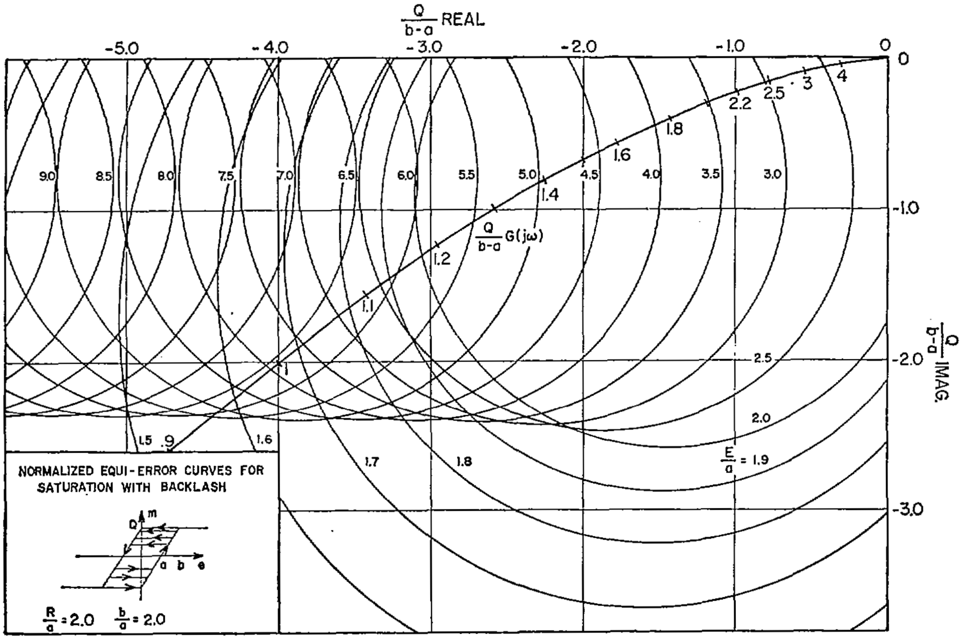

Fig. 2. Equi-error circles and $G(j\omega)$ curve for system of example.

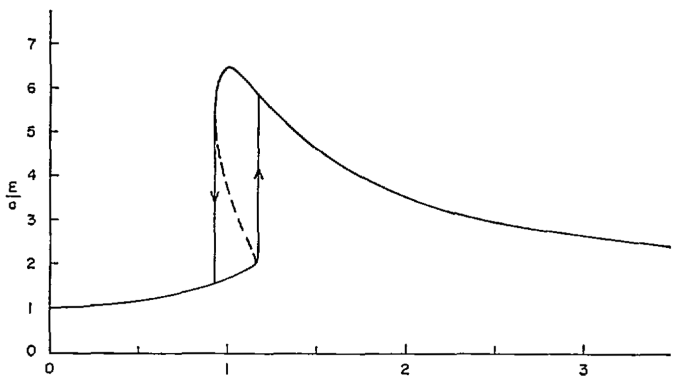

Fig. 3. E versus ω curve for system of example.

Conclusion

A direct method for finding the closed-loop response of nonlinear feedback control systems has been presented. It involved the intersection of a normalized nonlinearity-defined family of circles and the polar plot of the linear portion of the system. It allowed immediate identification of the regions of jump resonance and provided the means of avoiding them. The same approach has also been used for other system configurations. In such cases the basic equations are changed, but the same normalized families of circles can still be used.

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An Iterative Procedure for Solving Convex Optimal Control Problems

***Abstract*—A doubly iterative procedure for computing optimal controls in linear systems with convex cost functionals is presented. The procedure is based on an algorithm due to Gilbert [3] for minimizing a quadratic form on a convex set. Each step of the procedure makes use of an algorithm due to Neustadt and Paiewonsky [1] to solve a strictly linear optimal control problem.**

I. Introduction

Recently some basic facts from convex analysis have been used to solve a certain class of optimal control problems. This work has been spearheaded by Neustadt and Paiewonsky [1], [2], Gilbert [3], and Barr [4]. A typical problem solvable by the methods of Neustadt and Gilbert is the minimum fuel problem for linear systems. The purpose of this correspondence is to show how these methods can be used to solve the general linear optimal control problem with convex cost.

II. The Optimal Control Problem

Assume that our system is described by the linear differential equation

$$\dot{x} = A(t)x + B(t)u(t), \quad x(0) = x_0 \tag{1}$$

where x and u are n- and r-dimensional vectors, respectively; A and B are $n \times n$ and $n \times m$ continuous matrices, respectively. Here u denotes a control parameter which is measurable and restricted to take values in a given convex compact set $\Omega \subset E_m$.

The problem of optimal control is to determine a control u^0 which minimizes the cost functional

$$C(u) = \int_0^T f(t,x) + h(t,u)\,dt \tag{2}$$

subject to the differential equation (1) and the terminal constraint $x(T) = x_1$.

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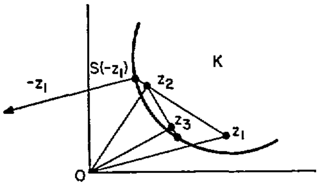

Fig. 1. Gilbert's algorithm.

For simplicity we shall restrict our discussion almost exclusively to the case where f and h are quadratic functions of x and u, respectively. Whenever necessary we shall indicate how the discussion should be modified to cover the case where f and h are general convex twice-continuously differentiable functions in the variables x and u. Thus we assume the cost functional (2) to be given by

$$C(u) = \int_0^T x \cdot Cx + u \cdot Du \, dt \tag{3}$$

where C and D are positive definite $n \times n$ and $m \times m$ matrices, respectively.

Let x_u denote the response of system (1) to the control u. The set

$$K = \{(x_u(\cdot),u(\cdot)) \mid x_u(T) = x_1\}$$

is convex in the space $E = L_C^2[0,T] \times L_D^2[0,T]$.

This follows from the fact that the set of all controls is convex and the system (1) is linear. $L_C^2[0,T]$ is the space of square integrable n-vector functions on $[0,T]$ with the inner product defined by

$$x \cdot y = \int_0^T x \cdot Cy \, dt$$

and $L_D^2[0,T]$ is defined similarly. Thus (3) gives the square of the norm of the element $(x,u) \in E$.

The problem of optimal control is clearly seen to be that of finding the point in K nearest the origin. An algorithm for finding the point in a convex set K nearest the origin has been described by Gilbert [3]. The algorithm proceeds as follows (see Fig. 1).

1) Choose a point z_1 in K arbitrarily.
2) Find a point $S(-z_1) \in K$ with the property that

$$\max_{z \in K} - z_1 \cdot z = -z_1 \cdot S(-z_1).$$

Observe that the vector $-z_1$ is normal to the set K at the point $S(-z_1)$.
3) Take $z_2 = z_1 + \beta(S(-z_1) - z_1)$ where β is chosen so that

$$C(z_2) = \min_{0 \leq \alpha \leq 1} C(z_1 + \alpha(S(-z_1) - z_1)).$$

For the quadratic problem z_2 is the point on the line segment $[z_1,S(-z_1)]$ nearest the origin.
4) Take z_2 to be the initial approximation and repeat steps 1)–3).

For the cost functional (3) the number β in 3) is given by

$$\beta = \min \left\{ \frac{z_1 \cdot (z_1 - S(-z_1))}{\| z_1 - S(-z_1) \|^2}, 1 \right\}.$$

Gilbert has shown that this procedure produces a sequence $z_1,z_2,\cdots$ converging to the point in K nearest the origin when such a point exists. Gilbert's attention was confined to the case of convex sets in finite-dimensional spaces but his proof is clearly valid in any Hilbert space and thus is applicable to the problem being considered here. The fact that the algorithm works for general convex functionals on convex sets can be deduced from the algorithm by Frank and Wolfe [5].

The only thing that remains is to provide a procedure for computing the point $S(-z_1)$ in 2) of the algorithm. Let $z_1 = (x_{u_1}(\cdot),u_1(\cdot))$. Then clearly $S(-z_1) = (x_{\tilde{u}}(\cdot),\tilde{u}(\cdot))$, where $\tilde{u}$ is the control which minimizes the linear functional

$$J(u) = \int_0^T Cx_{u_1}(t) \cdot x_u + Du_1(t) \cdot u \, dt \tag{4}$$

subject to $(x_u(\cdot),u(\cdot)) \in K$. This is a linear optimal control problem with terminal constraints. Its solution has been given by Neustadt and Paiewonsky [1].

In the general case, the linear functional (4) is replaced by the functional

$$J(u) = \int_0^T \frac{\partial f}{\partial x}(t,x_{u_1}(t)) \cdot x_u + \frac{\partial h}{\partial u}(t,u_1(t)) \cdot u \, dt.$$

Again we have a linear optimal control problem which can be solved by Neustadt's algorithm. A method for accelerating the convergence of Neustadt's algorithm is given by Pshenichnii [6].

III. Conclusion

An algorithm for solving linear optimal control problems with convex cost functionals and convex constraints on the controls has been presented. The algorithm is based largely on a quadratic programming algorithm due to Gilbert. Unlike most quadratic programming algorithms, Gilbert's algorithm is applicable to problems with general convex (not just linear) constraints on the variables. This perhaps is the greatest advantage of the method presented here.

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On the Numerical Computation of a Class of Distributed-Parameter Systems

***Abstract*—It is shown that the iterative computation of optimal control based on a modified steepest descent method in Hilbert space is very efficient for a class of distributed-parameter systems.**

We consider a dynamical system described by the vector partial differential equation

$$\partial v / \partial t = \mathfrak{D} v(t,x) + B\Psi(t,x), \quad x \in \Omega \subset R^r \tag{1}$$

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