[11] M. Polycarpou and P. Ioannou, "Modeling, identification and stable adaptive control of continuous-time nonlinear dynamical systems using neural networks," in *Proc. 1992 Amer. Contr. Conf.*, June 1992, pp. 36–40.
[12] R. M. Sanner and J.-J.E. Slotine, "Gaussian networks for direct adaptive control," *IEEE Trans. Neural Networks*, vol. 3, no. 6, pp. 837–863, Nov. 1992.
[13] E. Kosmatopoulos, M. Polycarpou, M. Christodoulou, and P. Ioannou, "High-order neural network structures for identification of dynamical systems," *IEEE Trans. Neural Networks*, vol. 6, no. 2, pp. 422–431, Mar. 1995.
[14] F. L. Lewis, K. Liu, and A. Yesildirek, "Multilayer neural net robot controller with guaranteed tracking performance," in *Proc. Conf. Decision Contr.*, Dec., 1993, pp. 2785–2791.
[15] E. Tzirkel-Hancock and F. Fallside, "Stable control of nonlinear systems using neural networks," *Int. J. Robust and Nonlinear Contr.*, vol. 2, no. 1, Apr. 1992.
[16] F.-C. Chen and H.K. Khalil, "Adaptive control of a class of nonlinear discrete-time systems using neural networks," *IEEE Trans. Automat. Contr.*, vol. 40, no. 5, pp. 791–801, May 1995.
[17] T. Poggio and F. Girosi "Regularization algorithms for learning that are equivalent to multilayer networks," *Sci.*, vol. 247, pp. 978–982, 1990.
[18] M. Corless, "Control of uncertain nonlinear systems," *Trans. ASME J. Dyn. Syst., Measur., Contr.*, vol. 115, pp. 362–372, June 1993.

On the Robustness of Receding-Horizon Control with Terminal Constraints

G. De Nicolao, L. Magni, and R. Scattolini

***Abstract*— Robustness properties of nonlinear receding-horizon controllers with zero terminal state constraints are investigated with respect to gain and additive perturbations. Some robustness margins are derived by extending to the receding-horizon case the analysis originally proposed by Geromel and da Cruz [1] for infinite-horizon controllers. In the linear case, it is shown that the zero terminal state receding-horizon controller exhibits worse robustness margins compared to standard infinite-horizon LQ control.**

I. Introduction

In recent years there has been a growing interest for the design of controllers based on receding-horizon optimization [2]. However, it is well known that a plain receding-horizon strategy does not guarantee closed-loop stability [3]. One can achieve stability by complementing the finite-horizon cost functional with terminal constraints on the state at the end of the optimization horizon [4]. This notion of zero terminal state receding-horizon (ZSRH) control is playing a major role in the development of predictive controllers with guaranteed stability, see e.g., [5]–[7]. Furthermore, the stabilizing properties of the ZSRH strategy hold also for nonlinear systems [8], [9].

Once stability has been guaranteed, it is natural to address the issue of its robustness in the face of system uncertainty. In the present paper, we investigate the robust stability of nonlinear systems controlled according to a ZSRH strategy. The main purpose is to show that ZSRH controllers enjoy robustness properties analogous to those of infinite-horizon optimal controllers [1]. In particular, robustness margins with respect to gain and additive perturbations are derived. In the linear quadratic case, a comparison with stationary linear quadratic (LQ) control shows that although the robustness margins of the ZSRH scheme are smaller, some LQ robustness margins can be asymptotically recovered as the optimization horizon lengthens.

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II. Receding-Horizon Control

Consider the discrete-time nonlinear system

$$x(t+1) = f(x(t), u(t))$$

where $f(\cdot,\cdot)$ is such that $f(0,0) = 0$, and $x \in \Re^n, u \in \Re^r$. At time t we determine $u(t)$ by minimizing

$$J(t) = \sum_{i=0}^{N-1} \{(l(x(t+i)) + m(u(t+i))\} \tag{1}$$

with respect to $u(t), u(t+1), \cdots, u(t+N-1)$, subject to the terminal constraint $x(t+N) = 0$. It is assumed that $l(x) > 0,\ \forall x \neq 0, l(0) = 0$, and $m(u) > 0,\ \forall u \neq 0, m(0) = 0$. At time $t+1, u(t+1)$ is found by minimizing $J(t+1)$ subject to $x(t+N+1) = 0$. As a result of this ZSRH strategy, at each step $u(t)$ will depend on the state $x(t)$ according to a control law of the type $u(t) = k(x(t))$. Correspondingly, the (nominal) closed-loop system will be $x(t+1) = f_c(x(t))$ where $f_c(x) = f(x, k(x))$.

The main purpose of this paper is to investigate some robustness properties of such a receding-horizon controller. Precisely, we consider two kinds of perturbations affecting the system dynamics, namely gain perturbations and additive perturbations. In the first case, instead of having $u(t) = k(x(t))$, the actual control law is $u(t) = \phi(k(x(t)))$ where $\phi(\cdot)$ is a given but unknown function. Consequently, the perturbed closed-loop dynamics equal $x(t+1) = f_g(x(t))$, where $f_g(\cdot) = f(x, \phi(k(x)))$. In the second case, we consider an additive perturbation on the control action, that is $u(t) = k(x(t)) + \psi(x)$. The perturbed closed-loop dynamics then becomes $x(t+1) = f_a(x(t))$, where $f_a(\cdot) = f(x, k(x) + \psi(x))$. Gain perturbations may arise due to actuator uncertainties, whereas additive perturbations may account for unmodeled dynamics; see [1].

III. Robustness Analysis

A. Preliminaries

Before proceeding to robustness analysis, we first review the basic stability result for the nominal closed-loop system. To this purpose, let $V(\xi, t, t+N-1)$ denote the optimal value of the cost functional (1) when $x(t) = \xi$, and define

$$V(x) = V(x, t, t+N-1)$$
$$V^*(x) = V(x, t+1, t+N-1).$$

Next, we show that

$$V(\xi) > V^*(f_c(\xi)) \geq V(f_c(\xi)), \quad \forall \xi \neq 0. \tag{2}$$

Indeed, in view of (1), $V(\xi) = V^*(f_c) + l(\xi) + m(k(\xi))$, so that $V(\xi) > V^*(f_c(\xi))$. To prove that $V^*(f_c(\xi)) \geq V(f_c(\xi))$, let $u(\tau), t+1 \leq \tau \leq t+N-1$ be the solution of the minimization problem over $[t+1, t+N-1]$ having $V^*(f_c(\xi))$ as the optimal cost. Then, the sequence $\{u(t+1), u(t+2), \cdots, u(t+N-1), 0\}$ is always an admissible control for the minimization problem over

$[t+1, t+N]$ having $V(f_c(\xi))$ as the optimal cost. Moreover, for this latter problem the cost associated with such an admissible sequence is just $V^*(f_c(\xi))$, so by optimality, $V^*(f_c(\xi)) \geq V(f_c(\xi))$.

In view of (2), the optimal cost $V(\xi)$ is a Lyapunov function ensuring the closed-loop asymptotic stability of the equilibrium point $x = 0$. This property can be exploited to investigate the robustness properties of the controller. Precisely, in the following, we will be interested in the characterization of the following sets:

$$D_g = \{\phi\colon V(f_g(\xi)) - V(\xi) < 0;\ \forall \xi \neq 0 \in \Re^n\}$$
$$D_a = \{\psi\colon V(f_a(\xi)) - V(\xi) < 0;\ \forall \xi \neq 0 \in \Re^n\}.$$

In fact, it is apparent that for $\phi \in D_g$ or $\psi \in D_a$, robust closed-loop stability of the equilibrium point $x = 0$ is guaranteed.

B. Gain Perturbation

Concerning the robustness in the face of gain perturbations, the following result holds.

Theorem 1: Assume that $V(\cdot) \in C^2, m(\cdot) \in C^2$, and $f(x,u) = f_1(x) + f_2(x)u$. If

$$\begin{aligned} &m(z) + m_u(z)'(\phi(z) - z) \\ &\quad > \frac{\alpha}{2}(\phi(z) - z)' m_{uu}(z)(\phi(z) - z), \quad \forall z \neq 0 \end{aligned} \tag{3}$$

where $\alpha \geq 0$ is a real number such that

$$f_2(x)' V_{xx}^*(\omega) f_2(x) \leq \alpha m_{uu}(k(x)), \quad \forall \omega, \forall x \tag{4}$$

then $\phi \in D_g$.

Proof: By equating to zero the derivative of (1) with respect to u, it follows that:

$$V_x^*(f_c(x))' f_2(x) = -m_u(k(x))'.$$

Now, in view of the definition of $V^*(f_c(x))$

$$V^*(f_g(x)) + V^*(f_c(x)) - V(x) = V^*(f_g(x)) - l(x) - m(k(x)).$$

Applying the Taylor formula to $V^*(f_g(x))$ and recalling that $V^*(f_c(x)) = V(x) - l(x) - m(k(x))$ yields

$$\begin{aligned} V^*(f_g(x)) &= V(x) - l(x) - m(k(x)) \\ &\quad + V_x^*(f_c(x))' f_2(x)[\phi(k(x)) - k(x)] \\ &\quad + \tfrac{1}{2}[\phi(k(x)) - k(x)]' f_2(x)' V_{xx}^*(\omega) f_2(x) \\ &\quad \cdot [\phi(k(x)) - k(x)] \end{aligned}$$

where ω belongs to the convex combination of $f_c(x)$ and $f_g(x)$. Hence, by (4)

$$\begin{aligned} &V^*(f_g(x)) - V(x) \\ &\quad \leq -l(x) - m(k(x)) - m_u(k(x))'[\phi(k(x)) - k(x)] \\ &\quad\quad + \frac{\alpha}{2}[\phi(k(x)) - k(x)]' m_{uu}(k(x))[\phi(k(x)) - k(x)] \end{aligned}$$

which, due to assumption (3), entails $V^*(f_g(x)) - V(x) \leq -l(x) < 0$. Since, in view of (2), $V(f_g(x)) \leq V^*(f_g(x))$, the thesis follows. ∎

When $m(\cdot)$ is quadratic, i.e., $m(u) = (1/2)u'Ru$ with $R = R' > 0$, then inequality (3) becomes

$$z'Rz + 2z'R[\phi(z) - z] > \alpha[\phi(z) - z]' R[\phi(z) - z], \quad \forall z \neq 0$$

or, equivalently

$$\phi(z)' R \phi(z) > (1 + \alpha)(\phi(z) - z)' R(\phi(z) - z), \quad \forall z \neq 0. \tag{5}$$

A sufficient condition for (5) to hold is that

$$\frac{\|\phi(z) - z\|^2}{\|\phi(z)\|^2} < \sigma(R)(1 + \alpha)^{-1} \tag{6}$$

where $\sigma(R) = \lambda_m(R)/\lambda_M(R)$ is the condition number of R. If $\sigma(R) = 1$ and $\phi(z)$ is linear, i.e., $\phi(z) = kz$, from (6) it follows that the equilibrium point $x = 0$ of the perturbed system is stable for $k_m < k < k_M$ with $k_m = (1 + \alpha - (1 + \alpha)^{1/2})/\alpha$ and $k_M = (1 + \alpha + (1 + \alpha)^{1/2})/\alpha$. Interestingly, following the same arguments as in [1], one can see that the parameters k_m, k_M characterize the robustness properties also in the case of nonlinear perturbations.

Corollary 1: Assume that $m(u) = (1/2)u'Ru$ with $R = R', \sigma(R) = 1$. If $\phi(z)$ satisfies (5), then

$$\operatorname*{Inf}_{\phi} \|\phi(z)\| = k_m \|z\|$$
$$\operatorname*{Sup}_{\phi} \|\phi(z)\| = k_M \|z\|, \quad \forall z \neq 0. \qquad ∎$$

Hence $\delta_g := k_M - k_m$ provides an index of the robustness of the controller with respect to gain perturbations. Notice also that $\delta_g(\alpha)$ is a decreasing function of α, so that the largest robustness margin corresponds to the smallest α satisfying (4).

C. Additive Perturbation

With reference to the class of additive perturbations, it is possible to derive the following robustness result.

Theorem 2: Assume that $V(\cdot) \in C^2, m(\cdot) \in C^2$, and $f(x,u) = f_1(x) + f_2(x)u$. If

i) $m_{uu}(z) > 0, \quad \forall z \neq 0,$
ii) $2m(z) \geq m_u(z)' m_{uu}(z)^{-1} m_u(z),$
iii) $l(x) > \frac{1}{2}(1 + \alpha)\psi(x)' m_{uu}(k(x))'\psi(x), \quad \forall x \neq 0,$ (7)

where $\alpha \geq 0$ is a real number satisfying (4), then $\psi \in D_a$.

Sketch of the Proof: By following a rationale analogous to the one of Theorem 1, one obtains

$$\begin{aligned} V^*(f_a(x)) - V(x) &\leq -l(x) - m(k(x)) - m_u(k(x))'\psi(x) \\ &\quad + \frac{\alpha}{2}\psi(x)' m_{uu}(k(x))\psi(x). \end{aligned}$$

In view of (7), this implies

$$\begin{aligned} V^*(f_a(x)) - V(x) &< -m(k(x)) - m_u(k(x))'\psi(x) \\ &\quad - \tfrac{1}{2}\psi(x)' m_{uu}(k(x))\psi(x). \end{aligned} \tag{8}$$

Using a result given in the proof of Theorem 3 in [1], it follows that

$$\begin{aligned} &-m_u(k(x))'\psi(x) - \tfrac{1}{2}\psi(x)' m_{uu}(k(x))\psi(x) \\ &\quad \leq \tfrac{1}{2} m_u(k(x))' m_{uu}(k(x))^{-1} m_u(k(x)) \end{aligned}$$

so that by assumption ii), the right-hand side of (8) is nonpositive, yielding $V^*(f_a(x)) < V(x)$. In analogy with (2), one can show that $V^*(f_a(x)) \geq V(f_a(x))$. Then, $V(f_a(x)) \leq V^*(f_a(x)) < V(x)$, so that $\psi \in D_a$. ∎

In particular, if $l(x) = (1/2)x'Qx, Q = Q' > 0$, and $m(u) = (1/2)u'Ru, R = R' > 0$, a sufficient condition for (7) to hold is that

$$\frac{\|\psi(x)\|^2}{\|x\|^2} < \frac{\lambda_m(Q)}{\lambda_M(R)}(1 + \alpha)^{-1}.$$

Consequently, $\delta_a := 2\{(1 + \alpha)^{-1}\lambda_m(Q)/\lambda_M(R)\}^{1/2}$ is an index of the robustness with respect to additive perturbations. Again, for a given system and cost function, the largest value of δ_a is obtained in correspondence of the smallest α satisfying (4).

IV. Linear Case and Comparison with Stationary LQ Control

In the linear quadratic case, $f(x,u) = Ax + Bu, l(x) = (1/2)x'Qx, Q > 0, m(u) = (1/2)u'Ru, R > 0$. We assume that the pair (A, B) is completely reachable. Let $\nu, \nu \leq n$ be the reachability index of (A, B) and assume that $N > \nu$. Then, it is possible to verify that $V(x)$ is a quadratic function of x [10]. Let $P(N)$ be the $n \times n$ symmetric matrix such that $V(x,t,t+N-1) = 1/2x'P(N)x$. In [10], it is shown that $P(t)$ is monotonic nonincreasing, i.e., $P(t+1) \leq P(t), t \geq \nu$. Moreover, it turns out that $V^*(x) = V(x,t+1,t+N-1) = x'P(N-1)x$.

Remark: Roughly speaking, $P(\cdot)$ can be viewed as the limit for $\gamma \to +\infty$ of the solution of the difference Riccati equation

$$P(t+1) = A'P(t)A - A'P(t)B[R+B'P(t)B]^{-1} \cdot B'P(t)A + Q \tag{9}$$

with initial condition $P(0) = \gamma I$ (it can be proven that in the limit $P(N-1)$, and hence $P(N)$, are finite). Then, in the reversible case, an alternative derivation of the monotonicity property, based on the analysis of the Riccati (9), can be found in [3]. ■

From (4) it is seen that the parameter α must satisfy the condition $B'P(N-1)B \leq \alpha R$. Then, the minimum value of α (leading to the largest robustness bounds) is $\alpha_N = \lambda_M(B'P(N-1)BR^{-1})$, where the subscript N emphasizes the dependence of α_N on the length of the optimization horizon. By exploiting the monotonicity property of the solution of (9), it follows that $\alpha_N > \alpha_N + 1$. As an important consequence, both the robustness indexes δ_g and δ_a are increasing functions of N. Recalling that for $N \to \infty$, the ZSRH controller tends to the infinite-horizon (stationary) LQ controller, we have the following comparison result (invertibility of A is not required).

Theorem 3: Let A, B, Q, R be given and assume that (A, B) is reachable and $Q > 0$. Let $\overline{\alpha} = \lambda_M(B'\overline{P}BR^{-1})$, where $\overline{P}$ is the (unique) constant equilibrium of (9). Then, $\lim_{N\to\infty} \alpha_N = \overline{\alpha}$ and $\overline{\alpha} \leq \alpha_N$, $\forall N > n$. ■

An important consequence of the above theorem is that the stationary LQ controller has better guaranteed robustness margins (measured by the indexes δ_g and δ_a) than the ZSRH one. Nevertheless, it is always possible to indefinitely approximate the robustness bounds of the stationary LQ controller by suitably lengthening the horizon N of the receding-horizon controller.

Finally, we consider the case $Q = \rho\overline{Q}$, where $\overline{Q}$ is a given nonnegative definite symmetric matrix, and the scalar ρ is a design parameter. By following the rationale of [1], it turns out that $\delta_a[\delta_g]$ is an increasing (decreasing) function of ρ. Consequently, there exists a design trade-off, since a larger weight on the state leads to better robustness bounds with respect to additive perturbations at the expense of worse robustness with respect to gain perturbations.

V. Concluding Remarks

Control strategies based on receding-horizon methods are becoming increasingly popular as witnessed by the widespread adoption of predictive control techniques [5], [11]. The results of the present paper clarify some issues concerning the robustness of receding-horizon control within a very general framework including linear as well as nonlinear control problems. In particular, it has been shown that the robustness margins of ZSRH control are similar to and asymptotically approach those of infinite-horizon LQ control. This is a remarkable result because receding-horizon control strategies are much easier to implement, since optimization has to be performed only over a finite time-interval.

References

[1] J. C. Geromel and J. J. Da Cruz, "On the robustness of optimal regulators for nonlinear discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 703–710, 1987.

[2] D. Q. Mayne and E. Polak, "Optimization based design and control," in *Proc. 12th IFAC World Congr.*, vol. III, 1993, pp. 129–138.

[3] R. R. Bitmead, M. Gevers, and W. Wertz, *Adaptive Optimal Control.* Englewood Cliffs, NJ: Prentice-Hall, 1990.

[4] W. H. Kwon and A. E. Pearson, "On feedback stabilization of time-varying discrete-linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 479–481, 1978.

[5] D. W. Clarke and R. Scattolini, "Constrained receding horizon predictive control," *Proc. IEE Part D*, vol. 138, pp. 347–354, 1991.

[6] E. Mosca and J. Zhang, "Stable redesign of predictive control," *Automatica*, vol. 28, pp. 1229–1233, 1992.

[7] G. DeNicolao and R. Scattolini, "Stability and output terminal constraints in predictive control," in *Advances in Model-Based Predictive Control*, D. Clarke, Ed. London: Oxford Univ., pp. 105–121, 1994.

[8] S. S. Keerthi and E. G. Gilbert, "Optimal, infinite-horizon feedback laws for a general class of constrained discrete-time systems," *J. Optimiz. Theory Appl.*, vol. 57, pp. 265–293, 1988.

[9] D. Q. Mayne and H. Michalska, "Receding horizon control of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 814–824, 1990.

[10] L. Chisci and E. Mosca, "Stabilizing predictive control: The singular transition matrix case," in: *Advances in Model-Based Predictive Control*, D. Clarke, Ed. London: Oxford Univ., pp. 122–130, 1994.

[11] T. W. Yoon and D. W. Clarke, "Adaptive predictive control of the benchmark plant," *Automatica*, vol. 30, pp. 621–628, 1994.

Quadratic Stability and Stabilization of Linear Systems with Frobenius Norm-Bounded Uncertainties

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***Abstract*—In this paper, a quadratic stability condition and a quadratic stabilizer are proposed for linear systems with Frobenius norm-bounded uncertainties. The necessary and sufficient condition for the quadratic stability of the uncertain system is given by a Riccati inequality. The proposed state feedback quadratic stabilizer can be obtained by solving a Riccati equation.**

I. Introduction

There have been many publications on the uncertain systems whose uncertainties are bounded by two-norm or the maximum singular value [1]–[8]. But there are few publications on the uncertain systems with Frobenius norm-bounded uncertainties. The Frobenius norm is better than two-norm, however, as a measure of uncertainties in some cases. For example, let us consider the following uncertainties:

$$\Delta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Delta_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The uncertainty Δ_2 has one more nonzero element at (2,2) than the uncertainty Δ_1. But, the maximum singular values of the two uncertainties Δ_1 and Δ_2 are the same. The Frobenius norm of the uncertainty Δ_2 is larger than that of the uncertainty Δ_1, however, where the Frobenius norm of the uncertainty Δ is defined as

$$\|\Delta\|_F = (\text{Trace } \Delta'\Delta)^{1/2}.$$

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