A Global Geometrical Input–Output Linearization Theory

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ABSTRACT

Input–output linearization is a method that uses nonlinear state feedback control to obtain linear equations between the input and output of a nonlinear control system . Its success depends crucially on the ability of the control to steer the state across the level sets of the output function. It is clear that this is essentially a global geometrical problem. In this work, we present a linearization theory that is geometrical. This means that the relation of the control vector fields to the output function is considered from a global geometrical perspective. We argue for general position, generic vector fields and output functions and linearization for typical trajectories. We also prove that the relative degree is generically one.

This approach succeeds in resolving the problem of singularities and is thus more complete than the existing literature. The unavoidable cost is that since the concept of strong relative degree cannot, in general, be defined globally, linearization may be piecewise. A simple example is included with a common type of output function and linearization is worked out in some detail as an illustration of some of the features of this novel approach.

1. Introduction

Input–output linearization theory has the task of obtaining linear equations between the input and output of a nonlinear system by using nonlinear state feedback. A necessary condition for achieving this is that we should have available control directions that can move the state across the level sets of the output function. To be more specific, at every point in the state space consider all possible paths through it that can be obtained using some control strategy. Then this set must contain, for every small number ϵ, a path that leaves the level set of the output function in time less than or equal to ϵ. The selection rule for the linearizing state feedback is such that we can use supplementary control to move across the level sets of the output function in a linear manner, for example driving it to zero with first–order dynamics. Control can also be chosen for other control purposes such as reference output tracking. Of course, since we are using full state feedback, we are assuming that the full state vector is observed. The output that is used in the linearization is some function of the state that we wish to stabilize or regulate.

In a sense, linearization theory is optimistic. It is an effort to transform nonlinear dynamics into linear ones, hoping that the transformation will preserve the most important features from the point of view of control (or at least that the parts that were left out will not interfere with control strategies). The unpleasant nonlinear dynamics having been relegated to the background, we then have familiar linear equations to work with. The exact state–space linearization theory posed by Brockett (1978) and solved independently by Hunt, Su and Meyer (1983) and Jakubczyk and Respondek (1980) gives conditions for the exact linearization of the state dynamics. A basic criticism of such work that will be made here is that the conditions used are not generic and will not work for many realistic control dynamics. The past few years have demonstrated that nonlinear dynamics are infinitely more subtle and complex than linear ones. At the same time, certain unifying concepts were brought to bear on the classification of possible nonlinear dynamical behavior. In fact, one can say that the interest in nonlinear dynamics is often in that part of their behaviour that cannot be captured by linear systems. (The distinction between local and global is a good example: linear systems 'look the same' close–up or viewed from a distance, in other words global dynamics can be inferred from the local ones; this is not the case for nonlinear systems, as the most simple system with two or more equilibrium points can demonstrate.)

The part of linearization theory that we discuss here, however, input–output linearization, has more modest aims. The elements of a comprehensive theory have been developed recently by Isidori, Byrnes and others. This research has led to the generalization of familiar concepts from linear systems, such as the system zeros (and zero dynamics) and minimum–phase systems.

The main results given in this paper are as follows. We prove that, generically (for 'almost all' possible control vector fields and output functions), the relative degree is one for systems with one output function. If more than one control is available, or if an appropriate Lyapunov function can be found, then we can control through the singular set $\{h(x) = 0\}$ for most trajectories. This can also be accomplished using control paths which have first–order contact with the singular set but not second–order contact. A consequence of our theory is that the singular set is generically a manifold and so are intersections of singular sets. Thus the zero dynamics can be defined piecewise globally on a smooth, closed manifold of appropriate dimension. So far, the existence of feedback controls that stabilize the output function is established, but there is no guarantee that the corresponding state trajectories are bounded. This led Isidori and Byrnes to the generalization of the concept of minimum phase: the requirement that the zero dynamics have a unique asymptotically stable equilibrium. In the present work, we make the more general assumption that the zero dynamics be dissipative (in the sense of Kappos (1986a) and (1986b)).

A criticism of exact input–output linearization theory is that it relies on exact cancellation. We show here that in fact the process is quite robust, since the vector field obtained from the stabilizing, linearizing feedback control has the output function as a global Lyapunov function. However, we make a more general criticism of previous nonlinear systems theory research in that the assumptions made in theses works are (a) influenced too much by linear theory, thus occasionally missing important aspects of nonlinear dynamics, (b) unlikely to hold globally, e.g. not generic and (c) often difficult to verify. Thus the resulting theories are often local at best. These points will be illustrated here primarily for the case of linearization theory, occasionally making reference to other parts of nonlinear systems theory.

In Section 2 below, we give an account of current input–output linearization theory as a first step in the construction of the geometric linearization theory of Section 3 for the case of one–dimensional output. Section 4 contains the proofs of the main results. In Section 5, using a three–dimensional example, we demonstrate the efficiency and practicality of our approach. The last section draws some conclusions.

2. Review of Input–Output Linearization Theory

In this section we review critically the existing linearization theory (Byrnes and Isidori (1984) and (1988) and Sastry and Isidori (1987)). In particular, we point out that it is not realistic to expect that the *strong relative degree* exists (is well defined) for a general nonlinear system. The concept of globally defined *zero dynamics* (as defined, for example, in Byrnes and Isidori (1988)) is therefore also put to question. The usual approach to linearization theory is thus shown to be imprecise or, at least, unlikely to be directly applicable. Some effort has been made to deal with this problem (Lamnabhi–Lagarrigue et. al. (1988), Hirschorn and Davis (1988)). In the next section, we give a complete treatment from a more geometrical approach. This approach is drawn from the global geometrical setting of Kappos (1986a) and (1986b) and Kappos and Sastry (1986), where it proved useful in discussing global control and stability problems (Lyapunov stability and stability related to large deviations).

Consider the nonlinear control system

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

and the observation equation

$$y = h(x), \tag{2}$$

with $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$, $f, g_1, \ldots, g_m$ smooth vector fields and h a smooth function defined on $\mathbf{R}^n$. The aim of linearization theory is to cast the dynamics in such a form that the control u can be used directly to steer the output of the system y; moreover, the equation relating input and output should be 'linear'. With $\mathcal{L}_{\mathbf{X}} h$ the *Lie derivative* of the function h in the direction of the vector field $\mathbf{X}$ (note that the Lie derivative evaluated on the function h is another smooth function on $\mathbf{R}^n$), we obtain by differentiation of the output

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f + \sum u_i g_i) \\ &= \mathcal{L}_f h + \sum u_i \mathcal{L}_{g_i} h \end{aligned}. \tag{3}$$

For a single–input, single–output system, suppose $\mathcal{L}_g h(x) \neq 0$ in an open set $U \subset \mathbf{R}^n$. Then the control

$$u(x, v) = -\frac{1}{\mathcal{L}_g h(x)}(\mathcal{L}_f h(x) - v) \tag{4}$$

yields the linear input–output equation

$$\dot{y} = v \tag{5}$$

Note that this only holds in U. We have no guarantee that the trajectories starting in U will stay there. In fact, they may leave U in finite time. In U, we are now free to choose the control v. To 'stabilize' the output, we set $v = ah(x) = ay$, with $a < 0$. In this case, the control u is a feedback control. If $\mathcal{L}_g h(x) = 0$ in U, then differentiate once more to get

$$\ddot{y} = \mathcal{L}_f \mathcal{L}_f h(x) + \mathcal{L}_g \mathcal{L}_f h(x) u. \tag{6}$$

Again, if $\mathcal{L}_g\mathcal{L}_f h(x) \neq 0$ in U, the control

$$u(x,v) = -\frac{1}{\mathcal{L}_g\mathcal{L}_f h(x)}(\mathcal{L}_f^2 h(x) - v) \tag{7}$$

yields the linear equation

$$\ddot{y} = v. \tag{8}$$

It is still possible to 'stabilize' the output by selecting, for example the feedback control $v = -a_1\dot{y} - a_2 y = -a_1\mathcal{L}_f h(x) - a_2 h(x)$, so that (8) is Hurwitz (stable). It is easy to see how to generalize this to the case where more differentiations are required for u to appear explicitly in the output equation. The **strong relative degree** is defined to be the least γ for which the control appears in $y^{(\gamma)}$.

For dynamics with strong relative degree γ it can be easily checked that the vector field

$$f - \frac{\mathcal{L}_f^\gamma h}{\mathcal{L}_g\mathcal{L}_f^{\gamma-1} h} g \tag{9}$$

leaves the set

$$\{x \in U \ ; \ h(x) = \mathcal{L}_f h(x) = \cdots = \mathcal{L}_f^{\gamma-1} h(x) = 0\} \tag{10}$$

invariant and thus defines a vector field on that set which is called the **zero dynamics**. The system is called **minimum phase** if the zero dynamics have a unique, asymptotically stable equilibrium point.

One can show that a change of coordinates $z = \phi(x)$ exists, $\phi : U \to U$ with

$$\begin{aligned} \phi_1(x) &= h(x) \\ \phi_2(x) &= \mathcal{L}_f h(x) \\ &\cdots \\ \phi_\gamma(x) &= \mathcal{L}_f^{\gamma-1} h(x) \end{aligned} \tag{11}$$

and such that

$$d\phi_i(g)(x) = 0 \text{ for } i = \gamma+1, \ldots, n.$$

In the new coordinates, the state equation is written in the '*normal form*' (see Byrnes and Isidori (1988) or Sastry and Isidori (1987)) whose first γ component are given by

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_\gamma &= \mathcal{L}_f^\gamma h(x) + \mathcal{L}_g\mathcal{L}_f^{\gamma-1} h(x) u \end{aligned} \tag{12}$$

The last $n-\gamma$ components of the new state vector give the zero dynamics (see Sastry and Isidori (1987)).

As we see in the next section, it is unlikely that the conditions—on which the above linearization theory depends—hold for $U = \mathbf{R}^n$. The condition $\mathcal{L}_g h \neq 0$ is an open one (it is satisfied in an open set) but this is not the case for the condition $\mathcal{L}_g h = 0$ (and similarly for higher–derivative equality and inequality conditions). It is possible that the set in which the higher–order linearizing control is applicable may have empty interior. In the typical case, we will have to contend with a non–empty singular set (the set of points where the Lie derivatives condition vanishes). Note that for linear systems these sets are always the whole of the state space or the empty set and thus the problem does not arise; this is an instance where reasoning by analogy with linear systems is misleading. In the typical case, as we will show, we have to contend with the fact that the set U may not be invariant for the flow corresponding to the linearizing feedback, but also with a partition of the state space into the singular set and it complement.

Thus the linearization theory of this section is at best local (if applicable in an open set) and it suggests no obvious ways of globalizing it. In particular, it does not give ways of dealing with the singular sets. We attempt in the next section to overcome these defficiencies by approaching the problem from a more geometrical, global point of view.

3. Geometrical Linearization Theory

For the manifold M define the set of all smooth functions $C^\infty(M)$ and the space of smooth vector fields $\mathcal{X}(M)$, considered with the Whitney C^∞ topology (see Hirsch (1976)). The manifold M is not necessarily compact, unless otherwise indicated. Most of our results will also be true if we assume that the manifold M considered is a compact manifold with boundary containing all the important limit set behaviour of the dynamics f which is assumed dissipative. This can be taken to be the interior, with respect to the flow of f of a Lyapunov surface that contains all the bounded α– and ω–limit sets of f (for precise definition of dissipative dynamics, see Kappos (1986a)). The standing assumption in this case will be that the sets involved are in the interior of this manifold. Since we are implicitly assuming that M^o is an open subset of $\mathbf{R}^n$, we will often write coordinate versions of the results. (Care must be taken not to disturb the transversality of the resulting controlled flow with the boundary; this can be arranged by smoothly zeroing the effect of the control near the boundary—in the case of stabilization, this does not alter the qualitative behaviour of the controlled flow.)

Define the **singular set** of a function h with respect to a vector field $\mathbf{X}$ by

$$M^{\mathbf{X}}(h) \equiv \{x \in M \ ; \ \mathcal{L}_{\mathbf{X}} h(x) = 0\} \tag{13}$$

(We can also view this as defining the zero set of the function obtained by evaluating the *one–form* dh along the vector field $\mathbf{X}$, $dh(\mathbf{X})$; in local coordinates, of course, this gives $\frac{\partial h}{\partial x}\mathbf{X}$.)

Next we discuss **piecewise smooth** feedback systems. Suppose the manifold M is partitioned into disjoint open sets $U_1, U_2, \ldots, U_k$ such that $M = \overline{(U_1 \cup U_2 \cup \ldots \cup U_k)}$, that is M is the closure of the union of these sets. We also assume, for simplicity, that $\overline{U}_i$ is a manifold with boundary and that this boundary coincides with the set–theoretic boundary of the set U_i. (This is done to avoid the case where the boundary of the closure of a set does not coincide with the set–theoretic boundary, see the interesting controversy in Zaborsky et.al. (1988) and Arapostathis et.al. (1982).) If $\mathbf{X}$ is a vector field in $\mathcal{X}(M)$, then it is also a vector field in each of the U_i. If $\mathbf{X}$ is a complete vector field in M, it need not be complete in the open sets U_i. From now on we assume given a vector field $\mathbf{X}_i$ for every set U_i, defined at least on $\overline{U}_i$. The equivalent differential equation will be defined, for every point x, on a maximal time interval depending on the point: (a_x, b_x) (this interval is always open). Suppose the common boundary of U_i and U_j is nonempty,

$$B \equiv \partial\overline{U}_i \cap \partial\overline{U}_j \neq \emptyset. \tag{14}$$

DEFINITION: *The solution of the differential equation corresponding to the vector field* $\mathbf{X}$ *can be* **continued across the common boundary** B *if for every point* x *in the boundary we can find points* $x_1 \in U_i$ *and* $x_2 \in U_j$ *such that*

(i) the closure of the maximal trajectories through x_1 *and* x_2 *are transverse to* B *and*

(ii) x *is the limit point of the trajectory through* x_1 *as we approach the upper endpoint* b_{x_1} *and the limit point of the trajectory through* x_2 *as we approach* a_{x_2} *(or the other way round, as we approach* a_{x_1} *and* b_{x_2} *respectively).*

If the solution of the differential equation can be continued across all the possible boundaries, suppose piecewise smooth trajectories exist globally (for all times). The resulting vector field is well–defined globally and will be called **piecewise complete**.

Another technical tool we need is the existence of **normal tubular neighborhoods** (see Hirsch (1976), p.109, Kappos (1986a)) for submanifolds N of $\mathbf{R}^n$, or of a manifold M. In $\mathbf{R}^n$, for small $\epsilon > 0$, an ϵ–normal tubular neighborhood is essentially an ϵ–thickening of the submanifold along its normal bundle, e.g. an annulus encircling a circle in the plane. More precisely, a *tubular* *neighborhood* of a submanifold N of $\mathbf{R}^n$ is a pair $(\beta, \mathbf{B})$, where $\mathbf{B} = (E, N, \pi)$ is a vector bundle over N and β is an embedding of E in $\mathbf{R}^n$ such that:

(a) β restricted to N is the identity map and

(b) $\beta(E)$ is an open neighborhood of N in $\mathbf{R}^n$.

We shall, by abuse of notation, refer to $\beta(E)$ as the tubular neighborhood of N. Note that the fibre over any $x \in N$ can be taken to be the normal space to the tangent space at x which, in $\mathbf{R}^n$ is identified with its orthogonal complement $T_x K^\perp$. In this case, $(\beta, \mathbf{B})$ is called a *normal tubular neighborhood* (n.t.n.) of N and we can take $\beta(E)$ to be the set $N_\epsilon(N)$, for some small ϵ.

We always assume from now on that the constant ϵ has been chosen small enough so that the neighborhood $N_\epsilon(N)$ exists as a normal tubular neighborhood.

The notion of genericity needs to be made precise. A subset of a set is called **residual** if it contains a Baire subset (ie the intersection of a countable collection of dense, open subsets). In the case when the set is compact, a residual subset is open and dense.

DEFINITION: *A property τ is* **generic** *for the set T if it holds in a residual subset of T.*

We will say that the control system (1), (2) is **piecewise linearizable** in the manifold M if there is a partition as above and a piecewise smooth feedback control that yields a linear input–output equation in every set of the partition. The main results of this paper are contained in the two theorems below that refer to a control system with a scalar output function.

THEOREM 1: *The relative degree of the generic single output system (1), (2) defined in the manifold M is one. This means that there is a residual subset of the set $C^\infty(M) \times \mathcal{X}(M) \times \cdots \mathcal{X}(M)$ (in the Whitney topology) of pairs (h, g) such that $\mathcal{L}_g(h) \neq 0$ in a nonemtpy subset of state space. Moreover, in the case of a single control vector field, the singular set $M^g(h)$ for the generic pair (h, g) is a closed, $(n-1)$–dimensional submanifold of M.*

Let $N_\epsilon(M^g(h))$ be an ϵ–normal tubular neighborhood of this manifold. Then, if for the dynamics f the function $|h|$ is a Lyapunov function in $\overline{N}_\epsilon(M^g(h))$, there exists a piecewise smooth feedback control $u^*(x)$ that stabilizes the system output globally and linearizes it outside $N_\epsilon(M^g(h))$. For the choice of a stabilizing control, the function $|h|$ is a global Lyapunov function.

If $m > 1$, i.e. if there are two or more controls available, let $M^{g_1}(h) \cap M^{g_2}(h)$ be the generic transversal intersection, an $(n-2)$–dimensional manifold. Then there is a piecewise smooth feedback control defined everywhere in $M \setminus N_\epsilon(M^{g_1}(h) \cap M^{g_2}(h))$ that piecewise linearizes the system. The control can be chosen so that the output function $y = h(x)$ is a strict Lyapunov function in this region.

To apply linearizing control near the singular set, there is also the option of differentiating the output function further. This does not, however, lead to an explicit formula for the linearizing control. We give a result on stabilization that requires the solving of a partial differential equation.

THEOREM 2: *For the case $m = 1$ and for a generic pair (h, g), suppose the singular manifold $M^g(h)$ is compact. If, for some $\epsilon_0 > 0$, we have that in $N_{\epsilon_0}(M^g(h))$*

(i) $\mathcal{L}_g^2 h \neq 0$ and

(ii) the discriminant condition

$$(\mathcal{L}_g\mathcal{L}_f h + \mathcal{L}_f\mathcal{L}_g h)^2 - 4\mathcal{L}_f^2 h\, \mathcal{L}_g^2 h > 0$$

is satisfied, then there exists a solution u_p of the polynomial equation (30) of section 4 in the set $N_{\epsilon_0}(M^g(h))$. Further suppose that the vector field $f + gu_p$ is transverse to $M^g(h)$.

Then there exists a piecewise smooth feedback control u^* defined in $M \setminus N_\epsilon(M^g(h))$ for some ϵ, $0 < \epsilon \leq \epsilon_0$ that piecewise stabilizes the system (1), (2), yielding the linear equation

$$\ddot{y} + a_1\dot{y} + a_0 y = 0$$

for some a_1, a_0 such that the above equation is Hurwitz stable.

In either theorem, the intersection of the singular sets may be empty. In such a case, the feedback is defined globally. The **strong** relative degree is not definable, in general. We have also lost the concept of globally definable zero dynamics.

Note that in both cases of theorem 1, choosing a stabilizing feedback control makes the controlled dynamics

$$f(x) + g(x)u^*(x)$$

have the output function h as a strict Lyapunov function in the indicated regions. This means that even though the feedback control seems to depend on **exact cancellation**, (a criticism of linearization theory in general, see for example Sastry and isidori (1987)), the stability results are stable (robust) under small perturbations in f and g. This is because if a vector field has a strict Lyapunov function globally in an open set M, then all vector fields sufficiently close have the same Lyapunov function (and also if we perturb the function slightly, we do not destroy the property of it strictly decreasing along trajectories of the vector field). A problem that remains, however, and may make the application of linearization methods problematic in practice is the fact that the linearizing control action may become very large as we approach the singular set $M^g(h)$ ϵ–closely.

The piecewise linear system of the last part of theorem 1 is defined eveywhere in the state space except a neighborhood of an $(n-2)$–dimensional set. This does not *'obstruct'* most of the trajectories from reaching the zero set of the output function. This is made precise in the following result given for M compact that asserts that the set of obstructed states can be made arbitrarily small. Define the obstructing set

$$\Sigma_\epsilon \equiv \overline{N_\epsilon(M^{g_1}(h) \cap M^{g_2}(h))},$$

and let ϕ^* be the piecewise maximal flow corresponding to the choice of a linearizing, stabilizing feedback control u^* defined in $M \setminus \Sigma_\epsilon$. Also define

$$S_\epsilon^- \equiv \{x \in M \setminus \Sigma_\epsilon\,;\; x = \phi^*(y,t)\,,\, y \in \partial\Sigma_\epsilon\,,\, t \leq 0\}$$

to be the set of states that do not get attracted to the zero of the output (that get obstructed by the set Σ_ϵ). Finally, assume Lebesgue measures m and μ are given on M and on $(n-1)$–dimensional submanifolds of M such that $m(N) = 0$, for N a submanifold.

THEOREM 3: *Suppose that for some ϵ_0 the obstructing set Σ_{ϵ_0} is compact with a smooth orientable boundary $\partial\Sigma_{\epsilon_0}$ and that Σ_{ϵ_0} does not intersect the zero set of h. Also suppose that h has no critical points in $\overline{\Sigma}_{\epsilon_0}$.*

Then, for any $\delta > 0$, there is an ϵ , $0 < \epsilon < \epsilon_0$ such that,

$$\frac{m(S_\epsilon^-)}{m(M)} < \delta.$$

Proof: Without loss of generality assume $h(x) > 0\,, \forall x \in \Sigma_{\epsilon_0}$. Let M^+ be the set where h is positive. We have that $m(M^+) > 0$. Use the gradient of the output function to extend the flow inside the set Σ_{ϵ_0}. The outward normal $n(x)$ is defined on the boundary $\partial\Sigma_{\epsilon_0}$. The sets $\partial\Sigma_{\epsilon_0}^-$ and $\partial\Sigma_{\epsilon_0}^+$ of points where the gradient of h, $\nabla h(x)$ points inwards (outwards) are non–empty and open. They are non–empty since otherwise the set $\overline{\Sigma}_{\epsilon_0}$ would be invariant under the gradient flow of h and hence h would have to have a minimum (maximum) there. They are open by the openness of the conditions

$$n(x) \cdot \nabla h(x) > 0 \; (< 0),$$

and they are therefore submanifolds of $\partial\Sigma_{\epsilon_0}$.

By the non–critical neck theorem, the gradient flow ∇h induces a diffeomorphism

$$F : \partial\Sigma_{\epsilon_0}^- \to \partial\Sigma_{\epsilon_0}^+$$

between the two subsets of the boundary. These two manifolds are *cobordant*. The set where $n(x) \cdot \nabla h(x) = 0\,,\, x \in \partial\Sigma_{\epsilon_0}$ is invariant under the gradient flow and hence does not affect the result.

We have thus obtained a piecewise smooth flow in the whole of the state space. This flow can be smoothed out if desired in the usual manner by considering neighborhoods of the separating boundaries and applying local smoothing using bump functions. What we end up with is a global gradient flow ϕ^* ($|h|$ is a global Lyapunov function), with the set $\{x \in M \;;\; h(x) = 0\}$ being the unique attracting limit set. (Note that in some cases there will be topological constraints on flows defined in M so that such gradient flows are not possible).

Next notice that

$$S^-_{\epsilon_0} = \phi^*(\partial\Sigma^-_{\epsilon_0}\,,\,(-\infty, 0]).$$

Now choose $k > 0$ such that the k–level set of h: $H_k \equiv \{x; h(x) = k\}$ is nonempty (choose k small, for example). H_k is in fact a submanifold of M of dimension $n-1$, since $\nabla h \neq 0$ in M^+. It is a *Lyapunov surface* for the flow ϕ^* (it is everywhere transverse to the flow). The projection map

$$\pi \;:\; M^+ \to H_k$$

is smooth and onto. Call Γ the intersection $M^{g_1}(h) \cap M^{g_2}(h)$. Now $\mu(\pi(\Gamma)) = 0$, since Γ is of dimension $n-2$. For any δ_0, we can choose $\epsilon_0 > 0$ such that $\mu\big(N_{\epsilon_0}(\pi(\Gamma))\big) < \delta_0\,\mu(H_k)$. (This neighborhood is not necessarily tubular, since the projection may not be an embedding). Also note that $m\big(\phi^*(\Gamma, (-\infty, \infty))\big) = 0$. By continuity, for any $\epsilon_0 > 0$, there is an $\epsilon > 0$ such that $\pi(N_\epsilon(\Gamma)) \subset N_{\epsilon_0}(\pi(\Gamma))$. This means that there is an ϵ such that $\mu\big(\pi(N_\epsilon(\Gamma))\big) < \delta_0\mu(H_k)$. A simple extension of this argument using the projection map shows that δ_0 can be chosen such that $m\big(\phi^*(\Sigma_\epsilon, (-\infty, \infty))\big) < \delta m(M)$, for the required δ. This proves the theorem, since S^-_ϵ is contained in the set $\phi^*(\Sigma_\epsilon, (-\infty, \infty))$.

The proof hinged on it being possible to define the projection map in M^+. This depends crucially on the fact that there are no critical points of h in the neighborhood $N_{\epsilon_0}(\Gamma)$.

We can do better if we use all available controls: for every control, generically, we reduce the dimension of the obstruction by one. Thus, for $n-1$ controls, the generic obstruction is simply a small ball. Of course, if the singular manifolds $M^{g_i}(h)\;,\; i = 1, \ldots, n-1$ have an empty intersection, we can define a piecewise linear system everywhere. A global stabilization result using linearizing feedback controls will be now given.

Suppose that we want to drive the output function $y = h(x)$ to zero and that it satisfies $h(x^0) = 0$ and $h(x) > 0$ in a neighborhood U of x_0. Equivalently, we want h to be, locally, a Lypunov function for the controlled dynamics, with x^0 an attractor.

Now the condition of **Lyapunov controllability** (introduced in Kappos (1986b) and Kappos and Sastry (1986)) is that

$$(\mathcal{L}_{g_1} h \cdots \mathcal{L}_{g_m} h)(x) \neq 0$$

This condition cannot be satisfied in all of U, since $\nabla(x^0) = 0$. Generically, it fails on a submanifold of dimension $n - m$. Suppose it is satisfied in $U \setminus \{x^0\}$. In particular this implies the existence of subsets K_i of U such that $\mathcal{L}_{g_i} h \neq 0$ in K_i and of functions β_i supported in the subsets K_i forming a partition of unity in U such that

$$g \equiv \sum_{i=1}^{m} \beta_i(x) g_i(x) \tag{15}$$

is a vector field transverse everywhere in U to the level sets of h. Then the local feedback control (in $U \setminus \{x_0\}$)

$$u_i^*(x, v) = -\beta_i(x) \big(\mathcal{L}_{g_i} h(x)\big)^{-1} \left[\mathcal{L}_f h(x) + v\right] \; , \; i = 1, \ldots, m \tag{16}$$

linearizes the input–output map, yielding

$$\dot{y} = \frac{dh}{dt} = v. \tag{17}$$

To stabilize the output, choose $v = -\alpha y = -\alpha h(x)$. We have thus proved the following result

THEOREM 4: *Given the function h defined in $U \ni x^0$ satisfying $h(x^0) = 0$, $h(x) > 0$, $x \neq x^0$, suppose it also satisfies the Lyapunov controllability condition.*

Then a feedback control law u^* exists that linearizes the input–output dynamics and can be chosen to stabilize the output, making h a local Lyapunov function for $f + \sum u_i g_i$ with x^0 an asymptotically stable equilibrium point.

We now come to the important concepts of zero dynamics and minimum phase systems. The stabilizing feedback controls of the two main theorems will produce state trajectories that asymptotically approach the zero level set of h, which is in general a closed manifold of dimension $n-1$. Even though the output function trajectory $h(x_t)$ is asymptotically stable for the resulting paths, the state trajectory may be unstable, i.e. it might run off to infinity. By analogy with linear systems we would like to call a nonlinear system **minimum phase** if this does not happen.

For nonlinear systems, the concept of asymptotic stability is more reasonably replaced by the global requirement that all possible future trajectories of the dynamics are bounded and are attracted to isolated attractors that are compact sets. These requirements, plus a structural stability assumption, lead to the definition of the class of **dissipative dynamics**, first defined and used in Kappos (1986a). For M compact, we let this class coincide with the class of **Morse–Smale** vector fields. For M not compact (eg $\mathbf{R}^n$), we assume the existence of an invariant oriented submanifold with boundary that is a global attractor for the dynamics outside it and inside which the flow is Morse–Smale. It is important to point out that dynamical systems in the above classes are structurally stable and possess global Lyapunov functions that are strict away from the recurrent sets.

We now proceed to give for the second case of Theorem 1 a result on minimum phase nonlinear systems and their zero dynamics. We denote by Γ the intersection $M^{g_1}(h) \cap M^{g_2}(h)$. Let 0 be a regular value of h and with

$$H_0 \equiv \{x \in M \ ; \ h(x) = 0\} \tag{18}$$

the zero level set of h, let $N_\epsilon(H_0)$ be an ϵ–neighborhood of it. Define the vector field

$$f_z \equiv f - \frac{\mathcal{L}_f h}{\mathcal{L}_{g_i} h} g \tag{19}$$

(where i is either 1 or 2 in the appopriate sets according to the theorem) in the set $H_0 \cap (M \setminus N_\epsilon(\Gamma))$. This vector field is called the **zero dynamics** of the system (1), (2).

THEOREM 5: *For the case when the conditions of the second part of theorem 1 are satisfied, suppose* 0 *is a regular value of the output function h, that the g_i , $i = 1, 2$ are bounded functions and that the zero dynamics (19) leave the set $H_0 \cap (M \setminus N_\epsilon(\Gamma)$ invariant, with the vector field pointing transversely inwards at the boundary. Further assume that the zero dynamics are dissipative in this set.*

Then the control system is minimum phase in the sense that all paths except those in the set $\phi^*(N_\epsilon(\Gamma), (-\infty, \infty))$ are future bounded, that is the forward time trajectories are bounded sets.

Remark: There is no a priori reason to expect that the zero dynamics will be dissipative. In this sense, the applicability of this resut is of limited value in practice.

4. Proofs of Main Results

In this section we give the proofs of Theorems 1,2 and 5. The first objective is to show that the singular set is generically non–empty.

It is clear that $\mathcal{L}_{\mathbf{X}} h$ is a smooth function on $\mathbf{R}^n$. If 0 is a regular value of this function, $M^{\mathbf{X}}(h)$ is a smooth, $(n-1)$–dimensional manifold. One reasonably expects that for 'most' functions h and for 'most' vector fields $\mathbf{X}$, this will be true. It is very important to remark that this does not follow directly from the usual transversality theorems. This is because we can only influence $\mathcal{L}_{\mathbf{X}}h$ by changing h and $\mathbf{X}$. Indeed, the point of the following result is to show that these perturbations of $\mathcal{L}_{\mathbf{X}}h$ are enough to move it to general position. Thus, the result is similar to the way the jet transversality theorem generalises the transversality theorem for maps.

THEOREM 6: *There is a residual subset K of $C^\infty(M) \times \mathcal{X}(M)$ such that for all pairs $(h, \mathbf{X}) \in K$, $M^{\mathbf{X}}(h)$ is a smooth, closed, $(n-1)$–dimensional manifold.*

In this case we will call $M^{\mathbf{X}}(h)$ the **singular manifold** of $\mathbf{X}$ with respect to h.

Examples:

If $h(x) = x_1$, then $M^{\mathbf{X}}(h)$ is just $\mathbf{X}_1$ and the result reduces to Sard's theorem for the single function $\mathbf{X}_1$, which is the first component of the vector field.

If h is a Morse function and $\mathbf{X}$ is the gradient field it defines, then $M^{\mathbf{X}}(h)$ is the set of critical points of h, which is not a manifold of dimension $(n-1)$. However, an arbitrarily small perturbation in either h or $\mathbf{X} = -\nabla h$ will yield a singular set that is a $(n-1)$–manifold, for example adding to the first vector field component a small number ϵ.

Proof of Theorem 6: The pair $(h, \mathbf{X})$, consisting of the function h and the vector field $\mathbf{X}$ gives a map from the product manifold $X \equiv M \times M$ to the product manifold $Y \equiv \mathbf{R} \times TM$, where TM is the tangent manifold of M.

Two functions h and g have **first–order contact** at a point x in M if they have the same function value and the same first–order derivatives (in any chosen local coordinates). Let $J^1(M, \mathbf{R})(x)$ be the set of equivalent classes of functions under the equivalence induced by having first–order contact. An element in that space is a **one–jet**. The **jet bundle** $J^1(M, \mathbf{R})$ is the space obtained by putting together the one–jets of all the points in M. It is a space of dimension $n+n+1 = 2n+1$ (the coordinates of the point x, the n derivatives and the value of the function h at x). The **one–jet** j^1h of some function h is the canonical map from M to $J^1(M, \mathbf{R})$, that associates to every point the function value $h(x)$ and the first–order derivatives of h at that point, in local coordinates $\frac{\partial h}{\partial x_1}(x)$, $\frac{\partial h}{\partial x_2}(x), \ldots, \frac{\partial h}{\partial x_n}(x)$. The zero jet manifold for the vector field, $J^0(M, TM)$ is simply the manifold $M \times TM$.

Consider the product jet bundle $J \equiv J^1(M, \mathbf{R}) \times J^0(M, TM)$. For the pair $(h, \mathbf{X})$, the jet $j \equiv (j^1h, j^0\mathbf{X})$ maps $X = M \times M$ to the jet bundle J above. Note that J has dimension $(2n + 1) + 3n = (5n + 1)$. Suppose we have selected local coordinates in J, in which the derivatives of a function (or, more properly, of a one–jet) correspond to coordinates $(p_1, p_2, \ldots, p_n)$ and the components of the vector field correspond to the coordinates $(q_1, q_2, \ldots, q_n)$. (For example j^1h will be mapped to $p_1 = \frac{\partial h}{\partial x_1}$ etc. and $j^0\mathbf{X}$ will be mapped to $q_1 = \mathbf{X}_1$ etc.) Define the set of all points in J *except the point* $(p,q) = 0$ for which

$$p_1 q_1 + p_2 q_2 + \cdots + p_n q_n = 0. \tag{20}$$

This set, call it W, is easily seen to be a closed submanifold of J of dimension $5n$.

We are now ready to apply the *jet transversality theorem* of Thom (Hirsch (1976), p.80, or Golubitsky and Guilllemin (1974), p.54)

TRANSVERSALITY THEOREM: *Consider the product spaces X and Y and the submanifold W of the jet bundle J defined above.*

Then the set

$$K \equiv \{(h,\mathbf{X}) \in C^\infty(M) \times \mathcal{X}(M) = C^\infty(X,Y)\ ;\ j(h,\mathbf{X}) \cap W \text{ transversely }\}$$

is a residual–and therefore dense—subset of $C^\infty(X,Y)$ in the Whitney C^∞ topology.

This theorem recasts the familiar jet transversality theorem of Thom, see Hirsch (1976), in a product space form. Its proof follows directly from that theorem and from elementary facts about product spaces. Since for elements of K the intersection with W is transverse, this means the inverse image of the intersection is a manifold, whose codimension in X is the same as the codimension of W in J. This is contained in the following result:

COROLLARY: *Let $(h,\mathbf{X})$ be an element of the set K. Let A be the intersection of $j(h,\mathbf{X})$ with W. Then A is a submanifold of J of codimension one and $j(h,\mathbf{X})^{-1}(A)$ is a submanifold of X, also of codimension one.*

Next consider the diagonal map

$$\Delta : M \to M \times M\ ;\ \Delta(x) = (x,x) \tag{21}$$

and the canonical projections back to M, π_1 and π_2, from the first and second copy of the manifold M in the product X. Composing with Δ, we have a map from M to X and then to Y. Moreover,

$$\pi_1\{j(h,\mathbf{X})^{-1}(A) \cap \Delta M\} = M^{\mathbf{X}}(h), \tag{22}$$

and $M^{\mathbf{X}}(h)$ is generically a manifold, also of codimension one in M and hence of dimension $(n-1)$. Thus, we have proved that for the generic pair of (output) function and vector field (i.e. in the dense subset K), the singular set is an $(n-1)$–dimensional manifold. This completes the proof of Theorem 6.

Given two vector fields, the intersection of their singular sets is generically a smooth, $(n-2)$–dimensional manifold:

THEOREM: *There is a residual subset K' of $C^\infty(M) \times \mathcal{X}(M) \times \mathcal{X}(M)$ such that for $(h, \mathbf{X}_1, \mathbf{X}_2) \in K'$, $M^{\mathbf{X}_1}(h)$ and $M^{\mathbf{X}_2}(h)$ intersect transversely in a smooth, closed, $(n-2)$–dimensional manifold.*

The proof of this Theorem proceeds along the same lines as the proof of Theorem 6. We omit the obvious generalisation to more than two vector fields.

Proof of Theorem 1: The proof of the first part is a direct consequence of Theorem 6: since for the generic pair (h, g_1), the singular set is an $(n-1)$–dimensional manifold, it obviously follows that it is not empty. Thus, the generic single–output system has relative degree one.

The linearizing controls for the other parts of the theorem can be constructed explicitly; by doing so, we give the proof of their existence. This will be done explicitly in the next section for a three dimensional example, the general case being similar. The fact that h is a (strict) Lyapunov function can be easily established and yields the last assertions of the theorem about the existence of global piecewise smooth trajectories.

Proof of Theorem 2: Differentiating the output twice yields:

$$\ddot{y} = \mathcal{L}_f^2 h + (\mathcal{L}_g \mathcal{L}_f h + \mathcal{L}_f \mathcal{L}_g h) u + \mathcal{L}_g^2 h\, u^2 + (\mathcal{L}_g h) \mathcal{L}_{f+gu} u. \tag{23}$$

For stabilization, we want this to equal

$$= -a_1 \dot{y} - a_0 y = -(\mathcal{L}_f h + \mathcal{L}_g h u) - a_0 h. \tag{24}$$

This means we would like to solve (locally, near $M^g(h)$) the **quasi–linear first order partial differential equation**

$$\mathcal{L}_{(\mathcal{L}_g h)(f+gu)} = -\left[\mathcal{L}_g^2 h\, u^2 + (\mathcal{L}_g \mathcal{L}_f h + \mathcal{L}_f \mathcal{L}_g h + a_1 \mathcal{L}_g h) u + \mathcal{L}_f^2 h + a_1 \mathcal{L}_f h + a_0 h\right]. \tag{25}$$

The general form of a quasi–linear, first order p.d.e. is

$$\mathcal{L}_{a(x,u)} u = b(x,u) \tag{26}$$

and the Cauchy problem for this equation is solved with the help of the characteristic vector field

$$A(x,u) = \begin{bmatrix} a(x,u) \\ b(x,u) \end{bmatrix} \tag{27}$$

as follows (Arnol'd (1982), Chapter 2, pp.61–66):

(i) suppose the function ψ is specified on an $(n-1)$–dimensional submanifold γ of state space. The graph of this function graph ψ is a submanifold of codimension 2 in the space $M \times \mathbf{R}$.

(ii) Assume that the component of the characteristic vector field a is nowhere tangent to γ. We then say that the initial condition (γ, ψ) is *non–characteristic* for the quasi–linear equation.

(iii) The solution u can then be obtained locally by integration of the characteristic direction field corresponding to the vector field A and is locally unique. This solution is obtained by using the direction field to extend the initial manifold graph ψ to an n–dimensional submanifold of $M \times \mathbf{R}$ that is, in fact, the graph of the solution function.

In our case, the components of the characteristic vector field A are

$$a(x,u) = (\mathcal{L}_g h)(f + gu) \tag{28}$$

and

$$b(x,u) = -\left[\mathcal{L}_g^2 h\, u^2 + (\mathcal{L}_g \mathcal{L}_f h + \mathcal{L}_f \mathcal{L}_g h + a_1 \mathcal{L}_g h)u + \mathcal{L}_f^2 h + a_1 \mathcal{L}_f h + a_0 h\right]. \tag{29}$$

We want a local solution near the singular manifold $\gamma \equiv M^g(h)$. Note that the $a(x,u)$ component vanishes on γ. The solution will be constructed by first examining the dynamics of the vector field A and patching up solutions around γ to give a graph of a control function u^* near γ.

First, we consider the polynomial equation in u obtained from equation (25) by setting $\mathcal{L}_g h = 0$. This is the second–order equation:

$$\mathcal{L}_g^2 h\, u^2 + (\mathcal{L}_g \mathcal{L}_f h + \mathcal{L}_f \mathcal{L}_g h + a_1 \mathcal{L}_g h)u + \mathcal{L}_f^2 h + a_1 \mathcal{L}_f h + a_0 h = 0. \tag{30}$$

Under conditions (i) and (ii) of the theorem, this equation can be solved for a_1, $a_0 = 0$ in a neighborhood $N_\epsilon(M^g(h))$. Because of the openness of these conditions, we can find a_1, a_0 small enough so that equation (30) is solved to yield a smooth feedback control u_p and furthermore the coefficients a_1, a_0 are such that the output equation is Hurwitz. This is because the set of pairs (a_1, a_0) that yield Hurwitz polynomials intersects any small neighborhood of zero.

Returning to the vector field A, we note that

$$a(x,u)|_\gamma = 0 \text{ , since } \mathcal{L}_g h|_\gamma = 0$$

and hence the set $\gamma \times \mathbf{R}$ is invariant under $A(x,u)$. Also

$$b(x, u_p) = 0$$

by the definition of $u_p(x)$. Thus graph$u_p|_\gamma$ is a set of equilibrium points for A.

By the assumption that $f + gu_p$ is transverse to γ we conclude that the function $\mathcal{L}_g h$ is locally a Lyapunov function for $a(x,u)$, with γ the attracting set. By the conditions (i) and (ii) of the theorem, the roots of the polynomial equation (30) are distinct and thus for u near $u_p(x)$, $b(x,u)$ changes sign in accordance with whether u is greater or smaller than u_p. Thus, $b(x,u)$ above the graph of u_p is of one sign and the same for the region below the graph. (In other words, $(u-u_p)\,b(x,u)$ is either greater or less than zero in $N_\epsilon(\gamma)$.) Putting these facts together, we now distinguish two possible cases:

(a) the set graph$u_p|_\gamma$ is attracting or repelling for A or

(b) the set graph$u_p|_\gamma$ is attracting in either of the x, u coordinates but repelling in the others.

By choosing the appropriate root, we can make $b(x,u)$ be attracting or repelling in the direction of γ according to whether $a(x,u)$ is attracting or repelling. Thus we only need to consider case (a). We take the case when it is attracting—the repelling case being exactly similar. The function

$$V(x,u) = \frac{1}{2}(u-u_p)^2 + \mathcal{L}_g h(x)$$

is a Lyapunov function in the neighborhood $N_\epsilon(\gamma)$. This is because

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} a(x,u) + (u-u_p)\,b(x,u) < 0.$$

Hence, we can take the set $\{\mathcal{L}_g h = \epsilon\} \cap$graph$u_p$ as the initial manifold for solving the p.d.e. This is because this set lies on the ϵ–level set of $V(x,u)$ and it is non–characteristic for the p.d.e. Repeating this on the other side of γ, we obtain a solution u^* in the neighborhood $N_\epsilon(\gamma)$ that coincides with u_p in γ. Note that even though γ fails the non-characteristic condition, we succeed in defining the control function in the whole of the neighborhood. This completes the proof of theorem 2.

Proof of Theorem 5 To guarantee the boundedness of the state trajectories, we divide the state space in two parts: far away from the zero level set of h and near it. The method we shall use employs global Lyapunov functions, which is natural in the context of nonlinear systems that are dissipative. Remember that Γ is the intersection $M^{g_1}(h) \cap M^{g_2}(h)$.

The control vector field $f + gu^*$ with u^* the stabilizing, linearizing feedback control defined in $M \setminus N_\epsilon(\Gamma)$ has H_0 as the unique attracting set. Since every state space trajectory has its α–limit set in H_0, there is a finite time T_x such that starting from a point x we reach the closure of the ϵ–tubular neighborhood of H_0 in time $T_x < \infty$. The state trajectory to that point is certainly bounded. When sufficiently close to H_0 though, the control vector field is very close to the vector field

$$f_z \equiv f - \frac{\mathcal{L}_f h}{\mathcal{L}_{g_i} h} g.$$

This is because $\mathcal{L}_g h$ is bounded away from zero in $M \setminus N_\epsilon(\Gamma)$, h is small and therefore the term $-\frac{ah}{\mathcal{L}_g h} g$ is small. By the properties of the normal tubular neighborhoods, there is a diffeomorphism (ψ, t) between $N_\epsilon(H_0)$ and the product $H_0 \times (-\epsilon, \epsilon)$ such that H_0 maps to $H_0 \times \{0\}$. The set $N_\epsilon(\Gamma)$ and its image under (ψ, t) can be excluded to give a diffeomorphism between $H_0 \setminus N_\epsilon(\Gamma)$ and its image.

Now f_z is dissipative. This implies: first, that there is a neighborhood of it in the space of vector fields $\mathcal{X}(M)$ of vector fields that are topologically orbitally equivalent to it and, second, that there exists a global Lyapunov function V satisfying $\mathcal{L}_{f_z} V < 0$ away from the recurrent set of f_z. Since a vector field $\tilde{f}$ in that neighborhood is equivalent to f_z, we can find a Lyapunov function $\tilde{V}$ of the same type (see Kappos (1986a)) for it that is close to V everywhere. The trick now is to pick the neighborhood of H_0 small enough so that the vector field $f + gu^*$ is close enough to f_z to be in the above neighborhood of equivalent vector fields. (All this of course is done after we push forward the vector fields to the product space.) Finally, we need to project $f + gu^*$ onto the layers $H_0 \times t$ that stratify the product space. Call $\tilde{V}_t$ the Lyapunov functions for this vector field in the layers ($t \in (-\epsilon, \epsilon)$). The t–component is pointing towards $H_0 \times \{0\}$, since $|h|$ is a Lyapunov function for $f + gu^*$. We can now make the claim that

t here is a smooth extension V^* of the Lyapunov function V defined in the set $N_\epsilon(H_0) \setminus N_\epsilon(\Gamma)$ that is such that $\mathcal{L}_{f+gu^*} V^* \leq 0$ there.

This extension is defined by

$$V^*(x) = \tilde{V}_{t(x)}(\psi(x)).$$

Finally, define the Lyapunov function

$$\mathcal{V} \equiv V + |h|.$$

We claim this is a global Lyapunov function for the dynamics $f + gu^*$ in $N_\epsilon(H_0) \setminus (H_0 \cup N_\epsilon(\Gamma))$. This is because (take the set M^+, for example)

$$\frac{d\mathcal{V}}{dt} = \mathcal{L}_{f+gu^*}(V^* + h) =$$

$$\mathcal{L}_{f+gu^*} V^* - ah^2 < 0.$$

Since the Lyapunov surfaces of V^* (its level sets) are compact and since the trajectories of $f + gu^*$ leave the interior of the Lyapunov surfaces of $\mathcal{V}$ invariant, it follows that the trajectories of $f + gu^*$ are bounded after they enter the ϵ neighborhood of H_0 and hence they are globally future bounded. This completes the proof of Theorem 5.

5. An Example

We give an example with one output and with the output function being the first state variable. This example demonstrates the following points: (a) that piecewise linearization (or at least stabilization) is frequently possible with a feedback control, (b) that linearization works for most trajectories, that is, outside an arbitrarily small set.

Throughout, we suppose that $x \in \mathbf{R}^3$ and that $y = x_1$. For any vector field $\mathbf{X}$, therefore, the Lie derivative of the output function with respect to the flow of that vector field is, in the given coordinates, $\mathcal{L}_{\mathbf{X}} h = \mathbf{X}_1$, the first component function of the vector field.

First, take the case of a single control and the following state equations

$$\begin{aligned} \dot{x}_1 &= -x_1 + (x_1^2 + x_2^2 + x_3^2 - c^2)u \\ \dot{x}_2 &= f_2(x) + g_2(x)u \\ \dot{x}_3 &= f_3(x) + g_3(x)u \end{aligned} \tag{31}$$

for some functions f_2, f_3, g_2 and g_3. Obviously, here $\dot{y} = \dot{x}_1$. We show now that linearization can be used to speed up the system dynamics for driving the state variable x_1 to zero. Note that we cannot make the response arbitrarily fast (the reasons will become clear below).

The Lie derivative $\mathcal{L}_g h(x) = g_1(x) = (x_1^2 + x_2^2 + x_3^2 - c^2)$, vanishes on the sphere of radius c. Thus:

$$M^g(h) = \{x \in \mathbf{R}^3\,;\, \mathcal{L}_g h(x) = 0\} = \{x \in \mathbf{R}^3\,;\, |x| = c\,\}. \tag{32}$$

Now, with ϵ a small positive number, try the linearizing feedback control

$$u(x_1, x_2, x_3) = \begin{cases} 0 & , \ c - \epsilon < |x| < c + \epsilon \\ -\frac{(a-1)x_1}{(x_1^2 + x_2^2 + x_3^2 - c^2)} & , \ |x| > c + \epsilon \text{ or } |x| < c - \epsilon. \end{cases} \tag{33}$$

This yields

$$\dot{y} = \begin{cases} -a x_1 = -a y & , \ c - \epsilon < |x| < c + \epsilon \\ -x_1 = -y & , \ |x| > c + \epsilon \text{ or } |x| < c - \epsilon \end{cases} \tag{34}$$

which, by adjusting the parameter a can drive y, i.e. the state variable x_1 to zero arbitrarily fast outside the ϵ–neighborhood of $M^g(h)$. The time to drive, say, an initial state $x_1 > c + \epsilon$ (and also assuming that $x_2^2 + x_3^2 < c^2$) to zero is bounded below by $\ln \frac{c+\epsilon}{c-\epsilon}$.

Next we turn to the case when the state dynamics of x_1 is unstable:

$$\dot{x}_1 = x_1 + (x_1^2 + x_2^2 + x_3^2 - c^2)u \tag{35}$$

As can be easily checked, it is not possible to stabilize the output equation by feedback linearization globally, since for initial conditions in the intersection of the cylinder $\{x\,;\, (x_2^2 + x_3^2)^{1/2} \le c + \epsilon\}$ and the outside of the ϵ–neighborhood of M^g, it is not possible to drive the state x_1 to zero.

In this case, we can turn to one of the two options given in the Theorem of section 3 above, namely differentiating further or using two controls to obtain stability, at least for most initial conditions.

First we take the case when a second control is present. Suppose that it enters the state equation for x_1 as follows

$$\dot{x}_1 = x_1 + (x_1^2 + x_2^2 + x_3^2 - c^2)u_1 + x_2 u_2. \tag{36}$$

The singular manifold of g_2 is

$$M^{g_2}(h) = \{x\,;\, x_2 = 0\} \tag{37}$$

and

$$M^{g_1}(h) \cap M^{g_2}(h) = \{x\,;\, |x| = c\,,\, x_2 = 0\} \tag{38}$$

which is a circle in the x_1–x_3–plane. The following feedback control will 'input–ouput linearize' the system (ϵ is a small positive number)

$$u_1(x_1,x_2,x_3) = \begin{cases} 0 & ,\ c-\epsilon < |x| < c+\epsilon \\ -\frac{(a+1)x_1}{(x_1^2+x_2^2+x_3^2-c^2)} & ,\ |x| > c+\epsilon \text{ or } |x| < c-\epsilon \end{cases} \tag{39a}$$

$$u_2(x_1,x_2,x_3) = \begin{cases} 0 & ,c-\epsilon < |x| < c+\epsilon\,,\ |x_2| < \epsilon \\ -\frac{(a+1)x_1}{x_2} & ,c-\epsilon < |x| < c+\epsilon\,,\ |x_2| > \epsilon \end{cases} \tag{39b}$$

This control yields the piecewise linear equation for the output

$$\dot{y} = \begin{cases} x_1 = y & ,\ c-\epsilon < |x| < c+\epsilon\,,\ |x_2| < \epsilon \\ -ax_1 = -ay & ,\ \text{everywhere else} \end{cases} \tag{40}$$

Note that the equation for y is unstable in this last set $N_\epsilon(M^{g_1}(h) \cap M^{g_2}(h))$ and, moreover, cannot be continued across the boundary into the first set $M \equiv \mathbf{R}^n - N_\epsilon(M^{g_1}(h) \cap M^{g_2}(h))$. However, the set of trajectories that converge to $y = 0$ lie in

$$\mathbf{R}^n - \{\phi(t,x)\,;\, t \in \mathbf{R}\,,\, x \in M\}, \tag{41}$$

where $\phi(t,x)$ is the piecewise smooth flow of the controlled vector field $f + gu$, with u the feedback control above.

Following theorem 2, we can differentiate y once more to obtain

$$\begin{aligned} \ddot{y} = \ddot{x}_1 &= \dot{x}_1 + 2(x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3)u \\ &= x_1 + (x_1^2 + x_2^2 + x_3^2 - c^2)u + 2\big[x_1[x_1 + (x_1^2 + x_2^2 + x_3^2 - c^2)u] + \\ &+ 2[x_2(f_2(x) + g_2(x)u) + x_3(f_3(x) + g_3(x)u)]\big]u + (x_1^2 + x_2^2 + x_3^2 - c^2)\dot{u}. \end{aligned} \tag{42a}$$

Let us rewrite this expression as

$$\ddot{y} = b_0(x_1, x_2, x_3) + b_1(x_1, x_2, x_3)u + b_2(x_1, x_2, x_3)u^2 + \mathcal{L}_{a(x,u)}u. \tag{42b}$$

If we have that, in the region $\{x \ ; \ c - \epsilon < |x| < c + \epsilon\}$

$$[2(x_1^2 + x_2 f_2(x) + x_3 f_3(x)]^2 - 8(x_2 g_2(x) + x_3 g_3(x)) > 0, \tag{43}$$

then for ϵ sufficiently small, $b_1^2 - 4b_0 b_2 > 0$, since then $(x_1^2 + x_2^2 + x_3^2 - c^2)$ is small. Furthermore, it is easy to see that we can find a_0 , a_1 small and such that the polynomial $s^2 + a_1 s + a_0$ has negative real roots and also the polynomial

$$(b_0(x) + a_1 \mathcal{L}_f h(x) + a_0 h(x)) + (b_1(x) + a_1 \mathcal{L}_g h(x))u + b_2(x)u^2 \tag{44}$$

still satisfies the condition for real roots.

We have checked conditions (i) and (ii) of theorem 2. It remains to construct u_p and check the transversality conditon on $f + gu_p$. These all depend on f_2, f_3, g_2 and g_3 and it will be left to the reader to see that there are choices that will lead to the conditions being satisfied.

6. Conclusions

We found that it is possible to push linearization through the singular sets, provided we are prepared to settle for piecewise smooth solutions for most initial conditions. The computation of the linearizing control is straightforward and makes use of the geometry of the singular sets.

The approach taken in this research yields a linearization theory that is global and that makes fewer assumptions on the dynamics. The assumptions made are in fact realistic, first because they are often generic and because they are satisfied by many practical systems (such as the assumption of dissipativeness, satisfied by many nonlinear circuits and power system models). It is hoped that this approach can lead to a geometrical linearization theory for outputs of higher dimension.

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