Calculus 1

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Contents

Differentiable Functions

An introduction to what it means for a function to be differentiable.

1 Differentiability

Zoom in closer and closer near a point $(a, f(a))$ on the graph of a common function f and you'll most likely notice that the graph looks more and more like the graph of a linear function (ie. like a non-vertical line). If so, we say the function $y = f(x)$ is *differentiable* at $x = a$. And the *derivative* of the function, evaluated at $x = a$, written as

$$\left.\frac{dy}{dx}\right|_{x=a},$$

is just the slope of that line or the rate of change of the linear function.

Exploration 1 *Use the graph of the function $y = f(x)$ below to answer the following questions.*

Desmos link: `https://www.desmos.com/calculator/ojdj4r3r9v`

151: Diff0

(a) *Zoom in toward the point A to determine if the function $y = f(x)$ is differentiable at $x = 1$. If so, approximate the derivative*

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{d}{dx}\left(f(x)\right)\Big|_{x=1}.$$

The derivative is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3.$$

(b) *Zoom in toward the point B to determine if the function $y = f(x)$ is differentiable at $x = 0$. If so, approximate the derivative*

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{d}{dx}\left(f(x)\right)\Big|_{x=0}.$$

The derivative is

$$\left.\frac{dy}{dx}\right|_{x=0} = -2.$$

Learning outcomes:
Author(s):

Exploration 2 *Use the graph of the function*

$$y = f(x) = |x|$$

below to evaluate the following derivatives if possible. Explain your reasoning.

Desmos link: `https://www.desmos.com/calculator/us2fruzbra`

151: Diff1

(a)
$$\left.\frac{d(|x|)}{dx}\right|_{x=2} = 1$$

(b)
$$\left.\frac{d(|x|)}{dx}\right|_{x=0} =$$

(c)
$$\left.\frac{d(|x|)}{dx}\right|_{x=-3} = -1$$

Exploration 3 *Use the graph of the function $y = f(x)$ below to determine if the function is differentiable at $x = 0$. Explain your reasoning.*

Desmos link: `https://www.desmos.com/calculator/ov8qt938ot`

151: Diff2

Exploration 4 (a) *Use the graph of the function $y = f(x)$ below to determine if the function is differentiable at $x = 0.2$. If so, approximate the derivative*

$$\left.\frac{dy}{dx}\right|_{x=0.2} = \left.\frac{d}{dx}\left(f(x)\right)\right|_{x=0.2}.$$

Explain your reasoning.

(b) *Use the graph of the function $y = f(x)$ below to determine if the function is differentiable at $x = 0$. If so, approximate the derivative*

$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{d}{dx}\left(f(x)\right)\right|_{x=0}.$$

Explain your reasoning.

Desmos link: https://www.desmos.com/calculator/tvwtbx9hco

151: Not Differentiable

Exploration 5 *Use the graph of $y = f(x)$ below to determine whether each of the following derivatives are negative, positive, or zero. Explain your reasoning.*

(a)
$$\left.\frac{d}{dx}\left(f(x)\right)\right|_{x=0.5}$$

(b)
$$\left.\frac{d}{dx}\left(f(x)\right)\right|_{x=1}$$

(c)
$$\left.\frac{d}{dx}\left(f(x)\right)\right|_{x=1.5}$$

(d)
$$\left.\frac{d}{dx}\left(f(x)\right)\right|_{x=2.29}$$

Desmos link: https://www.desmos.com/calculator/ks2yui6ofs

151:Diff 7

Question 6 (a) *Summarize your understanding of the main ideas of this section.*

(b) *What questions do you have about this section?*

Free Response:

2 Using Algebra to Compute Derivatives

Exploration 7 *The aim of this problem is to use numerical methods to approximate and algebra to evaluate the derivative*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1}.$$

(a) *First use the graph of the function $y = f(x) = x^3$ below to approximate or guess the value of the above derivative by zooming in on the appropriate point.*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1} = 3.$$

Desmos link: `https://www.desmos.com/calculator/rto22qzlvm`

151: Cubing Function

(b) *The idea to compute the derivative algebraically is this: Fix the point $P(1,1)$ on the graph of $y = f(x) = x^3$. Then choose a variable point Q on the graph, different from P, with coordinates (v, v^3). When Q is sufficiently close to P, the line PQ approximates the curve $y = x^3$ near $P(1,1)$ and the slope of this line approximates the derivative above.*

(i) *Our first step is to find an expression for the slope of line PQ as a function of v. The slope is*

$$\begin{aligned} m(v) &= \frac{\Delta y}{\Delta v} \\ &= \frac{f(v) - f(1)}{v - 1} \\ &= \frac{v^3 - 1}{v - 1} \end{aligned}$$

all assuming $v \neq 1$.

(ii) *Now we'll use the slope (or average rate of change) function $m(x)$ to create a table of slopes for the lines PQ. Reveal the contents of the Table folder in Line 1 of the worksheet above by clicking the Right Arrow just to the left of "Table".*

i. *Add a few entries to the table to get* better *approximations to the above derivative.*

ii. *The slopes $m(v)$ should appear should appear to approach some number as v approaches* 1*. What is that number?*

iii. *This suggests that*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1} = \lim_{v\to 1}\frac{v^3-1}{v-1} = 3.$$

(iii) *Another approach to approximating the derivative is to graph the slope function*

$$m(v) = \frac{v^3-1}{v-1}\,,\; v \neq 1.$$

Activate the Average Rate of Change folder on Line 11 by clicking the camera icon to the left of the line to see the graph of this function. Activate also the Table folder in Line 1.

i. *Drag the slider v on Line 4. Describe the relationship between the graph of the slope function and the line PQ.*

ii. *There is a hole in the graph of the function $y = m(x)$. Where is it? Why is it there?*

(iv) *To verify the numerical and graphical evidence that*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1} = \lim_{v\to 1}\frac{v^3-1}{v-1} = 3,$$

we'll use algebra to evaluate the above limit.
The idea is to factor the numerator $v^3 - 1$. Since

$$\left(v^3-1\right)\Big|_{v=1} = 1^3 - 1 = 0,$$

we know that $v-1$ is a factor of v^3-1. To simplify the quotient

$$\frac{v^3-1}{v-1}\,,\; v \neq 1,$$

we could use long division or factor the difference of two cubes:

$$a^3 - b^3 = (a-b)(a^2+ab+b^2).$$

Either way, the result is that

$$v^3 - 1 = (v-1)(v^2+v+1).$$

So

$$\frac{v^3-1}{v-1} = v^2+v+1\,,\; v \neq 1.$$

Putting this all together, we get

$$\begin{aligned}\frac{d}{dx}\left(x^3\right)\Big|_{x=1} &= \lim_{v\to 1}\frac{v^3-1}{v-1}\\ &= \lim_{v\to 1}\frac{(v-1)(v^2+v+1)}{v-1}\\ &= \lim_{v\to 1}(v^2+v+1)\\ &= 1^2+1+1\\ &= 3.\end{aligned}$$

Exploration 8 *Repeat all parts of Exploration 2 for the following derivatives.*

(a)

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=2} = 12.$$

(b)

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=a} = 3a^2.$$

Use the demonstration below to check your work by dragging the sliders v (Line 4) and a (Line 16). Then activate the Derivative Function folder on Line 17.

Desmos link: `https://www.desmos.com/calculator/ewy7jqij6s`

151: Cubing Function 2

(c)

$$\frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=1} = -1.$$

(d)

$$\frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=a} = -1/a^2.$$

(e)

$$\frac{d}{dx}\left(\frac{1}{x^2}\right)\Big|_{x=3} = -2/27.$$

(f)
$$\frac{d}{dx}\left(\frac{1}{x^2}\right)\Big|_{x=a} = -2/a^3.$$

(g)
$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\Big|_{x=1} = -\frac{1}{2}.$$

(h)
$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\Big|_{x=a} = -\frac{2a}{(1+a^2)^2}.$$

3 Applications

Question 9 *The function*

$$h = f(t)\,,\ 0 \le t \le 2.2,$$

expresses the height of a balloon (in hundreds of feet) in terms of the number of minutes past noon.

The graph of the function $h = f(t)$ is shown below.

Desmos link: `https://www.desmos.com/calculator/yd4xm6x6ub`

151: Balloon

(a) *Find an expression for the function $r = m(v)$ that gives the balloon's average rate of ascent (measured in hundreds of feet per minute) between time v minutes past noon and 12:02 pm. Include also the function's domain.*

(b) *How is the average rate of ascent function in part (a) related to the line PQ in the demonstration above?*

(c) *Use the graph of the function $h = f(t)$ above to sketch a rough graph of the function $r = m(v)$.*

(d) *Activate the folder (avg. rate of change function) in Line 11 to check your graph of the function $r = m(v)$.*

(e) *Open the Table in Line 1 by clicking the* right arrow *at the left of the line.*

(i) *What is the balloon's average rate of ascent between 12:02:00 pm and 12:02:06 pm?*

(ii) *Use the graph of the function $h = f(t)$ above to estimate the balloon's rate of ascent at 12:02pm by zooming in on the appropriate point.*

(iii) *What does the table suggest about the balloon's rate of ascent at 12:02pm? Explain.*

(f) *Now suppose*

$$f(t) = -2t^3 + 7t^2 - 8t + 8.$$

(i) *Find a simplified expression for the function $r = m(v)$. Include the appropriate domain.*

(ii) *Use your simplified expression to compute the balloon's rate of ascent at 12:02pm.*

Differentiable Functions, Part 2

Using limits to compute the derivative of a function at a general input.

1 Using Limits to Compute Derivatives

Exploration 10 *In part (b) of Exploration 7 from the previous chapter where we computed*

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=1}.$$

Now we'll use limits to evaluate the derivative

$$\frac{d}{dx}\left(x^3\right)\Big|_{x=a}.$$

In other words, we'll compute the derivative of the function $f(x) = x^3$ at a general input $x = a$. Think about zooming in on the graph of $y = x^3$ sufficiently close to the point (a, a^3) so that the graph looks like a straight line. We'll compute the slope of that line.

The algebra here is nearly identical to what we did earlier. You should compare the two computations.

Desmos link: `https://www.desmos.com/calculator/8eiffwbgt5`

151: Cubing Function 3

The idea to compute the derivative algebraically is this: Fix the point $P(a, a^3)$ on the graph of $y = f(x) = x^3$. Then choose a variable point Q on the graph, different from P, with coordinates (v, v^3). When Q is sufficiently close to P, the line PQ approximates the curve $y = x^3$ near $P(a, a)$ and the slope of this line is above derivative.

(a) *Our first step is to find an expression for the slope of line PQ as a function*

Learning outcomes:
Author(s):

of v. The slope is

$$\begin{aligned} m(v) &= \frac{\Delta y}{\Delta v} \\ &= \frac{f(v) - f(a)}{v - a} \\ &= \frac{v^3 - a^3}{v - a} \end{aligned}$$

all assuming $v \neq a$.

(b) *The next step is to factor the numerator $v^3 - a^2$. Since*

$$(v^3 - a^3)\Big|_{v=a} = a^3 - a^3 = 0,$$

we know that $v - a$ is a factor of $v^3 - a^3$. To simplify the quotient

$$\frac{v^3 - a^3}{v - a}, \, v \neq a,$$

we could use long division or factor the difference of two cubes:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2).$$

Either way, the result is that

$$v^3 - a^3 = (v - a)(v^2 + va + a^2).$$

So

$$\frac{v^3 - a^3}{v - a} = v^2 + va + a^2 \, , \, v \neq a.$$

Putting this all together, we get

$$\begin{aligned} \frac{d}{dx}\left(x^3\right)\Big|_{x=a} &= \lim_{v \to a} \frac{v^3 - a^3}{v - a} \\ &= \lim_{v \to a} \frac{(v - a)(v^2 + va + a^2)}{v - a} \\ &= \lim_{v \to a} (v^2 + va + a^2) \\ &= a^2 + a^2 + a^2 \\ &= 3a^2. \end{aligned}$$

(c) *More simply put, we just write the derivative*

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

as a function of x.

(d) *We can check our result using the desmos worksheet above. Do this as follows:*

(i) *Input the correct expression for the derivative $d(x^3)/dx$ on Line 17.*

(ii) *For a differentiable function $f(x)$ and values of h near zero, we can approximate the derivative $d(f(x))/dx$ as the slope*

$$\frac{d}{dx}\left(f(x)\right) = \frac{f(x+h) - f(x-h)}{2h}$$

of the line through the points $(x-h, f(x-h))$ and $(x+h, f(x+h))$. Now activate the folder Approximation to the derivaitve function *on Line 19. Then drag the slider h on Line 21 to get a better approximation to the derivative.*

Free Response: *Describe what happens to the above approximation as $h \to 0$.*

Exercise 11 (a) *Follow the method of Exploration 1 for the function $f(x) = x^4$ to compute the derivative*

$$\frac{d}{dx}\left(x^4\right)\Big|_{x=a}.$$

(b) *Modify the desmos worksheet below for the function $f(x) = x^4$.*

151: Cubing Function 3

Then to check your work, do part (d) of Exploration 1 for $f(x) = x^4$. Describe also what happens to the approximation in part (d) as $h \to 0$. Include a screenshot to help with your description.

Exercise 12 *Repeat all parts of Exercise for the function $f(x) = 1/x^4$.*

Exercise 13 *Repeat all parts of Exercise for the function $f(x) = 1 + x^2$.*

Exercise 14 *Repeat all parts of Exercise for the function $f(x) = 1/(1+x^2)$.*

Free Fall

Rocks in free fall.

Question 15 *Part (a) of the following question is an important takeaway from this chapter. It is not necessary to answer this question now.*

A rock is dropped near the surface of a planet without an atmosphere.

(a) *Through what fraction of its initial height does the rock fall during the last 1/10 of the time it takes to hit the surface? Give a quick estimate.*

(b) *What is the exact fraction?*

Exploration 16 *Play the slider u in Line 1 of the desmos worksheet below to watch two motions. One is a rock falling to the surface of Mars when dropped from a height of 190 meters above the surface. The other is a constant-speed motion.*

Desmos link: `https://www.desmos.com/calculator/j2ciuemii7`

151: Free Fall 1

(a) *Use the animation to sketch a graph of the function $s = g(t)$ that expresses the distance (in meters) of the rock from its starting point in terms of the number of seconds since the rock was released. Include the appropriate units and variable names on the axes.*

(b) *Activate the folder* Graph of distance function *in Line 17 above to see how you did.*

(c) *Our aim now is to compute the (instantaneous) speed of the rock (in m/s) at time $t = u$ seconds. We'll assume that*

$$s = g(t) = 1.9t^2\,,\ 0 \le t \le 10.$$

Do this as follows.

(i) *Find an expression that gives the rock's average speed between times $t = u$ and $t = v$.*

Learning outcomes:
Author(s):

(ii) *Evaluate the appropriate limit to find the speed of the rock at time $t = u$ seconds. Click the Hint tab above for help.*

Hint: *The speed at time $t = u$ is*

$$\begin{aligned} \frac{ds}{dt}\bigg|_{t=u} &= \lim_{v\to u} \frac{1.9v^2 - 1.9u^2}{v-u} \\ &= \lim_{v\to u} 1.9(v+u) \\ &= 3.8u. \end{aligned}$$

(iii) *Use the result of part (ii) to find the rock's speed as it hits the ground. Assume the rock is dropped from a height of 190 meters.*

(iv) *Compare the rock's speed as it hits the ground with its average speed during the entire time of its fall.*

(v) *Use the result of part (iv) to approximate the fraction of the total distance of 190 meters the rock falls through during the last second of its fall?*

(vi) *Compare the rock's speed at time $t = u$ with its average speed during the first u seconds of its fall.*

(vii) *Use point-slope to find an equation of the tangent line to the curve $s = 1.9t^2$ at the point $(u, 1.9u^2)$. Enter this equation in Line 21 of the worksheet above.*
The point-slope equation is

$$s = 1.9u^2 + 3.8u(t-u).$$

(viii) *Activate the folder* Average speed *in Line 23 above. How is the slope of the tangent line related to the slope of the line OP?*

(ix) *Find the coordinates of the point where the tangent line intersects the horizontal axes. Enter these in Line 30 above.*
The coordinates are

$$(t, s) = \left(\frac{u}{2}, 0\right).$$

Question 17 *Play the slider u in Line 2 of the desmos worksheet below to watch two motions. One shows a rock falling to the surface of a planet when it is dropped from rest near the surface. The other is a constant-speed motion.*

Desmos link: `https://www.desmos.com/calculator/dmlrxahkld`

151: Free Fall 2

(a) *What does the animation suggest about the speed of the rock as it hits the surface compared with its average speed during the time it takes to hit the ground?*

(b) *What does the animation suggest about the fraction of its initial height the rock falls through during the last* $1/20$ *of the time it takes to hit the surface? Give a quick estimate.*

(c) *Now suppose the rock is dropped from a height of* H *meters and that it takes* T *seconds to hit the surface. Assume also that the planet has no atmosphere so that the function expressing the rock's distance from its starting point (in meters) in terms of the number of seconds since it was dropped is of the form*

$$s = f(t) = at^2$$

for some constant a.

(i) *Express the constant* a *in terms of* H and T.

(ii) *What are the units of the constant* a*?*

(iii) *Explain the meaning of the constant* a*. Do* not *use the word* acceleration.

(iv) *Find the domain of the function* f.

(v) *Use calculus to express the speed of the rock (in metes/sec) in terms of* H and T*. Check that your expression has the correct units.*

(vi) *Compare the speed of the rock as it hits the ground with its average speed during the time it falls.*

(vii) *Through what* exact *fraction of its initial height the rock fall through during the last* $1/20$ *of the time it takes to hit the surface? Compare this with your earlier estimate.*

Purchasing Power

How a small change in price affects the quantity of an item you can buy.

1 Purchasing Power

Question 18 *This problem investigates how a small change in the price of apples would affect the number of pounds we can buy with \$10.*

(a) *We'll start by finding a function*

$$n = f(p)\,,\ p > 0,$$

that expresses the number of pounds of apples we can buy with \$10 in terms of the price (measured in dollars/pound).

The function is

$$n = f(p) = 10/p,\ p > 0.$$

(b) *Would you expect the derivative dn/dp to be postive, negative, or zero? Explain.*

(c) *Use limits to find an expression for the derivative*

$$\left.\frac{dn}{dp}\right|_{p=a}.$$

The derivative is (click the Hint tab above for help)

Hint:

$$\begin{aligned}\left.\frac{dn}{dp}\right|_{p=a} &= \lim_{p\to a}\frac{1}{p-a}\left(f(p)-f(a)\right)\\ &= \lim_{p\to a}\frac{1}{p-a}\left(\frac{10}{p}-\frac{10}{a}\right)\\ &= \lim_{p\to a}\frac{10}{p-a}\left(\frac{a-p}{pa}\right)\\ &= \lim_{p\to a}\frac{-10}{pa}\\ &= \frac{-10}{a^2}.\end{aligned}$$

Learning outcomes:
Author(s):

(d) *Evaluate the derivative*

$$\left.\frac{dn}{dp}\right|_{p=2}.$$

(i) *What are the units of the above derivative?*

(ii) *What do you get by "simplifying" the units? Explain how simpifying the units of this derivative gives you insight into its meaning.*

(e) *Interpret the meaning of the above derivative. Click the arrow to the lower right for help.*

For a way to interpret the meaning of the derivative

$$\left.\frac{dn}{dp}\right|_{p=2}.$$

we can ask ourselves what happens to the number of pounds of apples we can buy with \$10 *if the price changes by a small amount from* \$2/*lb. For this, let* Δp *be a small change in price from* \$2/*lb (measured in dollars/lb) and let* Δn *be the corresponding change in the number of pounds we can buy with* \$10.

For $\Delta p \sim 0$, *we have*

$$-2.5 = \left.\frac{dn}{dp}\right|_{p=2} \sim \frac{\Delta n}{\Delta p}$$

and (type "Delta" for Δ*)*

$$\Delta n \sim -2.5\Delta p.$$

This tells us that if the price increases from \$2/*lb to say* \$2.10/*lb, then*

$$\Delta n \sim \left(-2.5\frac{lb}{\$/lb}\right)\left(0.10\frac{\$}{lb}\right) = -0.25 \text{ lbs.}$$

So if the price increases from \$2/*lb to say* \$2.10/*lb, we can buy about* 0.25 *fewer pounds of apples with ten dollars.*

(f) *Next we'll use the worksheet below to visualize the approximate change*

$$\begin{aligned}\Delta n &\sim \left(\left.\frac{dn}{dp}\right|_{p=2}\right)\Delta p\\ &= \left(-2.5\frac{lb}{\$/lb}\right)\left(0.10\frac{\$}{lb}\right)\\ &= -0.25 \text{ lbs.}\end{aligned}$$

in the number of pounds of apples we can buy with \$10 *as the price increases from* \$2/*lb to* \$2.10/*lb.*

Desmos link: `https://www.desmos.com/calculator/qw7wislq0c`

151: Apples

(i) *To get started, find an equation (in point-slope form) of the tangent line to the curve* $n = f(p)$ *at the point* P *with coordinates* $(2, 5)$*. The tangent line has a slope equal to the derivative*

$$\left.\frac{dn}{dp}\right|_{p=2}.$$

So its equation is

$$n = 5 + -2.5(p - 2).$$

(ii) *Enter your equation of the tangent line on Line 17 of the desmos worksheet.*

(iii) *Activate the folders* tangent line *and* linear approximation *on Lines 18 and 22.*

(iv) *Explain why the difference in the* n*-coordinates of points* P *and* R' *(ie.* n*-coordinate of* R' *minus* n*-coordinate of* P*) is equal to our approximation of* Δn *above when* Q *as coordinates* $(2.10, f(2.10))$*.*

(v) *Drag Slider* v *on Line 2 to make* Q *approach* P*. What so you think happens to the ratio of* Δn *to our approximation of* Δn *as* $v \to 2$*?*

(g) *We can get a better understanding of the changes*

$$\Delta p = 2.1 - 2 = 0.10 \textit{ dollars/pound}$$

and

$$\Delta n = f(2.1) - f(2) \sim 0.25 \textit{ pounds}$$

by thinking about relative instead of absolute change.

An increase in price from \$2/lb *to* \$2.10/lb *is a relative change of*

$$\frac{\Delta p}{p} = \frac{\$0.1/lb}{\$2/lb} = 0.05 = 5\%.$$

This causes a relative change in the number of pounds we can buy of approximately

$$\frac{\Delta n}{n} \sim \frac{-0.25lbs}{5lbs} = -0.05 = -5\%.$$

(h) *We can get a the same result relating the two relative changes of the previous question by working in general.*

Suppose we increase the price of apples by $Q\%$*, where* $Q \sim 0$*. What can we say about the relative change in the number of pounds of apples we can buy with* \$10*?*

Well, if the change in price Δp is near zero and Δn is the corresponding change in the number of pounds we can buy, then

$$\frac{\Delta n}{\Delta p} \sim \frac{dn}{dp} = -10/p^2.$$

So

$$\Delta n \sim \left(\frac{-10}{p^2}\right) \Delta p.$$

Dividing both sides by $n = 10/p$ tells us that

$$\frac{\Delta n}{n} \sim \left(\frac{-10}{p^2}\right)\left(\frac{\Delta p}{\frac{10}{p}}\right) = -\frac{\Delta p}{p} = -Q\%.$$

So if we increase the price by $Q\% \sim 0$, then the number of apples we can buy decreases by approximately that same $Q\%$.

(i) *We can think about this relationship between the relative changes geometrically. To do this, remove your equation of the tangent line on Line 17 and activate the folder* tangent line *in the worksheet above. Then drag slider a on Line 4 to move point P. What do you notice about the ratio $PB : PA$ of the distances PA and PB as P moves? More on this later.*

Question 19 *Due to a printing error, the graph of the function*

$$n = f(p)\,,\, p > 0,$$

expressing the number of pounds of an item we can buy with \$100 in terms of the price (measured in \$/lb) is not shown below. All we can see is a point A on the graph.

Desmos link: `https://www.desmos.com/calculator/y5nqs8fkvj`

151: Printers Error 1

The problem is to draw the tangent line to the graph at A without *sketching the graph or doing any kind of computation. Click the arrow below for a hint.*

The key idea is to relate the slope of the tangent line at A to the slope of the line OA through the origin and A.

We know that

$$n = f(p) = \frac{100}{p}\,,\, p > 0,$$

and

$$\frac{dn}{dp} = -100/p^2.$$

Now write this derivative in terms of both n and p,

$$\frac{dn}{dp} = -\frac{100}{p^2} = -\frac{n}{p}.$$

Now relate this to the slope of line OA and draw the tangent line at A.

2 Weight in Space

Question 20 *The weight of an object is the gravitational force that the earth exerts on the object's mass and varies with the object's height above the surface. The function*

$$W = f(h) = \frac{k}{(h+4)^2}\,,\ h \geq 0,$$

expresses the weight (in pounds) of an object in terms of its height (in thousands of miles) above the surface of the earth. We'll suppose the object weighs 200 pounds on the surface.

Desmos link: `https://www.desmos.com/calculator/zklsucctjp`

151: Weight in Space

(a) *Find the value of the constant k. What are its units?*

(b) *Find the average rate of change of W with respect to h over the interval between heights of h and v thousands of miles.*

(c) *Use part (b) to find an expression for the derivative dW/dh.*

(d) *Use the graph of the function $W = f(h)$ above to approximate the derivative*

$$\left.\frac{dW}{dh}\right|_{h=4}.$$

Include units

(e) *Evaluate the derivative*

$$\left.\frac{dW}{dh}\right|_{h=4}$$

using your expression for dW/dh and compare this with your estimate.

(f) *What are the units of the derivative above? Interpret it meaning.*

(i) *At approximately what altitude does your weight decrease by 4 pounds when your altitude increases by 200 miles? Use your expression for the derivative dW/dh to help.*

(ii) *Compute the exact change in your weight over the interval you found in part (i).*

3 A Rock in Free Fall

Question 21 *A rock is dropped from a height of 100 meters on the planet Krypton.*

The function

$$v = f(h)\,,\ 0 \le h \le 100,$$

expresses the speed (in meters/sec) of the rock in terms of its height (in meters) above the surface.

Desmos link: `https://www.desmos.com/calculator/hcqb5nxc18`

151: Free Fall Speed and Height

(a) *Would you expect the derivative*

$$\left.\frac{dv}{dh}\right|_{h=36}$$

to be postive or negative? Why?

(b) *Suppose*

$$v = f(h) = 4\sqrt{100-h}\,,\ 0 \le h \le 100$$

and find an expression for the average rate of change of the rock's speed (measured in meters/sec) with respect to its height (in meters) between heights of v meters and a meters.

(c) *Use your expression from part (b) to find an expression for the derivative*

$$dv/dh\Big|_{h=a}.$$

(d) *Use the graph above to approximate the derivative*

$$dv/dh\Big|_{h=64}.$$

Include units.

(e) *Evaluate the derivative*

$$dv/dh\Big|_{h=64}$$

and compare the exact value with your estimate.

(f) *What are the units of the derivative above? What is its meaning?*

(g) *"Simplify the units of the derivative". What insight does this give you into its meaning?*

(h) *Use part (e) to approximate the rocks's speed at a height of* 61 *feet. Compare your estimate with the actual speed.*

(i) *Use part (c) to approximate the interval over which the rock falls two meters as its speed increases by* 0.5m/s.

4 Distance to the Horizon

Question 22 *The function*

$$s = f(h) = 1.22\sqrt{h}\,,\; 0 \le h \le 10,000$$

expresses the distance to the horizon (measured in miles) in terms of your alitude (measured in feet).

(a) *Find an expression for the average rate of change in the distance to the horizon with respect to altitude between altitudes h feet and v feet.*

(b) *Use your expression from part (a) to find an expression for the derivative ds/dh.*

(c) *Evaluate the derivative*

$$\frac{ds}{dh}\Big|_{h=25}.$$

(d) *What are the units of the derivative above? Interpret its meaning.*

(e) *Use the result of part (c) to approximate the distance to the horizon at an altitude of* 24 *feet. Then compare this approximation with the actual distance.*

(f) *Approximate the relative change in the distance to the horizon in terms of a small relative change in altitude.*

(g) *Use your result from part (b) to approximate the altitude at which moving* 10 *feet higher increase the distance to the horizon by* 0.5 *miles.*

5 Intensity of Sound

Question 23 *The* intensity *of sound is measured in Watts per square meter and is a function of the distance from the source.*

Suppose for a jet taking off, this function is

$$I = f(r) = \frac{100}{r^2}\,,\ r \geq 10,$$

where r is the distance (in meters) from the jet.

(a) *Find an expression for the average rate of change of the sound intensity (in Watts/m^2) with respect to the distance from the source (in meters) between distances r and a meters.*

(b) *Use your expression from part (a) to find an expression for the derivative*

$$dI/dr\Big|_{r=a}$$

(c) *Evaluate the derivative*

$$dI/dr\Big|_{r=20}.$$

(d) *What are the units of the derivative above? What is its meaning?*

(e) *Approximate the relative change in the sound intensity in terms of a small relative change in the distance to the source.*

Practice Quiz 1

First practice quiz, Weeks 1-2

Directions:

(a) Show all work.

(b) Give brief explanations for each problem. Include these explanations in the flow of the solution.

(c) Show all units in all computations.

1 Part 1

Question 24 *Explain what it means intuitively for a function to be differentiable at some input in terms of the graph of that function.*

Question 25 *The function*

$$s = f(h) = 1.22\sqrt{h}\,,\ 0 \leq h \leq 10,000,$$

expresses the distance to the horizon (measured in miles) in terms of your altitude (measured in feet).

(a) *Find an expression for the average rate of change in the distance to the horizon with respect to altitude between altitudes b feet and v feet.*

(b) *Use your expression from part (a) to find an expression for the derivative*

$$\left.\frac{ds}{dh}\right|_{h=b}.$$

(c) *Evaluate the derivative*

$$\left.\frac{ds}{dh}\right|_{h=25}.$$

(d) *What are the units of the derivative above? Interpret its meaning.*

Learning outcomes:
Author(s):

(e) *Use the result of part (c) to approximate the distance to the horizon at an altitude of* 24 *feet. Then compare this approximation with the actual distance.*

(f) *Approximate the relative change in the distance to the horizon in terms of a small relative change in altitude.*

(g) *Use your result from part (b) to approximate the altitude at which moving* 10 *feet higher increase the distance to the horizon by* 0.5 *miles.*

Question 26 *The* intensity *of sound is measured in Watts/(square meter) and is a function*

$$I = f(r) = \frac{100}{r^2}\ ,\ r \geq 0.5,$$

of the distance from the source.

Suppose r is measured in meters and that the intensity of the sound emitted by a vacuum cleaner is 10^{-4}watts/m^2 at a distance of 0.5 *meters.*

(a) *Compute the value of the constant k. What are its units?*

(b) *Find an expression for the average rate of change of the sound intensity (in Watts/m^2) with respect to the distance from the source (in meters) between distances b and r meters.*

(c) *Use your expression from part (b) to find an expression for the derivative*

$$\left.\frac{dI}{dr}\right|_{r=b}.$$

(d) *Evaluate the derivative*

$$\left.\frac{dI}{dr}\right|_{r=2}.$$

(e) *What are the units of the derivative above? What is its meaning?*

(f) *Approximate the relative change in the sound intensity in terms of a small relative change in the distance to the source.*

Question 27 *Between speeds of* 70 *miles/hr and* 84 *miles/hr, the gas mileage of a car (in miles/gal) is a one-to-one function $G = f(v)$ of its speed (in miles/hour). The car gets* 10 *miles/gal at a speed of* 80 *miles/hour.*

(a) *Which of the following is more likely to be true? Explain your reasoning.*

(i) $\left.\dfrac{dG}{dv}\right|_{v=80} = 0.25$ *or*

(ii) $\left.\dfrac{dG}{dv}\right|_{v=80} = -0.25$

(b) *What are the units of the correct derivative above? Explain its meaning.*

(c) *Assuming the correct choice in part (b), evaluate the derivative*

$$\left.\frac{dv}{dG}\right|_{G=10}.$$

(d) *Simplify the units of the derivative in part (c). What does this tell you about its meaning?*

(e) *At what rate (in gal/hour) does the car burn gas at a speed of* 80 *miles/hour?*

Question 28 *Between speeds of* 55 *miles/hour and* 70 *miles/hour the gas milesage of a car is a linear function of its speed. The car gets* 40 *miles/gal at a speed of* 55 *miles/hour and* 30 *miles/gal at a speed of* 70 *miles/hour.*

(a) *Find a function*

$$r = f(v)\,,\; 55 \le v \le 70,$$

that express the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour). Note: *This function is* not *linear*

(b) *Find an expression for the average rate of change of r with respect to v between speeds of b miles/hour and w miles/hour.*

(c) *Use your expression from part (b) to find an expression for the derivative*

$$\left.\frac{dr}{dv}\right|_{v=b}.$$

(d) *Evaluate the derivative*

$$\left.\frac{dr}{dv}\right|_{v=60}.$$

(e) *What are the units of the derivative above? Interpret its meaning.*

Introduction to Motion

An introduction to motion

Exploration 29 *Desmos link:*
`https://www.desmos.com/calculator/bk1z9cwbhb`

151: Sound Squaring Function

Learning outcomes:
Author(s):

Introduction

Introduction to Differential Calculus

This course is listed as Calculus I in the catalogue, but it really should be called Differential Calculus. *Differential* has the same root as *difference* and *calculus* the same root as *calculate*. So this class is really about calculating differences, or more simply put it's about subtracting. But it's about subtracting in the context of functions.

Pick a specific input to specific function and you'll likely find that the changes in the function's output are approximately proportional to *small* changes in the input. If so, we say that the function is *differentiable* at that input and we call the proportionality constant the *derivative*.

For example, let's look at the behavior of the function

$$A = f(s) = s^2, s \geq 0,$$

near the input $s = 5$. To emphasize the importance of units, let's define the input s to be the side length of a square measured in feet and the output $f(s)$ to be the area of that square, measured in square feet. The problem before us is to describe a simple relationship between a small change in the side length

$$\Delta s = s - 5$$

of the square and the change

$$\Delta A = f(s) - f(5)$$

in its area.

Question 30 *(a) We'll first take a numerical approach and compute some small changes and their ratios. Fill in the missing entries in the table below.*

s (ft)	$A = s^2$ (ft^2)	$\Delta s = s - 5$ (ft)	$\Delta A = s^2 - 25$ (ft^2)	$\Delta A/\Delta s$ (ft^2/ft)
4.9	24.01	-0.1	-0.99	9.9
4.99	24.9001	-0.01	-0.0999	9.99
5	*25*	0	0	–
5.01	25.1001	0.01	0.1001	10.01
5.1	26.01	0.1	1.01	10.1

Learning outcomes:
Author(s):

(b) The data in the table above suggests an approximate proportional relationship between ΔA and Δs. We can guess the constant of proportionality from the fifth column. As $s \to 5$ (as s approaches 5), it looks like the ratio $\Delta A/\Delta s$ approaches some number, the constant of proportionality.

i) What is that number? 10

ii) What are its units?

Free Response:

So for $\Delta s \sim 0$, we suspect that

$$\Delta A \sim 10\Delta s.$$

(c) The constant of proportionality is called the derivative, in this case of the function $A = s^2$, at the input $s = 5$. We write this as

$$\left.\frac{dA}{ds}\right|_{s=5} = 10.$$

(d) We could have taken an algebraic approach to determe this constant of proportionality instead. The idea is to first simplfiy the quotient $\Delta A/\Delta s$ as

$$\begin{aligned}\frac{\Delta A}{\Delta s} &= \frac{s^2 - 25}{s - 5} \\ &= \frac{(s+5)((s-5)}{s-5} \\ &= s + 5 \text{ if } s \neq 5.\end{aligned}$$

So, for example, if $s = 4.99$, then

$$\frac{\Delta A}{\Delta s} = 4.99 + 5 = 9.99$$

as shown in the last column of the second row of the above table.

The advantage of this algebraic approach is that we can now compute the proportionality constant as a limit:

$$\begin{aligned}\left.\frac{dA}{ds}\right|_{s=5} &= \lim_{s\to 5} \frac{\Delta A}{\Delta s} \\ &= \lim_{s\to 5}(s+5) \\ &= 5 + 5 \\ &= 10.\end{aligned}$$

(d) We can also use the graph of the function $A = f(s) = s^2$ to intrepret the ratios

$$\frac{\Delta A}{\Delta s} = \frac{f(s) - f(5)}{s - 5}$$

geometrically. Move the slider s in the demonstration below and describe

(i) how the line through the points P and Q is related to the ratio $\Delta A/\Delta s$ show on Line 2,

(ii) what happens to the line PQ as $s \to 5$, and

(iii) what happens to the line PQ when $s = 5$.

Free Response:

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/vz9ud5txva`

Continuing with the above demonstration,

(i) Open the Code folder in Line 3 and turn off the line PQ in Line 7.

(ii) Write an equation for the line through the point P with slope equal to the proportionality constant in the line below and on Line 8 in the desmos worksheet:

$$A = L(s) = 25 + 10(s - 5).$$

(iii) Zoom in close enough to the point P to make the graph of the function $A = f(s)$ look like a line. How do the graph of the function and the graph of the line $A = L(s)$ compare in this close-up view?

Free Response:

(e) **Summary:**

- *If we change the side of a square from a length of 5 feet to a length of $s \sim 5$ feet, then the area of the square changes by approximately*

  $$\Delta A = s^2 - 25 \sim 10\Delta s = 10(s - 5)$$

  square feet. The proportionality constant 10 has units $ft^2/ft = ft$.

- *Zoom in close enough to the graph of the function $A = f(s) = s^2$ near the point $P(5, 25)$ and the graph looks like a line with slope equal to the proportionality constant.*

- *We can compute the proportionality constant as the limit*

$$\left.\frac{dA}{ds}\right|_{s=5} = \lim_{s\to 5} \frac{f(s)-f(5)}{s-5}.$$

- *Suppose for example, we wanted to approximate the side length s of a square with area 25.06 ft^2. Then*

$$\Delta A = s^2 - 25 = 25.06 - 25 = 0.06.$$

And since

$$\Delta A \sim 10\Delta s = 10(s-5),$$

$$0.06 \sim 10\Delta s.$$

So

$$\Delta s \sim 0.006$$

and a square with area 25.06 ft^2 has an approximate side length (measured in feet) of

$$s = 5 + \Delta s \sim 5.006.$$

Question 31 *On a clear day with an unobstructed view (like you might have at the beach or in a hot air balloon), the distance to the horizon is limited by the curvature of the earth as illustrated in the demonstration below.*

In fact, as long as you are not too high above the surface of the earth, the function

$$s = f(h) = 1.22\sqrt{h}, 0 \le h \le 20,000,$$

gives a good approximation to the distance to the horizon (the length of the red arc AT below, measured in miles) in terms of your height above the ground (the distance AP below, measured in feet).

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

Desmos activity available at

151:Distance to Horizon 1

Our aim is to approximate the change in the distance to the horizon (in miles) in terms of a small change in height (in feet) from a height of 25 feet.

(i) To start, what are the units of the constant 1.22 above? Explain how you know.

Free Response:

(ii) Go through a similar analysis as in parts (a)-(e) of Example 1, to approximate the change $\Delta s = s - f(25)$ in the distance to the horizon in terms of the change $\Delta h = h - 25$ in your height above the ground. Start by completing the column headings (with units) and the missing entries in the table below.

h (ft)	$s = 1.22\sqrt{h}$ (miles)	$\Delta h = h - 25$ (ft)	$\Delta s = f(h) - f(25)$ (miles)	$\Delta s/\Delta h$ (units?)
4.9^2				
4.99^2				
25		0	0	–
5.01^2				
5.1^2				

Question 32 *This question is similar to the last, but suppose instead we are looking down on the earth from the space station or a rocket. Then the approximation to the distance to the horizon from the previous problem will* not *work.*

So our first step is to find a function

$$s = f(h), h \geq 0,$$

that expresses the distance to the horizon (still measured in miles) in terms of our height above the earth's surface, now measured in miles instead of feet. *We'll suppose the earth to be a perfect sphere of radius* 3960 *miles. The distance to the horizon is the arclength AT below, meaured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance AP.*

Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

(a) Find an expression for the above function.

Hint: *Use right triangle ΔOTP to find an expression for the radian measure of angle $\angle POT$. Then use this angle to find an expression for the arclength AT.*

Here are more details.

(i) Enter the two side lengths, measured in feet, in right triangle ΔOPT below.

$$OT = 3960$$

and

$$OP = h + 3960.$$

(ii) Let θ be the radian measure of $\angle TOP$. Write an equaion with a trigonometric function of θ that relates the two lengths in part (i). Use the Math Editor tab to enter the trig function and the angle θ.

$$\cos\theta = \frac{3960}{h + 3960}.$$

(iii) Now solve the equation from part (ii) for θ in terms of h. Then use what you know about measuring arclength along a circle to find an expression for the function f. Use the Math Editor tab to help.

$$s = f(h) = 3960 \arccos\left(\frac{3960}{h + 3960}\right).$$

(b) Now suppose we are 165 miles above the surface of the earth and we wish to approximate how a small change in our altitude changes the distance to the horizon.

To do this, fill in the missing entries in the table below.

h *(miles)*	$s = f(h)$ *(miles)*	$\Delta h = h - 165$ *(miles)*	$\Delta s = f(h) - f(165)$ *(miles)*	$\Delta s/\Delta h$ *(units?)*
162				
163				
164				
165				
166				
167				
168				

(c) Do the data above suggest that the quotients $\Delta s/\Delta h$ approach some number as h approaches 165? If so, use the data to approximate that number. If not, explain why not.

(d) Make your own table similar to the one above to get a better approximation, correct to the nearest thousandth, to

$$\lim_{h\to 165} \frac{f(h) - f(165)}{h - 165}.$$

(e) Use your result from part (d), rounded to the nearest thousandth, to approximate Δs in terms of Δh and enter your result below.

$$\Delta s \sim 3.291\Delta h, \text{ for } \Delta h \sim 0.$$

(f) Explain the meaning of the proportionality constant in parts (d) and (e). Be sure to include units in your explanation.

Small Changes

We explore how small changes to the input of a function change the output.

The main idea of differential calculus is to approximate the change in the output of a function in terms of a small change in the input. For some functions, called *differentiable*, the change in the output is approximately proportional to the (small) change in the input. The proportionality factor is called the derivative. In this chapter we explore this idea.

1 Odometer Readings

Example 1. The graph of the function

$$s = f(t), 0 \leq t \leq 2,$$

that expresses the trip odometer reading (measured in miles) on your car in terms of the number of hours past noon during a two-hour trip is shown below.

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/iiw69lr1bc`

Desmos activity available at

151: Odometer

Our goal is to approximate the car's speed at 12:30pm in three ways:

(1) geometrically, using the above graph as is.

(2) geometrically, by zooming in on the above graph.

(3) arithmetically, using the specific expression for the function f.

(a) Start by using the graph above to describe how the speed of the car varies over the two-hour period. Explain your reasoning. Then play the Slider u in Line 2 and use the animation of the motion to check if your description was accurate. Explain.

Learning outcomes:
Author(s):

(b) Set the slider in Line 6 to $n = 20$. Use *only* the graph (with $u = 2$) to create a table with five columns showing the values of t, s (approximate this), $\Delta t = t - 0.5$, $\Delta s = f(t) - f(0.5)$, and $\Delta s/\Delta t$. Include units in the heading of each column. The table should include five rows, with $t = 0.3, 0.5, \ldots 0.7$.

t (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.3				
0.4				
0.5				
0.6				
0.7				

(c) Explain the meaning of the fifth column in the table of part (b). What do the entries in this column suggest about the speed of the car at 12:30pm?

(d) Now we'll use the fact that

$$s = f(t) = 48t^2 - 16t^3, 0 \le t \le 2,$$

to construct another table like the one in part (b). Do this by finding expressions for

$$\Delta s = f(t) - f(0.5)$$

and

$$r = g(t) = \frac{\Delta s}{\Delta t} = \frac{f(t) - f(0.5)}{t - 0.5},$$

both in terms of t (and *not* Δt). Use these functions to fill in the missing entries in the table below.

t (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.49				
0.499				
0.4999				
0.5				
0.5001				
0.501				
0.51				

(e) Does your table from part (d) suggest that the ratios $\Delta s/\Delta t$ approach some number as $t \to 0.5$? If so, what would be your guess for the exact value of this number? What are its units? What is its meaning?

(f) Activate the folder "Graph of average speed" on Line 9.

(i) Use the graph to check some your entries in the fifth column of your table from part (d). Explain.

(ii) How is the line PQ, through the fixed point $P(0.5, 10)$ and the variable point $Q(t, s)$ on the graph of the function $r = g(t)$ related to the ratio $\Delta s/\Delta t$?

(iii) Describe what happens to the line PQ as point Q approaches point P.

(g) For a quicker way to approximate car's speed at 12:30pm, zoom in sufficiently close to point P in the graph above to make the graph of $s = f(t)$ look like a line. Use the coordinates of point P and a second point in the window far away from P to estimate the car's speed at 12:30pm. Explain your method.

(h) Summarize your conclusions by comparing your three estimates for the car's speed at 12:15pm. Which estimate do you think is most accurate? Least accurate?

Example 2. This is a continuation of the previous example where we'll algebraically compute the exact speed of the car at 12:30pm, using the odometer function

$$s = f(t) = 48t^2 - 16t^3, 0 \le t \le 2.$$

The idea is to first find an algebraic expression for the car's average speed between time t and time $t = 0.5$ hours past noon. Then we'll evaluate the limit of this average speed as $t \to 0.5$ to find the (instantaneous) speed at 12:30pm.

Question 33 *First we'll find the average speed between time t and time $t =$ 0.5.*

(a) Explain in general how to compute a car's average speed over some time interval. What do you need to know? What is the computation? Make up your own specific example.

(b) Now for our particular odometer function above, the average speed $v_{\text{avg}}(t)$, measured in miles/hour, between time t and time $t = 0.5$ is

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{48t^2 - 16t^3 - 10}{t - 0.5} \\ &= \frac{(2t-1)(-8t^2 + 20t + 10)}{t - 0.5} \\ &= -16t^2 + 40t + 20 \text{ if } t \neq 0.5. \end{aligned}$$

The key step in the computation above is in the third line. How did we know $2t - 1$ was a factor of

$$f(t) - f(0.5) = 48t^2 - 16t^3 - 10?$$

The reason is that $t = 0.5$ is a root of the polynomial $f(t) - f(0.5)$ and therefore $t - 0.5$ is a factor. And so

$$2(t - 0.5) = 2t - 1$$

is also a factor. Then we can use long division to find the quotient.

(c) Show the steps in the long division.

(d) The final step in computing the car's speed v (in miles/hour) at 12:30pm is to evaluate the limit of these average speeds as $t \to 0.5$. We get

$$\begin{aligned} v &= \lim_{t\to 0.5}(-16t^2 + 40t + 20) \\ &= 36. \end{aligned}$$

(e) Here's another way to simplify the average speed in part (b). Fill in the missing steps.

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{(48t^2 - 16t^3) - (48(0.5)^2 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{(48t^2 - 48(0.5)^2) - (16t^3 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{48(t^2 - (0.5)^2) - 16(t^3 - (0.5)^3)}{t - 0.5} \\ &= \frac{48(t - 0.5)(t + 0.5)) - 16(t - 0.5)(t^2 + 0.5t + 0.25)}{t - 0.5} \\ &= 48(t + 0.5)) - 16(t^2 + 0.5t + 0.25) \ \textit{if}\ t \neq 0.5. \end{aligned}$$

(f) Use the above expression for the average speed function to compute the (instantaneous) speed of the car at 12:30pm by evaluating the approriate limit.

(g) Sketch by hand *a graph of the average speed function $y = v_{avg}(t)$ over the appropriate domain. Be sure also to state this function's domain.*

2 A Projectile

Example 3. Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/l4fknr0hpl`

Desmos activity available at

151: Projectile

3 The Falling Ladder, Part 1

Example 4. The top end of a ten-foot ladder leans against a vertical wall and the bottom end rests on the horizontal floor. We analyze how a small change in the distance between the wall and the bottom of the ladder affects the height of the ladder's top above the floor.

Access Desmos interactives through the online version of this text at

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Desmos link: https://www.desmos.com/calculator/dvyuifyyg4

Desmos activity available at

151: Ladder 1B

Question 34 *(a) The slider s in Line 1 of the demonstration above controls the distance between the wall and the bottom of the ladder, measured in feet. Use the slider s to describe* qualitatively *how a small change in s changes the height h (also measured in feet) of the ladder's top end above the floor.*

(i) Do the small changes have the same or opposite signs?

(ii) At what positions of the ladder does a small change in s result in a comparatively large change in h?

(b) Now let's focus on the particular position of the ladder where the bottom end is $s = 8$ feet from the wall. For this, turn on the "one position" folder in Line 3.

(i) Drag the slider s close to $s = 8$ and use the cordinates of the endpoints of the ladder to construct a table with five columns showing the values of s, h, $\Delta s = s - 8$, $\Delta h = f(s) - f(8)$, and $\Delta h/\Delta s$. Include units in the heading of each column. The table should include seven rows, with $s = 7.7, 7.8, \ldots 8.2, 8.3$. Here $h = f(s)$ is the function described in part (ii) below.

(ii) Find a function

$$h = f(s), 0 \leq s \leq 10,$$

that expresses the height of the ladder's top end above the ground (in feet) in terms of the distance of its bottom end from the wall (in feet).

(iii) Use your function f to construct another table, like the one in part (i), with $s = 7.9, 7.99, 7.999, 8, 8.0001, 8.01, 8.1$. Do this by finding expressions for

$$\Delta h = f(s) - f(8)$$

and

$$r = g(s) = \frac{\Delta h}{\Delta s} = \frac{f(s) - f(8)}{s - 8},$$

both in terms of s (and not *Δs).*

(iv) Does your table from part (iii) suggest that the ratios $\Delta h/\Delta s$ approach some number as $s \to 8$? If so, what would is your guess for the exact value of this limit? What are its units? What is its meaning?

(v) Activate the folder "graph of function" on Line 8. How is the line PQ, through the fixed point $P(8, 6)$ and the variable point $Q(s, h)$ on the graph of the function $h = f(s)$ related to the ratios $\Delta h/\Delta s$?

(vi) Change the bounds for s in Line 1 to run between $s = 7.9$ and $s = 8.1$. Then activate the folder "graph: average rate of change function" on Line 18. Move the slider s and use the graph of the function $r = g(s)$ to check your computations in part (iii).

(vii) Use the result of part (iv) to write an approximation for the change in height

$$\Delta h = h - 8$$

in terms of the change

$$\Delta s = s - 8.$$

(viii) The graph of the function $h = f(s)$ suggests another, geometric way to find the proportionality constant (of part (vii)) that relates Δh to Δs. Explain how.

4 The Falling Ladder, Part 2

Example 5. A tree leans precariously with its trunk making an angle of $\phi = \pi/6$ radians with the ground. One end of a ten-foot ladder leans against the trunk, the other rests on the horizontal ground. We analyze how a small change in the distance between the bottom of the ladder and the base of the trunk changes the distance between the top of the ladder and the base of the trunk.

Access Geogebra interactives through the online version of this text at

.

Geogebra link: `https://www.geogebra.org/m/qmke5y7x`

We'll let t be the distance between the top of the ladder and the base of the trunk (measured in feet) and s the distance between the bottom of the ladder and the base of the trunk (also measured in feet).

The slider θ in the demonstration above controls the angle that the ladder makes with the ground, but this angle does not come into play in our problem.

(a) Use the slider θ to describe qualitatively how a small change in s (the length of segment GC) changes t (the length of segment GB):

(i) For what positions of the ladder do these small changes have the same signs? Opposite signs?

(ii) For what positions of the ladder does a small change in s result in a comparatively large change in t?

(b) Now let's focus on the particular position of the ladder when the bottom end C is 16 feet from the trunk's base and the top end D is about 8 feet $f$rom the base *as illustrated above*. Our first step is to find a function

$$t = f(s)$$

that expresses t in terms of s for values of s and t near $s = 16$ and $t = 8$ respectively.

To do this, first use the law of cosines to write an equation relating s and t. Then *complete the square* to solve this equation for t in terms of s to find the function $t = f(s)$. Keep in mind that when $s = 16$, we must have $t \sim 8$.

Hint: *Here is an outline of the steps to check your work:*

(i) Use the law of cosines to relate s and t.

$$s^2 + t^2 - \sqrt{3}st = 100.$$

(ii) Solve the above equation for t in terms of s as follows:

First rewrite the equation with the two terms with t as a factor on the left side and the other two terms on the right.

$$t^2 - \sqrt{3}st = 100 - s^2.$$

Then complete the square by adding the same perfect square to each side.

$$t^2 - \sqrt{3}st + \left(\frac{\sqrt{3}s}{2}\right)^2 = 100 - s^2 + \left(\frac{\sqrt{3}s}{2}\right)^2.$$

Then factor the LHS and simplify the RHS.

$$\left(t - \frac{\sqrt{3}s}{2}\right)^2 = 100 - \frac{s^2}{4}.$$

Next, solve for t in terms of s.

$$t = \frac{\sqrt{3}}{2}s \pm \sqrt{100 - \frac{s^2}{4}}$$

Finally, make the correct choice of $\pm$ to solve for t in terms of s, when s is near 16 and t near 8.

$$t = \frac{\sqrt{3}}{2}s - \sqrt{100 - \frac{s^2}{4}}.$$

(c) Use your function from part (b) to find an expressions for

$$\Delta t = f(s) - f(16)$$

and for the function

$$r = g(s) = \frac{\Delta t}{\Delta s} = \frac{f(s) - f(16)}{s - 16}.$$

Explain what the output of the function g measures. What are its units?

(d) Use the results of part (c) to construct a table with five columns showing the values of s, t, $\Delta s = s - 16$, $\Delta t = f(s) - f(16)$, and $\Delta t/\Delta s$. Include units in the heading of each column. The table should include seven rows, with $s = 15.9, 15.99, 15.999, 16, 16.001, 16.01, 16.1$.

(e) Does your table from part (d) suggest that the ratio $\Delta t/\Delta s$ approaches some number as $s \to 16$? If so, approximate the value of this number. What are its units?

(f) Check the box "GraphofRelation" in the demonstration above and explain how the line EF is related to part (d).

(g) Use the result of part (e) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of s near 16. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 16.4 feet from the trunk's base. Compare your approximation with the exact distance.

5 Riding a Ferris Wheel

Suppose you ride a ferris wheel

Limits

Limits in context.

1 Limits and Tangent Lines

Example 6. Let
$$g(x) = \frac{x^2 - 9}{3x - 9}.$$

(a) Evaluate each of the following expressions.

(i) $g(7)$

(ii) $\lim_{x \to 7} g(x)$

(iii) $g(3)$

(iv) $\lim_{x \to 3} g(x)$

(b) Simplify and then graph the function $g(x)$.

(c) Interpret the expressions in part (a) geometrically by considering the graph of the function $f(x) = x^2/3$ as in the demonstration below.

Desmos link: `https://www.desmos.com/calculator/u0uvuchnrk`

Desmos activity available at 151: Parabola Basic

2 Limits and Gas Mielage

Example 7. The function
$$G = f(s) = \frac{2}{5} + \frac{1}{5000}(40(s+2)^2 - (s+2)^3),\ 0 \le s \le 23,$$
expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/xzkknfpkw3`

Learning outcomes:
Author(s):

Desmos activity available at 151: Gas as a Function of Distance

(a) Use the graph of the function f shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Find your average gas mileage (in miles/gallon) over the interval $s \in [8, 18]$.

(c) Use the graph to approximate your gas mileage at the moment you are 18 miles from home. Do this by zooming in on the appropriate point.

(d) Use the algebra of limits to determine your exact gas mileage at the moment you are 18 miles from home.

(e) Use the result of part (d) to approximate the change in the volume of gas

$$\Delta G = f(s) - f(18)$$

in terms of the change

$$\Delta s = s - 18$$

in your distance from home for values of s near 18 miles. What are the units of the proportionality constant?

(f) Use the result of part (e) to approximate your distance from home when there are 1.9 gallons of gas in your tank.

3 Limits, Gas Mileage and Speed

Example 8. Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and that the car gets 38 miles/gallon at a speed of 40 miles/hour.

(a) Find an expression for the function

$$G = f(v)\,,\ 30 \leq v \leq 70,$$

that gives the gas mileage (in miles/gal) in terms of the speed (in miles/hour).

(b) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the car's speed at 60 miles/hour gives an approximately proportional change in its gas mileage.

(c) Use the results of part (b) to approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find the exact value of the proportionality constant.

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta G = g - f(60)$$

in gas mileage in terms of a small change

$$\Delta v = v - 60$$

in the car's speed.

(g) Use part (f) to approximate the speed at which the car gets 36 miles/gallon.

(h) Would you expect your approximation in part (g) to be greater or less than the actual speed? Explain your reasoning with a graph.

(i) Simplify the units of the proportionality constant. What might these units suggest about a way to interpret the constant?

(j) At what rate (in gal/hr) does the car burn gas at a speed of 60 miles/hour?

(k) How is the rate in part (j) related to the proportionality constant?

4 Limits, Speed and Altitude

Question 35 *A rock dropped from a height of* 100 *feet falls to the surface of Planet Krypton without air resistance.*

(a) By considering only the physical situation and without *doing any computations, sketch a graph of the function*

$$v = g(h)\,, 0 \le h \le 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet).

(b) Use the results from part (a) to choose a reasonable expression for the function g from the list below.

Multiple Choice:

(a) $g(t) = 100 - 9t^2$, $0 \le t \le 10/3$

(b) $g(h) = 100 - 9h^2$, $0 \le h \le 100$

(c) $g(h) = 0.005(100 - h)^2$, $0 \le h \le 100$

(d) $g(h) = 6\sqrt{100 - h}$, $0 \le h \le 100$ ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 *feet gives an approximately proportional change in its speed.*

(d) Use the results of part (c) to approximate the proportionality constant. What are its units?

(e) Use the algebra of limits to find the exact value of the proportionality constant.

(f) Explain the meaning of the proportionality constant.

(g) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(h) Use part (g) to approximate the rock's speed at a height of 63 *feet.*

(i) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(j) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

5 Limits and Gas Mileage

6 Limits and Purchasing Power

The Derivative

Computing derivatives with limits.

Example 9. Suppose

$$y = f(x) = x^2$$

and let's use limits to evaluate

$$f'(3) = \left.\frac{dy}{dx}\right|_{x=3} = \left.\frac{d(x^2)}{dx}\right|_{x=3}.$$

We have

$$\begin{aligned}
f'(3) = \left.\frac{d(x^2)}{dx}\right|_{x=3} &= \lim_{v\to 3} \frac{f(v) - f(3)}{v - 3} \\
&= \lim_{v\to 3} \frac{v^2 - 9}{v - 3} \\
&= \lim_{v\to 3} \frac{(v-3)(v+3)}{v - 3} \\
&= \lim_{v\to 3} (v + 3) \\
&= (3 + 3) \\
&= 6.
\end{aligned}$$

Next let's do almost the same thing and compute

$$f'(x) = \frac{dy}{dx}$$

for the function

$$y = f(x) = x^2$$

by replacing 3 in the above computation with x.

Learning outcomes:
Author(s):

We get

$$\begin{aligned}
\frac{dy}{dx} = \frac{d(x^2)}{dx} &= \lim_{v\to x} \frac{f(v)-f(x)}{v-x} \\
&= \lim_{v\to x} \frac{v^2-x^2}{v-x} \\
&= \lim_{v\to x} \frac{(v-x)(v+x)}{v-x} \\
&= \lim_{v\to x} (v+x) \\
&= (x+x) \\
&= 2x.
\end{aligned}$$

Just as a check, when $x = 3$,

$$\left.\frac{dy}{dx}\right|_{x=3} = (2x)\Big|_{x=3} = 2(3) = 6.$$

Question 36 *Use the result of the previous example to solve each of the following problems. Do* not *use limits.*

(a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(-4, 16)$.

(b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).

(c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.

(d) Let $\mathcal{L}$ be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where $\mathcal{L}$ intersects the y-axis.

(e) Repeat parts (a)-(d) above for the tangent line to the parabola $y = x^2$ at the general point (a, a^2). What do you notice?

Question 37 *(a) Use the method of Example 1 for the function*

$$y = g(x) = 1/x^2$$

to compute

$$g'(3) = \left.\frac{dy}{dx}\right|_{x=3} = \left.\frac{d(1/x^2)}{dx}\right|_{x=3}$$

and

$$g'(x) = \frac{dy}{dx}.$$

(b) Use the result of part (a) to find an equation of the tangent line to the curve $y = 1/x^2$ at the point $(3, 1/9)$.

Question 38 *(a) Use numerical methods to estimate the slope of the tangent line to the curve*

$$y = f(x) = x^3$$

at the point $(2, 8)$. Include enough data to suggest a progression toward a limit.

(b) Use the algebra of limits to find the exact slope of the tangent line in part (a).

(c) Use algebra to find the coordinates of the all pointw where the tangent line in part (a) intersects the curve $y = x^3$.

(d) Suppose you measure the edge length of a cube to be 2cm and then use this measurement to compute the volume of the cube. Use the result of part (b) to approximate your error in computing the volume in terms of your error in measuring the edge length. Assume the latter error is small.

Then compare your exact error in computing the volume with your approximation for some specific edge length near 2cm. You should start this problem defining a function with meaningful variable names (do not use x and y).

Question 39 *(a) Find a function*

$$s = g(V), V \geq 0$$

that expresses the edge length (measured in cm) of a cube in terms of its volume (measured in cubic centimeters).

(b) Use the algebra of limits to evaluate the derivative

$$g'(V_0) = \left.\frac{ds}{dV}\right|_{V=V_0}.$$

(c) What are the units of the derivative in part (b)? Explain how you know.

(d) Suppose you submerge the cube in water and measure its volume to be 8 cm^3. You then use this measurement to compute the edge length of the cube. Use the result of part (b) to approximate your error in computing the edge length in terms of your error in measuring the volume. Assume the latter error is small.

Start this problem by defining the errors Δs and ΔV in terms of the actual edge length s of the cube.

(e) Use the result of part (d) to approximate your error in computing the edge length of the cube if the cube's actual volume is 7.7cm^3.

(f) Explain how this question is related to Question 4.

Question 40 *Suppose that between speeds of 60 miles/hr and 72 miles/hr, the gas mileage of a car is a linear function of its speed. Suppose also that the car gets 36 miles/gallon at a speed of 60 miles/hour and 32 miles/gallon at a speed of 72 miles/hour.*

(a) Find a function

$$r = f(v), 60 \leq v \leq 72,$$

that expresses the rate (measured in gal/hr) at which the car burns gas in terms of its speed (measured in miles/hour). Explain your reasoning. This function is not *linear.*

Hint:

$$r = f(v) = \frac{3v}{168 - v}, 60 \leq v \leq 72.$$

(b) Use numerical methods to estimate the value of the derivative

$$f'(63) = \left.\frac{dr}{dv}\right|_{v=63}.$$

Make a table that shows enough data to suggest a progression toward a limit. Include units in all column headings.

(c) Use the algebra of limits to find an expression for the derivative

$$f'(v) = \frac{dr}{dv}.$$

Then use this expression to find the exact value of the derivative in part (b).

(d) What are the units of the derivative in part (b)? Explain its meaning.

(e) Use the result of part (c) to approximate the change

$$\Delta r = f(v) - f(63)$$

in the rate at which your car burns gas in terms of the change

$$\Delta v = v - 63$$

of the car's speed. Assume $\Delta v \sim 0$.

Question 41 *Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that* *the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and 34 miles/gallon at a speed of 30 miles/hour.*

(a) Find a function

$$r = h(v)\,,\; 30 \le v \le 70,$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour).

Hint: *(i) At what rate does the car burn gas at a speed of 50 miles/hour?* 25/21 *gal/mile*

(ii) Find a function that expresses the gas mileage G (measured in miles/gallon) in terms of the speed (measured in miles/hr).

$$G = 42 - 0.02(v-50)^2\,,\; 30 \le v \le 70.$$

(ii) The rate (in gal/hr) at which the car burns gas as a function of its speed (in miles/hr) is

$$r = h(v) = \frac{v}{42 - 0.02(v-50)^2}\,,\; 30 \le v \le 70.$$

(b) Use the algebra of limits to evaluate the derivative

$$h'(40) = \left.\frac{dr}{dv}\right|_{v=40}.$$

(c) What are the units of the above derivative? How do you know?

(d) Express the meaning of the derivative in the context of small changes.

Question 42 *At 10:00am on April 18, the wholesale price of Cosmic Crisp apples is \$2.00/lb and is decreasing at the rate of \$0.10/$lb/hour$.*

Use the algebra of limits to determine the rate (in pounds/hour) at which the number of pounds of apples a store can purchase with \$1000 is changing at this time.

Start this question by defining a function that expresses the number of pounds of apples the store can buy with \$1000 in terms of the price (in \$/lb). Choose meaningful variable names (not x and y). Do not *assume the price is decreasing at a constant rate.*

1 Limits, Speed and Altitude

Question 43 *A rock dropped from a height of* 100 *feet falls to the surface of Planet Krypton without air resistance.*

(a) By considering only the physical situation and without *doing any computations, sketch a graph of the function*

$$v = g(h)\,, 0 \leq h \leq 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet). Explain your reasoning.

(b) Use the results from part (a) to choose a reasonable expression for the function g from the list below.

Multiple Choice:

(a) $g(t) = 100 - 9t^2$, $0 \leq t \leq 10/3$

(b) $g(h) = 100 - 9h^2$, $0 \leq h \leq 100$

(c) $g(h) = 0.005(100 - h)^2$, $0 \leq h \leq 100$

(d) $g(h) = 6\sqrt{100 - h}$, $0 \leq h \leq 100$ ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 *feet gives an approximately proportional change in its speed. Then approximate the proportionality constant. What are its units?*

(d) Use the algebra of limits to find an expression for the derivative

$$g'(h) = \frac{dv}{dh}.$$

Then use this expression to find the exact value of the proportionality constant in part (c).

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(g) Use part (g) to approximate the rock's speed at a height of 63 *feet.*

(h) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(i) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

Derivatives of Polynomials

Working with polynomials and their derivatives.

1 Differentiating Polynomials

Question 44 *Let $f(x) = x^5$.*

(a) *Use the algebra of limits to find an expression for the derivative*

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(x^5\right).$$

(b) *Use the result of part (a) to find an expression for the derivative*

$$\frac{d}{dx}(f^{-1}(x)) = \frac{d}{dx}\left(x^{1/5}\right).$$

Use set-builder notation to state the domains of $f^{-1}(x)$ and its derivative.

(c) *Use the result of part (a) and the algebra of limits to find an expression for the derivative*

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{d}{dx}\left(x^{-5}\right).$$

(d) *Use the results of parts (a)-(c) to find equations of the tangent lines to the three curves $y = f(x)$, $y = f^{-1}(x)$, and $y = 1/f(x)$ at the point $(1, 1)$. Graph the curves and their tangent line in Desmos to check your work.*

Question 45 *(a) Use the results of Question 1 to make a conjecture about the derivative*

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^n)$$

of the function $f(x) = x^n$.

(b) What do you get for the derivative when $n = 0$? When $n = 1$? Are these results correct? Explain.

Learning outcomes:
Author(s):

Question 46 *Due to a printing error, the graph of the function $y = f(x) = 234x^5$ is missing below. All we see is a point A on the graph of the function.*

Desmos link: `https://www.desmos.com/calculator/pjyqjtdcxm`

151: Printers Error 2

Without *sketching a graph, drag the slider m to draw the tangent line to the curve $y = 234x^5$. Explain your reasoning.*

Question 47 *(a) Suppose one giraffe is always twice as tall as another. What can you say about their growth rates at any instant?*

(b) Suppose one giraffe is always two feet taller than another. What can you say about their growth rates at any instant?

(c) What do parts (a) and (b) suggest about how to compute the derivatives

$$\frac{d}{dx}\left(f(x) + b\right),$$

$$\frac{d}{dx}\left(af(x)\right),$$

and

$$\frac{d}{dx}\left(af(x) + b\right)$$

for constants $a, b \in \mathbb{R}$?

(d) Make up your own scenario that suggests how to compute the derivative

$$\frac{d}{dx}\left(f(x) + g(x)\right)$$

of the sum of two functions.

Question 48 *Find an equation of the tangent line to the curve*

$$y = -x^3 + 4x^2 - 3x + 1$$

at the point $(2, 3)$. Graph the curve and its tangent line on Desmos.

2 Stock Price

Question 49 *The function*

$$P = f(t) = 0.1t^3 - t^2 + t + 20\,,\; 0 \le t \le 8,$$

expresses the price (in dollars/share) of a stock in terms of the number of hours past 9am. The graph of the function is shown below.

Desmos link: `https://www.desmos.com/calculator/gpqicgxvk3`

151: Stock Price

(a) *Is the price increasing or decreasing at 10am? Use the graph to estimate this rate as follows:*

   (i) *First drag the slider u (another name for t) on Line 5 to the appropriate value.*

   (ii) *Then drag the slider m (the slope of the orange line) so that the orange line looks parallel to the (red) tangent line to the curve $P = f(t)$.*

(b) *Use the function to compute the exact rate at which the price is changing at 10am. Include units.*

(c) *At what relative rate is the price changing at 10am? Include units.*

(d) *At what rate is the number of shares you can buy with \$1000 changing at 10am? Do* not *use the quotient rule or the chain rule (we have not learned these yet). Use limits instead, but work in general (ie. not with this specific price function). See the Explanation below for help.*

(e) *Use the sliders m and b to control the line $P = b + mt$ and approximate when the price is decreasing at the rate of (\$1.8/share)/hr. Then use calculus and algebra to compute the exact time(s).*

(f) *Use calculus and technology to approximate the time(s) when the price is decreasing at the relative rate of 10%/hour.*

(g) *Use the graph to approximate the minimum and maximum prices of the stock during the eight-hour period. Then use calculus and algebra to compute these exact prices. Click the arrow below for a hint.*

   What is the value of the derivative dP/dt when the price is a maximum/minimum?

(h) *Use the slider u (another name for the variable t) to approximate the time(s) when the price is decreasing at the fastest rate. Then use calculus to compute the exact time(s). Click the arrow below for a hint.*

It helps to graph the fuction

$$dP/dt = 0.03t^2 - 2t + 1\,,\ 0 \le t \le 8,$$

Then think about how you would use calculus *to compute the exact time when the price is decreasing at the fastest rate. This is similar to the previous question.*

(i) *Use the graph above to approximate the end of the time interval beginning at 9am over which the price decreases at the greatest average rate. Do this by activating the folder* secant line *on Line 6 (click the circle on the left) and dragging the slider u.*

(j) *The use calculus and algebra to find the end of the time interval beginning at 9am over which the price decreases at the greatest average rate. Click the arrow below for a hint.*

Think about the function

$$m(t) = \frac{f(t) - f(0)}{t - 0}\,,\ 0 < t \le 8,$$

that expresses the average rate of change of the price (with respect to time) over the time interval between 9am and time t hours past 9am. Then use this function and the ideas of parts (g) and (h).

Explanation. (d) Here's the solution to part (d).

- Start by finding a function

$$S = g(t)\,,\ 0 \le t \le 8,$$

  that expresses the number of shares we can buy with $\$1,000$ in terms of the number of hours past 9am. Click the arrow to the lower right to check.

  This function is

$$S = g(t) = \frac{1000}{f(t)}\,,\ 0 \le t \le 8$$

- Next we'll use limits to find an expression for the derivative

$$\left.\frac{dS}{dt}\right|_{t=a}$$

  as follows.

$$
\begin{aligned}
\left.\frac{dS}{dt}\right|_{t=a} &= \lim_{v\to a} \frac{g(v)-g(a)}{v-a} \\
&= \lim_{v\to a} \left(\frac{1000}{v-a}\right)\left(\frac{1}{f(v)} - \frac{1}{f(a)}\right) \\
&= \lim_{v\to a} \left(\frac{1000}{v-a}\right)\left(\frac{f(a)-f(v)}{f(v)f(a)}\right) \\
&= \lim_{v\to a} \left(\frac{-1000}{f(v)f(a)}\right)\left(\frac{f(v)-f(a)}{v-a}\right) \\
&= \lim_{v\to a} \frac{-1000}{f(v)f(a)} \cdot \lim_{v\to a} \frac{f(v)-f(a)}{v-a}
\end{aligned}
$$

Let's pause here for a moment. The last equality above follows from the fact that both limits in the product exist. The first limit is easy enough to evaluate. The second limit exists because *it is equal to the derivative*

$$\left.\frac{dP}{dt}\right|_{t=a}.$$

That's the key point. Now we know that

$$
\begin{aligned}
\left.\frac{dS}{dt}\right|_{t=a} &= \lim_{v\to a} \frac{-1000}{f(v)f(a)} \cdot \lim_{v\to a} \frac{f(v)-f(a)}{v-a} \\
&= \left(\frac{-1000}{(f(a))^2}\right)\left(\left.\frac{dP}{dt}\right|_{t=a}\right)
\end{aligned}
$$

This is enough to compute the rate at which the number of shares we can buy with \$1000 is changing at 10am.

But we can get a better understanding of this by writing the above expression in a more meaningful way. Remember that $f(a) = P$ is the price at time t. So

$$
\begin{aligned}
\left.\frac{dS}{dt}\right|_{t=a} &= -\left(\frac{1000}{(f(a))^2}\right)\left(\left.\frac{dP}{dt}\right|_{t=a}\right) \\
&= -\left(\frac{1000}{P^2}\right)\left(\left.\frac{dP}{dt}\right|_{t=a}\right) \\
&= -\left(\frac{1000}{P}\right)\left(\frac{1}{P}\cdot\left.\frac{dP}{dt}\right|_{t=a}\right) \\
&= -S\left(\frac{1}{P}\cdot\left.\frac{dP}{dt}\right|_{t=a}\right),
\end{aligned}
$$

where $S = 1000/P$ is the number of shares we can buy with $\$1,000$ at a price of P dollars/share.

Now remember that the product

$$\frac{1}{P} \cdot \frac{dP}{dt}\Big|_{t=a}$$

is the *relative* rate of change in the price (with respect to time) at time $t = a$ hours past 9am.

From part (c) we know at 10am this relative rate is equal to

$$\frac{1}{P} \cdot \frac{dP}{dt}\Big|_{t=1} = -\frac{0.7}{20.1}\text{hr}^{-1} \sim -3.4826\%/\text{hour}$$

So at 10am we can buy

$$S = \frac{\$1000}{\$20.1/\text{share}} \sim 47.7512 \text{ shares}$$

with $\$1,000$. And the number of shares we can buy with is increasing at the rate of approximately

$$(47.7512 \text{ shares})\,(-3.4826\%/\text{hour}) \sim 1.733 \text{ shares/hour}.$$

at 10am.

Thinking about Parabolas

Question 50 *(a) Find an equation of the tangent line to the parabola $y = x^2$ at the point $(-3, 9)$.*

(b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).

(c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.

(d) Let $\mathcal{L}$ be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where $\mathcal{L}$ intersects the y-axis.

(e) Repeat parts (a)-(d) above for the tangent line to the parabola $y = x^2$ at the general point (b, b^2). What do you notice? Enter your work in the Desmos activity below.

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/qe7mgnu5sv`

Question 51 *Let*

$$y = f(x) = ax^2$$

where $a \in \mathbb{R}$ is a constant, and the variables x, y are measured in meters.

(a) What are the units of the constant a? How do you know?

(b) Answer part (e) of the previous question for this function. Modify the desmos activity in the previous question to check your work.

Question 52 *In the absense of air resistance, a rock released from rest near the surface of the earth falls $s = 16t^2$ feet during the first t seconds of its fall.*

Compare the speed of the rock when it hits the ground with the average speed of the rock during the entire time interval of its fall.

Question 53 *(a) The demonstration below shows two normal lines to a parabola and their point of intersection P. What do you think happens to P as point B approaches A? Answer this question* without *dragging the slider b.*

(b) Now drag the slider b near $a = 2$ and observe what happens to point P. Were you correct?

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/ybaivhc2tl`

Access this activity online at 151: Normals to Parabola

The parabola has equation $y = x^2/4$, point A has coordinates $(2, 1)$, and point B has coordinates $(b, b^2/4)$.

(c) Find an equation of the normal lines to the parabola at A and B.

(d) Use algebra to find an equation of the point P where the normal lines intersect.

(e) The point P approaches some point Q as B appraoches A. Use the algebra of limits to find the coordinates of Q.

(f) Find an equation of the circle centered at Q through A. Do this by first using vector algebra to find the coordinates of the center of the circle.

(g) Repeat parts (c)-(g), replacing the point $A(2, 1)$ with the point $A(a, a^2)$.

Applications

Question 54 *Play the slider u in the animation below to watch the motion of a balloon as it leaves behind track's at equal time intervals. Ignore the balloon's rightward jogs. Their purpose is to prevent the tracks from overlapping.*

Desmos link: `https://www.desmos.com/calculator/h91txxjcmi`

151: Balloon Motion 7

(a) Use the animation to sketch by hand the graph of the function

$$h = f(t)\,,\ -1 \le t \le 1.8,$$

that expresses the balloon's height (in thousands of feet) in terms of the number of hours past noon.

(b) Use the animation or your graph from part (a) to sketch by hand the graph of the function

$$r = g(t)\,,\ -1 \le t \le 1.8,$$

that expresses the balloon's rate of ascent (in thousands of feet/hour in terms of the number of hours past noon.

(c) Use your graph(s) or the animation to approximate when the balloon descends at the fastest rate.

Question 55 *The function*

$$h = f(t) = 5 - 2t - \frac{t^2}{4} + t^3\,,\ -1 \le t \le 1.8,$$

expresses the height (in thousands of feet) of a balloon in terms of the number of hours past noon.

(a) Find the balloon's average rate of ascent between 11:00am and 11:30am.

(b) Is the balloon rising or falling at 1:00pm? At what rate? Use the graph of the function $h = f(t)$ below to approximate the rate. Then compute the exact rate.

(c) When is the balloon descending at the rate of 1000 ft/hour? Use the sliders m anb b below to approximate the time(s). Then compute the exact time(s).

(d) Use the graph below to approximate when the balloon is descending at the fastest rate. Approximate this rate from the graph. Then compute the exact time and rate.

(e) Use the graph below to approximate when the balloon is at its lowest point. Then compute the exact time.

(f) Use the graph below to approximate when the balloon is at its highest point between 11am and 1:36pm. Then compute the exact time.

(g) Use algebra to find all half-hour time intevals during which the balloon descends at an average rate of 500 ft/hour.

Desmos link: `https://www.desmos.com/calculator/bvtukd0vlc`

Access this activity online at 151: Height of Balloon

Question 56 *The function*

$$G = f(s) = \frac{11}{5} + \frac{1}{5000}\left(s^3 - 50s^2 + 300s\right),\ 3 \le s \le 28,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/cphmgnrtm7`

Desmos activity available at 151: Gas as a Function of Distance 2c

(a) Use the graph of the function f shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Zoom in on the graph to approximate your gas mileage at the moment you are 20 miles from home. Show a screenshot to help explain how you got your approximation. Then compute the exact gas mileage.

(c) Use the sliders m and b_1 in the graph to approximate your distance from home at the moment your car gets 30 miles/gallon. Show a screenshot to help explain how you got your approximation. Then compute the exact distances.

(d) Use the sliders m and b_1 in the graph to approximate an interval beginning or ending when you are 20 miles from home over which your average gas mileage is equal to you gas mileage at the moment you are 20 miles from home. Show a screenshot to help explain how you got your approximation. Then compute the exact interval.

(e) Sensors on your car measure both the (instantaneous) gas mileage and the number of gallons of gas in your tank at each instant. A computer then uses these measurements to estimate the number of additional miles you can drive before running out of gas. Use this idea to find a function

$$m = g(s),\ 3 \le s \le 28,$$

that expresses the number of miles you can drive before running out of gas (assuming your gas mileage remains constant for the remainder of your trip) in terms of your distance from home. Explain your reasoning.

Question 57 *(a) Make up your own* quadratic *function*

$$v = f(G) = aG^2 + bG + c,$$

with a,b, and c all not equal to zero, that expresses the speed of a car (measured in miles/hour) in terms of its gas mileage (measured in miles/gallon). Be sure to include a domain. Explain why you think your function is reasonable.

(b) Compute the derivative dv/dG and evaluate it at a spefic gas mileage. Include units.

(c) Evaluate the derivative dv/dG at a specific gas mileage and its meaning. Include units in your explanation.

(d) Use your function from part (a) to find a function

$$r = h(G)$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its gas mileage (in miles/gal). Explain your logic thoroughly.

(e) Evaluate $h(G)$ at the same gas mileage, say G_0, you used in part (c). Compare the units of $h(G_0)$ and the derivative $dv/dG\Big|_{G=G_0}$. Are these two numbers related? What does this tell you about simplifying the units of a derivative?

(f) Use the ideas of this chapter (ie. derivatives of polynomials, and nothing beyond) to find an expression for the derivative dr/dG .

(g) What are the units of the derivative dr/dG?

(h) Evalute the derivative dr/dG at a specific gas mileage and explain its meaning. Include units in your explanation.

(i) Make up and answer your own question about the derivative dr/dG at a specific gas mileage.

A Few More Problems

Question 58 *The function*

$$h = f(t) = kt^3, t \geq 0,$$

expresses the height of a balloon (in thousands of feet) in terms of the number of hours past noon. Here $k > 0$ is a positive constant.

(a) What are the units of the constant k? Explain how you know.

(b) Find the balloon's rate of ascent (measured in thousands of ft/hr) at time $t = 3$ hours past noon. Then find its average rate of ascent (measured in thousands of ft/hr) between noon and 3pm. How are these rates related? Note that both will be expressed in terms of k.

(c) Find a function

$$r = g(a)$$

that expresses the balloon's rate of ascent (measured in thousands of ft/hr) at time $t = u$ hours past noon in terms of its average rate of ascent (measured in thousands of ft/hr) over the time interval $t \in [0, u]$. Assume $u > 0$.

(d) Interpret your result from part (b) geometrically on the graph of the function f. Follow the directions on Lines 4, 6, and 8 in the demonstration below to help explain your interpretation.

(e) Which of the following expresses the balloon's rate of ascent at time t hours past noon in terms of t and $h = f(t)$?

Multiple Choice:

(a) $\dfrac{h}{t}$

(b) $\dfrac{h}{3t}$

(c) $\dfrac{3h}{t}$ ✓

Desmos link: `https://www.desmos.com/calculator/0byjcy77yw`

Desmos activity available at 151: Subtangent 1

Question 59 *The function*

$$v = f(G)$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon) for speeds between 55 miles/hour and 70 miles/hour.

Suppose $f(30) = 60$.

(a) Which of the following is more likely to be true?

$$\left.\frac{dv}{dG}\right|_{G=30} = 4$$

or

$$\left.\frac{dv}{dG}\right|_{G=30} = -4?$$

Explain your reasoning.

(b) What are the units of the above derivative? Do not simplify the units and do not write "per" in place of "/".

(c) Explain the meaning of the derivative in part (a). It is not *enough to say "the rate of change of something with respect to something else." Remember this class is all about small changes and your explanation should be about an approximate relationship between small changes in this setting.*

(d) Simplify the units of the derivative. What does this suggest about its meaning?

(e) At what rate does the car burn gas (in gal/hour) at a speed of 60 *miles/hour?*

(f) What does this problem suggest about simplifying the units of a derivative?

Question 60 *The function*

$$v = f(G), 20 \leq G \leq 40,$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon). Ue the graph of the function f to find approximate answers to the following questions. Change the position and slope of line AB by dragging either the line or the points A or B. Change the position of the tangent line by dragging the slider G.

(a) Label the axes with the appropriate variable names and units.

(b) At what speed does the car burn gas as the fastest rate?

Hint: *One of questions (b), (c) is related to the tangent lines to the curve, the other is related to the lines through the origin and the points of the curve.*

(c) At what speed does increasing the speed by 0.1 *miles/hour result in the greatest change in the gas mileage? Approximate that change.*

Geogebra link: `https://www.geogebra.org/m/vjdf6x6z`

Geogebra activity available at 151: Gas Mileage

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3 Homework Solution

Desmos link: `https://www.desmos.com/calculator/jku58rp2ve`

Gas Consumption

Gas consumption and derivatives.

Question 61 *The function*

$$\begin{aligned} G &= f(v) \\ &= -\frac{v^2}{20} + 5v - 90\,,\ 35 \leq v \leq 70, \end{aligned}$$

graphed below expresses the gas mileage (in miles/gal) of a car in terms of its speed (in miles/hour).

Desmos link: `https://www.desmos.com/calculator/fapdhcqptl`

151: Burning Gas

(a) *Use the graph to determine the rate (in gal/hr) at which the car burns gas at a speed of 50 miles/hour.*

(b) *Drag the slider v in Line 1 to approximate the speeds between 35 miles/hour and 70 miles/hour at which the car burns gas at the maximum and minimum rates (measured in gal/hr). Explain your reasoning.*

(c) *Use calculus to determine the exact speeds in part (b). Find a way that avoids using the quotient rule.*

Question 62 *The function*

$$G = f(s) = \frac{11}{5} + \frac{1}{5000}\left(s^3 - 50s^2 + 300s\right),\ 3 \leq s \leq 28,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/pb8v4t3cxg`

Desmos activity available at 151: Gas as a Function of Distance 33

Learning outcomes:
Author(s):

(a) *Use the graph above to determine if you are driving toward or away from home. Explain your reasoning.*

(b) *Sensors on your car measure both the (instantaneous) gas mileage and the number of gallons of gas in your tank at each instant. A computer then uses these measurements to estimate the number of additional miles you can drive before running out of gas.*

   (i) *Use the graph and the slider* s_0 *above to approximate the reading for the number of additional miles you can drive when you are* 10 *miles from home and* 20 *miles from home. Explain your reasoning.*

   (ii) *Find a function*

$$m = g(s)\,,\; 3 \le s \le 28,$$

   that expresses the number of miles you can drive before running out of gas (assuming your gas mileage remains constant for the remainder of your trip) in terms of your distance from home. Explain your reasoning.

   (iii) *Find a function*

$$r = h(s)\,,\; 3 \le s \le 28,$$

   that expresses the relative *rate of change in the function* f *with respect to and in terms of the number of miles from home.*

   i. Evaluate $h(10)$ *and interpet its meaning. Include units.*

   ii. Compare the functions $r = h(s)$ *and* $m = g(s)$*. How are they related? Explain the logic behind this relation.*

The Quotient and Product Rules

Quotient and product rules.

Here are examples of how to use the Leibniz notation correctly in computing the derivative of a product and quotient of functions.

Example 10. Find expressions for the derivatives

(a)

$$\frac{d}{dx}\big((5x^3-2)(4-6x)\big)$$

(b)

$$\frac{d}{dt}\left(\frac{5t^3-2}{4-6t}\right)$$

Explanation. (a)

$$\frac{d}{dx}\big((5x^3-2)(4-6x)\big) = (4-6x)\frac{d}{dx}\big(5x^3-2\big) + (5x^3-2)\frac{d}{dx}(4-6x)$$

$$= (4-6x)15x^2 + (5x^3-2)(-6).$$

(b)

$$\frac{d}{dt}\left(\frac{5t^3-2}{4-6t}\right) = \frac{1}{(4-6t)^2}\left((4-6t)\frac{d}{dt}\big(5t^3-2\big) - (5t^3-2)\frac{d}{dt}(4-6t)\right)$$

$$= \frac{(4-6t)15t^2 - (5t^3-2)(-6)}{(4-6t)^2}.$$

1 Relative Rates of Change

Suppose $P = f(t)$ and $Q = g(t)$ are differentiable functions of t and $g(t) \leq 0$. Then the quotient and product rules are better written in the forms

$$\frac{d}{dt}\left(\frac{P}{Q}\right) = \frac{P}{Q}\left(\frac{1}{P}\cdot\frac{dP}{dt} - \frac{1}{Q}\cdot\frac{dQ}{dt}\right) \tag{1}$$

and

$$\frac{d}{dt}(PQ) = PQ\left(\frac{1}{P}\cdot\frac{dP}{dt} + \frac{1}{Q}\cdot\frac{dQ}{dt}\right).$$

Learning outcomes:
Author(s):

Question 63 (a) *Verify the statements above.*

(b) *What do they say about the relative rate of change in the quotient of two functions? In their product?*

2 Exercises

Exercise 64 *Between speeds of 45 miles/hr and 75 miles/hr, the function*

$$G = f(v)\,,\ 45 \le v \le 75,$$

expresses the gas mileage (in miles/gal) of a car in terms of its speed (in miles/hour).

Suppose $f(50) = 25$ *and*

$$\left.\frac{dG}{dv}\right|_{v=50} = 0.8.$$

(a) *What are the units of the derivative above? What is the meaning? Explain in terms of small changes.*

(b) *What are the simplified units of the derivative above? What insight do these units give you about the derivative's meaning?*

(c) *Approximate the gas mileage at a speed of 48 miles/hour.*

(d) *Let*

$$r = g(v)\,,\ 45 \le v \le 75,$$

be the function that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hr).

(i) *Use Equation (1) above to evaluate the derivative*

$$\left.\frac{dr}{dv}\right|_{v=50}.$$

Include units for all numbers.

(ii) *What are the units of the derivative above? What is the meaning? Explain in terms of small changes.*

(iii) *What are the simplified units of the derivative above? What insight do these units give you about the derivative's meaning?*

(iv) *Approximate the rate (in gal/hr) at which the car burns gas at a speed of 48 miles/hr.*

Exercise 65 *At 9:00am on February 23, 2023, the price of oil was decreasing at the relative rate of 2%/hour. At what relative rate was the number of gallons of oil you could buy with $100,000 changing at that time? Use calculus to justify your assertion.*

Exercise 66 *The function*

$$P = f(t) = 5 - 3t + t^2\,,\; 0 \le t \le 4,$$

expresses the price in $/share of a stock in terms of the number of hours past 9am.

(a) Use the graphs of the function $P = f(t)$ and the function $r = f'(t)/f(t)$ to estimate when the stock price is increasing at the greatest relative rate.

(b) Use algebra to find the exact time when the stock price is increasing at the greatest relative rate.

Hint: *What is the value of the derivative dr/dt at this time? But start by finding an expression for the instantaneous relative rate of change in the stock price.*

Desmos link: `https://www.desmos.com/calculator/xuupp3srqv`

Desmos activity available at 151: Stock Price 4

Exercise 67 *An h-foot tall giraffe walks directly toward a spotlight on the ground as the light casts its shadow on a vertical wall as illustrated below. The wall is b feet from the light.*

Desmos link: `https://www.desmos.com/calculator/2eiyjjpu9n`

Desmos activity available at 151: Spotlight

Suppose at a certain instant the giraffe is w feet from the spotlight and is walking at a speed of v ft/sec.

(a) *Is the length of the giraffe's shadow increasing or decreasing at this instant?*

(b) *At what rate?*

(c) *Check that your answer in part (b) has the correct units.*

(d) *Assume v is constant and describe what happens to the rate in part (b) as the giraffe approaches the spotlight. How is your conclusion revealed in the animation?*

Exercise 68 (a) *On January 1, 2024 the national debt of a country was decreasing at the rate of 3%/yr and the population was increasing at the rate of 2%/yr. Was the per-captita (ie. per person) share of the national debt increasing or decreasing at this time? At what relative rate?*

(b) *During the year 2024 the national debt of a country decreased 3% and the population increased 2%. Did the per-captita share of the national debt increase or decrease during the year? By what percent?*

(c) *Compare the two questions above and their answers.*

Test 1

Test 1.

(a) (6 points) The function

$$W = f(r) = \frac{2000}{r^2}, r \geq 4,$$

expresses the weight of an astronaut (measured in pounds) in terms of her distance from the center of the earth (measured in thousands of miles).

(i) (2 points) Find an expression for the average rate of change in the astronaut's weight with respect to her distance from the earth's center between distances $r = b$ and $r = c$ thousands of miles from the center. Assume $b, c \geq 4$ and that $b \neq c$.

(ii) (4 points) Use your *expression from part (a) directly and the definition of the derivative* to find an expression (fully simplified) for the derivative

$$\left.\frac{dW}{dr}\right|_{r=b}.$$

In particular, do *not* use the power rule. Show all work in its mathematically correct form. Write vertically, one equal sign per line. No need to explain algebra.

Explanation. (i) The average rate of change is

$$\begin{aligned}\frac{\Delta W}{\Delta h} &= \frac{f(c) - f(b)}{c - b}\\ &= \left(\frac{1}{c-b}\right)\left(\frac{2000}{c^2} - \frac{2000}{b^2}\right)\end{aligned}$$

(ii) (Some algebra omitted, left for you to fill in).

$$\begin{aligned}\left.\frac{dW}{dr}\right|_{r=b} &= \lim_{c \to b}\left(\frac{1}{c-b}\right)\left(\frac{2000}{c^2} - \frac{2000}{b^2}\right)\\ &= \lim_{c \to b}\frac{-2000(c+b)}{c^2b^2}\\ &= -\frac{2000(b+b)}{b^2b^2}\\ &= -\frac{4000}{b^3}.\end{aligned}$$

Learning outcomes:
Author(s):

(b) (5 points) At 3pm, you can buy 5 pounds of cod with \$100. And at 3pm the number of pounds of cod you can buy with \$100 is decreasing at the rate of 0.4 lbs/hour.

At what rate (with respect to time) is the price of cod changing at 3pm? Explain your reasoning and be sure to end with a concluding sentence.

Explanation. Let $P = f(t)$ be the function that expresses the price of cod (in dollars/pound) in terms of the number of hours since 3pm.

Let $Q = g(t)$ be the function that expresses the number of pounds of cod you can buy with \$100 in terms of the number of hours since 3pm.

Then
$$P = \frac{100}{Q}$$
and the quotient rule gives
$$\begin{aligned}\frac{dP}{dt} &= \frac{1}{Q^2}\left(Q\frac{d}{dt}(100) - 100\frac{dQ}{dt}\right)\\ &= -\frac{100}{Q^2}\cdot\frac{dQ}{dt}\end{aligned}$$

Since at 3pm the number of pounds we can buy with \$100 is decreasing at the rate of 0.4 lbs/hour, we know that
$$\left.\frac{dQ}{dt}\right|_{t=0} = -0.4 \text{ lb/hr}.$$

And since we can buy $Q = g(0) = 5$ pounds with \$100 at 3pm,
$$\begin{aligned}\left.\frac{dP}{dt}\right|_{t=0} &= -\left.\left(\frac{100}{Q^2}\cdot\frac{dQ}{dt}\right)\right|_{t=0}\\ &= \left(-\frac{100 \text{ dollars}}{25 \text{ lb}^2}\right)\left(-0.4\frac{\text{lb}}{\text{hr}}\right)\\ &= (1.6 \text{ dollars/lb})/\text{hr}.\end{aligned}$$

So at 3pm the price of cod is increasing at the rate of (1.6 dollars/lb)/hr.

(c) (5 points) Between speeds of 60 miles/hour and 80 miles/hour, the function
$$v = f(G)\,,\ 8 \le G \le 20,$$
expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon).

Suppose $f(12) = 72$.

(i) (1 point) Which would be more likely, that

$$\left.\frac{dv}{dG}\right|_{G=12} = 4 \qquad \text{or that} \qquad \left.\frac{dv}{dG}\right|_{G=12} = -4?$$

Explain your reasoning.

(ii) (1 point) What are the units of the correct derivative above?

(iii) (2 points) Explain the meaning of the correct derivative above using the language of *small changes*.

(iv) (1 point) Simplify the units of the correct derivative above. What do these simplified units tell you about the derivative's meaning?

Explanation. (i) At the high speed of 72 miles/hour it is likely that a small increase in the speed will decrease the gas mileage. So we would expect the derivative to be negative, so

$$\left.\frac{dv}{dG}\right|_{G=12} = -4.$$

(ii) Since v has units miles/hour and G has units miles/gal, the derivative dv/dG has units

$$\frac{\text{miles/hr}}{\text{miles/gal}}.$$

(iii) We know that

$$\left.\frac{dv}{dG}\right|_{G=12} = -\frac{0.4}{0.1}\frac{\text{miles/hr}}{\text{miles/gal}} \sim \frac{\Delta v}{\Delta G}$$

if $\Delta G \sim 0$. So increasing the speed by 0.4 mile/hour (from 72 miles/hr to 72.4 miles/hr) would decrease the gas mileage by about 0.1 miles/gal (from 12 miles/gal to about 11.9 miles/gal).

(iv) The units of the simplified derivative are

$$\frac{\text{miles/hr}}{\text{miles/gal}} = \frac{\text{gal}}{\text{hr}}.$$

Although this suggests the derivative is related to the rate at which the car burns gas, this is *not* true. Our interpretation of the derivative had nothing to do with this rate and in fact the car burns gas at the rate of 6 gal/hour at a speed of 72 miles/hour. So the simplified units are misleading and tell us *nothing* about the derivative's meaning.

(d) (5 points) The function

$$G = f(s) = \frac{3s + 25}{s + 5}\ ,\ 0 \le s \le 10,$$

expresses the number of gallons of gas in a car in terms of the trip odometer reading (measured in miles).

Sensors on the car measure both the number of gallons of gas in the tank and the current (instantaneous) gas mileage (measured in miles/gallon).

A computer then calculates the number of miles you have left to drive assuming the car's gas mileage remains constant for the remainder of your trip (and equal to the current gas mileage). This number is displayed on the dashboard.

Determine the exact trip odometer reading when the dashboard reading indicates that you have 13 miles left to drive. Explain your reasoning and be sure to end with a concluding sentence.

Explanation. I will not solve this problem, but will give you a hint instead by solving a closely related problem.

Suppose, for example, that we wish to find the dashboard reading (for the number of miles left to drive) at the odometer reading $s = 5$ miles.

First, we would find the number of gallons in the tank at this odometer reading. At this time, the tank has

$$f(5) = \frac{3(5) + 25}{5 + 5} = 4$$

gallons of gas.

Now at $s = 5$, the derivative dG/ds is (left for you to verify)

$$\left.\frac{dG}{ds}\right|_{s=5} = -0.10 \frac{\text{gal}}{\text{mile}}.$$

This means that five miles into our trip the car is burning gas at the rate of 0.10 gallons/mile. So if this rate were to remain constant for the remainder of the trip, we would be able to drive an additional

$$\frac{4 \text{ gallons}}{0.10 \frac{\text{gal}}{\text{mile}}} = 40 \text{ miles}$$

before running out of gas.

Derivatives of Exponential Functions

Working with exponential functions and their derivatives.

Relative Changes and Relative Rates of Change

Relative changes and relative errors are often more meaningful than absolute changes and errors. For example, I might measure the distance from Shoreline's Central Market to the Richmond beach library to be 5 km with an error of at most 0.2 km, while NASA might measure the between the earth and the moon on the first day of spring to be $384,400$ km with an error of at most 100 km. The relative error in my measurement is at most

$$\frac{0.2 \text{ km}}{5 \text{ km}} = 0.04 = 4\%,$$

while the relative error in NASA's measurement is at most

$$\frac{100 \text{ km}}{384,400 \text{ km}} \sim 0.00026 = 0.026\%.$$

Relatively speaking, NASA's measurement was about 150 times more accutate than mine.

Question 69 *At 10:00am the preices of Stock A and Stock B are both increasing at the rate of (\$2/share)/hour. At 10:00am Stock A sells for \$50/share and Stock B for \$10/share. Compare the relative rates at which the share prices are changing at 10:00am.*

Question 70 *The function*

$$P = f(t) = 5 - 3t + t^2 \, , \, 0 \le t \le 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

(a) At what relative rate is the price of the stock changing at 10am?

(b) When is the share price increasing at a rate of 60%/hr?

(c) During what time interval is the price of the stock increasing?

Learning outcomes:
Author(s):

(d) During what time interval is the relative rate of change in the price of the stock increasing?

Desmos link: `https://www.desmos.com/calculator/hhkveu6lxp`

Desmos activity available at 151: Stock Price

Question 71 *The function*

$$P = f(t)\,,\ 1 \leq t \leq 3.6,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am. Use the graph below to approximate the answers to the following questions without *putting a scale on the P axis.*

(a) At what relative rate is the price changing at 11am? At 12:30pm?

(b) When is the stock price increasing at its maximum relative rate? At its minimum relative rate? Approximate these rates.

Desmos link: `https://www.desmos.com/calculator/jyebaj5jif`

Desmos activity available at 151: Stock Price 2

Question 72 *The function*

$$W = f(t) = 200 + 4t + 2t^2, 0 \leq t \leq 12,$$

expresses the weight (in pounds) of a baby elephant in terms of its age (in months).

(a) Find the average rate at which the elephant gained weight between ages 4 and 10 months.

(b) Find the relative average rate at which the elephant gained weight between ages 4 and 10 months.

(c) Find the relative instantaneous rate at which the elephant is gaining weight at age 4 month.

(d) Find the relative instantaneous rate at which the elephant is gaining weight at age 10 months.

(e) Use the graph below to interpret your answers to parts (b)-(d) geometrically.

Desmos link: `https://www.desmos.com/calculator/2xj6xy7ggo`

Desmos activity available at 151: Elephant

Exponential Growth

Question 73 *(a) What does it mean for a population to grow exponentially?*

(b) Is it possible for a population to increase by 20% every year and not grow exponentially?

Question 74 *Suppose between noon and 10pm a colony of bacteria grows exponentially. The population is 200,000 at noon and 242,200 at 1pm.*

(a) Describe how the population grows. Keeping Question 2(b) in mind, is your description sufficient?

(b) How might we find a complete description of the exponential growth?

(c) Determine the relative average growth rate between noon and 12:30pm. Between 1pm and 1:30pm. Over any half-hour time period. Use the slider u in the graph below to intrepret these rates geometrically.

(d) Approximate the instantaneous relative growth rates in the population at noon, at 1pm, and at 2pm. Modify the definition of $v = u + 1/2$ in the demonstration below and interpret these rates geometrically.

(e) Use limits to write an expression that gives the instantaneous growth rate at time t hours past noon. What can you conclude?

(f) Try to answer part (b) again.

Desmos link: `https://www.desmos.com/calculator/wvpsotdhby`

Desmos activity available at 151: Exp Growth 1

Question 75 *The function*

$$P = f(t) = 3(2)^t, -2 \leq t \leq 4,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) Describe how the population grows. Is your description sufficient?

The population *doubles* every *hour*.

(b) Find an expression for the relative growth rate between time $t = u$ hours past noon and time $t = u + h$ hours past noon. Measure the rate relative to the population at time $t = u$. Is this question asking about an average or an instantaneous relative growth rate?

The relative growth rate is

$$\frac{1}{P}\left(\frac{\Delta P}{\Delta t}\right) = \frac{1}{f(u)}\left(\frac{f(u+h) - f(u)}{h}\right)$$

$$= \frac{1}{3(2)^u}\left(\frac{3(2)^{u+h} - 3(2)^u}{h}\right)$$

$$= \frac{1}{3(2)^u}\left(\frac{3(2)^u(2^h - 1}{h}\right)$$

$$= \frac{2^h - 1}{h}$$

(c) What are the units of the relative average growth rate in part (b)?

(d) Input your function from part (b) on Line 5 in the worksheet below.

(e) What do you notice about the distance between points R and S as you drag the slider u below. How is this distance related to the relative average growth rate in part (b)?

(f) Use limits to write an expression for the relative growth rate at time $t = u$ hours past noon. Simplify this expression as much as possible. What can you conclude about how the relative instantaneous growth rate varies with time?

The relative growth rate is

$$\lim_{h \to 0} \frac{2^h - 1}{h}.$$

(g) Interpret your expression from part (f) as the derivative of a specific function evaluated at a specific input. What does this tell you about the relative growth rate of this particular population?

The relative instantaneous growth rate of the population is equl to the derivative of the function

$$f(x) = 2^x$$

evaluated at $x = 0$.

(h) Use part (f) to numerically approximate the relative (instantaneous) growth rate of the population. Show a table that suggests a progression toward a limit.

(i) Use a similar method to approximate the relative instantaneous growth rate of the population

$$P = f(t) = 5(3)^t.$$

Desmos link: https://www.desmos.com/calculator/omjbec2hpu

Desmos activity available at 151: Exponential Growth 1

Question 76 *Parts (h) and (i) of the previous question suggest that there is a number e between 2 and 3 that makes the relative growth rate of the function*

$$P = f(t) = P_0 e^t, -3 \leq t \leq 5$$

equal to 100%/hr, where we assume here that t is measured in hours.

(a) What is the one-hour growth factor for this population?

(b) Describe what happens to the population every hour.

(c) At what relative rate is the population increasing at 1:00pm?

(d) Suppose at 1:00pm the population is $500,000$. Approximate the population at 1:03pm and compare your approximation to the actual population at that time.

Exponential Functions with Bases other than e

Question 77 *The function*

$$P = g(t) = P_0 e^{t/2}, -6 \leq t \leq 10$$

expresses the population (Colony B) of bacteria in terms of the number of hours past noon.

(a) Describe a transformation that takes the graph of the population function

$$P = f(t) = P_0 e^t, -3 \leq t \leq 5$$

for Colony A (where t is also the number of hours past noon) to the graph of $P = g(t)$.

(b) Suppose that the population of Colony A is $400,000$ at 4:00pm.

(i) When is the population of Colony B equal to $400,000$?

(ii) What are the growth rates of the two populations when they each have respective populations of 400,000 bacteria?

(iii) What are the relative growth rates of the two populations when they each have respective populations of 400,000 bacteria?

Question 78 *Here's another way to think about differentiating the function*

$$P = g(t) = P_0 e^{t/2}, -6 \leq t \leq 10$$

that expresses the population of a colony of bacteria in terms of the number of hours past noon.

We'll let $u = t/2$ be the number of two-hour periods since noon.

(a) Express the population in terms of u.

(b) Use what you know about the exponential function base e to express the growth rate of the population in terms of u.

(c) Use part (b) to find the growth rate of the population at 6pm. Pay careful attention to units.

(d) Use the idea of part (c) to express the growth rate

$$\frac{dP}{dt} = g'(t)$$

in terms of t.

(e) Suppose instead that the population grows exponentially and doubles every hour. Find the relative instantaneous growth rate of the population.

Question 79 *Between 11am and 8pm, a population of bacteria grows exponentially. The population is 4 million at noon and 5 million at 1pm.*

(a) What is the one-hour growth factor? 1.25

(b) Describe how the population grows.

The population increases by 25% *every hour.*

(c) Use your description from part (b) to find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon. Do not use e in your function. Define meaningful variables and include a domain.

The function

$$P = f(t) = 4(5/4)^t, \ -1 \leq t \leq 8$$

expresses the population (in millions of bacteria) in terms of the number of hours past noon.

(d) Use the fact that $k = e^{\ln k}$ for $k > 0$ to express your function from part (c) using an exponential function with base e.

$$P = f(t) = 4e^{t \ln(1.25)}, \ -1 \leq t \leq 8$$

(e) Use u-substitution and the chain rule (show all steps) along with the fact that

$$\frac{d}{du}(e^u) = e^u$$

to find the relative instantaneous growth rate of the population. Include units in your conclusion.

The relative instantaneous growth rate of the population is

$$\frac{1}{P}\left(\frac{dP}{dt}\right) = \ln(5/4) \sim 22.314\%/hour.$$

(f) At what rate is the population growing when there are 10 million bacteria? Do this the easy way. Note that the question is not *asking for a relative rate.*

(g) Find the population when the population is increasing at the rate of 3 million bacteria/hour. Do this the easy way.

Question 80 *Between 11am and 8pm, a population of bacteria decreases exponentially. The population is 5 million at noon and 4 million at 4pm.*

(a) What is the four-hour growth factor? (This is the number you multiply the current popluation by to get the population four hours later). 0.8

(b) What is the one-hour growth factor? $(4/5)^{1/4}$

(c) Describe how the population decreases.

The population decreases by 20% *every* 4 *hours.*

(d) Use your description from part (c) to find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon. Do *not use e in your function. Start by define meaningful variables. Include a domain with your function.*

The function

$$P = f(t) = 5(4/5)^{t/4}\,,\ -1 \le t \le 8$$

expresses the population (in millions of bacteria) in terms of the number of hours past noon.

(e) Express your function from part (c) using an exponential function with base e. Then use u-substitution and the chain rule to find the relative instantaneous growth rate of the population. Show all steps in using the chain rule.

The population function is

$$P = f(t) = 5e^{0.25\ln(0.8)t}\,,\ -1 \le t \le 8.$$

The relative intstantaneous growth rate is

$$\frac{1}{P}\left(\frac{dP}{dt}\right) = 0.25\ln(0.8) \sim -5.58\%/hour.$$

(f) At what rate is the population decreasing when there are 2 million bacteria? Do this the easy way.

(g) Find the population when the population is decreasing at the rate of 300, 000 *bacteria/hour. Do this the easy way.*

(h) Use the result of part (f) to approximate the population 4 minutes after there are 2 *million bacteria.*

Relative Rates Again

Question 81 *The function*

$$P = f(t), 0 \leq t \leq 2,$$

expresses the balance (in dollars) in an account in terms of the number of years since the start of 2022. Suppose

$$\frac{1}{P}\frac{dP}{dt}\bigg|_{P=5000} = 0.08.$$

(a) What are the units of the above derivative? How do you know?

(b) Interpret the meaning of the above derivative.

(c) Approximate the balance in the account four days after the account has \$5, 000. *Explain your reasoning.*

Question 82 *The function*

$$q = f(p) = 0.5(p-18)^2\ , 6 \leq p \leq 15,$$

expresses the average number of burgers/day sold at Five Guys of Edmonds in terms of the price (in \$*/burger).*

(a) At what relative rate does the quantity sold (q) change with respect to the price (p) at a price of \$10*/burger?*

(b) What are the units of the above relative rate of change?

(c) Explain the meaning of the relative rate of change in part (a).

(d) Use the graph of the function $q = f(p)$ and the slider u in the desmos activity below to interpret the relative rate of change in part (a) geometrically. Explain your reasoning.

(e) Use the result of part (a) to approximate the relative change in the average number of burgers sold per day if the Five Guys increases the price from \$10/burger to \$10.25/burger. Explain your reasoning.

(f) Use the result of part (a) to approximate the relative change $\Delta a\%$ in the average number of burgers sold per day in terms of a small relative change $\Delta r\%$ in the price from \$10/burger. Explain your reasoning.

Desmos link: `https://www.desmos.com/calculator/ylgk03oaza`

Desmos activity available at 151: Burgers 1

Question 83 *The function*

$$P = f(t), -2 \leq t \leq 5,$$

expresses the population of a colony (call it Colony A) of bacteria in terms of the number of hours past noon.

The function

$$P = g(t) = f(t/2), -4 \leq t \leq 10,$$

expresses the population of Colony B in terms of the number of hours past noon.

The populations do not *necessarily grow exponentially.*

(a) Compare the populations at noon.

(b) Suppose Colony A as 50,000 bacteria at 3:00pm. When does Colony B have 50,000 bacteria? Explain.

(c) Suppose the population of Colony A takes three hours to grow from 20,000 to 50,000. How long does it take the population of Colony B to grow from 20,000 to 50,000?

(d) Suppose the population of Colony A is increasing at the rate of 10,000 bac/hr at 3pm. What is the growth rate of Colony B at 6pm? Explain.

(e) What is the relative growth rate of Colony A at 3pm? What is the relative growth rate of Colony B at 6pm? Explain.

Question 84 *The function*

$$P = f(u) = 10e^u, -2 \leq u \leq 5$$

expresses the population (in millions of bacteria) of a colony of bacteria in terms of the number of hours since noon.

(a) What are the units of the input to the exponential function in the above expression for P?

(b) At what rate is the population growing when there are 30 million bacteria?

(c) At what rate is the population growing when there are P million bacteria? Do the units of your answer make sense?

(d) What is the relative instantaneous growth rate of the population?

(e) Find a function

$$P = g(t)$$

that expresses the population (in millions of bacteria) of the colony of bacteria in terms of the number of minutes *since noon. Include a domain.*

(f) Use common sense to evaluate the derivative

$$\left.\frac{dP}{dt}\right|_{P=30}.$$

Explain your reasoning.

(g) Use the idea of part (f) to find an expression for the derivative

$$\frac{dP}{dt} = g'(t)$$

at time t minutes past noon.

Question 85 *The function*

$$P = P_0 e^{kt}, -4 \leq t \leq 5,$$

expresses the population of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of the constant P_0?

(b) What are the units of the constant k?

(c) Use the ideas of the previous question to find an expression for the growth rate of the population at time t hours past noon. Include units in your answer.

(d) Find an expression for the relative growth rate at time t hours past noon. Include units in your answer.

Question 86 *One of the two functions graphed below is an exponential function. Which one? How do you know?*

The Derivative as a Magnification Factor

Exploration 87 *It sometimes helps to think of the derivative as a magnification factor that maps a small interval around an input to a function to a corresponding interval around the output.*

(a) Use this idea for the function $y = f(x)$ graphed below to approximate the derivatives

$$\frac{d}{dx}(f(x))\Big|_{x=4} \text{ and } \frac{dy}{dx}\Big|_{x=5}.$$

Desmos link: `https://www.desmos.com/calculator/la4f5ots3r`

Desmos activity available at 151: Magnification Factor 1

Differentiating the Exponential Function e^x

Exploration 88 *(a) Use the graph of the function $y = f(x)$ below to approximate the derivatives*

$$\frac{dy}{dx}\Big|_{y=k} \text{ for } k = 1, 2, \ldots, 6.$$

Note the above derivatives are evaluated at the outputs *of the function f.*

(b) What do you notice?

Desmos link: `https://www.desmos.com/calculator/k08dphtuca`

Desmos activity available at 151: Magnification Factor 2

Question 89 *The function*

$$P = 400e^t, 0 \leq t \leq 2$$

expresses the population (in thousands) of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of the factor 400?

(b) What are the units of the exponent in the factor e^t? Be careful.

(c) Find the (instantaneous) growth rate of the population when there are 1,200,000 bacteria.

(d) Find the relative (instantaneous) growth rate when there are 1,200,000 bacteria.

(e) Find the relative (instantaneous) growth rate at any time.

(f) Approximate the population 30 seconds after there are 1,200,000 bacteria.

Question 90 *Let $k > 0$ be a constant and let*

$$f(x) = ke^x.$$

For $a \in \mathbb{R}$ let point P with coordinates $(a, f(a))$ be a curve on the curve $y = f(x)$. Let Q be the point where the tangent line to the curve intersects the x-axis and let R be the point with coordinates $(a, 0)$

(a) Find the length of segment $\overline{QR}$.

(b) How is part (a) related to part (e) of the previous question?

Derivatives of Trigonometric Functions

Working with trigonometric functions and their derivatives.

1 Exercises

Question 91 *The function*

$$h = f(t) = 1200 + 500\cos t + 300\sin t\,,\ 0 \le t \le 10,$$

expresses the altitude (in feet) of a balloon in terms of the number of hours past noon.

Desmos link: `https://www.desmos.com/calculator/f1ldi6yrek`

151: Trig 1

(a) *Use calculus to find the minimum and maximum heights of the balloon between noon and 6pm. Do not use a calculator.*

(b) *Find the balloon's maximum rate of ascent. No calculator.*

Question 92 *The function*

$$h = f(t) = 800 - 200\tan t\,,\ -1.3 \le t \le 1.3,$$

expresses the altitude (in feet) of a balloon in terms of the number of hours past noon.

Desmos link: `https://www.desmos.com/calculator/re6nqofgs0`

151: Trig 2

Find exact answers to the following questions without using a calculator.

(a) *When is the balloon descending at the rate of 150 ft/hour?*

Learning outcomes:
Author(s):

(b) *How high is the balloon when it is descending at the rate of* 500 *ft/hour?*

(c) *Express the balloon's rate of ascent (in feet/hour) in terms of its altitude.*

Question 93 *The function*

$$h = f(t) = 11 - 7\cos t\,,\ 0 \leq t \leq 6,$$

expresses the depth of the water (in feet) at the Edmonds Pier in terms of the number of two-hour periods *past noon.*

Desmos link: `https://www.desmos.com/calculator/4bh7kimi7f`

151: Trig 3

Find exact answers to the following questions without using a calculator.

(a) *Is the depth of the water increasing or decreasing at 2pm? At what rate?*

(b) *At what rate is the depth of the water changing when the water is* 6 *feet deep?*

Question 94 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) *Use the figure below to find a function*

$$h = f(\theta)\,,\ 0 \leq \theta \leq 2\pi,$$

that expresses your height above the ground in terms of the rotation angle of the wheel (measured in radians since the time you boarded). Use the cosine *function,* not *the sine function.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

(b) *The wheel stops when you are* 100 *feet above the ground for the* second *time. It then starts again and turns through a small angle of* $\Delta\theta$ *radians before stopping again.*

Desmos link: `https://www.desmos.com/calculator/gi8obqmnav`

151: Ferris Wheel 34

(i) *Zoom in on point P above to approximate the change in your height (in feet) if $\Delta\theta = 0.1$.*

(ii) *Use a derivative to approximate the change in your height if $\Delta\theta = 0.1$.*

(iii) *Use a deriviative to approximate the change Δh in your height (measured in feet) as the wheel turns through the small angle $\Delta\theta$ radians.*

2 Visualizing Derivatives

Question 95 *You ride a ferris wheel for one revolution and get off. Let*

$$h = f(s)\,, 0 \leq s \leq ??,$$

be the function that expresses your height above the ground (measured in feet) in terms of your distance traveled, measured (in feet) along your path from your starting point.

(a) Choose a radius for the ferris wheel that you think is reasonable and fill in the missing upper bound for the domain of f above.

(b) Use the demonstration below to sketch by hand a graph of the function

$$r = \frac{dh}{ds} = f'(s).$$

Do not make any computations. Just use the demonstration below. Here are the key points to keep in mind:

- *Approximate the derivative dh/ds by the ratio $\Delta h/\Delta s$.*
- *The length of the red arclength (when the ferris wheel is on the way up) is the input s.*
- *The length of the purple segment is the output h.*
- *The lengths of the orange arc and orange segments are Δs.*
- *The (signed) length of the green segment is Δh.*

Explain your reasoning thoroughly. Be sure to include at least the following points:

- *The units of the input and output to the derivative*
- *Scales on the vertical and horizontal axes*

- *A discussion of how a small change in the input to the function f changes the output at various positions along your ride.*
- *A discussion of where a small change in the input to f gives the greatest positive change in the output and a consideration of the ratios of these changes*
- *A discussion of where a small change in the input to f gives the negative change in the output with the greatest magnitude and a consideration of the ratios of these changes*
- *A discussion of where a small change in the input to f barely changes the output and a consideration of the ratios of these changes*

(c) How would your graph of the derivative dh/ds change if you doubled the radius of the ferris wheel? Sketch the new graph.

Desmos link: `https://www.desmos.com/calculator/pxsmo04nmg`

3 The Derivative of the Sine Function

Question 96 *Desmos link:*
`https://www.desmos.com/calculator/jcmcyrpndw`

Desmos activity available at 151: Derivative of Sine

Question 97 *(a) Describe a transformation that takes the graph of the function*

$$y = f(\theta) = \sin\theta$$

to the graph of the function

$$y = g(\theta) = \cos\theta.$$

(b) Does that same transformation take the graph of the derivative of the sine function to the graph of the derivative of the cosine function? Do not use any particular facts about these functions to answer this question. Instead, give a general answer that would apply to all pairs of functions similarly related.

(c) Use your answer to part (b) and the fact that

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

to find an expression for the derivative

$$\frac{d}{d\theta}(\cos\theta)$$

of the cosine function.

4 Applications 1

Question 98 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) Find a function

$$h = f(\theta)\,,\ 0 \leq \theta \leq 2\pi,$$

that expresses your height above the ground in terms of the rotation angle of the wheel, measured in radians. Use the cosine *function,* not *the sine function.*

(b) The wheel stops when you are 100 *feet above the ground and on the way up. It then starts again and turns through a small angle of* $\Delta\theta$ *radians before stopping again. Use the appropriate linear approximation to estimate the change* Δh *in your height (measured in feet) as the wheel turned through the angle* θ*.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

5 Transformations of the Sine Function

Question 99 *(a) Describe a transformation that takes the graph of the function*

$$y = f(\theta) = \sin\theta$$

to the graph of the function

$$y = g(\theta) = \sin(2\theta).$$

(b) How does that same transformation affect the slope of a line?

(c) Use your answer to part (b) to find an expression for the derivative

$$g'(\theta) = \frac{d}{d\theta}(\sin(2\theta))\,.$$

6 Applications 2

Question 100 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You travel at a constant speed of 5 ft/sec as you ride the ferris wheel.*

(a) Find a function

$$h = f(t)\,, t \geq 0$$

that expresses your height above the ground in terms of the number of seconds since you got on. Use the cosine *function,* not *the sine function.*

(b) Are you ascending or descending the second time you are 90 feet above the ground? At what rate? Use the methods of this class, not vectors, to answer this question.

(c) Find your height when you are descending at the rate of 4.8 feet/sec. Give all possibilities. Do not use a calculator except to do arithmetic.

7 MATH 142

Question 101 *The graph below shows the x-coordinate function of a beetle moving around a circle at a constant speed.*

Desmos link: `https://www.desmos.com/calculator/qi6c9xbhnw`

142: Edmonds Pier 2

Use the graph to answer the following questions. Be sure to include units.

(a) *Find the x-coordinate of the circle's center.*

(b) *Find the radius of the circle.*

(c) *Find the period of the motion. This the time it takes the beetle to make one revolution about the center of its circular path.*

(d) *Find a time when the beetle's x-coordinate is a maximum.*

(e) *Use (a)-(d) to find an expression $x = f(t)$ for the function that expresses the x-coordinate of the beetle (measured in feet) in terms of the number of minutes past noon. Include the domain.*

(f) *Check your expression from part (e) by substituting the two times given in the graph.*

The Chain Rule

An introduction to the chain rule.

The chain rule tells us how to differentiate the composition of two functions. It say that the derivative of the composition is the product of the derivatives of the two functions.

Here is a more formal statement.

Theorem 1. (The Chain Rule) If $y = f(u)$ is a differentiable function of u and $u = g(t)$ is a differentiable function of t, then the composition

$$y = f(g(t))$$

is a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}.$$

Or paying closer attention to the inputs,

$$\left.\frac{dy}{dt}\right|_{t=t_0} = \left.\frac{dy}{du}\right|_{u=g(t_0)} \cdot \left.\frac{du}{dt}\right|_{t=t_0}.$$

We'll go through some examples to get an understanding of how to use the chain rule and also why it works.

Examples

Example 11. Find an equation of the tangent line to the curve

$$y = f(x) = \left(2x^3 + 1\right)^2$$

at the point $(1, 9)$.

Explanation. Let

$$y = \left(2x^3 + 1\right)^2$$

and

$$u = 2x^3 + 1.$$

Then

$$y = u^2$$

Learning outcomes:
Author(s):

and

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= \frac{d}{du}\left(u^2\right)\cdot\frac{d}{dx}\left(2x^3+1\right)\\ &= 2u(6x^2)\\ &= 2(2x^3+1)(6x^2).\end{aligned}$$

Then the slope of the tangent line to the curve $y=(2x^3+1)^2$ at the point $(1,9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 2(3)(6) = 36,$$

and an equation of the tangent line is

$$y-9 = 36(x-1).$$

We can get the same result without the chain rule, by rewriting the original function as

$$y = f(x) = (2x^3+1)^2 = 4x^6+4x^3+1.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(4x^6+4x^3+1\right)\\ &= 4\frac{d}{dx}\left(x^6\right)+4\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}(1)\\ &= 24x^5+12x^2.\end{aligned}$$

Then the slope of the tangent line to the curve $y=(2x^3+1)^2$ at the point $(1,9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 14+12 = 36,$$

as before.

Example 12. Find an equation of the tangent line to the curve

$$x^2+y^2=25$$

at the point $P(4,-3)$.

Explanation. Solve the above equation for y in terms of x to get

$$y = \pm\sqrt{25-x^2}.$$

While this equation does not define y as a function of x, near the point $(4,-3)$, the equation

$$y = -\sqrt{25-x^2}$$

does define y as a function of x.

Now to find the derivative of the function

$$y = -\sqrt{25 - x^2},$$

let

$$u = 25 - x^2.$$

Then

$$y = \sqrt{u} = -u^{1/2}$$

and

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}\left(-u^{1/2}\right) \cdot \frac{d}{dx}\left(25 - x^2\right) \\
&= -\frac{1}{2\sqrt{u}}(-2x) \\
&= \frac{x}{\sqrt{25 - x^2}}.
\end{aligned}$$

So the slope of the tangent line to the curve

$$x^2 + y^2 = 25$$

at the point $(P(4, -3)$ is

$$\left.\frac{dy}{dx}\right|_{x=4} = \left(\frac{x}{\sqrt{25 - x^2}}\right)\Big|_{x=4} = \frac{4}{3}$$

and an equation of the tangent line is

$$y + 3 = \frac{4}{3}(x - 4).$$

To find the slope of the tangent line without appealing to the chain rule, note that the curve

$$x^2 + y^2 = 25$$

is a circle centered at the origin. So the tangent line at $P(4, -3)$ is perpendicular to the radius $\overline{OP}$ from the origin to P. Since $\overline{OP}$ has slope $m_1 = -3/4$, the tangent line at $P(4, -3)$ has slope

$$m_2 = -1/m_1 = 4/3.$$

Question 102 *(a) Describe a transformation that takes the circle*

$$x^2 + y^2 = 25$$

to the ellipse

$$4x^2 + y^2 = 25. \tag{2}$$

(b) Find the image (call it Q) of the point $P(4, -3)$ under the transformation in part (a).

(c) Use the result of Example 2 to find an equation of the tangent line to the ellipse (2) at Q.

(d) Use the chain rule to first find the slope of the tangent line to the ellipse (2) at Q. Then find an equation of the tangent line.

Question 103 *The function*

$$P = f(t) = 22 + \frac{t}{2} - \frac{3t^2}{4}, 0 \leq t \leq 3,$$

expresses the price of a stock (in dollars/share) in terms of the number of hours past 9am.

(a) Is the price of the stock increasing or decreasing at 11am? At what rate? At what relative rate?

(b) Is the number of shares you can buy with \$1000 *increasing or decreasing at 11am? Approximate the rate numerically.*

(c) Compute the exact rate in part (b). Compute also the relative rate of change (with respect to time) in the number of shares you can buy with \$1000 *at 11am.*

Start your solution to this problem by defining a new function with a new function name and a new depedent variable.

Question 104 *At 10am the price of a stock is increasing at the relative rate of p%/hr. Is the number of shares you can buy with* \$1000 *increasing or decreasing at 10am? At what relative rate?*

Answer this question in two ways:

(i) Going back to limits and using the definition of the derivative.

(ii) Using the chain rule.

Either way, start your solution with defintions as in the previous problem.

Question 105 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a 14-foot ladder leans against the trunk, the other rests on the horizontal ground.*

Let t be the distance between the top of the ladder and the base of the trunk (measured in feet) and s the distance between the bottom of the ladder and the base of the trunk (also measured in feet).

Geogebra link: `https://www.geogebra.org/m/ctf2bcqz`

Geogebra activity available at 151: Ladder and Tree Part 2

(a) Use the law of cosines to write an equation relating t and s.

(b) Use the result of part (a) to find the two possible values of t when $s = 16$.

For the remainder of this problem we'll focus on positions of the ladder when $s \sim 16$ and $t \sim 10$.

(c) For these positions, complete the square to find a function

$$t = f(s)$$

that expresses the distance (in feet) from the top of the ladder to the base of the trunk in terms of the distance (in feet) from the bottom of the ladder to the trunk's base.

(d) Drag the slider θ in the worksheet above to approximate the value of the derivative

$$\left.\frac{dt}{ds}\right|_{s=16}.$$

Include units and explain your reasoning.

(e) Use your function from part (c) to find an expression for the derivative dt/ds. Show every step *in computing the derivative as in Examples 1 and 2 above.*

(f) Evaluate the derivative

$$\left.\frac{dt}{ds}\right|_{s=16}$$

and compare the exact value with your estimate from part (d).

(g) What are the units of the derivative in part (f)? Explain the meaning of the derivative, not *by giving a standard response about the rate of change of some quantity with respect to another, but by relating small changes.*

(h) Use your response to part (f) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of s near $s = 16$. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 15.6 feet from the base of the trunk. Compare your approximation with the exact distance.

(i) Let $\theta = m\angle GBA$ be the radian measure of the angle the ladder makes with the trunk.

part (i) Find an expression for the derivative $d\theta/ds$.

part (ii) Evaluate the derivative

$$\left.\frac{d\theta}{ds}\right|_{s=16}$$

and interpret its meaning in terms of small changes. Be sure to explain the significance of the derivative's sign as well.

Example 13. (a) Find the slope of the tangent line to the curve

$$y = f(\theta) = 6\sin\theta$$

at the point with coordinates $P(2\pi/3, 3\sqrt{3})$. We do not need the chain rule for this.

(b) Describe a transformation that takes the graph of $y = f(\theta)$ to the graph of the function

$$y = g(\theta) = 6\sin(\theta/2).$$

(c) Find the image of the point $P(2\pi/3, 3\sqrt{3})$ under the above transformation.

(d) Use the results of parts (a) - (c) and the demonstration below to guess the slope of the tangent line to the curve $y = g(\theta)$ at the point $Q(4\pi, 3\sqrt{3})$.

Desmos link: `https://www.desmos.com/calculator/mqjxpsqyo5`

Desmos activity available at 151: Chain Rule 1

(e) Use the chain rule to confirm your guess from part (d).

Example 14. Find an equation of the tangent line to the curve

$$y = f(t) = 4\sin^2\left(\frac{\pi}{6}t\right)$$

at the point $(2, 3)$.

Explanation. Let

$$y = 4\sin^2\left(\frac{\pi}{6}t\right) = 4\left(\sin\left(\frac{\pi}{6}t\right)\right)^2$$

and

$$u = \sin\left(\frac{\pi}{6}t\right).$$

Then

$$y = 4u^2$$

and

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{du} \cdot \frac{du}{dt} \\
&= \frac{d}{du}\left(4u^2\right) \cdot \frac{d}{dt}\left(\sin\left(\frac{\pi}{6}t\right)\right) \\
&= 8u\cos\left(\frac{\pi}{6}t\right)\frac{d}{dt}\left(\frac{\pi}{6}t\right) \\
&= 8\sin\left(\frac{\pi}{6}t\right)\cos\left(\frac{\pi}{6}t\right)\frac{\pi}{6}.
\end{aligned}$$

Question 106 *(a) The third equality above uses the chain rule again. Fill in the missing details of this computation by first letting*

$$z = \sin\left(\frac{\pi}{6}y\right)$$

and making an explicit substitution using the variable v in place of u.

(b) Complete the solution by finding an equation of the tangent line to the curve at the point $(2, 3)$.

Question 107 *Use the chain rule to compute each of the following derivatives. Show* all *steps as in Examples 1, 2, and 8 above.*

(a) $\dfrac{d}{dx}\left(\dfrac{12}{1+x^2}\right)$

(b) $\dfrac{d}{dt}\left(12(3)^{t/5}\right)$

(b) $\dfrac{d}{dt}\left(e^{3\sin(4t)}\right)$

(b) $\dfrac{d}{d\theta}\left(12 + 5\cos(\theta/4)\right)$

Example 15. The function

$$P = f(t) = 10e^{\frac{1}{4}t}, \; -4 \le t \le 12,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of 1/4 in the function above? How do you know?

(b) Use the chain rule to find an expression for the growth rate of the population at time t hours past noon. What are the units of the growth rate?

(c) Express the growth rate from part (b) in terms of the population $P = f(t)$ at time t hours past noon.

(d) What is the relative instantaneous growth rate of the population? Include units.

(e) Find the relative average growth rate of the population over a one-hour period.

(f) Describe what happens to the population every hour.

Example 16. (a) Use the chain rule to find the relative instantaneous growth rate of the population function

$$P = 10(2^t)\,,\ -2 \leq t \leq 5,$$

where P is measured in millions of bacteria and t is the number of hours past noon.

(b) Describe what happens to the population every hour.

(c) Estimate the population 2 minutes after there are 30 million bacteria.

Question 108 *This problem suggests a way to think about the chain rule geometrically.*

You ride a ferris wheel for one revolution and get off. The function

$$h = f(\theta) = \,,\ 0 \leq \theta \leq 2\pi,$$

expresses your height (in feet) in terms of the wheel's angle of rotation (measured in radians from the moment you boarded).

The function

$$\theta = g(t)\,,\ 0 \leq t \leq 44,$$

expresses the rotation angle of the wheel in terms of the number of seconds since you boarded.

Our goal is to use the graphs of the function f and g below (take the times t to be postive, not negative as shown) to approximate your rate of ascent $t = 16$ seconds after boarding. We'll do this in two different ways.

(a) The first way involves a few steps and uses the graphs of both f and g.

(i) Approximate the value of the derivative

$$\left.\frac{dh}{d\theta}\right|_{\theta=g(16)}$$

and interpet its meaning. Include units.

(ii) Approximate the value of the derivative

$$\left.\frac{d\theta}{dt}\right|_{t=16}$$

and interpet its meaning. Include units.

(iii) Use the results of (i) and (ii) and common sense (do not appeal to the chain rule directly) to approximate the value of the derivative

$$\left.\frac{dh}{dt}\right|_{t=16} = h'(16),$$

where $h(t) = f(g(t))$. Explain your logic and include all units in your computation. Interpret the meaning of this derivative.

(b) The second way has just one step and that's to use the graph of $h = f(\theta)$ to approximate your rate of ascent at time $t = 16$ using the fact that the horizontal blue lines are drawn at intervals of $\Delta t = 2$ seconds. Be sure to explain your logic.

Desmos link: `https://www.desmos.com/calculator/fkgfpsowe8`

Desmos activity available at 151: Ferris Wheel 2

Question 109 *This is a continuation of the previous question.*

Approximate the time between times $t = 5$ and $t = 22$ seconds when you are ascending at the slowest rate. Approximate this rate of ascent.

Question 110 *The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.*

(a) Use the geometry of the ferris wheel (see the picture below) to find a function

$$h = f(\theta)\,,\; 0 \leq \theta \leq 2\pi,$$

that expresses your height (in feet) above the ground in terms of the rotation angle of the wheel, measured in radians. Use the cosine *function,* not *the sine function.*

Geogebra link: `https://www.geogebra.org/m/tn75cq93`

(b) Suppose you move at the constant speed of 10 ft/sec as you ride the ferris wheel.

(i) Use the result of part (a) to find a function $h = g(t)$ that expresses your height (in feet) in terms of the number of seconds since you boarded. Include the appropriate domain.

(ii) Find a function $r = h(t)$ that expresses your rate of ascent (in ft/sec) in terms of the number of seconds since you boarded. Include the appropriate

domain. Use vectors to interpret this rate geometrically, in terms of the angle your velocity makes with the horizontal and your speed.

(iii) Write an equation that relates your height h (in feet) and your rate of ascent r (in ft/sec) at any instant. Graph the relation by hand.

(iv) Are you going up or down the second time you are 100 feet above the ground? At what rate?

(v) What is your height when you are descending at the rate of 4 ft/sec? Find all possibilities.

Question 111 *The function*

$$\theta = f(t), t \geq 0,$$

expresses the radian measure of a ferris wheel's rotation angle in terms of the number of seconds since you boarded. The wheel has radius r feet and its center is b feet above the ground.

(a) Find a function

$$h = g(t), t \geq 0,$$

that expresses your height (in feet) in terms of the number of seconds since you boarded.

(b) Find an expression for your rate of ascent (in ft/sec) at time t seconds after you boarded. Assume g is a differentiable function of t.

(c) Interpret your rate of ascent (part (b)) in terms of your speed at time t and the angle your velocity vector makes with the horizontal. Assume here that $\theta = f(t)$ is an increasing function of t.

Exercises

Exercise 112 *Use the facts that*

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

and

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

to compute each of the following derivatives. Show all steps in using the chain rule as in Examples 1 and 2 above.

(a) $\dfrac{d}{d\theta}(\sec\theta)$

(b) $\dfrac{d}{d\theta}(\csc\theta)$

(c) $\dfrac{d}{dt}\left(\sqrt{\sin^2 t + (4-\cos t)^2}\right)$

Hint: *Simplify the function first.*

(d) $\dfrac{d}{d\theta}\left(\sec^2\theta\right)$

(e) $\dfrac{d}{dt}\left(\dfrac{1}{5+3\cos(t/2)}\right)$

Explanation. (d) Let

$$y = \sec^2\theta = (\sec\theta)^2$$

and let

$$u = \sec\theta.$$

Then

$$y = u^2$$

and

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= \frac{d}{du}\left(u^2\right)\cdot\frac{d}{d\theta}(\sec\theta)\\ &= 2u(\sec\theta\tan\theta)\\ &= 2(\sec\theta)(\sec\theta\tan\theta).\end{aligned}$$

(e) Let

$$y = \frac{1}{5+3\cos(t/2)}$$

and let

$$u = 5+3\cos(t/2).$$

Then

$$y = u^{-1}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du}\cdot\frac{du}{dt}\\ &= \frac{d}{du}\left(u^{-1}\right)\cdot\frac{d}{dt}(5+3\cos(t/2))\\ &= -\frac{1}{u^2}\left(-3\sin\left(\frac{t}{2}\right)\frac{d}{dt}\left(\frac{t}{2}\right)\right)\\ &= \frac{3\sin\left(\frac{t}{2}\right)}{2(5+3\cos(t/2))^2}.\end{aligned}$$

Exercise 113 *You jog around a circular track of radius r feet at the constant speed of speed of v ft/sec. A flagpole lies b feet due east of the track's center.*

(a) Use the animation below (and nothing else) to sketch by hand a graph of the function

$$s = f(\theta)\,, \theta \geq 0,$$

that expresses your distance (in feet) to the flagpole in terms of your angle $\theta = \angle FOJ$ of rotation about the track's center, measured in radians from the time you start. Assume you start at the point A on the track nearest the flagpole. Be sure to include variable names, units, and scales on your axes. Explain your reasoning. For this particular graph assume that $r = 40$ and $b = 96$ (be sure to adjust the sliders in the worksheet to have these values).

Desmos link: `https://www.desmos.com/calculator/4pndurvhdd`

Worksheet available at 151: Jogger

(b) Using only your graph of the function f from part (a), sketch by hand a graph of the derivative

$$y = f'(\theta) = ds/d\theta$$

when $r = 40$ and $b = 96$. Be sure to include variable names, units, and scales on your axes. Explain your reasoning.

(c) Using only the animation (stop the motion), approximate the value of the derivative

$$\left.\frac{ds}{d\theta}\right|_{\theta=3\pi/2}$$

when $r = 40$ and $b = 96$. Include units. Explain your reasoning.

(d) Use trigonometry to find an expression for the function $s = f(\theta)$ in terms of the parameters r and b (ie. do not *assume $r = 40$ and $b = 96$).*

(e) Find an expression for the derivative $ds/d\theta$. Show every *step in using the chain rule as in Examples 1 and 2. Do* not *assume $r = 40$ and $b = 96$*

(f) Use your result from part (e) to compute the exact value of the derivative

$$\left.\frac{ds}{d\theta}\right|_{\theta=3\pi/2}$$

when $r = 40$ and $b = 96$. Compare this with your estimate from part (c).

(g) What are the units of the derivative in part (f)? Explain the meaning of the derivative, not *by giving a standard response about the rate of change of some quantity with respect to another, but by relating small changes.*

(h) Use your response to part (f) with $r = 40$ and $b = 96$ to write an approximation for the change

$$\Delta s = f(\theta) - f(3\pi/2)$$

in terms of the change

$$\Delta\theta = \theta - 3\pi/2$$

for values of θ *near* $\theta = 3\pi/2$*.*

(i) Find a function

$$s = g(t)\,, t \geq 0,$$

that expresses your distance (in feet) to the flagpole in terms of the number of seconds since you started jogging. Assume you start at the point on the track nearest the flagpole.

(j) Find a function that expresses the rate of change (with respect to time) in your distance to the flagpole in terms of the number of seconds since you began jogging.

(k) Express the rate of change in part (b) in terms of your speed and the angle between the following two vectors:

- *the vector that gives your position relative to the flagpole*
- *the vector that points in the direction of your motion*

Exercise 114 *High tide of 10.83 feet at Edmonds at 12:28am, May 15, low tide at 6:58am of ?? feet, lowest tide of 1.64 feet at 5:32pm.*

Desmos link: `https://www.desmos.com/calculator/zta9tkzzmx`

Worksheet available at 151: Edmonds Tides

Exercise 115 *A pendulum of length* L *feet is* $L + 5$ *oscillates sinuosoidally between angles* $-\theta_0$ *and* θ_0 *with period* $2\pi\sqrt{L/g}$ *seconds, where* g *is a constant. The angles* $\pm\theta_0$ *are measured in radians from the downward vertical.*

(a) Find a function

$$h = f(\theta)\,, -\theta_0 \leq \theta \leq \theta_0,$$

that expresses the height of the pendulum above its stagble equilibrium position (ie. its lowest point) in terms of the angle of rotation. Use the cosine function.

(b) Assume now that the pendulum is released from rest from the displacement angle $\theta = \theta_0$ *at time* $t = 0$ *seconds. Find a function*

$$\theta = g(t)\,, t \geq 0,$$

that expresses the displacement angle (in radians) in terms of the number of seconds since the pendulum was released. Use the cosine function.

(c) Find a function

$$h = w(t)\,,\, t \geq 0$$

that expresses the height (in feet) of the pendulum above equilibrium in terms of the number of seconds since the pendulum was released.

(d) Find an expression for the derivative dh/dt and interpret its meaning.

Exercise 116 *The function*

$$h = f(s) = 3 + 2\cos s\,,\; 0 \leq s \leq 3,$$

expresses the altitude (in thousand of feet) along a mountain trail in terms of the distance (measured in miles) from the summit. The distance is measured along the trail.

The function

$$s = g(t) = t + \frac{t^2}{4}\,,\; 0 \leq t \leq 2,$$

expresses your distance from the summit (in miles) in terms of the number of hours past noon.

The goal of this problem is to compute the rate (with respect to time) at which your altitude is changing at 1pm.

Desmos link: `https://www.desmos.com/calculator/twfsqmmgyb`

Worksheet available at 151: Mountain Road

(a) *Find an expression for a function $h = k(t)$ that expresses your altitude (in thousands of feet) in terms of the number of hours past noon. Include a domain.*

(b) *Evaluate the derivative*

$$\left.\frac{ds}{dt}\right|_{t=1}$$

and interpret its meaning.

(c) *Evaluate the derivative*

$$\left.\frac{dh}{ds}\right|_{s=g(1)}$$

and interpret its meaning.

(d) *Use the results of (b) and (c) to compute the rate (with respect to time) at which your altitude is changing at 1pm.*

(e) *Express the rate of part (d) as a derivative.*

(f) *Express the derivative*

$$\left.\frac{dh}{dt}\right|_{t=1}$$

in terms of the derivatives

$$\left.\frac{ds}{dt}\right|_{t=1}$$

and

$$\left.\frac{dh}{ds}\right|_{s=g(1)}.$$

Exercise 117 *A weight is attached to the midpoint of a 10-foot rope.*

You hold the ends of the rope 10 feet apart at the same height. You then move the ends directly toward each other, each at the constant speed of 2 ft/sec.

Desmos link: `https://www.desmos.com/calculator/grostrmdln`

Worksheet available at 151: Weighted Rope

(a) *Find a function*

$$v = f(t)\,,\ 0 \le t \le 2.5,$$

that expresses the speed of the weight (in feet/sec) in terms of the number of seconds since you began moving your hands.

(b) *Describe how the speed of the weight varies.*

(c) *Comments?*

Exponential Functions, Part 2

More on exponential functions and their derivatives.

Question 118 *What type of functions change at*

(a) *a constant rate?*

(b) *a constant relative rate?*

Question 119 *The function*

$$P = f(t) = 50e^{\frac{t}{10}}, \; 0 \leq t \leq 20,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past midnight. Graph shown below.

Desmos link: `https://www.desmos.com/calculator/kffjsoepmo`

151: Exp Function 5

(a) *Use the graph of the function $P = f(t)$ above to approximate the growth rate when there are 200 million bacteria.*

(b) *Find the relative (instantaneous) growth rate of the population.*

(c) *Find the* exact *growth rate when there are 200 million bacteria. No calculator.*

(d) *Find a function $r = g(P)$ that expresses the growth rate (measured in millions of bacteria/hour) in terms of the popuation (measured in millions). Include a domain.*

(e) *What is the population when it is increasing at the rate of 8 million bacteria/hour?*

Learning outcomes:
Author(s):

Question 120 *The function*

$$N = f(t) = \frac{20}{1+5e^{-t/5}}\ ,\ 0 \le t \le 30,$$

models the spread of a virus throughout a population. It takes as an input the number of months since January 1st and returns as an output the number of infected individuals (measured in millions). Graph shown below.

Desmos link: `https://www.desmos.com/calculator/ikis451wxu`

151: Logistic Model

(a) *Use the sliders in the worksheet above to approximate the number of infected individuals when the virus is spreading at the rate of* $800,000$ *people/month.*

(b) *Find a function*

$$r = g(N)$$

that expresses the infection rate (in millions of people/month) in terms of the population.

(c) *Use calculus and algebra to determine the number of infected individuals when the virus is spreading at the rate of* $800,000$ *people/month.*

(d) *Use the sliders in the worksheet above to approximate the number of infected individuals when the virus is spreading at its fastest rate. Also approximate this rate.*

(e) *Use calculus and algebra to determine the number of infected individuals when the virus is spreading at its fastest rate. Then find this exact rate.*

Question 121 *Use the chain rule and your knowledge of the derivative* $d(e^x)/dx$ *to compute each of the following derivatives.*

(a)

$$\frac{d}{dt}\left(100 \cdot 2^t\right)$$

(b)

$$\frac{d}{dt}\left(100 \cdot 2^{t/5}\right)$$

Question 122 *A colony of bacteria has a population of* 20 *million at noon and a population of* 25 *million at 2pm. The population grows exponentially between noon and midnight.*

(a) *Describe precisely how the population grows.*

(b) *Use your description from part (a) to find a function*

$$P = f(t)\,,\ 0 \le t \le 12,$$

that expresses the population (in millions) in terms of the number of hours past noon. Do not *use* e.

(c) *Find the relative instantaneous growth rate of the population.*

Question 123 *Between ground level and an altitude of* 10 *km, atmospheric pressure on the planet Krypton is an exponential function of altitude. The pressure is 100 kPa at ground level and 80 kPa at an altitude of* 0.75 *km.*

Desmos link: `https://www.desmos.com/calculator/j5h8kaj8xs`

151: Atomspheric Pressure

(a) *Describe exactly how the pressure decreases.*

(b) *Use your description from part (a) to find a function*

$$P = f(h)\,,\ 0 \le h \le 10,$$

that expresses the atmospheric pressure (in kPa) in terms of the altitude (in kilometers). Do not *use the number* e.

(c) *Input your expression for* f *in Line 2 of the worksheet above.*

(d) *Evaluate the derivative*

$$\left.\frac{dP}{dh}\right|_{h=5}$$

and interpret its meaning.

(e) *Find a function* $r = g(P)$ *that expresses the derivative* dP/dh *in terms of* P.

(f) *Use calculus to approximate the altitude at which the pressure is* 0.2 *kPa less at an altitude* 0.1 *km higher.*

Practice Quiz 2

First practice quiz, Weeks 4-6

Directions:

(a) Show all work.

(b) Give brief explanations for each problem. Include these explanations in the flow of the solution.

(c) Show all units in all computations.

(d) Show all steps when computing derivatives and use the Leibniz notation when doing so. Here is an example.

Learning outcomes:
Author(s):

Derivatives of Inverse Functions

Using the chain rule to compute the derivative of the inverse of a function.

Introduction

We saw at the beginning of the class the relationship between the derivative of a function and the derivative its inverse. Recall the two problems:

- You measure the edge length of a cube to be 2 cm and use this measurement to compute its volume. How is the error in computing the volume related to the error (assumed to be small) in your measurement of the edge length?
- You submerge a cube in water and measure its volume to be 8 cm^3. You then use this measurement to compute the edge length of the cube. How is the error in computing the edge length related to the error (assumed to be small) in your measurement of the volume?

The key to the first problem was to differentiate the function

$$V = f(s) = s^3\,, s > 0$$

that expresses the volume (in cubic cm) of the cube in terms of its edge length (in cm). We used limits and found that

$$\left.\frac{dV}{ds}\right|_{s=2} = 12.$$

Question 124 *(a) What are the units of the above derivative? How do you know?*

(b) Interpret the meaning of the above derivative in terms of the geometry of the cube.

Question 125 *From here, we let*

$$\Delta s = s - 2$$

Learning outcomes:
Author(s):

be the error in our measurement (measured in cm) and

$$\Delta V = s^3 - 8$$

the error in the computed volume (measured in cubic cm). Then for small errors $\Delta s \sim 0$,

$$\frac{\Delta V}{\Delta s} \sim \left.\frac{dV}{ds}\right|_{s=2} = 12,$$

and so

$$\Delta V \sim 12\Delta s.$$

For the second problem we used limits to differentiate the function

$$s = f^{-1}(V)\,,\; V > 0,$$

and found that

$$\left.\frac{ds}{dV}\right|_{V=8} = 1/12.$$

Question 125.1 *What are the units of the above derivative? How do you know?*

Then for small errors ΔV *in our measurement of the volume,*

$$\Delta s \sim \frac{1}{12}\Delta V$$

But all this work took us back to the same approximate relation between the errors ΔV *and* Δs *that we had already found in solving the first problem.*

Expressed another way, having found that

$$\left.\frac{dV}{ds}\right|_{s=2} = 12,$$

we should have known immediately (without any computations) that

$$\left.\frac{ds}{dV}\right|_{V=8} = 1/12.$$

For the same reason, this same relationship between the derivative of any function and the derivative of its inverse holds. That is, for any differentiable function $y = f(x)$*, the derivative* dx/dy *of the inverse function* $x = f^{-1}(y)$ *is*

$$\begin{aligned}\frac{d}{dy}\left(f^{-1}(y)\right) &= \frac{dx}{dy}\\ &= \frac{1}{dy/dx}.\end{aligned}$$

Well almost. We need to take some care in evaluating the above derivatives at the correct inputs.

Supposing that $b = f(a)$, a more precise statement of the relation between the derivative of a function $y = f(x)$ and the derivative dx/dy of its inverse is

$$\begin{aligned}\frac{d}{dy}\left(f^{-1}(y)\right)\Big|_{y=b} &= \frac{dx}{dy}\Big|_{y=b} \\ &= \frac{1}{\frac{dy}{dx}\Big|_{x=a}}.\end{aligned}$$

That is, the derivative of the inverse (of a function) is the reciprocal of the derivative (of that function).

Well, not quite. The above relationship is true if

$$\frac{dy}{dx}\Big|_{x=a} \neq 0.$$

This condition also guarantees that the function $y = f(x)$ is one-to-one in a sufficiently small neighborhood near $x = a$ and is therefore invertible in that neighborhood.

Here are some examples.

The Derivative of the Natural Log Function

Example 17. (a) Use the graph of the function $y = f(x)$ below to estimate the value of the derivative

$$\frac{dy}{dx}\Big|_{x=1.1} = \frac{dy}{dx}\Big|_{y=3} \sim 3$$

(b) Use the result of part (a) and the graph below to estimate the value of the derivative

$$\frac{dx}{dy}\Big|_{x=1.1} = \frac{dx}{dy}\Big|_{y=3} \sim 1/3.$$

Desmos link: `https://www.desmos.com/calculator/nnshzdh6jp`

(c) Set $u = 0.6931$ and $n = 90$ in the demonstration above and estimate the value of the derivative

$$\frac{dy}{dx}\Big|_{x=0.6931} = \frac{dy}{dx}\Big|_{y=2} \sim 2$$

(d) Use the result of part (c) and the graph above to estimate the value of the derivative

$$\left.\frac{dx}{dy}\right|_{x=0.6931} = \left.\frac{dx}{dy}\right|_{y=2} \sim 0.5$$

Question 126 *Parts (a) and (c) and perhaps a few more derivatives suggest that*

$$f(x) = e^x.$$

(e) Use the results of parts (b) and (d) to evaluate the derivatives

$$\left.\frac{dx}{dy}\right|_{y=3} = \left.\frac{d}{dy}(\ln y)\right|_{y=3} = 1/3,$$

$$\left.\frac{dx}{dy}\right|_{y=2} = \left.\frac{d}{dy}(\ln y)\right|_{y=2} = 1/2,$$

and more generally

$$\left.\frac{dx}{dy}\right|_{y=a} = \left.\frac{d}{dy}(\ln y)\right|_{y=a} = 1/a.$$

Question 127 *Our conclusion is that*

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Geogebra activity available at 151: Magnification Factor 3

Question 128 *Here's an equivalent, but more computational way to show that*

$$\frac{d}{dx}(\ln x) = 1/x.$$

The key is to recognize that the chain rule tells us that if $u = g(x)$ is a differentiable function of x, then

$$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}.$$

Now to compute the derivative above, we know that since the functions $f(x) = \ln x$ *and* $g(x) = e^x$ *are inverses of one another,*

$$e^{\ln x} = x.$$

Then differentiate both sides of this equation with respect to x *to get*

$$\frac{d}{dx}\left(e^{\ln x}\right) = \frac{dx}{dx}.$$

And by the chain rule we can rewrite this equation as

$$\left(e^{\ln x}\right)\frac{d}{dx}(\ln x) = 1.$$

And since $e^{\ln x} = x$*,*

$$\frac{d}{dx}(\ln x) = 1/x.$$

Question 129 *Find an equation of the tangent line to the curve* $y = \ln x$ *at the point* $(4, \ln 4)$*.*

$$y - \ln 4 = \frac{1}{4}(x-4).$$

Question 130 *(a) Use the chain rule to compute the derivative*

$$\frac{d}{dx}(\ln(4x)).$$

(b) Compute the same derivative without *using the chain rule.*

Explanation. Let

$$y = \ln(4x)$$

and

$$u = 4x.$$

Then

$$y = \ln u$$

and

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\
&= \frac{d}{du}(\ln u)\cdot\frac{d}{dx}(4x)\\
&= \frac{4}{u}\\
&= \frac{1}{x}.
\end{aligned}$$

(b) We can compute the derivative

$$\frac{d}{dx}\left(\ln(4x)\right)$$

wihout the chain rule as follows:

$$\begin{aligned}\frac{d}{dx}\left(\ln(4x)\right) &= \frac{d}{dx}\left(\ln 4 + \ln x\right)\\ &= \frac{d}{dx}\left(\ln 4\right) + \frac{d}{dx}\left(\ln x\right)\\ &= 0 + \frac{1}{x}.\end{aligned}$$

Question 131 *(a) Compute the derivative*

$$\frac{d}{dx}\left(\ln(x^2)\right)$$

both with and without the chain rule. Follow the steps exactly *as in the previous example when using the chain rule.*

(b) What are the domains of the functions $f(x) = \ln x$ and $g(x) = \ln(x^2)$ and their derivatives.

(c) Graph the function $g(x) = \ln(x^2)$ and its derivative on the same coordinate system by hand.

Question 132 *(a) Use the chain rule to compute the derivative*

$$\frac{d}{dx}\left(\ln|x|\right).$$

Do this by noting that

$$|x| = \begin{cases} x, & \text{if } x \geq 0\\ -x, & \text{if } x < 0\end{cases}$$

and using the chain rule to compute

$$\frac{d}{dx}\left(-x\right).$$

(b) A reflection about the y-axis takes the graph of a function $y = f(x)$ to the graph of a function $y = g(x)$. Describe the transformation that takes the graph of $y = f'(x)$ to the graph of $y = g'(x)$. Explain your reasoning.

Question 133 *(a) Find a function*

$$T = f(k),\ k \geq 1,$$

that expresses the time (in years) it takes an investment to grow by a factor of k. Assume the investment grows at a constant relative instantaneous rate of i%/yr. So, for example if $i = 5$, $f(3)$ would be the time it takes an investment to triple at an relative instantaneous growth rate of 5%/yr.

Hint: *Let B_0 be the initial investment. Then the value of the investment T years later is*

$$kB_0 = B_0 e^{\frac{i}{100}T}.$$

Solve this equation for T to find an expression for the function f. You should get that

$$T = f(k) = (100 \ln k)/i$$

(b) Suppose $i = 5$ and evaluate the derivative

$$\left.\frac{dT}{dk}\right|_{k=3}.$$

(c) What are the units of the above derivative? Explain its meaning in terms of small changes.

I'll leave the units up to you. But the meaning is that it takes an investment (increasing at the relative instantaneous rate of 5%/yr) about 0.067 years longer to increase by 201% than it does to triple.

(d) Use the result of parts (b) and (c) to estimate how much longer it would take an investment growing at an relative instantaneous rate of 5%/yr to increase by 210% than it would take to triple. Just use simple arithmetic (ie. multiplication), no calculator.

Question 134 *The function*

$$G = f(v) = 40 - 0.08\left(\frac{v}{2} - 25\right)^2 \quad , 25 \leq v \leq 65,$$

expresses the gas mileage of a car (in miles/gallon) in terms of its speed (in miles/hr).

(a) Explain the meaning of the derivative

$$\frac{d}{dv}\left(\ln(f(v))\right).$$

Include units in your explanation. Also, what are the units of 25 in the above expression? How do you know?

(b) Evaluate the above derivative at $v = 30$ and explain its meaning in terms of small changes.

Question 135 *The function*

$$v = g(h)\,,\ 50 \le h \le 200,$$

expresses the speed (in ft/sec) of a hawk in terms of its altitude (in feet) during a portion of its flight.

Suppose that $f(150) = 80$ and

$$\left.\frac{dv}{dh}\right|_{h=150} = -2.$$

(a) What are the units of the above derivative? Explain the meaning of the derivative in the context of small changes in this particular scenario. Be specific.

I'll leave the units up to you, but if we assume the hawk is descending, then its speed would increase by about 2 ft/sec as it falls from 150 to 149 feet.

(b) Evaluate the derivative

$$\left.\frac{d}{dh}\left(\ln(g(h))\right)\right|_{h=150}.$$

Show all work and explain your reasoning.

The key here is to use the chain rule to evaluate the derivative. So with

$$v = g(h)$$

and

$$w = \ln(g(h)),$$

we have

$$w = \ln v.$$

Then by the chain rule,

$$\begin{aligned}\frac{dw}{dh} &= \frac{dw}{dv} \cdot \frac{dv}{dh} \\ &= \left(\frac{1}{v}\right)\left(\frac{dv}{dh}\right),\end{aligned}$$

and continue from here by evaluating the above expression at $h = 150$.

(c) What are the units of the derivative in part (b)? Explain the meaning of the derivative in the context of small changes in this particular scenario. Be specific by answering this question much like part (a) above (appropriately modified).

The Inverse Sine Function

Let
$$\theta = g(y) = \arcsin y = \sin^{-1}(y).$$

(a) What is the domain of the function g?

(b) What is its range?

(c) Explain the meaning of $\arcsin y$.

(d) True or false: The inverse sine function is the inverse of the sine function.

(e) What function is the inverse of the inverse sine function?

Question 136 *The graph of the function*
$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \le \theta \le \pi/2,$$
is shown below.

Desmos link: `https://www.desmos.com/calculator/lxwoeir1pt`

Worksheet available at 151: Arc Sine

(a) Use the graph to estimate the derivative
$$\frac{d}{dy}\left(\arcsin y\right)\Big|_{y=0.8}.$$
Explain your reasoning.

(b) Use the graph and the slider u to estimate the derivative
$$\frac{d}{dy}\left(\arcsin y\right)\Big|_{y=0.8}.$$

(c) Express the derivative of the function
$$y = f(\theta) = \sin\theta\,,\ -\pi/2 \le \theta \le \pi/2,$$
in terms of y.

(d) Use part (c) and the ideas in parts (a),(b) to find an expression for the derivative
$$\frac{d}{dy}\left(\arcsin y\right).$$

(e) Check your answer to part (d) by evaluating the derivative at $y = 0.6, 0.8$.

(f) What is the domain of the derivative in part (d)?

Question 137 *The top of a 25-foot long ladder slides down a vertical wall at the constant rate of 4 ft/sec.*

(a) Find a function

$$\theta = f(t)\,,\; 0 \le t \le 6.25,$$

that expresses the angle the ladder makes with the ground (measured in radians) in terms of the number of seconds since the ladder was in the vertical position.

(b) Find the rotation rate of the ladder when the top of the ladder is

(i) 15 *feet above the ground.*

(ii) 24 *feet above the ground.*

(iii) 24.9 *feet above the ground.*

(c) Solve this problem again by working directly with the sine function, not the arcsine function.

Desmos link: `https://www.desmos.com/calculator/5c4lssovbi`

Worksheet available at 151: Ladder and ArcSine

Question 138 *The bottom end of a 25-foot long ladder slides across a horizontal floor at the constant rate of 4 ft/sec as the top end slides down a vertical wall.*

(a) Find a function

$$\theta = g(u)\,,\; 0 \le u \le 25,$$

that expresses the angle the ladder makes with the wall (measured in radians) in terms of the distance (in feet) between the wall and the bottom end of the ladder.

Hint: *Let A be the bottom end of the ladder, B the top end, and point O the point on the wall closest to A (as in the demonstration below). Use right triangle trigonometry in ΔAOB.*

(b) Use the slider u in the animation below to approximate each of the following derivatives. Include units. Note that the tick marks on the radian protractor are spaced at intervals of 0.1 *radians.*

(i) $\left.\dfrac{d\theta}{du}\right|_{u=1}$

(ii) $\left.\dfrac{d\theta}{du}\right|_{u=15}$

(iii) $\left.\dfrac{d\theta}{du}\right|_{u=24.9}$

(c) What do the above derivatives tell you?

(d) Find an expression for the derivative $d\theta/du$. Use your expression to evaluate the three derivatives in part (b) and compare these with your estimates.

(e) Find a function

$$\theta = f(t)\,,\; 0 \le t \le 6.25,$$

that expresses the angle the ladder makes with the ground (measured in radians) in terms of the number of seconds since the ladder was in the vertical position.

Hint: *Express the distance u (in feet) between the bottom end of the ladder and the wall in terms of t. Then substitute this expression for u in your function from part (a).*

(f) Find an expression for the derivative

$$r = \omega(t) = d\theta/dt.$$

Interpret its meaning. Include units.

(g) Find the rotation rate of the ladder when the bottom end of the ladder is

(i) 1 foot from the wall.

(ii) 15 feet from the wall.

(iii) 24.9 feet above the wall.

(h) Evaluate the limit

$$\lim_{t \to 6.25} \omega(t)$$

and interpret its meaning.

Desmos link: `https://www.desmos.com/calculator/egolipj5qg`

Worksheet available at 151: Ladder and ArcSine 2

Exercises

Question 139 *(a) Simplfiy the derivative*

$$\frac{d}{d\theta}\left(\arcsin(\sin\theta)\right)$$

(b) Use the result of part (a) to graph the function

$$y = \arcsin(\sin\theta).$$

Explain your reasoning.

Question 140 *Find the measure of the acute angle that the tangent line to the curve*

$$y = f(\theta) = \ln|\sec\theta|$$

at the point $(\pi/7, f(\pi/7))$ makes with the x-axis Do not *use a calculator.*

Question 141 *The function*

$$q = f(p) = 0.2\,(2p - 40)^2\ ,\ 5 \leq p \leq 12,$$

expresses the average number of burgers sold per day at Five Guys in Edmonds in terms of the price (in dollars/burger).

(a) Evaluate the derivative

$$\frac{d}{dp}\left(\ln(f(p))\right)\Big|_{p=7.5}$$

(b) What are the units of the derivative above? Explain its meaning.

Question 142 *The bottom end of a 25-foot ladder lies 24 feet from the base of a vertical wall. Use the appropriate linear approximation to estimate the angle through which the ladder rotates when the bottom end is pulled an additional 0.1 feet away from the wall along a horizontal floor.*

Solve this problem twice, first using an inverse trig function and again without an inverse trig function.

Start by defining your variables, with units.

Use a calculator if need be, but only *for arithmetic and not to evaluate any trigonometric functions.*

Compare your estimate with the actual angle of rotation.

Question 143 *A tree leans precariously with its trunk making an angle of $\phi = \pi/3$ radians with the ground. One end of a 14-foot ladder leans against the trunk, the other rests on the horizontal ground. The bottom end of the ladder is pulled away from the trunk at the constant speed of 4 ft/sec. At what rate is the ladder rotating when the bottom and tops ends are respectively 16 and 10 feet from the base of the trunk?*

Hint: *Use the law of sines.*

Desmos link: `https://www.desmos.com/calculator/rpms2jqfpm`

Desmos activity available at 151: Tree and Ladder

Question 144 *(a) The animation below shows water draining from a tank. Play the animation and sketch by hand a graph of the function $V = f(t)$ that expresses the depth of the water as a function of time. Explain your reasoning. Label the axes with units and the appropriate variable names.*

Access Desmos interactives through the online version of this text at

.

Desmos link: `https://www.desmos.com/calculator/pdghky6tie`

(b) Torricelli's law says that the rate, say in cm^3*/sec, at which water drains out of a small hole in the bottom of a tank is proportional to the square root of the depth of the water. So if $V = f(h)$ is a function that expresses the volume (in* cm^3*) of water in the tank in terms of the depth (in feet) and $h = g(t)$ is a function that expresses the depth of the water (in feet) in terms of number of seconds past noon, then*

$$\frac{dV}{dt} = -k\sqrt{h}$$

for some positive constant k.

For a cylindrical tank of radius r cm,

$$V = f(h) = \pi r^2 h$$

and

$$\frac{dV}{dh} = \pi r^2.$$

So by the chain rule

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt}\\ &= \pi r^2 \frac{dh}{dt}.\end{aligned}$$

So for the cylindrical tank we can write Torricelli's law as

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} = -k\sqrt{h}.$$

Or equivalently as

$$\frac{dh}{dt} = -\frac{k}{\pi r^2}\sqrt{h} = -k_2\sqrt{h},$$

where

$$k_2 = \frac{k}{\pi r^2}$$

is a positive constant.

(c) Which of the following functions might express the depth of water in a cylindrical tank (in terms of time) as the water drains out of a small hole in the bottom of the tank? Justify your reasoning.

(i) $h = g(t) = 2(5-t)^3, 0 \le t \le 5$

(ii) $h = g(t) = 2(5-t)^2, 0 \le t \le 5$

Solution:

The key idea is to express the derivative dh/dt *as a function of the depth* h *of the water.*

(i) If $h = 2(5-t)^3$*, then*

$$\begin{aligned}\frac{dh}{dt} &= \frac{d}{dt}\left(2(5-t)^3\right)\\ &= 2(3)(5-t)^2 \cdot \frac{d}{dt}(5-t)\\ &= -6(5-t)^2.\end{aligned}$$

Now to express dh/dt *in terms of* h*, solve the equation*

$$h = 2(5-t)^3$$

for t *to get*

$$t = 5 - \left(\frac{h}{2}\right)^{1/3}.$$

Then, substitute this expression for t *into the derivative* dh/dt*:*

$$\begin{aligned}\frac{dh}{dt} &= -6(5-t)^2\\ &= -6\left(\frac{h}{2}\right)^{2/3}\\ &= -\left(\frac{6}{2^{2/3}}\right)h^{2/3}.\end{aligned}$$

This tells us that the rate of change in the depth of the water is not *proportional to the square root of the water's depth as Torricelli's law requires. So the function*

$$h = 2(5-t)^3, 0 \le t \le 5$$

is not a possible depth function for water draining from a cylindrical tank.

Free Response: *Give a similar analysis for the depth function of part (ii).*

Question 145 *This is a continuation of the previous problem.*

Now we pour water into a cylindrical tank at a constant rate, while at the same time water leaks out through a small hole in the bottom of the tank. We'll suppose that the tank starts with some initial volume of water.

Free Response: *(a) What do you think happens to the water level in the tank initially?*

(b) What do you think happens to the water level in the long run?

To model this situtation, we need to modify Torricelli's law. For this, let's suppose that we pour water into the cylindrical tank (of radius r) at the constant rate of k_3 cm^3/sec. Then with the same notation as before,

$$\frac{dV}{dt} = k_3 - k\sqrt{h}.$$

But since

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \cdot \frac{dh}{dt} \\ &= \pi r^2 \cdot \frac{dh}{dt},\end{aligned}$$

the above modification of Torricelli's law becomes

$$\begin{aligned}\frac{dh}{dt} &= \frac{k_3}{\pi r^2} - \frac{k}{\pi r^2}\sqrt{h} \\ &= k_1 - k_2\sqrt{h},\end{aligned}$$

where

$$k_1 = \frac{k_3}{\pi r^2}$$

and

$$k_2 = \frac{k}{\pi r^2}$$

are positive constants.

Free Response: *(a) What are the units of k_1 and k_2? How do you know?*

(b) Explain the meaning of k_1.

The equation

$$\frac{dh}{dt} = k_1 - k_2\sqrt{h}$$

expresses the rate of change in the water's depth as a function of the depth. It is called a differential equation and you will learn a little about how to solve equations like this next quarter.

For this particular differential equation, it is not possible to express the depth of the water explicitly as a function of time. But assuming that the depth of the water is h_0 cm at time $t = 0$, it turns out that the function

$$t = g^{-1}(h) = -\frac{2}{k_2}\left(\sqrt{h} - \sqrt{h_0} + \frac{k_1}{k_2}\ln\left|\frac{k_1 - k_2\sqrt{h}}{k_1 - k_2\sqrt{h_0}}\right|\right)\ , t \geq 0,$$

expresses the time (in seconds) in terms of the depth of the water (in cm). We can check that this is indeed correct as follows:

(a) Show algebraically that the depth of the water at time $t = 0$ is $h = h_0$.

(b) Use the above expression for $t = g^{-1}(h)$ to compute and then simplify the derivative dt/dh.

(c) Use the result of part (b) to show that

$$\frac{dh}{dt} = k_1 - k_2\sqrt{h}.$$

Desmos link: `https://www.desmos.com/calculator/c78kv7wifv`

Worksheet available at 151: Draining Cylinder 2

(d) Experiment with the sliders above and summarize your observations about how the graph of the function $h = g(t)$ changes depending on the initial depth of the water and the constants k_1 and k_2.

(e) Express the equilibrium depth in terms of k_1 and k_2. Check that your expression has the correct units. The equilibrium depth is the depth at which the water level remains constant. It is also the depth which the water level approaches (independent of the initial depth).

(f) What happens to the equilibrium depth when k_1 increases (and k_2) is held constant? When k_2 changes and k_1 is held constant?

The Quotient Rule

An introduction to the quotient rule.

Discussion Questions

Question A: (a) At 9am on May 29, the balance in an account is increasing at the relative rate of 8%/yr. At the same time, the price of a stock in increasing at the rate of 5%/yr. Is the number of shares you can buy with the balance in the account increasing or decreasing at this time? At what relative rate? No computations. Just explain what you think.

(b) How would your answer to part (a) change if instead the stock price were decreasing at the rate of 5%/yr?

Question B: Let $B = f(t)$ and $P = g(t)$ be functions that respectively express the balance (in dollars) in an account and the price (in dollars/share) of a stock in terms of the number of years past 9am on May 29. Let $S = h(t)$ be the number of shares of the stock you can buy with the balance in the account at time t years past 9am on May 29.

(a) Interpret the meanings of the following derivatives. Include units.

(i) $\frac{d}{dt}(\ln(f(t))) = \frac{d}{dt}(\ln B))$

(ii) $\frac{d}{dt}(\ln(g(t))) = \frac{d}{dt}(\ln P))$

(iii) $\frac{d}{dt}(\ln(h(t))) = \frac{d}{dt}(\ln S))$

(b) Express the derivative

$$\frac{d}{dt}(\ln S)) = \frac{d}{dt}\left(\ln\left(\frac{B}{S}\right)\right)$$

in terms of the derivatives

$$\frac{d}{dt}(\ln B))$$

and

$$\frac{d}{dt}(\ln P)).$$

Learning outcomes:
Author(s):

The Relative Quotient Rule

Had we never learned about relative changes and relative rates of change, the following questions would have almost forced these ideas upon us.

Question 146 *(a) Suppose over a six-month perioid, the national debt of a small country increases from* $\$4$ *billion to* $\$4.16$ *billion while the population increases from* 20 *million to* 21 *million. Does the per-capita (ie. per-person) share of the national debt increase or decrease during this period?*

(b) At noon on July 1, the national debt of a small country with a population of 20 *million is* $\$4$ *billion.At that same instant the population is increasing at the rate of 1 million people/yr while the national debt is increasing at the rate of* $\$0.16$ *billion/yr. Is the per-capita share of the national debt increasing or decreasing at this instant? At what rate?*

Explanation. (a) The question is really to determine which is greater,

$$c_1 = \frac{4 \times 10^9}{2 \times 10^7} \frac{\text{dollars}}{\text{person}}$$

or

$$c_2 = \frac{4.16 \times 10^9}{2.1 \times 10^7} \frac{\text{dollars}}{\text{person}} = \frac{4 \times 10^9 \times 1.04}{2 \times 10^7 \times 1.05} \frac{\text{dollars}}{\text{person}} = \left(\frac{1.04}{1.05} \right) c_1.$$

And because the total debt increased by 4% while the population increased by 5%, the per-capita share of the national debt decreased during the six-month period.

But the per-capita share of the debt did *not* decrease by 1%, but rather by about (to the nearest thousandth of a percent) 0.952%.

(b) For the instantaneous rate, let

$$Q = f(t)$$

be the total national debt in billions of dollars at time t years past noon on July 1 and

$$P = g(t)$$

the population in millions at time t. Also, let

$$C = h(t) = \frac{f(t)}{g(t)}$$

be the per-capita share of the debt in thousands of dollars/person at time t years past noon on July 1. We wish to find the value of the derivative

$$\left. \frac{dC}{dt} \right|_{t=0}.$$

Part (a) suggests we express the relative instantaneous rate of change

$$\frac{1}{C} \cdot \frac{dC}{dt}\bigg|_{t=0}$$

in terms of the relative rates

$$\frac{1}{Q} \cdot \frac{dQ}{dt}\bigg|_{t=0} = 0.04$$

and

$$\frac{1}{P} \cdot \frac{dP}{dt}\bigg|_{t=0} = 0.05$$

Perhaps the easiest way to do this, although not entirely complete, is to assume that the population and national debt each grow exponentially. In that case, the relative growth rates of each are constant and

$$Q = f(t) = 4e^{0.04t}$$

and

$$P = g(t) = 20e^{0.05t}.$$

This tells us that

$$C = \frac{f(t)}{g(t)} = 0.2e^{-0.01t}$$

and

$$\frac{1}{C} \cdot \frac{dC}{dt}\bigg|_{t=0} = -0.01$$

So at noon on July 1, we suspect that the per-capita share of the debt is decreasing at the relative rate of 1%/yr and at the absolute rate of \$2/person/yr.

To be sure this is true even if the relative rates of change in the population and total debt are *not* constant, we can be more general and compute the relative instantaneous rate of change in the per-capita share of the national debt as

$$\begin{aligned}
\frac{1}{C} \cdot \frac{dC}{dt} &= \frac{d}{dt}\left(\ln\left(\frac{Q}{P}\right)\right) \\
&= \frac{d}{dt}\left(\ln Q - \ln P\right) \\
&= \frac{d}{dt}\left(\ln Q\right) - \frac{d}{dt}\left(\ln P\right) \\
&= \frac{1}{Q} \cdot \frac{dQ}{dt} - \frac{1}{P} \cdot \frac{dP}{dt}.
\end{aligned}$$

Theorem 2. (a) (The Relative Quotient Rule) If $Q = f(t)$ and $P = g(t)$ are differentiable functions of t, then if $g(t) \neq 0$,

$$C = \frac{f(t)}{g(t)} = \frac{Q}{P}$$

is a differentiable function of t and

$$\frac{1}{C} \cdot \frac{dC}{dt} = \frac{1}{Q} \cdot \frac{dQ}{dt} - \frac{1}{P} \cdot \frac{dP}{dt}.$$

The relative rate of change in a quotient of two functions is equal to the difference in the relative rates of change in the functions.

(b) (The Quotient Rule) With the same hypotheses (and obtained by multiplying both side of the previous equation by $C = Q/P$),

$$\frac{d}{dt}\left(\frac{Q}{P}\right) = \frac{1}{P} \cdot \frac{dQ}{dt} - \frac{Q}{P^2} \cdot \frac{dP}{dt}.$$

The Tangent Fuction

Question 147 *Compute the derivative*

$$\frac{d}{d\theta}(\tan\theta)$$

Explanation. We compute this derivative from scratch by letting

$$y = \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Then

$$\ln|y| = \ln|\sin\theta| - \ln|\cos\theta|$$

and

$$\frac{d}{d\theta}(\ln|y|) = \frac{d}{d\theta}(\ln|\sin\theta| - \ln|\cos\theta|).$$

So

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{d\theta} &= \frac{1}{\sin\theta} \cdot \frac{d}{d\theta}(\sin\theta) - \frac{1}{\cos\theta} \cdot \frac{d}{d\theta}(\cos\theta) \\ &= \cot\theta + \tan\theta.\end{aligned}$$

Then mulitplying both sides by $y = \tan\theta$ gives

$$\frac{d}{d\theta}(\tan\theta) = 1 + \tan^2\theta.$$

Question 148 *(a) Use the graph of the function*

$$y = f(\theta) = \tan(2\theta)$$

below to estimate the y-coordinates of all points on the curve where the tangent lines are parallel to the lines

(i) $6x - y = 12$.

(ii) $6x + y = 12$.

(b) Find the exact y-coordinates without *using a calculator.*

Desmos link: `https://www.desmos.com/calculator/obz6ghw3ej`

Desmos activity available at 151: Tangent Graph

Question 149 *You stand* 50 *feet from the base of a tree and measure the angle of elevation to the top of the tree with an error of at most* $\pm 2^\circ$. *You then compute the height of the tree above eye level to be* 100 *feet.*

Desmos link: `https://www.desmos.com/calculator/yjyghsoeog`

Desmos activity available at 151: Angle of Elevation 1

(a) Use the demonstration above to approimate your error in computing the tree's height.

(b) Use the appropriate linear approximation to estimate your error in computing the tree's height. Compare this with your estimate from part (a). Do not use a calculator. Do this as follows:

- *Find a function*

  $$h = f(\theta)\,,\ 0 < \theta < \pi/2,$$

  that expresses the computed height of the tree above eye level (measured in feet) in terms of the measured angle of elevation (in radians). Draw a picture to help with your explanation.

  This function is

  $$h = f(\theta) = 50\tan\theta\,,\ 0 < \theta < \pi/2.$$

- *Next find an expression for the derivative $dh/d\theta$ and evaluate the derivative*

  $$\left.\frac{dh}{d\theta}\right|_{h=100}.$$

- *Now we'll take the exact height of the tree above eye level to be* 100 *feet and let*

$$\Delta h = h - 100$$

be the error in the computed height and

$$\Delta\theta = \theta - \arctan(2)$$

be the error in the measured angle.

Then if $\Delta\theta \sim 0$,

$$\left.\frac{dh}{d\theta}\right|_{h=100} \sim \frac{\Delta h}{\Delta\theta}$$

and so

$$\Delta h \sim 250\,(\Delta\theta)\,.$$

- *I'll let you continue from here.*

The Inverse Tangent Function

Question 150 *(a) Explain the meaning of the function*

$$\theta = f(y) = \arctan y.$$

In particular, what does it take as an input and what does it return as an output? Include the domain and range of the function as part of your explanation.

(b) Express the derivative

$$\frac{d}{d\theta}(\tan\theta) = 1 + \tan^2\theta$$

of the function

$$y = \tan\theta$$

in terms of y.

(c) Use part (b) to find an expression for the derivative

$$\frac{d\theta}{dy} = \frac{d}{dy}(\arctan y) = \frac{d}{dy}\left(\tan^{-1} y\right).$$

(d) Evaluate the derivatives

$$\left.\frac{d}{d\theta}(\tan\theta)\right|_{\theta=\pi/4}$$

and

$$\frac{d}{dx}(\arctan x)\Big|_{x=1}.$$

Comments?

Question 151 *You stand 60 feet from the base of a tree and measure the angle of elevation to the top of an 85-foot tall tree. You then move 4 feet closer to the tree and measure the same angle. Assume your eyes are five feet above the ground.*

Desmos link: `https://www.desmos.com/calculator/qqgvq3noah`

Desmos activity available at 151: Tree 3

(a) Drag the slider u in the demonstration above to estimate the chnage in the angle of elevation after you move 4 feet closer to the tree. Note that the angles of elevation are $\angle TEB$ and $\angle TFB$ and that $\angle ETF$ measures their difference. Consecutive tick marks on the protractor are spaced at intervals of 0.01 radians.

(b) Use derivatives to approximate the change in the angle of elevation and compare your approximation with your estimate in part (a) and with the actual change. Use a calculator with only addition, subtradction, multiplication, and division.

Go about this by first finding a function

$$\theta = f(s)\,, s > 0,$$

that expresses the angle of elevation to the top of the tree (measured in radians) in terms of your distance from the tree (measured in feet). Use the arctangent function in your expression. Do not *use the inverse cotangent function.*

The function is

$$\theta = f(s) = \arctan(80/s)\,, s > 0.$$

Then continue in a manner similar to Question 4.

Two Motions

Question 152 *(a) Play the slider s in the demonstration below to show the motion of a beetle crawling along the y-axis as it leaves behind tracks spaced at equal time intervals.*

(b) Use the animation to sketch (by hand) a graph of the function

$$s = f(t)$$

that expresses the position (in this case the y-coordinate) of the beetle as a function of time. Label the axes with the appropriate variable names and units. Then activate the folder in Line 2 to see how you did.

(c) Use the animation to sketch (by hand) a graph of the function

$$v = g(t)$$

that expresses the beetle's velocity (in this case the rate of change, with respect to time, in the beetle's y-coordinate). Label the axes with the appropriate variable names and units. Then activate the folder in Line 7 to see how you did.

Desmos link: `https://www.desmos.com/calculator/srotstrdzm`

Desmos activity available at 151: Tangent Motion

Question 153 *Repeat parts (a)-(c) of the previous question for the motion below.*

Desmos link: `https://www.desmos.com/calculator/h6vq21lfql`

Desmos activity available at 151: ArcTangent Motion

Exercises

Question 154 *(a) Use the website below to compute an accurate estimate of the current rate (in dollars/year) at which the U.S. national debt is changing. Explain your method.*

National Debt Clock

(b) Use the website below to compute an accurate estimate (in people/yr) at which the U.S. population is currently increasing.

Popluation Clock

(c) Use your estimates above and the current national debt and U.S. population to compute

(i) the current rate of change in the per-capita share of the national debt.

(ii) the current relative rate of change in the per-captia share of the national debt.

Include units in every number in each step of your computations.

Question 155 *The function*

$$C = f(t) = At^4 e^{-kt}\,,\ t \geq 0,$$

expresses the concentration of a drug (measured in mg/L) in the bloodstream in terms of the number of hours since the drug was injected. Here A and k are postive constants.

Desmos link: `https://www.desmos.com/calculator/lvsxu3wa9a`

Desmos activity available at 151: Drug Concentration

(a) What are the units of the constant A? How do you know?

(b) What are the units of k? How do you know?

(c) Use the graph of $C = f(t)$ above to sketch by hand a graph of the derivative $r = dC/dt$. Be sure to label the axes with the appropriate variable names and units. Do not *use technology. Explain your reasoning.*

(d) Use calculus and algebra to find an expression in terms of k for the time when the concentration is a maximum. Work in general. Do not *use a specific value of k. Check your work by following the directions, Lines 4 and 5, in the desmos demonstration. Label also the coordinates of the corresponding point on your hand-drawn graph of the derivative.*

(e) Find expressions (in terms of k) for the times when the concentration is increasing and decreasing at the maximum rates. Check that your expressions have the correct units. Work in general. Do not *use a specific value of k. Check your work by following the directions, Lines 6-9, in the desmos demonstration. Label also the coordinates of the corresponding points on your hand-drawn graph of the derivative.*

(f) Find an expression (in terms of k) for the relative rate at which the concentration is changing t hours after the injection. Check that your expression has the correct units. Work in general. Do not *use a specific value of k.*

(g) Suppose the concentration is at its maximum five hours after injection and determine when the concetration is increasing at the rate of 50%/hr. Determine also when the concentration is decreasing at the rate 50%/hr.

(h) Observations?

Question 156 *The function*

$$P = f(t) = 5 - 3t + t^2\,,\ 0 \leq t \leq 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

(a) Use the graphs of the function $P = f(t)$ and the function $r = f'(t)/f(t)$ to estimate when the stock price is increasing at the greatest relative rate.

(b) Use algebra to find the exact time when the stock price is increasing at the greatest relative rate.

Hint: *What is the value of the derivative dr/dt at this time? But start by finding an expression for the instantaneous relative rate of change in the stock price.*

Desmos link: `https://www.desmos.com/calculator/xuupp3srqv`

Desmos activity available at 151: Stock Price 4

Question 157 *You jog once around a circular track of radius r meters at the constant speed of speed of v m/sec. A flagpole lies b meters due east of the track's center.*

(a) Find a function

$$s = f(t)\,, 0 \le t \le 2\pi r/v,$$

that expresses your distance (in meters) to the flagpole in terms of the time (measured in seconds) since you started running. Assume you start at the point A on the track nearest the flagpole. Explain your reasoning. Work with the general parameters r, v, and b, not *with any specific values for these parameters.*

(b) Find an expression fro the time when your distance to the flagpole is increasing at the greatest rate. Try to give a geometric interpretation of your position at this time.

Desmos link: `https://www.desmos.com/calculator/bxofhvfbfs`

Demonstration available at Math 151: Jogger 3

Implicit Differentiation

An introduction to implicit differentiation.

Discussion Questions

Question 158 *Which of the following equations define y implicty as a function of x in a sufficiently small neighborhood of the given point? Supplement your reasoning with a graph of each relation.*

(a) $x^2 + y^2 = 25$ *near the point* $(4, -3)$

(b) $x^2 + y^2 = 25$ *near the point* $(0, -5)$

(c) $x^2 + y^2 = 25$ *near the point* $(-5, 0)$

(d) $x^2 - xy + y^2 = 1$ *near the point* $(1, 1)$.

(e) $x^2 - xy + y^2 = 3$ *near the point* $(1, 2)$.

(f) $x^2 - xy + y^2 = 3$ *near the point* $(2, 1)$.

Introduction to Implicit Differentiation

Question 159 *Let $p = f(t)$ be a differentiable function of t. Find expressions for each of the following derivatives, first supposing that*

$$p = f(t) = t^3 + 1,$$

and then more generally, not assuming any particular expression for the function f.

(a) $\dfrac{d}{dp}\left(p^5\right)$

(b) $\dfrac{d}{dt}\left(p^5\right)$

(c) $\dfrac{d}{dt}\left(e^{2p}\right)$

(d) $\dfrac{d}{dt}\left(t^4 \sin(p)\right)$

Learning outcomes:
Author(s):

Question 160 *(a) Find equations of the tangent lines to the ellipse*

$$x^2 - xy + 2y^2 = 9$$

at the points where the ellipse intersects the x-axis.

(b) Find equations of all horizontal tangent lines to the ellipse.

(c) Find equations of all vertical tangent lines.

Desmos link: `https://www.desmos.com/calculator/ox3wvfitqk`

Desmos activity available at 151: Implicit Ellipse

Question 161 *(a) Find an equation of the tangent line to the curve*

$$4xe^y - 3y\cos(2xy) = 0$$

at the origin.

(b) Enter your equation in Line 5 of the worksheet below to check your answer.

Desmos link: `https://www.desmos.com/calculator/1qprxg226m`

Desmos activity available at 151: Implicit 1

Astroids

Question 162 *(a) The coordinate axes cut out a segment from the tangent line to the curve*

$$x^{2/3} + y^{2/3} = 5$$

at the point $P(8, 1)$. Find the length of that segment. Do this by using implicit differentiation to help find an equation of the tangent line at P and go from there. But first show that P lies on the curve.

(b) As a bug crawls around the above curve and passes the point $P(8, 1)$, it is moving away from the x-axis at the rate of 4 cm/sec. Is the bug moving toward or away from the y-axis at this time? At what rate?

(c) Find the slope of the tangent line in part (a) without using implicit differentiation.

Question 163 *(a) Use implicit differentiation to show that segments cut by the coordinate axes from the tangent lines to the astroid*

$$x^{2/3} + y^{2/3} = a^{2/3}$$

all have the same length. Here $a > 0$ is a constant.

(b) Prove the same result by using trigonometric functions to parameterize the astroid.

Desmos link: `https://www.desmos.com/calculator/vrythrvjuc`

Desmos activity available at 151: Astroid

Ellipses and Related Curves

Question 164 *(a) Use implicit differentiation to find an equation of the tangent line to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point P with coordinates (x_0, y_0). Here a, b are positive constants.

(b) Input your equation from part (a) on Line 3 of the desmos worksheet below. Drag point P to check your equation is correct.

(c) Solve part (c) again without *calculus by considering a composition of transformations that takes the circle*

$$x^2 + y^2 = 1$$

to the ellipse in part (a).

Desmos link: `https://www.desmos.com/calculator/t1e9v7ncpv`

Desmos activity available at 151: Tangents to Ellipse

Question 165 *An ellipse through the point $P(0, b)$ has focal points F_1 at the origin and F_2 at the point $A(a, 0)$.*

(a) Use the definition of an ellipse as the set of points whose distances to the foci have a constant sum to find an equation of the ellipse.

(b) Use implicit differentiation to find the slope of the tangent line to the ellipse at P.

(c) Find an equation of the line normal to the ellipse at Q.

(d) Find the coordinates of the point Q where the normal line intersects the x-axis.

(e) Express the ratio $F_1Q : QF_2$ in terms of a and b. Interpret the ratio geometrically.

The ratio is

$$\frac{F_1Q}{QF_2} = \frac{b}{\sqrt{a^2+b^2}}.$$

(f) Check your work in the demonstration below.

Desmos link: `https://www.desmos.com/calculator/xqywotxxf9`

Desmos activity available at 151:Normal to an Ellipse

Question 166 *Let F_1 and F_2 be respectively the origin and the point with coordinates $(a, 0)$. The curve $\mathcal{C}$ passes through the point $P(0, b)$. The curve is defined by the property that the sum of the distance from a point Q of $\mathcal{C}$ to F_1 and k times its distance to F_2 is a constant.*

(a) Find an equation of the curve $\mathcal{C}$.

(b) Find an equation of the tangent line to $\mathcal{C}$ at P.

(c) Find an equation of the line normal to $\mathcal{C}$ at P.

(d) Find the coordinates of the point Q where the normal line intersects the x-axis.

(e) Express the ratio $F_1Q : QF_2$ in terms of a and b. Interpret the ratio geometrically.

The ratio is

$$\frac{F_1Q}{QF_2} = \frac{bk}{\sqrt{a^2+b^2}}.$$

(f) Check your work in the demonstration below.

Desmos link: `https://www.desmos.com/calculator/uz7w5uh0vl`

Desmos activity available at 151: Generalized Ellipse

The Sliding Ladder

Question 167 *You walk in along a straight path, moving either directly toward or directly away from a tree at the end of the path.*

Let $s = f(t)$ be a function that expresses your distance to the tree in terms of the number of minutes past noon and suppose that

$$\left.\frac{ds}{dt}\right|_{t=5} = -240.$$

(a) Interpret the meaning of the above derivative.

(b) What is your speed at 12:05pm?

(c) Find an expression for your speed at time t minutes past noon.

Question 168 *A tree leans precariously with its trunk making an angle of ϕ radians with the ground. One end of a ladder with length L feet leans against the trunk, the other rests on the horizontal ground. You slide the bottom end of the ladder along the ground at random.*

(a) Let s and t be the respective distances (measured in feet) from the base of the trunk to the bottom and top ends of the ladder. Find an equation that relates these distances. Use the parameters L and ϕ, not *their particular values in the demonstration below.*

(b) Let u be time measured in seconds and suppose that s and t are (unknown) functions of u. Express the derivative dt/du in terms of the derivative ds/du. Interpret the meanings of the these derivatives.

(c) Switch s and t in your equation from part (b). What do you notice? Explain why.

(d) Use part (b) to express the speed of the top end of the ladder in terms of the speed of its bottom end.

(e) Suppose $\phi = \pi/3$ and that at some instant the bottom end of the ladder is three times as far from the trunk's base as the top end. What can you say about the speeds of the ends of the ladder at this instant?

(f) Still supposing $\phi = \pi/3$, find all possible angles the ladder makes with the ground when the top end of the ladder is moving four times as fast as the bottom end. Characterize each angle in one of two ways:

(i) One end of the ladder is moving toward the trunk's base and the other end away from the base.

(ii) Both ends of the ladder are either moving toward or way from the trunk's base.

Give exact angles. Then use a calculator to approximate their radian measures to the nearest hundredth.

(g) Use the animation below to check that your answers to parts (e) and (f) are reasonable. The graph is of the relation

Desmos link: `https://www.desmos.com/calculator/58jqjo0inl`

Geogebra activity available at 151: Tree and Ladder

Question 169 *A tree leans precariously with its trunk making an angle of ϕ radians with the ground. One end of a ladder with length L feet leans against the trunk, the other rests on the horizontal ground. You slide the bottom end of the ladder along the ground at random.*

Suppose at some instant the ladder makes an angle of β_0 radians with the trunk and that the bottom end is s_0 feet from the trunk's base. Suppose also that the ladder is rotating at the rate of ω_0 rad/sec at this same moment.

(a) Find an expression for rate of change in the distance from the bottom end of the ladder to the base of the trunk at this time.

(b) Find an expression for rate of change in the distance from the top end of the ladder to the base of the trunk at this time.

(c) Find expressions for the speeds of the ends of the ladder at this time.

Hint: *Use the law of sines in ΔBOT in the demonstration from the previous question.*

Question 170 *The function*

$$h = f(\delta) = \frac{24}{\pi}\arccos(-\tan\phi\tan\delta)\,,\ -\pi/2+\phi<\delta<\pi/2-\phi,$$

expresses the number of hours of daylight per day at latitude ϕ in terms of the declination of the sun. The declination of the sun (δ) is the angle the sun's rays make with the plane of the equator, taken to be positive between the spring and fall equinoxes in the northern hemisphere. The latitude ϕ, $-\pi/2 \le \phi \le \pi/2$, is positive at points in the northern hemipshere.

Geogebra link: `https://www.geogebra.org/m/vnhrutwu`

Geogebra activity available at 151: Declination of Sun 2 *(a) Use the graph of the function f below (at latitude $\phi \sim 1.1$) to sketch a graph of the derivative*

$$\frac{dh}{d\delta} = f'(\delta).$$

(b) Suppose the latitude ϕ is held constant and find an expression for the derivative

$$\frac{dh}{d\delta} = \frac{d}{d\delta}\left(-\tan\phi\tan\delta\right).$$

(c) What are the units of the derivative in part (b)?

(d) Input your expression for the derivative in Line 4 of the worksheet below (follow the directions there). Then vary the slider ϕ to see how the function f and its derivative vary with latitude. Summarize your observations.

(e) Find an expression for the derivative

$$\left.\frac{dh}{d\delta}\right|_{\delta=0} = \left.\frac{d}{d\delta}\left(-\tan\phi\tan\delta\right)\right|_{\delta=0}$$

in terms of the latitude ϕ.

(f) Evaluate the derivative in part (e) at a latitude of $\phi = \pi/4$. Interpret its meaning in terms of small changes.

Desmos link: `https://www.desmos.com/calculator/ifomatkcta`

Desmos activity available at 151: Length of Day 1

(g) Suppose now that each month has 30 days so that there are 360 days in one year. Suppose also that the declination of the sun varies sinusoidally as a function of time, that the maximum declination of $\delta = 23.5°$ occurs on the summer solstice (say June 21st) and the minimum declination $\delta = -23.5°$ occurs on the winter solstice (December 21st).

Find an expression for a function

$$\delta = k(t)\,,\ t \geq 0,$$

that gives the declination of the sun (measured in radians) in terms of the number of days since the spring equinox.

(h) Find a function

$$r = g(\phi)\,,\ -\pi/2 < \phi < \pi,$$

that expresses the rate of change in the number of minutes of daylight per day (measured in (minutes of daylight/day)/day)) on the spring equinox in terms of the latitude ϕ. Input this function in Line 1 of the demonstration below.

(i) Use the result of part (f) to approximate the rate in (hours of daylight/-day)/day) at which the number of daylight hours per day is changing at a latitude of $\phi = \pi/4$ radians on the spring equinox and on the fall equinox.

(j) Evaluate the rates from part (i) in Fairbanks, Alaska, latitude $64.8°$*N.*

Desmos link: `https://www.desmos.com/calculator/nf8n5uphhl`

Desmos activity available at 151: Length of Day 2

Waves

Question 171 *The function*

$$y = f(x,t) = a\sin(kx - \omega t), t \geq 0, \tag{3}$$

describes a wave on a string. The functions expresses the displacement (in meters) of a point on the string in terms of the position x *(in meters) of the point and time* t*, measured in seconds since the motion began.*

(a) Experiment with the sliders k*,* ω*, in the demonstration below, playing the slider* u *(*u *is just another name for* t*). Summarize your observations. In particular, be sure to turn off Line 1 to be better able to see the motion of the individual points of the string.*

(b) What are the units of k *and* ω*? How do you know?*

(c) Find an expression for the wavelength λ *in terms of* k*,* ω*.*

(d) Find an expression for the period of oscillation T *in terms of* k*,* ω*.*

(e) Hold y *constant and differentiate each side of equation (3) with respect to* t *to find an expression for the speed of the wave. Turn on the graph in Line 9. Explain the logic behind the computation.*

Desmos link: `https://www.desmos.com/calculator/9xmkig9hwi`

Desmos activity available at 151: Traveling Wave 1

Motion

An introduction to motion.

Question 172 *Play the slider u in the activity below to see the motion of a balloon. Use the animation to sketch graphs of*

(a) the altitude of the balloon as a function of time.

(b) the balloon's rate of ascent as a function of time.

(c) Activate the folders in Lines 13 and 18 to see how you did.

Desmos link: `https://www.desmos.com/calculator/amv52b9ljt`

Desmos activity available at 151: Balloon

Question 173 *The function*

$$h = f(t) = 10 - \frac{1}{2}t^2 e^{-t/5}\,,\ 0 \le t \le 40,$$

expresses the altitude of a balloon (in thousands of feet) in terms of the number of hours since noon on August 31, 2023.

Use the graph of this function in Question 1 to first approximate answers to the following questions. Then use calculus and algebra to determine the exact times.

(a) When is the balloon at its minimum height? At its maximum height?

(b) When is the balloon ascending at the fastest rate? Descending at the fastest rate?

To compute these times you will end up solving a quadratic equation that may be written in the form

$$t^2 - 20t + 50 = 0.$$

So the balloon is ascending at its fastest rate at time (give the exact times and then approximations to the nearest hundredth of an hour)

$$t = 10 + \sqrt{50} \sim 17.07$$

Learning outcomes:
Author(s):

hours past noon and descending at its fastest rate at time

$$t = 10 - \sqrt{50} \sim 2.93$$

hours past noon.

Review of Differentiation

Derivative Review.

Examples

Example 18. Find an equation of the tangent line to the curve

$$y = f(x) = \left(2x^3 + 1\right)^2$$

at the point $(1, 9)$.

Explanation. Let

$$y = \left(2x^3 + 1\right)^2$$

and

$$u = 2x^3 + 1.$$

Then

$$y = u^2$$

and

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}\left(u^2\right) \cdot \frac{d}{dx}\left(2x^3 + 1\right) \\
&= 2u(6x^2) \\
&= 2(2x^3 + 1)(6x^2).
\end{aligned}$$

Then the slope of the tangent line to the curve $y = (2x^3+1)^2$ at the point $(1, 9)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 2(3)(6) = 36,$$

and an equation of the tangent line is

$$y - 9 = 36(x - 1).$$

Example 19. Find an expression for the derivative

$$\frac{d}{d\theta}\left(\theta\cos(5\theta)\right).$$

Learning outcomes:
Author(s):

Explanation. We use the product rule first to get

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \frac{d}{d\theta}\left(\theta\right)\cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right) \\ &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right).\end{aligned}$$

Now we use the chain rule to compute

$$\frac{d}{d\theta}\left(\cos(5\theta)\right).$$

For this we hide the composition by letting

$$y = \cos(5\theta)$$

and

$$u = 5\theta.$$

Then

$$y = \cos u$$

and by the chain rule

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= \frac{d}{du}\left(\cos u\right)\cdot\frac{d}{d\theta}\left(5\theta\right) \\ &= (-\sin u)(5) \\ &= -5\sin(5\theta).\end{aligned}$$

The final result is that

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right) \\ &= \cos(5\theta) - 5\theta\sin(5\theta).\end{aligned}$$

Here's a shorter version of the same solution.

We use the product rule first to get

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \frac{d}{d\theta}\left(\theta\right)\cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right) \\ &= \cos(5\theta) + \theta\frac{d}{d\theta}\left(\cos(5\theta)\right).\end{aligned}$$

Then we use the chain rule to differentiate $\cos(5\theta)$, giving

$$\begin{aligned}\frac{d}{d\theta}\left(\theta\cos(5\theta)\right) &= \frac{d}{d\theta}\left(\theta\right)\cos(5\theta)+\theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta)+\theta\frac{d}{d\theta}\left(\cos(5\theta)\right)\\ &= \cos(5\theta)+\theta(-\sin(5\theta))\frac{d}{d\theta}\left(5\theta\right)\\ &= \cos(5\theta)-5\theta\sin(5\theta).\end{aligned}$$

Exercises

Directions: Follow the method of Example 1 *exactly* for each of the following problems.

Question 174 *Find expressions for each of the following derivatives. Simplify your expressions for the derivatives. Do* not *simplify the function being differentiated.*

Part 1:

(a) $\frac{d}{dw}\left(\arctan w+\arctan(1/w)\right)$

(b) $\frac{d}{dy}\left(\arcsin\left(\sqrt{1-y^2}\right)\right)$

Part 2:

Any comments or observations on the derivatives above?

Question 175 *Desmos link:*
https://www.desmos.com/calculator/zjqsdhvz4p

Desmos activity available at 151: Building Temperature

Question 176 *Water is poured into a cylindrical tank at a constant rate. At the same time, water flows out of a small hole in the bottom of the tank. The tank is empty at noon.*

The function

$$t=f(h)=-\frac{2}{k_2}\left(\sqrt{h}+\frac{k_1}{k_2}\ln\left|\frac{k_1-k_2\sqrt{h}}{k_1}\right|\right),\ t\geq 0,$$

expresses the time (measured in minutes past noon) in terms of the depth (measured in cm) of water in the tank. Here k_1, k_2 are postive constants.

(a) Use the above function to verify that the tank is empty at noon.

(b) Find a simplified expression for the derivative dt/dh.

(c) Use the result of part (b) to show that

$$\frac{dh}{dt} = k_1 - k_2\sqrt{h}.$$

(d) What are the units of k_1, k_2? How do you know?

(e) For much more on this problem, see Questions 23 and 24 of the chapter Derivatives of Inverse Functions.

Question 177 *The function*

$$s = f(t) = Ae^{-k_1 t}\cos(k_2 t)\,, \; t \geq 0,$$

expresses the displacement (in meters) from equilibrium of an oscillating mass on a spring in terms of the number of seconds since the mass was released from rest.

(a) What are the units of the constants A, k_1, and k_2? Explain how you know.

(b) Find an expression for the velocity ds/dt of the mass.

(c) Find an expression for the acceleration

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

of the mass.

(d) Use algebra to show that

$$\frac{d^2s}{dt^2} = -\left(2k_1\frac{ds}{dt} + (k_1^2 + k_2^2)s\right).$$

Desmos link: `https://www.desmos.com/calculator/ygikqgj7af`

Desmos activity available at 151: Damped Harmonic Oscillator

Question 178 *(a) Find a function*

$$s = f(h), h \geq 0,$$

that expresses the distance (in miles) to the horizon in terms of your altitude (in miles). We'll suppose the earth to be a perfect sphere of radius $R = 4000$ *miles. The distance to the horizon is the arclength* AT *below, meaured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance* AP.

The function is

$$s = f(h) = 4000 \arccos\left(\frac{4000}{4000+h}\right),\ h \geq 0.$$

Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

(b) Use the graph of the function $s = f(h)$ *show below to skectch a graph of the derivative*

$$y = \frac{ds}{dh} = f'(h).$$

by hand. What is the domain of the derivative? Note that the curve $s = f(h)$ *has a horizontal asymptote. What is an equation of this asymptote?*

Desmos link: `https://www.desmos.com/calculator/nof5l3mtfy`

Then check your sketch by activating the folder in Line 1 of the worksheet below.

Desmos link: `https://www.desmos.com/calculator/pdbjfao316`

(c) Find an expression for the derivative

$$\frac{ds}{dh} = \frac{d}{dh}\left(\arccos\left(\frac{R}{R+h}\right)\right).$$

Hint: Start by using the graphs of the functions $y = \arcsin x$ *and* $y = \arccos(x)$ *below and your knowledge of the derivative* $d/dx(\arcsin x)$ *to find an expression for* $d/dx(\arccos x)$. *Or equivalently, recognize that*

$$\arcsin x + \arccos x = \pi/2.$$

Desmos link: `https://www.desmos.com/calculator/zuq9rf1j4d`

(d) With $R = 4000$, *evaluate the derivative*

$$\left.\frac{ds}{dh}\right|_{h=16}.$$

(e) Interpret the meaning of the derivative in part (d) in terms of specific *small changes.*

(f) You take a ride on Blue Origin and in two minutes are boosted straight up to an altitude of 32 miles. Suppose that the function

$$h = g(t) = \begin{cases} 10t^2 + 6t^3 \text{ , } 0 \le t \le 1 \\ 32 - 10(2-t)^2 - 16(2-t)^3 \text{ , } 1 < t \le 2, \end{cases}$$

expresses your altitude (in miles) in terms of the number of minutes since launch for the first two minutes of your flight.

(i) At what rate (with respect to time) is your distance to the horizon changing when you are one minute into the flight? Use the graph of the function $s = f(g(t))$ shown below to first approximate this rate.

(ii) At what rate (with respect to time) is your distance to the horizon changing at the start of the flight? Use the graph of the function $s = f(g(t))$ shown below to first approximate this rate. Are you surprised given your graph in part (b)?

Desmos link: `https://www.desmos.com/calculator/jchwjpxugf`

Review Problems

Some problems for review.

Question 179 *This question is about how the temperature inside a building changes in response to changes in the outdoor temperature. We assume the building has no internal heating or cooling system.*

We'll suppose that the function

$$f(t) = M - B\cos\left(\frac{\pi}{12}t\right),\ t \geq 0,$$

expresses the outdoor temperature (in Fahrenheit degrees) in terms of the number of hours past 4am.

Newton's law of cooling models the rate at which the indoor temperature is changing at any time. It says that this rate of change is proportional to the difference in the indoor and outdoor temperatures. So if the function

$$T = g(t)\ t \geq 0,$$

expresses the outdoor temperature (in Fahrenheit degrees) in terms of the number of hours past 4am, Newton's law says that

$$\frac{dT}{dt} = k(f(t) - g(t)) \tag{4}$$

for some constant k.

(a) What are the units of k? How do you know?

(b) Is k postive or negative? How do you know?

(c) Experiment with the sliders in the demonstration below. Summarize your observations.

Desmos link: `https://www.desmos.com/calculator/oag9lhvgo5`

Desmos activity available at 151: Building Temperature

Next quarter you will learn how to use the above equation to determine the indoor temperature at any time given the temperature at some specific time.

Learning outcomes:
Author(s):

For now, we'll just claim that the indoor temperature is given by the function

$$\begin{aligned}
T &= g(h) \\
&= M + Ce^{-kt} - \frac{B}{1+(\frac{\pi}{12k})^2}\left(\cos\left(\frac{\pi}{12}t\right) + \frac{\pi}{12k}\sin\left(\frac{\pi}{12}t\right)\right) \\
&= M + Ce^{-kt} - \frac{B}{\sqrt{1+(\frac{\pi}{12k})^2}}\cos\left(\frac{\pi}{12}t - \phi\right),
\end{aligned}$$

where

$$\phi = \arctan\left(\frac{\pi}{12k}\right).$$

Hanging Chains

Comparing a catenary with a weighted chain.

Question 180 *Desmos link:* https://www.desmos.com/calculator/cifsqaas5j

Desmos activity available at 151: Weighted Chain

Question 181 *Desmos link:* https://www.desmos.com/calculator/sv70z21j2j

Desmos activity available at 151: Catenary

Learning outcomes:
Author(s):

Electron in a Crossed Field

Electron.

Crossed Fields and the Hodograph

The force on an a charge q with velocity $\mathbf{v}$ in a crossed magnetic/electric field is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Our problem is to parameterize the motion of a charged particle when the fields are uniform and mutually orthogonal. We suppose the magnetic and electric fields point in the respective directions $\mathbf{k}$ and $\mathbf{j}$, and that the charge has velocity $\mathbf{v}_0$ perpendicular to $\mathbf{k}$ at time $t = 0$.

Then the charge, assumed to be positve, has acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = k_1\mathbf{j} + k_2\mathbf{v} \times \mathbf{k},$$

for some constants $k_1, k_2 \geq 0$, with respective units m/sec^2 and sec^{-1}.

This is a differential equation that is easy enough to solve algebraically. But our aim here is to take a geoemetric approach that uses some key ideas from Calculus 3.

In sketching motion we typically draw the path and perhaps draw a few position vectors at equally-spaced time intervals. These vectors allow us to approximate a few velocity vectors, or at least their relative lengths. But instead of drawing the velocity vectors tangent to the path, it is usually more useful to draw them with their tails pinned at a common point. The curve traced by these tips of these pinned vectors is called the *hodograph* of a motion.

For starters, the hodograph gives us a way to visualize the motion's acceleration as it is tangent to the hodograph. But to get some idea of the accerlation's magnitude, we need to consider not only the direction and magnitude of the velocity vector but also its rotation rate.

For a uniform circular motion that rotates around a circle of radius r meters at a constant rate of ω rad/sec, for example, we *see* from the hodograph (a circle of radius $v = \omega r$ that the acceleration vector points directly toward the center of the path and rotates at the same rate as both the position and velocity vectors.

Learning outcomes:
Author(s):

So without any derivatives, we get the acceleration's magnitude to be

$$|\mathbf{a}| = \omega|\mathbf{v}| = \omega^2 r.$$

Turning off the Electric Field

Returning to our charged particle, with the electric field turned off ($k_1 = 0$), the acceleration and velocity vectors are perpendicular, so the charge moves with constant speed $|\mathbf{v}_0| = v_0$. But then because $\mathbf{v}$ is perpendicular to $\mathbf{k}$,

$$|\mathbf{a}| = k_2|v| = k_2 v_0$$

is also constant.

So we are looking for a plane motion having constant speed and an acceleration with constant magnitude. One choice would be uniform circular motion, where

$$|\mathbf{a}| = v_0\omega,$$

where $\omega = d\theta/dt$ is the (constant) rotation rate of the pinned velocity vector. Then $k_2 = \omega$ and the trajectory has radius

$$r = \frac{v_0}{\omega} = \frac{v_0}{k_2}.$$

To see that there are no other possible motions we could probably appeal to some uniqueness theorem of differential equations. But for a more geometric approach, consider what we know.

(a) that the charge moves with constant speed and

(b) that the acceleration vector rotates at a constant rate.

Question 182 *What do these conditions imply about the trajectory?*

Hint: *Think about curvature.*

Explanation. Click the arrow to the lower right for the solution.

Let θ be the angle from $\mathbf{i} = \langle 1, 0, 0\rangle$ to $\mathbf{v}$. Then because $\mathbf{v}$ and $\mathbf{a}$ are orthogonal, these vectors rotate at the same rate $\omega = d\theta/dt$. And because the speed is constant, the path's radius of curvature

$$r = \frac{|v|}{\left|\frac{d\theta}{dt}\right|} = \frac{v_0}{k_2}$$

is also constant. And because the path is a plane curve, it is a circle with this radius.

1 Newton's Law of Cooling

Exploration 183 *Desmos link:*
`https://www.desmos.com/calculator/yebumxuwms`

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