,$$

expresses the number of hours of daylight per day at latitude ϕ in terms of the declination of the sun. The declination of the sun (δ) is the angle the sun's rays make with the plane of the equator, taken to be positive between the spring and fall equinoxes in the northern hemisphere. The latitude ϕ, $-\pi/2 \le \phi \le \pi/2$, is positive at points in the northern hemipshere.

Geogebra link: `https://www.geogebra.org/m/vnhrutwu`

Geogebra activity available at 151: Declination of Sun 2 *(a) Use the graph of the function f below (at latitude $\phi \sim 1.1$) to sketch a graph of the derivative*

$$\frac{dh}{d\delta} = f'(\delta).$$

(b) Suppose the latitude ϕ is held constant and find an expression for the derivative

$$\frac{dh}{d\delta} = \frac{d}{d\delta}\left(-\tan\phi\tan\delta\right).$$

(c) What are the units of the derivative in part (b)?

(d) Input your expression for the derivative in Line 4 of the worksheet below (follow the directions there). Then vary the slider ϕ to see how the function f and its derivative vary with latitude. Summarize your observations.

(e) Find an expression for the derivative

$$\left.\frac{dh}{d\delta}\right|_{\delta=0} = \left.\frac{d}{d\delta}\left(-\tan\phi\tan\delta\right)\right|_{\delta=0}$$

in terms of the latitude ϕ.

(f) Evaluate the derivative in part (e) at a latitude of $\phi = \pi/4$. Interpret its meaning in terms of small changes.

Desmos link: `https://www.desmos.com/calculator/ifomatkcta`

Desmos activity available at 151: Length of Day 1

(g) Suppose now that each month has 30 days so that there are 360 days in one year. Suppose also that the declination of the sun varies sinusoidally as a function of time, that the maximum declination of $\delta = 23.5°$ occurs on the summer solstice (say June 21st) and the minimum declination $\delta = -23.5°$ occurs on the winter solstice (December 21st).

Find an expression for a function

$$\delta = k(t)\,,\ t \geq 0,$$

that gives the declination of the sun (measured in radians) in terms of the number of days since the spring equinox.

(h) Find a function

$$r = g(\phi)\,,\ -\pi/2 < \phi < \pi,$$

that expresses the rate of change in the number of minutes of daylight per day (measured in (minutes of daylight/day)/day)) on the spring equinox in terms of the latitude ϕ. Input this function in Line 1 of the demonstration below.

(i) Use the result of part (f) to approximate the rate in (hours of daylight/-day)/day) at which the number of daylight hours per day is changing at a latitude of $\phi = \pi/4$ radians on the spring equinox and on the fall equinox.

(j) Evaluate the rates from part (i) in Fairbanks, Alaska, latitude 64.8°*N.*

Desmos link: `https://www.desmos.com/calculator/nf8n5uphhl`

Desmos activity available at 151: Length of Day 2

Waves

Question 271 *The function*

$$y = f(x,t) = a\sin(kx - \omega t), t \geq 0, \tag{3}$$

describes a wave on a string. The functions expresses the displacement (in meters) of a point on the string in terms of the position x (in meters) of the point and time t, measured in seconds since the motion began.

(a) Experiment with the sliders k, ω, in the demonstration below, playing the slider u (u is just another name for t). Summarize your observations. In particular, be sure to turn off Line 1 to be better able to see the motion of the individual points of the string.

(b) What are the units of k and ω? How do you know?

(c) Find an expression for the wavelength λ in terms of k, ω.

(d) Find an expression for the period of oscillation T in terms of k, ω.

(e) Hold y constant and differentiate each side of equation (3) with respect to t to find an expression for the speed of the wave. Turn on the graph in Line 9. Explain the logic behind the computation.

Desmos link: `https://www.desmos.com/calculator/9xmkig9hwi`

Desmos activity available at 151: Traveling Wave 1

The Ladder and the Tree

Implicit differentiation.

Question 272 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a ladder leans against the trunk, the other rests on the horizontal ground. We analyze how a small change in the distance between the bottom of the ladder and the base of the trunk changes the distance between the top of the ladder and the base of the trunk.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/u8bxskxyhl`

151: Ladder and Tree 20

We'll let u be the distance OT between the top of the ladder and the base of the trunk (measured in feet) and s the distance OB between the bottom of the ladder and the base of the trunk (also measured in feet).

We'll focus on the particular position of the ladder when the bottom end is 5 feet from the trunk's base and the top end is 8 feet from the base as illustrated above.

(a) *Use the slider s approximate the derivative*

$$\left.\frac{ds}{du}\right|_{(s,u)=(5,8)}.$$

Be sure to include units.

(b) *Find the length of the ladder*

(c) *Find an equation that relates s and u. Then activate the Folder in Line 20. Explain the significance of the curve and the slope of the tangent line to the curve at P.*

(d) *Find an expression for the derivative ds/du.*

(e) *Use your expression for ds/du to evaluate the derivative*

$$\left.\frac{ds}{du}\right|_{(s,u)=(5,8)}.$$

Learning outcomes:
Author(s):

(f) *Interpret the meaning of the derivative using the language of small changes.*

(g) *Approximate the distance of the bottom of the ladder from the base of the tree when the top of the ladder is 7.95 feet from the tree's base.*

Question 273 *This is a continuation of the previous question.*

(a) *Use the slider s to approximate the distance $s = OB$ when moving the top of the ladder a small distance away from the base of the tree makes the bottom end move twice as far toward the base. Then use calculus and algebra to apprximate that distance.*

(b) *Activate the folder in Line 31 of the worksheet above. Let θ be the angle (measured in radians) the ladder makes with the ground as marked. Take $\theta = 2\pi/3$ when the ladder lies along the tree.*

(i) *Use the slider s to approximate the derivative*

$$\left.\frac{d\theta}{ds}\right|_{s=5}.$$

Include units.

(ii) *Write an equation relating θ and s.*

(iii) *Find an expression for the derivative $ds/d\theta$ and use it to compute the exact value of the derivative*

$$\left.\frac{d\theta}{ds}\right|_{s=5}.$$

(iv) *Interpret the meaning of the above derivative using the language of small changes.*

(v) *Find the exact value of the derivative*

$$\left.\frac{d\theta}{du}\right|_{s=5}.$$

Statuary Hall and Implicit Differentiation

Statuary Hall and John Quincy Adams

Question 274 *YouTube link:*
https://www.youtube.com/watch?v=watch?v=FX6rUU_74kk

Statuary Hall

Up until the middle of the 19th centuary, Statuary Hall in the U.S. Capitol was the meeting place of the House of Representatives. The room is in the shape of an ellipse and John Quincy Adams had his desk at one of the foci. Legend has it that he was able to eavesdrop on the whispered conversations of his political opponents when they were standing at the other focus.

The aim of this problem is to explain this and prove the reflective property of ellipses.

A light wave (or a sound wave) emitted from one focus of an ellipse passes through the other focus after reflecting off the ellipse.

We need to know the definition of an ellipse:

An ellipse is the set of points, the sum of whose distances from two fixed points (the foci) is constant.

Desmos link: https://www.desmos.com/calculator/6kxtojk72r

Access Desmos interactives through the online version of this text at

151: Statuary Hall.

In Calculus 3 you'll learn about the gradient vector and a short way to prove the reflective property of ellipses in general. But we'll work with a specific example, and take our ellipse to have focal points $F_1(0,0)$ and $F_2(4,0)$. We'll also suppose the ellipse passes through $P(0,3)$. Our problem is to first find an equation of the normal line to the ellipse at P. Then we'll show that this line bisects angle F_1PF_2.

(a) *Start by using the definition of the ellipse to write an equation of the ellipse.*

Learning outcomes:
Author(s):

(b) *Use implicit differentiation to find the slope of the tangent line to the ellipse at $P(0, 3)$.*

(c) *Find an equation of the normal line at P. Enter this equation on Line 16 in the worksheet above.*

(d) *For a shorter way to describe the normal line, do the following.*

(i) *Use vector arithmetic to find a vector parallel to the normal line at P.*

(ii) *Use your vector from part (i) to parameterize the normal line at P.*

Question 275 *Another way to prove the reflective property of ellipse is related to an optimization problem.*

We are given an ellipse (dotted) with focal points A and B, a point P on the ellipse, and the tangent line L to the ellipse at P, as shown below.

Desmos link: `https://www.desmos.com/calculator/ncyrefzwwg`

Access Desmos interactives through the online version of this text at

151: Minimization Property of Ellipses.

To prove the reflective property of the ellipse, it's enough to show that the marked angles at P are congruent.

To do this we temporarily forget about the ellipse and the point P. All that's left are the points A, B, and the line L. Now our optimization problem is this. Given points A and B, to find the point on the given line L that minimizes the sum

$$\text{dist}(P, A) + \text{dist}(P, B)$$

of the distances from P to the points A and B.

There are two key ideas.

(a) *The first is to show that the point P that minimizes the above sum lies on the given ellipse with focal points A and B that is tangent to L. To see why, drag the slider s in the worksheet above. Then explain the logic behind this claim.*

(b) *The second is to forget about the ellipse and solve the optimization problem to show that the marked angles at P are congruent for the point P on the line that minimizes the above sum. Try this.*

The Ladder and the Tree, Part 2

Implicit differentiation.

Question 276 *The bottom end of a seven-foot ladder slides across a horizontal floor as its top end slides down a vertical wall.*

Desmos link: `https://www.desmos.com/calculator/4nmxshey0e`

151: Ladder and Tree 23

(a) *Let θ (in radians) be the angle the ladder makes with the ground and s the distance (in feet) between the bottom of the ladder and the wall.*

Drag the slider s in the worksheet above to approximate the derivative

$$\left.\frac{d\theta}{ds}\right|_{s=3}.$$

Include units. The markings on the protractor are spaced 0.01 radians apart.

(b) *Suppose the bottom of the ladder is sliding toward the wall at a speed of 5 ft/sec when the bottom is 3 feet from the wall. At what rate is the ladder rotating at this instant?*

(c) *Drag the slider s in Line 1 of the worksheet below to approximate the angle the ladder makes with the ground when its bottom end is moving twice as fast as its top end.*

(d) *Find the exact angle in part (c).*

Access Desmos interactives through the online version of this text at

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Question 277 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a ladder leans against the trunk, the other rests on the horizontal ground.*

At the moment the bottom end of the ladder is 8 feet from the tree's base (in the position shown below), its bottom end is moving toward the tree's base at a

Learning outcomes:
Author(s):

speed of 5 ft/sec. At this same moment the top end of the ladder is 3 feet from the tree's base.

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/xrgftlr9dx`

151: Ladder and Tree 29

(a) *Is the ladder rotating clockwise or counterclockwise at this moment? Guess first, then drag the slider s above to see if you were correct.*

(b) *At what rate? Do* not *assume that the bottom end moves at a constant speed. Solve this problem twice. First, with implicit differentiation and then again without. Do* not *use a calculator.*

Question 278 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a seven-foot ladder leans against the trunk, the other rests on the horizontal ground.*

(a) *Find all possible angles the ladder makes with the ground when its bottom end is moving twice as fast as its top end.*

(b) *Drag the slider ϕ_2 in Line 2 of the worksheet below to check your answer.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/oftz4vb9qj`

151: Ladder and Tree 22

Question 279 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a ladder leans against the trunk, the other rests on the horizontal ground.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/bscbpblji1`

151: Ladder and Tree 21

(a) *Find a function*

$$v_B = f(\theta)\,,\ 0 \le \theta \le 2\pi/3,$$

that expresses the speed of the ladder's bottom end in terms of the angle $\theta = \angle OTP$ the ladder makes with the tree and the ladder's rotation rate ω (measured in rad/sec). Do not assume the rotation rate is constant.

(b) *Check that your expression for the speed has the correct units.*

(c) *Similarly find a function*

$$v_T = g(\theta)\,,\ 0 \le \theta \le 2\pi/3,$$

that expresses the speed of the ladder's top end in terms of θ and ω.

(d) *Play the slider in Line 2 of the worksheet above. Then use the animation to sketch by hand graphs of the speed functions in parts (a) and (c) assuming the ladder rotates at a constant rate.*

(e) *Activate the* speed functions *folder in Line 41 to check your graphs.*

(f) *Find the ratio v_B/v_T of the speeds when the bottom end of the ladder is five feet from the tree's base.*

(g) *What angle does the ladder make with the tree when the ladder's bottom end is moving twice as fast as its top end? Find all possibilities.*

(h) *Answer part (g) when the tree is perpendicular to the ground instead.*

More Related Rates Problems

Related rates.

Directions:

- Start by defining all variables, each in a complete sentence with units. Be sure to precisely define the variable for time as well.
- Answer each problem with a concluding sentence.

1 Shadow of a Building

Question 280 *On the vernal equinox at a location on the equator, the sun rises due east at 6am, passes directly overhead at noon, and sets due west at 6pm.*

151: Building Shadow

Desmos link: `https://www.desmos.com/calculator/55pixnvvqz`

(a) *Play the slider u in Line 2 of the worksheet to see how the length of building's shadow changes during the afternoon. Summarize your observations.*

Free Response:

(b) *Use the slider in the worksheet above to approximate the relative rate of change (with respect to time) in the length of the shadow when the shadow's length is twice the building's height*

(c) *Find a function*

$$s = f(\theta)\,, 0 < \theta \le \pi/2,$$

that expresses the length of the shadow cast by an h-foot tall building in terms of the angle of elevation of the sun (this is the angle the sun's rays may with the ground).

(d) *Find the relative rate of change (with respect to time) in the length of the shadow when the shadow's length is twice the building's height and the sun is setting. How does this compare with your estimate from part (b)?*

Learning outcomes:
Author(s):

(e) *During the afternoon, when is the length of the shadow increasing at its minimum relative rate? Justify your assertion and find this rate.*

2 Kite Flying

Question 281 *A kite drifts due east as it maintains a constant altitude. Play the slider u in Line 1 of the worksheet below to watch the motion.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/qdrzdt4erc`

151: Kite

(a) *Which do you think is greater, the rate at which the string is being let out or the speed of the kite? No computations, just go with your intuition.*

(b) *Express the speed of the kite (not assumed constant) in terms of the rate at which the string is being let out (not assumed constant) and the angle θ the string makes with the horizontal.*

(c) *What can you say if $\theta = \pi/3$?*

(d) *Was your intuition correct?*

3 A Crawling Beetle

Question 282 *As a beetle crawling in the xy-plane passes the point $(-4, -3)$ (coordinates in centimeters) its distance from the x-axis is increasing at the rate of 2 cm/sec and its distance from the origin is increasing at the rate of 3 cm/sec.*

Is the beetle rotating clockwise or counterclockwise at this instant? At what rate?

4 A Mechanical Motion

Question 283 *As a rod OP of length R meters rotates about its endpoint O, the other endpoint P drags along a second rod PA of length L meters as illustrated below.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/r20xwnok5r`

151: Engine

(a) *Express the speed of A in terms of R, L, the angle θ shown above, and the rotation rate ω (measured in radians/sec) of rod OP. Do* not *assume ω is constant. Work in general, not with the specific values of L and R in the worksheet.*

(b) *Check that your expression in part (a) has the correct units.*

(c) *Express the signed rotation rate of PA in terms of the same parameters. Measure the rate to be postive when PA rotates counterclockwise. Do not assume ω is constant.*

(d) *Check that your expression in part (c) has the correct units.*

5 Tracking a Helicopter

Sensors on the ground four hundred feet apart track a helicopter. At the instant shown below, the sensors at A and B are rotating counterclocwise at the respective rates of 2 rad/min and 3 rad/min. And the marked angles at A and B have respective measures $\angle A = \pi/6$ and $\angle B = \pi/3$.

Is the helicopter ascending or descending at this instant? At what rate?

Desmos link: `https://www.desmos.com/calculator/xl8t3toppg`

151: Tracking a Helicopter

6 Distance to the Horizon

Question 284 *The distance to the horizon is limited by the curvature of the earth as illustrated in the demonstration below. This distance is the length of the (red) arc AT on the earth's surface.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/8qwt6mfirt`

Desmos activity available at

151:Distance to Horizon 11

Let

$$s = f(h)$$

be the function that expresses the distance to the horizon (in thousands of miles) in terms of your altitude above the surface (in thousands of miles).

(a) *Sketch by hand a graph of the function f. Drag the slider h in the worksheet above to help with your sketch. Then activate the Graph folder in Line 17 to see how you did.*

(b) *Find an expression for the function f.*

(c) *Use the graph of the function f to approximate the derivative*

$$\left.\frac{ds}{dh}\right|_{h=1}.$$

Include units.

(d) *Use your expression for the function $s = f(h)$ to find the exact value of the derivative above.*

(e) *Interpret the meaning of the derivative in terms of small changes.*

(f) *Suppose at some instant you are in a rocket 1000 miles above the earth and descending at the rate of v miles/hour. Find an expression (in terms of v) for the rate (in miles/hour) at which your distance to the horizon (the length of the red arc AT above) is changing at this instant. Do* not *assume v is constant.*

7 Jar Lid and Rubber Band

Question 285 *You wrap a rubber band around a circluar jar lid and pull the band tight as illustrated below.*

Desmos link: https://www.desmos.com/calculator/jbku3wrtdq

151: String and Jar Lid

Suppose at some instant you are pulling the point P of the rubber band directly away from the center of the lid at a speed of v inches/sec. Express the rate (in inches/sec) at which the length of the rubber band is changing at this instant in terms of v and the marked angle θ between the straight segments of the band at P.

Hint: *Express the length of the rubber brand in terms of θ and the distance h (mesaured in inches) between P and the center of the lid. Keep in mind the band has a curved section in addition to the straight parts.*

8 Disco Dancing

Question 286 *A spotlight in a circular dance hall of radius r meters is located b meters from the center of the hall.*

At some instant the light is rotating at the rate of ω rad/sec.

Desmos link: https://www.desmos.com/calculator/m2o267u9ur

151: Disco Dancing

(a) *Find an expression for the speed of the light beam as it moves along the wall at this instant. Your expression should be in terms of ω, r, b, and the angle θ marked below.*

(b) *Check that your expression in part (a) has the correct units.*

The Natural Log Function

Log functions and their derivatives.

1 The Derivative of the Natural Log Function

Question 287 (a) *Explain the meaning of*

$$\ln 5 = \log_e 5.$$

(b) *Simplify the function*

$$f(x) = e^{\ln x}.$$

Include the appropriate domain.

(c) *Use the result of part (b) to find an expression for the derivative*

$$\frac{d}{dx}\left(\ln x\right).$$

Explanation. To compute the derivative in part (c), we know that since the functions $f(x) = \ln x$ and $g(x) = e^x$ are inverses of one another,

$$e^{\ln x} = x.$$

Then differentiate both sides of this equation with respect to x to get

$$\frac{d}{dx}\left(e^{\ln x}\right) = \frac{dx}{dx}.$$

And by the chain rule we can rewrite this equation as

$$\left(e^{\ln x}\right)\frac{d}{dx}\left(\ln x\right) = 1.$$

And since $e^{\ln x} = x$,

$$\frac{d}{dx}\left(\ln x\right) = 1/x.$$

Learning outcomes:
Author(s):

Question 288 *Find simplified expressions for each of the following derivatives. Include an appropriate domain for each.*

(a)
$$\frac{d}{dx}\left(\ln\left(x/2\right)\right)$$

(b)
$$\frac{d}{dx}\left(\ln\left(x^2\right)\right)$$

Question 289 *Find an equation of the tangent line to the curve*

$$y = 3\ln\left(\frac{x^2+1}{x^3+1}\right)$$

at the point on the curve with x-coordinate $x = 1$.

Question 290 (a) *Graph the function*

$$y = f(x) = \ln|x|$$

by hand.

(b) *Use the graph above to sketch (by hand) a graph of the derivative*

$$y = \frac{d}{dx}\left(\ln|x|\right).$$

(c) *Use the chain rule to compute the derivative*

$$\frac{d}{dx}\left(\ln|x|\right).$$

Do this by noting that

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

and using the chain rule to compute

$$\frac{d}{dx}\left(-x\right).$$

(d) *A reflection about the y-axis takes the graph of a function $y = f(x)$ to the graph of a function $y = g(x)$. Describe the transformation that takes the graph of $y = f'(x)$ to the graph of $y = g'(x)$. Explain your reasoning.*

Question 291 *Find an equation of the tangent line to the curve*

$$y = 3\ln\left|\frac{x^2+1}{x^3-1}\right|$$

at the point on the curve with x-coordinate $x = -1$.

Question 292 *Let $\mathcal{L}$ be the tangent line to the curve*

$$y = \ln|\sec\theta|\ ,\ -\pi/2 < \theta < \pi/2,$$

at the point on the curve with $\theta = \pi/7$.

Find the acute angle $\mathcal{L}$ makes with the x-axis.

2 Relative Rates of Change

Question 293 *The function $P = f(t)$, $0 \le t \le 12$, expresses the population of a colony of bacteria in terms of the number of hours past noon.*

Suppose that

$$\frac{d}{dt}(\ln P)\Big|_{t=4} = \frac{d}{dt}(\ln(f(t)))\Big|_{t=4} = \frac{3}{10}.$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative with and without the language of small changes.*

Question 294 *At a certain instant the population of a country is decreasing at the rate of 3%/yr. At this same instant the country's national debt is increasing at the rate of 5%/yr.*

(a) *At what relative rate is the per capita (ie. per person) share of the national debt changing at this time?* Hint: *Use the idea of the previous question.*

(b) *If the per capita share of the national debt is* \$200,000*/person at this time, at what (absolute) rate is the per-capita share of the debt changing at this instant?*

Question 295 *The function*

$$G = f(v) = 40 - 0.08\left(\frac{v}{2} - 25\right)^2 \quad , 25 \leq v \leq 65,$$

expresses the gas mileage of a car (in miles/gallon) in terms of its speed (in miles/hr).

(a) Explain the meaning of the derivative

$$\frac{d}{dv}\left(\ln(f(v))\right).$$

Include units in your explanation. Also, what are the units of 25 *in the above expression? How do you know?*

(b) Evaluate the above derivative at $v = 30$ *and explain its meaning in terms of small changes.*

Derivatives and the Shapes of Curves

First and second derivatives and their implications.

1 A Surge Function

Question 296 *The function*

$$C = f(t) = Ae^{-kt}\,,\; t \geq 0,$$

expresses the concentration of a drug (in mg/L) in terms of the number of hours since the drug was injected. Here A and k are postive constants.

(a) *What are the units of A? Of k? Explain how you know.*

(b) *Find an expression for the time when the concentration is a maximum.*

  (i) *Justify your assertion without relying on a graph.*

  (ii) *Check that your expression has the correct units.*

(c) *Find expressions for the times when*

  (i) *the concentration is increasing at its maximum rate.*

  (ii) *the concentration is decreasing at its maximum rate.*

  Justify your assertions without relying on a graph.

(d) *Determine all time intervals during which*

  (i) *the concentration is increasing at an increasing rate.*

  (ii) *the concentration is increasing at an decreasing rate.*

  (iii) *the concentration is decreasing at an increasing rate.*

  (iv) *the concentration is decreasing at an decreasing rate.*

  Justify your assertions without relying on a graph.

Learning outcomes:
Author(s):

2 Changes and Rates of Change, Relative Changes and Relative Rates of Change

Question 297 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\left.\frac{dP}{dt}\right|_{t=4} = 400,000.$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm the population ...

Question 298 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\left.\frac{d}{dt}(\ln P)\right|_{t=4} = \left.\frac{d}{dt}(\ln(f(t)))\right|_{t=4} = \frac{3}{10}.$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm the population ...

Question 299 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\left.\frac{d^2P}{dt^2}\right|_{t=4} = -6,000. \tag{4}$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm ...

(d) *Suppose also that*

$$\left.\frac{dP}{dt}\right|_{t=4} = 400,000.$$

Explain the meaning of the second derivative above without *using the language of small changes by completing the following sentence.*

At 4:01pm ...

Question 300 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\left.\frac{d^2(\ln P)}{dt^2}\right|_{t=4} = -\frac{3}{25}. \tag{5}$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm ...

(d) *Suppose also that*

$$\left.\frac{d}{dt}(\ln P)\right|_{t=4} = \left.\frac{d}{dt}(\ln(f(t)))\right|_{t=4} = \frac{3}{10}.$$

Explain the meaning of the second derivative above without *using the language of small changes by completing the following sentence.*

At 4:01pm ...

Question 301 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln P)\right|\right)\bigg|_{t=4} = -\frac{3}{5}. \tag{6}$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm ...

(d) *Suppose also that*

$$\frac{d}{dt}(\ln P)\bigg|_{t=4} = \frac{d}{dt}(\ln(f(t)))\bigg|_{t=4} = \frac{3}{10}.$$

Explain the meaning of the second derivative above without *using the language of small changes by completing the following sentence.*

At 4:01pm ...

Optimization

Optimization.

Question 302 *Use calculus to find an expression for the x-coordinate of the turning point of the parabola*

$$y = ax^2 + bx + c\,,\ c \neq 0.$$

Question 303 *Determine the minimum and maximum values of the function*

$$y = f(x) = x^3 - 9x\,,\ -3 \leq x \leq 0.$$

Question 304 *The function $G = f(v)$ graphed below expresses the gas mileage (in miles/gal) of a car in terms of its speed (in miles/hour).*

Desmos link: `https://www.desmos.com/calculator/bcj0k1fymu`

151: Burning Gas 2

(a) *Use the graph to determine the rate (in gal/hr) at which the car burns gas at a speed of 50 miles/hour.*

(b) *Use the slider v to sketch by hand a graph of the function $r = g(v)$ that expresses the rate (in gal/hour) at which your car burns gas in terms of its speed (in miles/hour). Then activate the folder in Line 8 to seee how you did.*

(c) *Drag the slider v in Line 1 to approximate the speeds between 35 miles/hour and 70 miles/hour at which the car burns gas at the maximum and minimum rates (measured in gal/hr). Explain your reasoning.*

(d) *Now suppose*

$$G = f(v) = -\frac{v^2}{20} + 5v - 90\,,\ 35 \leq v \leq 70.$$

Use calculus to determine the exact speeds in part (c).

Learning outcomes:
Author(s):

Question 305 *The function*

$$P = f(t) = 5 - 3t + t^2 \, , \, 0 \le t \le 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

(a) Use the graphs of the function $P = f(t)$ and the function $r = f'(t)/f(t)$ to estimate when the stock price is increasing at the greatest relative rate.

(b) Use calculus and algebra to find the exact time when the stock price is increasing at the greatest relative rate. Justify your assertion.

Desmos link: `https://www.desmos.com/calculator/y78fnyy7s3`

Desmos activity available at 151: Stock Price 4

Question 306 *The function*

$$P = f(t) = t^2 - 4t + 8 \, , \, 3 \le t \le 6,$$

expresses the price (in dollars/share) of a stock in terms of the number of hours past noon.

(a) *Use calculus,* not precalculus, *to show that the stock price increases between 3pm and 6pm.*

(b) *When is the stock price increasing at the maximum* relative *rate? When is it increasing at the minimum* relative *rate?*

(c) *Find the minumum and maximum relative rates of change in the stock price.*

(d) *Justify your answers to parts (b) and (c). Explain your reasoning in complete sentences.*

(e) *Preface each computation block with a brief description of what you are about to compute.*

Question 307 *A vertical wall b feet high runs parallel to a tall building. The wall is a feet from the building. A ladder reaching from the ground to the building rests on the top of the wall as shown below.*

Desmos link: `https://www.desmos.com/calculator/4ak46ub8ay`

151: Shortest Ladder

(a) *Find a function that express the length of the ladder (measured in feet) in terms of the angle the ladder makes with the ground. Include an appropriate domain.*

(b) *Use part (a) to express the length of the shortest such ladder in terms of a and b. Justify your assertion.*

(c) *Check that your expression has the correct units.*

Work in general and not *with the particular values of a and b in the worksheet above.*

Question 308 *The bottom and top edges of a painting are respectively a and b feet above eye level.*

Desmos link: `https://www.desmos.com/calculator/dkqndsegod`

151: Viewing Angle

(a) *Find a function that expresses the viewing angle in terms of your distance from the painting. Include an appropriate domain.*

(b) *How far from the paiting should you stand to maximimize the viewing angle marked above? Justify your assertion.*

(c) *Check that your expression has the correct units.*

Measuring Distances on the Earth

Shortest paths on the earth.

1 Trigonometry: Along a Circle of Latitude

Question 309 *Pick points A and B at the same latiitude ϕ on the earth's surface and let θ (measured in radians) be the difference in their longitudes, with $0 < \theta \leq \pi$.*

Desmos3D link: `https://www.desmos.com/3d/hhpog6ijnr`

151: Distances on Earth

The purpose of this problem is to compare two distances in traveling from A to B, one along their common circle of latitude, the other along the great circle through A and B. We'll assume the earth is a sphere of radius R miles.

To compute each distance, we need to see inside the earth. You can do this by deactivating the Sphere *folder in Line 3.*

(a) *First the distance along the circle of latitude.*

  (i) *Start by expressing the radius of the circle of latitude in terms of the latitude ϕ and the radius of earth R. Work in general, not with the specific values in the worksheet above.*

  (ii) *Then express the distance between A and B along the circle of latitude in terms of ϕ, R, and θ.*

(b) *Next we'll compute the distance along the great circle through A and B. The center of this circle coincides with the center of the sphere.*

  (i) *Try to do this on your own. Here are two triangles (from inside the sphere as illustrated above) shown in two dimensions to help.*

  Desmos link: `https://www.desmos.com/calculator/wkdhhbojii`

  151: Distances Earth 2D

Learning outcomes:
Author(s):

(c) *Use the results of parts (a) and (b) to compare these two distances betweeen San Francisco (latitude 38° N, longtitude 122° W) and Washington DC (latitude 38° N, longtitude 77° W). Take the radius of the earth to be* 3960 *miles.*

(d) *Find two other locations at approximately the same latitude and compare the two distances.*

2 Calculus: Along a Circle of Latitude

This is a continuation of the previous problem.

The question is this: Fix the difference in longitude θ between two points at the same latitude and determine an expression for the latitude at which the postive difference in the two distances (one along the circle of latitude, the other along the great circle through the points) is a maximum.

Desmos link: `https://www.desmos.com/calculator/muhahdawza`

151: Distances Earth 2D Part 2

The difference in the two distances is a maximum at latitude

$$\phi = \arccos\left(\sqrt{\csc^2\left(\frac{\theta}{2}\right) - \frac{4}{\theta^2}}\right).$$

Disco Dancing, an Optimization Problem

Revolving light in a dance hall.

Question 310 *A spotlight in a circular dance hall of radius r meters is located b meters from the center of the hall.*

The light rotates at the constant rate of ω rad/sec.

> *Desmos link:* `https://www.desmos.com/calculator/vtzsgtxswm`

151: Disco Dancing 25

(a) *Find an expression for the speed of the light beam as it moves along the wall at this instant. Your expression should be in terms of ω, r, b.* Hint: *Use the law of sines in ΔLOB to first find an expression for the measure of angle $\angle OLB$.*

(b) *Check that your expression in part (a) has the correct units.*

(c) *Activate the Distance Function folder in Line 27 of the worksheet above. Then try to guess the position of the spotlight on the wall when its speed is increasing at the greatest rate.*

(d) *Determine the position of the spotlight on the wall when its speed is increasing at the greatest rate.*

Learning outcomes:
Author(s):

Clock Hands

A twist on a familiar problem.

Question 311 *The question is to determine the angle between the minute and hour hands of a clock when the distance between the tips of the hands is increasing at the fastest rate.*

We'll assume the hour and minute hands have respective lengths a cm and b cm.

Desmos link: `https://www.desmos.com/calculator/efhex45zkr`

151: Hands of a Clock 3

(a) *Play the slider u in Line 2 above and reconcile the graph of the function $c = f(t)$ that expresses the distance between the hands (in cm) in terms of the nuumber of minutes past noon.*

(b) *Change the lengths of the hands in Lines 4 and 6 to see how the graph of the function f changes.*

(c) *The distance function looks sinusoidal, but open the folder in Line 26 to see the graph of the derivative dispels that notion.*

(d) *Find an expression for the function*

$$c = h(\theta)\,,\ 0 \leq \theta \leq 2\pi,$$

that expresses the distance between the tips of the hands in terms of the angle between the hands. Work in general with the parameters a and b, not *with their specific values in the worksheet.*

(e) *Find an expression for the derivative*

$$\frac{dc}{d\theta} = \frac{d}{d\theta}\left(h(\theta)\right).$$

Include units.

(f) *Use your expression for the derivative $dc/d\theta$ to find an expression for the derivative*

$$\frac{dc}{dt} = \frac{d}{dt}\left(f(t)\right).$$

Learning outcomes:
Author(s):

(g) *Use your expression for the derivative $dc/d\theta$ to determine the angle between the hands when the distance between their tips is increasing at the fastest rate. There are three cases to consider.*

(i) *the minute hand is longer than the hour hand ($b > a$)*

(ii) *the hour hand is longer ($a > b$)*

(iii) *the hands have the same length ($a = b$)*

(h) *What can you say about the angle between the shorter hand and the segment AB when the distance is increasing at the fastest rate?*

A Computational Solution

Question 312 *Determine the angle between the hour and minute hands of a clock when the distance between the tips of these hands is increasing at the fastest rate. Suppose the hour and minute hands have respective lengths a and b inches, with $a < b$.*

Explanation. We'll let a and b denote the respective lengths of the hour and minute hands, θ the angle (in radians) between the hands, and c the distance. Since the hands each turn at a constant rate, our aim is to maximize the derivative

$$\frac{dc}{d\theta} = \frac{ab\sin\theta}{c} = \frac{ab\sin\theta}{\sqrt{a^2+b^2-2ab\cos\theta}}.$$

Differentiating again and doing some algebra gives

$$\frac{d^2c}{d\theta^2} = \frac{ab}{c^3}\left(c^2\cos\theta - ab\sin^2\theta\right).$$

Then setting the second derivative equal to zero and substituting

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

leads to the equation (symmetric in a and b)

$$ab\cos^2\theta - (a^2+b^2)\cos\theta + ab = 0$$

or

$$(a\cos\theta - b)(b\cos\theta - a) = 0.$$

Assuming the minute hand is longer than the hour hand ($b > a$) and $0 \le \theta \le 2\pi$, we conclude that the distance is increasing at the fastest rate when $\theta = \arccos(a/b)$ and decreasing at the fastest rate when $\theta = 2\pi - \arccos(a/b)$.

Question 313 (a) *What can you say if $a < b$? If $a = b$?*

(b) *Supposing $a \neq b$, find an expression for the maximum rate at which the distance between the tips of the hands is increasing.*

A More Geometrical Solution

We could have saved ourselves the trouble of taking the second derivative by using the law of sines in ΔABC. Then

$$\left|\frac{dc}{d\theta}\right| = \frac{ab\sin C}{c} = \frac{ab\sin B}{B} = a\sin B$$

so that the rate of change is maximized when $B = \pi/2$ or when $\theta = \arccos(a/b)$ as above.

A Solution with Vectors

Mars and the Outer Planets

The retrograde motion of Mars and the outer planets.

Exploration 314 *YouTube link:*
https://www.youtube.com/watch?v=https://www.youtube.com/watch?v=1nVSzzYCAYk

Retrograde Motion of Mars

Exploration 315 *Assume for this problem that the planets rotate about the sun at constant rates in circular orbits that lie in the same plane. Suppose also that earth has an orbital period of 365 days and an orbital radius of 1 astronomical unit.*

Mars has an orbital period of Mars to be 687 earth days and an orbital radius of 1.524 astronomical units. Earth has an orbital period of 365 days and an orbital radius of 1 astronomical unit.

(a) *Parameterize the motions of earth and mars about the sun in terms of the number of earth days. Assume*

- *The sun is at the origin.*
- *The planets rotate counterclockwise at constant rates about the sun.*
- *The earth has coordinates $(1, 0)$ a time $t = 0$ days.*
- *The vector from the sun to Mars has polar angle θ_0 at time $t = 0$.*

(b) *Express the position of Mars relative to Earth in terms of the vectors $\overrightarrow{SE}$ and $\overrightarrow{SM}$ that respectively give the positions of Earth and Mars relative to the sun.*

(c) *Use part (b) to parameterize the motion of mars relative to the earth (with the earth fixed at the point $(0, -4)$.*

(d) *Check your work using the demonstration below by inputting the correct functions in Lines 3, 4, 6, 7, 9, and 10.*

Access Desmos interactive at

Mars Retrograde Motion

Learning outcomes:
Author(s):

Desmos link: `https://www.desmos.com/calculator/htc4xgrjxs`

(e) *Compute the synodic period of Mars as seen from Earth. This is the time, measured in earth days, it takes Earth to "lap" Mars. It is the time it takes Mars to return to the same position relative to the sun as seen from Earth.*

(f) *Sketch* by hand *graphs of the x-coordinate functions for the motions of Earth and Mars on the same coordinate system. Label the exact coordinates of at least two turning points on each graph. Label the axes with the appropriate variable names and units.*

(g) *Sketch* by hand *graphs of the y-coordinate functions for the motions of Earth and Mars on the same coordinate system. Label the exact coordinates of at least two turning points on each graph. Label the axes with the appropriate variable names and units.*

Access Geogebra interactive at

Mars Retrograde Motion

Geogebra link: `https://www.geogebra.org/m/addm38j6`

The Tangent Function and its Inverse

The tangent function and its inverse.

1 The Tangent Function

The most common way to find the derivative of the function $y = \tan(\theta)$ is to write

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and use the quotient rule. But this is not all that insightful, and we have not learned the quotient rule yet. Here are two other approaches.

Question 316 *We'll use the figure below to find an expression for the derivative*

$$\frac{d}{d\theta}(\tan\theta)$$

as follows.

Desmos link: https://www.desmos.com/calculator/2la5tvxn56

151: Tangent Derivative

(a) *First convince yourself that the marked angles $\theta = \angle BOA$ and $\angle QBC$ are congruent. Well almost, assuming $d\,\theta \sim 0$.*

(b) *Express the length OB in terms of θ.*

(c) *Express the length QB in terms of θ and $\Delta\theta$.*

(d) *Express the length BC in terms of θ and $\Delta\theta$.*

(e) *Then find an expression for*

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right).$$

Learning outcomes:
Author(s):

Hours of Daylight

Modeling the number of hours of daylight.

Question 317 *The function*

$$h = f(\delta) = \frac{24}{\pi} \arccos(-\tan\phi \tan\delta)\,,\ -\pi/2 + \phi < \delta < \pi/2 - \phi,$$

expresses the number of hours of daylight per day at latitude ϕ in terms of the declination of the sun. The declination of the sun (δ) is the angle the sun's rays make with the plane of the equator, taken to be positive between the spring and fall equinoxes in the northern hemisphere. The latitude ϕ, $-\pi/2 \le \phi \le \pi/2$, is positive at points in the northern hemipshere. The graph of the function $h = f(\delta)$ for latitude $\phi \sim 1.1$ is shown below.

Don't be confused by the angle δ (delta). It's just the lower case Greek letter for Δ.

Desmos link: `https://www.desmos.com/calculator/ifomatkcta`

Desmos activity available at 151: Length of Day 1

(a) *Use the graph to approximate the number of hours of daylight/day at a latitude of $\phi = 1.1$ radians on March 1, when the sun is about 8° below the plane of the equator. What about at our latitude? At the equator?*

(b) *Evaluate the derivative*

$$\left.\frac{dh}{d\delta}\right|_{\delta=0}.$$

(c) *What are the units of the derivative above? Interpret its meaning.*

(d) *Suppose now that each month has 30 days so that there are 360 days in one year. Suppose also that the declination of the sun varies sinusoidally as a function of time, that the maximum declination of $\delta = 23.5^\circ$ occurs on the summer solstice (say June 21st) and the minimum declination $\delta = -23.5^\circ$ occurs on the winter solstice (December 21st).*

(i) *Find an expression for a function*

$$\delta = k(t)\,,\ t \ge 0,$$

that gives the declination of the sun (measured in radians) in terms of the number of days since the spring equinox.

Learning outcomes:
Author(s):

(ii) *Evaluate the derivative*

$$\frac{d}{dt}\left(f(k(t))\right)\Big|_{t=0} = \frac{dh}{dt}\Big|_{t=0}.$$

(iii) *What are the units of the derivative above? Interpret its meaning.*

Question 318 *The function*

$$h = f(\delta) = \frac{24}{\pi}\arccos(-\tan\phi\tan\delta)\,,\ -\pi/2+\phi<\delta<\pi/2-\phi,$$

expresses the number of hours of daylight per day at latitude ϕ in terms of the declination of the sun. The declination of the sun (δ) is the angle the sun's rays make with the plane of the equator, taken to be positive between the spring and fall equinoxes in the northern hemisphere. The latitude ϕ, $-\pi/2 \le \phi \le \pi/2$, is positive at points in the northern hemipshere.

Geogebra link: `https://www.geogebra.org/m/vnhrutwu`

Geogebra activity available at 151: Declination of Sun 2

(a) *Use the graph of the function f below (at latitude $\phi \sim 1.1$) to sketch a graph of the derivative*

$$\frac{dh}{d\delta} = f'(\delta).$$

(b) *Suppose the latitude ϕ is held constant and find an expression for the derivative*

$$\frac{dh}{d\delta}$$

(c) *What are the units of the derivative in part (b)?*

(d) *Input your expression for the derivative in Line 4 of the worksheet below (follow the directions there). Then vary the slider ϕ to see how the function f and its derivative vary with latitude. Summarize your observations.*

(e) *Find an expression for the derivative*

$$\frac{dh}{d\delta}\Big|_{\delta=0}$$

in terms of the latitude ϕ.

(f) *Evaluate the derivative in part (e) at a latitude of $\phi = \pi/4$. Interpret its meaning in terms of small changes.*

Desmos link: `https://www.desmos.com/calculator/ifomatkcta`

Desmos activity available at 151: Length of Day 1

(g) *Suppose now that each month has 30 days so that there are 360 days in one year. Suppose also that the declination of the sun varies sinusoidally as a function of time, that the maximum declination of $\delta = 23.5°$ occurs on the summer solstice (say June 21st) and the minimum declination $\delta = -23.5°$ occurs on the winter solstice (December 21st).*

Find an expression for a function

$$\delta = k(t)\,,\ t \geq 0,$$

that gives the declination of the sun (measured in radians) in terms of the number of days since the spring equinox.

(h) *Find a function*

$$r = g(\phi)\,,\ -\pi/2 < \phi < \pi,$$

that expresses the rate of change in the number of hours of daylight per day (measured in (hours of daylight/day)/day)) on the spring equinox in terms of the latitude ϕ. Input this function in Line 1 of the demonstration below.

(i) *On the spring equinox, all latitudes receive 12 hours of daylight/day. Use the result of part (h) to approximate how many extra minutes of daylight we would get on the following day living in*

(i) *Shoreline, latitude $47.75°N$*

(ii) *Fairbanks, Alaska, latitude $64.8°N$*

Desmos link: `https://www.desmos.com/calculator/nf8n5uphhl`

Desmos activity available at 151: Length of Day 2

Motion

An introduction to motion.

Question 319 *Play the slider u in the activity below to see the motion of a balloon. Use the animation to sketch graphs of*

(a) the altitude of the balloon as a function of time.

(b) the balloon's rate of ascent as a function of time.

(c) Activate the folders in Lines 13 and 18 to see how you did.

Desmos link: `https://www.desmos.com/calculator/amv52b9ljt`

Desmos activity available at 151: Balloon

Question 320 *The function*

$$h = f(t) = 10 - \frac{1}{2}t^2 e^{-t/5}\,,\ 0 \le t \le 40,$$

expresses the altitude of a balloon (in thousands of feet) in terms of the number of hours since noon on August 31, 2023.

Use the graph of this function in Question 1 to first approximate answers to the following questions. Then use calculus and algebra to determine the exact times.

(a) When is the balloon at its minimum height? At its maximum height?

(b) When is the balloon ascending at the fastest rate? Descending at the fastest rate?

To compute these times you will end up solving a quadratic equation that may be written in the form

$$t^2 - 20t + 50 = 0.$$

So the balloon is ascending at its fastest rate at time (give the exact times and then approximations to the nearest hundredth of an hour)

$$t = 10 + \sqrt{50} \sim 17.07$$

Learning outcomes:
Author(s):

hours past noon and descending at its fastest rate at time

$$t = 10 - \sqrt{50} \sim 2.93$$

hours past noon.

Linear Approximation

The derivative as a scaling factor.

1 The Derivative as a Scaling Factor

We started this class by interpreting the derivative as a dimensionless, local stretching factor of a rubber band. Perhaps stretching factor was not the best description because a rubber band might get stretched in some regions but compressed in others.

As we saw many times, a better, more universal description would be to think of the derivative as a *local scaling factor*. Multiply a small change in the input to a function by the derivative (at that input) and the result is a good approximation to the change in the function's output. Good in the sense that the error in the approximation approaches zero faster than the change in the input.

This interpretation of the derivative is often more useful than interpreting the derivative as the slope of a tangent line or an instantaneous rate of change. For one, the idea of a slope does not generalize to higher dimensions. And if you pause to really think about things, you might find it hard to make precise sense of what exactly anyone means by an instanteous rate of change.

The idea of linear approximation is not new. We've been talking about it all quarter. And you knew about linear approximation well before you took this class. There's no need to learn anything more, or to use a special formula. It's really just common sense.

Here's a simple example.

Example 36. The function

$$h = f(t)\,,\; 0 \le t \le 60,$$

expresses the altitude of the balloon (measured in feet) in terms of the number of minutes past noon.

Suppose

(a) $f(10) = 1500$ and

(b) $\left.\dfrac{dh}{dt}\right|_{t=10} = -50.$

Learning outcomes:
Author(s):

Approximate the height of the balloon at 12:08pm.

Explanation. We know at 12:10pm the balloon is descending at the rate of 50 ft/min. So during the time interval (backward in time) from 12:10pm to 12:08pm of

$$\Delta t = (8 - 10) \text{ min} = -2 \text{ min},$$

the change in the balloon's altitude is approximately

$$\begin{aligned}
\Delta h &= f(8) - f(10) \\
&\sim \left(\frac{dh}{dt}\bigg|_{t=10} \right)(\Delta t) \\
&= \left(-50 \frac{\text{ft}}{\text{min}} \right)(-2 \text{ min}) \\
&= 100 \text{ ft}.
\end{aligned}$$

Since the balloon's altitude is 1500 feet at 12:10pm, its altitude at 12:08pm was approximately

$$\begin{aligned}
f(8) &= f(10) + \Delta h \\
&\sim (1500 + 100) \text{ ft} \\
&= 1600 \text{ ft}.
\end{aligned}$$

2 Interpreting the Derivative

2.1 Speed

A key idea in this class has been to interpret *speed* as a derivative. This has been a bit difficult, especially with the mindset that the derivative is always a rate of change. We typically don't think about speed this way.

But speed is a rate of change.

Speed is the rate of change, with respect to time, of the the function that records the cumulative distance traveled.

The image of a cumulative distance function easiest for me to understand is the odometer reading on a car. It records the distance the car has traveled since it came off the lot. The speedometer records the derivative of this function with respect to time (measured in hours).

Here's an example. Don't be confused by the variable s. It does not stand for speed. It's commonly used, as it is here, to measure distance. Or as we'll see later, and as you'll see in integral calculus and in your physics classes, for *signed distance.*

Don't be confused either by the variable $v = ds/dt$. It is *not* velocity, but rather speed. Speed is a scalar, velocity is a vector. Speed is the lenght of the velocity vector.

Example 37. The function

$$s = f(t) = 2t + 10t^2 - t^3 \,,\; 0 \le t \le 6,$$

expresses the mileage on the trip odometer reading in terms of the number of hours since the start of a car trip.

Desmos link: `https://www.desmos.com/calculator/t1ruocrgm4`

151: Odometer Reading 34

(a) Evaluate the derivative
$$\left.\frac{ds}{dt}\right|_{t=2}$$
and interpret its meaning. Include units.

(b) Interpet the meaning of the above derivative in terms a scaling factor and small changes.

(c) Use the result of part (b) to approximate the reading on the odometer at 2:01pm.

(d) Use the result of part (b) to approximate when the odometer reads 35 miles.

(e) When is the speed of the car a maximum? Use the graphs above to approximate the time. Then find the exact time and maximum speed.

3 Weight and Distance

Example 38. The function

$$W = f(h) = \frac{2000}{(h+4)^2} \,,\; h \ge 4,$$

expresses the weight of an astronaut (in pounds) in terms of her distance above the earth's surface (meausred in thousands of miles).

(a) Evaluate the derivative
$$\left.\frac{dW}{dh}\right|_{h=1}.$$
Include units.

(b) Interpet the meaning of the above derivative in terms a scaling factor and small changes.

(c) Approximate the astronaut's weight at an altitude of 1010 miles.

(d) At what approximate altitude does the astronaut weigh 126 pounds?

4 Speed and Height

Example 39. The function

$$v = f(h)\,,\; 0 \leq h \leq 100,$$

expresses the speed of a rock (measured in meters/sec) dropped from rest on the planet Krypton in terms of its height above the surface (measured in meters).

(a) Which of the following is more likey?

(i)
$$\left.\frac{dv}{dh}\right|_{h=60} = 0.5$$
or

(ii)
$$\left.\frac{dv}{dh}\right|_{h=60} = -0.5?$$

Explain your reasoning. Include units for the derivative.

(b) Intepret the meaning of the more likely derivative in terms of a scaling factor and small changes.

(c) Suppose that $f(60) = 40$.

(i) Interpret the meaning of this statement.

(ii) Assume the more likely derivative from part (a) and approximate the rock's speed at a height of 59.9 meters.

(iii) Suppose the speed of the rock increases at the constant rate of 20 (m/sec)/sec as it falls and find the rock's exact speed when it is 59.9 meters above the surface. Compare this with your estimate.

5 Distance to the Horizon

Example 40. An astronaut above the surface of a planet sees only a fraction of the surface as suggested by the figure below.

Desmos link: `https://www.desmos.com/calculator/8shf1msp4m`

151: Distance to Horizon 44

The visible part of the surface is a spherical disk with spherical radius BC above. We can think of this distance (an arc of a circle) as the distance to the horizon.

(a) Find a function
$$s = f(h)\,,\ h \geq 0,$$
that expresses the distance to the horizon on a planet of radius R kilometers in terms of the altitude of the astronaut (in km) above the surface.

(b) Evaluate the derivative
$$\left.\frac{ds}{dh}\right|_{h=2R/3}.$$
Include units.

(c) Explain the meaning of the derivative in terms of a scaling factor and small changes.

(d) Approximate the change in the distance to the horizon when the altitude above the surface increases from $h = (2/3)R$ to $h = (0.01 + 2/3)R$.

6 Gas Mileage

Example 41. The one-to-one function
$$G = f(v)\,,\ 50 \leq v \leq 80,$$
expresses the gas mileage of a car (in miles/gal) in terms of its speed (in miles/hr).

(a) Which of the following is more likely?

(i)
$$\left.\frac{dv}{dG}\right|_{v=70} = 2$$
or

(ii)

$$\left.\frac{dv}{dG}\right|_{v=70} = -2?$$

Explain your reasoning. Include units for the derivatives.

(b) Suppose $f(70) = 24$

(i) Interpret the meaning of this statement.

(ii) Assume the more likely derivative from part (a) and approximate the gas mileage at a speed of 73 miles/hour.

(iii) Simplify the units of the derivative in part (a). What do these units suggest about its meaning? Is this correct? Explain.

Linear Approximation, Part 2

Linear approximation and relative changes.

1 Review of the Natural Log Fuction

Question 321 *Find expressions for each of the following derivatives.*

(a) $\frac{d}{dx}(\ln x)$

(b) $\frac{d}{dx}(\ln(4x))$

(c) $\frac{d}{dx}(\ln(-x))$

(d) $\frac{d}{dx}(\ln|x|)$

(e) $\frac{d}{dx}\left(\ln(x^5)\right)$

(f) $\frac{d}{dx}\left(\ln(1+x^2)\right)$

2 First Derivatives

Question 322 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\left.\frac{dP}{dt}\right|_{t=4} = 400,000.$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

Learning outcomes:
Author(s):

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm the population ...

Question 323 *The function $P = f(t)$, $0 \leq t \leq 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\frac{d}{dt}(\ln P)\Big|_{t=4} = \frac{d}{dt}(\ln(f(t)))\Big|_{t=4} = \frac{3}{10}.$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm the population ...

Question 324 *The function*

$$G = f(s)\,,\ 0 \leq s \leq 130,$$

graphed below expresses the number of gallons of gas in a car in terms of the trip odometer reading (measured in miles).

Desmos link: `https://www.desmos.com/calculator/gdeybih5t2`

151: Relative Rates Gas

(a) *A reading on the car's dashboard shows the number of miles left to drive before running out of gas assuming the car continues to burn gas at the current rate. Drag the slide u (another name for s) in Line 1 to approximate this reading when the trip odometer reads 80 miles.*

(b) *Drag the slider u in Line 1 to approximate the derivative*

$$\frac{d}{ds}\left(\ln\left(\frac{G}{1}\right)\right)\Big|_{s=80}.$$

(c) *What are the units of the derivative above?*

(d) *Explain the meaning of the derivative.*

(e) *Approximate the derivative $d/ds(\ln G)$ for some other values of s. What do you notice? What does this suggest about the function f?*

(f) *In light of part (e), how would you describe what it means for a population of bacteria to decrease at the constant relative rate of 5%/hour? The rate is instantaneous.*

3 Second Derivatives

Question 325 *The function $P = f(t)$, $0 \le t \le 12$, expresses the population of a colony of bacteria (measured in the number of bacteria) in terms of the number of hours past noon.*

Suppose that

$$\left.\frac{d^2P}{dt^2}\right|_{t=4} = -6,000. \tag{7}$$

(a) *What are the units of this derivative?*

(b) *Explain the meaning of the derivative without using the language of small changes.*

(c) *Explain the meaning of the derivative using the language of small changes by completing the following sentence.*

Between 4:00pm and 4:01pm ...

(d) *Suppose also that*

$$\left.\frac{dP}{dt}\right|_{t=4} = 400,000.$$

Explain the meaning of the second derivative above without *using the language of small changes by completing the following sentence.*

At 4:01pm ...

Review of Differentiation

Derivative Review.

Examples

Example 42. Find an equation of the tangent line to the curve

$$y = f(x) = \left(2x^3 + 1\right)^2$$

at the point $(1, 9)$.

Explanation. Let

$$y = \left(2x^3 + 1\right)^2$$

and

$$u = 2x^3 + 1.$$

Then

$$y = u^2$$

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}\left(u^2\right) \cdot \frac{d}{dx}\left(2x^3 + 1\right) \\ &= 2u(6x^2) \\ &= 2(2x^3 + 1)(6x^2).\end{aligned}$$

Then the slope of the tangent line to the curve $y = (2x^3 + 1)^2$ at the point $(1, 9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 2(3)(6) = 36,$$

and an equation of the tangent line is

$$y - 9 = 36(x - 1).$$

Example 43. Find an expression for the derivative

$$\frac{d}{d\theta}\left(\theta \cos(5\theta)\right).$$

Learning outcomes:
Author(s):

Explanation. We use the product rule first to get

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \frac{d}{d\theta}\left(\theta\right)\cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right).\end{aligned}$$

Now we use the chain rule to compute

$$\frac{d}{d\theta}\left(\cos(5\theta)\right).$$

For this we hide the composition by letting

$$y = \cos(5\theta)$$

and

$$u = 5\theta.$$

Then

$$y = \cos u$$

and by the chain rule

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= \frac{d}{du}\left(\cos u\right)\cdot\frac{d}{d\theta}\left(5\theta\right)\\ &= (-\sin u)(5)\\ &= -5\sin(5\theta).\end{aligned}$$

The final result is that

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta) - 5\theta\sin(5\theta).\end{aligned}$$

Here's a shorter version of the same solution.

We use the product rule first to get

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \frac{d}{d\theta}\left(\theta\right)\cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right).\end{aligned}$$

Then we use the chain rule to differentiate $\cos(5\theta)$, giving

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \frac{d}{d\theta}(\theta)\cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta) + \theta(-\sin(5\theta))\frac{d}{d\theta}(5\theta)\\ &= \cos(5\theta) - 5\theta\sin(5\theta).\end{aligned}$$

Exercises

Directions: Follow the method of Example 1 *exactly* for each of the following problems.

Question 326 *Find expressions for each of the following derivatives. Simplify your expressions for the derivatives. Do* not *simplify the function being differentiated.*

Part 1:

(a) $\dfrac{d}{dw}\left(\arctan w + \arctan(1/w)\right)$

(b) $\dfrac{d}{dy}\left(\arcsin\left(\sqrt{1-y^2}\right)\right)$

Part 2:

Any comments or observations on the derivatives above?

Question 327 *Desmos link:*
https://www.desmos.com/calculator/zjqsdhvz4p

Desmos activity available at 151: Building Temperature

Question 328 *Water is poured into a cylindrical tank at a constant rate. At the same time, water flows out of a small hole in the bottom of the tank. The tank is empty at noon.*

The function

$$t = f(h) = -\frac{2}{k_2}\left(\sqrt{h} + \frac{k_1}{k_2}\ln\left|\frac{k_1 - k_2\sqrt{h}}{k_1}\right|\right),\ t \geq 0,$$

expresses the time (measured in minutes past noon) in terms of the depth (measured in cm) of water in the tank. Here k_1, k_2 are postive constants.

(a) Use the above function to verify that the tank is empty at noon.

(b) Find a simplified expression for the derivative dt/dh.

(c) Use the result of part (b) to show that

$$\frac{dh}{dt} = k_1 - k_2\sqrt{h}.$$

(d) What are the units of k_1, k_2? How do you know?

(e) For much more on this problem, see Questions 23 and 24 of the chapter Derivatives of Inverse Functions.

Question 329 *The function*

$$s = f(t) = Ae^{-k_1 t}\cos(k_2 t)\,,\, t \geq 0,$$

expresses the displacement (in meters) from equilibrium of an oscillating mass on a spring in terms of the number of seconds since the mass was released from rest.

(a) What are the units of the constants A, k_1, and k_2? Explain how you know.

(b) Find an expression for the velocity ds/dt of the mass.

(c) Find an expression for the acceleration

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

of the mass.

(d) Use algebra to show that

$$\frac{d^2s}{dt^2} = -\left(2k_1\frac{ds}{dt} + (k_1^2 + k_2^2)s\right).$$

Desmos link: `https://www.desmos.com/calculator/ygikqgj7af`

Desmos activity available at 151: Damped Harmonic Oscillator

Question 330 *(a) Find a function*

$$s = f(h), h \geq 0,$$

that expresses the distance (in miles) to the horizon in terms of your altitude (in miles). We'll suppose the earth to be a perfect sphere of radius $R = 4000$ *miles. The distance to the horizon is the arclength* AT *below, meaured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance* AP*.*

The function is

$$s = f(h) = 4000 \arccos\left(\frac{4000}{4000+h}\right), \; h \geq 0.$$

Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

(b) Use the graph of the function $s = f(h)$ *show below to skectch a graph of the derivative*

$$y = \frac{ds}{dh} = f'(h).$$

by hand. What is the domain of the derivative? Note that the curve $s = f(h)$ *has a horizontal asymptote. What is an equation of this asymptote?*

Desmos link: `https://www.desmos.com/calculator/nof5l3mtfy`

Then check your sketch by activating the folder in Line 1 of the worksheet below.

Desmos link: `https://www.desmos.com/calculator/pdbjfao316`

(c) Find an expression for the derivative

$$\frac{ds}{dh} = \frac{d}{dh}\left(\arccos\left(\frac{R}{R+h}\right)\right).$$

Hint: Start by using the graphs of the functions $y = \arcsin x$ *and* $y = \arccos(x)$ *below and your knowledge of the derivative* $d/dx(\arcsin x)$ *to find an expression for* $d/dx(\arccos x)$*. Or equivalently, recognize that*

$$\arcsin x + \arccos x = \pi/2.$$

Desmos link: `https://www.desmos.com/calculator/zuq9rf1j4d`

(d) With $R = 4000$*, evaluate the derivative*

$$\left.\frac{ds}{dh}\right|_{h=16}.$$

(e) Interpret the meaning of the derivative in part (d) in terms of specific *small changes.*

(f) You take a ride on Blue Origin and in two minutes are boosted straight up to an altitude of 32 miles. Suppose that the function

$$h = g(t) = \begin{cases} 10t^2 + 6t^3\,,\ 0 \le t \le 1 \\ 32 - 10(2-t)^2 - 16(2-t)^3\,,\ 1 < t \le 2, \end{cases}$$

expresses your altitude (in miles) in terms of the number of minutes since launch for the first two minutes of your flight.

(i) At what rate (with respect to time) is your distance to the horizon changing when you are one minute into the flight? Use the graph of the function $s = f(g(t))$ shown below to first approximate this rate.

(ii) At what rate (with respect to time) is your distance to the horizon changing at the start of the flight? Use the graph of the function $s = f(g(t))$ shown below to first approximate this rate. Are you surprised given your graph in part (b)?

Desmos link: `https://www.desmos.com/calculator/jchwjpxugf`

Final, Math 151

Directions for our final.

1 Directions

Here are the directions for our final.

- Use only the material of our class. In particular, *do not use vectors or other ideas that were not part of our class.*
- Use the Leibniz differential notation for derivatives and their evaluations. Do *not* use the prime notation.
- Use the Leibniz notation to show *all* steps when using the the chain, product, and quotient rules. See the next section *Examples of How to Show Work* for more details.
- Define, in complete sentences and with units any variables or unknowns you introduce.
- Include a domain for each function you introduce.
- Include units for *each* number in *each* numerical computation.
- Give explanations of your reasoning along with your solutions to each problem.
- Show all your work.
- Simplify each of your answers as much as possible.
- Answer each word problem with a concluding sentence.
- Write LARGE and neatly.
- Leave plenty of space.
- Work vertically. Do *not* split a page into multiple columns.
- No technology permitted. Put away all cell phones and calculators. An open cell phone will result in an automatic score of 0 for the exam.

Learning outcomes:
Author(s):

- A few derivatives:

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

2 Examples of How to Show Work

Here are some examples of how to show your work when taking derivatives. These examples will *not* be included with the exam.

Example 44. The chain rule (do *not* use the quotient rule for a derivative like this one):

$$\begin{aligned}
\frac{d}{d\theta}\left(\frac{4}{1+\tan\theta}\right) &= 4 \cdot \frac{d}{d\theta}\left((1+\tan\theta)^{-1}\right) \\
&= -4(1+\tan\theta)^{-2} \cdot \frac{d}{d\theta}(\tan\theta) \\
&= -4(1+\tan\theta)^{-2}(\sec^2\theta).
\end{aligned}$$

Or, if you prefer, you could make the u-substitution explicit by letting

$$y = 4(1+\tan\theta)^{-1}$$

and

$$u = 1+\tan\theta.$$

Then

$$y = u^{-1}$$

and by the chain rule

$$\begin{aligned}
\frac{dy}{d\theta} &= \frac{dy}{du} \cdot \frac{du}{d\theta} \\
&= \frac{d}{du}(u^{-1}) \cdot \frac{d}{d\theta}(\tan\theta) \\
&= (-u^{-2})(\sec^2\theta) \\
&= -4(1+\tan\theta)^{-2}(\sec^2\theta).
\end{aligned}$$

Example 45. The quotient rule with the chain rule:

$$\begin{aligned}
\frac{d}{dw}\left(\frac{w^2}{1+e^{3w}}\right) &= \frac{(1+e^{3w})\frac{d}{dw}(w^2) - w^2\frac{d}{dw}(1+e^{3w})}{(1+e^{3w})^2} \\
&= \frac{(1+e^{3w})(2w) - w^2 e^{3w}\frac{d}{dw}(3w)}{(1+e^{3w})^2} \\
&= \frac{(1+e^{3w})(2w) - 3w^2 e^{3w}}{(1+e^{3w})^2}.
\end{aligned}$$

Review Problems

Some problems for review.

Question 331 *This question is about how the temperature inside a building changes in response to changes in the outdoor temperature. We assume the building has no internal heating or cooling system.*

We'll suppose that the function

$$f(t) = M - B\cos\left(\frac{\pi}{12}t\right),\ t \geq 0,$$

expresses the outdoor temperature (in Fahrenheit degrees) in terms of the number of hours past 4am.

Newton's law of cooling models the rate at which the indoor temperature is changing at any time. It says that this rate of change is proportional to the difference in the indoor and outdoor temperatures. So if the function

$$T = g(t)\ t \geq 0,$$

expresses the outdoor temperature (in Fahrenheit degrees) in terms of the number of hours past 4am, Newton's law says that

$$\frac{dT}{dt} = k(f(t) - g(t)) \tag{8}$$

for some constant k.

(a) What are the units of k? How do you know?

(b) Is k postive or negative? How do you know?

(c) Experiment with the sliders in the demonstration below. Summarize your observations.

Desmos link: `https://www.desmos.com/calculator/oag9lhvgo5`

Desmos activity available at 151: Building Temperature

Next quarter you will learn how to use the above equation to determine the indoor temperature at any time given the temperature at some specific time.

Learning outcomes:
Author(s):

For now, we'll just claim that the indoor temperature is given by the function

$$\begin{aligned} T &= g(h) \\ &= M + Ce^{-kt} - \frac{B}{1+(\frac{\pi}{12k})^2}\left(\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{12k}\sin\left(\frac{\pi}{12}t\right)\right) \\ &= M + Ce^{-kt} - \frac{B}{\sqrt{1+(\frac{\pi}{12k})^2}}\cos\left(\frac{\pi}{12}t - \phi\right), \end{aligned}$$

where

$$\phi = \arctan\left(\frac{\pi}{12k}\right).$$

Hanging Chains

Comparing a catenary with a weighted chain.

Question 332 *Desmos link:* `https://www.desmos.com/calculator/cifsqaas5j`

Desmos activity available at 151: Weighted Chain

Question 333 *Desmos link:* `https://www.desmos.com/calculator/sv70z21j2j`

Desmos activity available at 151: Catenary

Learning outcomes:
Author(s):

Electron in a Crossed Field

Electron.

Crossed Fields and the Hodograph

The force on an a charge q with velocity $\mathbf{v}$ in a crossed magnetic/electric field is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Our problem is to parameterize the motion of a charged particle when the fields are uniform and mutually orthogonal. We suppose the magnetic and electric fields point in the respective directions $\mathbf{k}$ and $\mathbf{j}$, and that the charge has velocity $\mathbf{v}_0$ perpendicular to $\mathbf{k}$ at time $t = 0$.

Then the charge, assumed to be positve, has acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = k_1\mathbf{j} + k_2\mathbf{v} \times \mathbf{k},$$

for some constants $k_1, k_2 \geq 0$, with respective units m/sec^2 and sec^{-1}.

This is a differential equation that is easy enough to solve algebraically. But our aim here is to take a geoemetric approach that uses some key ideas from Calculus 3.

In sketching motion we typically draw the path and perhaps draw a few position vectors at equally-spaced time intervals. These vectors allow us to approximate a few velocity vectors, or at least their relative lengths. But instead of drawing the velocity vectors tangent to the path, it is usually more useful to draw them with their tails pinned at a common point. The curve traced by these tips of these pinned vectors is called the *hodograph* of a motion.

For starters, the hodograph gives us a way to visualize the motion's acceleration as it is tangent to the hodograph. But to get some idea of the accerlation's magnitude, we need to consider not only the direction and magnitude of the velocity vector but also its rotation rate.

For a uniform circular motion that rotates around a circle of radius r meters at a constant rate of ω rad/sec, for example, we *see* from the hodograph (a circle of radius $v = \omega r$ that the acceleration vector points directly toward the center of the path and rotates at the same rate as both the position and velocity vectors.

Learning outcomes:
Author(s):

So without any derivatives, we get the acceleration's magnitude to be

$$|\mathbf{a}| = \omega|\mathbf{v}| = \omega^2 r.$$

Turning off the Electric Field

Returning to our charged particle, with the electric field turned off ($k_1 = 0$), the acceleration and velocity vectors are perpendicular, so the charge moves with constant speed $|\mathbf{v}_0| = v_0$. But then because $\mathbf{v}$ is perpendicular to $\mathbf{k}$,

$$|\mathbf{a}| = k_2|v| = k_2 v_0$$

is also constant.

So we are looking for a plane motion having constant speed and an acceleration with constant magnitude. One choice would be uniform circular motion, where

$$|\mathbf{a}| = v_0\omega,$$

where $\omega = d\theta/dt$ is the (constant) rotation rate of the pinned velocity vector. Then $k_2 = \omega$ and the trajectory has radius

$$r = \frac{v_0}{\omega} = \frac{v_0}{k_2}.$$

To see that there are no other possible motions we could probably appeal to some uniqueness theorem of differential equations. But for a more geometric approach, consider what we know.

(a) that the charge moves with constant speed and

(b) that the acceleration vector rotates at a constant rate.

Question 334 *What do these conditions imply about the trajectory?*

Hint: *Think about curvature.*

Explanation. Click the arrow to the lower right for the solution.

Let θ be the angle from $\mathbf{i} = \langle 1, 0, 0\rangle$ to $\mathbf{v}$. Then because $\mathbf{v}$ and $\mathbf{a}$ are orthogonal, these vectors rotate at the same rate $\omega = d\theta/dt$. And because the speed is constant, the path's radius of curvature

$$r = \frac{|v|}{\left|\frac{d\theta}{dt}\right|} = \frac{v_0}{k_2}$$

is also constant. And because the path is a plane curve, it is a circle with this radius.

1 Newton's Law of Cooling

Exploration 335 *Desmos link:*
https://www.desmos.com/calculator/yebumxuwms

Electron 11

Question 336 *The function*

$$P = 40e^{-t/20}\,,\ 0 \leq t \leq 15,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) *Find a function*
$$r = g(P)$$
that expresses the growth rate of the population (in millions of bacteria/hour) in terms of the popuation. Include a domain.

(b) *Use the result of part (a) to determine the population when it is decreasing at the rate of* $1,820,000$ *bacteria/hour. End with a concluding sentence. No credit for other methods.*

Short Quizzes Math 151

Short Quizzes Math 151.

Question 337 *The function*

$$h = f(v)\,,\, 80 \le v \le 120,$$

expresses the height of a helicopter (measured in feet) in terms of its speed (measured in ft/sec).

Suppose

$$\left.\frac{dh}{dv}\right|_{v=100} = -20.$$

(a) *What are the units of the above derivative?*

(b) *Explain the meaning of the derivative using the language of small changes.*

Question 338 *The function*

$$P = f(t) = 40e^{-t/20}\,,\, 0 \le t \le 20,$$

expresses the popuation (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) *Find a function*

$$r = g(P),$$

that expresses the growth rate (in millions of bacteria/hour) of the population in terms of the population (in millions of bacteria).

(b) *Use the result of part (a) to determine the population when it is decreasing at the rate of* $1,820,000$ *bacteria/hour.*

(i) *No credit for other methods.*

(ii) *End with a concluding sentence.*

Learning outcomes:
Author(s):

Question 339 *Find an equation of the tangent line to the curve*

$$x^3 \sin(5y) + 4e^{xy} = 4x$$

at the point $(1, 0)$.

(a) *Show all work.*

(b) *Use the Leibniz notation for derivatives and their evaluations.*

(c) *Include a brief explanation with your work.*

(d) *End with a concluding sentence.*

Short Quizzes Math 142

Short Quizzes Math 142

Question 340 *Let Q be the point on the circle of radius 40 meters that is 90 meters from the point $A(40, 0)$ (coordinates measured in meters). The distance is measured* clockwise *around the circle from A to Q.*

Between 12:06pm and 1:00pm a beetle crawls counterclockwise around this circle at a constant speed of 5 meters/min, passing the point Q at 12:43pm.

(a) *Find an expression for the function*

$$\theta = f(t)\,,\ 6 \le t \le 60,$$

that expresses the polar angle (meausred in radians) of the beetle (more precisely of the vector giving the beetle's position relative to the origin) in terms of the number of minutes past noon being sure to do the following:

(i) *Explain your reasoning. This means to include a brief description of what you are computing for* each *of your computations.*

(ii) *Include units for each number in each computation.*

(iii) *Include a graph (drawn by hand) of the function $\theta = f(t)$ to help with your explanation.*

(iv) *Include a sketch (drawn by hand) of the circle on a coordinate system with appropriately labeled points and arclengths. Label the axes with the appropriate variable names and units.*

(b) *Find functions*

$$x = g_1(t)\,,\ 6 \le t \le 60,$$

and

$$y = g_2(t)\,,\ 6 \le t \le 60,$$

that express the coordinates (in meters) of the beetle in terms of the number of minute past noon. Include a brief explanation.

(c) *Input the correct coordinate functions on Lines 2 and 3 of the worksheet below. Then play the slider u (another name for t, the number of minutes past noon) in Line 1 to see if your functions are correct.*

Desmos link: `https://www.desmos.com/calculator/lkrunhfgxi`

Learning outcomes:
Author(s):

142: Short Quiz 1

(i) *Explain how you know that your functions are correct.*

(ii) *Include at least two screenshots of the animation at different times to help with your explanation.*

Question 341 *Solve the equation*

$$9 - 3\sin\theta = 11.$$

(a) *Explain your reasoning thoroughly, in complete sentences.*

(b) *Include a picture as in class to help with your explanation.*

(c) *End with a concluding sentence that expresses the solution as a set.*

Sinusoidal Modeling, Math 142

Implicit differentiation.

Question 342 *Suppose that during a five-day period, beginning at midnight Monday morning, the depth of the water at the Edmond's Pier is a sinusoidal function of time.*

Suppose also that a low tide of 7 feet occurs at 1:00am Monday morning and that the following high tide of 23 feet occurs at 6:45am that same morning.

(a) *Sketch by hand a graph of the function*

$$h = f(t)$$

that expresses the depth of the water (in feet) as a function of the number of hours since midnight, Monday morning.

(b) *Find an expression for the above function. Include a domain.*

(c) *Find all times during the week when the water is 10 feet deep. Give exact times. Do not use a calculutor.*

(d) *Approximate the clock times (to the nearest minute) on Friday when the water is 10 feet deep.*

Question 343 *Assume for this question that each month has 30 days and that the number of hours of daylight/day in Seattle is a sinusoidal function of time. Assume also that on June 21, Seattle gets a maximum of 16 hours of daylight/day and that on December 21, Seattle gets a minimu of 8 hours of daylight/day.*

(a) *Find a function*

$$H = f(t)\,,\ 0 \leq t \leq 12,$$

that expresses the number of daylight hours/day in Seattle in terms of the number of months since June 21. Use the cosine function. Start by sketching a graph. Explain your reasoning.

(b) *Use your function to determine the number of hours of daylight/day that Seattle gets on March 1.*

Learning outcomes:
Author(s):

(c) *About how many more minutes of daylight/day do we get tomorrow than today?*

(d) *Use your function to determine the day(s) of the year when Seattle gets 14 hours of daylight/day.*

Simple Harmonic Motion, Law of Cosines, Math 142

Simple harmonic motion.

Question 344 *Suppose for this problem that the earth is a ball with uniform density of radius 4000 miles. Now imagine drilling a straight tunnel through the earth from the North Pole to the South Pole.*

A rock dropped from rest at the north pole falling through the tunnel would then oscillate in simple harmonic motion between the poles and return to the north pole every 84 minutes. This means we can think of the rock as being dragged along by a point moving around the earth at constant speed as illustrated below.

Desmos link: `https://www.desmos.com/calculator/ij8dqowgza`

Desmos activity available at 142: Simple Harmonic Motion

(a) *Activate the* Protractor *folder in Line 15 to see the protractor, where consecutive tick marks subtend equal angles of* $\pi/100$ *radians about the earth's center. Use the protractor but* not *any trigonometry to answer the following questions.* Hint: *Think proportionately.*

(i) *Estimate the distance of the rock from the South Pole at time* $t = 16.8$ *minutes after the rock is released.*

(ii) *Estimate the first two times when the rock is* 1000 *miles from the South Pole.*

(b) *Sketch by hand one period of the graph of the function*

$$s = f(t) \;\; t \geq 0,$$

that expresses the distance of the rock (measured in thousands of miles) from the South Pole in terms of the number of minutes since the rock was released. Assume the rock is dropped at midnight on July 1, 2085.

(c) *Find an expression for the function* $s = f(t)$*. Start by expressing the distance* s *in terms of the polar angle* θ *marked above.*

Learning outcomes:
Author(s):

(d) *Use your function to determine the exact distance between the rock and the South Pole at time $t = 16.8$ minutes past noon. Then use a calculator to approximate this distance to the nearest mile and compare it with your estimate in part (i).*

(e) *Use your function to find all (exact) times when the rock is 1000 miles from the South Pole. Do this by first finding all (exact) polar angles when the rock is 1000 miles from the South Pole.*

(f) *Find the (exact) first two times when the rock is 1000 miles from the South Pole. Then use a calculator to estimate these clock times to the nearest minute*

(g) *Find all (exact) times between 8pm and 12pm, July 1, 2085 when the rock is 1000 miles from the South Pole. Then use a calculator to estimate these clock times to the nearest minute*

Question 345 *In ΔMAT, angle $\angle MAT$ has measure $\pi/3$ radians. Sides AM and AT have respective lengths 2 and 5 inches. Determine the exact measure of angle $\angle AMT$. Do* not *use a calculator.*

Question 346 *You measure the angle of eleveation to the top of a tree to be θ_1 radians. You then walk an additional c feet directly away from the tree and measure the angle of elevation to be θ_2 radians.*

(a) *Express the height of the tree above eye level in terms of θ_1, θ_2, and c.*

(b) *Check that your expression has the correct units.*

Question 347 *A tree leans precariously with its trunk inclined at an angle of $\pi/3$ radians to the ground. The top of a seven-foot ladder rests against the tree and the bottom of the ladder lies eight feet from the tree's base. How far is the top of the ladder from the base of the tree? Do not use a calculator except for arithmetic. Do* not *rely on the worksheet below.*

Desmos link: `https://www.desmos.com/calculator/sjmjseyqyp`

Desmos activity available at 142: Ladder and Tree 45

Quiz Solutions

Solutions to daily quizzes.

1 Quiz 5A

Question 348 *The function $G = f(s)$, $0 \leq s \leq 80$, expresses the number of gallons of gas in a car in terms of the trip odometer reading (measured in miles).*

(a) *Use the Leibniz evaluation notation to write an expression for the derivative of G with respect to s evaluated at an odometer reading of 10 miles.*

(b) *Now suppose*

$$G = f(s) = \frac{1}{2000}\left(s^2 - 200s + 10000\right), \; 0 \leq s \leq 80.$$

(i) *Evaluate the derivative in part (a). Include units. Interpret the meaning of the derivative in a way that someone without any knowledge of caclulus would understand. In particular, do* not *use the phrasing* rate of change of ... with respect to ....

(ii) *At what odometer reading(s), if any, is the (instantaneous) gas mileage 17 miles/gal? Explain your reasoning in a few complete sentences. End with a concluding sentence.*

Explanation. (a) The derivative is

$$\left.\frac{dG}{ds}\right|_{s=10}.$$

(b) Differentiating, we get

$$\begin{aligned}\frac{dG}{ds} &= \frac{d}{ds}\left(\frac{1}{2000}\left(s^2 - 200s + 10000\right)\right)\\ &= \frac{1}{2000}\left(\frac{d}{ds}\left(s^2 - 200s + 10000\right)\right)\\ &= \frac{2s - 200}{2000}\\ &= \frac{s - 100}{1000}.\end{aligned}$$

Learning outcomes:
Author(s):

So

$$\left.\frac{dG}{ds}\right|_{s=10} = \left.\frac{s-100}{1000}\right|_{s=10} = \frac{10-100}{1000} = -0.09\frac{\text{gal}}{\text{mile}}$$

This means that at an odometer reading of 10 miles, the car is burning gas at the rate of 0.09 gal/mile. The derivative is negative because as the odometer reading increases the volume of gas decreases.

Alternatively, we could say that as you drive one mile, between odoemeter readings $s = 10$ and $s = 11$, the car burns *approximately* 0.09 gallons of gas.

(c) When the car is getting 17 miles/gal, it is buring gas at the rate of (1/17) gal/mile and

$$\frac{dG}{ds} = \frac{s-100}{1000} = -\frac{1}{17}.$$

Solving this equation for s gives

$$17s - 1700 = -1000$$

and

$$s = \frac{700}{17}.$$

So when the car is getting 17 miles/gal, the odometer reads $s = 700/17$ miles.

Introduction to Integration, Part 1

Introduction to integral calculus.

The *differential* in Differential Calculus has the same root as *difference*, and *calculus* the same root as *compute*. Differential calculus is all about *computing differences.* Expressed more simply, it is about *subtraction.*

Integral Calculus, on the other hand, is about addition. *Integral* shares the same root as *integer* and comes from the Latin *integrare*, to make whole.

Addition and subtraction are inverse operations. They undo each other. Similarly, differentiation undoes integration. But we need to be more precise to claim that integration undoes differentiation. More about this later.

It would be a mistake to think integral calculus is primarily about finding anti-derivatives. While anti-differentiation plays a role in integration, this class is more about recongizing *when* to integrate rather than *how* to integrate. Many of your STEM classes use definite integrals, with an emphasis on setting up and interpreting integrals, not computing their values.

It would also be a mistake to think this class is about using definite integrals to compute *numbers*, like the distance travelled over a certain time interval or the period of a pendulum. Rather, the emphasis here will be on using definite integrals to create *functions.* Expressing the position of a pendulum as a function of time, for example, gives a lot more information that just computing its period.

Using definite integrals to write functions generalizes the point-slope equation of a line. You used this equation in differential calculus is to approximate a function in the neighborhood of a point. To capture the local nature of the approximation (ie. the approximation is usually accurate in just a small neighborhood of the point of tangency), it's the point-slope equation, *not* slope-intercept, that works best. Here's an example that illustrates this idea.

Example 46. Between 12:03pm and 1:00pm a balloon ascends at a constant rate of 73 ft/min. The balloon is 1900 feet high at 12:10pm.

(a) Find an expression for the function

$$h = g(t)\ ,\ 0 \le t \le 60,$$

that expresses the balloon's height (measured in feet) in terms of the number of minutes past noon.

Learning outcomes:
Author(s):

(b) Explain the logic behind your expression.

Explanation. The function is

$$h = g(t) = 1900 + 73(t - 10)\,,\ 0 \leq t \leq 60.$$

The logic behind the expression is this. To find the height at time t minutes past noon, we start with the height (1900 feet) at 12:10pm and add the change in height from 12:10pm to time t minutes past noon. The change in height,

$$\Delta h = 73(t - 10)$$

is the product of the rate of ascent (73 ft/min) and the number of minutes

$$\Delta t = t - 10$$

from 12:10pm to time t minutes past noon.

In the context of the last example, the main topic for this class is how to compute the height function *if the rate of ascent is not constant.*

The logic behind the solution is almost identical to point-slope. Supppose we're given a function

$$r = f(t)\ ,\ 0 \leq t \leq 60,$$

that gives the balloon's rate of ascent (in ft/min) at a general time t minutes past noon. To find the height at time t, we start with the height of the balloon at 12:10 pm and add to that the change in height from 12:10pm to time t. But because the rate of ascent is not necessarily constant, we'll almost certainly have (infinitely) many rates. Our job is to use these rates to compute, or at least approximate the change in height. The next example illustrates this idea.

Example 47. Suppose we know the balloon is 1900 feet high at 12:10pm and that we're given a graph of the function

$$r = f(t)\,,\ 0 \leq t \leq 60,$$

(assumed continuous) that expresses its rate of ascent (measured in ft/min) in terms of the number of minutes past noon. This graph is shown below.

Desmos link: `https://www.desmos.com/calculator/h6cworakdw`

152:Balloon Introduction 2

Our problem is to approximate the balloon's height at 12:50pm.

We'll start by computing a rough approximation of the change in height from 12:10pm and 12:50pm. For this, we'll partition the 40-minute interval into five equal subintervals of length

$$\Delta t = \frac{40 \text{ min}}{5} = 8 \text{ min}.$$

Now we'll suppose that during each subinterval the balloon ascends at a constant rate. While this is not true, the rate certainly varies less over an 8-minute interval than it does over the entire 40-minute interval. So we should get a better appoximation this way than if we assumed just one constant rate of ascent between 12:10pm and 12:50pm.

What rate we choose for each 8-minute interval doesn't really matter as long as we choose the actual rate of ascent at some time during that interval. To make things simple, we'll choose the rate at the start of each subinterval.

Then between 12:10pm and 12:18pm, the approximate change in height (measured in feet) is

$$f(10)(8) \sim (31 \text{ ft/min})(8 \text{ min}) = 248 \text{ ft}.$$

Between 12:18pm and 12:26pm, the approximate change in height (measured in feet) is

$$f(18)(8) \sim (12 \text{ ft/min})(8 \text{ min}) = 96 \text{ ft}.$$

And we continue similarly to approximate the changes in height (this time negative) over the remaining three 8-minute intervals. So from 12:10pm to 12:50pm the approximate change in height (in feet) is

$$\begin{aligned} \Delta h_5 &= h - 1900 \\ &\sim (f(10) + f(18) + f(26) + f(34) + f(42))(8) \\ &\sim -448. \end{aligned}$$

Verify this computation by typing this expression in Line 38 of the worksheet above.

This sum is on Line 7 of the worksheet.

To see the previous computation geometrically, move the slider a (the start of our time interval) in Line 2 above to $a = 10$ and the slider b (the end of our time interval) in Line 3 to $b = 50$. Then the signed area of a shaded retangle measures the approximate change in height during that subintevral. And the approximate change in height from 12:10pm to 12:50pm is just the sum of these signed areas. The sign of the area and the rate of ascent are the same. So rectangles below the horizontal axis contribute a negative signed area (ie. a decrease in height), while those above make a positive contribution (ie. an increase in height).

Our first approximation to the balloon's height at 12:50pm is then

$$(1900 - 448) \text{ ft} \sim 1452 \text{ ft}.$$

To get a better approximation, we partition the 40-minute interval from 12:10pm to 12:50pm into more subintervals of equal length. To do this, drag the slider n on Line 5 (the number of equal subintervals) to the right.

(a) Describe what happens to the approximate change in height (s_1 in Line 5) and how the picture changes as n grows without bound.

(b) Then approximate the exact change in height and interpret that change geometrically.

Free Response:

Takeaways:

Here's some notation along with a summary of the main idea for the last example.

The exact height of the balloon at 12:50pm is

$$1900 + \int_{10}^{50} f(t)\, dt.$$

This expression starts with the height at 12:10pm (1900 feet) and adds to it the change in height

$$\int_{10}^{50} f(t)\, dt$$

from 12:10pm to 12:50pm. The integral sign $\int$ stands for the letter S and means *sum*. You can think of the product

$$f(t)\, dt$$

as a small change in height. It multiplies the rate of ascent $f(t)$ and a small (differential) time interval dt to give a small (differential) change in height. The integral then sums these changes to get the exact change in height.

More generally, the height of the balloon as a function of time (measured in minutes past noon) is

$$h = f(t) = 1900 + \int_{10}^{t} f(u)\, du$$

and the logic is the same.

Compare this with the expression (point-slope) from Example 1 for the height

$$h = 1900 + 73(t - 10).$$

Free Response:

Introduction to Integration, Part 2

Introduction to numerical integration.

This chapter is a continuation of the introduction where we saw how to approximate a balloon's change in height over an interval of time with a sum of small approximate changes. We develop that theme here with summation notation and take advantage of technology to get accurate approximations.

But first a question for review.

1 Review

Example 48. Between 12:14pm and 1:00pm the temperature of a beaker of water increases at a constant rate of $\dfrac{33}{13}^{\circ}$ C/sec. The temperature is 23°C at 12:23pm.

(a) Find an expression for the function

$$C = f(t)\ ,\ 14 \le t \le 60,$$

that expresses the temperature of the water (in Celsius degrees) in terms of the number of minutes past noon.

(b) Explain the logic behind your expression.

Free Response:

2 Summation Notation and Technology

Back to the problem from the introduction. But first a few conceptual questions.

Example 49. The function

$$r = f(t)\,,\, 0 \le t \le 60,$$

expresses a balloon's rate of ascent (measured in ft/min) in terms of the number of minutes past noon. Its graph is shown below.

Learning outcomes:
Author(s):

Desmos link: `https://www.desmos.com/calculator/tgi5yiuzab`

152:Balloon Introduction

(a) When is the balloon ascending? Descending? Give all times for each and explain your reasoning.

(b) Do you think the balloon is higher at noon or at 1:00pm? Explain your reasoning. Try to prove your assertion using only arithmetic and simple reasoning.

Free Response:

Now suppose the balloon is 1900 feet above the ground at 12:10pm.

(a) Use the graph to approximate the balloon's height 30 seconds later. Explain your reasoning.

(b) Is your estimate greater or less than the actual height? Explain your reasoning.

Free Response:

We can turn the previous question of differential calculus into an addition problem of integral calculus by estimating the balloon's height at a time farther removed from 12:10.

Example 50. We'll assume as before that the balloon is 1900 feet above the ground at 12:10pm, and we'll approximate its height at 12:50pm.

Desmos link: `https://www.desmos.com/calculator/h6cworakdw`

152:Balloon Introduction 2

We'll start by computing a rough approximation of the change in height between 12:10pm and 12:50pm. For this, we'll partition the 40-minute time interval into five equal subintervals and suppose that during each subinterval the balloon ascends at a constant rate.

What rate we choose for each interval doesn't really matter as long as we choose the actual rate of ascent at some time during that interval. To make things simple, we'll choose the rate at the start of each subinterval as illustrated in the worksheet above.

(a) Use summation notation to express the approximate change in height from 12:10pm to 12:50pm. Type this expression in Line 5 of the worksheet above. Compare this value with your estimate from part (a).

Using the rate of ascent at the start of each 8-minute time interval, the approximate change in height from 12:10pm to 12:50pm is

$$\sum_{i=0}^{4} 8f(10+8i)$$

and the approximate height at 12:50pm is

$$1900 + \sum_{i=0}^{4} 8f(10+8i).$$

(b) Repeat part (a) using the rate of ascent at the *end* of each subinterval instead. Turn off the folder in Line 11 and activate the folder in Line 18 above. As a check, type your expression for the approximate height in Line 5 above.

The approximate change in height from 12:10pm to 12:50pm is

$$\sum_{i=1}^{5} 8f(10+8i)$$

and the approximate height at 12:50pm is

$$1900 + \sum_{i=1}^{5} 8f(10+8i).$$

(c) Repeat part (a)using the rate of ascent at the *midpoint* of each subinterval instead. Turn off the folder in Line 18 and activate the folder in Line 24 above. As a check, type your expression for the approximate height in Line 5 above.

The approximate change in height from 12:10pm to 12:50pm is

$$\sum_{i=0}^{4} 8f(14+8i)$$

and the approximate height at 12:50pm is

$$1900 + \sum_{i=0}^{4} 8f(14+8i).$$

(d) Compare your three estimates for the change in height. Which do you think is most accurate? Explain why.

Example 51. (a) Drag the slider in Line 4 of Example 3 to $n = 20$ and repeat parts (a) - (c) of Example 4 with $n = 20$ equal subintervals.

(b) Try parts (c) - (e) of Example 3 in general, with n-equal subintervals.

Using left endpoints, the approximate height at 12:50pm is

$$f(50) \sim 1900 + \frac{40}{n} \sum_{i=0}^{n-1} f\left(10 + \frac{40}{n} i\right).$$

With right endpoints,

$$f(50) \sim 1900 + \frac{40}{n} \sum_{i=1}^{n} f\left(10 + \frac{40}{n} i\right).$$

And with midpoints,

$$f(50) \sim 1900 + \frac{40}{n} \sum_{i=0}^{n-1} f\left(10 + \frac{20}{n} + \frac{40}{n} i\right).$$

Addition

When do we need to add?

1 Two Balloon Problems

Exercise 349 (a) *The table below shows the height of a balloon at several times.*

Time	Height (feet)	Average Rate of Ascent (ft/min)
12:30	420	——-
12:40	500	
12:44	550	
12:50	400	

Find the balloon's average rate of ascent over each of the three time intervals between consecutive times.

(b) *The table below shows the rate of ascent (in ft/min) of another balloon at several times.*

Time	Rate of ascent (ft/min)	Approximate height (ft)
12:30	40	
12:40	25	1800
12:44	-30	
12:50	-50	

The balloon is 1800 *feet high at 12:40pm.*

Estimate the heights of the balloon at 12:30pm, 12:44pm, and 12:50pm. Do this three times. First assume the balloon ascends at a constant rate over each time inerval equal to the rate of ascent at the start of each interval. Then at a constant rate equal to the rate at the end of each interval. And finally at a constant rate equal to the average of the rates at the endpoints of each interval.

(c) *Compare your computations in parts (a) and (b). How are they related?*

Learning outcomes:
Author(s):

2 Time, Distance, and Mileage Problems

Exercise 350 *The table below shows your speed at several times as you drive north on I-5 on part of a two-hour trip.*

Time	*Speed (miles/hour)*
12:30	*42*
12:40	*50*
12:44	*55*
12:50	*40*

The trip odometer reading is 110 *miles at 12:30pm.*

(a) *Use the data to give two approximations to the trip odometer reading at 12:50pm. Once assuming a constant speed over each subinterval equal to the speed at the start of the interval. And again assuming a constant speed over each subinterval equal to the speed at the end of the interval. Explain your reasoning.*

(b) *Activate the folder in Line 5 of the worksheet below and explain how what you see is related to part (a). Then turn the folder off and activate the folder in Line 11. Explain.*

(c) *To practice summation notation, let* t_i, $i = 1, 2, 3, 4$ *be the times, in order, in the above table measured in hours past 12:30pm and let* v_i *be the correponding speeds. Then using the speed at start of each subinterval , the trip odometer reading at 12:50pm is approximately*

$$110 + \sum_{i=1}^{4} (t_{i+1} - t_i) v_i.$$

Using the speed at the end of each subinterval, the odometer reading is approximately

$$110 + \sum_{i=1}^{4} (t_{i+1} - t_i) v_{i+1}.$$

(d) *Input the sums from part (c) in Lines 16 and 18 of the worksheet below to check your arithmetic. You'll need to write* $t[i]$ *in place of* t_i *and so on.*

Desmos link: `https://www.desmos.com/calculator/jjobuctzsa`

152: Distance Travelled

Exercise 351 *The table below shows your gas mileage on a car trip as you pass different exits while driving north on I-5. The exit numbers record the distance to the Columbia river along I-5, measured in miles.*

Exit Number	Mileage (miles/gallon)
176	40
200	32
210	30
248	25

(a) *Is this enough information to compute upper and lower bounds for the number of gallons of gas your car burned between Exits 176 and 248? If so, explain why. If not, what other information do you need?*

(b) *Using the given information and your additional hypothesis from part (a), if needed, find the best possible upper and lower and upper bounds for the number of gallons of your car burned between Exits 176 and 248.*

Desmos link: `https://www.desmos.com/calculator/csjpd0u3sw`

152: Gas Consumed 34

(c) *Let e_i and g_i, $i = 1, 2, 3, 4$ denote the respective exit numbers and gas mileages in the table above. Use summation notation to write your estimates in part (b) above.*

Assuming the gas mileage is a decreasing function, the number of gallons of gas burned between Exits 176 and 248 is at least

$$\sum_{i=1}^{3} (e_{i+1} - e_i) \left(\frac{1}{g_i} \right).$$

This is the best possible (ie. greatest possible) lower bound.

With the same assumption, the number of gallons of gas burned is at most

$$\sum_{i=1}^{3} (e_{i+1} - e_i) \left(\frac{1}{g_{i+1}} \right).$$

This is the best possible (ie. least possible) upper bound.

(d) *Use the desmos worksheet above to represent the sums in parts (a) and (b) geometrically by activating the appropriate folder in Line 5 (Left endpoints) or Line 11 (Right endpoints).*

Explanation

(a) *The table does not give enough information to determine such bounds because we don't know how the gas mileage varies during each interval. But with the additional assumption that the gas mileage is a decreasing function of the exit number we can compute these bounds.*

(b) *The key, as always in this class, is to consider the constant case. Here that means to assume the gas mileage is constant, say 40 miles/gallon. Then over a 200-mile trip, for example, your car would burn*

$$\frac{200 \text{ miles}}{40 \text{ miles/gal}} = 5 \text{ gallons}$$

of gas.

The car burns less gas when its gas mileage is greater. So for the greatest possible lower bound (ie. to find the minimum number of gallons of gas the car burned), we assume the gas mileage is the greatest possible during each interval. Here that means choosing left endpoints. That is, during each interval we assume the gas mileage is constant and equal to the gas mileage at the start of the interval (ie. equal to its maximum over the interval).

So between Exits 176 and 248 the car burns at least

$$\frac{(200-176) \text{ miles}}{40 \text{ miles/gal}} + \frac{(210-200) \text{ miles}}{32 \text{ miles/gal}} + \frac{(248-210) \text{ miles}}{30 \text{ miles/gal}} > 2.179 \text{ gallons.}$$

Similar reasoning shows that between Exits 176 and 248 the car burns at most

$$\frac{(200-176) \text{ miles}}{32 \text{ miles/gal}} + \frac{(210-200) \text{ miles}}{30 \text{ miles/gal}} + \frac{(248-210) \text{ miles}}{25 \text{ miles/gal}} < 2.604 \text{ gallons.}$$

Exercise 352 *The table below shows your speed and gas mileage at several times during the course of a two-hour trip.*

Time	*Speed (miles/hr)*	*Mileage (miles/gal)*
1:00pm	*40*	*30*
1:45pm	*48*	*36*
2:15pm	*60*	*40*
3:00pm	*70*	*28*

(a) *Give two estimates for the distance travelled during the trip. Explain your reasoning.*

(b) *Give two estimates for the number of gallons of gas your car burned during the trip. Explain your reasoning.*

3 On Trails

Exercise 353 *The table below shows the inclination angle of the Mt. Washington Cog Railway at several distances from Waumbek Station (the departure point, elevation 4000 feet) on its way to the summit (elevation 6288 feet).*

Distance (miles)	Inclination Angle (radians)
0.5	0.25
1.0	0.30
1.5	0.32
2.0	0.38

(a) *Use the data to find two estimates for the elevation change from mile marker* 0.5 *to mile marker* 2.0*. Explain your reasoning.*

(b) *Let s_i and ϕ_i denote the respective diestances and angles in the table above. Use summation notation to write the steimate in part (a)*

Assuming the steepness of the track increases, a lower bound for the change in height (in miles) is

$$h_1 = \sum_{i=1}^{3} (s_{i+1} - h_i)(\sin \phi_i).$$

An upper bound is

$$s_1 = \sum_{i=1}^{3} (h_{i+1} - h_i)(\sin \phi_{i+1}).$$

(c) *Input your expressions for the lower and upper bounds in the worksheet below.*

Desmos link: `https://www.desmos.com/calculator/ehnbprdu6q`

152: Cog Railroad

Exercise 354 *The table below shows the slope along a section of a trail at several altitudes.*

Altitude (feet)	Slope (ft/ft)
1000	0.05
1200	0.10
1350	0.12
1500	0.15

(a) *Use the data to find two estimates for the length of the trail between altitudes 1000 ft and 1500 feet. Explain your reasoning.*

(b) *Let h_i and m_i, $i = 1, 2, 3, 4$ denote the respective altitudes and slopes in the table above. Use summation notation to write your estimates in part (a).*

Assuming the steepness of the trail increases between altitudes 1000 feet and 1500 feet, a lower bound for the trail's length (in feet) is

$$s_1 = \sum_{i=1}^{3}(h_{i+1} - h_i)\left(\sqrt{1 + (1/m_{i+1})^2}\right).$$

An upper bound is

$$s_1 = \sum_{i=1}^{3}(h_{i+1} - h_i)\left(\sqrt{1 + (1/m_i)^2}\right).$$

(c) *Input your expressions for the lower and upper bounds in the worksheet below.*

Desmos link: `https://www.desmos.com/calculator/mmi4rx2rqp`

152: Length of Trail

4 A Rollercoaster Problem

Exercise 355 *The table below shows several speeds of a roller coaster as it slides down a cicular loop-the-loop with a radius of 80 feet. The bottom of the loop is 20 feet above the ground.*

Height above ground	*Speed (ft/sec)*
100	*40*
80	*48*
50	*64*
20	*80*

(a) *Is this enough information to compute upper and lower bounds for the time it takes for the roller coaster to fall from a height of 100 feet to a height of 20 feet? If so, explain why. If not, what other information do you need? Do not assume any particular values for the speed at other heights.*

(b) *Using the given information and your additional hypothesis from part (a) if needed, find the best possible upper and lower and upper bounds for the time it takes for the roller coaster to fall from a height of 100 feet to a height of 20 feet. Do not use a calculator for this part. But your bounds should be in a form that could be input directly into a calculator.*

(c) *Use a calculator to find approximations to your bounds from part (b). Then give your best estimate of the time together with bounds for the error in your estimate.*

Explain your reasoning throughly in the form of a letter to a friend with no knowledge of calculus or the ideas of Section 5.1. Include units in all steps of your computations. Be careful. In particular, take note that the roller coaster moves along a circular arc. It does not fall straight down.

Exercise 356 *Assume for this problem that each month has 30 days and that the number of hours of daylight/day in Seattle throughout the course of a year is a sinusoidal function of time. Suppose also that Seattle receives a maximum of* 16 *hours of daylight/day on June 21 and a minimum of* 8 *hours of daylight/day on December 21.*

(a) *Find an expression for a function*

$$h = f(t)\,,\ 0 \le t \le 360,$$

that expresses the number of hours of daylight/day in terms of the number of days since June 21. Use the cosine *function.*

The function is

$$h = f(t) = 12 + 4\cos\left(\frac{\pi}{180}t\right).$$

(b) *Use summation notation to write an expression for the total number of hours of daylight Seattle gets during the summer.*

The total number of hours of daylight during the summer is

$$\sum_{i=0}^{89} f(i).$$

(c) *Enter the sum in Line 1 of the worksheet below.*

Desmos link: `https://www.desmos.com/calculator/jvwyqn0vbj`

152: Hour of Daylight 1

(d) *Compute the average number of hours of daylight/day in Seattle during the summer?*

(e) *Is the number of hours of daylight/day in Seattle increasing or decreasing on March 21st? At what rate?*

(f) *Does the previous question make sense? Explain.*

5 The Mercator Map

Look at a map of Seattle and you'll see a scale, probably near the bottom. It might be 0.5 miles/inch, or something like that. But you will not see a scale like this on a map of the earth for the simple reason that one does not exist. The scale varies from place to place and is usually a function of latitude. Even at a specific location, there is often not one but infinitely many scaling factors that depend on direction.

But for some maps, the *conformal* ones, the scaling factor is independent of direction and depends only on latitude. The Mercator map is one of these maps. The next few exercises are about this map.

Exercise 357 *The worksheet below shows a Mercator map of the earth. The evenly-spaced vertical lines are the meridians (half-circles running between the north and south poles) and divide the earth into its 24 time zones. The horizontal lines are the parallels (circles of latitude). These are unevenly spaced.*

We'll suppose the map is $2\pi \sim 6.3$ *feet wide and that the earth is a perfect sphere of radius* 3960 *miles. The scale along the equator is then*

$$\frac{2\pi(3960\ miles)}{2\pi\ ft} = 3960\frac{miles}{ft}$$

So the 24 *meridians are spaced at intervals of*

$$\frac{2\pi\ ft}{24} = \frac{\pi}{12}\ ft$$

on the map. And along the equator of the earth, they are spaced at intervals of

$$\frac{2\pi(3960)\ miles}{24} = 330\pi\ miles \sim 1037\ miles.$$

Desmos link: `https://www.desmos.com/calculator/cr8vg7urnc`

152: Mercator Map Riemann Sum 2

Segment AB indicates the map scale. It represents a length of $330\pi \sim 1037$ *miles along any parallel. Along the equator AB has length* $\pi/12 \sim 0.26$ *feet. But drag point A vertically and you'll see how the scale varies with latitude as the length of AB changes.*

Integrating Power Functions

Power functions.

1 Two Balloon Problems

Learning outcomes:
Author(s):

Riemann Sums

Riemann sums and summation notation.

Question 358 *The function*

$$v = g(s) = 40 + \frac{s}{5}\,,\ 20 \le s \le 100,$$

expresses the speed (in miles/hr) of a car in terms of the trip odometer reading (measured in miles) during part of a 200-mile trip.

(a) *Use summation notation for a Riemann sum with* 10 *equal intervals of distance to find an upper bound for the time it takes to travel the* 80 *miles in this portion of the trip. Enter this in Line 1 of the worksheet below to get an approximation for the time. Activate the appropriate folder in Line 12 (Left endpoints) or Line 19 (Right Endpoints) to interpret the sum geometrically.*

(b) *Use summation notation for a Riemann sum with* 10 *equal intervals of distance to find a lower bound for the time it takes to travel the* 80 *miles in this portion of the trip. Enter this in Line 1 of the worksheet below to get an approximation for the time. Activate the appropriate folder in Line 12 (Left endpoints) or Line 19 (Right Endpoints) to interpret the sum geometrically.*

(c) *Use summation notation for a Riemann sum with n equal intervals of distance to find an upper bound for the time it takes to travel the* 80 *miles in this portion of the trip. Enter this in Line 1 below. Then drag the slider n in Line 2 to get a more accurate upper bound.*

(d) *Use summation notation for a Riemann sum with n equal intervals of distance to find a lower bound for the time it takes to travel the* 80 *miles in this portion of the trip. Enter this in Line 1 below. Then drag the slider n in Line 2 to get a more accurate upper bound.*

(e) *Use either part (c) or part (d) to write an expression (as a limit) for the* exact *time it takes to travel the* 80 *miles.*

(f) *Write a definite integral for the time it takes to travel the* 80 *miles and enter this on Line 39 of the worksheet below.*

Learning outcomes:
Author(s):

Desmos link: https://www.desmos.com/calculator/k10phbukon

152: Distance Speed 2

Question 359 *The function*

$$r = f(t) = 70\cos^3\left(\sqrt{t+40}\right),\ 0 \le t \le 48,$$

expresses a balloon's rate of ascent (in ft/min) in terms of the number of minutes past noon. The balloon was 300 feet high at 12:04pm.

(a) *Use summation notation for an expression with* 10 *equal time intervals to find a lower bound for the balloon's height at 12:44pm. Enter this expression in Line 1 of the worksheet below to get an approximation for the height. Activate the appropriate folder in Line 12 (Left endpoints) or Line 19 (Right Endpoints) to interpret the sum geometrically.*

(b) *Use summation notation for an expression with* 10 *equal time intervals to find an upper bound for the balloon's height at 12:44pm. Enter this expression in Line 1 of the worksheet below to get an approximation for the height. Activate the appropriate folder in Line 12 (Left endpoints) or Line 19 (Right Endpoints) to interpret the sum geometrically.*

(c) *Use summation notation for an expression with n equal time intervals to find a lower bound for the balloon's height at 12:44pm. Enter this expression in Line 1 of the worksheet below to get an approximation for the height. Enter this in Line 1 below. Then drag the slider n in Line 2 to get a more accurate upper bound.*

(d) *Use summation notation for an expression with n equal time intervals to find an upper bound for the balloon's height at 12:44pm. Enter this expression in Line 1 of the worksheet below to get an approximation for the height. Enter this in Line 1 below. Then drag the slider n in Line 2 to get a more accurate upper bound.*

(e) *Use either part (c) or part (d) to write an expression (as a limit) for the* exact *height at 12:44pm.*

(f) *Write an expression with a definite integral for the height at 12:44pm and enter this on Line 39 of the worksheet below.*

Desmos link: https://www.desmos.com/calculator/qwuqprd9vf

152: Balloon 56

Question 360 *The function*

$$\theta = g(s) = \frac{s}{2} - \frac{s^3}{12}, \; 0 \le s \le 3,$$

expresses the inclination angle (in radians) of the trail to Nada Lake in terms of the distance from the trailhead (measured along the trail in miles). The inclination angle is the angle the trail makes with the horizontal. It is positive (negative) when the trail slopes upward (downward) in the direction away from the trailhead.

The trail is at an elevation of 3200 *feet* 0.5 *miles from the trailhead.*

(a) *Use summation notation for an expression with* 10 *equal intervals of distance to find a lower bound for the trail's elevation (in feet) at Nada Lake (3 miles from the trailhead). Enter this in Line 1 of the worksheet below to get an approximation for the elevation. Activate the appropriate folder in Line 12 (Left endpoints) or Line 19 (Right Endpoints) to interpret the sum geometrically.*

(b) *Use summation notation for an expression with* 10 *equal intervals of distance to find an upper bound for the trail's elevation (in feet) at Nada Lake. Enter this in Line 1 of the worksheet below to get an approximation for the elevation. Activate the appropriate folder in Line 12 (Left endpoints) or Line 19 (Right Endpoints) to interpret the sum geometrically.*

(c) *Use summation notation for an expression with* n *equal intervals of distance to find an upper bound for the trail's elevation (in feet) at Nada Lake. Enter this in Line 1 below. Then drag the slider* n *in Line 2 to get a more accurate upper bound.*

(d) *Use summation notation for an expression with* n *equal intervals of distance to find a lower bound for the trail's elevation (in feet) at Nada Lake. Enter this in Line 1 below. Then drag the slider* n *in Line 2 to get a more accurate upper bound.*

(e) *Use either part (c) or part (d) to write an expression (as a limit) for the exact elevation of Nada Lake.*

(f) *Write an expression with a definite integral for the elevation of Nada Lake.*

Desmos link: `https://www.desmos.com/calculator/qhyco8cuou`

152: Inclination Angle 34

Question 361 *Repeat Question 3 with the function*

$$m = g(s) = \frac{s}{2} - \frac{s^3}{12}\,,\ 0 \leq s \leq 3,$$

that expresses the slope of a trail in terms of the distance from the trailhead (measured along the trail in miles). The slope is measured with respect to the horizontal. It is positive (negative) when the trail slopes upward (downward) in the direction away from the trailhead.

Question 362 *The function*

$$v = g(t) = 20 + 40t - 3t^2\,,\ 0 \leq t \leq 1,$$

expresses the speed of a car (in miles/hr) in terms of the number of hours past noon.

The function

$$G = h(t) = 30 + 10t\,,\ 0 \leq t \leq 1$$

expresses the gas mileage (in miles/gal) of that same car in terms of the number of hours past noon.

The car has 10 gallons of gas in its tank at 12:15pm.

Answer the six parts of the previous questions in regard to approximating the number of gallons of gas in the tank at 1:00pm.

Geometric Integration

Evaluating or approximating definite integrals geometrically.

Directions: Do *not* use the Fundamental Theorem of Calculus for these problems. Instead evaluate an integral geometrically or approximate an integral with a Riemann sum with 100 equally-spaced intervals.

Question 363 *For each definite integral, draw a graph of the appropriate function and shade the region of integration. Then use geometry to evaluate the integral.*

(a) $\displaystyle\int_3^8 12\,du$

(b) $\displaystyle\int_8^3 12\,dt$

(c) $\displaystyle\int_a^b k\,dx,$ where a, b, k are constants

(d) $\displaystyle\int_4^9 \frac{w}{2}\,dw$

(e) $\displaystyle\int_a^b \frac{z}{k}\,dz,$where a, b, k are constants

(f) $\displaystyle\int_{-10}^2 (4 - |x+3|)\,dx$

(g) $\displaystyle\int_0^4 \sqrt{25 - x^2}\,dx$

(h) $\displaystyle\int_{-5}^5 \left(2 - \sqrt{25 - x^2}\right)\,dx$

Question 364 *Make up a question in an applied scenario whose answer is the integral in Question 1(b) above and explain why the answer makes sense.*

Learning outcomes:
Author(s):

Question 365 *The function*

$$r = f(t)\,,\ 0 \le t \le 60,$$

expresses a balloon's rate of ascent (measured in ft/min) in terms of the number of minutes past noon. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/tgi5yiuzab`

152:Balloon Introduction

The balloon is 1900 *feet high at 12:10pm.*

(a) *Write an expression for the balloon's height at 12:40pm.Then use the graph to approximate this height.*

(b) *Write an expression for the balloon's height at 12:04pm.Then use the graph to approximate this height.*

(c) *Find an expression for the function*

$$h = g(t)\,,\ 0 \le t \le 60,$$

that gives the balloon's height at time t minutes past noon.

Question 366 *The function*

$$r = f(t)\,,\ 0 \le t \le 60,$$

expresses a balloon's rate of ascent (measured in ft/min) in terms of the number of minutes past noon. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/vfk9toeayo`

152:Balloon 53

The balloon is 3200 *feet high at 12:40pm.*

(a) *Ignore the graph and write an expression with a for the balloon's height at 12:30pm.Then use the graph to find this height.*

(b) *Ignore the graph and write an expression for the balloon's height at 12:10pm.Then use the graph to find this height.*

(c) *When is the balloon the same height as it is at 12:09pm?*

(d) *When is the balloon* 20 *feet higher than it is at 12:18pm?*

(e) *Ignore the graph and find an expression for the function*

$$h = g(t)\,,\; 0 \leq t \leq 60,$$

that gives the balloon's height at time t minutes past noon. Then use the graph and find an explicit expression for this piece-wise defined function.

The Fundamental Theorem of Calculus, Part 2

FTC

Question 367 *The function*

$$v = f(t)\,,\ 0 \le t \le 8,$$

expresses the speed (in feet/min) of a beetle in terms of the number of minutes past noon.

(a) *Write an expression for the distance the beetle crawls from 12:03pm to 12:07pm. Use the graph of the function $v = f(t)$ below to approximate this distance.*

(b) *Write an equation you would solve to determine when the beetle is halfway to its destination. Start by defining an unknown (in a complete sentence, with units). Do not use t, it is already taken.*

(c) *Use the graph of the function $v = f(t)$ below to approximate the time in part (b). But first determine whether this time is before or after 12:04pm.*

Desmos link: `https://www.desmos.com/calculator/gd54bqgk6l`

152:Beetle 1

(d) *Suppose*

$$v(t) = \frac{20}{(t+2)^2}.$$

and determine the exact time when the beetle is halfway to its destination.

Question 368 *The function*

$$r = f(t)\,,\ 2 \le t \le 36,$$

gives the net rate (in gal/min) at which water flows into a tank at time t minutes past noon.

Learning outcomes:
Author(s):

(a) *Write an expression that gives the change in volume from 12:15pm to 12:32pm.*

(b) *Write an equation you would solve to find when the tank holds* 22 *fewer gallons than it does at 12:09pm. Start by defining an unknown (in a complete sentence, with units). Do not use* t*, it is already taken.*

(c) *Use the graph of the function*

$$r = f(t)\,, \, 2 \leq t \leq 36,$$

shown below to approximate the time(s) when the tank holds 22 fewer gallons than it does at 12:09.

Desmos link: `https://www.desmos.com/calculator/rcsfzdhsax`

152:Tank 1

(d) *Suppose*

$$f(t) = 12 - 3\sqrt{t}, 2 \leq t \leq 36.$$

Use calculus and algebra to find the exact time(s) in part (c). Then use a calculator to approximate these times to the nearest second.

Question 369 *The function*

$$m = f(s)\,, \, 30 \leq s \leq 300$$

expresses your gas mileage (in miles/gallon) in terms of your distance from home. The distance is measured along your route in miles. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/v34aycmz93`

152:Mileage 89

Your tank has 12 *gallons of gas when you are* 50 *miles from home.*

(a) *Suppose you are driving away from home.*

(i) *Use the graph above to approximate the number of gallons of gas in your tank when you are* 150 *miles from home.*

(ii) *Find a function* $G_1 = g_1(s)$ *that expresses the number of gallons in your tank in terms of your distance (measured along your route in miles) from home.*

(b) *Suppose you are driving home.*

(i) *Use the graph above to approximate the number of gallons of gas in your tank when you are 150 miles from home.*

(ii) *Find a function $G_2 = g_2(s)$ that expresses the number of gallons in your tank in terms of your distance (measured along your route in miles) from home.*

Question 370 *The function*

$$r = f(t) = 100/t\,,\ a \le t \le b$$

expresses the rate (in gal/min) at which water flows into a tank in terms of the number of minutes past noon. The tank is empty at time $t = a$ minutes past noon and full at time $t = b$ minutes past noon.

(a) *Sketch a graph of the function $r = f(t)$. Label the axes with the appropriate units and variable names.*

(b) *Would you expect the tank to be half-full before or after time $t = (a+b)/2$? Explain.*

(c) *When is the tank half-full?*

(d) *When does water flow into the tank at a rate equal the average rate at which it ows into the tank between $t = a$ and $t = b$?*

Question 371 *The function*

$$r = f(t)\,,\ 0 \le t \le 12,$$

expresses a balloon's rate of ascent (in ft/min) in terms of the number of minutes past noon. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/yp3awkyck8`

152:Balloon 11

(a) *When is the balloon the same height as it is at 12:02pm? Explain.*

(b) *Use the graph to approximate the time(s) when the balloon is 20 feet higher than it was at 12:02pm.*

(c) *Write an equation whose solution(s) give the time(s) when the balloon is 20 feet higher than it was at 12:02pm.*

(d) *Suppose*

$$f(t) = 80 \sin\left(\frac{\pi}{6} t\right).$$

Use algebra to find exact solution(s) to your equation from part (c). Then use a calculator to approximate the clock time(s) to the nearest second.

The Fundamental Theorem of Calculus, Part 1

FTC

Question 372 *The function $f(t)$, $0 \leq t \leq 12$, gives the rate (in ft/hour) at which the depth of the water in a large tank is changing at time t hours past noon. The water is 40 feet deep at 5:00pm.*

Desmos link: `https://www.desmos.com/calculator/gnwlraiqhd`

152:Beetle 1

(a) *The graph of the function $r = f(t)$ is shown above. Label the axes with the appropriate variable names and units.*

(b) *Find a function $h = g(t)$ that expresses the depth of the water (in feet) in terms of the number of hours past noon.*

(c) *Sketch the graph of the function*

$$\frac{dh}{dt} = \frac{d}{dt}\left(g(t)\right).$$

Label the axes with the appropriate variable names and units.

(d) *Find expressions that gives the minimum and maximum depths of water in the tank. Use the graph to approximate these depths.*

(e) *Sketch a graph of the function $h = g(t)$. Label the axes with the appropriate variable names and units.*

(f) *Write an equation you would solve to determine the time(s) when the water level is 10 feet lower than it is at 10:00pm. Use the graph to estimate this time(s). Explain your reasoning.*

Question 373 *The function*

$$v = f(t), \; 0 \leq t \leq 6,$$

expresses the speed (in inches/sec) of a beetle in terms of the number of seconds past noon.

Learning outcomes:
Author(s):

(a) *Find a function $s = g(t)$ that expresses the distance (in inches) of the beetle to its destination in terms of the number of seconds past noon. The distance is measured along the route.*

(b) *Find an expression for the derivative ds/dt. Give two answers to this question. First, an intuitive explanation that someone without any knowledge of calculus might understand. Then again, using the Fundamental Theorem.*

(c) *Now Suppose*

$$v = f(t) = \sqrt{6t - t^2}\,,\ 0 \le t \le 6.$$

Desmos link: `https://www.desmos.com/calculator/cr9oays9zo`

152:Beetle Destination

(i) *Use the graph above to make a reasonably accurate graph of the function $s = g(t)$. Explain your reasoning.*

(ii) *Use the geometry of the curve $v = f(x)$ to find an explicit expression (without the integral) for the function $s = g(t)$.*

(iii) *Differentiate the function $s = g(t)$ to confirm your expression is correct.*

Question 374 *The function*

$$V = \int_0^h A(w)\,dw, 0 \le h \le 6,$$

expresses the volume (in ft^3) of water in a tank in terms of the water's depth (measured in feet). The graph of the function A is shown below.

Desmos link: `https://www.desmos.com/calculator/ae516hl7j6`

152:Beetle 1

(a) *Is the volume V above a function of w or h? Explain.*

(b) *What are the units of $A(w)$? How do you know?*

(c) *Label the axes on the graph with their appropriate variable names and units.*

Suppose the function $h = p(t)$, $0 \le t \le 12$, expresses the depth of the water (in ft) in terms of the number of hours past noon.

(d) *Find a function* $V = k(t)$ *that expresses the volume of water in terms of the number of hours past noon.*

(e) *Find an expression for the rate of change in the volume of the water with respect to time. Explain this expression intuitively.*

(f) *Now suppose*

$$p(t) = 3 + 2\cos\left(\frac{\pi}{12}t\right),\ 0 \leq t \leq 12.$$

(i) *Is the volume increasing or decreasing at 6 pm? At what rate?*

(ii) *Is the volume increasing or decreasing when the tank holds 44* ft^3 *of water? At what approximate rate?*

1 Gas Consumption

Question 375 *The function*

$$r = g(s),\ 0 \leq s \leq 200,$$

expresses the rate, in gallons/mile, at which your car burns gas in terms of the distance (in miles) from your home, as measured along your route on a 200-mile road trip to Spokane. When you are 50 miles from home, your tank has 10 gallons of gas. The trip takes four hours, and the fuction

$$v = w(t) = 64 - 6(t-3)^2,\ 0 \leq t \leq 4.$$

expresses yours speed (in miles/hour) in terms of the number of hours past noon.

(a) *Find a function* $G = f(s)$ *that expresses the volume of gas (in gallons) your tank holds in terms of your distance from home. Include its domain.*

(b) *Find a function* $G = Q(t)$ *that expresses the volume of gas in terms of the number of hours past noon. Include its domain.*

(c) *Use the Fundamental Theorem to find an expression for the derivative* dG/dt*. Then explain this expression intuitively, in a way that a student without a knowledge of calculus could understand.*

(d) *Use the graph of the function* $r = g(s)$ *shown below to find approximate answers to the following questions.*

Desmos link: `https://www.desmos.com/calculator/tzmqpsznou`

152: FTC1 Gas

(i) *Label the axes above with the appropriate variable names and units.*

(ii) *How many gallons of gas are in your tank at 2:00pm?*

(iii) *At what rate, with respect to time, is your car burning gas when you are 62 miles from Spokane?*

(iv) *What is your gas mileage at 2:00pm?*

(v) *At what rate (with respect to time) is your gas mileage changing at 2:00pm?*

2 On Trails

Question 376 *The function*

$$h = f(s) = 5\sin\left(\frac{s}{5}\right),\ 0 \le s \le 15,$$

expresses the altitude (in meters) of a beetle's trail in terms of the distance from the start (measured along the trail in meters).

The function

$$v = g(t) = \frac{t}{90} + \frac{t^2}{900},\ 0 \le t \le 30,$$

expresses the speed (in meters/min) of the beetle in terms of the number of minutes past noon (when it began its journey).

(a) *How far from the start is the beetle at 12:15pm?*

(b) *Find the inclination angle of the trail beneath the beetle at 12:15pm.*

(c) *How fast is the beetle crawling at 12:15pm?*

(d) *Is the beetle gaining or losing altitude at 12:15pm? At what rate?*

(e) *Express the rate at which the beetle's altitude is changing at 12:15pm in terms of some of the following derivatives. Include units for each derivative and explain their meanings.*

(i) $\left.\dfrac{dh}{ds}\right|_{s=15}$

(ii) $\left.\dfrac{dh}{ds}\right|_{s=g(15)}$

(iii) $\left.\dfrac{dh}{ds}\right|_{s=2.5}$

(iv) $\left.\dfrac{dv}{dt}\right|_{t=15}$

(v) $\dfrac{d}{dt}\displaystyle\int_0^t g(u)\,du$

Question 377 *The function*

$$\theta = f(s),\ 0 \le s \le 15,$$

expresses the inclination angle (in radians) of a beetle's trail in terms of the distance from the start (measured along the trail in meters). The trail is 200 *meters above sea level at the start.*

The function

$$v = g(t) = 5\sin(t/15)\,,\ 0 \le t \le 30,$$

expresses the speed (in meters/min) of the beetle in terms of the number of minutes past noon (when it began its journey).

(a) *Find an expression for the function*

$$h = w(t)\,,\ 0 \le t \le 30,$$

that expresses the beetle's elevation in terms of the number of minutes past noon.

(b) *Find an expression for the derivative dh/dt and interpret its meaning.*

(c) *Compare this question with the Question 5.*

Question 378 *The function*

$$h = f(s)\,,\ 0 \le s \le 60,$$

expresses the elevation (in meters) of a mountain road in terms of the distance (in kilometers) from the start.

The function

$$T = g(h), 0 \le h \le 5000,$$

expresses the temperature (in Celsius degrees) in terms of altitude (in meters) along the road.

(a) *Use the functions f and g to write an expression for the function*

$$T = w(s)\,,\ 0 \le s \le 60,$$

that expresses the temperature (in Celsius degrees) along the road in terms of the distance (in kilometers) from the start.

(b) *Use the Leibniz differential notation to write an expression for the derivative*

$$\left.\frac{dT}{ds}\right|_{s=54}$$

in terms of the appropriate derivatives of the functions f and g.

(c) *Use the graphs of the functions f and g below to approximate the temperature at the point on the road 54 kilometers from the start.*

(d) *Use the sliders in Lines 1 and the graphs of the functions f and g below to approximate the derivative*

$$\left.\frac{dT}{ds}\right|_{s=54}.$$

Include units.

(e) *Use the language of small changes to interpret the meaning of the above derivative.*

Desmos link: `https://www.desmos.com/calculator/gcmfa2a6jz`

152: Elevation and Temperature

Desmos link: `https://www.desmos.com/calculator/ohqk3ak539`

152: Elevation and Temperature 2

Question 379 *The function*

$$h = f(s)\,,\ 0 \le s \le 60,$$

expresses the elevation (in meters) of a mountain road in terms of the distance (in kilometers) from the start.

The function

$$r = k(h), 0 \le h \le 5000,$$

expresses the temperature gradient (in Celsius degrees/meter) in terms of altitude (in meters) along the road.

The temperature is $16.5^\circ C$ at an elevation of 3400 meters.

(a) *Use the functions f and k to write an expression for the function*

$$T = w(s)\,,\ 0 \le s \le 60,$$

that expresses the temperature (in Celsius degrees) along the road in terms of the distance (in kilometers) from the start.

(b) *Use the graphs of the functions f and k below to approximate the temperature at the point on the road 54 kilometers from the start.*

(c) *Find an expression for the derivative*

$$\left(\frac{d}{ds} \int_{3400}^{f(s)} k(h)\, dh \right) \Big|_{s=54}$$

and interpret its meaning. Do this twice. First using common sense. Then again using the Fundamental Theorem of Calculus, Part 1. Include units.

(d) *Use the sliders in Lines 1 and the graphs of the functions f and g below to approximate the derivative*

$$\frac{dT}{ds}\Big|_{s=54}.$$

Include units.

Desmos link: `https://www.desmos.com/calculator/gcmfa2a6jz`

152: Elevation and Temperature

Desmos link: `https://www.desmos.com/calculator/vvr0qrgoh1`

152: Elevation and Temperature 3

Substitution and Parametrically-defined Curves

Substitution and parameterized curves

1 The Basics

Question 380 *Which of the following definite integrals are equal to* $\int_2^6 f(x)\,dx$*? Avoid using a u-substitution. Instead think geometrically by describing the transformations that take the graph of* $y = f(x)$ *to the graph of each integrand. Assume* f *is continuous.*

(a) $\int_4^{12} 2f(2x)\,dx$

(b) $\int_4^{12} (1/2)f(x/2)\,dx$

(c) $\int_1^{3} (1/2)f(2x)\,dx$

(d) $\int_1^{3} 2f(2x)\,dx$

Question 381 *The function*

$$r = f(t)\,,\ 0 \le t \le 12,$$

expresses a balloon's rate of ascent (in ft/min) in terms of the number of minutes past noon. Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/yp3awkyck8`

152:Balloon 11

(a) *When is the balloon the same height as it is at 12:02pm? Explain.*

Learning outcomes:
Author(s):

(b) *Use the graph to approximate the time(s) when the balloon is 20 feet higher than it was at 12:02pm.*

(c) *Write an equation whose solution(s) give the time(s) when the balloon is 20 feet higher than it was at 12:02pm.*

(d) *Suppose*

$$f(t) = 80 \sin\left(\frac{\pi}{6}t\right).$$

Use a u-substitution and algebra to find exact solution(s) to your equation from part (c). Then use a calculator to approximate the clock time(s) to the nearest second.

Question 382 *Between noon and 8pm the growth rate of a population of bacteria decreases exponentially. At noon the population increases at the rate of 700,000 bac/hr and at 3pm the population increases at the rate of 400,000 bac/hr.*

The population is 2 million at noon.

(a) *Sketch by hand a rough graph of the function*

$$r = f(t)\,,\ 0 \le t \le 8,$$

that expresses the growth rate (in hundreds of thousand of bacteria/hr) in terms of the number of hours past noon.

(b) *Find an expression for the function in part (a). Do not use e.*

(c) *Use part (a) to sketch by hand a graph of the function*

$$P = g(t)\,,\ 0 \le t \le 8,$$

that expresses the population in terms of the number of hours past noon.

(d) *Find an expression for the function in part (c).*

Question 383 *Identify for which of the following definite integrals a u-substitution would work. Make the substitution for these and evaluate the integral. Do this by changing the lower and upper bounds. Do* not *substitute back to the original variable of integration.*

(a) $\displaystyle\int_0^{\sqrt{\pi}} \theta \sin(\theta^2)\, d\theta$

(b) $\displaystyle\int_0^{\pi} \theta^2 \sin(\theta)\, d\theta$

(c) $\displaystyle\int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta$

(d) $\displaystyle\int_0^{\pi/4} \tan\theta \sec^2\theta \, d\theta$

(e) $\displaystyle\int_0^{\pi/4} \tan\theta \sec^3\theta \, d\theta$

(f) $\displaystyle\int_0^{2} x^2(1+x^2)^2 \, dx$

(g) $\displaystyle\int_0^{2} x^2(1+x^3)^2 \, dx$

(h) $\displaystyle\int_1^{2} \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}} \, dx$

(i) $\displaystyle\int_1^{2} \sqrt{x}\cos\left(\sqrt{x}\right) \, dx$

(j) $\displaystyle\int_0^{1} \frac{e^x}{(1+e^x)^2} \, , dx$

(k) $\displaystyle\int_0^{1} \frac{e^x}{1+e^x} \, , dx$

(l) $\displaystyle\int_0^{1} \frac{e^x}{1+e^{2x}} \, , dx$

(m) $\displaystyle\int_2^{3} \frac{x}{\ln x} \, dx$

(n) $\displaystyle\int_2^{3} \frac{\ln x}{x} \, dx$

2 A Geometric Interpretation

To understand what u-substitution does geometrically, it helps to think about making a *reverse* u-substitution. This means starting with a simple integral and making it more complicated with a substitution. Here's an example.

Example 52. We'll start with the integral

$$I = \int_0^{\pi} \sin u \, du$$

and make the substitution

$$u = t^2.$$

This substitution really reparameterizes the curve

$$y = f(u) = \sin u \, , \, 0 \le u \le \pi$$

from the standard parameterization

$$(x, y) = (u, \cos u) \, , \, 0 \le u \le \pi$$

to the new parameterization

$$(x, y) = (t^2, \cos(t^2) \, , \, 0 \le t \le \sqrt{\pi}.$$

The substitution maps the rectangle with height $y = \sin u$ and width du to the rectangle with the same height $y = \sin(t^2)$ and width

$$du = 2t \, dt.$$

The effect is to transform a partition of the interval of integration, originally $u \in [0, \pi]$ with equal subintervals into a partition of the new interval of integration $t \in [0, \sqrt{2\pi}]$ with *unequal* subintervals $du = 2t \, dt$. You can see this transformation by dragging the slider w in Line 2 of the worksheeet below from $w = 0$ to $w = 1$.

Desmos link: `https://www.desmos.com/calculator/toiwzzqkva`

152: Sub 2

The next example puts this idea in more a more meaningul context.

Example 53. The function

$$r = f(s) \, , \, 0 \le s \le 240,$$

expresses the rate at which a car burns gas (in gal/mile) in terms of the trip odometer reading (in miles) during a 240-mile trip.

The car has 14 gallons of gas at odometer reading 100 miles.

The function

$$s = g(t) \, , \, 0 \le t \le 4,$$

expresses the car's trip odometer reading (in miles) in terms of the number of hours since the start of the trip.

(a) Find a function

$$G = w(s)\,,\ 0 \le s \le 240,$$

that expresses the number of gallons of gas in the car in terms of the trip odometer reading.

(b) Make a substitution in your expression from part (a) to find a function

$$G = h(t)\,,\ 0 \le s \le 240,$$

that expresses the number of gallons of gas in the car in terms of the number of hours since the start of the trip.

(c) Your function for part (b) should have been

$$G = h(t) = 14 - \int_{100}^{t} f(g(u))g'(u)\,du.$$

Interpret the integral in this expression in two different ways.

(i) By writing it as

$$\int_{100}^{t} f(g(u))(g'(u)\,du)$$

and describing the meaning of the product $g'(u)\,du$.

(ii) By keeping its original form and describing the meaning of the product $f(g(u))g'(u)$.

(d) Drag the slider w in Line 2 of the worksheet below to change the odometer reading function g. Describe what happens.

Desmos link: `https://www.desmos.com/calculator/gsdzd0lbva`

Substitution and Gas Comsumption 2

Example 54. The previous example assumed the car's gas mileage depended only on the trip odometer reading and not on its speed.

A more realistic example would look at the elevation change over a section of trail, the change being independent of how you traverse the trail.

Make up your own example like this. Start with the function $\theta = f(s)$ that expresses the trail's inclination angle in terms of the distance from the trailhead (measured along the trail in km).

The Limits to Growth

Depletion of the world's resources.

The Project on the Predicament of Mankind was initiated in 1968 by a group of 30 individuals having a wide range of expertise with the intent of examining the complex problems troubling all nations at that time and that still persist today. Within a year thay had created a model of the five basic factors they identified as limiting growth - population, agricultural production, natural resources, industrial production, and polution. The model provided valuable insights into how these areas interact with each other to limit growth. The findings are summarized in the 1972 classic *The Limits to Growth*, available at

Limits to Growth .

Here we focus on their analysis of the depletion of the world's chromium reserves as illustrated in the figure below.

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/9kqmbczco7`

Chromium is just one of 19 non-renewable natural resources analyzed. For each, the authors provide the following information.

- The known global reserves in 1970; for chromium 7.75×10^8 tons
- The static index. This is the number of years it the global reserves would last at the current global consumption rate. The static index for chromium is 420 years.
- The projected relative growth rate in the global consumption rate. For chromium, this relative instantaneous growth rate is 2.6%/yr, compounded continuously.

For each resource, the authors compute the *exponential index*. This is the number of years the known current reserves (in 1970) would last with consumption growing exponentially at assumed relative rate. They also compute the exponential index using five times the known reserves.

Learning outcomes:
Author(s):

Question 384 *Let's work in general (like the authors) and let r be the relative instantaneous growth rate (measured in yr^{-1}) in the global consumption rate and let s be the static index. Our goal here is to express the exponential index in terms of r and s.*

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/er2jq9vzeq`

Access this worksheet at

Limits to Growth 2

We'll let R_0 be the 1970 known global reserves (in units of hundreds of millions of tons). Use this parameter along with the parameters r and s as need be, to find expressions for the following:

(a) *The usage rate function*

$$u = f(t), 0 \leq t \leq 240,$$

that expresses the usage rate (measured in 10^8 tons/year) in terms of the number of years since 1970. Recall the assumption that the usage rate increases at the relative instantaneous rate of r yr^{-1}. This means that

$$u = f(t) = u_0 e^{rt}, 0 \leq t \leq 240$$

where u_0 is the usage rate in 1970. All that you need to do is to express u_0 in terms of R_0 and the static index s.

This function is graphed above, but it does not really belong on the same coordinate system as the others. Why not?

(b) *The remaining reserves function*

$$R = g(t), 0 \leq t \leq 240,$$

that expresses the remaining global reserves (in hundreds of millions of tons) in terms of the number of years since 1970, assuming the usage rate function in part (a).

(c) *The exponential index in terms of r and s.*

  (i) *Check that your expression for the exponential index is dimensionally correct.*

  (ii) *Use the values of r and s for Chromium given above to determine when the model predicts the world's Chromium reserves will run out. Check your result with the appropriate graph above.*

Area

Computing the area of regions in the plane bounded by graphs of functions.

Directions: When computing the area of a region, be sure to do the following.

- State the direction in which you decide to slice the region.
- Sketch a differential rectangle and label the coordinates of its endpoints.
- State the area of the differential rectangle.
- Explain the role of the definite integral in computing the area of the region.

1 Parabolas and Power Functions, Part 1

Question 385 *Let $a, b > 0$ be constants, measured in meters.*

(a) *Find the area of the region bounded by the x-axis and the parabola*

$$y = b - \frac{x^2}{a},$$

where x and y are measured in meters.

Include units and check that your answer is dimensionally correct. Do this twice:

(i) *first, by slicing the region perpendicular to the y-axis,*

(ii) *and again, by slicing the region perpendicular to the x-axis*

Question 386 *Express the area of the region shown below bounded by a parabola and a line perpendicular to its axis of symmetry in terms of the height h and the length of the base b.*

Desmos link: `https://www.desmos.com/calculator/kkdfpr66rt`

152: Parabola 1

Learning outcomes:
Author(s):

Question 387 *Use ideas of differentilal calculus,* not *the Fundamental theorem to answer Question 2 by comparing the areas of the differential rectangles shown below.*

Desmos link: `https://www.desmos.com/calculator/pvskgbpnew`

152: Parabola 2

Desmos link: `https://www.desmos.com/calculator/0w37bixe79`

152: Parabola 3

Question 388 *Let $b, h > 0$ be constants with units of meters and let $a = b/2$. Suppose the curve $y = cx^8$ passes through the point A with coordinates $(b/2, h) = (a, h)$. We wish to express the area of the shaded region below bounded by the curve and the line $y = h$ in terms of its base b and height h. Do this as follows.*

(a) *Express c in terms of a and h.*

(b) *Express the area of the region first in terms of a and h, and then in terms of b and h.*

(c) *Check that your expression for the area has the correct units.*

(d) *Interpret your expression for the area geometrically. Does it seem reasonable? Compare it with the area of a triangle and with the area of a rectangle.*

(e) *Find an equation of the horizontal line that bisects the area of the region. Type your equation in Line 2 above as a check.*

Desmos link: `https://www.desmos.com/calculator/egdpe7cccl`

152: Power Function 3

Question 389 *Let $a, b > 0$ be constants with units of meters.*

(a) *Use the Fundamental Theorem to evaluate the definite integral*

$$\int_0^b \frac{x^2}{a}\,dx.$$

Include units.

(b) *Interpret the result of part (a) geometrically.*

Question 390 *Let $a, b, c \in \mathbb{R}$ be constants with units of meters. Find the area of the region bounded by the x-axis and the parabola*

$$y = c + \frac{(x-b)^2}{a},$$

where the coordinates (x, y) are also measured in meters.

2 Parabolas, Part 2

Question 391 (a) *Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x + 3$. Start by deciding how to slice the region and computing the area of a differential slice.*

(b) *Drag the slider w in Line 1 of the worksheet below from $w = 0$ to $w = 1$ and describe what you see.*

Desmos link: `https://www.desmos.com/calculator/vci24vc4f1`

152: Area Parabolic Segment

(c) *Find an expression for the area of the region bounded by the parabola $y = x^2$ and the line through the points (a, a^2) and (b, b^2).*

Question 392 *Let $\mathcal{P}$ be the parabola symmetric about the y-axis with its vertex at the origin that passes through the point (a, h). Assume $a, h > 0$.*

(a) *Express the area of the region bounded by the parabola and the line $y = h$ in terms of its height h and its base $b = 2a$.*

(b) *Check that your expression for the area has the correct units.*

(c) *Interpret your expression for the area geometrically. Does it seem reasonable? Compare it with the area of the surrounding rectangle.*

(d) *Find an equation of the horizontal line that bisects the area of the region bounded by $\mathcal{P}$ and the line $y = h$.*

(e) *Check that your equation is dimensionally correct.*

(f) *Enter your equation for the bisecting line in Line 18 of the worksheet below as a check.*

Desmos link: `https://www.desmos.com/calculator/mbxrsmdho8`

152: Parabola 23

Question 393 *Let $\mathcal{P}$ be the parabola symmetric about the y-axis with its vertex at the origin that passes through the point (a, h). Assume $a, h > 0$.*

(a) *Find an expression (in terms of a and h) for the area of the region bounded by the x-axis, the parabola $\mathcal{P}$, and the tangent line to the parabola at P. Check that your expression is dimensionally correct.* Start by slicing the region perpendicular to the y-axis.

(b) *Find an expression for the area of the region in part (a) in terms of its base $b = OQ$ and height h.*

(c) *What fraction of triangle ΔAQP does the region in part (a) occupy?*

(d) *Compare the area of the region in part (a) with the area of the region bounded by the parabola and the segment OP.*

Desmos link: `https://www.desmos.com/calculator/avl2r9dubt`

152: Parabola and Tangent Line 3

Explanation. (a) Let $y = cx^2$ be an equation of the parabola. Then since the point (a, h) lies on the parabola

$$h = ca^2$$

and

$$c = \frac{h}{a^2}.$$

So the parabola has equation

$$y = \frac{h}{a^2}x^2.$$

The left and right endpoints of a horizontal slice AB of the region share the same y-coordinate. Call it y. Since the left endpoint $A(x, y)$ lies on the parabola,

$$x = \pm\sqrt{\frac{ya^2}{h}}.$$

But because A lies in the first quadrant, $x \geq 0$ and A has coordinates

$$\left(\sqrt{\frac{ya^2}{h}}, y\right).$$

To express the x-coordinate of the right endpoint B in terms of y, we need an equation of the tangent line to the parabola at the point $P(a, h)$.

The slope of this tangent line is

$$\begin{aligned}\frac{d}{dx}\left(\frac{h}{a^2}x^2\right)\Big|_{x=a} &= \frac{2h}{a^2}x\Big|_{x=a} \\ &= \frac{2h}{a}.\end{aligned}$$

So an equation of the tangent line is

$$y = h + \frac{2h}{a}(x - a) = \frac{2h}{a}x - h$$

and B has coordinates

$$\left(\frac{a}{2h}y + \frac{a}{2}, y\right).$$

Now the differential recangle AB has length

$$\frac{a}{2h}y + \frac{a}{2} - \sqrt{\frac{ya^2}{h}},$$

width dy and area

$$dA = \left(\frac{a}{2h}y + \frac{a}{2} - \sqrt{\frac{ya^2}{h}}\right)\, dy.$$

Finally, the area of the region bounded by the x-axis, the parabola, and the tangent line at (a, h) is

$$\begin{aligned}\int_0^h \left(\frac{a}{2h}y + \frac{a}{2} - \sqrt{\frac{ya^2}{h}}\right) dy &= \left(\frac{a}{4h}y^2 + \frac{a}{2}y - \frac{2}{3}\sqrt{\frac{a^2}{h}}y^{3/2}\right)\Big|_{y=0}^{y=h} \\ &= \frac{ah}{4} + \frac{ah}{2} - \frac{2ah}{3} \\ &= \frac{1}{12}ah.\end{aligned}$$

3 Which Area is Greatest? Least?

Question 394 *The vertical segments below are equally-spaced. The horizontal segments cut the shaded region into 20 smaller regions. Which of these has the greatest area? The least?*

Desmos link: `https://www.desmos.com/calculator/wsgteg55ta`

152: Self-calibrating area

Question 395 *The vertical segments below are equally-spaced. The horizontal segments cut the shaded region into 20 smaller regions. Which of these has the greatest area? The least?*

Desmos link: `https://www.desmos.com/calculator/dir9hpzvse`

152: Self-calibrating area 2

4 Trigonometric Functions

Question 396 *Let $a, b, k > 0$ be constants with units of cm.*

Find a simplified expression for the area of the shaded region below bounded by the y-axis and the curves $y = a\cos(x/k)$ and $y = b\sin(x/k)$. Check your expression is dimensionally correct.

The area is

$$k(\sqrt{a^2 + b^2} - b).$$

Desmos link: `https://www.desmos.com/calculator/ps1m0hksl0`

152: Sine Cosine Area

Explanation:

Suppose the point of intersection has coordinates

$$(x_0, y_0) = (x_0, a\cos\theta_0) = (x_0, b\sin\theta_0),$$

where

$$\theta_0 = x_0/k.$$

Then

$$a \cos \theta_0 = b \sin \theta_0$$

and

$$\tan \theta_0 \;= \frac{a}{b}.$$

Since $0 < \theta_0 < \pi/2$, *we can draw a right triangle with* θ_0 *as one of the acute angles,* a *the length of the side opposite this angle and* b *the length of the adjacent side. From this we know that*

$$\cos \theta_0 = \frac{b}{\sqrt{a^2 + b^2}}$$

and

$$\sin \theta_0 = \frac{a}{\sqrt{a^2 + b^2}}.$$

Now slice the region into differential rectangles perpendicular to the x*-axis. Then the general differential rectangle* PQ *with endpoints* $P(x, a\cos(x/k))$ *and* $Q(x, b\sin(x/k))$ *has differential area*

$$dA = (a\cos(x/k) - b\sin(x/k))\, dx.$$

So the area of the region is

$$\begin{aligned} A &= \int_0^{x_0} \left(a \cos\left(\frac{x}{k}\right) - b \sin\left(\frac{x}{k}\right)\right) dx \\ &= \left(ak \sin\left(\frac{x}{k}\right) + bk \cos\left(\frac{x}{k}\right)\right)\Big|_{x=0}^{x=x_0} \\ &= k\left(\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}} - b\right) \\ &= k(\sqrt{a^2+b^2} - b). \end{aligned}$$

5 The Rectangular Elastica

Let $a > 0$ be a constant with units of meters and let

$$f(x) = \int_x^a \frac{u^2}{\sqrt{a^4 - u^4}}\, du.$$

Question 397 *Desmos link:* `https://www.desmos.com/calculator/c3vderglut`

152: Elastica

(a) *Express the coordinates of the* x *intercept of the curve* $y = f(x)$ *as some multiple of* a*. Use a Riemann sum with* 1000 *equal subintervals to approximate the multiple.*

(b) *Express the exact and approximate coordinates of the* y*-intercept in terms of* a*.*

(c) *Find an equation of the tangent line to the curve* $u = f(x)$ *at the point where the line* $x = a/2$ *intersects the curve.*

(d) *Find an expression for the area bounded by the curve* $y = f(x)$ *and the* y*-axis. Activate the* Hint *folder in Line 1 of the worksheet above for a hint.*

Volumes of Solids in Three-Dimensions

Volume.

1 Square Cross Sections

Question 398 *The base of a solid is a disk of radius a meters. Parallel cross-sections perpendicular to the base are squares.*

(a) *Find the volume of the solid.*

(b) *Compare the solid's volume with the volume of the surrounding cylinder.*

Desmos3D link: `https://www.desmos.com/3d/q5jo9tip5q`

152: Circular Base Square Cross Sections

Explanation.

It is not necessary to visualize the solid to compute its volume. Instead, the best approach is to sketch its base.

Desmos link: `https://www.desmos.com/calculator/mnjevsoffp`

152: Solid Base 1

We can image the square cross-sections coming out of the page, but the key is to start by computing the area of the cross-section. The square cross-section has side length $s = 2y$ and area

$$A = s^2 = 4y^2.$$

The differential slice of the solid has width dx and differential volume

$$dV = A\,dx = 4y^2\,dx.$$

Before integrating, we'll need to eliminate the mixed variables and express the differential volume in terms of x and dx. To do this we need to use the fact that the base is bounded by the circle with equation

$$x^2 + y^2 = a^2$$

Learning outcomes:
Author(s):

in the above coordinate system. So

$$y^2 = a^2 - x^2$$

and the differential slice has volume

$$\begin{aligned} dV &= 4y^2\,dx \\ &= 4(a^2 - x^2)\,dx. \end{aligned}$$

We add these differential volumes to get the volume of the solid,

$$\begin{aligned} V &= \int_{-a}^{a} 4(a^2 - x^2)\,dx \\ &= 8\int_0^a (a^2 - x^2)\,dx \\ &= 8\left(a^2x - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=a} \\ &= \frac{16}{3}a^3. \end{aligned}$$

Activate the folder in Line 1 above to see the cylinder that surrounds the solid. This cylinder has volume

$$W = (2a)(\pi a^2) = 2\pi a^3 \sim 6.28a^3.$$

So the solid occupies

$$\frac{V}{W} = \frac{16}{6\pi} \sim 84.9\%$$

the volume of the cylinder.

Question 399 (a) *Solve Question 1 if the cross-sections are semi-circles (with their diameters in the base) instead of squares.*

(b) *Describe the solid in this case.*

(c) *What fraction of the cylinder's volume below is occupied by the tennis balls?*

Desmos link: `https://www.desmos.com/calculator/5ip866uqox`

152: Tennis Balls

Question 400 *Let $a, b > 0$ be constants.*

The base of a solid is the region bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Cross-sections perpendicular to the x-axis are squares.

Desmos3D link: `https://www.desmos.com/3d/zbcupsnbjv`

152: Elliptical Base Square Cross Sections

(a) *Find the volume of the solid.*

(b) *Check that your expression for the volume is dimensionally correct.*

(c) *Compare the solid's volume with the volume of the surrounding cylinder.*

Question 401 *The base of a tent is a regular hexagon with side length a meters. Three poles bent into semicircles run between opposite vertices. Cross sections parallel to the base are regular hexagons having their vertices on the poles.*

Desmos3D link: `https://www.desmos.com/3d/wczn2fml4l`

152: Tent

(a) *Find an expression for the volume of the tent.*

(b) *Compare the volume with the volume of the surrounding cylinder.*

(c) *The tent is partially filled with water that takes up half the tent's volume. What is the approximate depth of the water?*

Question 402 *The base of a solid is the parabolic region below with base $b = 2a$ and height h. Cross sections perpendicular to the y-axis are rectangles half as tall as they are wide*

Desmos link: `https://www.desmos.com/calculator/mbxrsmdho8`

152: Parabola 23

Desmos3D link: `https://www.desmos.com/3d/gdebhnchrz`

152: Parabolic Base

(a) *Find an expression for the volume of the solid.*

(b) *Check that your expression is dimensionally correct.*

(c) *Compare the solid's volume with the volume of the surrounding cylinder.*

2 The Hoof of Archimedes

Question 403 *A solid circular cylinder of radius a and height h is sliced by a plane passing through a diameter of the base and intersecting the top of the cylinder in exactly one point. The plane cuts the cylinder into two pieces. Our goal here is to find an expression in h and a for the volume of the lower (smaller) piece.*

Desmos3D link: `https://www.desmos.com/3d/cnspcoiggo`

152: Archimedes Hoof 2

We'll determine the volume of the lower piece (the hoof of Archimedes) by taking slices perpendicular to the coordinate axes . Turn off the folder in Line 2 and activate the appropriate folder in Lines 5, 13, or 22 if you need help visualizing the cross sections.

Slice the solid

(a) *perpendicular to the x-axis (activate the folder in Line 5),*

(b) *perpendicular to the y-axis (activate the folder in Line 13), and*

(c) *perpendicular to the z-axis (activate the folder in Line 22).*

(a) *Check that your expression for the volume is dimensionally correct.*

(b) *What fraction of the original cylinder does the hoof occupy? Activate the folder in Line 33 to see the cylinder.*

3 Archimedes' Bicylinder

Question 404 *Two solid right circular cylinders of radius a cm intersect orthogonally as shown below (the cylinders' axes of symmetry also intersect). We'll determine the volume of the solid common to both (Archimedes' bicylinder).*

Desmos3D link: `https://www.desmos.com/3d/wzzhod4ya6`

152: Bicylinder

(a) *Find an expression for the volume of the bicylinder and check that it is dimensionally correct.*

(b) *Compare the volume with its surrounding cylinders to make sure your answer is reasonable.*

(c) *Compare the volume of the bicylinder with the volume of the solid in Question 1 above. Then drag the slider w_2 in the worksheet below and describe what you see.*

Desmos3D link: `https://www.desmos.com/3d/cwcjqkjndw`

152: Square Cross-sections to Bicylinder

Question 405 *Turn off the folders in Line 1 (*Bicylinder*) and Line 5 (*Hoofs*) below. Then use your expression for the volume of the hoof of Archimedes to compute the volume of the bicylinder.*

Desmos3D link: `https://www.desmos.com/3d/phy3gfnhcj`

152: Bicylinder and Hoof

4 Triangles and Cones

Question 406 *Use integration to find an expression for the area of ΔABC below in terms of its base $b = AC$ and height $h = BZ$.*

Do this as follows:

(a) *Establish a one-dimensional coordinate system as shown below.*

(b) *Use similar triangles to express the length QR in terms of $w = BT$, h, and b.*

(c) *Find an expression for the differential area of the slice QR.*

(d) *Integrate to find the area of the triangle.*

Desmos link: `https://www.desmos.com/calculator/jly5a1jr9d`

152: Triangle Area

Question 407 *Use integration to express the volume of the red cone in terms of the area A of its base and its height h.*

Do this as follows:

(a) *Establish a one-dimensional coordinate system as shown below.*

(b) *Use similar cones to express the area of the cross-section at coordinate w in terms of w, h, and A.*

(c) *Find an expression for the differential volume of the slice.*

(d) *Integrate to find the volume of the cone.*

Desmos3D link: `https://www.desmos.com/3d/6pzlieihca`

152: Cone

Desmos3D link: `https://www.desmos.com/3d/mwkyamaixz`

Proof Without Words: Pyramid with Square Base

The Hoof of Archimedes

Finding the volume of the hoof, several ways.

Question 408 *A solid circular cylinder of radius a and height h is sliced by a plane passing through a diameter of the base and intersecting the top of the cylinder in exactly one point. The plane cuts the cylinder into two pieces. Our goal here is to find an expression in h and a for the volume of the lower (smaller) piece.*

Desmos3D link: `https://www.desmos.com/3d/cnspcoiggo`

152: Archimedes Hoof 2

We'll determine the volume of the lower piece (the hoof of Archimedes) by taking slices perpendicular to the coordinate axes . Turn off the folder in Line 2 and activate the appropriate folder in Lines 5, 13, or 22 if you need help visualizing the cross sections.

(a) *Compute the volume by slicing the solid perpendicular to the x-axis. To see the cross-sections, turn off the folder* Slanted Face *in Line 2 and activate the folder* Cross-sections perpendicular to the x-axis *in Line 5).*

Explanation. Let the top vertex of the red triangle have coordinates (x, y, z). Then the triangle has area

$$A = yz/2$$

and the thin slice has differential volume

$$dV = yz/2\,dx.$$

To express the differential volume in terms of x, first use the Pythagorean theorem to express y in terms of x to get

$$y = \sqrt{a^2 - x^2}.$$

Then use similar triangles (red and green) to get

$$\frac{z}{y} = \frac{h}{a}.$$

Learning outcomes:
Author(s):

and express z in terms of y as

$$z = \frac{yh}{a}.$$

Now substitute the previous expression for y to express the area of the cross-section in terms of x,

$$A = \frac{h}{2a}\left(a^2 - x^2\right).$$

So the differential volume of the thin slice is

$$dA = \frac{h}{2a}\left(a^2 - x^2\right)\,dx.$$

The volume of the hoof is then

$$V = 2\int_0^a \frac{h}{2a}\left(a^2 - x^2\right)\,dx = \frac{2}{3}a^2h.$$

(b) *Now compute the volume by slicing perpendicular to the y-axis. To see the slices turn off the folder in Line 5 and activate the folder in Line 13.*

Explanation. In terms of the coordinates (x, y, z) of the top right vertex of the rectangular cross-section, its area is

$$A = 2xz.$$

So the volume of the differential slice is

$$dV = 2xz\,dy.$$

In terms of y, the differential volume is

$$dV = \frac{2h}{a}y\sqrt{a^2 - y^2}\,dy.$$

So the volume of the hoof is

$$V = \frac{2h}{a}\int_0^a y\sqrt{a^2 - y^2}\,dy = \frac{2}{3}a^2h.$$

(c) *Lastly, compute the volume by slicing perpendicular to the z-axis. To see the slices turn off the folder in Line 13 and activate the folder in Line 22.*

Explanation. A common error is to think the cross-sections are semicircles. They are not. Drag the slider z in Line 2 of the worksheet below to see how the cross-sections vary with the z-coordinate of the slice.

Desmos link: `https://www.desmos.com/calculator/gxhq1yelvv`

152: Hoof Cross Sections

The first step in finding an expression for the area of the cross-section in terms of the coordinates (x, y, z) of the point on the surface directly above A is to find an expression for the angle $\theta = \angle AOW$ above.

This angle has measure
$$\theta = \arccos\left(\frac{y}{a}\right).$$

So the sector of the circle subtended by $\angle AOB$ has area
$$A_1 = a^2 \arccos\left(\frac{y}{a}\right)$$

and ΔAOB area
$$A_2 = y\sqrt{a^2 - y^2}.$$

Then the slice with differential thickness dz has differential volume
$$dV = a^2 \left(\arccos\left(\frac{y}{a}\right) - y\sqrt{a^2 - y^2}\right)\, dz.$$

Now to get rid of the mixed variables, we'll do something different and express dz in terms of dy. We know
$$z = \frac{yh}{a},$$

so
$$dz = \frac{h}{a}\, dy.$$

Then the hoof has volume
$$V = \frac{h}{a}\int_0^a \left(\arccos\left(\frac{y}{a}\right) - y\sqrt{a^2 - y^2}\right)\, dy.$$

(d) *Check that your expression for the volume is dimensionally correct.*

(e) *What fraction of the original cylinder does the hoof occupy? Activate the folder in Line 33 to see the cylinder.*

Volumes, Part 2

Volume.

1 Paraboloid of Revolution

Question 409 (a) *Find an expression for the area of the region below bounded by the parabola and the line $y = h$ in terms of the height h and base b of the region.*

Desmos link: `https://www.desmos.com/calculator/ftsereyrj4`

152: Parabola 23B

(b) *What fraction of the area of the surrounding rectangle does the parabolic region occupy?*

(c) *Rotate the region above about the y axis and you get a paraboloid of revolution as illustrated below.*

Desmos3D link: `https://www.desmos.com/3d/1bculfxxtt`

152: Paraboloid

(d) *Find an expression for the volume of the paraboloid in terms of its height h and the radius a of its base.*

(e) *Click the wrench in the upper right corner of the worksheet above and check the* Translucent Surfaces *box. Then activate the folder* Cylinder *in Line 3.*

(f) *What fraction of the volume of the surrounding cylinder does the paraboloid occupy? Does this look correct?*

(g) *How can you explain the difference between the fractions in parts (b) and (f)?*

(h) *One last question. Suppose the paraboloid is a tank that is partly filled with water that takes up half the volume of the tank. Express the depth of the water in terms of the height h of the tank.*

Drag the slider b in Line 3 of the workhsheet below to check your work.

Learning outcomes:
Author(s):

Desmos3D link: `https://www.desmos.com/3d/ejuarpozjx`

152: Paraboloid 2

Explanation. To answer part (h), let b be the depth of the the water (in meters) that takes up half the volume of the tank and let $y = cx^2$ be an equation of the parabola in the first worksheet above.

The segment AB with endpoints $A(0, y)$ on the y-axis and $B(x, y)$ on the parabola sweeps out a disk with radius x and area

$$A = \pi x^2.$$

So the corresponding differential slice with thickness dy has volume

$$dV = \pi x^2 \, dy.$$

But since $B(x, y)$ lies on the parabola $y = cx^2$,

$$x^2 = y/c.$$

and

$$dV = \pi x^2 \, dy = \frac{\pi}{c} y \, dy.$$

Summing these differential volumes gives the volume

$$V_t = \int_0^h \frac{\pi}{c} y \, dy$$

of the tank.

Similarly the volume of the water with depth b is

$$V_w = \int_0^b \frac{\pi}{c} y \, dy.$$

Since the water takes up half the volume of the tank,

$$\int_0^b \frac{\pi}{c} y \, dy = \frac{1}{2} \int_0^h \frac{\pi}{c} y \, dy.$$

So

$$\int_0^b y \, dy = \frac{1}{2} \int_0^h y \, dy$$

and

$$\frac{b^2}{2} = \frac{h^2}{4}.$$

Then since $b, h > 0$,

$$b = \frac{h}{\sqrt{2}}.$$

So the tank is half full when the depth of the water is $h/\sqrt{2} \sim 0.707h$ meters.

2 Napkin Ring

Question 410 (a) *Start with a solid sphere of radius a as shown below.*

Desmos3D link: `https://www.desmos.com/3d/dw5gjr0es4`

152: Napkin Ring

(b) *Click the wrench in the upper right corner of the worksheet above and check the* Translucent Surfaces *box.*

(c) *Now drill a circular hole of height $2h$ through the center of the sphere. To see this:*

(i) *Activate the folder* Cylinder *in Line 3 above.*

(ii) *Then activate the folder* Napkin Ring *in Line 5.*

(iii) *Then turn off the folders* Sphere *and* Cylinder *in Lines 1 and 3.*

(d) *We now go about expressing the volume of the remaining solid (a napkin ring) in terms of a and h.*

(i) *We start by slicing the solid perpendicular to the axis of rotation as illustrated below. Drag the slider w in Line 2 to translate the cross-section. Turn off the folder* Napkin Ring *in LIne 7 to better see the cross-sections.*

Desmos3D link: `https://www.desmos.com/3d/ubyhmxtccz`

152: Napkin Ring 2

(ii) *To find the area of a cross-section, it helps to drop down a dimension and work in the plane.*

Desmos link:
`https://www.desmos.com/calculator/6gylqexlh5`

152: Napkin Ring Cross Section

(iii) *Express the area of the annulus swept out by segment AB as it rotates about the z-axis in terms of the common z-coordinate of points A and B.*

(iv) *Then find an expression for the differential volume swept out by the differential rectangle AB with thickness dz.*

(v) *Finally, find an expression for the volume of the napkin ring.*

(e) *Drag the slider a in Line 12 of the first worksheet to change the radius of the original sphere. How does the volume of the napkin ring change as you vary a? Try to explain this geometrically by opening the* Tangent Cluster *folder and dragging the same slider a in Line 6 of the second worksheet.*

The Mylar Balloon

The mylar balloon.

1 The Rectangular Elastica

Question 411 *Let $a > 0$ be a constant with units of meters and for $x > 0$ define*

$$f(x) = \int_x^a \frac{u^2}{\sqrt{a^4 - u^4}}\, du.$$

Desmos link: `https://www.desmos.com/calculator/c3vderglut`

152: Elastica

(a) *Find the coordinates of the x-intercept of the curve $y = f(x)$.*

(b) *Find the exact coordinates of the y-intercept. Then make a substitution to express the x-intercept as a multiple of a. Then use Line 3 in the worksheet above to approximate the multiple.*

(c) *Find an equation of the tangent line to the curve $u = f(x)$ at the point where the line $x = a/2$ intersects the curve. Enter this equation in Line 4 above. Use the point-slope equation of a line, not slope-intercept.*

(d) *Find an expression for the area bounded by the curve $y = f(x)$ and the coordinate axes. Activate the* Hint *folder in Line 1 of the worksheet above for a hint.*

(e) *Compare the area in part (d) with the area of the surrounding rectangle. Does the ratio seem reasonable?*

(f) *Activate the folder* Rectangle *in Line 5. Compare the area of this rectangle with the area in part (d).*

Explanation. (a) The coordinates of the x-intercept are the roots of the equation

$$f(x) = \int_x^a \frac{u^2}{\sqrt{a^4 - u^4}}\, du = 0\,,\; x \geq 0.$$

Since the integrand is positive for $u > 0$, the integral is zero if and only if $x = a$. So the x-intercept has coordinates $(a, 0)$.

Learning outcomes:
Author(s):

(b) The y-intercept has y-coordinate

$$f(0) = \int_0^a \frac{u^2}{\sqrt{a^4 - u^4}}\,du = \frac{1}{a^2}\int_0^a \frac{u^2}{\sqrt{1 - \left(\frac{u}{a}\right)^4}}\,du.$$

Making the substitution $v = u/a$ and some algebra leads to

$$f(0) = a\int_0^1 \frac{v^2}{\sqrt{1 - v^4}}\,dv \sim 0.599a.$$

So the y-intercept has coordinates

$$\left(0, a\int_0^1 \frac{v^2}{\sqrt{1 - v^4}}\,dv\right) \sim (0, 0.599a).$$

(c) Since

$$y = f(x) = \int_x^a \frac{u^2}{\sqrt{a^4 - u^4}}\,du = -\int_a^x \frac{u^2}{\sqrt{a^4 - u^4}}\,du,$$

by the fundamental theorem

$$\begin{aligned}\frac{dy}{dx} &= -\frac{d}{dx}\int_a^x \frac{u^2}{\sqrt{a^4 - u^4}}\,du \\ &= -\frac{x^2}{\sqrt{a^4 - x^4}}.\end{aligned}$$

So at the point $(a/2, f(a/2))$, the tangent line has slope

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=a/2} &= -\frac{\left(\frac{a}{2}\right)^2}{\sqrt{a^4 - \left(\frac{a}{2}\right)^4}} \\ &= -1/\sqrt{15}.\end{aligned}$$

So at the point $(a/2, f(a/2))$ the tangent line to the curve has equation

$$y = \int_{a/2}^2 \frac{u^2}{\sqrt{a^4 - u^4}}\,du - \frac{1}{\sqrt{15}}\left(x - \frac{a}{2}\right).$$

(d) Slicing the region perpendicular to the x-axis, the area of a differential slice is

$$dA = y\,dx = \left(\int_x^a \frac{u^2}{\sqrt{a^4 - u^4}}\,du\right)dx.$$

So the area of the region bounded by the curve $y = f(x)$ and the coordinate axes is

$$A = \int_0^a \left(\int_x^a \frac{u^2}{\sqrt{a^4 - u^4}}\,du\right)dx.$$

But now we run into two problems. The first is that double integrals our not part of this class. The second is that even if they were, it's still not clear how we would evaluate this double integral.

So instead we slice the region perpendicular to the y-axis. Then the area of a differential slice is

$$dA = x\,dy.$$

But since

$$\frac{dy}{dx} = \frac{-x^2}{\sqrt{a^4 - x^4}},$$

$$dy = \frac{-x^2}{\sqrt{a^4 - x^4}}\,dx$$

and

$$dA = x\,dy = \frac{-x^3}{\sqrt{a^4 - x^4}}\,dx.$$

Letting the y-intercept have coordinates $(0, b) \sim (0, 0.599a)$, the area of the region is

$$\begin{aligned} A &= \int_0^b x\,dy \\ &= \int_a^0 \frac{-x^3}{\sqrt{a^4 - x^4}}\,dx \\ &= \frac{1}{2}a^2, \end{aligned}$$

where the last equality follows by making the substitution $u = x^4$ (the details are left for you).

2 Happy Birthday!

Question 412 *Rotate two copies of the rectangular elastica of Question 1 about the y-axis and you get a surface that models a mylar balloon.*

Desmos link: `https://www.desmos.com/calculator/wfmmpdd2tk`

152: Balloon

Desmos link: `https://www.desmos.com/calculator/u1vk5xr0ep`

152: Elastica 2

Desmos3D link: `https://www.desmos.com/3d/ztdgovbaqd`

152: Mylar Balloon 2

(a) *Find an expression for the volume of the balloon in the form $V = ka^3$, for some constant k.*

(b) *Use Line 9 of the worksheet above to approximate the constant k.*

Explanation. Slicing the solid perpendicular to the y-axis through the point $(x, f(x), 0)$, the cross-section is a circle with radius x and area

$$A = \pi x^2.$$

So the differential slice with thickness dy has differential volume

$$\begin{aligned} dV &= \pi x^2 \, dy \\ &= \frac{-x^4}{\sqrt{a^4 - x^4}} \, dx. \end{aligned}$$

And with $(0, b)$ the coordinates of the y-intercept of the curve $y = f(x)$, the solid has volume

$$\begin{aligned} V &= \pi \int_0^b x^2 \, dy \\ &= \pi \int_a^0 \frac{-x^4}{\sqrt{a^4 - x^4}} \, dx \\ &= \pi a^3 \int_0^1 \frac{v^4}{\sqrt{1 - v^4}} \, dv \\ &\sim 0.437\pi a^3. \end{aligned}$$

As a check, we compare this with the volume V_c of the surrounding right circular cylinder with radius a and height $b \sim 0.6a$. The ratio of volumes is

$$\frac{V}{V_c} \sim \frac{0.437\pi a^3}{0.6\pi a^3} \sim 0.73.$$

To better understand this ratio, we compare it with the ratio of the area $A = 0.5a^2$ of the region of rotation and the area $A_R \sim 0.6a^2$ of the surrounding rectangle. This ratio of areas is

$$\frac{A}{A_R} \sim \frac{0.5a^2}{0.6a^2} \sim 0.83.$$

I'm not quite sure about this, but it should be that

$$\left(\frac{A}{A_R}\right)^{3/2} < \frac{V}{V_c} < \frac{A}{A_R},$$

which is the case here since

$$\left(\frac{A}{A_R}\right)^{3/2} \sim 0.54.$$

Which Tanks Empty Faster?

Draining tanks.

1 Preliminaries

2 Power Functions

Desmos link: `https://www.desmos.com/calculator/zltxdekv7z`

152: Draining Cylinders

3 A Hemispherical Tank

Desmos link: `https://www.desmos.com/calculator/ggqodhz3jv`

152: Draining Cylinders 2

Learning outcomes:
Author(s):

Parametrically-defined curves and Area

Curves defined parametrically and signed area.

1 The Basics

In a sense there is nothing new here. Making a u-substitution is equivalent to reparameterizing the graph of a function.

To see why and to understand what u-substitution does geometrically, it helps to think about making a *reverse* u-substitution. This means starting with a simple integral and making it more complicated with a substitution. Here's an example.

Example 55. We'll start with the integral

$$I = \int_0^\pi \sin u \, du$$

and make the substitution

$$u = t^2.$$

This substitution really reparameterizes the curve

$$y = f(u) = \sin u \, , \, 0 \le u \le \pi$$

from the standard parameterization

$$(x, y) = (u, \cos u) \, , \, 0 \le u \le \pi$$

to the new parameterization

$$(x, y) = (t^2, \cos(t^2) \, , \, 0 \le t \le \sqrt{\pi}.$$

The substitution maps the rectangle with height $y = \sin u$ and width du to the rectangle with the same height $y = \sin(t^2)$ and width

$$du = 2t \, dt.$$

The effect is to transform a partition of the interval of integration, originally $u \in [0, \pi]$ with equal subintervals into a partition of the new interval of integration $t \in [0, \sqrt{2\pi}]$ with *unequal* subintervals $du = 2t \, dt$. You can see this transformation by dragging the slider w in Line 2 of the worksheeet below from $w = 0$ to $w = 1$.

Learning outcomes:
Author(s):

Desmos link: `https://www.desmos.com/calculator/toiwzzqkva`

152: Sub 2

2 The Area of a Circle

Computing the area of a circle of radius a with integration can be a challenging problem when approached in the standard way.

Starting with an equation

$$\sqrt{x^2 + y^2} = a$$

of a circle centerd at the origin, we'll find the area of the region bounded by the semicircle

$$y = f(x) = \sqrt{a^2 - x^2}$$

and the x-axis. On the one hand, the area of a differential rectangle having width dx and endpoints $(x, 0)$ and $(x, y) = (x, f(x))$ is

$$dA = y\,dx = \sqrt{a^2 - x^2}\,dx.$$

So the area of the semicircle is

$$A = 2\int_0^a \sqrt{a^2 - x^2}\,dx.$$

Now we're faced with the problem of finding an anti-derivative of $\sqrt{a^2 - x^2}$. The standard method is to use a trigonmetric substitution. But this really amounts parameterizing the circle.

So let's start over and parameterize the circle $x^2 + y^2 = a^2$ in terms of the polar angle θ of the vector $\overrightarrow{OP}$ from the origin to a point P on the circle with coordinates (x, y).

From the *definitions* of the cosine and sine functions we have

$$x = a\cos\theta \text{ and } y = a\sin\theta\,,\ 0 \le \theta \le 2\pi.$$

We'll slice the upper-semicircle perpendicular to the x-axis as before. Now since $0 \le \theta \le \pi$ for the upper semicircle,

$$dx = d(a\cos\theta) = -a\sin\theta \le 0,$$

the area of the differential rectangle is

$$\begin{aligned} dA &= -y\,dx \\ &= -(a\sin\theta)d(a\cos\theta) \\ &= -(a\sin\theta)(-a\sin\theta\,d\theta) \\ &= a^2\sin^2\theta\,d\theta. \end{aligned}$$

Geometrically, the effect of this parameterization is to transform the original parametrization

$$(x, y) = (x, \sqrt{a^2 - x^2}), \ -a \le x \le a$$

of the upper-semicircle

Desmos link: https://www.desmos.com/calculator/eu8hz9nmgz

152: Circle Area

Parametrically-defined Curves, CW

Curves defined parametrically and signed area.

1 The Basics

Not all curves are the graphs of functions.

The green curve below in the first quadrant, for example, is neither the graph of y as a function of x (ie. $y = f(x)$) nor of x as a function of y (ie. $x = f(y)$).

Why not?

Free Response:

Desmos link: `https://www.desmos.com/calculator/vio0b8ztnl`

152: Parametric 2

But we can describe the curve *parametrically.* This means to give functions

$$x = f(t) \text{ and } y = g(t)$$

that express the coordinates (x, y) of a point on the curve in terms of some parameter (t here) which is often helpful to think of as being time. We can then think dynamically of the *motion* of a point along its path instead of just a static path.

These coordinate functions for the green curve above are graphed in the fourth ($x = f(t)$) and second ($y = g(t)$) quadrants and share the domain $0 \leq t \leq 4$ (you should ignore the negative signs on the vertical and horizontal axes). To see the motion of the point along its path, play the slider $u = t$ in Line 2.

The question we would like to address here is to compute the unsigned (ie. positive) area of the region bounded by the curve

$$(x, y) = (f(t), g(t))\,,\ 0 \leq t \leq 4.$$

We'll use with the demonstration below, where the graphs of the coordinate functions are hidden.

Desmos link: `https://www.desmos.com/calculator/z9ndh3wvvv`

Learning outcomes:
Author(s):

152: Parametric 2B

As usual, our first choice is to decide which way to slice the region and we'll start by slicing perpendicular to the x-axis. But this is not really an accurate description because the differential slices run from the x-axis to the curve and do not always lie within the region.

To see this, drag the slider the slider u in Line 2 to move the differential rectangle. Sometimes the rectangle lies below the region, other times it lies partly within and partly below the region, and at other times completely within the region. Nevertheless, we can use these differential rectangles to compute the area of the region.

The key idea is to work with their *signed* areas. Cutting perpendicular to the x-axis, the differential rectangles have signed area

$$\begin{aligned} dA &= y\,dx \\ &= g(t)d(f(t)) \\ &= g(t)(f'(t)\,dt) \\ &= g(t)f'(t)\,dt. \end{aligned}$$

And so the region bounded by the green curve and the x-axis has signed area

$$A = \int_0^4 g(t)f'(t)\,dt.$$

Let's stop and think a minute about what we mean by signed area. The sign of the differential area dA depends upon *both* the signs of $y = g(t)$ and $dx = f'(t)\,dt$. For our example $y = g(t) \geq 0$, $dt > 0$, and so dA has the same sign as dx.

(a) Drag the slider u in Line 2 from $u = 0$ to $u = b = 4$. For what values of u is dA positive? Negative?

(b) Sketch by hand a rough graph of the cumulative signed area function

$$A_1(t) = \int_0^t g(u)f'(u)\,du.$$

Then drag the slider u in Line 2 to $u = 0$ and activate the folder *Graph of signed area function, slices perp to x-axis* in Line 7. Drag u to $u = b = 4$ and see the graph of the signed area function. Compare it with your graph.

(c) How is the integral

$$A_1(4) = \int_0^4 g(t)f'(t)\,dt$$

related to the area of the region bounded by the x-axi and the curve $(x, y) = (f(t), g(t))$? Explain your reasoning.

Free Response:

We can also take our differential rectangles perpendicular to the y=axis. Then

$$dA = x\,dy = f(t)g'(t)\,dt.$$

(d) Repeat parts (a)-(c) above for these differential rectangles and the signed area function

$$A_2(t) = \int_0^t f(u)g'(u)\,du.$$

(e) Now use the actual coordinate functions

$$x = f(t) = t^2 - 3t + 3\,,\; 0 \le t \le 4$$

and

$$y = g(t) = 4t - t^2\,,\; 0 \le t \le 4$$

to compute the area of the region bounded by the x-axis and the curve

$$(x, y) = (f(t), g(t))\,,\; 0 \le t \le 4.$$

Do this twice. Once with slices perpendicular to the x-axis and again with slices perpendicular to the y-axis.

2 An Application: Gas Consumption

Example 56. The function

$$r = f(t)\,,\; 0 \le t \le 4,$$

expresses the rate (in gal/mile) at which a car burns gas in terms of the number of hours past noon during a 240-mile trip.

The function

$$s = g(t)\,,\; 0 \le t \le 4,$$

expresses the car's trip odometer reading (in miles) in terms of the number of hours past noon.

The car has 14 gallons of gas at 1:00pm.

The functions f (second quadrant) and g (fourth quadrant) along with the curve $(x, y) = (g(t), f(t))$ (first quadrant) are graphed below.

Desmos link: `https://www.desmos.com/calculator/akedzpal8z`

Substitution and Gas Comsumption 22

(a) Approximate the rate, measured in gal/hour, at which the car burns gas at 1:54pm. Explain your reasoning and include units for all numbers in your computation.

(b) Approximate the number of gallons of gas in the tank at 1:54pm.

(c) Find a function

$$G = w(s)\,,\ 0 \leq t \leq 4,$$

that expresses the number of gallons of gas in the car in terms of the number of hours past noon.

3 Exercises

Exercise 413 *This question is about the circle*

$$(x, y) = (f(\theta), g(\theta)) = (a\cos\theta, b + a\sin\theta)\,,\ 0 \leq \theta \leq 2\pi.$$

Here a, b are positive constants measured in meters.

Desmos link: `https://www.desmos.com/calculator/weihqr1fxu`

152: Parametric 3

(a) *Use the worksheet below to sketch by hand the signed area functions*

$$A_1(\theta) = \int_0^t g(t)f'(t)\,dt$$

and

$$A_2(\theta) = \int_0^t f(t)g'(t)\,dt.$$

Then check your graphs by activating the folders in Lines 7 and 23.

Desmos link: `https://www.desmos.com/calculator/bg3tmflc1f`

152: Parametric 3B

(b) *Evaluate the integrals*

$$A_1(2\pi) = \int_0^{2\pi} g(\theta)f'(\theta)\,d\theta$$

and

$$A_2(2\pi) = \int_0^{2\pi} f(\theta)g'(\theta)\,d\theta$$

geometrically *by using the graph of the function*

$$A = a^2 \sin^2 \theta \, , \, 0 \le \theta \le \pi$$

shown below. Work in general, not with the specific value of a in the worksheet.

Desmos link: `https://www.desmos.com/calculator/dsoqsyyfba`

152: Sine Squared

Exercise 414 *Use the parameteriziation*

$$(x, y) = (a \cos \theta, a \sin \theta) \, , \, 0 \le \theta \le 2\pi,$$

of the circle with radius a meters centered at the origin to find an expression for the volume of a sphere with the same radius.

Exercise 415 *The circle with center $(x, z) = (b, 0)$ and radius a is rotated about the z-axis to sweep out the torus shown below.*

Desmos3D link: `https://www.desmos.com/3d/baapxxf50y`

152: Torus

The aim of this problem is to express the volume of the torus in terms of the parameters a and b, measured in meters. We'll do this by parameterizing the circle in terms of the polar angle of the vector $\overrightarrow{CP}$ from the center of the circle to a point P on the circle with coordinates (x, y). See the figure below.

Desmos link: `https://www.desmos.com/calculator/qjpoe06x3y`

152: Torus 2D

(a) *Express the coordinates of point P in terms of the polar angle θ.*

(b) *Activate the folder in Line 1 of the worksheet above to see the cross-section of the torus swept out by PQ. The cross-section should really be drawn peprendicular to the page. You can see the cross-section in the top worksheet by deactivating the folder* Torus *in Line 4.*

(c) *Express the area of the cross-section in terms of a, b, and θ.*

(d) *Express the differnential volume of the corresponding slice in terms of a, b, and θ.*

(e) *Add these differential volumes to find the volume of the torus.*

Exercise 416 *Do Question 2 in the chapter*

Volumes, Part 2

of our class notes by using trig functions to parameterize the circle of radius a centered at the origin.

Exercise 417 *Find the area bounded by the astroid*

$$(x, y) = (a\cos^3\theta, a\sin^3\theta), \; 0 \leq \theta \leq 2\pi,$$

where $a > 0$ is a constant measured in meters.

Exercise 418 *Find the area bounded by the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b > 0$ are constants measured in meters.

Start by parameterizing the ellipse.

Integration by Parts

Integration by parts, geometrically.

1 The Basics

Learning outcomes:
Author(s):

Arclength of Curves in the Plane

Arclength.

1 The Ideas

We'll speak of our curves as trails and assume they lie in a vertical plane (they don't twist around a mountain like a real trail usually does).

The constant case for computing the length of a trail is when the trail makes a constant angle ϕ with the horizontal. This means that the trail is a line segment and looks just like a ramp.

In this case, if the endpoints of the ramp have coordinates (x_1, y_1) and (x_2, y_2), the ramp has length

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

where

$$\Delta x = x_2 - x_1$$

and

$$\Delta y = y_2 - y_1.$$

It can be helpful to express the length in terms of the inclination angle ϕ of the ramp. Here we have two possibilities assuming $\Delta s \geq 0$. They are

$$\Delta s = |\Delta x| \sec \phi$$

or

$$\Delta s = |\Delta y| \csc \phi.$$

For a general trail, where the inclination angle is not necessarily constant, the Pythagorean theorem tells us the differential arclength is

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

We can write this in terms of the inclination angle ϕ (that now can vary along the curve) as

$$ds = \sec \phi \, dx$$

or

$$ds = \csc \phi \, dy.$$

Learning outcomes:
Author(s):

When the equation of a trial is expressed as a function $y = f(x)$, we usually write

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, |dx| \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \end{aligned}$$

if $dx > 0$.

I like to think of

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \geq 1,$$

as a dimensionless scaling factor that converts a differential change dx along the horizontal to a differential arclength along the trial.

Question 419 (a) *Express the derivative dy/dx in terms of the inclination angle ϕ.*

(b) *Express the scaling factor $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ in terms of ϕ.*

Finally, the length of the trail between points (a, b) and (c, d) (with $c > a$) is the sum

$$\int_a^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

of these differential arclengths.

2 Circles

Example 57. This problem is about the semicircle

$$y = f(x) = \sqrt{a^2 - x^2},$$

where the constant $a > 0$ is measured in meters.

(a) Use integration to find an expression for the function

$$s = g(w), -a \leq w \leq a,$$

that gives the distance from the point $A(a, 0)$ to the point P with coordinates $(w, f(w))$ on the semicircle

$$y = f(x) = \sqrt{a^2 - x^2}.$$

Measure the distance from A to P along the semicircle.

Start by expressing the differential arclength ds along the semicircle in terms of the differential change dx.

Desmos link: `https://www.desmos.com/calculator/rmeuvbnjds`

152: ArcLength Semicircle

(b) Interpret your result in part (a) geometrically.

(c) Sketch by hand a graph of the function $s = g(x)$. Then activate the folder *Arclength Function* in Line 11 above to see how you did.

(d) Express $g'(x)$ in terms of $f'(x)$.

(e) Express the coordinates of P in terms of its distance s from $(a, 0)$ as measured along the semi-circle.

Example 58. Let $a > 0$ be a constant measured in meters.

Use integration to find an expression for the function

$$s = f(\phi)\,,\ 0 \le \phi < 2\pi,$$

that gives the distance from the point $A(a, 0)$ to the point P with coordinates $(a\cos\phi, a\sin\phi)$ on the circle

$$(x, y) = (a\cos\theta, a\sin\theta)\,,\ 0 \le \theta < 2\pi,$$

in terms of ϕ. Measure the distance from A to P counterclockwise along the circle.

Start by expressing the differential arclength ds along the semicircle in terms of the differential change $d\theta$.

3 Parameterizing an Astroid by Arclength

d

Example 59. This example is about the curve (an astroid)

$$(x, y) = (a\cos^3\theta, a\sin^3\theta)\,,\ 0 \le \theta \le 2\pi,$$

where $a > 0$ is a constant measured in meters.

Desmos link: `https://www.desmos.com/calculator/up6jvtmspl`

152: Astroid

(a) Find an expression for the function

$$s = f(\phi)\,,\ 0 \le \phi < 2\pi,$$

that gives the distance from the point $A(a, 0)$ to the point P with coordinates $(a\cos^3\phi, a\sin^3\phi)$ on the astroid

$$(x, y) = (a\cos^3\theta, a\sin^3\theta)\,,\ 0 \le \theta < 2\pi,$$

in terms of ϕ. Measure the distance from A to P counterclockwise along the curve.

Start by expressing the differential arclength ds along the astroid in terms of the differential change $d\theta$.

(b) Evaluate $f(\pi/2)$ and interpret its meaning. Compare this distance with the straight-line distance from $(a, 0)$ to $(0, a)$.

(c) Express the coordinates of a point P on the curve in terms if its distance s from A measured counterclockwise along the astroid.

4 Trail Profiles

Circles, Spheres, Area, Volume

Thoughts on the hoof of Archimedes, spheres, circles.

1 The Ideas

Learning outcomes:
Author(s):

Hyperbolic Trigonometry

An introduction to the hyperbolic trigonometric functions.

1 The Hyperbolic Cosine and Sine Functions

The function *hyperbolic cosine* is defined as

$$\cosh(x) = 0.5(e^x + e^{-x}).$$

Its graph is shown below.

Desmos link: `https://www.desmos.com/calculator/nubdwzgsa9`

152: Hyp Cosine

(a) Drag the slider u in Line 2 of the worksheet above. Explain how to get the graph of the function $y = \cosh x$ from the graphs of the functions $y = e^x$ and $y = e^{-x}$.

Free Response:

(b) What is the range of the function $f(x) = \cosh x$?

(c) Activate the folder *Hyperbolic Sine Function* in Line 15 above to see the graph of the hyperbolic sine function

$$\sinh(x) = 0.5(e^x - e^{-x}).$$

(d) Drag the slider u in Line 2 of the worksheet above. Explain how to get the graph of the function $y = \cosh x$ from the graphs of the functions $y = e^x$ and $y = e^{-x}$.

Free Response:

(e) What is the range of the function $g(x) = \sinh x$?

(f) Prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

Hint: For a quick way, first factor the above expression.

Learning outcomes:
Author(s):

(g) Show that
$$\frac{d}{dx}(\cosh x) = \sinh x$$
and
$$\frac{d}{dx}(\sinh x) = \cosh x$$

2 Hanging Chains

Hold the ends of a chain and let the chain sag under its own weight. The resulting curve looks like a parabola, and so it was thought. But it turns out that this is not correct.

Desmos link: `https://www.desmos.com/calculator/tj3dz2cnf0`

152: Hanging Chain 1

(a) To see that the dashed (orange) chain in the worksheet above is *not a parabola* drag the slider c in Line 2.

(b) It turns out that a chain with uniform density hanging in a uniform graviatational field assumes the shape of the hyperbolic cosine function $y = \cosh x$ (we'll see why later in the course). All that's needed to describe a particular chain is to scale the size of the graph by some factor. Test this by dragging the slider a in Line 6 to make the graph of the function
$$f(x) = -a + a\cosh(x/a)$$
match the highlighted (orange) chain below.

3 The Hyperbolic Tangent Function

The hyperbolic tangent function is defined as
$$\begin{aligned}\tanh x &= \frac{\sinh x}{\cosh x} \\ &= \frac{e^x - e^{-x}}{e^x + e^{-x}}.\end{aligned}$$

(a) Find the domain and range of the function $y = \tanh x$.

(b) Sketch by hand a graph of the function $y = \tanh x$.

(c) Use your graph from part (b) to sketch a graph of the derivative
$$y = \frac{d}{dx}(\tanh x).$$

(d) Find an expression for the derivative in part (c) in terms of $\cosh x$ and/or $\sinh x$.

4 Inverse Functions

The key idea here is that *the derivative of the inverse of a function is the reciprocal of the function's derivative.* This should be intuitive.

So to find the derivative of the inverse of a one-to-one function $y = f(x)$, we should first express dy/dx in terms of y. Then take the reciprocal.

(a) Express the derivaive of each function below in terms of its output.

(i)
$$y = \sinh x$$

(ii)
$$y = \cosh x\,,\ x \geq 0$$

(iii)
$$y = \tanh x$$

(b) Use the results of part (a) to find expressions for the derivatives

(i)
$$\frac{d}{dy}(\sinh^{-1} y)$$

(ii)
$$\frac{d}{dy}(\cosh^{-1} y)$$

(iii)
$$\frac{d}{dy}(\tanh^{-1} y)$$

Review of the Chain Rule

Derivative review.

1 Example

Here's an example of how to use u-substitution with the chain rule. Just as importantly, it's an example of how to show your work clearly and precisely.

Example 60. (a) Ues the chain rule to find an expression for the derivative

$$\frac{d}{dx}\left((2x^3+1)^2\right).$$

(b) Use part (a) to find an equation of the tangent line to the curve

$$y = (2x^3+1)^2$$

at the point $(1, 9)$.

(c) Answer part (a) without the chain rule.

Explanation. (a) Let

$$y = \left(2x^3+1\right)^2$$

and

$$u = 2x^3+1.$$

Then

$$y = u^2$$

and from the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{d}{du}\left(u^2\right)\cdot\frac{d}{dx}\left(2x^3+1\right)\\ &= 2u(6x^2)\\ &= 2(2x^3+1)(6x^2).\end{aligned}$$

Learning outcomes:
Author(s):

(b) The slope of the tangent line to the curve $y = (2x^3+1)^2$ at the point $(1,9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 2(3)(6) = 36,$$

and an equation of the tangent line is

$$y - 9 = 36(x-1).$$

(c) We can get the same result without the chain rule, by rewriting the original function as

$$y = f(x) = (2x^3+1)^2 = 4x^6 + 4x^3 + 1.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(4x^6 + 4x^3 + 1\right)\\ &= 4\frac{d}{dx}\left(x^6\right) + 4\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}(1)\\ &= 24x^5 + 12x^2.\end{aligned}$$

Then the slope of the tangent line to the curve $y = (2x^3+1)^2$ at the point $(1,9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 14 + 12 = 36,$$

as before.

2 Exercises

Directions: Follow all the steps of *Example 1* (a) exactly for each of the following. Show all work as above. Do not skip steps.

Exercise 420 *Find simplified expressions for each of the following derivatives.*

(a)

$$\frac{d}{d\theta}\left(\ln|\cot(\theta/2)|\right)$$

(b)

$$\frac{d}{d\theta}\left(\ln|\sec\theta + \tan\theta|\right)$$

(c)

$$\frac{d}{d\theta}\left(\operatorname{arctanh}(\sin(\theta))\right)$$

(d)

$$\frac{d}{d\theta}\left(\operatorname{arcsinh}(\tan(\theta))\right)$$

(e)

$$\frac{d}{dt}\left(\tan\left(k \arctan(t/2)\right)\right),$$

where k is a constant.

Exercise 421 *Find the acute angle the tangent line to the curve*

$$y = \ln|\sec\theta|$$

at the point $P(\pi/7, \ln(\sec(\pi/7))$ makes with the x-axis.

Exercise 422 *Find simplified expressions for each of the following derivatives. Follow all steps of* Example 1(a).

(a)

$$\frac{d}{dx}\left(x \arctan x - \ln\sqrt{1+x^2}\right)$$

(b)

$$\frac{d}{dt}\left(t\ln|t| - t\right)$$

(c)

$$\frac{d}{dx}\left(\ln(x+\sqrt{x^2-1})\right)$$

(d)

$$\frac{d}{dx}\left(\ln(x+\sqrt{x^2+1})\right)$$

(e)

$$\frac{d}{dx}\left(\ln(x+\sqrt{x^2-1})\right)$$

(f)

$$\frac{d}{dt}\left(\ln\sqrt{\frac{1+t}{1-t}}\right)$$

Trails and Hanging Chains, Part 1

Trails and catenaries.

1 Angles of Inclination

Question 423 *Find the measure of the acute (between* 0 *and* $\pi/2$*) angle the tangent line to the curve*

$$y = \ln|\sec\theta|$$

at the point $(2\pi/7, \ln|\sec(2\pi/7)|)$ *makes with the* x*-axis. Give an exact answer without using a calculator.*

Desmos link: `https://www.desmos.com/calculator/yyyj3pjmk2`

152: Log Secant

Question 424 *Find the measure of the acute angle the tangent lines to the curve*

$$y = 4\tan(x/20)$$

make with the x*-axis at the points with* y*-coordinate* $y = 3$*. Give an exact answer, fully simplified, without using a calculator.*

Desmos link: `https://www.desmos.com/calculator/df7hjgabzb`

152: Angle of Inclination

Explanation. Since we are given the y-coordinate of the point of tangency, we should express the derivative of our function not in terms of its input, but in terms of its output.

As a start, note that if $z = \tan\theta$, then

$$\begin{aligned}\frac{dz}{d\theta} &= \frac{d}{d\theta}(\tan\theta)\\ &= \sec^2\theta\\ &= 1+\tan^2\theta\\ &= 1+z^2.\end{aligned}$$

Learning outcomes:
Author(s):

For our function $y = 4\tan(x/20)$, let $u = x/20$. Then

$$y = 4\tan u$$

and by the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{d}{du}(4\tan u)\cdot\frac{d}{dx}\left(\frac{1}{20}x\right)\\ &= 4(1+\tan^2 u)\left(\frac{1}{20}\right)\\ &= \frac{1+(y/4)^2}{5}.\end{aligned}$$

So the tangent lines at the points on the curve $y = 4\tan(x/20)$ with y-coordinate $y = 3$ have slope

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{y=3} &= \left.\left(\frac{1+(y/4)^2}{5}\right)\right|_{y=3}\\ &= 5/16.\end{aligned}$$

These tangent lines therefore cut the x-axis at the acute angle $\phi = \arctan(5/16)$.

2 A Climber's Trail

Question 425 *A loading ramp of length L meters runs from the ground to a truck. The ramp makes an angle of ϕ radians with the ground.*

Find an expression for the function

$$h = f(s)\,,\ 0 \le s \le L,$$

that expresses the height of the ramp (in meters) in terms of the distance along the ramp, measured in meters from the bottom of the ramp.

Question 426 *The function*

$$h = f(s) = 100 + 250s - 10s^3\,,\ 0 \le s \le 5$$

expresses the altitude (in meters) of a climber's trail in terms of the distance from the trailhead (in km).

(a) *Evaluate the derivative*

$$\left.\frac{dh}{ds}\right|_{s=1}.$$

Include units.

(b) *Interpret the meaning of the above derivative using the language of small changes for a* specific *small change.*

(c) *Find the exact angle the trail makes with the horizontal one kilometer from the trailhead.*

Desmos link: `https://www.desmos.com/calculator/ljcugfqmee`

152: Hiking Trail 54

Explanation. (a) The derivative is

$$\begin{aligned}\frac{dh}{ds} &= \frac{d}{ds}(100 + 250s - 10s^3)\\ &= 250 - 30s^2\end{aligned}$$

and

$$\left.\frac{dh}{ds}\right|_{s=1} = 220 \text{ meters/km}.$$

(b) The derivative relates a small change

$$\Delta s = s - 1$$

in distance along the trail from kilometer marker $s = 1$ and the corresponding small change

$$\Delta h = h - f(1) = h - 340$$

in altitude (measured in meters).

More specifically, if $\Delta s \sim 0$, then

$$\begin{aligned}\Delta h &\sim \left.\frac{dh}{ds}\right|_{s=1} \cdot \Delta s\\ &= (220 \text{ meters/km})(\Delta s \text{ km})\\ &= 220\Delta s \text{ meters}.\end{aligned}$$

So for example, if we walk 1 meter (ie. 0.001 km) farther along the trail from kilometer marker $s = 1$ km, then our altitude increases by approximately

$$\Delta h \sim 220(0.001) \text{ meters} = 0.2 \text{ meters}.$$

(c) The description of the derivative's meaning tells us how to compute the trail's inclination angle at kilometer marker $s = 1$ km. Since we gain altitude at the rate of

$$220 \text{ meters/km} = 0.22 \text{ meters in elevation/meter walked}$$

one kilometer from the trailhead, the trail looks locally like a ramp inclined at the angle

$$\phi = \arcsin(0.22) \text{ rad}$$

above the horizontal there.

Remark: It's important to distinguish between the graph of the function $h = f(s)$ that expresses altitude in terms of distance along the trail and the trail profile. The latter is like a picture of the actual trail, assuming the trail stays confined to a vertical plane. For most trails, the two are indistinguishable. The worksheet below, for example, shows our original graph $h = f(s)$ along with the corresponding trail profile (in red).

Desmos link: `https://www.desmos.com/calculator/e5kuf1je8p`

152: Hiking Trail 54B

But for steeper trails, like one an ant might climb, there is a clear difference between the two.

Desmos link: `https://www.desmos.com/calculator/lktn53d7ju`

152: Hiking Trail 54C

3 A Hanging Chain

Question 427 *Which of the following curves have the same shape as the curve*

$$y = \sin\theta?$$

Select all that apply.

Select All Correct Answers:

(a) $y = 2\sin\theta$

(b) $y = 2\sin(2\theta)$

(c) $y = 2\sin(\theta/2)$ ✓

(d) $y = 0.2\sin(0.2\theta)$

(e) $y = 0.2\sin(5\theta)$ ✓

Note: Two curves have the same shape if you can scale a photograph of one to get an exact copy of the other.

Question 428 *A chain of uniform density hanging under its own weight in a uniform gravitational field takes the shape of a the curve $y = \cosh x$ (we'll see why later in the course).*

Desmos link: `https://www.desmos.com/calculator/iifkdeyq3y`

152: Hanging Chain 2

(a) *All that's needed to describe a particular chain is to scale the size of the curve $y = \cosh x$ by some factor. Test this by dragging the slider a in Line 6 to make the graph of the function*

$$f(x) = -a + a\cosh(x/a)$$

match the highlighted (orange) chain below.

(b) *What are the units of a? How do you know?*

(c) *Using the value of a from part (a), find the angle the chain makes with the horizontal at the points two meters above the low point of the chain. Give an exact expression, fully simplified, without using a calculator.*

See the chapter

Hyperbolic Trigonometry

of our class notes for information on the hyperbolic trig functions.

The Mercator Map

The Mercator map and its relation to Lambert's equal-area map.

1 Scaling Factors and Elastic Bands

The interpretation of the derivative as a rate of change is limited. A more universal description of the derivative, and one that generalizes to higher dimensions, is as a scaling factor. Multiplying the derivative by a sufficiently small change in the input to a function gives a good approximation to the change in the output. Of all possible scaling factors the derivative is the best in the sense that the error in the approximation approaches zero faster than the change in the output.

We can visualize the derivative as a scaling factor by regarding a function's domain as an elastic band and the function as acting on that band by stretching or compressing it.

For example, imagine a thin elastic band of length 4 meters running along the horizontal L-axis from $L = 0$ meters to $L = 4$ meters. Now hold the left end fixed at $L = 0$ and stretch the band by moving right end an additional four meters to the right . Then the function

$$H = g(L) = 2L\,,\ 0 \leq L \leq 4,$$

describes this stretching action. It takes as an input the distance (in meters) of a point on the band from the origin ($L = 0$) and returns as an output the distance between the origin and the corresponding point on the stretched band. The exploration below shows this stretching action.

Exploration 429 *Drag the slider k in Line 2 below from $k = 1$ to $k = 2$ to illustrate the stretching action for the function g.*

Desmos link: `https://www.desmos.com/calculator/qejivz36ui`

151: Rubber Band 1

The *global stretching factor* for a linear function like the one above,

$$H = g(L) = 2L\,,\ 0 \leq L \leq 4,$$

Learning outcomes:
Author(s):

is the slope of its graph. We can calculate this factor as an average stretching factor (ie. an average rate of change) between the points $L = a$ meters and $L = b$ meters from the origin. For the function $H = f(L) = 2L$, the stretching factor is

$$\begin{aligned}\frac{\Delta H}{\Delta L} &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{2(b-a)}{b-a} \\ &= 2,\end{aligned}$$

This tells us that any two points on the stretched band are twice as far apart as they were on the unstretched band.

Question 430 (a) *What are the units of the stretching factor?*

Free Response:

(b) *Find an expression for the inverse function*

$$L = f^{-1}(H).$$

Include a domain.

(c) *Interpret the inverse function as a deformation of a thin elastic band. What is the global stretching factor for this function?*

Free Response:

Example 61. Here's an example

$$H = f(L) = 10 - \sqrt{100 - L^2}\,,\ 0 \le L \le 10,$$

of a non-linear stretching function (where L and H are measured in meters as before). Like most functions in this class, it acts like a linear function near almost all points in its domain.

To stretch the elastic band in the demonstration below, drag the slider u in Line 2 from $u = 0$ to $u = 1$. Then zoom in close enough to the point $H = f(9.5)$ in the stretched band (highlighted in black) to make the stretching function look linear.

Desmos link: `https://www.desmos.com/calculator/xk8dvcfgwi`

152: Rubber Band Ladder 5B

(a) Use the close-up view of the stretched band to approximate the local stretching factor at the input $L = 9.5$.

(b) Parts of the elastic band get stretched, others compressed. Identify these.

2 Lambert's Equal-Area Map

You probably saw the idea of a stretching (scaling) factor long before taking calculus. Look at a map of Seattle and you'll see a scale, probably near the bottom. It might be 2 miles/inch, or something like that. But you will not see such a scale on a map of the earth for the simple reason that one does not exist. The scale varies from place to place and is usually a function of latitude. Even at a specific location, there is often not one but infinitely many scaling factors that depend on the direction in which you head.

The worksheet below shows a way to create a map of the earth by wrapping a cylinder around its equator. The map sends each point of the earth to its nearest point on the cylinder (ie. straight out, directly away from the earth's axis of rotation). Unwrapping the cylinder gives Lambert's equal-area map, from the earth to the points in a rectangle.

Exploration 431 *Drag the slider in Line 6 from $r_1 = 0$ to $r_1 = 1$ to uwrap the cylinder. Drag sliders theta$_1$, ϕ_1 in LInes 2 and 4 to vary the longitude and latitude of the point on the sphere.*

Desmos3D link: `https://www.desmos.com/3d/ehyvvttdeo`

152: Mercator 0

Suppose the earth has radius one. Lambert's map sends the circle of latitude ϕ (the angle between the plane of the equator and the segments from the sphere's center to points on the circle) to a circle on the cylinder with unit radius. So the map stretches the circle of latitude ϕ, with radius $\cos\phi$, by the factor

$$\lambda_p = \frac{1}{\cos\phi}.$$

This is the scaling factor along the circle in the east-west direction at latitude ϕ.

In the north-south direction the scaling factor is also a function of latitude. The key point is that the map sends a point on the circle of latitude ϕ to signed height

$$h = f(\phi) = \sin\phi$$

above the plane of the equator. So the scaling factor at latitude ϕ in the north-south direction is

$$\lambda_m = \frac{dh}{d\phi} = \cos\phi.$$

Drag the slider m in Line 2 of the worksheet below to visualize the scaling factor.

Desmos link: `https://www.desmos.com/calculator/kbryuogdaj`

151: Lambert Map Cosine Function

3 The Mercator Map

Watch this video.

Equal Earth Projection

4 Loxodromes

Rhumb Lines on the Sphere

For a curve on a sphere of radius a that cuts the meridians at a fixed angle α,

$$d\theta = \frac{\tan\alpha\, d\phi}{\cos\phi},$$

where ϕ is the latitude and $\theta thelongitude. Taking \theta = 0$ at the equator where $\phi = 0$,

$$\begin{aligned}\theta &= \tan\alpha \int_0^{\phi^*} \sec\phi^*\, d\phi^* \\ &= \tan\alpha \operatorname{arctanh}(\sin\phi).\end{aligned}$$

So

$$\sin\phi = \tanh(k\theta),$$

where $k = \cot\alpha$.

Because $-\pi/2 < \phi < \pi/2$,

$$\cos\phi = \frac{1}{\cosh(k\theta)},$$

a parameterization of the rhumb line by longitude is

$$(x, y, z) = \left(\frac{a\cos\theta}{\cosh(k\theta)}, \frac{a\sin\theta}{\cosh(k\theta)}, a\tanh(k\theta)\right).$$

And a parameterization by latitude,

$$(x, y, z) = (a\cos\phi, a\cos\phi, a\sin\phi)$$

Desmos3D link: `https://www.desmos.com/3d/6cqnjsuew7`

152: Mercator 1

Desmos link: `https://www.desmos.com/calculator/rbcbxdrvcp`

Geogebra link: `https://www.geogebra.org/m/mvp9zvge`

152: Mercator

Riemann Sums with Geometric Partitions

Using Riemann sums with geometric partitions to evaluate definite integrals.

1 Integrating $1/x$

Before using Riemann sums and limits to find an expression for the integral

$$g(u) = \int_1^u \frac{1}{x}\,dx,$$

let's first compare the integrals

$$\int_1^2 \frac{1}{x}\,dx \tag{9}$$

and

$$\int_3^6 \frac{1}{x}\,dx. \tag{10}$$

We'll do this geometrially, without a computation.

The idea is to first describe a compositions of transformations that takes the curve $y = 1/x$ to itself and the interval $[1, 2]$ to the interval $[3, 6]$. Then we'll look at how this composition affects are

Exploration 432 (a) *Drag the slider v in Line 2 of the worksheet below from $v = 1$ to $v = 3$ and describe what happens to the graph of $y = 1/x$.*

Free Response:

(b) *What does the transformation do the area of the shaded rectangle (or any rectangle)?*

It mutiplies the area by a factor of 3.

(c) *Express a equation of the new curve (dotted) in terms of v.*

An equation is

$$y = v/x.$$

Learning outcomes:
Author(s):

(d) *Next drag the slider w in Line 4 of the worksheet from $w = 1$ to $w = 3$ and describe what happens to the graph of $y = v/x$.*

Free Response:

(e) *What does the transformation do the area of the shaded rectangle (or any rectangle)?*

It mutiplies the area by a factor of $1/3$.

(f) *What is your conclusion about the relationship between the integrals (9) and (10) above? Explain your reasoning.*

Free Response:

Desmos link: `https://www.desmos.com/calculator/al95rowj4h`

152: Log 1 Transformations

https://www.desmos.com/calculator/al95rowj4h

2 Computing

To evaluate

$$\int_1^2 \frac{1}{x}\,dx$$

with a limit of Riemann sums, we will *not* partition the interval $[1, 2]$ of integration into subintervals of equal length. Instead, we'll use the partition

$$x_i = 2^{i/n}$$

Desmos link: `https://www.desmos.com/calculator/rbcbxdrvcp`

152: Log 1

Quiz Solutions Fall 2025

Quiz solutions.

1 Quiz1B

Question 433 *The table below shows your gas mileage at different readings of the trip ododometer reading (in miles) during a stretch of a 200-mile car trip.*

Odometer Reading (miles)	*Gas Mileage (miles/gallon)*
23	*23*
35	*25*
40	*30*
50	*32*

Assume the gas mileage is an increasing function of the odometer reading and find the best possible (ie. the least) upper bound for the number of gallons of gas your car burned between odometer readings 23 miles and 50 miles. Explain your reasoning thoroughly. Your concluding sentence should start with "Between odometer readings 23 miles and 50 miles the car burned less than ..."

Explanation. We begin by considering the constant case. Here that means assuming your gas mileage is constant. Suppose for example our car gets 25 miles/gallon over a 50 mile trip. Then over these 50 miles our car burns

$$\frac{50 \text{ miles}}{25 \text{ miles/gal}} = 2 \text{ gallons}$$

of gas.

Now we would burn the most gas when our car gets the worst (ie. least) gas mileage. So to find the least possible upper bound we should assume the gas mileage is constant over each interval of distance and equal to its minimum over that interval. Since the gas mileage is assumed to increase throughout the trip, this means we should choose the left endpoint of each interval.

So between odometer readings 23 miles and 50 miles the car burned less than

$$\frac{(35-23) \text{ miles}}{23 \text{ miles/gal}} + \frac{(40-35) \text{ miles}}{25 \text{ miles/gal}} + \frac{(50-40) \text{ miles}}{30 \text{ miles/gal}} < 1.06 \text{ gallons}$$

Learning outcomes:
Author(s):

of gas.

2 Quiz 2A

Question 434 *The table below shows the inclination angle of a mountain trail at several elevations along the trail as it ascends to Snow Lake. The inclination angle is the angle the trail makes with the horizontal.*

Inclination Angle (radians)	*Elevation (feet)*
0.25	*2300*
0.2	*2340*
0.18	*2400*

Assume the inclination angle is a decreasing function of altitude and find the best possible (ie. the greatest) lower bound for the length of the trail between elevations 2300 feet and 2400 feet. Explain your reasoning thoroughly.

Explanation. We begin by considering the constant case. Here that means supposing the inclination angle is constant. Call the angle θ and consider a portion of the trail with length Δs miles and change in elevation Δh miles. Then (draw a picture) since

$$\sin\theta = \frac{\Delta h}{\Delta s},$$

we know that

$$\Delta s = \frac{\Delta h}{\sin\theta}.$$

Now the steeper the trail, the shorter the distance it takes to gain a given change in elevation. So to find a lower bound for the trail length, over each interval we should assume the trail has a constant inclination angle equal to the greatest angle in that interval. Since we've assumed the inclination angle to be a decreasing function over above section of trail, we should choose the left endpoints in our Riemann sum.

So between elevations 2300 feet and 2400 feet the length of the trail is at least

$$\frac{(2340 - 2300)\text{ ft}}{\sin(0.25)} + \frac{(2400 - 2340)\text{ ft}}{\sin(0.20)} > 463.68 \text{ ft}.$$

3 Quiz 2B

Question 435 *The function*

$$\theta = g(s) = \frac{s}{2} - \frac{s^3}{12}\,,\ 0 \le s \le 3,$$

expresses the inclination angle (in radians) of the trail to Nada Lake in terms of the distance from the trailhead (measured along the trail in miles). The inclination angle is the angle the trail makes with the horizontal. It is positive (negative) when the trail slopes upward (downward) in the direction away from the trailhead.

The trail is at an elevation of 3200 *feet* 1.6 *miles from the trailhead.*

Use summation notation for an expression with n equal intervals of distance to find a lower bound for the trail's elevation (in feet) at Nada Lake (3 miles from the trailhead).

Use the graph of the function $\theta = g(s)$ shown below to help with your explanation.

Desmos link: `https://www.desmos.com/calculator/nzhemrajow`

152: Quiz 3A

Explanation. We begin by considering the constant case. Here that means supposing the inclination angle is constant. Call the angle θ and consider a portion of the trail with length Δs miles and change in elevation Δh miles. Then (draw a picture) since

$$\sin\theta = \frac{\Delta h}{\Delta s},$$

we know that

$$\Delta h = \Delta s \sin\theta.$$

Now the less steep the trail, the less the elevation gain over a given distance along the trail. So to find a lower bound for the elevation of Nada Lake, over each interval we should assume the trail has a constant inclination angle equal to the smallest angle in that interval. Since the inclination angle is a decreasing function over the trail from mile-marker 1.6 to Nada lake, we should choose the right endpoints in our Riemann sum.

With n-equal subintervals over the trail distance

$$\Delta s = (3 - 1.6) \text{ miles} = 1.4 \text{ miles},$$

each subinterval has length

$$\frac{\Delta s}{n} \text{ miles} = \frac{1.4}{n} \text{ miles}.$$

So the change in elevation from the 1.6-mile marker to Nada Lake is at least

$$\frac{1.4}{n}\sum_{i=1}^{n} \sin\left(g\left(1.6 + \frac{1.4}{n} i\right)\right)$$

miles.

Since the elevation of the trail is 3200 feet at the 1.6 mile-marker and there are 5280 feet in one mile, the elevation of Nada Lake is at least

$$3200 + 5280\left(\frac{1.4}{n}\right)\sum_{i=1}^{n} \sin\left(g\left(1.6 + \frac{1.4}{n}i\right)\right)$$

feet.

4 Quiz 3A

Question 436 *The function*

$$r = g(t) = 12 - 2|t - 11|, 0 \le t \le 20,$$

expresses a balloon's rate of ascent (in meters/min) in terms of the number of minutes past noon.

(a) *Use geometry, not the Fundamental Theorem, to evaluate the integral*

$$\int_{20}^{3} g(t)\, dt.$$

Include a graph of the function to help with your explanation. Shade the region of integration and indicate the direction of integration.

(b) *Interpret the meaning of the integral, in its form above, in this particular scenario.*

Explanation. (a) The key to graphing the function is to realize $g(t)$ has its maximum value when $2|t-11|$ is as small as possible. This happens when $t = 11$.

Desmos link: https://www.desmos.com/calculator/wioqqjwyt9

152: Quiz 3A

Adding the signed area of the three triangles and integrating from right to left gives

$$\begin{aligned}\int_{20}^{3} g(t)\, dt &= \frac{1}{2}(-3 \text{ min })(-6 \text{ meters/min}) + \frac{1}{2}(-12 \text{ min })(12 \text{ meters/min}) + \\ &= \frac{1}{2}(-2 \text{ min })(-4 \text{ meters/min}) \\ &= -59 \text{ ft}.\end{aligned}$$

(b) The meaning of the integral is that going backward in time from 12:20pm to 12:03pm the balloon loses 59 feet of altitude. More simply stated, *the balloon is 59 feet lower at 12:03pm than it is at 12:20pm.*

5 Quiz 3B

Question 437 *Evaluate the definite integral*

$$\int_1^4 \frac{2-\sqrt{t}}{t^2}\, dt.$$

Explanation. The key is to rewrite the integrand as

$$\frac{2-\sqrt{t}}{t^2} = 2t^{-2} - t^{-3/2}.$$

That way we use the Fundamental Theorem and undo the power rule. Here's the computation.

$$\begin{aligned}
\int_1^4 \frac{2-\sqrt{t}}{t^2}\, dt &= \int_1^4 (2t^{-2} - t^{-3/2})\, dt \\
&= \left(-2t^{-1} + 2t^{-1/2}\right)\Big|_1^4 \\
&= (-\frac{1}{2} + 2) - (-1 + 2) \\
&= \frac{1}{2}.
\end{aligned}$$

6 Quiz 4A

This was almost identical to Quiz 3B.

7 Quiz 4B

Question 438 *Use the graph of the function*

$$y = \int_2^x f(t)\, dt$$

shown below to answer the following questions. Explain your reasoning thoroughly in complete sentences.

Desmos link: `https://www.desmos.com/calculator/zoyrvkqbv8`

152: Quiz 4B

(a) *Solve the inequality $f(x) > 0$. Explain your reasoning.*

(b) *For what value of $x \in [0, 5]$ is $f(x)$ a minimum? Explain your reasoning.*

(c) *Find the minimum value of $f(x)$, $x \in [0, 5]$. Explain your reasoning and show your work.*

Explanation. The key point to answering these questions is to recognize that

$$\frac{d}{dx}\left(\int_2^x f(t)\,dt\right) = f(t).$$

(a) Solving the inequality $f(x) > 0$ means finding the values of x where the function graphed above is increasing. The solution set is therefore

$$\{0 < x < 1 \text{ or } 4 < x \leq 5\}.$$

(b) The minimum value of $f(x)$ occurs where the function graphed above is decreasing at the fastest rate. This occurs at approximate $x = 3$.

(c) The minimum value of $f(x)$ is the slope of the tangent line to the above graph at $x = 3$. Using the approximate coordinates of the points $(3, -1.6)$ and $(3.2, -2)$ on the graph, the minimum value is approximately

$$\frac{\Delta y}{\Delta x} \sim \frac{-0.4}{0.2} = -2.$$

We can make better sense of this problem by adding some context. So we'll suppose the function $f(t)$ expresses the rate of ascent (in ft/sec) of a balloon in terms of the number of seconds past noon. Then (changing the variable from x to t) the function

$$y = g_1(t) = \int_2^t f(x)\,dx,$$

expresses the balloon's change in height (measured in feet) from time $t = 2$ to time t seonds past noon. Adding the balloon's height at time $t = 2$, say h_0 feet, gives us the height (in feet)

$$h = g_2(t) = h_0 + \int_2^t f(x)\,dx$$

at time t seconds past noon.

Then since the functions g_1 and g_2 differ only by a constant, the derivatives

$$dh/dt = dy/dt$$

give the balloon's rate of ascent $f(t)$ function.

So the questions are asking us to determine

(a) when the balloon is ascending

(b) when the balloon is descending at the fastest rate

(c) the fastest rate of descent, expressed as a negative rate.

8 Quiz 5A

Question 439 (a) *Evaluate the definite integral*

$$\int_1^2 \frac{5x}{1+9x^2}\,dx.$$

Use substitution as in the homework. Do not skip steps. No credit without using substitution.

Explanation. It helps to write the integral as

$$\int_1^2 \frac{5x}{1+9x^2}\,dx = \int_1^2 (1+9x^2)^{-1} 5x\,dx.$$

Then the integrand is a product of two functions,

$$(1+9x^2)^{-1}$$

and

$$5x.$$

The first, $(1+9x^2)^{-1}$, is a composition of two functions. And the second $5x$ is a scalar multiple of the derivative

$$\frac{d}{dx}\left(1+9x^2\right) = 18x$$

of the inside function of the composition.

This tells us that the substitution

$$u = 1+9x^2$$

will undo the chain rule.

So with $u = 1+9x^2$, we have

$$du = 18x\,dx$$

and

$$5x\,dx = \frac{5}{18}\,du.$$

When $x = 1$,
$$u = 1 + 9(1)^2 = 10$$
and when $x = 2$,
$$u = 1 + 9(2)^2 = 37.$$

So
$$\begin{aligned}
\int_1^2 \frac{5x}{1+9x^2}\,dx &= \int_1^2 (1+9x^2)^{-1}(5x\,dx) \\
&= \int_{10}^{37} u^{-1}\left(\frac{5}{18}\,du\right) \\
&= \frac{5}{18}\int_{10}^{37} u^{-1}\,du \\
&= \frac{5}{18}\ln|u|\Big|_{u=10}^{u=37} \\
&= \frac{5}{18}\left(\ln|37| - \ln|10|\right) \\
&= \frac{5}{18}\ln(3.7)\,.
\end{aligned}$$

9 Quiz 5B

Question 440

Evaluate the definite integral
$$\int_0^1 \frac{8e^{-3t}}{\sqrt{9-4e^{-3t}}}\,dt.$$

Explanation. It helps to write the integral as
$$\int_0^1 \frac{8e^{-3t}}{\sqrt{9-4e^{-3t}}}\,dt = \int_0^1 8e^{-3t}\left(9-4e^{-3t}\right)^{-1/2}\,dt.$$

Then the integrand is a product of two functions,
$$\left(9-4e^{-3t}\right)^{-1/2}$$
and
$$8e^{-3t}.$$

The first, $(9-4e^{-3t})^{-1/2}$, is a composition of two functions. And the second, $8e^{-3t}$, is a scalar multiple of the derivative
$$\frac{d}{dt}\left(9-4e^{-3t}\right) = 12e^{-3t}$$

of the inside function of the composition.

This tells us that the substitution

$$u = 9 - 4e^{-3t}$$

will undo the chain rule.

So with $u = 9 - 4e^{-3t}$, we have

$$du = 12e^{-3t}\,dt$$

and

$$8e^{-3t}\,dt = \frac{2}{3}\,du.$$

When $t = 0$,

$$u = 9 - 4e^{0} = 5$$

and when $t = 1$,

$$u = 9 - 4e^{-3}.$$

So

$$\begin{aligned}
\int_0^1 \frac{8e^{-3t}}{\sqrt{9-4e^{-3t}}}\,dt &= \int_5^{9-4e^{-3}} (9-4e^{-3t})^{-1/2}(8e^{-3t}\,dt) \\
&= \int_5^{9-4e^{-3}} u^{-1/2}\left(\frac{2}{3}\,du\right) \\
&= \frac{2}{3}\int_5^{9-4e^{-3}} u^{-1/2}\,du \\
&= \frac{2}{3}(2)(u^{1/2})\Big|_{u=5}^{u=9-4e^{-3}} \\
&= \frac{4}{3}\left(\sqrt{9-4e^{-3}} - \sqrt{5}\right).
\end{aligned}$$

10 Quiz 6A

Question 441 *A parabola with its vertex at the origin and symmetric about the y-axis passes through the point $P(-3, 6)$.*

Find the area of the region bounded by the parabola, the tangent line to the parabola at P, and the x-axis.

Do this by slicing the region perpendicular to the y-axis.

Explanation. Since the parabola has its vertex at the origin and issymmetric about the y-axis, its equation is of the form

$$y = cx^2$$

for some constant $c \neq 0$.

Since the point $(-3, 6)$ lies on the parabola

$$6 = c(-3)^2$$

and

$$c = 2/3.$$

So the parabola has equation

$$y = \frac{2}{3}x^2.$$

To find an equation of the tangent line at $(-3, 6)$ we first compute the slope by evaluating the derivative dy/dx at $x = -3$. The slope is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=-3} &= \frac{d}{dx}\left(\frac{2}{3}x^2\right)\Big|_{x=-3} \\ &= \left.\frac{4}{3}x\right|_{x=-3} \\ &= -4. \end{aligned}$$

So an equation of the tangent line at $(-3, 6)$ is

$$y = 6 - 4(x + 3)$$

or

$$y = -6 - 4x.$$

Now slice the region perpendicular to the y-axis to get the differential rectangle AB. The endpoints share the same y-coordinate. Call it y.

The left endpoint A lies on the tangent line with equation

$$y = -6 - 4x$$

and has x-coordinate

$$x = -\frac{6 + y}{4}.$$

So A has coordinates

$$\left(-\frac{6 + y}{4}, y\right).$$

The right endpoint B lies on the parabola

$$y = \frac{2}{3}x^2.$$

So
$$x = \pm\sqrt{\frac{3}{2}y}.$$
But since the region is in the second quadrant, the x-coordinate of B is negative. So B has x-coordinate
$$x = -\sqrt{\frac{3}{2}y}$$
and coordinates
$$\left(-\sqrt{\frac{3}{2}y}, y\right).$$

So the differential rectangle AB has width dy, length
$$L = -\sqrt{\frac{3}{2}y} + -\frac{6+y}{4}$$
and differential area
$$dA = L\,dy = \left(-\sqrt{\frac{3}{2}y} + \frac{6+y}{4}\right)\,dy.$$

The area of the region bounded is then the sum
$$\int_0^6 \left(-\sqrt{\frac{3}{2}y} + \frac{6+y}{4}\right)\,dy.$$
of the differential areas.

Using the fundamental theorem, the area is
$$\begin{aligned}\int_0^6 \left(-\sqrt{\frac{3}{2}y} + \frac{6+y}{4}\right)\,dy &= \left(-\frac{2}{3}\sqrt{\frac{3}{2}}y^{3/2} + \frac{(6+y)^2}{4}\right)\Big|_{y=-3}^{y=0} \\ &= 3/2.\end{aligned}$$

11 Quiz 6B

Question 442

A region is bounded by the coordinate axes, the line $x = -3$ and the curve
$$y = \frac{70}{4x-15}.$$

(a) *Sketch a reasonably accurate graph of the curve and shade the region.*

(b) *Write an integral that gives the area of the region.*

(c) *Use substitution to evalaute the integral and compute the area of the region.*

Explanation. (a) The curve

$$y = f(x) = \frac{70}{4x - 15}$$

is a linear transformation of the curve $y = 1/x$ with a vertical asymptote at $x = 15/4$ and no x-intercept. So to graph the function over the interval $x \in [-3, 0]$ we need only plot the points

$$(0, f(0)) = (1, -14/3)$$

and

$$(-3, f(-3)) = (-3, -70/27).$$

This tells us the function is negative over the interval $x \in [-3, 0]$. So a differential rectangle perpendicular to the x-axis has upper endpoint $A(x, 0)$ and lower endpoint $B(x, 70/(4x - 15)$. So with a width dx and length

$$L = 0 - \frac{70}{4x - 15} = \frac{-70}{4x - 15},$$

the rectangle has differential area

$$dA = L\,dx = \frac{-70}{4x - 15}\,dx.$$

(b) The area of the region is the sum

$$A = \int_{-3}^{0} \frac{-70}{4x - 15}\,dx$$

of the differential areas dA.

(c) To evalaute the above integral, we make the substitution

$$u = 4x - 15.$$

Then

$$du = 4x\,dx$$

and

$$dx = \frac{1}{4}\,du.$$

When $x = -3$,

$$u = 4(-3) - 15 = -27$$

and when $x = 0$,

$$u = 4(0) - 15 = -15.$$

So the area of the region is

$$\begin{aligned}
\int_{-3}^{0} \frac{-70}{4x - 15}\, dx &= -\int_{-27}^{-15} \frac{70}{u} \left(\frac{1}{4}\, du\right) \\
&= -\frac{35}{2} \ln |u| \Big|_{u=-27}^{u=-15} \\
&= \frac{35}{2} \left(\ln |-27| - \ln |-15|\right) \\
&= \frac{35}{2} \ln(27/15).
\end{aligned}$$