3. Planar kinematics

Mechanics of Manipulation

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon

Outline.

- One more general theorem: Displacement $=$ translation $\circ$ rotation.
- Some fundamental theorems on planar motion.
- Main result: every planar motion has a rotation center in projective plane.
- Centrodes.

Decomposition of displacements

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So then $T \circ R = T \circ T^{-1} \circ D = D$ is the desired decomposition. QED

Decomposition of displacements

Note:

- Instead of $R = T^{-1} \circ D$, $D = T \circ R$,
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- The decomposition is not unique: it depends on the choice of O.
- The proof extends to arbitrary $\mathbb{E}^n$.
- Note how simple it is to prove using group theory!

Planar kinematics

That is all we will do on “general” kinematics. On to planar kinematics.
What can we say about rigid motions of $\mathbb{E}^2$?
Theorem 2.3: A planar displacement is completely determined by the motion of any two points.

Proof: Construct a coordinate frame ...

Planar kinematics: every D is an R or a T

Now for the big one:

Theorem 2.4: Every planar displacement is either a translation or a rotation.

Not a proof:

Pick two points A and B.

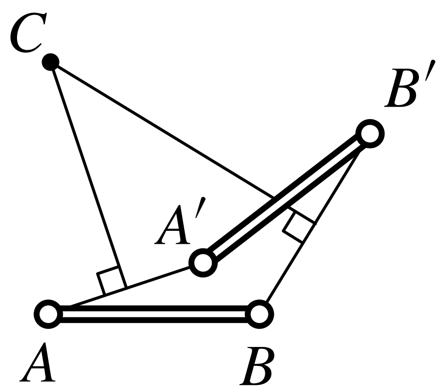
Let A' and B' be the images.

Construct perpendicular bisectors.

Intersection gives fixed point.

Why? Preserves distance from A and from B, so ...

Okay, not a proof, but a useful construction.

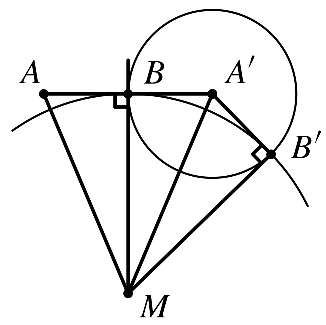


Planar kinematics: every D is an R or a T

Theorem 2.4: Every planar displacement is either a translation or a rotation.

Proof:

Pick any point A. We can assume $A \neq A'$.



Pick B the midpoint of the line segment $\overline{AA'}$. We can assume B' is not on $\overline{AA'}$.

Construct perp to AB at B, and perp to $A'B'$ at B'. They are not parallel. Let M be their intersection.

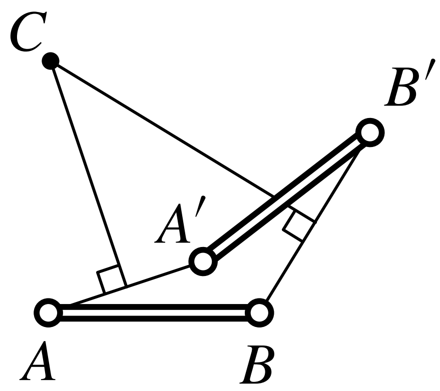
Consider the rotation that maps A to A' and M to itself. Where does it map B? Preservation of distance gives two candidates, and we can exclude one.

So our rotation maps B to B'. It is the given displacement. QED.

Planar kinematics. Rotation centers

So, every planar displacement is a rotation or a translation.

Consider again construction of rotation centers from the motion of two points. How does it fail when $\overline{AA'}$ is parallel to $\overline{BB'}$?



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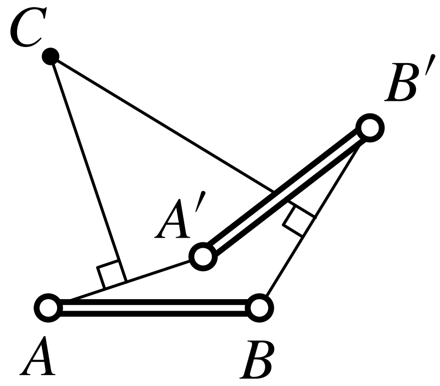

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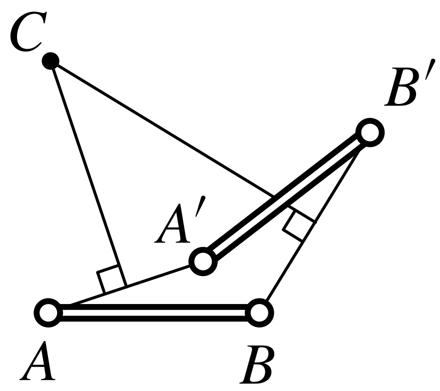

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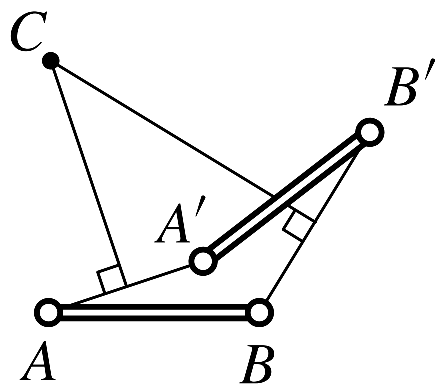


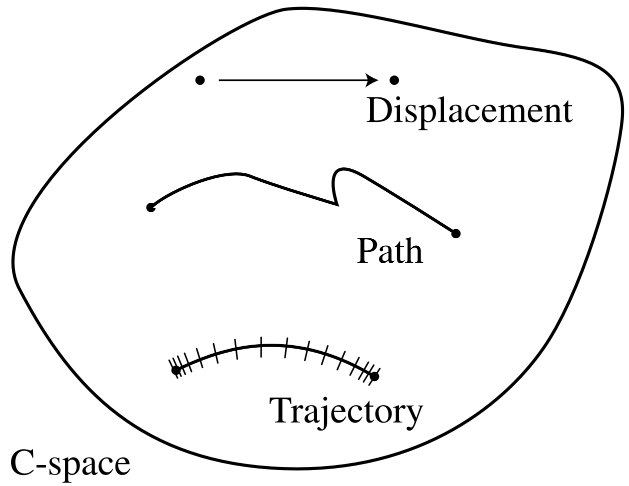
Every planar displacement is a rotation about a point in the projective plane.

Displacements, paths, trajectories.

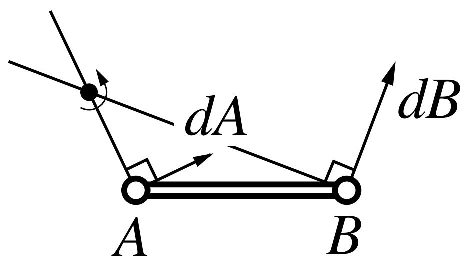
Displacement Discontinuous change of configuration.

Trajectory Configuration a continuous function of time: a continuous curve $q(t)$ in configuration space.

Path A curve $q(s)$ in configuration space parameterized perhaps by arc length.



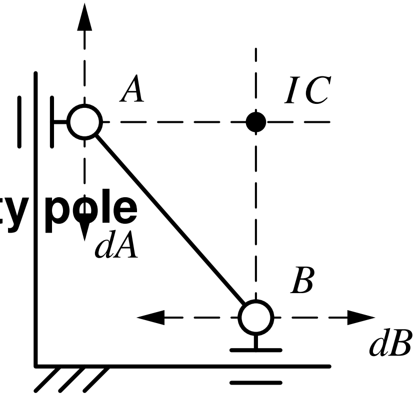
For differentiable trajectory $q(t)$ or a path $q(s)$ we have *velocity* dq/dt or *differential change in configuration* dq.
To construct rot'n center for diff'l displacement:



Planar kinematics. ICs for 4bar linkages.

Terminology:

Rotation pole; rotation center for displacements.



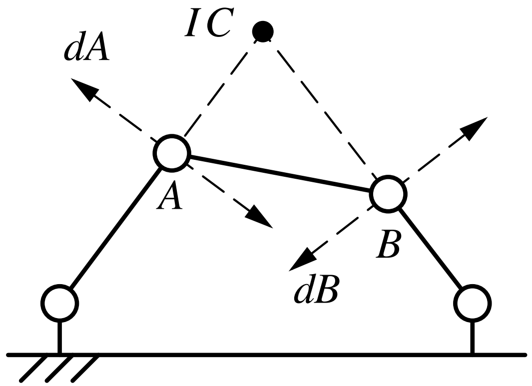
Instantaneous center; IC; velocity center; velocity pole for velocities or differential displacements.

Example: Four bar linkages!

Base link is fixed.

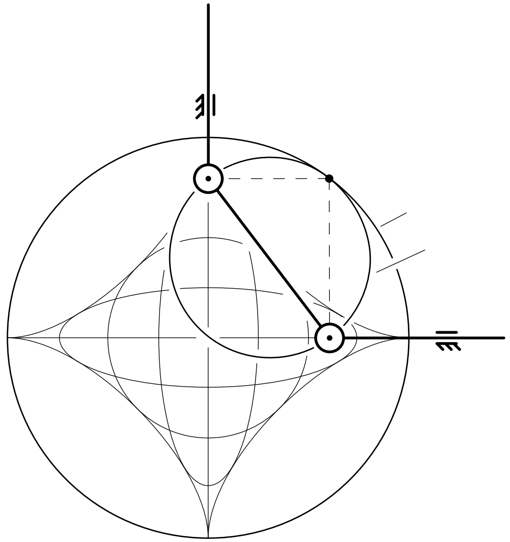
Two links that either translate or rotate w.r.t. base link.

The *coupler link*, which can make all sorts of interesting motions.



Construct the ICs for two different four-bar linkages.

Planar kinematics. Centrodes.



Take an arbitrary continuous planar motion. Generally the IC moves.

Plot the IC in the fixed plane. That gives the *fixed centrode*.

Plot the IC in the moving plane. That gives the *moving centrode*.

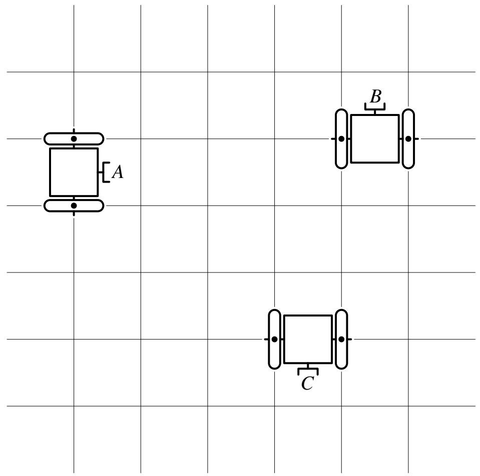
For any given time, the two curves must touch at the IC. And the moving plane rotates about the IC.

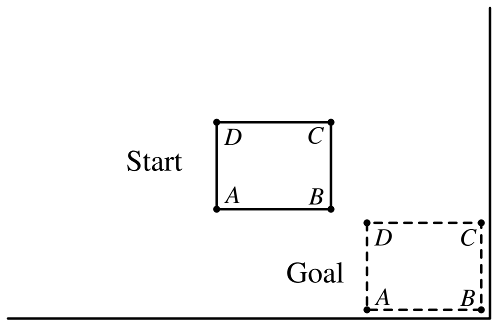
I.e. the moving centrode rolls without slipping on the fixed centrode.

It even works for discontinuous motions. If the centrodes are

Problem set 1.

1. DOFs of a line in $\mathbb{E}^3$.
2. Centrodes (central polygons) for mobile robot.
3. Centrodes (central polygons) for fridge.
4. Centrode for Chebyshev's linkage. *Show them my solution for Reuleaux's example.*





Centrodes for Watt's linkage

Planar kinematics. False ICs.

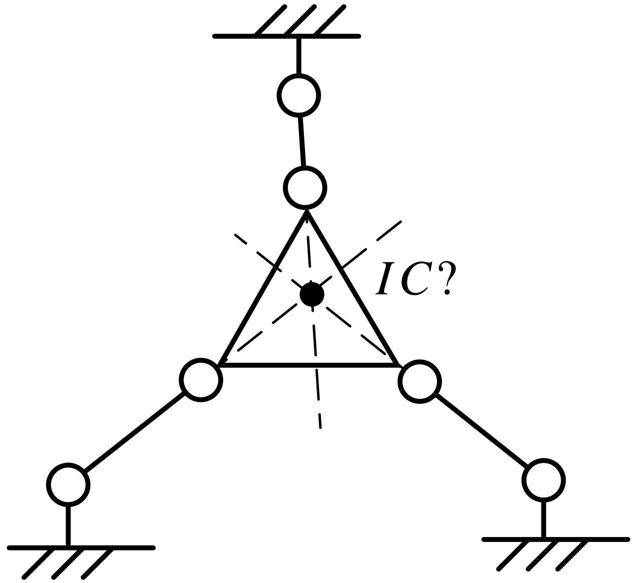
Review procedure for linkage IC construction

1. Reduce the constraints to point-velocity constraints.
2. Construct perpendiculars to allowed velocities at each point.
3. Intersection of perpendiculars are candidate ICs

   No intersection means no ICs. It must be immobile.

   But parallel lines intersect at infinity.

But existence of intersection does not imply mobility!!!



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