

# ON MOBILE ROBOTS : A PROBABILISTIC MODEL FOR THE REPRESENTATION AND MANIPULATION OF SPATIAL UNCERTAINTY

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## ABSTRACT

A mobile robot needs to represent and to reason about the approximate relationships between itself and other objects. In addition, the robot must be able to use sensor information to reduce locational uncertainty. In this paper we present a general probabilistic model which permits to represent the motions of 2-D or 3-D mobile robots and to estimate the distribution of a chain of relative transforms without strong assumptions.

done recently in this field (see [1], [2], [3], [4], [6]). The relative locations of objects are represented by coordinate frames which may be known only indirectly through a series of spatial relationships with its associated errors. A mobile robot needs to represent and to reason about the approximate relationships between itself and other objects. In addition, the robot must be able to use sensor information to reduce locational uncertainty. In order to see the difficulty of this problem we represent in (figure 1) the general software architecture for mobile robots. Uncertainty arises from sensing errors, control errors and uncertainty in the geometric models of the environment and of the robot. Therefore it's treatment is an essential part of the software. In general Kalman filtering techniques, approximation

## 1. Introduction

In many applications it is necessary to reason about spatial relationships among objects based on inaccurate information. Several works have been

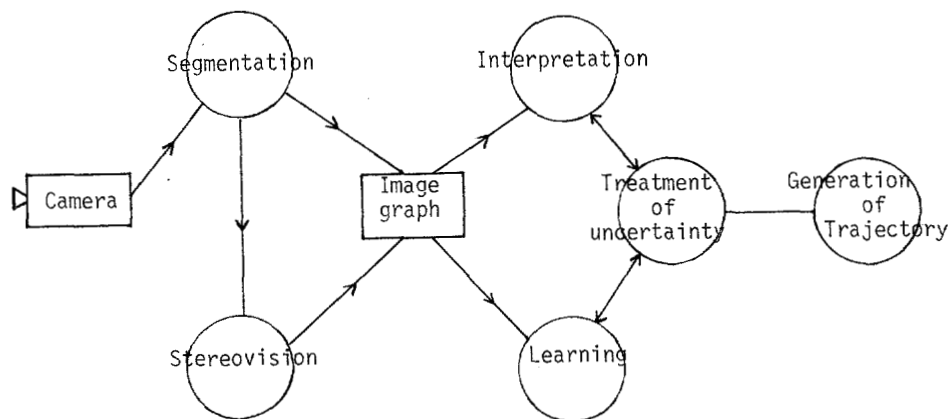


Figure 1

by Gaussian random vectors or sums of independent random variables are used. In this paper related to the mobile robot project titled ORASIS we present a general probabilistic model which permits to represent the motions of 2-D or 3-D mobile robots (section 2), we make some comments about the probabilistic approach used in [2], [3], [4] (section 3). Section 4 has a more theoretical interest, using properties of random walks on the motion group we show, without strong assumptions, that the distribution of a chain of relative transforms can be approximated by a Gaussian distribution whose covariance matrix is very simple: this proves in particular that some assumptions made in [2], [3], [4] are justified.

## 2. Representation of motions

In the ORASIS project, the elements to which we try to apply spatial reasoning are line segments. To each segment  $S$  corresponds a frame  $R$ . This can be done in the following way:

- the origin  $O_R$  of  $R$  is the middle point of  $S$ ,
- the vector  $J_R$  is normal to the surface of the object,
- the vector  $I_R$  is such that  $(I_R, J_R, K_R)$  is a direct frame.

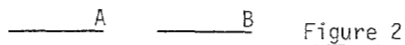
The spatial reasoning needs the choice of a representation for the spatial relations between frames. Two solutions are possible:

1. Use an absolute reference frame  $A$  and locate all the other frames with respect to  $A$ .
2. Use only the relative positions and orientations between the frames.

The following simple example proves that the solution 1 is not the best.

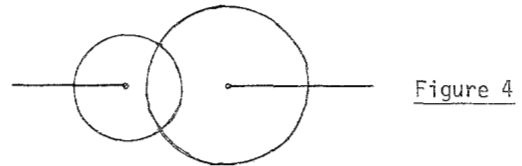
### Example.

Consider two walls with ends  $A$  and  $B$  (figure 2) and suppose that the breadth of the robot is equal to one unit



$A$  has been located with respect to the abso-

lute frame  $R$  with uncertainty equal to one unit (figure 3)



The robot estimates the distance between  $A$  and  $B$  to two units with uncertainty 0,25 and realize that he can go through  $A$  and  $B$ . But the figure 3 does not give this information since the two uncertainty disks intersect.

We shall express the spatial relations between frames in terms of motions: in our case we have to deal with plan motions. Nevertheless we introduce a formalism which permits also the study of space motions.

### The motion group.

#### Definition.

The motion group of  $\mathbb{R}^d$  ( $d=2$  for the plan  $d=3$  for the space) denoted  $G_d = SO(d) \times \mathbb{R}^d$  ( $SO(d)$  is the rotation group of  $\mathbb{R}^d$ ) is defined by the following rule

$$\begin{aligned} g \cdot g' &= (r, \vec{V}) \cdot (r', \vec{V}') \\ &= (r \circ r', \vec{V} + r(\vec{V}')) \end{aligned}$$

$$V(g, g') \in G_d \times G_d$$

$$g = (r, \vec{V}) \quad r \in SO(d), \quad \vec{V} \in \mathbb{R}^d.$$

$r(\vec{V}')$  is the vector resulting from the action of the rotation  $r$  on  $\vec{V}'$ .

### Example

For the plan motions

$$SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in \mathbb{R} \right\}, \quad \vec{V} = (X, Y),$$

$$r(\vec{V}) = \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

The different transformations between any two frames of reference can be computed.

If  $R_1, R_2$  are two frames, the transform  $D_{12} : R_1 \rightarrow R_2$  is an element of  $Gd : D_{12} = (r, \vec{V})$   $r \in SO(d), \vec{V} \in \mathbb{R}^d$ .

#### a) Compounding of transforms

Let  $R_i, R_j, R_k$  be three frames and  $D_{ij} : R_i \rightarrow R_j, D_{jk} : R_j \rightarrow R_k, D_{ij} = (r, \vec{V}), D_{jk} = (r', \vec{V}')$ .

Then the transform  $D_{ik} : R_i \rightarrow R_k$  is given by

$$D_{ik} = D_{ij} \circ D_{jk}$$

$$D_{i,k} = (r_3, \vec{V}_3) = (r_1, \vec{V}_1) \circ (r_2, \vec{V}_2) = (r_1 \circ r_2, \vec{V}_1 + r_1(\vec{V}_2)).$$

More generally :

$$\begin{aligned} D_{1,n} &= D_{1,2} \circ D_{2,3} \circ \dots \circ D_{n-1,n} \\ &= (r_1, \vec{V}_1) \circ (r_2, \vec{V}_2) \circ \dots \circ (r_{n-1}, \vec{V}_{n-1}) \\ &= (r_1 \circ r_2 \circ \dots \circ r_{n-1}, \vec{V}_1 + r_1(\vec{V}_2) + \\ &\quad r_1 \circ r_2(\vec{V}_3) + \dots + r_1 \circ r_2 \circ \dots \circ r_{n-2}(\vec{V}_{n-2})). \end{aligned}$$

#### Example

In the plan  $D_{ij} = (\theta_i, X_i, Y_i)$

$$r_i(\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

$$\vec{V}_i(X_i, Y_i) \text{ for } i \in \{1, 2\}$$

and

$$\begin{aligned} D_{i,k} &= D_{ij} \circ D_{jk} = (r_1 \circ r_2, \vec{V}_1 + r_1(\vec{V}_2)) \\ &= (\theta_1 + \theta_2, X_3, Y_3) \end{aligned}$$

$$X_3 = X_2 \cos\theta_1 - Y_2 \sin\theta_1 + X_1$$

$$Y_3 = X_2 \sin\theta_1 + Y_2 \cos\theta_1 + Y_1.$$

These formulas were given in [4].

#### b) Reversed transform

$$\begin{aligned} \text{If } D_{ij} = (r, \vec{V}) \text{ then } D_{i,j}^{-1} &= (r, \vec{V})^{-1} \\ &= (r^{-1}, -r^{-1}(\vec{V})). \end{aligned}$$

#### Example

In the plan  $D_{i,j} = (\theta, X, Y)$

$$D_{i,j}^{-1} = (-\theta, -X\cos\theta - Y\sin\theta, X\sin\theta - Y\cos\theta).$$

#### c) Merging of transforms

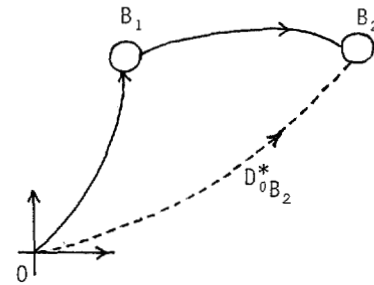
If for example  $D_{i,j}^1 : R_i \rightarrow R_j$  and  $D_{i,j}^2 : R_i \rightarrow R_j$  then  $D_{i,j}^1$  and  $D_{i,j}^2$  are combined to give a more accurate transformation denoted  $D_{i,j}^1 \cap D_{i,j}^2$  (see [ ]).

#### Application

##### Example 1

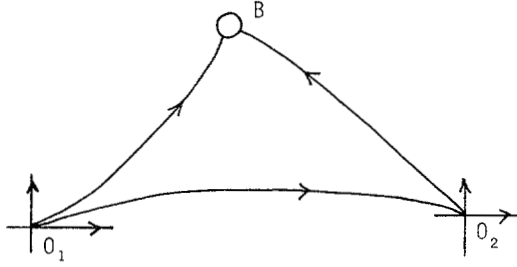
$B_1, B_2$  are two beacons, the robot is at the position 0. We know estimates for  $D_{B_1 B_2}$  and  $D_{0 B_1}$  then we can estimate the transform  $D_{0 B_2}$  by:

$$D_{0 B_2}^* = D_{0 B_1} \circ D_{B_1 B_2}$$



##### Example 2

$B$  is a beacon, the robot moves from point  $O_1$  to  $O_2$ . The robot knows estimates for  $D_{O_1 B}$ ,  $D_{O_1 O_2}$ ,  $D_{O_2 B}$ . The problem is to find a better estimate for the position of  $B$  with respect to the find position  $O_2$  of the robot:



$$D_{0_2 B}^* = D_{0_2 B} \cap (D_{0_1 0_2}^{-1} \circ D_{0_1 B / 0_2})$$

$D_{0_1 B / 0_2}$  means that the transform  $D_{0_1 B}$  is expressed in the frame  $0_2$ .

### 3. The probabilistic approach

The problem is to find the error probability distribution.

R. Smith and P. Cheeseman ([4]) make the assumption that mean and covariance are sufficient to characterize this distribution and calculated these two moments for the different transforms.

Let  $D = (\theta, X, Y)$  be a transform in the plan.  $\theta, X, Y$  are random variables.

The covariance matrix is given by:

$$C = E((\Delta X, \Delta Y, \Delta \theta)^t \cdot (\Delta X, \Delta Y, \Delta \theta))$$

where  $\Delta A = A - E(A)$ ,  $E$  is the expectation.

The covariance matrices of compounding, merging and reversing transforms are easy to calculate.

We remember below the results of [4] concerning plan transforms.

#### a) Compounding

$$D_1 = (\theta_1, X_1, Y_1), D_2 = (\theta_2, X_2, Y_2)$$

$C_i = (i=1,2)$  the corresponding covariance matrix.

The covariance matrix  $C_3$  of  $D_3 = D_1 \circ D_2$  is given by:

$$C_3 = [J] \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} [J]^t$$

where  $[J]$  is the jacobian of the transform :

$$\begin{pmatrix} \Delta X_3 \\ \Delta Y_3 \\ \Delta \theta_3 \end{pmatrix} = [J] (\Delta X_1, \Delta Y_1, \Delta \theta_1, \Delta X_2, \Delta Y_2, \Delta \theta_2)^t.$$

#### b) Reversed transform

$D = (\theta, X, Y)$ ,  $C$  the covariance matrix.

$C'$  the covariance matrix of  $D^{-1}$ .

$$C' = [J] C [J]^t.$$

#### c) Merging of transforms

$$D_3 = D_1 \cap D_2.$$

Extended Kalman filter theory gives

$$C_3 = C_1 - K \cdot C_1^t \text{ where } K = C_1 \cdot (C_1 + C_2)^{-1}.$$

#### Remark

This method is very interesting since we have only to do matricial calculus.

For the HILARE robot ([3]) only diagonal covariance matrices are considered and the motions are represented by Gaussian random vectors.

With these assumptions, the probability distribution is of the following type :

$$f(U) = (2\pi)^{-3/2} (\det C)^{-1/2} \exp \left[ -\frac{1}{2} (U C^{-1} U^t) \right]$$

$$U = (\theta, X, Y)$$

$$C = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \text{var}(\theta) \end{pmatrix} \quad \begin{array}{l} \alpha = \text{variance } X = \\ \text{variance } Y \end{array}$$

Only  $\alpha$  and  $\text{var}(\theta)$  have to be estimate. Therefore this approach is easy to use but great overestimates of the uncertainties can result as proved with the following simple example.

#### Example

The problem is to estimate the position of the end  $E$  of the segment  $S$  partially hidden by the object  $O$  whose length is one unit

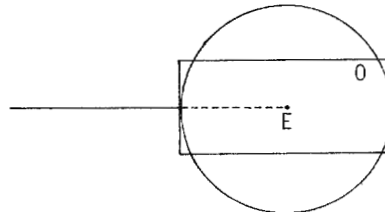


Figure 5

For the associated covariance matrix  $C$  we must have  $\alpha \approx 1$  and this gives rise to a bad overestimate of the domain in which we can find  $E$  with high probability (see figure 5).

#### Remarks

The covariance matrices are calculated using first order approximation. It appears that for the ORASIS project this is not sufficient since we have to treat positioning errors, measurement errors and also incomplete information. This conclusion is in accordance with [4].

#### 4. Chains of relative transformations

A chain (or series) of relative transformations can be computed by compounding the first two transformations to form a new transformation then compounding this new transformation with the third transformation etc...

Using the motion group representation we find that (see § 2,a):

$$D_{1,2} \circ D_{2,3} \circ \dots \circ D_{n-1,n} = D_{1,n}$$

$$= (r_1, \vec{V}_1) \cdot (r_2, \vec{V}_2) \cdot \dots \cdot (r_{n-1}, \vec{V}_{n-1})$$

where  $D_{i,i+1} = (r_i, \vec{V}_i) \in G_d, d = 2 \text{ or } 3$

$$= (r_1 \circ r_2 \circ \dots \circ r_{n-1}, \vec{V}_1 + r_1(\vec{V}_2) +$$

$$(r_1 \circ r_2)(\vec{V}_3) + \dots + r_1 \circ r_2 \circ \dots \circ r_{n-2}(\vec{V}_{n-2}))$$

$$= (R_{n-1}, \vec{S}_{n-1}) .$$

If necessary the components of the random vector  $\vec{S}_{n-1}$  can be given explicitly.

#### Advertisement

This section has an interest which is more theoretical. We try to show that is the probability distribution of the random vector  $\vec{S}_{n-1}$  if no special assumption is made about the angular errors, the covariance matrices of different transformations etc...

The practical conclusions are the following:

- i) the probability distribution of the random vector  $\vec{S}_{n-1}$  is approximately Gaussian if  $n$  is large (this is not trivial at all

since  $\vec{S}_{n-1}$  is the sum of  $(n-1)$  non independent random variables!)

- ii) as you can see the invariance matrices of different increments of  $\vec{S}_{n-1}$  are not independent nevertheless the covariance matrix of  $S_{n-1}$  can be approximated by a matrix which has a very simple form (namely a diagonal matrix!) but this matrix is not the same as if we had assumed  $S_{n-1}$  is the sum of  $(n-1)$  independent random variables
- iii) also when the angular errors are large, probabilistic estimates can be given
- iv) some assumptions made in [4] are justified, in particular the results of this section agree with simulations made by R. Smith and P. Cheeseman.

It's out of the scope for me to develop here all the arguments which permit to find the results (i) and (ii). The interested reader can see [7] in which the general theory of random walks on Lie groups is developed. The trouble is that [7] is not a handbook (i. e. not easy to read and to understand) for people working on mobile robot projects.

I try to explain in a most simpler way the different steps of the proofs.

#### Proposition

Assume that  $(r_i, \vec{V}_i)$  is a sequence of independent random variables of  $G_d$  with the same law  $\mu$  and that  $\mu$  has a second moment (i. e.  $\int_{G_d} |\vec{V}(g)|^2 d\mu(g) < +\infty$ ,  $|\vec{V}|$  = euclidian norm) then:

- i) the covariance matrix of  $\frac{S_n}{\sqrt{n}}$  tends to the diagonal matrix  $\theta \text{ Id}$  where  $\theta$  is a strictly positive constant and  $\text{Id}$  is the identity matrix
- ii)  $\frac{S_n}{\sqrt{n}}$  converges in law (as  $n \rightarrow +\infty$ ) to  $N(0, \theta \text{ Id})$  the Gaussian distribution in  $\mathbb{R}^d$  whose mean is zero and the covariance matrix  $\theta \text{ Id}$ .

### Proof of (i)

Let  $K_{\nu}$  and  $\overline{K}_{n-1}$  be respectively the covariance matrices of the random vectors  $\vec{V}_i$  and  $(r_1 \circ \dots \circ r_{n-2}(\vec{V}_{n-1}))$ :

$$\overline{K}_{n-1} = \int_{SO(d)} g K_{\nu} g^t d\sigma_{n-2}(g)$$

where  $\sigma_{n-2}$  is the law of  $r_1 \circ r_2 \circ \dots \circ r_{n-2}$   
 $g^t$  is the transpose of  $g$ .

A general result about measure theory on compact groups asserts that:

$\sigma_{n-2}$  converges to the Haar measure of the compact group  $SO(d)$  as  $n \rightarrow +\infty$  (the Haar measure plays the same role as the classical Lebesgue measure on  $\mathbb{R}^d$ )

$$\overline{K}_n \xrightarrow{n \rightarrow +\infty} K = \int_{SO(d)} g K_{\nu} g^t d\sigma(g)$$

But  $K$  has the following property:

$$r_i K r_i^{-1} = K \quad \forall r_i \in SO(d).$$

This means that  $K = \theta \text{Id}$  where  $\theta$  is a strictly positive constant and  $\text{Id}$  the identity matrix of  $\mathbb{R}^d$ .

### Proof of (ii)

Some notations are necessary.

If  $A$  is a  $d \times d$  matrix,  $A^*$  means the transposed matrix of  $A$ .

$$\|A\| = d \sup_{1 \leq i, j \leq d} |A_{ij}|, A = (A_{ij}) \quad i, j \in \{1, 2, \dots, d\}$$

if  $\vec{V} \in \mathbb{R}^d$ ,  $|\vec{V}|$  is the euclidian norm.  
 $\langle \vec{V}, \vec{V}' \rangle$  is the usual scalar product of the vectors  $\vec{V}, \vec{V}'$ .

$\phi_n(t) = E(\exp(i \langle t, \frac{S_n}{\sqrt{n}} \rangle))$ ,  $E$  means the expectation.

We try to prove now that:

$$\lim_{n \rightarrow +\infty} \phi_n(t) = \exp(-\frac{1}{2} \theta |t|^2) \quad (1)$$

The right member of (1) corresponds to a Gaussian variable with mean is zero and the covariance matrix  $\theta \text{Id}$ .

### Step 1

The part (i) of the proposition asserts that  $\|K_k - \theta \text{Id}\| \leq \epsilon_k$  where  $\epsilon_k \xrightarrow{k \rightarrow +\infty} 0$ .

For simplicity assume that  $E(\vec{V}_i) = 0 \quad \forall i$  (this is only a technical assumption) then:

$$\phi_k(t) = 1 - \frac{1}{2} t^* K_k t + o(|t|^2) \quad (2)$$

$$\phi_k(vt) = 1 - \frac{1}{2} t^* v^* K_k vt + o(|t|^2)$$

therefore

$$\begin{aligned} |\phi_k(t) - \phi_k(vt)| &= \left| \frac{1}{2} t^* v^* (K_k - \theta \text{Id}) vt - \frac{1}{2} t^* (K_k - \theta \text{Id}) t + \right. \\ &\quad \left. \frac{1}{2} \theta t^* v^* \text{Id} vt - \frac{1}{2} \theta t^* \text{Id} t + o(|t|^2) \right| \\ &\leq C_1 \epsilon_k |t|^2 + o(|t|^2) \quad \text{since } \|K_k - \text{Id}\| \leq \epsilon_k \\ &\leq 2 C_1 \epsilon_k |t|^2 \quad \text{for } t \text{ small.} \end{aligned}$$

$C_1$  is a constant independent of  $k$ .

### Step 2

Similar calculus as in step 1 permits to prove that:

$$|E(\exp(i \langle t, \frac{S_{nk}}{\sqrt{k}} \rangle) - \{\phi_k(\frac{t}{\sqrt{n}})\}^n)| \leq C_1 n \epsilon_k |t|^2.$$

### Step 3

From the equality (2) we deduce that:

$$\lim_{n \rightarrow +\infty} \{\phi_k(\frac{t}{\sqrt{n}})\}^n = \exp(-\frac{1}{2} t^* K_k t).$$

Apply now the result of the step 2:

$$\begin{aligned} \lim_{n \rightarrow +\infty} |E(\exp(i \langle \frac{t}{\sqrt{n}}, \frac{S_{nk}}{\sqrt{k}} \rangle) - \{\phi_k(\frac{t}{\sqrt{n}})\}^n)| &\leq C_1 n \epsilon_k \frac{|t|^2}{n} \\ &\leq C_1 \epsilon_k |t|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n \rightarrow +\infty} |\phi_{nk}(t) - \exp(-\frac{1}{2} t^* K_k t)| &= \\ \lim_{n \rightarrow +\infty} |E(\exp(i \langle \frac{t}{\sqrt{n}}, \frac{1}{\sqrt{k}} S_{nk} \rangle) - \exp(-\frac{1}{2} t^* K_k t))| &= \\ &\leq C_1 \epsilon_k |t|^2. \end{aligned}$$

Since  $\|K_k - \theta \text{Id}\| \leq \epsilon_k$ , we obtain

$$\lim_{n \rightarrow +\infty} |\phi_{nk}(t) - \exp(-\frac{1}{2} \theta |t|^2)| \leq C_2 \epsilon_k |t|^2 \quad (3)$$

For  $t$  in a compact set,  $C_2$  is a positive constant.

But the inequality (3) remains true for all  $t$ . Finally:

$$\lim_{n \rightarrow +\infty} |\phi_n(t) - \exp(-\frac{1}{2} \theta |t|^2)| \leq C_3 \varepsilon_k |t|^2$$

$C_3 = \text{constant}$ .

If  $k \rightarrow +\infty$ , we obtain the announced result.

## 5. Conclusion

In many cases the probabilistic approach is very interesting for the representation and manipulation of spatial uncertainty. Nevertheless the model must be developed, in particular a better characterization for the error probability distribution is needed.

A convex hull approach has been proposed in [8] but it seems that this method is not yet applicable in the ORASIS project.

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