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On the Quadratic Stability of Constrained Model Predictive Control

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Abstract

Analytic and numerical methods are developed in this paper for the analysis of the quadratic stability of Constrained Model Predictive Control (CMPC). According to the CMPC algorithm, each term of the closed-form of control law corresponding to an active constraint situation can be decomposed to have an uncertainty block, which is time varying over the control period. By analytic method, if a quadratic Lyapunov function can be found for the CMPC closed-loop system with uncertainty blocks in the feedback control law by solving a Riccati type equation, then the control system is quadratic stable. Since no rigorous solving method has been found, this Riccati type equation is solved by a trial-and-error method in this paper. A numerical method that does not solve the Riccati type equation, the Linear Matrix Inequality (LMI) technique, was found useful in solving this quadratic stability problem. Several examples are given to show the CMPC quadratic stability analysis results. It is also noticeable that the quadratic stability implies a similarity to a contraction.

1 Introduction

The quadratic stability analysis for continuous time control system has been studied in Khargonekar's paper [Khargonekar, Petersen, and Zhou, 1990]. The analysis method is for the closed-loop system which is affine function of time varying uncertainty blocks. Since this method is an analytic approach method, solving a continuous algebraic Riccati equation is needed. A generalized eigenvalue method [Dooren, 1980] was proposed as to solve this Riccati equation. Based on the study method in continuous time system, a quadratic stability analysis method by solving a discrete Riccati type equation for Constrained Model Predictive Control (CMPC) will be developed in this paper.

The CMPC system is a discrete time control system. It is based on the process model and quadratic objective function to predict future output and to minimize the penalty predicted error

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and/or input and/or its rate of change subject to hard constraints on the predicted output and/or input and/or its rate of change. At each sampling time, under the constraints, the optimization problem is solved to give an optimal control of the process. Because of the constraints, the whole control system dynamics is nonlinear even though the controlled process is assumed to be linear around the operating point. Several tuning parameters of the on-line objective function can be tuned to handle the stability and performance of the control system. In reality, this nonlinear control can be thought of as a piecewise linear type control. It contains a sequence of linear operators handling different active constraint situations over the whole control period. To analyze the quadratic stability problem of this nonlinear control system, the state space model is used and a closed-form linear control law with uncertainty blocks corresponding to different types of active constraint situations is constructed based on the CMPC algorithm. Then, the whole CMPC closed-loop control system can be described as an affine function of uncertainty blocks. By analytic approach to solve the quadratic stability problem of CMPC, a discrete Riccati type equation is consequently derived. Since no efficient solving method for this Riccati type equation has been found, a trial-and-error method is suggested to solve it. By numerical method, Linear Matrix Inequality (LMI) technique is applied to solve this quadratic stability problem of CMPC without solving the Riccati type equation. Several examples are given for the analytic and numerical methods.

Finally, this paper shows you that the quadratic stability of CMPC implies a similarity to a contraction.

2 Preliminaries

Although Model Predictive Control (MPC) algorithms have been applied to systems with nonlinear dynamic models [Garcia, 1984; Eaton *et al*, 1989], it is usually assumed that the dynamics are linear, the nonlinearity of the problem arising from the hard constraints. The properties of the controller are independent of the type of model description used for the plant [see, e.g., Morari *et al.*, 1989] . Consider a discrete state space model with disturbance directly adding to the output for a process given as:

$$\begin{aligned} x(k+1) &= \phi x(k) + \Theta u(k) \\ y(k) &= Cx(k) + d(k) \end{aligned} \tag{1}$$

where k is time index; $x(\cdot)$ is the state variable vector of the model; $u(k)$ and $y(k)$ are the manipulated variable (input) and output vectors of the model respectively; $d(k)$ is the disturbance vector; ϕ, Θ, C are the coefficient matrices of the model. Use the state space model (1) to predict the plant outputs over the prediction horizon (P) and assume that the predicted plant output is equal to the model output and that $d(k)$ is constant over the whole prediction horizon ($\Delta d(k+i) = d(k+i) - d(k+i-1) = 0,\ i = 1, \cdots, P$). For $P \geq M$, we also assume that $u(k+M-1) = u(k+M) = u(k+M+1) = \ldots = u(k+P-1)$.

Based on the model (1) and above assumptions, the prediction model can be written as:

$$\hat{Y}(k+1) = M_P \tilde{Y}(k) + Sv(k) \tag{2}$$

where

$$\hat{Y}(k+1) = \begin{bmatrix} \hat{y}(k+1) \\ \hat{y}(k+2) \\ \vdots \\ \hat{y}(k+P) \end{bmatrix}, \quad v(k) = \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+M-1) \end{bmatrix}, \quad M_P = I$$

$$S = \begin{bmatrix} C\Theta & 0 & \cdots & \cdots & 0 \\ C\phi\Theta & C\Theta & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C\phi^{M-1}\Theta & C\phi^{M-2}\Theta & \ddots & \ddots & C\Theta \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C\phi^{P-1}\Theta & C\phi^{P-2}\Theta & \cdots & C\phi^{P-M+1}\Theta & C\phi^{P-M}\Theta + \cdots + C\Theta \end{bmatrix}$$

and

$$\bar{Y} = \begin{bmatrix} C\phi x(k) + d(k) \\ C\phi^2 x(k) + d(k) \\ \vdots \\ C\phi^P x(k) + d(k) \end{bmatrix} = \eta x(k) + \alpha d(k)$$

where $\eta = [\ (C\phi)^T\ (C\phi^2)^T\ \cdots\ (C\phi^P)^T\ ]^T$, and $\alpha = [\ I_i\ \cdots\ I_P\ ]$ (I_i is an identity matrix).

Based on the prediction model (2), the QDMC-type algorithms [Garcia and Morshedi, 1986; Garcia and Morari, 1985b] use a quadratic objective function that includes the square of the weighted norm of the predicted error (e = setpoint minus predicted output ($\hat{Y}(k+1)$)) over a finite horizon in the future ($k+1, \cdots, k+P$) as well as penalty terms on input or its rate of change:

$$\min_{u(k),\cdots,u(k+M-1)} \sum_{l=1}^{P} [e^T(k+l)\Gamma^2 e(k+l) + u^T(k+l-1)B^2 u(k+l-1)$$
$$+ \Delta u^T(k+l-1)D^2 \Delta u(k+l-1)] \tag{3}$$

where $\Delta u(k) = u(k) - u(k-1)$.

The minimization of the objective function is carried out over the values of $u(k), \cdots, u(k+M-1)$, where M is the control horizon. The minimization is subject to possible constraints on the input u, its rate of change Δu, the predicted output $\hat{y}$ and other process variables usually referred to as associated variables. The details on the formulation of the optimization problem can be found in Prett and Garcia (1988). After the problem is solved on-line at k, only the optimal value for the first input vector $u(k)$ is implemented and the problem is solved again at $k+1$.

The optimization problem of the QDMC algorithm can be written as a standard Quadratic Programming problem:

$$\min_v q(v) = \frac{1}{2} v^T G v + g^T v \tag{4}$$

subject to

$$A^T v \geq b \tag{5}$$

where

$$v = [\ u^T(k) \quad \cdots \quad u^T(k+M-1)\ ]^T \tag{6}$$

and the matrices G, A, and vectors g, b are functions of the tuning parameters (weights, horizon P, M), and some bounds of the constraints. Also, the vector g, b are linear functions of $x(k)$, $u(k-1)$, $d(k)$.

For the optimal solution v^* we have [Fletcher, 1980]:

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{b} \end{bmatrix} \tag{7}$$

where $\hat{A}^T$, $\hat{b}$ consist of the rows of A^T, b that correspond to the constraints that are active at the optimum and λ^* is the vector of the Lagrange multipliers corresponding to these constraints. The optimal $u(k)$ corresponds to the first m elements of the v^* that solves (7), where m is the dimension of u.

The special form of the LHS matrix in (7) allows the numerically efficient computation of its inverse in a partitioned form [Fletcher, 1980]:

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H & -T \\ -T^T & U_L \end{bmatrix} \tag{8}$$

Then

$$v^* = -Hg + T\hat{b} \tag{9}$$

$$\lambda^* = T^T g - U_L \hat{b} \tag{10}$$

From equation (9), the unconstrained and constrained feedback control laws of CMPC are:

$$\begin{aligned} u(k) &= -[\,I\,0 \cdots 0\,]G^{-1}g \\ &= -[\,I\,0 \cdots 0\,]G^{-1}S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\ &\quad +[\,I\,0 \cdots 0\,]G^{-1}\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned} \tag{11}$$

and

$$\begin{aligned} u(k) &= [\,I\,0 \cdots 0\,](-Hg + T\hat{b}) \\ &= -[\,I\,0 \cdots 0\,][G^{-1} - G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T G^{-1}]S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\ &\quad +[\,I\,0 \cdots 0\,]G^{-1}\hat{A}(\hat{A}G^{-1}\hat{A})^{-1}\varpi^T \begin{bmatrix} \Delta\bar{u}(k) + u(k-1) \\ \Delta\bar{u}(k+1) \\ \vdots \\ \Delta\bar{u}(k+M-1) \\ \bar{u}(k) \\ \vdots \\ \bar{u}(k+M-1) \\ \bar{\eta}x(k) + \bar{\alpha}d(k) + \bar{y} \end{bmatrix} \end{aligned}$$

$$
+[\, I\ 0 \cdots 0\,][G^{-1} - G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T G^{-1}]\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{12}
$$

where $R(k+1)$ is a future setpoint vector; ϖ^T is an extraction matrix to extract sub-row-matrix from its right hand side matrix corresponding to an active constraint situation at the optimum; $\bar{\eta}, \bar{\alpha}$ are the matrices consisting of the rows from the η, α corresponding to the predicted output constraint window in which the constraint of predicted output is set, and vectors $\bar{y}$, $\Delta\bar{u}$, $\bar{u}$ are the upper bounds or lower bounds of $\hat{y}$, Δu, u respectively;

$$
\Pi = \begin{bmatrix} I & 0 & \cdots & 0 \\ -I & I & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & -I & I \end{bmatrix}
$$

3 The Closed-Loop Systems for CMPC

According to the quadratic stability analysis method, the control law of CMPC with state feedback part is considered (the past input feedback ($u(k-1)$) can be looked as an extra state feedback and $R(k+1) = 0$, $d(k) = 0$). From constrained control law (12), we see that the control law not only have state feedback but also have bounds ($\bar{u}\ \Delta\bar{u}\ \bar{y}$) feedback which are from the constraints of $\hat{y}$ and/or u and/or Δu. These bounds ($\bar{u}$, $\Delta\bar{u}$, $\bar{y}$) are neglected in the inner closed-loop stability analysis with a conservative compromise. For details, please see the technique report "The Closed-Form Control Laws of CMPC"[Chiou and Zafiriou, 1993]. Hence, in the following stability analysis, we do not include these bounds in the CMPC control law.

3.1 The Closed-Loop System for the Input Constraint Cases

First of all, the CMPC closed-loop system is constructed for the cases with constraints only on input (u) and/or rate change of input (Δu). The corresponding closed-form of CMPC control law with uncertainty blocks can be formulated by following several facts:

1. G^{-1} is a positive definite matrix.
2. H is a positive semidefinite matrix [Fletcher, 1980].
3. $G^{-1} \geq H$.
4. $G^{-1} = E^{-1}U^TUE^{-T}$, where E is a nonsingular matrix and U is an unitary matrix.
5. $H = E^{-1}U^T\Delta_c UE^{-T}$, where Δ_c is a diagonal matrix with the value of each entry element between 0 and 1.
6. The unitary matrix U is time varying dependent of the active constraint situation.

The facts 4,5,6 are referred to [Horn and Johnson, 1990].

Then, the control law with on external inputs ($d(k) = 0$, $R(k+1) = 0$) can be given as:

$$\begin{aligned} u(k) &= -[\, I\, 0 \cdots 0\,]Hg \\ &= -[\, I\, 0 \cdots 0\,]E^{-1}U^T\Delta_c U E^{-T} S^T \Gamma^T \Gamma \eta x(k) \\ &\quad +[\, I\, 0 \cdots 0\,]E^{-1}U^T\Delta_c U E^{-T}\Pi^T D^T D[\, u^T(k-1)\, 0 \cdots 0\,]^T \\ &= -[\, I\, 0 \cdots 0\,]E^{-1}\bar{F}(k)E^{-T}S^T\Gamma^T\Gamma\eta x(k) \\ &\quad +[\, I\, 0 \cdots 0\,]E^{-1}\bar{F}(k)E^{-T}\Pi^T D^T D[\, u^T(k-1)\, 0 \cdots 0\,]^T \\ &= e_a\bar{F}(k)E_b x(k) - e_a\bar{F}(k)E_c u(k-1) \end{aligned}$$

where $\bar{F}(k) = U^T\Delta_c U$ and $\bar{F}^T(k)\bar{F}(k) \leq I$; $e_a = -[\, I\, 0 \cdots 0\,]E^{-1}$ and $E_b = E^{-T}S^T\Gamma^T\Gamma\eta$; $E_c = E^{-T}\Pi^T D^T D\,[\, I\, 0 \cdots 0\,]^T$.

Let $x_1(k+1) = u(k)$ then $x_1(k) = u(k-1)$. The close-loop control system can be written as:

$$\begin{bmatrix} x(k+1) \\ x_1(k+1) \end{bmatrix} = \begin{bmatrix} \phi + \Theta e_a\bar{F}(k)E_b & -\Theta e_a\bar{F}(k)E_c \\ e_a\bar{F}(k)E_b & -e_a\bar{F}(k)E_c \end{bmatrix} \begin{bmatrix} x(k) \\ x_1(k) \end{bmatrix}$$

$$\Rightarrow \quad \bar{x}(k+1) = (\Phi + E_1 F(k) E_2)\bar{x}(k) \tag{13}$$

where

$$F(k) = 2\bar{F}(k) - U^T U = U^T\bar{\Delta}_c U$$

each element of the diagonal matrix $\bar{\Delta}_c$ has entry between -1 and 1.

$$\bar{x}(k+1) = [\, x^T(k+1)\;\; {x_1}^T(k+1)\,]^T;\;\; \Phi = \begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix} + E_1E_2;\;\; E_1 = 0.5[\, \Theta e_a \;\; e_a]^T;\;\; E_2 = [\, E_b \;\; -E_c]$$

3.2 The Closed-Loop System for the Input and Output Constraint Cases

The closed-loop system of cases with constraints set on input and output is constructed to fit for the CMPC quadratic stability analysis work. To do so, let

$$\hat{A}^T = \varpi^T\bar{s}$$

where $\bar{s}$ is a full rank submatrix of A^T, then several facts should be followed:

1. $(\bar{s}G^{-1}\bar{s}^T)^{-1} \geq \varpi(\hat{A}^TG^{-1}\hat{A})^{-1}\varpi^T$ (Proof is in Appendix A),
2. $(\bar{s}G^{-1}\bar{s}^T)^{-1} = E^{-1}U^TUE^{-T}$, where E and U are a nonsingular and an unitary matrix respectively,
3. $\varpi(\hat{A}^TG^{-1}\hat{A})^{-1}\varpi^T = E^{-1}U^T\Delta_c UE^{-T}$, where Δ_c is a diagonal matrix with the value of each entry element between 0 and 1,
4. The unitary matrix U is variably dependent of the active constraint situation.

The control law with no external input ($d(k)=0,\ R(k+1)=0$) can be given as:

$$
\begin{aligned}
u(k) &= -[\,I\,0\cdots0\,][G^{-1}-G^{-1}\hat{A}(\hat{A}^TG^{-1}\hat{A})^{-1}\hat{A}^TG^{-1}]S^T\Gamma^T\Gamma\eta x(k)\\
&\quad+[\,I\,0\cdots0\,]G^{-1}\hat{A}(\hat{A}G^{-1}\hat{A})^{-1}\varpi^T[\,0\ \cdots\ 0\,(\bar{\eta})^T\,]^Tx(k)\\
&\quad+[\,I\,0\cdots0\,][G^{-1}-G^{-1}\hat{A}(\hat{A}^TG^{-1}\hat{A})^{-1}\hat{A}^TG^{-1}]\Pi^TD^TD[\,u^T(k-1)\,0\cdots0\,]^T\\
&= -[\,I\,0\cdots0\,][G^{-1}-G^{-1}\bar{s}^T\varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T\bar{s}G^{-1}]S^T\Gamma^T\Gamma\eta x(k)\\
&\quad+[\,I\,0\cdots0\,]G^{-1}\bar{s}^T\varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T[\,0\ \cdots\ 0\,(\bar{\eta})^T\,]^Tx(k)\\
&\quad+[\,I\,0\cdots0\,][G^{-1}-G^{-1}\bar{s}^T\varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T\bar{s}G^{-1}]\Pi^TD^TD[\,u^T(k-1)\,0\cdots0\,]^T\\
&= -[\,I\,0\cdots0\,][G^{-1}-G^{-1}\bar{s}^TE^{-1}U^T\Delta_cUE^{-T}\bar{s}G^{-1}]S^T\Gamma^T\Gamma\eta x(k)\\
&\quad+[\,I\,0\cdots0\,]G^{-1}\bar{s}^TE^{-1}U^T\Delta_cUE^{-T}[\,0\ \cdots\ 0\,(\bar{\eta})^T\,]^Tx(k)\\
&\quad+[\,I\,0\cdots0\,][G^{-1}-G^{-1}\bar{s}^TE^{-1}U^T\Delta_cUE^{-T}\bar{s}G^{-1}]\Pi^TD^TD[\,u^T(k-1)\,0\cdots0\,]^T\\
&= (\psi_1+h_a\bar{F}(k)H_b)x(k)+(\psi_2-h_a\bar{F}(k)H_c)u(k-1)
\end{aligned}
\tag{14}
$$

where

$$
\begin{gathered}
\psi_1=-[\,I\,0\cdots0\,]G^{-1}S^T\Gamma^T\Gamma\eta,\quad h_a=[\,I\,0\cdots0\,]G^{-1}\bar{s}^TE^{-1}\\
H_b=E^{-T}(\bar{s}G^{-1}S^T\Gamma^T\Gamma\eta+[\,0\ \cdots\ 0\,(\bar{\eta})^T\,]^T),\quad \bar{F}(k)=U^T\Delta_cU,\quad \bar{F}(k)^T\bar{F}(k)\le I\\
\psi_2=[\,I\,0\cdots0\,]G^{-1}\Pi^TD^TD[\,I\,0\cdots0\,]^T,\quad H_c=E^{-T}\bar{s}G^{-1}\Pi^TD^TD[\,I\,0\cdots0\,]^T
\end{gathered}
$$

Follow the same method in the previous input constrained case, the close-loop control system can be written as:

$$\bar{x}(k+1)=(\Phi+H_1F(k)H_2)\bar{x}(k) \tag{15}$$

where $\bar{x}(k+1)=[\,x^T(k+1)\ \ x_1{}^T(k+1)\,]^T$; $x_1(k)=u(k-1)$;

$$\Phi=\begin{bmatrix}\phi+\Theta\psi_1 & \Theta\psi_2\\ \psi_1 & \psi_2\end{bmatrix}+H_1H_2;\ \ H_1=0.5[\,\Theta h_a\ \ h_a\,]^T;\ \ H_2=[\,H_b\ \ -H_c\,]$$

3.3 The Closed-Loop System for the Input Constraint and Softened Output Constraint Cases

Based on the state space prediction model (2), and the formulation of the standard quadratic programming problem (4) with constraints (5), and softened output constraints as following, the closed-loop system can be constructed for the cases with input constraint and output softened constraint.

$$y_L-\epsilon\le\hat{y}(k+l)\le y_U+\epsilon,\quad w_b\le l\le w_e \tag{16}$$

where $y_L,\ y_U$ are the lower and upper limits respectively; $w_b,\ w_e$ are the beginning and ending points of output constraint window. The term $\epsilon^TW^2\epsilon$ is added to the objective function, where W is the softening weight that determines the extent of softening. For $W=\infty$ we get hard constraints. $W=0$ corresponds to completely removing the constraints.

The objective function and constraints can be reformulated as:

$$\frac{1}{2}\bar{v}^T(k)\begin{bmatrix}G & 0\\ 0 & W^TW\end{bmatrix}\bar{v}(k)+[g^T\ 0]\bar{v}(k)$$

where $\bar{v}^T(k) = [v^T(k)\ \epsilon^T]$. Let

$$\bar{G} = \begin{bmatrix} G & 0 \\ 0 & W^T W \end{bmatrix}$$

and

$$\bar{g}^T = [g^T\ 0]$$

subject to

$$\begin{bmatrix} A_1^T & 0 \\ A_2^T & I \end{bmatrix} \bar{v}^T(k) \geq \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $A_1^T,\ b_1$ and $A_2^T,\ b_2$ correspond to the constraints of $u,\ \Delta u$ and $\hat{y}$ respectively.

Following the same procedures as in the previous subsections, we can have the control law as:

$$\begin{aligned} u(k) &= -\tau \bar{G}^{-1} \begin{bmatrix} S^T \Gamma^T \Gamma \eta x(k) \\ 0 \end{bmatrix} \\ &+ \tau \bar{G}^{-1} \bar{s}^T E^{-1} U^T \Delta_c U E^{-T} [\bar{s} \bar{G}^{-1} \begin{bmatrix} S^T \Gamma^T \Gamma \eta x(k) \\ 0 \end{bmatrix} + [\,0 \cdots 0\ (\bar{\eta} x(k))^T\,]^T] \\ &+ \tau \bar{G}^{-1} [I - \bar{s}^T E^{-1} U^T \Delta_c U E^{-T} \bar{s} \bar{G}^{-1}] \begin{bmatrix} \Pi^T D^T D \begin{bmatrix} u^T(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \end{aligned} \tag{17}$$

and the closed-loop system as:

$$\bar{x}(k+1) = (\bar{\Phi} + \bar{H}_1 F(k) \bar{H}_2) \bar{x}(k)$$

where $\bar{\Phi},\ \bar{H}_1,\ \bar{H}_2$ have the similar structure as $\Phi,\ H_1,\ H_2$ respectively, and they are also functions of the softening weight.

4 Quadratic Stability: Analytic Method

The analytic quadratic stability theorem now can be set up for the general CMPC closed-loop control system from previous section:

$$x(k+1) = (\Phi + E_1 F(k) E_2) x(k)$$

as:

Theorem 1 *If there exit positive definite matrices $\mathcal{P}$, $\mathcal{Q}$, R , $\tilde{\mathcal{P}}$, $\tilde{R}$ such that*

$$\Phi^T \tilde{\mathcal{P}} \Phi + \Phi^T \tilde{\mathcal{P}} E_1 (\tilde{R} + E_1^T \tilde{\mathcal{P}} E_1)^{-1} E_1{}^T \tilde{\mathcal{P}} \Phi + 2 E_2{}^T E_2 - \tilde{\mathcal{P}} = -\mathcal{Q} < 0 \tag{18}$$

with

$$R \geq \tilde{R} \, \| R + E_1{}^T \mathcal{P} E_1 \|_2 \tag{19}$$

and

$$\mathcal{P} = \tilde{\mathcal{P}} \, \| R + E_1{}^T \mathcal{P} E_1 \|_2 \tag{20}$$

then the control system is quadratic stable.

Proof is in Appendix A.

A suggested algorithm to solve the above problem (18), (19), (20) is given as:

Algorithm 1

BEGIN:

(1) *Guess a* $\tilde{R} < I$, *a* $Q > 0$, *a* $\tilde{\mathcal{P}}_1 > 0$ *and a large enough n.*

(2) *For i* = 1 *to n*

$$\tilde{\mathcal{P}} = \Phi^T \tilde{\mathcal{P}}_1 \Phi + \Phi^T \tilde{\mathcal{P}}_1 E_1 (\tilde{R} + E_1^T \tilde{\mathcal{P}}_1 E_1)^{-1} E_1{}^T \tilde{\mathcal{P}}_1 \Phi + 2E_2{}^T E_2 + Q$$

If $\tilde{\mathcal{P}} \approx \tilde{\mathcal{P}}_1$ *go to step (3)*

else if $n > i$ *then* $\tilde{\mathcal{P}}_1 = \tilde{\mathcal{P}}$ *continue the loop*

else go to step (1)

Continue

(3) *Guess a* $\mathcal{P}_1$ *and a R and a large enough* n_1.

(4) *For i* = 1 *to* n_1

$$\mathcal{P} = \tilde{\mathcal{P}} \; \| \, R + E_1{}^T \mathcal{P}_1 E_1 \|_2$$

If $\mathcal{P} \approx \mathcal{P}_1$ *go to step (5)*

else if $n_1 > i$ *then* $\mathcal{P}_1 = \mathcal{P}$ *continue the loop*

else go to step (3)

Continue

(5) *Check* $R \geq \tilde{R} \; \| \, R + E_1{}^T \mathcal{P} E_1 \|_2$ *or not. If it is not then go to step (1) or (3), else go to the end.*

END.

4.1 Examples for Illustrating the Feasibility of the Analytic Method

Several examples are given to show the feasibility of the analytic method, even though it may be conservative to find a feasible solution satisfying the stability condition. However, it is a simple and direct method to look for the quadratic stability condition of CMPC.

Example 1

A SISO distillation process model [Lang, 1989] is given as:

$$\tilde{p}(s) = \frac{-0.7s^2 - 1.1s - 79.7}{s^3 + 7.8s^2 + 94.2s + 40.6}$$

Select the following tuning parameters:

$$P = 10,\; M = 5,\; B = 0,\; D = 1,\; \Gamma = 1,\; T_s = 1$$

where T_s is the sampling time, and set the constraints on Δu, u over control horizon (M). By the algorithm 1, we obtain

$$\tilde{R} = 0.2I,\; Q = 0.1I,\; R = 0.3I$$

and

$$\tilde{\mathcal{P}} = \begin{bmatrix} 1.0588 & 0.0311 & 0.0005 & -0.3368 \\ 0.0311 & 0.2013 & 0 & -0.0133 \\ 0.0005 & 0 & 0.1 & -0.0002 \\ -0.3368 & -0.0133 & -0.0002 & 0.9123 \end{bmatrix} > 0$$

$$\mathcal{P} = \begin{bmatrix} 0.3616 & 0.0106 & 0.0002 & -0.1150 \\ 0.0106 & 0.0687 & 0 & -0.0045 \\ 0.0002 & 0 & 0.0342 & -0.0001 \\ -0.1150 & -0.0045 & -0.0001 & 0.3116 \end{bmatrix} > 0$$

$$\| R + E_1^T \mathcal{P} E_1 \|_2 \tilde{R} = 0.0683I \leq R$$

We can conclude that the control system is quadratic stable according to theorem 1.

Example 2

A SISO multieffect evaporator process model [Ricker *et al.*, 1989] is given as:

$$\tilde{p}(s) = \frac{2.69(-6s+1)e^{-1.5s}}{100s^2 + 25s + 1}$$

(I) Set constraints on Δu, u over the control horizon (M), and select tuning parameters:

$$P = 10,\ M = 2,\ B = 0,\ D = 5,\ \Gamma = 1,\ T_s = 3$$

By the algorithm 1, we obtain

$$\tilde{R} = 0.1I,\ Q = 0.0001I,\ R = 0.9I$$

and

$$\tilde{\mathcal{P}} = \begin{bmatrix} 1.2498 & -0.7007 & 0 & -3.7960 \\ -0.7007 & 0.3933 & 0 & 2.1741 \\ 0 & 0 & 0.0001 & 0 \\ -3.7960 & 2.1741 & 0 & 53.2547 \end{bmatrix} > 0$$

$$\mathcal{P} = \begin{bmatrix} 1.7977 & -1.0079 & 0 & -5.4603 \\ -1.0079 & 0.5657 & 0 & 3.1272 \\ 0 & 0 & 0.0001 & 0 \\ -5.4603 & 3.1272 & 0 & 76.6032 \end{bmatrix} > 0$$

$$\| R + E_1^T \mathcal{P} E_1 \|_2 \tilde{R} = 0.2069I \leq R$$

(II) Set constraints set on $\Delta u(k)$, $u(k)$, and $\hat{y}(k+13)$, and select tuning parameters:

$$P = 15,\ M = 2,\ B = 0,\ D = 5,\ \Gamma = 1,\ T_s = 3$$

By the algorithm 1, we obtain

$$\tilde{R} = 0.1I,\ Q = 0.0001I,\ R = 0.9I$$

and

$$\tilde{\mathcal{P}} = \begin{bmatrix} 0.9738 & -0.5397 & 0 & -2.7712 \\ -0.5397 & 0.2995 & 0 & 1.5649 \\ 0 & 0 & 0.0001 & 0 \\ -2.7712 & 1.5649 & 0 & 37.3142 \end{bmatrix} > 0$$

$$\mathcal{P} = \begin{bmatrix} 1.1067 & -0.6133 & 0 & -3.1495 \\ -0.6133 & 0.3404 & 0 & 1.7785 \\ 0 & 0 & 0.0001 & 0 \\ -3.1495 & 1.7785 & 0 & 42.4077 \end{bmatrix} > 0$$

$$\| R + E_1^T \mathcal{P} E_1 \|_2 \tilde{R} = 0.1137I \leq R$$

Hence, the control system under these two sets of tuning parameters and constraints is quadratic stable according to theorem 1.

Example 3

A 2×2 process model of a subsystem of the Shell Control Problem [Prett and Garcia, 1988] is given as:

$$\tilde{p}(s) = \begin{bmatrix} \frac{4.05e^{-27s}}{50s+1} & \frac{1.77e^{-28s}}{60s+1} \\ \frac{5.39e^{-18s}}{50s+1} & \frac{5.72e^{-14s}}{60s+1} \end{bmatrix}$$

(I) Set constraints on Δu, u over the control horizon (M), and select tuning parameters:

$$P = 30,\ M = 2,\ B = 0,\ D = 5I,\ \Gamma = I,\ T_s = 10$$

By the algorithm 1, we obtain

$$\tilde{R} = 0.1I,\ Q = 0.001I,\ R = 0.9I$$

and

$$\tilde{\mathcal{P}} = \begin{bmatrix}
1.0001e-03 & 4.1094e-05 & 3.5242e-06 & 2.3854e-05 & -1.0399e-05 & 0.0000e+00 \\
4.1094e-05 & 4.1273e-01 & 2.7610e-02 & 3.0026e-04 & -4.6335e-04 & 0.0000e+00 \\
3.5242e-06 & 2.7610e-02 & 4.0909e+00 & 1.8982e-04 & -2.4574e-04 & 0.0000e+00 \\
2.3854e-05 & 3.0026e-04 & 1.8982e-04 & 8.6988e-03 & -3.5369e-03 & 0.0000e+00 \\
-1.0399e-05 & -4.6335e-04 & -2.4574e-04 & -3.5369e-03 & 3.0152e-03 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.0000e-03 \\
0.0000e+00 & 1.7220e-11 & -1.0340e-11 & 1.5609e-03 & -8.0078e-04 & 0.0000e+00 \\
1.1215e-04 & 1.1621e+00 & -3.5542e+00 & 5.0026e-08 & -1.0586e-05 & 0.0000e+00 \\
-5.3609e-05 & -5.5551e-01 & -2.0334e+00 & 4.4211e-07 & -5.9541e-05 & 0.0000e+00
\end{bmatrix.$$

$$\left. \begin{matrix}
0.0000e+00 & 1.1215e-04 & -5.3609e-05 \\
1.7220e-11 & 1.1621e+00 & -5.5551e-01 \\
-1.0340e-11 & -3.5542e+00 & -2.0334e+00 \\
1.5609e-03 & 5.0026e-08 & 4.4211e-07 \\
-8.0078e-04 & -1.0586e-05 & -5.9541e-05 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
2.0000e-03 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 2.7865e+01 & -5.3243e+00 \\
0.0000e+00 & -5.3243e+00 & 3.0174e+01
\end{matrix} \right] > 0$$

$$\mathcal{P} = \begin{bmatrix}
1.1189e-03 & 4.5976e-05 & 3.9428e-06 & 2.6688e-05 & -1.1634e-05 & 0.0000e+00 \\
4.5976e-05 & 4.6176e-01 & 3.0890e-02 & 3.3593e-04 & -5.1839e-04 & 0.0000e+00 \\
3.9428e-06 & 3.0890e-02 & 4.5769e+00 & 2.1236e-04 & -2.7494e-04 & 0.0000e+00 \\
2.6688e-05 & 3.3593e-04 & 2.1236e-04 & 9.7321e-03 & -3.9571e-03 & 0.0000e+00 \\
-1.1634e-05 & -5.1839e-04 & -2.7494e-04 & -3.9571e-03 & 3.3734e-03 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.1188e-03 \\
0.0000e+00 & 1.9266e-11 & -1.1568e-11 & 1.7463e-03 & -8.9591e-04 & 0.0000e+00 \\
1.2547e-04 & 1.3001e+00 & -3.9764e+00 & 5.5969e-08 & -1.1843e-05 & 0.0000e+00 \\
-5.9978e-05 & -6.2151e-01 & -2.2749e+00 & 4.9464e-07 & -6.6615e-05 & 0.0000e+00
\end{bmatrix.$$

$$\left.\begin{array}{ccc}
0.0000e+00 & 1.2547e-04 & -5.9978e-05 \\
1.9266e-11 & 1.3001e+00 & -6.2151e-01 \\
-1.1568e-11 & -3.9764e+00 & -2.2749e+00 \\
1.7463e-03 & 5.5969e-08 & 4.9464e-07 \\
-8.9591e-04 & -1.1843e-05 & -6.6615e-05 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
2.2376e-03 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 3.1175e+01 & -5.9568e+00 \\
0.0000e+00 & -5.9568e+00 & 3.3758e+01
\end{array}\right] > 0$$

$$\| R + E_1^T \mathcal{P} E_1 \|_2 \tilde{R} = 0.1119I \leq R$$

(II) Set constraints set on $\Delta u(k)$, $u(k)$, and $\hat{y}_1(k+13)$, $\hat{y}_2(k+12)$, and select tuning parameters:

$$P = 30,\ M = 2,\ B = 0,\ D = 5I,\ \Gamma = I,\ T_s = 10$$

By the algorithm 1, we obtain

$$\tilde{R} = 0.1I,\ Q = 0.01I,\ R = 0.9I$$

and

$$\tilde{\mathcal{P}} = \left[\begin{array}{cccccc}
1.0001e-02 & 7.5271e-05 & -1.6218e-05 & 2.3854e-04 & -1.0399e-04 & 0.0000e+00 \\
7.5271e-05 & 6.4909e-01 & -2.5709e-01 & 3.0026e-03 & -4.6283e-03 & 0.0000e+00 \\
-1.6218e-05 & -2.5709e-01 & 3.7344e+00 & 1.8979e-03 & -2.4370e-03 & 0.0000e+00 \\
2.3854e-04 & 3.0026e-03 & 1.8979e-03 & 8.6988e-02 & -3.5369e-02 & 0.0000e+00 \\
-1.0399e-04 & -4.6283e-03 & -2.4370e-03 & -3.5369e-02 & 3.0153e-02 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.0000e-02 \\
0.0000e+00 & 1.7220e-10 & -1.0340e-10 & 1.5609e-02 & -8.0078e-03 & 0.0000e+00 \\
9.7934e-05 & 1.0148e+00 & -3.6206e+00 & 4.8279e-07 & -9.9297e-05 & 0.0000e+00 \\
-5.1298e-05 & -5.3177e-01 & -2.2420e+00 & 4.8046e-06 & -5.7948e-04 & 0.0000e+00
\end{array}\right.$$

$$\left.\begin{array}{ccc}
0.0000e+00 & 9.7934e-05 & -5.1298e-05 \\
1.7220e-10 & 1.0148e+00 & -5.3177e-01 \\
-1.0340e-10 & -3.6206e+00 & -2.2420e+00 \\
1.5609e-02 & 4.8279e-07 & 4.8046e-06 \\
-8.0078e-03 & -9.9297e-05 & -5.7948e-04 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
2.0000e-02 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 2.7859e+01 & -5.4342e+00 \\
0.0000e+00 & -5.4342e+00 & 3.0117e+01
\end{array}\right] > 0$$

$$\mathcal{P} = \left[\begin{array}{cccccc}
1.1218e-02 & 8.4432e-05 & -1.8192e-05 & 2.6758e-04 & -1.1665e-04 & 0.0000e+00 \\
8.4432e-05 & 7.2809e-01 & -2.8838e-01 & 3.3680e-03 & -5.1916e-03 & 0.0000e+00 \\
-1.8192e-05 & -2.8838e-01 & 4.1889e+00 & 2.1290e-03 & -2.7336e-03 & 0.0000e+00 \\
2.6758e-04 & 3.3680e-03 & 2.1290e-03 & 9.7575e-02 & -3.9674e-02 & 0.0000e+00 \\
-1.1665e-04 & -5.1916e-03 & -2.7336e-03 & -3.9674e-02 & 3.3822e-02 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.1217e-02 \\
0.0000e+00 & 1.9316e-10 & -1.1599e-10 & 1.7509e-02 & -8.9825e-03 & 0.0000e+00 \\
1.0985e-04 & 1.1383e+00 & -4.0613e+00 & 5.4155e-07 & -1.1138e-04 & 0.0000e+00 \\
-5.7542e-05 & -5.9649e-01 & -2.5148e+00 & 5.3893e-06 & -6.5001e-04 & 0.0000e+00
\end{array}\right.$$

$$\left.\begin{array}{rrr} 0.0000e+00 & 1.0985e-04 & -5.7542e-05 \\ 1.9316e-10 & 1.1383e+00 & -5.9649e-01 \\ -1.1599e-10 & -4.0613e+00 & -2.5148e+00 \\ 1.7509e-02 & 5.4155e-07 & 5.3893e-06 \\ -8.9825e-03 & -1.1138e-04 & -6.5001e-04 \\ 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\ 2.2434e-02 & 0.0000e+00 & 0.0000e+00 \\ 0.0000e+00 & 3.1249e+01 & -6.0957e+00 \\ 0.0000e+00 & -6.0957e+00 & 3.3783e+01 \end{array}\right] > 0$$

$$\| R + E_1^T \mathcal{P} E_1 \|_2 \tilde{R} = 0.1122I \leq R$$

Hence, the control system under these two sets of tuning parameters and constraints is quadratic stable according to theorem 1.

5 Quadratic Stability: Numerical Method

The numerical method can be set up for the general CMPC closed-loop system:

$$\bar{x}(k+1) = (\Psi + E_1 F(k) E_2)\bar{x}(k) \tag{21}$$

Let $\bar{q}(k) = E_2\bar{x}(k)$ and $\bar{p}(k) = F(k)\bar{q}(k)$.

Then, the following theorem shows how to analyze the quadratic stability of CMPC numerically.

Theorem 2 *If there exists a $\mathcal{P}$ such that*

$$\begin{bmatrix} \Psi^T \mathcal{P} \Psi - \mathcal{P} + E_2^T E_2 & \Psi^T \mathcal{P} E_1 \\ E_1^T \mathcal{P} \Psi & E_1^T \mathcal{P} E_1 - I \end{bmatrix} \leq 0 \tag{22}$$

then the CMPC system is quadratic stable.

Proof is in Appendix B.

An alternative way to check quadratic stability of CMPC without solving the above LMI problem is given as following.

Theorem 3 *If $\| E_2(zI - \Psi)^{-1} E_1 \|_\infty < 1$ and $\bar{x}(0) = 0$, then the CMPC system is quadratic stable.*

Proof is in Appendix B.

According to the above theorems, some examples are given in the following to test the quadratic stability of the CMPC system.

5.1 Examples for Illustrating the Feasibility of the Numerical Method

Example 4

A SISO multieffect evaporator process model [Ricker *et al.*, 1989] is given as:

$$\tilde{p}(s) = \frac{2.69(-6s+1)e^{-1.5s}}{100s^2 + 25s + 1}$$

(I) Set constraints on Δu, u over the control horizon (M), and select tuning parameters:

$$P=7,\ M=5,\ B=0,\ D=0.5,\ \Gamma=1,\ T_s=3$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P}=\begin{bmatrix} 112.4930 & -157.7059 & 52.4928 & -111.3498 \\ -157.7059 & 222.0851 & -74.3482 & 156.5486 \\ 52.4928 & -74.3482 & 25.1607 & -52.2174 \\ -111.3498 & 156.5486 & -52.2174 & 112.3428 \end{bmatrix} > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_{\infty}= 0.8373$$

(II) Set the constraint on predicted output $\hat{y}(k+2)$, and select the tuning parameters as:

$$P=7,\ M=5,\ B=0,\ D=0,\ \Gamma=1,\ T_s=3$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P}=\begin{bmatrix} 0.5337 & -0.0233 & -0.1163 \\ -0.0233 & 0.3281 & -0.0124 \\ -0.1163 & -0.0124 & 0.1910 \end{bmatrix} > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_{\infty}= 0.3673$$

(III) Set constraints set on $\Delta u(k)$, $u(k)$, $\hat{y}(k+7)$, and select tuning parameters:

$$P=10,\ M=2,\ B=0,\ D=1.0,\ \Gamma=1,\ T_s=3$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P}=\begin{bmatrix} 26.4645 & -35.2683 & 10.8566 & -23.5474 \\ -35.2683 & 49.4125 & -16.4221 & 31.8673 \\ 10.8566 & -16.4221 & 6.2463 & -9.8603 \\ -23.5474 & 31.8673 & -9.8603 & 27.2858 \end{bmatrix} > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_{\infty}= 0.8956$$

(IV) Set the constraint on predicted output $\hat{y}(k+2)$, and select the tuning parameters as:

$$P=7,\ M=5,\ B=0,\ D=0,\ \Gamma=1,\ T_s=3$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P}=\begin{bmatrix} 0.5337 & -0.0233 & -0.1163 \\ -0.0233 & 0.3281 & -0.0124 \\ -0.1163 & -0.0124 & 0.1910 \end{bmatrix} > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_\infty = 0.3673$$

(V) Soften the constraint on predicted output $\hat{y}(k+1)$, and select the tuning parameters as:

$$P=7,\ M=5,\ B=0,\ D=0,\ \Gamma=1,\ W=11.396,\ T_s=3$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P} = \begin{bmatrix} 108.6959 & -47.1126 & -0.2251 \\ -47.1126 & 30.4383 & -0.2448 \\ -0.2251 & -0.2448 & 1.3499 \end{bmatrix} > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_\infty = 0.2870$$

We can conclude that the CMPC system is stable to control this process under the constraints set on u, Δu over the control horizon M or $u(k)$, $\Delta u(k)$, $\hat{y}(k+7)$ or $\hat{y}(k+2)$ or softened constraint set on $\hat{y}(k+1)$.

Example 5

A 2×2 process model of a subsystem of the Shell Control Problem [Prett and Garcia, 1988] is given as:

$$\tilde{p}(s) = \begin{bmatrix} \frac{4.05e^{-27s}}{50s+1} & \frac{1.77e^{-28s}}{60s+1} \\ \frac{5.39e^{-18s}}{50s+1} & \frac{5.72e^{-14s}}{60s+1} \end{bmatrix}$$

(I) Set constraints on Δu, u over the control horizon (M), and select tuning parameters:

$$P=6,\ M=5,\ B=0,\ D=1.5I,\ \Gamma=I,\ T_s=6$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P} = \left[\begin{array}{cccccc}
3.9259e+05 & -2.0639e+04 & 3.5546e+05 & 5.2381e+05 & 1.3321e+02 & 1.1754e+05 \\
-2.0639e+04 & 1.0885e+03 & -1.8783e+04 & -2.6871e+04 & -6.8735e+00 & -6.1113e+03 \\
3.5546e+05 & -1.8783e+04 & 3.2465e+05 & 4.6379e+05 & -1.3774e+00 & 1.0558e+05 \\
5.2381e+05 & -2.6871e+04 & 4.6379e+05 & 1.4726e+06 & 2.7004e+02 & 2.5108e+05 \\
1.3321e+02 & -6.8735e+00 & -1.3774e+00 & 2.7004e+02 & 3.7060e+02 & -4.8399e+01 \\
1.1754e+05 & -6.1113e+03 & 1.0558e+05 & 2.5108e+05 & -4.8399e+01 & 4.6761e+04 \\
1.3341e+01 & -1.8454e+00 & 5.7661e+01 & 4.4199e+01 & -4.5896e+01 & 2.3289e+01 \\
-2.4722e+05 & 1.2853e+04 & -2.2196e+05 & -5.2811e+05 & -1.3831e+02 & -9.8284e+04 \\
5.5063e+00 & -3.5717e-01 & 4.6471e+00 & 9.9374e+00 & 1.0789e+01 & -2.0920e-01 \\
-4.8280e+01 & 4.1358e+00 & -1.0521e+02 & -8.9116e+01 & 4.3319e+01 & -3.6319e+01 \\
-5.2380e+05 & 2.6871e+04 & -4.6379e+05 & -1.4726e+06 & -2.6752e+02 & -2.5108e+05 \\
-2.7323e+05 & 1.4204e+04 & -2.4524e+05 & -5.8365e+05 & -1.9854e+02 & -1.0860e+05
\end{array}\right.$$

$$\left.\begin{array}{cccccc}
1.3341e+01 & -2.4722e+05 & 5.5063e+00 & -4.8280e+01 & -5.2380e+05 & -2.7323e+05 \\
-1.8454e+00 & 1.2853e+04 & -3.5717e-01 & 4.1358e+00 & 2.6871e+04 & 1.4204e+04 \\
5.7661e+01 & -2.2196e+05 & 4.6471e+00 & -1.0521e+02 & -4.6379e+05 & -2.4524e+05 \\
4.4199e+01 & -5.2811e+05 & 9.9374e+00 & -8.9116e+01 & -1.4726e+06 & -5.8365e+05 \\
-4.5896e+01 & -1.3831e+02 & 1.0789e+01 & 4.3319e+01 & -2.6752e+02 & -1.9854e+02 \\
2.3289e+01 & -9.8284e+04 & -2.0920e-01 & -3.6319e+01 & -2.5108e+05 & -1.0860e+05 \\
1.2481e+01 & -2.0651e+01 & -3.7083e+00 & -5.9652e+00 & -4.4199e+01 & -2.2540e+01 \\
-2.0651e+01 & 2.0674e+05 & -5.9748e+00 & 4.4429e+01 & 5.2811e+05 & 2.2847e+05 \\
-3.7083e+00 & -5.9748e+00 & 1.5418e+00 & 4.6093e-01 & -9.9374e+00 & -6.6030e+00 \\
-5.9652e+00 & 4.4429e+01 & 4.6093e-01 & 1.2343e+01 & 8.9178e+01 & 4.8133e+01 \\
-4.4199e+01 & 5.2811e+05 & -9.9374e+00 & 8.9178e+01 & 1.4726e+06 & 5.8366e+05 \\
-2.2540e+01 & 2.2847e+05 & -6.6030e+00 & 4.8133e+01 & 5.8366e+05 & 2.5251e+05
\end{array}\right] > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_{\infty} = 0.8373$$

(II) Set the constraints on predicted output $\hat{y}_1(k+6), \hat{y}_2(k+4)$, and select the tuning parameters as:

$$P=7,\ M=2,\ B=0,\ D=0,\ \Gamma=I,\ T_s=6$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P} = \left[\begin{array}{rrrrrr}
1.4223e+02 & -3.2417e+00 & -4.7621e-01 & 9.0582e-02 & 4.3273e+00 & -1.0109e+00 \\
-3.2417e+00 & 8.3873e-02 & -3.2576e-01 & 2.8365e-02 & 8.0830e-02 & -6.1209e-02 \\
-4.7621e-01 & -3.2576e-01 & 1.2785e+01 & -5.1221e-01 & -8.5525e+00 & 3.7959e+00 \\
9.0582e-02 & 2.8365e-02 & -5.1221e-01 & 2.3956e+00 & 1.4669e-02 & -2.6706e-01 \\
4.3273e+00 & 8.0830e-02 & -8.5525e+00 & 1.4669e-02 & 1.0748e+01 & -3.5835e+00 \\
-1.0109e+00 & -6.1209e-02 & 3.7959e+00 & -2.6706e-01 & -3.5835e+00 & 1.7815e+00 \\
-2.7348e+00 & 4.2211e-02 & 9.4847e-01 & 1.2213e-03 & -1.6456e+00 & 4.6625e-01 \\
-9.8033e-01 & 1.3847e-01 & -4.2334e+00 & 5.6584e-01 & 6.8169e-01 & -1.8387e+00 \\
-7.8967e-01 & 2.2941e-02 & -2.4256e-01 & 6.1552e-04 & 4.2525e-01 & -1.1980e-01 \\
-1.4463e+00 & 8.2572e-03 & 1.2004e+00 & -2.0406e-03 & -8.0787e-01 & 3.9640e-01
\end{array}\right.$$

$$\left.\begin{array}{rrrr}
-2.7348e+00 & -9.8033e-01 & -7.8967e-01 & -1.4463e+00 \\
4.2211e-02 & 1.3847e-01 & 2.2941e-02 & 8.2572e-03 \\
9.4847e-01 & -4.2334e+00 & -2.4256e-01 & 1.2004e+00 \\
1.2213e-03 & 5.6584e-01 & 6.1552e-04 & -2.0406e-03 \\
-1.6456e+00 & 6.8169e-01 & 4.2525e-01 & -8.0787e-01 \\
4.6625e-01 & -1.8387e+00 & -1.1980e-01 & 3.9640e-01 \\
6.5538e-01 & -2.0122e-02 & -1.9267e-01 & -1.1880e-01 \\
-2.0122e-02 & 9.2150e+00 & 6.6473e-05 & 4.1160e-02 \\
-1.9267e-01 & 6.6473e-05 & 2.2094e-01 & 6.5687e-03 \\
-1.1880e-01 & 4.1160e-02 & 6.5687e-03 & 6.9274e+00
\end{array}\right] > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_{\infty} = 0.1200$$

(III) Set the constraints on $u(k)$, $\Delta u(k)$ and predicted output $\hat{y}_1(k+8), \hat{y}_2(k+7)$, and select the tuning parameters as:

$$P=10,\ M=2,\ B=0,\ D=0,\ \Gamma=I,\ T_s=6$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P} = \left[\begin{array}{rrrrrr}
2.0858e+02 & -4.8944e+00 & 6.5917e+00 & 1.8069e+00 & 1.1199e+00 & 1.2164e+00 \\
-4.8944e+00 & 1.3621e-01 & -9.9255e-01 & -7.1154e-02 & 5.9043e-01 & -3.0638e-01 \\
6.5917e+00 & -9.9255e-01 & 3.4364e+01 & 1.2199e+00 & -2.8004e+01 & 1.1894e+01 \\
1.8069e+00 & -7.1154e-02 & 1.2199e+00 & 3.6020e+00 & -2.4942e-02 & 4.3959e-01 \\
1.1199e+00 & 5.9043e-01 & -2.8004e+01 & -2.4942e-02 & 3.2647e+01 & -1.1303e+01 \\
1.2164e+00 & -3.0638e-01 & 1.1894e+01 & 4.3959e-01 & -1.1303e+01 & 4.4139e+00 \\
-4.9641e+00 & 3.9567e-02 & 3.5026e+00 & -3.0312e-03 & -5.6018e+00 & 1.6462e+00 \\
-3.2162e-01 & 1.2822e-01 & -4.3751e+00 & -8.8866e-01 & 1.6553e+00 & -1.2505e+00 \\
3.4974e-01 & 9.2389e-03 & -8.2607e-01 & 1.6463e-03 & 1.3595e+00 & -3.9417e-01 \\
-2.8598e-01 & 1.0436e-01 & -4.6397e+00 & -4.1307e-03 & 5.4895e+00 & -1.8852e+00
\end{array}\right.$$

$$
\begin{bmatrix}
-4.9641e+00 & -3.2162e-01 & 3.4974e-01 & -2.8598e-01 \\
3.9567e-02 & 1.2822e-01 & 9.2389e-03 & 1.0436e-01 \\
3.5026e+00 & -4.3751e+00 & -8.2607e-01 & -4.6397e+00 \\
-3.0312e-03 & -8.8866e-01 & 1.6463e-03 & -4.1307e-03 \\
-5.6018e+00 & 1.6553e+00 & 1.3595e+00 & 5.4895e+00 \\
1.6462e+00 & -1.2505e+00 & -3.9417e-01 & -1.8852e+00 \\
1.6477e+00 & -1.2636e-02 & -5.1073e-01 & -7.0844e-01 \\
-1.2636e-02 & 1.3059e+00 & 1.3387e-03 & 9.8353e-02 \\
-5.1073e-01 & 1.3387e-03 & 2.5485e-01 & 8.4884e-02 \\
-7.0844e-01 & 9.8353e-02 & 8.4884e-02 & 1.4746e+00
\end{bmatrix} > 0
$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_\infty = 0.9438$$

(IV) Set the constraints on $u_1(k)$, $\Delta u_1(k)$ and soften $\hat{y}_1(k+5)$, and select the tuning parameters as:

$$P = 73,\ M = 1,\ B = 0,\ D = 0,\ \Gamma = I,\ W = 10.0835,\ T_s = 6$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$
\mathcal{P} = \begin{bmatrix}
1.9947e+02 & -9.9266e+00 & 1.5478e+02 & 5.6452e+00 & 1.5077e+01 & 2.8061e+01 \\
-9.9266e+00 & 1.5923e+00 & -3.4320e+01 & 1.5369e+01 & -3.9207e+00 & -4.5906e+00 \\
1.5478e+02 & -3.4320e+01 & 9.4560e+02 & -4.1002e+02 & -1.6235e+02 & 1.7840e+02 \\
5.6452e+00 & 1.5369e+01 & -4.1002e+02 & 4.3533e+03 & 3.0216e+01 & -1.1046e+01 \\
1.5077e+01 & -3.9207e+00 & -1.6235e+02 & 3.0216e+01 & 8.6856e+02 & -1.7029e+02 \\
2.8061e+01 & -4.5906e+00 & 1.7840e+02 & -1.1046e+01 & -1.7029e+02 & 8.1020e+01 \\
1.7718e+00 & -1.4203e+00 & -6.2749e-01 & 9.7338e+00 & -7.5477e+01 & 7.7218e+00 \\
-5.8749e+01 & 7.4396e+00 & -1.0718e+02 & -6.0303e+01 & -1.2566e+02 & -2.2244e+01 \\
-4.6337e-01 & -2.4712e-01 & -5.2041e-01 & 4.1031e+00 & -2.8594e+00 & -1.6362e+00 \\
1.5194e+01 & -4.0196e+00 & 2.4716e+01 & -2.2050e+01 & 2.1893e+02 & -3.5755e+01
\end{bmatrix}
$$

$$
\begin{bmatrix}
1.7718e+00 & -5.8749e+01 & -4.6337e-01 & 1.5194e+01 \\
-1.4203e+00 & 7.4396e+00 & -2.4712e-01 & -4.0196e+00 \\
-6.2749e-01 & -1.0718e+02 & -5.2041e-01 & 2.4716e+01 \\
9.7338e+00 & -6.0303e+01 & 4.1031e+00 & -2.2050e+01 \\
-7.5477e+01 & -1.2566e+02 & -2.8594e+00 & 2.1893e+02 \\
7.7218e+00 & -2.2244e+01 & -1.6362e+00 & -3.5755e+01 \\
1.1081e+02 & -9.9005e+00 & -2.8068e+01 & 2.7723e+01 \\
-9.9005e+00 & 2.2767e+02 & 1.6810e+00 & -4.1042e+01 \\
-2.8068e+01 & 1.6810e+00 & 4.3477e+01 & -1.1557e+01 \\
2.7723e+01 & -4.1042e+01 & -1.1557e+01 & 1.9527e+02
\end{bmatrix} > 0
$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_\infty = 0.5665$$

We can conclude that the CMPC system is stable to control this process under the constraints set on u, Δu over the control horizon M or $\hat{y}_1(k+6), \hat{y}_2(k+4)$ or $u(k)$, $\Delta u(k)$, $\hat{y}_1(k+8)$, $\hat{y}_2(k+7)$ and $u_1(k)$, $\Delta u_1(k)$, softened constraint set on $\hat{y}_1(k+5)$.

Example 6

A SISO process model of a subsystem of the Shell Control Problem [Prett and Garcia, 1988] is given as:

$$\frac{4.05e^{-27s}}{50s+1}$$

(I) Set softened constraint on $\hat{y}(k+7)$, and select tuning parameters:

$$P = 60,\ M = 1,\ B = 0,\ D = 0,\ W = 290,\ \Gamma = 1,\ T_s = 4$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P} = \left[\begin{matrix} 4.0342e+03 & 3.1960e-02 & -1.4894e-04 & 6.7493e-07 & -2.9368e-09 & 1.1983e-11 \\ 3.1960e-02 & 7.6964e+00 & 2.7394e-02 & -1.2411e-04 & 5.3993e-07 & -2.2026e-09 \\ -1.4894e-04 & 2.7394e-02 & 6.5969e+00 & 2.2828e-02 & -9.9290e-05 & 4.0495e-07 \\ 6.7493e-07 & -1.2411e-04 & 2.2828e-02 & 5.4974e+00 & 1.8262e-02 & -7.4467e-05 \\ -2.9368e-09 & 5.3993e-07 & -9.9290e-05 & 1.8262e-02 & 4.3980e+00 & 1.3697e-02 \\ 1.1983e-11 & -2.2026e-09 & 4.0495e-07 & -7.4467e-05 & 1.3697e-02 & 3.2985e+00 \\ -4.3468e-14 & 7.9883e-12 & -1.4684e-09 & 2.6997e-07 & -4.9645e-05 & 9.1312e-03 \\ 1.1829e-16 & -2.1734e-14 & 3.9942e-12 & -7.3419e-10 & 1.3498e-07 & -2.4822e-05 \end{matrix} \right.$$

$$\left. \begin{matrix} -4.3468e-14 & 1.1829e-16 \\ 7.9883e-12 & -2.1734e-14 \\ -1.4684e-09 & 3.9942e-12 \\ 2.6997e-07 & -7.3419e-10 \\ -4.9645e-05 & 1.3498e-07 \\ 9.1312e-03 & -2.4822e-05 \\ 2.1990e+00 & 4.5656e-03 \\ 4.5656e-03 & 1.0995e+00 \end{matrix} \right] > 0$$

and

$$\| E_2(zI - \Psi)^{-1} E_1 \|_\infty = 0.9204$$

We can conclude that the CMPC system is stable to control this process under the softened constraint set on $\hat{y}(k+7)$.

5.2 Quadratic Robust Stability Analysis of CMPC with Scalar Uncertainty in the Process Model

From section 3, we can have a general control law of CMPC as in equation (14).

$$u(k) = (\psi_1 + h_a F(k) H_b) x(k) + (\psi_2 - h_a F(k) H_c) u(k-1)$$

Suppose we have a process described by state space model with scalar uncertainty in the input matrix as following:

$$\begin{aligned} x(k+1) &= \Phi x(k) + (\Theta_n + W_n \Delta_p) u(k) \\ y(k) &= C x(k) \end{aligned}$$

where Θ_n is the nominal input matrix; W_n is the uncertainty weighting matrix; Δ_p is the scalar uncertainty. The the closed-loop of the CMPC system can be written as:

$$\bar{x}(k+1) = (\Psi + E_1 \bar{\Delta} E_2) \bar{x}(k) \tag{23}$$

where Ψ, E_1, E_2 are matrices in which every element is function of ψ_1, h_a, H_b, ψ_2, H_c, W_n, B_n;

$$\bar{\Delta} = \begin{bmatrix} F(k) & 0 & 0 \\ 0 & \Delta_p & 0 \\ 0 & 0 & \Delta_p F(k) \end{bmatrix}$$

The equation (23) is used to analyze the quadratic robust stability of CMPC as shown by the following SISO examples.

5.3 Examples for Illustrating the Quadratic Robust Stability Analysis

All examples are for SISO only. The same method can be extended to MIMO, if the process can be described by the state space model with coefficients which are affine functions of uncertainty blocks.

Example 7

A SISO process model of a subsystem of the Shell Control Problem [Prett and Garcia, 1988] is given as a state space model:

$$\begin{aligned} x(k+1) &= \phi x(k) + (4.05 + 2.11\Delta_p)\Theta u(k) \\ y(k) &= Cx(k) \end{aligned}$$

where Δ_p is a time varying scalar uncertainty;

$$\phi = \begin{bmatrix} 0.9231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Theta = [\,1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]^T$$

$$C = [\,0\ 0\ 0\ 0\ 0\ 0\ 0.0198\ 0.0571\,]$$

(I) Set constraint set on u, Δu over the control horizon M, and select tuning parameters:

$$P = 100,\ M = 3,\ B = 0,\ D = 4,\ \Gamma = 1,\ T_s = 4$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P} = \left[\begin{matrix} 7.1848e-01 & -6.6786e-01 & 9.7455e-03 & 3.2509e-03 & 1.1278e-03 & 4.9086e-04 \\ -6.6786e-01 & 1.1620e+00 & -5.3055e-01 & 2.3389e-02 & 8.1613e-03 & 2.8115e-03 \\ 9.7455e-03 & -5.3055e-01 & 9.5054e-01 & -4.5238e-01 & 1.5219e-02 & 5.0039e-03 \\ 3.2509e-03 & 2.3389e-02 & -4.5238e-01 & 7.7179e-01 & -3.6227e-01 & 1.1455e-02 \\ 1.1278e-03 & 8.1613e-03 & 1.5219e-02 & -3.6227e-01 & 5.9741e-01 & -2.7049e-01 \\ 4.9086e-04 & 2.8115e-03 & 5.0039e-03 & 1.1455e-02 & -2.7049e-01 & 4.2371e-01 \\ 1.8681e-04 & 9.7801e-04 & 1.6279e-03 & 3.5623e-03 & 8.4049e-03 & -1.7841e-01 \\ -1.9371e-04 & 4.3866e-04 & 5.7644e-04 & 1.1497e-03 & 2.4363e-03 & 5.4658e-03 \\ -2.2769e+00 & 1.3914e+00 & 5.1137e-01 & 1.8916e-01 & 6.9817e-02 & 2.4050e-02 \end{matrix}\right.$$

$$\left.\begin{matrix} 1.8681e-04 & -1.9371e-04 & -2.2769e+00 \\ 9.7801e-04 & 4.3866e-04 & 1.3914e+00 \\ 1.6279e-03 & 5.7644e-04 & 5.1137e-01 \\ 3.5623e-03 & 1.1497e-03 & 1.8916e-01 \\ 8.4049e-03 & 2.4363e-03 & 6.9817e-02 \\ -1.7841e-01 & 5.4658e-03 & 2.4050e-02 \\ 2.5009e-01 & -8.6295e-02 & 8.3077e-03 \\ -8.6295e-02 & 7.6735e-02 & 9.1753e-03 \\ 8.3077e-03 & 9.1753e-03 & 6.7735e+01 \end{matrix}\right] > 0$$

and

$$\| E_2(zI-\Psi)^{-1}E_1 \|_{\infty}= 0.9819$$

(II) Set constraint set on $\hat{y}(k+11)$, and select tuning parameters:

$$P=60,\ M=1,\ B=0,\ D=0,\ \Gamma=1,\ T_s=4$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P}=\begin{bmatrix}
7.0132e-02 & -1.8152e-02 & 1.1698e-05 & 1.0569e-05 & 1.0279e-05 & 8.4813e-06 & 6.1904e-06 & 4.1903e-06 \\
-1.8152e-02 & 3.4158e-02 & -1.5561e-02 & 8.1744e-06 & 7.2434e-06 & 7.1251e-06 & 5.8752e-06 & 4.1934e-06 \\
1.1698e-05 & -1.5561e-02 & 2.8947e-02 & -1.2966e-02 & 7.9326e-06 & 6.9780e-06 & 6.4711e-06 & 5.0447e-06 \\
1.0569e-05 & 8.1744e-06 & -1.2966e-02 & 2.3735e-02 & -1.0371e-02 & 7.5693e-06 & 6.5814e-06 & 5.7024e-06 \\
1.0279e-05 & 7.2434e-06 & 7.9326e-06 & -1.0371e-02 & 1.8523e-02 & -7.7768e-03 & 7.1049e-06 & 6.0945e-06 \\
8.4813e-06 & 7.1251e-06 & 6.9780e-06 & 7.5693e-06 & -7.7768e-03 & 1.3312e-02 & -5.1823e-03 & 6.5217e-06 \\
6.1904e-06 & 5.8752e-06 & 6.4711e-06 & 6.5814e-06 & 7.1049e-06 & -5.1823e-03 & 8.0999e-03 & -2.5880e-03 \\
4.1903e-06 & 4.1934e-06 & 5.0447e-06 & 5.7024e-06 & 6.0945e-06 & 6.5217e-06 & -2.5880e-03 & 2.8879e-03
\end{bmatrix} > 0$$

(II) Soften constraint on $\hat{y}(k+7)$, and select tuning parameters:

$$P=60,\ M=1,\ B=0,\ D=0,\ W=74,\ \Gamma=1,\ T_s=4$$

By the theorem 2,3 and LMI tool to check the stability condition, we obtain

$$\mathcal{P}=\begin{bmatrix}
5.4356e-02 & -5.2726e-04 & 1.5202e-07 & 1.3651e-07 & 1.1985e-07 & 1.0227e-07 & 8.4508e-08 & 6.5514e-08 \\
-5.2726e-04 & 1.0905e-03 & -4.5193e-04 & 1.3456e-07 & 1.2038e-07 & 1.0528e-07 & 8.9743e-08 & 7.6739e-08 \\
1.5202e-07 & -4.5193e-04 & 9.2523e-04 & -3.7660e-04 & 1.2459e-07 & 1.1009e-07 & 9.4930e-08 & 8.0089e-08 \\
1.3651e-07 & 1.3456e-07 & -3.7660e-04 & 7.5998e-04 & -3.0126e-04 & 1.1410e-07 & 9.9319e-08 & 8.4149e-08 \\
1.1985e-07 & 1.2038e-07 & 1.2459e-07 & -3.0126e-04 & 5.9473e-04 & -2.2592e-04 & 1.0312e-07 & 8.8115e-08 \\
1.0227e-07 & 1.0528e-07 & 1.1009e-07 & 1.1410e-07 & -2.2592e-04 & 4.2948e-04 & -1.5059e-04 & 9.1725e-08 \\
8.4508e-08 & 8.9743e-08 & 9.4930e-08 & 9.9319e-08 & 1.0312e-07 & -1.5059e-04 & 2.6422e-04 & -7.5252e-05 \\
6.5514e-08 & 7.6739e-08 & 8.0089e-08 & 8.4149e-08 & 8.8115e-08 & 9.1725e-08 & -7.5252e-05 & 9.8972e-05
\end{bmatrix} > 0$$

We can conclude that the CMPC system is stable to control this process under the constraint set on u, Δu over the M or $\hat{y}(k+11)$ or softened constraint set on $\hat{y}(k+7)$.

6 Similarity to A Contraction

The quadratic stability of CMPC implies a similarity to a contraction of the control system.

Theorem 4 *The close-loop system of CMPC is quadratic stable, then it is similar to a contraction.*

Proof is in Appendix A.

7 Conclusions

Appropriate stability analysis approach to Constrained Model Predictive Control (CMPC) has been accomplished by solving quadratic stability problem. Both analytic and numerical methods can work on the quadratic stability analysis. Since no efficient Riccati solver has been found for analytic method, trial-and-error method algorithm could be easy but conservative. A numerical method, Linear Matrix Inequality (LMI) technique, was found useful on solving the quadratic stability problem of CMPC. It would be less conservative and more efficient than the analytic method. Several examples show the feasibility of using both methods. Noticeably, the quadratic stability implies a similarity to a contraction.

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Appendix A: The Proofs of Theorems 1, 4 and Fact 1 in Sub-Section 3.2

(1) Proof of theorem 1

Basically, the quadratic stability is a Lyapunov stability. If we can find a Lyapunov function:

$$V(k) = x^T(k)\mathcal{P}x(k), \;\; \mathcal{P} > 0$$

with time invariant $\mathcal{P}$ for the closed-loop system:

$$x(k+1) = (\Phi + E_1F(k)E_2)x(k)$$

such that

$$\Delta V = V(k+1) - V(k) < 0$$

then the control system is quadratic stable. The detail derivatives can be followed:

$$\begin{aligned}
\Delta V &= V(k+1) - V(k) \\
&= x^T(k+1)\mathcal{P}x(k+1) - x^T(k)\mathcal{P}x(k) \\
&= x^T(k)(\Phi + E_1F(k)E_2)^T\mathcal{P}(\Phi + E_1F(k)E_2)x(k) - x^T(k)\mathcal{P}x(k) \\
&= x^T(k)\Phi^T\mathcal{P}\Phi x(k) + 2x^T(k)\Phi^T\mathcal{P}E_1F(k)E_2x(k) + x^T(k){E_2}^TF^T(k){E_1}^T\mathcal{P}E_1F(k)E_2x(k) \\
&\quad -x^T(k)\mathcal{P}x(k)
\end{aligned}$$

because

$$\begin{aligned}
&\| \sqrt{e}{E_1}^T\mathcal{P}\Phi x(k) - \frac{1}{\sqrt{e}}F(k)E_2x(k) \|^2 \\
&= ex^T(k)\Phi^T\mathcal{P}E_1{E_1}^T\mathcal{P}\phi x(k) - 2x^T(k)\Phi^T\mathcal{P}E_1F(k)E_2x(k) + \frac{1}{e}x^T(k){E_2}^TF^T(k)F(k)E_2x(k) \geq 0
\end{aligned}$$

where e is a given constant.

$$ex^T(k)\Phi^T\mathcal{P}E_1{E_1}^T\mathcal{P}\Phi x(k) + \frac{1}{e}x^T(k){E_2}^TF^T(k)F(k)E_2x(k) \geq 2x^T(k)\Phi^T\mathcal{P}E_1F(k)E_2x(k)$$

and

$$F^T(k)F(k) \leq I$$

such that

$$\frac{1}{e}x^T(k){E_2}^TF^T(k)F(k)E_2x(k) \leq \frac{1}{e}x^T(k){E_2}^TE_2x(k)$$

Hence,

$$\begin{aligned}
\Delta V \;\leq\; & x^T(k)\Phi^T\mathcal{P}\Phi x(k) + ex^T(k)\Phi^T\mathcal{P}E_1{E_1}^T\mathcal{P}\Phi x(k) + \frac{1}{e}x^T(k){E_2}^TE_2x(k) \\
& +x^T(k){E_2}^TF^T(k){E_1}^T\mathcal{P}E_1F(k)E_2x(k) - x^T(k)\mathcal{P}x(k)
\end{aligned}$$

Next, try to eliminate the uncertainty $F(k)$.

$$\begin{aligned}
&x^T(k)E_2{}^T F^T(k)E_1{}^T\mathcal{P}E_1F(k)E_2x(k)\\
&\leq x^T(k)E_2{}^T \parallel F^T(k)E_1{}^T\mathcal{P}E_1F(k) \parallel_2 E_2x(k)\\
&\leq x^T(k)E_2{}^T \parallel F^T(k) \parallel_2 \parallel E_1{}^T\mathcal{P}E_1 \parallel_2 \parallel F(k) \parallel_2 E_2x(k)\\
&\leq x^T(k)E_2{}^T \parallel E_1{}^T\mathcal{P}E_1 \parallel_2 E_2x(k)\\
&< x^T(k)E_2{}^T \parallel R + E_1{}^T\mathcal{P}E_1 \parallel_2 E_2x(k)
\end{aligned}$$

where $R > 0$. Then,

$$\Delta V < x^T(k)[\Phi^T\mathcal{P}\Phi + e\Phi^T\mathcal{P}E_1E_1{}^T\mathcal{P}\Phi + \frac{1}{e}E_2{}^TE_2 + \parallel R + E_1{}^T\mathcal{P}E_1 \parallel_2 E_2{}^TE_2 - \mathcal{P}]x(k)$$

and let

$$\mathcal{P} = \tilde{\mathcal{P}} \parallel R + E_1{}^T\mathcal{P}E_1 \parallel_2 = \tilde{\mathcal{P}}\sigma, \quad e = \frac{1}{\sigma}$$

where $\tilde{\mathcal{P}} > 0$. Hence,

$$\begin{aligned}
\Delta V &< \sigma x^T(k)[\Phi^T\tilde{\mathcal{P}}\Phi + \Phi^T\tilde{\mathcal{P}}E_1 \parallel \sigma^{-1}R + E_1{}^T\tilde{\mathcal{P}}E_1\parallel_2{}^{-1}E_1{}^T\tilde{\mathcal{P}}\Phi + 2E_2{}^TE_2 - \tilde{\mathcal{P}}]x(k)\\
&\leq \sigma x^T(k)[\Phi^T\tilde{\mathcal{P}}\Phi + \Phi^T\tilde{\mathcal{P}}E_1(\sigma^{-1}R + E_1{}^T\tilde{\mathcal{P}}E_1)^{-1}E_1{}^T\tilde{\mathcal{P}}\Phi + 2E_2{}^TE_2 - \tilde{\mathcal{P}}]x(k)
\end{aligned}$$

and let

$$\sigma^{-1}R \geq \tilde{R} > 0$$

then

$$\Delta V < \sigma x^T(k)[\Phi^T\tilde{\mathcal{P}}\Phi + \Phi^T\tilde{\mathcal{P}}E_1(\tilde{R} + E_1{}^T\tilde{\mathcal{P}}E_1)^{-1}E_1{}^T\tilde{\mathcal{P}}\Phi + 2E_2{}^TE_2 - \tilde{\mathcal{P}}]x(k)$$

Let

$$\Phi^T\tilde{\mathcal{P}}\Phi + \Phi^T\tilde{\mathcal{P}}E_1(\tilde{R} + E_1{}^T\tilde{\mathcal{P}}E_1)^{-1}E_1{}^T\tilde{\mathcal{P}}\Phi + 2E_2{}^TE_2 - \tilde{\mathcal{P}} = -Q < 0, \quad Q > 0$$

then

$$\Delta V < -\sigma x^T(k)Qx(k) < 0$$

such that the control system is quadratic stable, if there exist positive matrices, $\tilde{\mathcal{P}}, \mathcal{P}, Q, R, \tilde{R}$ such that $\Delta V < 0$, and $\sigma^{-1}R \geq \tilde{R} \leq I, \ \mathcal{P} = \tilde{\mathcal{P}} \parallel R + E_1{}^T\mathcal{P}E_1 \parallel_2$. □

(2) Proof of theorem 4

If the closed-loop control system:

$$x(k+1) = (\bar{\Psi} + E_1\Delta_F(k)E_2)x(k)$$

satisfy the quadratic stability condition, then let $\mathcal{C}^T\mathcal{C} = \mathcal{P}$; $\mathcal{C}$ is a nonsingular matrix, and $\bar{x}(k) = \mathcal{C}x(k)$. We have:

$$\begin{aligned}
& x(k)^T[(\bar{\Psi} + E_1\Delta_F(k)E_2)^T\mathcal{P}(\bar{\Psi} + E_1\Delta_F(k)E_2) - \mathcal{P}]x(k) < 0\\
\Rightarrow\ & x(k)^T[(\bar{\Psi} + E_1\Delta_F(k)E_2)^T\mathcal{C}^T\mathcal{C}(\bar{\Psi} + E_1\Delta_F(k)E_2) - \mathcal{C}^T\mathcal{C}]x(k) < 0\\
\Rightarrow\ & \parallel \mathcal{C}(\bar{\Psi} + E_1\Delta_F(k)E_2)x(k) \parallel_2 - \parallel \mathcal{C}x(k) \parallel_2 < 0\\
\Rightarrow\ & \parallel \mathcal{C}(\bar{\Psi} + E_1\Delta_F(k)E_2)\mathcal{C}^{-1}\bar{x}(k) \parallel_2 - \parallel \bar{x}(k) \parallel_2 < 0\\
\Rightarrow\ & \parallel \mathcal{C}(\bar{\Psi} + E_1\Delta_F(k)E_2)\mathcal{C}^{-1} \parallel_2 < 1
\end{aligned}$$

We can conclude that if the control system satisfy the quadratic stability condition, then the quadratic stability implies a similarity to a contraction. □

(3) Proof of fact 1 in sub-section 3.2

The inequality as following is obviously held:

$$\varpi(\hat{A}^T G^{-1}\hat{A})^{-1}\varpi^T \geq \varpi(\hat{A}^T G^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\varpi^T$$

We are going to prove:

$$(\bar{s}G^{-1}\bar{s}^T)^{-1} \geq \varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T$$

and the extraction matrix has the following property

$$\varpi^T\varpi = I$$

From the properties of positive definite matrix [Horn and Johnson, 1990], we know:

$$\begin{aligned}
& \varpi^T(\bar{s}G^{-1}\bar{s}^T)^{-1}\varpi \geq (\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1} > 0 \\
\Longrightarrow\ & \varpi^T(\bar{s}G^{-1}\bar{s}^T)^{-1}\varpi \geq \varpi^T\varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T\varpi > 0 \\
\Longrightarrow\ & \varpi^T[\,(\bar{s}G^{-1}\bar{s}^T)^{-1} - \varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T\,]\varpi \geq 0 \\
\Longrightarrow\ & (\bar{s}G^{-1}\bar{s}^T)^{-1} \geq \varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T
\end{aligned} \tag{24}$$

or

$$(\bar{s}G^{-1}\bar{s}^T)^{-1} > 0,\ \ (\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi) > 0$$

the inequality (24) implies

$$\begin{bmatrix} \varpi^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} (\bar{s}G^{-1}\bar{s}^T)^{-1} & \varpi \\ \varpi^T & (\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi) \end{bmatrix}\begin{bmatrix} \varpi & 0 \\ 0 & I \end{bmatrix} \geq 0$$

It also implies

$$\begin{bmatrix} (\bar{s}G^{-1}\bar{s}^T)^{-1} & \varpi \\ \varpi^T & (\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi) \end{bmatrix} \geq 0$$

Hence, $(\bar{s}G^{-1}\bar{s}^T)^{-1} \geq \varpi(\varpi^T\bar{s}G^{-1}\bar{s}^T\varpi)^{-1}\varpi^T$ □

Appendix B: The Proofs of Theorems 2, 3

(1) Proof of theorem 2
For the system:

$$\bar{x}(k+1) = (\Psi + E_1 F(k) E_2)\bar{x}(k)$$

Let $\bar{q}(k) = E_2\bar{x}(k)$ and $\bar{p}(k) = F(k)\bar{q}(k)$, then we can rewrite the system as:

$$\begin{aligned} \bar{x}(k+1) &= \Psi\bar{x}(k) + E_1\bar{p}(k) \\ \bar{q}(k) &= E_2\bar{x}(k) \\ \bar{p}(k) &= F(k)\bar{q}(k) \\ F(k) &\leq 1 \end{aligned} \tag{25}$$

If we can find a Lyapunov function:

$$V(k) = x^T(k)\tilde{\mathcal{P}}x(k), \quad \tilde{\mathcal{P}} > 0$$

with time invariant $\tilde{\mathcal{P}}$ for the closed-loop system (25) such that

$$\Delta V = V(k+1) - V(k) < 0$$

whenever

$$\bar{p}^T(k)\bar{p}(k) \leq \bar{q}^T(k)\bar{q}(k)$$

then the control system is quadratic stable. The detail derivatives is following:

$$\begin{aligned} \Delta V &= V(k+1) - V(k) \\ &= \bar{x}^T(k+1)\tilde{\mathcal{P}}\bar{x}(k+1) - \bar{x}^T(k)\tilde{\mathcal{P}}\bar{x}(k) \\ &= (\Psi\bar{x}(k) + E_1\bar{p}(k))^T\tilde{\mathcal{P}}(\Psi\bar{x}(k) + E_1\bar{p}(k)) - \bar{x}^T(k)\tilde{\mathcal{P}}\bar{x}(k) \\ &= \begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix}^T \begin{bmatrix} \Psi^T\tilde{\mathcal{P}}\Psi - \tilde{\mathcal{P}} & \Psi^T\tilde{\mathcal{P}}E_1 \\ E_1{}^T\tilde{\mathcal{P}}\Psi & E_1{}^T\tilde{\mathcal{P}}E_1 \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix} < 0 \end{aligned}$$

whenever

$$\begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix}^T \begin{bmatrix} -E_2{}^TE_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix} \leq 0$$

By $\mathcal{S}$ procedure, there exists a positive $\bar{\lambda}$ such that

$$\begin{bmatrix} \Psi^T\tilde{\mathcal{P}}\Psi - \tilde{\mathcal{P}} & \Psi^T\tilde{\mathcal{P}}E_1 \\ E_1{}^T\tilde{\mathcal{P}}\Psi & E_1{}^T\tilde{\mathcal{P}}E_1 \end{bmatrix} - \bar{\lambda}\begin{bmatrix} -E_2{}^TE_2 & 0 \\ 0 & I \end{bmatrix} \leq 0$$

Let $\tilde{\mathcal{P}}/\bar{\lambda} = \mathcal{P}$ then we have

$$\begin{bmatrix} \Psi^T\mathcal{P}\Psi - \mathcal{P} + E_2{}^TE_2 & \Psi^T\mathcal{P}E_1 \\ E_1{}^T\mathcal{P}\Psi & E_1{}^T\mathcal{P}E_1 - I \end{bmatrix} \leq 0$$

□

(2) Proof of theorem 3

In the proof of theorem 2, we know that

$$\begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix}^T \begin{bmatrix} \Psi^T \mathcal{P} \Psi - \mathcal{P} + E_2{}^T E_2 & \Psi^T \mathcal{P} E_1 \\ E_1{}^T \mathcal{P} \Psi & E_1{}^T \mathcal{P} E_1 - I \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix} \leq 0$$

This implies

$$\bar{x}^T(k+1)\mathcal{P}\bar{x}(k+1) - \bar{x}^T(k)\mathcal{P}\bar{x}(k) \leq \begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix}^T \begin{bmatrix} -E_2{}^T E_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{p}(k) \end{bmatrix}$$

Hence

$$\bar{x}^T(k+1)\mathcal{P}\bar{x}(k+1) - \bar{x}^T(k)\mathcal{P}\bar{x}(k) \leq \bar{p}^T(k)\bar{p}(k) - \bar{q}^T(k)\bar{q}(k)$$

and

$$\sum_{k=0}^{N_f-1} x^T(k+1)\mathcal{P}\bar{x}(k+1) - \sum_{k=0}^{N_f-1} \bar{x}^T(k)\mathcal{P}\bar{x}(k) \leq \sum_{k=0}^{N_f-1} \bar{p}^T(k)\bar{p}(k) - \sum_{k=0}^{N_f-1} \bar{q}^T(k)\bar{q}(k)$$

where N_f is the final sampling point. Because we assume zero initial state ($\bar{x}(0) = 0$), we can have

$$0 < \sum_{k=0}^{N_f-1} \bar{p}^T(k)\bar{p}(k) - \sum_{k=0}^{N_f-1} \bar{q}^T(k)\bar{q}(k)$$

$$\frac{\| \bar{q}(k) \|_2}{\| \bar{p}(k) \|_2} < 1$$

Therefore,

$$\| E_2(zI - \Psi)E_1 \|_\infty < 1$$

□

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