7. Quaternions

Mechanics of Manipulation

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Outline.

- What is a quaternion
- Representing rotation
- Geometric view
- Transformations to other representations
- Topological and metric properties

Why can't we invert vectors in $\mathbf{R}^3$?

We can invert reals. $x \times \frac{1}{x} = 1$.

We can invert elements of $\mathbf{R}^2$ using complex numbers. $z \times z^* = 1$, where * is complex conjugate.

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Hamilton's quaternions are to $\mathbf{R}^3$ what complex numbers are to $\mathbf{R}$.

Complex numbers versus quaternions

To define complex numbers:

- Basis elements 1 and i;
- Vector space over reals: elements have the form $x + iy$;
- One more axiom required: i^2 = -1.

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To define quaternions:

- Basis elements 1, i, j, k;
- Vector space over reals: elements have the form $q_0 + q_1 i + q_2 j + q_3 k$;
- Six more axioms:

$$
\begin{aligned}
i^2 = j^2 = k^2 &= -1 \\
ij &= k \\
jk &= i \\
ki &= j
\end{aligned}
$$

Quaternion notation

We can write a quaternion several ways:

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

$$q = (q_0, q_1, q_2, q_3)$$

$$q = q_0 + \mathbf{q}$$

where q_0 is the *scalar part* and $\mathbf{q}$ is the *vector part*

Quaternion product

We can write a quaternion product several ways:

$$
\begin{aligned}
pq &= (p_0 + p_1 i + p_2 j + p_3 k)(q_0 + q_1 i + q_2 j + q_3 k) \\
&= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + \ldots i + \ldots j + \ldots k \\
pq &= (p_0 + \mathbf{p})(q_0 + \mathbf{q}) \\
&= (p_0 q_0 + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{pq})
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$$

So what is $\mathbf{pq}$? Cross product? Dot product?

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$$
pq = (p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})
$$

Conjugate, length

Quaternion **conjugate**:

$$q^* = q_0 - q_1 i - q_2 j - q_3 k$$

Note that

$$\begin{aligned} qq^* &= (q_0 + \mathbf{q})(q_0 - \mathbf{q}) \\ &= q_0^2 + q_0\mathbf{q} - q_0\mathbf{q} - \mathbf{q}\mathbf{q} \\ &= q_0^2 + \mathbf{q} \cdot \mathbf{q} - \mathbf{q} \times \mathbf{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{aligned}$$

Quaternion **length**:

$$|q| = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Quaternion inverse

Note that every quaternion other than the additive identity 0 has an inverse:

$$q^{-1} = \frac{q^*}{|q|^2}$$

That means quaternions are a linear algebra and a field. Hamilton's dream. Quaternions are the only extension of complex numbers that is both a linear algebra and a field. If 1D numbers are the reals, and 2D numbers are the complex numbers, then 4D numbers are quaternions, and that's all there is. (Frobenius?)

Rotation using unit quaternions

Let q be a unit quaternion, i.e. $|q| = 1$.

It can be expressed as

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

Let $x = 0 + \mathbf{x}$ be a "pure vector".

Let $x' = qxq^*$.

Then x' is the pure vector $\mathrm{rot}(\theta, \hat{\mathbf{n}})\mathbf{x}$!!!

Proof that unit quaternions work

Expand the product qxq^*;

Apply half angle formulas;

Simplify;

to obtain Rodrigues's formula.

Why $\theta/2$? Why qxq^* instead of qx?

Two puzzling things. In analogy with complex numbers, why not use

$$p = \cos\theta + \hat{\mathbf{n}}\sin\theta$$

$$\mathbf{x}' = p\mathbf{x}$$

To explore that idea, define a map $L_p(q) = pq$ with p a unit pure vector. Note that $L_p(q)$ can be written:

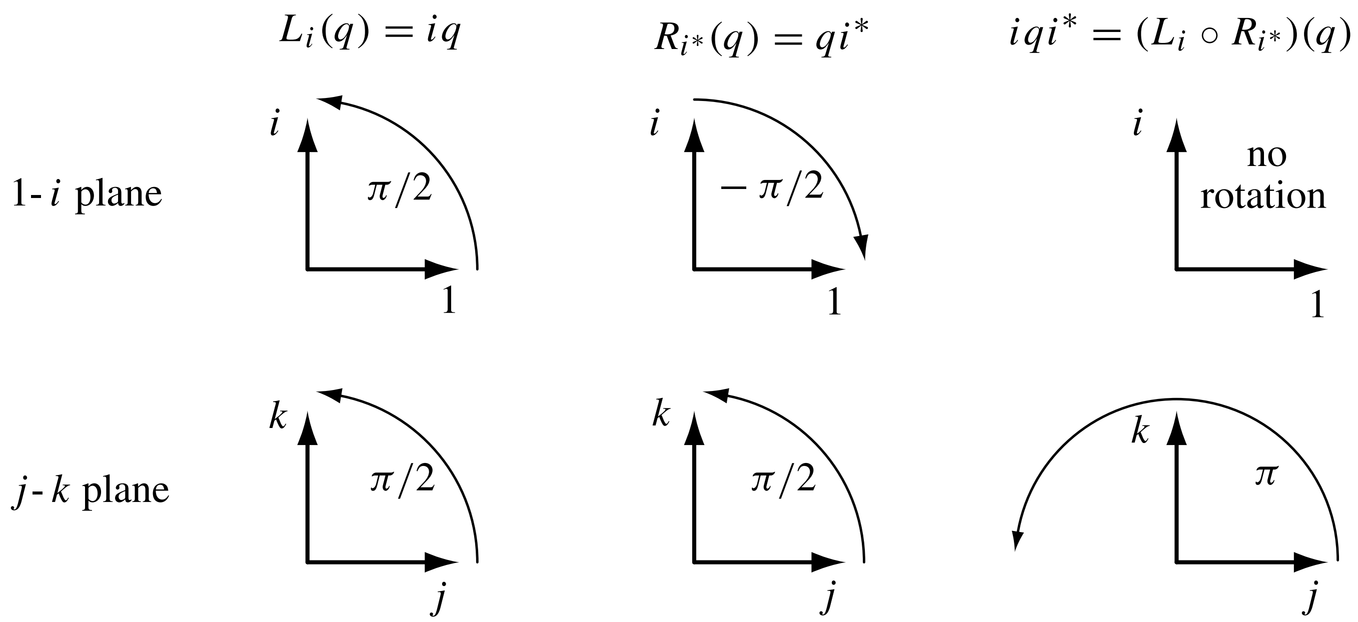
$$L_p(q) = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Note that the matrix above is orthonormal. L_p is a rotation of Euclidean 4 space! (Without even using the fact that p is a pure vector.)

Geometrical explanation

Although $L_p(q)$ rotates the 4D space of quaternions, it is *not* a rotation of the 3D subspace of pure vectors. Some of the 3D subspace leaks into the fourth dimension.

Consider an example using $p = i$. Is it a rotation about i of $\pi/2$?



What do we do with a representation?

Rotate a point: qxq^*.

Compose two rotations:

$$q(p\mathbf{x}p^*)q^* = (qp)\mathbf{x}(qp)^*$$

Convert to other representations:

From axis-angle to quaternion:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

From quaternion to axis-angle:

$$\theta = 2\tan^{-1}(|\mathbf{q}|, q_0)$$
$$\hat{\mathbf{n}} = \mathbf{q}/|\mathbf{q}|$$

assuming θ is nonzero.

From quaternion to rotation matrix

Just expand the product

$$qxq^* = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix} \mathbf{x}$$

From rotation matrix to quaternion

Given $R = (r_{ij})$, solve expression on previous page for quaternion elements q_i

Linear combinations of diagonal elements seem to solve the problem:

$$
\begin{aligned}
q_0^2 &= \frac{1}{4}(1 + r_{11} + r_{22} + r_{33}) \\
q_1^2 &= \frac{1}{4}(1 + r_{11} - r_{22} - r_{33}) \\
q_2^2 &= \frac{1}{4}(1 - r_{11} + r_{22} - r_{33}) \\
q_3^2 &= \frac{1}{4}(1 - r_{11} - r_{22} + r_{33})
\end{aligned}
$$

so take four square roots and you're done? You have to figure the signs out. There is a better way ...

Look at the off-diagonal elements

$$
\begin{aligned}
q_0 q_1 &= \frac{1}{4}(r_{32} - r_{23}) \\
q_0 q_2 &= \frac{1}{4}(r_{13} - r_{31}) \\
q_0 q_3 &= \frac{1}{4}(r_{21} - r_{12}) \\
q_1 q_2 &= \frac{1}{4}(r_{12} + r_{21}) \\
q_1 q_3 &= \frac{1}{4}(r_{13} + r_{31}) \\
q_2 q_3 &= \frac{1}{4}(r_{23} + r_{32})
\end{aligned}
$$

Given any one q_i, could solve the above for the other three.

The procedure

1. Use first four equations to find the largest q_i^2. Take its square root.
2. Use the last six equations (well, three of them anyway) to solve for the other q_i.

   That way, only have to worry about getting one sign right.

   Actually q and $-q$ represent the same rotation, so no worries about signs.

   Taking the largest square root avoids division by small numbers.

Properties of unit quaternions

Unit quaternions live on the unit sphere in $\mathbf{R}^4$.

Quaternions q and $-q$ represent the same rotation.

Inverse of rotation q is the conjugate q^*.

Null rotation, the identity, is the quaternion 1.

Metrics and topologies

Quaternions have the right metric. Consider unit quaternion

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

Shortest path on the unit sphere joining $\pm q$ with 1 has length $\theta/2$.

What is the shortest distance on the sphere from $\pm p$ to $\pm q$? The same as the distance from $\pm pq^*$ to 1. I.e. $\alpha/2$, where α is the rotation angle required from p to q.

The right metric matters. Uniform distribution on the three-sphere maps to uniform distribution on $SO(3)$. Hence problem 2 on the problem set.

What is the topology of $SO(3)$? Since unit quaternion representation has the right metric, it also has the right topology. What do we call the topology of a three-sphere with antipodes identified?