Receding Horizon Control of Nonlinear Systems

DAVID Q. MAYNE, FELLOW, IEEE, AND HANNAH MICHALSKA

Abstract—**The receding horizon control strategy provides a relatively simple method for determining feedback control for linear or nonlinear systems; the method is especially useful for the control of slow nonlinear systems, such as chemical batch processes, where it is possible to solve, sequentially, open-loop fixed-horizon, optimal control problems on line. The method has been shown to yield a stable closed-loop system when applied to time-invariant or time-varying linear systems. In this paper we show that the method also yields a stable closed-loop system when applied to nonlinear systems.**

I. Introduction

There exist many methods, including classical frequency-domain techniques, for designing stabilizing control laws for time-invariant linear systems. In contrast, there exist relatively few methods for time-varying linear systems, and fewer still for nonlinear systems. The major difficulty in the design of feedback control laws for nonlinear systems arises from the necessity to explore the whole state space. The problem of the design of feedback controls for nonlinear systems has found a general solution only in the case of systems which are feedback equivalent to linear systems. Since many types of nonlinear systems are not feedback equivalent to linear systems, alternative approaches to nonlinear design which do not require the construction of diffeomorphic state-feedback transformations should be developed. This motivated our study of receding horizon control—an optimal-control based method for the construction of stabilizing feedback control laws. The notion of the receding horizon control has been long known, but the stabilizing property of the receding horizon control law has so far been proven only for linear systems (cf. Kwon and Pearson [1] and Kwon, Bruckstein, and Kailath [2]).

If optimal control is employed, determination of the optimal feedback control law requires solution of a partial differential equation (the Hamilton–Jacobi-Bellman equation) in x and t (where x is the state and t is the time); this is generally difficult if the dimension of x is large. Determination of an optimal open-loop control, for a given initial state is, on the other hand, relatively simple, and this fact makes receding horizon control attractive in many situations. Additionally, in the case of linear systems, it is known that many types of feedback control laws (e.g., the well-known proportional navigation in the linear pursuer–evader problem) can be embedded into a suitably defined class of receding horizon control laws (cf. [12]).

In receding horizon control (cf. [1] and [2]), the current control at state x and time t is obtained by determining on-line (the open-loop) optimal control $\hat{u}$ over the interval $[t, t+T]$ and setting the current control equal to $\hat{u}(t)$. Repeating this calculation continuously yields a feedback control (since $\hat{u}(t)$ clearly depends on the current state x). The optimal control problem, $P(x, t)$, is usually posed as minimizing a quadratic function (over the interval $[t, t+T]$) subject to the terminal constraint $x(t+T) = 0$ and solved on line. For a linear time-varying system defined by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1}$$

where $u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$ for all t, and receding horizon cost

$$V(x, t; u) \triangleq \frac{1}{2}\int_t^{t+T} [x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)]\, d\tau, \tag{1.2}$$

the optimizing control can be determined as follows. For given μ in $\mathbb{R}^n$, determine the feedback control u^μ (if it exists) which minimizes $V(x, t; u) + \mu^T x^u(t+T; x, t)$ (where $x^u(\cdot\,; x, t)$ denotes the solution of (1.1) due to control u and initial condition $x(t) = x$). This control generates a corresponding trajectory $x^\mu(\cdot\,; x, t)$; if (1.1) is uniformly controllable, the map $\mu \mapsto x^\mu(t+T; x, t)$ is bijective (for all x and t) and μ can be chosen so that $x^\mu(t+T; x, t) = 0$. The corresponding control u^μ solves $P(x, t)$ (i.e., minimizes $V(x, t; u)$ subject to the terminal constraint $x^u(t+T; x, t) = 0$). The existence of a control minimizing (1.2) (for each μ) is ensured if the corresponding Riccati equation has no conjugate points. These conditions (the controllability condition and the absence of conjugate points in the Riccati equation) also occur in the analysis of nonlinear systems.

In this paper we consider the application of receding horizon control to the time-invariant nonlinear system denoted by

$$\dot{x}(t) = f(x(t), u(t)) \tag{1.3}$$

where $(x, u) \mapsto f(x, u): \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ satisfies $f(0, 0) = 0$. The receding horizon cost is, as before, defined by (1.2), and the receding horizon control at (x, t) is the value at t of the control which solves $P(x, t)$. Because the problem is time invariant, we can, without loss of generality, assume that $t = 0$. Clearly, the value at t of the control u which solves $P(x, t)$ is equal to the value at 0 of the control which solves $P(x, 0)$.

We show that, under reasonable conditions, the receding horizon controller stabilizes system (1.3). Our result should serve as a first basis for future research on properties and construction of implementable feedback laws for nonlinear systems by using optimal control methods.

The less-important proofs can be found in the Appendix.

II. Assumptions and Preliminary Results

We shall use the symbol $\|\cdot\|$ to denote any vector norm in $\mathbb{R}^n$ (where the dimension n will follow from the context). The same symbol will also be used to denote a matrix norm induced by a given vector norm in $\mathbb{R}^n$. $\mathbb{L}_\infty^m[0, T]$ will denote the space of all Lebesgue measurable and essentially bounded functions $f:[0, T] \mapsto \mathbb{R}^m$ with the norm defined in the usual way, i.e., $\|f\|_\infty \triangleq \operatorname{ess\,sup}_{t\in[0,T]} \|f(t)\|$. $\mathbb{C}^n[0, T]$ will denote the space of all continuous functions $f:[0, T] \mapsto \mathbb{R}^n$ with the norm denoted by the same symbol $\|f\|_\infty$ but meaning $\max_{t\in[0,T]} \|f(t)\|$. The symbol $B(x; \rho)$ ($\bar{B}(x; \rho)$) will denote an open (closed) ball in $\mathbb{R}^n$ with center x and radius ρ. We shall use the notation $B_\infty(x; \rho)$ ($\bar{B}_\infty(x; \rho)$ for balls in $\mathbb{L}_\infty^m[0, T]$ or in $\mathbb{C}^n[0, T]$.

For any matrix A, the expression $A > 0$ ($A \geq 0$) will mean that A is positive definite (positive semidefinite). For Hermitian matrices, the expression $A \geq \alpha I$ will stand for $\lambda_{\min}(A) \geq \alpha$ and $A \leq \alpha I$ for $\lambda_{\max}(A) \leq \alpha$, where I is the identity matrix and $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote, respectively, the smallest and largest eigenvalues of the matrix A.

For each x_0 in $\mathbb{R}^n$, and any control $u \in \mathbb{L}_\infty^m[0, T]$ let $x(\cdot\,; x_0, 0)$ denote the corresponding trajectory of (1.3) with initial condition $x(0) = x_0$. For each x_0 in $\mathbb{R}^n$, let the optimal solu-

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The authors are with the Department of Electrical Engineering, Imperial College of Science and Technology, Exhibition Road, London SW7 2BT, U.K.

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tion to $P(x_0, 0)$ in the space $\mathbb{L}_\infty^m[0, T]$ be denoted by $\hat{u}(\cdot; x_0, 0)$ (i.e., $\hat{u}$ minimizes $V(x_0, 0; u)$ subject to $x(T; x_0, 0) = 0$) and the associated trajectory by $\hat{x}(\cdot; x_0, 0)$. We shall also use brief notation $x(\cdot)$, $\hat{x}(\cdot)$, and $\hat{u}(\cdot)$.

Our initial assumptions are as follows.

A1: $(x, u) \mapsto f(x, u): \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is two times continuously differentiable and $f(0, 0) = 0$.

A2: $Q > 0$ and $R > 0$.

A3: For any $\rho > 0$, there exists an $M(\rho) \in (0, \infty)$ such that

$$\|f(x, u)\| \leq M(\rho)(1 + \|x\|) \qquad \text{for all } x \in \mathbb{R}^n \quad \text{and all } u \in \bar{B}(0; \rho).$$

A4: For each x_0 in $\mathbb{R}^n$, a solution $\hat{u} \in \mathbb{L}_\infty^m[0, T]$ to $P(x_0, 0)$ exists and is unique (in the sense of equivalent classes in $\mathbb{L}_\infty^m[0, T]$, i.e., two optimal controls, $\hat{u}_1 \in \mathbb{L}_\infty^m[0, T]$ and $\hat{u}_2 \in \mathbb{L}_\infty^m[0, T]$, differ only on a set $\Omega \subset [0, T]$ of measure zero). Additionally, for any $\rho \in (0, \infty)$, there exists an $N(\rho) \in (0, \infty)$ such that for any initial state $x_0 \in \bar{B}(0; \rho)$, the optimal control $\hat{u}$, steering the system to $x(T; x_0, 0) = 0$, satisfies $\hat{u} \in \bar{B}_\infty(0; N(\rho))$.

A5: (A constraint qualification.) For any x_0 and corresponding optimal trajectories $\hat{u}$ and $\hat{x}$, the linearized system (1.3)

$$\delta\dot{x}(t) = f_x(\hat{x}(t), \hat{u}(t))\delta x(t) + f_u(\hat{x}(t), \hat{u}(t))\delta u(t) \qquad \text{for } t \in [0, T] \tag{2.1}$$

is completely uniformly controllable in that:

i) it is completely controllable (i.e., can be driven from the origin to any point in $\mathbb{R}^n$ using a control in $\mathbb{L}_\infty^m[0, T]$); and furthermore

ii) there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ (independent of x_0) such that

$$\alpha_1 I \leq W(0, T) \leq \alpha_2 I$$

where W denotes the controllability Grammian of (2.1); and

iii) there exists a function $\alpha_3: \mathbb{R} \mapsto \mathbb{R}$, independent of x_0, and bounded on bounded intervals, such that

$$\|\Phi(\tau, t)\| \leq \alpha_3(|t - \tau|) \qquad \text{for all } \tau,\ t \in [0, T],$$

where Φ is the transition matrix for the linearized system (2.1).

To facilitate proof of stability for the receding-horizon strategy using the Lyapunov approach, continuous differentiability of the value function will be established. Differentiability will be studied by analyzing the necessary conditions of optimality provided by the Minimum Principle for problem $P(x_0, 0)$, as follows.

Proposition 1 (See [3, Theorems 12.1 and 13.1, pp. 84 and 101] for a more general case): Suppose that Assumptions A1–A5 are satisfied and that $\hat{u}$ is optimal for $P(x_0, 0)$ and $\hat{x}$ is the corresponding optimal trajectory. Then there exists an absolutely continuous function $t \mapsto \hat{\lambda}(t): [0, T] \mapsto \mathbb{R}^n$ such that

$$-\dot{\hat{\lambda}}(t) = \nabla_x H(\hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) \qquad \text{a.e. on } [0, T] \tag{2.2}$$

$$\min_{u \in \mathbb{R}^m} H(\hat{x}(t), u, \hat{\lambda}(t)) = H(\hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) \qquad \text{a.e. on } [0, T] \tag{2.3}$$

where $(x, u, \lambda) \mapsto H(x, u, \lambda): \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$ is defined by

$$H(x, u, \lambda) \triangleq \frac{1}{2}[x^T Q x + u^T R u] + \lambda^T f(x, u). \tag{2.4}$$

The following is also valid. ■

Proposition 2: Suppose A1–A5 are satisfied. Then for each x_0 in $\mathbb{R}^n$, the multiplier $\hat{\lambda}$ corresponding to the solution $(\hat{x}, \hat{u})$ of $P(x_0, 0)$ is unique. ■

This result follows from Assumption A6 which ensures that the constraint set for problem $P(x_0, 0)$ (i.e., the system equation (1.3) and boundary condition $x(T; x_0, 0) = 0$) is regular at every optimal $(\hat{x}, \hat{u}) \in \mathbb{C}^n[0, T] \times \mathbb{L}_\infty^m[0, T]$. A proof of this result, stated in one form or other, is available from the literature or can be found in [9].

Comment 1: Assumption A3 guarantees that for any x_0 and any control $u \in \mathbb{L}_\infty^m[0, T]$, a unique, absolutely continuous solution, $x \in \mathbb{C}^n[0, T]$, of (1.3) exists. By a straightforward application of the Bellman–Gronwall Lemma, it is easy to show that for every ρ_0 and ρ_1, there exists a ρ_2 such that $x_0 \in \bar{B}(0; \rho_0)$ and $u \in \bar{B}_\infty(0; \rho_1)$ imply that $x \in \bar{B}_\infty(0; \rho_2)$. Assumption A3 can be substituted by any other assumption ensuring the above.

Let us now make the following additional assumptions.

A6: For any initial state x_0 and a corresponding optimal state-control-costate triple $(\hat{x}, \hat{u}, \hat{\lambda})$, there exists an $\epsilon > 0$ such that the set $\arg\min\{H(x, u, \lambda) | u \in \mathbb{R}^m\}$ is a singleton, $\{u(x, \lambda)\}$, for any $(x, \lambda) \in T((\hat{x}, \hat{\lambda}); \epsilon)$; where the set $T((\hat{x}, \hat{\lambda}); \epsilon)$ is an "ϵ-tube" defined by

$$T((\hat{x}, \hat{\lambda}); \epsilon) \triangleq \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n | \exists\, t \in [0, T], \quad (x, \lambda) \in B((\hat{x}(t), \hat{\lambda}(t)); \epsilon)\}; \tag{2.5}$$

and additionally, there exists a constant $M(\epsilon, x_0)$ such that

$$\|u(x, \lambda)\| \leq M(\epsilon, x_0) \qquad \text{for all } (x, \lambda) \in T((\hat{x}, \hat{\lambda}); \epsilon). \tag{2.6}$$

A7: There exists a $\delta > 0$ such that for any initial state x_0 and a corresponding optimal state-control-costate triple $(\hat{x}, \hat{u}, \hat{\lambda})$

$$\nabla_{uu} H(\hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) \geq \delta I \qquad \text{a.e. on } [0, T]. \tag{2.7}$$

Comment 2: From Assumption A7, it follows that "feedback control function" $(x, \lambda) \mapsto u(x, \lambda): T((\hat{x}, \hat{\lambda}); \epsilon) \mapsto \mathbb{R}^m$ uniquely describes the point

$$H(x, u(x, \lambda), \lambda) = \min_{u \in \mathbb{R}^m} H(x, u, \lambda) \tag{2.8}$$

and is continuous in its region of definition $T((\hat{x}, \hat{\lambda}); \epsilon)$. (The latter because $H(\cdot, \cdot, \cdot)$ is continuous, the set

$$\arg\min_{u \in \mathbb{R}^m} H(x, u, \lambda) = \arg\min_{u \in \bar{B}(0;\, M(\epsilon, x_0))} H(x, u, \lambda) \tag{2.9}$$

is a singleton, and the ball $\bar{B}(0; M(\epsilon, x_0))$ is compact in $\mathbb{R}^m$.) Hence, $u(\hat{x}(\cdot), \hat{\lambda}(\cdot))$ is continuous in $[0, T]$.

Comment 3: Suppose that Assumptions A1–A7 are satisfied. Then, by Proposition 1, $\hat{u}$ satisfies

$$H(\hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) = \min_{u \in \mathbb{R}^m} H(\hat{x}(t), u, \hat{\lambda}(t)) \qquad \text{a.e. on } [0, T] \tag{2.10}$$

and hence,

$$\hat{u}(t) \in \arg\min_{u \in \mathbb{R}^m} H(\hat{x}(t), u, \hat{\lambda}(t)) \qquad \text{a.e. on } [0, T]. \tag{2.11}$$

Since the set

$$\arg\min_{u \in \mathbb{R}^m} H(\hat{x}(t), u, \hat{\lambda}(t))$$

is a singleton, we must have

$$\hat{u}(t) = u(\hat{x}(t), \hat{\lambda}(t)) \qquad \text{a.e. on } [0, T], \tag{2.12}$$

by uniqueness of solutions of $P(x_0, 0)$. Hence, $\hat{u}$ generates the same trajectory as u which is continuous with respect to t. This means that the optimal control $\hat{u}$ can be considered continuous.

Comment 4: Provided that Assumptions A1–A7 are satisfied, an argument similar to that presented in Comment 1 applies to the costate equation (2.2) treated as an initial value problem (i.e., for a given initial value of the costate vector). Since all the functions included in the Hamiltonian are twice continuously differentiable, we have that, whenever $u \in \bar{B}(0; \rho) \subset \mathbb{R}^m$ and $x \in \bar{B}(0; \rho) \subset \mathbb{R}^n$, there exists a constant $M(\rho)$ such that

$$\|\nabla_x H(x, u, \lambda)\| = \|Qx + f_x(x, u)^T \lambda\| \leq \|Q\|\rho + L(\rho)\|\lambda\| \leq M(\rho)(1 + \|\lambda\|) \quad (2.13)$$

where $L(\rho)$ is an upper bound for $\|f_x(x, u)^T\|$ on $\bar{B}(0; \rho) \times \bar{B}(0; \rho) \subset \mathbb{R}^n \times \mathbb{R}^m$. Inequality (2.13) implies that the right-hand side of the costate equation satisfies a condition similar to that of A3, where x and u are both treated as "inputs." Hence, for any x_0 and $\lambda(0)$, and any control $u \in \mathbb{L}_\infty^m[0, T]$ with corresponding trajectory $x \in \mathbb{C}^n[0, T]$ of (1.3), a unique, absolutely continuous solution of the costate equation, $\lambda \in \mathbb{C}^n[0, T]$, exists. Also, for every ρ_0 and ρ_1, there exists a ρ_2 such that if $(x_0, \lambda(0)) \in \bar{B}(0; \rho_0) \times \bar{B}(0; \rho_0)$ and $u \in \bar{B}_\infty(0; \rho_1)$ then $\lambda \in \bar{B}_\infty(0; \rho_2)$.

The following auxiliary results will be used for proving continuous differentiability of the value function.

Proposition 3: Suppose Assumptions A1–A7 are satisfied, and let x_0 be any initial state and $(\hat{x}, \hat{u}, \hat{\lambda})$ the corresponding state-control-costate, optimal triple. Then there exists an $\epsilon' \in (0, \epsilon)$ such that $u(x, \lambda)$ is of class $\mathbb{C}^1$ (continuously differentiable) on $T((\hat{x}, \hat{\lambda}); \epsilon')$ where ϵ and $u(\cdot, \cdot)$ are defined in Assumption A6 and Comment 2. ∎

We now introduce our final assumption.

A8: For any initial state x_0 and a corresponding optimal state-control-costate triple $(\hat{x}, \hat{u}, \hat{\lambda})$, Assumption A7 holds together with

$$[\nabla_{xx}H - (\nabla_{xu}H)(\nabla_{uu}H)^{-1}(\nabla_{ux}H)](\hat{x}(t), \hat{u}(t), \hat{\lambda}(t)) \geq 0 \quad \text{a.e. on } [0, T]. \quad (2.15)$$

Comment 5: Since continuous differentiability of the optimal value function is a very strong requirement, sufficient conditions for it would be expected to be fairly strict. Hence, it is not surprising that it is difficult to provide easily verifiable conditions for nonlinear systems. Indeed, sufficient conditions for continuous differentiability of the optimal value function for the optimal control problem here considered have never been given before. This difficulty is reflected in Assumptions A4, A5, A7, and A8 which, in the case of a general nonlinear system, may often be difficult to verify directly. However, they are almost trivially satisfied by controllable linear systems, and hence should hold for a large class of systems which may be regarded as "nearly linear." Assumption A6 and Proposition 3 are redundant if equation (2.3) is analytically solvable with respect to u, which is the case for a large class of systems of the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)) \quad (2.14)$$

where u_i denotes the ith component of the control vector u.

As we shall see later, the advantage of having specified Assumptions A1–A8 is not only that these ensure continuous differentiability of the optimal value functions but that they also ensure continuity of the receding horizon control law. The latter is much more important because the behavior of solutions of nonlinear differential equations with only piecewise continuous right-hand sides is known to be complex. The application of the methods available for the analysis of such equations leads to difficulties when the solution encounters a line (or surface) of discontinuity of the right-hand side of the differential equation infinitely often.

Comment 6: Since $\hat{u}$ can be considered continuous (see Comment 3), the conditions (2.7) and (2.15) of A7 and A8 are valid for all $t \in [0, T]$. (The latter follows directly from the extension principle for continuous mappings since the set of points for which (2.7) and (2.15) are satisfied is dense in $[0, T]$.)

As is well known (e.g., see [4]), a consequence of A8 is that the matrix Riccati equation arising from a second variation analysis of $P(x_0, 0)$ about an optimal trajectory $(\hat{x}, \hat{u})$ has no conjugate points in $[0, T]$. This permits a perturbational analysis of (1.3), (2.2), and (2.3) which leads to Proposition 4.

Proposition 4: Let assumptions A1–A8 be satisfied. Let the initial state vector x_0 be arbitrary, and let $(\hat{x}, \hat{\lambda})$ be the corresponding optimal state and costate trajectories for $P(x_0, 0)$. Further, let $(x(\cdot; v, 0), \lambda(\cdot; v, 0))$ denote the solution of the "state-costate" system

$$\dot{x} = f(x, u(x, \lambda)) \quad (2.16)$$

$$-\dot{\lambda} = \nabla_x H(x, u(x, \lambda), \lambda), \quad (2.17)$$

with $u(x, \lambda)$ defined, and of class $\mathbb{C}^1$ in some $T((\hat{x}, \hat{\lambda}); \epsilon)$ (see Assumption A6) and Proposition 3), and initial conditions

$$x(0) = v_x \quad \text{and } \lambda(0) = v_\lambda. \quad (2.18)$$

Here $v \triangleq (v_x, v_\lambda) \in B((x_0, \hat{\lambda}(0)); \delta)$ where δ is supposed to be sufficiently small for the solution x and λ to lie in $T((\hat{x}, \hat{\lambda}); \epsilon)$. Under these conditions, the partial derivative $x_{\lambda 0}((x_0, \hat{\lambda}(0))(T)$ [which is the derivative of the partial mapping $v_{\lambda 0} \mapsto x(T; v, 0): B(\hat{\lambda}(0); \delta) \mapsto \mathbb{R}^n$ evaluated at the point $(x_0, \hat{\lambda}(0))$] is a nonsingular matrix. ∎

Proposition 5: Suppose Assumptions A1–A8 are satisfied. Let $\hat{x}(\cdot; x_0, 0)$, $\hat{\lambda}(\cdot; x_0, 0)$ denote the optimal state-costate vectors for the initial state x_0. Under these conditions, the mappings

$$x_0 \mapsto \hat{x}(\,\cdot\,; x_0, 0): \mathbb{R}^n \mapsto \mathbb{C}^n[0, T]$$

$$x_0 \mapsto \hat{\lambda}(\,\cdot\,; x_0, 0): \mathbb{R}^n \mapsto \mathbb{C}^n[0, T] \quad (2.19)$$

are continuously differentiable with respect to the initial condition x_0. The derivatives, $\hat{x}_{x0}$ and $\hat{\lambda}_{x0}$, of these mappings are continuous. (The derivatives are continuous as mappings $x_0 \mapsto \hat{x}_{x0}(x_0): \mathbb{R}^n \mapsto \mathcal{L}[\mathbb{R}^n; \mathbb{C}^n[0, T]]$ and $x_0 \mapsto \hat{\lambda}_{x0}(x_0): \mathbb{R}^n \mapsto \mathcal{L}[\mathbb{R}^n; \mathbb{C}^n[0, T]]$, where $\mathcal{L}$ is the space of all linear and bounded operators acting from $\mathbb{R}^n$ into $\mathbb{C}^n[0, T]$ with the induced operator norm $\|\cdot\|_{\mathcal{L}}$). ∎

III. Differentiability Properties of the Value Function for Problem $P(x_0, 0)$

The results of Section II will now be used to establish the required differentiability of the value function and also of the feedback for the receding horizon strategy.

Proposition 6: Under the assumptions of Proposition 5, the optimal control $\hat{u}(\cdot; x_0, 0) \triangleq u(\hat{x}(\cdot; x_0, 0), \hat{\lambda}(\cdot; x_0, 0))$ is continuously differentiable with respect to x_0. The derivative of the mapping $x_0 \mapsto \hat{u}(\cdot; x_0, 0): \mathbb{R}^n \mapsto \mathbb{C}^m[0, T]$ with respect to x_0, denoted by $\hat{u}_{x0}$, is continuous in x_0. ($\hat{u}_{x0}$ is continuous as a mapping $x_0 \mapsto \hat{u}_{x0}(x_0): \mathbb{R}^n \mapsto \mathcal{L}[\mathbb{R}^n; \mathbb{C}^m[0, T]]$ where $\mathcal{L}$ is the space of all linear and bounded operators acting from $\mathbb{R}^n$ into $\mathbb{C}^m[0, T]$ with the induced operator norm.) ∎

Now let us define precisely the receding horizon strategy. The control which solves $P(x, t)$ is denoted $\hat{u}(\cdot; x, t)$, and the corresponding state trajectory is $\hat{x}(\cdot; x, t)$. The receding horizon strategy yields a stationary control law $h^*: \mathbb{R}^n \mapsto \mathbb{R}^m$ [i.e., the control action at state x is $h^*(x)$]. Clearly, h^* satisfies

$$h^*(x) = \hat{u}(0; x, 0) \quad (= \hat{u}(t; x, t) \text{ for any } t) \quad \text{for all } x \in \mathbb{R}^n. \quad (3.1)$$

From Proposition 6 follows Corollary 1.

Corollary 1: If Assumptions A1–A8 are satisfied, then the receding horizon policy h^* is a continuously differentiable function of the current state x of the controlled system. ∎

We can now establish that the optimal value function (for $P(x, t)$),

$$x \mapsto \hat{V}(x) \triangleq \inf\{V(x, 0; u) | u \in \mathbb{L}_\infty^m[0, T], \quad x(T; x, 0) = 0\} : \mathbb{R}^n \mapsto \mathbb{R}, \quad (3.2)$$

is sufficiently smooth to justify its use as a Lyapunov function for establishing the stability of the closed-loop system.

Proposition 7: Suppose Assumptions A1–A8 are satisfied. Then the optimal value function $\hat{V}: \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable. ■

We are now in a position to prove that the receding horizon strategy, when applied to a nonlinear system, yields a stable closed-loop system.

IV. Closed-Loop Stability

Theorem 1: Suppose Assumptions A1–A8 are satisfied. Then the closed-loop system $\dot{x} = f(x, h^*(x))$, using the receding horizon strategy, is asymptotically stable (there exists a ball $B(0; \rho) \in \mathbb{R}^n$, $\rho > 0$, such that for any initial condition $x_0 \in B(0; \rho)$, the solution of the closed-loop system equation tends to zero as $t \to \infty$). ■

Proof: We have established (Proposition 7) that $\hat{V}$ is continuously differentiable. Clearly, $\hat{V}(0) = 0$ (since $\hat{u}(0; 0, 0) = 0$). Since Q is positive definite, it follows that $\hat{V}(x) > 0$ for all $x \neq 0$. Hence, $\hat{V}$ is a Lyapunov function.

Let x^* and u^* denote the state and control resulting from the application of the receding horizon strategy when the initial state is x_0 at $t = 0$ [i.e., x^* is the solution of $\dot{x} = f(x, h^*(x))$ with initial condition $x(0) = x_0$ and $u^*(t) = h^*(x^*(t))$]. We wish to evaluate $(d/dt)\hat{V}(x^*(t))$ at an arbitrary, fixed instant of time. By definition,

$$\hat{V}(x^*(t)) = \frac{1}{2}\int_t^{t+\Delta t} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\, d\tau + \frac{1}{2}\int_{t+\Delta t}^{t+T} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\, d\tau \quad (4.1)$$

where $\hat{x}(\tau) \triangleq \hat{x}(\tau; x^*(t), t)$ and $\hat{u}(\tau) \triangleq \hat{u}(\tau; x^*(t), t)$ are optimal for $P(x^*(t), t)$. Consider a control $\tilde{u}: [t + \Delta t, t + T + \Delta t] \mapsto \mathbb{R}^m$ defined as follows:

$$\tilde{u}(\tau) \triangleq \begin{cases} \hat{u}(\tau; x^*(t), t) & \text{for } \tau \in [t + \Delta t, t + T] \\ 0 & \text{for } \tau \in (t + T, t + T + \Delta t]. \end{cases} \quad (4.2)$$

Let $\tilde{x}(\cdot) = \tilde{x}(\cdot : \hat{x}(t+\Delta t), t+\Delta t) : [t+\Delta t, t+T+\Delta t] \mapsto \mathbb{R}^n$ denote the corresponding trajectory of (1.3), with initial condition $\tilde{x}(t + \Delta t) = \hat{x}(t + \Delta t; x^*(t), t)$. Clearly,

$$\tilde{x}(\tau) = \begin{cases} \hat{x}(\tau; x^*(t), t) & \text{for } \tau \in [t + \Delta t, t + T] \\ 0 & \text{for } \tau \in (t + T, t + T + \Delta t) \end{cases} \quad (4.3)$$

because $\tilde{x}(t + T; \hat{x}(t + \Delta t), t + \Delta t) = \hat{x}(t + T; x^*(t), t) = 0$ and $\tilde{u} = 0$ for $\tau > t + T$. Since $\tilde{u}$ is not necessarily optimal for $P(\hat{x}(t + \Delta t), t + \Delta t)$, it follows that

$$\begin{aligned} \hat{V}(x^*(t)) &= \frac{1}{2}\int_t^{t+\Delta t} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\, d\tau + V(\hat{x}(t + \Delta t), t + \Delta t; \tilde{u}) \\ &\geq \frac{1}{2}\int_t^{t+\Delta t} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\, d\tau + \hat{V}(\hat{x}(t + \Delta t)) \end{aligned} \quad (4.4)$$

so that

$$\hat{V}(\hat{x}(t + \Delta t)) - \hat{V}(x^*(t)) \leq -\frac{1}{2}\int_t^{t+\Delta t} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\, d\tau. \quad (4.5)$$

Since $\hat{V}$ is continuously differentiable, it follows from the Mean Value Theorem that

$$\frac{\hat{V}(\hat{x}(t + \Delta t)) - \hat{V}(x^*(t))}{\Delta t} = \nabla_x \hat{V}(x^*(t) + \theta(\Delta t)(\hat{x}(t + \Delta t) - x^*(t)))\frac{(\hat{x}(t + \Delta t) - x^*(t))}{\Delta t} \quad (4.6)$$

for $\theta(\Delta t) \in (0, 1)$. Since

$$x^*(t) = \hat{x}(t; x^*(t), t) = \hat{x}(t), \qquad u^*(t) = \hat{u}(t; x^*(t), t) = \hat{u}(t) \quad (4.7)$$

and $\hat{u}$ is continuous at t, we have that

$$\lim_{\Delta t \to 0_+} \frac{\hat{x}(t + \Delta t) - x^*(t)}{\Delta t} = f(\hat{x}(t), \hat{u}(t)) = f(x^*(t), u^*(t)). \quad (4.8)$$

Since $\nabla_x \hat{V}$ and $\hat{x}$ are continuous, then from (4.6), (4.7), and (4.8), it follows that

$$\lim_{\Delta t \to 0_+} \frac{\hat{V}(\hat{x}(t + \Delta t)) - \hat{V}(x^*(t))}{\Delta t} = \nabla_x \hat{V}(x^*(t))f(x^*(t), u^*(t)). \quad (4.9)$$

By continuity of $\hat{x}$ and $\hat{u}$ and the Mean Value Theorem for integrals, we also have

$$\begin{aligned} \lim_{\Delta t \to 0_+} -\frac{1}{2\Delta t}\int_t^{t+\Delta t} [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\, d\tau &\leq \lim_{\Delta t \to 0_+} -\frac{1}{2\Delta t}\int_t^{t+\Delta t} [\hat{x}^T(\tau)Q\hat{x}(\tau)]\, d\tau \\ &= -\frac{1}{2}[x^*(t)^T Q x^*(t)]. \end{aligned} \quad (4.10)$$

Dividing both sides of (4.5) by $\Delta t > 0$ and taking the limit as $\Delta t \to 0+$ yields

$$\nabla_x \hat{V}(x^*(t))f(x^*(t), u^*(t)) \leq -\frac{1}{2}[x^*(t)^T Q x^*(t)] < 0. \quad (4.11)$$

Hence,

$$(d/dt)\hat{V}(x^*(t)) = \nabla_x \hat{V}(x^*(t))f(x^*(t), u^*(t)) < 0 \quad (4.12)$$

unless $x^*(t) = 0$. Hence, the system $\dot{x} = f(x, h^*(x))$ is asymptotically stable (see [8, Theorem 4.5, p. 137]). □

The above theorem guarantees only (local) asymptotic stability. We shall prove asymptotic stability in the large under the following additional assumption.

A9: The rate of growth of $f(x, u)$ as $\|(x, u)\| \to \infty$ is not faster than quadratic. More precisely, there exists a constant K and a radius $r > 0$ such that

$$\|f(x, u)\| \leq K(\|x\|^2 + \|u\|^2) \qquad \text{for every } (x, u) \notin B(0; r). \quad (4.13)$$

Theorem 2: Suppose the assumptions of Theorem 1 hold and, additionally, Assumption A9 is satisfied. Then the closed-loop system is asymptotically stable in the large (i.e., for every initial condition x_0, the solution of the closed-loop system equation tends to zero as $t \to \infty$). ■

Proof: By the Barbashin–Krassowskii Theorem (see [8, Theorem 4.7, p. 137]), it is enough to prove that $\hat{V}(x) \to \infty$ as $\|x\| \to \infty$, i.e., that $\hat{V}$ is radially unbounded. First, we notice that, by A9, if x is the solution of (1.3) satisfying boundary conditions $x(0) = x_0$ and $x(T) = 0$, then

$$\begin{aligned}\|x_0\| &= \left\|\int_0^T \dot{x}(\tau)\,d\tau\right\| = \left\|\int_0^T f(x(\tau), u(\tau))\,d\tau\right\| \\ &\le \int_0^T \|f(x(\tau), u(\tau))\|\,d\tau \\ &\le \int_{I_B} \|f(x(\tau), u(\tau))\|\,d\tau + \int_{I_B^C} \|f(x(\tau), u(\tau))\|\,d\tau \end{aligned} \tag{4.14}$$

where I_B is the subset of $[0, T]$ such that $(x, u) \in \bar{B}(0; r)$ and I_B^C is the subset of $[0, T]$ such that $(x, u) \notin B(0; r)$. Since f is continuous, we have the following bound:

$$\sup_{t \in I_B} \|f(x(t), u(t))\| \le \max_{(x,u)\in\bar{B}(0;r)} \|f(x, u)\| = M_f. \tag{4.15}$$

From (4.14), it now follows that

$$\begin{aligned}\|x_0\| &\le M_f \mu(I_B) + K \int_{I_B^C} (\|x(\tau)\|^2 + \|u(\tau)\|^2)\,d\tau \\ &\le M_f \mu(I_B) + K \int_0^T (\|x(\tau)\|^2 + \|u(\tau)\|^2)\,d\tau \end{aligned} \tag{4.16}$$

where $\mu(I_B) \le T$ is the measure of the set I_B. From the latter, it follows that

$$\int_0^T (\|x(\tau)\|^2 + \|u(\tau)\|^2)\,d\tau \to \infty \qquad \text{as } \|x_0\| \to \infty \tag{4.17}$$

where x is a solution of (1.3) with boundary conditions $x(0) = x_0$ and $x(T) = 0$ and any input u that realizes the satisfaction of these.

Without loss of generality, we may assume that the matrices Q and R in the definition of the performance index V are symmetric. Then

$$\begin{aligned}\hat{V}(x_0) &= \frac{1}{2}\int_0^T [\hat{x}^T(\tau)Q\hat{x}(\tau) + \hat{u}^T(\tau)R\hat{u}(\tau)]\,d\tau \\ &\ge \frac{1}{2}[\lambda_{\min}(Q) + \lambda_{\min}(R)] \int_0^T (\|\hat{x}(\tau)\|_2^2 + \|\hat{u}(\tau)\|_2^2)\,d\tau \end{aligned} \tag{4.18}$$

where $\hat{u}$ and $\hat{x}$ are optimal and $\|\cdot\|_2$ denotes the Euclidean vector norm. Since $\lambda_{\min}(Q)$ and $\lambda_{\min}(R)$ are positive, from the above and (4.17), it follows immediately that $\hat{V}(x_0) \to \infty$ as $\|x_0\| \to \infty$ and the proof is done. □

The receding horizon control strategy, as defined in (3.1), is not directly implementable because it involves the exact open-loop optimal control for the nonlinear system (over the interval $[t, t+T]$) being recomputed at every current value of the state (i.e., infinitely often). An implementable strategy should yield stabilization despite i) using an approximation to the optimal control on $[t, t+T]$, and ii) the optimal open-loop control being updated only discretely in time, say at the beginning of each time interval of length Δ_D (usually $\Delta_D \ll T$). Two results are presented below which show how to deal with these problems separately. A rigorous theory for a complete solution to both problems simultaneously is under development. The examples of Section V show that a heuristic implementable algorithm does actually yield stabilization and indeed robust stabilization.

We may define a "discrete-time" receding horizon strategy as follows. Let $u_D^*(\cdot)$ and $x_D^*(\cdot)$ denote the "discrete-time" receding horizon control and the "discrete-time" receding horizon trajectory (emanating from x_0 at time t_0), respectively. Then, we have

$$u_D^*(\tau) \triangleq \hat{u}(\tau; x_D^*(t_i), t_i) \qquad \text{for } \tau \in [t_i, t_{i+1}) \text{ and for all } i \in \{0\} \cup \mathbb{N}, \tag{4.19}$$

where $\hat{u}(\tau; x_D^*(t_i), t_i)$ solves $P(x_D^*(t_i), t_i)$, $x_D^*(t_0) = x_0$, and $t_{i+1} \triangleq t_i + \Delta_D$ for all i. The corresponding closed-loop receding horizon trajectory is given by

$$x_D^*(\tau) \triangleq \hat{x}(\tau; x_D^*(t_i), t_i) \qquad \text{for } \tau \in [t_i, t_{i+1}) \text{ and for all } i \in \{0\} \cup \mathbb{N}. \tag{4.20}$$

Obviously,

$$x_D^*(t_{i+1}) = \hat{x}(t_{i+1}; x_D^*(t_i), t_i). \tag{4.21}$$

Using the approach of [10], it can be shown that, under reasonable conditions, the "discrete-time" receding horizon control law stabilizes system (1.3). The result is given in detail in Theorem 3 below, and the proof can be found in [11].

Theorem 3 [11]: Let Assumptions A1–A9 be valid and, additionally, for any $\rho \in (0, \infty)$ let there exist an $N(\rho) \in (0, \infty)$ such that for any initial state $x_0 \in \bar{B}(0; \rho)$, the optimal control $\hat{u}(\cdot; x_0, 0)$, steering the system to $x(T; x_0, 0) = 0$, satisfies $\|\hat{u}\|_\infty \le N(\rho)\|x_0\|$ (the latter assumption is only slightly stronger than A4). Then the "discrete-time" receding horizon control strategy is globally asymptotically stabilizing for every $\Delta_D \in (0, T]$. ■

The consequences of using an approximation to each open-loop optimal control function in the (continuous) receding horizon formulation are considered now. This is relevant because exact calculation of an optimal control in a finite number of computational steps is not possible. Hence, only suboptimal open-loop controls can be applied when constructing the receding horizon control. A natural question is: how large a discrepancy between the optimal and suboptimal solutions can be tolerated in order to maintain the stabilizing property of the "approximate" receding horizon control strategy.

To put it rigorously, for any initial point $x \in \mathbb{R}^n$, let $\hat{u}_{\text{sub}}(\cdot; x, 0)$ denote a suboptimal control for problem $P(x, 0)$, and let $\hat{x}_{\text{sub}}(\cdot; x, 0)$ denote the corresponding suboptimal trajectory. Let the "approximate" receding horizon control law employing the suboptimal controls be denoted by $h_{\text{sub}}^*: \mathbb{R}^n \mapsto \mathbb{R}^m$ and the corresponding receding horizon control trajectory by $x_{\text{sub}}^*(\cdot)$ i.e.,

$$h_{\text{sub}}^*(x) = \hat{u}_{\text{sub}}(0; x, 0) \qquad (= \hat{u}_{\text{sub}}(t; x, t) \text{ for any } t) \text{ for all } x \in \mathbb{R}^n. \tag{4.22}$$

To ensure continuity of h_{sub}^*, we shall assume that the suboptimal open-loop controls $\hat{u}_{\text{sub}}$ are calculated in such a way as to ensure that the mapping $x \to \hat{u}_{\text{sub}}(0; x, 0): \mathbb{R}^n \mapsto \mathbb{R}^m$ is continuous (this assumption is only necessary for the "continuous-time" analysis of the receding horizon control). It will also be understood that the resulting receding horizon control law h_{sub}^* yields a closed-loop system

$$\dot{x} = f(x, h_{\text{sub}}^*(x))$$

which has unique solutions (on the interval $[0, \infty)$) for every $x \in \mathbb{R}^n$.

With the above definitions, the following result can be shown which validates the use of suboptimal controls in constructing receding horizon control laws. The proof of this result is based on the approach used in the proof of Theorem 1 and can be found in [11].

Theorem 4 [11]: Let Assumptions A1 and A2 and A4–A8 be valid. Replace Assumption A3 by the assumption that the function $(x, u) \mapsto f(x, u): \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is globally Lipschitz continuous. Additionally, for any $\rho \in (0, \infty)$, let there exist an $N(\rho) \in (0, \infty)$ such that for any initial state $x_0 \in \bar{B}(0; \rho)$, the optimal control $\hat{u}(\cdot; x_0, 0)$, steering the system to $x(T; x_0, 0) = 0$, satisfies $\|\hat{u}\|_\infty \leq N(\rho)\|x_0\|$. Suppose the suboptimal open-loop controls $\hat{u}_{\text{sub}}$ are calculated in such a way as to ensure that for all $x \in \mathbb{R}^n$, $\tau \mapsto \hat{u}_{\text{sub}}(\tau; x, 0): [0, T] \mapsto \mathbb{R}^m$ is continuous at $\tau = 0$. Under these conditions, for any $\delta > 0$, there exists an $\alpha > 0$ such that if the controls $\hat{u}_{\text{sub}}$ are additionally calculated with a precision satisfying

$$\|\hat{u}_{\text{sub}}(\cdot; x, 0) - \hat{u}(\cdot; x, 0)\|_\infty \leq \alpha\|x\| \qquad \text{for all } x \in \mathbb{R}^n, \tag{4.23}$$

then the receding horizon control law h^*_{sub} is asymptotically stabilizing for system (1.3) on $\bar{B}(0; \delta)$. ∎

A complete implementable algorithm for stabilization using the receding horizon approach would have to incorporate a mechanism for iterating on the control until condition (4.23) is satisfied. The design of such a mechanism seems to be a nontrivial task. The significance of the above result is that it shows that exact optimization is not essential. In fact, the approximate optimal controls used in the following example were found by just solving the associated two-point boundary value problem to a prespecified precision—and that worked well in practice and gave stabilization to within a small neighborhood of the origin.

V. Examples

The stabilizing properties of the receding horizon control, which were discussed in Section IV, were tested by simulation on the following systems:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + u \begin{bmatrix} \mu + (1-\mu)x_1 \\ \mu - 4(1-\mu)x_2 \end{bmatrix}; \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{S1}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(1+\epsilon x_2)x_2 \\ (1+\epsilon x_1)x_1 \end{bmatrix} + u \begin{bmatrix} x_1 \\ -4x_2 \end{bmatrix}, \tag{S2}$$

where μ and ϵ took the values: $\mu = 0.5$ and 0.1, $\epsilon = 0.001$, 0.5, and 1.

The free system $(S1)$ is an undamped oscillator, hence stabilization was nontrivial. The receding horizon control was calculated using suboptimal solutions to the optimal control problem $P(x, 0)$ for system $(S1)$ (in the sequel denoted by $P_{S_1}(x, 0)$) with the performance functional:

$$0.5 \int_0^T (0.1x_1^2 + 0.1x_2^2 + u^2)\, d\tau \qquad \text{with } T = 7.5 \text{ seconds.}$$

The suboptimal controls were calculated by solving the associated two-point boundary value problem to fixed precision. The optimal control was updated discretely in time yielding a "discrete-time" receding horizon control. A few trajectories of system $(S1)$ with $\mu = 0.5$, and the "discrete-time" receding horizon control with $\Delta_D = 0.05$ s, are shown in Fig. 1. The robustness of the receding horizon control law was first tested by applying the receding horizon control generated with reference to system $(S1)$, with $\mu = 0.5$ (which will be referred to as the "model"), to system $(S2)$ with $\epsilon = 0.001$ (referred to as the "actual system").

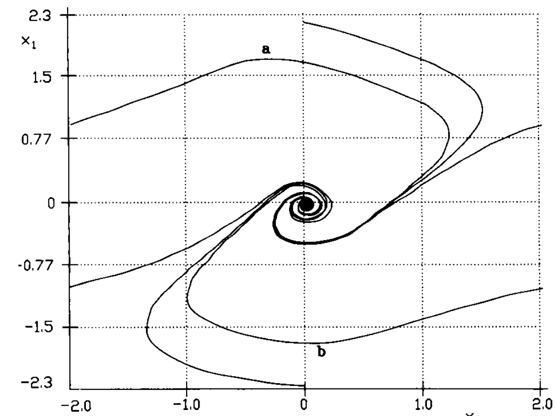

Fig. 1. Trajectories for system $(S1)$ with $\mu = 0.5$, using the "discrete-time" receding horizon control for system $(S1)$.

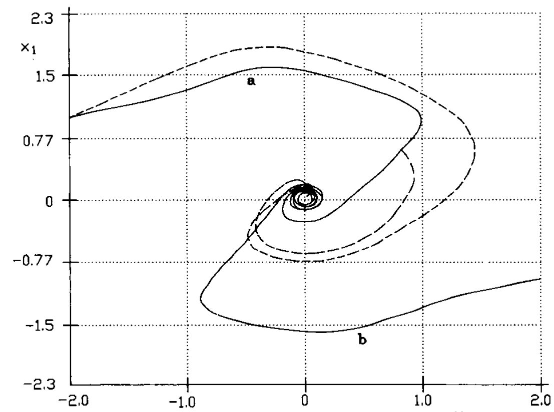

Fig. 2. The continuous curves show the trajectories for system $(S2)$ with $\epsilon = 0.001$, using the "discrete-time" receding horizon control for system $(S1)$ with $\mu = 0.5$. The discontinuous curves show the suboptimal open-loop trajectories for system $(S1)$ with $\mu = 0.5$ computed at two different current values of the state of system $(S2)$.

The latter was done as follows: at any instant of time $t = \Delta_D i$, $i \in \mathbb{N}$, a suboptimal control for $P_{S_1}(x_{S_2}(\Delta_D i); \Delta_D i)$ was calculated and applied to system $(S2)$ for $t \in [\Delta_D i, \Delta_D(i+1)]$, where $x_{S_2}(\Delta_D i)$ denotes the actual ("measured") state of system $(S2)$ at time $\Delta_D i$. Two trajectories of system $(S2)$, calculated using the receding horizon control scheme described above, are shown in Fig. 2. These are plotted as continuous lines, while the discontinuous lines correspond to the open-loop suboptimal trajectories (for $P_{S_1}(x_{S_2}(\tau); \tau)$) shown for times $\tau = 0.25$, and $\tau = 1$ s. The curves marked (a) and (b) in Fig. 2 should be compared to curves (a) and (b) in Fig. 1, which correspond to the situation when the "model" coincides with the "actual system."

Yet a better robustness study was performed with reference to system $(S1)$ with $\mu = 0.1$ as the "model." Fig. 3 shows three receding horizon trajectories emanating from the same initial state and generated by a "discrete-time" receding horizon control law with $\Delta_D = 0.5$ s. The three curves correspond to the trajectories of the "actual system" being: (A) $(S1)$ with $\mu = 0.1$, (B) $(S2)$ with $\epsilon = 0.5$, and (C) $(S2)$ with $\epsilon = 1$.

The simulations performed indicate that the receding horizon control approach yields a strongly robust feedback stabilization strategy. It is a feedback strategy because all controls are computed as a function of the current state. Since fast converging optimization algorithms are now available, and because of the

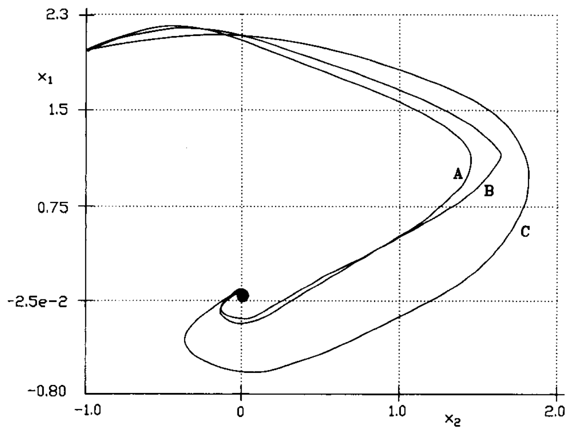

Fig. 3. Trajectories for systems: (A) ($S1$) with $\mu = 0.1$, (B) ($S2$) with $\epsilon = 0.5$, and (C) ($S2$) with $\epsilon = 1$, for "discrete-time" receding horizon controls computed for system ($S1$) with $\mu = 0.1$.

absence of other general methods for the construction of stabilizing control laws for nonlinear systems, the receding horizon control should find practical applications.

Appendix

Proof of Proposition 3: First we will show that given any $t \in [0, T]$, there exists a $\delta(t) > 0$ such that $u(x, \lambda)$ is of class $\mathbb{C}^1$ on $B((\hat{x}(t), \hat{\lambda}(t)); \delta(t))$. To see this, let us observe that $(\hat{x}(t), \hat{\lambda}(t)) \in T((\hat{x}, \lambda); \epsilon)$ and

$$\{u(\hat{x}(t), \hat{\lambda}(t))\} = \arg\min_{u \in \mathbb{R}^m} H(\hat{x}(t), u, \hat{\lambda}(t))$$

and hence,

$$\nabla_u H(\hat{x}(t), u(\hat{x}(t), \hat{\lambda}(t)), \hat{\lambda}(t)) = 0. \tag{A.1}$$

By Assumption A1, $\nabla_u H$ is of class $\mathbb{C}^1$ with respect to (x, u, λ). Since

$$\nabla_{uu} H(\hat{x}(t), u(\hat{x}(t), \hat{\lambda}(t)), \hat{\lambda}(t)) \geq \delta I > 0 \tag{A.2}$$

(see Assumption A7 and Comment 6), we can apply the Implicit Function Theorem by which there exist neighborhoods $B((\hat{x}(t), \hat{\lambda}(t)); \delta_1(t))$ and $B(u(\hat{x}(t), \hat{\lambda}(t)); \delta_2(t))$ such that for every $(x, \lambda) \in B((\hat{x}(t), \hat{\lambda}(t)); \delta_1(t))$, the equation $\nabla_u H(x, u, \lambda) = 0$ is *uniquely solvable* with respect to u in $B(u(\hat{x}(t), \hat{\lambda}(t)); \delta_2(t))$ (the uniqueness is restricted to $B(u(\hat{x}(t), \hat{\lambda}(t)); \delta_2(t))$. Denoting this unique solution by $\bar{u}(x, \lambda)$, we also have that $\bar{u}(x, \lambda)$ is of class $\mathbb{C}^1$ in $B((\hat{x}(t), \hat{\lambda}(t)); \delta_1(t))$. By continuity of $\hat{x}$, $\hat{\lambda}$, and $u(\cdot, \cdot)$ (see Comment 2), there exists a $\delta_3(t) < \epsilon$ such that for all $(x, \lambda) \in B((\hat{x}(t), \hat{\lambda}(t)); \delta_3(t))$, we have that $u(x, \lambda) \in B(u(\hat{x}(t), \hat{\lambda}(t)); \delta_2(t))$ and obviously $\nabla_u H(x, u(x, \lambda), \lambda) = 0$ (because $B((\hat{x}(t), \hat{\lambda}(t)); \delta_3(t)) \subset T((\hat{x}, \hat{\lambda}); \epsilon)$). Hence, taking $\delta(t) = \min(\delta_1(t), \delta_3(t))$, we deduce that

$$u(x, \lambda) \equiv \bar{u}(x, \lambda) \qquad \text{for all } (x, \lambda) \in B((\hat{x}(t), \hat{\lambda}(t)); \delta(t))$$

which shows that $u(x, \lambda)$ is $\mathbb{C}^1$ on $B((\hat{x}(t), \hat{\lambda}(t)); \delta(t))$, as required.

Suppose that for any $t \in [0, T]$, $\hat{\delta}(t)$ denotes the maximal radius $\delta(t)$ for which $u(x, \lambda)$ is $\mathbb{C}^1$ on the set $B((\hat{x}(t), \hat{\lambda}(t)); \hat{\delta}(t))$, and take

$$\epsilon' \triangleq \inf\{\hat{\delta}(t) | t \in [0, T]\}. \tag{A.3}$$

It remains to show that $\epsilon' > 0$.

Suppose that this is false, i.e., there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$, $t_i \in [0, T]$ such that $\hat{\delta}(t_i) \to 0$ as $i \to \infty$. Since $[0, T]$ is compact, there exists a subset $\mathbb{K} \subset \mathbb{N}$ and a $t^* \in [0, T]$ such that

$$t_i \to t^* \text{ and } \hat{\delta}(t_i) \to 0 \qquad \text{as } i \in \mathbb{K} \text{ and } i \to \infty \tag{A.4}$$

and obviously $\hat{\delta}(t^*) > 0$. But $\hat{x}$ and $\hat{\lambda}$ are continuous, hence, there exists an index n_1 such that for all $i \geq n_1$, $i \in \mathbb{K}$

$$(\hat{x}(t_i), \hat{\lambda}(t_i)) \in B((\hat{x}(t^*), \hat{\lambda}(t^*)); \hat{\delta}(t^*)/2).$$

Hence, for $i \geq n_1$, $i \in \mathbb{K}$

$$B((\hat{x}(t_i), \hat{\lambda}(t_i)); \hat{\delta}(t^*)/4) \subset B((\hat{x}(t^*), \hat{\lambda}(t^*)); \hat{\delta}(t^*)).$$

But on $B((\hat{x}(t^*), \hat{\lambda}(t^*)); \hat{\delta}(t^*))\, u(\cdot, \cdot)$ is $\mathbb{C}^1$, so that

$$\hat{\delta}(t_i) \geq \hat{\delta}(t^*)/4 \qquad \text{for all } i \geq n_1, \qquad i \in \mathbb{K}.$$

This contradicts (A.4) which proves that $\epsilon' > 0$. □

Auxiliary Results and Comments Needed for the Proof of Proposition 4: In order to prove Proposition 4, we have to give some facts concerning the continuity and differentiability of solutions to differential equations with perturbation to the initial conditions. These facts are known from literature, but have been proved there with the assumption that the right-hand sides of the differential equations are globally Lipschitz continuous on some compact and convex set containing the nominal initial condition. In our case, the right-hand sides of the state and costate equations are only defined locally ($u(\cdot, \cdot)$ is only defined, and of class $\mathbb{C}^1$, on the set $T((\hat{x}, \hat{\lambda}); \epsilon')$). The set $T((\hat{x}, \hat{\lambda}); \epsilon')$ is generally nonconvex, and therefore, even though the feedback function u is of class $\mathbb{C}^1$ (and hence locally Lipschitz continuous), it may not be globally Lipschitz continuous on $T((\hat{x}, \hat{\lambda}); \epsilon')$. To suit our particular case, we had to modify slightly standard proofs concerning the continuity and differentiability of solutions to differential equations with perturbation to the initial conditions (see [9]). We have not included these proofs since they are almost identical to those presented in [6]. We will bear in mind that

$$\dot{z}(t) = F(x(t)) \tag{A.5}$$

will later play the role of the state and costate equations treated as a pair. Hence, z will later stand for the state–costate pair (x, λ) and $\hat{z}$ for the optimal $(\hat{x}, \hat{\lambda})$.

Lemma A1: Let $\hat{z} \in \mathbb{C}^p[0, T]$ be an absolutely continuous trajectory $t \mapsto \hat{z}(t): [0, T] \mapsto \mathbb{R}^p$ and suppose that a function $z \mapsto F(z): \mathbb{R}^p \mapsto \mathbb{R}^p$ is of class $\mathbb{C}^1$ on a tube $T(\hat{z}; \epsilon)$ for some $\epsilon > 0$. Then F is globally Lipschitz continuous on any closed subtube $\bar{T}(\hat{z}; \epsilon')$ with $\epsilon' \in (0, \epsilon)$. ■

Proposition A1: Suppose the equation defined by (A.5) has a solution $t \mapsto \hat{z}(t; z_0, 0): [0, T] \mapsto \mathbb{R}^p$ [shortly $\hat{z}(\cdot)$] satisfying the initial condition $\hat{z}(0) = z_0$. Suppose further that for some $\epsilon > 0$, F is of class $\mathbb{C}^1$ in $T(\hat{z}; \epsilon)$. Then there exists a $\delta > 0$ such that for any initial value $v = z_0 + \delta z_0$ with $\|\delta z_0\| \leq \delta$, the initial value problem given by (A.5) with $z(0) = v$ has a solution $t \mapsto z(t; v, 0): [0, T] \mapsto \mathbb{R}^p$ and $z(t; v, 0) \in T(\hat{z}; \epsilon)$ for all $t \in [0, T]$. Furthermore, z is continuous in v uniformly with respect to t. ■

Let z_v denote the strong derivative of the mapping $v \mapsto z(\cdot; v, 0): \mathbb{R}^p \mapsto \mathbb{C}^p[0, T]$, where $z(\cdot; v, 0)$ denotes the solution of (A.5) with initial condition $z(0) = v$. The following proposition guarantees strong differentiability of the solution z and gives a way to compute the derivative z_v.

Proposition A2: Suppose the equation defined by (A.5) has a solution $t \mapsto \hat{z}(t; z_0, 0): [0, T] \mapsto \mathbb{R}^p$ [shortly $\hat{z}(\cdot)$] satisfying the initial condition $\hat{z}(0) = z_0$. Suppose further that, for some $\epsilon > 0$, F is of class $\mathbb{C}^1$ in $T(\hat{z}; \epsilon)$. Then the strong derivative $z_v(\cdot)$ exists for all $v = z_0 + \delta z_0$ for which the solution $z(\cdot; v, 0)$ exists and lies in $T(\hat{z}; \epsilon)$. The derivative $z_v(\cdot)$ is continuous in v (treated as a mapping $v \mapsto z_v(v): \mathbb{R}^p \to \mathfrak{L}[\mathbb{R}^p; \mathbb{C}^p[0, T]]$, where $\mathfrak{L}$ is the space of all linear and bounded operators acting from $\mathbb{R}^p$ into

$\mathfrak{C}^p[0, T]$ with the induced operator norm $\|\cdot\|_{\mathcal{L}}$). Furthermore,

$$z_v(v) = y_v(0) \tag{A.6}$$

where $y_v(0)$ is the strong derivative of the mapping $h \mapsto y(\cdot; h, 0)$ at $h = 0$, and $y(\cdot; h, 0)$ is the solution of the following linear equation:

$$\dot{y}(t) = F_z(z(t; v, 0))y(t) \qquad \text{with initial condition } y(0) = h. \tag{A.7}$$ ■

Suppose x_0 is given. Let $(\hat{x}, \hat{\lambda})$ be the optimal state–costate pair corresponding to x_0. Hence, $\hat{x}$ solves the state equation (1.3) with initial condition $\hat{x}(0) = x_0$ and terminal condition $\hat{x}(T; x_0, 0) = 0$, and $\hat{\lambda}$ satisfies the costate equation (2.2). The optimal state–costate pair is thus a solution to a boundary value problem. By our assumptions, $\hat{\lambda}$ is unique so, equivalently, we may think of $(\hat{x}, \hat{\lambda})$ as a solution to an initial value problem with an augmented initial condition $(x_0, \hat{\lambda}(0))$ where $\hat{\lambda}(0)$ is chosen to yield the satisfaction of $\hat{x}(T; x_0, 0) = 0$. This approach to the state–costate boundary value problem will allow for an easy analysis of differentiability of its solutions.

In this spirit, let us view the state and costate equations as one system of equations of the form given in (A.5) by defining

$$(x, \lambda) \triangleq z$$

$$\begin{aligned} F(z) &\triangleq (F_1(z), F_2(z)) \\ &\triangleq (\nabla_\lambda H(x, u(x, \lambda), \lambda), -\nabla_x H(x, u(x, \lambda), \lambda)) \end{aligned} \tag{A.8}$$

with the additional assumption that the initial condition $z(0) = v \triangleq (v_x, v_\lambda)$ is sufficiently close to the initial condition $(x_0, \hat{\lambda}(0)) \triangleq z_0$ (which yields the optimal pair $(\hat{x}, \hat{\lambda}) \triangleq \hat{z}$) to ensure that the corresponding solution lies in $T(\hat{z}; \epsilon')$. Here, by Assumption A6 and Proposition 3, u is the feedback control function defined and continuously differentiable in the neighborhood $T(\hat{z}, \epsilon')$ of the optimal solution $\hat{z}$. Obviously, $\hat{\lambda}(0)$ satisfies $\hat{z}(T; z_0, 0) = 0$. To express the dependence of the solution z on the initial condition v, we will use the notation $(x(\cdot; v, 0), \lambda(\cdot; v, 0)) \triangleq z(\cdot; v, 0)$.

Proof of Proposition 4: By applying Proposition A.2, the derivative $z_v(z_0)(T)$ for the system defined by (A.5) and (A.8) (i.e., the derivative of the mapping $v \mapsto z(T; v, 0)$ at z_0) exists and is the value of the transition matrix $\Phi(T, 0)$ for the linearized equations (2.16) and (2.17), i.e.,

$$\dot{y} = F_z(z(t; z_0, 0))y. \tag{A.9}$$

Let $x_{\lambda 0}(z_0)$ denote the derivative of the partial mapping $v_\lambda \mapsto x(\cdot; (v_x, v_\lambda), 0)$ at z_0, and $x_{\lambda 0}(z_0)(T)$ the derivative of the partial mapping $v_\lambda \mapsto x(T; (v_x, v_\lambda), 0)$ at z_0.

In order to compute $x_{\lambda 0}(z_0)(T)$, let us first compute $F_z(z(t; z_0, 0))$ in (A.9). Since $u(\hat{x}, \hat{\lambda})$ satisfies

$$\nabla_u H(\hat{x}, \hat{u}(\hat{x}, \hat{\lambda}), \hat{\lambda}) = Ru + f_u^T(\hat{x}, \hat{u}(\hat{x}, \hat{\lambda}))\hat{\lambda} = 0, \tag{A.10}$$

then by Assumption A6, and the Implicit Function Theorem, the derivatives of u are given by

$$u_x(\hat{x}, \hat{\lambda}) = -(\nabla_{uu}H)^{-1}(\hat{x}, \hat{u}(\hat{x}, \hat{\lambda}), \hat{\lambda})\nabla_{ux}H(\hat{x}, \hat{u}(\hat{x}, \hat{\lambda}), \hat{\lambda}) \tag{A.11}$$

$$u_\lambda(\hat{x}, \hat{\lambda}) = -(\nabla_{uu}H)^{-1}(\hat{x}, \hat{u}(\hat{x}, \hat{\lambda}), \hat{\lambda})\nabla_{u\lambda}H(\hat{x}, \hat{u}(\hat{x}, \hat{\lambda}), \hat{\lambda}). \tag{A.12}$$

By direct computation, we then get that

$$F_{1x}(\hat{z}) = [f_x - f_u(\nabla_{uu}H)^{-1}\nabla_{ux}H](\hat{x}, u(\hat{x}, \hat{\lambda}), \hat{\lambda}) \tag{A.13}$$

$$F_{1\lambda}(\hat{z}) = -[f_u(\nabla_{uu}H)^{-1}f_u^T](\hat{x}, u(\hat{x}, \hat{\lambda}), \hat{\lambda}) \tag{A.14}$$

$$F_{2x}(\hat{z}) = -[\nabla_{xx}H - \nabla_{xu}H(\nabla_{uu}H)^{-1}\nabla_{ux}H](\hat{x}, u(\hat{x}, \hat{\lambda}), \hat{\lambda}) \tag{A.15}$$

$$F_{2\lambda}(\hat{z}) = -[f_x^T - \nabla_{xu}H(\nabla_{uu}H)^{-1}f_u^T](\hat{x}, u(\hat{x}, \hat{\lambda}), \hat{\lambda}). \tag{A.16}$$

Defining

$$A(t) \triangleq F_{1x}(\hat{z}), \qquad B(t) \triangleq f_u(\hat{x}, u(\hat{x}, \hat{\lambda}), \hat{\lambda}) \tag{A.17}$$

$$\tilde{R}(t) \triangleq \nabla_{uu}H(\hat{x}, u(\hat{x}, \hat{\lambda}), \hat{\lambda}), \qquad C(t) \triangleq -F_{2x}(\hat{z}) \tag{A.18}$$

and partitioning y into (y_x, y_λ), (A.9) becomes

$$\dot{y}_x(t) = A(t)y_x(t) - B\tilde{R}^{-1}B^T(t)y_\lambda(t) \tag{A.19}$$

$$\dot{y}_\lambda(t) = -C(t)y_x(t) - A^T(t)y_\lambda(t) \tag{A.20}$$

where $C(t)$ is symmetric and $C(t) \geq 0$ for $t \in [0, T]$ by Assumption A8 and Comment 6. Using arbitrary initial conditions

$$\begin{aligned} y_x(0) &= h_x \\ y_\lambda(0) &= h_\lambda, \end{aligned} \tag{A.21}$$

we shall solve (A.19) and (A.20) directly by using the following general transformation:

$$y_\lambda(t) = \beta(t) + S(t)y_x(t), \tag{A.22}$$

where β is an n-vector, and S a symmetric $n \times n$ matrix. If y_λ and y_x are related by (A.22), then (A.19) and (A.20) give the following dependence:

$$\begin{aligned} &\dot{\beta}(t) - S(t)B\tilde{R}^{-1}B^T(t)\beta(t) + A^T(t)\beta(t) \\ &\quad = [-\dot{S}(t) - S(t)A(t) - A^T(t)S(t) \\ &\qquad + S(t)B\tilde{R}^{-1}B^T(t)S(t) - C(t)]y_x(t). \end{aligned} \tag{A.23}$$

Equation (A.23) can be satisfied for an arbitrary h_x (hence y_x) if both sides of (A.23) are equal to zero, i.e., if

$$\dot{S}(t) = -S(t)A(t) - A^T(t)S(t) + S(t)B\tilde{R}^{-1}B^T(t)S(t) - C(t) \tag{A.24}$$

$$\dot{\beta}(t) = [S(t)B\tilde{R}^{-1}B^T(t) - A^T(t)]\beta(t). \tag{A.25}$$

From the above transformations, we see that if there exists a symmetric matrix S_T such that the Riccati equation (A.24) has a bounded solution $S(t)$ for $t \in [0, T]$, which passes through $S(T) = S_T$, then, using this, we can solve the linear equation (A.25) with initial condition

$$\beta(0) = h_\lambda - S(0)h_x. \tag{A.26}$$

Next, substituting (A.22) into (A.19) (with known β and S), we can solve for y_x using initial condition h_x. The resulting y_x and y_λ will satisfy (A.19)–(A.21). The existence of a symmetric matrix S_T is guaranteed by the absence of conjugate points in the interval $[0, T]$, which in turn is ensured by A8 (see [7, p. 159]).

Let S denote a known solution of (A.24), and $\Psi^T(0, t)$ the transition matrix of the linear equation in β, (A.25). Hence,

$$\beta(t) = \Psi^T(0, t)(h_\lambda - S(0)h_x) \qquad \text{for all } t \in [0, T]. \tag{A.27}$$

Substituting (A.22) into (A.19) we get that

$$\dot{y}_x(t) = [A - B\tilde{R}^{-1}B^TS](t)y_x(t) - B\tilde{R}^{-1}B^T(t)\beta(t) \tag{A.28}$$

hence, the transition matrix for (A.28) is $\Psi(t, 0)$ (the transition matrix for the adjoint system; see [7, p. 44]). Hence, for $t \in [0, T]$,

$$y_x(t) = \Psi(t, 0)h_x - \int_0^t \Psi(t, \tau)B\tilde{R}^{-1}B^T(\tau) \cdot \Psi^T(0, \tau)\, d\tau (h_\lambda - S(0)h_x), \quad \text{(A.29)}$$

so

$$y_x(T; (h_x, h_\lambda), 0) = \Psi(T, 0)h_x - \Psi(T, 0) \cdot W_\Psi(0, T)(h_\lambda - S(0)h_x), \quad \text{(A.30)}$$

where $W_\Psi(0, T)$ is the controllability matrix for system (A.28) defined by

$$W_\Psi(0, t) \triangleq \int_0^t \Psi(0, \tau)B\tilde{R}^{-1}B^T(\tau)\Psi^T(0, \tau)\, d\tau. \quad \text{(A.31)}$$

Let the partitions of the transition matrix Φ for system (A.9) be given by

$$\begin{bmatrix} \Phi_{1x} & \Phi_{1\lambda} \\ \Phi_{2x} & \Phi_{2\lambda} \end{bmatrix}.$$

From (A.30) we have that

$$x_{\lambda 0}(z_0)(T) = \Phi_{1\lambda} = -\Psi(T, 0)W_\Psi(0, T). \quad \text{(A.32)}$$

Since $\tilde{R}^{-1}(t)$ is strictly positive definite and symmetric for all $t \in [0, T]$, we deduce that $\tilde{R}^{-1/2}(t)$ exists for all $t \in [0, T]$ and is symmetric. Hence, W_Ψ can be treated as the controllability matrix for the system described by the following matrix pair:

$$[(A - B\tilde{R}^{-1}B^TS)(t);\ B\tilde{R}^{-1/2}(t)]. \quad \text{(A.33)}$$

From the definitions of $A(t)$ and $B(t)$, it follows that the system described by the matrix pair $[A(t); B(t)]$ is controllable since it is feedback equivalent with $[f_x(\hat{x}, u(\hat{x}, \hat{\lambda}))(t); f_u(\hat{x}, u(\hat{x}, \hat{\lambda}))(t)]$ which is controllable by Assumption A6 (see [7, Theorem 2, p. 118]). For the same reason, the system described by the pair $[(A - B\tilde{R}^{-1}B^TS)(t); B(t)]$ is controllable. Hence, the system described by (A.33) is also controllable, since $\tilde{R}^{-1/2}(t)$ is invertible for all $t \in [0, T]$ (if u is the control that brings the system $[(A - B\tilde{R}^{-1}B^TS)(t); B(t)]$ from point x_a to point x_b along trajectory x, then the control $\tilde{R}^{1/2}u$ steers the system described by (A.33) from x_a to x_b along the same trajectory, which proves controllability). Therefore, $W_\Psi(0, T)$ is strictly positive definite. By (A.32), $x_\lambda(z_0)(T)$ is nonsingular since $\Psi(T, 0)$ is nonsingular (the latter because, by our assumptions, the system matrix $(A - B\tilde{R}^{-1}B^TS)(\cdot)$ is a continuous matrix function on the interval $[0, T]$, and hence must have a bounded trace on $[0, T]$). □

Before we can prove Proposition 5, we need the following.

Proposition A3: Suppose Assumptions A1–A8 are satisfied. Let $\mathbb{R}^n \supset \{x_{0i}\}_{i\in\mathbb{N}} \mapsto x_0$ be any convergent sequence of initial conditions for problems $P(x_{0i}, 0)$, and let the corresponding sequences of optimal solutions, trajectories, and costate trajectories of $P(x_{0i}, 0)$ be $\{\hat{u}_i\}_{i\in\mathbb{N}} \subset \mathbb{C}^m[0, T]$, $\{\hat{x}_i\}_{i\in\mathbb{N}} \subset \mathbb{C}^n[0, T]$, and $\{\hat{\lambda}_i\}_{i\in\mathbb{N}} \subset \mathbb{C}^n[0, T]$. Under these conditions:

i) the sequence of the initial costate vectors $\{\hat{\lambda}_i(0)\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$ is bounded, and

ii) the sequences $\{\hat{u}_i\}_{i\in\mathbb{N}}$, $\{\hat{x}_i\}_{i\in\mathbb{N}}$, and $\{\hat{\lambda}_i\}_{i\in\mathbb{N}}$ are equibounded and equicontinuous families of functions on the interval $[0, T]$. □

Proof:

i) Under our assumptions, the optimal trajectories $\hat{x}_i$, $\hat{\lambda}_i$, and $\hat{u}_i$ are continuous functions on the closed interval $[0, T]$, and they satisfy the state and costate equations, i.e., for all i, and for all $t \in [0, T]$

$$\dot{\hat{x}}_i(t) = f(\hat{x}_i(t), \hat{u}_i(t)) \quad \text{with } \hat{x}_i(T) = 0 \quad \text{(A.34)}$$

$$-\dot{\hat{\lambda}}_i(t) = \nabla_x H(\hat{x}_i(t), \hat{u}_i(t), \hat{\lambda}_i(t)) = Q\hat{x}_i(t) + f_x(\hat{x}_i(t), \hat{u}_i(t))^T\hat{\lambda}_i(t) \quad \text{(A.35)}$$

and

$$\nabla_u H(\hat{x}_i(t), \hat{u}_i(t), \hat{\lambda}_i(t)) = R\hat{u}_i(t) + f_u^T(\hat{x}_i(t), \hat{u}_i(t))\hat{\lambda}_i(t) = 0. \quad \text{(A.36)}$$

Since $\{x_{0i}\}_{i\in\mathbb{N}}$ is bounded, then by Assumptions A3 and A4 and Comment 1, there exist constants M_1 and M_2 such that, for all i,

$$\|\hat{u}_i\|_\infty \le M_1 \quad \text{and consequently} \quad \text{(A.37)}$$

$$\|\hat{x}_i\|_\infty \le M_2 \quad \text{for all } i. \quad \text{(A.38)}$$

By continuous differentiability of f, this implies the existence of a constant M_3 such that for all i

$$\|f_u^T(\hat{x}_i, \hat{u}_i)\|_\infty \le M_3. \quad \text{(A.39)}$$

Let Φ_i denote the transition matrix for the linearized state equation $\delta\dot{x}_i(t) = f_x(\hat{x}_i(t), \hat{u}_i(t))\delta x_i(t) + f_u(\hat{x}_i(t), \hat{u}_i(t))\delta u_i(t)$. Considering equation (A.36), we can write formally

$$\hat{u}_i(t) = -R^{-1}f_u^T(\hat{x}_i(t), \hat{u}_i(t)) \cdot \left(\Phi_i^T(0, t)\hat{\lambda}_i(0) - \int_0^t \Phi_i^T(\tau, t)Q\hat{x}_i(\tau)\, d\tau\right). \quad \text{(A.40)}$$

Multiplying the above equation by $\Phi_i(0, t)f_u(\hat{x}_i(t), \hat{u}_i(t))$ and integrating over $[0, T]$, we get, for all i,

$$\int_0^T \Phi_i(0, t)f_u(\hat{x}_i(t), \hat{u}_i(t)) \cdot \left(\hat{u}_i(t) - R^{-1}f_u^T(\hat{x}_i(t), \hat{u}_i(t))\int_0^t \Phi_i^T(\tau, t)Q\hat{x}_i(\tau)\, d\tau\right) dt = -W_i(0, T)\hat{\lambda}_i(0) \quad \text{(A.41)}$$

where $W_i(0, T)$ is the weighted (with weight R^{-1}) controllability Grammian corresponding to the linearized state equation. By A5, ii) and iii), $W_i(0, T)$ is invertible for all i, and there exists a bound M_4 such that $\|W_i^{-1}(0, T)\| \le M_4$ and $\|\Phi_i(\tau, t)\| \le M_4$ for all τ, $t \in [0, T]$, and all i. Thus, (A.37)–(A.39) and (A.41) imply that, for all i,

$$\|\hat{\lambda}_i(0)\| \le M_4^2 M_3 T(M_1 + \|R^{-1}\|M_3\|Q\|M_2M_4T)$$

which proves that $\{\hat{\lambda}_i(0)\}_{i\in\mathbb{N}}$ is bounded.

ii) From part i) of this proposition, it follows that a convergent subsequence $\{(x_{0k}, \hat{\lambda}_k(0))\}_{k\in\mathbb{K}} \to (x_0, \lambda_0)$ can be selected. The corresponding trajectories $\hat{x}_k$ and $\hat{u}_k$ are equibounded by (A.37) and (A.38). By Comment 4, the corresponding costate trajectories, $\hat{\lambda}_k$, are also equibounded, i.e., there exists a constant M_5 such that

$$\|\hat{\lambda}_k\|_\infty \le M_5.$$

It is now easy to see that the sequences $\{\hat{x}_k\}_{k\in\mathbb{K}}$ and $\{\hat{\lambda}_k\}_{k\in\mathbb{K}}$ are equicontinuous on the interval $[0, T]$. (This is because by (A.37), (A.38), and Assumption A3, for all $k \in \mathbb{K}$,

$$\|\dot{\hat{x}}_k\|_\infty = \|f(\hat{x}_k, \hat{u}_k)\|_\infty \le M(M_1)(1 + M_2)$$

where $M(\cdot)$ is defined in Assumption A3. Similarly, making use of Comment 4, we have that, for all k,

$$\|\dot{\hat{\lambda}}_k\|_\infty = \|Q\hat{x}_k + f_x(\hat{x}_k, \hat{u}_k)^T\hat{\lambda}_k\|_\infty \le \|Q\|M_2 + M_5M_6,$$

where, regarding the continuity of f_x, M_6 is chosen to satisfy $\|f_x(\hat{x}_k, \hat{u}_k)^T\|_\infty \le M_6$.)

By Proposition 3, any optimal $\hat{u}_k$ can be treated as a continuously differentiable function of $\hat{x}_k$ and $\hat{\lambda}_k$; hence, by the Implicit Function Theorem and the chain rule for differentiation, we get that

$$\begin{aligned}\dot{\hat{u}}_k(t) = &-(\nabla_{uu}H)^{-1}(\hat{x}_k(t), \hat{u}_k(t), \hat{\lambda}_k(t))\\ &\cdot \nabla_{ux}H(\hat{x}_k(t), \hat{u}_k(t), \hat{\lambda}_k(t))\dot{\hat{x}}_k(t)\\ &-(\nabla_{uu}H)^{-1}(\hat{x}_k(t), \hat{u}_k(t), \hat{\lambda}_k(t))\\ &\cdot \nabla_{u\lambda}H(\hat{x}_k(t), \hat{u}_k(t), \hat{\lambda}_k(t))\dot{\hat{\lambda}}_k(t).\end{aligned}$$

Considering Assumption A7, and using the same argument as in part i) of this proof, $(\nabla_{uu}H)^{-1}$ exists and is uniformly bounded for all optimal triples $(\hat{x}_k, \hat{u}_k, \hat{\lambda}_k)$ and, as a consequence, there exists a bound M_7 such that $\|(\nabla_{uu}H)^{-1}(\hat{x}_k(t), \hat{u}_k(t), \hat{\lambda}_k(t))\| \le M_7$ for all k, and all $t \in [0, T]$. By Assumption A1, the matrix functions $\nabla_{u\lambda}H$ and $\nabla_{ux}H$ are continuous in their arguments (x, u, λ) (in the sense of induced matrix norms) and hence bounded on bounded sets. Since $\hat{x}_k$, $\hat{u}_k$, and $\hat{\lambda}_k$ are equibounded, there exists a constant M_8, such that for all k, $\|(\hat{x}_k, \hat{u}_k, \hat{\lambda}_k)\|_\infty \le M_8$. Therefore, for all k, we have the following bound:

$$\|\dot{\hat{u}}_k\|_\infty \le M_7\left(\sup_{|(x,u,\lambda)|\le M_8}\|\nabla_{ux}H(x, u, \lambda)\|\,\|\dot{\hat{x}}_k\|_\infty + \sup_{|(x,u,\lambda)|\le M_8}\|\nabla_u\lambda H(x, u, \lambda)\|\,\|\dot{\hat{\lambda}}_k\|_\infty\right).$$

This proves that the family of functions $\{\hat{u}_k\}_{k\in\mathbb{K}}$ is equicontinuous on $[0, T]$. □

Proof of Proposition 5: Let x_0 be an arbitrarily initial state, and $(\hat{x}(\cdot; x_0, 0), \hat{\lambda}(\cdot; x_0, 0)) \triangleq \hat{z}(\cdot; x_0, 0)$ the corresponding optimal state-costate pair. By Proposition 3, there exists an ϵ-tube, $T(\hat{z}; \epsilon)$, in which a feedback control function $u(x, \lambda)$ is defined which globally minimizes the Hamiltonian, i.e., uniquely describes the point defined by (2.8) and, additionally, $u(x, \lambda)$ is of class $\mathbb{C}^1$ in $T(\hat{z}; \epsilon)$. Let us use the following notation:

$$v = (v_x, v_\lambda) \triangleq (x_0 + \delta x_0, \hat{\lambda}(0) + \delta\lambda_0) \triangleq z_0 + \delta z_0, \quad \text{(A.42)}$$

where z_0 and δz_0 are defined by $(x_0, \hat{\lambda}(0)) \triangleq z_0$, $(\delta x_0, \lambda_0) \triangleq \delta z_0$.

In view of our assumptions, by Proposition A1, there exists a $\delta > 0$ such that for any initial v, with $\|\delta z_0\| \le \delta$, the initial value problem

$$\dot{x} = \nabla_\lambda H(x, u(x, \lambda), \lambda) \quad \text{(A.43)}$$

$$\dot{\lambda} = -\nabla_x H(x, u(x, \lambda), \lambda) \quad \text{(A.44)}$$

$$x(0) = v_x,\ \lambda(0) = v_\lambda \quad \text{(A.45)}$$

has unique solutions $x(\cdot; v, 0)$ and $\lambda(\cdot; v, 0)$ in $[0, T]$, and additionally, $(x(t; v, 0), \lambda(t; v, 0)) \in T(\hat{z}; \epsilon)$ for all $t \in [0, T]$. By Proposition A2, these solutions are continuously differentiable with respect to v, and the derivatives x_v, λ_v are continuous in $B(z_0; \delta)$. Hence, the mapping $v \mapsto x(T; v, 0): B(z_0; \delta) \mapsto \mathbb{R}^n$ is continuously differentiable. Obviously, z_0 satisfies the terminal constraint

$$x(T; z_0, 0) = 0. \quad \text{(A.46)}$$

Consider (A.43) and (A.44) with the initial conditions substituted by boundary conditions

$$x(0; v, 0) = x(0; (v_x, v_\lambda), 0) = v_x \quad \text{(A.47)}$$

$$x(T; v, 0) = x(T; (v_x, v_\lambda), 0) = 0. \quad \text{(A.48)}$$

By Proposition 4, $x_{\lambda 0}(z_0)(T)$ is nonsingular so, by the Implicit Function Theorem, there exists a neighborhood $B(x_0; \zeta_1)$, such that for every $v_x \in B(x_0; \zeta_1)$, the value $v_\lambda(v_x)$ is uniquely defined and solves (A.48) uniquely in some neighborhood $B(\hat{\lambda}(0); \zeta_2)$. Additionally, the function $v_x \mapsto v_\lambda(v_x): B(x_0; \zeta_1) \mapsto B(\hat{\lambda}(0); \zeta_2)$ is of class $\mathbb{C}^1$. Suppose that ζ_1 and ζ_2 are taken such that $B(x_0; \zeta_1) \times B(\hat{\lambda}(0); \zeta_2) \subset B(z_0; \delta)$. Then, the boundary value problem defined by (A.43) and (A.44) together with (A.47) and (A.48) is locally, uniquely solvable for $v_x \in B(x_0; \zeta_1)$, and the solutions are identical with the solutions of the initial value problem defined by (A.43) and (A.44) with initial conditions "parametrized" by v_x:

$$\begin{aligned}x(0) &= v_x\\ \lambda(0) &= v_\lambda(v_x).\end{aligned} \quad \text{(A.49)}$$

From the above argument it follows that the initial conditions, parameterized in such a way, can be treated as a continuously differentiable vector function from $B(x_0; \zeta_1)$ into $B(z_0; \delta)$. However, in $B(z_0; \delta)$, the solution of the initial value problem with (A.43) and (A.44) is continuously differentiable with respect to the initial conditions, and hence, $x(\cdot; (v_x, v_\lambda(v_x)), 0)$ and $\lambda(\cdot; (v_x, v_\lambda(v_x)), 0)$ are continuously differentiable with respect to v_x in $B(x_0; \zeta_1)$ (see Proposition A2). Additionally, it is easy to see that the functions $x(\cdot; (v_x, v_\lambda(v_x)), 0)$ and $\lambda(\cdot; (v_x, v_\lambda(v_x)), 0)$ are also extremal for problem $P(v_x, 0)$.

It remains to prove that, for sufficiently small $\delta > 0$, $x(\cdot; (v_x, v_\lambda(v_x)), 0)$ and $\lambda(\cdot; (v_x, v_\lambda(v_x)), 0)$, obtained in the above way, are not only extremal but optimal for $P(v_x, 0)$. Suppose this is not true, i.e., however close to x_0, there exist v_x for which the feedback control function $u(\cdot, \cdot)$ defined in $T(\hat{z}; \epsilon)$, together with the initial conditions $(v_x, v_\lambda(v_x))$, do not result in optimal trajectories. To see whether this situation is possible, let us take any sequence of initial vectors $\{x_{0i}\}_{i\in\mathbb{N}} \subset B(x_0; \zeta_1)$ such that $x_{0i} \to x_0$ as $i \to \infty$, and let the corresponding optimal trajectories be $\{\hat{x}_i\}_{i\in\mathbb{N}}$, $\{\hat{\lambda}_i\}_{i\in\mathbb{N}} \subset \mathbb{C}^n[0, T]$, and $\{\hat{u}_i\}_{i\in\mathbb{N}} \subset \mathbb{C}^m[0, T]$ (the optimal controls will exist and be continuous in view of our assumptions). Suppose also that, for all i,

$$\begin{aligned}&V(x(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0), 0; u(x(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0), \lambda\\ &\quad\cdot(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0))) > V(\hat{x}_i(\cdot), 0; \hat{u}_i(\cdot)).\end{aligned} \quad \text{(A.50)}$$

By the results of Proposition A.3, the sequences $\{\hat{u}_i\}_{i\in\mathbb{N}}$, $\{\hat{x}_i\}_{i\in\mathbb{N}}$, and $\{\hat{\lambda}_i\}_{i\in\mathbb{N}}$ are equibounded and equicontinuous families of functions in their respective spaces and, hence, by the Ascoli–Arzela Theorem, convergent subsequences $\{\hat{x}_k\}_{k\in\mathbb{K}}$, $\{\hat{u}_k\}_{k\in\mathbb{K}}$, and $\{\hat{\lambda}_k\}_{k\in\mathbb{K}}$, $\mathbb{K} \subset \mathbb{N}$, can be selected (where convergence is understood in the sense of the $\mathbb{C}^n[0, T]$, $\mathbb{C}^m[0, T]$, and $\mathbb{C}^n[0, T]$ topologies). Let $\hat{x}^* \in \mathbb{C}^n[0, T]$, $\hat{u}^* \in \mathbb{C}^m[0, T]$, $\hat{\lambda}^* \in \mathbb{C}^n[0, T]$ denote the limits of these subsequences. By continuity of f, f_x, and f_u, the limit functions will satisfy the state and costate equations (1.3) and (2.2) together with the terminal condition (A.48). (This is because $\hat{x}_k(\cdot) \to \hat{x}^*(\cdot)$, $\hat{u}_k(\cdot) \to \hat{u}^*(\cdot)$ uniformly in $\tau \in [0, T]$, and thus, $f(\hat{x}_k(\cdot), \hat{u}_k(\cdot)) \to f(\hat{x}^*(\cdot), \hat{u}^*(\cdot))$ uniformly in $\tau \in [0, t] \subset [0, T]$ for all $t \in [0, T]$.)

From the latter, it follows that $(\hat{x}^*, \hat{u}^*, \hat{\lambda}^*)$ are admissible for problem $P(x_0, 0)$ in that they satisfy the state-costate boundary value problem. Similarly, making $k \to \infty$ in (A.50), we obtain that

$$\begin{aligned}&V(x(\cdot; (x_0, v_\lambda(x_0)), 0), 0; u(x(\cdot;\\ &\quad (x_0, v_\lambda(x_0)), 0), \lambda(\cdot; (x_0, v_\lambda(x_0)), 0)))\\ &\quad = V(\hat{x}(\cdot; x_0, 0), 0; u(\hat{x}(\cdot; x_0, 0), \hat{\lambda}(\cdot; x_0, 0)))\\ &\quad \ge V(\hat{x}^*(\cdot), 0; \hat{u}^*(\cdot)).\end{aligned} \quad \text{(A.51)}$$

(This is because, by Proposition A1, the convergence

$x(\cdot; (x_{0k}, v_\lambda(x_{0k})), 0) \to x(\cdot; (x_0, v_\lambda(x_0)), 0)$ and $\lambda(\cdot; (x_{0k}, v_\lambda(x_{0k})), 0) \to \lambda(\cdot; (x_0, v_\lambda(x_0)), 0)$ is uniform on $[0, T]$). If for all $t \in [0, T]$ $(\hat{x}(t; x_0, 0), \hat{\lambda}(t; x_0, 0)) = (\hat{x}^*(t), \hat{\lambda}^*(t))$, then, by convergence of the sequences $\{\hat{x}_k\}_{k\in\mathbb{N}}$ and $\{\hat{\lambda}_k\}_{k\in\mathbb{N}}$, there exists an index k_1 such that for all $k > k_1$, and for all $t \in [0, T]$ $(\hat{x}_k(t), \hat{\lambda}_k(t)) \in T(\hat{z}; \epsilon)$, and also $\hat{\lambda}_k(0) \in B(\hat{\lambda}(0); \zeta_2)$. Since the corresponding controls $\hat{u}_k$ are optimal, they must globally minimize the Hamiltonian, which means that $\hat{u}_k(t) = u(\hat{x}_k(t), \hat{\lambda}_k(t))$ for all $k > k_1$, and for all $t \in [0, T]$ [because the feedback function u is uniquely defined in $T(\hat{z}; \epsilon)$]. Hence, the right-hand sides of the state and costate equations are identical with (A.43) and (A.44), so that, by unique solvability of the boundary condition (A.48) in $B(\hat{\lambda}(0); \zeta_2)$, we must have $\lambda_k(0) = v_\lambda(x_{0k})$ for all $k > k_1$. By uniqueness of solutions of the initial value problem given by (A.43), (A.44), and (A.49), this implies that for all $k > k_1$, the optimal triple $(\hat{x}_k, \hat{u}_k, \hat{\lambda}_k)$ is identical to $(x(\cdot; (x_{0k}, v_\lambda(x_{0k}))), 0)$, $u(x(\cdot; (x_{0k}, v_\lambda(x_{0k})), 0), \lambda(\cdot; (x_{0k}, v_\lambda(x_{0k})), 0))$, $\lambda(\cdot; (x_{0k}, v_\lambda(x_{0k})), 0))$. Next, suppose that for some $t \in [0, T]$ $(\hat{x}(t; x_0, 0), \hat{\lambda}(t; x_0, 0)) \neq (\hat{x}^*(t), \hat{\lambda}^*(t))$ (by continuity of trajectories this must hold on some subset of $[0, T]$ of nonzero measure). If, despite this, for all $t \in [0, T]$ $u(\hat{x}(t; x_0, 0), \hat{\lambda}(t; x_0, 0)) = \hat{u}^*(t)$, then by uniqueness of solutions of the state equation, we have that for all $t \in [0, T]$, $\hat{x}(t; x_0, 0) = \hat{x}^*(t)$, and hence, there are two different costate trajectories, $\hat{\lambda}(\cdot; x_0, 0)$, and $\hat{\lambda}^*(\cdot)$, corresponding to the optimal solution of $P(x_0, 0)$. Under our assumptions, this is not possible, so for some $t \in [0, T]$ $u(\hat{x}(t; x_0, 0), \hat{\lambda}(t; x_0, 0)) \neq \hat{u}^*(t)$. In view of (A.51) and continuity of the optimal control, the latter contradicts the uniqueness of the optimal solution of $P(x_0, 0)$.

We have thus shown that there do not exist sequences of initial conditions $\{x_{0i}\}_{i\in\mathbb{N}}$ which are converging to x_0, and for which the corresponding optimal triples $(\hat{x}_i, \hat{u}_i, \hat{\lambda}_i)$ are different from $(x(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0)$, $u(x(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0)$, $\lambda(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0))$, $\lambda(\cdot; (x_{0i}, v_\lambda(x_{0i})), 0))$ for sufficiently large indexes i. Hence, we have proven that for sufficiently small δ, $x(\cdot; (v_x, v_\lambda(v_x)), 0)$, $u(x(\cdot; (v_x, v_\lambda(v_x)); 0)$, $\lambda(\cdot; (v_x, v_\lambda(v_x)), 0))$, and $\lambda(\cdot; (v_x, v_\lambda(v_x)), 0)$ are not only extremal but optimal for $P(v_x, 0)$. This proves that the mappings

$$v_x \mapsto \hat{x}(\cdot; v_x, 0): B(x_0; \delta) \mapsto \mathfrak{C}^n[0, T]$$

$$v_x \mapsto \hat{\lambda}(\cdot; v_x, 0): B(x_0; \delta) \mapsto \mathfrak{C}^n[0, T]$$

are continuously differentiable. Since x_0 was arbitrary, this implies continuous differentiability everywhere. □

Proof of Proposition 6: The proof follows directly from Proposition 5 and the fact that $u(\cdot, \cdot)$ is $\mathfrak{C}^1$ in some neighborhood of $(\hat{x}, \hat{\lambda})$. More precisely, for any given x_0 (and the corresponding optimal pair $(\hat{x}, \hat{\lambda})$) and for sufficiently small ϵ, the feedback control function u, treated as a mapping $(x, \lambda) \mapsto u(x, \lambda): \mathbb{R}^n \times \mathbb{R}^n \supset T((\hat{x}, \hat{\lambda}); \epsilon) \mapsto \mathbb{R}^m$, is of class $\mathfrak{C}^1$. Thus, the same u, treated as an operator $(x(\cdot), \lambda(\cdot)) \mapsto u(x(\cdot), \lambda(\cdot)): \mathfrak{C}^n[0, T] \times \mathfrak{C}^n[0, T] \supset B_\infty((\hat{x}, \hat{\lambda}); \epsilon) \mapsto \mathfrak{C}^m[0, T]$, is continuously differentiable (this can be checked by direct calculation of the strong derivative; the continuity of the derivative is to be verified employing the induced operator norm in $\mathfrak{L}[B_\infty(\hat{x}, \hat{\lambda}); \epsilon); \mathfrak{C}^m[0, T]]$). For a sufficiently small ϵ', which also guarantees that $B_\infty(\hat{x}; \epsilon') \times B_\infty(\hat{\lambda}; \epsilon') \subset B_\infty((\hat{x}, \hat{\lambda}); \epsilon)$, the mappings $x_0 \mapsto \hat{x}(\cdot; x_0, 0): \mathbb{R}^n \supset B(x_0; \epsilon') \mapsto B_\infty(\hat{x}; \epsilon') \subset \mathfrak{C}^n[0, T]$ and $x_0 \mapsto \hat{\lambda}(\cdot; x_0, 0): \mathbb{R}^n \supset B(x_0; \epsilon') \mapsto B_\infty(\hat{\lambda}; \epsilon') \subset \mathfrak{C}^n[0, T]$ are continuously differentiable by Proposition 5. Hence, the mapping $x_0 \mapsto u(\hat{x}(\cdot; x_0, 0), \hat{\lambda}(\cdot; x_0, 0)): \mathbb{R}^n \supset B(x_0; \epsilon') \mapsto \mathfrak{C}^m[0, T]$ is continuously differentiable as it is a composition of continuously differentiable mappings. □

Proof of Proposition 7: By direct computation of the strong derivative, it is easily verified that the right-hand side of (1.2), treated as a functional $(\hat{x}, \hat{u}) \mapsto V(\hat{x}, 0; \hat{u}): \mathfrak{C}^n[0, T] \times \mathfrak{C}^m[0, T] \mapsto \mathbb{R}$, is continuously differentiable (in the induced operator norm of $\mathfrak{L}[\mathfrak{C}^n[0, T] \times \mathfrak{C}^m[0, T]; \mathbb{R}]$). Continuity of the mappings $x \mapsto \hat{x}_{x0}(x): \mathbb{R}^n \mapsto \mathfrak{L}[\mathbb{R}^n; \mathfrak{C}^n[0, T]]$ and $x \mapsto \hat{u}_{x0}(x): \mathbb{R}^n \mapsto \mathfrak{L}[\mathbb{R}^n; \mathfrak{C}^m[0, T]]$ follows from Propositions 5 and 6. Hence, $x \mapsto \hat{V}(x) \triangleq V(\hat{x}(\cdot; x, 0), 0; \hat{u}(\cdot; x, 0))$ is continuously differentiable as a composition of continuously differential mappings. □

Note Added in Proof: The authors have, subsequent to the acceptance of this paper, become aware of [13] which establishes similar results for nonlinear discrete-time systems.

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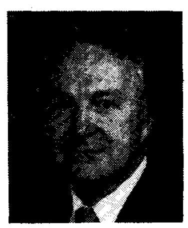
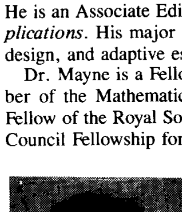


David Q. Mayne (S'75–SM'77–F'81) was born in South Africa. He received the B.Sc. and M.Sc. degrees in engineering from the University of Witwatersrand and the Ph.D. and D.Sc. degrees from the University of London.

During the period 1950 to 1959 he held the posts of Lecturer at the University of Witwatersrand and, for two years, Research and Development Engineer at the British Thomson Houston Company, Rugby, England. Since 1960 he has been with the Imperial College of Science and Technology, London, England. He became a Reader in 1967 and a Professor in 1971. He was a Visiting Research Fellow at Harvard University in 1971, and has spent several summers at the University of California, Berkeley, in collaborative research with Professor E. Polak. He is the joint author, with D. H. Jacobson, of *Differential Dynamic Programming* (New York: Elsevier) and joint Editor, with R. W. Brockett, of *Geometric Methods in System Theory* (Dordrecht: Reidel). He is an Associate Editor of the *Journal of Optimization Theory and Applications*. His major research interests are in optimization, computer-aided design, and adaptive estimation and control.

Dr. Mayne is a Fellow of the Institute of Electrical Engineers and a member of the Mathematical Programming Society. He was recently elected a Fellow of the Royal Society. He was also awarded a Senior Science Research Council Fellowship for the period 1979 to 1980.



Hannah Michalska received the M.Sc. degree in applied mathematics from the Technical University of Warsaw, Poland, and the Ph.D. degree in control systems from Imperial College of Science, Technology and Medicine, London, U.K., in 1989.

She is currently employed as a Research Fellow at the Imperial College Interdisciplinary Research Centre on Process Simulation, Integration and Control. Her research interests include control theory of nonlinear systems and optimization, with emphasis on the design of nonlinear feedback systems and stabilization.