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Discontinuous feedback stabilization of wheeled mobile robots [1]

F. U. Rehman , H. Michalska

Department of Electrical Engineering,
McGill University,
3480 University Street, Montréal, P.Q. H3A 2A7 Canada.
Email: michalsk@cim.mcgill.ca

Abstract

The paper presents a systematic approach to the construction of feedback control for stabilization to set points for a class of nonholonomic wheeled mobile robots. This approach does not necessitate conversion of the system model into a "chained form", and thus does not rely on any special transformation techniques. The feedback controls are piece-wise constant and the method is based on the introduction of a set of guiding functions whose sum vanishes only at the origin. The guiding functions are not Lyapunov functions, however, a comparison of their values allows to determine a desired direction of system motion and permits to construct a sequence of controls such that the sum of the guiding functions decreases in an average sense.

Keywords: nonlinear control, feedback stabilization, variable structure systems.

1. Introduction

Recently, there is a growing interest in the problem of feedback control design for nonholonomic systems such as, for example, mobile robots with constraints which arise when the wheels of the robots are restricted to roll without slipping. A variety of models for this type of mobile robots have been developed, and their dynamics has been studied, see [1, 2] to find other references.

Stabilization of of such systems raises practical, and theoretically challenging issues. It is well known, that nonholonomic systems (usually modelled as drift-free systems) cannot be stabilized by smooth, static feedback controls, so that feedback stabilization can be accomplished either by employing time-varying, or else discontinuous control laws.

This paper presents a simple and effective approach to the construction of such discontinuous feedback controls with application to stabilization of a class of nonholonomic mobile robots about a desired set point. Three types of robots are considered, distinguished by different degrees of mobility and steeribility, see [1] for relevant definitions. Our approach is similar to that of [3], and permits construction of stabilizing controls without resorting to any specific coordinate transformations such as the ones which bring the system to a "chained form".

The basic new concept involved is that of "guiding functions". The latter are only required to be semipositive definite and are not restricted to decrease monotonically along the controlled system trajectories. Generally, (numerous simplifications are possible) the number of guiding functions needed is equal to the number of controls in the system, and on-line information about their behaviour along the trajectories of the controlled system permits to determine stabilizing feedback control actions.

2. Stabilizing feedback construction

2.1. Assumptions

The control strategy to be presented applies to drift-free systems in the form:

$$\dot{x} = \sum_{i=1}^{m} f_i(x)v_i, \tag{1}$$

where $f_i, i \in \underline{m} \overset{def}{=} \{1, ..., m\}$, $m < n$, are linearly independent, smooth vector fields in $\mathbb{R}^n$, and v_i are Lebesgue integrable control functions on the interval $[0, \infty)$. Further assumptions are needed for the construction of a set of guiding functions whose sum can be regarded as a "Lyapunov function" (which, however, is not expected to decrease monotonically along the controlled system trajectories) :

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A1. System (1) is completely controllable in that it satisfies the Lie algebraic rank condition for controllability (LARC). Furthermore, the controllability Lie algebra for (1) is spanned only by the vector fields f_i, $i \in \underline{m}$, and their Lie brackets of depth one, so that

$$span\{f_i(x), [f_i, f_m](x), i \in \underline{m}\} = \mathbb{R}^n \quad \forall\, x \in \mathbb{R}^n$$

A2. The following distributions

$$\Delta_1(x) \stackrel{def}{=} span\{f_i(x), i \neq 1, i \in \underline{m}, [f_k, f_m](x), k \in \underline{m}\}$$

$$\Delta_2(x) \stackrel{def}{=} span\{f_i(x), i \neq 2, i \in \underline{m}, [f_k, f_m](x), k \in \underline{m}\}$$

$$\dots$$

$$\Delta_m(x) \stackrel{def}{=} span\{f_i(x), i \in \underline{m-1}\}$$

are involutive and their respective codistributions are *globally* spanned by exact differentials.

Without the loss of generality, the construction of a stabilizing feedback is carried out assuming that the desired set point is the origin of the configuration space (a suitable translation of the coordinate system may be introduced otherwise).

2.2. Guiding function construction

As guaranteed by assumption A2 , let $x \mapsto \lambda_i(x)$, $i \in \underline{m-1}$, be scalar functions such that their differentials $d\lambda_i(x)$, $i \in \underline{m-1}$ span the codistributions $\Delta_i^{\perp}(x)$, $i \in \underline{m-1}$, , at each $x \in \mathbb{R}^n$, respectively. Further, let $x \mapsto \lambda_k(x)$, $k = m, ..., n$, be such that $d\lambda_m(x), ..., d\lambda_n(x)$ span the codistribution $\Delta_m^{\perp}(x)$, for all $x \in \mathbb{R}^n$ (globally). The latter is stated as:

$$d\lambda_i(x) \perp \Delta_i(x), \quad \text{for all } i \in \underline{m-1}, \quad x \in \mathbb{R}^n$$
$$d\lambda_k(x) \perp \Delta_m(x), \quad \text{for all } k = m, ..., n, \quad x \in \mathbb{R}^n$$

The scalar functions λ_i, $i \in \underline{m}$, are used in the following definitions of semi-positive definite guiding functions :

$$V_i(x) \stackrel{def}{=} \frac{1}{2}[\lambda_i(x) - \lambda_i(0)]^2, \quad i \in \underline{m-1} \quad (2)$$

$$V_m(x) \stackrel{def}{=} \frac{1}{2}\sum_{k=m}^{n}[\lambda_k(x) - \lambda_k(0)]^2 \quad (3)$$

It can be shown, which is important for analysis, that the sum of all guiding functions $V(x) \stackrel{def}{=} \sum V_i(x)$ possesses the standard properties of a Lyapunov function, such as positive definiteness and properness.

Additionally, we assume that the following property holds which is related to the speed of convergence of the resulting feedback control, and is not restrictive:

A3. $L_{f_i}^2 V_i(x) \neq 0$ for all $x \in \mathbb{R}^n$, $i \in \underline{m}$.

The above guiding functions can serve as a tool for stabilization, as explained next.

2.3. Stabilizing feedback strategy

It is clear that for systems of type (1) the set $S \stackrel{def}{=} \{x \in \mathbb{R}^n : L_{f_i}V(x) = 0, i \in \underline{m}\}$ contains points of "impasse" in that if $p \in S$, then $\frac{d}{dt}V(x) = 0$, regardless of the controls, which makes any further instantaneous decrease in V along the trajectories of the controlled system impossible. It is precisely at such points, when the guiding functions are of help. The basic idea can be most briefly explained as follows.

It can be verified that the values of the guiding functions V_i, $i \in \underline{m}$, can be changed independently of each other by the respective controls v_i, $i \in \underline{m}$. Hence "temporary" increments in V_i, $i \in \underline{m-1}$, are of no harm, and the stabilizing strategy can be focused on decreasing V_m alone. This motivates a control strategy in which the controls v_i, $i \in \underline{m-1}$, are used to steer the system away from the set of "impasse" S, letting V_i, $i \in \underline{m-1}$, increase, while keeping the value of V_m intact (see Step 2b below). The latter is possible because, assuming piece-wise constant controls v_i, $i \in \underline{m}$, with $v_m \equiv 0$, and employing the Jacobi identity, yields

$$\frac{d}{dt}L_{f_m}V_m(x) = \sum_{i \in \underline{m}, i \neq m} L_{f_i}L_{f_m}V(x)v_i$$
$$= \sum_{i \in \underline{m}, i \neq m} L_{[f_i, f_m]}V(x)v_i \quad (4)$$

along the controlled system trajectories, where, by construction, $L_{f_m}L_{f_i}V_m(x) = 0$, for $i \in \underline{m-1}$. Equation (4) implies that the value of $L_{f_m}V_m$ can be changed from zero to non-zero (the system can be steered away from S) by employing only the controls v_i, $i \in \underline{m-1}$, which do not effect the value of V_m itself. (This is because when $p \in S$, then, by virtue of the LARC condition for complete controllability, there exists an index $k \in \underline{m-1}$ such that $L_{[f_k, f_m]}V_m(x) \neq 0$.)

When the system's state satisfies $L_{f_m}V_m(x) \neq 0$, the value of V_m can be decreased easily (see Step 2c below). After $L_{f_m}V_m(x)$ returns to zero, the remaining functions V_i, $i \in \underline{m-1}$, can be restored to their previous values (see Step 2d).

Repeating the above results in monotonic decrease in V_m while the other functions V_i, $i \in \underline{m-1}$, are oscillating in value. These oscillations can finally be "modulated" by the current value of the global cost function V, see exit condition of Step 2b.

The above translates into the following algorithmic feedback strategy :

Stabilizing feedback strategy

- **Data:** $\alpha \geq 1$.

- **1** If $x \in I\!R^n \setminus S$, then for each $i \in \underline{m}$ employ the control

$$v_k(x) = \begin{cases} -sign[L_{f_i}V_i(x)], & \text{for } k = i \\ 0, & \text{for } k \neq i \end{cases}$$

until $L_{f_i}V_i(x) = 0$.

- **2** Define $p \stackrel{def}{=} x(t)$ in which t is the time at the exit of Step 1 (when the set S is traversed). If $p = 0$ then stop, else if $p \neq 0$, then

- **2a** Select a set of indices $I \in \underline{m-1}$, such that

$$i \in I \quad \text{if} \quad L_{[f_i,f_m]}V_m(p) \neq 0,$$

- **2b** Employ the controls

$$v_k(x) = \begin{cases} 1, & \text{for } k \in I \\ 0, & \text{for } k \notin I \end{cases}$$

until, for each $i \in I$: $L_{[f_i,f_m]}V_m(x) = 0$, or else until $V_i(x) \geq \alpha V(p)$.

- **2c** Until $L_{f_m}V_m(x) = 0$, employ the controls

$$v_k(x) = \begin{cases} -sign[L_{f_m}V_m(x)], & \text{for } k = m \\ 0, & \text{for } k \neq m \end{cases}$$

- **2d** For each of the indices $i \in I$, employ the controls

$$v_k(x) = \begin{cases} -sign[L_{f_i}V_i(x)], & \text{for } k = i \\ 0, & \text{for } k \neq i \end{cases}$$

until $L_{f_i}V_i(x) = 0$, for each i. Repeat Step 2.

A quantitative analysis of the decrements in V_m yields a stabilization result.

Theorem 1 *[3] Under assumptions A1-A3, for any value of the parameter $\alpha \in (1, \infty)$, the stabilization feedback strategy is well defined, in that all steps are feasible and are exited in finite time. In particular, the controlled system converges to S in finite time.*

Every trajectory of the controlled system converges to the origin asymptotically (the origin is globally attractive).

3. Application to control of mobile robots

In the sequel, the abbreviation "WMR of type (δ_m, δ_s)" is used to denote wheeled mobile robots of degree of mobility δ_m and degree of steeribility δ_s. The application of the strategy to three classes of WMR is discussed below.

3.1. Control of WMR of type $(2,1)$

The kinematic model of WMR of type $(2,1)$ is given by, (see [1]):

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} &= \begin{pmatrix} -sin(\theta+\beta) & 0 \\ cos(\theta+\beta) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ \dot{\beta} &= \eta_3 \end{aligned} \qquad (5)$$

The notation $(x, y, \theta, \beta) = (x_1, x_2, x_3, x_4)$ and $(\eta_1, \eta_2, \eta_3) = (u_3, u_2, u_1)$, is used for simplicity, so that (5) becomes:

$$\dot{x} = X_1(x)u_1 + X_2(x)u_2 + X_3(x)u_3, \quad x \in I\!R^4 \qquad (6)$$

$$X_3(x) = -sin(x_3 + x_4)\,\frac{\partial}{\partial x_1} + cos(x_3 + x_4)\,\frac{\partial}{\partial x_2}$$

$$X_2(x) = \frac{\partial}{\partial x_3}, \quad X_1(x) = \frac{\partial}{\partial x_4} \qquad (7)$$

The Lie bracket multiplication table for the controllability Lie algebra $L(X_1, X_2, X_3)$ is given by :

$$\begin{aligned} &[X_1, X_3] = X_4,\ [X_2, X_3] = X_4,\ [X_1, X_2] = 0 \qquad (8) \\ &[X_3, X_4] = 0,\ [X_1, X_4] = -X_3,\ [X_2, X_4] = -X_3 \end{aligned}$$

where

$$X_4 = -cos(x_3 + x_4)\,\frac{\partial}{\partial x_1} - sin(x_3 + x_4)\,\frac{\partial}{\partial x_2} \qquad (9)$$

The controllability condition A1 is hence satisfied and, it follows directly from the multiplication table (8), that the following distributions are involutive :

$$\begin{aligned} \Delta_1(x) &\stackrel{def}{=} span\{X_2, X_3, X_4\}(x) \\ \Delta_2(x) &\stackrel{def}{=} span\{X_1, X_3, X_4\}(x) \\ \Delta_3(x) &\stackrel{def}{=} span\{X_1, X_2\}(x) \end{aligned}$$

From the Frobenius Theorem, it then follows that the corresponding codistributions have the following expressions as linear spans of exact differentials :

$$\begin{aligned} &\Delta_1^{\perp}(x) = span\{d\lambda_1(x)\}, \quad \Delta_2^{\perp}(x) = span\{d\lambda_2(x)\} \\ &\Delta_3^{\perp}(x) = span\{d\lambda_3(x), d\lambda_4(x)\} \end{aligned}$$

The choices for the scalar functions $\lambda_i, i = 1, ..., 4$, are immediate:

$$\lambda_1(x) = x_4,\ \lambda_2(x) = x_3,\ \lambda_3(x) = x_2,\ \lambda_4(x) = x_1$$

The guiding functions are hence given by:

$$\begin{aligned} &V_1(x) \stackrel{def}{=} \frac{1}{2}x_4^2, \quad V_2(x) \stackrel{def}{=} \frac{1}{2}x_3^2 \\ &V_3(x) \stackrel{def}{=} \frac{1}{2}(x_1^2 + x_2^2) \end{aligned} \qquad (10)$$

and are used in the application of the stabilizing strategy to model (5). The results of a simulation are shown in Figures 1-3. It can be seen that the stabilization control task is performed in finite time.

3.2. Control of WMR of type $(1,2)$

The kinematic model of the WMR of type $(1,2)$ is given by, see [1] :

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -2L cos\theta sin\beta_1 sin\beta_2 - L sin\theta sin(\beta_1+\beta_2) \\ -2L sin\theta sin\beta_1 sin\beta_2 + L cos\theta sin(\beta_1+\beta_2) \\ sin(\beta_2-\beta_1) \end{bmatrix} \eta_1$$
$$\dot{\beta}_1 = \xi_1$$
$$\dot{\beta}_2 = \xi_2 \qquad (11)$$

where (x,y) are the Cartesian coordinates of a point P of the wheeled robot platform, θ is the orientation of the platform with respect to the horizontal axis, β_i, $i=1,2$ are the orientation angles of the independent steering wheels, and L is the distance between P and the centre of the master wheel.

In this case, let the desired rest point be given by:

$$(x,y,\theta,\beta_1,\beta_2) = (0,0,0,\pi/2,\pi/2)$$

Consider the following stabilization problem for a WMR of type $(1,2)$:

Find a feedback control which stabilizes the system described by (11) on the manifold $\mathcal{M}$:

$$\mathcal{M} \stackrel{def}{=} \{(x,y,\theta,\beta_1,\beta_2)^T \in I\!R^5 : \beta_1,\beta_2 \neq 0[\pi]\} \qquad (12)$$

to the set point $x_0 = (0,0,0,\pi/2,\pi/2) \in \mathcal{M}$.

It should be noted that restricting the motion of the robot to manifold $\mathcal{M}$ is necessary for controllability purposes; at points $(0,0,0,0[\pi],0[\pi])$ the system fails to satisfy the Lie algebraic rank controllability condition.

For simplicity, we assume that $L=1$, and define:

$$\begin{aligned} (x,y,\theta,\beta_1,\beta_2) &= (x_1,x_2,x_3,x_4+\pi/2,x_5+\pi/2) \\ (u_3,u_2,u_1) &= (\eta_1,\xi_1,\xi_2) \end{aligned}$$

In this notation, (11) takes the following vector form :

$$\dot{x} = X_1(x)u_1 + X_2(x)u_2 + X_3(x)u_3, \quad x \in \mathcal{M} \qquad (13)$$
$$X_3(x) = \{-2cosx_3cosx_4cosx_5 + sinx_3 sin(x_4+x_5)\}\frac{\partial}{\partial x_1}$$
$$-\{2sinx_3cosx_4cosx_5 + cosx_3 sin(x_4+x_5)\}\frac{\partial}{\partial x_2}$$
$$+sin(x_5-x_4)\frac{\partial}{\partial x_3},$$
$$X_2(x) = \frac{\partial}{\partial x_4}, \quad X_1(x) = \frac{\partial}{\partial x_5}$$

It can be verified that, although (13) is completely controllable, there does not exist an ordering of vector fields $X_1,...,X_3$ such that the involutiveness condition A2 holds. It is thus impossible to construct guiding functions directly for (13).

Motivated by this fact, we consider an approximation to model (13) which is also completely controllable, and is obtained by substituting the nonlinear terms in the expression for the vector field X_3 by their truncated (of order one) Taylor series expansions at zero. In doing so, $sinx \approx x$, $cosx \approx 1$ and $kx \approx 0$ for $k \geq 2$, which results in the following approximate system:

$$\dot{x} = \tilde{X}_1\tilde{u}_1 + \tilde{X}_2\tilde{u}_2 + \tilde{X}_3\tilde{u}_3 \qquad (14)$$
$$\tilde{X}_3(x) = -2\frac{\partial}{\partial x_1} - (x_4+x_5)\frac{\partial}{\partial x_2} + (x_5-x_4)\frac{\partial}{\partial x_3}$$
$$\tilde{X}_2(x) = \frac{\partial}{\partial x_4} \qquad \tilde{X}_1(x) = \frac{\partial}{\partial x_5}$$

The Lie bracket multiplication table for the controllability Lie algebra $L(\tilde{X}_1,\tilde{X}_2,\tilde{X}_3)$ is given by:

$$\begin{aligned} &[\tilde{X}_1,\tilde{X}_3] = \tilde{X}_4,\ [\tilde{X}_2,\tilde{X}_3] = \tilde{X}_5,\ [\tilde{X}_1,\tilde{X}_2] = 0 \\ &[\tilde{X}_i,\tilde{X}_j] = 0,\ i=1,..3,\ j=4,5 \end{aligned} \qquad (15)$$

where the new vector fields $\tilde{X}_4$ and $\tilde{X}_5$ are linearly independent of the original vector fields $\tilde{X}_1,...,\tilde{X}_3$, for all $x \in I\!R^5$, and are given by :

$$\begin{aligned} \tilde{X}_4 &\stackrel{def}{=} [\tilde{X}_1,\tilde{X}_3] = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \\ \tilde{X}_5 &\stackrel{def}{=} [\tilde{X}_2,\tilde{X}_3] = -\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \end{aligned} \qquad (16)$$

The controllability assumption A1 is hence satisfied and, from the multiplication table (15) it follows that the distributions :

$$\begin{aligned} \Delta_1(x) &\stackrel{def}{=} span\{\tilde{X}_2,\tilde{X}_3,\tilde{X}_4,\tilde{X}_5\} \\ \Delta_2(x) &\stackrel{def}{=} span\{\tilde{X}_1,\tilde{X}_3,\tilde{X}_4,\tilde{X}_5\}, \\ \Delta_3(x) &\stackrel{def}{=} span\{\tilde{X}_1,\tilde{X}_2\} \end{aligned}$$

are all involutive. The corresponding codistributions thus have the following expressions:

$$\Delta_1^{\perp}(x) = span\{d\lambda_1(x)\}, \quad \Delta_2^{\perp}(x) = span\{d\lambda_2(x)\}$$
$$\Delta_3^{\perp}(x) = span\{d\lambda_3(x), d\lambda_4(x), d\lambda_5(x)\}$$

where the choices for the scalar functions $\lambda_i, i=1,...,5$, are again immediate: $\lambda_1(x) = x_5$, $\lambda_2(x) = x_4$,

$$\lambda_3(x) = x_3,\ \lambda_4(x) = x_2,\ \lambda_5(x) = x_1$$

The resulting guiding functions are hence given by:

$$V_1(x) \stackrel{def}{=} \frac{1}{2}x_5^2, \quad V_2(x) \stackrel{def}{=} \frac{1}{2}x_4^2$$
$$V_3(x) \stackrel{def}{=} \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \qquad (17)$$

The above guiding functions are next incorporated into the stabilizing strategy which is applied to the original system (11). Large deviations in the value of the guiding functions are prevented by a suitable choice of the parameter α of the strategy; the latter is also a way to prevent the system from leaving the manifold $\mathcal{M}$.

Simulations show that also in this case the set point is reached in a finite number of steps (in finite time). The simulated trajectories are depicted in Figures 4-5.

3.3. Control of WMR of type $(1,1)$

The kinematic state space model of WMR of type $(1,1)$ is given by, see [1]

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} &= \begin{pmatrix}-Lsin\theta sin\beta\\ Lcos\theta sin\beta\\ cos\beta\end{pmatrix}\eta_1\\ \dot{\beta} &= \xi_1\end{aligned} \tag{18}$$

where (x,y) are the Cartesian coordinates of a point P of the wheeled mobile robots platform, θ is the orientation of the platform with respect to the horizontal axis, β is the orientation angle of the independent steering wheel, and L is the distance between P and the centre of wheel. For simplicity, we take $L=1$ and denote :

$$(x,y,\theta,\beta) = (x_1,x_2,x_3,x_4), \qquad (u_1,u_2) = (\eta_1,\xi_1)$$

so that (18) is written as :

$$\dot{x} = Y_1(x)u_1 + Y_2(x)u_2, \qquad x \in I\!R^4 \tag{19}$$

$$Y_1(x) = -sinx_3sinx_4\frac{\partial}{\partial x_1} + cosx_3sinx_4\frac{\partial}{\partial x_2}$$

$$+cosx_4\frac{\partial}{\partial x_3}, \qquad Y_2(x) = \frac{\partial}{\partial x_4}$$

As before, it is readily verified that although (19) satisfies the LARC controllability condition, the controllability Lie algebra $L(Y_1,Y_2)$ is spanned by a bracket of depth one and two, so the conditions A1-A2 are not satisfied. To simplify matters, we again seek a controllable approximation to (19), such as, for example,

$$\dot{x} = \tilde{Y}_1v_1 + \tilde{Y}_3v_3 \tag{20}$$

where $Y_1(x) \approx \tilde{Y}_3(x), Y_2(x) = \tilde{Y}_1(x), v_1 = u_2, v_3 = u_1,$

$$\tilde{Y}_3(x) = -x_3x_4\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, \quad \tilde{Y}_1(x) = \frac{\partial}{\partial x_4}$$

The Lie bracket multiplication table for $L(\tilde{Y}_1,\tilde{Y}_3)$ shows that :

$$\begin{aligned}&[\tilde{Y}_3,\tilde{Y}_1] = \tilde{Y}_2,\ [\tilde{Y}_3,\tilde{Y}_2] = \tilde{Y}_4,\ [\tilde{Y}_1,\tilde{Y}_2] = 0\\ &[\tilde{Y}_3,\tilde{Y}_4] = [\tilde{Y}_1,\tilde{Y}_4] = [\tilde{Y}_2,\tilde{Y}_4] = 0\end{aligned} \tag{21}$$

where the new vector fields $\tilde{Y}_2$ and $\tilde{Y}_4$ are given by

$$\tilde{Y}_2(x) = x_3\,\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \qquad \tilde{Y}_4(x) = \frac{\partial}{\partial x_1} \tag{22}$$

so that $span\{\tilde{Y}_i, i=1,...,4\} = I\!R^4$, as needed for complete controllability. However, the vector field $\tilde{Y}_4$ is a bracket of depth two: $\tilde{Y}_4 = [\tilde{Y}_3,[\tilde{Y}_3,\tilde{Y}_1]]$, so the guiding function strategy is not applicable directly. To extend its application also to this case we consider an "extended" system to (20) which is obtained by complementing (20) with the "missing" bracket $\tilde{Y}_2$. Such extended system is then given by :

$$\dot{x} = \tilde{Y}_1v_1 + \tilde{Y}_2v_2 + \tilde{Y}_3v_3 \tag{23}$$

in which the control v_2 (or equivalently motion along the Lie bracket direction $\tilde{Y}_2 = [\tilde{Y}_3,\tilde{Y}_1]$) is realized only approximately and indirectly, by employing, for example, sinusoidal inputs of the type :

$$v_3 = u_1 = sin(\frac{2\pi}{T}t), \quad v_1 = u_2 = cos(\frac{2\pi}{T}t) \tag{24}$$

where T is some constant.

Clearly, the guiding function approach can be applied to the "extension" (23), without change, as the multiplication table (21) indicates that the distributions

$$\begin{aligned}\Delta_1(x) &= span\{\tilde{Y}_2,\tilde{Y}_3,\tilde{Y}_4\}(x)\\ \Delta_2(x) &= span\{\tilde{Y}_1,\tilde{Y}_2,\tilde{Y}_4\}(x),\ \Delta_3(x) = span\{\tilde{Y}_1,\tilde{Y}_2\}\end{aligned}$$

are involutive. The corresponding codistributions are spanned by the exact differentials of the following scalar functions

$$\lambda_1(x) = x_4,\ \lambda_2(x) = x_2,\ \lambda_3(x) = x_3,\ \lambda_4(x) = x_1 + x_2x_3$$

and result in the guiding functions:

$$\begin{aligned}&V_1(x) \stackrel{def}{=} \frac{1}{2}x_4^2,\ V_2(x) \stackrel{def}{=} \frac{1}{2}x_2^2,\\ &V_3(x) \stackrel{def}{=} \frac{1}{2}((x_1+x_2x_3)^2 + x_3^2)\end{aligned}$$

Figures 6-8 are obtained as the above are applied to the original system as part of a modified stabilization strategy, in which the controls (24) are used whenever system motion along the Lie bracket $[Y_1,Y_2]$ is required. The value $T=1$ was used. Also in this case our approach proves very effective.

References

[1] d'Andrea-Novel, B., Campion, G., and Bastin, G., "Control of nonholonomic wheeled mobile robots by state feedback linearization" *The International Journal of Robotics Research*, Vol.14, No.6, December 1995, pp. 543-559.

[2] Campion, G., Bastin, G., d'Andrea-Novel, B., "Structural properties and classification of kinematic and dynamic models of wheeled mobile robots", *IEEE Trans. on Robotics and Automat.*, Vol. 12, No.1, 1996, pp. 47-62.

[3] Michalska, H., Rehman, F. U., "Stabilization of a class of nonlinear systems through Lyapunov function decomposition", *International Journal of Control*, 1997, vol 67, No. 3, pp. 381-409.

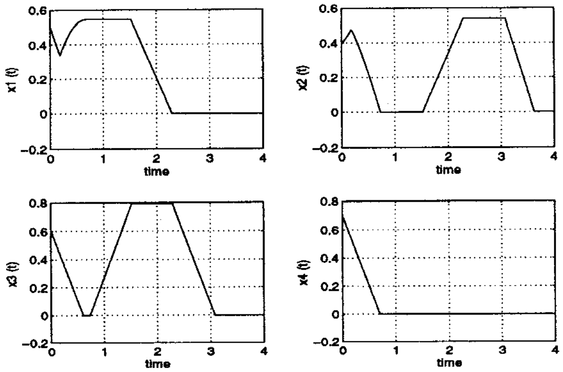

Figure 1: *WMR of type (2,1):* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_4(t))$ versus time.

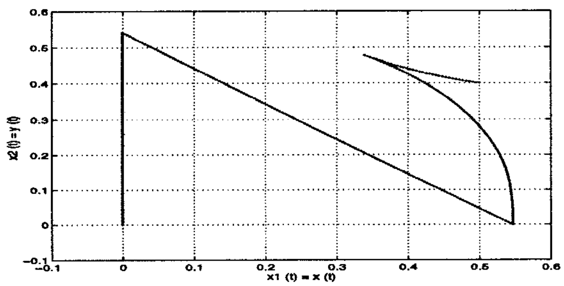

Figure 2: *WMR of type (2,1):* Plot of the controlled state trajectory $x_1(t)$ versus $x_2(t)$.

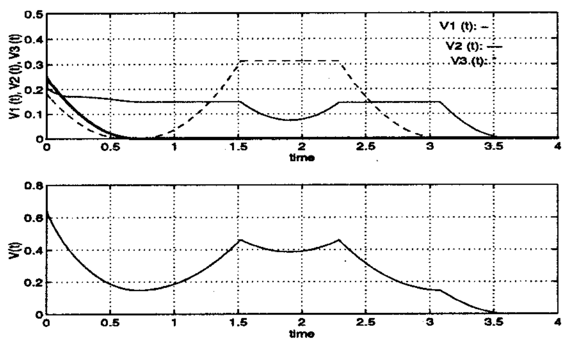

Figure 3: *WMR of type (2,1):* Plots of the guiding functions $V_1(t)$, $V_2(t)$, $V_3(t)$ and their sum $V(t)$ versus time.

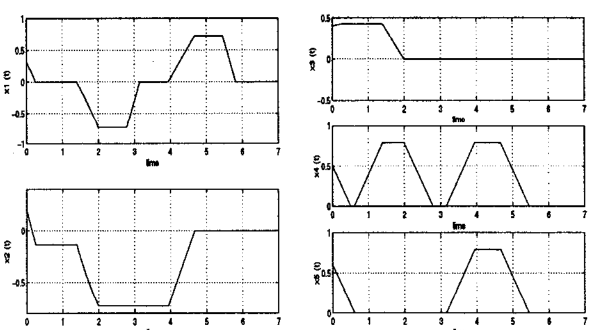

Figure 4: *WMR of type (1,2):* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_5(t))$ versus time.

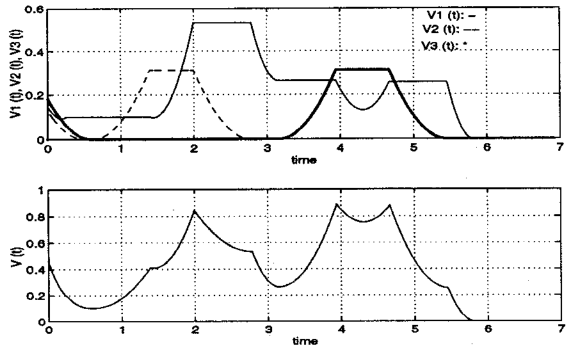

Figure 5: *WMR of type (1,2):* Plots of the guiding functions $V_1(t)$, $V_2(t)$, $V_3(t)$ and their sum $V(t)$ versus time.

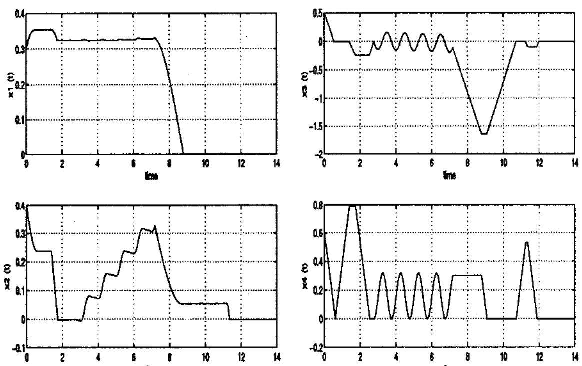

Figure 6: *WMR of type (1,1):* Plots of the controlled state trajectories $t \mapsto (x_1(t), ..., x_4(t))$ versus time.

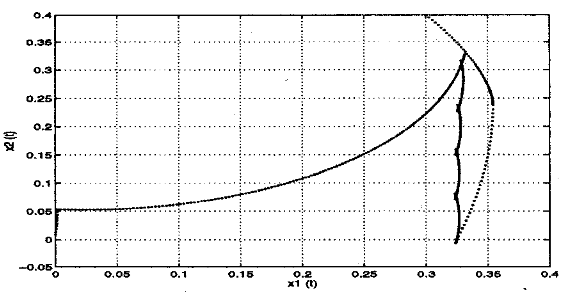

Figure 7: *WMR of type (1,1):* Plots of the controlled state variables $x_1(t)$ versus $x_2(t)$.

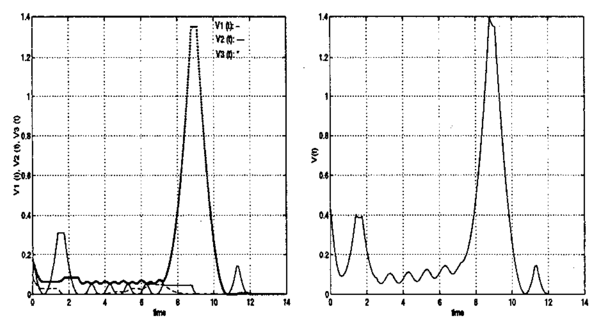

Figure 8: *WMR of type (1,1):* Plots of the guiding functions $V_1(t)$, $V_2(t)$, and their sum $V(t)$ v.s time.