Calculus 1

April 14, 2024

Contents

Introduction

Introduction to Differential Calculus

This course is listed as Calculus I in the catalogue, but it really should be called Differential Calculus. *Differential* has the same root as *difference* and *calculus* the same root as *calculate*. So this class is really about calculating differences, or more simply put it's about subtracting. But it's about subtracting in the context of functions.

Pick a specific input to specific function and you'll likely find that the changes in the function's output are approximately proportional to *small* changes in the input. If so, we say that the function is *differentiable* at that input and we call the proportionality constant the *derivative*.

For example, let's look at the behavior of the function

$$A = f(s) = s^2, s \geq 0,$$

near the input $s = 5$. To emphasize the importance of units, let's define the input s to be the side length of a square measured in feet and the output $f(s)$ to be the area of that square, measured in square feet. The problem before us is to describe a simple relationship between a small change in the side length

$$\Delta s = s - 5$$

of the square and the change

$$\Delta A = f(s) - f(5)$$

in its area.

Question 1 *(a) We'll first take a numerical approach and compute some small changes and their ratios. Fill in the missing entries in the table below.*

s (ft)	$A = s^2$ (ft^2)	$\Delta s = s - 5$ (ft)	$\Delta A = s^2 - 25$ (ft^2)	$\Delta A/\Delta s$ (ft^2/ft)
4.9	$\boxed{24.01}$	$\boxed{-0.1}$	$\boxed{-0.99}$	$\boxed{9.9}$
4.99	$\boxed{24.9001}$	$\boxed{-0.01}$	$\boxed{-0.0999}$	$\boxed{9.99}$
5	*25*	0	0	–
5.01	$\boxed{25.1001}$	$\boxed{0.01}$	$\boxed{0.1001}$	$\boxed{10.01}$
5.1	$\boxed{26.01}$	$\boxed{0.1}$	$\boxed{1.01}$	$\boxed{10.1}$

Learning outcomes:
Author(s):

(b) The data in the table above suggests an approximate proportional relationship between ΔA and Δs. We can guess the constant of proportionality from the fifth column. As $s \to 5$ (as s approaches 5), it looks like the ratio $\Delta A/\Delta s$ approaches some number, the constant of proportionality.

i) What is that number? $\boxed{10}$

ii) What are its units?

Free Response:

So for $\Delta s \sim 0$, we suspect that

$$\Delta A \sim \boxed{10}\Delta s.$$

(c) The constant of proportionality is called the derivative, in this case of the function $A = s^2$, at the input $s = 5$. We write this as

$$\left.\frac{dA}{ds}\right|_{s=5} = \boxed{10}.$$

(d) We could have taken an algebraic approach to determe this constant of proportionality instead. The idea is to first simplfiy the quotient $\Delta A/\Delta s$ as

$$\begin{aligned} \frac{\Delta A}{\Delta s} &= \frac{s^2 - 25}{s - 5} \\ &= \frac{(s + \boxed{5})((s - \boxed{5})}{s - 5} \\ &= \boxed{s + 5} \text{ if } s \neq \boxed{5}. \end{aligned}$$

So, for example, if $s = 4.99$, then

$$\frac{\Delta A}{\Delta s} = \boxed{4.99} + 5 = \boxed{9.99}$$

as shown in the last column of the second row of the above table.

The advantage of this algebraic approach is that we can now compute the proportionality constant as a limit:

$$\begin{aligned} \left.\frac{dA}{ds}\right|_{s=5} &= \lim_{s \to 5} \frac{\Delta A}{\Delta s} \\ &= \lim_{s \to 5} (s + \boxed{5}) \\ &= \boxed{5} + \boxed{5} \\ &= \boxed{10}. \end{aligned}$$

(d) We can also use the graph of the function $A = f(s) = s^2$ to intrepret the ratios

$$\frac{\Delta A}{\Delta s} = \frac{f(s) - f(5)}{s - 5}$$

geometrically. Move the slider s in the demonstration below and describe

(i) how the line through the points P and Q is related to the ratio $\Delta A/\Delta s$ show on Line 2,

(ii) what happens to the line PQ as $s \to 5$, and

(iii) what happens to the line PQ when $s = 5$.

Free Response:

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/vz9ud5txva`

Continuing with the above demonstration,

(i) Open the Code folder in Line 3 and turn off the line PQ in Line 7.

(ii) Write an equation for the line through the point P with slope equal to the proportionality constant in the line below and on Line 8 in the desmos worksheet:

$$A = L(s) = \boxed{25} + \boxed{10}(s - \boxed{5}).$$

(iii) Zoom in close enough to the point P to make the graph of the function $A = f(s)$ look like a line. How do the graph of the function and the graph of the line $A = L(s)$ compare in this close-up view?

Free Response:

(e) **Summary:**

- *If we change the side of a square from a length of 5 feet to a length of $s \sim 5$ feet, then the area of the square changes by approximately*

  $$\Delta A = s^2 - 25 \sim 10\Delta s = 10(s - 5)$$

  square feet. The proportionality constant 10 has units $ft^2/ft = ft$.

- *Zoom in close enough to the graph of the function $A = f(s) = s^2$ near the point $P(5, 25)$ and the graph looks like a line with slope equal to the proportionality constant.*

- *We can compute the proportionality constant as the limit*

$$\frac{dA}{ds}\Big|_{s=5} = \lim_{s\to 5} \frac{f(s)-f(5)}{s-5}.$$

- *Suppose for example, we wanted to approximate the side length s of a square with area 25.06 ft^2. Then*

$$\Delta A = s^2 - 25 = 25.06 - 25 = 0.06.$$

And since

$$\Delta A \sim 10\Delta s = 10(s-5),$$

$$0.06 \sim 10\Delta s.$$

So

$$\Delta s \sim 0.006$$

and a square with area 25.06 ft^2 has an approximate side length (measured in feet) of

$$s = 5 + \Delta s \sim 5.006.$$

Question 2 *On a clear day with an unobstructed view (like you might have at the beach or in a hot air balloon), the distance to the horizon is limited by the curvature of the earth as illustrated in the demonstration below.*

In fact, as long as you are not too high above the surface of the earth, the function

$$s = f(h) = 1.22\sqrt{h}, 0 \le h \le 20,000,$$

gives a good approximation to the distance to the horizon (the length of the red arc AT below, measured in miles) in terms of your height above the ground (the distance AP below, measured in feet).

Access Desmos interactives through the online version of this text at

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Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

Desmos activity available at

151:Distance to Horizon 1

Our aim is to approximate the change in the distance to the horizon (in miles) in terms of a small change in height (in feet) from a height of 25 feet.

(i) To start, what are the units of the constant 1.22 above? Explain how you know.

Free Response:

(ii) Go through a similar analysis as in parts (a)-(e) of Example 1, to approximate the change $\Delta s = s - f(25)$ in the distance to the horizon in terms of the change $\Delta h = h - 25$ in your height above the ground. Start by completing the column headings (with units) and the missing entries in the table below.

h (ft)	$s = 1.22\sqrt{h}$ (miles)	$\Delta h = h - 25$ (ft)	$\Delta s = f(h) - f(25)$ (miles)	$\Delta s/\Delta h$ (units?)
4.9^2				
4.99^2				
25		0	0	–
5.01^2				
5.1^2				

Question 3 *This question is similar to the last, but suppose instead we are looking down on the earth from the space station or a rocket. Then the approximation to the distance to the horizon from the previous problem will* not *work.*

So our first step is to find a function

$$s = f(h), h \geq 0,$$

that expresses the distance to the horizon (still measured in miles) in terms of our height above the earth's surface, now measured in miles instead of feet. *We'll suppose the earth to be a perfect sphere of radius* 3960 *miles. The distance to the horizon is the arclength AT below, meaured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance AP.*

Desmos link: `https://www.desmos.com/calculator/ewowig5sgk`

(a) Find an expression for the above function.

Hint: *Use right triangle ΔOTP to find an expression for the radian measure of angle $\angle POT$. Then use this angle to find an expression for the arclength AT.*

Here are more details.

(i) Enter the two side lengths, measured in feet, in right triangle ΔOPT below.

$$OT = \boxed{3960}$$

and

$$OP = \boxed{h + 3960}.$$

(ii) Let θ be the radian measure of $\angle TOP$. Write an equaion with a trigonometric function of θ that relates the two lengths in part (i). Use the Math Editor tab to enter the trig function and the angle θ.

$$\boxed{\cos\theta = \frac{3960}{h+3960}}.$$

(iii) Now solve the equation from part (ii) for θ in terms of h. Then use what you know about measuring arclength along a circle to find an expression for the function f. Use the Math Editor tab to help.

$$s = f(h) = \boxed{3960\arccos(\frac{3960}{h+3960})}.$$

(b) Now suppose we are 165 miles above the surface of the earth and we wish to approximate how a small change in our altitude changes the distance to the horizon.

To do this, fill in the missing entries in the table below.

h *(miles)*	$s = f(h)$ *(miles)*	$\Delta h = h - 165$ *(miles)*	$\Delta s = f(h) - f(165)$ *(miles)*	$\Delta s/\Delta h$ *(units?)*
162				
163				
164				
165				
166				
167				
168				

(c) Do the data above suggest that the quotients $\Delta s/\Delta h$ approach some number as h approaches 165? If so, use the data to approximate that number. If not, explain why not.

(d) Make your own table similar to the one above to get a better approximation, correct to the nearest thousandth, to

$$\lim_{h\to 165} \frac{f(h) - f(165)}{h - 165}.$$

(e) Use your result from part (d), rounded to the nearest thousandth, to approximate Δs in terms of Δh and enter your result below.

$$\Delta s \sim \boxed{3.291}\,\Delta h, \text{ for } \Delta h \sim 0.$$

(f) Explain the meaning of the proportionality constant in parts (d) and (e). Be sure to include units in your explanation.

Small Changes

We explore how small changes to the input of a function change the output.

The main idea of differential calculus is to approximate the change in the output of a function in terms of a small change in the input. For some functions, called *differentiable*, the change in the output is approximately proportional to the (small) change in the input. The proportionality factor is called the derivative. In this chapter we explore this idea.

Odometer Readings

Example 1. *The graph of the function*

$$s = f(t), 0 \leq t \leq 2,$$

that expresses the trip odometer reading (measured in miles) on your car in terms of the number of hours past noon during a two-hour trip is shown below.

Access Desmos interactives through the online version of this text at

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Desmos link: https://www.desmos.com/calculator/iiw69lr1bc

Desmos activity available at

151: Odometer

Our goal is to approximate the car's speed at 12:30pm in three ways:

(1) geometrically, using the above graph as is.

(2) geometrically, by zooming in on the above graph.

(3) arithmetically, using the specific expression for the function f.

(a) Start by using the graph above to describe how the speed of the car varies over the two-hour period. Explain your reasoning. Then play the Slider u in Line 2 and use the animation of the motion to check if your description was accurate. Explain.

Learning outcomes:
Author(s):

(b) Set the slider in Line 6 to $n = 20$. Use only *the graph (with $u = 2$) to create a table with five columns showing the values of t, s (approximate this), $\Delta t = t - 0.5$, $\Delta s = f(t) - f(0.5)$, and $\Delta s/\Delta t$. Include units in the heading of each column. The table should include five rows, with $t = 0.3, 0.5, \ldots 0.7$.*

t (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.3				
0.4				
0.5				
0.6				
0.7				

(c) Explain the meaning of the fifth column in the table of part (b). What do the entries in this column suggest about the speed of the car at 12:30pm?

(d) Now we'll use the fact that

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2,$$

to construct another table like the one in part (b). Do this by finding expressions for

$$\Delta s = f(t) - f(0.5)$$

and

$$r = g(t) = \frac{\Delta s}{\Delta t} = \frac{f(t) - f(0.5)}{t - 0.5},$$

both in terms of t (and not *Δt). Use these functions to fill in the missing entries in the table below.*

t (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.49				
0.499				
0.4999				
0.5				
0.5001				
0.501				
0.51				

(e) Does your table from part (d) suggest that the ratios $\Delta s/\Delta t$ approach some number as $t \to 0.5$? If so, what would be your guess for the exact value of this number? What are its units? What is its meaning?

(f) Activate the folder "Graph of average speed" on Line 9.

(i) Use the graph to check some your entries in the fifth column of your table from part (d). Explain.

(ii) How is the line PQ, through the fixed point $P(0.5, 10)$ and the variable point $Q(t, s)$ on the graph of the function $r = g(t)$ related to the ratio $\Delta s/\Delta t$?

(iii) Describe what happens to the line PQ as point Q approaches point P.

(g) For a quicker way to approximate car's speed at 12:30pm, zoom in sufficiently close to point P in the graph above to make the graph of $s = f(t)$ look like a line. Use the coordinates of point P and a second point in the window far away from P to estimate the car's speed at 12:30pm. Explain your method.

(h) Summarize your conclusions by comparing your three estimates for the car's speed at 12:15pm. Which estimate do you think is most accurate? Least accurate?

Example 2. *This is a continuation of the previous example where we'll algebraically compute the exact speed of the car at 12:30pm, using the odometer function*

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2.$$

The idea is to first find an algebraic expression for the car's average speed between time t and time $t = 0.5$ hours past noon. Then we'll evaluate the limit of this average speed as $t \to 0.5$ to find the (instantaneous) speed at 12:30pm.

Question 4 *First we'll find the average speed between time t and time $t = 0.5$.*

(a) Explain in general how to compute a car's average speed over some time interval. What do you need to know? What is the computation? Make up your own specific example.

(b) Now for our particular odometer function above, the average speed $v_{\text{avg}}(t)$, measured in miles/hour, between time t and time $t = 0.5$ is

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{48t^2 - 16t^3 - 10}{t - 0.5} \\ &= \frac{(2t-1)(\boxed{-8t^2 + 20t + 10})}{t - 0.5} \\ &= \boxed{-16t^2 + 40t + 20} \text{ if } t \neq \boxed{0.5}. \end{aligned}$$

The key step in the computation above is in the third line. How did we know $2t - 1$ was a factor of

$$f(t) - f(0.5) = 48t^2 - 16t^3 - 10?$$

The reason is that $t = 0.5$ is a root of the polynomial $f(t) - f(0.5)$ and therefore $t - 0.5$ is a factor. And so

$$2(t - 0.5) = 2t - 1$$

is also a factor. Then we can use long division to find the quotient.

(c) Show the steps in the long division.

(d) The final step in computing the car's speed v (in miles/hour) at 12:30pm is to evaluate the limit of these average speeds as $t \to 0.5$. We get

$$\begin{aligned} v &= \lim_{t \to 0.5} (\boxed{-16t^2 + 40t + 20}) \\ &= \boxed{36}. \end{aligned}$$

(e) Here's another way to simplify the average speed in part (b). Fill in the missing steps.

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{(48t^2 - 16t^3) - (48(0.5)^2 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{(48t^2 - 48(0.5)^2) - (16t^3 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{48(t^2 - (0.5)^2) - 16(t^3 - (0.5)^3)}{t - 0.5} \\ &= \frac{48(t - 0.5)(\boxed{t + 0.5})) - 16(t - 0.5)(\boxed{t^2 + 0.5t + 0.25})}{t - 0.5} \\ &= 48(\boxed{t + 0.5})) - 16(\boxed{t^2 + 0.5t + 0.25}) \text{ if } t \neq \boxed{0.5}. \end{aligned}$$

(f) Use the above expression for the average speed function to compute the (instantaneous) speed of the car at 12:30pm by evaluating the approriate limit.

(g) Sketch by hand *a graph of the average speed function $y = v_{avg}(t)$ over the appropriate domain. Be sure also to state this function's domain.*

A Projectile

Example 3. *Access Desmos interactives through the online version of this text at*

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Desmos link: `https://www.desmos.com/calculator/l4fknr0hpl`

Desmos activity available at

151: Projectile

The Falling Ladder, Part 1

Example 4. *The top end of a ten-foot ladder leans against a vertical wall and the bottom end rests on the horizontal floor. We analyze how a small change in the distance between the wall and the bottom of the ladder affects the height of the ladder's top above the floor.*

Access Desmos interactives through the online version of this text at

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Desmos link: https://www.desmos.com/calculator/dvyuifyyg4

Desmos activity available at

151: Ladder 1B

Question 5 *(a) The slider s in Line 1 of the demonstration above controls the distance between the wall and the bottom of the ladder, measured in feet. Use the slider s to describe* qualitatively *how a small change in s changes the height h (also measured in feet) of the ladder's top end above the floor.*

(i) Do the small changes have the same or opposite signs?

(ii) At what positions of the ladder does a small change in s result in a comparatively large change in h?

(b) Now let's focus on the particular position of the ladder where the bottom end is $s = 8$ feet from the wall. For this, turn on the "one position" folder in Line 3.

(i) Drag the slider s close to $s = 8$ and use the cordinates of the endpoints of the ladder to construct a table with five columns showing the values of s, h, $\Delta s = s - 8$, $\Delta h = f(s) - f(8)$, and $\Delta h/\Delta s$. Include units in the heading of each column. The table should include seven rows, with $s = 7.7, 7.8, \ldots 8.2, 8.3$. Here $h = f(s)$ is the function described in part (ii) below.

(ii) Find a function

$$h = f(s), 0 \leq s \leq 10,$$

that expresses the height of the ladder's top end above the ground (in feet) in terms of the distance of its bottom end from the wall (in feet).

(iii) Use your function f to construct another table, like the one in part (i), with $s = 7.9, 7.99, 7.999, 8, 8.0001, 8.01, 8.1$. Do this by finding expressions for

$$\Delta h = f(s) - f(8)$$

and

$$r = g(s) = \frac{\Delta h}{\Delta s} = \frac{f(s) - f(8)}{s - 8},$$

both in terms of s *(and* not Δs*).*

(iv) Does your table from part (iii) suggest that the ratios $\Delta h/\Delta s$ *approach some number as* $s \to 8$*? If so, what would is your guess for the exact value of this limit? What are its units? What is its meaning?*

(v) Activate the folder "graph of function" on Line 8. How is the line PQ*, through the fixed point* $P(8, 6)$ *and the variable point* $Q(s, h)$ *on the graph of the function* $h = f(s)$ *related to the ratios* $\Delta h/\Delta s$*?*

(vi) Change the bounds for s *in Line 1 to run between* $s = 7.9$ *and* $s = 8.1$*. Then activate the folder "graph: average rate of change function" on Line 18. Move the slider* s *and use the graph of the function* $r = g(s)$ *to check your computations in part (iii).*

(vii) Use the result of part (iv) to write an approximation for the change in height

$$\Delta h = h - 8$$

in terms of the change

$$\Delta s = s - 8.$$

(viii) The graph of the function $h = f(s)$ *suggests another, geometric way to find the proportionality constant (of part (vii)) that relates* Δh *to* Δs*. Explain how.*

The Falling Ladder, Part 2

Example 5. *A tree leans precariously with its trunk making an angle of* $\phi = \pi/6$ *radians with the ground. One end of a ten-foot ladder leans against the trunk, the other rests on the horizontal ground. We analyze how a small change in the distance between the bottom of the ladder and the base of the trunk changes the distance between the top of the ladder and the base of the trunk.*

Access Geogebra interactives through the online version of this text at

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Geogebra link: `https://tube.geogebra.org/m/qmke5y7x`

We'll let t *be the distance between the top of the ladder and the base of the trunk (measured in feet) and* s *the distance between the bottom of the ladder and the base of the trunk (also measured in feet).*

The slider θ in the demonstration above controls the angle that the ladder makes with the ground, but this angle does not come into play in our problem.

(a) Use the slider θ to describe qualitatively how a small change in s (the length of segment GC) changes t (the length of segment GB):

(i) For what positions of the ladder do these small changes have the same signs? Opposite signs?

(ii) For what positions of the ladder does a small change in s result in a comparatively large change in t?

(b) Now let's focus on the particular position of the ladder when the bottom end C is 16 feet from the trunk's base and the top end D is about 8 feet from the base as illustrated above. *Our first step is to find a function*

$$t = f(s)$$

that expresses t in terms of s for values of s and t near $s = 16$ and $t = 8$ respectively.

To do this, first use the law of cosines to write an equation relating s and t. Then complete the square *to solve this equation for t in terms of s to find the function $t = f(s)$. Keep in mind that when $s = 16$, we must have $t \sim 8$.*

Hint: *Here is an outline of the steps to check your work:*

(i) Use the law of cosines to relate s and t.

$$s^2 + t^2 - \boxed{\sqrt{3}st} = \boxed{100}.$$

(ii) Solve the above equation for t in terms of s as follows:

First rewrite the equation with the two terms with t as a factor on the left side and the other two terms on the right.

$$t^2 - \boxed{\sqrt{3}st} = \boxed{100 - s^2}.$$

Then complete the square by adding the same perfect square to each side.

$$t^2 - \boxed{\sqrt{3}st} + \left(\boxed{\frac{\sqrt{3}s}{2}}\right)^2 = 100 - s^2 + \left(\boxed{\frac{\sqrt{3}s}{2}}\right)^2.$$

Then factor the LHS and simplify the RHS.

$$\left(\boxed{t - \frac{\sqrt{3}s}{2}}\right)^2 = \boxed{100 - \frac{s^2}{4}}.$$

Next, solve for t in terms of s.

$$t = \boxed{\frac{\sqrt{3}}{2}s} \pm \boxed{\sqrt{100 - \frac{s^2}{4}}}$$

Finally, make the correct choice of $\pm$ to solve for t in terms of s, when s is near 16 and t near 8.

$$t = \boxed{\frac{\sqrt{3}}{2}s - \sqrt{100 - \frac{s^2}{4}}}.$$

(c) Use your function from part (b) to find an expressions for

$$\Delta t = f(s) - f(16)$$

and for the function

$$r = g(s) = \frac{\Delta t}{\Delta s} = \frac{f(s) - f(16)}{s - 16}.$$

Explain what the output of the function g measures. What are its units?

(d) Use the results of part (c) to construct a table with five columns showing the values of s, t, $\Delta s = s - 16$, $\Delta t = f(s) - f(16)$, and $\Delta t/\Delta s$. Include units in the heading of each column. The table should include seven rows, with $s = 15.9, 15.99, 15.999, 16, 16.001, 16.01, 16.1$.

(e) Does your table from part (d) suggest that the ratio $\Delta t/\Delta s$ approaches some number as $s \to 16$? If so, approximate the value of this number. What are its units?

(f) Check the box "GraphofRelation" in the demonstration above and explain how the line EF is related to part (d).

(g) Use the result of part (e) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of s near 16. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 16.4 feet from the trunk's base. Compare your approximation with the exact distance.

Riding a Ferris Wheel

Suppose you ride a ferris wheel

Limits

Limits in context.

Limits and Tangent Lines

Example 6. *Let*

$$g(x) = \frac{x^2 - 9}{3x - 9}.$$

(a) Evaluate each of the following expressions.

(i) $g(7)$

(ii) $\lim_{x \to 7} g(x)$

(iii) $g(3)$

(iv) $\lim_{x \to 3} g(x)$

(b) Simplify and then graph the function $g(x)$*.*

(c) Interpret the expressions in part (a) geometrically by considering the graph of the function $f(x) = x^2/3$ *as in the demonstration below.*

Desmos link: `https://www.desmos.com/calculator/u0uvuchnrk`

Desmos activity available at 151: Parabola Basic

Limits and Gas Mielage

Example 7. *The function*

$$G = f(s) = \frac{2}{5} + \frac{1}{5000}(40(s+2)^2 - (s+2)^3),\ 0 \leq s \leq 23,$$

expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.

Desmos link: `https://www.desmos.com/calculator/xzkknfpkw3`

Learning outcomes:
Author(s):

Desmos activity available at 151: Gas as a Function of Distance

(a) Use the graph of the function f shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Find your average gas mileage (in miles/gallon) over the interval $s \in [8, 18]$.

(c) Use the graph to approximate your gas mileage at the moment you are 18 miles from home. Do this by zooming in on the appropriate point.

(d) Use the algebra of limits to determine your exact gas mileage at the moment you are 18 miles from home.

(e) Use the result of part (d) to approximate the change in the volume of gas

$$\Delta G = f(s) - f(18)$$

in terms of the change

$$\Delta s = s - 18$$

in your distance from home for values of s near 18 miles. What are the units of the proportionality constant?

(f) Use the result of part (e) to approximate your distance from home when there are 1.9 gallons of gas in your tank.

Limits, Gas Mileage and Speed

Example 8. *Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and that the car gets 38 miles/gallon at a speed of 40 miles/hour.*

(a) Find an expression for the function

$$G = f(v)\,,\, 30 \leq v \leq 70,$$

that gives the gas mileage (in miles/gal) in terms of the speed (in miles/hour).

(b) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the car's speed at 60 miles/hour gives an approximately proportional change in its gas mileage.

(c) Use the results of part (b) to approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find the exact value of the proportionality constant.

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta G = g - f(60)$$

in gas mileage in terms of a small change

$$\Delta v = v - 60$$

in the car's speed.

(g) Use part (f) to approximate the speed at which the car gets 36 *miles/gallon.*

(h) Would you expect your approximation in part (g) to be greater or less than the actual speed? Explain your reasoning with a graph.

(i) Simplify the units of the proportionality constant. What might these units suggest about a way to interpret the constant?

(j) At what rate (in gal/hr) does the car burn gas at a speed of 60 *miles/hour?*

(k) How is the rate in part (j) related to the proportionality constant?

Limits, Speed and Altitude

Example 9. *A rock dropped from a height of* 100 *feet falls to the surface of Planet Krypton without air resistance.*

(a) By considering only the physical situation and without *doing any computations, sketch a graph of the function*

$$v = g(h)\,, 0 \le h \le 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet).

(b)

Question 6 *Use the results from part (a) to choose a reasonable expression for the function g from the list below.*

Multiple Choice:

(a) $g(t) = 100 - 9t^2$, $0 \le t \le 10/3$

(b) $g(h) = 100 - 9h^2$, $0 \le h \le 100$

(c) $g(h) = 0.005(100 - h)^2$, $0 \le h \le 100$

(d) $g(h) = 6\sqrt{100 - h}$, $0 \le h \le 100$ ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 *feet gives an approximately proportional change in its speed.*

(d) Use the results of part (c) to approximate the proportionality constant. What are its units?

(e) Use the algebra of limits to find the exact value of the proportionality constant.

(f) Explain the meaning of the proportionality constant.

(g) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(h) Use part (g) to approximate the rock's speed at a height of 63 *feet.*

(i) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(j) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

Limits and Gas Mileage

Limits and Purchasing Power

The Derivative

Computing derivative with limits.

Nothing yet.

Learning outcomes:
Author(s):