Proceedings of the American Control Conference
Albuquerque, New Mexico June 1997
0-7803-3832-4/97/$10.00 © 1997 AACC

Dynamic Programming and Model Predictive Control

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Abstract

This paper presents a new criterion for assessing the stability of nonlinear model predictive control (MPC). The criterion is based on the monotonicity of the MPC cost function as a function of horizon length, and is derived using dynamic programming theory in a deterministic setting.

1. Introduction

Model predictive control (MPC) refers to the class of controllers in which control inputs are selected based on an optimization criterion that is formulated over a prediction horizon, using an explicit model to predict the effect of future inputs on internal states or outputs. MPC incorporates feedback by dynamically updating the optimization problem to include the effects of process measurements.

This general definition can include a variety of different model forms, optimization criteria and feedback mechanisms. Since MPC is implemented via computer, discrete-time formulations are preferred; this paper therefore adopts the following process model:

$$x_{k+1} = f(x_k, u_k)$$

The optimization criterion is

$$J_0(x_{k+N|k}) + \sum_{j=0}^{N-1} L(x_{k+j|k}, u_{k+j|k})$$

in which N is the *prediction horizon*, L is the *stage cost* and J_0 is the *terminal state penalty.* The stage cost and terminal state penalty are required to be non-negative. Without loss of generality, we can assume that $f(0,0) = 0$. Additional restrictions concerning f, J_0 and L will be discussed subsequently in terms of their effect on stability of MPC. Concerning notation, it is a commonly accepted convention in the MPC literature to use the double subscript to distinguish a future value within the prediction horizon (e.g. $x_{k+j|k}$) from a past or current variable (x_k). With this convention, the index $k|k$ is equivalent to k.

Using the above definitions, the MPC controller is determined as the solution to the following optimization problem:

$$J_N^*(x_k) = \min_{u_{k+j|k}} J_0(x_{k+N|k}) + \sum_{j=0}^{N-1} L(x_{k+j|k}, u_{k+j|k}) \quad (1)$$

The optimization indicated in Equation 1 is subject to the system equation constraints

$$x_{k+j+1|k} = f(x_{k+j|k}, u_{k+j|k})$$

and may be subject to additional constraints

$$\begin{aligned} x_{k+j|k} &\in \mathcal{X} \\ u_{k+j|k} &\in \mathcal{U} \end{aligned}$$

in which $\mathcal{X}$ and $\mathcal{U}$ represent constraint regions in the state and control spaces, respectively. These are often defined by linear inequality constraints.

The first element $u^*_{k|k}$ of the minimizing control sequence $\left\{u^*_{k|k}, u^*_{k+1|k}, \ldots, u^*_{k+N-1|k}\right\}$ gives the MPC controller. The optimal control sequence depends upon the current state x_k, the prediction horizon N and the functions f, L and J_0, as well as any constraints that may be present. Since f, L, and J_0 are usually fixed within the context of a specific problem, the dependence of the optimal control on these will not be explicit and the MPC control law is denoted by $\mu_N(x_k)$. This form emphasizes that feedback is incorporated into the MPC controller via the use of the current state value as the initial condition in the prediction horizon. With this selection of controller, the system evolves according to the following:

$$\begin{aligned} x_{k+1} &= f(x_k, \mu_N(x_k)) \\ &\triangleq h(x_k) \end{aligned} \quad (2)$$

Our control objective is to regulate the process to the origin. More precisely, we wish to choose the control horizon N, the stage cost L and the terminal cost J_0 such that the origin is an asymptotically stable fixed point of the closed-loop system of Equation 3 with a non-negligible region of attraction.

Clearly, if the state were already at the origin, we would select a control sequence $\{0, 0, \ldots, 0\}$ to maintain the state at the origin. It is desirable to choose J_0 and L such that this condition satisfy $J_N^*(0) = 0$ for all N. Since J_0 and L are non-negative, we therefore require that $J_0(0) = 0$ and $L(0,0) = 0$, which together imply that $J_N^*(0) = 0$.

Starting with the set of conditions

- $L(x,u) > 0 \;\; \forall \, (x,u) \neq (0,0)$
- $L(0,0) = 0$

this paper presents some conditions on J_0 and L such that an asymptotically stable closed-loop system (Equation 3) is obtained.

2. Model Predictive Control as a Dynamic Programming Problem

As applied to deterministic systems, the dynamic programming (DP) algorithm approach to the solution of Equation 1 is to decompose the N-stage problem into a series of subproblems:

$$\begin{aligned} J_1^*(x) &= \min_{u \in \mathcal{U}} \{L(x,u) + J_0(f(x,u))\} \\ J_2^*(x) &= \min_{u \in \mathcal{U}} \{L(x,u) + J_1^*(f(x,u))\} \\ &\vdots \\ J_N^*(x) &= \min_{u \in \mathcal{U}} \{L(x,u) + J_{N-1}^*(f(x,u))\} \end{aligned}$$

Using the notation of Bertsekas [1], the right-hand sides of the above equations may be represented as an operator T:

$$T(J)(x) = \min_{u \in \mathcal{U}} \{L(x,u) + J(f(x,u))\} \tag{3}$$

Notice that T accepts two arguments: J and x. (As in the MPC problem formulation, the obvious dependence on L and f is not placed in evidence.) The expressions $T(J)(x)$ provides the value at x of the objective of a one-stage dynamic programming problem, beginning with the terminal state penalty function J. The composition of T with itself $T[T(J)](x)$ provides the value of the two-stage DP objective function evaluated at x. We can continue the process for an arbitrary number of stages to form

$$T^N(J)(x) = T\left[T^{N-1}(J)\right](x)$$

At the N-th stage, the feedback control law is given by

$$\mu_N(x) = \arg T^N(J)(x) \tag{4}$$

Although they consider similar problems, the MPC and the DP approaches emphasize different aspects of the optimization problem:

- Because MPC gained its current popularity as an industrially developed computer algorithm, the MPC formulation deemphasizes the underlying functional relationships between u_k and x_k, and between x_k and the objective function $J_N^*(x_k)$. From this perspective, the critical problem to be solved is to provide a value for u_k based upon the current value of x_k via numerical optimization.
- In the DP algorithm, the objective function at each stage is of central interest, since it is the means of carrying information between subproblems. Specific values of the objective and the control are deemphasized, as highlighted by the form of Equations 3 and 4. The state x is left in evidence, but specific state values (x_k in MPC) are not required for analysis of the algorithm, since the solution to the DP problem provides μ_N, a function from the state space to the control space and not u_k.

These differences in emphasis obscure similarities between DP and MPC. For finite horizon, deterministic problems, the MPC problem is *identical* to the DP problem, which can be shown using the Principle of Optimality. A proof of this intuitively obvious result is provided by Bertsekas [1, Chapter 1] in the context of stochastic systems, of which the deterministic problem may be considered a special case.

3. Stability Condition for Finite Horizon MPC

Consider the infinite horizon MPC (or deterministic DP) problem

$$J^*(x_k) = \min \lim_{N \to \infty} \sum_{j=0}^{N-1} L(x_{k+j|k}, u_{k+j|k})$$

The minimum is taken with respect to the infinite sequence $\{u_{k|k}, u_{k+1|k}, u_{k+2|k}, \ldots\}$. Note that the infinite horizon MPC objective is represented without subscript N. With non-negativity of L, it may be shown [1, Chapter 5, Proposition 8] that J^* satisfies the (deterministic) Bellman Equation:

$$J^*(x) = \min_{u \in \mathcal{U}} \{L(x,u) + J^*[f(x,u)]\} \tag{5}$$

Denoting the infinite horizon feedback control law arising from the solution of Equation 5 by $\mu(x)$, we can rewrite Equation 5 as follows:

$$J^*(x_{k+1}) - J^*(x_k) = -L(x_k, \mu(x_k)) \leq 0 \tag{6}$$

In Equation 6, we take x_{k+1} to be the state resulting from implementation of μ, i.e., $x_{k+1} = f(x_k, \mu(x_k))$. Since $J^*(x_k) \geq 0$ and $J^*(0) = 0$, it would seem that the infinite horizon objective function would almost satisfy the requirements for a Lyapunov function. If we add the following additional conditions

- J^* is continuous at the origin
- $L(x,u)$ is locally lower bounded by a class $\mathcal{K}$ function [6]

then in fact J^* is a local Lyapunov function for x_k and x_k and we conclude that Equation 3 is asymptotically stable.

The above result is not new. A number of researchers have presented similar results [3, 4]; however, this example illustrates the potential of applying the large body of existing results in dynamic programming to analysis of the MPC problem.

Consider now the finite horizon case. The N-stage optimal objective satisfies the dynamic programming equation:

$$J_N^*(x) = \min_u \left\{ L(x,u) + J_{N-1}^* \left[f(x,u) \right] \right\} \tag{7}$$

Beginning at x_k and implementing the optimal control for the N-stage problem, we can rewrite Equation 7 as follows:

$$J_N^*(x_k) = L(x_k, \mu_N(x_k)) + J_{N-1}^*(x_{k+1})$$

Subtracting $J_N^*(x_{k+1})$ from both sides and changing the sign provides

$$\begin{aligned} J_N^*(x_{k+1}) - J_N^*(x_k) = & -L(x_k, \mu_N(x_k)) \\ & - \left[J_{N-1}^*(x_{k+1}) - J_N^*(x_{k+1}) \right] \end{aligned} \tag{8}$$

Equation 8 is an N-stage analog of Equation 6. If J_N^* is continuous at the origin and if the function $\tilde{L}_N$, defined by

$$\tilde{L}_N(x_k) = L(x_k, \mu_N(x_k)) + \left[J_{N-1}^*(x_{k+1}) - J_N^*(x_{k+1}) \right]$$

is locally lower bounded by a class-$\mathcal{K}$ function, then we will have sufficient conditions for asymptotic stability of a nonlinear system controlled via finite horizon MPC. The above result is a nonlinear generalization of the condition provided by Bitmead *et al.* [2, Chapter 4] for stability of MPC for linear systems with quadratic criteria.

As in the infinite horizon case, stability is assured if $\tilde{L}_N(x)$ is locally lower bounded by a class-$\mathcal{K}$ function. Following Vidyasagar [6], a class-$\mathcal{K}$ function $\alpha : \Re_+ \to \Re_+$ satisfies $\alpha(0) = 0$, $\alpha(r) > 0$ if $r \neq 0$ and is non-decreasing. (This is a slight restriction of the general idea of a positive definite function in that positive definite functions are not necessarily non-decreasing.) In the remainder of this article, the stability assurance provided by the positivity conditions on $\tilde{L}_N(x_k)$ will be applied in two ways: (1) Prescriptively, to *force* $\tilde{L}$ to be non-negative and thereby guarantee stability, and (2) descriptively, to determine whether a given finite-horizon MPC controller provides nominal stability.

Bertsekas [1] provides the following result, which is relevant to the search for a condition that forces $\tilde{L}_N$ to be non-negative:

Theorem 1 (Bertsekas [1]) *For any functions $J(x)$ and $J'(x)$ mapping the state space into $\Re$, $J(x) \leq J'(x)$ for all x implies*

$$T^k(J)(x) \leq T^k(J')(x)$$

for all x and $k \geq 0$.

Using this theorem, it can be shown that a particular choice of J_0 to begin the DP iterations can provide asymptotic stability for any finite horizon controller. Let $\mathcal{J}$ be the set of functions mapping the state space into $\bar{\Re}_+$ (the non-negative extended real numbers) such that $J(0) = 0$ for all $J \in \mathcal{J}$. Using the dynamic programming equation and $f(0,0) = 0$, it is easy to show that $T^k(J)(0) = 0$ for any $k \geq 0$ and $J \in \mathcal{J}$. Choose J_0 to be the largest element of $\mathcal{J}$, i.e., $J_0(x) \geq J(x)$ for all $x \in \Re^n$ and $J \in \mathcal{J}$. The function $T(J_0)(\cdot)$ is a member of $\mathcal{J}$ and since J_0 is the largest member of $\mathcal{J}$, we have

$$T(J_0)(x) \leq J_0(x)$$

Applying Theorem 1, this results in

$$T^N(J_0)(x) \leq T^{N-1}(J_0)(x) \tag{9}$$

As noted previously, the notation $T^N(J_0)(x)$ simply represents the objective of an N-stage MPC problem with terminal penalty $J_0(x)$ and Equation 9 may be expressed as

$$J_{N-1}^*(x) - J_N^*(x) \geq 0$$

with the understanding that J_N^* and J_{N-1}^* include the effect of the final state penalty J_0. Therefore, with this choice of final state penalty function, $\tilde{L}_N(x)$ of Equation 3 is positive for all N. If $L(x,u)$ is lower bounded by a class-$\mathcal{K}$ function and $J_N^*(x)$ is continuous at the origin, then MPC provides a stabilizing control with this choice of J_0 for any value of N.

What is this J_0 that produces stability? Since it must be larger than all other possible J's, it must be the extended real-valued function

$$J_0(x) = \left\{ \begin{array}{cc} 0 & x = 0 \\ \infty & x \neq 0 \end{array} \right] \tag{10}$$

which seems to be rather vaguely defined and not a very good candidate for performing numerical calculations. However, reverting back to conventional MPC notation, we can interpret this final state penalty function as a final state constraint instead: $x_{k+N|k} = 0$. The stabilizing properties of this final state constraint are well-known.

4. An Example

Finite horizon MPC was applied to the scalar dynamic system

$$x_{k+1} = 1.2 x_k + \sin u_k \tag{11}$$

This system is unstable and is not stabilizable for $|x_k| \geq 5$. Using $L(x,u) = x^2 + u^2$, MPC control laws and objective function were computed for various horizon lengths, both with and without the final state constraint.

Unconstrained MPC: The ability of unconstrained MPC to stabilize the system can be assessed by examining $\tilde{N}_N(x_k)$. Figure 1 shows values of $\tilde{N}_N(x_k)$ for $N = 2$ through $N = 5$. (Since $\tilde{L}_N(x_k)$ is symmetric for this example, only $\tilde{L}_N(|x_k|)$ is presented.) For $N = 1$, $\tilde{L}_N(x_k)$

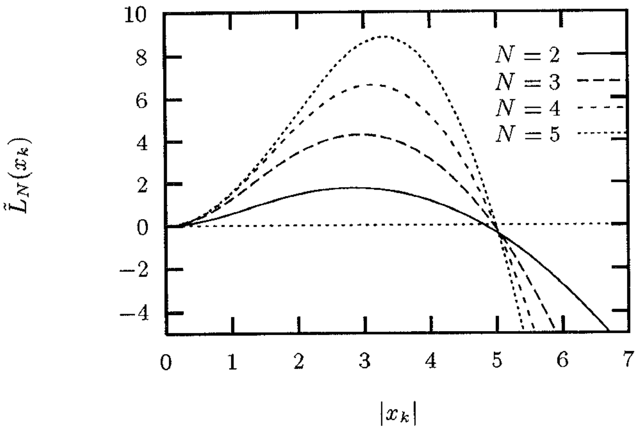

Figure 1: $\tilde{L}_N(x_k)$ versus $|x_k|$

is identically zero and stability can not be inferred based on $\tilde{N}_N$ alone; however, for any $N > 2$, $\tilde{L}_N$ can be locally lower bounded by a class-K function, which is sufficient for asymptotic stability under MPC.

Asymptotic stability is essentially a local concept, but practical controllers require a non-negligible region of attraction to be of value. The Lyapunov theory upon with the $\tilde{L}_N$ criterion is based does not predict the region of attraction without additional assumptions (see e.g., Vidyasagar[6]). However, for this simple example, the region of attraction *can* be predicted based on $\tilde{L}_N(x_k)$. Figure 2 show $\tilde{L}_N(x_k)$ near the boundary of the stabilizable region at $x_k = 5$. From Figure 2, $\tilde{L}_2(x_k)$ changes

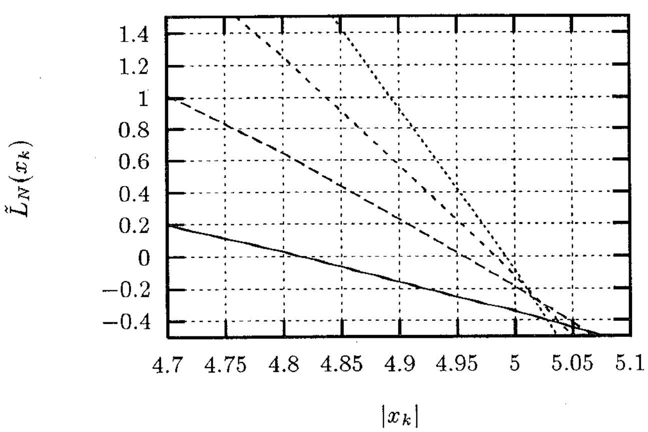

Figure 2: $\tilde{L}_N(x_k)$ versus $|x_k|$ (close view)

sign at approximately $|x_k| = 4.82$. Starting from various initial states distributed near $x_k = 4.82$, Figure 3 shows the evolution of the state under MPC control with $N = 2$. A straight line is included in Figure 3 to show

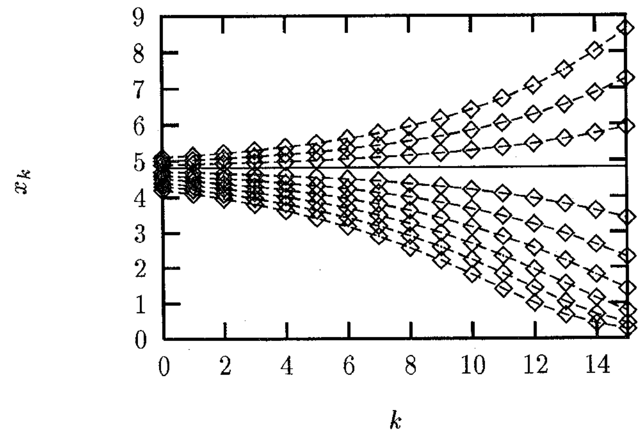

Figure 3: State trajectories under unconstrained MPC with $N = 2$

$x_k = 4.82$. Trajectories beginning at initial states less than 4.82 demonstrate typically stable behavior while those with initial states greater than 4.82 show unstable behavior.

MPC with final state constraint: Figure 4 shows the objective function calculated using the final state constraint $x_N = 0$. As required by the monotonicity result of Theorem 1, the objection function $J_N^*(x_k)$ is monotonically decreasing with N, resulting in positive (i.e., locally lower bounded) $\tilde{L}_N(x_k)$ for any N.

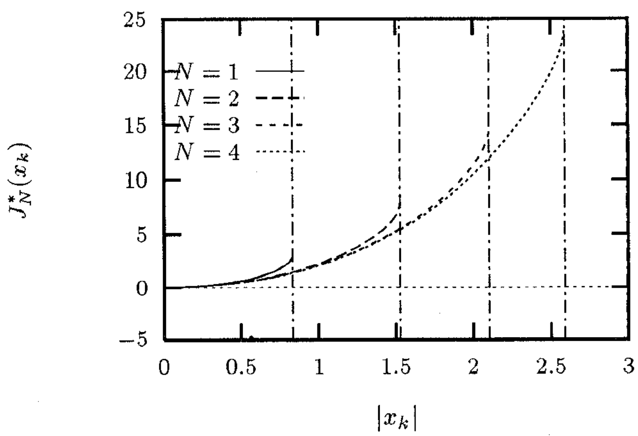

Figure 4: MPC Objective Function with Final State Constraint

The final state constraint cannot be satisfied for all values of x_k. This can be interpreted as an infinite value for the objective function and the MPC control law is not defined for these regions. For regions where the objective function is finite, then the MPC control is defined and the region of attraction consists of those point in the state space for which $J_N^*(x_k) < \infty$ [4]. As a function of horizon length N, the regions of attraction for the example problem are given by $\left\{x_k \,\middle|\, |x_k| < \sum_{j=1}^{N}\left(\frac{5}{6}\right)^j\right\}$. The boundaries of these regions are indicated by broken vertical lines in Figure 4.

Convergence of constrained and unconstrained MPC: Figure 5 shows that objective function for the unconstrained MPC controller. A careful comparison with Figure 4 shows that the unconstrained and constrained objective function converge pointwise as $N \to \infty$; numerical results are similar for the corresponding control law (not pictured). This need not be true in the general case. For the dynamic system $x_k = x_k^2 + u_k^2 - \left(x_k^2 + u_k^2\right)^2$ [5] using the same $L(x,u)$ as this example, the objective function converges to a discontinuous function if the final state constraint is used. The corresponding unconstrained objective is smooth.

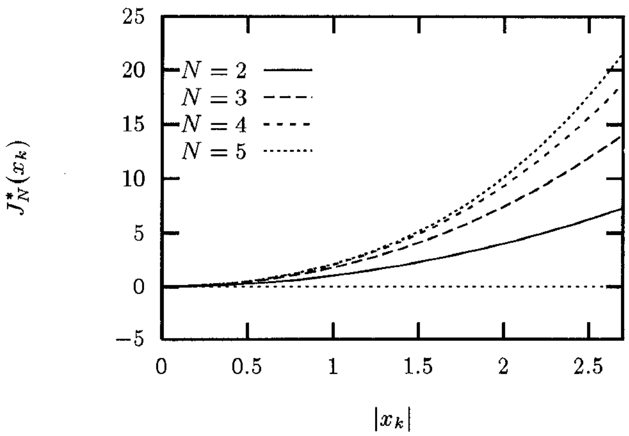

Figure 5: MPC Objective Function with Final State Constraint

5. Conclusions

In this article, a new criterion for assessing the stability of finite horizon MPC. The development of the $\tilde{L}_N$ criterion was motivated by two paradoxical observations: (1) Sufficient conditions for the *general* nonlinear MPC problem depend on the zero final state constraint, a very strong condition; and (2) MPC can usually produce stabilizing controllers for nonlinear systems rather easily without the final state constraint, but it has not been possible to determine *a priori* what horizon length is necessary.

Calculating $\tilde{L}_N(x_k)$ is computationally expensive. At this point $\tilde{L}_N$ should not be viewed as a general purpose tool; however, it is hoped that the new stability criterion might serve as a basis for the development of simpler means to assess MPC stability without the imposition of the final state constraint, which is an unnecessarily strong condition in most cases.

The results in the example illustrate a typical problem in applying the final state constraint $x_N = 0$ to guarantee stability: the difficulty in achieving acceptable domains of attraction. In the unconstrained case, the horizon length $N = 2$ stabilized more than 96 percent of the stabilizable region. When the final state constraint is included $N = 2$ includes less than 30 percent. To achieve stability in 96 percent of the stabilizable region would require $N = 18$ with the final state constraint.

Although not explored here, the condition developed in this article also points to another possible choice of J_0 that results in stability: $J_0(x) = J^*(x)$. Since $J_0(x)$ satisfies the Bellman Equation (Equation 5), we have

$$\tilde{L}(x) = L\left(x, \mu\left(x\right)\right)$$

which satisfies the stability condition. Unfortunately, a closed form solution of the Bellman equation for $J^*(x)$ is rarely available; therefore, use of $J^*(x)$ as a final state penalty function in finite horizon MPC is not now an available strategy. Although not used in this work, a promising idea for adaptively computing an approximation of $J^*(x)$ with artificial neural networks has been suggested by Werbos [7].

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