Control of Chained Systems Application to Path Following and Time-Varying Point-Stabilization of Mobile Robots

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Abstract— **Chain form systems have recently been introduced to model the kinematics of a class of nonholonomic mechanical systems. The first part of the study is centered on control design and analysis for nonlinear systems which can be converted to the chain form. Solutions to various control problems (open-loop steering, partial or complete state feedback stabilization) are either recalled, generalized, or developed. In particular, globally stabilizing time-varying feedbacks are derived, and a discussion of their convergence properties is provided. Application to the control of nonholonomic wheeled mobile robots is described in the second part of the study by considering the case of a car-pulling trailers.**

I. Introduction

FROM a theorem due to Brockett [5], it is known that nonholonomic wheeled mobile robots with restricted mobility (such as unicycle-type and car-like vehicles) cannot be stabilized to a desired configuration (or posture) via differentiable, or even continuous, pure-state feedback [3], [24], [1]. Nonsmooth feedback has been proposed as an alternative solution (see [4], [6], [32], for example). Another alternative, first pointed out by the author in [25], consists of using smooth time-varying feedbacks, i.e., feedbacks which explicitly depend on the time variable. Such feedbacks had previously been very little studied in Control Theory. The result given in [25], for a unicycle-type vehicle, has subsequently motivated research work to explore the potentialities of time-varying feedbacks [8], [9], [11], derive explicit design methods [19], [34], and extend their use in robotics applications [26], [29].

The possibility of modeling the kinematic equations of wheeled mobile robots by so-called canonical chain form equations (a particular class of nonlinear nilpotent systems) has been pointed out in [17] when treating the case of car-like vehicles. It was known before that the equations of unicycle-type vehicles (a simpler case) could be written in this form, but this had not been used explicitly at the control design level. More recently, it has been shown [31] that the equations of vehicles with trailers could also be locally converted into a chain form.

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In [17], the authors aimed essentially at exploring methods for open-loop steering of nonholonomic systems by using sinusoidal inputs. More recently, the authors of [34] have realized that chained systems could also be put under another canonical form, called "power form," and that power form systems belonged to the class of systems considered in [19] for which explicit smooth time-varying stabilizing feedbacks can be derived. The method proposed in [34], for deriving such controls, is applied to the problem of locally stabilizing a car-like system to a desired posture. A global solution to this problem had previously been given by the author in [26], [27] by using another approach.

Since the power form is mathematically equivalent to the chain form, one may question the interest of using one form rather than the other one when dealing with practical applications and mobile robot control in particular. To the author's knowledge, no single and definitive answer can be brought to this question because the two forms present complementary advantages and drawbacks. For example, it seems that model equations of nonholonomic vehicles using physical coordinates, such as Cartesian coordinates and angles between articulated bodies, are naturally closer to the chain form than to the power form. In this respect, the logic of using the chain form when addressing the problem of path following should be evident from the present paper. Concerning the theoretically more difficult problem of point stabilization, the answer is not as clear. As shown in this paper, a rather simple solution to smooth feedback stabilization can be obtained by complementing path following solutions, thus making the chain form intuitively attractive in this case. Smooth time-varying stabilization, however, does not seem to be compatible with fast convergence [11]. For this reason, recent studies on point stabilization have focused on the possibility of achieving faster (exponential) convergence by using nonsmooth feedback: M'Closkey and Murray have been working with the power form (see [14] for example), with the probable reason that their approach based on the use of homogeneous system coordinates does not apply well to the chain form, while Sørdalen has proposed an original solution, mixing open-loop and feedback control strategies, by using the chain form [32]. The question is therefore far from being settled at the time being and will certainly motivate future developments.

This paper is organized as follows. Section II focuses on the control of chain form systems. After pointing out some facts about these systems and recalling some results about the open-loop steering problem, it is shown that chain form systems can themselves be converted into a slightly different form, named here skew-symmetric chain form, particularly well adapted to subsequent Lyapunov design and analysis of globally stabilizing time-varying feedbacks. Whenever this is possible, as it is the case here, finding adequate Lyapunov functions at an early stage of the analysis presents some advantages in terms of simplicity of the control design and stability proofs. A comparison with Center Manifold techniques, as used in [34] for example, is in this respect illustrative. In the process of deriving smooth feedback control laws for chained systems, useful connections with more classical linear control techniques are carried out. Several solutions to the point-stabilization problem are then tentatively compared by analyzing the type of stability associated with each of them. This motivates a short discussion about the practical relevance of the notion of asymptotical rate of convergence.

In Section III, the results of Section II are applied to a car pulling -n- trailers, seen as an extension of the unicycle and car cases. The same approach as in [29], in which stabilization to a desired configuration is treated as an extension of the path following problem, is considered. This approach involves a specific parameterization of the vehicle's posture which facilitates the decoupling of the path following problem from translational velocity control. The interest of this parameterization for solving path planning issues is also pointed out in connection with the approach developed in [21]. Finally, a modification in the modeling of the system's equations is proposed so as to broaden the control stability domain.

II. Control of Chain Form Systems

A. About Chained Systems

Let us consider a chain form system which may be written as

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_1 x_3 \\ \dot{x}_3 &= u_1 x_4 \\ &\vdots \\ \dot{x}_{n-1} &= u_1 x_n \\ \dot{x}_n &= u_2 \end{aligned} \tag{1}$$

or, equivalently

$$\dot{X} = h_1(X)u_1 + h_2 u_2 \quad h_1(X) = \begin{bmatrix} 1 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2}$$

With respect to the notations used in [17], the components of the state vector have just been ordered differently.

In the second part of the study, where system (1) will be used to model the kinematic equations of wheeled mobile robots, x_1 will represent the distance covered by the vehicle along a path to be followed, and x_2 will represent the lateral distance between the vehicle and the path. Path following will therefore mainly consist in regulating x_2 to zero, independently of the values taken by x_1 (and thus u_1), while stabilization to a desired configuration will further involve the regulation of x_1 to zero by utilizing also the input u_1. The possibility of directly relating the first state components of the chained form to physical euclidean-type coordinates justifies the preference given here to this form over the theoretically equivalent power form introduced in [34].

It is worth noting that a chained system like (1), although it is nonlinear, has a strong underlying linear structure. This clearly appears when u_1 is taken as a function of time and no longer as a control variable. In this case, the system becomes a single-input time-varying linear system which may be written as

$$\begin{aligned} \dot{\tilde{x}}_1 &= 0 \\ \dot{X}_2 &= \begin{bmatrix} 0 & u_1(t) & 0 & \cdots & \cdots & 0 \\ 0 & 0 & u_1(t) & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & u_1(t) & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & u_1(t) \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} X_2 + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \end{aligned} \tag{3}$$

with

$$\tilde{x}_1 = x_1 - \int_0^t u_1(\tau)\, d\tau \quad \text{and} \quad X_2^T = [x_2, x_3, \cdots, x_n].$$

Putting a two-input nonlinear system in the chain form, when it is possible, is thus equivalent to linearizing this system with respect to one of its inputs. Since the chained system is controllable, controllability of the original system is a necessary condition. A necessary and sufficient condition is given in [18].

When the input u_1 is taken as a function of time, the system is clearly no longer controllable due to the first equation. Under certain conditions upon the choice of $u_1(t)$, however, the second part of the system involving X_2 remains controllable.

This type of property is very useful. It is used further in the study for the derivation of smooth time-varying feedbacks which asymptotically stabilize the point $X = 0$. It can also be utilized to solve the open-loop steering problem, i.e., the problem of determining open-loop control inputs that steer the system to a desired configuration X_{desired}, chosen equal to zero without loss of generality. The method basically consists of two steps: i) choose an integrable function $u_1(t)$ which ensures controllability of the second part of the system, and determine a control $u_2(t)$ which drives $X_2(t)$ to zero in finite time (usually done by integrating the system's equations on some time interval and solving a set of algebraic equations), and ii) once X_2 is at zero, keep u_2 equal to zero so as to leave X_2 unchanged, and determine $u_1(t)$ so as to drive $x_1(t)$ to zero in finite time.

This method has been used in [16], with $u_1(t)$ and $u_2(t)$ chosen as piecewise constant inputs. In this case, the first step corresponds to discretizing the system's equations with u_1 being kept constant (and different from zero) over $n-1$ sampling time intervals Δ and applying a dead-beat control strategy (the poles of the controlled discretized system are set equal to zero) to determine the values of u_2 on the time intervals $[k\Delta, (k+1)\Delta]$ $(0 \le k \le n-2)$. At time $t = (n-1)\Delta$, X_2 has reached zero and u_1 may then be chosen equal to $-x_1((n-1)\Delta)/\Delta$ so as to have x_1 equal to zero at time $t = n\Delta$. Note that by working more on the choice of u_1, feedback versions of this technique can be obtained. It may also be shown, as a complement to [16], that multiplying the piecewise constant inputs by $(1 - \cos(\omega t))$, with $\omega = (2\pi/\Delta)$, does not change the values of X at the sampling instants. In this way piecewise constant inputs are transformed into time-continuous inputs that achieve the same result.

The solution proposed in [17], with $u_1(t)$ and $u_2(t)$ being composed of sinusoids at integrally related frequencies, may also be viewed as a variant of this method. Obviously, the same method applies to other inputs. A more geometrical method for open-loop steering of nonholonomic vehicles will also be pointed out further in Section III-B.

When u_1 is constant and different from zero, the above system becomes time-invariant and the second part of the system is clearly controllable. By applying classical linear control techniques, it is then possible to derive linear feedbacks $u_2(X_2)$ which stabilize the origin $X_2 = 0$ exponentially.

In fact, even if $u_1(t)$ is not constant but only piecewise continuous, bounded, and strictly positive (or negative), it is quite simple to derive stabilizing feedbacks $u_2(X_2)$ for the second part of the system. Indeed, since $x_1(t)$ varies monotonically with time, differentiation with respect to time can be replaced by differentiation with respect to x_1. From now on we will refer to this change of variable as the u_1-time-scaling procedure. Then, the second part of the system may equivalently be written

$$\begin{aligned} x_2^{(1)} &= \mathrm{sign}(u_1)x_3 \\ x_3^{(1)} &= \mathrm{sign}(u_1)x_4 \\ &\vdots \\ x_{n-1}^{(1)} &= \mathrm{sign}(u_1)x_n \\ x_n^{(1)} &= \mathrm{sign}(u_1)v_2 \end{aligned} \tag{4}$$

with

$$x_i^{(j)} = \mathrm{sign}(u_1)\frac{\partial^j x_i}{\partial x_1^j} \quad \text{and} \quad v_2 = u_2/u_1(t).$$

This is the equation of a linear invariant system, an equivalent input–output representation of which is

$$x_2^{(n-1)} = \mathrm{sign}(u_1)^{n-1}v_2. \tag{5}$$

One falls upon a controllable invariant linear system which admits exponentially stabilizing linear feedbacks in the form

$$v_2(X_2) = -\mathrm{sign}(u_1)^{n-1}\sum_{i=1}^{i=n-1} g_i x_2^{(i-1)} \qquad (g_i > 0,\ \forall i) \tag{6}$$

the control gains g_i being chosen so as to satisfy the classical Routh–Hurwitz stability criterion (the positivity of g_i is necessary but not sufficient).

Hence, the time-varying control

$$u_2(X_2, t) = u_1(t)v_2(X_2) \tag{7}$$

globally asymptotically stabilizes the origin $X_2 = 0$ in this case. Moreover, the trajectories followed by the system's solutions are invariant with respect to variations of $u_1(t)$.

This "feedback linearization" technique, associated with u_1-time-scaling, has in fact been used by other authors working on mobile robot control. For example, Sampei *et al.* [22] have applied it to the problem of following a straight line in the case of a car pulling a single trailer. Their solution differs, however, from the one given further in the article in that they took the car's steering wheel angle as a control, instead of the angle's velocity.

In the earlier work of Dickmanns and Zapp [10], on vision-based roadline following, u_1-time-scaling is also implicitly used together with tangent linearization of the system's equations, instead of exact feedback linearization. In their work, u_1 has the physical meaning of the car's translational velocity.

Extension of the path following problem to the point-stabilization problem to achieve smooth time-varying feedback stabilization of a unicycle-type vehicle to a given posture, based on u_1-time-scaling, has been first proposed in [29]. The present study may be seen as a generalization of the results described in this paper.

B. Skew-Symmetric Chain Form and Lyapunov Control Design

We show next, by introducing the skew-symmetric chain form evoked before, and via a Lyapunov-like analysis, that control (7) globally stabilizes the origin $X_2 = 0$ for the second part of the chained system, provided that $|u_1(t)|$ and $|\dot{u}_1(t)|$ are bounded, and $u_1(t)$ does not asymptotically tend to zero. An important difference with the result stated previously is that $u_1(t)$ is now allowed to pass through zero.

From there it will be simple to complement the analysis and derive smooth time-varying feedbacks which globally stabilize the origin $X = 0$ of the complete system.

To this purpose, let us consider the following change of coordinates $\phi_1 : X \mapsto Z$ in R^n

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= x_3 \\ &\vdots \\ z_{j+3} &= k_j z_{j+1} + L_{h_1} z_{j+2} \qquad 1 \le j \le n-3 \end{aligned} \tag{8}$$

where

- k_j $(1 \le j \le n-3)$ is a real positive number;
- $L_{h_1} z_j = \frac{\partial z_j}{\partial X} h_1(X)$: the Lie derivative of z_j along h_1;
- $L_{h_1}^k = L_{h_1}^{k-1} L_{h_1}$: the Lie differentiation operator of order k along h_1.

One easily verifies that the Jacobian matrix $(\partial \phi_1 / \partial X)$ is a constant lower triangular matrix with ones on the diagonal. It is therefore a regular linear change of coordinates in R^n.

Moreover: $L_{h_2} z_i = 0$ $(1 \le i \le n-1)$, and $L_{h_2} z_n = 1$.

Taking the time derivative of z_{j+3} and using 2

$$\begin{aligned}\dot{z}_{j+3} &= \frac{\partial z_{j+3}}{\partial X} \dot{X} \\ &= (L_{h_1} z_{j+3}) u_1 + (L_{h_2} z_{j+3}) u_2. \end{aligned} \tag{9}$$

Also, from 8

$$L_{h_1} z_{j+3} = -k_{j+1} z_{j+2} + z_{j+4}. \tag{10}$$

Hence

$$\dot{z}_{j+3} = -k_{j+1} u_1 z_{j+2} + u_1 z_{j+4} \qquad (0 \le j \le n-4) \tag{11}$$

and

$$\dot{z}_n = L_{h_1} z_n u_1 + u_2. \tag{12}$$

The original chained system has thus been converted to the following skew-symmetric chained system

$$\begin{aligned}
\dot{z}_1 &= u_1 \\
\dot{z}_2 &= u_1 z_3 \\
\dot{z}_3 &= -k_1 u_1 z_2 + u_1 z_4 \\
&\vdots \\
\dot{z}_{j+3} &= -k_{j+1} u_1 z_{j+2} + u_1 z_{j+4} \qquad (0 \le j \le n-4) \\
&\vdots \\
\dot{z}_n &= -k_{n-2} u_1 z_{n-1} + w_2
\end{aligned} \tag{13}$$

with

$$w_2 = (k_{n-2} z_{n-1} + L_{h_1} z_n) u_1 + u_2. \tag{14}$$

The interest of this form is that it naturally lends itself to Lyapunov control design and analysis, as illustrated by the following proposition.

Proposition 2.1: Assume that $|u_1(t)|$ and $|\dot{u}_1(t)|$ are bounded, and consider the control

$$w_2 = -k_{w_2}(u_1) z_n \tag{15}$$

where $k_{w_2}(\cdot)$ is a continuous application strictly positive on $R - \{0\}$. If this control is applied to system (13), then the positive function

$$V(Z_2) = 1/2\left(z_2^2 + (1/k_1) z_3^2 + (1/k_1 k_2) z_4^2 + \cdots + \left(1/\prod_{j=1}^{j=n-2} k_j\right) z_n^2\right) \tag{16}$$

is nonincreasing along the closed-loop system's solutions, and asymptotically converges to some limit value $V_{\lim}$ (which *a priori* depends on the initial conditions).

Moreover $u_1(t) V(Z_2(t))$ asymptotically tends to zero.

Therefore, if $u_1(t)$ does not asymptotically tend to zero, then $V_{\lim} = 0$ and the manifold $Z_2 = 0$ is globally asymptotically stable.

The proof of this proposition, as of subsequent propositions, uses an extended version of Barbalat's Lemma stating that if a given differentiable function $f(x)$ from R^+ to R converges to some limit value when x tends to infinity, and if the derivative $(df/dx)(x)$ of this function is the sum of two terms, one being uniformly continuous and the other one tending to zero when x tends to infinity, then $(df/dx)(x)$ tends to zero when x tends to infinity.

Proof of Proposition 2.1: Taking the time derivative of V and using the system's $(n-1)$ last equations, one obtains

$$\dot{V} = \left(1/\prod_{j=1}^{j=n-2} k_j\right) z_n w_2. \tag{17}$$

Thus, if control (15) is used

$$\dot{V} = -\left(k_{w_2}(u_1)/\prod_{j=1}^{j=n-2} k_j\right) z_n^2 \qquad (\le 0). \tag{18}$$

The considered Lyapunov-like function is thus nonincreasing.

This in turn implies that $\|Z_2(t)\|$ is bounded, uniformly with respect to the initial conditions. Existence and uniqueness of the system's solutions over R^+ also follows.

Now, since V is nonincreasing, $V(t)$ converges to some limit value $V_{\lim}$ (≥ 0). Since $k_{w_2}(\cdot)$ is continuous and since $|u_1(t)|$ and $|\dot{u}_1(t)|$ are bounded, $k_{w_2}(u_1(t))$ is uniformly continuous. Hence, the right-hand side member of equality (18) is uniformly continuous along any system's solution, and, by application of Barbalat's lemma, $\dot{V}(t)$ tends to zero. Therefore, $k_{w_2}(u_1(t)) z_n(t)$ tends to zero. This in turn implies, using the properties of the function $k_{w_2}(\cdot)$ and the boundedness of $|u_1(t)|$ and $|z_n(t)|$, that $u_1(t) z_n(t)$ tends to zero.

From now on, the time index will often be omitted to simplify the notations. Taking the time derivative of $u_1^2 z_n$ and using the convergence of $u_1 z_n$ to zero, gives

$$\frac{d}{dt}(u_1^2 z_n) = -k_{n-2} u_1^3 z_{n-1} + o(t) \quad \text{with} \lim_{t \to +\infty} o(t) = 0 \tag{19}$$

$u_1^3 z_{n-1}$ is uniformly continuous along a system's solution since its time derivative is bounded. Therefore, in view of (19) and since $u_1^2 z_n$ tends to zero, $(d/dt)(u_1^2 z_n)$ also tends to zero (by application of the aforementioned extended version of Barbalat's lemma). Hence, $u_1^3 z_{n-1}$, and thus $u_1 z_{n-1}$, tend to zero.

Taking the time derivative of $u_1^2 z_j$ and repeating the above procedure iteratively, one obtains that $u_1 z_j$ tends to zero for $2 \le j \le n$. In view of the system's equations, we note that this in turn implies the convergence of $\dot{Z}_2$ to zero.

Summing up the squared values of $u_1(t) z_j(t)$, for $2 \le j \le n$, it appears that $u_1(t)^2 V(t)$ tends to zero. And so does $u_1(t)^2 V_{\lim}$ (from the already established convergence of $V(t)$ to $V_{\lim}$). $\Diamond$

Remarks:

- One can verify that, with the particular choice $k_{w_2}(u_1) = k'_{w_2}|u_1|$, the set of controls u_2 given by (14) and (15) with $k_i > 0$ $(i = 1, \cdots, n-2)$ and $k'_{w_2} > 0$, coincides with the set of stable linear controls (7) previously associated with the linear invariant system (4). More precisely, there is a one-to-one correspondence between the elements of the two sets. One can thus apply classical linear control design methods to determine adequate values for the parameters k_i $(i = 1, \cdots, n-2)$ and k'_{w_2} and optimize the control performance near $Z_2 = 0$, as illustrated in [29]. This correspondence also underlies the connection existing between the Routh–Hurwitz criterion and the operation consisting in transforming the original chain form into a skew-symmetric chain form.
- Nonconvergence of $u_1(t)$ to zero, under the assumption that $|\dot{u}_1(t)|$ is bounded, implies that $\int_0^t |u_1(\tau)| d\tau$ tends to infinity with t. Divergence of this integral is in fact necessary to the asymptotical convergence of $\|Z_2(t)\|$ to zero, when using control (15) with $k_{w_2}(u_1) = k'_{w_2}|u_1|$. This appears clearly when interpreting this control as a stabilizing linear control for the linear invariant system (4) obtained by replacing the time variable by the aforementioned integral. This integral may still diverge, however, when $u_1(t)$ tends to zero slowly "enough" (like $t^{-\frac{1}{2}}$, for example). This indicates that $\|Z_2(t)\|$ may still converge to zero when $u_1(t)$ does.

Proposition 2.1 is not only of interest in solving the path following problem for mobile robots, it also suggests a way of determining smooth time-varying feedbacks which globally asymptotically stabilize the origin $Z = 0$ (or $X = 0$) of the whole system. In this case, u_1 is used as a control the role of which is to complement the action of the control u_2 (or w_2) in order to also obtain asymptotical convergence of z_1 (or x_1) to zero. Since chained systems like (1) cannot be asymptotically stabilized by using smooth pure state feedbacks (by application of a Brockett's theorem [5]), smooth feedback stabilization can only be achieved by using another type of control. A time-varying control law will be considered in the present case.

Proposition 2.2: Consider the same control as in Proposition 2.1

$$w_2 = -k_{w_2}(u_1) z_n \tag{20}$$

complemented with the following time-varying control

$$u_1 = -k_{u_1} z_1 + h(Z_2, t) \tag{21}$$

where

- k_{u_1} is a positive number;
- $h(Z_2, t)$ is a function of class C^{p+1} $(p \geq 1)$, uniformly bounded with respect to t, with all successive partial derivatives also uniformly bounded with respect to t, and such that C_1: $h(0, t) = 0, \forall t$ C_2: There is a time-diverging sequence $\{t_i\}_{i \in N}$, and a positive continuous function $\alpha(\cdot)$ such that

$$\|Z_2\| \geq l > 0 \Rightarrow \sum_{j=1}^{j=p} \left(\frac{\partial^j h}{\partial t^j}(Z_2, t_i) \right)^2 \geq \alpha(l) > 0, \forall i.$$

Controls (20) and (21) globally asymptotically stabilizable the origin $Z = 0$.

Proof of Proposition 2.2: It has already been shown that the positive function $V(Z_2)$ used in Proposition 2.1 is nonincreasing along the closed-loop system's solutions, implying that $\|Z_2(t)\|$ is bounded uniformly with respect to initial conditions. Note that the boundedness of $|u_1(t)|$ and $|\dot{u}_1(t)|$ is not needed to prove this fact.

The first equation of the controlled system is

$$\dot{z}_1 = -k_{u_1} z_1 + h(Z_2, t). \tag{22}$$

This is the equation of a stable linear system subjected to the bounded additive perturbation $h(Z_2(t), t)$. Therefore, $|z_1(t)|$ is also bounded uniformly with respect to the initial conditions.

Existence and uniqueness of the solutions over R^+ is thus ensured.

From the expression of u_1, it is then found that $u_1(t)$ (taken as a function of time along a system's solution) is bounded. And so is its first derivative [by using the regularity properties imposed upon $h(Z_2, t)$].

Proposition 2.1 thus applies. In particular, $V(Z_2(t))$ tends to some limit value $V_{\lim}$ (≥ 0), $\|\dot{Z}_2(t)\|$ tends to zero, and $Z_2(t)$ tends to zero if $u_1(t)$ does not.

We now proceed by contradiction.

Assume that $u_1(t)$ does not tend to zero. Then, $\|Z_2(t)\|$ tends to zero. By uniform continuity and since $h(0, t) = 0$ (condition C_1), $h(Z_2(t), t)$ also tends to zero. Equation (22) then becomes the equation of a stable linear system subjected to an additive perturbation which asymptotically vanishes. As a consequence, $z_1(t)$ tends to zero. From the expression of u_1, this in turn implies that $u_1(t)$ tends to zero, yielding a contradiction.

Therefore, $u_1(t)$ must asymptotically tend to zero.

Differentiating the expression of u_1 with respect to time and using the convergence of $u_1(t)$ and $\|\dot{Z}_2(t)\|$ to zero, we get

$$\dot{u}_1(t) = \frac{\partial h}{\partial t}(Z_2(t), t) + o(t) \quad \text{with} \lim_{t \to +\infty} o(t) = 0. \tag{23}$$

Since $(\partial h/\partial t)(Z_2(t), t)$ is uniformly continuous (its time derivative is bounded), $\dot{u}_1(t)$, and thus $(\partial h/\partial t)(Z_2(t), t)$, tend to zero (Barbalat's Lemma).

By using similar arguments, one obtains that the time-derivative of $(\partial h/\partial t)(Z_2(t), t)$ and $(\partial^2 h/\partial t^2)(Z_2(t), t)$ tend to zero.

By repeating the same procedure as many times as necessary, we show that $(\partial^j h/\partial t^j)(Z_2(t), t)$ tends to zero, $(1 \leq j \leq p)$. Therefore

$$\lim_{t \to \infty} \sum_{j=1}^{j=p} \left(\frac{\partial^j h}{\partial t^j}(Z_2(t), t) \right)^2 = 0. \tag{24}$$

Assume now that $V_{\lim}$ is different from zero. This implies that $\|Z_2(t)\|$ remains larger than some positive real number l (which can be calculated from $V_{\lim}$). The previous convergence result is then not compatible with the condition C_2 imposed on the function $h(Z_2, t)$.

Therefore, $V_{\lim}$ is equal to zero, and $Z_2(t)$ asymptotically converges to zero. Then, by uniform continuity and using condition C_1, $h(Z_2(t), t)$ tends to zero.

In view of the expression of u_1, asymptotical convergence of $z_1(t)$ to zero readily follows. ◊

Remark: Dependence of the function $h(Z_2, t)$ upon the last state variable z_n is not required when the function $k_{w_2}(\cdot)$ is strictly positive on R. The reason is that $z_n(t)$ unconditionally converges to zero in this case, due to the convergence of $\dot{V}(Z_2(t))$ to zero (cf. proof of Proposition 2.1).

It can be noted that only the control input u_1 depends on time explicitly via the function $h(Z_2, t)$. We will refer to this function as the heat-function, to establish a parallel with well-known probabilistic global minimization methods and underline the primary role of this term in the control, i.e., forcing "motion" as long as the system has not reached the desired equilibrium point, thus preventing the system's state from converging to other equilibrium points.

According to Proposition 2.1, when one is only interested in the regulation of Z_2 (as in the case of mobile robot path following), any sufficiently regular input $u_1(t)$, which does not asymptotically tend to zero, can be used. This leaves the user with some freedom concerning the choice of this input. For instance, uniform exponential convergence of $\|Z_2(t)\|$ to zero is obtained when $|u_1(t)|$ remains larger than some positive number. Other sufficient conditions for exponential convergence of $\|Z_2(t)\|$ to zero, which do not require $u_1(t)$ to have always the same sign, may also be derived. For example, if $|\dot{u}_1(t)|$ is bounded, it is sufficient to have $|u_1(t)|$ periodically larger than some positive number.

If the application further requires the regulation of z_1 (stabilization of a mobile robot about a fixed desired configuration, for example), then Proposition 2.2 suggests implementing a time-varying feedback u_1. In both cases, the same control law u_2 (or w_2) based on u_1-time-scaling can be used.

The conditions imposed by Theorem 2.2 upon the heat-function are not severe and can easily be met. For example, the following three functions

$$h(Z_2, t) = \|Z_2\|^2 \sin(t) \tag{25}$$

$$h(Z_2, t) = \sum_{j=0}^{j=n-2} a_j \sin(\beta_j t) z_{2+j} \tag{26}$$

$$h(Z_2, t) = \sum_{j=0}^{j=n-2} a_j \frac{\exp(b_j z_{2+j}) - 1}{\exp(b_j z_{2+j}) + 1} \sin(\beta_j t) \tag{27}$$

(with $a_j \neq 0$, $b_j \neq 0$, $\beta_j \neq 0$, and $\beta_i \neq \beta_j$ when $i \neq j$) satisfy these conditions. For the first function, this is obvious. For the second function, the proof is given in [26]. The same proof basically applies to the third function which presents the additional feature of being uniformly bounded with respect to all its arguments. It can be noted that it is not necessary to use sinusoids at integrally related frequencies, as opposed to the solution proposed in [34]. In fact, the theorem indicates that it is not even necessary to use time-periodic functions, as it is assumed in most time-varying feedback stabilization studies.

For practical purposes, the choice of the heat-function is important because the overall control performance (asymptotical convergence rate, time needed to enter a small ball centered on zero, sensitivity with respect to perturbations, etc.) critically depends upon this choice. This has been checked by the author in simulation. By performing a complementary analysis in the three-dimensional case, based on Center Manifold techniques, it is also possible to explain why the functions (26) and (27) are better than (25) with respect to the induced asymptotical convergence rate. For the last two functions, the parameters a_j and b_j, which characterize the "slope" of $h(Z_2, t)$ near $Z_2 = 0$ have been found to have much influence on the transient time needed for the system's solutions to get close to zero. Basically, the larger these parameters are, the shorter the transient time is.

C. Stability and Asymptotical Rate of Convergence

From Section II-B, we already know that, when using the smooth time-varying control law (20), (21) with $k_{w_2}(u_1) = k'_{w_2}|u_1|$ $(k'_{w_2} > 0)$, the convergence of $\|Z(t)\|$ to zero cannot be exponential. Indeed, $u_1(t)$ would otherwise converge to zero exponentially and the integral $\int_0^t |u_1(\tau)| d\tau$ would not diverge. This would be in contradiction with the fact (pointed out earlier) that divergence of this integral is necessary to the asymptotical convergence of $\|Z_2(t)\|$ to zero.

From the simulation of a smooth time-varying feedback control applied to a unicycle-type vehicle, it has also been observed in [28] that the norm of the state vector did not converge to zero faster than $t^{-\frac{1}{2}}$ for most initial configurations. This is much slower than the uniform exponential rate of convergence that can be obtained in the case of nonlinear systems the linear tangent approximation of which is controllable. It is claimed in [11] that it is not possible to achieve exponential stability for nonholonomic systems by using smooth (differential everywhere) time-periodic feedbacks. In mathematical terms, this means that the system's trajectories cannot satisfy the following inequality

$$\|X(t)\| \leq K\|X(0)\| \exp(-\lambda t) \quad \forall X(0) \text{ in some open ball centered on zero} \tag{28}$$

for some positive real numbers K and λ.

The practical significance of this relation, when it is satisfied, is two-fold: i) the ratio $\|X(t)\|/\|X(0)\|$ between transient and initial errors is uniformly bounded, and ii) all solutions end up converging to zero exponentially.

It is worth noting that these two properties, regrouped under the strong concept of exponential stability, do not necessarily hold together.

For example, the piecewise-continuous time-invariant feedback law proposed by Canudas and Sørdalen in [6], for posture stabilization of a unicycle, only yields the following result

$$\|X(t)\| \leq (K_1 + K_2\|X(0)\|) \exp(-\lambda t). \tag{29}$$

Each solution converges to zero exponentially, but the slightest initial error, or perturbation, may produce transient deviations the size of which is larger than some constant. Note that this sensitivity to small perturbations, observed for this particular example, is not necessarily indicative of nonsmooth feedbacks as a whole.

In [32], Sørdalen proposes another interesting control which may be seen as a time-varying mix of open-loop and feedback strategies, continuous with respect to time, but nonsmooth in the state vector. He shows that this control, which applies to any chain form system, yields the following property

$$\|X(t)\| \le g(\|X(0)\|)\exp(-\lambda t) \tag{30}$$

where $g(\cdot)$ is a class $\mathcal{K}$-function (i.e., strictly increasing and such that $g(0) = 0$) which is not Lipschitz around zero. Precisely, the derivative of $g(x)$ tends to infinity when x tends to zero.

This property has been called $\mathcal{K}$-exponential stability. It is weaker than the usual exponential stability notion in the sense that the ratio between transient and initial deviations is not uniformly bounded. Nevertheless, it is better than (29) in the sense that the deviations are not lowerbounded by some positive constant. As in the previous case, all solutions tend to zero exponentially in the absence of perturbations acting on the system.

By using the properties of homogeneous systems [12], time-periodic feedbacks which are continuous with respect to both time and state, but not differentiable at the origin, have been proposed in [14], [20], [30] and have been shown to achieve the same type of exponential stability.

Concerning the case of smooth time-varying feedbacks, such as the ones derived in the Section II-B, it is simple to verify that we have

$$\|X(t)\| \le K\|X(0)\| \tag{31}$$

for some positive constant K the size of which may be taken as close to one as desired via a suitable choice of the control parameters.

Smooth time-varying feedbacks thus are, in some sense, less sensitive to initial errors than the aforementioned nonsmooth feedbacks. The "price" paid for this type of robustness is that the system's solutions do not converge to zero as fast as exponentially. In fact, in view of the above discussion, one can only expect to have

$$\|X(t)\| \le K\|X(0)\|f(t); \quad f(0) = 1, \quad \lim_{t\to+\infty} f(t) = 0 \tag{32}$$

where $f(t)$ is a decreasing function which does not tend to zero as fast as exponentially. For example: $f(t) = (1+t)^{-\frac{1}{2}}$, in the case of the control considered in [28], as it may be rigorously established either by applying Center Manifold techniques [7] or by invoking two-time scale techniques, as done in [14].

The purpose of the above discussion was to summarize our actual knowledge concerning the stability properties of controlled nonholonomic mobile robots (with restricted mobility) in the case of point-stabilization and to point out the difficulty in objectively comparing smooth and nonsmooth feedback solutions. So far, exponential stability, in the usual sense, has not been obtained and is most likely out of reach. Exponential convergence of the solutions to zero, in the absence of perturbations and modeling errors, is possible, however, by using either piecewise-continuous time-invariant feedbacks or continuous time-varying feedbacks which are not differentiable at zero. Smooth time-varying feedbacks are less efficient in terms of asymptotical rate of convergence, but are also potentially not very sensitive to initial conditions and perturbations in the vicinity of zero. A slow asymptotical convergence rate still does not mean that the system's solutions cannot be rapidly steered to an arbitrarily small neighborhood of zero, as pointed out in [34] (for example) and illustrated by simulation results in [29]. Nevertheless, this type of performance has not been obtained for small values of K. This again reflects the apparently unavoidable compromise between performance and robustness. It should also be noted that perturbations acting on nonholonomic systems are not of equal importance depending on the state component which is primarily affected: a deviation in a direction compatible with the vehicle's mobility is clearly not as severe as a deviation which violates one of the system's kinematic constraints (lateral skidding of a car, for example).

Further clarification of these issues is thus needed in relation to a rather fundamental question, seldom addressed in the control literature: Is the asymptotical rate of convergence a good measurement of the overall control performance? Answering this question is not simple, knowing that regulation errors are physically unavoidable and that what often really matters in practice is to keep these errors as small as possible under realistic adverse experimental conditions. While connections between robustness issues and asymptotical rate of convergence of the controlled system have been much studied in the case of linear systems (or nonlinear systems that can be approximated by controllable linear systems), they are still not well understood in other cases.

To illustrate the difficulty with a concrete example, a control law similar to (20)–(21) has been simulated for a three-dimensional unicycle-type vehicle with the following nonsmooth heat-function

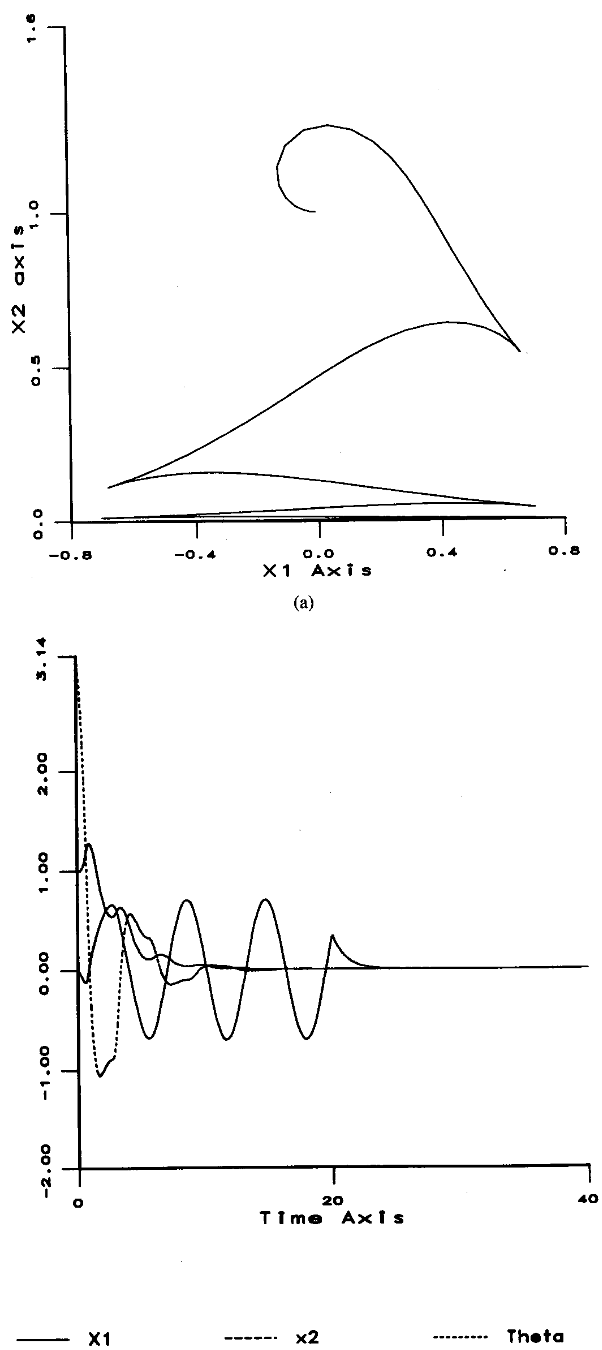
$$h(x_2, x_3, t) = \begin{cases} \sin(t) & \text{if } x_2^2 + (1/k_1)x_3^2 \ge \epsilon^2 \\ 0 & \text{if } x_2^2 + (1/k_1)x_3^2 < \epsilon^2. \end{cases} \tag{33}$$

Note that this function does not satisfies the conditions imposed in Proposition 2.2. The corresponding feedback control is time-varying, but not even continuous.

The $(x_1(t), x_2(t))$ Cartesian position of the vehicle is represented in Fig. 1(a).

The time-evolution of $x_1(t)$, $x_2(t)$, and the vehicle's orientation angle $\theta(t)$ $(\approx x_3(t))$ is shown in Fig. 1(b).

After 25 seconds, all variables "seem" to have converged to zero. In reality, it is possible to show that $x_1(t)$ and both control inputs converge to zero exponentially, while



(a)

(b)

Fig. 1.

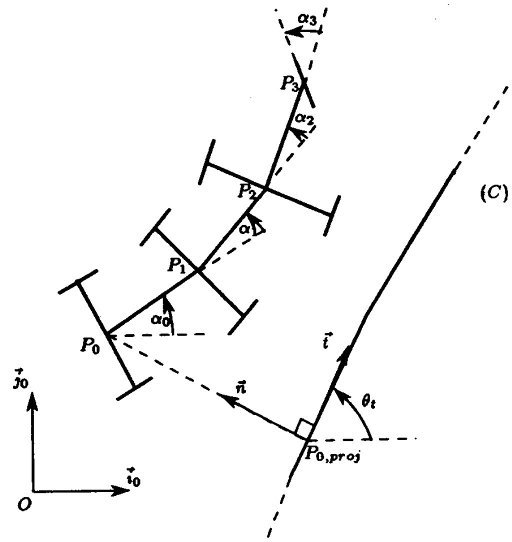


Fig. 2.

$2V(X_2(t)) = x_2(t)^2 + (1/k_1)x_3(t)^2$ can only be shown to become smaller than ϵ^2 ($= 10^{-6}$, in the simulation) after a finite time.

This control therefore does not asymptotically stabilize the system to the desired equilibrium, so that the notion of asymptotical rate of convergence does not even apply here. Is this fact sufficient in itself to assert that this is not a good control?

III. Application to the Control of a Car with -n- Trailers

A. Modeling Equations and Notations

We consider a car with -n- trailers as represented in Fig. 2. The system is assumed to move on flat ground. The wheels are allowed to roll and spin, but not slip.

The vehicle counted first is the last trailer and the following notations are used:

- l_i is the distance between P_i and P_{i+1}
- α_i $(1 \leq i \leq n)$ is the angle between $P_{i-1}P_i$ and P_iP_{i+1} which characterizes the orientation of the vehicle $(i+1)$ with respect to the previous vehicle
- α_0 gives the orientation of the first vehicle with respect to a fixed frame. For instance, we may choose: $\alpha_0 = \text{angle}(\vec{i}_0, \vec{P_0P_1})$
- α_{n+1} is the angle of the car's driving front wheel with respect to the car's body.
- v_i $(0 \leq i \leq n+1)$ is the intensity of the velocity of the point P_i. This is the translational velocity of the $(i+1)$ vehicle.
- r is the radius of the car's front steering wheel, and ω the angular velocity of this wheel about its horizontal axis so that $v_{n+1} = r\omega$.

In what follows, only velocity control is considered, and ω and $(d/dt)\alpha_{n+1}$ are chosen as control variables.

Kinematic equations of this system, with respect to a fixed frame, have been derived by various authors. See [13], [22], [35], for example. We will work here with a more general set of equations expressing the system's motion relatively to an arbitrary given path.

A first set of equations is simply obtained by using the classical identity

$$\frac{d}{dt}\vec{P}_{i+1} = \frac{d}{dt}\vec{P}_i + \vec{\omega}_i \wedge \overrightarrow{P_iP_{i+1}} \quad \text{for } 0 \leq i \leq n \qquad (34)$$

where $\vec{\omega}_i$ is the angular velocity about the vertical axis of the ith vehicle's body.

This yields the following equations

$$\begin{aligned} v_{n+1} &= r\omega \\ v_i &= v_{i+1}\cos(\alpha_{i+1}) \qquad (0 \le i \le n) \\ \dot{\alpha}_0 &= v_0 \frac{\tan(\alpha_1)}{l_1} \\ \dot{\alpha}_i &= v_i\left(\frac{\tan(\alpha_{i+1})}{l_{i+1}} - \frac{\sin(\alpha_i)}{l_i}\right) \qquad (1 \le i \le n). \end{aligned} \tag{35}$$

We note that the set of variables α_i entirely characterizes the relative positioning of each vehicle with respect to the others.

The remaining equations must describe the motion of one of the vehicles with respect to the path we would like this vehicle to follow. To this purpose, we choose the first vehicle (i.e., the last trailer) and the position coordinates of the point P_0.

The path to be followed by this point is denoted as (C). For the sake of simplicity, we consider a smooth simple curve defined by one of its point, the unitary tangent vector at this point, and its curvature $\text{curv}(s)$, with s being the curvilinear coordinate along the curve. Moreover, it is assumed that

- $\text{curv}(s)$ is differentiable $(n+1)$ times. This is necessary for P_0 to be able to remain on (C) without stopping, as it will later appear.
- The radius of any circle tangenting (C) at two or more points and the interior of which does not contain any point of the curve is lowerbounded by some positive real number denoted as $r_{\min}$. The set of the circles' centers so defined is the Voronoi diagram associated with the curve [33]. This assumption implies in particular that $|\text{curv}(s)| \le 1/r_{\min}$, $\forall s$. For example, if (C) is a straight line, then $r_{\min} = +\infty$ and $\text{curv}(s) = 0$. If (C) is a circle, then $r_{\min}$ is the circle's radius and $\text{curv}(s) = 1/r_{\min}$.

Under these assumptions, if the distance between P_0 and (C) is smaller than $r_{\min}$, there is a unique point on (C), denoted as $P_{0,\text{proj}}$, so that $\|P_0 P_{0,\text{proj}}\|$ is equal to this distance (see Fig. 2).

Let s denote the curvilinear coordinate at the point $P_{0,\text{proj}}$, and $(P_{0,\text{proj}};\ \vec{t},\ \vec{n})$ the Frenet frame on the curve at this point. The position of P_0 in the plane is completely characterized by the pair of variables $(s,\ y)$ where y is the intensity of the vector $\vec{P}_{0,\text{proj}}P_0$, i.e.,

$$\overrightarrow{P_{0,\text{proj}}\, P_0} = y\vec{n}. \tag{36}$$

Note that in the particular case where (C) is a straight line, s and y coincide with classical Cartesian coordinates. For other curves, one of the control objectives will be to keep the coordinate y smaller than $r_{\min}$ all the time so as to avoid any ambiguity when using the parameterization $(s,\ y)$.

This parameterization has previously been proposed in [29] for the control of a unicycle-type vehicle. While it is primarily used here for feedback control purposes, it is also related to the approach developed in [21] for path planning, as shown in the next section.

Let:

- θ_t denote the angle between $\vec{i}_0$ and $\vec{t}(s)$
- $\theta = \alpha_0 - \theta_t$ the angle between the first vehicle's body and the curve's tangent vector $\vec{t}$. When the first vehicle follows (C) exactly, with a nonzero translational velocity, θ can only take values equal to $k\pi$ $(k \in Z)$. Without loss of generality one may assume that the desired value for the angle θ is zero.

The following equations for the first vehicle (see also [29]) are then easily derived

$$\begin{aligned} \dot{s} &= v_0 \frac{\cos(\theta)}{1 - \text{curv}(s)y} \\ \dot{y} &= v_0 \sin(\theta) \\ \dot{\theta}_t &= \text{curv}(s)\dot{s} \\ &= v_0 \frac{\text{curv}(s)\cos(\theta)}{1 - \text{curv}(s)y}. \end{aligned} \tag{37}$$

By regrouping (35) and (37) one obtains the following control system

$$\dot{X} = g_1(X)v_0 + g_2 v_2 \tag{38}$$

with

$$X = \begin{bmatrix} s \\ y \\ \theta \\ \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \quad \dim(X) = n+4$$

$$\begin{aligned} v_0 &= r\omega \prod_{i=4}^{i=n+4} \cos(x_i) \\ u_2 &= \dot{x}_{n+4} \end{aligned}$$

$$\begin{aligned} g_{1,1}(X) &= \frac{\cos(x_3)}{1 - \text{curv}(x_1)x_2} \\ g_{1,2}(X) &= \sin(x_3) \\ g_{1,3}(X) &= \frac{\tan(x_4)}{l_1} - \frac{\text{curv}(x_1)\cos(x_3)}{1 - \text{curv}(x_1)x_2} \\ g_{1,4}(X) &= \frac{1}{\cos(x_4)}\left(\frac{\tan(x_5)}{l_2} - \frac{\sin(x_4)}{l_1}\right) \\ &\vdots \\ g_{1,j}(X) &= \frac{1}{\prod_{l=4}^{l=j}\cos(x_j)}\left(\frac{\tan(x_{j+1})}{l_{j-2}} - \frac{\sin(x_j)}{l_{j-3}}\right) \\ &\vdots \\ g_{1,n+4}(X) &= 0 \end{aligned}$$

$$g_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

This control system characterizes our mechanical system, as long as X belongs to the set

$$\Omega = R\times] - r_{\min}, +r_{\min}[\times R \times (] - \frac{\pi}{2}, +\frac{\pi}{2}[)^{n+1}. \tag{39}$$

Although controllability of the mechanical system in $R^3 \times SO_1^{n+1}$ (i.e., the possibility of steering the system between any two configurations in finite time) has been theoretically established (see [13]), for example), the control design and analysis is hereafter limited to the set Ω. In particular, the angles α_i $(1 \leq i \leq n+1)$ are bound to stay in the interval $]-\pi/2, +\pi/2[$. For most practical purposes, this is not restrictive. Moreover, undesirable "jack-knife" effects will systematically be avoided in this way.

B. About Path Planning

As already mentioned, the angle θ must be equal to zero (or π) when P_0 moves along the desired path. This also means that the time-derivative of this angle must be equal to zero. Thus, according to the third equation of the above system, and since y is also equal to zero along the path, one must have

$$\tan(\alpha_1) = \begin{cases} +l_1 \operatorname{curv}(s) & \text{if } \theta = 0 \\ -l_1 \operatorname{curv}(s) & \text{if } \theta = \pi \end{cases}. \quad (40)$$

By taking the time derivative of this last equation and comparing the result with the fourth system's equation, one obtains after simple calculations

$$\tan(\alpha_2) = \begin{cases} \frac{l_2}{l_1^2 \operatorname{curv}(s)^2+1}\left(\operatorname{curv}(s) + \frac{l_1 \operatorname{curv}^{(1)}(s)}{l_1^2 \operatorname{curv}(s)^2+1}\right) \\ \text{if } \theta = 0 \\ \frac{l_2}{l_1^2 \operatorname{curv}(s)^2+1}\left(\operatorname{curv}(s) - \frac{l_1 \operatorname{curv}^{(1)}(s)}{l_1^2 \operatorname{curv}(s)^2+1}\right) \\ \text{if } \theta = \pi \end{cases}. \quad (41)$$

where $\operatorname{curv}^{(1)}(s)$ is the first derivative (with respect to the curvilinear coordinate) of the path's curvature.

Repeating the above procedure $(n-1)$ times, one would obtain that, along the desired path and for each of the two possible values of θ, the angles α_i $(1 \leq i \leq n+1)$ are functions of the path's curvature and its successive derivatives up to the order $(i-1)$. Moreover, one may verify that the correspondence from $(]-\pi/2, +\pi/2[)^{n+1}$ onto R^{n+1} between the set of angles $\{\alpha_i\}_{1\leq i\leq n+1}$ and the set $\{\operatorname{curv}^{(i)}(s)\}_{0\leq i\leq n}$ is one-to-one.

A direct consequence of this fact is that the problem of steering the system between any two configurations (satisfying the aforementioned condition imposed on the range of the angles α_i) can be addressed as a purely geometrical problem consisting of finding a planar path of class C^{n+3} which connects two given points (corresponding to the initial and final position of the point P_0), with given tangents at these points (corresponding to initial and final values of α_0), and conditions imposed on the curvature and its n successive derivatives at both extremities of the path (corresponding to initial and final values of angles α_i $(1 \leq i \leq n+1)$). Since such a path obviously exists, one finds again in this way that the system is controllable in Ω.

The possibility of parameterizing the vehicle's motion by the curvature, and its derivatives, of the path drawn by the first vehicle has been here derived from the system's error equation (38). In [21], this possibility, which yields the above formulation of the path planning problem, is presented as a more general consequence of the system's flatness.

It may also be noted that there is an abundant literature dealing with this type of geometrical problem. Solutions proposed in this domain, such as widely used Bezier's polynomial curves (splines) for example [2], could thus be of interest for people working on mobile robot path planning problems and yield methods complementary to those already explored.

C. Smooth Feedback Controls for Path Following and Stabilization about a Fixed Desired Configuration

To apply the results of Section II, system (38) should first be converted into the chain form (1), or, equivalently, to the skew-symmetric chain form (13).

We first rewrite the original system as follows

$$\dot{X} = f_1(X)u_1 + g_2\dot{x}_{n+4} \quad (42)$$

with

$$u_1 = v_0 \frac{\cos(x_3)}{1 - \operatorname{curv}(x_1)x_2}$$

$$\begin{aligned} f_{1,1} &= 1 \\ f_{1,2}(X) &= \frac{1-\operatorname{curv}(x_1)x_2}{\cos(x_3)} g_{1,2}(x_3) \\ f_{1,3}(X) &= \frac{1-\operatorname{curv}(x_1)x_2}{\cos(x_3)} g_{1,3}(x_1, x_2, x_3, x_4) \\ f_{1,4}(X) &= \frac{1-\operatorname{curv}(x_1)x_2}{\cos(x_3)} g_{1,4}(x_4, x_5) \\ &\vdots \\ f_{1,n+3}(X) &= \frac{1-\operatorname{curv}(x_1)x_2}{\cos(x_3)} g_{1,n+3}(x_4, \cdots, x_{n+4}) \\ f_{1,n+4} &= 0. \end{aligned}$$

This control system is equivalent to the original one within the reduced set

$$\Omega_{\text{reduced}} = R \times]-r_{\min}, +r_{\min}[\times (]-\frac{\pi}{2}, +\frac{\pi}{2}[)^{n+2} \quad (\subset \Omega).$$

In particular, due to the choice of the input u_1, the variable x_3 (i.e., the orientation error θ) has to be kept in the interval $]-\frac{\pi}{2}, +\frac{\pi}{2}[$.

This control system is directly converted into a skew-symmetric chain form via the change of coordinates ϕ_2: $X \mapsto Z$, with

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= h_y(x_2) \\ z_3 &= f_{1,2}(x_1, x_2, x_3)\frac{\partial h_y}{\partial x_2} \\ z_4 &= k_1 z_2 + L_{f_1} z_3 \\ &\vdots \\ z_{j+3} &= k_j z_{j+1} + L_{f_1} z_{j+2} \qquad (j \leq 2 \leq n+1) \\ &\vdots \end{aligned} \quad (43)$$

where $h_y(x_2)$ is a smooth monotonic function which maps $]-r_{\min}, +r_{\min}[$ onto R, with first derivative strictly larger than a positive real number, and such that $h_y(0) = 0$. For example, $h_y(x_2) = (2r_{\min}/\pi)\tan(\pi y/2r_{\min})$ is a possible candidate when $r_{\min} < +\infty$. If $r_{\min} = +\infty$, the simplest choice is $h_y(x_2) = x_2$. This function is introduced here to force $|y(t)|$ $(= |x_2(t)|)$ to remain smaller than $r_{\min}$ in the subsequent control analysis.

One can verify that the Jacobian matrix $(\partial\phi_2/\partial X)$ is a lower triangular matrix with nonzero components $d_i$on the diagonal given by

$$\begin{aligned} D_1 &= 1 \\ d_2 &= \frac{\partial h_y}{\partial x_2} \\ d_3 &= \frac{1-\mathrm{curv}(x_1)x_2}{\cos(x_3)^2}\frac{\partial h_y}{\partial x_2} \\ d_4 &= \frac{(1-\mathrm{curv}(x_1)x_2)^2}{l_1\cos(x_3)^3\cos(x_4)^2}\frac{\partial h_y}{\partial x_2} \\ &\vdots \\ d_{n+4} &= \frac{(1-\mathrm{curv}(x_1)x_2)^{n+2}}{(\prod_{i=1}^{i=n+1} l_j)(\prod_{j=3}^{j=n+4}\cos(x_j)^{n+6-j})}\frac{\partial h_y}{\partial x_2} \end{aligned}$$

All other components are well defined in Ω_{reduced}. This matrix is thus defined and nonsingular on Ω_{reduced}. Moreover, $\phi_2(\Omega_{\text{reduced}}) = R^{n+4}$. Thus, according to the theorem 0.5 in [23, p. 13], ϕ_2 induces a diffeomorphism of class C^{n+1} ($n+1$ being the degree of differentiability of $\mathrm{curv}(x_1)$) between Ω_{reduced} and R^{n+4}.

Then, by using

$$L_{g_2}L_{f_1}^j z_{i+3} = 0 \qquad (0 \le i \le n,\ 0 \le j \le n-i) \quad (44)$$

and

$$L_{g_2}L_{f_1}^{n+1}z_3 = \frac{(1-\mathrm{curv}(x_1)x_2)^{n+2}}{(\prod_{i=1}^{i=n+1} l_j)(\prod_{j=3}^{j=n+4}\cos(x_j)^{n+6-j})}\frac{\partial h}{\partial x_2}(x_2) \quad (45)$$

it is easy to verify that the control system, with Z as state vector, has the form (13), with the auxiliary control input w_2 defined by

$$w_2 = (k_{n+2}z_{n+3} + L_{f_1}z_{n+4})u_1 + (L_{g_2}L_{f_1}^{n+1}z_3)u_2. \quad (46)$$

Once the system has been put into the skew-symmetric chain form, there only remains to apply Proposition 2.1 to determine a control input w_2 which stabilizes the point $Z_2 = 0$ and thus make the mechanical system follow the path (C). One can also apply Proposition 2.2 to determine smooth time-varying feedbacks which make the system converge to a given configuration on the path.

Note that stability is only local in this case since the state vector X must belong to Ω_{reduced}. This implies that the angles θ and α_i $(1 \le n+1)$ must have initial values in $]-\frac{\pi}{2}, +\frac{\pi}{2}[$, and that the initial distance $|y(0)|$ must be smaller than $r_{\min}$.

Remark: The method described above allows asymptotically stabilizing the mechanical system to any configuration so that $|\alpha_i| < (\pi/2)$ $(1 \le n+1)$. In other studies ([22], [32]), for example), only configurations with zero angles α_i (all trailers aligned) have been considered.

D. A Modification to Broaden the Stability Domain

A practical shortcoming of the above control design method is the necessity of starting with an orientation error $|\theta|$ smaller than $\pi/2$.

This limitation can be removed by considering a more global change of coordinates which converts the initial control system to a modified chain form.

This modification was implicitly used in [29] in the particular case of a unicycle-type vehicle, and simulation results can be found in the same reference.

The new transformation ϕ_3: $X \mapsto Z$ that is considered is the following

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= h_y(x_2) \\ z_3 &= x_3 \\ z_4 &= k_1\frac{\sin(x_3)}{x_3}\frac{\partial h_y}{\partial x_2}(x_2)z_2 + g_{1,3}(x_1, x_2, x_3, x_4) \\ z_5 &= k_2 z_3 + L_{g_1}z_4 \\ &\vdots \\ z_{j+4} &= k_{j+1}z_{j+2} + L_{g_1}z_{j+3} \qquad (2 \le j \le n) \\ &\vdots \end{aligned} \quad (47)$$

where

- $g_1(X)$ is the vector field involved in the system's representation (38)
- k_j $(1 \le j \le n+1)$ are positive real numbers
- $h_y(x_2)$ is the monotonic function introduced before.

Remark: Instead of $z_3 = x_3$, one may also take $z_3 = h_\theta(x_3)$, with $h_\theta(x_3)$ being a smooth monotonic function, alike $h_y(x_2)$, which maps an open interval containing $]-\pi, +\pi[$ into R. For example, $h_\theta(x_3) = \tan(kx_3)$ with $k < (1/2)$ can be used. In this case the coordinate z_4 becomes $z_4 = k_1(\sin(x_3)/h_\theta(x_3))(\partial h_y/\partial x_2)(x_2)z_2 + g_{1,3}(x_1, x_2, x_3, x_4)$, and all subsequent coordinates are modified accordingly.

One can verify that $\phi_3(\Omega) = R^{n+4}$ and that the Jacobian matrix $(\partial\phi_3/\partial X)$ is a lower-triangular matrix with nonzero components on the diagonal equal to 1, $(\partial h_y/\partial x_2)(x_2)$, 1, $1/(l_1\cos(x_4)^2)$, $1/(l_1l_2\cos(x_4)^3\cos(x_5)^2), \cdots, 1/((\prod_{i=1}^{i=n+1} l_i)$ $(\prod_{j=4}^{j=n+4}\cos(x_j)^{n+6-j}))$.

The change of coordinates ϕ_3 thus induces a diffeomorphism of class C^{n+1} between Ω and R^{n+4}.

Then, by using

$$L_{g_2}L_{g_1}^j z_{i+3} = 0 \qquad (0 \le i \le n,\ 0 \le j \le n-i) \quad (48)$$

$$L_{g_2}L_{g_1}^n z_4 = \frac{1}{(\prod_{i=1}^{i=n+1} l_j)(\prod_{j=4}^{j=n+4}\cos(x_j)^{n+6-j})} \quad (49)$$

with the following auxiliary control input w_2

$$w_2 = (k_{n+2}z_{n+3} + L_{g_1}z_{n+4})v_0 + (L_{g_2}L_{g_1}^n z_4)u_2 \quad (50)$$

it is simple to verify that the control system, expressed in terms of the new coordinates, has the following skew-symmetric chain form

$$\begin{aligned}
\dot{z}_1 &= v_0 \frac{\cos(z_3)}{1-\mathrm{curv}(z_1)x_2} \\
\dot{z}_2 &= v_0 \frac{\sin(z_3)}{z_3}\frac{\partial h_y}{\partial x_2}(x_2) z_3 \\
\dot{z}_3 &= -k_1 v_0 \frac{\sin(z_3)}{z_3}\frac{\partial h_y}{\partial x_2}(x_2) z_2 + v_0 z_4 \\
\dot{z}_4 &= -k_2 v_0 z_3 + v_0 z_5 \\
&\vdots \\
\dot{z}_{j+4} &= -k_{j+2} v_0 z_{j+3} + v_0 z_{j+5} \qquad (1 \le j \le n-1) \\
&\vdots \\
\dot{z}_{n+4} &= -k_{n+2} v_0 z_{n+3} + w_2
\end{aligned} \tag{51}$$

with $x_2 = h_y^{-1}(z_2)$, and $v_0 = r\omega \prod_{i=4}^{i=n+4} \cos(x_i)$.

Although this system is not exactly the same as the skew-symmetric chained system (13), a result very similar to Proposition 2.1 can be derived for path following.

Proposition 3.1: If $|v_0(t)|$ and $|\dot{v}_0(t)|$ are bounded, and if the control

$$w_2 = -k_{w_2}(v_0) z_{n+4}$$
$$(k_{w_2}(\cdot): \text{continuous application strictly positive on} R - \{0\}) \tag{52}$$

is applied to system (51), then the positive function

$$V(Z_2) = 1/2(z_2^2 + (1/k_1)z_3^2 + (1/k_1k_2)z_4^2 + \cdots + \left(1/\prod_{j=1}^{j=n+2} k_j\right) z_{n+4}^2) \tag{53}$$

is nonincreasing along the system's solutions, and thus asymptotically converges to some limit value $V_{\lim}$ (which *a priori* depends on the initial conditions).

Moreover, $v_0(t)V(Z_2(t))$ asymptotically converges to zero.

Therefore, if $v_0(t)$ does not converge to zero, then $V_{\lim} = 0$. The submanifold $Z_2 = 0$ is thus globally asymptotically stabilized in this case.

Proof of Proposition 3.1: The proof is quite similar to the proof of Proposition 2.1 except that one has to show at some point that the convergence of $v_0(\sin(z_3)/z_3)(\partial h_y/\partial x_2)(x_2)z_2$ and $v_0 z_3$ to zero yields the convergence of $v_0 z_2$ to zero.

Since $|z_2(t)|$ is bounded (from the boundedness of the Lyapunov function), $(\partial h_y/\partial x_2)(x_2(t))$ is also upperbounded, and $v_0 z_3 (\partial h_y/\partial x_2)(x_2)z_2$ thus tends to zero. Therefore, $v_0^2(((\sin(z_3)/z_3))^2 + z_3^2)((\partial h_y/\partial x_2)(x_2))^2 z_2^2$ also tends to zero.

By assumption, $(\partial h_y/\partial x_2)(x_2)$ is bounded from below by a positive real number. Moreover, the function $((\sin(z_3)/z_3))^2 + z_3^2$ is itself larger than some positive real number. Along a system's solution, it is also bounded from above, since $|z_3(t)|$ is bounded. Using these bounds in the previous convergence result, it is found that $v_0^2 z_2^2$ tends to zero. ◊

The remarks made after Proposition 2.1 also hold in this case. In particular, adequate values for the control "gains" k_j $(1 \le j \le n+2)$ can be determined by comparing, in the neighborhood of $Z_2 = 0$, the control u_2 provided by the proposition and relation (50), with the linearizing feedback (7). Since these gains do not *a priori* depend on the path's shape, one may also use, for this comparison, a simpler linear control calculated from the system's tangent linear approximation about the equilibrium $(Z_2 = 0, u_2 = 0)$, assuming that the path to be followed is a straight line and that the velocity v_0 is constant.

The problem of stabilizing the system to a fixed desired configuration requires asymptotical convergence of the full state vector Z to zero. A smooth time-varying feedback solution is given in the next complementary proposition.

Proposition 3.2: Consider the same control as in Proposition 3.1

$$w_2 = -k_{w_2}(v_0) z_{n+4} \tag{54}$$

complemented with the following time-varying control

$$v_0 = -k_{v_0} h_s(z_1) + h(Z_2, t) \tag{55}$$

where:

- k_{v_0} is a positive real number.
- $h_s(\cdot)$ is a function of class C^2 which maps R into a bounded interval of R, and such that: i) $h_s(0) = 0$, ii) $0 < h_s^{(1)}(x) < +\infty, \forall x$, and iii) $|h_s^{(2)}(x)| < +\infty, \forall x$. Take for example, the sigmoid function: $h_s(x) = (\exp(ax) - 1/\exp(ax) + 1)$ $(a > 0)$.
- $h(Z_2, t)$ is a function with the same properties as in Proposition 2.2.

This control globally asymptotically stabilizes the origin $Z = 0$ of the system (51).

Proof of Theorem 3.2: The first part of the proof consists in showing that $v_0(t)$, and its time derivative are bounded along any system's solution.

Since $\|Z_2(t)\|$ is bounded (due to the boundedness of the Lyapunov function considered in Theorem 3.1), it is clear, from the expression of the control v_0 and the properties of the functions $h_s(z_1)$ and $h(Z_2, t)$, that $|v_0(t)|$ is bounded. As a consequence, $\|\dot{Z}(t)\|$ is also bounded.

Taking the time derivative of the v_0 control law expression

$$\dot{v}_0 = \left[-k_{v_0} h_s^{(1)}, \frac{\partial h}{\partial Z_2}\right]\dot{Z} + \frac{\partial h}{\partial t} \tag{56}$$

which, in view of the boundedness of $\|Z_2(t)\|$ and $\|\dot{Z}(t)\|$, implies that $|\dot{v}_0(t)|$ is bounded.

Although v_0 is not, strictly speaking, a function of time only (since it is a feedback control), it can be viewed as such along any system's solution, and the results of Proposition 3.1 do apply.

In particular, if $v_0(t)$ does not tend to zero, then $\|Z_2(t)\|$ tends to zero. In this case, $h(Z_2(t), t)$ tends to zero (from condition C_1 and by uniform continuity). From the first system's equation, we also have

$$\dot{z}_1(t) = -k_{v_0} h_s(z_1(t)) + o(t) \quad \text{with} \lim_{t\to\infty} o(t) = 0. \tag{57}$$

Using the properties of the function $h_s(\cdot)$, the equation implies that the following proposition is true

$$\forall \epsilon > 0,\ \exists \eta > 0,\ \exists t_0: (t > t_0 \text{ and } |z_1(t)| \geq \epsilon) \Rightarrow (z_1(t)\dot{z}_1(t) < -\eta). \quad (58)$$

Since $z_1^2(t)$ cannot remain larger than ϵ^2 with a negative derivative smaller than -2η, there is a time t_1 ($\geq t_0$) such that $|z_1(t_1)| < \epsilon$. Moreover, after the time t_1, $|z_1(t)|$ remains smaller than ϵ (since ϵ^2 cannot be reached from below by $z_1^2(t)$ with a negative derivative). The above proposition thus implies

$$\forall \epsilon > 0,\ \exists t_1: (t > t_1) \Rightarrow (|z_1(t)| < \epsilon). \quad (59)$$

This is a characterization of the convergence of $z_1(t)$ to zero. In view of the expression of v_0, this in turn implies that $v_0(t)$ tends to zero (contradiction).

Therefore $v_0(t)$ must tend to zero, implying in turn that $w_2(t)$ and $\dot{z}(t)$ tend to zero. Now, in view of (56)

$$\dot{v}_0(t) = \frac{\partial h}{\partial t}(Z_2(t), t) + o(t).$$

Since $(\partial h/\partial t)(Z_2(t), t)$ is uniformly continuous (its time derivative is bounded), $\dot{v}_0(t)$ tends to zero (Barbalat's lemma). Therefore, $(\partial h/\partial t)(Z_2(t), t)$ also tends to zero. From there, the proof goes on like the proof of Proposition 2.2. $\Diamond$

IV. Conclusion

New results about feedback control of chained systems and their application to path following and point stabilization of nonholonomic mobile robots have been presented.

Throughout the paper, feedback control design and analysis have been performed via explicit Lyapunov techniques which apply naturally once the original chain form has been transformed into an equivalent form termed here skew-symmetric chain form. Asymptotical stabilization of the origin of a n-dimensional chained system has been achieved via a two-step approach which allows path following and point stabilization of mobile robots to be treated within the same framework. This approach yields a simple way of determining globally stabilizing smooth time-varying feedbacks the underlying structure of which is easily interpreted. It also applies to the determination of Hölder continuous time-periodic feedbacks which ensure $\mathcal{K}$-exponential stabilization of the origin, as illustrated in [30] in the three-dimensional case. Several smooth and nonsmooth solutions to the point stabilization problem have then been tentatively compared by recalling and commenting upon the type of stability associated with each of them.

Application to path following and point stabilization of a car pulling trailers has been addressed in the second part of the study, based on a specific parameterization of the system's configuration which facilitates the decoupling of the path following problem from translational velocity control and yields general model equations in the chain form. Finally, it has been shown how, by an adequate choice of the system's state coordinates and a slight modification of the original chain form, it is possible to derive feedback control laws endowed with a larger stability domain.

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