Control and Stabilization of Nonholonomic Dynamic Systems

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***Abstract*—A theoretical framework is established for the control of nonholonomic dynamic systems, i.e., dynamic systems with nonintegrable constraints. In particular, we emphasize control properties for nonholonomic systems that have no counterpart in holonomic systems. A model for nonholonomic dynamic systems is first presented in terms of differential-algebraic equations defined on a phase space. A reduction procedure is carried out to obtain reduced-order state equations. Feedback is then used to obtain a nonlinear control system in a normal form. The assumptions guarantee that the resulting normal form equations necessarily contain a nontrival drift vector field. Conditions for smooth (C^∞) asymptotic stabilization to an m-dimensional equilibrium manifold are presented; we also demonstrate that a single equilibrium solution cannot be asymptotically stabilized using continuous state feedback. However, any equilibrium is shown to be strongly accessible and small time locally controllable. Finally, an approach using geometric phases is developed as a basis for the control of Caplygin dynamical systems, i.e., nonholonomic systems with certain symmetry properties which can be expressed by the fact that the constraints are cyclic in certain variables. The theoretical development is applied to physical examples of systems that we have studied in detail elsewhere: the control of a knife edge moving on a plane surface and the control of a wheel rolling without slipping on a plane surface. The results of the paper are also applied to the control of a planar multibody system using angular momentum preserving control inputs since the angular momentum may be viewed as a nonholonomic constraint which is an invariant of the motion.**

I. Introduction

NUMEROUS papers have been published in recent years on the control of systems with holonomic constraints. The work of the authors includes McClamroch and Bloch in [17], McClamroch and Wang in [18]. The earliest work on control of nonholonomic systems (that we are aware of) is by Brockett in [6]. Bloch in [2] has examined several control theoretic issues which pertain to both holonomic and nonholonomic systems in a very general form. Related work in robotics [14], [15], [20] and multibody systems [10], [11], [12], [25], [29] has recently appeared. Our recent work in [3], [4], [22], [23] has also emphasized several classes of physical problems. All of this work has demonstrated that there is a common theoretical framework for a large class of control problems for mechanical systems with nonholonomic constraints. In this paper, we identify that common theoretical framework. Our development is based on the formulation of nonholonomic dynamics by Neimark and Fufaev [21] and the modern formulation of nonlinear geometric control.

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II. Models of Nonholonomic Systems

We consider the class of nonholonomic systems described by the equations

$$M(q)\ddot{q} + F(q,\dot{q}) = J'(q)\lambda + B(q)u \tag{1}$$

$$J(q)\dot{q} = 0. \tag{2}$$

Note that a "prime" denotes transpose. We refer to q as an n-vector of generalized configuration variables, $\dot{q}$ as an n-vector of generalized velocity variables, and $\ddot{q}$ as an n-vector of generalized acceleration variables; in addition, λ is an m-vector of constraint multipliers and u is an r-vector of control input variables, where $r \geq n - m$. The $n \times n$ matrix function $M(q)$ is assumed to be symmetric and positive definite, $F(q,\dot{q})$ is an n-vector function, $J(q)$ denotes an $m \times n$ matrix function which is assumed to have full rank and $B(q)$ is a full rank $n \times r$ matrix function. All of these functions are assumed to be smooth (C^∞) and defined on an appropriate open subset of the $(q,\dot{q})$ phase space. The formulation could be given in terms of a system defined on the tangent bundle of a C^∞ manifold; we have not made such a generalization since it is direct. Various assumptions about the control input variables are indicated subsequently.

Differential-algebraic equations of the above form are known to arise for (uncontrolled) nonholonomic systems; see [1] and [21] for many examples. Here, we note that the classical approach for the formulation of constrained dynamics as described in [21] is used. This is in contrast to the variational approach, or "vakonomic" theory (see e.g., [1]). We also note that a Hamiltonian formulation can be developed.

We have assumed that the $m \times n$ matrix $J(q)$ has full rank; hence, there is no loss of generality in assuming that the configuration variables are ordered so that the last m columns of the matrix $J(q)$ constitute an $m \times m$ locally invertible matrix function, i.e., the matrix $J(q)$ can be

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expressed as $[J_1(q)J_2(q)]$, where $J_1(q)$ is an $m \times (n-m)$ matrix function and $J_2(q)$ is an $m \times m$ locally nonsingular matrix function. The columns of the $n \times (n-m)$ matrix function

$$C(q) = \begin{bmatrix} I \\ -\bar{J}(q) \end{bmatrix} \tag{3}$$

where I is the $(n-m) \times (n-m)$ identity matrix and $\bar{J}(q) = J_2^{-1}(q)J_1(q)$ is a locally smooth $m \times (n-m)$ matrix function, span the null space of $J(q)$. Formally, the rows of $J(q)$ constitute m linearly independent smooth covector fields defined on the configuration space; these covector fields span a codistribution Ω and the annihilator of the codistribution Ω, denoted $\Omega^{\perp}$, is spanned by $n-m$ linearly independent smooth vector fields

$$\tau_j = \sum_{i=1}^{n} C_{ij}(q)\frac{\partial}{\partial q_i}, \qquad j = 1,\cdots,n-m. \tag{4}$$

We present the following definition.

Definition 1 [30]: Consider the following nondecreasing sequence of locally defined distributions

$$N_1 = \Omega^{\perp}$$
$$N_k = N_{k-1} + \operatorname{span}\{[X,Y] | X \in N_1, Y \in N_{k-1}\}.$$

There exists an integer k^* such that

$$N_k = N_{k^*}$$

for all $k > k^*$. If $\dim N_{k^*} = n$ and $k^* > 1$, then the constraints (2) are called completely nonholonomic and the smallest (finite) number k^* is called the degree on nonholonomy.

In this paper, it is assumed that constraint equations (2) are completely nonholonomic with nonholonomy degree k^*. Note that for this to hold $n-m$ must be strictly greater than one. Note also that since the constraints are nonholonomic, there is in fact no explicit restriction on the values of the configuration variables.

We also assume that the matrix product $C'(q)B(q)$ is full rank. As will be seen in Section IV, this assumption guarantees that all $n-m$ degrees of freedom can be (independently) actuated.

The constraints (2) define a $(2n-m)$-dimensional smooth submanifold

$$\boldsymbol{M} = \{(q,\dot{q}) | J(q)\dot{q} = 0\} \tag{5}$$

of the phase space. This manifold $\boldsymbol{M}$ plays a critical role in the concept of solutions and the formulation of control and stabilization problems associated with (1) and (2).

We begin by making it clear that (1) and (2) do represent well-posed models in the sense that the associated initial value problem has a unique solution, at least locally.

Definition 2: A pair of vector functions $(q(t), \lambda(t))$ defined on an interval $[0,T)$ is a solution of the initial value problem defined by (1) and (2) and the initial data $(q_0, \dot{q}_0)$ if $q(t)$ is at least twice differentiable, $\lambda(t)$ is integrable, the vector functions $(q(t), \lambda(t))$ satisfy the differential-algebraic equations (1) and (2) almost everywhere on their domain of definition, and the initial conditions satisfy $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$.

The following existence and uniqueness result has been obtained.

Theorem 1 [3]: Assume that the control input function $u: [0,T) \to R^r$ is a given bounded and measurable function for some $T > 0$. If the initial data satisfy $(q_0, \dot{q}_0) \in \boldsymbol{M}$, then there exists a unique solution (at least locally defined) of the initial value problem corresponding to (1) and (2) which satisfies $(q(t), \dot{q}(t)) \in \boldsymbol{M}$ for each t for which the solution is defined.

Since the differential-algebraic equations (1) and (2) define a smooth vector field on $\boldsymbol{M}$, a number of other results could be stated, including conditions for continuous dependence of the solution on initial conditions and parameters, conditions for nonexistence of finite escape times, etc. Such results are important, but they are not given here since they are easily obtained. We subsequently use the notation $(Q(t,q_0,\dot{q}_0), \Lambda(t,q_0,\dot{q}_0))$ to denote the solution of (1) and (2) at time $t \geq 0$ corresponding to the initial conditions $(q_0, \dot{q}_0)$. Thus, for each initial condition $(q_0, \dot{q}_0) \in \boldsymbol{M}$ and each bounded, measurable input function $u: [0,T) \to R^r$, $(Q(t,q_0,\dot{q}_0), \dot{Q}(t,q_0,\dot{q}_0)) \in \boldsymbol{M}$ holds for all $t \geq 0$ where the solution is defined.

A particularly important class of solutions are the equilibrium solutions of (1) and (2). A solution is an equilibrium solution if it is a constant solution; note that if (q^e, λ^e) is an equilibrium solution, we refer to q^e as an equilibrium configuration. The following result should be clear.

Theorem 2: Suppose that $u(t) = 0$, $t \geq 0$. The set of equilibrium configurations of (1) and (2) is given by

$$\{q | F(q,0) - J'(q)\lambda = 0 \text{ for some } \lambda \in R^m\}.$$

An equivalent expression for the set of equilibrium configurations is

$$\{q | C'(q)F(q,0) = 0\}.$$

III. Closed-Loop Models of Nonholonomic Systems

We are interested in feedback control of the form $u = U(q,\dot{q})$ where U: $\boldsymbol{M} \to R^r$; the corresponding closed loop is described by

$$M(q)\ddot{q} + F(q,\dot{q}) = J'(q)\lambda + B(q)U(q,\dot{q}) \tag{6}$$
$$J(q)\dot{q} = 0. \tag{7}$$

We point out the obvious fact that the closed loop is still defined in terms of the nonholonomic constraint equations.

Suppose $U(q,\dot{q})$ is a smooth function; if the initial conditions satisfy $(q_0, \dot{q}_0) \in \boldsymbol{M}$, then there exists a unique solution $(q(t), \lambda(t))$ (at least locally defined) of the initial value problem corresponding to (6) and (7) which satisfies $(q(t), \dot{q}(t)) \in \boldsymbol{M}$ for each t for which the solution is defined.

The set of equilibrium configurations of (6) and (7) is given by

$$\{q \mid F(q,0) - J'(q)\lambda = B(q)U(q,0) \text{ for some } \lambda \in R^m\}$$

which is a smooth submanifold of the configuration space. An equivalent expression for the equilibrium submanifold of the configuration space is

$$\{q \mid C'(q)[F(q,0) - B(q)U(q,0)] = 0\}.$$

We remark that generically the equilibrium manifold has dimension at least m. On the other hand, for certain cases, there may not be even a single equilibrium configuration (e.g., the uncontrolled dynamics of a ball on an inclined plane). However, since we have assumed that $C'(q)B(q)$ is full rank, we can always introduce an equilibrium manifold of dimension at least m by appropriate choice of input.

We now formulate a stabilization problem for nonholonomic systems described by (1) and (2). A suitable stability definition for the closed-loop system described by (6) and (7) is first introduced.

Definition 3: Assume that $u = U(q, \dot{q})$. Let $\boldsymbol{M}_s = \{(q, \dot{q}) \mid \dot{q} = 0\}$ be an embedded submanifold of $\boldsymbol{M}$. Then $\boldsymbol{M}_s$ is locally stable if for any neighborhood $\boldsymbol{U} \supset \boldsymbol{M}_s$ there is a neighborhood $\boldsymbol{V}$ of $\boldsymbol{M}_s$ with $\boldsymbol{U} \supset \boldsymbol{V} \supset \boldsymbol{M}_s$ such that if $(q_0, \dot{q}_0) \in \boldsymbol{V} \cap \boldsymbol{M}$ then the solution of (6) and (7) satisfies $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \in \boldsymbol{U} \cap \boldsymbol{M}$ for all $t \geq 0$. If, in addition, $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \to (q_s, 0)$ as $t \to \infty$ for some $(q_s, 0) \in \boldsymbol{M}_s$ then we say that $\boldsymbol{M}_s$ is a locally asymptotically stable equilibrium manifold of (6) and (7).

Note that if $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \to (q_s, 0)$ as $t \to \infty$ for some $(q_s, 0) \in \boldsymbol{M}_s$, it follows that there is $\lambda_s \in R^m$ such that $\Lambda(t, q_0, \dot{q}_0) \to \lambda_s$ as $t \to \infty$.

The usual definition of local stability corresponds to the case that $\boldsymbol{M}_s$ is a single equilibrium solution; the more general case is required in the present paper.

The existence of a feedback function so that a certain equilibrium manifold is asymptotically stable is of particular interest; hence, we introduce the following.

Definition 4: The system defined by (1) and (2) is said to be locally asymptotically stabilizable to a smooth equilibrium manifold $\boldsymbol{M}_s$ in $\boldsymbol{M}$ if there exists a feedback function $U: \boldsymbol{M} \to R^r$ such that, for the associated closed-loop equations (6) and (7), $\boldsymbol{M}_s$ is locally asymptotically stable.

If there exists such a feedback function which is smooth on $\boldsymbol{M}$ then we say that (1) and (2) are smoothly asymptotically stabilizable to $\boldsymbol{M}_s$; of course it is possible (and we subsequently show that it is generic in certain cases) that (1) and (2) might be asymptotically stabilizable to $\boldsymbol{M}_s$ but not smoothly (even not continuously) asymptotically stabilizable to $\boldsymbol{M}_s$.

IV. Normal Form Equations for Nonholonomic Control Systems

A number of approaches have been suggested for eliminating the constraint multipliers so that a minimum set of differential equations is obtained: the reduced differential equations characterize the control dependent motion on the constraint manifold.

We first emphasize that the reduced state space is $2n - m$ dimensional. The state of the system can be specified by the n-vector of configuration variables and an $(n - m)$-vector of kinematic variables. Let $q = (q_1, q_2)$ be a partition of the configuration variables corresponding to the partitioning of the matrix function $J(q)$ introduced previously. Then consider the following relation

$$\dot{q} = C(q)\dot{q}_1$$

where $C(q)$ is defined by (3). Taking time derivatives yields

$$\ddot{q} = C(q)\ddot{q}_1 + \dot{C}(q)\dot{q}_1$$

where $\dot{C}(q)$ denotes the time derivative of $C(q)$. Substituting this into (1) and multiplying both sides of the resulting equation by $C'(q)$ gives

$$C'(q)M(q)C(q)\ddot{q}_1 = C'(q)\left[B(q)u - F(q, C(q)\dot{q}_1) - M(q)\dot{C}(q)\dot{q}_1\right]. \tag{8}$$

Note that $C'(q)M(q)C(q)$ is an $(n - m) \times (n - m)$ symmetric positive definite matrix function.

We also assume that $r = n - m$ (for simplicity). Then the matrix product $C'(q)B(q)$ is locally invertible. Consequently for any $u \in R^r$ there is unique $v \in R^{n-m}$ which satisfies

$$C'(q)\left[B(q)u - F(q, C(q)\dot{q}_1) - M(q)\dot{C}(q)\dot{q}_1\right] = C'(q)M(q)C(q)v. \tag{9}$$

(Note that if $r > n - m$ then v can be chosen to depend smoothly on the variables $(q, \dot{q}_1, u)$). This assumption guarantees that the reduced configuration variables satisfy the linear equations

$$\ddot{q}_1 = v.$$

Define the following state variables

$$x_1 = q_1,$$
$$x_2 = q_2,$$
$$x_3 = \dot{q}_1.$$

Then the normal form equations are given by

$$\dot{x}_1 = x_3 \tag{10}$$

$$\dot{x}_2 = -\bar{J}(x_1, x_2)x_3 \tag{11}$$

$$\dot{x}_3 = v. \tag{12}$$

Equations (10)–(12) define a drift vector field $f(x) = (x_3, -\bar{J}(x_1, x_2)x_3, 0)$ and control vector fields $g_i(x) = (0, 0, e_i)$, where e_i is the ith standard basis vector in R^{n-m}, $i = 1, \cdots, n - m$, according to the standard control system form

$$\dot{x} = f(x) + \sum_{i=1}^{n-m} g_i(x)v_i. \tag{13}$$

We consider local properties of (10)–(12), near an equilibrium solution $(x_1^e, x_2^e, 0)$.

Note that the normal form equations (10)–(12) are a special case of the normal form equations in [8]. In particular, the zero dynamics equation of (10) and (12), corresponding to the output x_1, is given by

$$\dot{x}_2 = 0$$

and it is not locally asymptotically stable. The fact that the zero dynamics is a linear system with all zero eigenvalues, means that (10)–(12) are critically minimum phase at the equilibrium; this has important implications in terms of local asymptotic stabilizability of the original equations (1) and (2).

V. Stabilization to an Equilibrium Manifold Using Smooth Feedback

In this section, we study the problem of stabilization of (1) and (2) to a smooth equilibrium submanifold of $\boldsymbol{M}$ defined by

$$N_e = \{(q, \dot{q}) | \dot{q} = 0, s(q) = 0\}$$

where $s(q)$ is a smooth $n - m$ vector function. We show that with appropriate assumptions, there exists a smooth feedback such that the closed loop is locally asymptotically stable to N_e.

The smooth stabilization problem is the problem of giving conditions so that there exists a smooth feedback function $U: \boldsymbol{M} \to R^r$ such that N_e is locally asymptotically stable. Of course, we are interested not only in demonstrating that such a smooth feedback exists but also in indicating how such an asymptotically stabilizing smooth feedback can be constructed.

Note that in this section, we consider nonholonomic control systems whose normal form equations satisfy the property that if $q_1(t)$ and $\dot{q}_1(t)$ are exponentially decaying functions, then the solution to

$$\dot{q}_2 = -\bar{J}(q_1(t), q_2)\dot{q}_1(t)$$

is bounded (all the physical examples of nonholonomic systems, of which we are aware, satisfy this assumption).

Note also that the first and second time derivatives of $s(q)$ are given by

$$\dot{s} = \frac{\partial s(q)}{\partial q} C(q)\dot{q}_1,$$

$$\ddot{s} = \frac{\partial}{\partial q}\left(\frac{\partial s(q)}{\partial q} C(q)\dot{q}_1\right) C(q)\dot{q}_1 + \frac{\partial s(q)}{\partial q} C(q) v.$$

Theorem 3: Assume that the above solution property holds. Then the nonholonomic control system, defined by (1) and (2) is locally asymptotically stabilizable to

$$N_e = \{(q, \dot{q}) | \dot{q} = 0, s(q) = 0\} \qquad (14)$$

using smooth feedback, if the transversality condition

$$\det\left(\frac{\partial s(q)}{\partial q_1}\right) \det\left(\frac{\partial s(q)}{\partial q} C(q)\right) \neq 0 \qquad (15)$$

is satisfied.

Proof: It is sufficient to analyze the system in the normal form (10)–(12). By the transversality condition, the change of variables from $(q_1, q_2, \dot{q}_1)$ to $(s, q_2, \dot{s})$ is a diffeomorphism.

Let

$$v = -\left(\frac{\partial s(q)}{\partial q} C(q)\right)^{-1} \left[\frac{\partial}{\partial q}\left(\frac{\partial s(q)}{\partial q} C(q)\dot{q}_1\right) C(q)\dot{q}_1 + K_1 \frac{\partial s(q)}{\partial q} C(q)\dot{q}_1 + K_2 s(q)\right]$$

where K_1 and K_2 are symmetric positive definite $(n - m) \times (n - m)$ constant matrices. Then, obviously

$$\ddot{s} + K_1\dot{s} + K_2 s = 0$$

is asymptotically stable so that $(s, \dot{s}) \to 0$ as $t \to \infty$. The remaining system variables satisfy (11) of the normal form equations (with $x_2 = q_2$), and, by our assumption on the constraint matrix $\bar{J}$, these variables remain bounded for all time. Thus $(q(t), \dot{q}(t)) \to N_e$ as $t \to \infty$.

Equations (1) and (2) can be smoothly asymptotically stabilized to the m dimensional equilibrium manifold specified by (14). Condition (15) depends on the specific partitioning of the configuration variables corresponding to the constraint equations (2).

VI. Stabilization to an Equilibrium Solution Using Piecewise Analytic Feedback

The results in the previous section demonstrate that smooth feedback can be used to asymptotically stabilize certain smooth manifolds N_e in $\boldsymbol{M}$, where the dimension of N_e is equal to the number m of independent constraints. Consequently, those results do not guarantee smooth asymptotic stabilization to a single equilibrium solution if $m \geq 1$.

In fact, there is no C^1 feedback which can asymptotically stabilize the closed-loop system to a single equilibrium solution. Suppose that there is a C^1 feedback which asymptotically stabilizes, for example, the origin. Then it follows that there is an equilibrium manifold of dimension m containing the origin; that is, the origin is not isolated, which contradicts the assumption that it is asymptotically stable. We state this formally.

Theorem 4: Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in $\boldsymbol{M}$. The nonholonomic control system, defined by (1) and (2), is not asymptotically stabilizable using C^1 state feedback to $(q^e, 0)$.

Proof: A necessary condition for the existence of a C^1 asymptotically stabilizing state feedback law for system (10)–(12) is that the image of the mapping

$$(x_1, x_2, x_3, v) \mapsto \left(x_3, -\bar{J}(x_1, x_2)x_3, v\right)$$

contains some neighborhood of zero (see Brockett [7]). No points of the form

$$\begin{pmatrix} 0 \\ \epsilon \\ \alpha \end{pmatrix}, \qquad \epsilon \neq 0 \text{ and } \alpha \in R^{n-m} \text{ arbitrary}$$

are in its image: it follows that Brockett's necessary condition is not satisfied. Hence, system (10)–(12) cannot be asymptotically stabilized to $(q_1^e, q_2^e, 0)$ by a C^1 state feedback law. Consequently, the nonholonomic control system, defined by (1) and (2), is not asymptotically stabilizable to $(q^e, 0)$ using a C^1 state feedback.

We remark that even C^0 (continuous) state feedback (which results in existence of unique trajectories) is ruled out since Brockett's necessary condition is not satisfied [31].

A corollary of Theorem 4 is that a single equilibrium solution of (1) and (2) cannot by asymptotically stabilized using linear feedback nor can it be asymptotically stabilized using feedback linearization or any other control design approach that uses smooth feedback. Of course, it may be that a single equilibrium solution simply cannot be asymptotically stabilized or it may be that any asymptotically stabilizing state feedback is necessarily not C^0. However, in the subsequent sections, we show that a single equilibrium can be asymptotically stabilized by use of piecewise analytic state feedback.

We first demonstrate that the system of normal form equations (10)–(12), and hence the nonholonomic control system defined by (1) and (2), does indeed satisfy certain strong local controllability properties. In particular, we show that the system is strongly accessible and that the system is small time locally controllable at any equilibrium. These results not only provide a theoretical basis for the use of inherently nonlinear control strategies but they also suggest constructive procedures for the desired control strategies.

Theorem 5: Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in $\boldsymbol{M}$. The nonholonomic control system defined by (1) and (2) is strongly accessible at $(q^e, 0)$.

Proof: It suffices to prove that system (10)–(12) is strongly accessible at the origin. Let I denote the set $\{1, \cdots, n-m\}$. The drift and control vector fields can be expressed as

$$f = \sum_{j=1}^{n-m} x_{3,j} \tau_j,$$

$$g_i = \frac{\partial}{\partial x_{3,i}}, \qquad i \in I,$$

where

$$\tau_j = \frac{\partial}{\partial x_{1,j}} - \sum_{i=1}^{n-m} \bar{J}_{ij}(x_1, x_2) \frac{\partial}{\partial x_{2,i}}, \qquad j \in I$$

are considered as vector fields on the (x_1, x_2, x_3) state space. It can be verified that

$$[g_{i_1}, f] = \tau_{i_1}, \qquad i_1 \in I;$$
$$\big[g_{i_2}, [f, [g_{i_1}, f]]\big] = [\tau_{i_2}, \tau_{i_1}], \qquad i_1, i_2 \in I;$$
$$\vdots$$
$$\big[g_{i_{k^*}}, \big[f, \cdots, \big[g_{i_2}, [f, [g_{i_1}, f]]\big] \cdots\big]\big] = \big[\tau_{i_{k^*}}, \cdots, [\tau_{i_2}, \tau_{i_1}] \cdots\big], \qquad i_k \in I, 1 \leq k \leq k^*$$

hold, where k^* denotes the nonholonomy degree. Let

$$\mathscr{G} = \operatorname{span}\{g_i, i \in I\},$$

$$\mathscr{H} = \operatorname{span}\Big\{[g_{i_1}, f], \cdots, \big[g_{i_{k^*}}, \big[f, \cdots, \big[g_{i_2}, [f, [g_{i_1}, f]]\big] \cdots\big]\big]; \; i_k \in I, 1 \leq k \leq k^*\Big\}.$$

Note that $\dim \mathscr{G}(0) = n - m$ and $\dim \mathscr{H}(0) = n$ since the distribution defined by the constraints is completely nonholonomic; moreover $\dim\{\mathscr{G}(0) \cap \mathscr{H}(0)\} = 0$. It follows that the strong accessibility distribution

$$\mathscr{L}_0 = \operatorname{span}\{X: X \in \mathscr{G} \cup \mathscr{H}\}$$

has dimension $2n - m$ at the origin. Hence, the strong accessibility rank condition [28] is satisfied at the origin. Thus system (10)–(12) is strongly accessible at the origin. Consequently, the nonholonomic control system, defined by (1) and (2), is strongly accessible at $(q^e, 0)$.

Theorem 6: Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in $\boldsymbol{M}$. The nonholonomic control system, defined by (1) and (2), is small time locally controllable at $(q^e, 0)$.

Proof: It suffices to prove that system (10)–(12) is small time locally controllable at the origin.

The proof involves the notion of the degree of a bracket. To make this notion well defined we consider, as in [27], a Lie algebra of indeterminates and an associated evaluation map (on vector fields) as follows.

Let $X = (X_0, \cdots, X_{n-m})$ be a finite sequence of indeterminates. Let $A(X)$ denote the free associative algebra over R generated by the X_j, let $L(X)$ denote the Lie subalgebra of $A(X)$ generated by $X_0, \cdots, X_{n-m}$ and let $Br(X)$ be the smallest subset of $L(X)$ that contains $X_0, \cdots, X_{n-m}$ and is closed under bracketing.

Now consider the vector fields $f, g_1, \cdots, g_{n-m}$ on the manifold $\boldsymbol{M}$. Each $f, g_1, \cdots, g_{n-m}$ is a member of $D(\boldsymbol{M})$, the algebra of all partial differential operators on $C^\infty(\boldsymbol{M})$, the space of C^∞ real-valued functions on $\boldsymbol{M}$. Now let $g_0 = f$, and let $\boldsymbol{g} = (g_0, \cdots, g_{n-m})$ and define the evaluation map

$$Ev(\boldsymbol{g}): A(X) \to D(\boldsymbol{M})$$

obtained by substituting the g_j for the X_j, i.e.,

$$Ev(\boldsymbol{g})\Big(\sum_I a_I X_I\Big) = \sum_I a_I g_I$$

where $g_I = g_{i_1} g_{i_2} \cdots g_{i_k}$, $I = (i_1, \cdots, i_k)$. Note that the kernel of $Ev(\boldsymbol{g})$: $A(X) \to A(\boldsymbol{g})$ is the set of all algebraic identities satisfied by the g_i while the kernel of $Ev(\boldsymbol{g})$: $L(X) \to L(\boldsymbol{g})$ is the set of Lie algebraic identities satisfied by g_i.

Now, let B be a bracket in $Br(X)$. We define the degree of a bracket to be $\delta(B) = \sum_{i=0}^{n-m}\delta^i(B)$, where $\delta^0(B), \delta^1(B), \cdots, \delta^{n-m}(B)$ denote the number of times $X_0, \cdots, X_{n-m}$, respectively, occur in B. The bracket B is called "bad" if $\delta^0(B)$ is odd and $\delta^i(B)$ is even for each i, $i = 1, \cdots, n-m$. The theorem of Sussmann tells us the system is STLC at the origin if it satisfies the accessibility rank condition; and if B is "bad" there exist brackets $C_1, \cdots, C_k$ of lower degree in $Br(X)$ such that

$$Ev_0(g)(\beta(B)) = \sum_{i=1}^{k} \xi_i Ev_0(g)(C_i)$$

where Ev_0 denotes the evaluation map at the origin and $(\xi_1, \cdots, \xi_k) \in R^k$. Here, $\beta(B)$ is the symmetrization operator, $\beta(B) = \sum_{\pi \in S_{n-m}} \bar{\pi}(B)$, where $\pi \in S_{n-m}$, the group of permutations of $\{1, \cdots, n-m\}$ and for $\pi \in S_{n-m}$, $\bar{\pi}$ is the automorphism of $L(X)$ which fixes X_0 and sends X_i to $X_{\pi(i)}$.

By Theorem 5, the system is accessible at the origin.

The brackets in $\mathscr{G}$ are obviously "good" (not of the type defined as "bad") and $\delta^0(h) = \sum_{j=1}^{n-m}\delta^j(h)\ \forall h \in \mathscr{H}$; thus $\delta(h)$ is even for all h in $\mathscr{H}$, i.e., $\mathscr{H}$ contains "good" brackets only. It follows that the tangent space $T_0 M$ to M at the origin is spanned by the brackets that are all "good." Next we show that the brackets that might be "bad" vanish at the origin. First note that f vanishes at the origin. Let B denote a bracket satisfying $\delta(B) > 1$. If B is a "bad" bracket then, necessarily, $\delta^0(B) \neq \sum_{j=1}^{n-m}\delta^j(B)$, i.e., $\delta(B)$ must be odd. It can be verified that if $\delta^0(B) < \sum_{j=1}^{n-m}\delta^j(B)$ then B is identically zero and if $\delta^0(B) > \sum_{j=1}^{n-m}\delta^j(B)$ the B is of the form $\sum_{i=1}^{n-m} r_i(x_3)Y_i(x_1, x_2)$, for some vector fields $Y_i(x_1, x_2)$, $i \in I$, where $r_i(x_3)$, $i \in I$, are homogeneous functions of degree $(\delta^0(B) - \sum_{j=1}^{n-m}\delta^j(B))$ in x_3; thus B vanishes at the origin. Consequently, the Sussmann condition is satisfied. Hence, system (10)–(12) is small time locally controllable at the origin. It follows that, the nonholonomic control system, defined by (1) and (2), is small time locally controllable at $(q^e, 0)$.

VII. Construction of Piecewise Analytic Stabilizing Controllers for Caplygin Systems

Our recent work on control of nonholonomic systems in [4], [22], [23] has identified a large class of physical systems, which are referred to as "controlled Caplygin systems." Our subsequent results are developed for this class of systems.

We first describe the class of controlled Caplygin systems. We use the notation introduced previously. If the functions used in defining (1) and (2) do not depend explicitly on the configuration variables q_2, so that the system is locally described by

$$M(q_1)\ddot{q} + F(q_1, \dot{q}) = J'(q_1)\lambda + B(q_1)u \qquad (16)$$

$$\bar{J}(q_1)\dot{q}_1 + \dot{q}_2 = 0 \qquad (17)$$

where $\bar{J}(q_1)$ is an $mx(n-m)$ matrix function, then the uncontrolled system is called a "Caplygin system" [21]. In terms of the Lagrangian formalism for the problem this corresponds to the Lagrangian of the free problem being cyclic in (i.e., independent of) the variables q_2 while the constraints are also independent of q_2. The cyclic property is an expression of symmetries in the problem, such symmetries occurring naturally in many physical examples. More generally, if a system can be expressed in the form (16) and (17) using feedback, then we refer to it as a "controlled Caplygin system."

For the Caplygin system described by (16) and (17), (8) becomes

$$C'(q_1)M(q_1)C(q_1)\ddot{q}_1 = C'(q_1)\left[B(q_1)u - F(q_1, C(q_1)\dot{q}_1) - M(q_1)\dot{C}(q_1)\dot{q}_1\right] \qquad (18)$$

which is an equation in the phase variables $(q_1, \dot{q}_1)$ only. As a consequence, q_1 constitutes a reduced configuration space for the system (16) and (17). This reduced configuration space is also referred to as the "base space" (or "shape space") of the system. The term shape space (see [10], [11], [12], [14], [15]) arises from the theory of coupled mechanical systems, where it refers to the internal degrees of freedom of the system. It is possible to consider control theoretic problems which can be expressed solely in the base space, which can be solved using classical methods. However, in our work, we are interested in the more general control problems associated with the complete dynamics defined by (16) and (17), which are reflected in (17) and (18). We remark that the dimension of the base space is unique, equal to the number of degrees of freedom; however the identity of the base space variables is not unique.

As in Section IV, we assume that $r = n - m$ and that the matrix product $C'(q_1)B(q_1)$ is locally invertible; this assumption is not restrictive. Consequently, it can be shown that the normal form equations for the system (16) and (17), following the development in Section IV, are given by

$$\dot{x}_1 = x_3 \qquad (19)$$

$$\dot{x}_2 = -\bar{J}(x_1)x_3 \qquad (20)$$

$$\dot{x}_3 = v \qquad (21)$$

where $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$ and v satisfies

$$C'(q_1)\left[B(q_1)u - F(q_1, C(q_1)\dot{q}_1) - M(q_1)\dot{C}(q_1)\dot{q}_1\right] = C'(q_1)M(q_1)C(q_1)v. \qquad (22)$$

Our basic approach is to make use of the normal form equations (19)–(21) to control the Caplygin system (16) and (17). Note that the theoretical results obtained in previous sections certainly apply to the system (16) and (17).

Clearly, there is no continuous state feedback which asymptotically stabilizes a single equilibrium. However, the controllability properties possessed by the system guarantee the existence of a piecewise analytic state feedback in the analytic case [26]. We now describe the ideas that are employed to construct such a feedback which does achieve the desired local asymptotic stabilization of a single equilibrium solution. These ideas are based on the use of geometric phase (holonomy) which has proved useful in a variety of kinematics and dynamics problems (see e.g., [10], [11], and [19]). More information concerning geometric phases can be found in the recent book [24] of Shapere and Wilczek, and a review article [16] of Marsden, Montgomery and Ratiu. Our use of geometric phase is, to the best of our knowledge, its first application to nonlinear control systems of the form (19)–(21) which contain nontrivial drift vector fields [5], [22], [23]. The key observation is that the geometric phase, the extent to which a closed path in the base space fails to be closed in the configuration space, depends only on the path traversed in the base space and not on the time history of traversal of the path. Related ideas have been used for a class of path planning problems, based on kinematic relations, in [14], [15], and [12].

For simplicity, we consider control strategies which transfer any initial configuration and velocity (sufficiently close to the origin) to the zero configuration with zero velocity. The proposed control strategy initially transfers the given initial configuration and velocity to the origin of the $(q_1, \dot{q}_1)$ base phase space. The main point then is to determine a closed path in the q_1 base space that achieves the desired geometric phase. We show that, in the analytic case, the indicated assumptions guarantee that this geometric phase construction can be made and that (necessarily piecewise analytic) feedback can be determined which accomplishes the desired control objective.

Let $x^0 = (x_1^0, x_2^0, x_3^0)$ denote an initial state. We now describe two steps involved in construction of a control strategy which transfers the initial state to the origin.

Step 1: Bring the system to the origin of the (x_1, x_3) base phase space, i.e., find a control which transfers the initial state (x_1^0, x_2^0, x_3^0) to $(0, x_2^1, 0)$ in a finite time, form some x_2^1.

Step 2: Traverse a closed path (or a series of closed paths) in the x_1 base space to produce a desired geometric phase in the (x_1, x_2) configuration space, i.e., find a control which transfers $(0, x_2^1, 0)$ to $(0,0,0)$.

The desired geometric phase condition is given by

$$x_2^1 = \oint_\gamma \bar{J}(x_1)\, dx_1 \tag{23}$$

where γ denotes a closed path traversed in the base space. The geometric phase is reflected in the fact that traversing a closed path in the base space yields a nonclosed path in the full configuration space. Note that here, for notational simplicity in presenting the main idea, we assume that the desired geometric phase can be obtained by a single closed path. In general, more than one loop may be required to produce the desired geometric phase; for such cases γ can be viewed as concatenation of a series of closed paths.

Under the assumptions mentioned previously, explicit procedures can be given for each of the above two steps. Step 1 is classical; it is Step 2, involving the geometric phase, that requires special consideration. Explicit characterization of a closed path γ which satisfies the desired geometric phase condition (23) can be given for several specific examples. In the next section, we present three such examples. However, some problems may require a general computational approach. An algorithm based on Lie algebraic methods as in [13] can be employed to approximately characterize the required closed path. Suppose the closed path γ which satisfies the desired geometric phase condition is chosen. Then a feedback algorithm which realizes the closed path in the base space can be constructed since the base space equations (19), (21) constitute decoupled $n - m$ double integrators on the base space.

This general construction procedure provides a strategy for transferring an arbitrary initial state of (19)–(21) to the origin. Implementation of this control strategy in a (necessarily piecewise analytic) feedback form can be accomplished as follows.

Let $a = (a_1, \cdots, a_{n-m})$ and $b = (b_1, \cdots, b_{n-m})$ denote displacement vectors in the x_1 base space and let $\gamma(a, b)$ denote the closed path (in the base space) formed by the line segments from $x_1 = 0$ to $x_1 = a$, from $x_1 = a$ to $x_1 = a + b$, from $x_1 = a + b$ to $x_1 = b$, and from $x_1 = b$ to $x_1 = 0$. Then the geometric phase of the parameterized family

$$\{\gamma(a, b) | a, b \in R^{n-m}\}$$

is determined by the geometric phase function $\gamma(a, b) \mapsto \alpha(a, b)$ given as

$$\alpha(a, b) = - \oint_{\gamma(a,b)} \bar{J}(x_1)\, dx_1.$$

Now let π_s denote the projection map $\pi_s\colon (x_1, x_2, x_3) \mapsto (x_1, x_3)$. In order to construct a feedback control algorithm to accomplish the above two steps, we first define a feedback function $V^{x_1^*}(\pi_s x)$ which satisfies: for any $\pi_s x(t_0)$ there is $t_1 \geq t_0$ such that the unique solution of

$$\begin{aligned} \dot{x}_1 &= x_3, \\ \dot{x}_3 &= V^{x_1^*}(\pi_s x), \end{aligned}$$

satisfies $\pi_s x(t_1) = (x_1^*, 0)$. Note that the feedback function is parameterized by the vector x_1^*. Moreover, for each x_1^*, there exists such a feedback function. One such feedback function $V^{x_1^*}(\pi_s x) = (V_1^{x_1^*}(\pi_s x), \cdots, V_{n-m}^{x_1^*}(\pi_s x))$ is given as

$$V_i^{x^*}(\pi_s x) = \begin{cases} -k_i \operatorname{sign}(x_{1,i} - x_{1,i}^* + x_{3,i}|x_{3,i}|2k_i), \\ \qquad (x_{1,i}, x_{3,i}) \neq (x_{1,i}^*, 0), \\ 0, \\ \qquad (x_{1,i}, x_{3,i}) = (x_{1,i}^*, 0), \end{cases}$$

where k_i, $i = 1, \cdots, n - m$, are arbitrary positive constants.

We specify the control algorithm, with values denoted by v^*, according to the following construction, where x denotes the "current state":

Control Algorithm for v^*:

Step 0: Choose (a^*, b^*) to achieve the desired geometric phase.

Step 1: Set $v^* = V^{a^*}(\pi_s x)$, until $\pi_s x = (a^*, 0)$; then go to Step 2;

Step 2: Set $v^* = V^{a^*+b^*}(\pi_s x)$, until $\pi_s x = (a^* + b^*, 0)$; then go to Step 3;

Step 3: Set $v^* = V^{b^*}(\pi_s x)$, until $\pi_s x = (b^*, 0)$; then go to Step 4;

Step 4: Set $v^* = V^0(\pi_s x)$, until $\pi_s x = (0, 0)$; then go to Step 0.

We assumed here that the desired geometric phase can be obtained by a single closed path. Clearly, the above algorithm can be modified to account for more complex cases.

Note that the control algorithm is constructed by appropriate switchings between members of the parameterized family of feedback functions. On each cycle of the algorithm the particular functions selected depend on the closed path parameters a^*, b^*, computed in Step 0, to correct for errors in x_2.

The control algorithm can be initialized in different ways. The most natural is to begin with Step 4 since v^* in that step does not depend on the closed path parameters; however, many other initializations of the control algorithm are possible. The original control u^* is computed using (22).

Justification that the constructed control algorithm asymptotically stabilizes the origin follows as a consequence of the construction procedure: switching between feedback functions guarantees that the proper closed path (or a sequence of closed paths) is traversed in the base space so that the origin $(0, 0, 0)$ is necessarily reached in a finite time. This construction of a stabilizing feedback algorithm represents an alternative to the approach by Hermes [9], which is based on Lie algebraic properties.

It is important to emphasize that the above construction is based on the *a priori* selection of simply parametrized closed paths in the base space. The above selection simplifies the tracking problem in the base space, but other path selections could be made and they would, of course, lead to a different feedback strategy from that proposed above.

We remark that the technique presented in this section can be generalized to some systems which are not Caplygin. For instance, this generalization is tractible to systems for which (20) takes the form

$$\dot{x}_2 = \rho(x_2)\bar{J}(x_1)$$

where $\rho(x_2)$ denotes a certain Lie group representation (see e.g., [16]). The geometric phase of a closed path for such systems is given as a path ordered exponential rather than a path integral.

VIII. Examples

Control of Knife Edge Using Steering and Pushing Inputs: We first consider the control of a knife edge moving in point contact on a plane surface [3]–[5]. Let x and y denote the coordinates of the point of contact of the knife edge on the plane and let ϕ denote the heading angle of the knife edge, measured from the x-axis. Then the equations of motion, with all numerical constants set to unity, are given by

$$\ddot{x} = \lambda \sin\phi + u_1 \cos\phi \tag{24}$$

$$\ddot{y} = -\lambda \cos\phi + u_1 \sin\phi \tag{25}$$

$$\ddot{\phi} = u_2 \tag{26}$$

where u_1 denotes the control force in the direction defined by the heading angle, u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the scalar nonholonomic constraint

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0 \tag{27}$$

which has nonholonomy degree two at any configuration. It is clear that the constraint manifold is a five-dimensional manifold and is defined by

$$\boldsymbol{M} = \left\{(\phi, x, y, \dot{\phi}, \dot{x}, \dot{y}) | \dot{x}\sin\phi - \dot{y}\cos\phi = 0\right\}$$

and any configuration is an equilibrium if the controls are zero.

Define the variables

$$\begin{aligned} x_1 &= x\cos\phi + y\sin\phi, \\ x_2 &= \phi, \\ x_3 &= -x\sin\phi + y\cos\phi, \\ x_4 &= \dot{x}\cos\phi + \dot{y}\sin\phi - \dot{\phi}(x\sin\phi - y\cos\phi), \\ x_5 &= \dot{\phi}, \end{aligned}$$

so that the reduced differential equations are given by

$$\begin{aligned} \dot{x}_1 &= x_4, \\ \dot{x}_2 &= x_5, \\ \dot{x}_3 &= -x_1 x_5, \\ \dot{x}_4 &= u_1 + u_2 x_3 - x_1 x_5^2, \\ \dot{x}_5 &= u_2. \end{aligned}$$

Consequently, (24)–(27) represent a controlled Caplygin system with base space equations which are feedback linearizable. The following conclusions are based on the analysis of the above reduced equations.

Proposition 1: Let $x^e = (x_1^e, x_2^e, x_3^e, 0, 0)$ denote an equilibrium solution of the reduced differential equations corresponding to $u = 0$. The knife edge dynamics described by (24)–(27) have the following properties:

1) There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth one dimensional

equilibrium manifold in $\boldsymbol{M}$ which satisfies the transversality condition.

2) There is no continuous state feedback which asymptotically stabilizes x^e.

3) The system is strongly accessible at x^e since the space spanned by the vectors

$$g_1, g_2, [g_1, f], [g_2, f], [g_2, [f, [g_1, f]]]$$

has dimension 5 at x^e.

4) The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.

Note that the base variables are (x_1, x_2). Consider a parameterized rectangular closed path γ in the base space with four corner points

$$(0,0), (x_1, 0), (x_1, x_2), (0, x_2)$$

i.e., $a = (x_1, 0)$ and $b = (0, x_2)$ following the notation introduced in the general development. By evaluating the integral in (23) in closed form for this case, the desired geometric phase condition is

$$x_3^1 = x_1 x_2.$$

This equation can be explicitly solved to determine a closed path $\gamma^* = \gamma(a^*, b^*)$ which achieves the desired geometric phase. One solution can be given as follows:

$$a^* = \left(\sqrt{|x_3^1|}\operatorname{sign} x_3^1, 0\right), \qquad b^* = \left(0, \sqrt{|x_3^1|}\right).$$

Note that the previously described feedback algorithm can be used to asymptotically stabilize the knife edge to the origin. A different feedback algorithm for this example is given in [4].

Control of Rolling Wheel Using Steering and Driving Inputs: As a second example, we consider the control of a vertical wheel rolling without slipping on a plane surface [3], [5]. Let x and y denote the coordinates of the point of contact of the wheel on the plane, let ϕ denote the heading angle of the wheel, measured from the x-axis and let θ denote the rotation angle of the wheel due to rolling, measured from a fixed reference. Then the equations of motion, with all numerical constants set to unity, are given by

$$\ddot{x} = \lambda_1 \tag{28}$$

$$\ddot{y} = \lambda_2 \tag{29}$$

$$\ddot{\theta} = -\lambda_1 \cos\phi - \lambda_2 \sin\phi + u_1 \tag{30}$$

$$\ddot{\phi} = u_2 \tag{31}$$

where u_1 denotes the control torque about the rolling axis of the wheel and u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the two nonholonomic constraints

$$\dot{x} = \dot{\theta}\cos\phi \tag{32}$$

$$\dot{y} = \dot{\theta}\sin\phi \tag{33}$$

which have nonholonomy degree three at any configuration. The constraint manifold is a six-dimensional manifold and is given by

$$\boldsymbol{M} = \left\{\left(\theta, \phi, x, y, \dot{\theta}, \dot{\phi}, \dot{x}, \dot{y}\right) | \dot{x} = \dot{\theta}\cos\phi, \dot{y} = \dot{\theta}\sin\phi\right\}$$

and any configuration is an equilibrium if the controls are zero.

Define the variables

$$x_1 = \theta, \quad x_2 = \phi, \quad x_3 = x, \quad x_4 = y, \quad x_5 = \dot{\theta}, \quad x_6 = \dot{\phi}$$

so that the reduced differential equations are given by

$$\begin{aligned}
\dot{x}_1 &= x_5,\\
\dot{x}_2 &= x_6,\\
\dot{x}_3 &= x_5 \cos x_2,\\
\dot{x}_4 &= x_5 \sin x_2,\\
\dot{x}_5 &= \frac{1}{2}u_1,\\
\dot{x}_6 &= u_2.
\end{aligned}$$

Consequently, (28)–(33) represent a controlled Caplygin system with base space equations which are feedback linearizable. The following conclusions are based on analysis of the above reduced equations.

Proposition 2: Let $x^e = (x_1^e, x_2^e, x_3^e, x_4^e, 0.0)$ denote an equilibrium solution of the reduced differential equations corresponding to $u = 0$. The rolling wheel dynamics described by (28)–(33) have the following properties:

1) There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth two-dimensional equilibrium manifold in $\boldsymbol{M}$ which satisfies the transversality condition.

2) There is no continuous state feedback which asymptotically stabilizes x^e.

3) The system is strongly accessible at x^e since the space spanned by the vectors

$$g_1, g_2, [g_1, f], [g_2, f], [g_2, [f, [g_1, f]]], \\ [g_2, [f, [g_1, [f, [g_2, f]]]]]$$

has dimension 6 at x^e.

4) The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.

Note that the base variables are (x_1, x_2). Consider a parameterized rectangular closed path γ in the base space with four corner points

$$(0,0), (x_1, 0), (x_1, x_2), (0, x_2).$$

By evaluating the integral in (23) in closed form for this case, the desired geometric phase conditions are

$$\begin{aligned}
x_3^1 &= x_1(\cos x_2 - 1),\\
x_4^1 &= x_1 \sin x_2.
\end{aligned}$$

These equations can be explicitly solved to determine a closed path (or a concatenation of closed paths) γ^* which achieves the desired geometric phase. One solution can be given as follows: if $x_3^1 \neq 0$ then γ^* is the closed path specified by

$$a^* = -\left((x_3^1)^2 + (x_4^1)^2\right)/2x_3^1, 0),$$
$$b^* = \left(0, -\sin^{-1}(2x_3^1x_4^1/\left((x_3^1)^2 + (x_4^1)^2\right)\right)$$

and if $x_3^1 = 0$ then γ^* is a concatenation of two closed paths specified by

$$a^* = (0.5x_4^1, 0), \qquad b^* = (0, 0.5\pi),$$
$$a^{**} = (-0.5x_4^1, 0), \qquad b^{**} = (0, -0.5\pi).$$

Note that the previously described feedback algorithm can be used (with the modification indicated in the general development) to asymptotically stabilize the rolling wheel to the origin.

Control of Planar Multibody Systems Using Angular Momentum Preserving Inputs: Another interesting class of physical examples is given by the control of a planar multibody system with angular momentum preserving control torques. For more details on the origin of this problem, and references to previous work, see [10] and [25]. Related papers are in [22], [23]. It is assumed that a system of N planar rigid bodies are interconnected by frictionless one degree of freedom joints in the form of an open kinematic chain. The configuration space of the N-body system is T^N, the N-dimensional torus. Define the vector of absolute angles of the N bodies

$$\theta = (\theta_1, \cdots, \theta_N)$$

and the vector of relative angles (or joint angles) corresponding to the $(N-1)$ joints

$$\psi = (\psi_1, \cdots, \psi_{N-1}).$$

The relationship between the vectors θ and ψ is given by

$$\psi = P\theta$$

where P is a constant $(N-1) \times N$ matrix. In the absence of potential energy, the equations of motion are given by

$$J(\theta)\ddot{\theta} + F(\theta, \dot{\theta}) = P'u \tag{34}$$

where the $N \times N$ matrix function $J(\theta)$ is invertible, and

$$F(\theta, \dot{\theta}) = \frac{d}{dt}[J(\theta)]\dot{\theta} - \frac{1}{2}\frac{\partial}{\partial\theta}\left(\dot{\theta}'J(\theta)\dot{\theta}\right)$$

in an N-vector function, and the control input u is the $N-1$ vector of joint torques. Assuming that the angular momentum is zero, it follows that

$$1'J(\theta)\dot{\theta} = 0$$

holds, where $1 = (1, \cdots, 1)'$. It can be shown that (35) is nonholonomic for $N \geq 3$. Define the variables

$$x_1 = \psi,$$
$$x_2 = \theta_1$$
$$x_3 = \dot{\psi}_1,$$

so that the reduced differential equations are given by

$$\dot{x}_1 = x_3,$$
$$\dot{x}_2 = -\bar{J}(x_1)x_3,$$
$$\dot{x}_3 = \bar{F}(x_1, x_2) + \bar{B}(x_1)u.$$

The indicated assumptions guarantee that (34) and (35) take the form of a controlled Caplygin system with shape space equations that are feedback linearizable.

The following conclusions are based on analysis of the above reduced equations.

Proposition 3: Let $x^e = (x_1^e, x_2^e, 0)$ denote a regular equilibrium of the reduced differential equations corresponding to $u = 0$, i.e., $(\partial\bar{J}_{i_0}(x_1^e)/\partial x_{1,j_0}^e) - (\partial\bar{J}_{j_0}(x_1^e)/\partial x_{1,i_0}^e) \neq 0$ for some (i_0, j_0). The dynamics of the planar multibody system described by (34) and (35) have the following properties if $N \geq 3$:

1) There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth one dimensional equilibrium manifold in $\boldsymbol{M}$ which satisfies the transversality condition.

2) There is no continuous state feedback which asymptotically stabilizes x^e.

3) The system is strongly accessible at x^e since the space spanned by the vectors

$$g_1, \cdots, g_{N-1}, [g_1, f], \cdots, [g_{N-1}, f], \left[g_{i_0}, \left[f, [g_{j_0}, f]\right]\right]$$

has dimension $2N-1$ at x^e.

4) The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.

If $N = 1$ or 2, then the system (34) and (35) is neither strongly accessible nor small time locally controllable. If the equilibrium solution x^e is not regular, higher order brackets are required to obtain the same conclusions.

Note that the shape variables are the $N-1$ joint angles x_1. Following the development in [22], the N bodies can be treated as three interconnected bodies by locking all the joints except the ones labelled (i_0, j_0). Consider a parameterized rectangular closed path γ in the $x_{1,i_0} - x_{1,j_0}$ plane with four corner points

$$(0,0), (x_{1,i_0}, 0), (x_{1,i_0}, x_{1,j_0}), (0, x_{1,j_0}).$$

In this case, the desired geometric phase condition can be written as

$$x_2^1 = \oint_\gamma \bar{s}_{i_0}(x_{1,i_0}, x_{1,j_0})\, dx_{1,i_0} + \bar{s}_{j_0}(x_{1,i_0}, x_{1,j_0})\, dx_{1,j_0}$$

where $\bar{s}_{i_0}(x_{1,i_0}, x_{1,j_0})$ and $\bar{s}_{j_0}(x_{1,i_0}, x_{1,j_0})$ are obtained by evaluating $\bar{J}_{i_0}(x_1)$ and $\bar{J}_{j_0}(x_1)$ at $x_{1,i} = 0$, for $i = 1, \cdots, N-1$, $i \neq i_0$, $i \neq j_0$. In this case, the path integral can be computed numerically as a function of the loop parameters x_{1,i_0}, x_{1,j_0} as in [23]. Further, loop parameters x_{1,i_0}^*, x_{1,j_0}^* can be computed numerically, thereby determining a closed path γ^* which achieves the desired geometric phase. Note that the previously described feedback algorithm can be used (with the modification indicated in the general development) to asymptotically stabilize the planar multibody to the origin.

IX. Conclusions

A class of inherently nonlinear control problems has been identified, the nonlinear features arising directly from physical assumptions about constraints on the motion of a mechanical system. In this paper, we have presented models for mechanical systems with nonholonomic constraints represented both by differential-algebraic equations and by reduced state equations. We have studied control issues for this class of systems and we have derived a number of fundamental results. Although a single equilibrium solution cannot be asymptotically stabilized using continuous state feedback, a general procedure for constructing a piecewise analytic state feedback which achieves the desired result has been suggested. The theoretical issues addressed in the paper have been illustrated through several classes of example problems.

The general approach described in this paper makes substantial use of the geometric approach to nonlinear control. However, the specific nonlinear control strategy suggested is substantially different, both conceptually and in detail, from the smooth nonlinear control strategies most commonly studied in the literature. It is hoped that this paper provides a foundation for future research on this important and challenging class of nonlinear control problems.

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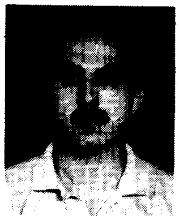

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