Discontinuous Feedback Stabilization Using Nonlinear Model Predictive Controllers

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Abstract

We propose a Model Predictive Control (MPC) framework to generate feedback controls for time-varying nonlinear systems with input constraints. One of the main features of this framework is to allow the feedback laws to be discontinuous and thereby enlarge the class of nonlinear systems that can be stabilized by continuous-time MPC. We consider a continuous-time MPC framework and perform a continuous-time stability analysis while considering that the inter-sampling times are nonzero and that the open-loop optimal control problems are solved at every sampling instant. The feedback law generated by MPC is not a function of the state at every instant of time, rather it is a function of the state at the last sampling instant. The trajectories resulting from this "sampling-feedback" are well-de£ned even when the feedback is discontinuous. Important classes of nonlinear systems that could not be stabilized by a continuous feedback, such as the nonholonomic systems, can now be addressed in a continuous-time MPC framework.

key word: Model predictive control; stabilizing design parameters; nonlinear stability analysis; discontinuous feedback.

1 Introduction

A common assumption in most previous MPC continuous-time approaches is the continuity of the controls resulting from the open-loop optimal control problems, as well as the continuity of the resulting feedback law. This continuity assumption, in addition to being very dif£cult to verify, is a major obstacle in enabling MPC to address a broader class of nonlinear systems. This is because some nonlinear systems cannot be stabilized by a continuous feedback as was £rst noticed in [1] and [9]. Systems that cannot be stabilized by a continuous feedback include some mechanical and nonholonomic systems with interest in practice.

The main dif£culty in relaxing this continuity assumption is that, if we allow discontinuous feedbacks, it would not be clear what should be the solution (in a classical sense) of the dynamic differential equation. Consider a time-varying feedback control $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and the system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), k(t, x(t))), \quad t \in \mathbb{R} \\ x(t_0) &= x_0. \end{aligned}$$

Assume that the function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and that $x \mapsto f(\cdot, x, \cdot)$ is locally Lipschitz. The classical de£nition of a trajectory of this system requires that the feedback $x \mapsto k(t,x)$ is continuous. This fact motivated the development of new concepts of solution to differential equations under a discontinuous feedback; from which the best known example is probably the concept of Filippov solutions. Previous attempts to deal with discontinuous controls in a MPC context are [7] using Filippov solutions and approaches avoiding the continuity problem by considering a discrete-time framework (e.g. [6]). Ryan [8] and Coron and Rosier [3] have shown that Filippov solutions cannot lead to stability results for general nonlinear systems. Therefore, the class of nonlinear systems that could be addressed by MPC was limited by the continuity assumption.

Recently, Clarke *et al.* [2] were able to stabilize a general class of nonlinear systems using discontinuous feedbacks. In their de£nition of trajectory, the feedback is not a function of the state on *every* instant of time, rather it is a function of the state at the last sampling instant. Our MPC framework constructs, as can be seen in the next section, a "sampling-feedback" law de£ning a trajectory in a way which is very similar to the concept introduced in [2]. These trajectories are, under mild conditions, well-de£ned even when the feedback law is discontinuous. Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. Let the function $t \mapsto \lfloor t \rfloor_\pi$ give the greatest sampling instant less than or equal to t, that is

$$\lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\}.$$

For a sequence π, the trajectory of the system under the feedback k is de£ned by

$$\dot{x}(t) = f(t, x(t), k(t, x(\lfloor t \rfloor_\pi))), \quad t \in \mathbb{R} \tag{1a}$$
$$x(t_0) = x_0. \tag{1b}$$

Apart from possible £nite escape times, the trajectory x is well-de£ned even in the case when $x \mapsto k(t,x)$ is discontinuous.

We proceed to de£ne stability in this context.

De£nition 1.1 The sampling-feedback k is said to *asymptotically stabilize* the system (1) on X_0 if there exists a suf£ciently small inter-sample time δ such that the following condition is satis£ed. For any $\gamma > 0$ we can £nd a scalar $M > 0$ such that for any pair $(t_0, x_0) \in \mathbb{R} \times X_0$ we have $\|x(s+t_0; t_0, x_0, k)\| < \gamma$ for $s \geq M$.

2 The Model Predictive Control Framework

We shall consider a plant with input constraints, where the evolution of the state after time t is predicted by the following nonlinear model.

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \geq t \tag{2a}$$
$$x(t) = x_t \tag{2b}$$
$$u(s) \in U(s). \tag{2c}$$

We assume that the set $U(t)$ contains the origin and that $f(t,0,0) = 0$ for all $t \in \mathbb{R}$. We proceed to de£ne a continuous-time MPC framework, perform a continuous-time stability analysis while considering that the inter-sampling times are nonzero and that the open-loop optimal control problems are solved at every sampling instant.

Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T)$ at each sampling instant t_i, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(x_t, T)$: Minimize

$$\int_t^{t+T} L(s, x(s), u(s))ds + W(t+T, x(t+T)) \tag{3}$$

subject to:

$$\dot{x}(s) = f(s, x(s), u(s)) \qquad \text{a.e. } s \in [t, t+T] \tag{4}$$
$$x(t) = x_t$$
$$u(s) \in U(s) \qquad \text{a.e. } s \in [t, t+T]$$
$$x(t+T) \in S. \tag{5}$$

The notation adopted here is as follows. The variable t represents real time while we reserve s to denote the time variable used in the prediction model. The vector x_t denotes the actual state of the plant measured at time t. The process (x,u) is a pair trajectory/control obtained from the model of the system. The trajectory is also denoted as $s \mapsto x(s; t, x_t, u)$ when we want to make explicit its dependency on the initial time, initial state, and control function. The pair $(\bar{x}, \bar{u})$ denotes an optimal solution to an open-loop optimal control problem (OCP). The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* to the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the time horizon T, the running and terminal costs functions L and W, and the terminal constraint set $S \subset \mathbb{R}^n$.

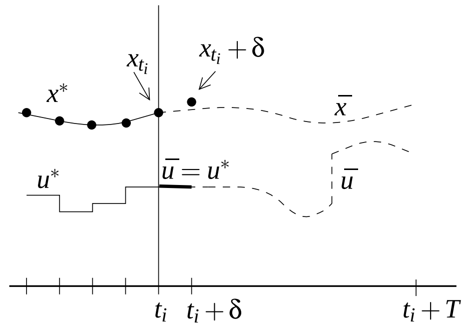

Figure 1: The MPC strategy.

The MPC conceptual algorithm consists of performing the following steps at a certain instant t_i (see Fig. 1).

1. Measure the current state of the plant x_{t_i}.
2. Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x_{t_i}, T)$.
3. The control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta)$ is applied to the plant (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).
4. The procedure is repeated from (1.) for the next sampling instant t_{i+1} (the index i is incremented by one unit).

The resultant control law is a "sampling-feedback" control since during each sampling interval, the control u^* is dependent on the state x_{t_i}:

$$\dot{x}(t) = f(t, x(t), k(t, x(\lfloor t \rfloor_\pi))), \quad t \in \mathbb{R}$$

where

$$k(t, x(\lfloor t \rfloor_\pi)) = u^*(t).$$

This allows our MPC framework to overcome the inherent dif£culty of de£ning solutions to differential equations with discontinuous feedbacks. In this way, the class of nonlinear systems potentially addressed by MPC is enlarged, including, for example, nonholonomic systems (see [5, 4] for a discussion of examples).

The use of this framework to stabilize nonlinear systems is illustrated in the following stability result.

Assume that we are able to choose a set of design parameters (time horizon T, objective functions L and W, and terminal constraint set S) to satisfy the following conditions:

C1 The function L is continuous, $L(\cdot,0,0)=0$, and there is a continuous positive de£nite and radially unbounded function $M:\mathbb{R}^n\to\mathbb{R}_+$ such that $L(t,x,u)\geq M(x)$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^m$.

C2 The function W is positive de£nite and continuously differentiable.

C3 The set S=$\{0\}$.

C4 There exist of solutions to the sequence of optimal control problems $\mathcal{P}(t_i,x_{t_i},T)$ for all positive integers i.

Conditions C1 and C2 are automatically satis£ed if we choose L and W to be positive de£nite quadratic functions. Condition C4 might be dif£cult to verify for some problems, specially if combined with S=$\{0\}$. In [5, 4] a stability analysis is carried out when the terminal set is chosen in a general way. The conditions for the existence of solutions to the sequence of optimal control problems $\mathcal{P}(t_i,x_{t_i},T)$ are also analysed.

A result on stability follows. This is an instance of a result in [5, 4] where we refer to for the proof.

Theorem 2.1 *If we choose the design parameters to satisfy C1–C4 then the closed loop system resulting from the application of the MPC strategy is asymptotically stable.*

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