Lecture Notes on

Model Predictive Control

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Any corrections/suggestions are very welcome.

Abstracts

Lecture 1: Overview of Optimal Control

This lecture starts by introducing the Open-loop optimal control problem and describe some of its main applications and limitations. The notable case of the linear quadratic regulator is explored. The efficiency of the optimization algorithms for this problem is discussed while the prominent role of the necessary conditions of optimality is emphasized. Closed-loop optimal control problems are discussed next. Solution techniques based on dynamic programming or the Hamilton-Jacobi equation are briefly overviewed. Their practical limitations for solving nonlinear or constrained problems of large size is identified.

Lecture 2: Model Predictive Control: An introduction

After the optimal control problems are overviewed and critically discussed, the scene is set to introduce Model Predictive Control (MPC). MPC is introduced as a method of generating closed-loop control laws by solving a sequence of open-loop optimal control problems. Some of the difficulties MPC practioners have are described. We review some of the MPC schemes available, linear and nonlinear, and discuss their merits.

Specialized seminar: Stability analysis of Nonlinear Model Predictive Controllers

A general Model Predictive Control (MPC) framework to generate feedback controls for time-varying systems with input constraints is formally described. A set of conditions on the design parameters that are sufficient to guarantee stability of the closed loop response are given. These sufficient conditions for stability allow MPC practitioners to verify a priori whether a given set of design parameters will lead to stability. The proof of the results involves Lyapunov stability arguments as well as results on the existence of solution to optimal control problems.

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Introduction

> *Every restriction corresponds to a law of nature, a regularization of the universe. The more restrictions there are on what matter and energy can do, the more knowledge human beings can attain."*
>
> Carl Sagan, *Broca's Brain*

The aim of this series of lectures is to address the design of controllers for nonlinear systems, specially those systems which are subject to constraints on the inputs. Among the methodologies well suited to deal with this problem we concentrate on the optimization-based ones: in particular open-loop optimal control methods and model predictive control.

The input constraints are present in most dynamical systems modelling physical phenomena. Virtually every actuator has a restriction on maximum displacement, force, or power. These restrictions must be accounted for in the model in many design problems.

Considering the model to be nonlinear might be beneficial, or even necessary in some applications. Common hard nonlinearities of a discontinuous nature do not allow linear approximation. Even for systems that are adequately described by a linear model in some range of operation, the use of a nonlinear model can broaden the valid range of operation as well as improve the accuracy of the model. These benefits often result in enhanced performances.

The methodologies to synthesise the control for nonlinear systems can be divided in two groups: the methods that use optimisation as a main procedure in finding the desired control; and those methods in which optimisation is not used or does not have a prominent role.

The latter methods typically explore particular structures of the system: known examples are passivity, sliding modes, and methods for affine or linear systems. Even general analysis methods like Lyapunov stability theory are not easily transformable into general synthesis methods and involve trial-and-error procedures. If we consider that the nonlinear system also has input or state constraints, the scarcity of adequate methods is even greater.

The optimisation-based methods typically are less restricted to particular structures of the model and naturally incorporate constraints. Moreover, the designer just has to translate the set of specifications and performance criteria into a set of constraints and an objective function; the remaining part of the process can be automated. Finally, the method not only finds a control that drives the system to meet the specifications, but also finds a good, if not the best, control according to criteria of our own choice.

These lectures concentrate on the optimisation-based methods to control nonlinear systems. Amongst the optimisation-based methods we distinguish the following:

- Open-loop optimal control methods: These include necessary conditions of optimality in the form of a Maximum Principle and Nonlinear Programming of the discretized model.
- Closed-loop optimal control methods: Dynamic Programming and the Hamilton-Jacobi equation are known tools to obtain the optimal closed-loop control.
- Model Predictive Control (MPC): It generates a "feedback" control by solving on-line a sequence of open-loop optimal control problems.

For open-loop optimal control problems there is a well-developed body of theory addressing a variety of features like nonsmooth data and state or control constraints. There are also several algorithms capable of solving even large dimension problems in an efficient way. The handicap is precisely that we obtain an open-loop and not a closed-loop control. An exception is the linear quadratic regulator.

The closed-loop methods have as one of their main advantages precisely the fact that they provide a feedback control. The superiority of the feedback control is a key idea, if not the key idea in control engineering: as the model is never an exact representation of the real dynamical system, it is preferable to have a feedback control that acts based on the information of "how the system is" rather than having an open-loop control that acts based on the information of "how the model predicts the system to be". However, the price we pay to obtain the supreme goal—the optimal feedback control—is computationally very high. The Hamilton-Jacobi partial differential equation as well as the Dynamic Programming Bellman recursion, are computationally very hard both in time and memory requirements. These requirements grow exponentially with the dimension of the problem and soon become impossible to satisfy with current technology. In addition, the Hamilton-Jacobi equation provides just sufficient conditions of optimality: in the cases when these sufficient conditions are never satisfied, they will be of no help in finding the optimum.

Model predictive control combines the best of both worlds. It generates a feedback control by using open-loop optimisation methods that are computationally efficient. The drawback is that these open-loop optimal control problems have to be solved on-line: hence MPC can only be applied when the time to solve these optimisation problems is low in comparison with the time constants of the system. Nevertheless, the number of applications of MPC, specially in the chemical process industries, is vast[1] and is rapidly increasing with the increase of the available computational power.

In the first lecture we introduce optimal control problems and review some of its most important solution tecniques. The second lecture is on Model Predictive Control. One of the main focus of this lecture and also of the specialized seminar will be the study of design techniques that guarantee that the Model Predictive Controllers are stabilizing.

When designing a controller the most basic, yet most important requirement, is stability. An unstable system is typically useless and might even be dangerous. Being able to guarantee stability is therefore, in many applications, a major concern. Despite that, the MPC strategy when naively used does not guarantee stability. The stability of MPC for linear systems is a well studied subject, but the nonlinear case is a different story altogether. The existent results on the stability of the MPC for nonlinear systems require conditions (typically terminal state constrained to the origin) that not only limit the class of systems that can be controlled in this way, but also increase the complexity of the optimisation algorithms solved on-line. There is clearly a need to find conditions under which we can guarantee stability without compromising the efficiency of the algorithms and, more importantly, without restricting the class of systems that can be addressed. This need is addressed in the specialized seminar: Stability analysis of nonlinear model predictive controllers.

[1]Industrial surveys [QB97, QB98] done with five MPC software vendors identified more than 2200 applications, 86 nonlinear, and the largest being a process with 603 outputs and 283 inputs.

1. Optimal Control

This problem, is it good to know that it is not, as it may seem, purely speculative and without practical use. Rather it even appears, and this may be hard to believe, that it is very useful also for other branches of science than mechanics

Johann Bernoulli, 1696, Posing the Brachystochrone Problem

Here we introduce the optimal control problem and the main methods that enable us to identify its solution. These methods form the main set of tools used to derive the results obtained throughout. We start by presenting the origins and the classical results in an informal setting.

1.1 Introduction

Origins: Optimal control theory can be regarded as a generalisation of the Calculus of Variations which was initiated when Bernoulli challenged in 1696 "the sharpest minds of the globe" with the Brachystochrone Problem[1]. The Calculus of Variations has, since then, been built up by numerous famous mathematicians. The most influential contributions, like the ones of Euler, Lagrange, Legendre, Hamilton, Jacobi, and Weierstrass can still be found (in different formats) in modern optimality conditions for control problems, which are identified by the names of their creators. (see e.g. [Bry96, SW97])

In the 1950s Bellman developed the concept of Dynamic Programming which can be used to solve discrete optimal control problems [Bel57]. Later Kalman [Kal60] solved a problem with linear dynamics and integral quadratic cost function, showing that the optimal control is a linear feedback. This important instance of the optimal control problem was later named the Linear Quadratic Regulator (LQR). Two years later, in 1962, Pontryagin and his collaborators published the monograph [PBGM62] with the first necessary conditions of optimality for nonlinear control problems, known as the Maximum Principle. These conditions were subsequently generalised in

[1] The problem of finding the shape of a curve in the vertical plane such that an object sliding along the curve under the influence of its own weight transverses the distance between two given endpoints in minimum time.

many ways, in particular to allow for the possibility of addressing problems with nondifferentiable data. Some of the most influential generalisations were the ones based on Nonsmooth Analysis initiated in the late 70s by Francis Clarke. The main results are compiled in the book [Cla83]. A recent thorough treatment of the subject is [Vin99].

Initially, in the 50s and 60s, most of the applications that drove research in Optimal Control came from the aerospace industry. Today the applications covered include a wide range of advanced industrial design problems from process systems to robotics.

The Optimal Control Problem: In the optimal control problem addressed here we are given a dynamical system modelled on some time interval $[a, b] \subset \mathbb{R}_+$ by a set of first order ordinary differential equations together with an initial condition

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(a) = x_0.$$

The evolution of the state $x(t) \in \mathbb{R}^n$ is determined (uniquely, if f satisfies some hypotheses) by the given function f, the given initial state x_0, and a control function $u \in \mathcal{U}([a, b])$ where $\mathcal{U}([a, b])$ is the set of functions on $[a, b]$ whose values are chosen from a given subset $\Omega(t)$ of $\mathbb{R}^m$. The Optimal Control Problem consists of choosing a control function $u \in \mathcal{U}([a, b])$ (thereby defining the trajectory x) in such a way that the pair (x, u) minimises a given performance criterion represented by a functional of the type

$$g(x(b)) + \int_a^b L(t, x(t), u(t))dt.$$

This functional is denoted the *objective function* and it can represent very general costs : running costs like energy/fuel consumption or time spent, and terminal costs like distance from target, among others. The control function u is merely required to be measurable and the set Ω can be defined in very general terms. Our wide freedom to specify the set of possible controls combined with the possibility of dealing with general objective functions gives this formulation the ability to cover a wide range of control engineering problems.

Additional features can be added to this basic formulation to further expand the range of problems that can be addressed. Namely, we can have the initial state x_0 to be chosen from a given set C_0 instead of being fixed *a priori*. We can also consider the free-time problem, where the problem is defined on an interval $[a, a + T]$ and T is a decision variable. Additional constraints can be added to the problem, for instance requiring the final state $x(b)$ to be within a given set C_1, or state constraints along the path like the ones represented by a functional inequality $h(t, x(t)) \leq 0$. For the moment we will refrain from these complications and proceed to present in an expository manner the optimality conditions for this problem in a smooth setting. First we introduce a necessary condition of optimality in the form of the celebrated Maximum Principle. After that, we give conditions based on dynamic programming. Dynamic programming arguments will lead us to the Hamilton-Jacobi equation, whose existence of a smooth solution provides a sufficient condition for optimality.

In later sections of this chapter we provide a formal specification of the optimal control problem and more modern optimality conditions involving much less restrictive hypotheses on the data.

Necessary Conditions of Optimality (NCO): The literature on Necessary conditions for optimal control problems, in the form of a Maximum Principle, is vast. (See e.g. [PBGM62], [War72],[FR75], [IT79], and [Cla83].) Early versions of the Maximum Principle for the basic optimal control problem described above assert (under smoothness hypotheses) that if $(\bar{x}, \bar{u})$ is a minimiser then there exists an absolutely continuous function p satisfying the Euler-Lagrange equation

$$-\dot{p}(t) = p(t) \cdot f_x\,(t, \bar{x}(t), \bar{u}(t)) - L_x(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t \in [a, b],$$

the transversality condition

$$p(b) = -g_x(\bar{x}(b)),$$

and the Maximisation of the Hamiltonian or Weierstrass condition: $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto p(t) \cdot f\,(t, \bar{x}(t), u) - L(t, \bar{x}(t), u)$$

for almost every $t \in [a, b]$.

If we now define the (unmaximised) Hamiltonian as

$$H(t, x, p, u) = p \cdot f(t, x, u) - L(t, x, u),$$

the problem of finding an optimal solution can be restated as solving the Hamiltonian system of equations

$$\dot{p}(t) = -H_x(t, \bar{x}(t), p(t), \bar{u}(t)) \tag{1.1}$$

$$\dot{x}(t) = H_p(t, \bar{x}(t), p(t), \bar{u}(t)), \tag{1.2}$$

with boundary conditions

$$x(a) = x_0 \tag{1.3}$$

$$p(b) = -g_x(\bar{x}(b)), \tag{1.4}$$

where $\bar{u}(t)$ maximises over $\Omega(t)$ the function

$$u \mapsto H(t, \bar{x}(t), p(t), u) \tag{1.5}$$

for almost every $t \in [a, b]$.

The problem of finding an actual pair of functions $(\bar{x}, p)$ satisfying the equations (1.1) – (1.4) above for some control function is known as the *two point boundary value problem* and its solution is a well studied subject in numerical analysis (see for example [Atk89]).

The proof of the Maximum Principle in its general form is long, but there are several interesting informal interpretations of this result. There are some authors that see it as an extension of the Euler-Lagrange condition in the calculus of variations (see e.g [Loe93]). Some explore the links with the more intuitive Dynamic Programming; the interpretation of [Dre65] in this sense is reproduced below. In a discrete time context some authors apply the Kuhn-Tucker conditions to a reformulation of the OCP as a mathematical programming problem [Var72]. Finally, some explore the interesting geometric interpretation that the adjoint vector p is an outward normal to a hyperplane moving along the optimal trajectory. This hyperplane is the support hyperplane of a convex cone constructed on the basis of the effects of perturbations to the optimal control [AF66]. In linear problems this hyperplane supports the set of reachable states [Var72].

Necessary Conditions of Optimality valid under much weaker hypothesis are provided in a later section.

Sufficient Conditions of Optimality: Having identified a set of candidates for minimisers for our problem (using the NCO or any other, perhaps *ad hoc*, method) we might be interested in verifying if a particular candidate is in fact a minimiser. This is when the sufficient conditions come into play.

Here, we concentrate on global sufficient conditions of optimality. We introduce the concepts involved starting from dynamic programming.

The concept of dynamic programming (see [Bel57]) is based on Bellman's Principle of Optimality which states:

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

The simplicity and power of this principle is striking. Applying the Principle of Optimality to continuous time optimal control problems, we obtain the Hamilton-Jacobi equation, and under some (somewhat strong) assumptions a very simple derivation of the Maximum Principle is possible.

Define the *Value Function* as the infimum cost from a initial pair time/state $(t_0, x(t_0))$

$$V(t_0, x(t_0)) = \inf_{u \in \mathcal{U}([t_0, b])} \left\{ g(x(b)) + \int_{t_0}^{b} L(s, x(s), u(s)) ds \right\}.$$

From the Principle of Optimality, we deduce that for any time subinterval $[t, t+\delta] \subset [a, b]$ $(\delta > 0)$ we have that for a trajectory x corresponding to u

$$V(t, x(t)) = \inf_{u \in \mathcal{U}([t,t+\delta])} \left\{ \int_t^{t+\delta} L(s, x(s), u(s))ds + V(t+\delta, x(t+\delta)) \right\}, \tag{1.6}$$

with the boundary condition

$$V(b, x(b)) = g(x(b)).$$

Assuming the existence of a process $(\bar{x}, \bar{u})$ defined on $[t, t+\delta]$ which is actually a minimiser for the equation (1.6), we can write

$$-V(t, \bar{x}(t)) + \int_t^{t+\delta} L(s, \bar{x}(s), \bar{u}(s))ds + V(t+\delta, \bar{x}(t+\delta)) = 0,$$

and for all pairs (x, u)

$$-V(t, x(t)) + \int_t^{t+\delta} L(s, x(s), u(s))ds + V(t+\delta, x(t+\delta)) \geq 0.$$

Assume that $\bar{u}$ and u are continuous from the right. Suppose also that V is continuously differentiable and L is continuous. Now, add and subtract $V(t+\delta, x(t))$ to the equations above. Dividing by δ, and taking the limit as $\delta \downarrow 0$, we obtain the Hamilton-Jacobi equation (HJE)

$$\begin{aligned} &V_t(t, x(t)) + \min_{u \in \Omega(t)} \{V_x(t, x(t)) \cdot f(t, x(t), u(t)) + L(t, x(t), u(t))\} = 0, \\ &V(b, x(b)) = g(x(b)). \end{aligned} \tag{1.7}$$

This is usually written as

$$V_t(t, x) - \max_{u \in \Omega(t)} H(t, x, -V_x(t, x), u) = 0, \quad V(b, x) = g(x),$$

where, as before,

$$H(t, x, p, u) = p \cdot f(t, x, u) - L(t, x, u)$$

is the (unmaximised) Hamiltonian.

The above analysis relates the Hamilton-Jacobi equation and the Value function. The Hamilton-Jacobi equation also features in the following sufficient condition for a pair $(\bar{x}, \bar{u})$ to be a minimiser.

If we can find a continuously differentiable function V such that the Hamilton-Jacobi equation is satisfied and

$$V_t(t, \bar{x}(t)) - H(t, \bar{x}(t), -V_x(t, \bar{x}(t)), \bar{u}(t)) = V_t(t, \bar{x}(t)) - \max_{u \in \Omega(t)} H(t, \bar{x}(t), -V_x(t, \bar{x}(t)), u),$$

then $(\bar{x}, \bar{u})$ is a local minimiser.

As we have seen, if the value function is continuously differentiable it is a natural candidate for the function V in the above sufficient condition. The main limitation of this result is that a continuously differentiable function V may not exist. To remedy this, some developments of the Hamilton-Jacobi theory using Nonsmooth Analysis techniques were carried out (see e.g. [AF90, CLSW98, Vin99]).

Even if we do not have just a small set of candidates for minimisers, we can still use the HJE to help us find the solution. In this case we should express the control as a function of the time and state defined by the relationship $u(t,x) = \arg\max_u H(t,x,-V_x,u)$, and solve the partial differential equation (1.7) (see [RV91a]). This has the advantage that we obtain the optimal control already in feedback form. However the analytical solution of the HJE is in general not possible (an exception is the Linear Quadratic Regulator), and a numerical solution is computationally very hard. Thus, in general, only very low dimensional problems can be solved in reasonable time. This constitutes the main practical limitation of the Hamilton-Jacobi / Dynamic-Programming approach.

Assuming that V is C^2, derivation of the conditions of the Maximum Principle is very simple [Dre65]. Let $(\bar{x},\bar{u})$ be a minimiser satisfying the Hamilton-Jacobi equation. Define

$$p(t) = -V_x(t,\bar{x}(t)).$$

Since for free terminal state OCP's the boundary condition can be written as $V(b,x) = g(x)$ for all x in the domain of g, we have that $V_x(b,\bar{x}(b)) = g_x(\bar{x}(b))$. It follows immediately that

$$p(b) = -g_x(\bar{x}(b)).$$

From the HJE, we obtain

$$p(t)\cdot f(t,\bar{x}(t),\bar{u}(t)) - L(t,\bar{x}(t),\bar{u}(t)) = \max_{u\in\Omega(t)} \{\, p(t)\cdot f(t,\bar{x}(t),u) - L(t,\bar{x}(t),u)\,\}.$$

We have deduced the transversality condition and the maximisation of the Hamiltonian condition. It remains to obtain the Euler-Lagrange equation. Since the HJE equals zero for any x, differentiating with respect to x we obtain

$$V_{tx}(t,\bar{x}(t)) + V_{xx}(t,\bar{x}(t))\cdot f(t,\bar{x}(t),\bar{u}(t)) + V_x(t,\bar{x}(t))\cdot f_x(t,\bar{x}(t),\bar{u}(t)) + L_x(t,\bar{x}(t),\bar{u}(t)) = 0. \quad (1.8)$$

It follows using (1.8) that the derivative of p is

$$\begin{aligned} \dot{p}(t) &= -\frac{d}{dt}V_x(t,\bar{x}(t)) \\ &= -[V_{tx}(t,\bar{x}(t)) + V_{xx}(t,\bar{x}(t))\cdot f(t,\bar{x}(t),\bar{u}(t))] \\ &= -[V_x(t,\bar{x}(t))\cdot f_x(t,\bar{x}(t),\bar{u}(t)) + L_x(t,\bar{x}(t))], \end{aligned}$$

or written equivalently we obtain the Euler-Lagrange equation

$$-\dot{p}(t) = p(t) \cdot f_x(t, \bar{x}(t), \bar{u}(t)) + L_x(t, \bar{x}(t), \bar{u}(t)).$$

1.2 The Optimal Control Problem

We address the fixed time open-loop optimal control problem (OCP), adopting the following formulation:

$$\begin{aligned}
(P) \quad & \text{Minimise} \quad g(x(0), x(1)) && (1.9)\\
& \text{subject to} && \\
& \qquad \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [0,1] && (1.10)\\
& \qquad x(0) \in C_0 && \\
& \qquad x(1) \in C_1 && \\
& \qquad u(t) \in \Omega(t) \qquad\qquad \text{a.e. } t \in [0,1]. && (1.11)
\end{aligned}$$

The data of this problem comprise functions $g : \mathbb{R}^n \to \mathbb{R}$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the sets $C_0, C_1 \in \mathbb{R}^n$, and a multifunction $\Omega : [0,1] \rightrightarrows \mathbb{R}^m$. The set of *control functions* for (P) is

$$\mathcal{U} := \{u : [0,1] \to \mathbb{R}^m : u \text{ is a measurable function, } u(t) \in \Omega(t) \text{ a.e. } t \in [0,1]\}.$$

A *state trajectory* is an absolutely continuous function which satisfies (1.10) for some control function u. The domain of the above optimisation problem is the set of *admissible processes*, namely pairs (x, u) comprising a control function u and a corresponding state trajectory x which satisfy the constraints of (P). We say that an admissible process $(\bar{x}, \bar{u})$ is a *strong local minimiser* if there exists $\delta > 0$ such that

$$g(\bar{x}(0), \bar{x}(1)) \leq g(x(0), x(1))$$

for all admissible processes (x, u) satisfying

$$\|x(t) - \bar{x}(t)\|_{L^\infty} \leq \delta.$$

In considering the objective function without the integral term we are not sacrificing generality since the Bolza/Lagrange formulation (with integral term) can be transformed into an equivalent Mayer formulation (without integral term) simply by considering an additional state x_{n+1} which is added to the objective function with dynamics $\dot{x}_{n+1}(t) = L(t, x(t), u(t))$. Also, in the case of a fixed time interval we can, without loss of generality, consider the interval $[0,1]$ as "time" domain for our problem. The free-time problem (for example minimum time problems) can be fitted to this formulation by means of state-augmentation. This transformation is not possible if $t \mapsto f(t, x, u)$ is not Lipschitz (because even using the weakest hypotheses for the NCO given below the function $x \mapsto f(t, x, u)$ is required to be Lipschitz). In this case, we can use NCO with a slightly different form, having an extra condition on the value of the Hamiltonian. (see e.g. [RV91b] for details)

1.3 Existence of Optimal Controls

In the search for an optimal solution, especially when the NCO are the tool of choice, to guarantee the existence of a solution is of prime importance. The logical chain of reasoning in what is called the *deductive method* in optimisation proceeds as follows [Cla89]:

1. A solution to the problem exists.
2. The necessary conditions are applicable, and they identify certain candidates – extrema.
3. Further elimination, if necessary, identifies a solution.

Obviously, if the first step is ignored we might end up selecting an element from the set of candidates given by the NCO when this set does not contain the minimiser we seek.

Among several results on existence of solutions that can be found in the literature, we provide the following well-known result that will be especially useful later in the framework of Model Predictive Control. For a proof we refer to [FR75, Vin87].

Theorem 1.3.1 *Assume that the data of problem (P) satisfy:*

1. *The function g is continuous.*
2. *The set C_0 is compact and C_1 is closed.*
3. *There exists at least one admissible process.*
4. *The function $t \mapsto f(t, x, u)$ is measurable for all (x, u), and $(x, u) \mapsto f(t, x, u)$ is continuous for all $t \in [0, 1]$.*
5. *The function $x \mapsto f(t, x, u)$ is globally Lipschitz for all $t \in [0, 1]$ and all $u \in \Omega(t)$. (With a Lipschitz constant k not depending on t nor u.)*
6. *The "velocity set" $f(t, x, \Omega(t)) := \{v \in \mathbb{R}^n : v = f(t, x, u), u \in \Omega(t)\}$ is convex for all $(t, x) \in [0, 1] \times \mathbb{R}^n$.*

Then there exists an optimal process.

The continuity, compactness, and non-emptiness requirements in the first four conditions are natural in any existence result based on the Weierstrass theorem. The last condition on the convexity of the velocity set has no parallel in finite dimensional optimisation problems, but ensures that the minimum is not achieved by limits of rapid switching controls that do not exist in a conventional sense. The measurability and Lipschitz properties of the velocity function f guarantee that the trajectory is uniquely defined by the control.

For problems with Bolza-type objective function

$$\text{Minimise } g(x(1)) + \int_0^1 L(t, x(t), u(t))\, dt,$$

subject to the same constraints as (P), existence of minimisers is guaranteed if we modify the last three conditions to (see [FR75, Thm. III.4.1])

4'. The functions $t \mapsto f(t, x, u)$ and $t \mapsto L(t, x, u)$ are measurable for all (x, u); and the functions $(x, u) \mapsto f(t, x, u)$ and $(x, u) \mapsto L(t, x, u)$ are continuous for all $t \in [0, 1]$.

5'. The functions $x \mapsto f(t, x, u)$ and $x \mapsto L(t, x, u)$ are globally Lipschitz for all $t \in [0, 1]$ and all $u \in \Omega(t)$. (With a Lipschitz constant k not depending on t nor u.)

6'. The "extended velocity set" $\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(t, x, u), \ell \geq L(t, x, u), u \in \Omega(t)\}$ is convex for all $(t, x) \in [0, 1] \times \mathbb{R}^n$.

We note that the last condition on the convexity of the "extended velocity set" can be easily established in the cases when L and Ω are convex and f is affine in the control (i.e. $f(t, x, u) = \alpha(t, x) + \beta(t, x) \cdot u$), and also in the case when $f(t, x, \Omega)$ is convex and L does not depend on u.

1.4 The Maximum Principle

We provide here a version of the maximum principle under much weaker hypotheses. In fact the hypotheses under which this problem is treated are the minimum hypothesis in which it makes sense to talk about a control problem [Cla76]. They are denoted here and throughout as the Basic Hypotheses.

Basic Hypotheses For the strong local minimizer $(\bar{x}, \bar{u})$ of interest, the following hypotheses will be invoked. There exists a positive scalar δ' such that:

H1 The function $(t, u) \mapsto f(t, x, u)$ is $\mathcal{L} \times \mathcal{B}$ measurable for each x. ($\mathcal{L} \times \mathcal{B}$ denotes the product σ-algebra generated by the Lebesgue subsets $\mathcal{L}$ of $[0, 1]$ and the Borel subsets of $\mathrm{I\!R}^m$.)

H2 There exists a $\mathcal{L} \times \mathcal{B}$ measurable function $k(t, u)$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|$$

for $x, x' \in \{\bar{x}(t)\} + \delta'\mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0, 1]$.

H3 The function g is Lipschitz continuous on $\{(\bar{x}(0), \bar{x}(1))\} + \delta'\mathbb{B}$.

H4 The end-point constraint sets C_0 and C_1 are closed.

H5 The graph of Ω is $\mathcal{L} \times \mathcal{B}$ measurable.

In the above, $\mathbb{B}$ denotes the closed unit ball, $\mathbb{B} := \{\xi \in \mathrm{I\!R}^n : \|\xi\| \leq 1\}$.

Proposition 1.4.1 (Necessary Conditions of Optimality) *Assume hypotheses H1 – H5. If $(\bar{x}, \bar{u})$ is an optimal process then there exist an absolutely continuous function $p : [0, 1] \to \mathrm{I\!R}^n$, and a scalar $\lambda \geq 0$ such that*

$$||p||_{L_\infty} + \lambda > 0, \tag{1.12}$$

$$-\dot{p}(t) \in \mathrm{co}\, \partial_x \left(p(t) \cdot f\left(t, \bar{x}(t), \bar{u}(t)\right)\right) \quad a.e.\ t \in [0, 1], \tag{1.13}$$

$$(p(0), -p(1)) \in N_{C_0}(\bar{x}(0)) \times N_{C_1}(\bar{x}(1)) + \lambda \partial g(\bar{x}(0), \bar{x}(1)), \tag{1.14}$$

and for almost every $t \in [0, 1]$, $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto p(t) \cdot f\left(t, \bar{x}(t), u\right). \tag{1.15}$$

Here, $N_C(x)$ denotes the limiting normal cone to the closed set $C \subset \mathrm{I\!R}^n$ at $x \in C$ defined as

$$\begin{aligned} N_C(x) := \{\lim y_i : \text{there exist } x_i \xrightarrow{C} x,\ \{M_i\} \subset \mathrm{I\!R}_+ \text{ s.t.} \\ y_i \cdot (z - x_i) \leq M_i \|z - x_i\|^2\ \forall z \in C,\ \forall i\}, \end{aligned}$$

and $\partial f(x)$ is the limiting subdifferential of lower semicontinuous function $f : \mathrm{I\!R}^n \to \mathrm{I\!R} \cup \{+\infty\}$ at a point $x \in \mathrm{dom}\, f$ defined as

$$\partial f(x) := \{y : (y, -1) \in N_{\mathrm{epi}\, f}(x, f(x))\}.$$

Further details of the nonsmooth analysis involved are provided in the appendix. For a proof see [Vin99].

2. Introduction to Model Predictive Control

Model Predictive Control (MPC), also known as Receding Horizon Control or Moving Horizon Control is an optimization based technique to construct feedback laws to control linear or nonlinear systems which might be subject to constraints.

The feedback control is obtained by solving a sequence of open-loop optimal control problems, each of them using the measured state of the plant as its initial state.

Among the advantages of using MPC we distinguish the following:

1. The ability to deal naturally with constraints within the open-loop OCPs. Most other control techniques do not apply or become considerably more complex when constraints on the inputs or states are present.

2. Each section of the control strategy is selected to from a function minimizing some objective function in the optimal control problem. It is thus natural that the resulting trajectory is a "good" one with respect to the desired criteria represented by the objective function. We note, however, that in all fixed, finite horizon approaches the resulting trajectory is not in general optimal with respect to the objective function of the OCPs.

3. The construction of feedback laws via the solution of open-loop optimal control problems that can be solved efficiently enables MPC to address problems of very high dimension: applications with 603 outputs and 283 inputs have been reported in the literature [QB97].

4. It is widely used in applications. Within the process industry it is the most widely used multivariable technique; only PID controllers are used more often. Industrial surveys of 1997 and 1998 done to five MPC software vendors identified more than 2200 applications. These applications are mainly in the process industry, with special incidence on refining. However,

the application in areas like the automotive, aerospace and defence industries, that typically involve faster systems, is also starting to appear. [QB97, QB98].

The history of MPC had independent important contributions from control theory researchers as well as the process industry practitioners. The first known reference to the technique is in [Pro63]. He proposed a moving horizon technique that became known as "open loop optimal feedback". The classical optimal control book by Lee and Markus [LM67] also describes a "technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers" analogous to the MPC strategy. These literature, however, seem unknown to the process control engineers community who rediscovered the MPC strategy in an attempt to address the needs and concerns of the petrochemical industry. In the late 1970s Richalet *et al.* [RRTP76, RRTP78] described a successful application of a technique he called "Model Heuristic Predictive Control" and in 1979 a group of engineers from Shell describe the "Dynamic Matrix Control" technique and the results of its application.

Even in recent literature different origins of each community could still be identified. The term "Receding Horizon Control" was until recently the most used term within the control theory community to describe MPC. In contrast "Model Predictive" or "Model Based" control were the preferred terms in the process engineering community. Model Predictive Control is now becoming the accepted term in workshops and papers involving both communities.

3. Stability of Nonlinear Model Predictive Controllers

In this work we present an algorithm to generate stabilizing feedback controls for a large class of time-varying nonlinear system which might be subject to constraints on the input. The algorithm is based on the Model Predictive Control (MPC) technique which naturally allows the controls to be constrained. The stability results for this technique are extended in several ways. The MPC framework proposed, besides covering important new classes of nonlinear systems and being able to analyse stability of most previous MPC schemes, also brings some important practical advantages. The sufficient stability condition presented makes possible a more flexible choice of the design parameters for the sequence of open-loop optimal control problems involved. This flexibility permits reduction of the terminal constraints traditionally imposed, resulting in optimal control problems that can be solved more efficiently by current optimisation algorithms.

3.1 Introduction

The method used here to construct stabilising feedbacks for general nonlinear systems is Model Predictive Control (MPC), also known as Receding-Horizon or Moving-Horizon Control. This method obtains the feedback control by solving a sequence of open-loop optimal control problems, each of them using the measured state of the plant as its initial state.

Contributions of this work can be seen both on the theoretical side by providing a general framework to analyse stability, and on the practical side in that the flexibility of the framework allows the use of open-loop optimal control problems that can be solved more efficiently.

The study of MPC stabilising schemes has been the subject of intense research in recent years. For continuous-time nonlinear systems the first stability results, using terminal equality constraints, were developed by Mayne and Michalska in 1990 [MM90]. These were succeeded by other important contributions like [MM93] with the dual-mode approach, [YP93] and [dOM97] using contractive constraints, and more recently [CA98b] with the quasi-infinite horizon. This last work uses the terminal cost as an important ingredient to prove stability. The importance of the terminal cost to guarantee stability was first noticed in [RM93] in a context of linear systems. The use of a terminal cost in the open loop optimal control problem is also of key importance in our approach. Recent surveys on nonlinear model predictive control schemes focusing on stability are [May97], [MRRS98] and [CA98a]. The success of MPC in dealing with some of the difficult control engineering problems can be confirmed in the industrial surveys of [QB97] and [QB98].

Traditionally, MPC schemes with guaranteed stability for nonlinear systems impose conditions on the open loop optimal control problem that either lead to some demanding hypotheses on the system or make the on-line computation of the open loop optimal control very hard. In previous works, these conditions take the form of a terminal state constrained to the origin, or an infinite horizon, or else impose some rather conservative controllability conditions on the system near the origin. These approaches considerably restrict the applicability of the MPC method, not only by narrowing the classes of systems to which it can be applied, but also by making very difficult to verify whether some hypotheses are satisfied for a particular nonlinear system.

Most practitioners of MPC methods know that, by appropriate choice of some parameters of the objective function (obtained by trial-and-error and some empirical rules), it is possible to obtain stabilising trajectories without imposing demanding artificial constraints. However, their achievements can not often be supported by any theoretical result to date, and "playing" with the design parameters is an option criticised by researchers (see e.g. [BGW90]). Here we intend to reduce this gap.

We propose a very general framework of MPC for systems satisfying very mild hypotheses. The *design parameters* of the MPC strategy are chosen in order to satisfy a certain (sufficient) *stability condition*, and hence the resulting closed-loop system will have the desirable stability properties guaranteed.

From a theoretical point of view, we provide a unifying framework for stable MPC schemes. Most MPC schemes can be constructed from our framework and their stability properties deducted from our stability results. But perhaps more important is that with the insight obtained by using the general framework we are able to construct novel MPC schemes capable of dealing with new classes of nonlinear systems.

From a practical point of view, we give a stability condition that can be verified *a priori* (i.e. one not requiring trial-and-error, simulations) and guarantees that a particular set of design parameters will lead to stability. In particular, the design parameters can always be chosen in such a way that the open-loop optimal control problem is a free terminal state one, which leads to more efficient optimisation algorithms to solve this problem. Alternatively, the design parameters can be very easily found, if we impose a mild constraint on the terminal state. Several examples are examined to show how to choose the design parameters in order to satisfy the stability condition. These include some instances of nonholonomic systems for which none of the cited MPC schemes is able to guarantee stability.

3.2 Preliminaries on Discontinuous Feedback and Stability

A common assumption in most previous MPC approaches was the continuity of the controls resulting from the open-loop optimal control problems. This assumption, in addition to being very difficult to verify, was a major obstacle in enabling MPC to address a broader class of nonlinear systems. This is because some nonlinear systems cannot be stabilised by a continuous feedback as was first noticed in [SS80] and [Bro83].

However, if we allow discontinuous feedbacks, it would not be clear what should be the solution (in a classical sense) of the dynamic differential equation. Consider a time-varying feedback control $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$. The classical definition of a trajectory of the system

$$
\begin{aligned}
&\dot{x}(t) = f(t, x(t), k(t, x(t))), \quad t \in \mathbb{R} \\
&x(t_0) = x_0,
\end{aligned}
$$

depends on certain properties of the function f, as well as on the requirement that the feedback $x \mapsto k(t, x)$ is continuous.

This motivated the development of new concepts of solution to differential equations under a discontinuous feedback. Previous attempts to deal with discontinuous controls in a MPC context are [MV94] using Filippov solutions and approaches avoiding the continuity problem by considering a discrete-time framework (e.g. [MHER95]). Recently, Ryan [Rya94] and Coron and Rosier [CR94] have shown that Filippov solutions cannot lead to stability results for general nonlinear systems. A successful approach that deals with discontinuous feedbacks to stabilise general nonlinear systems is to use the "sampling-feedbacks" of Clarke *et al.*[CLSS97]. In their definition of trajectory, the feedback is not a function of the state on *every* instant of time, rather it is a function of the state at the last sampling instant. But this coincides with the trajectories defined by our MPC framework. Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. Let the function $t \mapsto \lfloor t \rfloor_\pi$ give the greatest sampling instant less than or equal to t, that is

$$\lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\}.$$

The π-trajectories of the system under the feedback k are obtained by

$$\dot{x}(t) = f(t, x(t), k(t, x(\lfloor t \rfloor_\pi))), \quad t \in \mathbb{R} \tag{3.1a}$$

$$x(t_0) = x_0. \tag{3.1b}$$

The π-trajectories are, under mild conditions, defined even for discontinuous feedback. We proceed to define stability in this framework.

Definition 3.2.1 The sampling-feedback $k(t, x(t_i)) \in U(t)$ is said to *asymptotically stabilise* the system (3.1) on X_0 if there exists a sufficiently small inter-sample time δ such that the following condition is satisfied. For any $\gamma > 0$ we can find a scalar $M > 0$ such that for any pair $(t_0, x_0) \in \mathbb{R} \times X_0$ we have $\|x(s + t_0; t_0, x_0, k)\| < \gamma$ for $s \geq M$.

Note that this definition does not imply existence of a feedback which makes the system stable in the usual Lyapunov sense. The concept of stabilisability defined above is better suited to many nonlinear systems for which a controller cannot simultaneously satisfy our objective of driving the state to the origin together with the Lyapunov notion of keeping the state arbitrarily close to the origin. A well known example is a car-like vehicle (investigated in section 3.6.2 below). It can be easily seen that even if we are arbitrarily close to our objective, we may have to manoeuvre the vehicle to a certain minimal distance away from our target in order to drive to it. This is what we usually do to park a car sideways.

In order to prove that a particular system can be stabilised we naturally have to assume that there exists at least an open-loop control driving the system to the origin. More precisely, we require *uniformly asymptotically controllability* defined as follows.

Definition 3.2.2 A system is said to be *uniformly asymptotically controllable on* some set $X_0 \subset \mathbb{R}^n$ if for any $\gamma > 0$, there exists a scalar $M > 0$ such that for all pairs $(t_0, x_0) \in \mathbb{R} \times X_0$, we can find a piecewise-continuous control function $u : [t_0, +\infty) \to \mathbb{R}^m$ such that $u(s + t_0) \in U(s + t_0)$ for all $s \geq 0$ and $\|x(s + t_0; t_0, x_0, u)\| \leq \gamma$ for all $s \geq M$.

Note that this is just an attractiveness requirement ($\|x(s+t_0; t_0, x_0, u)\| \to 0$ as $s \to \infty$ uniformly on t_0 and x_0) and, as before, there is no requirement on Lyapunov stability.

3.3 The Model Predictive Control Framework

We shall consider a nonlinear plant with input constraints, where the evolution of the state after time t is predicted by the following model.

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \geq t \tag{3.2a}$$

$$x(t) = x_t \tag{3.2b}$$

$$u(s) \in U(s). \tag{3.2c}$$

The data of this model comprise a set $X_0 \subset \mathbb{R}^n$ containing all possible initial states, a vector $x_t \in X_0$ that is the state of the plant measured at time t, a given function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a multifunction $U : \mathbb{R} \rightrightarrows \mathbb{R}^m$ of possible sets of control values. These data combined with a particular measurable control function $u : [t, +\infty) \to \mathbb{R}^m$ define an absolutely continuous trajectory $x : [t, +\infty) \to \mathbb{R}^n$.

We assume this system to be *uniformly asymptotically controllable on* X_0.

Our objective is to obtain a feedback law that (asymptotically) drives the state of our plant to the origin. This task is accomplished by using a MPC strategy. Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T)$ at each sampling instant t_i, every time using the current measure of the state of the plant x_{t_i}.

$$\mathcal{P}(t, x_t, T) \quad \text{Minimise} \quad \int_t^{t+T} L(s, x(s), u(s))ds + W(t+T, x(t+T)) \tag{3.3}$$

subject to

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \in [t, t+T] \tag{3.4}$$

$$x(t) = x_t$$

$$u(s) \in U(s) \quad \text{a.e. } s \in [t, t+T]$$

$$x(t+T) \in S. \tag{3.5}$$

The domain of this optimisation problem is the set of admissible processes, namely pairs (x, u) comprising a measurable control function u and the corresponding absolutely continuous state trajectory x which satisfy the constraints of $\mathcal{P}(t, x_t, T)$. A process $(\bar{x}, \bar{u})$ is said to solve $\mathcal{P}(t, x_t, T)$ if it globally minimises (3.3) among all admissible processes.

We pause to clarify the notation adopted here. The variable t will represent real time while we reserve s to denote the time variable used in the prediction model. The vector x_t will denote the actual state of the plant measured at time t. The process (x, u) is a pair trajectory/control obtained from the model of the system. The trajectory will sometimes be denoted as $s \mapsto x(s; t, x_t, u)$ when we want to make explicit the dependence on the initial time, initial state, and control function. The pair $(\bar{x}, \bar{u})$ denotes an optimal solution to an open-loop optimal control problem (OCP). The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the time horizon T, the running and terminal costs functions L and W, and the terminal constraint set $S \subset \mathbb{R}^n$.

Figure 3.1: The MPC strategy.

The MPC conceptual algorithm consists of performing the following steps at a certain instant t_i (see Fig. 3.1).

1. Measure the current state of the plant x_{t_i}.

2. Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x_{t_i}, T)$.

3. The control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta)$ is applied to the plant, (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).

4. The procedure is repeated from (1.) for the next sampling instant t_{i+1} (the index i is incremented by one unit).

The resultant control law, called the MP controller, is a feedback control since during each sampling interval, the control u^* is dependent on the state x_{t_i}.

It is a well-known fact that (for fixed finite horizon) the closed-loop trajectory of the system (x^*) does not necessarily coincide with the open-loop trajectory ($\bar{x}$) solution to the OCP. Hence, the fact that MPC will lead to a stabilising closed-loop system is not guaranteed *a priori*, and is highly dependent on the *design parameters* of the MPC strategy.

We will show that we can guarantee stability of the resultant closed loop system, by choosing the design parameters to satisfy a certain *stability condition.* We anticipate here some of the key steps in our result. The crucial element of the stability condition is the requirement that the design parameters are chosen in such away that for all states x belonging to the set of possible terminal states of the OCP (which is, of course, a subset of S), there exists a control value $\tilde{u}$ such that

$$W_x(x) \cdot f(t, x, \tilde{u}) \ \leq \ -L(t, x, \tilde{u}). \tag{3.6}$$

This condition is important in establishing that a certain function V^δ constructed from value functions of the OCP's involved is "decreasing". Then, using Lyapunov-type arguments, we are able to prove the stability of the closed-loop system.

It is interesting to note that Mayne *et al* [MRRS98], in an independent development, identified a condition similar to (3.6) for discrete time systems as a "common ingredient" to most stabilising MPC schemes. Here, we also show how stability of other MPC schemes can be verified with the help of (3.6). But a perhaps more important consequence is that this generalisation enables us to construct new MPC schemes guaranteeing stability for new important classes of nonlinear systems, like for example nonholonomic systems.

In the next section, we show that we can guarantee stability of the resultant closed-loop trajectory for all systems complying with the following hypotheses.

H1 For all $t \in \mathbb{R}^n$ the set $U(t)$ contains the origin, and $f(t, 0, 0) = 0$.

H2 The function f is continuous, and $x \mapsto f(t, x, u)$ is locally Lipschitz continuous for every pair (t, u).

H3 The set $U(t)$ is compact for all t, and for every pair (t, x) the set $f(t, x, U(t))$ is convex.

H4 The function f is compact on compact sets of x, more precisely given any compact set $X \subset \mathbb{R}^n$, the set $\{ \|f(t, x, u)\| : t \in \mathbb{R}, x \in X, u \in U(t) \}$ is compact.

H5 Let $X_0 \subset \mathbb{R}^n$ be a compact set containing all possible initial states. The system (3.2) is uniformly asymptotically controllable on X_0.

Remark 3.3.1 *A controllability hypothesis, such as H5, is inevitable to obtain stability of the closed-loop system. It should be noted however that in other respects, all the hypotheses are expressed directly in terms of the data of the nonlinear model. This has not been the case in most previous MPC literature where assumptions on the feasibility of the OCP or on properties of the value function are standard.*

Hypothesis H1 should not be seen as a restrictive one, since most systems can be made to satisfy it after an appropriate change of coordinates similar to the one discussed in Section 3.5.1.

The assumption H3 (together with H2) is necessary to guarantee the existence of solution to the OCP. Some variations are possible. For example, if we allow relaxed controls (see [War72, You69, Art83]), the convexity of the velocity set $f(t, x, U(t))$ is no longer required.

The condition in H4 is automatically satisfied for time-invariant systems since the image of compact sets under a continuous function is compact.

3.4 Stability Results

In this section, we state two of our results. The main result asserts that the feedback controller resulting from the application of the MPC strategy is a stabilising controller, as long as the design parameters satisfy the stability condition below. The other result states that we can always find design parameters satisfying the stability condition for systems complying with assumptions H1–H5.

The important consequence of these two results, expressed in the corollary below, is that we are able to design a stabilising feedback controller for any nonlinear system belonging to the large class of systems that satisfies the assumptions H1–H5, using the MPC strategy.

Consider the following stability condition SC:

SC For system (3.2) the design parameters: time horizon T, objective functions L and W, and terminal constraint set S, satisfy:

SC1 The set S is closed and contains the origin.

SC2 The function L is continuous, $L(\cdot, 0, 0) = 0$, and there is a continuous positive definite and radially unbounded function $M : \mathbb{R}^n \to \mathbb{R}_+$ such that $L(t, x, u) \geq M(x)$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^m$. Moreover, the "extended velocity set" $\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(t, x, u), \ell \geq L(t, x, u), u \in U(t)\}$ is convex for all (t, x).

SC3 The function W is positive semi-definite and continuously differentiable.

SC4 The time horizon T is such that, the set S is reachable in time T from any initial state and from any point in the generated trajectories: that is, there exists a set X containing

X_0 such that for each pair $(t_0, x_0) \in \mathbb{R} \times X$ there exists a control $u : [t_0, t_0 + T] \to \mathbb{R}^m$ satisfying

$$x(t_0 + T; t_0, x_0, u) \in S.$$

Also, for all control functions u in the conditions above

$$x(t; t_0, x_0, u) \in X \quad \text{for all } t \in [t_0, t_0 + T].$$

SC5 There exists a scalar $\epsilon > 0$ such that for every time $t \in [T, \infty)$ and each $x_t \in S$, we can choose a control function $\tilde{u}$ continuous from the right at t satisfying

$$W_t(t, x_t) + W_x(t, x_t) \cdot f(t, x_t, \tilde{u}(t)) \leq -L(t, x_t, \tilde{u}(t)), \qquad (SC5a)$$

and

$$x(t + r; t, x_t, \tilde{u}) \in S \quad \text{for all } r \in [0, \epsilon]. \qquad (SC5b)$$

The main result on stability is the following.

Theorem 3.4.1 *Assume the system satisfies hypotheses H1–H5. If we choose the design parameters to satisfy SC then the closed loop system resulting from the application of the MPC strategy is asymptotically stable.*

Next, we state a result, that says that it is always possible to choose the design parameters satisfying SC. Furthermore, the result has the appealing consequence that it is always possible to design a stabilising MPC strategy using free terminal state optimal control problems.

Theorem 3.4.2 *Assume H1–H5. Then it is always possible to find design parameters S, T, L, and W to satisfy SC. In particular this choice can be made with $S = \mathbb{R}^n$,(i.e. free end-point problem).*

The proof of these results is supplied in a later section. The following result is an immediate consequence of the two theorems above.

Corollary 3.4.3 *Using MPC we are able to construct an asymptotic stabilising feedback for any nonlinear system complying with hypotheses H1-H5.*

Remark 3.4.4 *The requirement in SC1 that the set S is closed is necessary to guarantee the existence of a solution to the open-loop optimal control problem.*

The first part of condition SC2 and condition SC3 are trivially satisfied for the usual quadratic objective function $L(x, u) = x^T Q x + u^T R u$, with $Q > 0$ and $R \geq 0$ and $W(x) = x^T P x$, with $P \geq 0$.

The second part of SC2 on the convexity of the "extended velocity set" is a well known requirement for existence of solution in OCP with integral cost term. Given H3, it is automatically satisfied if L *is convex and* f *depends linearly on* u *(i.e. affine systems, see [FR75]) or if* L *does not depend on* u*. The latter is a consequence of both* $f(t, x, U(t))$ *and* $\{\ell : \ell \geq L(t, x)\}$ *being convex sets.*

Condition SC4 is obviously necessary for the existence of solution to the sequence of OCP's.

Condition SC5a is a key requirement for establishing the existence of a decreasing "Lyapunov" function, and thus asymptotic stability. It can be interpreted as the existence of a control $\tilde{u}$ *that drives the state towards inner level sets of* W *at rate* L*. In the case of quadratic functions* W *the level sets are ellipsoids centred at the origin and the velocity vector should point inwards. However,* W *need not to be restricted to quadratic functions, and this freedom can be used in our advantage as we shall see below.*

The condition SC5b, which states that the trajectory does not leave the set S *immediately, can be written as the easier to verify condition*

$$\zeta \cdot f(t, x_t, \tilde{u}(t)) \leq 0 \quad \textit{for all } \zeta \in N_S^P(x(t)),$$

where $N_S^P(x)$ *denotes the proximal normal cone to the set* S *at* $x \in S$ *defined as*

$$N_S^P(x) = \{\zeta \in \mathbb{R}^n : \textit{there exists } \sigma \geq 0 \textit{ s.t. } \zeta \cdot (y - x) \leq \sigma \|y - x\|^2,\ y \in S\}.$$

See [CLSW98] for details.

The task of choosing design parameters to satisfy all the conditions of SC might seem formidable at first. But one should not be discouraged by the generality of SC. In fact, the stability condition greatly simplifies for some standard choices of part of the design parameters. Typically, we might choose the objective function to be quadratic (making SC2 and SC3 trivially satisfied); choose the set S to be the whole space $\mathbb{R}^n$ (makes SC4 trivially satisfied); or make SC5 trivially satisfied by choosing S to be the set of points that satisfy SC5. This issue will be explored in a later section where we show, with the help of some examples, how stabilizing design parameters can be easily chosen.

3.5 Tracking and Economic Objectives

Up until now we have considered our objective to be regulating the state to a particular point we called the origin. This approach, adopted for reasons of simplicity might hide the full potential of the MPC framework given here. In fact, using straighforward changes of coordinates, the problem of tracking (i.e. following a specific trajectory) can be dealt with. As we will see, the time-varying capabilities of the framework play an important role, since the tracking problem of even time-invariant systems is converted into a time-varying regulating (i.e. drive to the origin) problem.

A criticism that is often directed at MPC frameworks such as ours, in which the design parameters are chosen to satisfy stability criteria, is that they do not give the designer the freedom to incorporate in the objective function the economic/performance objectives that he ultimately wants to achieve. We argue that this criticism is unjustified in MPC frameworks that are capable of dealing with the tracking problem. The reason is that the economic objectives can be expressed in an objective function of an OCP that is solved initially off-line. The solution to this initial OCP will be a process (y^{ref}, v^{ref}) that maximises our economic objectives. The MPC strategy will then enter at a later stage. The design parameters (including the objective function of the OCP's to be solved on-line) are then chosen to obey stability criteria to ensure that the system is actually tracking the desired reference process (y^{ref}, v^{ref}). This procedure guarantees that the process that maximises our economic criteria is closely followed.

3.5.1 Tracking

Suppose our objective is to make the system

$$\dot{y}(t) = \phi(y(t), v(t)) \quad \text{a.e.} \tag{3.7a}$$

$$y(0) = y_0 \tag{3.7b}$$

$$v(t) \in \Omega(t) \quad \text{a.e.} \tag{3.7c}$$

track a given (feasible) reference trajectory $y^{ref} : [0, +\infty) \to \mathbb{R}^n$. This problem is easily convertible into one of driving to the origin the trajectory x defined as

$$x(t) := y(t) - y^{ref}(t) \quad \text{for all } t.$$

Associated with the reference trajectory y^{ref} we select a reference control function v^{ref} (such that $\dot{y}^{ref}(t) = \phi(y^{ref}(t), v^{ref}(t))$ a.e.) and define a new control function $u : [0, +\infty) \to \mathbb{R}^m$ as

$$u(t) := v(t) - v^{ref}(t).$$

The dynamics of the process are given by

$$\begin{aligned} \dot{x}(t) = f(t, x(t), u(t)) \quad &:= \quad \dot{y}(t) - \dot{y}^{ref}(t) \\ &= \quad \phi(x(t) + y^{ref}(t), u(t) + v^{ref}(t)) - \phi(y^{ref}(t), v^{ref}(t)). \end{aligned}$$

Define also

$$x_0 = y_0 - y^{ref}(0),$$

and

$$U(t) := \{u \in \mathbb{R}^m : u + v^{ref}(t) \in \Omega(t)\}.$$

We have constructed the system (3.2) in such a way that $f(t, 0, 0) = 0$, the origin is in $U(t)$ for all t (hypothesis H1 is satisfied), and our objective is now driving the trajectory x to the origin.

3.5.2 Economic Objectives

Suppose that our plant is still represented by the model (3.7), and that our ultimate objective is to maximise a given economic or performance criteria represented by a functional J of the trajectory and control on some suitable time interval I.

The first step is to solve off-line the following initial optimal control problem (iOCP).

$$
\begin{array}{lll}
(iOCP) & \text{Maximise} & J(y, v) \\
& \text{subject to} & \\
& & \dot{y}(t) = \phi(y(t), v(t)) \qquad \text{a.e. } t \in I \\
& & y(0) = y_0 \\
& & v(t) \in \Omega(t) \qquad \text{a.e. } t \in I.
\end{array}
$$

Next, define the process (y^{ref}, v^{ref}) as a solution to this problem. Finally, apply the MPC strategy to track this reference trajectory as seen in the previous subsection. The stability results of the MPC strategy guarantee that the process (y^{ref}, v^{ref}) that maximises the criterion J is closely followed.

3.6 Choosing Stabilising Design Parameters

We analyse three different strategies for choosing a set of design parameters satisfying the stability condition SC. We start by the natural and easiest choice; setting the objective function to be quadratic and the terminal set as the whole space. This simplified framework works for general linear systems and for some nonlinear systems as is shown in the examples below. The quadratic objective function and, most significantly, the free terminal state of the open-loop optimal control problem make possible a more efficient computation of the optimal controls. Therefore, it might be worth checking, for the particular system we have at hand, whether we can satisfy SC just using this simplified framework. A more thorough discussion of this choice is provided in [FV00]. There, we identify a class of nonlinear systems that can be stabilised with this simplified framework, and explore additional properties that can be established, such as a prescribed degree of exponential stability.

Although Thm. 3.4.2 guarantees the existence of design parameters satisfying SC for $S = \mathbb{R}^n$, it might be difficult to find these parameters for certain nonlinear systems with such a large set S. This task, in particular choosing W to satisfy SC5, can be simplified if we restrict the set S to be just a subset of $\mathbb{R}^n$ containing the origin, for example a linear subspace or a closed ball centred at the origin.

We propose two strategies to define a convenient set S. The first of these strategies is based on physical knowledge of the system. This insight combined with a geometric interpretation of SC5 enable us to immediately identify a set of states where SC5 is satisfied and that can be reached in a known finite time. Some examples, including nonholonomic systems, are presented.

The last method we propose for choosing a set of stabilising design parameters is to define S simply to be the set in which SC5 is satisfied. This procedure, as we will see later, is much less restrictive than the contractive constraint methods. If this choice is made, then some care must be taken to ensure that the choice of T satisfies SC4.

The examples provided below are of academic nature and deliberately simple; there are undoubtedly methods, some ad-hoc, to design adequate stabilising controllers for each of the examples. The purpose of these examples is to illustrate firstly how to choose a convenient set of design parameters. Secondly, the examples illustrate how our MPC framework succeeds in guaranteeing stability, where previous ones fail. Finally, we also illustrate how a single framework can address all the examples in a unified way.

3.6.1 Method A: Quadratic Objective Function and Free Terminal State

Here we set S to be the whole space $S = \mathbb{R}^n$, and the objective function to be a quadratic positive definite function — where $L(x, u) = x^T Q x + u^T R u$ (with $Q > 0$ and $R \geq 0$), and $W(x) = k x^T P x$ (with k a positive scalar and P a positive definite matrix). If f is affine in u or R is chosen to be zero then conditions SC2 and SC3 are satisfied. If additionally, we set S to be $\mathbb{R}^n$ then SC4 and SC5b are also trivially satisfied and SC reduces to the following simplified version of SC5a.

SC' The positive scalar k and the positive definite symmetric matrix P are such that for each $x \in \mathbb{R}^n$ belonging to the set of possible terminal states of the OCP, we can choose a control value $u \in U$ satisfying

$$2k x^T P f(x, u) \leq -(x^T Q x + u^T R u).$$

In this simplified framework, the stability result, Thm. 3.4.1 holds with SC replaced by SC'.

Next we present some examples intended to show how we can easily choose the design parameters in order to satisfy SC', with the appealing property of free terminal state open-loop optimal control problems (i.e. $S = \mathbb{R}^n$). The first one, a linear system, has the purpose of giving the first insight into how SC' can be achieved, and confirms the results of [RM93].

In Example 2, a deliberately simple nonlinear system, we can start to see the power of this framework. For this system, none of the previously cited MPC methods is able to guarantee stability, since the origin cannot be reached in finite time (equality end-point constraint methods [MM90] fail) and the linearization of the system is uncontrollable at the origin ([MM93], [YP93], [dOM97] and [CA98b] require stabilizability of the linearization of the system at the origin, thus cannot be used).

Example 1: A Linear System Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{a.e. .}$$

We start by choosing $S = \mathbb{R}^n$. The horizon T can then be chosen as an arbitrary positive number. Choose $L(x, u) = x^T Qx + u^T Ru$ with Q and R any symmetric positive definite matrices. Select also $W(x) = kx^T Px$ in which the scalar k and the matrix P will be defined below.

From assumption H5 and by the definition of stabilizable linear system there exist a matrix F and a linear feedback control $u = Fx$ such that the closed-loop system $\dot{x}(t) = \mathcal{A}x(t)$ (with $\mathcal{A} = A + BF$) is stable. Since $\dot{x}(t) = \mathcal{A}x(t)$ is stable, the matrix P can be chosen as a positive definite solution to the Lyapunov equation

$$P\mathcal{A} + \mathcal{A}^T P = -I.$$

This last expression can be written as

$$2x^T P\mathcal{A}x = -\|x\|^2.$$

Now if we choose k satisfying

$$k \geq \lambda_{max}(Q + F^T RF),$$

we have that

$$2kx^T P\mathcal{A}x = -k\|x\|^2 \leq -x^T(Q + F^T RF)x,$$

or written equivalently as

$$2kx^T P\mathcal{A}x \leq -x^T Qx + u^T Ru,$$

we obtain SC', guaranteeing stability of the closed-loop system for this choice of parameters.

An alternative way, without having to find the matrix F *a priori*, would be to set

$$F = -R^{-1}B^T P.$$

The condition SC' would then be

$$2kx^T P(A + BF)x \leq -x^T(Q + F^T RF)x$$

or equivalently

$$2kx^T(PA - PBR^{-1}B^T P)x \leq -x^T(Q + PBR^{-1}B^T P)x.$$

Selecting $k = 1$ we get

$$x^T(2PA - PBR^{-1}B^T P + Q)x \leq 0. \tag{3.8}$$

We can easily see that for instance choosing $P > 0$ satisfying the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0,$$

we satisfy (3.8) as equality confirming previous stability results on the infinite horizon MPC for linear systems and the results in [RM93]. However, here we have a much wider choice for the design parameters, being able to guarantee stability under weaker conditions. This freedom, in particular the possibility of choosing $k > 1$ is explored in [FV00].

Example 2: A simple nonlinear system Consider the nonlinear system with control constraints:

$$\dot{x}(t) = x(t) \cdot u(t), \qquad u(t) \in [-1, 1].$$

We can easily see that it is impossible to drive the state to the origin in a finite time, and that the linearization of the system is uncontrollable at the origin. Hence, trying to satisfy a terminal-state constraint such as is required in the classical stability results for MPC would fail. Also, the dual-mode, contractive constraint, or the quasi-infinite approach cannot be used since they require stabilizability of the linearised system at the origin.

Despite that, we can easily find design parameters such that SC' is satisfied for this system. For example, if we set the design parameters

$$Q = I_n, \quad R = 0, \quad P = I_n, \text{ and } k = \frac{1}{2},$$

then, there exist a control $u = -1$ such that the stability condition SC' is satisfied:

$$\begin{aligned} 2kx^T Pf(x,u) &= -x^T Px = -\|x\|^2 \\ &= -x^T Qx. \end{aligned}$$

Thus, this simple choice of design parameters guarantees closed-loop stability.

3.6.2 Method B: Set S Chosen Using Physical Knowledge of the System

Recall the geometric interpretation of SC5a given in Remark 3.4.4. Of course, for some systems whose linearization around the origin gives a stabilizable system we might be able to choose S to be a neighbourhood of the origin, and W to be a Lyapunov function for the linearised system, using some convenient stabilising controller. This is the approach followed in [MM93] and [CA98b]. One of the powerful features of our framework is the ability to choose different types of terminal-set, and thus be able to tackle a large class of systems whose linearization is not stabilizable, including some interesting instances of nonholonomic systems.

Example 3: A nonlinear system Consider the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + 2x_1(t)u(t), \end{cases}$$

with the control constraint

$$u(t) \in [0, 1] \quad \text{a.e. } t.$$

This system cannot be driven to the origin in finite time, hence MPC schemes having terminal state constrained to the origin do not apply. Moreover, linearising around the origin we obtain the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x,$$

having poles over the imaginary axis at $\{-j, j\}$. Hence it is not stabilizable, and all the other cited MPC schemes fail to guarantee closed loop stability.

However, our framework enables us to almost trivially find the feedback control for this system. Simply notice that in the subspace

$$S := \{(x_1, x_2) : x_1 = -x_2\}$$

the control $u = 1$ drives the system towards the origin through S. Furthermore, this subspace can be reached with a well-determined finite horizon (see Fig. 3.2)

$$T = 2\pi,$$

because choosing $u = 0$, the trajectories are

$$\begin{cases} x_1(t) = x_1(0)\sin(t) \\ x_2(t) = x_2(0)\cos(t). \end{cases}$$

Figure 3.2: Example 3: Reaching set S.

Hence the trajectories will certainly meet at S within time 2π.

As to the objective function, the simple choice

$$L(x) = \|x\|^2 \text{ and } W(x) = \|x\|^2$$

is able to satisfy SC5 since

$$\dot{W} = 2x^T \cdot f(x, 1) = -4x_1^2 \leq -L(x) = -2x_1^2,$$

and

$$x(t + r, t, x_t, 1) \in S \quad \text{for all } r \geq 0,$$

if $x_t \in S$.

It follows from our main stability result that this choice of design parameters guarantees the stability of the closed-loop trajectory.

From this example we may conclude that using the terminal set to be a neighbourhood of the origin (as done in most MPC schemes) is clearly not the best choice for some systems, namely the important class of nonholonomic systems. This same conclusion can also be drawn from the next example.

Example 4: A car-like vehicle Consider the car-like vehicle of Fig. 3.3, steered by two front directional wheels, represented by the following model.

$$\begin{cases} \dot{x} = v \cdot \cos\theta \\ \dot{y} = v \cdot \sin\theta \\ \dot{\theta} = v \cdot c \end{cases}$$

where the control inputs v and c satisfy

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

Figure 3.3: A car-like vehicle.

Here (x, y) represents the location in the plane of a point in the car (the mid-point of the axle between the two rear wheels), and θ the angle of the car body with the x axis. The control v represents the linear velocity and c the curvature which is the inverse of the turning radius. It should be noted that the vehicle has a minimum turning radius ($R_{min} = c_{max}^{-1}$).

Our objective is to find a feedback controller to drive the vehicle to the origin ($x = y = 0$ and also $\theta = 0$). This objective cannot be achieved by any of the MPC methods cited: firstly because the linearization of the system around the origin is not stabilizable; secondly because the system cannot be stabilised by a continuous feedback, since it is a nonholonomic system. (This last issue makes the stabilisation of this system challenging; see [Ast96] for a discussion of this point and [Ast95] for a non MPC controller for a car-like vehicle.)

To define S we should look for possible trajectories approaching the origin. One possibility for such a set is represented in Fig. 3.4.

We define S to be the union of all semi-circles with radius greater than or equal to R_{min}, with centre lying on the y axis, passing through the origin, and lying in the left half-plane. In order to make this set reachable in finite time from any point in the space, we add the set of trajectories that are horizontal lines of distance more than $2R_{min}$ from the x axis, and lie in the right half-plane. More precisely

$$S := S_1 \cup S_2 \cup S_3$$

where

$$\begin{aligned} S_1 &:= \{(x, 0, 0) : x \leq 0\} \\ S_2 &:= \Big\{(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : x \leq 0,\ x^2 + (y - r)^2 = r^2,\ r \geq R_{min}, \end{aligned}$$

Figure 3.4: Set S of trajectories approaching the origin.

$$\tan\left(\theta - \frac{\pi}{2}\right) = \frac{y-r}{x}\Bigg\}$$

$$S_3 \;:=\; \{(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : x > 0,\ |y| \geq 2R_{min},\ \theta = \pi\}$$

The set S as defined above can be reached in a well-determined finite time-horizon. This horizon is the time to complete a circle of minimum radius at maximum velocity, that is

$$T = 2\pi R_{min}/v_{max}.$$

Conditions SC1 and SC4 are satisfied.

Choose L as

$$L(x, y, \theta) := x^2 + y^2 + \theta^2,$$

and W as

$$W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t))dt$$

where $\bar{t}$ is the time to reach the origin with the controls chosen to be the maximum velocity and the curvatures depicted in Fig. 3.4. Of course, if we are already at the origin the value chosen for the velocity is zero. That is

$$v_{x,y,\theta} = \begin{cases} v_{max} & \text{if } (x, y, \theta) \neq 0 \\ 0 & \text{if } (x, y, \theta) = 0, \end{cases}$$

and

$$c_{x,y,\theta} = \begin{cases} 0 & \text{if } (x,y,\theta) \in S_1 \cup S_3 \\ -\text{sign}(\theta)/r & \text{if } (x,y,\theta) \in S_2 \cup \bar{S}, \end{cases}$$

where

$$r = \frac{x^2+y^2}{2|y|}.$$

An explicit formula for W is, as is derived below

$$W(x,y,\theta) = \begin{cases} \dfrac{-x^3}{3v_{max}} & \text{if } (x,y,\theta) \in S_1 \\ \dfrac{r}{3v_{max}}[6r^2\theta + \theta^3 - 6rx + 3\theta x^2 + 3\theta y^2 \\ \qquad +6r\,(x-\theta y)\cos(\theta) + 6r\,(-r+\theta x + y)\sin(\theta)] & \text{if } (x,y,\theta) \in S_2 \\ \dfrac{r}{3v_{max}}\left[x^2 + 3\pi^2 x + r\pi^3 + 30\pi r^2\right] & \text{if } (x,y,\theta) \in S_3. \end{cases}$$

We can easily see that SC2 and SC3 are satisfied and as we show below SC5 also is fulfilled.

It follows from our main stability result that this choice of design parameters guarantees the stability of the closed-loop trajectory.

We proceed to a detailed verification of SC5. We verify this condition separately for each of the subsets S_1, S_2, and S_3. Starting by S_1, we choose the controls

$$\begin{cases} v = v_{max} \\ c = 0 \end{cases}$$

from which follows immediately that

$$f(x,y,\theta) = \begin{cases} \dot{x} = v_{max} \\ \dot{y} = 0 \\ \dot{\theta} = 0, \end{cases}$$

the trajectories are

$$\begin{cases} x(t) = x_0 + v_{max}t \\ y(t) = 0 \\ \theta(t) = 0. \end{cases}$$

and the time to reach the origin is

$$\bar{t} = -x_0/v_{max},$$

satisfying the second part of SC5, since starting in S_1 with these controls we remain inside this set.

Expanding W we obtain

$$W(x_0, y_0, \theta_0) = \int_0^{-x_0/v_{max}} x^2(t)\, dt = \frac{-x_0^3}{3v_{max}},$$

and

$$\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 \leq -L(x, y, \theta),$$

satisfying SC5a.

At S_2 we choose the controls

$$\begin{cases} v = v_{max} \\ c = -\text{sign}(\theta)/r, \end{cases}$$

We analyse the case in which θ is positive, the remaining case can be analysed in a similar way. It follows that

$$f(x, y, \theta) = \begin{cases} \dot{\theta}(t) = -v_{max}/r \\ \dot{x} = v_{max} \cos\theta(t) \\ \dot{y} = v_{max} \sin\theta(t), \end{cases}$$

the trajectories are

$$\begin{cases} \theta(t) = \theta_0 - v_{max}t/r \\ x(t) = x_0 + r\sin\theta_0 - r\sin(\theta_0 - v_{max}t/r) \\ y(t) = x_0 - r\cos\theta_0 + r\cos(\theta_0 - v_{max}t/r), \end{cases}$$

and the time to reach the origin is

$$\bar{t} = \theta_0 r/v_{max},$$

satisfying the second part of SC5, since starting in S_2 with these controls, we remain inside S_2.

Expanding W we obtain

$$\begin{aligned} W(x, y, \theta) &= \int_0^{\theta_0 r/v_{max}} [x^2(t) + y^2(t) + \theta^2(t)]\, dt && (3.9) \\ &= \frac{r}{3v_{max}} \left[6r^2\theta + \theta^3 - 6rx + 3\theta x^2 + 3\theta y^2 \right. \\ &\qquad \left. + 6r\,(x - \theta y)\cos(\theta) + 6r\,(-r + \theta x + y)\sin(\theta)\right] && (3.10) \end{aligned}$$

and

$$\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 - y^2(t) - \theta^2(t) \leq -L(x, y, \theta),$$

satisfying SC5a.

Finally, if we are in S_3, we choose the controls

$$\begin{cases} v = v_{max} \\ c = 0 \end{cases}$$

from which follows immediately that

$$f(x,y,\theta) = \begin{cases} \dot{x} = -v_{max} \\ \dot{y} = 0 \\ \dot{\theta} = 0, \end{cases}$$

the trajectories are

$$\begin{cases} x(t) = x_0 - v_{max}t \\ y(t) = y_0 \\ \theta(t) = \pi, \end{cases}$$

and the time to reach the y axis is

$$\tilde{t} = x_0/v_{max},$$

satisfying the second part of SC5, since starting in S_3 with these controls we remain inside it for some interval of time.

Expanding W we obtain

$$W(x_0, y_0, \theta_0) = \int_0^{\tilde{t}} x^2(t) + \pi^2\, dt + W(0, 2r, \pi) = \frac{r}{3v_{max}} \left[x_0^2 + 3\pi^2 x_0 + r\pi^3 + 30\pi r^2\right],$$

where $W(0, 2r, \pi)$ is given by the expression of W in S_2 (3.10), when the state is on the y axis. Finally, we confirm SC5a since

$$\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 \leq -L(x, y, \theta).$$

3.6.3 Method C: Set S Defined as Satisfying SC5

It might happen that we are faced with a nonlinear system whose complexity does not allows us to easily use the methodologies described above to choose the design parameters. Alternatively we may want to build a MPC package that works for several nonlinear systems with minimal intervention by the user, who might not necessarily have the knowledge to perform an analysis similar to the one above. In this case we define the terminal set to be the set of states that satisfy SC5, eliminating one major decision: to select the design parameter S.

One possible approach is to choose L and W as some functions satisfying SC2 and SC3 (for example quadratic). Define the function h as

$$h(t, x, u) = W_x(x) \cdot f(t, x, u) + L(t, x, u),$$

and solve the following optimal control problem for some small scalar $r > 0$

$$\mathcal{P}_r(s, x_s) \quad \text{Minimise} \qquad \int_s^{s+T} L(t, x(t), u(t))dt + W(x(s+T))$$

subject to

$$\begin{aligned}
&\dot{x}(t) = f(t, x(t), u(t)) && \text{a.e. } t \in [s, s+T+r] \\
&x(s) = x_s && \\
&u(t) \in U(t) && \text{a.e. } t \in [s, s+T+r] \\
&h(t, x(t), u(t)) \leq 0 && \text{for all } t \in [s+T, s+T+r].
\end{aligned} \tag{3.11}$$

Note that the satisfaction of constraint (3.11) implies that SC5 holds. The first part of SC5 is guaranteed by $h(t, x(t), u(t)) \leq 0$ for $t = s + T$, and the second part by $h(t, x(t), u(t)) \leq 0$ for $t > s + T$.

Only condition SC4 is not yet guaranteed. But if SC4 does not hold, this is immediately evident when we try to solve $\mathcal{P}_r(s, x_s)$ because it would not have a feasible solution.

However, we still have one parameter to play with — the horizon T. Of course, we can try to analyse the feasibility of problem $\mathcal{P}_r(s, x_s)$ to try to find a suitable horizon; but if we want an analysis-free framework/package, with minimal user intervention, the parameter T can always be chosen by trial-and-error until $\mathcal{P}_r(s, x_s)$ is feasible for a significant set of possible initial states.

We should note that in this last situation, stability is only guaranteed if $\mathcal{P}_r(s, x_s)$ is feasible for *all* possible initial states. However, all other MPC methods with fixed-horizon and guaranteed stability also impose a terminal constraint that in some cases can make the required feasibility of the OCP difficult to verify. The advantage of our method is that the terminal constraint used, (SC5) or (3.11), is less restrictive than in other MPC approaches, increasing the possibilities of feasibility. This point is discussed in detail in the next section.

The described method, method C, can be used for all systems that can be addressed by methods A and B, provided we choose T, L and W in the same way.

3.7 Comparison with alternative MPC approaches

In this section we show how most of the previous approaches can be seen as particular cases of the general framework proposed, and the prominent role of the stability conditions, mainly SC5, for the stabilising properties of the MPC strategy.

Terminal state constrained to the origin This is also known as the classical approach. The stabilising properties for continuous-time nonlinear systems were first proved in [MM90]. To cover this approach by our framework we would choose

$$L(x, u) = x^T Q x + u^T R u,$$

$$W = 0,$$
$$S = \{0\},$$

The stabilising properties of this approach can be confirmed by using our stability conditions. Condition SC5 is satisfied since $L(0,\cdot) = 0 = \dot{W}$, and so stability is guaranteed provided that S is reachable in time T. The main drawback of this approach is precisely the terminal state constraint, because the assumption on the existence of an admissible solution to the open loop optimal control problem is not always easy to verify, and also because of the difficulty of computing an exact solution to the constrained optimal control problem on-line.

Infinite horizon This approach was mainly discussed in the context of linear systems (see e.g. [BGW90]). We have

$$L(x, u) = x^T Q x + u^T R u,$$
$$W = 0,$$
$$S = \mathbb{R}^n,$$
$$T \to \infty.$$

Here, the assumption on the existence of an admissible solution implies a finite cost solution, therefore we must have $\lim_{T\to\infty} x(T) = 0$. We would then obtain an equivalent open loop optimal control problem if we set $S = \{0\}$, and SC would also be satisfied as in the previous case, confirming the stabilising properties of the approach. The disadvantage of this approach is that (apart from the problem concerning existence of solutions to the OCP) computing the solution to an infinite horizon nonlinear (or constrained linear) optimal control problem is very hard, which limits the applicability of this method.

Dual mode approach This approach was first described in [MM93]. In this case, outside a neighbourhood S of the origin, we have to solve the open loop optimal control problem with

$$L(x, u) = x^T Q x + u^T R u,$$
$$W = 0,$$
$$S = \epsilon \mathrm{B}$$
$$T \text{ free}$$

After the set S is reached we switch to a linear stabilising feedback controller for the linearised system.

Before we reach S, we have a free time problem. It follows therefore from the Bellman principle of optimality, that the trajectory resulting from the MPC strategy coincides with the open loop trajectory of the solution to the optimal control problem. This has two important consequences: firstly it is certain that the closed loop trajectory reaches the set S and secondly, in terms of performance, the MPC trajectory is actually the one that minimises the objective function of the optimal control problem.

Naturally, we would have to define S in such a way that the linear feedback controller stabilises the actual system, which may not be easy if we have strong nonlinearities, or even impossible if the linearization of the system is not stabilizable. The other disadvantage is that having to solve free-time optimal control problems leads to less efficient computational procedures since apart from having to deal with an extra decision variable, the solution may lead to a very long horizon.

Despite the above disadvantages, this approach was the most promising until recently. The quasi-infinite approach discussed next is an evolution of this concept that does not require switching between controllers. But we can easily see that our general framework subsumes this particular approach since satisfying SC5 within a small ball S centred at the origin is a much weaker requirement than being able to find a stabilising linear feedback controller (see Example 1.)

Quasi-infinite horizon The quasi-infinite horizon approach that was recently developed by Chen and Allgower [CA98b] also uses a terminal cost in the objective function as a key element to achieve stability. Their approach has a very intuitive explanation that might also help to give some insight in our analytical results. The central idea is to choose the terminal cost such that it exceeds the running cost till infinity,

$$W(x(t+T)) \geq \int_{t+T}^{\infty} L(x(s), u(s))\, ds. \tag{3.12}$$

This would imply that the value function for this problem is greater than the value function for the infinite horizon problem. Thus – using the value function as a Lyapunov function – stability can be easily proved. As determining W satisfying the above inequality might be difficult in general, W is computed in some neighbourhood of the origin (which we choose to be the terminal constraint set S), on which the controller is a stabilising linear feedback for the linearization of the system around the origin.

Condition (3.12) is closely related to our condition in SC5a. Let $(\tilde{x}, \tilde{u})$ be a stabilising process defined in $[t_f, \infty)$, and

$$W(x(t_f)) = \int_{t_f}^{\infty} L(\tilde{x}(s), \tilde{u}(s))\, ds.$$

Differentiating with respect to time, we obtain

$$\dot{W}(x(t_f)) = -L(\tilde{x}(t_f), \tilde{u}(t_f)),$$

which yields our condition SC5a.

There are two main disadvantages in this quasi-infinite approach. Firstly, it might be difficult to find a finite horizon T such that the chosen neighbourhood of the origin S is reachable from any possible initial condition. Secondly, if the linearization of the system around the origin is not stabilizable then this approach cannot be used.

Contraction constraints This is the approach followed in [YP93] and [dOM97]. In this approach a constraint is introduced into the OCP which requires that

$$w(x(t+T)) \leq \alpha w(x(t)).$$

Here $\alpha \in (0,1)$ and $w(x) = x^T P x$ for some positive definite matrix P. Moreover, there is an assumption that there exist a scalar $\beta \in [1, \infty)$ such that

$$\sup_{s \in [t, t+T]} \|\bar{x}(s)\| \leq \beta \|\bar{x}(t)\| \quad \text{for all } t.$$

With these conditions it might be very difficult to guarantee *a priori* feasibility of the OCP for a fixed horizon T (see [May97]). We argue that our condition SC5a is weaker since, as was explained in Remark 3.4.4, ours is a local and pointwise condition merely requiring the state to be driven towards inner level sets of W. The contractive constraint requires the terminal state to be in an inner ellipsoid to the one that passes through the initial state. That is, some components of the state have to decrease and this requirement is global, hence to be met it certainly had to be satisfied locally in some interval of time of non-zero measure. More precisely, the contractive constraint can be written as

$$w(x(t+T)) - w(x(t)) \leq -(1-\alpha) w(x(t))$$

or equivalently

$$\int_t^{t+T} \dot{w}(x(s)) ds \leq -(1-\alpha) w(x(t)).$$

The interpretation is that $\dot{w}$ has to be lower than a negative quadratic term in an interval of time with non-zero measure. This will of course be recognised as a stronger requirement than our condition SC5a: when we choose W and L to be a positive quadratic terms our condition merely requires that $\dot{W}$ is less than a negative quadratic term at a specific point, namely the final state.

3.8 Proof of the results

3.8.1 Existence of Design Parameters satisfying SC (Thm.3.4.2)

Set $S = \mathbb{R}^n$ and T an arbitrary positive number. The conditions SC1, SC4 (with $X = \mathbb{R}^n$) and SC5b are trivially satisfied.

For any pair $(t_0, x_0) \in \mathbb{R} \times X_0$ define v_{t_0,x_0} to be a right-continuous control function that drives $x(t; t_0, x_0, v_{t_0,x_0})$ asymptotically to the origin and satisfies

$$v_{t_1,x_1}(t) = v_{t_0,x_0}(t) \quad \text{for all } t \in [t_1, \infty)$$

if $x_1 = x(t_1; t_0, x_0, v_{t_0,x_0})$. (i.e. the choices of control functions are consistent with previous ones). From H5 these control functions exist. The right-continuity of the controls does not cause any difficulty since the existence of a piecewise-continuous stabilising control u implies the existence of a right-continuous one v, that is also stabilising. To see this, consider the sequence $\{t_i\}$ of all points of discontinuity of u. Let $v(t_i) = \lim_{t \downarrow t_i} u(t)$ for all i, and $v(t) = u(t)$ on all remaining points. As the points of discontinuity form a set of null Lebesgue measure, $v(t) = u(t)$ a.e. , and therefore the trajectories corresponding to each of these controls coincide.

The construction of L and W to satisfy the remaining conditions of SC has similarities with the proof of converse Lyapunov theorems. We will borrow a definition and couple of lemmas from this area. See e.g. [Vid93, Chap. 5].

A function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of *class L* if it is continuous, strictly decreasing, bounded and $\sigma(r) \to 0$ as $r \to \infty$.

Lemma 3.8.1 *If $\|x(s + t_0; t_0, x_0, v_{t_0,x_0})\| \to 0$ as $s \to \infty$ uniformly on t_0 and x_0, then there exists a function σ of class L such that*

$$\|x(s + t_0; t_0, x_0, v_{t_0,x_0})\| \le \sigma(s) \quad \textit{for all } s \ge 0 \textit{ and all } x_0 \in X_0.$$

Lemma 3.8.2 (Massera) *Let σ be a given function of class L. There exist a C^1, positive definite, strictly increasing and radially unbounded function γ such that*

$$\int_0^\infty \gamma(\sigma(r))\, dr < \infty.$$

An attentive reader might note that the radially unbounded property is not part of the usual Massera's lemma. However, the modification is trivial. Pick a point $r' > 0$ and define the function $\gamma'(y)$ to be equal to $\gamma(y)$ for $y \le \sigma(r')$ and some radially unbounded C^1 extension of γ for $y > \sigma(r')$

(say, $\gamma'(y) = \gamma(y) + (y - \sigma(r'))^2$ for $y > \sigma(r')$). Obviously, if γ satisfies Massera's Lemma then γ' will satisfy the lemma above since

$$\int_0^\infty \gamma'(\sigma(r))\,dr = \int_0^{r'} \gamma'(\sigma(r))\,dr + \int_{r'}^\infty \gamma(\sigma(r))\,dr < \infty.$$

Let $L(\cdot, x, \cdot) := \gamma(\|x\|)$. By the previous lemmas SC2 is satisfied. Define

$$W(t_0, x_0) := \int_{t_0}^\infty \gamma(\|x(s; t_0, x_0, v_{t_0,x_0})\|)ds.$$

Since γ is an increasing function we have that

$$W(t_0, x_0) \le \int_{t_0}^\infty \gamma(\sigma(s - t_0))ds < \infty.$$

The time-derivative of W is given by

$$\begin{aligned}
\frac{d}{dt}W(t, x(t; t_0, x_0, v_{t_0,x_0})) &= \frac{d}{dt}\int_t^\infty \gamma(\|x(s; t, x(t; t_0, x_0, v_{t_0,x_0}), v_{t,x(t)})\|)ds \\
&= \frac{d}{dt}\int_t^\infty \gamma(\|x(s; t_0, x_0, v_{t_0,x_0})\|)ds \\
&= -\gamma(\|x(t; t_0, x_0, v_{t_0,x_0})\|) \\
&= -L(\cdot, x(t), \cdot).
\end{aligned}$$

This implies condition SC5a, since the controls were chosen to be right-continuous. As L is continuous, $\frac{d}{dt}W$ is also continuous. As W is clearly positive semidefinite, SC3 is satisfied as well. All conditions of SC are satisfied.

3.8.2 Existence of Solutions to the OCP's

Proposition 3.8.3 (Existence of Solution) *Assume hypotheses H1–H5. Assume also that the design parameters satisfy SC. Then for any $(t_0, x_0) \in \mathbb{R} \times X_0$ a solution to the open loop optimal control problem $\mathcal{P}(t_0, x_0, T)$ exists.*

Consider the sequence of pairs $\{t_i, x_i\}$ such that x_i is the value at instant t_i of a trajectory solving $\mathcal{P}(t_{i-1}, x_{i-1}, T)$, then a solution to $\mathcal{P}(t_i, x_i, T)$ for all $i \ge 1$ also exists.

Moreover, the trajectory x^* constructed by MPC has no finite escape times.

Proof.

Consider first $\mathcal{P}(t_0, x_0, T)$. Noticing that f and the objective function are continuous, $x \mapsto f(t, x, u)$ is Lipschitz, U is compact and non-empty, the "extended velocity set" is convex, the terminal set S is closed and nonempty, and SC4 guarantees the existence of an admissible process, we are in conditions to apply a well-known existence result on solution to OCP's (see e.g. [FR75]). The first assertion follows.

Assume that the solution to $\mathcal{P}(t_{i-1}, x_{i-1}, T)$ exists and that $x_{i-1} \in X$ (X defined as in SC4), pick a pair (t_i, x_i) from the trajectory solving this latter problem. Then from SC4, we have that $x_i \in X$, and we also satisfy all the conditions for $\mathcal{P}(t_i, x_i, T)$ to have existence of solution guaranteed. The second assertion follows by induction.

It remains to prove the third assertion. Notice that implicit in the existence of solution, we have that if $\bar{x}$ is a trajectory from a solution to $\mathcal{P}(t_i, x_i, T)$ then

$$\|\bar{x}\|_{L^\infty[t_i, t_i+T]} < \infty \quad \text{for all } i \geq 0.$$

As the MPC trajectory x^* is constructed with the concatenation of solutions to a sequence of problems $\mathcal{P}(t_i, x_i, T)$ in the conditions of the second assertion, we deduce that for all $M \geq t_0$ there exists $M_2 \in \mathbb{R}$ such that

$$\|x^*\|_{L^\infty[t_0, M]} < M_2,$$

as required. $\square$

Next, we shall prove that the closed loop system resulting from the MPC strategy is asymptotically stable.

3.8.3 Main Stability Result (Thm. 3.4.1)

We show that the "MPC value function" $V^\delta(t, x)$ constructed with value functions of OCP's, satisfies a decrescence condition implying that the closed loop system is asymptotically stable as required.

Consider the sampling interval $[t_i, t_i + \delta)$. Choose $(\bar{x}, \bar{u})$ to be a solution to $\mathcal{P}(t_i, x_{t_i}, T)$. By definition of the MPC strategy we have that

$$u^*(t) = \bar{u}(t) \quad \text{for all } t \in [t_i, t_i + \delta).$$

Assuming the plant behaves as predicted by the model in this interval (we are considering just nominal stability, not robust stability) we have also that

$$x^*(t) = \bar{x}(t) \quad \text{for all } t \in [t_i, t_i + \delta). \tag{3.13}$$

For $t \in [t_i, t_i + \delta)$ define

$$V_{t_i}(t, x_t)$$

to be the value function for problem $\mathcal{P}(t, x_t, T - (t - t_i))$ (the usual OCP but where we shrink the horizon by $t - t_i$). We have that $V_{t_i}(t, x^*(t)) = V_{t_i}(t, \bar{x}(t))$ and by Bellman's Principle of Optimality the solution to $\mathcal{P}(t, \bar{x}(t), T-(t-t_i))$ coincides with the remaining trajectory of $(\bar{x}, \bar{u})$ (because for all $t \in [t_i, t_i+\delta)$ all these problems terminate at the same instant $t_i + T$), therefore

$$
\begin{aligned}
V_{t_i}(t, x^*(t)) &= \int_t^{t_i+T} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i+T, \bar{x}(t_i+T)) \\
&= V_{t_i}(t_i, \bar{x}(t_i)) - \int_{t_i}^{t} L(s, \bar{x}(s), \bar{u}(s))ds. \qquad (3.14)
\end{aligned}
$$

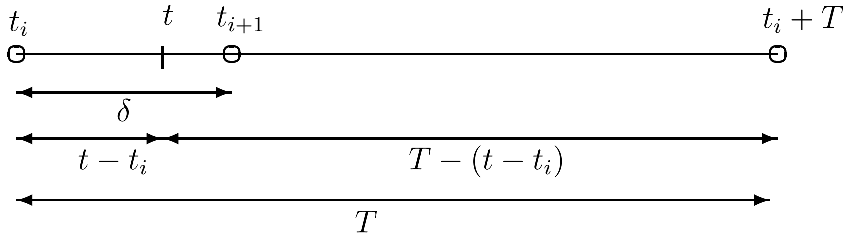

Figure 3.5: Time intervals involved in problems $\mathcal{P}(t, x_t, T-(t-t_i))$.

Finally define the "MPC Value function" to be

$$V^\delta(t, x_t) := V_\tau(t, x_t)$$

where τ is the sampling instant immediately before t, that is $\tau = \max_i\{t_i : t_i \leq t\}$.

We show that for the closed-loop trajectory x^* the function $t \mapsto V^\delta(t, x^*(t))$ converges to zero as $t \to \infty$ and thus x^* converges to zero as well. From (3.14) we know that this function is decreasing on each interval $(t_i, t_i+\delta)$ for any i. The next lemma establishes that V^δ is smaller at t_{i+1} than at t_i (see Fig. 3.6).

Figure 3.6: "Decreasing" behaviour of the MPC value function $V^\delta(t, x^*(t))$.

Lemma 3.8.4 *There exists an inter-sample time $\delta > 0$ small enough such that*

$$V_{t_{i+1}}(t_{i+1}, x^*(t_{i+1})) - V_{t_i}(t_i, x^*(t_i)) \leq -\int_{t_i}^{t_{i+1}} M(x^*(s))ds \quad \textit{for all } i \in \mathbb{N}.$$

Proof. First notice that due to the assumption on the accuracy of the model (3.13)

$$V_{t_{i+1}}(t_{i+1}, x^*(t_{i+1})) - V_{t_i}(t_i, x^*(t_i)) = V_{t_{i+1}}(t_{i+1}, \bar{x}(t_{i+1})) - V_{t_i}(t_i, \bar{x}(t_i)).$$

The value function for $\mathcal{P}(t_i, x_{t_i}, T)$ is

$$V_{t_i}(t_i, \bar{x}(t_i)) = \int_{t_i}^{t_i+T} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T, \bar{x}(t_i + T)).$$

Choose δ smaller than ϵ of SC5. Extend the process $(\bar{x}, \bar{u})$ to $[t_i, t_i + T + \delta])$ in such a way that $\bar{u} : [t_i + T, t_i + T + \delta] \to \mathbb{R}^m$ satisfies SC5. To this control will correspond the trajectory $\bar{x} : [t_i + T, t_i + T + \delta] \to \mathbb{R}^n$. Since this process is not necessarily optimal for $\mathcal{P}(t_i + \delta, x_{t_i+\delta}, T)$ we have

$$V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) \leq \int_{t_i+\delta}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T + \delta, \bar{x}(t_i + T + \delta)),$$

whence

$$\begin{aligned} V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) - V_{t_i}(t_i, \bar{x}(t_i)) \quad \leq \quad & -\int_{t_i}^{t_i+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ & + \int_{t_i+T}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ & + W(t_i + T + \delta, \bar{x}(t_i + T + \delta)) \\ & - W(t_i + T, \bar{x}(t_i + T)). \end{aligned}$$

Our choice of δ and SC5b imply

$$\bar{x}(t_i + T + r) \in S \quad \text{for all } r \in [0, \delta].$$

Integrating SC5a we obtain

$$W(t_i + T + \delta, \bar{x}(t_i + T + \delta)) - W(t_i + T, \bar{x}(t_i + T)) + \int_{t_i+T}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \leq 0.$$

Finally, recalling the condition on M in SC2 we obtain

$$\begin{aligned} V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) - V_{t_i}(t_i, \bar{x}(t_i)) \quad & \leq \quad -\int_{t_i}^{t_i+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ & \leq \quad -\int_{t_i}^{t_i+\delta} M(x^*(s))ds. \end{aligned}$$

The lemma is proved. □

Lemma 3.8.5

$$V^\delta(t, x^*(t)) + \int_0^t M(x^*(s))ds \le V^\delta(0, x^*(0)), \quad \textit{for all } t \ge 0. \tag{3.15}$$

Proof.

Let $t_i = i \cdot \delta$. From Lemma 3.8.4 we easily obtain

$$V^\delta(t_i, x^*(t_i)) - V^\delta(0, x^*(0)) \le -\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} M(x^*(s))ds$$

or

$$V^\delta(t_i, x^*(t_i)) \le V^\delta(0, x^*(0)) - \int_0^{t_i} M(x^*(s))ds.$$

Using equality (3.14)

$$\begin{aligned} V^\delta(t, x^*(t)) &= V_{t_i}(t, x^*(t)) \\ &\le V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^t M(x^*(s))ds \\ &\le V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^t M(x^*(s)) \\ &\le V^\delta(0, x^*(0)) - \int_0^t M(x^*(s))ds \end{aligned}$$

□

Now, from the last lemma, since M is positive definite, the function $t \mapsto V^\delta(t, x^*(t))$ is bounded for all $t \in [0, \infty)$. We may also deduce from (3.15) that $\int_0^t M(x(s))ds$ is bounded as well. We have that x^* is bounded and from the properties of f that $\dot{x}^*$ is also bounded. These facts combine with the following well known lemma (the proof of which can be found in e.g. [MV94]) to prove asymptotic convergence.

Lemma 3.8.6 *Let M be a continuous, positive definite function and x be an absolutely continuous function on $\mathbb{R}_+$. If*

$$\|x(\cdot)\|_{L^\infty(0,\infty)} < \infty, \qquad \|\dot{x}(\cdot)\|_{L^\infty(0,\infty)} < \infty, \qquad \textit{and} \lim_{T\to\infty} \int_0^T M(x(t))\,dt < \infty,$$

then $x(t) \to 0 \quad as \quad t \to \infty$.

The result of Thm. 3.4.1 follows immediately.

Appendix A

Background

The material provided here is intended as a concentrated summary of definitions and results that are used throughout this thesis. It also aims to define the notation used.

This material is by no means new and can be found spread in several textbooks on mathematics and optimisation like [KT66, Roy88, Lue69, HL69, Bil68, Bil86] where we can find a more detailed description of the concepts of functional analysis below. The books [AF90, Cla83, Cla89, Loe93, Vin99] were the main references used for nonsmooth analysis. Finally, we provide a summary of stability theory extracted mainly from [Hah67, Vid93].

A.1 Functional Analysis

An import tool in optimisation, namely in establishing existence of solution in function spaces, is the study of convergence, continuity and compacticity in topologies weaker than the usual *strong topology* induced by the Euclidean norm in $\mathbb{R}^n$, it is then natural to start this chain of concepts by this basic definition of topology.

A.1.1 Topologies

A **topology** $\mathcal{T}$ for a set X is a family of subsets of X satisfying:

1. The empty set $\emptyset$, and X belong to $\mathcal{T}$;

2. The intersection of a finite collection of sets of $\mathcal{T}$ is again in $\mathcal{T}$;

3. The union of a countable collection of sets of $\mathcal{T}$ is again in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called **open sets** and $(X, \mathcal{T})$ is called a **topological space**.

Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, $\mathcal{T}_1$ is said to be a **weaker topology** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case $\mathcal{T}_2$ is said to be a **stronger topology** than $\mathcal{T}_1$.

This notion of open set enables us to define closed sets, neighbourhoods, continuity, convergence, and compactness.

A set F is said to be a **closed set** if its complement $X \backslash F$ is an open set. We will denote by $\mathrm{cl}A$ the **closure** of a set A, that is the smallest closed set containing A.

An open set containing a point x is called a **neighbourhood** of x.

Consider the topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$. A function $f : X \mapsto Y$ is **continuous at a point** x in X if to each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U) \subset V$. The function is called **continuous** if it is continuous at each $x \in X$, or equivalently, if the inverse image of every open set is open, that is $f^{-1}(O) \in \mathcal{T}$ for every $O \in \mathcal{S}$.

A sequence $\{x_n\}$ in X (i.e. $\{x_n \in X : n = 1, 2, \cdots\}$) is said to **converge** to a point x in X (denoted $x_n \to x$, or $\lim_{n\to\infty} x_n = x$) if for any neighbourhood V of x there is a positive integer such that $x_n \in V$ for all $n > m$.

Let S be a subset of X. A family $\mathcal{C}$ of open sets of X is said to be a **open covering** of S if the union of the elements of $\mathcal{C}$ contains the set S. The set S is **compact** if every open covering of S includes a finite subfamily which covers S. The set S is called **sequentially compact** if every sequence $\{x_n\}$ in S contains a subsequence $\{x_{n_k}\}$ convergent to an element $x \in S$.

It turns out that for metric spaces (defined below) sequentially compactness is equivalent to compactness providing an alternative and perhaps more useful definition. In finite dimensional spaces this definition simplifies even further – a compact set is one that is simply closed and bounded.

A.1.2 Metric Spaces

In most applications the topology on a space X is determined by a distance function or a **metric**, which is a real-valued function d defined on pairs of elements of X satisfying:

1. $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The metric topology is defined by **open balls** $\mathbb{B}_\epsilon(x) := \{y : d(x, y) < \epsilon\}$ [1] which are neighbourhoods of x. A set O is open (i.e. is in the metric topology) if for every $x \in X$ there is some $\epsilon > 0$ such

[1] In the context of set-valued analysis we will use the notation $\{x\} + \epsilon\mathbb{B}$ where $\mathbb{B}$ denotes the open unit ball centred at the origin.

that $B_\epsilon(x) \subset O$. The space (X, d) defined this way is called a **metric space**.

A sequence $\{x_n\}$ in a metric space is called a **Cauchy sequence** if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

If a sequence is convergent then it is Cauchy, by the triangle inequality. A metric space (X, d) is said to be **complete** if the converse also holds, namely if every Cauchy sequence converges to an element of X.

A.1.3 Linear Spaces

A set X is a **linear space** if operations of addition and scalar multiplication are defined and if X is closed under these operations, that is for any pair of elements x, y in X, and for any pair of scalars α, β, the element $\alpha x + \beta y$ is again in X.

The set X is a **normed linear space** if there is a function, called the **norm**, that assigns a nonnegative real number $\|x\|$ to each element x of X and satisfies

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

We also denote the norm by $\|x\|_X$ when we want to make explicit the normed space we are using.

The norm defines a metric $d(x, y) = \|x - y\|$ and hence a metric topology. If the normed linear space X is complete in this metric, then it is a **Banach space**.

Examples of Banach spaces include the Euclidean spaces $\mathbb{R}^n$ with the usual Euclidean norm $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, and some function spaces such as L^p for $p = 1, 2, \ldots$, that are further described below.

A set D is said to be **dense** in a normed space X if there are point of D arbitrarily close to each point of X, or equivalently if the closure of D is X.

Examples of dense sets are the set of rationals in the real line, and the space of polynomials that is dense in the space of continuous functions. This latter result is known as the Weierstrass approximation theorem.

A normed space is **separable** if it contains a countable dense set.

Most of the spaces to be used are separable but L^∞ is an example of one that is not.

Dimension of a linear space is the minimum number of elements of the space such that their linear combination can generate any element of the space.

Consider the normed linear space X. A **functional** on X is a scalar valued linear function defined on X. A **linear functional** f is a functional satisfying $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

The set of all linear continuous functionals on X is also a linear space. This space is called the **normed dual** (or simply **dual**) of X and is denoted X^*. The norm of an element $f \in X^*$ is

$$\|f\| = \sup_{\|x\| \le 1} |f(x)|.$$

Now, we define the **inner product**. Given two vector vectors $x, y \in X$, the inner product $x \cdot y$ is a real number, and satisfies the following:

1. $x \cdot y = y \cdot x$;
2. $(x + y) \cdot z = x \cdot y + x \cdot z$;
3. $(\lambda x) \cdot y = \lambda(x \cdot y)$;
4. $x \cdot x \ge 0$, and $x \cdot x = 0$ if and only if $x = 0$.

A Banach space X together with an inner product defined on $X \times X$ is called a **Hilbert space**.

In a Hilbert space X, the functional $f(x) = x \cdot y$ for a fixed $y \in X$ defines a bounded linear functional on x. Thus, in Hilbert Spaces the elements of the dual can be generated by elements of the space itself. Some well known spaces that illustrate this result are $\mathbb{R}^n$ and L_2 which are dual of themselves.

A.1.4 Continuity and Differentiability

Consider a function $f : X \mapsto Y$. If both X and Y are normed linear spaces, then continuity of f at a point $x_0 \in X$ is equivalent to the following popular assertion: for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - x_0\|_X < \delta$ implies $\|f(x) - f(x_0)\|_Y < \epsilon$.

We say that f is **uniformly continuous** on $X' \subset X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X'$ with $\|x - y\|_X < \delta$ we have $\|f(x) - f(y)\|_Y < \epsilon$.

Consider a sequence of functions $\{f_n\}$ and a function f, each one a map from X to Y. The sequence $\{f_n\}$ is said to **converge pointwise** to f on X if for every $x \in X$ we have $f(x) = \lim f_n(x)$; that is, if, for any $x \in X$ and any $\epsilon > 0$, there is an N such that for all $n \ge N$ we have $\|f(x) - f_n(x)\|_Y < \epsilon$. If we can choose N as above independently of x we say that the sequence $\{f_n\}$ **converges uniformly** to f on X; that is, if, for any $\epsilon > 0$, there is an N such that for all $x \in X$ and all $n \ge N$ we have $\|f(x) - f_n(x)\|_Y < \epsilon$.

Consider now f to be a real valued function. We define

$$\limsup_{x \to y} f(x) = \inf_{\delta > 0} \left\{ \sup_{0 < |x - y| < \delta} f(x) \right\}, \text{ and}$$

$$\liminf_{x \to y} f(x) = \sup_{\delta > 0} \left\{ \inf_{0 < |x-y| < \delta} f(x) \right\}.$$

We have that $\liminf_{x \to y} f(x) \leq \limsup_{x \to y} f(x)$ with equality if and only if the standard limit $\lim_{x \to y} f(x)$ exists.

The function f is called **lower semi-continuous** at a point y of its domain if $f(y) \leq \liminf_{x \to y} f(x)$. It is called upper-semicontinuous if $(-f)$ is lower semi-continuous.

Let the **epigraph** of f (denoted $\operatorname{epi} f$) be the set of points on and above the graph of f. More precisely $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. The $\operatorname{epi} f$ is a closed set if and only if f is lower semi-continuous. This last result is of importance in the context of nonsmooth analysis of semi-continuous functions.

A real-valued function f defined on a real interval $[a, b]$ is said to be **absolutely continuous** if given any $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \text{ implies } \sum_{i=1}^{n} |f(b_i) - (a_i)| < \epsilon$$

for every finite collection of disjoint intervals (a_i, b_i) contained in $[a, b]$. This notion extends to vector-valued functions in the obvious way, by imposing that each of its component functions is absolutely continuous. This concept is of prime importance in control theory since the solutions to ordinary differential equations are absolutely continuous, and from this some important properties can be deduced. Namely if f is absolutely continuous, it is continuous, it is a function of bounded variation, it is differentiable almost everywhere, and is equal to the indefinite integral of its derivative. An absolutely continuous function $f : [a, b] \mapsto \mathbb{R}^n$ is also called an **arc**.

Another very import notion is the slightly stronger condition of Lipschitz continuity which is usually easier to verify. A function $f : X \mapsto \mathbb{R}$ is **Lipschitz continuous** on $A \subset \mathbb{R}$ if there is some nonnegative scalar K satisfying

$$|f(x) - f(y)| \leq K\|x - y\| \quad \text{for all } x, y \in A.$$

The function f is **globally Lipschitz** if it is Lipschitz on X, it is **locally Lipschitz** if it is Lipschitz on any compact subset A of X, and it is called **Lipschitz near** $x \in X$ if it is Lipschitz for some neighbourhood of x. The locally Lipschitz property is equivalent to being Lipschitz near every x in X.

A locally Lipschitz continuous function is also absolutely continuous, therefore it is also differentiable almost everywhere. This last result is known as the Rademacher's theorem and formed the basis for the nonsmooth analysis in its earlier developments.

A.1.5 Measure Theory

Consider a subset A of the real numbers. Consider also countable collections of open intervals $\{I_n\}$ that cover A. The length of each of this intervals is a real number: the length of an interval (a, b) is simply $b-a$. For each collection consider the sum of the lengths of all the intervals in the collection. We define the **Lebesgue** (outer) measure of A to be the infimum amongst all collections of all such sums. That is, the Lebesgue measure of A is given by $\inf\{\sum \text{length}(I_n) : A \subset \bigcup I_n\}$.

We say that an equation or property is satisfied **almost everywhere**, or a.e., on a set $A \subset \mathbb{R}$, if it is satisfied on every point of A except on a set a Lebesgue measure zero. We use the notation μ-a.e. , if a measure μ is used instead of the Lebesgue measure.

We say that a real-valued function f defined on $[0, 1]$ is **Lebesgue measurable**, or simply measurable, if there exists a sequence of continuous functions $\{f_i\}$ such that

$$\lim f_i(t) = f(t) \quad \text{a.e. } t \in \mathbb{R}.$$

A set S is said to be the **support of a measure** μ if and only if S is smallest closed set whose complement has μ measure 0.

To define measurability in a more abstract sense, we need the following definitions. A collection $\mathcal{A}$ of sets in X is called an algebra in X, if (i) X is in $\mathcal{A}$, (ii) any finite number of unions in $\mathcal{A}$ is again in $\mathcal{A}$, and (iii) the complement of any set in $\mathcal{A}$ is again in $\mathcal{A}$. If, in addition, any countable union in $\mathcal{A}$ is again in $\mathcal{A}$, it is called a **σ-algebra**.

If $\mathcal{A}$ is a **σ-algebra** in X, then $(X, \mathcal{A})$, or simply X, is called a measurable space, and the members of $\mathcal{A}$ are called the **measurable sets** in X.

If $(X, \mathcal{A})$ is a measurable space, Y a topological space, and f is a mapping of X into Y, then f is said to be measurable on $(X, \mathcal{A})$ provided that $f^{-1}(V)$ is a measurable set in $(X, \mathcal{A})$ for every open set V in Y.

Note that the intersection of any collection of closed sets, or the union of a finite collection of closed sets is closed. However, a countable union of closed sets might not be closed. Thus if we are interested in σ-algebras of closed sets, we must consider more general types of closed sets: the Borel sets. A collection $\mathcal{B}$ of **Borel sets** is the smallest σ-algebra which contains all the open sets.

A function $f : [0, 1] \times \mathbb{R}^m \mapsto \mathbb{R}$ is said to be $\mathcal{L} \times \mathcal{B}$ **measurable** if it is measurable on the space $([0, 1] \times \mathbb{R}^m, \mathcal{L} \times \mathcal{B})$, where $\mathcal{L} \times \mathcal{B}$ denotes the σ-algebra of subsets of $[0, 1] \times \mathbb{R}^m$ generated by product sets of the Lebesgue measurable subsets of $[0, 1]$ and the Borel subsets of $\mathbb{R}^m$.

A.1.6 Function Spaces

We define $L_p([0,1];\mathbb{R}^n)$, also denoted simply L_p as the set of (Lebesgue) measurable functions $x : [0,1] \mapsto \mathbb{R}^n$ for which $\int_0^1 \|x(t)\|^p dt < \infty$. The norm in this spaces is given by $\|f\|_{L_p} := \left(\int_0^1 \|x(t)\|^p dt\right)^{1/p}$. The space L_1 is simply the set of Lebesgue integrable functions on $[0,1]$. We denote by L_∞ the space of all bounded measurable functions on $[0,1]$. The norm in this space is defined as $\|f\|_{L_\infty} := \text{ess sup}\,\|f(t)\|$. The L_p spaces are Banach spaces.

Let p and q be two positive extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p$ and L_q, then

$$\int_0^1 |f(t) \cdot g(t)| dt \leq \|f\|_{L_p} \|g\|_{L_q}.$$

This result is known as the **Hölder inequality**.

The dual of L_p is L_q since fixing an element $y \in L_q$ we can define for all $x \in L_q$ the bounded linear functional

$$f(x) = \int_0^1 x(t) \cdot y(t) dt,$$

having norm $\|f\| = \|y\|_{L_q}$.

We denote by $C^k([0,1],\mathbb{R}^n)$, or C^k, $k = 0, 1, \cdots, \infty$ the space of all k-times continuously differentiable functions from $[0,1]$ to $\mathbb{R}^n$. The space C^0 is simply denoted as C. These spaces are also Banach Spaces.

Another space of particular interest is the dual space of $C([0,1],\mathbb{R})$. A Radon measure is defined by linear functionals on $C([0,1],\mathbb{R})$ (i.e. defined in $C^*([0,1],\mathbb{R})$) through the **Riesz Representation Theorem**: *Let ϕ be a positive linear functional on $C([0,1],\mathbb{R})$. Then there exists a σ-algebra $\mathcal{A}$ in $[0,1]$ which contains all Borel sets in $[0,1]$, and there exists a unique positive measure μ on $\mathcal{A}$ which represents ϕ in the sense that for every $f \in C$*

$$\phi(f) = \int_{[0,1]} f d\mu.$$

The measure μ defined as above is a regular Borel measure or **Radon measure**.

A.1.7 Weak and Weak* Convergence

Let X be a normed linear space. Consider the following **Weierstrass Existence Theorem**: *A lower-semicontinuous function defined on a compact set $K \in X$ attains its minimum on K.*

Since compactness in some functional spaces is difficult to establish in the norm topology (the closed unit ball in an infinite-dimensional normed linear space is not compact in the norm topology), it might be useful to consider weaker topologies. The weaker the topology we choose the less number of open sets it will have, therefore there are more convergent sequences and the compactness will be easier to establish. On the other hand continuity will be harder to establish in weaker topologies.

Consider the **weak topology** generated by X^*, which is the weakest topology for X under which the elements of X^* are still continuous. In this topology the ϵ-neighbourhoods of a point $x_0 \in X$ are the sets

$$\{x \in X : x' \cdot (x - x_0) < \epsilon, x' \in X^*\}.$$

Also of interest is the **weak* topology**. In this topology the closed unit ball of X^* is compact. In the weak* topology the ϵ-neighbourhoods of a point $x'_0 \in X^*$ are the sets

$$\{x' \in X^* : (x' - x'_0) \cdot x < \epsilon, x \in X\}.$$

In this way a sequence $\{x_i\}$ of elements of X is **weak convergent** to $x \in X$ if for all $x' \in X^*$, the sequence of real numbers $x' \cdot x_i$ converges to $x' \cdot x$.

Similarly a sequence $\{x'_i\}$ of elements of X^* is **weak* convergent** to $x' \in X^*$ if for all $x \in X$, the sequence of real numbers $x'_i \cdot x$ converges to $x' \cdot x$.

A set S in X^* is **weak* compact** if all sequences of elements in S contain a subsequence that is weak* convergent. A useful result, known as the **Alaoglu's Theorem**, is the following: *the closed unit ball in X^* is a weak* compact set.*

A.1.8 Cones and Convexity

A set K in X is said to be a **convex set** if given two points $x, y \in K$ the "line segment" $\alpha x + (1-\alpha)y$, for $\alpha \in [0, 1]$ also belongs to K.

The **convex hull** of a set K, denoted $\operatorname{co} K$, is the smallest convex set containing K. In other words, $\operatorname{co} K$ is the intersection of all convex sets containing K.

A function f is **convex** if the "line segment" joining two points of its graph is on or above the graph. That is, if the epigraph of f is a convex set, or alternatively if for all $x, y \in \operatorname{dom} f$ and for all $\alpha \in [0, 1]$, we have that $\alpha f(x) + (1 - \alpha)f(y) \geq f(\alpha x + (1 - \alpha)y)$.

A set K in X is said to be a **cone** with vertex at the origin if $x \in K$ implies that $\alpha x \in K$ for all $\alpha \geq 0$.

A convex cone N is said to be **pointed** if for any nonzero elements $x_1, x_2 \in N \setminus \{0\}$, we have $x_1 + x_2 \neq 0$.

A.1.9 Differential Equations

Consider the system of first order ordinary differential equations

$$\dot{x}(t) = f(t, x(t)) \quad \text{a.e. } [t_0, T], \tag{A.1}$$

with initial condition

$$x(t_0) = x_0, \tag{A.2}$$

where $x : [t_0, T] \mapsto \mathbb{R}^n$, $f : [t_0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$, and $\dot{x}(t)$ denotes $\frac{d}{dt}x(t)$.

Assume that f is measurable in t and continuous in x. Assume additionally the existence of a real-valued function $\phi \in L_1$ such that $\sup_x \|f(t,x)\| \leq \phi(t)$. This suffices to apply a **local existence theorem**, which establishes the existence of arc satisfying (A.1) and (A.2) on an interval $[t_0, t_0 + \delta)$ for some $\delta > 0$.

However, to guarantee **uniqueness of solution** we have to assume in addition that f is locally Lipschitz as a function of x.

In order to guarantee **existence of a global solution**, i.e. solution on $[t_0, +\infty)$, we have to assume linear growth, that is that there exist nonnegative constants a and b such that $\|f(t,x)\| \leq a\|x\| + b$ for all (t, x).

The last is a consequence of the Gronwall-Bellman lemma. Since this lemma is also of independent interest it is provided below.

Lemma A.1.1 (Gronwall-Bellman) *Consider the functions $\Phi : T \mapsto \mathbb{R}$ nonnegative and integrable, h and u continuous on T, where h is also nonnegative and $T = [t_0, t_1]$. Let*

$$u(t) \leq \int_{t_0}^{t} \Phi(s)u(s)\, ds + h(t) \quad \textit{for all } t \in T,$$

then

$$u(t) \leq h(t) + C \int_{t_0}^{t_1} \Phi(s)h(s)\, ds \quad \textit{for all } t \in T,$$

with $C = \exp\left(\int_{t_0}^{t_1} \Phi(s)\, ds\right)$.

A.2 Nonsmooth Analysis

We start by providing some basic concepts of set-valued analysis. Consider two sets A and B and a scalar λ. The scalar multiplication and (Minkowski) set addition are understood in the following way:

$$\lambda A := \{\lambda a : a \in A\},$$

and

$$A + B := \{a + b : a \in A,\, b \in B\}.$$

A **multifunction** F from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a mapping such that for each $x \in \mathbb{R}^n$, $F(x)$ is a subset of $\mathbb{R}^m$.

A.2.1 Normal Cones

Consider a closed set S and a point y not in S. If x is the point in S that is the nearest point to y, then the direction $y - x$ is called a **proximal normal direction** to S at x. Any nonnegative multiple $\lambda(y - x)$, $\lambda \geq 0$, of such direction is also a proximal normal to S at x. It turns out that if the boundary of S is not smooth at a certain point then the set might admit more than one (or none) proximal normal directions at that point. The set of all proximal normals of S at x forms a cone and is called the **proximal normal cone**, denoted as $N_S^P(x)$.

$$N_S^P(x) = \{\, \zeta \in \mathbb{R}^n : \zeta = \lambda(y - x),\, \lambda > 0,\, \exists\, y \in \mathbb{R}^n \text{ s.t. } \|y - x\| = \min\{\|y - x'\| : x' \in S\}\,\}.$$

An alternative definition valid in a Hilbert Space X is the following

$$N_S^P(x) = \{\zeta \in X : \text{there exists } \sigma \geq 0 \text{ s.t. } \zeta \cdot (y - x) \leq \sigma \|y - x\|^2,\ \forall y \in S,\, \forall i\}.$$

Another normal cone obtained from taking limits of a sequence of proximal normal cones is the **limiting normal cone**. We define the limiting normal cone $N_S(\bar{x})$ to the closed set $S \subset \mathbb{R}^n$ at $\bar{x} \in S$ as:

$$\begin{aligned} N_S(x) := \{\lim \zeta_i : \text{there exist } x_i \xrightarrow{S} x,\, \{\sigma_i\} \subset \mathbb{R}_+ \text{ s.t.} \\ \zeta_i \cdot (y - x_i) \leq \sigma_i \|y - x_i\|^2 \text{ for all } y \in S\}. \end{aligned}$$

Finally we define the **Clarke's normal cone**, which can be obtained from the limiting by taking convex hull and closure. The Clarke's normal cone $\bar{N}_S(\bar{x})$ to the closed set $S \subset \mathbb{R}^n$ at $\bar{x} \in S$ is defined as

$$\bar{N}_S(x) := \operatorname{cl} \operatorname{co} N_S(x).$$

We obviously have that $N_S^P(x) \subset N_S(x) \subset \bar{N}_S(x)$.

A.2.2 Subdifferentials

Let f be a continuously differentiable real-valued function. It is a well known calculus fact that the gradient of a function is related to a normal to its graph: the normal vectors to the epigraph of f at the point $(x, f(x))$ are given by $\lambda(\nabla f(x), -1)$ for $\lambda > 0$.

Similarly we can define generalized gradients via normal directions for all functions for which the epigraph is a closed set, i.e. for all lower semicontinuous functions.

Based on the normal cones above we define the corresponding proximal, limiting and Clarke's **subdifferentials** (denoted $\partial^P f(x)$, $\partial f(x)$, and $\bar{\partial} f(x)$ respectively) of lower semicontinuous function $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ to be the following multifunctions.

$$\partial^P f(x) := \left\{ \zeta : (\zeta, -1) \in N^P_{\operatorname{epi} f}(x, f(x)) \right\},$$

$$\partial f(x) := \{ \zeta : (\zeta, -1) \in N_{\operatorname{epi} f}(x, f(x)) \},$$

$$\bar{\partial} f(x) := \left\{ \zeta : (\zeta, -1) \in \bar{N}_{\operatorname{epi} f}(x, f(x)) \right\}.$$

We call subgradients to the elements of a subdifferential.

If f is Lipschitz the Clarke's subdifferential can also be given by the following limit of gradients:

$$\bar{\partial} f(x) := \operatorname{co} \left\{ \lim_{i \to \infty} \nabla f(x_i) : x_i \to x, x \notin A, x \notin B \right\}$$

where A is the set points in the neighbourhood of x where f fails to be differentiable (since f is Lipschitz this set has measure zero) and B any other set of measure zero.

In the context of optimal control problems with inequality state constraints we consider also $\partial^>_x h(t, x)$ which denotes the "hybrid" partial subdifferential of h in the x-variable and is defined as

$$\begin{aligned} \partial^>_x h(t, x) := \operatorname{co}\{ \xi : & \text{ there exists } (t_i, x_i) \to (t, x) \text{ s.t.} \\ & h(t_i, x_i) > 0 \, \forall i, \; h(t_i, x_i) \to h(t, x), \text{ and } \nabla_x h(t_i, x_i) \to \xi \}. \end{aligned}$$

A.2.3 Nonsmooth Calculus

Consider two Lipschitz functions f and g and a scalar λ. Some important calculus rules using the Clarke's subdifferential are the following

- **Scalar multiples:**

$$\bar{\partial}(\lambda f(x)) = \lambda \bar{\partial} f(x).$$

- **Sum rule:**

$$\bar{\partial}(f + g)(x) \subset \bar{\partial} f(x) + \bar{\partial} g(x).$$

- **Lebourg Mean Value Theorem**: There exist a point u in the line segment (x, y) such that

$$f(y) - f(x) \in \bar{\partial} f(u) \cdot (y - x).$$

- If $\bar{x}$ is a **local minimum** for f, then $0 \in \bar{\partial} f(\bar{x})$.
- **Constrained local minimum:** Let $\bar{x}$ be a local minimum for f when $\bar{x}$ is constrained to be in C. Then $0 \in \bar{\partial} f(\bar{x}) + \bar{N}_C(\bar{x})$.

One of the advantages of the limiting subdifferential over the Clarke's one is that the former distinguishes better between maximum and minimum and between f and $-f$. For example $\partial(|x|) = \bar{\partial}(|x|) = \bar{\partial}(-|x|) = [-1, 1]$, but $\partial(-|x|) = \{-1, 1\}$. However the Clarke's subdifferential has a much cleaner calculus rules mainly due to the fact of being convex.

A.3 Stability

Consider once again the dynamical system (A.1) and (A.2). Let 0 be an equilibrium point of this system, that is $f(t, 0) = 0$ for all t. The equilibrium point 0 is **Lyapunov Stable** at t_0 if for any $R > 0$, there exists a positive scalar $r = r(R, t_0)$ such that

$$\|x(t_0)\| < r \text{ implies } \|x(t)\| < R \quad \text{for all } t \geq t_0.$$

The equilibrium 0 is **attractive** if $x(t + t_0; t_0, x_0)$ converges to 0 as $t \to \infty$. It is **uniformly attractive** if $x(t + t_0; t_0, x_0)$ converges to 0 uniformly in t_0 and x_0.

Traditional literature defines a system to be Asymptotic Stable at zero, if zero is both attractive and Lyapunov stable [Hah67, Vid93]. Attractiveness it typically established using a continuous differentiable Lyapunov functions and this regularity gives also Lyapunov Stability as a bonus. Here we consider stability just with attractiveness as in [CLSW98, pag. 208], which, we believe, is better suited to the large class of nonlinear systems we consider.

Bibliography

[AF66] M. Athans and P. L. Falb. *Optimal Control.* McGraw Hill, New York, 1966.

[AF90] J. P. Aubin and H. Frankowska. *Set-Valued Analysis.* Birkhauser, Boston, 1990.

[Art83] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Analysis, Theory, Methods and Applications*, 7(11):1163–1173, 1983.

[Ast95] A. Astolfi. Exponential stabilization of a car-like vehicle. In *Conference on Robotics and Automation*, Nagoya, Japan, 1995.

[Ast96] A. Astolfi. Discontinuous control of nonholonomic systems. *Systems and Control Letters*, 27:37–45, 1996.

[Atk89] K. E. Atkinson. *An Introduction to Numerical Analysis.* John Wiley, New York, 1989.

[Bel57] R. Bellman. *Dynamic Programming.* Princeton University Press, Princeton, 1957.

[BGW90] R. R. Bitmead, M. Gevers, and V. Wertz. *Adaptive Optimal Control: The Thinking Man's GPC.* Prentice Hall, New York, 1990.

[Bil68] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.

[Bil86] P. Billingsley. *Probability and Measure.* 2nd Edition, 1986.

[Bro83] R. W. Brockett. Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman, and H. S. Sussmann, editors, *Differential Geometric Control Theory*, pages 181–191. Birkhouser, Boston, 1983.

[Bry96] A. E. Bryson. Optimal control — 1950 to 1985. *IEEE Control Systems*, pages 26–33, June 1996.

[CA98a] H. Chen and F. Allgöwer. Nonlinear model predictive control schemes with guaranteed stability. In R. Berber, editor, *Nonlinear Model Based Process Control.* Kluwer, 1998.

[CA98b] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10), 1998.

[Cla76] F. H. Clarke. The maximum principle under minimal hypotheses. *SIAM Journal of Control and Optimization*, 14(6):1078–1091, 1976.

[Cla83] F. H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley, New York, 1983.

[Cla89] F. H. Clarke. *Methods of Dynamic and Nonsmooth Optimization.* SIAM, 1989.

[CLSS97] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Transactions on Automatic Control*, 42(10):1394–1407, 1997.

[CLSW98] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth Analysis and Control Theory.* Springer Verlag, New York, 1998.

[CR94] J. M. Coron and L. Rosier. A relation between time-varying and discontinuous feedback stabilization. *Journal of Math. Systems, Estimation and Control*, 4:67–84, 1994.

[dOM97] S. L. de Oliveira and M. Morari. Contractive model predictive control for constrained nonlinear systems. *To appear in IEEE Transactions on Automatic Control*, 1997.

[Dre65] S. E. Dreyfus. *Dynamic Programming and the Calculus of Variations.* Academic Press, New York, 1965.

[FR75] W. H. Fleming and R. W. Rishel. *Deterministic and Stochastic Optimal Control.* Springer Verlag, NewYork, 1975.

[FV00] F. A. C. C. Fontes and R. B. Vinter. Nonlinear model predictive control: Specifying rates of exponential stability without terminal state constraints. In *Proceedings of ADCHEM 2000 - IFAC Meeting on Advanced Control of Chemical Processes*, Pisa, Italy, June 2000.

[Hah67] W. Hahn. *Stability of motion.* Springer-Verlag, New York, 67.

[HL69] H. Hermes and J. P. LaSalle. *Functional Analysis and Time Optimal Control.* Academic Press, New York, 1969.

[IT79] A.D. Ioffe and V.M. Tihomirov. *Theory of extremal problems.* North-Holland, Amsterdam, 1979.

[Kal60] R. E. Kalman. Contributions to the theory of optimal control. *Bol. de Soc. Math. Mexicana*, 1960.

[KT66] J. F. C. Kingman and S. J. Taylor. *Introduction to Measure and Probability*. Cambridge University Press, 1966.

[LM67] E. B. Lee and L. Markus. *Foudations of Optimal Control Theory*. Wiley, New York, 1967.

[Loe93] P. D. Loewen. *Optimal Control via Nonsmooth Analysis*. American Mathematical Society, Providence, RI, 1993.

[Lue69] D. G. Luenberguer. *Optimization by Vector Space Methods*. John Wiley, New York, 1969.

[May97] D. Q. Mayne. Nonlinear model predictive control: An assessment. In *5th International Conference on Chemical Process Control*, volume 93 of *AIChe Symposium Series*, pages 217–231, 1997.

[MHER95] E. S. Meadows, M. A. Henson, J. E. Eaton, and J. B. Rawlings. Receding horizon control and discontinuous state feedback stabilization. *International Journal of Control*, 62(5):1217–1229, 1995.

[MM90] D. Q. Mayne and H. Michalska. Receding horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, 35:814–824, 1990.

[MM93] H. Michalska and D. Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38:1623–1633, 1993.

[MRRS98] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Model predictive control: A review. *Automatica*, to appear, 1998.

[MV94] H. Michalska and R. B. Vinter. Nonlinear stabilization using discontinuous moving-horizon control. *IMA Journal of Mathematical Control and Information*, 11:321–340, 1994.

[PBGM62] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Wiley Interscience, New York, 1962.

[Pro63] A. I. Propoi. Use of lp methods for synthesizing sampled-data automatic systems. *Automation and Remote Control*, 24(7):837–844, 1963.

[QB97] S. J. Qin and T. A. Badgwell. An overview of industrial model predictive control technology. In *Fifth International Conference on Chemical Process Control*, volume 93 of *AIChe Symposium Series*, pages 232–256, 1997.

[QB98] S. J. Qin and T. A. Badgwell. An overview of nonlinear model predictive control applications. In *Nonlinear Model Predictive Control Workshop*, Ascona, Switzerland, 1998.

[RM93] J. B. Rawlings and K. R. Muske. The stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38(10):1512–1516, 1993.

[Roy88] H. L. Royden. *Real Analysis*. Macmillan, New York, 1988.

[RRTP76] J. Richalet, A. Rault, J. L. Testud, and J. Papon. Algorithmic control of industrial processes. In *Proceedings of the 4th IFAC Symposium on Identification and System Parameter Estimation*, pages 1119–1167, 1976.

[RRTP78] J. Richalet, A. Rault, J. L. Testud, and J. Papon. Model predictive heuristic control: applications to industrial processes. *Automatica*, 14:413–428, 1978.

[RV91a] J. D. L. Rowland and R. B. Vinter. Construction of optimal feedback controls. *Systems and Control Letters*, 16:357–367, 1991.

[RV91b] J. D. L. Rowland and R. B. Vinter. A maximum principle for free endtime optimal control problems with data discontinuous in time. *IEEE Transactions on Automatic Control*, 36:603–608, 1991.

[Rya94] E. P. Ryan. On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback. *SIAM Journal of Control and Optimization*, 32:1597–1604, 1994.

[SS80] E. D. Sontag and H. J. Sussman. Remarks on continuous feedback. In *IEEE Conference on Decision and Control, Albuquerque*, pages 916–921, 1980.

[SW97] H. J. Sussman and J. C. Willems. 300 years of optimal control: From the brachystochrone to the maximum principle. *IEEE Control Systems*, pages 32–44, June 1997.

[Var72] P. Varaya. *Notes on Optimization*. Van Nostrand Reinhold, 1972.

[Vid93] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice Hall, Englewood Cliffs, NJ, 1993.

[Vin87] R. B. Vinter. Optimal control theory. Lecture Notes, Summer School “Optimization, Control and Related Subjects”, Nice, 1987.

[Vin99] R. B. Vinter. *Optimal Control*. Birkhouser, Boston, 1999. in press.

[War72] J. Warga. *Optimal Control of Differential and Functional Equations*. Academic Press, New York, 1972.

[You69] L. C. Young. *Lectures on the calculus of variations and optimal control theory*. Saunders, Philadelphia, 1969.

[YP93] T. H. Yang and E. Polak. Moving horizon control of nonlinear systems with input saturation, disturbances and plant uncertainty. *International Journal of Control*, 58(4):875–903, 1993.