Control of Nonholonomic Systems with Drift Terms

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Abstract

In the present paper nonholonomic systems with drift terms are studied. The discussion is focused on a class of Lagrangian systems with a cyclic coordinate. We present an approach to open–loop path planning in which the system evolution is studied on manifolds of dimension equal to the number of control inputs. A control algorithm is derived and it is applied to the planar diver. A similar algorithm is derived for the study of what states can be reached within a given time. An exponentially stabilizing feedback controller is derived for tracking of the planned trajectories. The results are illustrated with simulations.

1 Introduction

Driftless nonholonomic control systems have been studied in recent years by Walsh and Sastry (1991), Teel *et al.* (1992), Murray and Sastry (1993), Bloch *et al.* (1993), Kolmanovsky and McClamroch (1995), and others. Several important results have been derived based on the structure of Lie algebras generated by the control vector fields. A dual point of view using exterior differential systems was developed by Murray (1994), Tilbury *et al.* (1995), and Tilbury and Sastry (1995). The discussion of nonholonomic systems with drift in the literature has been concentrated on the so-called dynamic extension of drift free systems by Kapitanovsky *et al.* (1993) and M'Closkey and Murray (1994). A dynamic extension is an addition of integrators to the velocity inputs. Walsh *et al.* (1993) have also considered steering on the group of rotations for left–variant systems with drift terms.

An approach to motion planning for nonlinear systems which has received increasing attention in the last few years is the concept of differential flatness (Rouchon *et al.*, 1993). Flatness is locally equivalent to feedback linearizability, and may be used to simplify the design of trajectories. A sufficient condition for flatness is given in (Martin, 1993) for systems where the dimension of the control vector is one less than the dimension of the state vector. Lagrangian systems of this class where the flat outputs depend on the configuration variables are called configuration flat (Rathinam and Murray, 1996). Controllability properties of Lagrangian systems have also been studied by Lewis and Murray (1995), and dissipative systems by Kelly and Murray (1996).

Classification of nonholonomic systems has also been explored in the literature. A powerful motivation for finding simple standardized forms is to generate reusable control schemes that can be applied to classes of nonlinear control systems. The search for canonical forms by (Murray and Sastry, 1993) defined the chained form, which has since gained much popularity. This form is equivalent to the power form (Pomet and Samson, 1993). Sinusoids have been used in motion planning for these systems because of their smoothness and periodic properties. Systems that can be put in chained form are in a subset of the larger class of nilpotentizable (Di Giamberardino *et al.*, 1996) systems. A system is nilpotent when the Lie algebra is finite dimensional and all Lie brackets of order higher than a finite integer are zero (Kawski, 1988). When the Lie algebra is nilpotent, it is possible to write the solution for the state as a composition of a finite number of solutions to possibly less complicated differential equations. If a system is nilpotent, or nilpotentizable by a diffeomorphic transformation, then Kawski (1993) gives a simple algorithm for transformation to a canonical nonlinear representation of the system. Nilpotent approximations are useful for local stabilization (Hermes *et al.*, 1984). An extension of the derivation of canonical forms to systems with drift terms is a case for further study.

A class of Lagrangian systems with a cyclic coordinate is investigated in this paper as a step toward extending the understanding of control of systems with drift terms. The eventual goal is an increased repertoire of canonical forms. From Noether's Theorem (Arnold, 1989) it follows that if a Lagrangian system admits a one parameter group of diffeomorphisms, i.e. there is some kind of symmetry in the system such that the Lagrangian is invariant under some mapping, then a conservation law and a first integral exist. A cyclic coordinate means that the Lagrangian is invariant under, for example, a translation of this coordinate. The first integral will be a conserved generalized momentum conjugate (Goldstein, 1980). These systems exhibit a subtriangular structure, as the Jacobian of the vector fields with respect to the state is subtriangular. This structure will be utilized in the design below. If the constant value of this first integral is nonzero, then there is drift in the system. The example considered here is the planar diver. In this example the cyclic coordinate is the body angle, and the generalized momentum is the angular momentum of the diver.

The main focus of the present paper is on path planning, the problem of finding inputs that steer the system from an initial state to a desired state. The proposed approach is to consider piecewise constant inputs. In physical systems there is usually an upper bound on how large inputs can be. Further, it is well known that minimum–time control of systems with constrained inputs often results in bang–bang control, in which the control values are at the boundary of the allowed set of inputs. An algorithm to find a path that connects a given initial state with a given desired state is presented. The algorithm uses bang–bang controls to generate trajectories, and includes both a computation of the maximum and minimum of a scalar function, and a one dimensional search for a solution. The solution is often not unique. A necessary condition for convergence of the algorithm is controllability. An algorithm to find a subset of the reachable set is given, which is a slight modification of the path planning algorithm. Obstacle avoidance is a traditional difficulty that must be considered in motion planning when moving in a cluttered environment. In that case the initial and desired configurations must lie in the same connected component of the free configuration space for the motion planning problem to be solvable (Murray and Sastry, 1993). Obstacle is however not discussed further in this paper.

Open–loop paths are very sensitive to initial condition errors. A feedback control law, which closes the loop around the planned trajectory is derived in order to render the control scheme more robust. The feedback control law provides exponential convergence to the planned trajectories under certain assumptions on the nominal trajectory. The combination of an open–loop path planner and an underlying feedback control law for continuous tracking can be seen as the lower modules in a hybrid hierarchical controller scheme for control of nonholonomic mechanical systems as in (Varaiya, 1993), for example.

The outline of the rest of the paper is as follows. In Section 2 controllability for nonlinear systems with drift terms is considered. The motivating class of nonholonomic systems with drift is derived in Section 3 from the Euler–Lagrange differential equations. The main result of the paper is the path planning algorithm given in Section 4. This approach is applied to the example of a planar diver in Section 5. The control loop is closed with a feedback control law for exponential tracking in Section 6.

2 Controllability

Consider an affine nonlinear control system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x}) u_i \tag{1}$$

with $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{x} \in \mathcal{M}$, under the assumption

Assumption 2.1 ((Sussmann, 1987))
The state space $\mathcal{M} \subset \mathbb{R}^n$ is assumed to be a smooth manifold of dimension n, the control input space $\mathcal{U} \subset \mathbb{R}^m$ is assumed compact and convex, the vector fields in the set $\mathcal{F} = \{\boldsymbol{f}, \boldsymbol{g}_1, \dots, \boldsymbol{g}_m\}$ are assumed to be real and analytic, and the input vector fields $\boldsymbol{g}_i$ are assumed to be linearly independent of each other.

Definition 2.1
(Nijmeijer and van der Schaft, 1990) The nonlinear system (1) is called ***controllable*** *if for any two points $(\boldsymbol{x}^0, \boldsymbol{x}^*)$ in $\mathcal{M}$ there exists a finite final time T and an admissible control function $\boldsymbol{u} : [0, T] \mapsto \mathcal{U}$, such that the solution is $\boldsymbol{x}(T, 0, \boldsymbol{x}^0, \boldsymbol{u}) = \boldsymbol{x}^*$.*

2.1 Accessibility

The **accessibility algebra** $\mathcal{C}$ (Nijmeijer and van der Schaft, 1990) is the smallest subalgebra of the Lie algebra of vector fields on $\mathcal{M}$ that contains $\mathcal{F}$. The **accessibility distribution** C is the distribution generated by (spanned by) $\mathcal{C}$. The **strong accessibility algebra** $\mathcal{C}_0$ is the smallest subalgebra of the Lie algebra of vector fields on $\mathcal{M}$ which contains $\mathcal{G} = \{\boldsymbol{g}_1, \dots, \boldsymbol{g}_m\}$ and satisfies $[\boldsymbol{f}, \boldsymbol{X}] \in \mathcal{C}_0$ for all $\boldsymbol{X} \in \mathcal{C}_0$. The **strong accessibility distribution** C_0 is the distribution generated by $\mathcal{C}_0$.

Definition 2.2
The ***Lie algebra rank condition (LARC)*** *at $\boldsymbol{x}$ is given by (Sussmann, 1987)*

$$\dim C(\boldsymbol{x}) = \dim \mathcal{M} = n \tag{2}$$

and the distribution C spans the whole tangent space $T_x\mathcal{M}$. A system is ***locally accessible*** *at* $\boldsymbol{x}$ *if* $\dim C(\boldsymbol{x}) = n$, *and* ***accessible*** *if* $\dim C(\boldsymbol{x}) = n$ *for all* $\boldsymbol{x} \in \mathcal{M}$. *It is* ***strongly accessible*** *if* $\dim C_0(\boldsymbol{x}) = n$ *for all* $\boldsymbol{x} \in \mathcal{M}$.

For driftless systems, controllability follows from LARC (Nijmeijer and van der Schaft, 1990). Controllability is, however, very different from accessibility for systems with drift. Consider the example system

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= \frac{1}{2}x_1^2\end{aligned} \tag{3}$$

taken from (Nijmeijer and van der Schaft, 1990), where

$$C(\boldsymbol{x}) = C_0(\boldsymbol{x}) = \mathrm{span}\{\boldsymbol{g}_1, \boldsymbol{g}_2, [[\boldsymbol{f}, \boldsymbol{g}_1], \boldsymbol{g}_1]\} = \mathbb{R}^3$$

However, x_3 is nondecreasing and all the states $\boldsymbol{x}$ where $x_3 < x_3(0)$, are not reachable. It can be concluded that even though this system is strongly accessible for all $\boldsymbol{x} \in \mathbb{R}^3$, it is not controllable.

2.2 Small time local controllability

Definition 2.3 ((Sussmann, 1987))
A system (1) is ***small time locally controllable (STLC)*** *(Sussmann, 1987) from* $\boldsymbol{x}^0 \in \mathcal{M}$, *if* $\boldsymbol{x}^0$ *is an interior point of the reachable set* $\mathcal{R}_T(\boldsymbol{x}^0)$ *consisting of all points* $\boldsymbol{x}$ *that can be reached from* $\boldsymbol{x}^0$ *in time* T, *defined by*

$$\begin{aligned}\mathcal{R}_T(\boldsymbol{x}^0) &= \bigcup_{0\le\tau\le T} \mathcal{R}(\boldsymbol{x}^0, \tau) \qquad (4)\\ \mathcal{R}(\boldsymbol{x}^0, \tau) &= \{\boldsymbol{x} | \exists \boldsymbol{u} : [0, \tau] \mapsto \mathcal{U} \text{ so that } \boldsymbol{x}(0) = \boldsymbol{x}^0 \text{ and } \boldsymbol{x}(\tau) = \boldsymbol{x}\}\end{aligned}$$

Proposition 2.1
A system cannot be STLC from $\boldsymbol{x}$ *unless it satisfies the LARC (2) at* $\boldsymbol{x}$ *(Sussmann, 1987).*

Proposition 2.2
Assume that $\boldsymbol{x}$ *is an equilibrium with* $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, $\mathcal{U} = \{|u_i| \le 1, i = 1, \ldots, m\}$, *and the vectors* $(\mathrm{ad}\, \boldsymbol{f})^k(\boldsymbol{g}_i)(\boldsymbol{x})$, *for all* $i \in \{1, \ldots, m\}$, $k \in \{0, 1, \ldots\}$ *together with the vectors* $[\boldsymbol{g}_i, \boldsymbol{g}_j](\boldsymbol{x})$, *for all pairs* $i, j \in \{1, \ldots, m\}$, *span* $T_x\mathcal{M}$. *Then the system (1) is STLC from* $\boldsymbol{x}$ *(Sussmann, 1987).*

Remark 2.1
From Proposition 2.2 it follows that the example (3) does not satisfy the condition for STLC.

The following lemma contributed by the authors is a generalization of Proposition 2.2 for systems with nonvanishing drift.

Lemma 2.1
Assume now that the drift term is bounded, but nonzero by $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{0}$, *and the vectors* $(\mathrm{ad}\, \boldsymbol{f})^k(\boldsymbol{g}_i)(\boldsymbol{x})$, *for all* $i \in \{1, \ldots, m\}$, $k \in \{0, 1, \ldots\}$ *together with the vectors* $[\boldsymbol{g}_i, \boldsymbol{g}_j](\boldsymbol{x})$, *for all pairs* $i, j \in \{1, \ldots, m\}$ *span* $T_x\mathcal{M}$. *Then the system (1) still is STLC from* $\boldsymbol{x}$ *if the controls are sufficiently* **large**, i.e. *with controls* $\lambda\boldsymbol{u}$, *and* $\boldsymbol{u} \in \mathcal{U} = \{|u_i| \le 1, i = 1, \ldots, m\}$ *for some large scalar* $\lambda > 0$.

Proof:
This is the same as Proposition 2.2 but with nonvanishing drift ($\boldsymbol{f} \neq \boldsymbol{0}$).

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}) + \lambda \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x}) u_i \tag{5}$$

Scaling the time with $\tau = \lambda t$ gives

$$\frac{d\boldsymbol{x}}{d\tau} = \frac{1}{\lambda}\boldsymbol{f}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{x})u_i \tag{6}$$

Increasing the size of the control inputs is seen to be analogous to reducing the impact of the drift $\boldsymbol{f}$. From this it can be concluded that if all directions are spanned by these brackets, then large controls will be able to compensate for the bounded drift term, and the system is STLC. The vector fields to consider with large controls are $\boldsymbol{f}, \lambda\boldsymbol{g}_i, \lambda(\mathrm{ad}\,\boldsymbol{f})^k(\boldsymbol{g}_i)$, and $\lambda^2[\boldsymbol{g}_i, \boldsymbol{g}_j]$, and it is seen that the magnitude of all these except $\boldsymbol{f}$ increase with increasing λ. □

3 A class of nonholonomic systems with drift terms

In this section a motivating class of systems for the discussion later in the paper is presented. The problem caused by the drift term which makes this class different from driftless systems, is considered.

Consider a conservative mechanical system with Lagrangian

$$L = L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = K(\boldsymbol{x}, \dot{\boldsymbol{x}}) - U(\boldsymbol{x}) \tag{7}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ are the generalized coordinates, $K = \sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x})\dot{x}_i\dot{x}_j$ is the kinetic energy and U is the potential energy. Assume there is a cyclic coordinate, which without loss of generality is chosen to be x_n, i.e.

$$\frac{\partial L}{\partial x_n} = 0 \tag{8}$$

If no generalized forces act in the direction of x_n ($Q_n = 0$), then the Euler-Lagrange differential equations (Goldstein, 1980) yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_n}\right) = 0 \tag{9}$$

This is a special result of Noether's Theorem (Arnold, 1989). A first integral μ of the motion of the mechanical system is given by

$$\mu = \frac{\partial L}{\partial \dot{x}_n} = \sum_{i=1}^{n} a_{in}(\boldsymbol{x})\dot{x}_i \tag{10}$$

where for all $i \in \{1, \ldots, n\}$

$$\frac{\partial a_{in}(\boldsymbol{x})}{\partial x_n} = 0 \tag{11}$$

Collect the first $(n-1)$ coordinates in a vector defined by

$$\boldsymbol{x}_a = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \tag{12}$$

The vector $\boldsymbol{x}_a$ is assumed to be controlled directly by the control inputs $\boldsymbol{u} = [u_1, \ldots, u_{n-1}]^T$ through single integrators, and the trajectories are designed so that $a_{nn}(\boldsymbol{x}_a) \neq 0$, then a control system

$$\begin{aligned} \dot{\boldsymbol{x}}_a &= \boldsymbol{u} \\ \dot{x}_n &= \frac{\mu}{a_{nn}(\boldsymbol{x}_a)} - \sum_{i=1}^{n-1} \frac{a_{in}(\boldsymbol{x}_a)}{a_{nn}(\boldsymbol{x}_a)} u_i \end{aligned} \tag{13}$$

is associated to the first integral (10). Define the vector fields $\boldsymbol{f}, \boldsymbol{g}_i : \mathcal{M} \mapsto \mathbb{R}^n$, $i \in \{1, \dots, n-1\}$

$$\boldsymbol{f} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ f \end{bmatrix}, \quad \boldsymbol{g}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ g_1 \end{bmatrix}, \quad , \dots, \quad \boldsymbol{g}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ g_i \end{bmatrix}, \quad , \dots, \quad \boldsymbol{g}_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ g_{n-1} \end{bmatrix} \tag{14}$$

where 1 is the i'th entry of $\boldsymbol{g}_i$, and with

$$f(\boldsymbol{x}_a) = \frac{\mu}{a_{nn}(\boldsymbol{x}_a)} \tag{15}$$

$$g_i(\boldsymbol{x}_a) = -\frac{a_{in}(\boldsymbol{x}_a)}{a_{nn}(\boldsymbol{x}_a)} \tag{16}$$

Proposition 3.1
System (13) is accessible by Definition 2.2 and STLC by Definition 2.3 for $\boldsymbol{x} \in \mathcal{M}$ if the controls are sufficiently large, and either from (15)-(16), $\frac{\partial f}{\partial x_i} \neq 0$ for some $i \in \{1, \dots, n-1\}$, or $\frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} \neq 0$ for a pair $i, j \in \{1, \dots, n-1\}$ for all $\boldsymbol{x} \in \mathcal{M}$.

Proof:
The Lie brackets are given by

$$[\boldsymbol{f}, \boldsymbol{g}_i] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\frac{\partial f}{\partial x_i} \end{bmatrix}, \quad [\boldsymbol{g}_i, \boldsymbol{g}_j] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial g_j}{\partial x_i} - \frac{\partial g_i}{\partial x_j} \end{bmatrix} \tag{17}$$

using the fact that the partial of $\boldsymbol{f}$ and $\boldsymbol{g}_i$ (14) with respect to x_n is zero. If one of these Lie brackets (17) is nonzero, then together with all $\boldsymbol{g}_i$, $i \in \{1, \dots, n-1\}$, it spans all $\mathbb{R}^n$ and system (1) is accessible by Definition 2.2. Lemma 2.1 ensures STLC provided that the controls are sufficiently large. □

4 Path planning

In this section an algorithm for open–loop path planning is derived for the class of systems presented in the previous section. The idea is to utilize the structure and to apply simple bang–bang controls in the planning.

Problem 4.1 (Path Planning Problem)
Given an initial state $\boldsymbol{x}(0) = \boldsymbol{x}^0$ and a desired final state $\boldsymbol{x}^*$, find trajectories $\boldsymbol{x}(t) \in \mathcal{M}$ and inputs $\boldsymbol{u}(t) \in \mathcal{U}$, such that forward integration of

$$\begin{aligned} \dot{\boldsymbol{x}}_a &= \boldsymbol{u} \\ \dot{x}_n &= f(\boldsymbol{x}_a) + \sum_{i=1}^{n-1} g_i(\boldsymbol{x}_a) u_i \end{aligned} \tag{18}$$

over the time interval $t \in [0, T]$, gives $\boldsymbol{x}(T) = \boldsymbol{x}^*$.

The final time T may be free or given. Initial time is set to zero without loss of generality, since no vector field depends explicitly on time.

Consider piecewise bang–bang control $\boldsymbol{u}(t)$ with unitary bound

$$\boldsymbol{u} \in \mathcal{U} = \left\{ \boldsymbol{u} \in \mathbb{R}^{n-1} : u_i \in \{-1, 0, 1\}, i \in \{1, \ldots, n-1\} \right\} \tag{19}$$

It is well known that bang–bang control often is time optimal, when the control inputs are bounded and the control system is affine. The amount of control available is a concern in the planning for this system due to the drift term. The class of bang–bang controls is often a sufficiently rich class of controls for analysis of nonlinear systems subject to the following remark by Nijmeijer and van der Schaft (1990).

Remark 4.1
From continuity of solutions it follows that an approximation of a more general control with a piecewise constant control gives a solution that approximates the solution with more general controls. In this sense many properties can be established by only considering the piecewise constant inputs. And as piecewise constant controls can approximate continuous controls, it can be stated that bang–bang controls can approximate bounded continuous controls.

The following theorem by Sussmann (1983) will be helpful in the proof of convergence of the path planning Algorithm 4.1. The theorem below has been modified to fit m-input systems from the original version with $m = 1$.

Theorem 4.1
Consider a real analytic system (1) with inputs constrained by $|u_i| \leq 1$. Suppose that for all $\boldsymbol{x} \in \mathcal{M}$ and for all integers $k > 0$, there is a neighborhood $\mathcal{N} \subset \mathcal{M}$ of $\boldsymbol{x}$, where for all $\boldsymbol{x} \in \mathcal{N}$ and for all $i \in \{1, \ldots, m\}$, the term $[\boldsymbol{g}_i, (\mathrm{ad}\, \boldsymbol{f})^k(\boldsymbol{g}_i)](\boldsymbol{x})$ is a linear combination written $\sum_{j=0}^{k+1} \alpha_j (\mathrm{ad}\, \boldsymbol{f})^j(\boldsymbol{g}_i)(\boldsymbol{x})$ with real analytic coefficients α_j and $|\alpha_{k+1}| < 1$. Then the real analytic system (1) has the property that whenever there exists a trajectory which connects two arbitrary points $\boldsymbol{x} \in \mathcal{N}$ in time T, then there exists a trajectory between the two same points in time T, which can be generated by a bang–bang control with a finite number of switches.

Lemma 4.1
The class of systems (18) considered here meet the sufficient conditions of Theorem 4.1 if there exist real analytic coefficients α_0 and α_1 such that for all (i, j)

$$\frac{\partial^2 f}{\partial x_i x_j} = \alpha_0 f + \alpha_1 \frac{\partial f}{\partial x_i} \tag{20}$$

The following remark by Sussmann (1987) is also useful for the verification of Algorithm 4.1.

Remark 4.2
The STLC property in Proposition 2.2 does not change if the discussion is restricted to piecewise constant controls, or even to bang–bang controls, under Assumption 2.1.

Hence it can be concluded that it is possible to steer the state to $\boldsymbol{x}$ with bang–bang controls if the conditions of Lemma (4.1) are satisfied, since the system (18) is STLC at $\boldsymbol{x}$ by Proposition 3.1.

Remark 4.3
The more general case where $u_i \in [-u_{i,\max}, u_{i,\max}]$ for some $u_{i,\max} > 0$, is covered by the diffeomorphic input and state transformations

$$\boldsymbol{u}' = \boldsymbol{D}^{-1}\boldsymbol{u}, \quad \boldsymbol{x}_a' = \boldsymbol{D}^{-1}\boldsymbol{x}_a \tag{21}$$

where $\boldsymbol{D} = \mathrm{diag}(u_{1,\max}, \ldots, u_{n-1,\max})$, so that the new inputs $u_i' \in [-1, 1]$. New scalars to consider are $f'(\boldsymbol{x}_a') = f(\boldsymbol{D}\boldsymbol{x}_a')$, and $g_i'(\boldsymbol{x}_a') = u_{i,\max} g_i(\boldsymbol{D}\boldsymbol{x}_a')$.

The path planning scheme illustrated below is inspired by the well–known technique of variable structure control (Utkin, 1977). In such systems the controller design problem is twofold: the definition of a manifold and the selection of a switching control. The manifold is designed so that a desired behaviour of the closed–loop system is achieved when the state is restricted to move on it. The switching control is designed to steer the initial state to a point on the manifold and then to maintain the motion on the manifold.

The time for steering the initial state $\boldsymbol{x}^0$ to the desired state $\boldsymbol{x}^*$ is partitioned into two parts as shown in Figure 1. The control law in the first part will be termed as *reaching control*, since here the state is steered from the initial state to an $(n-1)$ dimensional manifold $\mathcal{S}_V$. The desidered behaviour of the system is to reach the desired state $\boldsymbol{x}^*$. Hence, the manifold $\mathcal{S}_V$ is designed so that $\boldsymbol{x}^*$ belongs to it and all the trajectories contained in it converge to $\boldsymbol{x}^*$. The control of the system on the manifold will be called *manifold control.*

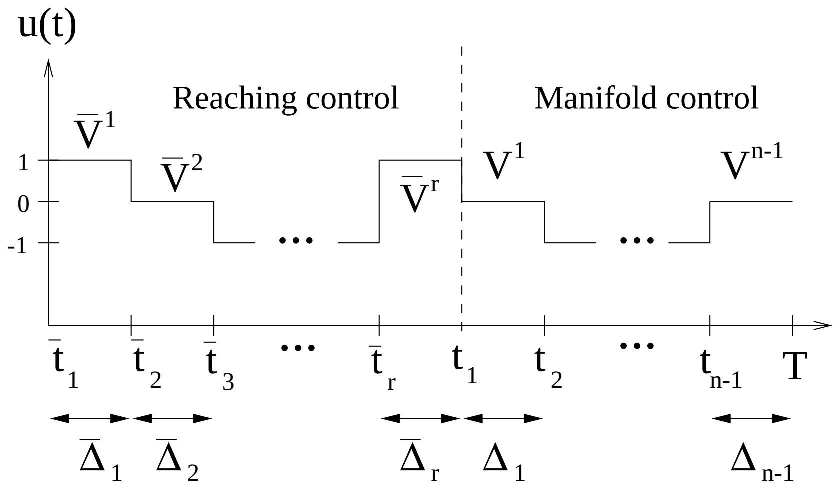

Figure 1: *Timing diagram of the sequence of control actions: reaching control for* $t \in [\bar{t}_1, t_1)$ *and manifold control for* $t \in [t_1, T]$.

4.1 Reaching control

Let the reaching control be defined by a sequence of $r \geq 1$ piecewise constant signals as follows

$$u_k(t) = \bar{v}_{k,i}, \quad t \in [\bar{t}_i, \bar{t}_{i+1}), \; i \in \{1, \ldots, r\} \tag{22}$$

for all $k \in \{1, \ldots, n-1\}$, where $\bar{v}_{k,i} \in \{-1, 0, 1\}$, $r \geq 1$ is the number of control switches in the reaching control segment, $\bar{t}_1 = 0$ is the starting time, $\bar{t}_{r+1}$ is the time where the reaching control ends and the manifold control starts. Let

$$\bar{\Delta}_i = \bar{t}_{i+1} - \bar{t}_i \geq 0, \quad \bar{\boldsymbol{\Delta}} = [\bar{\Delta}_1, \ldots, \bar{\Delta}_r]^T \tag{23}$$

Define the $(n-1) \times r$ *reaching control matrix* $\bar{\boldsymbol{V}}$ by

$$\bar{\boldsymbol{V}} = \begin{bmatrix} \bar{v}_{1,1} & \ldots & \bar{v}_{1,r} \\ \vdots & \ddots & \vdots \\ \bar{v}_{n-1,1} & \ldots & \bar{v}_{n-1,r} \end{bmatrix} \tag{24}$$

so that the control $\boldsymbol{u}$ during the time interval $t \in [\bar{t}_i, \bar{t}_{i+1}]$ is given by the column $\bar{\boldsymbol{V}}^i$ of $\bar{\boldsymbol{V}}$.

Forward integration of the system (18) with initial state $\boldsymbol{x}^0$ gives $\boldsymbol{x}(t_1) = \boldsymbol{x}^1$ at time $t_1 = \bar{t}_{r+1} = \sum_{i=1}^{r} \bar{\Delta}_i$,

$$\begin{aligned} \boldsymbol{x}_a^1 &= \boldsymbol{x}_a^0 + \bar{\boldsymbol{V}} \bar{\boldsymbol{\Delta}} \\ x_n^1 &= x_n^0 + \sum_{i=1}^{r} \int_0^{\bar{\Delta}_i} f(\boldsymbol{x}_a^0 + \bar{\boldsymbol{V}} \boldsymbol{Z}^{i-1} \bar{\boldsymbol{\Delta}} + \bar{\boldsymbol{V}}^i \tau) d\tau \\ &\quad + \sum_{i=1}^{r} \sum_{j=1}^{n-1} \bar{v}_{j,i} \int_0^{\bar{\Delta}_i} g_j(\boldsymbol{x}_a^0 + \bar{\boldsymbol{V}} \boldsymbol{Z}^{i-1} \bar{\boldsymbol{\Delta}} + \bar{\boldsymbol{V}}^i \tau) d\tau \end{aligned} \tag{25}$$

where $\boldsymbol{Z}^i$ is the $r \times r$ matrix whose first i columns are the first i columns of the $r \times r$ identity matrix while the others are zero columns.

4.2 Manifold control

Let the manifold controls be given by the $(n-1)$ piecewise constant signals

$$u_k(t) = v_{k,i}, \quad t \in [t_i, t_{i+1}\rangle, \ i \in \{1, \ldots, n-1\} \tag{26}$$

for all $k \in \{1, \ldots, n-1\}$, where $v_{k,i} \in \{-1, 0, 1\}$, and $t_n = T$ is the time available. Let

$$\Delta_i = t_{i+1} - t_i \geq 0, \quad \boldsymbol{\Delta} = [\Delta_1, \ldots, \Delta_{n-1}]^T \tag{27}$$

Define the $(n-1) \times (n-1)$ *manifold control matrix* $\boldsymbol{V}$ by

$$\boldsymbol{V} = \begin{bmatrix} v_{1,1} & \ldots & v_{1,n-1} \\ \vdots & \ddots & \vdots \\ v_{n-1,1} & \ldots & v_{n-1,n-1} \end{bmatrix} \tag{28}$$

so that the control $\boldsymbol{u}$ during the time interval $t \in [t_i, t_{i+1}]$ is given by the column $\boldsymbol{V}^i$ of $\boldsymbol{V}$.

Imposing $\boldsymbol{x}(T) = \boldsymbol{x}^*$, forward integration of the system (18) with intial state $\boldsymbol{x}^1$, at final time $T = t_1 + \sum_{i=1}^{n-1} \Delta_i$, gives

$$\begin{aligned} \boldsymbol{x}_a^* &= \boldsymbol{x}_a^1 + \boldsymbol{V}\boldsymbol{\Delta} \\ x_n^* &= x_n^1 + \sum_{i=1}^{n-1} \int_0^{\Delta_i} f(\boldsymbol{x}_a^1 + \boldsymbol{Z}^{i-1}\boldsymbol{\Delta} + \boldsymbol{V}^i\tau)d\tau \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} v_{j,i} \int_0^{\Delta_i} g_j(\boldsymbol{x}_a^1 + \boldsymbol{V}\boldsymbol{Z}^{i-1}\boldsymbol{\Delta} + \boldsymbol{V}^i\tau)d\tau \end{aligned} \tag{29}$$

Note that the component $\boldsymbol{x}_a^1$ is a vector living in the cone (Brøndsted, 1983) with vertex $\boldsymbol{x}_a^*$ and negatively spanned by the columns $\boldsymbol{V}^i$ of $\boldsymbol{V}$.

Define an $(n-1)$ dimensional manifold $\mathcal{S}_V$ by

$$\mathcal{S}_V = \{\boldsymbol{x} \in \mathcal{M} | \sigma_V(\boldsymbol{x}) = 0\} \tag{30}$$

where

$$\begin{aligned} \sigma_V(\boldsymbol{x}) &= x_n - x_n^* + \sum_{i=1}^{n-1} \int_0^{\Delta_i} f(\boldsymbol{x}_a + \boldsymbol{V}\boldsymbol{Z}^{i-1}\boldsymbol{\Delta} + \boldsymbol{V}^i\tau)d\tau \\ &\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} v_{j,i} \int_0^{\Delta_i} g_j(\boldsymbol{x}_a + \boldsymbol{V}\boldsymbol{Z}^{i-1}\boldsymbol{\Delta} + \boldsymbol{V}^i\tau)d\tau \end{aligned} \tag{31}$$

Note that $\sigma_V(\boldsymbol{x})$ only is defined for those $\boldsymbol{x}$ where all entries Δ_i in

$$\boldsymbol{\Delta} = \boldsymbol{V}^{-1}(\boldsymbol{x}_a^* - \boldsymbol{x}_a) \tag{32}$$

are nonnegative.

$\boldsymbol{V}$ can always be selected so that $\Delta_i \geq 0$ in (32) and $\det(\boldsymbol{V}) \neq 0$. This can be obtained by $\boldsymbol{V} = \operatorname{diag}\left(\operatorname{sgn}(x_1^* - x_1), \ldots, \operatorname{sgn}(x_{n-1}^* - x_{n-1})\right)$, so that $\Delta_i = |x_i^* - x_i| \geq 0$ for $i \in \{1, \ldots, n-1\}$.

With the definition (30) of $\mathcal{S}_V$, the following is given by design.

Proposition 4.1
Any state $\boldsymbol{x} \in \mathcal{S}_V$ can be steered to the desired state $\boldsymbol{x}^*$ by a sequence of switching controls (26), so that $\boldsymbol{x}(t_1) = \boldsymbol{x}^1 \in \mathcal{S}_V \Rightarrow \boldsymbol{x}(T) = \boldsymbol{x}^*$.

Remark 4.4
The motion on $\mathcal{S}_V$ can be described as a motion on manifolds of decreasing dimension, where the dimension decreases at each switching time. Even if this motion is not optimal in general, it is similar to the one found in the regular synthesis in (Boltyanskii, 1966) for optimal control problems.

By inserting (25) into (31) and setting $\sigma_V = 0$, the result is

$$\begin{aligned}
x_n^* - x_n^0 &= \sum_{i=1}^{r} \int_0^{\bar{\Delta}_i} f(\boldsymbol{x}_a^0 + \bar{\boldsymbol{V}}\boldsymbol{Z}^{i-1}\bar{\boldsymbol{\Delta}} + \bar{\boldsymbol{V}}^i\tau)d\tau \\
&+ \sum_{i=1}^{r}\sum_{j=1}^{n-1} \bar{v}_{j,i} \int_0^{\bar{\Delta}_i} g_j(\boldsymbol{x}_a^0 + \bar{\boldsymbol{V}}\boldsymbol{Z}^{i-1}\bar{\boldsymbol{\Delta}} + \bar{\boldsymbol{V}}^i\tau)d\tau \\
&+ \sum_{i=1}^{n-1} \int_0^{\Delta_i} f(\boldsymbol{x}_a^0 + \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}} + \boldsymbol{V}\boldsymbol{Z}^{i-1}\boldsymbol{\Delta} + \boldsymbol{V}^i\tau)d\tau \\
&+ \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} v_{j,i} \int_0^{\Delta_i} g_j(\boldsymbol{x}_a^0 + \boldsymbol{V}\bar{\boldsymbol{\Delta}} + \boldsymbol{V}\boldsymbol{Z}^{i-1}\boldsymbol{\Delta} + \boldsymbol{V}^i\tau)d\tau \\
&= \Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a^0)
\end{aligned} \tag{33}$$

where $\boldsymbol{\Delta}$ is given by

$$\boldsymbol{\Delta} = \boldsymbol{V}^{-1}(\boldsymbol{x}_a^* - \boldsymbol{x}_a^0 - \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}}) \tag{34}$$

The equation (33) can be solved with respect to a $\bar{\boldsymbol{\Delta}}$ with nonnegative entries for a given pair $(\boldsymbol{x}^0, \boldsymbol{x}^*)$ by Newton's method (Luenberger, 1984). Thereafter, $\boldsymbol{\Delta}$ is found from (34).

From this it can be concluded that the control (24) will steer the state $\boldsymbol{x}$ from the initial $\boldsymbol{x}^0$ to the manifold $\mathcal{S}_V$ if $x_n^* \in [x_n^0 + \Psi^{\min}_{\boldsymbol{V},\bar{\boldsymbol{V}}}, x_n^0 + \Psi^{\max}_{\boldsymbol{V},\bar{\boldsymbol{V}}}]$. These bounds can be obtained by solving the optimization problems

$$\begin{aligned}
\Psi^{\min}_{\boldsymbol{V},\bar{\boldsymbol{V}}} &= \begin{cases} \text{minimize} & \Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a^0) \\ \text{subject to all} & \bar{\Delta}_i \geq 0 \end{cases} \\
\Psi^{\max}_{\boldsymbol{V},\bar{\boldsymbol{V}}} &= \begin{cases} \text{maximize} & \Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a^0) \\ \text{subject to all} & \bar{\Delta}_i \geq 0 \end{cases}
\end{aligned} \tag{35}$$

Methods for solving (35) can be found in (Luenberger, 1984). Note that the result of the optimization might be infinite.

The result of this analysis can be summarized as follows:

Proposition 4.2
Any x_n in the interval

$$\mathcal{I} = [x_n^0 + \Psi^{\min}_{\boldsymbol{V},\bar{\boldsymbol{V}}}, x_n^0 + \Psi^{\max}_{\boldsymbol{V},\bar{\boldsymbol{V}}}] \tag{36}$$

can be reached with the reaching control $\bar{\boldsymbol{V}}$ (24) and the manifold control $\boldsymbol{V}$ (28). Further it can be concluded by Proposition (4.1) that if time is unconstrained and the system is STLC by Definition (2.3) everywhere, then the union of $\mathcal{I}$ for all pairs $(\boldsymbol{V}, \bar{\boldsymbol{V}})$ is the whole $\mathbb{R}$.

4.3 Path planning algorithm

An algorithm is derived from the discussion above which provides solutions for the path planning problem for the pair of an initial state $\boldsymbol{x}^0$ and a desired final state $\boldsymbol{x}^*$.

Algorithm 4.1

- **Initialization.**

```
Initialize r = 1.
Enumerate all possible nonsingular (n-1) x (n-1) matrices...
  V of type (28) by V_k, 1 <= k <= k_max, where the number...
  of possibilities is bounded by k_max <= 3^{(n-1)^2}.
In the same way enumerate all possible (n-1) x r matrices...
  V̄ of type (24) by V̄_l, where 1 <= l <= l_max <= 3^{(n-1)r}.
Initialize k = l = 1.
```

- **Step 1.**

```
Dothe min/max optimization (35) and establish...
  the following reachable set with controls given by V̄_l and V_k:
  I^{k,l} = [x_n^0 + Ψ^min_{V_k,V̄_l}, x_n^0 + Ψ^max_{V_k,V̄_l}]
```

- **Step 2.**

```
If x_n^* ∈ I^{k,l}, then...
   solve (33) with respect to Δ̄ with nonnegative entries Δ̄_i,...
   compute Δ from (34),...
   and stop, the algorithm is successful.
else if l < l_max, then
        l = l + 1,
        and go to step 1.
else if k < k_max, then
        l = 1, k = k + 1,
        and go to step 1.
else
     r = r + 1,
     reenumerate all possible (n-1) x r matrices...
     V̄ of type (24) by V̄_l, where 1 <= l <= l_max < 3^{(n-1)r},
     l = k = 1,
     and go to step 1.
```

From the discussion above the following can be concluded.

Proposition 4.3
Algorithm 4.1 solves Problem 4.1 and converges if the controls are sufficiently large, the system is STLC from any $\boldsymbol{x} \in \mathcal{M}$ by Definition 2.3, and satisfies the condition of Lemma 4.1.

Remark 4.5
A more flexible procedure regarding the order of investigation of the different pairs ($\boldsymbol{V}_k$, $\bar{\boldsymbol{V}}_l$) would improve the efficiency of Algorithm 4.1. The next pair ($\boldsymbol{V}_k$, $\bar{\boldsymbol{V}}_l$) should be selected in view of the results of the algorithm so far.

4.4 Reachability analysis

A slight modification of Algorithm 4.1 allows a computation of a subset of the reachable set $\mathcal{R}_T(\boldsymbol{x}^0)$ defined by (4). This set is restricted when time T available is restricted by some physical constraint, for example.

The boundary of the reachable set $\mathcal{R}_T(\boldsymbol{x}^0)$ are reached by minimum–time trajectories. This follows from the fact that if a nonminimum–time trajectory can reach a point, then there exists another trajectory which can reach the same point in less time, and hence has time left to reach other states. For affine control systems with bounded inputs, minimum–time trajectories often correspond to bang-bang controls. In the solution of minimum–time problems, it is well known that infinite switching may appear with so-called singular trajectories. This problem is avoided by Theorem 4.1, however. A maximum integer r_{max} for the length of the sequence of reaching control is introduced in order to ensure convergence of the algorithm, i.e. $r \in \{1, \ldots, r_{max}\}$.

For a given pair of matrices ($\boldsymbol{V}$, $\bar{\boldsymbol{V}}$), the projection of the set of points reachable from $\boldsymbol{x}^0$ within time T to the $(n-1)$ dimensional subspace where $\boldsymbol{x}_a$ lives, is given by the convex polygon

$$\mathcal{P}_T(\boldsymbol{x}^0) = \left\{\boldsymbol{x}_a \in \mathbb{R}^{n-1} \middle|\ \boldsymbol{x}_a = \boldsymbol{x}_a^0 + \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}} + \boldsymbol{V}\boldsymbol{\Delta}\right\} \tag{37}$$

with $\Delta_i \geq 0$, $\overline{\Delta}_j \geq 0$, and $\sum_{j=1}^{r} \overline{\Delta}_j + \sum_{i=1}^{n-1} \Delta_i \leq T$. The set $\mathcal{P}_T(\boldsymbol{x}^0)$ is a subset of an $(n-1)$ dimensional box with side lengths $2T$ and with center $\boldsymbol{x}_a^0$.

The control laws (22) and (26) will in a time interval of length $\tau \leq T$ steer the initial state $\boldsymbol{x}^0$ to a final state $\boldsymbol{x}$ expressed by

$$\boldsymbol{x}_a \in \mathcal{P}_T(\boldsymbol{x}^0), \quad x_n = x_n^0 + \Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a) \tag{38}$$

where $\Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a)$ is defined as in (33), and $\bar{\boldsymbol{\Delta}}$ has nonnegative entries $\overline{\Delta}_i$, for $i \in \{1, \ldots, r\}$, and

$$\| \boldsymbol{V}^{-1}(\boldsymbol{x}_a - \boldsymbol{x}_a^0 - \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}})\|_1 + \|\bar{\boldsymbol{\Delta}}\|_1 \leq T \tag{39}$$

where the 1–norm $\| \cdot \|_1$ is given by the sum of absolute values of the entries of the vector.

The reachable set for x_n for a given $\boldsymbol{x}_a \in \mathcal{P}_T(\boldsymbol{x}^0)$, can be obtained by solving the following optimization problems:

$$\begin{aligned}
\Psi^{\min}_{\boldsymbol{V},\bar{\boldsymbol{V}}}(T, \boldsymbol{x}_a) &= \left\{ \begin{array}{lll} \text{minimize} & \Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a) & \\ \text{subject to} & \overline{\Delta}_i \geq 0 & \text{for } i = 1, \ldots, r \\ & \| \boldsymbol{V}^{-1}(\boldsymbol{x}_a - \boldsymbol{x}_a^0 - \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}})\|_1 + \|\bar{\boldsymbol{\Delta}}\|_1 \leq T & \end{array} \right. \\
\Psi^{\max}_{\boldsymbol{V},\bar{\boldsymbol{V}}}(T, \boldsymbol{x}_a) &= \left\{ \begin{array}{lll} \text{maximize} & \Psi_{\boldsymbol{V},\bar{\boldsymbol{V}}}(\bar{\boldsymbol{\Delta}}, \boldsymbol{x}_a) & \\ \text{subject to} & \overline{\Delta}_i \geq 0 & \text{for } i = 1, \ldots, r \\ & \| \boldsymbol{V}^{-1}(\boldsymbol{x}_a - \boldsymbol{x}_a^0 - \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}})\|_1 + \|\bar{\boldsymbol{\Delta}}\|_1 \leq T & \end{array} \right.
\end{aligned} \tag{40}$$

The following algorithm returns a subset $\mathcal{R}_T^s(\boldsymbol{x}^0)$ of the reachable set $\mathcal{R}_T(\boldsymbol{x}^0)$. The projection $\mathcal{P}_T^s(\boldsymbol{x}^0)$ of $\mathcal{R}_T^s(\boldsymbol{x}^0)$ into the $(n-1)$ dimensional subspace where $\boldsymbol{x}_a$ lives, is computed.

Algorithm 4.2

- **Initialization.**

  Initialize $r = 1$.
  Enumerate all possible nonsingular $(n-1) \times (n-1)$ matrices...
  $\boldsymbol{V}$ of type (28) by $\boldsymbol{V}_k$, $1 \leq k \leq k_{\max} = 3^{(n-1)^2}$.
  In the same way enumerate all possible $(n-1) \times r$ matrices...
  $\bar{\boldsymbol{V}}$ of type (24) by $\bar{\boldsymbol{V}}_l$, where $1 \leq l \leq l_{\max} = 3^{(n-1)r}$.
  Initialize $k = l = 1$ and $\mathcal{R}_T^s(\boldsymbol{x}^0) = \{\boldsymbol{x}^0\}$, $\mathcal{P}_T^s(\boldsymbol{x}^0) = \{\boldsymbol{x}_a^0\}$.

- **Step 1.**

  Compute $\mathcal{P}_T^{k,l}(\boldsymbol{x}_0)$ with $\boldsymbol{V}_k$, $\bar{\boldsymbol{V}}_l$ as in (37).
  $\mathcal{P}_T^s(\boldsymbol{x}^0) = \mathcal{P}_T^s(\boldsymbol{x}^0) \cup \mathcal{P}_T^{k,l}(\boldsymbol{x}^0)$.
  Discretize $\mathcal{P}_T^{k,l}(\boldsymbol{x}^0)$ as a grid with space $\delta > 0$ between each point.
  Enumerate all points $\boldsymbol{\zeta}_i \in \mathcal{P}_T^{k,l}(\boldsymbol{x}^0) \subset \mathbb{R}^{n-1}$, $i \in \{1, \ldots, (2T/\delta + 1)^{n-1}\}$, from the grid.
  For all $\boldsymbol{\zeta}_i$
      Solve the min/max optimization (40) with the pair $(\boldsymbol{V}_k, \bar{\boldsymbol{V}}_l)$.
      $\mathcal{R}_T^{k,l}(\boldsymbol{x}^0) = \{\boldsymbol{x} = [\boldsymbol{\zeta}_i^T, x_n]^T \in \mathcal{M} : x_n^0 + \Psi^{\min}_{\boldsymbol{V}_k,\bar{\boldsymbol{V}}_l}(T, \boldsymbol{\zeta}_i) \leq x_n \leq x_n^0 + \Psi^{\max}_{\boldsymbol{V}_k,\bar{\boldsymbol{V}}_l}(T, \boldsymbol{\zeta}_i)\}$
  end
  $\mathcal{R}_T^s(\boldsymbol{x}^0) = \mathcal{R}_T^s(\boldsymbol{x}^0) \cup \mathcal{R}_T^{k,l}(\boldsymbol{x}^0)$

- **Step 2.**

  If $l < l_{\max}$, then
      $l = l + 1$
      and go to step 1.
  else if $k < k_{\max}$, then
          $l = 1$, $k = k + 1$
          and go to step 1.
  else if $r < r_{\max}$, then
          $l = k = 1$, $r = r + 1$,
          reenumerate all possible $(n-1) \times r$ matrices
          $\bar{\boldsymbol{V}}$ of type (24) by $\bar{\boldsymbol{V}}_l$, where $1 \leq l \leq l_{\max}$,
          and go to step 1.
  else stop.

Remark 4.6
With $r = 1$ no min/max optimization is necessary in order to find the reachable set. This follows from the fact that the boundary of the reachable set is defined by

$$\| \boldsymbol{V}^{-1}(\boldsymbol{x}_a - \boldsymbol{x}_a^0 - \bar{\boldsymbol{V}}\bar{\boldsymbol{\Delta}})\|_1 + \|\bar{\boldsymbol{\Delta}}\|_1 = T \tag{41}$$

This equality condition reduces the number of free variables by one, and for $r = 1$ this means that there are no free variables for optimization.

Remark 4.7
The approximation of the reachable set will get better and better with an improved density of the grid (smaller δ), and it will converge towards the theoretical reachable set as $\delta \to 0$ and $r_{\max} \to \infty$.

5 Path planning for the planar diver

A planar diver (Crawford and Sastry, 1995) has the structure of a Lagrangian system in $\mathbb{R}^3$ with a cyclic coordinate (13), where the functions a_{i3}, $i = 1, 2, 3$, are highly nonlinear (trigonometric). The drift is caused by the constant angular momentum $\mu \neq 0$. A typical dive is a $1\frac{1}{2}$ somersault pike. The steering problem is to lead the diver through the somersaults driven by the actuating arms and legs, and to enter the water in a fully extended vertical configuration. The time T available to complete the dive is predetermined by gravity g, the initial vertical velocity v_0, and the distance h the diver's center of mass must fall to reach the water:

$$T = \frac{v_0 + \sqrt{v_0^2 + 2gh}}{g} \tag{42}$$

The configuration of the planar diver can be described by means of $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T$, where θ_1 is the angle between the legs and the trunk of the diver, θ_2 is the angle between the arms and the trunk, and θ_3 is the orientation of the trunk with respect to the vertical axis. Details can be found in (Crawford and Sastry, 1995). The motion planning problem for a dive is then to find a trajectory from a given initial state $\boldsymbol{\theta}^0$ to the desired final state $\boldsymbol{\theta}^* = [0, \pi, (2k+1)\pi]^T$, where $k \in \mathbb{Z}$ defines the number of somersaults in the dive.

In this system the drift term $\mu > 0$ is *good* if the diver makes *forward* somersaults ($k > 0$). If however the diver with the same initial spin $\mu > 0$ would want to make *backward* somersaults ($k < 0$), then the drift must be counteracted with heavy backward arm rotations in the opposite direction. In a real dive this is not possible due to physical constraints on the inputs. This is a good example of how the STLC property for driftless systems holds also with drift when the controls are sufficiently large by Lemma 2.1. However, if the constrained inputs are not sufficiently large for the diver, the system is not STLC at $\boldsymbol{x}^0$. Backward dives characterized by those $\boldsymbol{x}^*$ where $\theta_3^* < \theta_3^0$, cannot be carried out. This means that the initial point is not an interior point of the reachable set. The scheme discussed in this paper will work, however, since only forward dives are considered below. Time available is of course a concern that must be considered.

A symmetric diver is considered in order to simplify the presentation. In the general case the same procedure can be carried out with a little more tedious computation. More advanced dives can be approached in a similar manner.

The dynamic equations for a symmetric diver modified from (Crawford and Sastry, 1995) are

$$\begin{aligned} \dot{\theta}_1 &= v_1 \\ \dot{\theta}_2 &= v_2 \\ \dot{\theta}_3 &= \tfrac{\mu}{a_{33}(\theta_1,\theta_2)} - \tfrac{a_{13}(\theta_1,\theta_2)}{a_{33}(\theta_1,\theta_2)} v_1 - \tfrac{a_{23}(\theta_1,\theta_2)}{a_{33}(\theta_1,\theta_2)} v_2 \end{aligned} \tag{43}$$

where

$$\begin{aligned} a_{13}(\theta_1, \theta_2) &= [\alpha + \beta\cos(\theta_1) + \zeta\cos(\theta_2 - \theta_1)] \\ a_{23}(\theta_1, \theta_2) &= [\alpha + \beta\cos(\theta_2) + \zeta\cos(\theta_2 - \theta_1)] \\ a_{33}(\theta_1, \theta_2) &= \alpha_3 + 2\beta[\cos(\theta_1) + \cos(\theta_2)] + 2\zeta\cos(\theta_2 - \theta_1) \end{aligned} \tag{44}$$

Let $\boldsymbol{D}$ be a scaling matrix given by

$$\boldsymbol{D} = \begin{bmatrix} v_{1,\max} & 0 \\ 0 & v_{2,\max} \end{bmatrix} \tag{45}$$

where $v_{1,\max} > 0$ is the maximum angular velocity of the legs, and $v_{2,\max} > 0$ is the maximum angular velocity of the arms. For simplicity let $v_{1,\max} = v_{2,\max} = A$.

The state and input transformations

$$\boldsymbol{x}_a = \boldsymbol{D}^{-1} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad x_3 = \theta_3, \quad \boldsymbol{u} = \boldsymbol{D}^{-1}\boldsymbol{v} \tag{46}$$

give

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= \tfrac{\mu}{a_{33}(Ax_1,Ax_2)} - \tfrac{a_{13}(Ax_1,Ax_2)}{a_{33}(Ax_1,Ax_2)}Au_1 - \tfrac{a_{23}(Ax_1,Ax_2)}{a_{33}(Ax_1,Ax_2)}Au_2 \end{aligned} \tag{47}$$

and the system is of the form (18), with scalars

$$\begin{aligned} f &= \frac{\mu}{a_{33}(Ax_1, Ax_2)} \\ g_1 &= -A\frac{a_{13}(Ax_1, Ax_2)}{a_{33}(Ax_1, Ax_2)} \\ g_2 &= -A\frac{a_{23}(Ax_1, Ax_2)}{a_{33}(Ax_1, Ax_2)} \end{aligned}$$

Using algorithm 4.1, a $\boldsymbol{V}$ can be found as

$$\boldsymbol{V} = \begin{bmatrix} \operatorname{sgn}(x_1^* - x_1) & 0 \\ 0 & \operatorname{sgn}(x_2^* - x_2) \end{bmatrix} \tag{48}$$

with the nice property $\boldsymbol{V}^{-1} = \boldsymbol{V}$. This $\boldsymbol{V}$ gives the following $\boldsymbol{\Delta}$.

$$\boldsymbol{\Delta} = \begin{bmatrix} |x_1^* - x_1| \\ |x_2^* - x_2| \end{bmatrix} \tag{49}$$

With (48), θ_1 is controlled first, and θ_2 last. Integration of (43) as in (31) gives

$$\begin{aligned} \sigma_V(\boldsymbol{x}) \;=\;& A\left(x_3 - x_3^* - \tfrac{1}{2}(x_1^* - x_1) - \tfrac{1}{2}(x_2^* - x_2)\right) \\ &+ \left(a(Ax_2) + \tfrac{b(Ax_2)}{A\operatorname{sgn}(x_1^*-x_1)}\right) \\ &\cdot \left(f_{c(Ax_2)}\left(Ax_1^* - \phi(Ax_2)\right) - f_{c(Ax_2)}\left(Ax_1 - \phi(Ax_2)\right)\right) \\ &+ \left(a(Ax_1^*) + \tfrac{b(Ax_1^*)}{A\operatorname{sgn}(x_2^*-x_2)}\right) \\ &\cdot \left(f_{c(Ax_1^*)}\left(Ax_2^* - \phi(Ax_1^*)\right) - f_{c(Ax_1^*)}\left(Ax_2 - \phi(Ax_1^*)\right)\right) \end{aligned}$$

where

$$f_c(x) = \arctan\left(c\tan(\frac{x}{2})\right) + \pi k(x) \qquad \text{with } c \in \mathbb{R} \tag{50}$$

and $k : \mathbb{R} \mapsto \mathbb{Z}$ is a piecewise constant mapping, such that $f_c(\cdot)$ is continuous over $\mathbb{R}$ and $f_c(0) = 0$. The functions $a(s), b(s), c(s), \phi(s)$ are given by

$$\phi(s) = \arctan 2(\zeta\sin(s), \beta + \zeta\cos(s))$$

$$\psi(s) = \sqrt{\zeta^2 + \beta^2 + 2\zeta\beta\cos(s)}$$

$$a(s) = \frac{\alpha_3 + 2\beta\cos(s) - 2\alpha}{\sqrt{(\alpha_3+2\beta\cos(s))^2 - 4\psi(s)^2}}$$

$$b(s) = \frac{2\mu}{\sqrt{(\alpha_3+2\beta\cos(s))^2 - 4\psi(s)^2}}$$

$$c(s) = \frac{\sqrt{(\alpha_3+2\beta\cos(s))^2 - 4\psi(s)^2}}{\alpha_3+2\beta\cos(s)+2\psi(s)}$$

with parameters $A = 10, \mu = 70, \alpha_3 = 11.234, \beta = 2.299, \alpha = 3.732$, and $\zeta = -0.207$.

First the effect of the drift on the system can be seen by letting $\bar{\boldsymbol{V}}$ be given by

$$\bar{\boldsymbol{V}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{51}$$

From (43) and with zero controls it is obtained

$$\begin{aligned} x_1^1 &= x_1^0 \\ x_2^1 &= x_2^0 \\ x_3^1 &= x_3^0 + \frac{\mu}{Aa_{33}(Ax_1^0, Ax_2^0)}\overline{\Delta} \end{aligned} \tag{52}$$

With this initial zero control interval, all initial states $\boldsymbol{x}^0$ in a set $\mathcal{Q}_I^0 \subset \mathbb{R}^3$ given by

$$\mathcal{Q}_I^0 = \{\boldsymbol{x} \in \mathcal{M} | \sigma_V(x_1, x_2, x_3 + \frac{\mu}{Aa_{33}(Ax_1, Ax_2)}\overline{\Delta}) = 0, 0 \leq \overline{\Delta} \leq \overline{\Delta}_{\max}\} \tag{53}$$

for some given $\overline{\Delta}_{\max}$, can be steered so that $\boldsymbol{x}(T) = \boldsymbol{x}^*$.

The set $\mathcal{Q}_I^0$ with $0 \leq \overline{\Delta} \leq 1$ is shown in Figure 2. All states between the top surface $\mathcal{S}_V$ and the bottom surface with $\overline{\Delta} = 1$ can reach the desired state with the control (48) and (51).

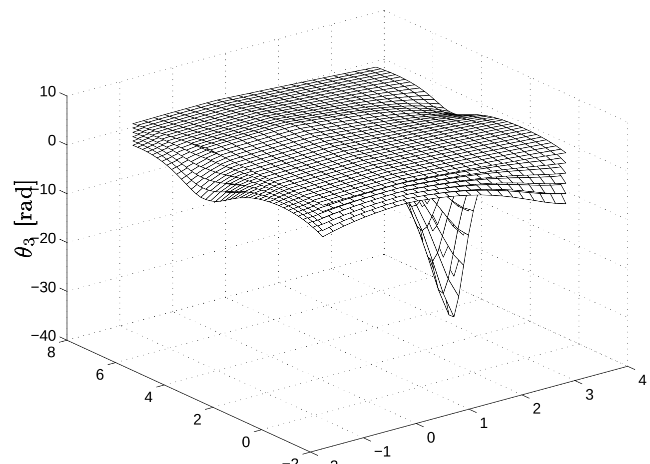

Figure 2: *The set* $\mathcal{Q}_I^0$ *for the diver example with* $0 \leq \overline{\Delta} \leq 1$. *The top surface is the original manifold* $\mathcal{S}_V$*, which corresponds to* $\overline{\Delta} = 0$.

The effect of applying another reaching control can be seen by selecting the following $\bar{\boldsymbol{V}}$

$$\bar{\boldsymbol{V}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{54}$$

which results in

$$\begin{aligned} x_1^1 &= x_1^0 \\ x_2^1 &= x_2^0 - \overline{\Delta} \\ x_3^1 &= x_3^0 + \frac{A\overline{\Delta}}{2} + \left(a(Ax_1^0) - \frac{b(Ax_1^0)}{A}\right) \\ &\quad \cdot \left(f_{c(Ax_1^0)}\left(Ax_2^0 - A\overline{\Delta} - \phi(Ax_1^0)\right) - f_{c(Ax_1^0)}\left(Ax_2^0 - \phi(Ax_1^0)\right)\right) \end{aligned} \tag{55}$$

With this initial reaching control applied in a time interval $[0, \overline{\Delta}]$, all initial states $\boldsymbol{x}^0$ in a set $\mathcal{Q}_I^1 \subset \mathbb{R}^3$ defined by

$$\mathcal{Q}_I^1 = \{\boldsymbol{x} \in \mathcal{M} | \sigma_V(\boldsymbol{x}^1) = 0, 0 \leq \overline{\Delta} \leq \overline{\Delta}_{\max}\} \tag{56}$$

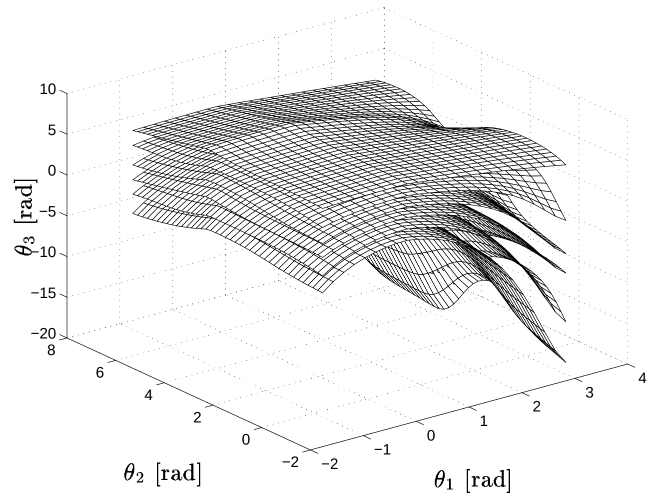

Figure 3: *The set* $\mathcal{Q}_I^1$ *for the diver example with* $0 \leq \overline{\Delta} \leq 1$. *The top surface is the original manifold* $\mathcal{S}_V$, *which corresponds to* $\overline{\Delta} = 0$.

with $\boldsymbol{x}^1 = [x_1^1, x_2^1, x_3^1]^T$ as in (55), and for some $\overline{\Delta}_{\max}$, can be steered so that $\boldsymbol{x}(T) = \boldsymbol{x}^*$.

The set $\mathcal{Q}_I^1$ with $0 \leq \overline{\Delta} \leq 1$ is shown in Figure 3. All states between the top surface $\mathcal{S}_V$ and the bottom surface ($\overline{\Delta} = 1$) can reach the desired state with the control matrices above.

A realistic initial condition to use for the diver model for a $1\frac{1}{2}$ somersault dive is $\boldsymbol{\theta} = [0, \pi, 0]^T$ and a desired final state $\boldsymbol{\theta}^* = [0, \pi, 3\pi]^T$. In order to generate a motion similar to the one of a real diver, this control sequence is applied:

$$\bar{\boldsymbol{V}} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{V} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{57}$$

The predetermined time T available to complete the dive puts an additional constraint on the choice of control intervals:

$$\sum_{i=1}^{3} \overline{\Delta}_i + \sum_{i=1}^{2} \Delta_i = T \tag{58}$$

This time constraint defines a subset of the manifold $\mathcal{S}_V$. The number of free variables decreases by one due to this constraint, and hence it is necessary to have $r \geq 2$. Free variables are $\overline{\Delta}_i \geq 0$ with $i \in \{1, \ldots, r\}$.

Using the equations presented above, an appropriate choice of control intervals was found to steer the diver from the initial to the final configuration in 1.7 seconds. The trajectory is shown in Figures 4 and 5.

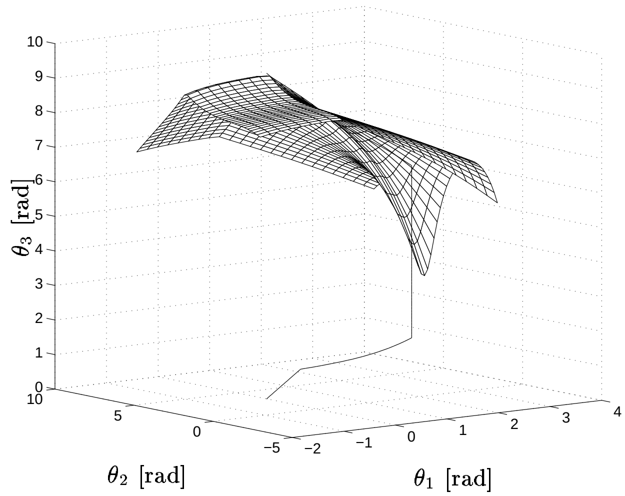

Figure 4: *Evolution of the diver state. Initially the state follows the line shown with reaching control until it hits the manifold $\mathcal{S}_V$. After that the trajectory stays on $\mathcal{S}_V$ until the desired state is reached.*

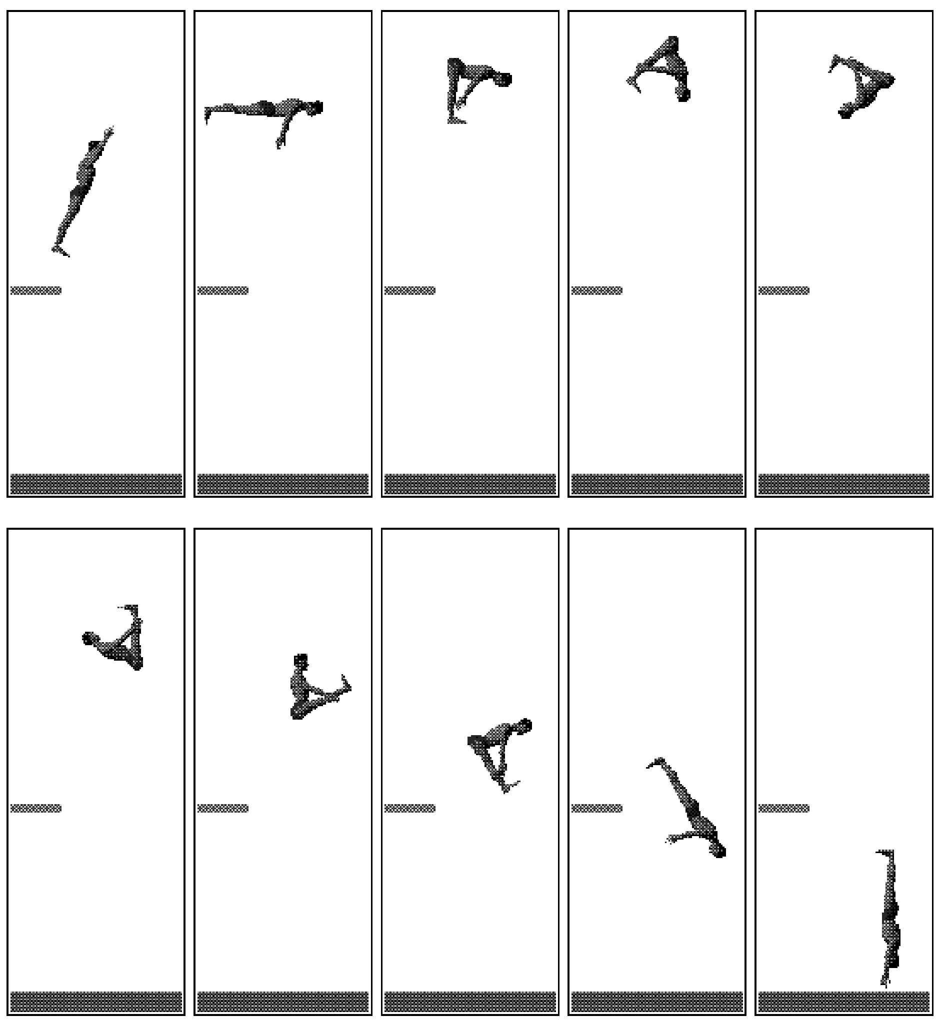

Figure 5: Frames from an animation of the simulation results with a symmetric model of a planar diver and bang–bang controls.

6 Tracking of nonholonomic control systems

The control problem for the class of systems (18) is divided into parts. The first and perhaps most difficult part has been solved above with the path planning algorithm 4.1. This algorithm provides nominal trajectories and inputs. The second part is a feedback tracking control law, which closes the loop and forces the actual state to follow the nominal trajectory under the influence of small disturbances. It is well-known that open–loop planners sometimes are very sensitive to initial errors (Murray and Sastry, 1993), and this motivates the design of feedback control laws.

In this section a new control law for stabilization of a class of nonholonomic systems to trajectories is derived. Such systems cannot be stabilized to a point using static state feedback, since they violate Brockett's necessary condition for stabilizability (Brockett, 1983). The control law presented here stabilizes systems (18) around predefined time-varying trajectories. It is assumed that a motion planner has generated a trajectory which satisfies the nonholonomic constraints in open–loop. A feedback control law is derived for tracking around this trajectory. A previous solution to this problem was given in (Walsh *et al.*, 1994). That solution was based on a linearization of the system around a nominal trajectory, and hence gave local results. It provided exponential convergence whenever a time varying matrix was positive and bounded. The area of attraction for the resulting controller was not discussed. In the present section nonlinear techniques are applied, and exponential stability is achieved under two assumptions.

6.1 Modeling

Consider nonholonomic systems of the form (18). Assume that the path planning algorithm 4.1 supplied a time varying nominal trajectory $\boldsymbol{y}(t)$ and a nominal input $\boldsymbol{v}(t)$, which satisfy

$$\begin{aligned} \dot{\boldsymbol{y}}_a &= \boldsymbol{v} && (59)\\ \dot{y}_n &= f(\boldsymbol{y}_a) + \sum_{i=1}^{n-1} g_i(\boldsymbol{y}_a) v_i && (60) \end{aligned}$$

The state deviation vector $\boldsymbol{e}$ and the feedback control $\boldsymbol{\tau}$ are defined by

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{x} - \boldsymbol{y} && (61)\\ \boldsymbol{u} &= \boldsymbol{\tau} + \boldsymbol{v} && (62) \end{aligned}$$

The control $\boldsymbol{u}$ hence consists of a nominal part $\boldsymbol{v}(t)$ and a feedback part $\boldsymbol{\tau}(\boldsymbol{e}, t)$ which will be derived below. Time differentiation of $\boldsymbol{e}$ gives

$$\begin{aligned} \dot{\boldsymbol{e}}_a &= \boldsymbol{\tau} \\ \dot{e}_n &= f(\boldsymbol{x}_a) + \sum_{i=1}^{n-1} g_i(\boldsymbol{x}_a) u_i - f(\boldsymbol{y}_a) - \sum_{i=1}^{n-1} g_i(\boldsymbol{y}_a) v_i && (63) \end{aligned}$$

The mean value theorem for multivariable scalar functions (Kreyszig, 1988) gives

$$\begin{aligned} f(\boldsymbol{x}_a) - f(\boldsymbol{y}_a) &= \boldsymbol{\nabla}_a f(\boldsymbol{s}_0) \boldsymbol{e}_a && (64)\\ g_i(\boldsymbol{x}_a) - g_i(\boldsymbol{y}_a) &= \boldsymbol{\nabla}_a g_i(\boldsymbol{s}_i) \boldsymbol{e}_a && (65) \end{aligned}$$

for all $i \in \{1, \ldots, n-1\}$, where

$$\boldsymbol{\nabla}_a = \frac{\partial}{\partial \boldsymbol{x}_a^T} \qquad (66)$$

and $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n-1}$ are vectors from the origin to some points on the line between $\boldsymbol{x}_a$ and $\boldsymbol{y}_a$. Then (63) can be written

$$\begin{aligned} \dot{\boldsymbol{e}}_a &= \boldsymbol{\tau} \\ \dot{e}_n &= \boldsymbol{p}^T \boldsymbol{e}_a + \boldsymbol{g}^T \boldsymbol{\tau} && (67) \end{aligned}$$

where

$$\boldsymbol{p} = \boldsymbol{\nabla}_a f(\boldsymbol{s}_0) + \sum_{i=1}^{n-1} \boldsymbol{\nabla}_a g_i(\boldsymbol{s}_i) v_i \tag{68}$$

$$\boldsymbol{g} = [g_1, \ldots, g_{n-1}]^T \tag{69}$$

Proposition 6.1
A sufficient condition for controllability for the linearized system (Kailath, 1980) and a necessary condition for stabilizability (Brockett, 1983) of system (67), is that $\boldsymbol{p} \neq \boldsymbol{0}$.

Proof:
The controllability matrix for system (63) is given by

$$\boldsymbol{Q}_c = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{g}^T & \boldsymbol{p}^T \end{bmatrix} \tag{70}$$

and the controllability rank condition is given by $\det(\boldsymbol{Q}_c \boldsymbol{Q}_c^T) = \|\boldsymbol{p}\|^2 \neq 0$.

If $\boldsymbol{p} = \boldsymbol{0}$, then the vector field on the right side of (67) is not onto any open neighborhoods around the origin, since points of the type $(0, \ldots, 0, l)$ (any small l) are not in any neighborhoods around the origin. $\square$

Assumption 6.1
It is assumed that $\boldsymbol{p}$ *is bounded by* $0 < M_1 \leq \|\boldsymbol{p}\| \leq M_2 < \infty$.

Assumption 6.2
It is assumed that the time derivative of the vector $\boldsymbol{p}$ *is upper bounded by* $\|\dot{\boldsymbol{p}}\| \leq L$ *for some* $L > 0$. *For bang–bang controls, this might be a problem, but the problem is solved by making smooth approximations of the derived trajectories by splines as described in (Bartels* et al., *1987).*

6.2 Feedback control law

Theorem 6.1
Exponential tracking around the nonholonomic trajectory $\boldsymbol{y}$ *is achieved under Assumptions 6.1 and 6.2 by the control law*

$$\boldsymbol{\tau} = -\left(\boldsymbol{I} + \alpha \boldsymbol{p} \boldsymbol{g}^T\right)^{-1} \left(\lambda \left(\boldsymbol{e}_a + \alpha \boldsymbol{p} e_n\right) + \alpha \boldsymbol{p} \boldsymbol{p}^T \boldsymbol{e}_a\right) \tag{71}$$

where $\lambda > 0$ *is a large control parameter and* $0 < \alpha < 1/\max_{t \geq 0}(\|\boldsymbol{p}\| \cdot \|\boldsymbol{g}\|)$ *is a small control parameter.*

Proof:
The state transformation $(\boldsymbol{e}_a, e_n) \mapsto (\boldsymbol{z}, e_n)$, where

$$\boldsymbol{z} = \boldsymbol{e}_a + \alpha \boldsymbol{p} e_n \tag{72}$$

and the control law (71) gives

$$\dot{\boldsymbol{z}} = -\lambda \boldsymbol{z} + \alpha \dot{\boldsymbol{p}} e_n \tag{73}$$

$$\dot{e}_n = \frac{-\alpha \|\boldsymbol{p}\|^2 e_n + (\boldsymbol{p} - \lambda \boldsymbol{g})^T \boldsymbol{z}}{1 + \alpha \boldsymbol{p}^T \boldsymbol{g}} \tag{74}$$

Consider the Lyapunov function candidate

$$V(\boldsymbol{e}, \boldsymbol{p}) = \frac{1}{2} \boldsymbol{z}^T \boldsymbol{z} + \frac{1}{2} \beta e_n^2 \tag{75}$$

where $\beta > 0$. V is decrescent when $\|\boldsymbol{p}\|$ is upper bounded, and radially unbounded when $\|\boldsymbol{p}\|$ is lower bounded.

Time differentiation along the system trajectories gives

$$\begin{aligned}
\dot{V}(\boldsymbol{e},\boldsymbol{p}) &= \boldsymbol{z}^T(-\lambda\boldsymbol{z}+\alpha\dot{\boldsymbol{p}}e_n)+\beta e_n\left(\frac{-\alpha\|\boldsymbol{p}\|^2 e_n+(\boldsymbol{p}-\lambda\boldsymbol{g})^T\boldsymbol{z}}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\right)\\
&= -\frac{1}{2}\begin{bmatrix}\boldsymbol{z}^T & e_n\end{bmatrix}\begin{bmatrix}2\lambda & -\alpha\dot{\boldsymbol{p}}-\beta\frac{(\boldsymbol{p}-\lambda\boldsymbol{g})}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\\ -\alpha\dot{\boldsymbol{p}}^T-\beta\frac{(\boldsymbol{p}-\lambda\boldsymbol{g})^T}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}} & \frac{2\alpha\beta\|\boldsymbol{p}\|^2}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}\\ e_n\end{bmatrix}\\
&< 0
\end{aligned}\tag{76}$$

since $\lambda > 0$, and the determinant

$$\begin{aligned}
\tfrac{4\lambda\alpha\beta\|\boldsymbol{p}\|^2}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}-\left\|\alpha\dot{\boldsymbol{p}}+\beta\tfrac{\boldsymbol{p}-\lambda\boldsymbol{g}}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\right\|^2 &\geq 2\left(\tfrac{2\lambda\alpha\beta\|\boldsymbol{p}\|^2}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}-\alpha^2\|\dot{\boldsymbol{p}}\|^2-\beta^2\tfrac{\|\boldsymbol{p}-\lambda\boldsymbol{g}\|^2}{(1+\alpha\boldsymbol{p}^T\boldsymbol{g})^2}\right)\\
&\geq \tfrac{2\lambda\alpha\beta}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\left(\|\boldsymbol{p}\|^2-\tfrac{\alpha L^2}{\beta\lambda}\left(1+\alpha\boldsymbol{p}^T\boldsymbol{g}\right)\right)\\
&\quad +\tfrac{2\lambda\alpha\beta}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\left(\|\boldsymbol{p}\|^2-\tfrac{\beta}{\alpha}\tfrac{\|\frac{1}{\lambda}\boldsymbol{p}-\boldsymbol{g}\|^2}{1+\alpha\boldsymbol{p}^T\boldsymbol{g}}\right)\\
&> 0
\end{aligned}$$

for small $\beta > 0$ and large $\lambda > 0$, since $\|\boldsymbol{p}\|$ is assumed to have a lower bound. And since $V > 0$ and $\dot{V} < 0$, it follows by Lyapunov arguments that both V and $\boldsymbol{e} \to \boldsymbol{0}$ exponentially. Note that β is not a design parameter, but only a parameter used in the proof. □

6.3 Tracking example - kinematic ship and sea current

Consider the kinematic model of a ship under the influence of sea current with constant velocity.

$$\dot{x} = v_1\cos\theta \tag{77}$$

$$\dot{y} = v_1\sin\theta + d \tag{78}$$

$$\dot{\theta} = v_2 \tag{79}$$

The state vector is $\boldsymbol{q} = [x, y, \theta]^T \in \mathcal{M} = SE(2) = \mathbb{R}^2 \times SO(2)$, where (x, y) is the position of the ship in an inertial coordinate system. This inertial system is chosen so that the constant current with known velocity d goes along the y–axis. The rotation of the body of the ship relative to the inertial system is given by θ. The forward velocity (surge) v_1 of the ship is considered as an input (speed controlled by the main propeller), and the angular velocity v_2 of the ship is considered as a second input (controlled by the rudder). The drift of this system is not caused by a first integral as discussed earlier in this chapter. The drift here is caused by an environmental disturbance. The structure is the same as for the Lagrangian systems discussed. The example of a kinematic ship influenced by sea current was selected because of its simple structure. The simplicity of the the structure makes it simple to find smooth nominal trajectories which satisfy the nonholonomic constraint.

The state transformation $\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{q})$ given by

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{\Phi}(\boldsymbol{q}) = \begin{bmatrix} \frac{x}{d} \\ \tan\theta \\ \frac{y}{d} \end{bmatrix} \tag{80}$$

along with the input transformation $\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{v}$ given by

$$\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{v} = \begin{bmatrix} \frac{\cos\theta}{d} & 0 \\ 0 & \frac{1}{\cos^2\theta} \end{bmatrix}\boldsymbol{v} \tag{81}$$

give

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ 1 + x_2 u_1 \end{bmatrix} \tag{82}$$

which is called the *chained form with unitary drift*, with $f=1$, $g_1=x_2$, and $g_2=0$.

The following nominal path represented by $(\boldsymbol{y},\boldsymbol{v})$ satisfies the nonholonomic contraints (82):

$$\boldsymbol{y}=\begin{bmatrix} 2a\sin(\frac{t}{ab}) \\ b\left(1-\cos(\frac{t}{ab})\right) \\ 2ab\sin(\frac{t}{ab})-\frac{ab}{2}\sin(\frac{2t}{ab}) \end{bmatrix} \tag{83}$$

$$\boldsymbol{v}=\begin{bmatrix} \frac{2}{b}\cos(\frac{t}{ab}) \\ \frac{1}{a}\sin(\frac{t}{ab}) \end{bmatrix} \tag{84}$$

so that $\dot{y}_1=v_1, \dot{y}_2=v_2, \dot{y}_3=1+y_2v_1$ as desired.

Remark 6.1
The nominal trajectory (83) and the corresponding nominal input (84) define a time-periodic figure eigth motion. The only equlibrium for this system is when the ship is heading into the current, but the desired configuration is the one where the current is coming from the side. Point stabilization to this configuration is not possible. Stabilization around a periodic small amplitude trajectory around the origin is a solution which approximates point stabilization, and may be a good solution when the control objective is to make the ship stay as close to the origin as possible with sea current acting in the y-direction. The amplitude of the nominal trajectory is limited by the achievable control inputs. This can be seen by letting the parameters a and b be small in (83). The result will be large inputs (84).

The deviation vector $\boldsymbol{e}$ (61) and the stabilizing control $\boldsymbol{\tau}$ (62) are defined by

$$\boldsymbol{e}=\begin{bmatrix} x_1-2a\sin(\frac{t}{ab}) \\ x_2-b\left(1-\cos(\frac{t}{ab})\right) \\ x_3-2ab\sin(\frac{t}{ab})+\frac{ab}{2}\sin(\frac{2t}{ab}) \end{bmatrix} \tag{85}$$

$$\boldsymbol{\tau}=\begin{bmatrix} u_1-\frac{2}{b}\cos(\frac{t}{ab}) \\ u_2-\frac{1}{a}\sin(\frac{t}{ab}) \end{bmatrix} \tag{86}$$

The vector $\boldsymbol{p}$ used in the control law is given by

$$\boldsymbol{p}=\begin{bmatrix} 0 \\ v_1 \end{bmatrix}=\begin{bmatrix} 0 \\ \frac{2}{b}\cos(\frac{t}{ab}) \end{bmatrix} \tag{87}$$

with norm $\|\boldsymbol{p}\|=\left|\frac{2}{b}\cos(\frac{t}{ab})\right|$, which equals $\left|\frac{1}{b}\right|>0$ in average. This $\boldsymbol{p}$ does not satisfy the lower bound Assumption 6.1. However, a $\boldsymbol{p}$ which has a lower bounded average, has shown to be sufficient for stabilization in the simulations run.

Remark 6.2
Trajectories as $\boldsymbol{y}=[0,0,t]^T$ with nominal input $\boldsymbol{v}=[0,0]^T$ cannot be tracked with the algorithm above, since then $\boldsymbol{p}=\boldsymbol{0}$ and Assumption 6.1 does not hold. The result is $\boldsymbol{\tau}=-\lambda\boldsymbol{e}_a$, $\boldsymbol{e}_a\rightarrow\boldsymbol{0}$ exponentially, but e_n is not accounted for.

7 Conclusions

The control problem for a class of nonholonomic systems with drift has been studied. A lemma for controllability for systems with nonvanishing drift was presented. The path planning problem was solved with an algorithm providing numeric solutions with bang–bang controls, and the algorithm is guarenteed to solve the problem under a necessary controllability assumption. This algorithm was applied to the mathematical model of a planar diver. The set of reachable sets was approximated with a slightly modified algorithm. A nonlinear feedback control law for exponential tracking was derived to close the loop, and to make the total controller scheme more robust.

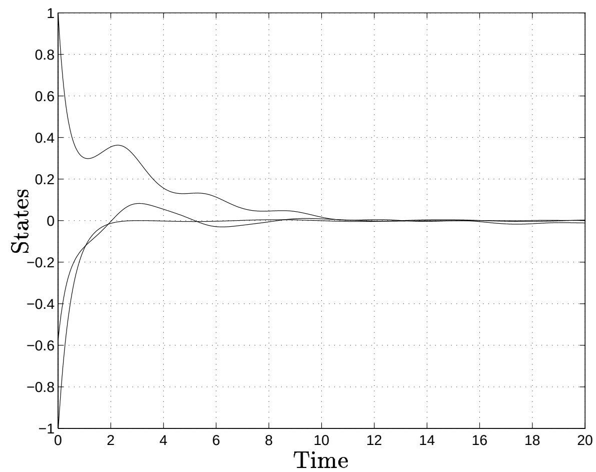

Figure 6: *The control deviation* $\boldsymbol{e}$ *for a simulation of tracking for the chained form with drift with nominal trajectory (83) with parameters* $a = b = 1$ *after an initial error* $\boldsymbol{e}(0) = [-1, -1/\sqrt{3}, 1]^T$. *The control parameters were* $\lambda = 2$ *and* $\alpha = 0.2$.

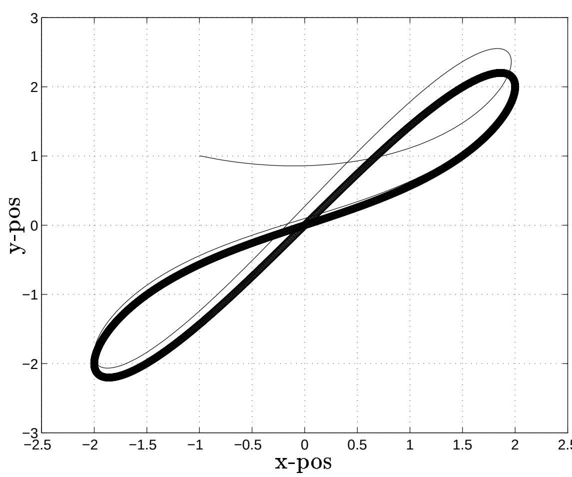

Figure 7: *The trajectories in the* x_1-x_3 *plane for the planned (*$\boldsymbol{y}$*=thick line) and the real (*$\boldsymbol{x}$*=thin line) trajectories. This figure eigth motion corresponds to the* $x - y$ *position of a nonholonomic ship under the influence of a constant sea current (*$d = 1$*) in the* y*-direction.*

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