for some constants $k_i > 0$, $i = 1, 2$, and all $t_2 > t_1 \geq 0$ (for the discrete-time case, $\dot{\theta}(t)$ or $\dot{\rho}(t)$ is replaced by $\theta(t+1) - \theta(t)$ or $\rho(t+1) - \rho(t)$, and $\int_{t_1}^{t_2} dt$ is replaced by $\sum_{t=t_1}^{t_2}$).

These properties are sufficient for establishing the closed-loop signal boundedness in the discrete-time case, as similar to that in [7] and [8]. For the continuous-time case, the signal boundedness proof in [6] for input dead-zones with equal slopes $m_r = m_l$ and their equal estimates $\widehat{m_r} = \widehat{m_l}$ can be used to show similar signal boundedness results for our adaptive inverse controllers here for input dead-zone, backlash, and hysteresis characteristics.

It is our main interest to show the order reduction of the closed-loop adaptive system with controller (3.5) for input nonlinearities. The block $\theta_4^T \frac{A_1(D)}{\Lambda_0(D)}$ has $(n - n_r - 1)p$ parameters and its dynamic order is $(n - n_r - 1)p$, where p is the dimension of θ_N^* (or $(\theta_N^{*T}, -1)^T$ for the hysteresis case). The similar block to this $\theta_4^T \frac{A_1(D)}{\Lambda_0(D)}$ in the adaptive control structure without using the knowledge of $R_0(D)$ has $(n-1)p$ parameters and is with a dynamic order $(n-1)p$. Hence, with our new controller (3.5), the adaptive system order reduction is $n_r p + n_r p = 2n_r p$. For example, in the hysteresis case, $p = 9$, the order reduction is $18n_r$.

To figure out the order reduction with the adaptive controller (3.12) for output nonlinearities, we see that the block $\theta_4^T \frac{A_2(D)}{\Lambda_0(D)}$ has $(n-1)p$ parameters and its dynamic order is $(n-1)p$, where p is the dimension of the parameter vector θ_N^*. The similar block in the adaptive control structure without using the knowledge of $R_0(D)$ has $(n + n_r - 1)p$ parameters and is with a dynamic order $(n + n_r - 1)p$. Hence, with our new controller (3.12), the adaptive system order reduction is again $n_r p + n_r p = 2n_r p$. For the output hysteresis case, $p = 8$, the order reduction is $16n_r$.

IV. Conclusions

We have developed several adaptive control schemes for partially known plants. For linear plants, we first proposed two modified model reference adaptive controllers: one for plants with partially known stable pole dynamics and the other for plants with partially known stable zero dynamics. We then developed two modified adaptive inverse controllers for plants with nonsmooth nonlinearities such as a dead-zone, backlash, or hysteresis characteristic: one for plants with an unknown nonlinearity at the input of a linear part whose zero dynamics is partially known and the other for plants with an unknown nonlinearity at the output of a linear part whose pole dynamics is partially known. Our adaptive controllers reduce the closed-loop system order by $2n_r p$, where n_r is the order of the known dynamics, and p is the number of nonlinearity parameters ($p = 1$ for plants being linear).

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Stability of Model Predictive Control with Mixed Constraints

Alex Zheng and Manfred Morari

Abstract—**We derive stability conditions for model predictive control (MPC) with hard constraints on the inputs and "soft" constraints on the outputs for an infinitely long output horizon. We show that with state feedback, MPC is globally asymptotically stabilizing if and only if all the eigenvalues of the open-loop system are in the closed unit disk. With output feedback, we show that the results hold if all the eigenvalues are strictly inside the unit circle. The on-line optimization problem defining MPC can be posed as a finite dimensional quadratic program even though the output constraints are specified over an infinite horizon.**

I. Introduction

Many practical control problems are dominated by constraints. There are generally two types of constraints—input constraints and output constraints. The input constraints are always present and are imposed by physical limitations of the actuators which cannot be exceeded under any circumstances. Often, it is also desirable to keep specific outputs within certain limits for reasons related to plant operation, e.g., safety, material constraints, etc. It is usually unavoidable to exceed the output constraints, at least temporarily, for example, when the system is subjected to unexpected disturbances.

Industry has embraced model predictive control (MPC), also referred to as moving horizon control and receding horizon control, as a powerful feedback strategy to control systems with constraints. The basic idea behind MPC is as follows: At sampling time k, m (called input horizon) future control moves are calculated such that an objective function over some output horizon is minimized subject to constraints. Only the first one of the m computed control moves is implemented. At the next sampling time, the measurement is used to update the state estimate and the same calculations are repeated.

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For linear time-invariant stable systems, assuming a feasible set of input and output constraints, several results [1], [2] on global asymptotic stability have been shown using finite input horizon and infinite output horizon, infinite input horizon and infinite output horizon, and finite input and output horizons with an end constraint. Output constraints, however, can lead to an infeasible optimization problem. Rawlings and Muske [2] proposed to remove the infeasible output constraints during the initial portion of the infinite horizon to make the optimization problem feasible. This may result, however, in undesirable performance: the violation of the output constraints during this initial portion of the infinite horizon can be very large to satisfy the constraints during the rest. Thus, large constraint violations may be experienced when the computed control actions are implemented.

An alternative way to handle the feasibility problem is to relax the infeasible output constraints for the entire horizon and to penalize the extent of the violation. This technique is referred to as "constraint softening" [13]. The problem is that global stability may not be guaranteed. Zafiriou and Chiou [9] have derived some conditions for stability for single-input single-output systems. These conditions, however, are generally conservative and difficult to check.

In this note, we show that global asymptotic stability can be guaranteed for linear time-invariant discrete time with poles inside the closed unit disk systems subject to mixed constraints—hard input constraints and soft output constraints. Furthermore, in the case that the state must be estimated, we show that global asymptotic stability for stable systems is preserved by using an asymptotic observer. Finally, we show that the optimization problem can be cast as a finite dimensional quadratic program even though the output constraints are specified over the infinite horizon.

II. State Feedback

Consider the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x(k) \in \Re^{n_x}, u(k) \in \Re^{n_u}$, and $y(k) \in \Re^{n_y}$. Define the objective function as

$$\begin{aligned} \Phi_k = &\sum_{i=1}^{\infty} x(k+i \mid k)^T Rx(k+i \mid k) \\ &+ \sum_{i=0}^{m} \Big[u(k+i \mid k)^T Su(k+i \mid k) \\ &+ \Delta u(k+i \mid k)^T P \Delta u(k+i \mid k)\Big] \end{aligned} \tag{2}$$

where $R > 0$, $S > 0$, $P \geq 0$, $\Delta u(k+i \mid k) = u(k+i \mid k) - u(k+i-1 \mid k)$ and m is finite. R, S, and P are symmetric. $(\cdot)(k+i \mid k)$ denotes the variable $(\cdot)$ at sampling time $k+i$ predicted at sampling time k and $(\cdot)(k) = (\cdot)(k \mid k)$. The control actions are generated by state feedback controller MPC which is defined as follows.

Definition 1—State Feedback Controller MPC: At sampling time k, the control move $u(k)$ equals the first element $u(k \mid k)$ of the sequence $\{u(k \mid k), u(k+1 \mid k), \cdots, u(k+m-1 \mid k)\}$ which is the minimizer of the optimization problem

$$J_k = \min_{u(k|k),\cdots,u(k+m-1|k),\epsilon(k)} \Phi_k + \epsilon(k)^T Q \epsilon(k)$$
$$\text{subject to} \begin{cases} |\Delta u(k+i|k)| \leq \Delta u^{\max} & i = 0, 1, \cdots, m \\ u^{\min} \leq u(k+i|k) \leq u^{\max} & i = 0, 1, \cdots, m-1 \\ u(k+i|k) = 0 & i = m, m+1, \cdots, \infty \\ Gx(k+i|k) \leq g + \epsilon(k) & i = 0, 1, \cdots, \infty \\ \epsilon(k) \geq 0 \end{cases} \tag{3}$$

where $G \in \Re^{n_G \times n_x}, g \in \Re^{n_G}$, and $Q > 0$ is diagonal.

The input constraints represent physical limitations on the actuators which cannot be violated under any circumstances. The output constraints are softened by the slack variables $\epsilon(k)$. They can be violated temporarily, if necessary. In the long term, the penalty term $\epsilon(k)^T Q \epsilon(k)$ in the objective function will drive the slack variables to zero. The optimization problem (3) can be cast as a quadratic program.

The control problem is to bring the state to the origin. To make it well posed, the feasible region for

$$\begin{cases} |\Delta u(k+i \mid k)| \leq \Delta u^{\max} & i = 0, 1, \cdots, m \\ u^{\min} \leq u(k+i \mid k) \leq u^{\max} & i = 0, 1, \cdots, m-1 \end{cases}$$

must contain $u(k+i \mid k) = 0, i = 0, 1, \cdots, m-1$, as an interior point. The feasible region for

$$\begin{cases} Gx(k+i \mid k) \leq g + \epsilon(k) & i = 0, 1, \cdots, \infty \\ \epsilon(k) = 0 \end{cases}$$

contains $x(k+i \mid k) = 0, i = 0, 1, \cdots, \infty$, as an interior point. Note that this implies $g > 0$. Then we have the following theorem which extends the results in [2] for $\epsilon(k) = 0 \forall k \geq 0$.

Theorem 1: The closed-loop system with state feedback controller MPC is globally asymptotically stable if and only if the optimization problem (3) is feasible for all $x(0) \in \Re^{n_x}$.

Proof: If the optimization problem (3) is not feasible, the controller is not defined. Feasibility of the optimization problem implies that J_1 is finite. At sampling time $k+1$, let

$$\begin{cases} u^*(k+i \mid k+1) = u(k+i \mid k) & i = 1, 2, \cdots, m \\ \epsilon^*(k+1) = \epsilon(k). \end{cases}$$

Thus, (u^*, ϵ^*) is a feasible solution but may not be optimal. We have

$$\begin{aligned} J_{k+1} \leq J_k &- x(k+1)^T Rx(k+1) - u(k)^T Su(k) \\ &- \Delta u(k)^T P \Delta u(k) \end{aligned}$$

which yields

$$\begin{aligned} J_{k+1} + &\sum_{i=1}^{k} [x(i+1)^T Rx(i+1) + u(i)^T Su(i) \\ &+ \Delta u(k)^T P \Delta u(k)] \leq J_1 < \infty. \end{aligned}$$

Note that we replaced $x(k+1 \mid k)$ with $x(k+1)$ since $x(k+1) = x(k+1 \mid k)$. This together with $R, S > 0$ implies that $x(k) \to 0$ and $u(k) \to 0$ as $k \to \infty$. □

Remark 1: Theorem 1 also holds if $S \geq 0$ provided that at steady state $x = 0$ if and only if $u = 0$ which is equivalent to that $(I-A)^{-1}B$ has full column rank. Also if $(I-A)^{-1}B$ has full column rank, Theorem 1 holds with $u(k+i \mid k) = 0, i = m, m+1, \cdots, \infty$ in (3) replaced by $\Delta u(k+i \mid k) = 0, i = m, m+1, \cdots, \infty$.

Remark 2: If $Q = \infty$, then the output constraints become hard, and the optimization problem (3) may not be feasible.

The following theorem states that for $Q < \infty$, feasibility of the optimization problem (3) is guaranteed for stable systems.

Theorem 2: If A is stable, i.e., all eigenvalues of A are strictly inside the unit circle, then the optimization problem (3) is feasible $\forall m \geq 1, Q < \infty$, and $x(0) \in \Re^{n_x}$.

Proof: All we have to do is to prove the feasibility of the optimization problem (3) at the first sampling time. We will prove this theorem by construction. Since A is stable, $x(k)$ is bounded $\forall\, k \geq 0$ for any initial condition. Then

$$\begin{cases} u^*(i \mid 1) = 0 & i = 1, 2, \cdots, m \\ \epsilon^*(1) = \max_{i \geq 1} |Gx(i \mid 1)|_\infty < \infty \end{cases}$$

satisfies all the constraints and results in $J_1 < \infty$. Thus it is a feasible solution. □

In light of results by Tsirukis and Morari [7] and Zheng *et al.* [10], we can show that Theorem 2 also holds for stabilizable systems with poles in the closed unit disk provided that m is sufficiently large. This is stated in the following theorem.

Theorem 3: Assume that $\{A, B\}$ is stabilizable and that all eigenvalues of A are in the closed unit disk. Given any $x(0) \in \Re^{n_x}$, the optimization problem (3) is feasible $\forall\, Q < \infty$ for a sufficiently large value of m.

Proof: See, for example, [7]. □

We have shown, that with m properly chosen, state feedback controller MPC globally asymptotically stabilizes any constrained stabilizable system with poles in the closed unit disk, using state feedback. When the inputs are constrained, i.e., $u^{\min} \le u(k) \le u^{\max}$ $\forall k$, Sontag [4] showed that there does not exist a controller that globally stabilizes any system with poles outside the unit circle.[1] Thus, the MPC controller globally stabilizes all constrained systems for which global stabilization is possible.

Remark 3: Since for systems with poles on the unit circle m depends on the initial condition, the term semiglobal asymptotic stability may be more appropriate here than global asymptotic stability. In [10], an alternative MPC algorithm in which m does not depend on the initial condition is proposed.

Remark 4: Theorems 1, 2, and 3 hold as well if other norms for softening the output constraints are used.

III. Output Feedback

In the previous section, we assumed that the state is measured. Since the closed-loop system may be nonlinear because of the constraints, we cannot apply the separation principle to prove global stability for the output feedback case. It is well known that, in general, a nonlinear closed-loop system with the state estimated via an exponentially converging observer can be unstable even though it is stable with state feedback. Although it is trivial to show local asymptotic stability here, proving global asymptotic stability is nontrivial. We will show in this section that global asymptotic stability of the closed-loop system generated by state feedback MPC controller and an exponentially converging observer is guaranteed for stable systems.

Denote the state (output) at sampling time $k+i$ estimated at sampling time k by $\hat{x}(k+i \mid k)$ ($\hat{y}(k+i \mid k)$). The state is estimated as follows

$$\begin{aligned}\hat{x}(k \mid k) &= A\hat{x}(k-1 \mid k-1) + Bu(k-1) \\ &\quad + L(y(k) - \hat{y}(k \mid k-1)) \\ \hat{x}(k+i \mid k) &= A\hat{x}(k+i-1 \mid k) + Bu(k+i-1) \quad i \ge 1 \end{aligned} \tag{4}$$

where L is the observer gain. Define output feedback controller MPC as follows.

Definition 2—Output Feedback Controller MPC: At sampling time k, the control move $u(k)$ equals the first element $u(k \mid k)$ of the sequence $\{u(k \mid k), u(k+1 \mid k), \cdots, u(k+m-1 \mid k)\}$ which is the minimizer of the optimization problem

$$\hat{J}_k = \min_{\epsilon(k), u(k|k), \cdots, u(k+m-1|k)} \hat{\Phi}_k + \epsilon(k)^T Q \epsilon(k)$$
$$\text{subject to} \begin{cases} |\Delta u(k+i \mid k)| \le \Delta u^{\max} & i = 0, 1, \cdots, m \\ u^{\min} \le u(k+i \mid k) \le u^{\max} & i = 0, 1, \cdots, m-1 \\ u(k+i \mid k) = 0 & i = m, m+1, \cdots, \infty \\ G\hat{x}(k+i \mid k) \le g + \epsilon(k) & i = 0, 1, \cdots, \infty \\ \epsilon(k) \ge 0 \end{cases} \tag{5}$$

[1] With constraints on Δu, we can also show that there does not exist a controller which globally stabilizes an unstable system.

where $G \in \Re^{n_G \times n_x}, g \in \Re^{n_G}$, $Q > 0$ diagonal, $\hat{x}(\cdot|\cdot)$ estimated via (4) and

$$\begin{aligned}\hat{\Phi}_k &= \sum_{i=1}^{\infty} \hat{x}(k+i \mid k)^T R \hat{x}(k+i \mid k) \\ &\quad + \sum_{i=0}^{m} \Big[u(k+i \mid k)^T S u(k+i \mid k) \\ &\quad + \Delta u(k+i \mid k)^T P \Delta u(k+i \mid k) \Big]. \end{aligned} \tag{6}$$

Combining (4) and (1) yields

$$e(k+1) = (I - LC)Ae(k) \tag{7}$$

where $e(k) = x(k) - \hat{x}(k \mid k)$. Thus (4) can be written as

$$\begin{aligned}\hat{x}(k \mid k) &= \hat{x}(k \mid k-1) + LCAe(k-1) \\ \hat{x}(k+i \mid k) &= A\hat{x}(k+i-1 \mid k) + Bu(k+i-1) \quad i \ge 1 \end{aligned} \tag{8}$$

which yields

$$\begin{aligned}\xi(k \mid k) &= LCAe(k-1) \\ \xi(k+i \mid k) &= A\xi(k+i-1 \mid k) \quad i \ge 1 \end{aligned} \tag{9}$$

where $\xi(k+i \mid k) = \hat{x}(k+i \mid k) - \hat{x}(k+i \mid k-1)$.

Remark 5: The overall system with output feedback controller MPC can be expressed as follows

$$\begin{cases} x(k+1) &= f(x(k), e(k)) \\ e(k+1) &= (I - LC)Ae(k) \end{cases} \tag{10}$$

where $x(k+1) = f(x(k), 0)$ represents the closed-loop system with state feedback and is globally asymptotically stable for stable systems. To prove global asymptotic stability for (10) is a special case of an actively studied problem (see, for example, [6] and [5]) that considers a more general set of equations

$$\begin{cases} \dot{x} &= f_1(x, e) \\ \dot{e} &= f_2(e) \end{cases} \tag{11}$$

where both $\dot{x} = f_1(x, 0)$ and $\dot{e} = f_2(e)$ are globally asymptotically stable.

Before we state the result on global asymptotic stability of the closed-loop system with output feedback controller MPC, let us first prove the following lemma.

Lemma 1: Assume that A and $(I - LC)A$ are stable, i.e., all the eigenvalues are strictly inside the unit circle. Define

$$\eta(k) = \max_{i \ge k} |GA^{i-k} e(i)|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \Re^{n_G}.$$

Then

$$\begin{aligned}&\sum_{k=1}^{\infty} \sqrt{\sum_{i=0}^{\infty} \xi(k+i \mid k)^T R \xi(k+i \mid k)} < \infty \\ &\sum_{k=1}^{\infty} \sqrt{\eta(k)^T Q \eta(k)} < \infty \\ &\sum_{k=1}^{\infty} \left(\sqrt{\sum_{i=0}^{\infty} \xi(k+i \mid k)^T R \xi(k+i \mid k)} + \sqrt{\eta(k)^T Q \eta(k)} \right)^2 < \infty \end{aligned} \tag{12}$$

where $0 < R, Q < \infty$.

Proof: From (7) and (9), we have

$$|e(k)|_2 \leq c_1 k^{\alpha_1 - 1}\rho_1^k |e(0)|_2$$

and

$$\begin{aligned}|\xi(k+i \mid k)|_2 &\leq c_2 i^{\alpha_2 - 1}\rho_2^i |\xi(k \mid k)|_2\\ &\leq c_1 c_2 i^{\alpha_2-1} k^{\alpha_1 - 1}\rho_2^i \rho_1^k |e(0)|_2\end{aligned}$$

where $\rho_1 = \lambda_{\max}((I - LC)A)$ and $\rho_2 = \lambda_{\max}(A)$; c_1 and c_2 are constant; α_1 and α_2 are the multiplicies associated with the largest eigenvalues[2] of $(I - LC)A$ and A, respectively. Here $\lambda_{\max}(A)$ denotes the spectral radius of A. Stability of A and $(I - LC)A$ implies that $\rho_1, \rho_2 < 1$. Thus

$$\begin{aligned}&\sum_{k=1}^{\infty}\sqrt{\sum_{i=0}^{\infty}\xi(k+i \mid k)^T R \xi(k+i \mid k)}\\ &\quad\leq c_1 c_2 |e(0)|_2 \bar{\sigma}(R^{\frac{1}{2}})\sqrt{\sum_{i=0}^{\infty} i^{2(\alpha_2 - 1)}\rho_2^{2i}}\sum_{k=1}^{\infty} k^{\alpha_1 - 1}\rho_1^k < \infty.\end{aligned}$$

The other two expressions can be proved similarly. □

Remark 6: If A is unstable or has poles on the unit circle, Lemma 1 clearly does not hold.

The following theorem states that global asymptotic stability with output feedback can be guaranteed for stable systems.

Theorem 4: Assume that A and $(I - LC)A$ are stable, i.e., all eigenvalues of A and $(I - LC)A$ are strictly inside the unit circle. Then the overall system with output feedback controller MPC is globally asymptotically stable.

Proof: Denote the weighted two-norm $\sqrt{x^T R x}$ by $|x|_R$. Let

$$\begin{cases} u^*(k+i \mid k+1) = u(k+i \mid k) & i = 1, 2, \cdots, m\\ \epsilon^*(k+1) = \epsilon(k) + \eta(k)\end{cases}$$

where $\eta(k)$ is as defined in Lemma 1. Thus, (u^*, ϵ^*) is a feasible solution but may not be optimal. Define

$$\begin{aligned}U &= \sum_{i=1}^{m}\left[|u(k+i \mid k)|_S^2 + |\Delta u(k+i \mid k)|_P^2\right]\\ V(k) &= |\hat{x}(k+1 \mid k)|_R^2 + |u(k)|_S^2 + |\Delta u(k)|_P^2.\end{aligned}$$

We have

$$\begin{aligned}\hat{J}_{k+1} &\leq \sum_{i=2}^{\infty}|\hat{x}(k+i \mid k+1)|_R^2 + U + |\epsilon^*(k+1)|_Q^2\\ &= \sum_{i=2}^{\infty}|\hat{x}(k+i \mid k) + \xi(k+i \mid k+1)|_R^2 + U\\ &\qquad + |\epsilon(k) + \eta(k)|_Q^2\\ &\leq \left(\sqrt{\sum_{i=2}^{\infty}|\hat{x}(k+i \mid k)|_R^2 + U + |\epsilon(k)|_Q^2}\right.\\ &\qquad \left. + \sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)^2\\ &= \left(\sqrt{\hat{J}_k - V(k)}\right.\\ &\qquad \left. + \sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)^2.\end{aligned}$$

[2] The largest eigenvalue is defined to be the eigenvalue with the largest absolute value.

Taking square root of both sides yields

$$\begin{aligned}\sqrt{\hat{J}_{k+1}} &\leq \sqrt{\hat{J}_k - V(k)}\\ &\qquad + \sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\\ &\leq \sqrt{\hat{J}_k} + \sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\end{aligned}$$

which in turn yields

$$\sqrt{\hat{J}_{k+1}} \leq \sqrt{\hat{J}_1} + \sum_{j=1}^{k}\left[\sqrt{\sum_{i=2}^{\infty}|\xi(j+i \mid j+1)|_R^2 + |\eta(j)|_Q}\right].$$

By Lemma 1, the second term on the right-hand side is bounded for all k. Therefore, we have

$$\hat{J}_k \leq J^{\max} < \infty \quad \forall\, k > 0.$$

From before, we have

$$\begin{aligned}\hat{J}_{k+1} &\leq \left(\sqrt{\hat{J}_k - V(k)} + \sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)^2\\ &= \hat{J}_k - V(k) + \left(\sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)^2\\ &\quad + 2\sqrt{\hat{J}_k - V(k)}\left(\sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)\\ &\leq \hat{J}_k - V(k)\\ &\quad + \left(\sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)^2\\ &\quad + 2\sqrt{J^{\max}}\left(\sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)\end{aligned}$$

which yields

$$\begin{aligned}&\hat{J}_{k+1} + \sum_{i=1}^{k}V(i) \leq \hat{J}_1\\ &\quad + \sum_{i=1}^{k}\left[\left(\sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)^2\right.\\ &\quad \left. + 2\sqrt{J^{\max}}\left(\sqrt{\sum_{i=2}^{\infty}|\xi(k+i \mid k+1)|_R^2 + |\eta(k)|_Q}\right)\right].\end{aligned}$$

By Lemma 1 and boundness of $J^{\max}$, the second term is bounded for all k. Thus

$$\begin{aligned}\hat{J}_{k+1} + \sum_{i=1}^{k}V(i) = \hat{J}_{k+1} + \sum_{i=1}^{k}&\left[|\hat{x}(i+1 \mid i)|_R^2\right.\\ &\left. + |u(i)|_S^2 + |\Delta u(i)|_P^2\right] < \infty.\end{aligned}$$

Following a similar argument as in the proof of Theorem 1, we can therefore conclude that $x(k) \to 0$ and $u(k) \to 0$ as $k \to \infty$. □

The following theorem shows that the output constraints over the infinite horizon can be replaced by the output constraints over the finite horizon. A similar result was derived by Rawlings and Muske [2].

Theorem 5: Assume that A is stable. Given any $\hat{x}(k \mid k)$ and $\epsilon(k) \geq 0$, there exists a finite N such that

$$G\hat{x}(k+i \mid k) \leq g + \epsilon(k) \quad \forall\, i \geq N.$$

Proof: We need only prove this theorem for $\epsilon(k) = 0$: since $\epsilon(k) \geq 0\ \forall\, k$, $G\hat{x}(k+i \mid k) \leq g\ \forall\, i \geq N$ implies $G\hat{x}(k+i \mid k) \leq g + \epsilon(k)\ \forall\, i \geq N$. WLOG, assume that A is nonsingular.[3] Consider a zero input, i.e., $u(k+i \mid k) = 0, i = 0, \cdots, m-1$, and denote the value of the objective function for this input sequence by $\hat{J}_k^*$. Then

$$\begin{aligned}\hat{J}_k \leq \hat{J}_k^* &= \hat{x}(k \mid k)^T \sum_{i=1}^{\infty} (A^T)^i R A^i \hat{x}(k \mid k)\\ &\equiv \hat{x}(k \mid k)^T \Pi \hat{x}(k \mid k)\end{aligned}$$

where Π is positive definite and bounded since A is nonsingular and stable. Also we have

$$\begin{aligned}\hat{J}_k &= \sum_{i=1}^{\infty} \hat{x}(k+i \mid k)^T R \hat{x}(k+i \mid k)\\ &\quad + \sum_{i=0}^{m-1} \Big[u(k+i \mid k)^T S u(k+i \mid k)\\ &\quad + \Delta u(k+i \mid k)^T P \Delta u(k+i \mid k)\Big] + \epsilon(k)^T Q \epsilon(k)\\ &\geq \sum_{i=m}^{\infty} \hat{x}(k+i \mid k)^T R \hat{x}(k+i \mid k)\\ &= \hat{x}(k+m \mid k)^T \Pi \hat{x}(k+m \mid k).\end{aligned}$$

Combining these two inequalities, we obtain

$$\hat{x}(k+m \mid k)^T \Pi \hat{x}(k+m \mid k) \leq \hat{x}(k \mid k)^T \Pi \hat{x}(k \mid k)$$

which yields

$$|\hat{x}(k+m \mid k)|_2 \leq \kappa(\Pi) |\hat{x}(k \mid k)|_2$$

where $\kappa(\Pi) < \infty$ denotes the condition number of Π. Finally

$$\begin{aligned}|G\hat{x}(k+m+N \mid k)|_\infty &= |GA^N \hat{x}(k+m \mid k)|_\infty\\ &\leq |GA^N \hat{x}(k+m \mid k)|_2\\ &\leq \bar{\sigma}(G)\bar{\sigma}(A^N)|\hat{x}(k+m \mid k)|_2\\ &\leq \bar{\sigma}(G)\bar{\sigma}(A^N)\kappa(\Pi)|\hat{x}(k \mid k)|_2\end{aligned}$$

where $\bar{\sigma}(G)$ denotes the largest singular value of G. If N is such that

$$\bar{\sigma}(G)\bar{\sigma}(A^{N+i})\kappa(\Pi)|\hat{x}(k \mid k)|_2 \leq \min_j(g_j) \quad \forall\, i \geq 0$$

then

$$G\hat{x}(k+i \mid k) \leq g \quad \forall\, i \geq N+m$$

$\min_j(g_j) > 0$ and stability of A imply that a finite N exists.

Fig. 2 shows responses with output feedback. The initial state estimate is $\hat{x}(0) = [0 \quad 0]^T$ and the observer gain is $L = [0.1 \quad 1]^T$.

[3] If A is singular, we can write $A = T^{-1}\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}T$ where $\Sigma_1 > 0$ and Σ_2 is nilpotent. Define $\tilde{x}(k) = Tx(k)$ and $\begin{bmatrix}\tilde{x}_1(k+1)\\ \tilde{x}_2(k+1)\end{bmatrix} = \begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1(k)\\ \tilde{x}_2(k)\end{bmatrix}$. Then, after a finite number of sampling times, $\tilde{x}_2$ becomes identically zero since Σ_2 is nilpotent. Thus it suffices to consider the reduced system with $\tilde{x}_1$ as its states.

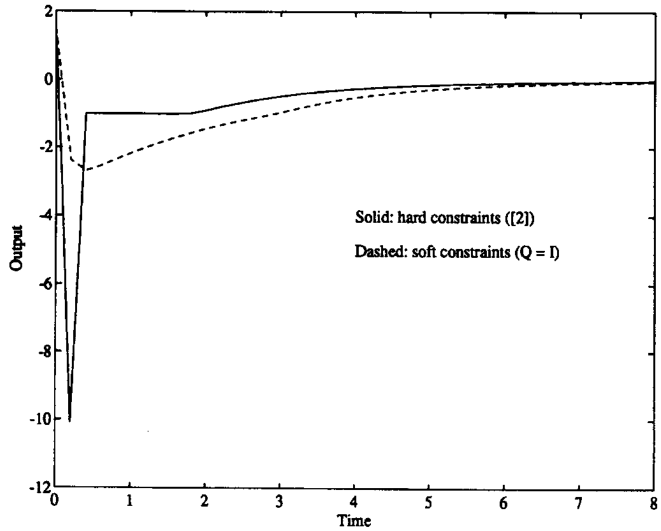

Fig. 1. Comparison of responses for the two approaches.

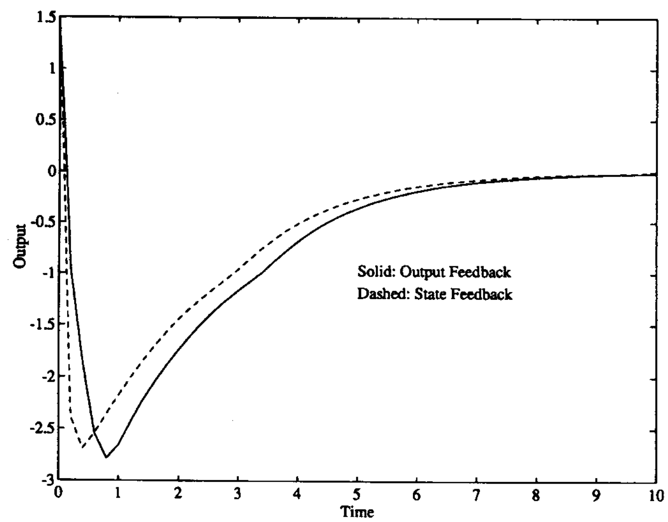

Fig. 2. Output feedback responses.

IV. Example

Consider the system

$$\begin{aligned}x(k+1) &= \begin{bmatrix}0.655 & -0.1673\\ 0.1673 & 0.9825\end{bmatrix}x(k) + \begin{bmatrix}0.1637\\ 0.0175\end{bmatrix}u(k)\\ y(k) &= [-2 \quad 1]x(k)\end{aligned} \tag{13}$$

which is obtained from the continuous-time transfer function $\frac{-2s+1}{(s+1)^2}$ with a sampling time of 0.2. The initial condition is $x(0) = [1.5 \quad 1.5]^T$. The output is constrained between ± 1. Since the system exhibits inverse response behavior, hard output constraints can cause stability problems [8]. To use the approach proposed in [2], the output constraint at the first sampling time must be ignored to make the optimization problem feasible. We can also use the approach presented in this note and soften the output constraints over the infinite horizon. The following parameter values are used

$$m = 5, \quad R = \begin{bmatrix}5 & -2\\ -2 & 2\end{bmatrix}, \quad S = 0.1I, \quad P = 0, \quad Q = I$$

where I is the identity matrix. Using the arguments leading to Theorem 5, one can show that the state constraints will be satisfied automatically after 35 time steps. Thus, the output constraints need only be enforced over a finite horizon of length 35. The responses for the two approaches are depicted in Fig. 1. A very large overshoot is observed for the controller designed via the approach proposed in [2] but the output comes within the constraints faster.

V. Conclusion

We analyzed the closed-loop stability for an infinite horizon MPC algorithm with soft and hard constraints. We showed that global stability can be guaranteed for both state feedback and output feedback. The on-line optimization problem can be cast as a finite dimensional quadratic program.

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A New Approach for Computing the State Feedback Gains of Multivariable Systems

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***Abstract*— This note presents some new results in linear systems. First, the relationship between the polynomial matrix description and the state-space representation of multivariable systems is clarified. Then, we show that once such a relationship is determined, the coprime matrix fraction description can be easily computed. And we can further develop a closed-form formula to solve the pole-assignment problem of multivariable systems. Such formula can be thought of as an extension of the Ackermann's formula for multi-input/multi-output (MIMO) systems. Thus this note potentially gives us a clearer insight into linear systems from the theoretical viewpoint.**

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I. Introduction

Analysis and synthesis based on the matrix fraction description (MFD) of multivariable systems have received a great deal of attention for the past few decades [1]–[4]. Accordingly, many numerical algorithms have been proposed to compute the coprime MFD and the corresponding Bezout identity solutions [5]–[7]. In these existing algorithms, one requires a state-space representation of multi-input/multi-output (MIMO) systems and a suitable choice of feedback and observer gains. There are also several numerical algorithms to obtain the required feedback gains of MIMO systems. Pole assignment in linear systems is also a topic of considerable interest in control literature [8]–[11].

This note presents a new approach to compute the coprime MFD and the state feedback gains of MIMO systems. Some new results in linear systems are proposed here. First, we explore the relationship between the polynomial matrix description (PMD) and the state-space representation of MIMO systems. Then, we show that once such relationship is determined, the right coprime MFD can be computed easily. Furthermore, we can further develop a closed-form formula to solve the pole-assignment problem of MIMO systems. Such a formula can be thought of as an extension of the Ackermann's formula for MIMO systems. From the proposed formula, the freedom to select the desired state feedback gains is clarified.

This note is organized as follows. In Section II, some results in linear system theory are revisited. Section III presents a new method to compute the right coprime MFD, and Section IV proposes a closed-form formula to determine the MIMO state feedback gains. In Section V, an example for illustration is given, and in Section VI, the conclusions are given.

II. Preliminaries

The plant considered in this note is a controllable and observable MIMO system

$$\dot{X}(t) = AX(t) + BU(t) \tag{2.1a}$$

$$Y(t) = CX(t) \tag{2.1b}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$, and $C \in \Re^{q\times n}$ are all real constant matrices. $X(t)$, $U(t)$, and $Y(t)$ are the state vector, the control input, and the measured output, respectively. The transfer-function matrix of system (2.1) can be given in a right coprime MFD

$$G(s) = C(sI - A)^{-1}B = N(s)D^{-1}(s) \tag{2.2}$$

where $N(s)$ and $D(s)$ are polynomial matrices of dimensions $q \times p$ and $p \times p$, respectively.

Let $B = [b_1 \quad b_2 \quad \cdots \quad b_p]$ be of full rank. Since the system in (2.1) is controllable, we can obtain a nonsingular controllability matrix M which is expressed as

$$M = \left[b_1\, Ab_1\, \cdots A^{m_1-1}b_1 \middle| b_2\, Ab_2 \cdots\, A^{m_2-1}b_2 \middle| \cdots \middle| b_p\, Ab_p\, \cdots\, A^{m_p-1}b_p\right] \tag{2.3a}$$

with

$$\sum_{j=1}^{p} m_j = n. \tag{2.3b}$$

Note that there are n linearly independent column vectors in the controllability matrix M which are the first n linearly independent column vectors in the sequence of vectors $b_1, b_2, \cdots, b_p$; $Ab_1, Ab_2, \cdots, Ab_p$; $A^2b_1, A^2b_2, \cdots$; and so forth. The set $\{m_1, m_2, \cdots, m_p\}$ is called the controllability indexes of $\{A, B\}$.