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# Control of Linear Unstable Systems with Constraints

Alex Zheng
Chemical Engineering, 210-41
California Institute of Technology
Pasadena, CA 91125, USA
zqz@bach.caltech.edu

Manfred Morari
Automatic Control Laboratory
ETH - Zürich ETL
CH-8092 Zürich, Switzerland
morari@aut.ee.ethz.ch

**Abstract**

In this note, we analyze and determine the domain of attraction for a linear unstable discrete-time system with bounded controls. An algorithm is proposed to construct the domain of attraction. A class of control laws is developed to stabilize all initial conditions in the domain of attraction.

## 1 Introduction

It is well known [5] that a linear system is *globally* stabilizable with bounded controls if and only if it is stabilizable and all the eigenvalues are inside the closed unit disk. For stable systems, control laws that optimize performance and guarantee global stability with both state feedback and output feedback have been developed [4, 9, etc]. Global stabilization of systems with poles on the unit disk has attracted much attention recently. Several researchers [6, 7, 8, etc] have constructed globally stabilizing control laws for stabilizable systems with poles on the closed unit disk. Since global stability is not possible for systems with poles outside the unit disk, it may be desirable to characterize the domain of attraction, i.e. the set of *all* initial conditions for which a stabilizing control law exists, but very little work has been done. Gilbert and Tang [1] characterized the region of attraction, called Maximal Output Admissible Sets, for *linear controllers* with both input and state constraints. However, their work is not applicable to unstable systems.

In this paper, we analyze and determine the domain of attraction for a linear unstable discrete time system with input saturation constraints. A class of controllers is then constructed to stabilize the system for all initial conditions in the domain of attraction.

## 2 Domain of Attraction

Consider the following linear time invariant discrete time system,

$$x(k+1) = Ax(k) + Bu(k), |u(k)|_\infty \leq 1, k \geq 0 \quad (1)$$

where $x(k) \in \Re^{n_x}, u(k) \in \Re^{n_u}$, and $A$ and $B$ are matrices of appropriate dimensions. The domain of attraction, $W$, is defined as follows.

**Definition 1** *The domain of attraction, denoted by $W$, is the set of* all *initial conditions for which there exists a sequence of controls $\{u(0), u(1), \cdots, \}, |u(i)|_\infty \leq 1 \ \forall\ i \geq 0$ such that the state approaches the origin asymptotically.*

**Remark 1** It is without loss of generality (WLOG) to assume that $|u|_\infty \leq 1$ in (1) instead of $u^{min} \leq u \leq u^{max}$. Let $P$ be a diagonal matrix whose diagonal elements equal $\frac{1}{2}(u^{max} - u^{min})$. By defining $u = P\tilde{u} + \frac{1}{2}(u^{max} + u^{min})$ and $x = \tilde{x} + \frac{1}{2}(I-A)^{-1}B(u^{max} + u^{min})$,[1] we can transform (1) with $u^{min} \leq u \leq u^{max}$ into $\tilde{x}(k+1) = A\tilde{x}(k) + \tilde{B}\tilde{u}(k)$ where $|\tilde{u}|_\infty \leq 1$ and $\tilde{B} = BP$.

The following result is immediate from Definition 1.

**Theorem 1** *There exists a control law such that the closed loop system is asymptotically stable if and only if the initial condition $x(0) \in W$.*

For stabilizable systems with $\rho(A) \leq 1$, where $\rho(A)$ denotes the spectral radius of $A$, Sontag proved that $W$ is $\Re^{n_x}$.

**Theorem 2 (Sontag 1984 [5])** $W = \Re^{n_x}$ *if and only if $(A, B)$ is stabilizable and $\rho(A) \leq 1$.*

Assume, WLOG, that the system is represented as follows.

$$\begin{bmatrix} x_s(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} u(k) \quad (2)$$

where $A_s \in \Re^{n_{x_s} \times n_{x_s}}$ has all eigenvalues inside the closed unit disk and $A_u \in \Re^{n_{x_u} \times n_{x_u}}$ outside the unit circle. By Theorem 2, the region of attraction for the system without any poles outside the unit circle is $\Re^{n_{x_s}}$. The following corollary states that the poles outside the unit circle do not change that.

**Corollary 1** *Consider the system described by (2) and assume $\{A, B\}$ is controllable.*[2] *The region of attrac-*

[1] Here we assume that $A$ does not have eigenvalues at 1.
[2] Corollary holds as well if the system is stabilizable.

*tion for $x_s$ is $\Re^{n_{xs}}$.*

*Proof:* Denote the region of attraction for $x_u$ by $W$. Let $x_u \in W$. Then there exists $K$ such that $x_u(K+1) = 0$. So it is WLOG to assume that $x_u(0) = 0$ and to show that the region of attraction for $x_s$ is $\Re^{n_{xs}}$, i.e. $[x_s(k)\ \ x_u(k)]^T \to 0$ as $k \to \infty$ for all $x_s(0) \in \Re^{n_{xs}}$ and $x_u(0) = 0$. Let $\alpha$ be some integer. We have

$$\begin{bmatrix} x_s(k+\alpha) \\ x_u(k+\alpha) \end{bmatrix} = \begin{bmatrix} A_s^\alpha & 0 \\ 0 & A_u^\alpha \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_s \\ \tilde{B}_u \end{bmatrix} v(k)$$

where

$$\begin{bmatrix} \tilde{B}_s \\ \tilde{B}_u \end{bmatrix} = \begin{bmatrix} A_s^{\alpha-1} B_s & \cdots & B_s \\ A_u^{\alpha-1} B_u & \cdots & B_u \end{bmatrix}$$

$$v(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+\alpha-1) \end{bmatrix}$$

Since the system is controllable, $\alpha$ exists such that $\begin{bmatrix} \tilde{B}_s \\ \tilde{B}_u \end{bmatrix}$ has full row rank. Consider a linear feedback control law $v(k) = F_s x_s(k)$. We have

$$\begin{bmatrix} x_s(k+\alpha) \\ x_u(k+\alpha) \end{bmatrix} = \begin{bmatrix} A_s^\alpha + \tilde{B}_s F_s & 0 \\ \tilde{B}_u F_s & A_u^\alpha \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_u(k) \end{bmatrix}$$

If $\tilde{B}_u F_s = 0$, then $x_u(k\alpha) = 0\ \forall\ k$ since $x_u(0) = 0$. If $A_s^\alpha + \tilde{B}_s F_s$ has all the eigenvalues inside the unit circle, then $x_s(k\alpha) \to 0$ as $k \to \infty$. If we can show that such an $F_s$ exists, we are done.

WLOG, assume that $\begin{bmatrix} \tilde{B}_s \\ \tilde{B}_u \end{bmatrix}$ is square and nonsingular: just set some rows of $F_s$ to zeros if $\begin{bmatrix} \tilde{B}_s \\ \tilde{B}_u \end{bmatrix}$ is non-square. Let $\tilde{B}_u^\perp$ be the orthogonal complement of $\tilde{B}_u$, i.e. $\begin{bmatrix} \tilde{B}_u^\perp \\ \tilde{B}_u \end{bmatrix}$ is square and nonsingular and $\tilde{B}_u^\perp \tilde{B}_u^T = 0$. Let $\tilde{B}_s = C_1 \tilde{B}_u^\perp + C_2 \tilde{B}_u$ and $F_s = (\tilde{B}_u^\perp)^T E$. Thus we have

$$\begin{aligned} \tilde{B}_u F_s &= 0\ \forall\ E \\ A_s^\alpha + \tilde{B}_s F_s &= A_s^\alpha + C_1 \tilde{B}_u^\perp (\tilde{B}_u^\perp)^T E \end{aligned}$$

Since $C_1$ and $\tilde{B}_u^\perp (\tilde{B}_u^\perp)^T$ are nonsingular,[3] $E$ exists such that $A_s^\alpha + \tilde{B}_s F_s$ has all eigenvalues inside the unit circle. From the results by Lin and Saberi [3], $E$ exists such that $|v(k)|_\infty \leq 1\ \forall\ k$ for any initial condition $x_s(0) \in \Re^{n_{xs}}$. Thus, the region of attraction for $x_s$ is $\Re^{n_{xs}}$. □

Therefore, we *only* need to determine the region of attraction for $x_u$. From now on, unless otherwise specified, we assume, WLOG, that $A$ has *all* the eigenvalues outside the unit circle.

[3]That $C_1$ is nonsingular can be seen as follows: $\begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{B}_u^\perp \\ \tilde{B}_u \end{bmatrix}$.

The state at time $k$ can be written as

$$x(k) = A^k x(0) + [A^{k-1}B\ \ \cdots\ \ B] \begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix} \quad (3)$$

Let $W_N$ be the set of all initial conditions for which there exists a sequence of controls $\{u(0), u(1), \cdots, u(N-1)\}, |u(i)|_\infty \leq 1\ \forall\ i \geq 0$ such that $x(N) = 0$. Thus, $W = \lim_{N\to\infty} W_N$. $W_N$ can be written as follows.

$$\begin{aligned} W_N = \ & \{z : z = [A^{-1}B\ \ \cdots\ \ A^{-N}B] \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix}, \\ & |u(i)|_\infty \leq 1, i \geq 0\} \end{aligned} \quad (4)$$

Some properties of $W_N$ and $W$ are stated here.

**Lemma 1** *$W_N$ and $W$ are bounded, convex, and symmetric.*

*Proof:* Since $A$ contains poles strictly outside the unit circle, $\rho(A^{-1}) < 1$. We have

$||A^{-i}||_\infty \leq c\gamma^i i^\beta, \gamma \in [0,1)$ for some integer $\beta$

Suppose $x(0) \in W_N$. We have

$$\begin{aligned} |x(0)|_\infty &= \left| \sum_{i=1}^N A^{-i} B u(i) \right|_\infty \leq \sum_{i=1}^N |A^{-i} B u(i)|_\infty \\ &\leq c_1 \sum_{i=1}^N ||A^{-i}||_\infty \leq c_1 c \sum_{i=1}^N \gamma^i i^\beta < \infty\ \forall\ N \end{aligned}$$

Thus, $W_N$ is bounded. The convexity of $W_N$ follows by observing that convexity is preserved for linear transformations. For $x(0) \in W_N$, there exists a sequence of controls $\{u(0), \cdots, u(N-1)\}, |u(i)|_\infty \leq 1\ \forall\ i$ such that $x(0) = \sum_{i=1}^N A^{-i} B u(i-1)$. Clearly, $-x(0) = \sum_{i=1}^N A^{-i} B(-u(i-1))$ must also belong to $W_N$. Therefore, $W_N$ is symmetric. The proof for $W$ follows by replacing $N$ with $\infty$. □

**Remark 2** *Although $W_N$ is closed, $W$ is open.*

In the next three subsections, we discuss several ways to characterize $W$.

### 2.1 Exact Characterization of $W_N$

In this section, we propose an algorithm which determines $W_N$. Let us first present some preliminary results.

**Lemma 2** *Consider the following sets.*

$$X_1 = \{z : H_1 z \leq h_1\}$$
$$X_2 = \{z : H_2 z \leq h_2\}$$

*Assume that both $X_1$ and $X_2$ are bounded. Denote the vertices of $X_1$ by $\mu_i, i = 1, \cdots, n_1$, and the vertices of $X_2$ by $\nu_i, i = 1, \cdots, n_2$. Let*

$$X = \{z : z = z_1 + z_2, z_1 \in X_1, z_2 \in X_2\}$$

*Then $X$ is bounded and is the smallest convex set which contains the points $\mu_i + \nu_j, i = 1, \cdots, n_1, j = 1, \cdots, n_2$. Furthermore, $X$ can be represented as follows:*

$$X = \{z : Hz \leq h\}$$

*Proof.* The convexity of $X$ can be shown as follows: Suppose $y, z \in X, y_1, z_1 \in X_1, y_2, z_2 \in X_2$ and $0 \leq \lambda \leq 1$. $\lambda y + (1-\lambda)z = \lambda(y_1 + y_2) + (1-\lambda)(z_1 + z_2) = (\lambda y_1 + (1-\lambda)z_1) + (\lambda y_2 + (1-\lambda)z_2) \in X$ since $X_1$ and $X_2$ are convex. $X$ is bounded since $X_1$ and $X_2$ are bounded.

Next we want to prove the following: If $X$ contains the points $\mu_i + \nu_j, i = 1, \cdots, n_1, j = 1, \cdots, n_2$, then $y_1 + y_2 \in X \; \forall \; y_1 \in X_1, y_2 \in X_2$. By convexity of $X$, for $0 \leq \lambda_1 \leq 1$, we have $\lambda_1(\mu_i + \nu_{j_1}) + (1-\lambda_1)(\mu_i + \nu_{j_2}) \in X \; \forall \; i, j_1, j_2$ which yields

$$\mu_i + \lambda_1 \nu_{j_1} + (1-\lambda_1)\nu_{j_2} \in X \; \forall \; i, j_1, j_2$$

Similarly, for $0 \leq \lambda_2 \leq 1$, we have

$$\lambda_2(\mu_{i_1} + \lambda_1 \nu_{j_1} + (1-\lambda_1)\nu_{j_2}) + (1-\lambda_2)(\mu_{i_2} + \lambda_1 \nu_{j_1} + (1-\lambda_1)\nu_{j_2}) \in X$$

which yields

$$(\lambda_2 \mu_{i_1} + (1-\lambda_2)\mu_{i_2}) + (\lambda_1 \nu_{i_1} + (1-\lambda_1)\nu_{i_2}) \in X$$
$$i_1, i_2 = 1, \cdots, n_1, j_1, j_2 = 1, \cdots, n_2$$

Thus, all points which are the sum of the points on edges of $X_1$ and $X_2$ belong to $X$. By similar arguments, one can show easily that $y_1 + y_2 \in X \; \forall \; y_1 \in X_1, y_2 \in X_2$. Clearly the smallest convex set which contains a finite number of points is a polytope. □

Recall

$$W_N = \left\{ x(0) : x(0) = \sum_{i=1}^{N} A^{-i} B u(i), |u(i)|_\infty \leq 1, i \geq 0 \right\}$$
$$= \left\{ x(0) : x(0) = \sum_{i=1}^{N} x_i, x_i \in X_i \right\}$$

where

$$X_i = \{z : z = A^{-i} B y, |y|_\infty \leq 1\}$$

$W_N$ can then be determined via the following algorithm.

**Algorithm 1** *Data: $A, B$, and $N$. Denote the set of vertices of the polytope $X_i$ by $V(X_i)$.*

**Step 0** *Set $i = 1$. Determine $V(X_1)$ and set $V(X) = V(X_1)$.*

**Step 1** *If $i = N$, go to* **Step 2**. *Otherwise, set $i = i+1$. Determine $V(X_i)$. Calculate $PV(X) = \{\mu : \mu = y + z, y \in V(X), z \in V(X_i)\}$. Eliminate all points from $PV(X)$ that are not vertices for the smallest polytope that covers all points in $PV(X)$. Set $V(X) = PV(X)$. Go to* **Step 1**.

**Step 2** *Construct the polytope with vertices $V(X)$.*

Let $PV(X) = \{\mu_1, \cdots, \mu_M\}$. We can determine if a point in $PV(X)$, say $\mu_i$, is a vertex by solving the following optimization problem, which can be cast as a linear program.

$$J = \min_{\alpha} \left| \mu_i - \sum_{j=1}^{M} \delta_j \mu_j \right|_\infty$$
$$\text{subject to } \; \delta_j \geq 0 \;\; \forall \; j, \delta_i = 0, \sum_{j=1}^{M} \delta_j = 1$$

It is clear that $\mu_i$ is a vertex if and only if $J < 0$.

**Remark 3** *Constructing $W_N$ this way requires to repeat* **Step 1** *$N-1$ times, i.e. $N-1$ operations in set addition. Since doing set addition may be computationally expensive, we can reduce the number of set addition as follows: Define $D_i$ of full row rank and $l(N) < N$ [4] such that*

$$[D_1 \;\; \cdots \;\; D_{l(N)}] = [A^{-1}B \;\; \cdots \;\; A^{-N}B]$$

*Then $W_N$ can be rewritten as*

$$W_N = \left\{ x(0) : x(0) = \sum_{i=1}^{l(N)} D_i v(i), |v(i)|_\infty \leq 1, i \geq 0 \right\}$$

*By defining $X_i$ similarly, we only have to repeat* **Step 1** *$l(N) - 1$ times, i.e. $l(N) - 1$ operations of set addition. Of course, in this case, it may take more computational time to determine the vertices of $X_i$.*

### 2.2 Subsets of $W$

Let $C = [A^{-1}B \;\; \cdots \;\; A^{-n}B]$, where $n$ is the smallest integer such that $C$ has full row rank.[5] We have

$$W = \left\{ x(0) : x(0) = \sum_{i=0}^{\infty} (A^{-n})^i C U_n(i), |U_n(i)|_\infty \leq 1 \right\}$$

where $U_n(i) = [u(i \cdot n) \;\; \cdots \;\; u((i+1)n-1)]^T$. Let the set $W_{in}$ be generated by assuming $U_n(i) = U_n(0) \; \forall \; i \geq 1$, i.e.

$$W_{in} = \left\{ x(0) : x(0) = \sum_{i=0}^{\infty} (A^{-n})^i C U_n(0), |U_n(0)|_\infty \leq 1 \right\}$$
$$= \left\{ x(0) : x(0) = (I - A^{-n})^{-1} C U_n(0), |U_n(0)|_\infty < 1 \right\}$$

Then we must have $W_{in} \subseteq W$. If $C$ is square and nonsingular,[6] then

$$W_{in} = \left\{ x(0) : |C^{-1}(I - A^{-n}) x(0)|_\infty < 1 \right\} \tag{5}$$

### 2.3 Supersets of $W$

From $x(0) = \sum_{i=0}^{\infty} (A^{-n})^i C U_n(i)$, we have

$$|Tx(0)|_\infty \leq \sum_{i=0}^{\infty} |T(A^{-n})^i C U_n(i)|_\infty \leq \sum_{i=0}^{\infty} |T(A^{-n})^i C|_\infty$$

where $T$ is some nonsingular weighting matrix. Thus, a superset of $W$, $W_{out}$, can be defined as follows:

$$W_{out} = \left\{ x(0) : |Tx(0)|_\infty \leq \sum_{i=0}^{\infty} |T(A^{-n})^i C|_\infty \right\} \supseteq W$$

[4] $D_i$'s and $l(N)$ are clearly not unique.

[5] Since $(A, B)$ is controllable, such an $n$ exists.

[6] For single input controllable systems, $C$ is always nonsingular if $C$ is square.

### 2.4 Characterization of $W$

$W_N$ can be characterized exactly and can be used to approximate $W$. Then the techniques presented in Section 2.2 and 2.3 can be used to bound the approximation error, i.e. we have the following relations:

$$W \supseteq \{x(0) : x(0) = y + z, y \in W_N, z \in W_{in_N}\} \quad (6)$$

$$W \subseteq \{x(0) : x(0) = y + z, y \in W_N, z \in W_{out_N}\} \quad (7)$$

where

$$W_{in_N} = \{x(0) : x(0) = A^{-N}(I - A^{-n})^{-1}CU_n(0), \\ |U_n(0)|_\infty < 1\}$$

$$W_{out_N} = \left\{ x(0) : |Tx(0)|_\infty \leq \sum_{i=0}^{\infty} |TA^{-N}(A^{-n})^i C|_\infty \right\}$$

## 3 Stabilizing Control Laws

For any initial condition $x(0) \in W$, Theorem 1 states the existence of a stabilizing control law. In this section, we construct a class of Model Predictive Control (MPC) laws and state a necessary and sufficient condition for them to be stabilizing. Define Controller MPC as follows.

**Definition 2 Controller MPC:** *At sampling time $k$, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+m-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+m-1|k)} \sum_{i=1}^{\infty} x(k+i|k)^T Rx(k+i|k) \\ + \sum_{i=0}^{m-1} u(k+i|k)^T Su(k+i|k) + \epsilon(k)^T Q\epsilon(k)$$

subject to

$$\begin{cases} |u(k+i|k)|_\infty \leq 1 & i = 0, 1, \cdots, m-1 \\ u(k+i|k) = 0 & i = m, m+1, \cdots, \infty \\ Gx(k+i|k) \leq g + \epsilon(k) & i = 0, 1, \cdots, \infty \\ \epsilon(k) \geq 0 & \end{cases} \quad (8)$$

*where $R > 0, S > 0, G \in \Re^{n_G \times n_x}$ and $Q > 0$ is diagonal. Assume also that each element of $g$ is strictly positive.*

Here $Gx(k+i|k) \leq g + \epsilon(k)$ denotes "soft" state constraints [9]. It was shown in [4] that MPC Controller is stabilizing if and only if the optimization problem (8) is feasible. If $x(0) \in W_N$, then the optimizing problem is feasible for all $m \geq N$. Thus we have the following result.

**Theorem 3** *Assume that all eigenvalues of $A$ are outside the unit circle. Then Controller MPC is stabilizing for all $x(0) \in W_N$ if and only if $m \geq N$.*

*Proof:* If $x(0) \in W_N$, then by definition there exists a sequence of controls $\{u(0), \cdots, u(N-1)\}$ such that $x(N) = 0$. The optimization problem (8) has a feasible solution for all $x(0) \in W_N$ *if and only if* $m \geq N$. Then $J_0$ is bounded. At sampling time $k+1$, the control sequence of $\{u(k+1|k), \cdots, u(k+m-1|k), 0\}$ results in a finite objective that equals $J_k - \left[x(k)^T Rx(k) + u(k)^T Su(k)\right]$. Thus, we have

$$J_{k+1} \leq J_k - x(k+1)^T Rx(k+1) - u(k+1)^T Su(k+1)$$

which yields

$$J_{k+1} + \sum_{i=0}^{k+1} \left[x(i)^T Rx(i) + u(i)^T Su(i)\right] \leq J_0 < \infty,$$

for all $k > 0$, which, in turn, implies that $x(k), u(k) \to 0$ as $k \to \infty$. □

If $A$ has poles inside the unit circle but no poles on the unit circle, with $u(k+i|k) = 0 \ \forall \ i \geq m$ the stable modes approach zero exponentially fast. With a control sequence that derives the unstable modes to zero, $J_0$ is bounded. Following similar arguments as in the proof of Theorem 3, we can show that Theorem 3 holds in this case.

If $A$ has poles on the unit circle, then the number of control moves $m$ necessary to drive the corresponding modes to zero depends on the initial condition [8]. Not every $m \geq N$ may work in this case. These discussions are summarized in the following corollary.

**Corollary 2** *Consider the system described by (2). Suppose $A_s$ has no poles on the unit circle, then Controller MPC is stabilizing for all $x_u(0) \in W_N$ and $x_s(0) \in \Re^{n_{x_s}}$ if and only if $m \geq N$. If $A_s$ also has poles on the unit circle, then Controller MPC is stabilizing for all $x_u(0) \in W_N$ and $x_s(0) \in \Re^{n_{x_s}}$ for a sufficiently large $m$.*

If $m \geq N$ and $A$ has all eigenvalues outside the unit circle, the infinite state horizon in *Controller MPC* can be replaced by a finite state horizon with the end constraint $x(k+N|k) = 0$ at each sampling time $k$.

**Theorem 4** *Suppose that $A$ has all eigenvalues outside the unit circle and that $m \geq N$. Then the optimization problem (8) is equivalent to the following.*

$$J_k = \min_{u(k|k),\cdots,u(k+m-1|k)} \sum_{i=1}^{m} x(k+i|k)^T Rx(k+i|k) \\ + \textstyle\sum_{i=0}^{m-1} u(k+i|k)^T Su(k+i|k) + \epsilon(k)^T Q\epsilon(k)$$

subject to

$$\begin{cases} x(k+m|k) = 0 & \\ |u(k+i|k)|_\infty \leq 1 & i = 0, 1, \cdots, m-1 \\ u(k+i|k) = 0 & i = m, m+1, \cdots, \infty \\ Gx(k+i|k) \leq g + \epsilon(k) & i = 0, 1, \cdots, m-1 \\ \epsilon(k) \geq 0 & \end{cases} \quad (9)$$

*Proof:* Since $A$ is unstable and $u(k+m+i|k) = 0, i \geq 0$, $J_k$ is finite if and only if $x(k+m|k) = 0$. Thus, $\sum_{i=m+1}^{\infty} x(k+i|k)^T Rx(k+i|k) = 0$. □

## 4 Examples

**Example 1** Consider a linear model approximating longitudinal dynamics at 3000 ft altitude and 0.6 mach velocity for a modified F-16 aircraft [2].

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} -0.0151 & -60.5651 & 0 & -32.174 \\ -0.0001 & -1.3411 & 0.9929 & 0 \\ -0.00018 & 43.2541 & -0.86939 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2.516 & -13.136 \\ -0.1689 & -0.2514 \\ -17.251 & -1.5766 \\ 0 & 0 \end{bmatrix}$$

The constraints on both inputs are $\pm 25$. The system is discretized with a sampling time of 0.1. Since the system contains only one unstable pole at 1.7252, the region of attraction for the system is equal to the region of attraction associated with the unstable pole.

$$\hat{x}(k+1) = 1.7252\hat{x}(k) + [-3.6435 \quad -0.6311]\hat{u}(k),$$

$$|\hat{u}(k)|_\infty \leq 1 \ \forall\ k$$

where $\hat{x} = [-0.0002 \quad 9.5168 \quad 1.4947 \quad 0.0013]x$ and $\hat{u} = \frac{u}{25}$. Straightforward calculations yield

$W = \{x : |[-0.0002 \quad 9.5168 \quad 1.4947 \quad 0.0013]x| < 147.33\}$[7]

**Example 2** Consider the following system.

$$A = \begin{bmatrix} -2 & -0.8 \\ -2 & 0.7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Shown in Figure 1 are $W_{15}$, $W_{in}$, and $W_{out}$ with $T = I$, the identity matrix, and $T = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$. Here $W_{in}$ and $W_{out}$ are determined via Equations (6) and (7). As one can see, $T$ can be chosen to make $W_{out}$ as small as possible and $W_{out}$ with $T = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $W_{in}$ are very close. Choosing $W = W_{in}$ is a good approximation. For comparison, we also show the region of attraction for the linear controller which places closed loop poles at 1 and 1.

## 5 Conclusions

In this note we have analyzed the region of attraction for unstable systems with input constraints. Several methods were presented to characterize thee region of attraction. A class of stabilizing MPC controllers was then constructed. In this paper, we have assumed that the state can be measured. Characterizing the region of attraction, given the bound on the initial estimation error, is currently under investigation.

[7]Notice that $W$ is *open*.

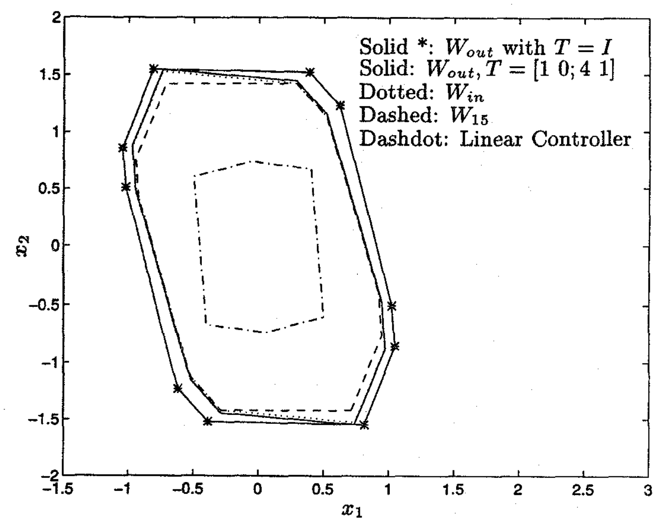


Figure 1: Domains of Attraction

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# GENERALIZED PREDICTIVE CONTROL IN THE DELTA-DOMAIN

Morten B. Lauritsen, Morten Rostgaard, Niels K. Poulsen
Institute of Mathematical Modelling

Technical University of Denmark
DK-2800 Lyngby, Denmark
mbl@imm.dtu.dk, mrj@imm.dtu.dk, nkp@imm.dtu.dk

## Abstract

This paper describes new approaches to generalized predictive control formulated in the delta ($\delta$) domain. A new $\delta$-domain version of the continuous-time emulator-based predictor is presented. It produces the optimal estimate in the deterministic case whenever the predictor order is chosen greater than or equal to the number of future predicted samples, however a "good" estimate is usually obtained in a much longer range of samples. This is particularly advantageous at fast sampling rates where a "conventional" predictor is bound to become very computationally demanding. Two controllers are considered: one having a well-defined limit as the sampling period tends to zero, the other being a close approximation to the conventional discrete-time GPC. Both algorithms are discrete in nature and well-suited for adaptive control. The fact, that $\delta$-domain model are used does not introduce an approximation since such a model could be obtained by an exact sampling of a continuous-time model.

## 1. Introduction

In recent years it has become widely accepted, that the shift operator is not well suited to describe sampled systems at fast sampling rates. It has been suggested to use the so-called delta-operator [1]

$$\delta = \frac{q-1}{T} \quad \text{or} \quad q = 1 + T\delta \tag{1}$$

where $T$ is the sampling period and $q$ is the forward shift-operator. Using system models parameterized by the $\delta$-operator gives a closer relationship between the sampled system and the underlying continuous-time system at fast sampling rates. This is due to the fact, that for a signal $x(t)$

$$\lim_{T\to 0} \delta x(t) = \frac{d}{dt}x(t) \tag{2}$$

In accordance with this relationship we will call $\delta^n x(t)$ the n'th order $\delta$-derivative of $x(t)$. The fact, that $\delta$-domain descriptions converge to their corresponding continuous-time descriptions as $T \to 0$ makes it possible to utilize physical a priori knowledge. Also non-minimum phase problems introduced by the sampling process can be handled more easily than in the $q$-domain. An important advantage in adaptive control systems is, that better numerical and convergence properties of least squares parameter estimators are obtained when formulated in the $\delta$-domain, see [2].

In order to make use of the above mentioned advantages in adaptive control systems we need to formulate control algorithms based on $\delta$-domain models. Examples of such control algorithms are given in e.g. [1, 3, 4]. In this paper $\delta$-domain versions of generalized predictive control will be considered. Some of the work has been presented in [5].

The $\delta$-operator offers the same flexibility (and restrictions) in modeling as the $q$-operator which makes it possible to transform any $q$-domain control algorithm to the $\delta$-domain. A directly transformed version of the basic GPC algorithm, see [6], has been presented in [3] for systems on $\delta$-domain ARMAX form. The calculations involve a reformulation of optimal output predictions from a $q$-domain formulation to a $\delta$-domain equivalent. This is obtained by means of the two Diophantine equations

$$C(\delta)(1+T\delta)^{k-1} = A(\delta)E_k(\delta) + F_k(\delta) \tag{3}$$

$$(1+T\delta)B(\delta)E_k(\delta) = C(\delta)G_k(\delta) + H_k(\delta) \tag{4}$$

These Diophantine equations can be solved in a recursive manner, see e.g. [4]. However, they become singular when $T \to 0$. This is impossible to avoid when considering optimal (MV) prediction because this is inherently connected to the shift-operator, i.e. the output is predicted a number of samples ahead rather than at absolute time instants independent from the sampling period.

In this paper a new discrete-time predictor which overcomes this problem is presented. The predictor is based on ideas from the continuous-time emulator-based predictor presented in [7] It is shown to have a certain optimality property in the deterministic case.

The algorithms presented here avoid the problems of constructing continuous-time estimators and implementing continuous-time control algorithms. The control algorithms are based on discrete-time $\delta$-domain models possibly obtained using a discrete-time identification algorithm and are in a sense exact—they are not derived as simple approximations of continuous-time algorithms. The $\delta$-domain models are also exact in the sense, that they should *not* be thought of as approximations of continuous-time models and are fully equivalent to $q$-domain models. The reason for spelling out these observations is that many people may think of the $\delta$-operator as simply a tool for implementation of continuous-time algorithms. However, as will be shown, the nature of the new $\delta$-domain emulator-based predictor makes it possible to construct algorithms that approximate both continuous-time algorithms with a well-defined limit as $T \to 0$ and discrete-time algorithms which unavoidably become unrealizable when $T \to 0$.