STABILIZING NONHOLONOMIC SYSTEMS BY MODEL PREDICTIVE CONTROL

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Abstract: Nonholonomic systems are a class of nonlinear systems frequently appearing in robotics and other practical problems. This class includes, for example, underactuated manipulators and wheeled vehicles. The design of stabilizing feedback controllers for these systems offers some interesting challenges. Namely, these systems are inherently nonlinear: they cannot be handled by any linear control method and are not transformable into linear systems (even locally) in any meaningful way. Furthermore, a (time-invariant) feedback law capable of stabilizing this class of systems must be allowed to be discontinuous. As a consequence, nonclassical definitions of a solution to a differential equation are needed to analyse the resulting trajectories. Despite these difficulties, a continuous-time Model Predictive Control (MPC) framework has been shown to be an appropriate methodology to generate stabilizing feedbacks for such systems. In this work, we review some characteristics of nonholonomic systems, describe our MPC framework and its stabilizing properties, and show its application to a unicycle and a car-like vehicle examples. *Copyright © Controlo 2002*

Keywords: Nonlinear control design; predictive control; receding horizon; discontinuous feedback, nonholonomic systems.

1. INTRODUCTION

Model Predictive Control (MPC) is an increasingly popular control technique that has been developed both by the systems theory community — where it is also known as Receding Horizon Control —, and by the process engineering community — where it is often referred to by commercial names such as Dynamic Matrix Control.

This technique constructs a feedback law by solving on-line a sequence of open-loop optimal control problems, each of them using the currently measured state of the plant as its initial state. Similarly to optimal control, MPC has an inherent ability to deal naturally with constraints on the inputs and on the state. Since the controls are obtained by optimizing some criterion, the method possesses some desirable performance properties, and also intrinsic robustness properties (Magni and Sepulchre, 1997). These facts can partially explain the substantial impact it has made on industry: surveys carried out with five MPC software vendors, identified more than 2200 industrial applications (Qin and Badgwell, 1997; Qin and Badgwell, 1998).

There has been, in the recent years, an intense research addressing a wide range of issues such as stability, robustness, performance analysis and state estimation; see e.g. survey paper (Mayne *et al.*, 2000). Most of the MPC approaches, however, address systems whose linearization is stabilizable, being unable to handle nonholonomic systems because these systems are not amenable to methods of linear control theory. An exception is the MPC framework introduced in (Fontes, 2001), that does not impose any controllability conditions on the linearized system, and is, therefore, the framework used in the examples studied here. Moreover, since nonholonomic systems cannot be stabilized by a time-invariant continuous feedback, some care is required when studying the trajectories resulting from MPC controllers that allow discontinuous feedbacks. The MPC framework used here overcomes such difficulties through the use of a natural "sampling feedback" concept.

2. NONHOLONOMIC SYSTEMS AND DISCONTINUOUS FEEDBACKS

Nonholonomic systems are typically completely controllable but instantaneously they cannot move in certain directions. Although these systems are allowed to move, eventually, in any direction, at a certain time or state there are constraints imposed on the motion – the so-called nonholonomic constraints. Some of the interesting examples are the wheeled vehicles, which, at a certain instant can only move in a direction perpendicular to the axle connecting the wheels.

Consider the unicycle mobile robot of Fig. 1 represented by the following model:

$$\begin{cases} \dot{x} = (u_1 + u_2) \cdot \cos\theta \\ \dot{y} = (u_1 + u_2) \cdot \sin\theta \\ \dot{\theta} = u_1 - u_2 \end{cases}$$

where $u_1, u_2(t) \in [-u_{max}, u_{max}]$.

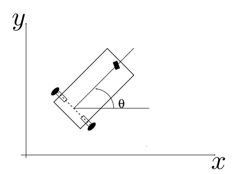

Fig. 1. A unicycle mobile robot.

The coordinates (x, y) are the position in the plane of the midpoint of the axle connecting the rear wheels, and θ denotes the heading angle measured from the x-axis. The controls u_1 and u_2 are the angular velocity of the right wheel and of the left wheel respectively. If the same velocity is applied to both wheels, the robot moves along a straight line (maximum forward velocity if $u_1 = u_2 = u_{max}$). The robot can turn by choosing $u_1 \neq u_2$ (if $u_1 = -u_2 = u_{max}$ the robot turns anticlockwise around the midpoint of the axle).

Assuming the wheels do not slip sideways, the velocity vector is always orthogonal to the wheel axis: the nonholonomic constraint

$$(\dot{x}, \dot{y})^T (\sin\theta, -\cos\theta) = 0.$$

When trying to obtain a linearization of this system around any operating point $(x_0, y_0, \theta_0, u_1 = 0, u_2 = 0)$ we can easily conclude that the resulting linear system is not controllable. Therefore, linear control methods cannot be used to handle this system.

Another example we investigate here is the car-like vehicle (see Fig. 2). In this case, the heading is controlled by the angle of two front directional wheels. In contrast with the previous example, this vehicle cannot turn with zero velocity; furthermore, it has a minimum turning radius. The system is represented by the following model:

$$\begin{cases} \dot{x} = v \cdot \cos\theta \\ \dot{y} = v \cdot \sin\theta \\ \dot{\theta} = v \cdot c \end{cases}$$

with control inputs v and c satisfying

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

(Minimum turning radius $R_{min} = c_{max}^{-1}$)

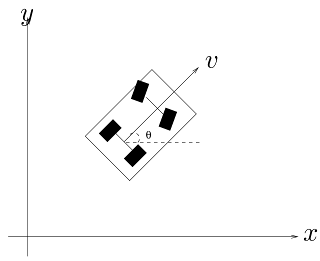

Fig. 2. A car-like vehicle.

For this system, it is also immediate to check that any linearization around the origin or any other operating point would lead to an uncontrollable system.

Another challenging characteristic of the nonholonomic systems is apparent in this example: it is not possible to stabilize it if just time-invariant continuous feedbacks are allowed.

To illustrate this characteristic suppose the vehicle is on the positive side of the y-axis, heading towards the left-half plane ($x = 0$, $y > 0$, $\theta = \pi$). If $y \geq R_{min}$, then it is possible to move towards the origin following a semi-circle. So, the decision in this case would be to turn left. On the other hand, if the car is too close to the origin, $y < R_{min}$, then it needs to move away from the origin to obtain the correct heading. The decision in this case is to turn right. Therefore, the decision of where to turn, deciding input c, is a discontinuous function of the state — a discontinuous feedback.

The need for discontinuous feedbacks to stabilize nonholonomic systems is known for some time (Brockett, 1983; Kolmanovsky and McClamroch, 1995). However, if we allow discontinuous feedbacks, it might not be clear what is the solution of the dynamic differential equation. The trajectory x, solution to

$$\dot{x}(t) = f(x(t), k(x(t))), \quad x(0) = x_0$$

is just defined, in classical sense, if k is assumed to be continuous. See (Clarke, 2001) for a further discussion of this issue.

There are a few alternative definitions of solutions to ODE with discontinuous right-hand side. The best known is the concept of Filippov solutions, but that one was proved not interesting for stabilization of some systems (Ryan, 1994; Coron and Rosier, 1994).

A solution concept that has been proved successful in dealing with stabilization by discontinuous feedbacks is the concept of CLSS solution introduced in (Clarke *et al.*, 1997). Consider a sequence of sampling instants $\pi := \{t_i\}_{i\geq 0}$ in $[0, +\infty)$ with $t_0 < t_1 < t_2 < \ldots$

$$\dot{x}(t) = f(x(t), k(x(\lfloor t \rfloor_\pi))), \quad x(0) = x_0$$

where $\lfloor t \rfloor_\pi := \max_i\{t_i \in \pi : t_i \leq t\}$

That is, the feedback is not a function of the state at every instant of time, rather it is a function of the state at the last sampling instant.

The difficulties of combining discontinuous feedbacks and MPC and how to overcome these difficulties is discussed in (Fontes, 2002). Our MPC framework (introduced in (Fontes, 2001)) constructs, as can be seen in the next section, a "sampling-feedback" law defining a trajectory in a way which is very similar to the concept introduced in (Clarke *et al.*, 1997). These trajectories are, under mild conditions, well-defined even when the feedback law is discontinuous.

3. THE MPC FRAMEWORK AND STABILITY RESULTS

Consider a plant with input constraints, where the evolution of the state after time t is predicted by the following nonlinear model.

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \geq t \tag{1a}$$
$$x(t) = x_t \tag{1b}$$
$$u(s) \in U(s). \tag{1c}$$

Assume that the set $U(t)$ contains the origin and that $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. We proceed to define a continuous-time MPC framework, perform a continuous-time stability analysis while considering that the inter-sampling times are nonzero and that the open-loop optimal control problems are solved at every sampling instant.

Consider a sequence of sampling instants $\pi := \{t_i\}_{i\geq 0}$ with a constant inter-sampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T)$ at each sampling instant t_i, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(t, x_t, T)$: Minimize

$$\int_t^{t+T} L(s, x(s), u(s))ds + W(t+T, x(t+T))$$

subject to:

$$\begin{aligned}
&\dot{x}(s) = f(s, x(s), u(s)) && \text{a.e. } s \in [t, t+T] \\
&x(t) = x_t \\
&u(s) \in U(s) && \text{a.e. } s \in [t, t+T] \\
&x(t+T) \in S.
\end{aligned}$$

The pair $(\bar{x}, \bar{u})$ denotes an optimal solution to an open-loop optimal control problem (OCP). The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* to the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the time horizon T, the running and terminal costs functions L and W, and the terminal constraint set $S \subset \mathbb{R}^n$.

The MPC algorithm consists of performing the following steps at a certain instant t_i.

(1) Measure the current state of the plant x_{t_i}.
(2) Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x_{t_i}, T)$.
(3) Apply to the plant the control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta)$ (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).
(4) Repeat the procedure from (1.) for the next sampling instant t_{i+1} using the new measure of the state of the plant $x_{t_{i+1}}$.

The resultant control law is a "sampling-feedback" control since during each sampling interval, the control u^* is dependent on the state x_{t_i}:

$$\dot{x}(t) = f(t, x(t), k(t, x(\lfloor t \rfloor_\pi))), \quad t \in \mathbb{R}$$

where

$$k(t, x(\lfloor t \rfloor_\pi)) = u^*(t).$$

This allows our MPC framework to overcome the inherent difficulty of defining solutions to differential equations with discontinuous feedbacks. In this way, the class of nonlinear systems potentially addressed by MPC is enlarged, including, for example, nonholonomic systems.

The use of this framework to stabilize nonlinear systems is illustrated in the following stability result, for which the proof can be found in (Fontes, 2001).

Theorem 1. Choose the design parameters: time horizon T, objective functions L and W, and terminal constraint set S, satisfying:

SC1 The set S is closed and contains the origin.
SC2 The function L is continuous, $L(\cdot, 0, 0) = 0$, and there is a continuous positive definite and radially unbounded function $M : \mathbb{R}^n \to \mathbb{R}_+$ such that $L(t, x, u) \geq M(x)$ for all $(t, u) \in \mathbb{R}_+ \times \mathbb{R}^m$. Moreover, the "extended velocity set" $\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(t, x, u), \ell \geq L(t, x, u), u \in U(t)\}$ is convex for all (t, x).
SC3 The function W is positive semi-definite and continuously differentiable.
SC4 The time horizon T is such that, the set S is reachable in time T from any initial state and from any point in the generated trajectories: that is, there exists a set X containing X_0 such that for each pair $(t_0, x_0) \in \mathbb{R}_+ \times X$ there exists a control $u : [t_0, t_0+T] \to \mathbb{R}^m$, with $u(s) \in U(s)$ for all $s \in [t_0, t_0 + T]$, satisfying

$$x(t_0 + T; t_0, x_0, u) \in S.$$

Also, for all control functions u in the conditions above

$$x(t; t_0, x_0, u) \in X \quad \text{for all } t \in [t_0, t_0 + T].$$

SC5 For each time $t \in [T, \infty)$ and each $x_t \in S$, we can choose a control function $\tilde{u} : [t, t + \delta] \to \mathbb{R}^m$ with $\tilde{u}(s) \in U(s)$ for all $s \in [t, t + \delta]$, satisfying

$$\begin{aligned} &W_t(t, x_t) + W_x(t, x_t) \cdot f(t, x_t, \tilde{u}(t)) \\ &\quad \leq -L(t, x_t, \tilde{u}(t)), \end{aligned} \tag{$SC5a$}$$

and

$$x(t + \delta; t, x_t, \tilde{u}) \in S. \tag{$SC5b$}$$

Then, for a sufficiently small inter-sample time δ, the closed-loop system resulting from the application of the MPC strategy is asymptotically stable.

In the next section, we analyse the selection of design parameters guaranteeing stability for the unicycle and car-like vehicle examples.

4. EXAMPLES AND STABILITY ANALYSIS

4.1 *Example 1: The unicycle*

Recall the unicycle systems of Fig. 1

$$\begin{cases} \dot{x}(t) = (u_1(t) + u_2(t)) \cdot \cos\theta(t) \\ \dot{y}(t) = (u_1(t) + u_2(t)) \cdot \sin\theta(t) \\ \dot{\theta} = (u_1(t) - u_2(t)). \end{cases}$$

where the heading angle $\theta(t) \in [-\pi, \pi]$, and the controls $u_1, u_2(t) \in [-1, 1]$.

Our objective is to drive this system to the origin $(x, y, \theta) = (0, 0, 0)$.

This system has been addressed by MPC in (Van Essen and Nijmeijer, 2001). There, simulation results showed the viability of the MPC methodology to address these systems, but no rigorous stability analysis was carried out.

We construct the design parameters based on a simple strategy we might use to drive the system to the origin. This auxiliary strategy is not actually implemented, it is just used to prove that the MPC controller is stabilizing. Obviously, the MPC controller, based on solving optimization problems, should perform much better than the auxiliary strategy.

A possible stabilizing strategy, not necessarily the best, might be

(1) Rotate the robot until its heading angle θ points to the origin of the plane $(x, y) = (0, 0)$.
(2) Move forward until reaching the origin of the plane.
(3) Rotate again until $\theta = 0$.

To apply this strategy it is convenient to define $\phi(x, y)$ to be the angle that points to the origin from position (x, y), that is

$$\phi(x, y) = \begin{cases} 0 & \text{if } x = 0, y = 0; \\ -(\pi/2)\ \mathrm{sign}(y) & \text{if } x = 0, y \neq 0; \\ \tan^{-1}(y/x) + \pi & \text{if } x > 0; \\ \tan^{-1}(y/x) & \text{if } x < 0. \end{cases}$$

(This function is easily implementable in Matlab, or other languages with a four-quadrant inverse tangent built-in function, as $\phi(x, y) = atan2(-y, -x)$.)

A feedback law implementing the strategy above is

$$k_{aux}(x, y, \theta) = \begin{cases} (0, 0) & \text{(Stop)} & \text{if } (x, y, \theta) = 0; \\ (1, 1) & \text{(Forward)} & \text{if } \theta = \phi(x, y); \\ (1, -1) & \text{(Anticlockwise)} & \text{if } \theta < \phi(x, y); \\ (-1, 1) & \text{(Clockwise)} & \text{if } \theta > \phi(x, y). \end{cases}$$

Now we are ready to define the design parameters. We define the terminal set S to be the set of states heading towards the origin of the plane.

$$\begin{aligned} S := \{&(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : \\ &\theta = \phi(x, y) \vee (x, y) = (0, 0)\}. \end{aligned}$$

This set can be reached for any state if we define the horizon to be the time to complete an 180 degrees turn.

$$T = \pi/2.$$

Define the terminal cost simply as

$$W(x, y, \theta) = x^2 + y^2 + (\theta - \phi(x, y))^2.$$

When trying to verify the stability condition SC5a for W as defined, we easily conclude that we are interested in running cost functions of the type

$$L(x, y, \theta) = \sqrt{x^2 + y^2} + |\theta - \phi(x, y)|.$$

It is an easy task to check that these design parameters satisfy conditions SC1 to SC4 and SC5b. We verify SC5a below.

In the case when $(x, y) = (0, 0)$ and $\theta \neq \phi(x, y)$, the control is such that $u_1 = -u_2 = -\text{sign}(\theta - \phi)$.

$$\begin{aligned}
&\nabla W(x, y, \theta) \cdot f(x, y, \theta, k_{aux}(x, y, \theta)) \\
&\quad = (2x, 2y, 2\theta - 2\phi)^T (0, 0, -2\,\text{sign}(\theta - \phi)) \\
&\quad = -4(\theta - \phi)\,\text{sign}(\theta - \phi)) \\
&\quad = -4|\theta - \phi| \\
&\quad \leq -L(0, 0, \theta) \\
&\quad \leq -L(x, y, \theta).
\end{aligned}$$

When $\theta = \phi(x, y)$

$$\begin{aligned}
&\nabla W(x, y, \theta) \cdot f(x, y, \theta, k_{aux}(x, y, \theta)) \\
&\quad = (2x, 2y, 2\theta - 2\phi)^T (\cos\theta, \sin\theta, 0) \\
&\quad = 2x\cos\theta + 2y\sin\theta|_{\theta=\phi(x,y)}.
\end{aligned}$$

In the case when $x > 0$ and using the fact that

$$\tan^{-1} a = \sin^{-1}\frac{a}{\sqrt{1 + a^2}} = \cos^{-1}\frac{1}{\sqrt{1 + a^2}},$$

we obtain

$$\begin{aligned}
&\nabla W(x, y, \theta) \cdot f(x, y, \theta, k_{aux}(x, y, \theta)) \\
&\quad = 2x\cos\left[\tan^{-1}(y/x) + \pi\right] \\
&\qquad +2y\sin\left[\tan^{-1}(y/x) + \pi\right] \\
&\quad = -\sqrt{x^2 + y^2} \\
&\quad \leq -L(x, y, \phi).
\end{aligned}$$

The remaining cases are also easily verified.

4.2 *Example 2: A car-like vehicle*

We now analyse the car-like vehicle described in Section 2, whose model is repeated here for convenience.

$$\begin{cases}
\dot{x} = v \cdot \cos\theta \\
\dot{y} = v \cdot \sin\theta \\
\dot{\theta} = v \cdot c
\end{cases}$$

The control inputs v and c satisfy

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

(Minimum turning radius $R_{min} = c_{max}^{-1}$.)

Our goal, as before, is to find a set of design parameters and verify that they satisfy the stability conditions SC1–SC5.

To define S, a possibility is to look for possible trajectories approaching the origin. One such set is represented in Fig. 3. This set is the union of

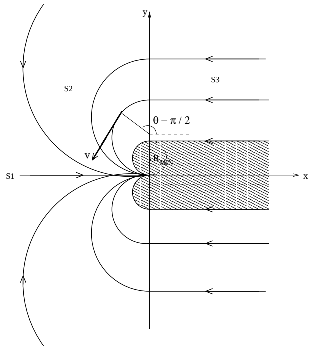

Fig. 3. Set S of trajectories approaching the origin.

all semi-circles with radius greater than or equal to R_{min}, with centre lying on the y axis, passing through the origin, and lying in the left half-plane. In order to make this set reachable in finite time from any point in the space, we add the set of trajectories that are horizontal lines of distance more than $2R_{min}$ from the x axis, and lie in the right half-plane. More precisely

$$S := S_1 \cup S_2 \cup S_3$$

where

$$\begin{aligned}
S_1 &:= \{(x, 0, 0) : x \leq 0\} \\
S_2 &:= \Big\{(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : x \leq 0, \\
&\qquad x^2 + (y - r)^2 = r^2,\ r \geq R_{min}, \\
&\qquad \tan\left(\theta - \frac{\pi}{2}\right) = \frac{y - r}{x}\Big\} \\
S_3 &:= \{(x, y, \theta) \in \mathbb{R}^2 \times [-\pi, \pi] : x > 0, \\
&\qquad |y| \geq 2R_{min},\ \theta = \pi\}
\end{aligned}$$

The set S as defined above can be reached in a well-determined finite time-horizon. This horizon is the time to complete a circle of minimum radius at maximum velocity, that is

$$T = 2\pi R_{min}/v_{max}.$$

Conditions SC1 and SC4 are satisfied.

Choose L as

$$L(x, y, \theta) := x^2 + y^2 + \theta^2,$$

and W as

$$W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t))dt$$

where $\bar{t}$ is the time to reach the origin with the controls chosen to be the maximum velocity and the curvatures depicted in Fig. 3. Of course, if we are already at the origin the value chosen for the velocity is zero. That is

$$v_{x,y,\theta} = \begin{cases} v_{max} & \text{if } (x,y,\theta) \neq 0 \\ 0 & \text{if } (x,y,\theta) = 0, \end{cases}$$

and

$$c_{x,y,\theta} = \begin{cases} 0 & \text{if } (x,y,\theta) \in S_1 \cup S_3 \\ -\text{sign}(\theta)/r & \text{if } (x,y,\theta) \in S_2 \cup \bar{S}, \end{cases}$$

where

$$r = \frac{x^2+y^2}{2|y|}.$$

An explicit formula for W is, as is derived below

$$W(x,y,\theta) = \begin{cases} \dfrac{-x^3}{3v_{max}} & \text{if } (x,y,\theta) \in S_1 \\ \dfrac{r}{3v_{max}}[6r^2\theta + \theta^3 - 6rx + 3\theta x^2 + 3\theta y^2 \\ \quad +6r\,(x-\theta y)\cos(\theta) \\ \quad +6r\,(-r+\theta x + y)\sin(\theta)] & \text{if } (x,y,\theta) \in S_2 \\ \dfrac{r}{3v_{max}}\left[x^2 + 3\pi^2 x + r\pi^3 + 30\pi r^2\right] & \text{if } (x,y,\theta) \in S_3. \end{cases}$$

We can easily see that SC2 and SC3 are satisfied and as we show below SC5 also is fulfilled.

It follows from our main stability result that this choice of design parameters guarantees the stability of the closed-loop trajectory.

We proceed to a detailed verification of SC5. We verify this condition separately for each of the subsets S_1, S_2, and S_3. Starting by S_1, we choose the controls

$$\begin{cases} v = v_{max} \\ c = 0 \end{cases}$$

from which follows immediately that

$$f(x,y,\theta) = \begin{cases} \dot{x} = v_{max} \\ \dot{y} = 0 \\ \dot{\theta} = 0, \end{cases}$$

the trajectories are

$$\begin{cases} x(t) = x_0 + v_{max}t \\ y(t) = 0 \\ \theta(t) = 0. \end{cases}$$

and the time to reach the origin is

$$\bar{t} = -x_0/v_{max},$$

satisfying the second part of SC5, since starting in S_1 with these controls we remain inside this set.

Expanding W we obtain

$$W(x_0,y_0,\theta_0) = \int_0^{-x_0/v_{max}} x^2(t)\,dt = \frac{-x_0^3}{3v_{max}},$$

and

$$\nabla W(x,y,\theta)\cdot f(x,y,\theta) = -x^2 \leq -L(x,y,\theta),$$

satisfying SC5a.

At S_2 we choose the controls

$$\begin{cases} v = v_{max} \\ c = -\text{sign}(\theta)/r. \end{cases}$$

We analyse the case in which θ is positive, the remaining case can be analysed in a similar way. It follows that

$$f(x,y,\theta) = \begin{cases} \dot{\theta}(t) = -v_{max}/r \\ \dot{x} = v_{max}\cos\theta(t) \\ \dot{y} = v_{max}\sin\theta(t), \end{cases}$$

the trajectories are

$$\begin{cases} \theta(t) = \theta_0 - v_{max}t/r \\ x(t) = x_0 + r\sin\theta_0 - r\sin(\theta_0 - v_{max}t/r) \\ y(t) = x_0 - r\cos\theta_0 + r\cos(\theta_0 - v_{max}t/r), \end{cases}$$

and the time to reach the origin is

$$\bar{t} = \theta_0 r/v_{max},$$

satisfying the second part of SC5, since starting in S_2 with these controls, we remain inside S_2.

Expanding W we obtain

$$\begin{aligned} W(x,y,\theta) &= \int_0^{\theta_0 r/v_{max}} [x^2(t) + y^2(t) + \theta^2(t)]\,dt \\ &= \frac{r}{3v_{max}}[6r^2\theta + \theta^3 \\ &\quad -6rx + 3\theta x^2 + 3\theta y^2 \\ &\quad +6r(x-\theta y)\cos(\theta) \\ &\quad +6r(-r+\theta x+y)\sin(\theta)] \qquad (2) \end{aligned}$$

and

$$\begin{aligned} \nabla W(x,y,\theta)\cdot f(x,y,\theta) &= -x^2 - y^2(t) - \theta^2(t) \\ &\leq -L(x,y,\theta), \end{aligned}$$

satisfying SC5a.

Finally, if we are in S_3, we choose the controls

$$\begin{cases} v = v_{max} \\ c = 0 \end{cases}$$

from which follows immediately that

$$f(x,y,\theta) = \begin{cases} \dot{x} = -v_{max} \\ \dot{y} = 0 \\ \dot{\theta} = 0, \end{cases}$$

the trajectories are

$$\begin{cases} x(t) = x_0 - v_{max}t \\ y(t) = y_0 \\ \theta(t) = \pi, \end{cases}$$

and the time to reach the y axis is

$$\tilde{t} = x_0/v_{max},$$

satisfying the second part of SC5, since starting in S_3 with these controls we remain inside it for some interval of time.

Expanding W we obtain

$$
\begin{aligned}
W(x_0,y_0,\theta_0) &= \int_0^{\tilde{t}} x^2(t) + \pi^2 \, dt + W(0, 2r, \pi) \\
&= \frac{r}{3v_{max}} \left[x_0^2 + 3\pi^2 x_0 + r\pi^3 + 30\pi r^2 \right],
\end{aligned}
$$

where $W(0, 2r, \pi)$ is given by the expression of W in S_2 (2), when the state is on the y axis. Finally, we confirm SC5a since

$$
\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 \leq -L(x, y, \theta).
$$

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