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Implementable Model Predictive Control in the State Space

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Abstract

Model predictive control is an optimal control based method for constrained feedback control. Previous work at The University of Texas at Austin has focussed on the development of model predictive controllers that are nominally asymptotically stable for all valid tuning parameters. This development eliminated the need for tuning to obtain nominal stability. However, some implementation issues were not addressed. This work provides a discussion of those issues and solutions that allow the application of a nominally stable linear model predictive controller to be more easily realized in practice. The algorithms discussed in this work are implemented using **octave** [1] and can be easily translated into other programming languages.

1. Problem Statement

This section reviews the nominally stable, constrained, receding horizon regulator proposed by Rawlings and Muske [2]. The model predictive controller based on this constrained regulator is discussed by Muske and Rawlings [3, 4]. The presentation here is necessarily brief and the reader is referred to the preceding references for a more detailed discussion.

We seek a regulator for the time-invariant, linear system described by the following discrete state-space model with $x \in \Re^n$, $u \in \Re^m$, and $y \in \Re^p$

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \tag{1}$$

that brings the state of the system to the origin subject to linear state and input constraints.

$$Hx_k \leq h \tag{2}$$

$$Du_k \leq d \tag{3}$$

All elements of $h \in \Re^q$ and $d \in \Re^r$ are taken to be strictly positive to ensure that the origin is contained in the interior of the feasible set.

A stabilizing regulator can be determined as the solution to an infinite horizon, open-loop optimal control problem of the following form.

$$\min_{\{v_j\}} \sum_{j=0}^{\infty} \left(z_j^T Q z_j + v_j^T R v_j \right) \tag{4}$$

$$\text{Subject to:} \quad \begin{aligned} z_0 &= x_k \\ z_{j+1} &= Az_j + Bv_j \\ Hz_j &\leq h \\ Dv_j &\leq d \end{aligned} \tag{5}$$

In Eqs. 4 and 5, z_j and v_j are the predicted open-loop state and control input trajectories which may be different from the actual closed-loop state and input trajectories denoted by x_k and u_k. In order to provide a computationally tractable minimization problem, a finite parameterization of the input is required. Rawlings and Muske [2] chose the parameterization $v_j = 0$ for $j \geq N$, in which N is some integer greater than zero. The objective function in Eq. 4 can then be divided into a finite control horizon and an infinite prediction horizon.

$$\min_{\{v_j\}} \sum_{j=0}^{N-1} \left(z_j^T Q z_j + v_j^T R v_j \right) + \sum_{j=N}^{\infty} z_j^T Q z_j \tag{6}$$

If A is stable, the infinite prediction horizon can be expressed as a terminal state penalty

$$\sum_{j=N}^{\infty} z_j^T Q z_j = z_N^T \bar{Q} z_N$$

in which $\bar{Q}$ satisfies the discrete matrix Lyapunov equation.

$$\bar{Q} - A^T \bar{Q} A = Q \tag{7}$$

Muske and Rawlings [3] proved that this formulation provides a stabilizing feedback controller for all $N > 0$, $Q \geq 0$, and $R > 0$ provided A is stable and the constraints in Eq. 5 are feasible.

2. Unstable Process Models

The discrete matrix Lyapunov equation in Eq. 7 has a unique solution if and only if A is stable. When A is unstable, Rawlings and Muske [2] restrict the open-loop state trajectory z_j to the stable subspace of A for $k \geq N$, which agrees with the intuitive understanding that unstable modes cannot be allowed to proceed over an unbounded time interval without control action. This restriction is imposed by the following equality constraint on the unstable modes, appended to Eq. 5, in which the matrix V is composed of basis vectors for the unstable subspace of A.

$$V^T z_N = 0 \tag{8}$$

2.1. Application of Schur Decomposition

Rawlings and Muske use the Jordan form of the A matrix to determine a basis for the unstable subspace for analysis purposes. However, the computation of Jordan forms is known to be numerically unstable. The Schur decomposition offers a numerically stable procedure for obtaining a basis for the unstable subspace of A. Every matrix $A \in \Re^{n\times n}$ has a real Schur decomposition [5] expressed as

$$U^T A U = T \tag{9}$$

in which U is orthogonal and T is upper block triangular. The dimension of the diagonal blocks of T is either 1×1 or 2×2 corresponding to real or complex conjugate eigenvalues, respectively. The blocks may be arbitrarily ordered. If the diagonal blocks of T are ordered with the smallest magnitude eigenvalues in the upper left block, U can be partitioned as follows in which the columns of U span the stable and unstable subspaces, respectively.

$$U = \begin{bmatrix} U_s & U_u \end{bmatrix} \tag{10}$$

If A has no stable modes, U_s is empty and $U = U_u$. The converse follows if A has no unstable modes. With U_u so defined, the final state equality constraint in Eq. 8 is expressed as

$$U_u^T z_N = 0 \tag{11}$$

and the terminal state penalty $\bar{Q}$ is given by

$$\bar{Q} - A_s^T \bar{Q} A_s = Q \tag{12}$$

in which A_s satisfies $A_s U_u = 0$ and $A_s U_s = A U_s$. The matrix A_s is determined by the partition of T corresponding to the stable and unstable eigenvalues. If the stable eigenvalues are contained in T_{11} and the unstable eigenvalues in T_{22}, A can be expressed as

$$A = \begin{bmatrix} U_s & U_u \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} U_s^T \\ U_u^T \end{bmatrix}$$

and A_s computed as follows.

$$A_s = U_s T_{11} U_s^T \tag{13}$$

2.2. Choice of Control Horizon Length N

An arbitrary choice of N may lead to the absence of a feasible solution to the control input constraints in Eq. 5 and the equality constraint on the unstable modes in Eq. 11. However, the authors are not aware of a simple test, such as a rank condition, that can be checked to verify feasibility of these constraints. Clearly, if (A, B) is not stabilizable, it will not be possible to achieve the constraint in Eq. 11 for any N. In this case, it is necessary to reconsider the control objectives and physical conditions that prevent stabilizability. In the presence of input constraints, stabilizability of (A, B) does not guarantee feasibility of Eq. 11. Muske and Rawlings [3] achieve constrained stabilizability by restricting the state space that can be considered for a desired control horizon. In this work, the minimum control horizon that results in feasibility of these constraints is determined for a given initial state. As the magnitude of the unstable modes increase, it requires a longer control horizon to stabilize the system in the presence of the input constraints. This control horizon is determined using the following algorithm.

1. Choose a desired and maximum acceptable control horizon, N_{des} and $N_{\max}$, and set $N = N_{\text{des}}$.

2. Solve the following linear program:

$$\min_{v_j, r_N} \Phi_N = [1 \dots 1] r_N$$

Subject to:

$$\begin{aligned} z_0 &= x_0 \\ z_{j+1} &= Az_j + Bv_j \\ Dv_j &\le d, \; j = 0, \ldots, N-1 \\ JU_u^T z_N &= r_N \\ r_N &\ge 0 \end{aligned}$$

3. If $\Phi_N > 0$ and $N < N_{\max}$, set $N = N + 1$ and repeat Step 2.

4. If $\Phi_N = 0$ and $N \le N_{\max}$, set the control horizon to N and stop.

5. If $\Phi_N > 0$ and $N = N_{\max}$, the system is not constrained stabilizable within the maximum acceptable control horizon.

An initial feasible solution to each of the linear programs is: $v_j = 0$, $j = 0, \ldots, N-1$ and $r_N = JU_u^T A^N x_0$. The diagonal matrix J ensures that each component of the initial r_N is nonnegative. This algorithm requires the solution of a sequence of linear programs for which efficient algorithms and software implementations are available.

3. State Constraints on an Infinite Horizon

Nominal stability of the constrained, receding horizon regulator requires that the state constraints in Eq. 5 be satisfied on an infinite horizon [2]. The series of state constraints is

$$HA_s^j z_N \le h, \quad j = 0, 1, \ldots \tag{14}$$

after the control input is set to zero. Despite the finite parameterization for the control input, the quadratic programming problem is not easily implemented with an infinite constraint horizon. For implementation purposes we wish to replace the infinite set of state constraints with a finite set. Rawlings and Muske [2] address this issue by showing that for any $\|z_N\| < \infty$, all active constraints are contained within a finite constraint horizon. However, their maximum constraint horizon length is given in terms of z_N which makes it impossible to specify *a priori* and, in most cases, is larger than the minimum constraint horizon necessary.

Gilbert and Tan [6] provide an algorithm to specify *a priori* the maximum constraint horizon in Eq. 14 that can be binding. Their approach involves the determination of the family of subsets of $\Re^n$ given by

$$O_t(A, H, h) = \left\{ x \in \Re^n \,\middle|\, HA^i x \le h, \;\; i = 0, 1, \ldots t \right\}$$

with the objective of characterizing O_∞. Gilbert and Tan show that if $O_t = O_{t+1}$ for some finite t, then $O_\infty = O_t$. Based on this result, they propose the following off-line numerical procedure for the determination of O_∞.

1. Set $M = 0$.

2. For $i = 1$ to q (the row dimension of H), solve each of the following linear programming problems:

   $$\max_x \; \Phi_i(x) = \left\{ HA_s^{M+1} \right\}_i x$$

   Subject to: $HA_s^j x \le h, \;\; j = 0, \ldots, M$

   $\{\cdot\}_i$ indicates the ith row of the matrix.

3. For each optimal Φ_i, if $\Phi_i - h_i > 0$ or is unbounded, set $M = M + 1$ and repeat Step 2.

4. If each optimal Φ_i satisfies $\Phi_i - h_i \le 0$, then $O_\infty = O_M$ and $M + N$ is the maximum constraint horizon.

This algorithm requires the solution of a sequence of linear programs that can be efficiently implemented. Sufficient conditions for convergence of this procedure are A_s asymptotically stable, (C, A_s) observable, and the state constraint space defined by $\{x \,|Hx \le h\}$ bounded [6]. Since A_s is asymptotically stable by construction and this procedure can be applied to the observable subspace as shown in [6], convergence is guaranteed provided the constraint space is bounded.

In most applications, however, the state constraint space is not bounded and O_∞ may not be finitely determined (see example 3.4 in [6]). In these cases, an upper bound for the constraint horizon length for A matrices with distinct stable eigenvalues is [2]

$$N + \max\{\ln\left(\frac{h_{\min}}{\kappa(V)\,\|H\|\;\|z_N\|}\right) / \ln(\lambda_{\max}), 0\} \tag{15}$$

in which $h_{\min} = \min_i h_i$, $\lambda_{\max}$ is the largest modulus eigenvalue of A_s, and $\kappa(V)$ is the condition number of V obtained from the Jordan canonical form $A_s = V^{-1}\Lambda V$. In practice, this constraint horizon is generally much larger than necessary which leads to additional computation requirements for the regulator quadratic program. The use of a shorter constraint horizon, which is normally sufficient, can efficiently be checked by the following inequality

$$\begin{bmatrix} H \\ HA_s \\ \vdots \\ HA_s^{j_2-N} \end{bmatrix} z_N \leq \begin{bmatrix} h \\ h \\ \vdots \\ h \end{bmatrix} \tag{16}$$

in which j_2 is computed from Eq. 15. If this inequality is not satisfied, the optimization problem can be resolved with the constraint horizon in Eq. 15.

4. Infeasibilities Arising From State Constraints

The use of the control horizon length determined from the algorithm in Section 2.2, guarantees the existence of a feasible solution for the control input and unstable mode constraints. It does *not* guarantee that all state constraints are satisfied. Rather than rule out these cases completely, Rawlings and Muske relax the state constraints at the beginning of the constraint horizon to achieve feasibility. State constraint relaxation does not affect the nominal stability properties of the regulator and provides a wider range of initial states that can be stabilized. For $\|x_k\| < \infty$, feasibility is achieved by removing a finite number of state constraints from the beginning of the constraint horizon and, in the nominal case, the start of the constraint horizon can be reduced by one each time the regulator is executed without loss of feasibility [2].

Minimization of the time that the constraints are violated requires that the state constraints be enforced at the earliest time in the prediction horizon that provides a feasible solution to the quadratic program. However, this procedure can result in large constraint violations for nonminimum phase systems. As shown in Muske and Rawlings [4], relaxing the state constraints for a longer time than the minimum necessary can result in a decrease in the magnitude of the constraint violations at the expense of an increase in the time that the constraints are violated.

5. Constrained Regulator Formulation

For the current state x_k, the quadratic program for the constrained, receding horizon regulator is the following.

$$\min_{v_j} \sum_{j=0}^{N-1} \left(z_j^T Q z_j + v_j^T R v_j \right) + z_N^T \bar{Q} z_N \tag{17}$$

Subject to:

$$\begin{array}{llcl} \text{(A)} & z_0 & = & x_k \\ \text{(B)} & z_{j+1} & = & Az_j + Bv_j \\ \text{(C)} & Dv_j & \leq & d,\ j = 0, \ldots, N-1 \\ \text{(D)} & Hz_j & \leq & h,\ j = j_1, \ldots, j_2 \\ \text{(E)} & U_u^T z_N & = & 0 \end{array}$$

For this quadratic program, the following summarizes the purpose of each constraint: (A) represents the state at time k for which a feedback controller is desired; (B) represents the linear system equations; (C) represents the specified control input constraints; (D) represents the specified state constraints in which $j_1 \geq 1$ is determined such that the state constraints are feasible and $j_2 \geq j_1$ is determined such that the state constraint horizon is equivalent to enforcing the state constraints on an infinite horizon; and (E) is the state stability constraint on the unstable modes (not present if A is stable). The quadratic program contains $N \times m$ decision variables in which m is the dimension of the control input vector u.

6. Conclusions

Rawlings and Muske [2] provide a constrained, receding horizon regulator formulation in which nominal stability is guaranteed by the existence of a feasible solution to a quadratic program. The quadratic program is derived from an infinite horizon, open-loop optimal control problem. No tuning of the weighting matrices is required for stability, but a correct choice of the control horizon is critical to stability when A is unstable. In this work, we provide a simple algorithm for determining the control horizon. Efficient implementation of the state inequality constraints on the infinite horizon was the most significant open issue remaining from the original work. Based on the results of Gilbert and Tan [6], this work provides an algorithm for reducing the infinite set of constraints to a finite set suitable for numerical implementation.

Although this paper stresses improvements made to facilitate numerical implementation, this regulator has also been shown to provide a feedback control law that is Lipschitz continuous in the state variable under certain restrictions [7]. An important consequence of Lipschitz continuity is stability of the combined observer-controller with an exponentially stable observer.

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