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Brief Paper

On Receding Horizon Feedback Control*

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Key Words—Receding horizon; nonlinear systems; inverse optimal control; stability; Lyapunov methods.

Abstract—Receding horizon feedback control (RHFC) was originally introduced as an easy method for designing stable state-feedback controllers for linear systems. Here those results are generalized to the control of nonlinear autonomous systems, and we develop a performance index which is minimized by the RHFC (inverse optimal control problem). Previous results for linear systems have shown that desirable nonlinear controllers can be developed by making the RHFC horizon distance a function of the state. That functional dependence was implicit and difficult to implement on-line. Here we develop similar controllers for which the horizon distance is an easily computed explicit function of the state.

1. *Introduction*

KLEINMAN (1970) proposed a simple method to stabilize a linear time-invariant controllable plant. Later, this method was also derived by using the concept of a receding horizon by Thomas (1975). Recently, Kwon and Pearson (1975) extended the concept of receding horizon to the stabilization of time-varying linear plants by considering Riccati equations only over finite time intervals.

In another direction, Shaw, Thomas and Sarlat (1977; 1978) have considered making the receding-horizon state dependent and getting nonlinear controllers for linear time-invariant plants. These controllers can respond relatively faster to large regulation errors than to small errors, when compared to linear controllers. Shaw and Chen (1980) extended the result to more general nonlinears controllers by making any scalar design parameter state dependent under appropriate conditions.

In this paper, we mainly accomplish three things: (i) We establish the asymptotic stability of receding horizon feedback controls for a general class of *nonlinear* autonomous systems; (ii) we weaken the required condition for asymptotic stability when the receding horizon is made state dependent; (an example is given in which the horizon distance is an explicit function of the state); and (iii) we solve the inverse optimal control problem for receding horizon feedback control.

The concept of receding horizon feedback control is defined with respect to a finite-time optimal control problem. Consider the system

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \tag{1}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Let $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(T) = \mathbf{0}$ and consider the cost functional

$$V(\mathbf{x}_0; T) = \int_0^T L(\mathbf{x}, \mathbf{u})\,\mathrm{d}t. \tag{2}$$

Assume there exists an optimal control function $\mathbf{u}^*(t; T, \mathbf{x}_0)$, $0 \le t < T$, which gives the minimal cost $V^*(\mathbf{x}_0; T)$ and yields the required zero terminal condition. Then, $\mathbf{u}+(\mathbf{x}; T) = \mathbf{u}^*(0; T, \mathbf{x})$ is called a receding horizon feedback control (RHFC) of plant (1) according to cost $L(\mathbf{x}, \mathbf{u})$ with the receding horizon distance T.

The physical interpretation of this definition is that the controller always assumes that it wants to drive the state vector to $\mathbf{0}$ after an additional time T, while minimizing the cost functional (2), with the current state vector as the initial condition. The name 'receding horizon' comes from this apparent terminal time which is always T units in the future. The time-invariant RHFC is faster in response if the parameter T is smaller.

In Section 2, we show that under appropriate conditions, the RHFC leads to an asymptotically stable closed-loop system. Even more, the receding horizon distance T can be made state-dependent, in an appropriate manner, without destroying the asymptotic stability. In Section 3, we apply our result to the design of nonlinear controllers for linear time-invariant plants. The appendix contains the mathematical proofs of the theorems presented in Section 2.

2. *Asymptotic stability of* RHFC

Here, we derive sufficient conditions for the closed-loop RHFC system to be asymptotically stable. Assume the optimal cost $V^*(\mathbf{x}; T)$ defined in the previous section satisfies the following conditions for any $T > 0$.

(a) $V^*(\mathbf{0}; T) = 0$ and $V^*(\mathbf{x}; T) > 0$ for $x \neq 0$;
(b) $V^*(\mathbf{x}; T) \to \infty$ when $\|\mathbf{x}\| \to \infty$;
(c) $(\partial V^*(\mathbf{x}; T)/\partial x_i)$ exists for every component x_i of vector $\mathbf{x}$; and the function $L(\mathbf{x}, \mathbf{u})$ satisfies the following conditions:
(d) $L(\mathbf{x}, \mathbf{u}) \ge 0$ for any $\mathbf{x}$, $\mathbf{u}$; $L(\mathbf{0}, \mathbf{0}) = 0$;
(e) If $\mathbf{u} \neq \mathbf{0}$, then $L(\mathbf{x}, \mathbf{u}) > 0$ for any $\mathbf{x}$.

Comments:
(i) (d) and (e) imply (a).
(ii) All the conditions mentioned above hold in the linear plant-quadratic cost RHFC problems studied earlier.
(iii) The derivatives in (c) are not necessarily continuous in time.

Now, we want to show that the scalar function $V^*(\mathbf{x}; T)$, defined through the optimal control problem, can be used as a Lyapunov function for the RHFC. This is achieved by comparing the behavior of $V^*(\mathbf{x}; T)$ when the original optimal finite-time control (OFTC) is applied, and when RHFC is applied.

First, we show that the scalar function $V^*(\mathbf{x}; T)$ decreases along $\mathbf{x}^*(t)$, the trajectory under OFTC. Because of the principle of optimality, the final portion of an optimal trajectory is itself optimal, so we have

$$V^*(\mathbf{x}_0; T) - V^*[\mathbf{x}^*(\Delta t); T - \Delta t] = \int_0^{\Delta t} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau; \mathbf{x}_0, T)]\,\mathrm{d}\tau. \tag{3}$$

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Subtracting $V^*[\mathbf{x}^*(\Delta t); T]$ from both sides, we get

$$V^*(\mathbf{x}_0; T) - V^*[\mathbf{x}^*(\Delta t); T] = \int_0^{\Delta t} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau; \mathbf{x}_0, T)]\,\mathrm{d}\tau + \{V^*[\mathbf{x}^*(\Delta t); T - \Delta t] - V^*[\mathbf{x}^*(\Delta t); T]\}. \quad (4)$$

If we can show the last term in (4) is nonnegative, then the scalar function $V^*(\mathbf{x}; T)$ is nonincreasing along the trajectory $\mathbf{x}^*(t)$. Actually, we have the following lemma.

Lemma 1. If $0 < T_1 < T_2$, then $V^*(\mathbf{x}; T_1) \geq V^*(\mathbf{x}; T_2)$ for any $\mathbf{x}$.

Proof: The control function $\mathbf{u}^*(t; \mathbf{x}, T_1)$, $0 \leq t < T_1$, gives the cost $V^*(\mathbf{x}; T_1)$ and the terminal condition $\mathbf{x}(T_1) = \mathbf{0}$. Construct a new control function as follows:

$$\mathbf{u}'(t; \mathbf{x}, T_1, T_2) = \begin{cases} \mathbf{u}^*(t; \mathbf{x}, T_1) & \text{for} \quad 0 \leq t < T_1 \\ 0 & \text{for} \quad T_1 \leq t < T_2. \end{cases}$$

Because $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$, and $L(\mathbf{0}, \mathbf{0}) = 0$; $\mathbf{u}'(t; \mathbf{x}, T_1, T_2)$ gives the cost $V^*(\mathbf{x}; T_1)$ and the terminal condition $\mathbf{x}(T_2) = \mathbf{0}$. From the definition of $V^*(\mathbf{x}; T_2)$, we have $V^*(\mathbf{x}; T_1) \geq V^*(\mathbf{x}; T_2)$.

The fact that $V^*(\mathbf{x}; T)$ is also decreasing along trajectories with RHFC is proved in the Appendix as part of the proof of Theorem 1.

Additional information is needed to prove that the closed-loop RHFC is asymptotically stable (i.e. that the decreasing V^* converges to zero). We impose the following assumption:

(f) any solution of the free-running system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{0})$ either is unbounded or converges to $\mathbf{0}$.

In the special case of a linear time-invariant plant, this assumption means there is no open-loop pole on the imaginary axis. More generally, limit cycles are not permitted in the free-running system.

Under the assumptions (a)–(f), we have the following theorem (proved in the appendix).

Theorem 1. For any fixed $T > 0$, the closed-loop system of RHFC is asymptotically stable in the large.

It is known that RHFC is asymptotically stable for linear plants when L is quadratic, without assumption (f). However, that is proved using controllability properties of the plant which are difficult to generalize to nonlinear plants.

Now, let us consider the possibility of making T state-dependent, instead of keeping it fixed. We want to know what is the sufficient condition for this kind of closed-loop system to be asymptotically stable in the large. [The utility of a state dependent $T(\mathbf{x})$ to achieve faster response for bigger errors has been described in Shaw (1979).]

Theorem 2. Under the assumptions (a)–(f), if the total time derivative of $T(\mathbf{x})$ along the trajectory is greater than or equal to -1, then the RHFC with the state dependent $T(\mathbf{x})$ makes the closed-loop system asymptotically stable in the large.

Section 3 shows how it is possible to easily construct nonlinear controllers for linear time-invariant plants, in which the state dependent $T(\mathbf{x})$ satisfies the sufficient condition on $\mathrm{d}T/\mathrm{d}t$.

3. *Nonlinear feedback control for the linear time-invariant plant*

In this section, we propose a class of nonlinear feedback controls for the linear time-invariant plant by applying Theorem 2.

Consider the linear time-invariant plant described by

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad (5)$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, and none of the eigenvalues of A are on the imaginary axis.

Designing the RHFC for this plant according to the cost $L(\mathbf{x}, \mathbf{u}) = \mathbf{u}'R\mathbf{u}$ with the horizon distance T, we get (Thomas, 1975)

$$\mathbf{u} = -R^{-1}B'W^{-1}(T)\mathbf{x} \quad (6)$$

where $W(T)$ is defined through

$$\frac{\mathrm{d}W(T)}{\mathrm{d}T} = -AW - WA' + BR^{-1}B', \quad W(0) = 0 \quad (7)$$

or explicitly

$$W(T) = \int_0^T \mathrm{e}^{-A\tau}BR^{-1}B'\,\mathrm{e}^{-A'\tau}\,\mathrm{d}\tau. \quad (8)$$

Now, we want to consider the case when T depends on the current state vector through a decreasing monotone function $T(\|\mathbf{x}\|_Q^2)$, where $\|\mathbf{x}\|_Q^2 = \mathbf{x}'Q\mathbf{x}$ and Q is a positive definite matrix. Intuitively, this will lead to a shorter horizon and faster response when $\|\mathbf{x}\|_Q^2$ is bigger. The matrix Q can be chosen to emphasize the relative importance of some components of $\mathbf{x}$. Then, the feedback control law becomes of the following nonlinear form

$$\mathbf{u} = -R^{-1}B'W^{-1}[T(\|\mathbf{x}\|_Q^2)]\mathbf{x}. \quad (9)$$

Consideration of performance for small $\mathbf{x}$ and limitations on $\|\mathbf{u}\|$ lead us to choose a T function satisfying

$$0 < T_1 \leq T(\|\mathbf{x}\|_Q^2) \leq T_2 \quad \text{for any } \mathbf{x} \in R^n. \quad (10)$$

Define $G(T)$ to be inverse function of $T(\|\mathbf{x}\|_Q^2)$, then we have

$$G[T(\|\mathbf{x}\|_Q^2)] = \|\mathbf{x}\|_Q^2. \quad (11)$$

Taking the time derivative on both sides of (11), we get

$$\frac{\partial G(T)}{\partial T}\dot{T} = 2\mathbf{x}'QA\mathbf{x} - 2\mathbf{x}'QBR^{-1}B'W^{-1}(T)\mathbf{x}. \quad (12)$$

Using the spectral norm, which is the maximum eigenvalue $[\lambda_{\max}(Q)]$ for a symmetric, positive definite Q, we get

$$\frac{\partial G(T)}{\partial T}\dot{T} \leq 2\|\mathbf{x}\|^2\|Q\|[\|A\| + \|BR^{-1}B'\| \cdot \|W^{-1}(T)\|] \quad (13)$$

$$\leq 2\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\|\mathbf{x}\|_Q^2[\|A\| + \|BR^{-1}B'\| \cdot \|W^{-1}(T)\|]. \quad (14)$$

But $\|\mathbf{x}\|_Q^2 = G(T)$, and $\|W^{-1}(T)\|$ is a nonincreasing function, so

$$\frac{\partial G(T)/\partial T}{G(T)}\dot{T} \leq 2\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}[\|A\| + \|BR^{-1}B'\| \cdot \|W^{-1}(T_1)\|]. \quad (15)$$

Using M to represent the right-side of (15), a sufficient condition to get $\dot{T} > -1$ is

$$\frac{\partial}{\partial T}\log[G(T)] < -M. \quad (16)$$

The following example demonstrates the feasibility of this approach. Consider $\log[G(T)]$ of the following form

$$\log[G(T)] = \frac{a[T - (T_1 + T_2)/2)]}{(T - T_1)(T - T_2)}; \quad T_1 < T < T_2. \quad (17)$$

Then it is easy to see that when $T \to T_1^+$, $\log[G(T)] \to \infty$, when $T \to T_2^-$, $\log[G(T)] \to -\infty$. Taking the derivative

$$\frac{\mathrm{d}}{\mathrm{d}T}\log[G(T)] = -\frac{a}{2}[\alpha^{-2} + (T_2 - T_1 - \alpha)^{-2}] \quad (18)$$

where $\alpha = (T - T_1)$ and $0 < \alpha < T_2 - T_1$. The bracketed term in (18) is >0, with a minimum at $\alpha=(T_2-T_1)/2$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}T}\log[G(T)]\le\frac{-4a}{(T_2-T_1)^2} \tag{19}$$

and (16) is assured if the parameter a is chosen so that

$$\frac{4a}{(T_2-T_1)^2}>M. \tag{20}$$

Now, we want to find the inverse function of $G(T)$. Let

$$y=\log[G(T)]=\log(\|\mathbf{x}\|_Q^2). \tag{21}$$

Then, we have

$$\bar{T}(y)=\frac{y(T_1+T_2)+a-\sqrt{[(T_2-T_1)^2y^2+a^2]}}{2y}. \tag{22}$$

Now, the nonlinear feedback control law is defined by (9), (21) and (22) with $T=\bar{T}(y)$.

Example. Consider the plant in (5) with

$$A=\begin{pmatrix}1 & 4\\ 0 & 2\end{pmatrix},\ B=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

In designing the nonlinear controller for this plant, we let

$$Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ R=1,\ T_1=0.4,\ T_2=1.2,\ \text{and } a=15.0.$$

The nonlinear response is demonstrated in Fig. 1, where curves for the larger initial conditions are plotted with a scaling of 0.01. The x_1 response has a relatively smaller overshoot for the larger initial condition. These results correspond to the following feedback matrix:

$$-R^{-1}B'W^{-1}=\left[\frac{3-12e^{-3\bar{T}(y)}+9e^{-4\bar{T}(y)}}{2-18e^{-2\bar{T}(y)}+32e^{-3\bar{T}(y)}-18e^{-4\bar{T}(y)}+2e^{-6\bar{T}(y)}}\right.$$
$$\left.\cdot\frac{12-72e^{-2\bar{T}(y)}+96e^{-3\bar{T}(y)}-36e^{-4\bar{T}(y)}}{2-18e^{-2\bar{T}(y)}+32e^{-3\bar{T}(y)}-18e^{-4\bar{T}(y)}+2e^{-6\bar{T}(y)}}\right].$$

4. *The inverse optimal control problem*

In this section, we show that the RHFC has an inverse optimal control solution when the stability conditions hold. In other words, we show that an asymptotically stable (in the large) RHFC minimizes some appropriate cost functional.

Theorem 3. If $\lim_{\mathbf{x}\to\mathbf{0}}T(\mathbf{x})=T_0>0$, if the RHFC stability conditions listed in Section 2 hold, and if the set of admissible control functions is composed of all those which yield $\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{0}$; then the RHFC minimizes the following cost functional

$$c(u)=\int_0^\infty\left\{L(x,u)-\frac{\partial V^*(x;T)}{\partial T}[1+D(x,u)]\right\}\mathrm{d}t$$

and the optimal cost is $V^*[\mathbf{x}(0);T(\mathbf{x}(0))]$. This is proved by using the finite-time Hamilton–Jacobi equation (see appendix) to show that any other admissible controller yields a greater value for the cost $c(u)$. See Chen (1981) for details.

There are two interesting comments on this result. The minimal value of $c(u)$ over the infinite interval is numerically equal to the minimal V^* in the related finite-time problem, if $\mathbf{x}(0)$ is the same in both models. In the linear case discussed in Section 3, with fixed T, $c(u)$ becomes

$$c(u)=\int_0^\infty[u'Ru+x'Q_1x]\,\mathrm{d}t$$

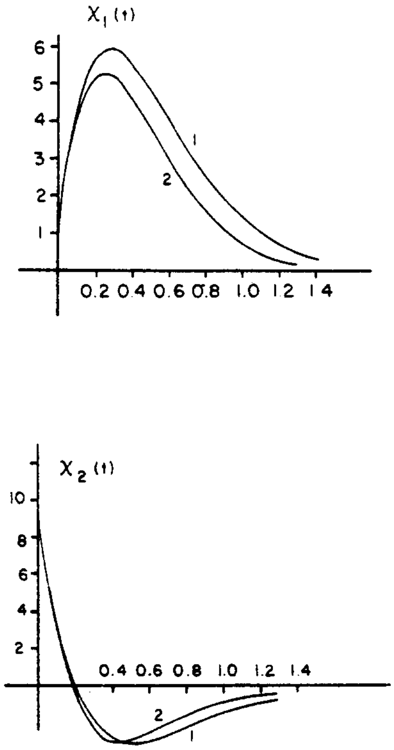

FIG. 1. Responses for two sets of initial conditions. Curve 1 has $\mathbf{x}'(0)=(1,10)$. Curve 2 has $\mathbf{x}'(0)=(100,1000)$.

where

$$Q_1=-\frac{\mathrm{d}W^{-1}(T)}{\mathrm{d}T}\ge0.$$

5. *Conclusion*

This paper has generalized previous stability results on receding horizon feedback controllers, for linear systems, to the case of autonomous nonlinear systems. In general, applying these results to design feedback control systems requires solving the nonlinear finite time optimal control problem which was discussed in Willemstein (1977). A suitable infinite time performance measure was found to be minimized by the RHFC. Finally, previous results on nonlinear controllers based on a state-dependent receding horizon were modified to get a much more practical class of controllers.

While the design procedure of Section 3 assures asymptotically stable controllers, any application would require simulation studies for a range of anticipated initial conditions in order to select appropriate T_1, T_2, Q, a, etc., with regard to the desired faster response to bigger errors and to implicit constraints on state variables and control amplitudes. It is straightforward to modify these regulator results to the servo tracking case, as in Shaw (1978), and to incorporate the use of observers when the full state is not available. While the regulator performance seems useful, servo examples designed in this way have exhibited poor initial transient response, apparently due to the method's emphasis on asymptotic behavior. The results from Vidyasagar (1980) on overall stability of a stable observer (Kuo, Elliott and Tarn, 1975) combined with a stable state feedback controller, apply in this case.

The method used in Section 3 to get an explicit $T(\mathbf{x})$ for a state-dependent feedback gain can be adapted in a parallel manner to the case of a nonlinear controller based on the modified receding horizon controls of Kwon and Pearson (1977) and Shaw and Chen (1980).

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Appendix: Proof of stability theorems

Proof of Theorem 1. The Hamilton–Jacobi equation for the optimal finite time problems gives (Lee and Marcus, 1967)

$$-\frac{\partial J^*(t,\mathbf{x},T)}{\partial t} = L + \left[\frac{\partial J^*(t,\mathbf{x},T)}{\partial \mathbf{x}}\right]' f[\mathbf{x},\mathbf{u}^*(t;x(0),T)] = \min\left\{L + \left[\frac{\partial J^*(t,\mathbf{x},T)}{\partial \mathbf{x}}\right]' f(\mathbf{x},\mathbf{u})\right\}. \tag{A.1}$$

Because the plant is autonomous, we have

$$-\frac{\partial J^*(t,\mathbf{x},T)}{\partial t}\bigg|_{t=0} = \frac{\partial V^*(\mathbf{x};T)}{\partial T}. \tag{A.2}$$

At the time instant $t = 0$, (A.1) becomes

$$\frac{\partial V^*(\mathbf{x};T)}{\partial T} = L + \left[\frac{\partial V^*(\mathbf{x};T)}{\partial \mathbf{x}}\right]' f[\mathbf{x},\mathbf{u}^*(0;x,T)]. \tag{A.3}$$

But $\mathbf{u}^*(0;\mathbf{x},T) = \mathbf{u}+(\mathbf{x};T)$ by the definition of RHFC. Therefore

$$\frac{\partial V^*(\mathbf{x};T)}{\partial T} = L + \left[\frac{\partial V^*(\mathbf{x};T)}{\partial \mathbf{x}}\right]' f[\mathbf{x},\mathbf{u}+(\mathbf{x},T)]. \tag{A.4}$$

We notice that

$$\left[\frac{\partial V^*(\mathbf{x};T)}{\partial \mathbf{x}}\right]' f[\mathbf{x},\mathbf{u}+(\mathbf{x},T)]$$

is just the total time derivative of $V^*(\mathbf{x};T)$ when $\mathbf{u}+(\mathbf{x};T)$, the RHFC, is applied. Thus

$$\dot V(\mathbf{x};T)\Big|_{\text{RHFC is applied}} = -L + \frac{\partial V^*(\mathbf{x};T)}{\partial T}. \tag{A.5}$$

Lemma 1 tells us that $(\partial V^*(\mathbf{x};T)/\partial T)$ is nonpositive. Thus

$$\dot V(\mathbf{x};T)\Big|_{\text{RHFC is applied}} \le 0. \tag{A.6}$$

Because of the assumption (b) on $V^*(\mathbf{x};T)$, (A.6) guarantees that the closed-loop trajectory $\mathbf{x}+(t)$ is bounded. Since the closed-loop system resulting from RHFC is autonomous, as the time goes to infinity, the bounded trajectory $\mathbf{x}+(t)$ converges to the largest invariant subset of S_0 (LaSalle, 1960), where S_0 is defined as

$$S_0\{\mathbf{x}|\,\dot V(\mathbf{x}) = 0\}. \tag{A.7}$$

Denote this largest invariant set of S_0 as N. Because of (A.6), Lemma 1, and the assumption (e) on L, we have

$$S_0 \subset S_1 = \{\mathbf{x}|\mathbf{u}+(\mathbf{x};T) = \mathbf{0}\}. \tag{A.8}$$

But we have assumed that the solution of the free-running system either is unbounded or converges to $\mathbf{0}$. Therefore, according to assumption (a) in Section 2, the only invariant set in S_0 is $\{\mathbf{0}\}$. As a consequence, $N = \{\mathbf{0}\}$ and this theorem is proved.

Proof of Theorem 2. First, we reconsider (A.1) when the control is $\mathbf{u}^*[t;\mathbf{x}(0),T(\mathbf{x}(0))]$. In this case, at $t = 0$, $x = x(0)$, or at any t, $x(t)$ when u^* is used, (A.4) becomes

$$\frac{\partial V^*(\mathbf{x};T)}{\partial T}\Big|_{T=T(\mathbf{x})} = L + \left[\frac{\partial V^*[\mathbf{x},T(y)]}{\partial \mathbf{x}}\Big|_{y=\mathbf{x}}\right] f[\mathbf{x},\mathbf{u}+(\mathbf{x};T(\mathbf{x}))]. \tag{A.9}$$

Introducing the dependence of T on $\mathbf{x}$

$$\frac{\mathrm{d}\,V^*[\mathbf{x};T(\mathbf{x})]}{\mathrm{d}\mathbf{x}} = \frac{\partial V^*(\mathbf{x};T)}{\partial \mathbf{x}} + \frac{\partial V^*(\mathbf{x};T)}{\partial T}\cdot\frac{\mathrm{d}T(\mathbf{x})}{\mathrm{d}\mathbf{x}} \tag{A.10}$$

and the definition

$$D(\mathbf{x},\mathbf{u}) = \left[\frac{\mathrm{d}T(\mathbf{x})}{\mathrm{d}\mathbf{x}}\right]' f(\mathbf{x},\mathbf{u}) = \frac{\mathrm{d}T(\mathbf{x})}{\mathrm{d}t} \tag{A.11}$$

we get

$\dot V[\mathbf{x};T(\mathbf{x})]$ RHFC is applied

$$= -L + [1 + D(\mathbf{x},\mathbf{u}+(\mathbf{x};T(\mathbf{x}))]\frac{\partial V^*(\mathbf{x};T)}{\partial T}\bigg|_{T=T(\mathbf{x})} \tag{A.12}$$

Using the assumptions and Lemma 1, $V \le 0$ and the rest of the proof follows that of Theorem 1, see Chen (1981) for details.