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# Calculus 1

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May 17, 2024

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# Introduction

## Introduction to Differential Calculus

This course is listed as Calculus I in the catalogue, but it really should be called Differential Calculus. *Differential* has the same root as *difference* and *calculus* the same root as *calculate*. So this class is really about calculating differences, or more simply put it's about subtracting. But it's about subtracting in the context of functions.

Pick a specific input to specific function and you'll likely find that the changes in the function's output are approximately proportional to *small* changes in the input. If so, we say that the function is *differentiable* at that input and we call the proportionality constant the *derivative*.

For example, let's look at the behavior of the function

$$A = f(s) = s^2, s \geq 0,$$

near the input  $s = 5$ . To emphasize the importance of units, let's define the input  $s$  to be the side length of a square measured in feet and the output  $f(s)$  to be the area of that square, measured in square feet. The problem before us is to describe a simple relationship between a small change in the side length

$$\Delta s = s - 5$$

of the square and the change

$$\Delta A = f(s) - f(5)$$

in its area.

**Question 1** (a) We'll first take a numerical approach and compute some small changes and their ratios. Fill in the missing entries in the table below.

$s$ (ft)	$A = s^2$ (ft <sup>2</sup> )	$\Delta s = s - 5$ (ft)	$\Delta A = s^2 - 25$ (ft <sup>2</sup> )	$\Delta A / \Delta s$ (ft <sup>2</sup> /ft)
4.9	24.01	-0.1	-0.99	9.9
4.99	24.9001	-0.01	-0.0999	9.99
5	25	0	0	—
5.01	25.1001	0.01	0.1001	10.01
5.1	26.01	0.1	1.01	10.1

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Learning outcomes:

Author(s):

(b) The data in the table above suggests an approximate proportional relationship between  $\Delta A$  and  $\Delta s$ . We can guess the constant of proportionality from the fifth column. As  $s \rightarrow 5$  (as  $s$  approaches 5), it looks like the ratio  $\Delta A/\Delta s$  approaches some number, the constant of proportionality.

i) What is that number?  $\boxed{10}$

ii) What are its units?

**Free Response:**

So for  $\Delta s \sim 0$ , we suspect that

$$\Delta A \sim \boxed{10} \Delta s.$$

(c) The constant of proportionality is called the derivative, in this case of the function  $A = s^2$ , at the input  $s = 5$ . We write this as

$$\left. \frac{dA}{ds} \right|_{s=5} = \boxed{10}.$$

(d) We could have taken an algebraic approach to determine this constant of proportionality instead. The idea is to first simplify the quotient  $\Delta A/\Delta s$  as

$$\begin{aligned} \frac{\Delta A}{\Delta s} &= \frac{s^2 - 25}{s - 5} \\ &= \frac{(s + \boxed{5})(s - \boxed{5})}{s - 5} \\ &= \boxed{s + 5} \text{ if } s \neq \boxed{5}. \end{aligned}$$

So, for example, if  $s = 4.99$ , then

$$\frac{\Delta A}{\Delta s} = \boxed{4.99} + 5 = \boxed{9.99}$$

as shown in the last column of the second row of the above table.

The advantage of this algebraic approach is that we can now compute the proportionality constant as a limit:

$$\begin{aligned} \left. \frac{dA}{ds} \right|_{s=5} &= \lim_{s \rightarrow 5} \frac{\Delta A}{\Delta s} \\ &= \lim_{s \rightarrow 5} (s + \boxed{5}) \\ &= \boxed{5} + \boxed{5} \\ &= \boxed{10}. \end{aligned}$$

(d) We can also use the graph of the function  $A = f(s) = s^2$  to interpret the ratios

$$\frac{\Delta A}{\Delta s} = \frac{f(s) - f(5)}{s - 5}$$

geometrically. Move the slider  $s$  in the demonstration below and describe

- (i) how the line through the points  $P$  and  $Q$  is related to the ratio  $\Delta A/\Delta s$  show on Line 2,
- (ii) what happens to the line  $PQ$  as  $s \rightarrow 5$ , and
- (iii) what happens to the line  $PQ$  when  $s = 5$ .

**Free Response:**

Access Desmos interactives through the online version of this text at

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*Desmos link:* <https://www.desmos.com/calculator/vz9ud5txva>

Continuing with the above demonstration,

- (i) Open the Code folder in Line 3 and turn off the line  $PQ$  in Line 7.
- (ii) Write an equation for the line through the point  $P$  with slope equal to the proportionality constant in the line below and on Line 8 in the desmos worksheet:

$$A = L(s) = \boxed{25} + \boxed{10}(s - \boxed{5}).$$

- (iii) Zoom in close enough to the point  $P$  to make the graph of the function  $A = f(s)$  look like a line. How do the graph of the function and the graph of the line  $A = L(s)$  compare in this close-up view?

**Free Response:**

(e) **Summary:**

- If we change the side of a square from a length of 5 feet to a length of  $s \sim 5$  feet, then the area of the square changes by approximately

$$\Delta A = s^2 - 25 \sim 10\Delta s = 10(s - 5)$$

square feet. The proportionality constant 10 has units  $ft^2/ft = ft$ .

- Zoom in close enough to the graph of the function  $A = f(s) = s^2$  near the point  $P(5, 25)$  and the graph looks like a line with slope equal to the proportionality constant.

- We can compute the proportionality constant as the limit

$$\left. \frac{dA}{ds} \right|_{s=5} = \lim_{s \rightarrow 5} \frac{f(s) - f(5)}{s - 5}.$$

- Suppose for example, we wanted to approximate the side length  $s$  of a square with area  $25.06 \text{ ft}^2$ . Then

$$\Delta A = s^2 - 25 = 25.06 - 25 = 0.06.$$

And since

$$\Delta A \sim 10\Delta s = 10(s - 5),$$

$$0.06 \sim 10\Delta s.$$

So

$$\Delta s \sim 0.006$$

and a square with area  $25.06 \text{ ft}^2$  has an approximate side length (measured in feet) of

$$s = 5 + \Delta s \sim 5.006.$$

**Question 2** On a clear day with an unobstructed view (like you might have at the beach or in a hot air balloon), the distance to the horizon is limited by the curvature of the earth as illustrated in the demonstration below.

In fact, as long as you are not too high above the surface of the earth, the function

$$s = f(h) = 1.22\sqrt{h}, 0 \leq h \leq 20,000,$$

gives a good approximation to the distance to the horizon (the length of the red arc  $AT$  below, measured in miles) in terms of your height above the ground (the distance  $AP$  below, measured in feet).

Access Desmos interactives through the online version of this text at

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**Desmos link:** <https://www.desmos.com/calculator/ewowig5sgk>

Desmos activity available at

[151:Distance to Horizon 1](#)

Our aim is to approximate the change in the distance to the horizon (in miles) in terms of a small change in height (in feet) from a height of 25 feet.

(i) To start, what are the units of the constant 1.22 above? Explain how you know.

**Free Response:**

(ii) Go through a similar analysis as in parts (a)-(e) of Example 1, to approximate the change  $\Delta s = s - f(25)$  in the distance to the horizon in terms of the change  $\Delta h = h - 25$  in your height above the ground. Start by completing the column headings (with units) and the missing entries in the table below.

$h$ (ft)	$s = 1.22\sqrt{h}$ (miles)	$\Delta h = h - 25$ (ft)	$\Delta s = f(h) - f(25)$ (miles)	$\Delta s/\Delta h$ (units?)
$4.9^2$				
$4.99^2$				
25		0	0	—
$5.01^2$				
$5.1^2$				

**Question 3** This question is similar to the last, but suppose instead we are looking down on the earth from the space station or a rocket. Then the approximation to the distance to the horizon from the previous problem will not work.

So our first step is to find a function

$$s = f(h), h \geq 0,$$

that expresses the distance to the horizon (still measured in miles) in terms of our height above the earth's surface, now measured in miles instead of feet. We'll suppose the earth to be a perfect sphere of radius 3960 miles. The distance to the horizon is the arclength  $AT$  below, measured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance  $AP$ .

**Desmos link:** <https://www.desmos.com/calculator/ewowig5sgk>

(a) Find an expression for the above function.

**Hint:** Use right triangle  $\triangle OTP$  to find an expression for the radian measure of angle  $\angle POT$ . Then use this angle to find an expression for the arclength  $AT$ .

Here are more details.

(i) Enter the two side lengths, measured in feet, in right triangle  $\triangle OPT$  below.

$$OT = \boxed{3960}$$

and

$$OP = \boxed{h + 3960}.$$

(ii) Let  $\theta$  be the radian measure of  $\angle TOP$ . Write an equation with a trigonometric function of  $\theta$  that relates the two lengths in part (i). Use the Math Editor tab to enter the trig function and the angle  $\theta$ .

$$\cos \theta = \frac{3960}{h + 3960}.$$

(iii) Now solve the equation from part (ii) for  $\theta$  in terms of  $h$ . Then use what you know about measuring arclength along a circle to find an expression for the function  $f$ . Use the Math Editor tab to help.

$$s = f(h) = 3960 \arccos\left(\frac{3960}{h + 3960}\right).$$

(b) Now suppose we are 165 miles above the surface of the earth and we wish to approximate how a small change in our altitude changes the distance to the horizon.

To do this, fill in the missing entries in the table below.

$h$ (miles)	$s = f(h)$ (miles)	$\Delta h = h - 165$ (miles)	$\Delta s = f(h) - f(165)$ (miles)	$\Delta s / \Delta h$ (units?)
162				
163				
164				
165				
166				
167				
168				

(c) Do the data above suggest that the quotients  $\Delta s / \Delta h$  approach some number as  $h$  approaches 165? If so, use the data to approximate that number. If not, explain why not.

(d) Make your own table similar to the one above to get a better approximation, correct to the nearest thousandth, to

$$\lim_{h \rightarrow 165} \frac{f(h) - f(165)}{h - 165}.$$

(e) Use your result from part (d), rounded to the nearest thousandth, to approximate  $\Delta s$  in terms of  $\Delta h$  and enter your result below.

$$\Delta s \sim \boxed{3.291} \Delta h, \text{ for } \Delta h \sim 0.$$

(f) Explain the meaning of the proportionality constant in parts (d) and (e). Be sure to include units in your explanation.



## Small Changes

*We explore how small changes to the input of a function change the output.*

The main idea of differential calculus is to approximate the change in the output of a function in terms of a small change in the input. For some functions, called *differentiable*, the change in the output is approximately proportional to the (small) change in the input. The proportionality factor is called the derivative. In this chapter we explore this idea.

## Odometer Readings

**Example 1.** *The graph of the function*

$$s = f(t), 0 \leq t \leq 2,$$

*that expresses the trip odometer reading (measured in miles) on your car in terms of the number of hours past noon during a two-hour trip is shown below.*

*Access Desmos interactives through the online version of this text at*

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*Desmos link:* <https://www.desmos.com/calculator/iw69lr1bc>

*Desmos activity available at*

*151: Odometer*

*Our goal is to approximate the car's speed at 12:30pm in three ways:*

- (1) geometrically, using the above graph as is.*
- (2) geometrically, by zooming in on the above graph.*
- (3) arithmetically, using the specific expression for the function  $f$ .*

*(a) Start by using the graph above to describe how the speed of the car varies over the two-hour period. Explain your reasoning. Then play the Slider  $u$  in Line 2 and use the animation of the motion to check if your description was accurate. Explain.*

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Learning outcomes:  
Author(s):

## Small Changes

(b) Set the slider in Line 6 to  $n = 20$ . Use only the graph (with  $u = 2$ ) to create a table with five columns showing the values of  $t$ ,  $s$  (approximate this),  $\Delta t = t - 0.5$ ,  $\Delta s = f(t) - f(0.5)$ , and  $\Delta s/\Delta t$ . Include units in the heading of each column. The table should include five rows, with  $t = 0.3, 0.5, \dots, 0.7$ .

$t$ (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.3				
0.4				
0.5				
0.6				
0.7				

(c) Explain the meaning of the fifth column in the table of part (b). What do the entries in this column suggest about the speed of the car at 12:30pm?

(d) Now we'll use the fact that

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2,$$

to construct another table like the one in part (b). Do this by finding expressions for

$$\Delta s = f(t) - f(0.5)$$

and

$$r = g(t) = \frac{\Delta s}{\Delta t} = \frac{f(t) - f(0.5)}{t - 0.5},$$

both in terms of  $t$  (and not  $\Delta t$ ). Use these functions to fill in the missing entries in the table below.

$t$ (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.49				
0.499				
0.4999				
0.5				
0.5001				
0.501				
0.51				

(e) Does your table from part (d) suggest that the ratios  $\Delta s/\Delta t$  approach some number as  $t \rightarrow 0.5$ ? If so, what would be your guess for the exact value of this number? What are its units? What is its meaning?

(f) Activate the folder "Graph of average speed" on Line 9.

(i) Use the graph to check some of your entries in the fifth column of your table from part (d). Explain.

(ii) How is the line  $PQ$ , through the fixed point  $P(0.5, 10)$  and the variable point  $Q(t, s)$  on the graph of the function  $r = g(t)$  related to the ratio  $\Delta s / \Delta t$ ?

(iii) Describe what happens to the line  $PQ$  as point  $Q$  approaches point  $P$ .

(g) For a quicker way to approximate car's speed at 12:30pm, zoom in sufficiently close to point  $P$  in the graph above to make the graph of  $s = f(t)$  look like a line. Use the coordinates of point  $P$  and a second point in the window far away from  $P$  to estimate the car's speed at 12:30pm. Explain your method.

(h) Summarize your conclusions by comparing your three estimates for the car's speed at 12:15pm. Which estimate do you think is most accurate? Least accurate?

**Example 2.** This is a continuation of the previous example where we'll algebraically compute the exact speed of the car at 12:30pm, using the odometer function

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2.$$

The idea is to first find an algebraic expression for the car's average speed between time  $t$  and time  $t = 0.5$  hours past noon. Then we'll evaluate the limit of this average speed as  $t \rightarrow 0.5$  to find the (instantaneous) speed at 12:30pm.

**Question 4** First we'll find the average speed between time  $t$  and time  $t = 0.5$ .

(a) Explain in general how to compute a car's average speed over some time interval. What do you need to know? What is the computation? Make up your own specific example.

(b) Now for our particular odometer function above, the average speed  $v_{\text{avg}}(t)$ , measured in miles/hour, between time  $t$  and time  $t = 0.5$  is

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{48t^2 - 16t^3 - 10}{t - 0.5} \\ &= \frac{(2t - 1)(-8t^2 + 20t + 10)}{t - 0.5} \\ &= \boxed{-16t^2 + 40t + 20} \text{ if } t \neq \boxed{0.5}. \end{aligned}$$

The key step in the computation above is in the third line. How did we know  $2t - 1$  was a factor of

$$f(t) - f(0.5) = 48t^2 - 16t^3 - 10?$$

The reason is that  $t = 0.5$  is a root of the polynomial  $f(t) - f(0.5)$  and therefore  $t - 0.5$  is a factor. And so

$$2(t - 0.5) = 2t - 1$$

is also a factor. Then we can use long division to find the quotient.

(c) Show the steps in the long division.

(d) The final step in computing the car's speed  $v$  (in miles/hour) at 12:30pm is to evaluate the limit of these average speeds as  $t \rightarrow 0.5$ . We get

$$\begin{aligned} v &= \lim_{t \rightarrow 0.5} \left( \boxed{-16t^2 + 40t + 20} \right) \\ &= \boxed{36}. \end{aligned}$$

(e) Here's another way to simplify the average speed in part (b). Fill in the missing steps.

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{(48t^2 - 16t^3) - (48(0.5)^2 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{(48t^2 - 48(0.5)^2) - (16t^3 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{48(t^2 - (0.5)^2) - 16(t^3 - (0.5)^3)}{t - 0.5} \\ &= \frac{48(t - 0.5)(\boxed{t + 0.5}) - 16(t - 0.5)(\boxed{t^2 + 0.5t + 0.25})}{t - 0.5} \\ &= 48(\boxed{t + 0.5}) - 16(\boxed{t^2 + 0.5t + 0.25}) \text{ if } t \neq \boxed{0.5}. \end{aligned}$$

(f) Use the above expression for the average speed function to compute the (instantaneous) speed of the car at 12:30pm by evaluating the appropriate limit.

(g) Sketch by hand a graph of the average speed function  $y = v_{\text{avg}}(t)$  over the appropriate domain. Be sure also to state this function's domain.

## A Projectile

**Example 3.** Access Desmos interactives through the online version of this text at

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*Desmos link:* <https://www.desmos.com/calculator/l4fknr0hpl>

Desmos activity available at

151: Projectile

## The Falling Ladder, Part 1

**Example 4.** *The top end of a ten-foot ladder leans against a vertical wall and the bottom end rests on the horizontal floor. We analyze how a small change in the distance between the wall and the bottom of the ladder affects the height of the ladder's top above the floor.*

Access Desmos interactives through the online version of this text at

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*Desmos link:* <https://www.desmos.com/calculator/dvyuifyy94>

Desmos activity available at

151: Ladder 1B

**Question 5** (a) The slider  $s$  in Line 1 of the demonstration above controls the distance between the wall and the bottom of the ladder, measured in feet. Use the slider  $s$  to describe qualitatively how a small change in  $s$  changes the height  $h$  (also measured in feet) of the ladder's top end above the floor.

(i) Do the small changes have the same or opposite signs?

(ii) At what positions of the ladder does a small change in  $s$  result in a comparatively large change in  $h$ ?

(b) Now let's focus on the particular position of the ladder where the bottom end is  $s = 8$  feet from the wall. For this, turn on the "one position" folder in Line 3.

(i) Drag the slider  $s$  close to  $s = 8$  and use the coordinates of the endpoints of the ladder to construct a table with five columns showing the values of  $s$ ,  $h$ ,  $\Delta s = s - 8$ ,  $\Delta h = f(s) - f(8)$ , and  $\Delta h / \Delta s$ . Include units in the heading of each column. The table should include seven rows, with  $s = 7.7, 7.8, \dots, 8.2, 8.3$ . Here  $h = f(s)$  is the function described in part (ii) below.

(ii) Find a function

$$h = f(s), 0 \leq s \leq 10,$$

that expresses the height of the ladder's top end above the ground (in feet) in terms of the distance of its bottom end from the wall (in feet).

(iii) Use your function  $f$  to construct another table, like the one in part (i), with  $s = 7.9, 7.99, 7.999, 8, 8.0001, 8.01, 8.1$ . Do this by finding expressions for

$$\Delta h = f(s) - f(8)$$

and

$$r = g(s) = \frac{\Delta h}{\Delta s} = \frac{f(s) - f(8)}{s - 8},$$

both in terms of  $s$  (and not  $\Delta s$ ).

(iv) Does your table from part (iii) suggest that the ratios  $\Delta h/\Delta s$  approach some number as  $s \rightarrow 8$ ? If so, what would be your guess for the exact value of this limit? What are its units? What is its meaning?

(v) Activate the folder “graph of function” on Line 8. How is the line  $PQ$ , through the fixed point  $P(8, 6)$  and the variable point  $Q(s, h)$  on the graph of the function  $h = f(s)$  related to the ratios  $\Delta h/\Delta s$ ?

(vi) Change the bounds for  $s$  in Line 1 to run between  $s = 7.9$  and  $s = 8.1$ . Then activate the folder “graph: average rate of change function” on Line 18. Move the slider  $s$  and use the graph of the function  $r = g(s)$  to check your computations in part (iii).

(vii) Use the result of part (iv) to write an approximation for the change in height

$$\Delta h = h - 8$$

in terms of the change

$$\Delta s = s - 8.$$

(viii) The graph of the function  $h = f(s)$  suggests another, geometric way to find the proportionality constant (of part (vii)) that relates  $\Delta h$  to  $\Delta s$ . Explain how.

## The Falling Ladder, Part 2

**Example 5.** A tree leans precariously with its trunk making an angle of  $\phi = \pi/6$  radians with the ground. One end of a ten-foot ladder leans against the trunk, the other rests on the horizontal ground. We analyze how a small change in the distance between the bottom of the ladder and the base of the trunk changes the distance between the top of the ladder and the base of the trunk.

Access Geogebra interactives through the online version of this text at

*Geogebra link:* <https://tube.geogebra.org/m/qmke5y7x>

We'll let  $t$  be the distance between the top of the ladder and the base of the trunk (measured in feet) and  $s$  the distance between the bottom of the ladder and the base of the trunk (also measured in feet).

The slider  $\theta$  in the demonstration above controls the angle that the ladder makes with the ground, but this angle does not come into play in our problem.

(a) Use the slider  $\theta$  to describe qualitatively how a small change in  $s$  (the length of segment  $GC$ ) changes  $t$  (the length of segment  $GB$ ):

(i) For what positions of the ladder do these small changes have the same signs? Opposite signs?

(ii) For what positions of the ladder does a small change in  $s$  result in a comparatively large change in  $t$ ?

(b) Now let's focus on the particular position of the ladder when the bottom end  $C$  is 16 feet from the trunk's base and the top end  $D$  is about 8 feet from the base as illustrated above. Our first step is to find a function

$$t = f(s)$$

that expresses  $t$  in terms of  $s$  for values of  $s$  and  $t$  near  $s = 16$  and  $t = 8$  respectively.

To do this, first use the law of cosines to write an equation relating  $s$  and  $t$ . Then complete the square to solve this equation for  $t$  in terms of  $s$  to find the function  $t = f(s)$ . Keep in mind that when  $s = 16$ , we must have  $t \sim 8$ .

**Hint:** Here is an outline of the steps to check your work:

(i) Use the law of cosines to relate  $s$  and  $t$ .

$$s^2 + t^2 - \boxed{\sqrt{3}st} = \boxed{100}.$$

(ii) Solve the above equation for  $t$  in terms of  $s$  as follows:

First rewrite the equation with the two terms with  $t$  as a factor on the left side and the other two terms on the right.

$$t^2 - \boxed{\sqrt{3}st} = \boxed{100 - s^2}.$$

Then complete the square by adding the same perfect square to each side.

$$t^2 - \boxed{\sqrt{3}st} + \left(\boxed{\frac{\sqrt{3}s}{2}}\right)^2 = 100 - s^2 + \left(\boxed{\frac{\sqrt{3}s}{2}}\right)^2.$$

Then factor the LHS and simplify the RHS.

$$\left(\boxed{t - \frac{\sqrt{3}s}{2}}\right)^2 = \boxed{100 - \frac{s^2}{4}}.$$

Next, solve for  $t$  in terms of  $s$ .

$$t = \frac{\sqrt{3}}{2}s \pm \sqrt{100 - \frac{s^2}{4}}$$

Finally, make the correct choice of  $\pm$  to solve for  $t$  in terms of  $s$ , when  $s$  is near 16 and  $t$  near 8.

$$t = \frac{\sqrt{3}}{2}s - \sqrt{100 - \frac{s^2}{4}}.$$

(c) Use your function from part (b) to find an expressions for

$$\Delta t = f(s) - f(16)$$

and for the function

$$r = g(s) = \frac{\Delta t}{\Delta s} = \frac{f(s) - f(16)}{s - 16}.$$

Explain what the output of the function  $g$  measures. What are its units?

(d) Use the results of part (c) to construct a table with five columns showing the values of  $s$ ,  $t$ ,  $\Delta s = s - 16$ ,  $\Delta t = f(s) - f(16)$ , and  $\Delta t/\Delta s$ . Include units in the heading of each column. The table should include seven rows, with  $s = 15.9, 15.99, 15.999, 16, 16.001, 16.01, 16.1$ .

(e) Does your table from part (d) suggest that the ratio  $\Delta t/\Delta s$  approaches some number as  $s \rightarrow 16$ ? If so, approximate the value of this number. What are its units?

(f) Check the box “GraphofRelation” in the demonstration above and explain how the line  $EF$  is related to part (d).

(g) Use the result of part (e) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of  $s$  near 16. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 16.4 feet from the trunk’s base. Compare your approximation with the exact distance.

## Riding a Ferris Wheel

Suppose you ride a ferris wheel



# Limits

*Limits in context.*

## Limits and Tangent Lines

**Example 6.** *Let*

$$g(x) = \frac{x^2 - 9}{3x - 9}.$$

(a) *Evaluate each of the following expressions.*

(i)  $g(7)$

(ii)  $\lim_{x \rightarrow 7} g(x)$

(iii)  $g(3)$

(iv)  $\lim_{x \rightarrow 3} g(x)$

(b) *Simplify and then graph the function  $g(x)$ .*

(c) *Interpret the expressions in part (a) geometrically by considering the graph of the function  $f(x) = x^2/3$  as in the demonstration below.*

*Desmos link:* <https://www.desmos.com/calculator/u0uvuchnrk>

*Desmos activity available at 151: Parabola Basic*

## Limits and Gas Mielage

**Example 7.** *The function*

$$G = f(s) = \frac{2}{5} + \frac{1}{5000}(40(s+2)^2 - (s+2)^3), \quad 0 \leq s \leq 23,$$

*expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.*

*Desmos link:* <https://www.desmos.com/calculator/xzknfjpkw3>

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Learning outcomes:  
Author(s):

*Desmos activity available at 151: Gas as a Function of Distance*

(a) Use the graph of the function  $f$  shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Find your average gas mileage (in miles/gallon) over the interval  $s \in [8, 18]$ .

(c) Use the graph to approximate your gas mileage at the moment you are 18 miles from home. Do this by zooming in on the appropriate point.

(d) Use the algebra of limits to determine your exact gas mileage at the moment you are 18 miles from home.

(e) Use the result of part (d) to approximate the change in the volume of gas

$$\Delta G = f(s) - f(18)$$

in terms of the change

$$\Delta s = s - 18$$

in your distance from home for values of  $s$  near 18 miles. What are the units of the proportionality constant?

(f) Use the result of part (e) to approximate your distance from home when there are 1.9 gallons of gas in your tank.

## Limits, Gas Mileage and Speed

**Example 8.** Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and that the car gets 38 miles/gallon at a speed of 40 miles/hour.

(a) Find an expression for the function

$$G = f(v), \quad 30 \leq v \leq 70,$$

that gives the gas mileage (in miles/gal) in terms of the speed (in miles/hour).

(b) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the car's speed at 60 miles/hour gives an approximately proportional change in its gas mileage.

(c) Use the results of part (b) to approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find the exact value of the proportionality constant.

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta G = g - f(60)$$

in gas mileage in terms of a small change

$$\Delta v = v - 60$$

in the car's speed.

- (g) Use part (f) to approximate the speed at which the car gets 36 miles/gallon.
- (h) Would you expect your approximation in part (g) to be greater or less than the actual speed? Explain your reasoning with a graph.
- (i) Simplify the units of the proportionality constant. What might these units suggest about a way to interpret the constant?
- (j) At what rate (in gal/hr) does the car burn gas at a speed of 60 miles/hour?
- (k) How is the rate in part (j) related to the proportionality constant?

## Limits, Speed and Altitude

**Question 6** A rock dropped from a height of 100 feet falls to the surface of Planet Krypton without air resistance.

(a) By considering only the physical situation and without doing any computations, sketch a graph of the function

$$v = g(h), 0 \leq h \leq 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet).

(b) Use the results from part (a) to choose a reasonable expression for the function  $g$  from the list below.

**Multiple Choice:**

- (a)  $g(t) = 100 - 9t^2, 0 \leq t \leq 10/3$
- (b)  $g(h) = 100 - 9h^2, 0 \leq h \leq 100$
- (c)  $g(h) = 0.005(100 - h)^2, 0 \leq h \leq 100$
- (d)  $g(h) = 6\sqrt{100 - h}, 0 \leq h \leq 100$  ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rock's height from 64 feet gives an approximately proportional change in its speed.

(d) Use the results of part (c) to approximate the proportionality constant. What are its units?

(e) Use the algebra of limits to find the exact value of the proportionality constant.

(f) Explain the meaning of the proportionality constant.

(g) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(h) Use part (g) to approximate the rock's speed at a height of 63 feet.

(i) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(j) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

## Limits and Gas Mileage

## Limits and Purchasing Power

# The Derivative

*Computing derivatives with limits.*

**Example 9.** *Suppose*

$$y = f(x) = x^2$$

*and let's use limits to evaluate*

$$f'(3) = \left. \frac{dy}{dx} \right|_{x=3} = \left. \frac{d(x^2)}{dx} \right|_{x=3}.$$

*We have*

$$\begin{aligned} f'(3) &= \left. \frac{d(x^2)}{dx} \right|_{x=3} = \lim_{v \rightarrow 3} \frac{f(v) - f(3)}{v - 3} \\ &= \lim_{v \rightarrow 3} \frac{v^2 - 9}{v - 3} \\ &= \lim_{v \rightarrow 3} \frac{(v - 3)(v + 3)}{v - 3} \\ &= \lim_{v \rightarrow 3} (v + 3) \\ &= (3 + 3) \\ &= 6. \end{aligned}$$

*Next let's do almost the same thing and compute*

$$f'(x) = \frac{dy}{dx}$$

*for the function*

$$y = f(x) = x^2$$

*by replacing 3 in the above computation with  $x$ .*

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Learning outcomes:  
Author(s):

We get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d(x^2)}{dx} = \lim_{v \rightarrow x} \frac{f(v) - f(x)}{v - x} \\
 &= \lim_{v \rightarrow x} \frac{v^2 - x^2}{v - x} \\
 &= \lim_{v \rightarrow x} \frac{(v - x)(v + x)}{v - x} \\
 &= \lim_{v \rightarrow x} (v + x) \\
 &= (x + x) \\
 &= 2x.
 \end{aligned}$$

Just as a check, when  $x = 3$ ,

$$\left. \frac{dy}{dx} \right|_{x=3} = (2x) \Big|_{x=3} = 2(3) = 6.$$

**Question 7** Use the result of the previous example to solve each of the following problems. Do not use limits.

- (a) Find an equation of the tangent line to the parabola  $y = x^2$  at the point  $(-4, 16)$ .
- (b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).
- (c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.
- (d) Let  $\mathcal{L}$  be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where  $\mathcal{L}$  intersects the  $y$ -axis.
- (e) Repeat parts (a)-(d) above for the tangent line to the parabola  $y = x^2$  at the general point  $(a, a^2)$ . What do you notice?

**Question 8** (a) Use the method of Example 1 for the function

$$y = g(x) = 1/x^2$$

to compute

$$g'(3) = \left. \frac{dy}{dx} \right|_{x=3} = \left. \frac{d(1/x^2)}{dx} \right|_{x=3}$$

and

$$g'(x) = \frac{dy}{dx}.$$

(b) Use the result of part (a) to find an equation of the tangent line to the curve  $y = 1/x^2$  at the point  $(3, 1/9)$ .

**Question 9** (a) Use numerical methods to estimate the slope of the tangent line to the curve

$$y = f(x) = x^3$$

at the point  $(2, 8)$ . Include enough data to suggest a progression toward a limit.

(b) Use the algebra of limits to find the exact slope of the tangent line in part (a).

(c) Use algebra to find the coordinates of the all pointw where the tangent line in part (a) intersects the curve  $y = x^3$ .

(d) Suppose you measure the edge length of a cube to be 2cm and then use this measurement to compute the volume of the cube. Use the result of part (b) to approximate your error in computing the volume in terms of your error in measuring the edge length. Assume the latter error is small.

Then compare your exact error in computing the volume with your approximation for some specific edge length near 2cm. You should start this problem defining a function with meaningful variable names (do not use  $x$  and  $y$ ).

**Question 10** (a) Find a function

$$s = g(V), V \geq 0$$

that expresses the edge length (measured in cm) of a cube in terms of its volume (measured in cubic centimeters).

(b) Use the algebra of limits to evaluate the derivative

$$g'(V_0) = \left. \frac{ds}{dV} \right|_{V=V_0}.$$

(c) What are the units of the derivative in part (b)? Explain how you know.

(d) Suppose you submerge the cube in water and measure its volume to be  $8 \text{ cm}^3$ . You then use this measurement to compute the edge length of the cube. Use the result of part (b) to approximate your error in computing the edge length in terms of your error in measuring the volume. Assume the latter error is small.

Start this problem by defining the errors  $\Delta s$  and  $\Delta V$  in terms of the actual edge length  $s$  of the cube.

(e) Use the result of part (d) to approximate your error in computing the edge length of the cube if the cube's actual volume is  $7.7 \text{ cm}^3$ .

(f) Explain how this question is related to Question 4.

**Question 11** Suppose that between speeds of 60 miles/hr and 72 miles/hr, the gas mileage of a car is a linear function of its speed. Suppose also that the car gets 36 miles/gallon at a speed of 60 miles/hour and 32 miles/gallon at a speed of 72 miles/hour.

(a) Find a function

$$r = f(v), 60 \leq v \leq 72,$$

that expresses the rate (measured in gal/hr) at which the car burns gas in terms of its speed (measured in miles/hour). Explain your reasoning. This function is not linear.

**Hint:**

$$r = f(v) = \boxed{\frac{3v}{168 - v}}, 60 \leq v \leq 72.$$

(b) Use numerical methods to estimate the value of the derivative

$$f'(63) = \left. \frac{dr}{dv} \right|_{v=63}.$$

Make a table that shows enough data to suggest a progression toward a limit. Include units in all column headings.

(c) Use the algebra of limits to find an expression for the derivative

$$f'(v) = \frac{dr}{dv}.$$

Then use this expression to find the exact value of the derivative in part (b).

(d) What are the units of the derivative in part (b)? Explain its meaning.

(e) Use the result of part (c) to approximate the change

$$\Delta r = f(v) - f(63)$$

in the rate at which your car burns gas in terms of the change

$$\Delta v = v - 63$$

of the car's speed. Assume  $\Delta v \sim 0$ .

**Question 12** Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that



the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and 34 miles/gallon at a speed of 30 miles/hour.

(a) Find a function

$$r = h(v), \quad 30 \leq v \leq 70,$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour).

**Hint:** (i) At what rate does the car burn gas at a speed of 50 miles/hour? 25/21 gal/mile

(ii) Find a function that expresses the gas mileage  $G$  (measured in miles/gallon) in terms of the speed (measured in miles/hr).

$$G = \boxed{42 - 0.02(v - 50)^2}, \quad 30 \leq v \leq 70.$$

(ii) The rate (in gal/hr) at which the car burns gas as a function of its speed (in miles/hr) is

$$r = h(v) = \boxed{\frac{v}{42 - 0.02(v - 50)^2}}, \quad 30 \leq v \leq 70.$$

(b) Use the algebra of limits to evaluate the derivative

$$h'(40) = \left. \frac{dr}{dv} \right|_{v=40}.$$

(c) What are the units of the above derivative? How do you know?

(d) Express the meaning of the derivative in the context of small changes.

**Question 13** At 10:00am on April 18, the wholesale price of Cosmic Crisp apples is \$2.00/lb and is decreasing at the rate of \$0.10/lb/hour.

Use the algebra of limits to determine the rate (in pounds/hour) at which the number of pounds of apples a store can purchase with \$1000 is changing at this time.

Start this question by defining a function that expresses the number of pounds of apples the store can buy with \$1000 in terms of the price (in \$/lb). Choose meaningful variable names (not  $x$  and  $y$ ). Do not assume the price is decreasing at a constant rate.

## Limits, Speed and Altitude

**Question 14** A rock dropped from a height of 100 feet falls to the surface of Planet Krypton without air resistance.

(a) By considering only the physical situation and without doing any computations, sketch a graph of the function

$$v = g(h), 0 \leq h \leq 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet). Explain your reasoning.

(b) Use the results from part (a) to choose a reasonable expression for the function  $g$  from the list below.

**Multiple Choice:**

- (a)  $g(t) = 100 - 9t^2, 0 \leq t \leq 10/3$
- (b)  $g(h) = 100 - 9h^2, 0 \leq h \leq 100$
- (c)  $g(h) = 0.005(100 - h)^2, 0 \leq h \leq 100$
- (d)  $g(h) = 6\sqrt{100 - h}, 0 \leq h \leq 100$  ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 feet gives an approximately proportional change in its speed. Then approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find an expression for the derivative

$$g'(h) = \frac{dv}{dh}.$$

Then use this expression to find the exact value of the proportionality constant in part (c).

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

- (g) Use part (g) to approximate the rock's speed at a height of 63 feet.
  - (h) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.
  - (i) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?
-

# Derivatives of Polynomials

*Working with polynomials and their derivatives.*

## Differentiating Polynomials

**Question 15** (a) Use the algebra of limits to find an expression for the derivative

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^5)$$

of the function  $f(x) = x^5$ .

(b) Use the result of part (a) to find an expression for the derivative

$$\frac{d}{dx}(f^{-1}(x)) = \frac{d}{dx}(x^{1/5})$$

of the function  $f^{-1}(x) = x^{1/5}$ . And use set-builder notation to state the domains of  $f^{-1}(x)$  and its derivative.

(c) Use the result of part (a) and the algebra of limits to find an expression for the derivative

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{d}{dx}(x^{-5})$$

of the function  $1/f(x) = x^{-5}$ .

(d) Use the results of parts (a)-(c) to find equations of the tangent lines to the three curves  $y = f(x)$ ,  $y = f^{-1}(x)$ , and  $y = 1/f(x)$  at the point  $(1, 1)$ . Graph the curves and their tangent line in Desmos to check your work.

**Question 16** (a) Use the results of Question 1 to make a conjecture about the derivative

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^n)$$

of the function  $f(x) = x^n$ .

(b) What do you get for the derivative when  $n = 0$ ? When  $n = 1$ ? Are these results correct? Explain.

Learning outcomes:  
Author(s):

**Question 17** (a) Suppose one giraffe is always twice as tall as another. What can you say about their growth rates at any instant?

(b) Suppose one giraffe is always two feet taller than another. What can you say about their growth rates at any instant?

(c) What do parts (a) and (b) suggest about how to compute the derivatives

$$\frac{d}{dx}(f(x) + b),$$

$$\frac{d}{dx}(af(x)),$$

and

$$\frac{d}{dx}(af(x) + b)$$

for constants  $a, b \in \mathbb{R}$ ?

(d) Make up your own scenario that suggests how to compute the derivative

$$\frac{d}{dx}(f(x) + g(x))$$

of the sum of two functions.

**Question 18** Find an equation of the tangent line to the curve

$$y = -x^3 + 4x^2 - 3x + 1$$

at the point  $(2, 3)$ . Graph the curve and its tangent line on Desmos.

## Thinking about Parabolas

**Question 19** (a) Find an equation of the tangent line to the parabola  $y = x^2$  at the point  $(-3, 9)$ .

(b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).

(c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.

(d) Let  $\mathcal{L}$  be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where  $\mathcal{L}$  intersects the  $y$ -axis.

(e) Repeat parts (a)-(d) above for the tangent line to the parabola  $y = x^2$  at the general point  $(b, b^2)$ . What do you notice? Enter your work in the Desmos activity below.

Access Desmos interactives through the online version of this text at

.

*Desmos link:* <https://www.desmos.com/calculator/qe7mgnu5sv>

**Question 20** Let

$$y = f(x) = ax^2$$

where  $a \in \mathbb{R}$  is a constant, and the variables  $x, y$  are measured in meters.

- (a) What are the units of the constant  $a$ ? How do you know?
- (b) Answer part (e) of the previous question for this function. Modify the desmos activity in the previous question to check your work.

**Question 21** In the absense of air resistance, a rock released from rest near the surface of the earth falls  $s = 16t^2$  feet during the first  $t$  seconds of its fall.

Compare the speed of the rock when it hits the ground with the average speed of the rock during the entire time interval of its fall.

**Question 22** (a) The demonstration below shows two normal lines to a parabola and their point of intersection  $P$ . What do you think happens to  $P$  as point  $B$  approaches  $A$ ? Answer this question without dragging the slider  $b$ .

- (b) Now drag the slider  $b$  near  $a = 2$  and observe what happens to point  $P$ . Were you correct?

Access Desmos interactives through the online version of this text at

.

*Desmos link:* <https://www.desmos.com/calculator/ybaivhc2t1>

Access this activity online at [151: Normals to Parabola](#)

The parabola has equation  $y = x^2/4$ , point  $A$  has coordinates  $(2, 1)$ , and point  $B$  has coordinates  $(b, b^2/4)$ .

- (c) Find an equation of the normal lines to the parabola at  $A$  and  $B$ .
- (d) Use algebra to find an equation of the point  $P$  where the normal lines intersect.

- (e) The point  $P$  approaches some point  $Q$  as  $B$  approaches  $A$ . Use the algebra of limits to find the coordinates of  $Q$ .
- (f) Find an equation of the circle centered at  $Q$  through  $A$ . Do this by first using vector algebra to find the coordinates of the center of the circle.
- (g) Repeat parts (c)-(g), replacing the point  $A(2,1)$  with the point  $A(a, a^2)$ .

## Applications

**Question 23** The function

$$h = f(t) = 5 - 2t - \frac{t^2}{4} + t^3, \quad -1 \leq t \leq 1.8,$$

expresses the height (in thousands of feet) of a balloon in terms of the number of hours past noon.

- (a) Find the balloon's average rate of ascent between 11:00am and 11:30am.
- (b) Is the balloon rising or falling at 1:00pm? At what rate? Use the graph of the function  $h = f(t)$  below to approximate the rate. Then compute the exact rate.
- (c) When is the balloon descending at the rate of 1000 ft/hour? Use the sliders  $m$  and  $b$  below to approximate the time(s). Then compute the exact time(s).
- (d) Use the graph below to approximate when the balloon is descending at the fastest rate. Approximate this rate from the graph. Then compute the exact time and rate.
- (e) Use the graph below to approximate when the balloon is at its lowest point. Then compute the exact time.
- (f) Use the graph below to approximate when the balloon is at its highest point between 11am and 1:36pm. Then compute the exact time.
- (g) Use algebra to find all half-hour time intervals during which the balloon descends at an average rate of 500 ft/hour.

**Desmos link:** <https://www.desmos.com/calculator/bvtukd0v1c>

Access this activity online at 151: [Height of Balloon](#)

**Question 24** The function

$$G = f(s) = \frac{11}{5} + \frac{1}{5000} (s^3 - 50s^2 + 300s), \quad 3 \leq s \leq 28,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

**Desmos link:** <https://www.desmos.com/calculator/cphmgnr7m7>

Desmos activity available at 151: Gas as a Function of Distance 2c

- (a) Use the graph of the function  $f$  shown above to determine if you are driving toward or away from home. Explain your reasoning.
- (b) Zoom in on the graph to approximate your gas mileage at the moment you are 20 miles from home. Show a screenshot to help explain how you got your approximation. Then compute the exact gas mileage.
- (c) Use the sliders  $m$  and  $b_1$  in the graph to approximate your distance from home at the moment your car gets 30 miles/gallon. Show a screenshot to help explain how you got your approximation. Then compute the exact distances.
- (d) Use the sliders  $m$  and  $b_1$  in the graph to approximate an interval beginning or ending when you are 20 miles from home over which your average gas mileage is equal to your gas mileage at the moment you are 20 miles from home. Show a screenshot to help explain how you got your approximation. Then compute the exact interval.
- (e) Sensors on your car measure both the (instantaneous) gas mileage and the number of gallons of gas in your tank at each instant. A computer then uses these measurements to estimate the number of additional miles you can drive before running out of gas. Use this idea to find a function

$$m = g(s), 3 \leq s \leq 28,$$

that expresses the number of miles you can drive before running out of gas (assuming your gas mileage remains constant for the remainder of your trip) in terms of your distance from home. Explain your reasoning.

**Question 25** (a) Make up your own quadratic function

$$v = f(G) = aG^2 + bG + c,$$

with  $a, b$ , and  $c$  all not equal to zero, that expresses the speed of a car (measured in miles/hour) in terms of its gas mileage (measured in miles/gallon). Be sure to include a domain. Explain why you think your function is reasonable.

- (b) Compute the derivative  $dv/dG$  and evaluate it at a specific gas mileage. Include units.
- (c) Evaluate the derivative  $dv/dG$  at a specific gas mileage and its meaning. Include units in your explanation.



(d) Use your function from part (a) to find a function

$$r = h(G)$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its gas mileage (in miles/gal). Explain your logic thoroughly.

(e) Evaluate  $h(G)$  at the same gas mileage, say  $G_0$ , you used in part (c). Compare the units of  $h(G_0)$  and the derivative  $dv/dG \Big|_{G=G_0}$ . Are these two numbers related? What does this tell you about simplifying the units of a derivative?

(f) Use the ideas of this chapter (ie. derivatives of polynomials, and nothing beyond) to find an expression for the derivative  $dr/dG$ .

(g) What are the units of the derivative  $dr/dG$ ?

(h) Evaluate the derivative  $dr/dG$  at a specific gas mileage and explain its meaning. Include units in your explanation.

(i) Make up and answer your own question about the derivative  $dr/dG$  at a specific gas mileage.

## A Few More Problems

**Question 26** The function

$$h = f(t) = kt^3, t \geq 0,$$

expresses the height of a balloon (in thousands of feet) in terms of the number of hours past noon. Here  $k > 0$  is a positive constant.

(a) What are the units of the constant  $k$ ? Explain how you know.

(b) Find the balloon's rate of ascent (measured in thousands of ft/hr) at time  $t = 3$  hours past noon. Then find its average rate of ascent (measured in thousands of ft/hr) between noon and 3pm. How are these rates related? Note that both will be expressed in terms of  $k$ .

(c) Find a function

$$r = g(a)$$

that expresses the balloon's rate of ascent (measured in thousands of ft/hr) at time  $t = u$  hours past noon in terms of its average rate of ascent (measured in thousands of ft/hr) over the time interval  $t \in [0, u]$ . Assume  $u > 0$ .

(d) Interpret your result from part (b) geometrically on the graph of the function  $f$ . Follow the directions on Lines 4, 6, and 8 in the demonstration below to help explain your interpretation.

(e) Which of the following expresses the balloon's rate of ascent at time  $t$  hours past noon in terms of  $t$  and  $h = f(t)$ ?

**Multiple Choice:**

- (a)  $\frac{h}{t}$
- (b)  $\frac{h}{3t}$
- (c)  $\frac{3h}{t}$  ✓

**Desmos link:** <https://www.desmos.com/calculator/0byjcy77yw>

Desmos activity available at 151: Subtangent 1

**Question 27** The function

$$v = f(G)$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon) for speeds between 55 miles/hour and 70 miles/hour.

Suppose  $f(30) = 60$ .

(a) Which of the following is more likely to be true?

$$\left. \frac{dv}{dG} \right|_{G=30} = 4$$

or

$$\left. \frac{dv}{dG} \right|_{G=30} = -4?$$

Explain your reasoning.

(b) What are the units of the above derivative? Do not simplify the units and do not write “per” in place of “/”.

(c) Explain the meaning of the derivative in part (a). It is not enough to say “the rate of change of something with respect to something else.” Remember this class is all about small changes and your explanation should be about an approximate relationship between small changes in this setting.

(d) Simplify the units of the derivative. What does this suggest about its meaning?

(e) At what rate does the car burn gas (in gal/hour) at a speed of 60 miles/hour?

(f) What does this problem suggest about simplifying the units of a derivative?

**Question 28** The function

$$v = f(G), 20 \leq G \leq 40,$$

expresses the speed of a car (in miles/hour) in terms of its gas mileage (in miles/gallon). Use the graph of the function  $f$  to find approximate answers to the following questions. Change the position and slope of line  $AB$  by dragging either the line or the points  $A$  or  $B$ . Change the position of the tangent line by dragging the slider  $G$ .

- (a) Label the axes with the appropriate variable names and units.
- (b) At what speed does the car burn gas as the fastest rate?

**Hint:** One of questions (b), (c) is related to the tangent lines to the curve, the other is related to the lines through the origin and the points of the curve.

- (c) At what speed does increasing the speed by 0.1 miles/hour result in the greatest change in the gas mileage? Approximate that change.

**Geogebra link:** <https://tube.geogebra.org/m/vjdf6x6z>

Geogebra activity available at 151: Gas Mileage

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# Derivatives of Exponential Functions

*Working with exponential functions and their derivatives.*

## Relative Changes and Relative Rates of Change

Relative changes and relative errors are often more meaningful than absolute changes and errors. For example, I might measure the distance from Shoreline's Central Market to the Richmond beach library to be 5 km with an error of at most 0.2 km, while NASA might measure the between the earth and the moon on the first day of spring to be 384,400 km with an error of at most 100 km. The relative error in my measurement is at most

$$\frac{0.2 \text{ km}}{5 \text{ km}} = 0.04 = 4\%,$$

while the relative error in NASA's measurement is at most

$$\frac{100 \text{ km}}{384,400 \text{ km}} \sim 0.00026 = 0.026\%.$$

Relatively speaking, NASA's measurement was about 150 times more accurate than mine.

**Question 29** At 10:00am the prices of Stock A and Stock B are both increasing at the rate of (\$2/share)/hour. At 10:00am Stock A sells for \$50/share and Stock B for \$10/share. Compare the relative rates at which the share prices are changing at 10:00am.

**Question 30** The function

$$P = f(t) = 5 - 3t + t^2, 0 \leq t \leq 4,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am.

- At what relative rate is the price of the stock changing at 10am?
- When is the share price increasing at a rate of 60%/hr?
- During what time interval is the price of the stock increasing?

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Learning outcomes:  
Author(s):

(d) During what time interval is the relative rate of change in the price of the stock increasing?

*Desmos link:* <https://www.desmos.com/calculator/hhkveu6lxp>

Desmos activity available at 151: Stock Price

**Question 31** The function

$$P = f(t), 1 \leq t \leq 3.6,$$

expresses the price in \$/share of a stock in terms of the number of hours past 9am. Use the graph below to approximate the answers to the following questions without putting a scale on the  $P$  axis.

- (a) At what relative rate is the price changing at 11am? At 12:30pm?
- (b) When is the stock price increasing at its maximum relative rate? At its minimum relative rate? Approximate these rates.

*Desmos link:* <https://www.desmos.com/calculator/jyebaj5jif>

Desmos activity available at 151: Stock Price 2

**Question 32** The function

$$W = f(t) = 200 + 4t + 2t^2, 0 \leq t \leq 12,$$

expresses the weight (in pounds) of a baby elephant in terms of its age (in months).

- (a) Find the average rate at which the elephant gained weight between ages 4 and 10 months.
- (b) Find the relative average rate at which the elephant gained weight between ages 4 and 10 months.
- (c) Find the relative instantaneous rate at which the elephant is gaining weight at age 4 month.
- (d) Find the relative instantaneous rate at which the elephant is gaining weight at age 10 months.
- (e) Use the graph below to interpret your answers to parts (b)-(d) geometrically.

*Desmos link:* <https://www.desmos.com/calculator/2xj6xy7ggo>

Desmos activity available at 151: Elephant

## Exponential Growth

**Question 33** (a) What does it mean for a population to grow exponentially?  
 (b) Is it possible for a population to increase by 20% every year and not grow exponentially?

**Question 34** Suppose between noon and 10pm a colony of bacteria grows exponentially. The population is 200,000 at noon and 242,200 at 1pm.

- (a) Describe how the population grows. Keeping Question 2(b) in mind, is your description sufficient?
- (b) How might we find a complete description of the exponential growth?
- (c) Determine the relative average growth rate between noon and 12:30pm. Between 1pm and 1:30pm. Over any half-hour time period. Use the slider  $u$  in the graph below to interpret these rates geometrically.
- (d) Approximate the instantaneous relative growth rates in the population at noon, at 1pm, and at 2pm. Modify the definition of  $v = u + 1/2$  in the demonstration below and interpret these rates geometrically.
- (e) Use limits to write an expression that gives the instantaneous growth rate at time  $t$  hours past noon. What can you conclude?
- (f) Try to answer part (b) again.

**Desmos link:** <https://www.desmos.com/calculator/wvpsotdhby>

Desmos activity available at 151: Exp Growth 1

**Question 35** The function

$$P = f(t) = 3(2)^t, -2 \leq t \leq 4,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

- (a) Describe how the population grows. Is your description sufficient?

The population doubles every hour.

- (b) Find an expression for the relative growth rate between time  $t = u$  hours past noon and time  $t = u + h$  hours past noon. Measure the rate relative to the population at time  $t = u$ . Is this question asking about an average or an instantaneous relative growth rate?

## Derivatives of Exponential Functions

The relative growth rate is

$$\begin{aligned}
 \frac{1}{P} \left( \frac{\Delta P}{\Delta t} \right) &= \frac{1}{f(u)} \left( \frac{f(u+h) - f(u)}{h} \right) \\
 &= \frac{1}{3(2)^u} \left( \frac{3(2)^{u+h} - 3(2)^u}{h} \right) \\
 &= \frac{1}{3(2)^u} \left( \frac{3(2)^u (2^h - 1)}{h} \right) \\
 &= \frac{2^h - 1}{h}
 \end{aligned}$$

- (c) What are the units of the relative average growth rate in part (b)?
- (d) Input your function from part (b) on Line 5 in the worksheet below.
- (e) What do you notice about the distance between points  $R$  and  $S$  as you drag the slider  $u$  below. How is this distance related to the relative average growth rate in part (b)?
- (f) Use limits to write an expression for the relative growth rate at time  $t = u$  hours past noon. Simplify this expression as much as possible. What can you conclude about how the relative instantaneous growth rate varies with time?

The relative growth rate is

$$\lim_{h \rightarrow 0} \frac{2^h - 1}{h}.$$

- (g) Interpret your expression from part (f) as the derivative of a specific function evaluated at a specific input. What does this tell you about the relative growth rate of this particular population?

The relative instantaneous growth rate of the population is equal to the derivative of the function

$$f(x) = 2^x$$

evaluated at  $x = 0$ .

- (h) Use part (f) to numerically approximate the relative (instantaneous) growth rate of the population. Show a table that suggests a progression toward a limit.

## Derivatives of Exponential Functions

(i) Use a similar method to approximate the relative instantaneous growth rate of the population

$$P = f(t) = 5(3)^t.$$

Desmos link: <https://www.desmos.com/calculator/omjbec2hpu>

Desmos activity available at 151: Exponential Growth 1

**Question 36** Parts (h) and (i) of the previous question suggest that there is a number  $e$  between 2 and 3 that makes the relative growth rate of the function

$$P = f(t) = P_0 e^t, -3 \leq t \leq 5$$

equal to 100%/hr, where we assume here that  $t$  is measured in hours.

- (a) What is the one-hour growth factor for this population?
- (b) Describe what happens to the population every hour.
- (c) At what relative rate is the population increasing at 1:00pm?
- (d) Suppose at 1:00pm the population is 500,000. Approximate the population at 1:03pm and compare your approximation to the actual population at that time.

## Exponential Functions with Bases other than $e$

**Question 37** The function

$$P = g(t) = P_0 e^{t/2}, -6 \leq t \leq 10$$

expresses the population (Colony B) of bacteria in terms of the number of hours past noon.

- (a) Describe a transformation that takes the graph of the population function

$$P = f(t) = P_0 e^t, -3 \leq t \leq 5$$

for Colony A (where  $t$  is also the number of hours past noon) to the graph of  $P = g(t)$ .

- (b) Suppose that the population of Colony A is 400,000 at 4:00pm.
  - (i) When is the population of Colony B equal to 400,000?
  - (ii) What are the growth rates of the two populations when they each have respective populations of 400,000 bacteria?



(iii) What are the relative growth rates of the two populations when they each have respective populations of 400,000 bacteria?

**Question 38** Here's another way to think about differentiating the function

$$P = g(t) = P_0 e^{t/2}, -6 \leq t \leq 10$$

that expresses the population of a colony of bacteria in terms of the number of hours past noon.

We'll let  $u = t/2$  be the number of two-hour periods since noon.

- (a) Express the population in terms of  $u$ .
- (b) Use what you know about the exponential function base  $e$  to express the growth rate of the population in terms of  $u$ .
- (c) Use part (b) to find the growth rate of the population at 6pm. Pay careful attention to units.
- (d) Use the idea of part (c) to express the growth rate

$$\frac{dP}{dt} = g'(t)$$

in terms of  $t$ .

- (e) Suppose instead that the population grows exponentially and doubles every hour. Find the relative instantaneous growth rate of the population.

**Question 39** Between 11am and 8pm, a population of bacteria grows exponentially. The population is 4 million at noon and 5 million at 1pm.

- (a) What is the one-hour growth factor? 1.25
- (b) Describe how the population grows.

The population increases by 25% every hour.

- (c) Use your description from part (b) to find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon. Do not use  $e$  in your function. Define meaningful variables and include a domain.

The function

$$P = f(t) = \boxed{4} \left( \boxed{5/4} \right)^t, \boxed{-1} \leq t \leq \boxed{8}$$

expresses the population (in millions of bacteria) in terms of the number of hours past noon.

# Derivatives of Exponential Functions

(d) Use the fact that  $k = e^{\ln k}$  for  $k > 0$  to express your function from part (c) using an exponential function with base  $e$ .

$$P = f(t) = \boxed{4} e^{\boxed{t \ln(1.25)}}, \boxed{-1} \leq t \leq \boxed{8}$$

(e) Use  $u$ -substitution and the chain rule (show all steps) along with the fact that

$$\frac{d}{du}(e^u) = e^u$$

to find the relative instantaneous growth rate of the population. Include units in your conclusion.

The relative instantaneous growth rate of the population is

$$\frac{1}{P} \left( \frac{dP}{dt} \right) = \boxed{\ln(5/4)} \sim \boxed{22.314} \% / \boxed{\text{hour}}.$$

(f) At what rate is the population growing when there are 10 million bacteria? Do this the easy way. Note that the question is not asking for a relative rate.

(g) Find the population when the population is increasing at the rate of 3 million bacteria/hour. Do this the easy way.

**Question 40** Between 11am and 8pm, a population of bacteria decreases exponentially. The population is 5 million at noon and 4 million at 4pm.

(a) What is the four-hour growth factor? (This is the number you multiply the current population by to get the population four hours later).  $\boxed{0.8}$

(b) What is the one-hour growth factor?  $\boxed{(4/5)^{1/4}}$

(c) Describe how the population decreases.

The population decreases by  $\boxed{20} \%$  every  $\boxed{4}$  hours.

(d) Use your description from part (c) to find a function that expresses the population (in millions of bacteria) in terms of the number of hours past noon. Do not use  $e$  in your function. Start by define meaningful variables. Include a domain with your function.

The function

$$P = f(t) = \boxed{5} \left( \boxed{4/5} \right)^{t/4}, \boxed{-1} \leq t \leq \boxed{8}$$

expresses the population (in millions of bacteria) in terms of the number of hours past noon.

(e) Express your function from part (c) using an exponential function with base  $e$ . Then use  $u$ -substitution and the chain rule to find the relative instantaneous growth rate of the population. Show all steps in using the chain rule.

The population function is

$$P = f(t) = \boxed{5} e^{\boxed{0.25 \ln(0.8)t}}, \boxed{-1} \leq t \leq \boxed{8}.$$

The relative instantaneous growth rate is

$$\frac{1}{P} \left( \frac{dP}{dt} \right) = \boxed{0.25 \ln(0.8)} \sim \boxed{-5.58} \% / \boxed{\text{hour}}.$$

(f) At what rate is the population decreasing when there are 2 million bacteria? Do this the easy way.

(g) Find the population when the population is decreasing at the rate of 300,000 bacteria/hour. Do this the easy way.

(h) Use the result of part (f) to approximate the population 4 minutes after there are 2 million bacteria.

## Relative Rates Again

**Question 41** The function

$$P = f(t), 0 \leq t \leq 2,$$

expresses the balance (in dollars) in an account in terms of the number of years since the start of 2022. Suppose

$$\left. \frac{1}{P} \frac{dP}{dt} \right|_{P=5000} = 0.08.$$

- (a) What are the units of the above derivative? How do you know?
- (b) Interpret the meaning of the above derivative.
- (c) Approximate the balance in the account four days after the account has \$5,000. Explain your reasoning.

**Question 42** The function

$$q = f(p) = 0.5(p - 18)^2, 6 \leq p \leq 15,$$

expresses the average number of burgers/day sold at Five Guys of Edmonds in terms of the price (in \$/burger).

- (a) At what relative rate does the quantity sold ( $q$ ) change with respect to the price ( $p$ ) at a price of \$10/burger?
- (b) What are the units of the above relative rate of change?
- (c) Explain the meaning of the relative rate of change in part (a).
- (d) Use the graph of the function  $q = f(p)$  and the slider  $u$  in the desmos activity below to interpret the relative rate of change in part (a) geometrically. Explain your reasoning.
- (e) Use the result of part (a) to approximate the relative change in the average number of burgers sold per day if the Five Guys increases the price from \$10/burger to \$10.25/burger. Explain your reasoning.
- (f) Use the result of part (a) to approximate the relative change  $\Delta a\%$  in the average number of burgers sold per day in terms of a small relative change  $\Delta r\%$  in the price from \$10/burger. Explain your reasoning.

Desmos link: <https://www.desmos.com/calculator/ylgk03oaza>

Desmos activity available at 151: Burgers 1

**Question 43** The function

$$P = f(t), -2 \leq t \leq 5,$$

expresses the population of a colony (call it Colony A) of bacteria in terms of the number of hours past noon.

The function

$$P = g(t) = f(t/2), -4 \leq t \leq 10,$$

expresses the population of Colony B in terms of the number of hours past noon.

The populations do not necessarily grow exponentially.

- (a) Compare the populations at noon.
- (b) Suppose Colony A has 50,000 bacteria at 3:00pm. When does Colony B have 50,000 bacteria? Explain.
- (c) Suppose the population of Colony A takes three hours to grow from 20,000 to 50,000. How long does it take the population of Colony B to grow from 20,000 to 50,000?
- (d) Suppose the population of Colony A is increasing at the rate of 10,000 bac/hr at 3pm. What is the growth rate of Colony B at 6pm? Explain.
- (e) What is the relative growth rate of Colony A at 3pm? What is the relative growth rate of Colony B at 6pm? Explain.

**Question 44** The function

$$P = f(u) = 10e^u, -2 \leq u \leq 5$$

expresses the population (in millions of bacteria) of a colony of bacteria in terms of the number of hours since noon.

- (a) What are the units of the input to the exponential function in the above expression for  $P$ ?
- (b) At what rate is the population growing when there are 30 million bacteria?
- (c) At what rate is the population growing when there are  $P$  million bacteria? Do the units of your answer make sense?
- (d) What is the relative instantaneous growth rate of the population?
- (e) Find a function

$$P = g(t)$$

that expresses the population (in millions of bacteria) of the colony of bacteria in terms of the number of minutes since noon. Include a domain.

- (f) Use common sense to evaluate the derivative

$$\left. \frac{dP}{dt} \right|_{P=30}.$$

Explain your reasoning.

- (g) Use the idea of part (f) to find an expression for the derivative

$$\frac{dP}{dt} = g'(t)$$

at time  $t$  minutes past noon.

**Question 45** The function

$$P = P_0 e^{kt}, -4 \leq t \leq 5,$$

expresses the population of a colony of bacteria in terms of the number of hours past noon.

- (a) What are the units of the constant  $P_0$ ?
- (b) What are the units of the constant  $k$ ?
- (c) Use the ideas of the previous question to find an expression for the growth rate of the population at time  $t$  hours past noon. Include units in your answer.
- (d) Find an expression for the relative growth rate at time  $t$  hours past noon. Include units in your answer.

**Question 46** One of the two functions graphed below is an exponential function. Which one? How do you know?

## The Derivative as a Magnification Factor

**Exploration 47** It sometimes helps to think of the derivative as a magnification factor that maps a small interval around an input to a function to a corresponding interval around the output.

(a) Use this idea for the function  $y = f(x)$  graphed below to approximate the derivatives

$$\left. \frac{d}{dx}(f(x)) \right|_{x=4} \quad \text{and} \quad \left. \frac{dy}{dx} \right|_{x=5}.$$

*Desmos link:* <https://www.desmos.com/calculator/la4f5ots3r>

Desmos activity available at 151: Magnification Factor 1

## Differentiating the Exponential Function $e^x$

**Exploration 48** (a) Use the graph of the function  $y = f(x)$  below to approximate the derivatives

$$\left. \frac{dy}{dx} \right|_{y=k} \quad \text{for } k = 1, 2, \dots, 6.$$

Note the above derivatives are evaluated at the outputs of the function  $f$ .

(b) What do you notice?

*Desmos link:* <https://www.desmos.com/calculator/k08dphtuca>

Desmos activity available at 151: Magnification Factor 2

**Question 49** The function

$$P = 400e^t, 0 \leq t \leq 2$$

expresses the population (in thousands) of a colony of bacteria in terms of the number of hours past noon.

- (a) What are the units of the factor 400?
  - (b) What are the units of the exponent in the factor  $e^t$ ? Be careful.
  - (c) Find the (instantaneous) growth rate of the population when there are 1,200,000 bacteria.
  - (d) Find the relative (instantaneous) growth rate when there are 1,200,000 bacteria.
  - (e) Find the relative (instantaneous) growth rate at any time.
  - (f) Approximate the population 30 seconds after there are 1,200,000 bacteria.
- 

**Question 50** Let  $k > 0$  be a constant and let

$$f(x) = ke^x.$$

For  $a \in \mathbb{R}$  let point  $P$  with coordinates  $(a, f(a))$  be a curve on the curve  $y = f(x)$ . Let  $Q$  be the point where the tangent line to the curve intersects the  $x$ -axis and let  $R$  be the point with coordinates  $(a, 0)$

- (a) Find the length of segment  $\overline{QR}$ .
  - (b) How is part (a) related to part (e) of the previous question?
-

# Derivatives of Trigonometric Functions

*Working with trigonometric functions and their derivatives.*

## Visualizing Derivatives

**Question 51** You ride a ferris wheel for one revolution and get off. Let

$$h = f(s), 0 \leq s \leq ??,$$

be the function that expresses your height above the ground (measured in feet) in terms of your distance traveled, measured (in feet) along your path from your starting point.

(a) Choose a radius for the ferris wheel that you think is reasonable and fill in the missing upper bound for the domain of  $f$  above.

(b) Use the demonstration below to sketch by hand a graph of the function

$$r = \frac{dh}{ds} = f'(s).$$

Do not make any computations. Just use the demonstration below. Here are the key points to keep in mind:

- Approximate the derivative  $dh/ds$  by the ratio  $\Delta h/\Delta s$ .
- The length of the red arclength (when the ferris wheel is on the way up) is the input  $s$ .
- The length of the purple segment is the output  $h$ .
- The lengths of the orange arc and orange segments are  $\Delta s$ .
- The (signed) length of the green segment is  $\Delta h$ .

Explain your reasoning thoroughly. Be sure to include at least the following points:

- The units of the input and output to the derivative
- Scales on the vertical and horizontal axes

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Learning outcomes:  
Author(s):



- A discussion of how a small change in the input to the function  $f$  changes the output at various positions along your ride.
- A discussion of where a small change in the input to  $f$  gives the greatest positive change in the output and a consideration of the ratios of these changes
- A discussion of where a small change in the input to  $f$  gives the negative change in the output with the greatest magnitude and a consideration of the ratios of these changes
- A discussion of where a small change in the input to  $f$  barely changes the output and a consideration of the ratios of these changes

(c) How would your graph of the derivative  $dh/ds$  change if you doubled the radius of the ferris wheel? Sketch the new graph.

Desmos link: <https://www.desmos.com/calculator/pxsmo04nmg>

## The Derivative of the Sine Function

**Question 52** Desmos link:

<https://www.desmos.com/calculator/jcmcyrpndw>

Desmos activity available at 151: Derivative of Sine

**Question 53** (a) Describe a transformation that takes the graph of the function

$$y = f(\theta) = \sin \theta$$

to the graph of the function

$$y = g(\theta) = \cos \theta.$$

(b) Does that same transformation take the graph of the derivative of the sine function to the graph of the derivative of the cosine function? Do not use any particular facts about these functions to answer this question. Instead, give a general answer that would apply to all pairs of functions similarly related.

(c) Use your answer to part (b) and the fact that

$$\frac{d}{d\theta} (\sin \theta) = \cos \theta$$

to find an expression for the derivative

$$\frac{d}{d\theta}(\cos \theta)$$

of the cosine function.

## Applications 1

**Question 54** The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.

(a) Find a function

$$h = f(\theta), 0 \leq \theta \leq 2\pi,$$

that expresses your height above the ground in terms of the rotation angle of the wheel, measured in radians. Use the cosine function, not the sine function.

(b) The wheel stops when you are 100 feet above the ground and on the way up. It then starts again and turns through a small angle of  $\Delta\theta$  radians before stopping again. Use the appropriate linear approximation to estimate the change  $\Delta h$  in your height (measured in feet) as the wheel turned through the angle  $\theta$ .

Geogebra link: <https://tube.geogebra.org/m/tn75cq93>

## Transformations of the Sine Function

**Question 55** (a) Describe a transformation that takes the graph of the function

$$y = f(\theta) = \sin \theta$$

to the graph of the function

$$y = g(\theta) = \sin(2\theta).$$

(b) How does that same transformation affect the slope of a line?

(c) Use your answer to part (b) to find an expression for the derivative

$$g'(\theta) = \frac{d}{d\theta}(\sin(2\theta)).$$

## Applications 2

**Question 56** The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You travel at a constant speed of 5 ft/sec as you ride the ferris wheel.

(a) Find a function

$$h = f(t), t \geq 0$$

that expresses your height above the ground in terms of the number of seconds since you got on. Use the cosine function, not the sine function.

(b) Are you ascending or descending the second time you are 90 feet above the ground? At what rate? Use the methods of this class, not vectors, to answer this question.

(c) Find your height when you are descending at the rate of 4.8 feet/sec. Give all possibilities. Do not use a calculator except to do arithmetic.

**Question 57** Nothing yet.

# The Chain Rule

*An introduction to the chain rule.*

The chain rule tells us how to differentiate the composition of two functions. It says that the derivative of the composition is the product of the derivatives of the two functions.

Here is a more formal statement.

**Theorem 1.** *(The Chain Rule) If  $y = f(u)$  is a differentiable function of  $u$  and  $u = g(t)$  is a differentiable function of  $t$ , then the composition*

$$y = f(g(t))$$

*is a differentiable function of  $t$  and*

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}.$$

*Or paying closer attention to the inputs,*

$$\left. \frac{dy}{dt} \right|_{t=t_0} = \left. \frac{dy}{du} \right|_{u=g(t_0)} \cdot \left. \frac{du}{dt} \right|_{t=t_0}.$$

We'll go through some examples to get an understanding of how to use the chain rule and also why it works.

## Examples

**Example 10.** *Find an equation of the tangent line to the curve*

$$y = f(x) = (2x^3 + 1)^2$$

*at the point  $(1, 9)$ .*

**Explanation.** *Let*

$$y = (2x^3 + 1)^2$$

*and*

$$u = 2x^3 + 1.$$

*Then*

$$y = u^2$$

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Learning outcomes:  
Author(s):

and

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
 &= \frac{d}{du}(u^2) \cdot \frac{d}{dx}(2x^3 + 1) \\
 &= 2u(6x^2) \\
 &= 2(2x^3 + 1)(6x^2).
 \end{aligned}$$

Then the slope of the tangent line to the curve  $y = (2x^3 + 1)^2$  at the point  $(1, 9)$  is

$$\left. \frac{dy}{dx} \right|_{x=1} = 2(3)(6) = 36,$$

and an equation of the tangent line is

$$y - 9 = 36(x - 1).$$

We can get the same result without the chain rule, by rewriting the original function as

$$y = f(x) = (2x^3 + 1)^2 = 4x^6 + 4x^3 + 1.$$

Then

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(4x^6 + 4x^3 + 1) \\
 &= 4 \frac{d}{dx}(x^6) + 4 \frac{d}{dx}(x^3) + \frac{d}{dx}(1) \\
 &= 24x^5 + 12x^2.
 \end{aligned}$$

Then the slope of the tangent line to the curve  $y = (2x^3 + 1)^2$  at the point  $(1, 9)$  is

$$\left. \frac{dy}{dx} \right|_{x=1} = 14 + 12 = 36,$$

as before.

**Example 11.** Find an equation of the tangent line to the curve

$$x^2 + y^2 = 25$$

at the point  $P(4, -3)$ .

**Explanation.** Solve the above equation for  $y$  in terms of  $x$  to get

$$y = \pm \sqrt{25 - x^2}.$$

While this equation does not define  $y$  as a function of  $x$ , near the point  $(4, -3)$ , the equation

$$y = -\sqrt{25 - x^2}$$

does define  $y$  as a function of  $x$ .

Now to find the derivative of the function

$$y = -\sqrt{25 - x^2},$$

let

$$u = 25 - x^2.$$

Then

$$y = \sqrt{u} = -u^{1/2}$$

and

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du} (-u^{1/2}) \cdot \frac{d}{dx} (25 - x^2) \\ &= -\frac{1}{2\sqrt{u}} (-2x) \\ &= \frac{x}{\sqrt{25 - x^2}}. \end{aligned}$$

So the slope of the tangent line to the curve

$$x^2 + y^2 = 25$$

at the point  $(P(4, -3))$  is

$$\left. \frac{dy}{dx} \right|_{x=4} = \left( \frac{x}{\sqrt{25 - x^2}} \right) \Big|_{x=4} = \frac{4}{3}$$

and an equation of the tangent line is

$$y + 3 = \frac{4}{3}(x - 4).$$

To find the slope of the tangent line without appealing to the chain rule, note that the curve

$$x^2 + y^2 = 25$$

is a circle centered at the origin. So the tangent line at  $P(4, -3)$  is perpendicular to the radius  $\overline{OP}$  from the origin to  $P$ . Since  $\overline{OP}$  has slope  $m_1 = -3/4$ , the tangent line at  $P(4, -3)$  has slope

$$m_2 = -1/m_1 = 4/3.$$

**Question 58** (a) Describe a transformation that takes the circle

$$x^2 + y^2 = 25$$

to the ellipse

$$4x^2 + y^2 = 25. \quad (1)$$

(b) Find the image (call it  $Q$ ) of the point  $P(4, -3)$  under the transformation in part (a).

(c) Use the result of Example 2 to find an equation of the tangent line to the ellipse (1) at  $Q$ .

(d) Use the chain rule to first find the slope of the tangent line to the ellipse (1) at  $Q$ . Then find an equation of the tangent line.

**Question 59** The function

$$P = f(t) = 22 + \frac{t}{2} - \frac{3t^2}{4}, 0 \leq t \leq 3,$$

expresses the price of a stock (in dollars/share) in terms of the number of hours past 9am.

(a) Is the price of the stock increasing or decreasing at 11am? At what rate? At what relative rate?

(b) Is the number of shares you can buy with \$1000 increasing or decreasing at 11am? Approximate the rate numerically.

(c) Compute the exact rate in part (b). Compute also the relative rate of change (with respect to time) in the number of shares you can buy with \$1000 at 11am.

Start your solution to this problem by defining a new function with a new function name and a new dependent variable.

**Question 60** At 10am the price of a stock is increasing at the relative rate of  $p\%/hr$ . Is the number of shares you can buy with \$1000 increasing or decreasing at 10am? At what relative rate?

Answer this question in two ways:

(i) Going back to limits and using the definition of the derivative.

(ii) Using the chain rule.

Either way, start your solution with definitions as in the previous problem.

**Question 61** A tree leans precariously with its trunk making an angle of  $\phi = \pi/3$  radians with the ground. One end of a 14-foot ladder leans against the trunk, the other rests on the horizontal ground.

Let  $t$  be the distance between the top of the ladder and the base of the trunk (measured in feet) and  $s$  the distance between the bottom of the ladder and the base of the trunk (also measured in feet).

**Geogebra link:** <https://tube.geogebra.org/m/ctf2bcqz>

Geogebra activity available at [151: Ladder and Tree Part 2](#)

- (a) Use the law of cosines to write an equation relating  $t$  and  $s$ .
- (b) Use the result of part (a) to find the two possible values of  $t$  when  $s = 16$ .

For the remainder of this problem we'll focus on positions of the ladder when  $s \sim 16$  and  $t \sim 10$ .

- (c) For these positions, complete the square to find a function

$$t = f(s)$$

that expresses the distance (in feet) from the top of the ladder to the base of the trunk in terms of the distance (in feet) from the bottom of the ladder to the trunk's base.

- (d) Drag the slider  $\theta$  in the worksheet above to approximate the value of the derivative

$$\left. \frac{dt}{ds} \right|_{s=16}.$$

Include units and explain your reasoning.

- (e) Use your function from part (c) to find an expression for the derivative  $dt/ds$ . Show every step in computing the derivative as in Examples 1 and 2 above.

- (f) Evaluate the derivative

$$\left. \frac{dt}{ds} \right|_{s=16}$$

and compare the exact value with your estimate from part (d).

- (g) What are the units of the derivative in part (f)? Explain the meaning of the derivative, not by giving a standard response about the rate of change of some quantity with respect to another, but by relating small changes.

- (h) Use your response to part (f) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of  $s$  near  $s = 16$ . Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 15.6 feet from the base of the trunk. Compare your approximation with the exact distance.



(i) Let  $\theta = m\angle GBA$  be the radian measure of the angle the ladder makes with the trunk.

part (i) Find an expression for the derivative  $d\theta/ds$ .

part (ii) Evaluate the derivative

$$\left. \frac{d\theta}{ds} \right|_{s=16}$$

and interpret its meaning in terms of small changes. Be sure to explain the significance of the derivative's sign as well.

**Example 12.** (a) Find the slope of the tangent line to the curve

$$y = f(\theta) = 6 \sin \theta$$

at the point with coordinates  $P(2\pi/3, 3\sqrt{3})$ . We do not need the chain rule for this.

(b) Describe a transformation that takes the graph of  $y = f(\theta)$  to the graph of the function

$$y = g(\theta) = 6 \sin(\theta/2).$$

(c) Find the image of the point  $P(2\pi/3, 3\sqrt{3})$  under the above transformation.

(d) Use the results of parts (a) - (c) and the demonstration below to guess the slope of the tangent line to the curve  $y = g(\theta)$  at the point  $Q(4\pi, 3\sqrt{3})$ .

*Desmos link:* <https://www.desmos.com/calculator/mqjxpsqyo5>

Desmos activity available at 151: Chain Rule 1

(e) Use the chain rule to confirm your guess from part (d).

**Example 13.** Find an equation of the tangent line to the curve

$$y = f(t) = 4 \sin^2 \left( \frac{\pi}{6} t \right)$$

at the point  $(2, 3)$ .

**Explanation.** Let

$$y = 4 \sin^2 \left( \frac{\pi}{6} t \right) = 4 \left( \sin \left( \frac{\pi}{6} t \right) \right)^2$$

and

$$u = \sin \left( \frac{\pi}{6} t \right).$$

Then

$$y = 4u^2$$

and

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du} \cdot \frac{du}{dt} \\ &= \frac{d}{du}(4u^2) \cdot \frac{d}{dt}\left(\sin\left(\frac{\pi}{6}t\right)\right) \\ &= 8u \cos\left(\frac{\pi}{6}t\right) \frac{d}{dt}\left(\frac{\pi}{6}t\right) \\ &= 8 \sin\left(\frac{\pi}{6}t\right) \cos\left(\frac{\pi}{6}t\right) \frac{\pi}{6}.\end{aligned}$$

**Question 62** (a) The third equality above uses the chain rule again. Fill in the missing details of this computation by first letting

$$z = \sin\left(\frac{\pi}{6}y\right)$$

and making an explicit substitution using the variable  $v$  in place of  $u$ .

(b) Complete the solution by finding an equation of the tangent line to the curve at the point  $(2, 3)$ .

**Question 63** Use the chain rule to compute each of the following derivatives. Show all steps as in Examples 1, 2, and 8 above.

(a)  $\frac{d}{dx} \left( \frac{12}{1+x^2} \right)$

(b)  $\frac{d}{dt} \left( 12(3)^{t/5} \right)$

(b)  $\frac{d}{dt} \left( e^{3 \sin(4t)} \right)$

(b)  $\frac{d}{d\theta} (12 + 5 \cos(\theta/4))$

**Example 14.** The function

$$P = f(t) = 10e^{\frac{1}{4}t}, \quad -4 \leq t \leq 12,$$

expresses the population (in millions) of a colony of bacteria in terms of the number of hours past noon.

(a) What are the units of  $1/4$  in the function above? How do you know?

(b) Use the chain rule to find an expression for the growth rate of the population at time  $t$  hours past noon. What are the units of the growth rate?

- (c) Express the growth rate from part (b) in terms of the population  $P = f(t)$  at time  $t$  hours past noon.
- (d) What is the relative instantaneous growth rate of the population? Include units.
- (e) Find the relative average growth rate of the population over a one-hour period.
- (f) Describe what happens to the population every hour.

**Example 15.** (a) Use the chain rule to find the relative instantaneous growth rate of the population function

$$P = 10(2^t), \quad -2 \leq t \leq 5,$$

where  $P$  is measured in millions of bacteria and  $t$  is the number of hours past noon.

- (b) Describe what happens to the population every hour.
- (c) Estimate the population 2 minutes after there are 30 million bacteria.

**Question 64** This problem suggests a way to think about the chain rule geometrically.

You ride a ferris wheel for one revolution and get off. The function

$$h = f(\theta), \quad 0 \leq \theta \leq 2\pi,$$

expresses your height (in feet) in terms of the wheel's angle of rotation (measured in radians from the moment you boarded).

The function

$$\theta = g(t), \quad 0 \leq t \leq 44,$$

expresses the rotation angle of the wheel in terms of the number of seconds since you boarded.

Our goal is to use the graphs of the function  $f$  and  $g$  below (take the times  $t$  to be positive, not negative as shown) to approximate your rate of ascent  $t = 16$  seconds after boarding. We'll do this in two different ways.

- (a) The first way involves a few steps and uses the graphs of both  $f$  and  $g$ .
- (i) Approximate the value of the derivative

$$\left. \frac{dh}{d\theta} \right|_{\theta=g(16)}$$

and interpret its meaning. Include units.

- (ii) Approximate the value of the derivative

$$\left. \frac{d\theta}{dt} \right|_{t=16}$$

and interpret its meaning. Include units.

(iii) Use the results of (i) and (ii) and common sense (do not appeal to the chain rule directly) to approximate the value of the derivative

$$\left. \frac{dh}{dt} \right|_{t=16} = h'(16),$$

where  $h(t) = f(g(t))$ . Explain your logic and include all units in your computation. Interpret the meaning of this derivative.

(b) The second way has just one step and that's to use the graph of  $h = f(\theta)$  to approximate your rate of ascent at time  $t = 16$  using the fact that the horizontal blue lines are drawn at intervals of  $\Delta t = 2$  seconds. Be sure to explain your logic.

**Desmos link:** <https://www.desmos.com/calculator/fkgfpsowe8>

Desmos activity available at 151: Ferris Wheel 2

**Question 65** This is a continuation of the previous question.

Approximate the time between times  $t = 5$  and  $t = 22$  seconds when you are ascending at the slowest rate. Approximate this rate of ascent.

**Question 66** The center of a ferris wheel with a radius of 50 feet is 60 feet above the ground. You ride the wheel for one revolution and get off.

(a) Use the geometry of the ferris wheel (see the picture below) to find a function

$$h = f(\theta), \quad 0 \leq \theta \leq 2\pi,$$

that expresses your height (in feet) above the ground in terms of the rotation angle of the wheel, measured in radians. Use the cosine function, not the sine function.

**Geogebra link:** <https://tube.geogebra.org/m/tn75cq93>

(b) Suppose you move at the constant speed of 10 ft/sec as you ride the ferris wheel.

(i) Use the result of part (a) to find a function  $h = g(t)$  that expresses your height (in feet) in terms of the number of seconds since you boarded. Include the appropriate domain.

(ii) Find a function  $r = h(t)$  that expresses your rate of ascent (in ft/sec) in terms of the number of seconds since you boarded. Include the appropriate

domain. Use vectors to interpret this rate geometrically, in terms of the angle your velocity makes with the horizontal and your speed.

(iii) Write an equation that relates your height  $h$  (in feet) and your rate of ascent  $r$  (in ft/sec) at any instant. Graph the relation by hand.

(iv) Are you going up or down the second time you are 100 feet above the ground? At what rate?

(v) What is your height when you are descending at the rate of 4 ft/sec? Find all possibilities.

**Question 67** The function

$$\theta = f(t), t \geq 0,$$

expresses the radian measure of a ferris wheel's rotation angle in terms of the number of seconds since you boarded. The wheel has radius  $r$  feet and its center is  $b$  feet above the ground.

(a) Find a function

$$h = g(t), t \geq 0,$$

that expresses your height (in feet) in terms of the number of seconds since you boarded.

(b) Find an expression for your rate of ascent (in ft/sec) at time  $t$  seconds after you boarded. Assume  $g$  is a differentiable function of  $t$ .

(c) Interpret your rate of ascent (part (b)) in terms of your speed at time  $t$  and the angle your velocity vector makes with the horizontal. Assume here that  $\theta = f(t)$  is an increasing function of  $t$ .

## Exercises

**Exercise 68** Use the facts that

$$\frac{d}{d\theta}(\sin \theta) = \cos \theta$$

and

$$\frac{d}{d\theta}(\cos \theta) = -\sin \theta$$

to compute each of the following derivatives. Show all steps in using the chain rule as in Examples 1 and 2 above.

(a)  $\frac{d}{d\theta}(\sec \theta)$

$$(b) \frac{d}{d\theta} (\csc \theta)$$

$$(c) \frac{d}{dt} \left( \sqrt{\sin^2 t + (4 - \cos t)^2} \right)$$

**Hint:** Simplify the function first.

$$(d) \frac{d}{d\theta} (\sec^2 \theta)$$

$$(e) \frac{d}{dt} \left( \frac{1}{5 + 3 \cos(t/2)} \right)$$

**Exercise 69** You jog around a circular track of radius  $r$  feet at the constant speed of  $v$  ft/sec. A flagpole lies  $b$  feet due east of the track's center.

(a) Use the animation below (and nothing else) to sketch by hand a graph of the function

$$s = f(\theta), \theta \geq 0,$$

that expresses your distance (in feet) to the flagpole in terms of your angle  $\theta = \angle FOJ$  of rotation about the track's center, measured in radians from the time you start. Assume you start at the point  $A$  on the track nearest the flagpole. Be sure to include variable names, units, and scales on your axes. Explain your reasoning. For this particular graph assume that  $r = 40$  and  $b = 96$  (be sure to adjust the sliders in the worksheet to have these values).

**Desmos link:** <https://www.desmos.com/calculator/4pndurvhdh>

Worksheet available at 151: Jogger

(b) Using only your graph of the function  $f$  from part (a), sketch by hand a graph of the derivative

$$y = f'(\theta) = ds/d\theta.$$

Be sure to include variable names, units, and scales on your axes. Explain your reasoning.

(c) Using only the animation (stop the motion), approximate the value of the derivative

$$\left. \frac{ds}{d\theta} \right|_{\theta=3\pi/2}.$$

Include units. Explain your reasoning.

(d) Use trigonometry to find an expression for the function  $s = f(\theta)$  in terms of the parameters  $r$  and  $b$  (ie. do not assume  $r = 40$  and  $b = 96$ ).

(e) Find an expression for the derivative  $ds/d\theta$ . Show every step in using the chain rule as in Examples 1 and 2.

(f) Use your result from part (e) to compute the exact value of the derivative

$$\left. \frac{ds}{d\theta} \right|_{\theta=3\pi/2}.$$

(g) What are the units of the derivative in part (f)? Explain the meaning of the derivative, not by giving a standard response about the rate of change of some quantity with respect to another, but by relating small changes.

(h) Use your response to part (f) to write an approximation for the change

$$\Delta s = f(\theta) - f(3\pi/2)$$

in terms of the change

$$\Delta\theta = \theta - 3\pi/2$$

for values of  $\theta$  near  $\theta = 3\pi/2$ .

(i) Find a function

$$s = g(t), t \geq 0,$$

that expresses your distance (in feet) to the flagpole in terms of the number of seconds since you started jogging. Assume you start at the point on the track nearest the flagpole.

(j) Find a function that expresses the rate of change (with respect to time) in your distance to the flagpole in terms of the number of seconds since you began jogging.

(k) Express the rate of change in part (b) in terms of your speed and the angle between the following two vectors:

- the vector that gives your position relative to the flagpole
- the vector that points in the direction of your motion

**Exercise 70** High tide of 10.83 feet at Edmonds at 12:28am, May 15, low tide at 6:58am of ?? feet, lowest tide of 1.64 feet at 5:32pm.

*Desmos link:* <https://www.desmos.com/calculator/zta9tkzzmx>

Worksheet available at 151: Edmonds Tides

**Exercise 71** A pendulum of length  $L$  feet is  $L + 5$  oscillates sinusoidally between angles  $-\theta_0$  and  $\theta_0$  with period  $2\pi\sqrt{L/g}$  seconds, where  $g$  is a constant. The angles  $\pm\theta_0$  are measured in radians from the downward vertical.

(a) Find a function

$$h = f(\theta), -\theta_0 \leq \theta \leq \theta_0,$$

that expresses the height of the pendulum above its stable equilibrium position (ie. its lowest point) in terms of the angle of rotation. Use the cosine function.

(b) Assume now that the pendulum is released from rest from the displacement angle  $\theta = \theta_0$  at time  $t = 0$  seconds. Find a function

$$\theta = g(t), t \geq 0,$$

that expresses the displacement angle (in radians) in terms of the number of seconds since the pendulum was released. Use the cosine function.

(c) Find a function

$$h = w(t), t \geq 0$$

that expresses the height (in feet) of the pendulum above equilibrium in terms of the number of seconds since the pendulum was released.

(d) Find an expression for the derivative  $dh/dt$  and interpret its meaning.

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