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# Calculus 1

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April 21, 2024

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# Introduction

## Introduction to Differential Calculus

This course is listed as Calculus I in the catalogue, but it really should be called Differential Calculus. *Differential* has the same root as *difference* and *calculus* the same root as *calculate*. So this class is really about calculating differences, or more simply put it's about subtracting. But it's about subtracting in the context of functions.

Pick a specific input to specific function and you'll likely find that the changes in the function's output are approximately proportional to *small* changes in the input. If so, we say that the function is *differentiable* at that input and we call the proportionality constant the *derivative*.

For example, let's look at the behavior of the function

$$A = f(s) = s^2, s \geq 0,$$

near the input  $s = 5$ . To emphasize the importance of units, let's define the input  $s$  to be the side length of a square measured in feet and the output  $f(s)$  to be the area of that square, measured in square feet. The problem before us is to describe a simple relationship between a small change in the side length

$$\Delta s = s - 5$$

of the square and the change

$$\Delta A = f(s) - f(5)$$

in its area.

**Question 1** (a) We'll first take a numerical approach and compute some small changes and their ratios. Fill in the missing entries in the table below.

$s$ (ft)	$A = s^2$ (ft <sup>2</sup> )	$\Delta s = s - 5$ (ft)	$\Delta A = s^2 - 25$ (ft <sup>2</sup> )	$\Delta A / \Delta s$ (ft <sup>2</sup> /ft)
4.9	24.01	-0.1	-0.99	9.9
4.99	24.9001	-0.01	-0.0999	9.99
5	25	0	0	—
5.01	25.1001	0.01	0.1001	10.01
5.1	26.01	0.1	1.01	10.1

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Learning outcomes:

Author(s):

(b) The data in the table above suggests an approximate proportional relationship between  $\Delta A$  and  $\Delta s$ . We can guess the constant of proportionality from the fifth column. As  $s \rightarrow 5$  (as  $s$  approaches 5), it looks like the ratio  $\Delta A/\Delta s$  approaches some number, the constant of proportionality.

i) What is that number?  $\boxed{10}$

ii) What are its units?

**Free Response:**

So for  $\Delta s \sim 0$ , we suspect that

$$\Delta A \sim \boxed{10} \Delta s.$$

(c) The constant of proportionality is called the derivative, in this case of the function  $A = s^2$ , at the input  $s = 5$ . We write this as

$$\left. \frac{dA}{ds} \right|_{s=5} = \boxed{10}.$$

(d) We could have taken an algebraic approach to determine this constant of proportionality instead. The idea is to first simplify the quotient  $\Delta A/\Delta s$  as

$$\begin{aligned} \frac{\Delta A}{\Delta s} &= \frac{s^2 - 25}{s - 5} \\ &= \frac{(s + \boxed{5})(s - \boxed{5})}{s - 5} \\ &= \boxed{s + 5} \text{ if } s \neq \boxed{5}. \end{aligned}$$

So, for example, if  $s = 4.99$ , then

$$\frac{\Delta A}{\Delta s} = \boxed{4.99} + 5 = \boxed{9.99}$$

as shown in the last column of the second row of the above table.

The advantage of this algebraic approach is that we can now compute the proportionality constant as a limit:

$$\begin{aligned} \left. \frac{dA}{ds} \right|_{s=5} &= \lim_{s \rightarrow 5} \frac{\Delta A}{\Delta s} \\ &= \lim_{s \rightarrow 5} (s + \boxed{5}) \\ &= \boxed{5} + \boxed{5} \\ &= \boxed{10}. \end{aligned}$$

(d) We can also use the graph of the function  $A = f(s) = s^2$  to interpret the ratios

$$\frac{\Delta A}{\Delta s} = \frac{f(s) - f(5)}{s - 5}$$

geometrically. Move the slider  $s$  in the demonstration below and describe

- (i) how the line through the points  $P$  and  $Q$  is related to the ratio  $\Delta A/\Delta s$  show on Line 2,
- (ii) what happens to the line  $PQ$  as  $s \rightarrow 5$ , and
- (iii) what happens to the line  $PQ$  when  $s = 5$ .

**Free Response:**

Access Desmos interactives through the online version of this text at

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*Desmos link:* <https://www.desmos.com/calculator/vz9ud5txva>

Continuing with the above demonstration,

- (i) Open the Code folder in Line 3 and turn off the line  $PQ$  in Line 7.
- (ii) Write an equation for the line through the point  $P$  with slope equal to the proportionality constant in the line below and on Line 8 in the desmos worksheet:

$$A = L(s) = \boxed{25} + \boxed{10}(s - \boxed{5}).$$

- (iii) Zoom in close enough to the point  $P$  to make the graph of the function  $A = f(s)$  look like a line. How do the graph of the function and the graph of the line  $A = L(s)$  compare in this close-up view?

**Free Response:**

(e) **Summary:**

- If we change the side of a square from a length of 5 feet to a length of  $s \sim 5$  feet, then the area of the square changes by approximately

$$\Delta A = s^2 - 25 \sim 10\Delta s = 10(s - 5)$$

square feet. The proportionality constant 10 has units  $ft^2/ft = ft$ .

- Zoom in close enough to the graph of the function  $A = f(s) = s^2$  near the point  $P(5, 25)$  and the graph looks like a line with slope equal to the proportionality constant.

- We can compute the proportionality constant as the limit

$$\left. \frac{dA}{ds} \right|_{s=5} = \lim_{s \rightarrow 5} \frac{f(s) - f(5)}{s - 5}.$$

- Suppose for example, we wanted to approximate the side length  $s$  of a square with area  $25.06 \text{ ft}^2$ . Then

$$\Delta A = s^2 - 25 = 25.06 - 25 = 0.06.$$

And since

$$\Delta A \sim 10\Delta s = 10(s - 5),$$

$$0.06 \sim 10\Delta s.$$

So

$$\Delta s \sim 0.006$$

and a square with area  $25.06 \text{ ft}^2$  has an approximate side length (measured in feet) of

$$s = 5 + \Delta s \sim 5.006.$$

**Question 2** On a clear day with an unobstructed view (like you might have at the beach or in a hot air balloon), the distance to the horizon is limited by the curvature of the earth as illustrated in the demonstration below.

In fact, as long as you are not too high above the surface of the earth, the function

$$s = f(h) = 1.22\sqrt{h}, 0 \leq h \leq 20,000,$$

gives a good approximation to the distance to the horizon (the length of the red arc  $AT$  below, measured in miles) in terms of your height above the ground (the distance  $AP$  below, measured in feet).

Access Desmos interactives through the online version of this text at

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**Desmos link:** <https://www.desmos.com/calculator/ewowig5sgk>

Desmos activity available at

[151:Distance to Horizon 1](#)

Our aim is to approximate the change in the distance to the horizon (in miles) in terms of a small change in height (in feet) from a height of 25 feet.

(i) To start, what are the units of the constant 1.22 above? Explain how you know.

**Free Response:**

(ii) Go through a similar analysis as in parts (a)-(e) of Example 1, to approximate the change  $\Delta s = s - f(25)$  in the distance to the horizon in terms of the change  $\Delta h = h - 25$  in your height above the ground. Start by completing the column headings (with units) and the missing entries in the table below.

$h$ (ft)	$s = 1.22\sqrt{h}$ (miles)	$\Delta h = h - 25$ (ft)	$\Delta s = f(h) - f(25)$ (miles)	$\Delta s/\Delta h$ (units?)
$4.9^2$				
$4.99^2$				
25		0	0	—
$5.01^2$				
$5.1^2$				

**Question 3** This question is similar to the last, but suppose instead we are looking down on the earth from the space station or a rocket. Then the approximation to the distance to the horizon from the previous problem will not work.

So our first step is to find a function

$$s = f(h), h \geq 0,$$

that expresses the distance to the horizon (still measured in miles) in terms of our height above the earth's surface, now measured in miles instead of feet. We'll suppose the earth to be a perfect sphere of radius 3960 miles. The distance to the horizon is the arclength  $AT$  below, measured along the surface of the earth (you can think of this distance as the radius of the spherical disk visible to us). Our height is the distance  $AP$ .

**Desmos link:** <https://www.desmos.com/calculator/ewowig5sgk>

(a) Find an expression for the above function.

**Hint:** Use right triangle  $\triangle OTP$  to find an expression for the radian measure of angle  $\angle POT$ . Then use this angle to find an expression for the arclength  $AT$ .

Here are more details.

(i) Enter the two side lengths, measured in feet, in right triangle  $\triangle OPT$  below.

$$OT = \boxed{3960}$$

and

$$OP = \boxed{h + 3960}.$$

(ii) Let  $\theta$  be the radian measure of  $\angle TOP$ . Write an equation with a trigonometric function of  $\theta$  that relates the two lengths in part (i). Use the Math Editor tab to enter the trig function and the angle  $\theta$ .

$$\cos \theta = \frac{3960}{h + 3960}.$$

(iii) Now solve the equation from part (ii) for  $\theta$  in terms of  $h$ . Then use what you know about measuring arclength along a circle to find an expression for the function  $f$ . Use the Math Editor tab to help.

$$s = f(h) = 3960 \arccos\left(\frac{3960}{h + 3960}\right).$$

(b) Now suppose we are 165 miles above the surface of the earth and we wish to approximate how a small change in our altitude changes the distance to the horizon.

To do this, fill in the missing entries in the table below.

$h$ (miles)	$s = f(h)$ (miles)	$\Delta h = h - 165$ (miles)	$\Delta s = f(h) - f(165)$ (miles)	$\Delta s / \Delta h$ (units?)
162				
163				
164				
165				
166				
167				
168				

(c) Do the data above suggest that the quotients  $\Delta s / \Delta h$  approach some number as  $h$  approaches 165? If so, use the data to approximate that number. If not, explain why not.

(d) Make your own table similar to the one above to get a better approximation, correct to the nearest thousandth, to

$$\lim_{h \rightarrow 165} \frac{f(h) - f(165)}{h - 165}.$$

(e) Use your result from part (d), rounded to the nearest thousandth, to approximate  $\Delta s$  in terms of  $\Delta h$  and enter your result below.

$$\Delta s \sim \boxed{3.291} \Delta h, \text{ for } \Delta h \sim 0.$$

(f) Explain the meaning of the proportionality constant in parts (d) and (e). Be sure to include units in your explanation.



## Small Changes

*We explore how small changes to the input of a function change the output.*

The main idea of differential calculus is to approximate the change in the output of a function in terms of a small change in the input. For some functions, called *differentiable*, the change in the output is approximately proportional to the (small) change in the input. The proportionality factor is called the derivative. In this chapter we explore this idea.

## Odometer Readings

**Example 1.** *The graph of the function*

$$s = f(t), 0 \leq t \leq 2,$$

*that expresses the trip odometer reading (measured in miles) on your car in terms of the number of hours past noon during a two-hour trip is shown below.*

*Access Desmos interactives through the online version of this text at*

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*Desmos link:* <https://www.desmos.com/calculator/iw69lr1bc>

*Desmos activity available at*

*151: Odometer*

*Our goal is to approximate the car's speed at 12:30pm in three ways:*

- (1) geometrically, using the above graph as is.*
- (2) geometrically, by zooming in on the above graph.*
- (3) arithmetically, using the specific expression for the function  $f$ .*

*(a) Start by using the graph above to describe how the speed of the car varies over the two-hour period. Explain your reasoning. Then play the Slider  $u$  in Line 2 and use the animation of the motion to check if your description was accurate. Explain.*

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Learning outcomes:  
Author(s):

## Small Changes

(b) Set the slider in Line 6 to  $n = 20$ . Use only the graph (with  $u = 2$ ) to create a table with five columns showing the values of  $t$ ,  $s$  (approximate this),  $\Delta t = t - 0.5$ ,  $\Delta s = f(t) - f(0.5)$ , and  $\Delta s/\Delta t$ . Include units in the heading of each column. The table should include five rows, with  $t = 0.3, 0.5, \dots, 0.7$ .

$t$ (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.3				
0.4				
0.5				
0.6				
0.7				

(c) Explain the meaning of the fifth column in the table of part (b). What do the entries in this column suggest about the speed of the car at 12:30pm?

(d) Now we'll use the fact that

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2,$$

to construct another table like the one in part (b). Do this by finding expressions for

$$\Delta s = f(t) - f(0.5)$$

and

$$r = g(t) = \frac{\Delta s}{\Delta t} = \frac{f(t) - f(0.5)}{t - 0.5},$$

both in terms of  $t$  (and not  $\Delta t$ ). Use these functions to fill in the missing entries in the table below.

$t$ (hours)	$s = f(t)$ (miles)	$\Delta t = t - 0.5$ (hrs)	$\Delta s = f(t) - f(0.5)$ (miles)	$\Delta s/\Delta t$ (miles/hr)
0.49				
0.499				
0.4999				
0.5				
0.5001				
0.501				
0.51				

(e) Does your table from part (d) suggest that the ratios  $\Delta s/\Delta t$  approach some number as  $t \rightarrow 0.5$ ? If so, what would be your guess for the exact value of this number? What are its units? What is its meaning?

(f) Activate the folder "Graph of average speed" on Line 9.

(i) Use the graph to check some of your entries in the fifth column of your table from part (d). Explain.

(ii) How is the line  $PQ$ , through the fixed point  $P(0.5, 10)$  and the variable point  $Q(t, s)$  on the graph of the function  $r = g(t)$  related to the ratio  $\Delta s / \Delta t$ ?

(iii) Describe what happens to the line  $PQ$  as point  $Q$  approaches point  $P$ .

(g) For a quicker way to approximate car's speed at 12:30pm, zoom in sufficiently close to point  $P$  in the graph above to make the graph of  $s = f(t)$  look like a line. Use the coordinates of point  $P$  and a second point in the window far away from  $P$  to estimate the car's speed at 12:30pm. Explain your method.

(h) Summarize your conclusions by comparing your three estimates for the car's speed at 12:15pm. Which estimate do you think is most accurate? Least accurate?

**Example 2.** This is a continuation of the previous example where we'll algebraically compute the exact speed of the car at 12:30pm, using the odometer function

$$s = f(t) = 48t^2 - 16t^3, 0 \leq t \leq 2.$$

The idea is to first find an algebraic expression for the car's average speed between time  $t$  and time  $t = 0.5$  hours past noon. Then we'll evaluate the limit of this average speed as  $t \rightarrow 0.5$  to find the (instantaneous) speed at 12:30pm.

**Question 4** First we'll find the average speed between time  $t$  and time  $t = 0.5$ .

(a) Explain in general how to compute a car's average speed over some time interval. What do you need to know? What is the computation? Make up your own specific example.

(b) Now for our particular odometer function above, the average speed  $v_{\text{avg}}(t)$ , measured in miles/hour, between time  $t$  and time  $t = 0.5$  is

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{48t^2 - 16t^3 - 10}{t - 0.5} \\ &= \frac{(2t - 1)(-8t^2 + 20t + 10)}{t - 0.5} \\ &= \boxed{-16t^2 + 40t + 20} \text{ if } t \neq \boxed{0.5}. \end{aligned}$$

The key step in the computation above is in the third line. How did we know  $2t - 1$  was a factor of

$$f(t) - f(0.5) = 48t^2 - 16t^3 - 10?$$

The reason is that  $t = 0.5$  is a root of the polynomial  $f(t) - f(0.5)$  and therefore  $t - 0.5$  is a factor. And so

$$2(t - 0.5) = 2t - 1$$

is also a factor. Then we can use long division to find the quotient.

(c) Show the steps in the long division.

(d) The final step in computing the car's speed  $v$  (in miles/hour) at 12:30pm is to evaluate the limit of these average speeds as  $t \rightarrow 0.5$ . We get

$$\begin{aligned} v &= \lim_{t \rightarrow 0.5} \left( \boxed{-16t^2 + 40t + 20} \right) \\ &= \boxed{36}. \end{aligned}$$

(e) Here's another way to simplify the average speed in part (b). Fill in the missing steps.

$$\begin{aligned} v_{\text{avg}}(t) &= \frac{f(t) - f(0.5)}{t - 0.5} \\ &= \frac{(48t^2 - 16t^3) - (48(0.5)^2 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{(48t^2 - 48(0.5)^2) - (16t^3 - 16(0.5)^3)}{t - 0.5} \\ &= \frac{48(t^2 - (0.5)^2) - 16(t^3 - (0.5)^3)}{t - 0.5} \\ &= \frac{48(t - 0.5)(\boxed{t + 0.5}) - 16(t - 0.5)(\boxed{t^2 + 0.5t + 0.25})}{t - 0.5} \\ &= 48(\boxed{t + 0.5}) - 16(\boxed{t^2 + 0.5t + 0.25}) \text{ if } t \neq \boxed{0.5}. \end{aligned}$$

(f) Use the above expression for the average speed function to compute the (instantaneous) speed of the car at 12:30pm by evaluating the appropriate limit.

(g) Sketch by hand a graph of the average speed function  $y = v_{\text{avg}}(t)$  over the appropriate domain. Be sure also to state this function's domain.

## A Projectile

**Example 3.** Access Desmos interactives through the online version of this text at

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*Desmos link:* <https://www.desmos.com/calculator/l4fknr0hpl>

Desmos activity available at

151: Projectile

## The Falling Ladder, Part 1

**Example 4.** *The top end of a ten-foot ladder leans against a vertical wall and the bottom end rests on the horizontal floor. We analyze how a small change in the distance between the wall and the bottom of the ladder affects the height of the ladder's top above the floor.*

Access Desmos interactives through the online version of this text at

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*Desmos link:* <https://www.desmos.com/calculator/dvyuifyy94>

Desmos activity available at

[151: Ladder 1B](#)

**Question 5** (a) The slider  $s$  in Line 1 of the demonstration above controls the distance between the wall and the bottom of the ladder, measured in feet. Use the slider  $s$  to describe qualitatively how a small change in  $s$  changes the height  $h$  (also measured in feet) of the ladder's top end above the floor.

- (i) Do the small changes have the same or opposite signs?
- (ii) At what positions of the ladder does a small change in  $s$  result in a comparatively large change in  $h$ ?

(b) Now let's focus on the particular position of the ladder where the bottom end is  $s = 8$  feet from the wall. For this, turn on the "one position" folder in Line 3.

- (i) Drag the slider  $s$  close to  $s = 8$  and use the coordinates of the endpoints of the ladder to construct a table with five columns showing the values of  $s$ ,  $h$ ,  $\Delta s = s - 8$ ,  $\Delta h = f(s) - f(8)$ , and  $\Delta h/\Delta s$ . Include units in the heading of each column. The table should include seven rows, with  $s = 7.7, 7.8, \dots, 8.2, 8.3$ . Here  $h = f(s)$  is the function described in part (ii) below.

- (ii) Find a function

$$h = f(s), 0 \leq s \leq 10,$$

that expresses the height of the ladder's top end above the ground (in feet) in terms of the distance of its bottom end from the wall (in feet).

- (iii) Use your function  $f$  to construct another table, like the one in part (i), with  $s = 7.9, 7.99, 7.999, 8, 8.0001, 8.01, 8.1$ . Do this by finding expressions for

$$\Delta h = f(s) - f(8)$$

and

$$r = g(s) = \frac{\Delta h}{\Delta s} = \frac{f(s) - f(8)}{s - 8},$$

both in terms of  $s$  (and not  $\Delta s$ ).

(iv) Does your table from part (iii) suggest that the ratios  $\Delta h/\Delta s$  approach some number as  $s \rightarrow 8$ ? If so, what would be your guess for the exact value of this limit? What are its units? What is its meaning?

(v) Activate the folder “graph of function” on Line 8. How is the line  $PQ$ , through the fixed point  $P(8, 6)$  and the variable point  $Q(s, h)$  on the graph of the function  $h = f(s)$  related to the ratios  $\Delta h/\Delta s$ ?

(vi) Change the bounds for  $s$  in Line 1 to run between  $s = 7.9$  and  $s = 8.1$ . Then activate the folder “graph: average rate of change function” on Line 18. Move the slider  $s$  and use the graph of the function  $r = g(s)$  to check your computations in part (iii).

(vii) Use the result of part (iv) to write an approximation for the change in height

$$\Delta h = h - 8$$

in terms of the change

$$\Delta s = s - 8.$$

(viii) The graph of the function  $h = f(s)$  suggests another, geometric way to find the proportionality constant (of part (vii)) that relates  $\Delta h$  to  $\Delta s$ . Explain how.

## The Falling Ladder, Part 2

**Example 5.** A tree leans precariously with its trunk making an angle of  $\phi = \pi/6$  radians with the ground. One end of a ten-foot ladder leans against the trunk, the other rests on the horizontal ground. We analyze how a small change in the distance between the bottom of the ladder and the base of the trunk changes the distance between the top of the ladder and the base of the trunk.

Access Geogebra interactives through the online version of this text at

*Geogebra link:* <https://tube.geogebra.org/m/qmke5y7x>

We'll let  $t$  be the distance between the top of the ladder and the base of the trunk (measured in feet) and  $s$  the distance between the bottom of the ladder and the base of the trunk (also measured in feet).

The slider  $\theta$  in the demonstration above controls the angle that the ladder makes with the ground, but this angle does not come into play in our problem.

(a) Use the slider  $\theta$  to describe qualitatively how a small change in  $s$  (the length of segment  $GC$ ) changes  $t$  (the length of segment  $GB$ ):

(i) For what positions of the ladder do these small changes have the same signs? Opposite signs?

(ii) For what positions of the ladder does a small change in  $s$  result in a comparatively large change in  $t$ ?

(b) Now let's focus on the particular position of the ladder when the bottom end  $C$  is 16 feet from the trunk's base and the top end  $D$  is about 8 feet from the base as illustrated above. Our first step is to find a function

$$t = f(s)$$

that expresses  $t$  in terms of  $s$  for values of  $s$  and  $t$  near  $s = 16$  and  $t = 8$  respectively.

To do this, first use the law of cosines to write an equation relating  $s$  and  $t$ . Then complete the square to solve this equation for  $t$  in terms of  $s$  to find the function  $t = f(s)$ . Keep in mind that when  $s = 16$ , we must have  $t \sim 8$ .

**Hint:** Here is an outline of the steps to check your work:

(i) Use the law of cosines to relate  $s$  and  $t$ .

$$s^2 + t^2 - \boxed{\sqrt{3}st} = \boxed{100}.$$

(ii) Solve the above equation for  $t$  in terms of  $s$  as follows:

First rewrite the equation with the two terms with  $t$  as a factor on the left side and the other two terms on the right.

$$t^2 - \boxed{\sqrt{3}st} = \boxed{100 - s^2}.$$

Then complete the square by adding the same perfect square to each side.

$$t^2 - \boxed{\sqrt{3}st} + \left(\frac{\boxed{\sqrt{3}s}}{2}\right)^2 = 100 - s^2 + \left(\frac{\boxed{\sqrt{3}s}}{2}\right)^2.$$

Then factor the LHS and simplify the RHS.

$$\left(t - \frac{\boxed{\sqrt{3}s}}{2}\right)^2 = \boxed{100 - \frac{s^2}{4}}.$$

Next, solve for  $t$  in terms of  $s$ .

$$t = \frac{\sqrt{3}}{2}s \pm \sqrt{100 - \frac{s^2}{4}}$$

Finally, make the correct choice of  $\pm$  to solve for  $t$  in terms of  $s$ , when  $s$  is near 16 and  $t$  near 8.

$$t = \frac{\sqrt{3}}{2}s - \sqrt{100 - \frac{s^2}{4}}.$$

(c) Use your function from part (b) to find an expressions for

$$\Delta t = f(s) - f(16)$$

and for the function

$$r = g(s) = \frac{\Delta t}{\Delta s} = \frac{f(s) - f(16)}{s - 16}.$$

Explain what the output of the function  $g$  measures. What are its units?

(d) Use the results of part (c) to construct a table with five columns showing the values of  $s$ ,  $t$ ,  $\Delta s = s - 16$ ,  $\Delta t = f(s) - f(16)$ , and  $\Delta t/\Delta s$ . Include units in the heading of each column. The table should include seven rows, with  $s = 15.9, 15.99, 15.999, 16, 16.001, 16.01, 16.1$ .

(e) Does your table from part (d) suggest that the ratio  $\Delta t/\Delta s$  approaches some number as  $s \rightarrow 16$ ? If so, approximate the value of this number. What are its units?

(f) Check the box “GraphofRelation” in the demonstration above and explain how the line  $EF$  is related to part (d).

(g) Use the result of part (e) to write an approximation for the change

$$\Delta t = f(s) - f(16)$$

in terms of the change

$$\Delta s = s - 16$$

for values of  $s$  near 16. Use this approximation to estimate the distance between the top of the ladder and the base of the trunk when the bottom of the ladder is 16.4 feet from the trunk’s base. Compare your approximation with the exact distance.

## Riding a Ferris Wheel

Suppose you ride a ferris wheel



# Limits

*Limits in context.*

## Limits and Tangent Lines

**Example 6.** *Let*

$$g(x) = \frac{x^2 - 9}{3x - 9}.$$

(a) *Evaluate each of the following expressions.*

(i)  $g(7)$

(ii)  $\lim_{x \rightarrow 7} g(x)$

(iii)  $g(3)$

(iv)  $\lim_{x \rightarrow 3} g(x)$

(b) *Simplify and then graph the function  $g(x)$ .*

(c) *Interpret the expressions in part (a) geometrically by considering the graph of the function  $f(x) = x^2/3$  as in the demonstration below.*

*Desmos link:* <https://www.desmos.com/calculator/u0uvuchnrk>

*Desmos activity available at 151: Parabola Basic*

## Limits and Gas Mielage

**Example 7.** *The function*

$$G = f(s) = \frac{2}{5} + \frac{1}{5000}(40(s+2)^2 - (s+2)^3), \quad 0 \leq s \leq 23,$$

*expresses the number of gallons of gas in your car in terms of your distance from home. The distance is measured in miles along your route.*

*Desmos link:* <https://www.desmos.com/calculator/xzknfjpkw3>

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Learning outcomes:  
Author(s):

*Desmos activity available at 151: Gas as a Function of Distance*

(a) Use the graph of the function  $f$  shown above to determine if you are driving toward or away from home. Explain your reasoning.

(b) Find your average gas mileage (in miles/gallon) over the interval  $s \in [8, 18]$ .

(c) Use the graph to approximate your gas mileage at the moment you are 18 miles from home. Do this by zooming in on the appropriate point.

(d) Use the algebra of limits to determine your exact gas mileage at the moment you are 18 miles from home.

(e) Use the result of part (d) to approximate the change in the volume of gas

$$\Delta G = f(s) - f(18)$$

in terms of the change

$$\Delta s = s - 18$$

in your distance from home for values of  $s$  near 18 miles. What are the units of the proportionality constant?

(f) Use the result of part (e) to approximate your distance from home when there are 1.9 gallons of gas in your tank.

## Limits, Gas Mileage and Speed

**Example 8.** Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and that the car gets 38 miles/gallon at a speed of 40 miles/hour.

(a) Find an expression for the function

$$G = f(v), \quad 30 \leq v \leq 70,$$

that gives the gas mileage (in miles/gal) in terms of the speed (in miles/hour).

(b) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the car's speed at 60 miles/hour gives an approximately proportional change in its gas mileage.

(c) Use the results of part (b) to approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find the exact value of the proportionality constant.

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta G = g - f(60)$$

in gas mileage in terms of a small change

$$\Delta v = v - 60$$

in the car's speed.

- (g) Use part (f) to approximate the speed at which the car gets 36 miles/gallon.
- (h) Would you expect your approximation in part (g) to be greater or less than the actual speed? Explain your reasoning with a graph.
- (i) Simplify the units of the proportionality constant. What might these units suggest about a way to interpret the constant?
- (j) At what rate (in gal/hr) does the car burn gas at a speed of 60 miles/hour?
- (k) How is the rate in part (j) related to the proportionality constant?

## Limits, Speed and Altitude

**Question 6** A rock dropped from a height of 100 feet falls to the surface of Planet Krypton without air resistance.

(a) By considering only the physical situation and without doing any computations, sketch a graph of the function

$$v = g(h), 0 \leq h \leq 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet).

(b) Use the results from part (a) to choose a reasonable expression for the function  $g$  from the list below.

**Multiple Choice:**

- (a)  $g(t) = 100 - 9t^2, 0 \leq t \leq 10/3$
- (b)  $g(h) = 100 - 9h^2, 0 \leq h \leq 100$
- (c)  $g(h) = 0.005(100 - h)^2, 0 \leq h \leq 100$
- (d)  $g(h) = 6\sqrt{100 - h}, 0 \leq h \leq 100$  ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rock's height from 64 feet gives an approximately proportional change in its speed.

(d) Use the results of part (c) to approximate the proportionality constant. What are its units?

(e) Use the algebra of limits to find the exact value of the proportionality constant.

(f) Explain the meaning of the proportionality constant.

(g) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

(h) Use part (g) to approximate the rock's speed at a height of 63 feet.

(i) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.

(j) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?

## Limits and Gas Mileage

## Limits and Purchasing Power

# The Derivative

*Computing derivatives with limits.*

**Example 9.** *Suppose*

$$y = f(x) = x^2$$

*and let's use limits to evaluate*

$$f'(3) = \left. \frac{dy}{dx} \right|_{x=3} = \left. \frac{d(x^2)}{dx} \right|_{x=3}.$$

*We have*

$$\begin{aligned} f'(3) &= \left. \frac{d(x^2)}{dx} \right|_{x=3} = \lim_{v \rightarrow 3} \frac{f(v) - f(3)}{v - 3} \\ &= \lim_{v \rightarrow 3} \frac{v^2 - 9}{v - 3} \\ &= \lim_{v \rightarrow 3} \frac{(v - 3)(v + 3)}{v - 3} \\ &= \lim_{v \rightarrow 3} (v + 3) \\ &= (3 + 3) \\ &= 6. \end{aligned}$$

*Next let's do almost the same thing and compute*

$$f'(x) = \frac{dy}{dx}$$

*for the function*

$$y = f(x) = x^2$$

*by replacing 3 in the above computation with  $x$ .*

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Learning outcomes:  
Author(s):

We get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d(x^2)}{dx} = \lim_{v \rightarrow x} \frac{f(v) - f(x)}{v - x} \\
 &= \lim_{v \rightarrow x} \frac{v^2 - x^2}{v - x} \\
 &= \lim_{v \rightarrow x} \frac{(v - x)(v + x)}{v - x} \\
 &= \lim_{v \rightarrow x} (v + x) \\
 &= (x + x) \\
 &= 2x.
 \end{aligned}$$

Just as a check, when  $x = 3$ ,

$$\left. \frac{dy}{dx} \right|_{x=3} = (2x) \Big|_{x=3} = 2(3) = 6.$$

**Question 7** Use the result of the previous example to solve each of the following problems. Do not use limits.

- (a) Find an equation of the tangent line to the parabola  $y = x^2$  at the point  $(-4, 16)$ .
- (b) Find an equation of the tangent line to the parabola perpendicular to the tangent line in part (a).
- (c) Find the coordinates of the point where the lines in parts (a) and (b) intersect.
- (d) Let  $\mathcal{L}$  be the line through the points of tangency of the lines in parts (a) and (b). Find the coordinates of the point where  $\mathcal{L}$  intersects the  $y$ -axis.
- (e) Repeat parts (a)-(d) above for the tangent line to the parabola  $y = x^2$  at the general point  $(a, a^2)$ . What do you notice?

**Question 8** (a) Use the method of Example 1 for the function

$$y = g(x) = 1/x^2$$

to compute

$$g'(3) = \left. \frac{dy}{dx} \right|_{x=3} = \left. \frac{d(1/x^2)}{dx} \right|_{x=3}$$

and

$$g'(x) = \frac{dy}{dx}.$$

(b) Use the result of part (a) to find an equation of the tangent line to the curve  $y = 1/x^2$  at the point  $(3, 1/9)$ .

**Question 9** (a) Use numerical methods to estimate the slope of the tangent line to the curve

$$y = f(x) = x^3$$

at the point  $(2, 8)$ . Include enough data to suggest a progression toward a limit.

(b) Use the algebra of limits to find the exact slope of the tangent line in part (a).

(c) Use algebra to find the coordinates of the all pointw where the tangent line in part (a) intersects the curve  $y = x^3$ .

(d) Suppose you measure the edge length of a cube to be 2cm and then use this measurement to compute the volume of the cube. Use the result of part (b) to approximate your error in computing the volume in terms of your error in measuring the edge length. Assume the latter error is small.

Then compare your exact error in computing the volume with your approximation for some specific edge length near 2cm. You should start this problem defining a function with meaningful variable names (do not use  $x$  and  $y$ ).

**Question 10** (a) Find a function

$$s = g(V), V \geq 0$$

that expresses the edge length (measured in cm) of a cube in terms of its volume (measured in cubic centimeters).

(b) Use the algebra of limits to evaluate the derivative

$$g'(V_0) = \left. \frac{ds}{dV} \right|_{V=V_0}.$$

(c) What are the units of the derivative in part (b)? Explain how you know.

(d) Suppose you submerge the cube in water and measure its volume to be  $8 \text{ cm}^3$ . You then use this measurement to compute the edge length of the cube. Use the result of part (b) to approximate your error in computing the edge length in terms of your error in measuring the volume. Assume the latter error is small.

Start this problem by defining the errors  $\Delta s$  and  $\Delta V$  in terms of the actual edge length  $s$  of the cube.

(e) Use the result of part (d) to approximate your error in computing the edge length of the cube if the cube's actual volume is  $7.7 \text{ cm}^3$ .

(f) Explain how this question is related to Question 4.

**Question 11** Suppose that between speeds of 60 miles/hr and 72 miles/hr, the gas mileage of a car is a linear function of its speed. Suppose also that the car gets 36 miles/gallon at a speed of 60 miles/hour and 32 miles/gallon at a speed of 72 miles/hour.

(a) Find a function

$$r = f(v), 60 \leq v \leq 72,$$

that expresses the rate (measured in gal/hr) at which the car burns gas in terms of its speed (measured in miles/hour). Explain your reasoning. This function is not linear.

**Hint:**

$$r = f(v) = \boxed{\frac{3v}{168 - v}}, 60 \leq v \leq 72.$$

(b) Use numerical methods to estimate the value of the derivative

$$f'(63) = \left. \frac{dr}{dv} \right|_{v=63}.$$

Make a table that shows enough data to suggest a progression toward a limit. Include units in all column headings.

(c) Use the algebra of limits to find an expression for the derivative

$$f'(v) = \frac{dr}{dv}.$$

Then use this expression to find the exact value of the derivative in part (b).

(d) What are the units of the derivative in part (b)? Explain its meaning.

(e) Use the result of part (c) to approximate the change

$$\Delta r = f(v) - f(63)$$

in the rate at which your car burns gas in terms of the change

$$\Delta v = v - 63$$

of the car's speed. Assume  $\Delta v \sim 0$ .

**Question 12** Suppose that between speeds of 30 miles/hour and 70 miles/hour the gas mileage of a car is a quadratic function of its speed. Suppose also that



the car gets a maximum of 42 miles/gal at a speed of 50 miles/hour and 34 miles/gallon at a speed of 30 miles/hour.

(a) Find a function

$$r = h(v), \quad 30 \leq v \leq 70,$$

that expresses the rate (in gal/hr) at which the car burns gas in terms of its speed (in miles/hour).

**Hint:** (i) At what rate does the car burn gas at a speed of 50 miles/hour? 25/21 gal/mile

(ii) Find a function that expresses the gas mileage  $G$  (measured in miles/gallon) in terms of the speed (measured in miles/hr).

$$G = \boxed{42 - 0.02(v - 50)^2}, \quad 30 \leq v \leq 70.$$

(ii) The rate (in gal/hr) at which the car burns gas as a function of its speed (in miles/hr) is

$$r = h(v) = \boxed{\frac{v}{42 - 0.02(v - 50)^2}}, \quad 30 \leq v \leq 70.$$

(b) Use the algebra of limits to evaluate the derivative

$$h'(40) = \left. \frac{dr}{dv} \right|_{v=40}.$$

(c) What are the units of the above derivative? How do you know?

(d) Express the meaning of the derivative in the context of small changes.

**Question 13** At 10:00am on April 18, the wholesale price of Cosmic Crisp apples is \$2.00/lb and is decreasing at the rate of \$0.10/lb/hour.

Use the algebra of limits to determine the rate (in pounds/hour) at which the number of pounds of apples a store can purchase with \$1000 is changing at this time.

Start this question by defining a function that expresses the number of pounds of apples the store can buy with \$1000 in terms of the price (in \$/lb). Choose meaningful variable names (not  $x$  and  $y$ ). Do not assume the price is decreasing at a constant rate.

## Limits, Speed and Altitude

**Question 14** A rock dropped from a height of 100 feet falls to the surface of Planet Krypton without air resistance.

(a) By considering only the physical situation and without doing any computations, sketch a graph of the function

$$v = g(h), 0 \leq h \leq 100$$

that expresses the rock's speed (in ft/sec) in terms of its height (in feet). Explain your reasoning.

(b) Use the results from part (a) to choose a reasonable expression for the function  $g$  from the list below.

**Multiple Choice:**

(a)  $g(t) = 100 - 9t^2, 0 \leq t \leq 10/3$

(b)  $g(h) = 100 - 9h^2, 0 \leq h \leq 100$

(c)  $g(h) = 0.005(100 - h)^2, 0 \leq h \leq 100$

(d)  $g(h) = 6\sqrt{100 - h}, 0 \leq h \leq 100$  ✓

(c) Give numerical and graphical evidence that either supports or refutes the claim that a small change in the rocks height from 64 feet gives an approximately proportional change in its speed. Then approximate the proportionality constant. What are its units?

(d) Use the algebra of limits to find an expression for the derivative

$$g'(h) = \frac{dv}{dh}.$$

Then use this expression to find the exact value of the proportionality constant in part (c).

(e) Explain the meaning of the proportionality constant.

(f) Approximate the change

$$\Delta v = v - g(64)$$

in the rock's speed in terms of a small change

$$\Delta h = h - 64$$

in its height.

- (g) Use part (g) to approximate the rock's speed at a height of 63 feet.
  - (h) Would you expect your approximation in part (h) to be greater or less than the actual speed? Explain your reasoning with a graph.
  - (i) Simplify the units of the proportionality constant. Does this simplification help to understand or obscure the meaning of the proportionality constant?
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# Derivatives of Polynomials

*Working with polynomials and their derivatives.*

Nothing yet

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Learning outcomes:  
Author(s):