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Global Exponential Setpoint Control of Mobile Robots

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Abstract: *This paper presents a new smooth time-varying controller for the stabilization problem about a fixed setpoint for a wheeled mobile robot. The structure of the proposed control algorithm relies on a previously designed controller for the induction motor. After the WMR kinematics/dynamics have been transformed into an advantageous form, a dynamic ocscillator, in lieu of explicit cosine or sine terms, is constructed to promulgate a standard exponential stability result for the transformed dynamics via a Lyapunov-type argument. Simulation results are presented to demonstrate the performance of the proposed controller.*

1 Introduction

Over the past twenty years wheeled mobile robots (WMRs) have become increasingly important in settings that range from shopping centers, hospitals, warehouses, and nuclear waste facilities for applications such as security, transportation, inspection, planetary exploration, *etc.* This increased demand for WMRs has led to greater research interest in the areas of electromechanical design, sensor integration techniques, path planning, and control design. As noted in [4] and [8], control research for nonholonomic systems (*i.e.*, the family of mechanical systems which includes WMRs as special cases) has been centered around the problem of *i*) stabilizing a robot about a time varying trajectory, *ii*) stabilizing the robot about a geometric path, and *iii*) stabilizing the WMR about a desired posture. It has also been noted that the problems of stabilizing a robot about a time varying trajectory and stabilizing about a geometric path can be solved with standard nonlinear control techniques; however, many researchers (see [1], [4], [13], and [16]) have pointed out that the problem of stabilization about a fixed point is more challenging due to the structure of the governing differential equations (*i.e.*, the control problem can not be solved via a smooth time-invariant state feedback law due to the implications of Brockett's condition [3]). With this technical obstacle in mind, researchers have proposed controllers that utilize discontinuous control laws, piecewise continuous control laws, smooth time-varying control laws, or a hybrid form of the previous controllers to achieve setpoint regulation. For an in-depth review of the previous work, the interested reader is referred to [12], [16], and the references therein. For brevity and for illuminating the motivation for this work, we confine our review to a much smaller subset of papers.

In [16], Samson illustrated how a smooth time-varying feedback controller could be utilized to asymptotically stabilize a mobile robot about a point. The work of [16] led to development of smooth time-varying controllers for a more general class of system (*e.g.*, see [5], [15], and [19]). In [2], Bloch *et al.* developed a piecewise analytic control structure for regulating a nonholonomic systems to a setpoint. In [4], Canudas de Wit *et al.* constructed a piecewise smooth controller to exponentially stabilize a WMR to a setpoint; however, due to the control structure, the orientation of the WMR is not arbitrary. More recently, in [17] Samson developed globally asymptotically stabilizing feedback controllers for a general class of nonholonomic systems in the chain form and provided a detailed discussion on the convergence issue. In [12], McCloskey *et al.* developed a set of sufficient conditions for generating ρ-exponential stabilizers from asymptotic stabilizers for a general class of driftless systems. In [9], Godhavn *et al.* constructed a local, continuous feedback control law with time-periodic terms that ρ-exponentially stabilized nonholonomic systems in the power form. Motivated by the desire to remove the exact model knowledge dependence of the aforementioned controllers, Dong *et al.* [7] proposed an adaptive control solution for chained nonholonomic systems with unknown constant inertia effects. In [10], Jiang et al. proposed a reference robot tracking controller; however, the tracking control problem does not reduce to the regulation problem. Recently in [14], Escobar *et al.* illustrated how the field oriented controller which has been derived for induction motors can redesigned to exponentially stabilize a nonholonomic double integrator control problem (*e.g.*, Heisenberg flywheel). Unfortuneately, the controller presented in [14] exhibited singularities in either the double integrator state or the output state.

In this paper, we illustrate how the previously designed controller for the induction motor control problem given in [6] can be reconfigured to globally exponentially stabilize a WMR to any constant setpoint. In contrast with [14], the proposed controller does not exhibit any singularities. In contrast, with much of the previous work on nonholonomic systems, the proposed controller does not utilize explicit sinusoidal terms in the feedback controller; rather, a dynamic oscillator with a tunable frequency of oscillation is constructed. Roughly speaking, the frequency of oscillation is used as auxiliary control input to cancel odious terms during the Lyapunov analysis. While the control synthesis and the error system development is slightly more involved than some of the previously designed controllers for the WMR problem, the stability analysis is a straightforward Lyapunov analysis which renders an exponential result for the transformed dynamics. The paper is organized as follows. In Section 2, we transform the kinematic and dynamic models of the WMR into a form which resembles the induction motor model. In Section 3, we present the differentiable, time-varying control law and the corresponding closed-loop error system. In Section 4, we provide a Lyapunov based stability analysis to illustrate global exponential stabilization about a fixed setpoint. In Section 5, we present some simulation results to illustrate the performance of the controller and in Section 6, we compare the proposed kinematic control algorithm to some previously designed kinematic control algorithms. Concluding remarks are presented in Section 7.

2 Wheeled Mobile Robot Model

2.1 Kinematic Transformation

As described in [8] and [11], the kinematic equations of motion of the center of mass (COM) of the WMR can be written as follows

$$\dot{q} = S(q)v \tag{1}$$

where $\dot{q}(t) \in \Re^3$ defined as

$$\dot{q}(t) = \begin{bmatrix} \dot{x}_c & \dot{y}_c & \dot{\theta} \end{bmatrix}^T \tag{2}$$

represents the time derivative of $q(t) \in \Re^3$, $v(t) \in \Re^2$ is vector of linear and angular WMR velocities denoted by $v_l(t) \in \Re^1$

and $\dot{\theta}(t) \in \Re^1$, respectively, as follows

$$v = [v_1 \quad v_2]^T = \begin{bmatrix} v_l & \dot{\theta} \end{bmatrix}^T, \tag{3}$$

$\dot{x}_c(t)$, $\dot{y}_c(t) \in \Re^1$ represent the time derivative of the Cartesian position of the COM denoted by $x_c(t)$, $y_c(t) \in \Re^1$, the transformation matrix $S(q) \in \Re^{3\times 2}$ is defined as

$$S(q) = \begin{bmatrix} \cos\theta & -d\sin\theta \\ \sin\theta & d\cos\theta \\ 0 & 1 \end{bmatrix} \tag{4}$$

and d is the distance between the driving wheels and the COM. In order to express the WMR model in a form that is more amenable to the subsequent control design and stability analysis, we define the following new variables, denoted by $z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T \in \Re^2$ and $w(t) \in \Re^1$, which are related to the Cartesian position/orientation of the COM via the following transformation

$$\begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \\ -\tilde{\theta}\cos\theta + 2\sin\theta & -\tilde{\theta}\sin\theta - 2\cos\theta & 2d \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \end{bmatrix} \tag{5}$$

where $\tilde{x}(t)$, $\tilde{y}(t)$, $\tilde{\theta}(t) \in \Re^1$ are defined as the difference between the actual Cartesian position/orientation of the COM and the desired constant position/orientation setpoints, denoted by x_{cd}, y_{cd}, $\theta_d \in \Re^1$, as follows

$$\tilde{x} = x_c - x_{cd}, \quad \tilde{y} = y_c - y_{cd}, \quad \tilde{\theta} = \theta - \theta_d. \tag{6}$$

After taking the time derivative of (5), and using (1), (2), (3), (4), (5), and (6), we have the following transformed kinematic equations

$$\begin{bmatrix} \dot{w} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 u_1 - z_1 u_2 \\ u_1 \\ u_2 \end{bmatrix} \tag{7}$$

where the new variable, denoted by $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T \in \Re^2$, is defined in terms of the Cartesian position/orientation of the COM and the linear WMR velocities as follows

$$\begin{aligned} u_1 &= v_2 \\ u_2 &= v_1 - v_2(\tilde{x}\sin\theta - \tilde{y}\cos\theta). \end{aligned} \tag{8}$$

Finally, we rewrite the expression given in (7) in the following compact form

$$\begin{aligned} \dot{w} &= u^T J^T z \\ \dot{z} &= u \end{aligned} \tag{9}$$

where $J \in \Re^{2\times 2}$ is a constant, skew symmetric matrix defined as follows

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{10}$$

2.2 Dynamic Transformation

The dynamic model for the 2-dimensional (*i.e.*, 3-dimensional configuration space with 1 constraint) nonholonomic *pure rolling and non slipping* WMR can be transformed to the following form[1] [8]

$$\bar{M}(q)\dot{v} + \bar{V}_m(q,\dot{q})v + \bar{F}(v) = \bar{B}\tau \tag{11}$$

where $v(t)$ was defined in (3), $\tau(t) \in \Re^2$ denotes the control torque input, $\bar{M}(q) \in \Re^{2\times 2}$ represents the symmetric, positive definite inertia matrix, $\bar{V}_m(q,\dot{q}) \in \Re^{2\times 2}$ represents the centripetal-Coriolis matrix, $\bar{F}(v) \in \Re^2$ represents the surface friction effects, and $\bar{B} \in \Re^{2\times 2}$ is a constant invertible matrix that depends on the distance between the driving wheels and the radius of the wheels.

As done in [8], we feedback linearize (11) with the following control torque input

$$\tau = \bar{B}^{-1}\left[\bar{M}(q)\begin{bmatrix} \tau_{c2} + f \\ \tau_{c1} \end{bmatrix} + \bar{V}_m(q,\dot{q})v + \bar{F}(v)\right] \tag{12}$$

where $\tau_c(t) = \begin{bmatrix} \tau_{c1}(t) & \tau_{c2}(t) \end{bmatrix}^T \in \Re^2$ is an auxiliary control input designed in the subsequent analysis, and the auxiliary function $f(\tilde{x}, \tilde{y}, \theta, v_2, \tau_{c1}) \in \Re^1$ is defined as follows

$$\begin{aligned} f(\cdot) &= \tau_{c1}(\tilde{x}\sin\theta - \tilde{y}\cos\theta) \\ &\quad + v_2\left(\dot{x}_c\sin\theta + \tilde{x}\dot{\theta}\cos\theta - \dot{y}_c\cos\theta + \tilde{y}\dot{\theta}\sin\theta\right). \end{aligned} \tag{13}$$

After substituting (12) into (11), we obtain the following expression

$$\dot{v} = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \tau_{c2} + f(\cdot) \\ \tau_{c1} \end{bmatrix}. \tag{14}$$

We now rewrite (14) in terms of the variable $u(t)$ defined in (8) by taking the time derivative of (8) to yield

$$\begin{aligned} \dot{u}_1 &= \dot{v}_2 \\ \dot{u}_2 &= \dot{v}_1 - \dot{v}_2(\tilde{x}\sin\theta - \tilde{y}\cos\theta) \\ &\quad - v_2\left(\dot{x}_c\sin\theta + \tilde{x}\dot{\theta}\cos\theta - \dot{y}_c\cos\theta + \tilde{y}\dot{\theta}\sin\theta\right) \end{aligned} \tag{15}$$

where (6) and (2) have been utilized. It is now easy to see from (13), (14), and (15) that the variable $u(t)$ defined in (8) and the auxiliary control input $\tau_c(t)$ are related via the following dynamic expression

$$\dot{u} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \tau_{c1} \\ \tau_{c2} \end{bmatrix} = \tau_c. \tag{16}$$

3 Control Development

Our control objective is to design an exponential setpoint controller for the transformed *pure rolling and non slipping* WMR model given by (9) and (16). To this end, we define an auxiliary error signal $\tilde{z}(t) \in \Re^2$ as the difference between the subsequently designed auxiliary signal $z_d(t) \in \Re^2$ and the transformed variable $z(t)$ defined in (5) as follows

$$\tilde{z} = z_d - z. \tag{17}$$

In addition, we define another auxiliary error signal $\eta(t) \in \Re^2$ as the difference between the subsequently designed auxiliary signal $u_d(t) \in \Re^2$ and the auxiliary signal $u(t)$ defined in (8) as shown below

$$\eta = u_d - u. \tag{18}$$

3.1 Control Formulation

Based on the kinematic equations given in (9) and the subsequent stability analysis, we design the auxiliary signal $u_d(t)$ as follows

$$u_d = u_a - kz \tag{19}$$

where the control term $u_a(t) \in \Re^2$ is defined as

$$u_a = \left(\frac{kw}{\delta_d^2}\right) J z_d + \Omega_1 z_d, \tag{20}$$

[1] See Appendix A for explicit expressions for $\bar{M}(q)$, $\bar{V}_m(q,\dot{q})$, and $\bar{B}$.

$z_d(t)$ is defined by the following oscillator-like relationship

$$\dot{z}_d = \frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{kw}{\delta_d^2} + w\Omega_1\right) J z_d, \qquad z_d^T(0) z_d(0) = \delta_d^2(0), \quad (21)$$

the auxiliary terms $\Omega_1(t)$, $\delta_d(t) \in \Re^1$ are defined as

$$\Omega_1 = k + \frac{\dot{\delta}_d}{\delta_d} + \frac{kw^2}{\delta_d^2} \quad (22)$$

and

$$\delta_d = \alpha_0 \exp(-\alpha_1 t) \quad (23)$$

respectively, and k, α_0, $\alpha_1 \in \Re^1$ are positive, constant control gains. Based on (16) and the subsequent stability analysis, we design the control signal $\tau_c(t)$ as follows

$$\tau_c = \dot{u}_d + k\eta + Jzw + \tilde{z} \quad (24)$$

where $\dot{u}_d(\eta, w, z_d, z) \in \Re^2$ denotes the time derivative of (19) (see Appendix B for an explicit expression for $\dot{u}_d(t)$).

Remark 1 *Motivation for the structure of (21) is obtained by taking the time derivative of $z_d^T(t)z_d(t)$ as follows*

$$\frac{d}{dt}\left(z_d^T z_d\right) = 2z_d^T \dot{z}_d = 2z_d^T \left(\frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{kw}{\delta_d^2} + w\Omega_1\right) J z_d\right) \quad (25)$$

where (21) has been utilized. After noting that the matrix J of (10) is skew symmetric, we can rewrite (25) as follows

$$\frac{d}{dt}\left(z_d^T z_d\right) = 2\frac{\dot{\delta}_d}{\delta_d} z_d^T z_d. \quad (26)$$

As result of the selection of the initial conditions given in (21), it is easy to verify that

$$z_d^T z_d = \|z_d\|^2 = \delta_d^2 \quad (27)$$

is a unique solution to the differential equation given in (26).

3.2 Error System Development

We begin the closed-loop error system by adding and subtracting the auxiliary control input $u_d(t)$ to the right-side of the dynamics for $w(t)$ given by (9) to obtain

$$\dot{w} = -u_d^T J z + \eta^T J z \quad (28)$$

where (18) has been utilized. After substituting (19) for $u_d(t)$ into (28), we can rewrite the dynamics for $w(t)$ as follows

$$\dot{w} = -u_a^T J z_d + u_a^T J \tilde{z} + \eta^T J z \quad (29)$$

where (17) and the skew symmetry of J defined in (10) have been used. After substituting (20) for only the first occurrence of $u_a(t)$ in (29), we obtain the final closed-loop dynamics for $w(t)$ as follows

$$\dot{w} = -kw + u_a^T J \tilde{z} + \eta^T J z \quad (30)$$

where (27), the skew symmetry of J defined in (10), and the fact that $J^T J = I_2$ have been used (Note that I_2 denotes the two by two identity matrix). To determine the closed-loop error system for $\tilde{z}(t)$, we take the time derivative of (17) and then substitute (21) and (9) to obtain

$$\dot{\tilde{z}} = \frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{kw}{\delta_d^2} + w\Omega_1\right) J z_d - u + u_d - u_d \quad (31)$$

where $u_d(t)$ has been added and subtracted to right-hand side of (31). After substituting (19) for only the last occurrence of $u_d(t)$ and then substituting (20) into the resulting expression, we have

$$\dot{\tilde{z}} = \frac{\dot{\delta}_d}{\delta_d} z_d + \left(\frac{kw}{\delta_d^2} + w\Omega_1\right) J z_d - \left(\left(\frac{kw}{\delta_d^2}\right) J z_d + \Omega_1 z_d - kz\right) + \eta \quad (32)$$

where (18) has been utilized. After substituting (22) for only the second occurrence of $\Omega_1(t)$, cancelling common terms, and then rearranging the resulting expression, we have

$$\dot{\tilde{z}} = -k\tilde{z} + wJ\left[\left(\frac{kw}{\delta_d^2}\right) J z_d + \Omega_1 z_d\right] + \eta \quad (33)$$

where (17) and the fact that $JJ = -I_2$ have been used. Since the bracketed term in (33) is equal to $u_a(t)$ defined in (20), we rewrite (33) as follows

$$\dot{\tilde{z}} = -k\tilde{z} + wJu_a + \eta. \quad (34)$$

To determine the closed-loop error system for $\eta(t)$, we take the time derivative of (18), substitute (16), and then substitute (24) to obtain

$$\dot{\eta} = -k\eta - Jzw - \tilde{z}. \quad (35)$$

4 Stability Analysis

Theorem 1 *Given the closed-loop system of (30), (34), and (35), the position/orientation setpoint error defined in (6) is globally exponentially regulated in the sense that*

$$\|\tilde{x}\|, \|\tilde{y}\|, \left\|\tilde{\theta}\right\| \leq \beta_0 \exp(-\beta_1 t) \quad (36)$$

for some positive constants β_0 and β_1 provided the control parameters α_1 and k are selected as follows

$$k > 2\alpha_1. \quad (37)$$

Proof: To prove *Theorem 1*, we define the following non-negative function

$$V(t) = \frac{1}{2}w^2 + \frac{1}{2}\eta^T\eta + \frac{1}{2}\tilde{z}^T\tilde{z} \quad (38)$$

which can be written in the following compact form

$$V(t) = \frac{1}{2}\|\Psi\|^2 \quad (39)$$

where $\Psi(t) \in \Re^5$ is defined as follows

$$\Psi = \begin{bmatrix} w & \eta^T & \tilde{z}^T \end{bmatrix}^T. \quad (40)$$

After taking the time derivative of (38), making substitutions for (30), (34), and (35), and cancelling common terms, we obtain

$$\dot{V} = -kw^2 - k\eta^T\eta - k\tilde{z}^T\tilde{z} + u_a^T J\tilde{z}w + \tilde{z}^T J u_a w. \quad (41)$$

After noting that $J^T = -J$ (see (10)), we can place an upper bound on $\dot{V}(t)$ of (41) as follows

$$\dot{V} \leq -k\|\Psi\|^2 \quad (42)$$

where $\Psi(t)$ was defined in (40). Based on (39) and (42), it is evident that standard Lyapunov arguments [18] can be utilized to place an upper bound on $\Psi(t)$ of (40) as follows

$$\|\Psi(t)\| \leq \|\Psi(0)\| \exp(-kt). \quad (43)$$

From (40) and (43), we have that $w(t)$, $\eta(t)$, $\tilde{z}(t) \in \mathcal{L}_\infty$. Utilizing (17), (27), and the fact that $\tilde{z}(t) \in \mathcal{L}_\infty$, we can conclude that $z(t)$, $z_d(t) \in \mathcal{L}_\infty$. Based on the definition of $\delta_d(t)$ in (23), there appear to be potential singularities in the auxiliary terms given by (20), (21), and (22). That is, since $\delta_d(t)$ goes to zero exponentially fast, the terms contained in (20), (21), and (22) given below

$$\frac{kw}{\delta_d^2} J z_d, \qquad \frac{kw^2}{\delta_d^2} J z_d, \qquad \frac{kw^3}{\delta_d^2} J z_d, \tag{44}$$

appear to be unbounded as $t \to \infty$; however, since $w(t)$ is driven to zero within the exponential envelope given in (43), it can be clearly seen that if the sufficient condition given in (37) holds then the potential singularities depicted in (44) are always avoided. Based on this fact, we can now use standard signal chasing arguments to show that $u_d(t)$, $u_a(t)$, $\dot{z}_d(t)$, $\Omega_1(t)$, $u(t) \in \mathcal{L}_\infty$. Based on the preceding arguments, we can now show that $\dot{u}_d(t) \in \mathcal{L}_\infty$ (see Appendix B for further details). Based on the fact that $\eta(t)$, $w(t)$, $\tilde{z}(t)$, $u(t)$, $\dot{u}_d(t) \in \mathcal{L}_\infty$, we can show that $\tau_c(t)$, $\tau(t) \in \mathcal{L}_\infty$.

To show that the Cartesian position/orientation of the COM defined in (1) are bounded, we note that it is easy to show that the inverse transformation for (5) is given by

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\theta} \end{bmatrix} = \begin{bmatrix} -d\sin\theta & \frac{1}{2}\left(\tilde{\theta}\sin\theta + 2\cos\theta\right) & \frac{1}{2}\sin\theta \\ d\cos\theta & -\frac{1}{2}\left(\tilde{\theta}\cos\theta - 2\sin\theta\right) & -\frac{1}{2}\cos\theta \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix}. \tag{45}$$

Since $z(t) \in \mathcal{L}_\infty$, we can see from (45) that $\tilde{\theta}(t) \in \mathcal{L}_\infty$ (and hence $\theta(t) \in \mathcal{L}_\infty$). Since $w(t)$, $z(t)$, $\tilde{\theta}(t) \in \mathcal{L}_\infty$, we can see from (45) that $\tilde{x}(t)$, $\tilde{y}(t) \in \mathcal{L}_\infty$ (and hence $x_c(t)$, $y_c(t) \in \mathcal{L}_\infty$). To show that the linear and angular velocities defined in (3) are bounded, it is easy to show that the inverse transformation for (8) is given by

$$\begin{aligned} v_2 &= u_1 \\ v_1 &= u_2 + u_1\left(\tilde{x}\sin\theta - \tilde{y}\cos\theta\right). \end{aligned} \tag{46}$$

Since $u(t)$, $\tilde{x}(t)$, $\tilde{y}(t) \in \mathcal{L}_\infty$, we can see from (46) that $v(t) \in \mathcal{L}_\infty$; therefore, it follows from (1), (2), (3), and (4) that $\dot{\theta}(t)$, $\dot{x}_c(t)$, $\dot{y}_c(t) \in \mathcal{L}_\infty$. Standard signal chasing arguments can now be used to show that all of the remaining signals in the control and the system are bounded during closed-loop operation.

In order to prove (36), we first show that $z(t)$ defined in (5) goes to zero exponential fast by applying the triangle inequality to (17) to obtain the following exponential bound for $z(t)$

$$\|z\| \le \|\tilde{z}(t)\| + \|z_d\| \le \|\Psi(0)\| \exp(-kt) + \alpha_0 \exp(-\alpha_1 t) \tag{47}$$

where (23), (27), and (43) have been utilized. The main result given by (36) now directly follows (40), (43), (45), and (47).□

5 Simulation Results

The proposed controller was simulated based on the transformed system given in (5), (9) and (16) where the distance between the driving wheels and the COM, denoted by d, was chosen as $1m$. The desired Cartesian position and orientation were chosen as follows

$$x_{dc} = 5 \text{ (m)}, \quad y_{dc} = -5 \text{ (m)}, \quad \theta_d = 1.57 \text{ (rad)},$$

respectively, while the initial conditions for the actual Cartesian positions and orientation were selected as follows

$$x_c(0) = 10.20 \text{ (m)}, \quad y_c(0) = -1.48 \text{ (m)}, \quad \theta(0) = -0.43 \text{ (rad)},$$

$$v_1(0) = 0.0 \text{ (m/sec)}, \text{ and } v_2(0) = 0.0 \text{ (m/sec)}.$$

The control gains were tuned until the best response was obtained and then recorded as follows

$$\alpha_0 = 6.5, \quad \alpha_1 = 0.5, \quad k = 1.0. \tag{48}$$

The x and y position setpoint errors are shown in Figure 1, the orientation setpoint error is shown in Figure 2, and the auxiliary torque control inputs are depicted in Figure 3.

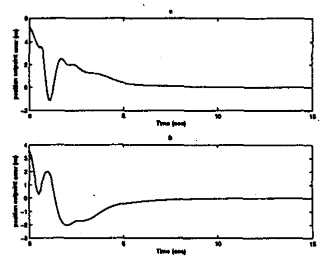

Figure 1: Setpoint errors a) $\tilde{x}$ and b) $\tilde{y}$

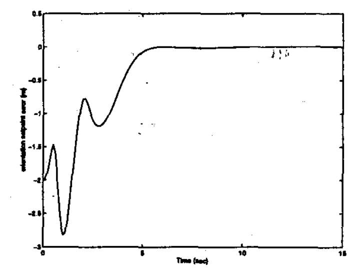

Figure 2: Orientation setpoint error

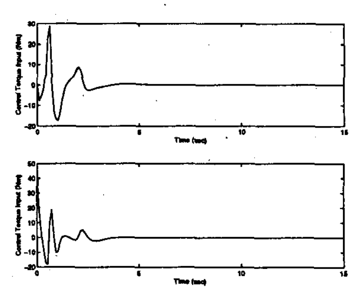

Figure 3: Auxillary torque control input a) τ_{c1} and b) τ_{c2}

6 Comparative Analysis

In order to contrast the performance of the proposed controller with several other proposed kinematic controllers, we now compare the simulation results from the kinematic portion of the proposed global exponential controller with the kinematic controllers presented in [12], [17], and [19]. Specifically, for the following kinematic model

$$\begin{aligned} \dot{p}_1 &= u_1 \\ \dot{p}_2 &= u_2 \\ \dot{p}_3 &= p_2 u_1 \end{aligned} \tag{49}$$

we compare the asymptotic controllers presented in [17] and [19] as shown below

$$\begin{aligned} u_1 &= -p_1 + p_3\cos(t) \\ u_2 &= -p_2 + p_3^2\sin(t) \end{aligned} \tag{50}$$

and

$$\begin{aligned} u_1 &= -p_1 + p_3^2\sin(t) \\ u_2 &= -p_2 - p_3\left(-p_1 + p_3^2\sin(t)\right) \end{aligned} \tag{51}$$

respectively, and the "ρ−exponential" controller of [12] given as

$$\begin{aligned} u_1 &= -p_1 + \frac{p_3}{(p_1^4 + p_2^4 + p_3^2)^{\frac{1}{4}}}\cos(t) \\ u_2 &= -p_2 + \frac{p_3^2}{(p_1^4 + p_2^4 + p_3^2)^{\frac{2}{4}}}\sin(t) \end{aligned} \tag{52}$$

with the controller proposed in (19), (20), (21), (22), and (23). We point out that the notation of the controllers described in (50), (51), and (52) have been rewritten in terms of the notation used in this paper for ease of readability. In order to portray an accurate comparison between the controllers, the initial conditions of each controller were chosen to be the same as follows

$$p_1(0) = 0, \qquad p_2(0) = 0, \qquad p_3(0) = 1 \tag{53}$$

(note the above initial values were chosen as the same values that were utilized in the comparative analysis performed by McCloskey *et al.* in [12]). Moreover, the control gains for the proposed controller were selected as follows

$$\alpha_0 = 2.45, \qquad \alpha_1 = 0.25, \qquad k = 0.5$$

in order to have a similar control input profile as the controller given in [12]. From Figure 4, it is straightforward to conclude that the proposed controller exhibits slightly favorable transient response in comparison with the controller of [12] (Note that it may be possible to tune the controllers to achieve different transient results). In addition to slightly improved transient performance of the proposed controller, Figure 5 illustrates that the proposed controller requires approximately the same amount of control energy in comparison with the controller of [12].

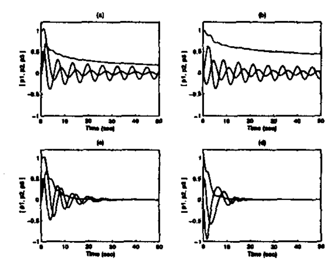

Figure 4: Comparison of the kinematic system response of the controllers presented in a) [19], b) [17], c) [12], and d) the proposed controller.

7 Conclusion

In this paper, we utilized our previous experience in the field of induction motor control to design a new differentiable time-varying controller for the stabilization about a fixed setpoint problem for the wheeled mobile robot. Through the use of a dynamic oscillator and a Lyapunov stability analysis, we show that the controller exponentially stabilizes the WMR about any fixed setpoint. Simulation results indicate a favorable comparison between the proposed kinematic control algorithm and several previously proposed kinematic control algorithms. It should be noted that in addition to the WMR problem, the kinematic portion of the proposed controller can be applied to other nonholonomic systems (for several other examples of kinematic systems the proposed controller can exponentially stabilize about a point see [2]). It should also be pointed out that the standard exponential stability result presented in this paper allows one to utilize existing Lyapunov control design tools to provide enhancements to the proposed control structure. Specifically, in a forthcoming paper, we will illustrate how the controller can be redesigned to provide for robustness against uncertainty in the dynamic model.

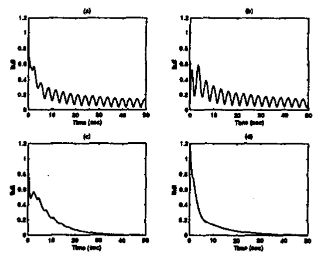

Figure 5: Comparison of the Euclidean norm of the control inpus for the controllers presented in a) [19], b) [17], c) [12], and d) the proposed controller.

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Appendix A

The explicit definitions of $\bar{M}(q)$, $\bar{V}_m(q,\dot{q})$, and $\bar{B}$ are given as follows

$$\bar{M}(q)=\begin{bmatrix} m & 0 \\ 0 & I-md^2 \end{bmatrix},\quad \bar{V}_m(q,\dot{q})=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\tag{54}$$

and

$$\bar{B}=\frac{1}{r}\begin{bmatrix} 1 & 1 \\ R & -R \end{bmatrix}\tag{55}$$

where $m\in\Re^1$ denotes the mass of the WMR, $I\in\Re^1$ is a function of the moment of inertia of the mobile platform and the moment wheels (for specific details see [11]), $r\in\Re^1$ denotes the radius of the wheels, $R\in\Re^1$ is half the distance between the driving wheels, and $\theta(t)$ and d are defined in (4).

Appendix B

First, to illustrate how $\dot{u}_d(\cdot)$ can be written in terms of the transformed states of the system (*i.e*, $\eta(t)$, $w(t)$, $z_d(t)$, $z(t)$), we take the time derivative of (19), substitute for the time derivative of $u_a(t)$ defined in (20), utilize (9) and then add/subtract ku_d to the left-side to obtain

$$\begin{aligned}\dot{u}_d &= \left(\frac{k\dot{w}}{\delta_d^2}\right)Jz_d-2\left(\frac{kw\dot{\delta}_d}{\delta_d^3}\right)Jz_d+\dot{\Omega}_1 z_d \\ &\quad+\left(\Omega_1+\left(\frac{kw}{\delta_d^2}\right)J\right)\dot{z}_d-k\left[u_d-\eta\right].\end{aligned}\tag{56}$$

After substituting (19), (20), (21), (22), (30) and the time derivative of $\Omega_1(t)$, we have

$$\begin{aligned}\dot{u}_d &= \left(\frac{k\eta^T Jz}{\delta_d^2}+\left(\frac{k^2 w}{\delta_d^4}\right)z_d^T\tilde{z}\right)Jz_d \\ &\quad+\left(\left(\frac{k^2}{\delta_d^2}+\frac{k\dot{\delta}_d}{\delta_d^3}+\frac{k^2w^2}{\delta_d^4}\right)z_d^T J\tilde{z}-\frac{k^2 w}{\delta_d^2}\right)Jz_d \\ &\quad-2\left(\frac{kw\dot{\delta}_d}{\delta_d^3}\right)Jz_d+\frac{\ddot{\delta}_d}{\delta_d}z_d-\frac{\dot{\delta}_d^2}{\delta_d^2}z_d+2\frac{kw\eta^T Jz}{\delta_d^2}z_d \\ &\quad+2kw\left(\left(\frac{kw}{\delta_d^4}\right)z_d^T J^T+\left(\frac{k}{\delta_d^2}+\frac{\dot{\delta}_d}{\delta_d^3}+\frac{kw^2}{\delta_d^4}\right)z_d^T\right)J\tilde{z}z_d \\ &\quad-2\frac{k^2w^2}{\delta_d^2}z_d-2\frac{kw^2\dot{\delta}_d}{\delta_d^3}z_d \\ &\quad+\left(k+\frac{\dot{\delta}_d}{\delta_d}+\frac{kw^2}{\delta_d^2}+\left(\frac{kw}{\delta_d^2}\right)J\right) \\ &\quad\left(\frac{\dot{\delta}_d}{\delta_d}z_d+\frac{kw}{\delta_d^2}Jz_d+w\left(k+\frac{\dot{\delta}_d}{\delta_d}+\frac{kw^2}{\delta_d^2}\right)Jz_d\right) \\ &\quad-k\left[\left(\frac{kw}{\delta_d^2}\right)Jz_d+\left(k+\frac{\dot{\delta}_d}{\delta_d}+\frac{kw^2}{\delta_d^2}\right)z_d-kz-\eta\right].\end{aligned}\tag{57}$$

Second, to illustrate that $\dot{u}_d(t)\in\mathcal{L}_\infty$, we utilize (10), (17), (23), and (27) to upper bound $\dot{u}_d(t)$ as follows

$$\begin{aligned}\|\dot{u}_d\| &\le k\|\eta\|+\frac{k\|\eta\|\|\tilde{z}\|}{\delta_d}+\left(\frac{k^2|w|}{\delta_d^2}\right)\|\tilde{z}\| \\ &\quad+\left(k^2+k\alpha_1+\frac{k^2w^2}{\delta_d^2}\right)\|\tilde{z}\|+\frac{k^2|w|}{\delta_d} \\ &\quad+2\left(\frac{k|w|\alpha_1}{\delta_d}\right)+2k|w|\|\eta\| \\ &\quad+2\frac{k|w|\|\eta\|\|\tilde{z}\|}{\delta_d}+\left(2\frac{k^2w^2}{\delta_d^2}\right)\|\tilde{z}\| \\ &\quad+\left(2k^2|w|+2k|w|\alpha_1+2\frac{k^2|w|^3}{\delta_d^2}\right)\|\tilde{z}\| \\ &\quad+2\frac{k^2w^2}{\delta_d}+2\frac{kw^2\alpha_1}{\delta_d}+\left(k+\alpha_1+\frac{kw^2}{\delta_d^2}+\left(\frac{k|w|}{\delta_d^2}\right)\right) \\ &\quad\left(\dot{\delta}_d+\frac{k|w|}{\delta_d}+|w|\left(k\alpha_0+\dot{\delta}_d+\frac{kw^2}{\delta_d}\right)\right) \\ &\quad+\left(\frac{k^2|w|}{\delta_d}\right)+\left(k^2\alpha_0+k\dot{\delta}_d+\frac{k^2w^2}{\delta_d}\right) \\ &\quad+k^2\alpha_0+k^2\|\tilde{z}\|+k\|\eta\|.\end{aligned}\tag{58}$$

By utilizing the definition of $\Psi(t)$ given in (40), substituting for the time derivative of (23), and simplifying the resulting inequality, we can now place the following upper bound on (58)

$$\begin{aligned}\|\dot{u}_d\| &\le k\|\Psi\|+2k^2\|\Psi\|+k(\alpha_1+1)\|\Psi\| \\ &\quad+2k^2\|\Psi\|^2+2k(\alpha_1+1)\|\Psi\|^2 \\ &\quad+2k\alpha_0\alpha_1(1+\|\Psi\|)+\alpha_0\alpha_1^2(1+\|\Psi\|) \\ &\quad+k^2\alpha_0(2+\|\Psi\|)+k\alpha_0\alpha_1\|\Psi\| \\ &\quad+\frac{4k\alpha_1\|\Psi\|}{\delta_d}+\frac{3k^2\|\Psi\|}{\delta_d}+\frac{k\|\Psi\|^2}{\delta_d} \\ &\quad+\frac{3k^2\|\Psi\|^2}{\delta_d}+\frac{4k\alpha_1\|\Psi\|^2}{\delta_d}+\frac{k^2(\alpha_0+1)\|\Psi\|^2}{\delta_d^2} \\ &\quad+\frac{k^2\|\Psi\|^2}{\delta_d^3}+\frac{2k(\alpha_1+1)\|\Psi\|^3}{\delta_d}+\frac{k^2\|\Psi\|^3}{\delta_d} \\ &\quad+\frac{k^2(\alpha_0+3)\|\Psi\|^3}{\delta_d^2}+\frac{k^2\|\Psi\|^3}{\delta_d^3} \\ &\quad+\frac{2k^2\|\Psi\|^4}{\delta_d^2}+\frac{k^2\|\Psi\|^4}{\delta_d^3}+\frac{k^2\|\Psi\|^5}{\delta_d^3}.\end{aligned}\tag{59}$$

Based on the definition of $\delta_d(t)$ given in (23), it appears that singularities may exist in the following terms found in (59)

$$\begin{gathered}\frac{\|\Psi\|}{\delta_d},\ \frac{\|\Psi\|^2}{\delta_d},\ \frac{\|\Psi\|^2}{\delta_d^2},\ \frac{\|\Psi\|^2}{\delta_d^3},\ \frac{\|\Psi\|^3}{\delta_d}, \\ \frac{\|\Psi\|^3}{\delta_d^2},\ \frac{\|\Psi\|^3}{\delta_d^3},\ \frac{\|\Psi\|^4}{\delta_d^2},\ \frac{\|\Psi\|^4}{\delta_d^3},\ \frac{\|\Psi\|^5}{\delta_d^3}.\end{gathered}\tag{60}$$

However, upper bounds for the terms above can be formulated as follows

$$\begin{gathered}\frac{\|\Psi(0)\|\exp(-kt)}{\alpha_0\exp(-\alpha_1 t)},\quad \frac{\|\Psi(0)\|^2\exp(-2kt)}{\alpha_0\exp(-\alpha_1 t)}, \\ \frac{\|\Psi(0)\|^2\exp(-2kt)}{\alpha_0^2\exp(-2\alpha_1 t)},\quad \frac{\|\Psi(0)\|^2\exp(-2kt)}{\alpha_0^3\exp(-3\alpha_1 t)}, \\ \frac{\|\Psi(0)\|^3\exp(-3kt)}{\alpha_0\exp(-\alpha_1 t)},\quad \frac{\|\Psi(0)\|^3\exp(-3kt)}{\alpha_0^2\exp(-2\alpha_1 t)}, \\ \frac{\|\Psi(0)\|^3\exp(-3kt)}{\alpha_0^3\exp(-3\alpha_1 t)},\quad \frac{\|\Psi(0)\|^4\exp(-4kt)}{\alpha_0^2\exp(-2\alpha_1 t)}, \\ \frac{\|\Psi(0)\|^4\exp(-4kt)}{\alpha_0^3\exp(-3\alpha_1 t)},\quad \frac{\|\Psi(0)\|^5\exp(-5kt)}{\alpha_0^3\exp(-3\alpha_1 t)};\end{gathered}\tag{61}$$

hence, it is easy to see that if k is chosen according to the sufficient condition given in (37), then the singularities are avoided. Based on the fact that the singularities are always avoided and the fact that $\eta(t)$, $w(t)$, $z_d(t)$, $z(t)\in\mathcal{L}_\infty$, it is straightforward to see from (57) that $\dot{u}_d(t)\in\mathcal{L}_\infty$.