[4] B. Armstrong-Helouvry, P. Dupont, and C. Canudas de Wit, "A survey of models, analysis tools and compensation methods for the control of machine with friction," *Automatica*, vol. 30, pp. 1083–1138, 1994.

[5] M. M. Bridges, D. M. Dawson, and J. Hu, "Adaptive control for a class of direct drive robot manipulators," *Int. J. Adapt. Contr. Signal Processing*, vol. 10, pp. 417–441, 1996.

[6] T. Burg, D. Dawson, J. Hu, and M. de Queiroz, "An adaptive partial state-feedback controller for RLED robot manipulators," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1024–1030, 1996.

[7] D. M. Dawson, J. J. Carroll, and M. Schneider, "Integrator backstepping control of a brush DC motor turning a robotic load," *IEEE Trans. Contr. Syst. Technol.*, vol. 2, pp. 233–244, 1994.

[8] A. De Luca and S. Panzieri, "An iterative scheme for learning gravity compensation in flexible robot arms," *Automatica*, vol. 30, pp. 993–1002, 1994.

[9] A. Jaritz and M. Spong, "An experimental comparison of robust control algorithm on a direct drive manipulator," *IEEE Trans. Contr. Syst. Technol.*, vol. 4, pp. 627–640, 1996.

[10] F. L. Lewis, C. T. Abdallah, and D. M. Dawson, *Control of Robot Manipulator*. New York: Macmillan, 1991.

[11] M. S. Mahmoud, "Robust control of robot arms including motor dynamics," *Int. J. Contr.*, vol. 58, pp. 853–873, 1993.

[12] H. Olsson, K. J. Astrom, C. Canudas de Wit, M. Gafvert, and P. Lischinsky, "Friction Models and Friction Compensation," Laboratoire d'Automatique de Grenoble and the Lund Univ., Contract TFR 95-759 and Contract EDF P29L13/2L3074/EP663, 1997.

[13] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: Wiley, 1989.

[14] T.-J. Tarn, A. K. Bejczy, and X. Yun, "Effect of motor dynamics on nonlinear feedback robot arm control," *IEEE Trans. Robot. Automat.*, vol. 7, pp. 114–122, 1991.

[15] P. Tomei, "A simple PD controller for robots with elastic joints," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1208–1213, 1991.

[16] M. Vidyasagar, *Nonlinear Systems Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 1993.


# Tracking Control of Uncertain Dynamic Nonholonomic System and Its Application to Wheeled Mobile Robots

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***Abstract*—This paper considers the tracking problem of dynamic nonholonomic systems with unknown inertia parameters. A new controller is proposed. The proposed controller not only ensures the entire state of the dynamic system asymptotically track the desired trajectory, but also is characterized by low dimension and the absence of singular points. Simulation results show that the proposed controller is effective.**

***Index Terms*—Adaptive control, nonholonomic system, nonlinear system, tracking control, uncertain system.**

## I. Introduction

Recently, there has been a growing interest in the design of feedback control laws for nonholonomic systems [7]. Due to Brockett's theorem, it is well known that a nonholonomic control system cannot be asymptotically stabilized to a rest configuration by differentiable pure-state feedback laws [1]. However, several approaches have been proposed for stabilizing them [7].

Another control problem of the nonholonomic system is the trajectory tracking problem. Depending on whether the nonholonomic system is represented by a kinematic or dynamic model, the tracking problem can be classified as either a kinematic tracking problem or a dynamic tracking problem. Several authors have studied the kinematic tracking problem (see [11], [3], [5], and [6] for reference). However, many nonholonomic systems in reality have significant dynamics. Therefore, it will be more realistic to consider the tracking problem of the dynamic models than that of the kinematic models. Recognizing the importance of considering the system dynamics, several researchers have started to pay attention to this problem in recent years. Su [10] studied the dynamic tracking problem of the nonholonomic system with unknown inertia parameters. Chen [2] discussed the dynamic tracking problem with uncertainty using $H_\infty$ techniques. Unfortunately, in these two papers the nonholonomic constraints of the system were not made use of in full, so that the dynamic tracking problem with uncertainty was not solved completely. In [4], the trajectory tracking control problem of the dynamic nonholonomic systems was discussed when the system dynamics are not known. Robust controllers were proposed. However, these results are only suitable for dealing with the nonholonomic systems with nonparameter uncertainty.

In this paper, we consider the dynamic tracking problem of a class of the nonholonomic system with unknown constant inertia parameters. A new controller is proposed. Comparing with the existing results of the tracking problem of the nonholonomic system, novelty of the proposed

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controllers lies in that the dynamic tracking problem of a class of nonholonomic systems with unknown constant inertia parameters is solved completely.

## II. Problem Statement

Consider the general mechanical system with nonholonomic constraints expressed in the following form [1]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B(q)\tau + J^T(q)\lambda \tag{1}$$
$$J(q)\dot{q} = 0 \tag{2}$$

where $q = [q_1, \ldots, q_n]^T$ is an $n$-vector of the generalized system variables, $M(q)$ is an $n \times n$ bounded positive-definite symmetric matrix, $C(q,\dot{q})\dot{q}$ presents an $n$-vector of centripetal and Coriolis torque, $G(q)$ is an $n$-vector of gravitational torque, $B(q)$ is an $n \times r$ input transformation matrix, $\tau$ is an $r$-vector of control input, $J(q)$ is an $(n-(m+1)) \times n$ full rank matrix with $m+1 = n - \text{Rank}(J(q))$, $2 \le m+1 < n$, $r \ge m+1$. $\lambda$ is an $(n-m-1)$-vector of Lagrange multiplier which expresses the constraint force, and the superscript $T$ denotes the transpose. Let $a$ be the inertia parameter vector of the system (1)–(2) [8], we assume $a$ is a constant vector and completely unknown. In the system (1)–(2), the constraint (2) is assumed to be completely nonholonomic [9]. Two simplifying properties should be noted about (1) [4], [8], [10].

*Property 1:* $\dot{M} - 2C$ is skew-symmetric for a proper definition of $C$.

*Property 2:* For any differentiable vector $\xi \in R^n$, $M(q)\dot{\xi} + C(q,\dot{q})\xi + G(q) = Y(q,\dot{q},\xi,\dot{\xi})a$, where the regressor matrix $Y(q,\dot{q},\xi,\dot{\xi})$ is a known matrix of $q$, $\dot{q}$, $\xi$, and $\dot{\xi}$.

Given a differentiable desired trajectory $y(t) \in R^n$, which satisfies the nonholonomic constraint

$$J(y)\dot{y} = 0 \tag{3}$$

the dynamic tracking problem discussed in this paper is defined as finding a feedback control law $\tau$ such that $\lim_{t\to\infty}(q-y) = 0$ and $\lim_{t\to\infty}(\dot{q}-\dot{y}) = 0$ when the inertia parameter vector $a$ is unknown.

To solve the dynamic tracking problem, we eliminate the Lagrange multiplier $\lambda$ first. Following [1] and [4], let the vector fields $g_1(q), \ldots, g_{m+1}(q)$ form a basis of the null space of $J(q)$. Then by (2), there exists an $(m+1)$-vector $v = [v_1, \ldots, v_{m+1}]^T$ such that

$$\dot{q} = g(q)v = g_1(q)v_1 + \cdots + g_{m+1}(q)v_{m+1} \tag{4}$$

where $g(q) = [g_1(q), \ldots, g_{m+1}(q)]^T$. Differentiating both sides of (4) gives $\ddot{q} = \dot{g}(q)v + g(q)\dot{v}$. Substituting it into (1) and multiplying both sides by $g^T(q)$, one obtains

$$M_1(q)\dot{v} + C_1(q,\dot{q})v + G_1(q) = B_1(q)\tau \tag{5}$$

where $M_1(q) = g^T(q)M(q)g(q)$, $C_1(q,\dot{q}) = g^T(q)M(q)\dot{g}(q) + g^T(q)C(q,\dot{q})g(q)$, $G_1(q) = g^T(q)G(q)$, $B_1(q) = g^T(q)B(q)$. In order to completely actuate the nonholonomic system, $B_1(q)$ is assumed to be a full rank matrix.

In order to make the dynamic tracking problem solvable, we assume that there exists a diffeomorphic transformation

$$\begin{aligned} x &= [x_1, x_{2,1}, \ldots, x_{n_1,1}, \ldots, x_{2,m}, \ldots, x_{n_m,m}]^T = \phi(q) \\ u &= [u_1, \ldots, u_{m+1}]^T = \varphi(q)v \end{aligned} \tag{6}$$

such that (4) can be converted into the extended one-generator multichain form [12]. That is, we assume that there exists the transformation (6) such that the system (4)–(5) can be converted into the canonical form

$$\begin{aligned} &\dot{x}_1 = u_1, \qquad \dot{x}_{i,j} = u_1 x_{i+1,j}, \qquad (2 \le i \le n_j - 1) \\ &\dot{x}_{n_j,j} = u_{j+1}, \qquad (1 \le j \le m) \end{aligned} \tag{7}$$

$$M_2(x)\dot{u} + C_2(x,\dot{x})u + G_2(x) = B_2(x)\tau \tag{8}$$

where $M_2 = \varphi^{-T}(q)M_1(q)\varphi^{-1}(q)|_{q=\phi^{-1}(x)}$, $C_2 = \varphi^{-T}(q)(C_1(q,\dot{q}) \cdot \varphi^{-1}(q) - M_1(q)\varphi^{-1}(q)\dot{\varphi}(q)\varphi^{-1}(q))|_{q=\phi^{-1}(x)}$, $G_2 = \varphi^{-T}(q) \cdot G_1(q)|_{q=\phi^{-1}(x)}$, $B_2 = \varphi^{-T}(q)B_1(q)|_{q=\phi^{-1}(x)}$, $\sum_{j=1}^m n_j - m + 1 = n$.

*Remark 1:* The necessary and sufficient conditions for the existence of (6) can be found in [9] and [12].

Based on Properties 1 and 2, it can be easily proved that (8) satisfies the following two properties.

*Property 3:* $\dot{M}_2 - 2C_2$ is skew-symmetric.

*Property 4:* For any differentiable vector $\xi \in R^{(m+1)}$, $M_2(x)\dot{\xi} + C_2(x,\dot{x})\xi + G_2(x) = Y_2(x,\dot{x},\xi,\dot{\xi})a$ where $Y_2(x,\dot{x},\xi,\dot{\xi})$ is a known matrix of $x$, $\dot{x}$, $\xi$, and $\dot{\xi}$.

Since $y(t)$ satisfies (3), $y(t)$ can be generated by the virtual system $\dot{y} = g(y)v^*$, where $v^* = [v_1^*, \ldots, v_{m+1}^*]^T$ is a time-varying reference control. Noting the assumption made about (4) and (5), there exist $z = [z_1, z_{2,1}, \ldots, z_{n_1,1}, \ldots, z_{2,m}, \ldots, z_{n_m,m}]^T = \phi(y)$ and $w = [w_1, \ldots, w_{m+1}]^T = \varphi(y)v^*$ such that

$$\begin{aligned} &\dot{z}_1 = w_1, \qquad \dot{z}_{i,j} = w_1 z_{i+1,j}, \qquad (2 \le i \le n_j - 1) \\ &\dot{z}_{n_j,j} = w_{j+1}, \qquad (1 \le j \le m). \end{aligned} \tag{9}$$

With the above assumption and transformations, the dynamic tracking problem in this paper is equivalent to find the control law $\tau$ of the system (7)–(8) such that

$$\lim_{t\to+\infty}(x-z) = 0 \qquad \lim_{t\to+\infty}(\dot{x}-\dot{z}) = 0. \tag{10}$$

## III. Controller Design

In this section, we design the control law $\tau$ of the system (7)–(8) with the unknown inertia parameter vector $a$ such that (10) is achieved in three steps. First, an appropriate tracking error is introduced to aid designing the control law. Then, the dynamical part (8) of the system (7)–(8) is neglected and $u$ is assumed to be the virtual control. We consider the kinematic tracking problem of the system (7) and design the control input $u = \eta$ (see (13) for $\eta$) such that (10) holds. Finally, the control law $\tau$ is designed based on the kinematic controller $\eta$ with ideas borrowed from the well-known backstepping technique.

Define the nonsingular constant matrices $\Psi^l = \{\psi_{i,j}^l\} \in R^{n_l \times n_l}(1 \le l \le m)$ by

$$\begin{aligned} &\psi_{i,i}^l = 1, \quad (i = 1,2) \qquad \psi_{1,j}^l = 0, \quad (2 \le j \le n_l) \\ &\psi_{2,j}^l = 0, \quad (3 \le j \le n_l) \qquad \psi_{i,1}^l = 0, \quad (2 \le i \le n_l) \\ &\psi_{i,j}^l = k_{i-3,l}\psi_{i-2,j}^l + \psi_{i-1,j-1}^l, \quad (3 \le i \le n_l, 2 \le j \le n_l) \end{aligned}$$

where $k_{0,l} = 1$ and $k_{i,l}$ are positive constants. $\Psi_1^l$ denotes the resulting matrix after eliminating the first row and the first column of the matrix $\Psi^l$. Let $\Psi = \text{diag}[\Psi^1, \Psi_1^2, \ldots, \Psi_1^m] \in R^{n\times n}$, the tracking error is defined as

$$e = [e_1, e_{2,1}, \ldots, e_{n_1,1}, \ldots, e_{2,m}, \ldots, e_{n_m,m}]^T = \Psi \cdot (x-z).$$

So (10) is equivalent to $\lim_{t\to\infty} e = 0$ and $\lim_{t\to\infty} \dot{e} = 0$.

As a precursor to the full dynamic system study, let us first neglect the dynamics of the system (7)–(8) and consider the kinematic tracking problem of the system (7) only. With the aid of the tracking error $e$, a new controller of the kinematic tracking problem is proposed in the following lemma.

*Lemma 1:* Consider the kinematic tracking problem of the system (7). Assume that $z_{i,j}(3 \le i \le n_j, 1 \le j \le m)$, $w(t)$ and $\dot{w}(t)$ are bounded. Then, the control law

$$u = \eta \tag{11}$$

$$\dot{p} = -\mu_2 p - \mu_1 e_1 - \sum_{l=1}^{m} \sum_{j=2}^{n_l - 1} \left[ \sum_{i=2}^{j} \frac{e_{j,l}\psi_{j,i}^{l} x_{i+1,l}}{k_{0,l}k_{1,l}\cdots k_{j-2,l}} + \frac{e_{n_l,l}\psi_{n_l,j}^{l} x_{j+1,l}}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}} \right] \tag{12}$$

can make $e$, $\dot{e}$, and $p$ uniformly bounded. Furthermore, if $w_1(t)$ does not converge to zero, the controller (11)–(12) can make $e$, $\dot{e}$, and $p$ asymptotically tend to zero, where $\eta$ is defined in (13), shown at the bottom of the page, the control parameters $k_{i,l} > 0 (1 \le i \le n_l, 1 \le l \le m)$, $\mu_i > 0 (1 \le i \le 2)$, and $\mu_{3,l} > 0 (1 \le l \le m)$.

*Proof:* See the Appendix. ■

Next, we design the more realistic control law $\tau$ of the dynamic system (7)–(8) with unknown constant inertia parameter vector $a$ such that (10) holds. Based on the result in Lemma 1, a natural idea is to design $\tau$ such that $u$ tends to $\eta$ asymptotically. However, since the system (7)–(8) is nonlinear, it is well known that (10) may not hold even if $u$ tends to $\eta$ fast enough due to the peaking-phenomenon. To overcome this, we make full use of the ideas of the well-known backstepping techniques and the passivity property of the mechanical systems. By introducing a cross-term $\Lambda$ (see Theorem 1) in the controller, the peaking-phenomenon is prevented. Since the inertia parameter vector $a$ is unknown, a suitable adaptive law is used to update the estimated inertia parameters in the controller. Let $\hat{a}$ be the estimated value of $a$, $\hat{M}_2$, $\hat{C}_2$, and $\hat{G}_2$ be the values of $M_2$, $C_2$, and $G_2$ corresponding to the estimated parameter vector $\hat{a}$, respectively, with aid of Lemma 1, the following theorem can be proved.

*Theorem 1:* Consider the system (7)–(8) with unknown constant inertia parameter vector $a$ and with the desired trajectory $z$ given by (9) under the following assumptions.

1) $z_{i,j}(3 \le i \le n_j, 1 \le j \le m)$, $w$, and $\dot{w}$ are bounded.

2) If one of $M_2(x)$, $C_2(x,\dot{x})$, and $G_2(x)$ is unbounded on $x_1$, $z_1$ is assumed to be bounded. If one of $M_2(x)$, $C_2(x,\dot{x})$ and $G_2(x)$ is unbounded on $x_{2,j}$, $z_{2,j}$ is assumed to be bounded, for $j = 1, 2, \ldots, m$, respectively.

Then, the control law

$$\tau = B_2^{\#}(x)[\hat{M}_2(x)\dot{\eta} + \hat{C}_2(x,\dot{x})\eta + \hat{G}_2(x) - K_p(u - \eta) - \Lambda] \tag{14}$$

and (12) with the adaptive law

$$\dot{\hat{a}} = -\Gamma^{-1} Y_2^T(x, \dot{x}, \eta, \dot{\eta})(u - \eta) \tag{15}$$

can make $e, \dot{e}, p$, and $\hat{a}$ uniformly bounded. Furthermore, if $w_1$ satisfies the following additional assumption.

3) $\lim_{t\to\infty} \inf |w_1(t)| = \epsilon > 0$.

The controller (14), (15) can make $e$, $\dot{e}$, and $p$ asymptotically tend to zero, where $^{\#}$ is any left inverse, $\eta$ is defined in (13), $\Lambda$ is defined as shown in the equation at the bottom of the page, and the control parameters $k_{i,j} > 0 (1 \le i \le n_j - 2, 1 \le j \le m)$, $\mu_i > 0 (1 \le i \le 2)$, $\mu_{3,l} > 0 (1 \le l \le m)$, $K_p$ and $\Gamma$ are positive-definite matrices.

*Proof:* See the Appendix. ■

*Remark 2:* The physical meaning of the controller (14)–(15) is that the controller consists of an inner-loop velocity controller and a outer-loop torque controller. The inner-loop velocity controller is used to make the system trajectory asymptotically converge to the desired trajectory. The ideal inner-loop control command is $\eta$. The outer-loop torque controller is used to realize the inner-loop control command $\eta$ by an adaptive control. In the controller (14)–(15), $\Lambda$ is a cross-term to prevent the peaking-phenomenon of the overall system. The adaptive law (15) is used to update the estimated inertia parameter vector $\hat{a}$. The control parameters are $k_{i,j}(1 \le i \le n_j - 2, 1 \le j \le m)$, $\mu_i > 0 (1 \le i \le 2)$, $\mu_{3,j} > 0 (1 \le l \le m)$, $K_p$, and $\Gamma$ which are only necessary to be positive or positive-definite. Large control parameters generally result in fast convergence of the tracking errors.

*Remark 3:* In the theorem, the controller (14)–(15) is an adaptive one. The assumptions made about the desired trajectory $z$ are easily satisfied. In the control, the estimated value $\hat{a}$ is bounded and does not necessarily converge to its real value $a$. The controller (14)–(15) can make $x$ and $\dot{x}$ asymptotically tend to $z$ and $\dot{z}$, respectively. Therefore,

$$\eta = \begin{bmatrix} w_1 + p \\ w_2 - \mu_{3,1}e_{n_1,1} - k_{n_1-2,1}w_1 e_{n_1-1,1} - w_1 \sum_{i=2}^{n_1-1} \psi_{n_1,i}^{1}(x_{i+1,1} - z_{i+1,1}) \\ \vdots \\ w_{m+1} - \mu_{3,m}e_{n_m,m} - k_{n_m-2,m}w_1 e_{n_m-1,m} - w_1 \sum_{i=2}^{n_m-1} \psi_{n_m,i}^{m}(x_{i+1,m} - z_{i+1,m}) \end{bmatrix} \tag{13}$$

$$\Lambda = [\Lambda_1, \Lambda_2, \ldots, \Lambda_{m+1}]^T = \begin{bmatrix} \mu_1 e_1 + \sum_{l=1}^{m} \sum_{j=2}^{n_l-1} \left[ \sum_{i=2}^{j} \frac{e_{j,l}\psi_{j,i}^{l} x_{i+1,l}}{k_{0,l}k_{1,l}\cdots k_{j-2,l}} + \frac{e_{n+l,l}\psi_{n_l,j}^{l} x_{j+1,l}}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}} \right] \\ \frac{e_{n_1,1}}{k_{1,1}k_{2,1}\cdots k_{n_1-2,1}} \\ \vdots \\ \frac{e_{n_m,m}}{k_{1,m}k_{2,m}\cdots k_{n_m-2,m}} \end{bmatrix}$$

the dynamic tracking problem of the nonholonomic system with unknown inertia parameters is solved completely. This is one of the contributions of this paper. There are only limited reports on the dynamic tracking problem of the nonholonomic system with uncertainty [10], [2], [4], and they cannot solve the dynamic tracking problem defined in this paper. Some corollaries can be obtained from the theorem, which are omitted here for space limit.

## IV. Application to a Wheeled Mobile Robot

In this section, the results obtained in Section III are applied to the tracking control of a wheeled mobile robot discussed in the simulation of [4]. Let $(x_Q, y_Q, \theta)$ be the configuration of the mobile robot, the dynamics of the system are described by (1)–(2) with $q = [x_Q, y_Q, \theta]^T$, $M(q) = \mathrm{diag}[m, m, I_Q]$, $C(q, \dot{q}) = 0$, $G(q) = 0$, $J(q) = [\cos\theta, \sin\theta, 0]$, $B(q) = (1/R)$ $[col\{-\sin\theta, \cos\theta, L\}, col\{-\sin\theta, \cos\theta, -L\}]$, and $a = [m, I_Q]^T$, where $m$ is the mass of the mobile robot, $I_Q$ is its inertia moment around the vertical axis at point Q, $R$ is the radius of the wheels, $2L$ is the length of the axis of the front wheels (see [4] for details).

Given a desired trajectory $y = [y_1, y_2, y_3]^T$ with $y_1 = \cos(1.5t)$, $y_2 = 4\sin(1.5t)$, $y_3$ is determined by the nonholonomic constraint $\dot{y}_1 \cos y_3 + \dot{y}_2 \sin y_3 = 0$. The dynamic tracking problem is to design $\tau$ such that $\lim_{t\to\infty} (q - y) = 0$ and $\lim_{t\to\infty} (\dot{q} - \dot{y}) = 0$ when the inertia parameter vector $a$ is unknown. Let $g(q) = [col\{-\sin q_3, \cos q_3, 0\}, col\{0, 0, 1\}]$ and the transformation (6) as $x_1 = q_3$, $x_{2,1} = q_1 \cos q_3 + q_2 \sin q_3$, $x_{3,1} = -q_1 \sin q_3 + q_2 \cos q_3$, $u_1 = v_2$ and $u_2 = v_1 - (q_1 \cos q_3 + q_2 \sin q_3)v_2$, by the procedure in Section II, the controller, which is omitted here for space limit, can be easily obtained by Theorem 1.

In the simulation, we suppose that the real inertia parameter vector $a$ = [10 kg, 10 kg·m$^2$]$^T$, $R = L = 1$ m, the initial condition $q(0)$ = [2.7 m, $-1.5$ m, 0.2 rad] and $\dot{q}(0)$ = [$-3.7$ m/s, 18.2527 m/s, 0.2 rad/s]. In the controller, we select the control parameters $\mu_1 = 12.25$, $\mu_2 = 7$, $\mu_{3,1} = 6$, $k_{1,1} = 1$, $K_p = \mathrm{diag}[5, 5]$, $\Gamma = \mathrm{diag}[1, 1]$, $p(0) = 0.5$, and the initial estimated inertia parameter vector $\hat{a}(0)$ = [12 kg, 12 kg·m$^2$]$^T$. Responses of $(q - y)$ and $(\dot{q} - \dot{y})$ are shown in Figs. 1 and 2, respectively. Fig. 3 shows the curves of $y_1$ versus $y_2$ and $q_1$ versus $q_2$. From the figures, the effectiveness of the controller is shown.

## Appendix

*Proof of Lemma 1:* With the control law (11)–(12), the closed loop of the system (7) can be written as follows:

$$\begin{cases} \dot{e}_1 = p, \dot{e}_{2,j} = w_1 e_{3,j} + p x_{3,j} \\ \dot{e}_{i+3,j} = w_1(-k_{i+1,j} e_{i+2,j} + e_{i+4,j}) + p \sum\limits_{l=2}^{i+3} \psi^j_{i+3,l} x_{l+1,j}, \\ \qquad (0 \le i \le n_j - 4; 1 \le j \le m) \\ \dot{e}_{n_j,j} = -\mu_{3,j} e_{n_j,j} - k_{n_j-2,j} w_1 e_{n_j-1,j} + p \sum\limits_{l=2}^{n_j-1} \psi^j_{n_j,l} x_{l+1,j} \\ \dot{p} = -\mu_2 p - \Lambda_1. \end{cases}$$

Let $V = (1/2)(p^2 + \mu_1 e_1^2) + (1/2) \sum_{j=1}^m \sum_{i=2}^{n_j} e_{i,j}^2/(k_{0,j} \cdots k_{i-2,j})$, differentiating $V$ along the closed-loop system yields

$$\dot{V} = -\mu_2 p^2 - \sum_{j=1}^m \mu_{3,j} e_{n_j,j}^2/(k_{1,j} \cdots k_{n_j-2,j}) \le 0.$$

Therefore, $V$ is nonincreasing and converges to a limiting value $V_{\lim} \ge 0$. Furthermore, $p$ and $e$ are bounded. From the assumption made about $z$, $x_{i,j}(3 \le i \le n_j, 1 \le j \le m)$ are bounded. Noting $w$ is bounded, $\dot{e}$ and $\dot{p}$ are bounded. Therefore, $p$ and $e$ are uniformly bounded, and $\dot{x}$ is bounded. By differentiating $\dot{e}$, we can prove that $\dot{e}$ is uniformly bounded.

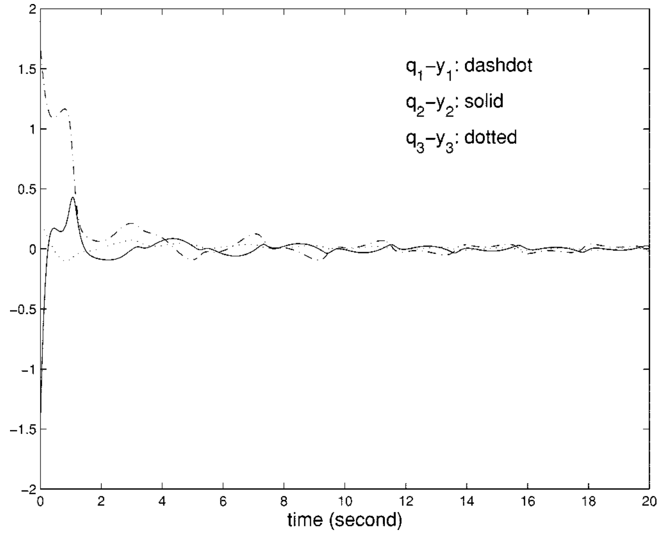

Fig. 1. Response of $(q - y)$.

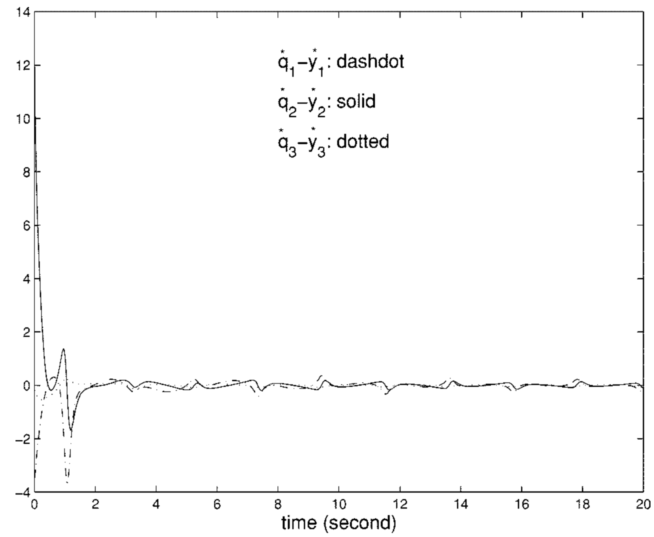

Fig. 2. Response of $(\dot{q} - \dot{y})$.

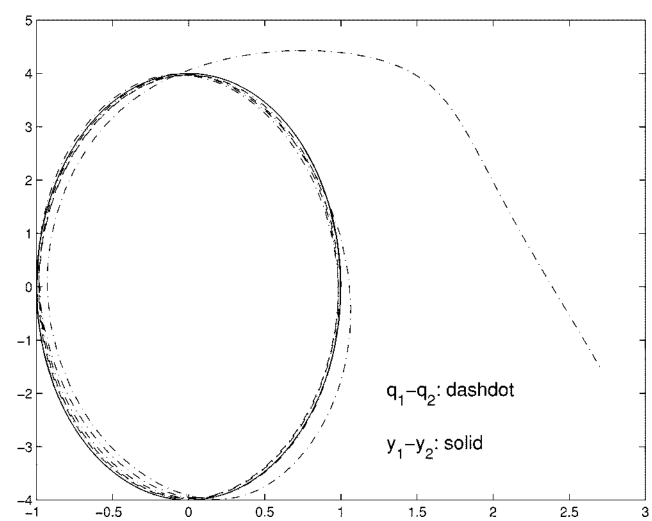

Fig. 3. Curves of $y_1$ versus $y_2$ and $q_1$ versus $q_2$.

Since $\ddot{V} = -2\mu_2 p\dot{p} - \sum_{j=1}^{m} 2\mu_{3,j} e_{n_j,j}\dot{e}_{n_j,j}/(k_{1,j}\cdots k_{n_j-2,j})$ is bounded, $\dot{V}$ is uniformly continuous. By [4, Lemma 1], $\dot{V}$ tends to zero. Therefore, $p$ and $e_{n_j,j}(1 \le j \le m)$ tend to zero, respectively. Since $w_1$ is bounded, $w_1^2 e_{n_j,j}(1 \le j \le m)$ tends to zero. Differentiating $w_1^2 e_{n_j,j}(1 \le j \le m)$ yields $(d/dt)(w_1^2 e_{n_j,j}) = -k_{n_j-2,j} w_1^3 e_{n_j-1,j} + [2w_1 e_{n_j,j}\dot{w}_1 - \mu_{3,j} w_1^2 e_{n_j,j} + w_1^2 p \sum_{i=2}^{n_j-1} \psi_{n_j,i}^j x_{i+1,j}]$ where the first term is uniformly continuous, and the other terms tend to zero. By [4, Lemma 1], $(d/dt)(w_1^2 e_{n_j,j})$ tends to zero; thus $w_1^3 e_{n_j-1,j}$ tends to zero. Furthermore, $w_1^2 e_{n_j-1,j}$ and $w_1 e_{n_j-1,j}$ converge to zero for $1 \le j \le m$. Differentiating $w_1^2 e_{i,j}(i = n_j - 1, \ldots, 2; 1 \le j \le m)$, and repeating the above reasoning, it can be proved that $w_1 e_{i,j}$ converges to zero. By differentiating $e$, $\dot{e}$ tends to zero. Differentiating $w_1^2 p$ yields $(d/dt)(w_1^2 p) = -\mu_1 w_1^2 e_1 + 2w_1\dot{w}_1 p - \mu_2 w_1^2 p - w_1^2 \sum_{l=1}^{m} [\sum_{j=2}^{n_l-1}\sum_{i=2}^{j} e_{j,l}\psi_{j,i}^l x_{i+1,l} / (k_{0,l}k_{1,l}\cdots k_{j-2,l}) + \sum_{i=2}^{n_l-1} e_{n_l,l}\psi_{n_l,i}^l x_{i+1,l}/(k_{1,l}\cdots k_{n_l-2,l})]$ where the first term is uniformly continuous, the other terms tend to zero. Noting that $w_1^2 p$ converges to zero, by application of [4, Lemma 1], $(d/dt)(w_1^2 p)$ converges to zero; therefore $w_1^2 e_1$ converges to zero. Furthermore, $w_1 e_1$ converges to zero.

Since $w_1 e_1$, $w_1 e_{i,j}(1 \le i \le n_j; 1 \le j \le m)$, and $p$ tend to zero, $w_1^2 V$ converges to zero. Since $V$ has a limit $V_{\lim}$, and $w_1$ does not converge to zero, $V_{\lim}$ is necessarily equal to zero. Therefore, $e$ converges to zero. In summary, we have proved that $p$, $\dot{e}$, and $e$ tend to zero, respectively. ■

*Proof of Theorem 1:* Let $\tilde{u} = u - \eta = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m+1}]^T$, $\tilde{a} = \hat{a} - a$, the closed-loop system can be written as

$$\begin{cases} \dot{e}_1 = p + \tilde{u}_1, \quad \dot{e}_{2,j} = w_1 e_{3,j} + (p + \tilde{u}_1)x_{3,j}, \\ \dot{e}_{i+3,j} = w_1(-k_{i+1,j}e_{i+2,j} + e_{i+4,j}) \\ \quad +(p+\tilde{u}_1)\sum_{l=2}^{i+3} \psi_{i+3,l}^j x_{l+1,j}, \qquad (0 \le i \le n_j - 4) \\ \dot{e}_{n_j,j} = -\mu_{3,j}e_{n_j,j} - k_{n_j-2,j}w_1 e_{n_j-1,j} + \tilde{u}_{j+1} \\ \quad +(p+\tilde{u}_1)\sum_{l=2}^{n_j-1} \psi_{n_j,l}^j x_{l+1,j}, \qquad (1 \le j \le m) \\ M_2\dot{\tilde{u}} = Y_2(x,\dot{x},\eta,\dot{\eta})\tilde{a} - C_2(x,\dot{x})\tilde{u} - K_p\tilde{u} - \Lambda \\ \dot{\tilde{a}} = -\Gamma^{-1}Y_2^T(x,\dot{x},\eta,\dot{\eta})\tilde{u}, \dot{p} = -\mu_2 p - \Lambda_1. \end{cases} \tag{16}$$

Consider the function $V = (1/2)(p^2 + \mu_1 e_1^2 + \tilde{u}^T M_2 \tilde{u} + \tilde{a}^T\Gamma\tilde{a}) + (1/2)\sum_{j=1}^{m}\sum_{i=2}^{n_j} e_{i,j}^2/(k_{0,j}\cdots k_{i-2,j})$ differentiating $V$ along (16) yields $\dot{V} = -\mu_2 p^2 - \sum_{j=1}^{m} \mu_{3,j}e_{n_j,j}^2/(k_{1,j}\cdots k_{n_j-2,j}) - \tilde{u}^T K_p\tilde{u} \le 0$. Therefore, $V$ is nonincreasing and converges to a limiting value $V_{\lim} \ge 0$, and $p$, $e$, $\tilde{u}$ and $\tilde{a}$ are bounded. In view of (16) and the assumptions in Theorem 1, $\dot{p}$, $\dot{e}$, $\ddot{e}$, $\dot{\tilde{a}}$, and $\dot{\tilde{u}}$ are bounded; thus $p$, $e$, $\dot{e}$, and $\hat{a}$ are uniformly bounded, and $\dot{x}$ is bounded.

Since $\ddot{V} = -2\mu_2 p\dot{p} - \sum_{j=1}^{m} 2\mu_{3,j}e_{n_j,j}\dot{e}_{n_j,j}/(k_{1,j}\cdots k_{n_j-2,j}) - \dot{\tilde{u}}^T K_p\tilde{u} - \tilde{u}^T K_p\dot{\tilde{u}}$ is bounded, $\dot{V}$ is uniformly continuous. So $\dot{V}$ tends to zero. Therefore, $p$, $e_{n_j,j}(1 \le j \le m)$, and $\tilde{u}$ tend to zero. Since $w_1$ is bounded, $w_1^2 e_{n_j,j}(1 \le j \le m)$ tends to zero. Differentiating $w_1^2 e_{n_j,j}(1 \le j \le m)$ yields $(d/dt)(w_1^2 e_{n_j,j}) = -k_{n_j-2,j}w_1^3 e_{n_j-1,j} + [2w_1 e_{n_j,j}\dot{w}_1 - \mu_{3,j}w_1^2 e_{n_j,j} + w_1^2\tilde{u}_{j+1} + w_1^2(p+\tilde{u}_1)\sum_{i=2}^{n_j-1}\psi_{n_j,i}^j x_{i+1,j}]$ where the first term is uniformly continuous, and the other terms tend to zero. By [4, Lemma 1], $(d/dt)(w_1^2 e_{n_j,j})$ tends to zero; thus $w_1^3 e_{n_j-1,j}$ tends to zero. Furthermore, $w_1^2 e_{n_j-1,j}$ and $w_1 e_{n_j-1,j}$ converge to zero for $1 \le j \le m$. Differentiating $w_1^2 e_{i,j}(i = n_j - 1, \ldots, 2; 1 \le j \le m)$, and repeating the above reasoning, it can be proved that $w_1 e_{i,j}$ converges to zero. By virtue of (16), $\dot{e}$ tends to zero. Noting Assumption 3, it is easy to prove that $e_{i,j}$ tend to zero. By differentiating $w_1^2 p$, it is easy to prove that $w_1^2 e_1$ converges to zero. Furthermore, $e_1$ converges to zero by Assumption 3. Therefore, we have proved that $p$, $\dot{e}$, and $e$ tend to zero, and $\hat{a}$ is bounded. ■

## References


[1] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch, "Control and stabilization of nonholonomic dynamic systems," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1746–1757, Nov. 1992.

[2] B. S. Chen, T. S. Lee, and W. S. Chang, "A robust $H^\infty$ model reference tracking design for nonholonomic mechanical control systems," *Int. J. Contr.*, vol. 63, pp. 283–306, 1996.

[3] B. d'Andrea-Novel, G. Campion, and G. Bastin, "Control of nonholonomic wheeled mobile robots by state feedback linearization," *Int. J. Robot. Res.*, vol. 14, pp. 543–559, 1995.

[4] W. Dong, W. Huo, and W. L. Xu, "Trajectory tracking control of dynamic nonholonomic systems with unknown dynamics," *Int. J. Robust Nonlinear Contr.*, vol. 9, no. 13, pp. 905–922, 1999.

[5] M. Fliess, J. Levine, P. Martin, and P. Rouchon, "Design of trajectory stabilizing feedback for driftless flat systems," in *Proc. Eur. Control Conf.*, 1995, pp. 1882–1887.

[6] Z.-P. Jiang and N. Nijmeijer, "A recursive technique for tracking control of nonholonomic systems in chained form," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 256–279, Jan. 1999.

[7] I. Kolmanovsky and N. H. McClamroch, "Development in nonholonomic control problems," *IEEE Contr. Syst. Mag.*, vol. 15, pp. 20–36, Jan. 1995.

[8] F. Lewis, C. Abdallah, and D. Dawson, *Control of Robot Manipulators*. New York: Macmillan, 1993.

[9] R. M. Murray and S. S. Sastry, "Nonholonomic motion planning: Steering using sinusoids," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 700–716, May 1993.

[10] C. Y. Su and Y. Stepanenko, "Robust motion/force control of mechanical systems with classical nonholonomic constraints," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 609–614, Mar. 1994.

[11] G. Walsh, D. Tilbury, S. S. Sastry, R. M. Murray, and J. P. Laumond, "Stabilization of trajectories for systems with nonholonomic constraints," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 216–222, Jan. 1994.

[12] G. Walsh and L. G. Bushnell, "Stabilization of multiple input chained form control systems," *Syst. Contr. Lett.*, vol. 25, pp. 227–234, 1995.