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Optimal Motion Planning for a Wheeled Mobile Robot*

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Abstract

This paper concerns time optimal motion planning problem under kinematic and dynamic constraints for a 2-DOF wheeled mobile robot(WMR). The dynamic model of a WMR is derived using Newton-Euler method and its constraints are analyzed. Kinematic constraints are imposed by its nonholonomy and structural limits while dynamic constraints are due to motor saturation. The motion planning problem is formulated as two stage planning. First, path planning under kinematic constraints is transformed into a pure geometric problem. The shortest path composed of circular arcs and straight lines is obtained. Then, combined with dynamic characteristics of the WMR, time optimal velocity profile is generated under dynamic constraints. Since constraints of a WMR are fully exploited, the proposed method is simple and effective on the feasibility of motion planning. Simulation results illustrate the capability of the planning scheme.

1 Introduction

Motion planning problems are concerned with obtaining boundary control inputs to steer the system from an initial state to a final state over finite time interval. A wheeled mobile robot (WMR) belongs to typical nonholonomic systems which have nonintegrable constraints. Although nonholonomic systems are controllable, their motion planning is still challenging. With holonomic systems, a set of independent generalized coordinates can be found to define a space in which arbitrary motion is feasible. But for nonholonomic systems, it is not the case, i.e. motion at any moment is not free but only motion satisfying nonholonomic constraints is possible. At present, main ideas to deal with motion planning of WMRs are categorized into two: one is to determine control inputs based on discrete kinematic equations after geometric path is planned [1-3], another is to generate net motion in the directions provided by the iterated Lie brackets using control inputs such as high frequency high gain periodic functions [4] and piecewise constant inputs [5], etc. However, the present methods only considered nonholonomic constraints, the planned motion can not be put into practice when there are extra constraints such as minimum turning radius or maximum curvature. In addition, control inputs are limited motor output torques due to motor saturation. Although trapezoidal planning scheme was used to meet the limits after torque limits have been transformed to acceleration limits [1,6], none has taken the dynamics of WMRs into account during planning. This paper formulates motion planning as a two stage planning. First, path satisfying kinematic constraints is planned using geometric methods. Then optimal trajectory is generated considering dynamic properties using the method analogous to the scheme in [8] for robot manipulators.

In the following section, kinematic and dynamic models are derived and constraints are discussed. In section 3, path composed of circular arcs and straight-line segments is to meet kinematic constraints. The details of trajectory generation combining with dynamic characteristics are presented in section 4 and illustrated by simulation results in section 5. In the concluding section, the results in this paper are summarized.

2 Models and Constraints

2.1 Kinematics and Kinematic Constraints

The WMR considered in this paper is a 2-DOF differ-

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entially drived vehicle moving on a flat plane, which has two independent driving wheels and one free caster. The typical structure is shown in Fig.1. The kinematic model is described as follows with the center of mass C as the reference point:

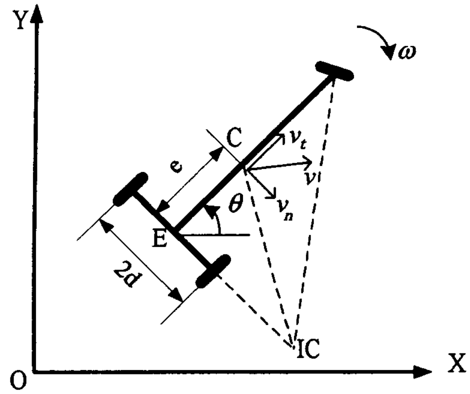

Fig.1 Typical structure of wheeled mobile robots

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_t \\ v_n \\ \omega \end{bmatrix} \quad (1)$$

Besides

$$\begin{bmatrix} v_t \\ v_n \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ er/2d & -er/2d \\ r/2d & -r/2d \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_r \end{bmatrix} \quad (2)$$

where $\dot{q}_1$ and $\dot{q}_r$ denote angular velocity of left and right driving wheels respectively, r denotes the wheel radius. From (2), the relation holds $v_n = e\omega$. The kinematics using v_t、ω as control inputs is written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -e\sin\theta \\ \sin\theta & e\cos\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_t \\ \omega \end{bmatrix} \quad (3)$$

From the former two equations in (3), we can obtain the nonholonomic constraint of WMRs:

$$\dot{y}\cos\theta - \dot{x}\sin\theta = \omega e \quad (4)$$

Let $e = 0$ in (4), i.e. using the center of the wheel's axes E as reference point, it becomes

$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0 \quad (5)$$

In this case, the physical significance of nonholonomic constraints lies in no possible motion in the axial direction, namely the direction of translational velocity is the tangent direction of the path. Additionally, limited by inertial structure of WMRs, planned path is restricted by minimum turning radius or maximum curvature.

$$\rho \geq R_{\min} \text{ or } k \leq K_{\max} \quad (6)$$

We categorize nonholonomic constraint in (5) and path constraint in (6) as kinematic constraints.

2.2 Dynamics and Dynamic Constraints

In this part, a WMR is divided into three parts: left, right wheel and vehicle frame. The front wheel is a free caster, whose translational force is beyond consideration. Then the dynamics of each part is analyzed using Newton-Euler method.

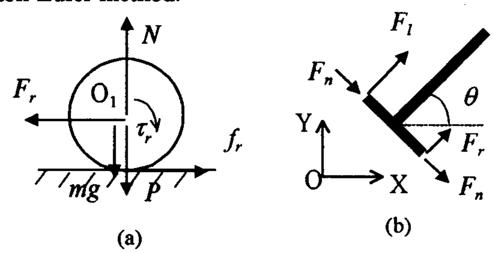

Fig.2 Dynamic analysis of a WMR

(1) left and right wheel

In practice, the coriolis items in Euler equation provided by the wheel rotation and the frame rotation is negligible for the wheel inertia can be ignored compared with the frame inertia. Shown as Fig.2(a), Newton-Euler equations of right wheel is expressed as

$$\begin{cases} f_r - F_r = m\ddot{x}_r \\ \tau_r - f_r \cdot r = J_w \ddot{q}_r \end{cases} \quad (7)$$

where f_r denotes the friction force resulting from the contact between the ground and the wheel, F_r denotes the reaction force applied on the right wheel by the frame, m denotes the wheel mass, τ_r denotes the driving torque acting on the right wheel, J_w denotes the wheel inertia. Under pure rolling without slipping condition, the following relation is obtained.

$$\dot{x}_r = r \cdot \dot{q}_r \quad (8)$$

Let μ be the maximum static friction factor between the ground and the wheel, P is the vertical reaction force applied on the wheel by the frame. The pure rolling without slipping condition is written as $f_r \leq \mu(P + mg)$, i.e. the driving torque is limited as

$$\tau_r \leq \frac{\mu(P + mg)(mr^2 + J_w) - F_r \cdot J_w}{mr} \quad (9)$$

In the same way, the force and moment equations for the left wheel yields:

$$\begin{cases} f_l - F_l = m\ddot{x}_l \\ \tau_l - f_l \cdot r = J_w \ddot{q}_l \\ \dot{x}_l = r \cdot \dot{q}_l \end{cases} \quad (10)$$

(2) Vehicle frame

$$\dot{\xi} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ k \end{bmatrix} v = \dot{\xi}_P v \tag{15}$$

Since the curvature of the above planned path is piecewise constant, derivative of (15) w.r.t time with $dk / dt = 0$ is given by

$$\ddot{\xi} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ k \end{bmatrix} a + \begin{bmatrix} -k\sin\theta \\ k\cos\theta \\ 0 \end{bmatrix} v^2 = \dot{\xi}_P a + \ddot{\xi}_P v^2 \tag{16}$$

where $a = dv / dt$. The parameters a, v only dependent on time are separated from the parameters $\dot{\xi}_P, \ddot{\xi}_P$ only dependent on path features. Trajectory planning is to generate velocity v and acceleration a under torque limits based on $\dot{\xi}_P, \ddot{\xi}_P$. Taking (15),(16) into dynamic equations (13) yields:

$$J(\xi)\tau = aM\dot{\xi}_P + [M\ddot{\xi}_P + C(\xi,\dot{\xi}_P)\dot{\xi}_P]v^2 + \eta \tag{17}$$

Let $J^+ = (J^T J)^{-1} J^T$, $Q = J^+ M\dot{\xi}_P$, $N = J^+\eta$, $P = J^+[M\ddot{\xi}_P + C(\xi,\dot{\xi}_P)\dot{\xi}_P]$, the formula (17) takes the form

$$\tau = Qa + Pv^2 + N \tag{18}$$

(1) Velocity limits

Velocity limit v_m is obtained when uniform velocity is possible, i.e. $a = 0$ in (18). In this case, equation (18) becomes $\tau_i = P_i v^2 + N_i$ (i =1,2; τ_1, τ_2 corresponding to τ_l, τ_r). When $\tau_i = \mathrm{sgn}(P_i)\cdot\alpha T_{\max}$, the velocity limit is given by

$$v_{mi} = \sqrt{[\mathrm{sgn}(P_i)\cdot\alpha T_{\max} - N_i] / P_i} \tag{19}$$

In order to both make the best use of the actuators and keep the torques under limited torques, the overall velocity limit is determined by $v_m = \min(v_{m1}, v_{m2})$.

(2) Acceleration bounds

From (18), boundary acceleration (deceleration) is determined by

$$a_{m+} = \min_{i=1,2} \frac{\mathrm{sgn}(Q_i)\cdot\alpha T_{\max} - v^2 P_i - N_i}{Q_i} \tag{20}$$

$$a_{m-} = \max_{i=1,2} \frac{-\mathrm{sgn}(Q_i)\cdot\alpha T_{\max} - v^2 P_i - N_i}{Q_i} \tag{21}$$

From above discussion, the velocity limits are dependent not only on motor saturation torques but also on path features on considering the dynamics. Furthermore, the acceleration bounds are even related to current velocity, not the constant values given in [1,6].

If the planning scheme in [1,6] is preferred, the velocity limit is chosen as the minimum velocity limit and the acceleration is also the minimum acceleration bound in the motion. The scheme guaranteed motion stability while the motors are not fully utilized. In the following, trajectory is planned to minimize the motion time.

Since the planned path takes the form CLC with curvature $k_C = 1/R_{\min}$ and $k_L = 0$, rotational velocity at the point where the arc and the straight-line segment intersects must be zero, i.e. $\omega = 0$. The fact the intersection point is also on the arc requires translational velocity to be zero ($v = \omega R_{\min}$). Therefore, there are three acceleration and deceleration processes during the whole path, corresponding to two arcs and one straight-line segment. In practical planning, discrete arc length is in use. The planning procedure is as follows: Step 1. Velocity limits are computed using (19); Step 2. Velocity profile is iteratively calculated using (20), (21) from both end points respectively with uniform acceleration (deceleration) during each sample arc length. The iterative calculation terminates when the velocity reaches its limit or the whole curve segment is traversed. Step 3. There are two cases concerning two velocity curves from different end points. One is the velocity curves planning from two end points intersect in admissible velocity region, as shown in Fig.3(a),(b). Then the resulting trajectory is acceleration motion during SC and deceleration motion during CS_0. Another is they have no intersection in admissible velocity region, as shown in Fig.3(c). The resulting trajectory in this case is acceleration motion during SA, uniform velocity motion during AB and deceleration motion during BS_0.

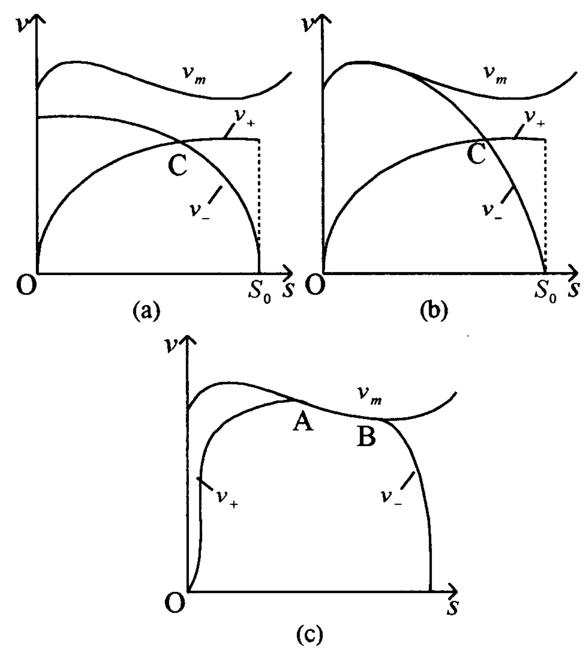

Fig.3 Optimal trajectory planning under limited torques

Shown as Fig.2(b), Newton-Euler equations of the frame are expressed as

$$\begin{cases}(F_l + F_r - F_f)\cos\theta + 2F_n \sin\theta = M_c\ddot{x}_c \\ (F_l + F_r - F_f)\sin\theta + 2F_n \cos\theta = M_c\ddot{y}_c \\ F_l \cdot d - F_r \cdot d - 2F_n \cdot e = J_c\ddot{\theta}\end{cases} \tag{11}$$

where (x_c, y_c, θ) is the pose of the center of mass w.r.t fixed coordinates, M_c is the frame mass, J_c is the frame inertia relative to point C, F_f is the reaction force applied on the frame by the front wheel, F_n is the force resulting from the normal friction force between the ground and the wheel.

From the kinematic equations described with point E as reference point, we can find that nonholonomic constraint of a WMR is of apparent physical significance. The constraint $tg\theta = dy / dx$ means that as long as the translational direction is coincident with the tangent direction of the path, its nonholonomic constraint is satisfied. In order to simplify the consideration of nonholonomic constraint during planning, point E is also preferred to be the reference point for the description of the dynamic model. The relation between point E and C is given by

$$\begin{cases}x_c = x_e + e\cos\theta \\ y_c = y_e + e\sin\theta\end{cases} \tag{12}$$

Let $\xi = (x_e, y_e, \theta)^T$, $\tau = (\tau_l, \tau_r)^T$, $J_0 = Jw + mr^2$, conservative forces eliminated from equations (2)-(3) and (7)-(12), the dynamic model is written as

$$M(\xi)\ddot{\xi} + C(\xi,\dot{\xi})\dot{\xi} + \eta = J(\xi)\tau \tag{13}$$

where

$$M(\xi) = \begin{bmatrix} (Mr^2 + 2J_0c^2)/r & 2J_0sc/r & -Mres \\ 2J_0sc/r & (Mr^2 + 2J_0s^2)/r & Mres \\ 0 & 0 & (J_cr^2 + 2d^2J_0)/r^2 \end{bmatrix}$$

$$C(\xi,\dot{\xi}) = \begin{bmatrix} -2J_0sc\dot{\theta}/r & 2J_0c^2\dot{\theta}/r & -M_crec\dot{\theta} \\ -2J_0s^2\dot{\theta}/r & 2J_0sc\dot{\theta}/r & -M_crec\dot{\theta} \\ 0 & 0 & 0 \end{bmatrix}$$

$$J(\xi) = \begin{bmatrix} c & c \\ s & s \\ d/r & -d/r \end{bmatrix} \qquad \eta = \begin{bmatrix} -2F_nrs \\ 2F_nrc \\ -2F_ne \end{bmatrix}$$

$$s = \sin\theta,\ c = \cos\theta$$

It is obvious equations (13) takes the similar form of the dynamics of robot manipulator, but they differ in the presence of the driving matrix $J(\xi)$ in equations (13). The matrix $J(\xi)$ is characteristic of the nonholonomy of WMRs. In practice, the motor output torques are limited. Because of the inaccuracies in the models and disturbances in motion, the dynamic constraint is formulated as follows in order to obtain strong robustness.

$$|\tau| \le \alpha T_{max} \tag{14}$$

where the certain value about models $\alpha \in (0, 1)$, T_{max} is motor saturation torque.

3 Path Planning under Kinematic Constraints

Kinematic constraints include nonholonomic constraint in (5) and minimum turning radius constraint in (6). When planning with the center of the wheel's axes as reference point, the nonholonomic constraint can be understood as the coincidence between the translational direction and the path tangent direction. Hence, so long as the tangent direction of the planned path is defined as the translational direction, the nonholonomic constraints can be naturally met. In addition, when circular arcs of radius R_{min} and straight line segments (of radius infinite) are employed to plan path, the minimum turning radius constraint is also satisfied. According to [7], the shortest curve with a constraint on maximum curvature takes the form CLC where C denotes the circular arc of radius of R_{min} and L denotes straight-line segment. So far, path-planning problem under kinematic constraints can be transformed to pure geometric problem. It is formulated as follows: given start point $S(x_s, y_s, \theta_s)$ and goal point $G(x_g, y_g, \theta_g)$, a C^1 curve composed of circular arcs of radius R_{min} and straight line segments is pursued to connect the two points. The idea of planning is analogous to driving cars. First, the car is steered to the nearest direction to the goal in maximum curvature, then approaches the goal in straight, finally adjust its direction and reaches the goal. The planning procedure is: there are two circles of radius R_{min} tangent to θ_s at point S, one is counterclockwise oriented Z_s, and another is clockwise oriented Y_s. The same is with point G (Z_g and Y_g). Then one circle is chosen from each set of circles , between which the tangent line is found, as illustrated in Fig.4 in section 5. The details are investigated by proposition 11 in [7].

4 Trajectory planning under dynamic constraints

In this section, a method analogous to the planning scheme for robot manipulators in [8] is proposed to generate time optimal trajectory. Taking $\omega = kv$ into (3) and let $e = 0$, we obtain

5 Simulation Results

The proposed methods have been simulated for our laboratory mobile robot with parameters given in Table 1.

Table 1 Vehicle Parameters

m = 0.25kg; M = 25kg; r = 10cm; e = 10cm;
d = 20cm; R_{min} = 40cm; J_0 = 31.25kg·cm^2;
J_c = 2.5×10^4 kg·cm^2; T_{max} = 6N·m

Without loss of generality, initial pose of the vehicle is set to $\xi_s = 0$. The path and trajectory are planned to reach the goal $x_g = (3.4,\ 3.4, pi/2)^T$. The planned path can be described as

$$\begin{cases} x^2 + (y-0.4)^2 = 0.16 & if\ 0 \le x \le 0.28 \\ y = x - 0.16 & if\ 0.28 < x \le 3.28 \\ (x-3)^2 + (y-3.4)^2 = 0.16 & if\ 3.28 < x \le 3.4 \end{cases}$$

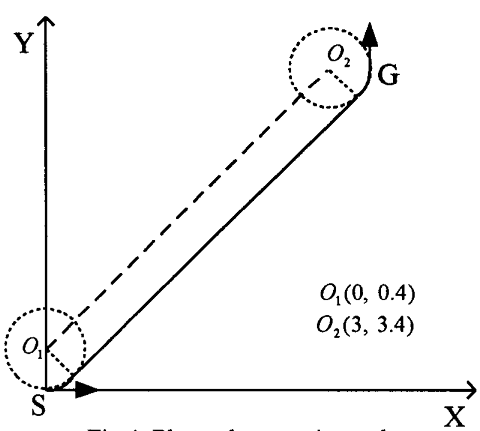

Fig.4 Planned geometirc path

Let $\alpha = 0.8$, the velocity limit curve computed from (19) is plotted in Fig.5. The nearly constant velocity limits are due to slightly variable P_i in (19). Under velocity limit curve, velocity profile is generated from (20),(21) with step size δ_1 = 0.63cm, δ_2 = 1.41cm, δ_3 = 0.63cm. The velocity curve and corresponding torque curves are shown in Fig.6. It indicates that at least one motor is saturated in motion.

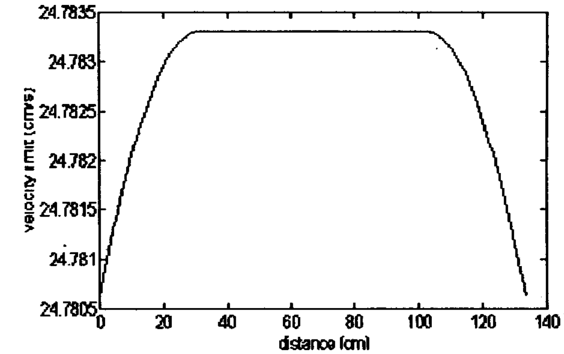

Fig.5 Velocity limit curve

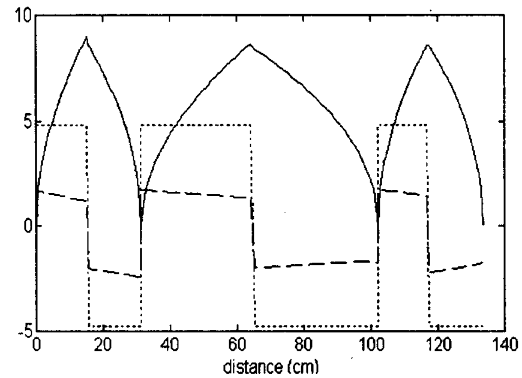

Fig.6 Velocity and torque curves
solid: velocity; dashed: τ_1 (Nm);dotted: τ_2 (Nm)

6 Conclusions

A method for planning optimal WMR motions has been presented based on the analysis of various constraints of WMRs. We classify the constraints into kinematic constraints and dynamic constraints. Kinematic constraints are imposed by nonholonomy of a WMR and its structural limits, while dynamic constraints are due to power limits in practice. In this paper, motion planning is completed in two stages. First, the path is planned using geometric methods under kinematic constraints, then the velocity limits are computed combining with the dynamics under which time optimal trajectory is generated. Since constraints are fully exploited in planning, the proposed method is not only simple and easily applicable but also effective on the feasibility of the motion planning.

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