Reduced Explicit Constrained Linear Quadratic Regulators

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***Abstract*—In this note, it is studied how structural properties of certain linear systems can be exploited to derive reduced dimension multiparametric quadratic programs that lead to explicit piecewise linear feedback solutions to the state and input constrained linear quadratic regulation problem. The reduced dimensionality typically results in suboptimal controllers of lower complexity, with associated computational advantages in the online implementation. At the heart of the methods are state-space projections based on the singular value decomposition.**

***Index Terms*—Constrained control, piecewise linear control, singular value decomposition.**

I. Introduction

We consider constrained linear quadratic regulators (LQRs) [1], [2]. Recently, explicit solutions in terms of piecewise linear (PWL) state feedbacks have been investigated [3]–[6]. In particular, numerical algorithms for multiparametric quadratic programming (mp-QP) has opened for the efficient and exact design of such PWL state feedback laws defined on polyhedral partitions of the state–space. This allows the conventional, but resource demanding, real-time optimization approach [1], [2] to be replaced by a simple PWL function evaluation, at least for problems of moderate complexity. However, the complexity of the polyhedral partition tend to increase rapidly with the number of constraints, and the dimension of the state vector. This has led to approximate algorithms for solving mp-QP problems being investigated, [7], [8], with significant reduction in complexity. Moreover, it has led to the investigation of efficient implementation of piecewise linear function evaluation [9]–[11] as well as input trajectory parameterization [10] and restrictions on the active constraint switching [12] in order to reduce the complexity.

In this note, we take on a different approach, which can be used in combination with any of the approaches previously mentioned. It is based on the idea that certain structural properties of linear systems may be exploited in order to define an approximate mp-QP problem on a subspace of the state (parameter) space. This results in a suboptimal PWL state feedback defined on a lower dimensional space, combined with a full linear state feedback. The benefit of this is that the mp-QP of reduced dimension typically requires less computer processing and memory, both offline and online. Two methods are suggested. The first method is useful only for systems where the constrained states are separated from the inputs by relatively few integrators. The resulting suboptimal control is shown to be stabilizing under some conditions on the error being introduced when the cost function is redefined on a lower dimensional space. The second method defines a lower dimensional approximate mp-QP by relaxing the constraints by allowing small violation. The resulting suboptimal control is shown to be stabilizing if the constraint relaxation is small, and is also proved to be of lower complexity.

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II. Explicit Constrained Linear Quadratic Regulator

Formulating the constrained LQR problem as an mp-QP is briefly described here; see [4]. Consider the linear system

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input. Define the infinite-horizon cost

$$J_\infty (u_t, u_{t+1}, \ldots, x(t)) = \sum_{k=0}^{\infty} \left(x_{t+k|t}^T Q x_{t+k|t} + u_{t+k}^T R u_{t+k} \right) \tag{2}$$

with predictions $x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k}$, output $y_{t+k|t} = Cx_{t+k|t}$ and $x_{t|t} = x(t)$. We assume symmetric Q, $R \succ 0$ (positive definite) and (A, B) is controllable. Introducing state and input constraints on the first N samples leads to the following constrained optimization problem:

$$V^*(x(t)) = \min_U J(U, x(t)) \tag{3}$$

$$\text{subject to} \quad y_{\min} \le y_{t+k|t} \le y_{\max}$$
$$u_{\min} \le u_{t+k-1} \le u_{\max}, \; k = 1, 2, \ldots, N \tag{4}$$

with $U = (u_t, \ldots, u_{t+N-1})$ and the cost function given by

$$J(U, x(t)) = \sum_{k=0}^{N-1} \left(x_{t+k|t}^T Q x_{t+k|t} + u_{t+k}^T R u_{t+k} \right) + x_{t+N|t}^T P x_{t+N|t}. \tag{5}$$

It is assumed that $y_{\min} < 0 < y_{\max}$, $u_{\min} < 0 < u_{\max}$ such that the origin is in the interior of the admissible set. The symmetric final cost matrix $P \succ 0$ is taken as the solution of the algebraic Riccati equation. With the assumption that no constraints are active for $k \ge N$ (see [1] and [2]) this finite-horizon problem is equivalent to minimizing the infinite-horizon LQ criterion (2). With proper definitions of the matrices Y, H, F, G, W, and E (see [4] and [12]), this and similar problems can be reformulated as follows. Minimize with respect to U

$$J(U, x) = \frac{1}{2} U^T H U + x^T F U + \frac{1}{2} x^T Y x \tag{6}$$

$$\text{subject to} \quad GU \le W + Ex. \tag{7}$$

It is shown in [4] that $H \succ 0$ due to $R \succ 0$, such that this problem is strictly convex. Completing squares in (6)–(7), the dependence on x is moved from the cost to the constraints, such that the problem is equivalent to the following problem (similar to the closed-loop prediction formulation suggested in [13])

$$V_z^*(x) = \min_z \frac{1}{2} z^T H z \tag{8}$$

$$\text{subject to} \quad Gz \le W + Sx \tag{9}$$

where $z = U + H^{-1}F^T x$ and $S = E + GH^{-1}F^T$. The unconstrained LQ optimal control is denoted $U_{LQ}(t) = -K_{LQ}x(t)$ where $K_{LQ} = H^{-1}F^T$ is an extended LQ gain matrix. The m first elements of $U_{LQ}(t)$ are denoted $u_{LQ}(t)$, and the corresponding m first rows of K_{LQ} are denoted k_{LQ}, the usual LQ gain matrix. Equations (8)–(9) define a strictly convex mp-QP in z parameterized by $x \in X$, where X is a given closed polyhedral set. This mp-QP can be solved explicitly using the algorithms described in [4] and [6], which give the solution $z^*(x)$ as an explicit PWL function of $x \in X$ with the following properties [4].

Theorem 1: Consider the mp-QP (8)–(9) with $H \succ 0$. The solution $z^*(x)$ (and $U^*(x) = z^*(x) - H^{-1}F^T x$) is a continuous PWL function, and $V_z^*(x)$ (and $V^*(x) = V_z^*(x) + x^T(Y - FH^{-1}F^T)x$) is a convex piecewise quadratic function. □

The complexity of solving the mp-QP and implementing the resulting PWL state feedback increases very rapidly with the number

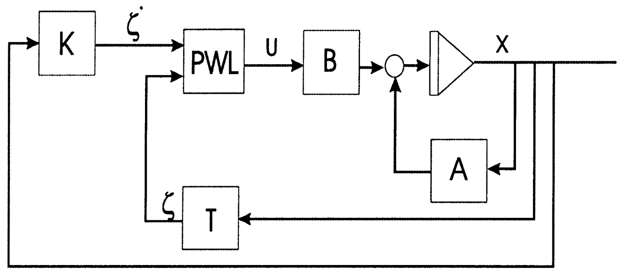

Fig. 1. Feedback structure I.

of constraints and the dimension of the state–space. In this note, we suggest some methods for reducing complexity, where we essentially replace the linear terms Ex and $F^T x$ in (6)–(7) or Sx in (9) with approximate linear terms defined on a subspace of the state space. This leads to new (suboptimal) mp-QPs defined on a lower dimensional parameter space, which usually has computational advantages.

III. Feedback Structure I

Consider the feedback structure in Fig. 1. It contains an inner PWL feedback loop, to be designed by solving an mp-QP, and a linear outer feedback loop, to be designed to achieve local LQ optimality. The idea is that the inner PWL loop relies on feedback from a reduced state $\zeta = Tx$, where the projection matrix $T \in \mathbb{R}^{p\times n}$, with $2p < n$, is chosen such that it contains the necessary information to guarantee close-to-optimal control of ζ to its specified setpoint ζ^*, while fulfilling all constraints. This amounts to solving an mp-QP with $2p$ parameters.

Lemma 1: Constraints (7) are equivalent to

$$GU \leq W + E_0\zeta \tag{10}$$

where $\zeta = Tx$ is defined by the projection matrix $T = V_0^T$, and $E_0 = U_0\Sigma_0$, where U_0, V_0, Σ_0 are the submatrices of the singular value decomposition (SVD) $E = U\Sigma V^T$ corresponding to nonzero singular values.

Proof: $Ex = U\Sigma V^T x = U_0\Sigma_0 V_0^T x = E_0 Tx = E_0\zeta$ [14]. □

Theorem 2: The row rank of the observability matrix $W_o = (C^T, (CA)^T, \ldots, (CA^{n-1})^T)^T$ of the system (A, C) is an upper bound on the number of nonzero singular values of E, such that $p = \dim(\zeta) = \text{rank}(E) \leq \text{rank}(W_o)$. For $N \geq n$, $p = \text{rank}(W_o)$.

Proof: For input constraints, the corresponding rows of E are zero. For a generic output constraint $y_{\min} \leq y(t+k) \leq y_{\max}$ the corresponding block of E is CA^k, see e.g. [12]. Hence, E can be written as

$$E = \begin{pmatrix} 0_{2Nm\times n} \\ W_N \\ -W_N \end{pmatrix} \tag{11}$$

where the first block corresponds to input constraints and the two last blocks corresponds to the output constraints, with W_N being the Krylov matrix $W_N = (C^T, (CA)^T, (CA^2)^T, \ldots, (CA^N)^T)^T$. For $N \geq n$, the row rank of W_N equals the row rank of W_o due to Cayley–Hamiltons theorem. The row rank of E equals the row rank of W_N, from (11), and the result follows by Lemma 1. □

Example: A laboratory model helicopter (Quanser 3-DOF Helicopter) is sampled with interval $T = 0.01\ s$, and the following state-space representation is obtained:

$$A = \begin{pmatrix} 1 & 0 & 0.01 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.01 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.01 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0000 & 0.0000 \\ 0.0001 & -0.0001 \\ 0.0019 & 0.0019 \\ 0.0132 & -0.0132 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The states of the system are x_1-elevation, x_2-pitch angle, x_3-elevation rate, x_4-pitch angle rate, x_5-integral of elevation error, and x_6-integral of pitch angle error. The inputs to the system are u_1-front rotor voltage and u_2-rear rotor voltage. Assume the state is to be regulated to the origin with the following constraints on the inputs and pitch and elevation rates $-1 \leq u_1 \leq 3$, $-1 \leq u_2 \leq 3$, $-0.44 \leq y_1 \leq 0.44$, and $-0.6 \leq y_2 \leq 0.6$ with

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We assume the horizon $N = 50$ and the input trajectory is a piecewise constant function of time parameterized by 3 parameters per input as in [10]. For this sixth-order system the observability matrix of (A, C) and E both have rank two, since there is one integrator between the inputs and each of the two constrained states in this cascaded system. Hence, $p = m = 2$ and the dimension of the mp-QP parameter space is reduced from $n = 6$ to $2p = 4$. Since $T = V_0^T = C$, the resulting cascaded control structure has a simple interpretation. The inner loop controls the pitch and elevation rates to their reference values, subject to the constraints. The outer loop is a linear position feedback with integral action. □

These results suggest that for the purpose of fulfilling the constraints it is sufficient to use information only about those modes of the system that are observable from the output $y = Cx$, which are the constrained modes. This usually leads to a complexity reduction since certain modes can be neglected. The neglected modes might otherwise have contributed with additional optimal combinations of active constraints that would have lead to additional polyhedral regions in the PWL mp-QP solution. Neglecting these modes does instead lead to suboptimality, because it is necessary to change the cost function such that it does not depend on the neglected modes. Obviously, the reformulation (10) makes sense only if it is possible to find a projection matrix with $p < n/2$, since otherwise there will be no reduction in the dimension of the parameter space. The approach is also meaningless if there are only input constraints or if $p < m$. On the other hand, it was suggested by the example that the idea is useful when the system possesses some structural properties, such as a cascade where all constrained states are "close to the inputs" in the sense that there are relatively few integrators between the inputs and the constrained states. The suggested feedback structure may also be useful in an approximate setting. In this case, p will equal the number of singular values of E that are significantly larger than zero.

In order to design the feedback laws, we introduce the similarity transform

$$\begin{pmatrix} \zeta \\ \varrho \end{pmatrix} = V^T x \tag{12}$$

where the vector ζ contains the p modes that are observable through the constrained states y, and ϱ the $n-p$ modes that are not. Hence, the following projections hold: $\zeta = V_0^T x$, $\varrho = V_1^T x$ with $V_0 \in \mathbb{R}^{n\times p}$ and $V_1 \in \mathbb{R}^{n\times(n-p)}$. Since V is orthogonal, the inverse transform is given by $x = V_0\zeta + V_1\varrho$. We define the following projected matrices $F_0 = V_0^T F$, and $F_1 = V_1^T F$. We are then in position to reformulate the cost function (6) into the form that reflects the objective of regulating $\zeta(t)$ to some setpoint ζ^*

$$J\left(U, \zeta(t), \varrho(t)\right) = \frac{1}{2}U^T H U + \left(\zeta(t) - \zeta^*\right)^T F_0 U + \frac{1}{2}x^T Y x + \zeta^{*T} F_0 U + \varrho^T(t) F_1 U. \tag{13}$$

We have introduced the new variable ζ^*, whose value does not influence the value of J. A suboptimal strategy is developed by isolating the two first terms into the optimization criterion

$$J_0\left(U, \zeta(t), \zeta^*\right) = \frac{1}{2}U^T H U + \left(\zeta(t) - \zeta^*\right)^T F_0 U \tag{14}$$

$$\text{subject to} \quad GU \le W + E_0\zeta(t). \tag{15}$$

Assuming $2p < n$, (14)–(15) define a reduced-dimension mp-QP on a $2p$-dimensional subspace of the state–space, and from the previous results it is guaranteed that for any ζ^* the original constraints (7) are fulfilled. When solving the mp-QP (14)–(15) a set $\Upsilon \times \Upsilon^*$ of possible (ζ, ζ^*) must be specified. Polyhedral Υ and Υ^* can be specified by projections of the polyhedral set X: $\Upsilon = \{\zeta \in \mathbb{R}^p | \zeta = Tx,\ x \in X\}$, $\Upsilon^* = \{\zeta^* \in \mathbb{R}^p | \zeta^* = Kx,\ x \in X\}$. Let the solution to (14)–(15) on $\Upsilon \times \Upsilon^*$ be denoted $U_0^*(\zeta, \zeta^*)$ and its first m elements $u_0^*(\zeta, \zeta^*)$. The receding horizon control is then given by

$$u(t) = u_0^*\left(\zeta(t), \zeta^*(t)\right). \tag{16}$$

The variable ζ^* is viewed as a reference signal to the inner loop, see Fig. 1. Since the constraints are guaranteed to be fulfilled with the PWL inner feedback loop described above, we restrict our attention to a (suboptimal) linear outer loop that determines $\zeta^* = Kx$. Let the gain matrix of the reduced-dimension unconstrained LQ design be denoted $k_0 \in \mathbb{R}^{m\times p}$ and given by the m first rows of the matrix $K_0 = H^{-1}F_0^T = K_{LQ}V_0$. Hence, $u = -k_0(\zeta - \zeta^*)$ coincides with the solution $u_0^*(\zeta, \zeta^*)$ of (14)–(15) in a neighborhood of the origin. Local LQ optimality follows if $K \in \mathbb{R}^{p\times n}$ is appropriately chosen.

Theorem 3: If $p \ge m$ and $\mathrm{rank}(k_0) = m$, there exists a gain matrix K solving the system of linear equations

$$k_0 K = k_0 T - k_{LQ} \tag{17}$$

and (1) in closed loop with (16) and $\zeta^*(t) = Kx(t)$ is locally (unconstrained) LQ optimal, with respect to (2).

Proof: Notice that (17) defines mn linear equations with pn unknowns, and recall that $p \ge m$.

$$\begin{pmatrix} k_0 & 0 & & 0 \\ 0 & k_0 & & 0 \\ & & \ddots & \\ 0 & 0 & & k_0 \end{pmatrix} \begin{pmatrix} K^1 \\ K^2 \\ \vdots \\ K^n \end{pmatrix} = \begin{pmatrix} (k_0T - k_{LQ})^1 \\ (k_0T - k_{LQ})^2 \\ \vdots \\ (k_0T - k_{LQ})^n \end{pmatrix}. \tag{18}$$

The superscript index denotes the corresponding column of a matrix. Due to $\mathrm{rank}(k_0) = m$ the matrix to the left has full-row rank, and there exists a K solving (18). There also exists a positively invariant set containing the origin where the optimal control $u(t)$ has no active constraints [1], and the closed-loop dynamics are given by

$$\begin{aligned} x(t+1) &= (A - Bk_0T)x(t) + Bk_0\zeta^*(t) \\ &= \left(A - B(k_0T - k_0K)\right)x(t) \\ &= (A - Bk_{LQ})x(t) \end{aligned} \tag{19}$$

and the result follows due to LQ optimality of (19). □

If $p = m$ the system of linear equations (18) has a unique solution, while there may be several solutions for $p > m$. One may then take the solution given by the Moore–Penrose pseudoinverse, [14]. The condition $\mathrm{rank}(k_0) = m$ is not restrictive since $k_0 = k_{LQ}V_0 = -(R + B^TPB)^{-1}B^TPAV_0$. It is sufficient with $\mathrm{rank}(B) = m$ and $\mathrm{rank}(A) = n$, which, in general, holds if there are no redundant inputs and (A, B) is the discretization of a continuous-time system.

Theorem 3 implies local asymptotic stability of the closed loop as a direct consequence of local LQ optimality. It is of interest to investigate nonlocal asymptotic stability and quantify the degree of suboptimality. These topics are closely interrelated and essentially depend on the cost function error that results from replacing $F^Tx = F_0^T\zeta + F_1^T\varrho$ with $F_0^T\zeta$. Define the optimal cost function of the reduced-dimension problem

$$V_0^*(x) = J_0\left(U_0^*(Tx, Kx), Tx, Kx\right) + \frac{1}{2}x^TYx \tag{20}$$

and its suboptimal cost

$$\hat{V}(x) = J\left(U_0^*(Tx, Kx), x\right). \tag{21}$$

Theorem 4: If $\zeta^* = Kx$, where K satisfies (17), then $0 \le \hat{V}(x) - V^*(x) \le \Delta(x)$ for all $x \in X$, with $\Delta(x) = x^T(V_1F_1 + K^TF_0)(U_0^*(Tx, Kx) - U^*(x))$.

Proof: The lower bound is due to feasibility and suboptimality of $U_0^*(Tx, Kx)$ in (21). Since $(\zeta^{*T}F_0 + \varrho^TF_1)U = x^T(V_1F_1 + K^TF_0)U$ and

$$J(U, \zeta, \varrho) = J_0(U, \zeta, \zeta^*) + x^T(V_1F_1 + K^TF_0)U + \frac{1}{2}x^TYx \tag{22}$$

we have

$$\hat{V}(x) = V_0^*(x) + x^T(V_1F_1 + K^TF_0)U_0^*(Tx, Kx) \tag{23}$$

$$V_0^*(x) = \min_U\left(J\left(U, Tx, V_1^Tx\right) - x^T(V_1F_1 + K^TF_0)U\right)$$
$$\text{subject to } GU \le W + E_0Tx. \tag{24}$$

Due to feasibility and suboptimality of $U^*(x)$, (24) gives

$$V_0^*(x) \le V^*(x) - x^T(V_1F_1 + K^TF_0)U^*(x). \tag{25}$$

Combining (23) and (25) gives the upper bound. □

Let $\mathbb{X}$ be the set of stabilizable initial states, i.e. those $x(t)$ for which there exists a U such that $GU \le W + E_0Tx(t)$ and $J_\infty(U, -k_{LQ}x_{t+N|t}, -k_{LQ}x_{t+N+1|t}, \ldots, x(t))$ is finite.

Theorem 5: Suppose $\mathbb{X}$ is compact, N is sufficiently large, the largest singular value $\overline{\sigma}(F_1)$ is sufficiently small, and $\zeta^* = Kx$ where K satisfies (17). Then, for all $x(0) \in \mathbb{X}$ the origin is an asymptotically stable equilibrium point for (1) in closed loop with (16).

Proof: The proof is similar to [7] and [15]. Let Ω be the maximal admissible set for the system $x(t+1) = (A - Bk_{LQ})x(t)$ with the constraint set $\overline{X} = \{x \in \mathbb{R}^n | y_{\min} \le Cx \le y_{\max},\ u_{\min} \le -k_{LQ}x \le u_{\max}\}$, as defined in [1] and [16]. Such a set with nonempty interior exists because $\overline{X}$ contains the origin in its interior and $Q \succ 0$. Since N is sufficiently large, the compactness of $\mathbb{X}$ implies that there exists a feasible $U_0^*(Tx(t), Kx(t))$ such that $x_{t+N|t} \in \Omega$, [1]. Because Ω is positively invariant [16], there exists a feasible U at time $t+1$ and from standard arguments

$$\begin{aligned} V^*(x(t+1)) - V^*(x(t)) &\le \hat{V}(x(t+1)) - V^*(x(t)) \\ &= \hat{V}(x(t)) - V^*(x(t)) \\ &\quad - x^T(t)Qx(t) - u^T(t)Ru(t) \\ &\le \Delta(x(t)) - x^T(t)Qx(t). \end{aligned} \tag{26}$$

From (17) and $K_0T - K_{LQ} = H^{-1}F_0^TV_0^T - H^{-1}(F_0^TV_0^T + F_1^TV_1^T) = -H^{-1}F_1^TV_1^T$ it follows that $\|K\|_2 \le c\overline{\sigma}(F_1)$ for some

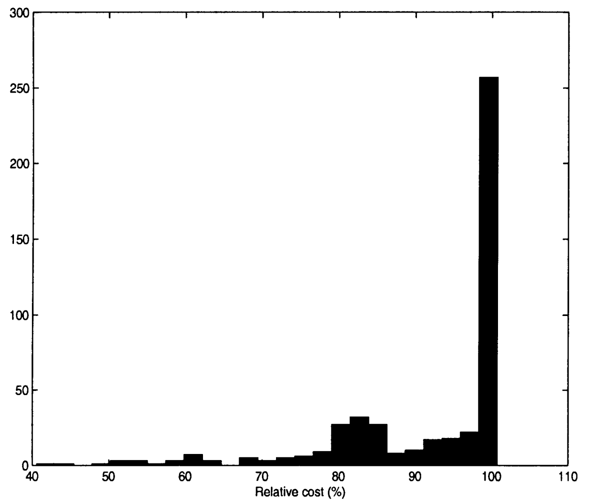

Fig. 2. Results from Monte Carlo simulation.

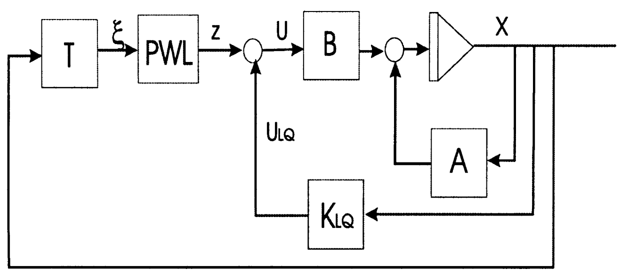

Fig. 3. Feedback structure II.

$c > 0$. Since $\overline{\sigma}(F_1)$ is sufficiently small and $Q \succ 0$, it follows from Theorem 4 that for $x(t) \notin \Omega$, $V^*(x(t+1)) - V^*(x(t)) < 0$. Recall that $\Delta(x(t)) = 0$ for $x(t) \in \Omega$ and Ω is positively invariant such that $V^*(x(t+1)) - V^*(x(t)) \leq -x^T Q x$ for $x \in \Omega$. Hence, the closed loop is asymptotically stable. □

If the method previously suggested leads to unacceptable performance degradation or loss of stability, one may augment the state-space projection T with appropriate rows from the right factor of the SVD of the F matrix, as this will reduce the error made when replacing $F^T x$ by $F_0^T x$ and, hence, improve the performance.

Example, Continued: Let the LQ cost function be given by $Q = \text{diag}(100, 100, 10, 10, 400, 160)$ and $R = I_{2\times 2}$. Recall that $p = m = 2$ and the dimension of the mp-QP parameter space is reduced from $n = 6$ to $2p = 4$. This leads to a reduction in the number of regions in the partition generated by the mp-QP algorithm [6] from 4279 to 1253. Evaluating the resulting PWL functions via binary search trees as suggested in [11], the maximum number of arithmetic operations per sample is reduced from 402 to 188 and the required computer memory is reduced from 814 to 36 kWords. The results of a Monte Carlo simulation over 469 random initial conditions that give admissible trajectories in the set $X = [-0.75, 0.75]^4 \times [-1, 1]^2$ is shown in Fig. 2. The histogram shows the relative cost (100% means that the optimal cost is achieved, and less than 100% indicates suboptimality). Sample curves reported in [17] indicate that the performance degradation is not prohibitive. □

IV. FEEDBACK STRUCTURE II

Consider the feedback structure II shown in Fig. 3. It contains an inner linear feedback that is prestabilizing and LQ-optimal for the unconstrained system, similar to [13], and a PWL outer feedback defined on a subspace of the state space. The outer loop is designed by solving an mp-QP similar to (8)–(9) to modify the unconstrained linear LQR feedback such that the constraints are fulfilled to some tolerance. Using arguments similar to Theorem 2, the number of nonzero singular values of S equals the rank of the observability matrix of the system $(A - Bk_{\text{LQ}}, C)$. We notice that in this case any structural properties of the system (A, B, C) will typically be lost due to the LQ feedback, and only in special cases will the observability matrix of the system $(A - Bk_{\text{LQ}}, C)$ not have full rank. However, one may still exploit projections to derive a reduced-dimension mp-QP if small violation of the constraints are allowed. This is easily achieved by defining a threshold on the singular values of S such that the constraints (9) are equivalently represented as

$$Gz \leq W + S_0\xi + \varepsilon(x) \tag{27}$$

with $\xi = T_0 x$, $T_0 = V_0^T$, $S_0 = U_0\Sigma_0$, where U_0, V_0, and Σ_0 are the submatrices of the SVD $S = U\Sigma V^T$ corresponding to singular values larger than a given threshold $\sigma_0 \geq 0$. Likewise, $\varepsilon(x) = U_1^T \Sigma_1 V_1^T x$, where U_1, V_1, Σ_1 are the submatrices of the SVD corresponding to singular values that are not larger than σ_0. In general $\dim(\xi) \leq \dim(x)$, and a uniform bound on ε follows directly from properties of the SVD [14]:

Lemma 2: Let σ_t be the largest singular value of S that satisfies $\sigma_t \leq \sigma_0$, and assume $X \subset \mathbb{R}^n$ is a compact set. Then $\max_{x\in X} \|\varepsilon(x)\|_2 \leq \sigma_t \max_{x \in X} \|x\|_2$. □

Hence, the term ε in (27) will be uniformly small if the threshold σ_0 is small, and may be neglected if small violations of the constraints are tolerated. This suggests the following reduced-dimension mp-QP, defined on the projection of X onto the subspace spanned by the rows of T_0, $\Xi = \{\xi \mid \xi = T_0 x,\ x \in X\}$

$$V_{z,0}^*(\xi) = \min_z \frac{1}{2} z^T H z \tag{28}$$

$$\text{subject to} \quad Gz \leq W + S_0 \xi. \tag{29}$$

The receding horizon control is chosen according to

$$u(t) = u_{LQ}(t) + z_{0,0}^*(\xi(t)) \tag{30}$$

where $z_{0,0}^*(\xi)$ denotes the m first components of the vector $z_0^*(\xi)$ that solves (28)–(29). When using the SVD, appropriate scaling is important. Essentially, the inequalities should be scaled according to some prioritization of the constraints.

As shown in the following lemma, the solution to the reduced mp-QP equals the solution of the original mp-QP, when restricted to a subspace of the parameter space.

Lemma 3: Define the subspace $\mathbb{L} = \{x \in \mathbb{R}^n \mid V_1^T x = 0\}$. Then, $z^*(x) = z_0^*(T_0 x)$, for all $x \in \mathbb{L}$.

Proof: Follows by inspection of the explicit PWL solutions [4].□

Corollary 1: The number of full-dimensional critical regions defining the PWL solution to the mp-QP (28)–(29) on Ξ is not larger than the number of full-dimensional critical regions defining the PWL solution to the mp-QP (8)–(9) on X.

Proof: The result follows trivially from Lemma 3, as every full-dimensional critical region in the solution to (28)–(29) is also a full-dimensional critical region in the solution to (8)–(9), restricted to the subspace $\mathbb{L}$. □

Corollary 1 shows that the complexity of the solution to the reduced problem is never larger than the complexity of the solution to the original problem. In fact, Lemma 3 strongly indicates that the complexity is typically smaller, since the solution to the original mp-QP (8)–(9) typically contains full-dimensional critical regions that do not intersect $\mathbb{L}$.

Let $\mathbb{X}_0$ be the set of stabilizable initial states for (1) subject to the constraints (29), i.e., all $x(t)$ for which there exists a z such that $Gz \leq W + S_0 T_0 x(t)$ and $J_\infty(z - H^{-1}F^T x(t), -k_{LQ}x_{t+N|t}, -k_{LQ}x_{t+N+1|t}, \ldots, x(t))$ is finite.

Theorem 6: Suppose $X = \mathbb{X}_0$ is compact, N is sufficiently large, and σ_0 sufficiently small. Then for all $x(0) \in X$ the receding horizon control (30) in closed loop with the system (1) makes the origin asymptotically stable.

Proof: Define the perturbed mp-QP

$$V_{z,0}^*(x, \varepsilon) = \min_z \frac{1}{2} z^T H z \quad \text{subject to} \quad Gz \leq W + Sx - \varepsilon \quad (31)$$

with the property $V_z^*(x) = V_{z,0}^*(x, 0)$. Assume without loss of generality that the mp-QP is not degenerate at x (see [4] and [6]) and moreover that x is an internal point of some critical region. Then, [18, Cor. 3.4.4] gives for ε in a neighborhood of the origin

$$\frac{\partial}{\partial \varepsilon} V_{z,0}^*(x, \varepsilon) = \lambda(x). \quad (32)$$

For $\varepsilon(x)$ sufficiently small (due to σ_0 sufficiently small)

$$\begin{aligned} V_{z,0}^*(x, \varepsilon(x)) - V_z^*(x) &= V_{z,0}^*(x, \varepsilon(x)) - V_{z,0}^*(x, 0) \\ &= \int_{\epsilon=0}^{\epsilon=\varepsilon(x)} \frac{\partial}{\partial \epsilon} V_{z,0}^*(x, \epsilon) d\epsilon \\ &= \lambda^T(x)\varepsilon(x). \end{aligned} \quad (33)$$

If x is not an internal point, (32) does not hold. Still, because V_z^* and $V_{z,0}^*$ are continuous functions and (32) fails to hold only on a set of measure zero, we argue that (33) holds for all $x \in X$. Equation (33), thus, provides an upper bound on the suboptimality

$$V_{z,0}^*(x, \varepsilon(x)) \leq V_z^*(x) + \sigma_0 \overline{\lambda} \|x\|_2 \quad (34)$$

(see Lemma 2) and we have defined $\overline{\lambda} = \max_{x \in X} \|\lambda(x)\|_2$ which exists because $\lambda(x)$ is PWL on the compact domain X, [4]. Since σ_0 is sufficiently small, asymptotic stability follows using standard arguments similar to [7] and [15]. □

It may be a requirement that certain constraints are not allowed to be violated. This is often the case for input constraints, which are usually physical limitations rather than operational constraints. In order to fulfill hard input constraints with the receding horizon control (30), information about $u_{LQ}(t)$ is sufficient.

Lemma 4: If span$(k_{LQ}) \subseteq$ span(T_0), then S_0 in (29) can be chosen such that the input constraints $u_{\min} \leq u(t) \leq u_{\max}$ are satisfied at the optimum for any $x(t) \in X$.

Proof: Let the submatrices $\tilde{G}$, $\tilde{W}$ and $\tilde{S}$ correspond to the constraints $u_{\min} \leq u(t) \leq u_{\max}$ in the form

$$\tilde{G}z(t) \leq \tilde{W} + \tilde{S}x(t). \quad (35)$$

It is straightforward to see that

$$\tilde{G} = \begin{pmatrix} I_{m \times m} & 0_{m \times m(N-1)} \\ -I_{m \times m} & 0_{m \times m(N-1)} \end{pmatrix}$$

$$\tilde{W} = \begin{pmatrix} u_{\max} \\ -u_{\min} \end{pmatrix}$$

$$\tilde{S} = \begin{pmatrix} k_{LQ} \\ -k_{LQ} \end{pmatrix}$$

since $S = E + K_{LQ}$ and $E = 0$ for input constraints. Now, consider the corresponding submatrices of the reduced constraints (29), i.e.,

$$\tilde{G}z(t) \leq \tilde{W} + \tilde{S}_0 \xi(t) = \tilde{W} + \tilde{S}_0 T_0 x(t.) \quad (36)$$

The result follows since the reduced and original constraints can be made equivalent by the choice $\tilde{S}_0^T = (\mathcal{X}^T, -\mathcal{X}^T)$, where $\mathcal{X} \in \mathbb{R}^{m \times p}$ is a matrix such that $\mathcal{X}T_0 = k_{LQ}$. $\mathcal{X}$ must exist and be of rank m since span$(k_{LQ}) \subseteq$ span(T_0). □

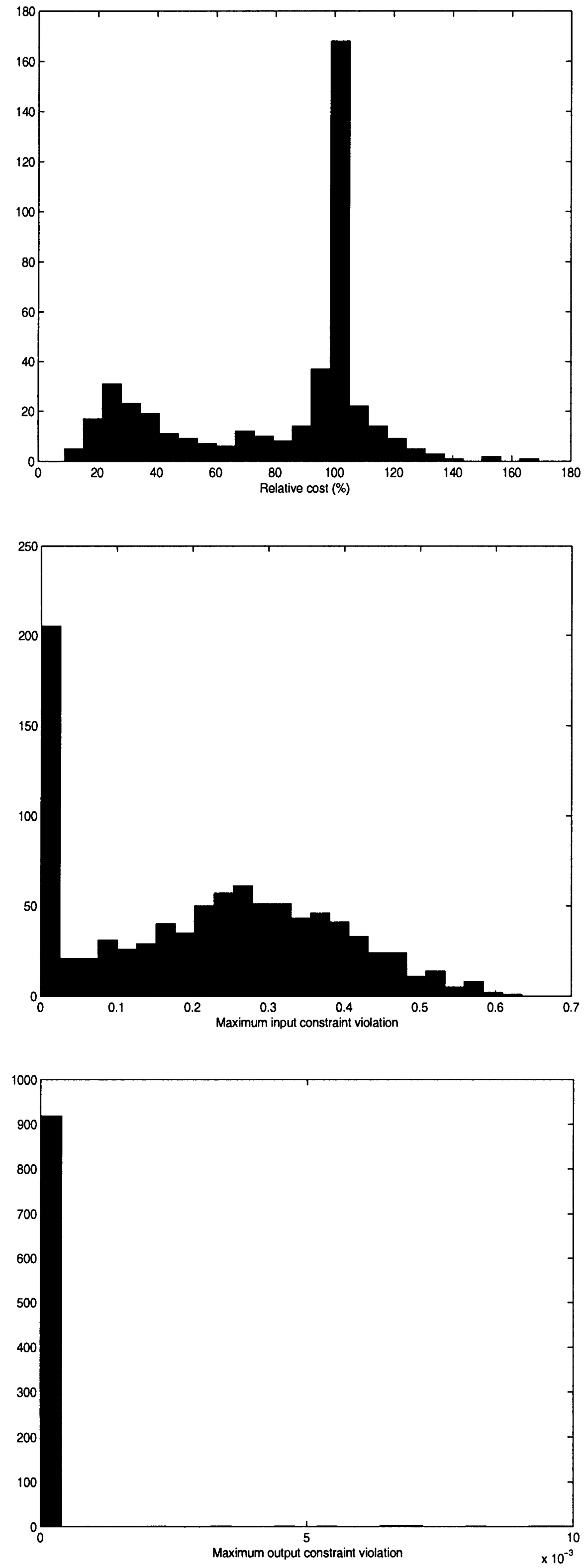

Fig. 4. Results from Monte Carlo simulation, without hard input constraints.

According to Lemma 4, the rows of the projection matrix should include the (scaled) rows of k_{LQ}. In order to minimize violation of the state constraints, we suggest the following procedure to choose additional rows in the projection matrix such that it includes the most significant directions of the orthogonal complement of the subspace spanned by k_{LQ}. Let the rows of $k_{LQ}^\perp$ contain a basis for null(k_{LQ}). Assuming

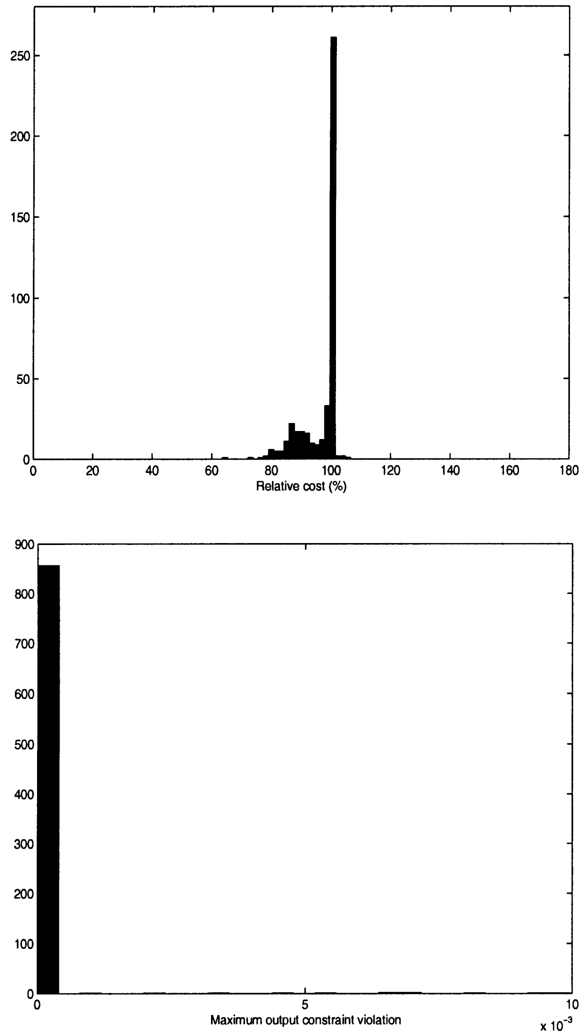

Fig. 5. Results from Monte Carlo simulation, with hard input constraints.

without loss of generality that k_{LQ} has row rank m such that its null space basis $k_{LQ}^{\perp}$ has rank $n-m$, we define

$$D = S\begin{pmatrix} k_{LQ} \\ k_{LQ}^{\perp} \end{pmatrix}^{-1}. \tag{37}$$

Hence, $S = D_1 k_{LQ} + D_2 k_{LQ}^{\perp}$ where D_1 contains the m first columns of D and D_2 the last $n-m$ columns. Consider the SVD $D_2 k_{LQ}^{\perp} = U\Sigma V^T$ which gives $Sx = \mathcal{S}_0 \xi + e$ where

$$\mathcal{S}_0 = (D_1, U_0\Sigma_0) \quad \mathcal{T}_0 = \begin{pmatrix} k_{LQ} \\ V_0^T \end{pmatrix} \tag{38}$$

and $e = U_1 \Sigma_1 V_1^T x$ where U_0, Σ_0, V_0 and U_1, Σ_1, V_1 are as before. With $\beta = \mathcal{T}_0 x$, this leads to the following mp-QP:

$$\mathcal{V}_{z,0}^{*}(\xi) = \min_{z} \frac{1}{2} z^T H z \quad \text{subject to} \quad Gz \le W + \mathcal{S}_0 \beta. \tag{39}$$

Example, Continued: With the same LQR criterion and input parameterization, the S matrix has the following singular values: 64.0025, 32.2419, 5.6246, 2.8686, 1.2025, and 1.0842. Assume we neglect the two smallest singular values, which yields an approximate mp-QP defined on a four-dimensional parameter space. The number of regions in the PWL feedback laws are 4279 in the original partition, 1930 in the partition without hard input constraints, and 1936 in the partition with hard input constraints, respectively. Hence, there is significant complexity reduction. The results of Monte Carlo simulations starting from 469 random initial states that give admissible trajectories in X are shown in Figs. 4 and 5 for the cases without and with hard input constraints, respectively. The histograms show the relative change in cost (100% indicates optimality), and maximum constraint violations. Notice that the ratio between the largest and smallest singular values is fairly small, such that some constraint violations and performance degradation appear in this example. Sample simulations are given in [17].

V. Conclusion

Methods for reducing the dimension of the parameter space of mp-QP problems associated with the explicit PWL solution of constrained LQR problems are investigated. It is shown that for systems with certain properties such dimension reduction can be achieved by state space projections that leads to mp-QPs that require less offline and online computations, and computer memory. Examples indicate that the performance degradation may be acceptable.

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