Notes on model predictive control

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Contents

Chapter 1

MPC: Problem Formulation

In this chapter, a generalized model predictive control (MPC) problem is defined and results about guaranteed stability are summarized. The survey paper [7] is used as a guide so the perspective considers constrained MPC, although the nonlinear unconstrained case is given some attention. The next chapter gives detailed proofs of stability for some of the MPC formulations.

1.1 Problem Setting

The general system under consideration will be of the form

$$\begin{aligned} &\dot{x} = f(x,u),\ x(t_0) = x_0,\ t \in [t_0, \infty),\ t_0 \in \mathbb{R}^+ \\ &f : \mathbb{X} \subset \mathbb{R}^n \times \mathbb{U} \subset \mathbb{R}^m \to \mathbb{R}^n. \end{aligned} \tag{1.1}$$

The control objective is to steer the state x to the origin, which in turn should be an equilibrium point for equation (1.1). For the MPC formulation, the following are standard assumptions:

A1 f is twice continuously differentiable (C^2) [4], [8]. For control affine forms of equation (1.1), i.e. $f(x,u) \mapsto f(x) + g(x)u$, other MPC formulations only require that f and g be C^1 functions [10], [9].

A2 $f(0,0) = 0$ and $(D_x f(0,0), D_u f(0,0))$ is controllable [4]. Only stabilizability of the linearization is assumed in [8].

A3 $u(t) \in \mathbb{U}$, a convex compact subset of $\mathbb{R}^m$ and $x(t) \in \mathbb{X}$, a convex closed subset of $\mathbb{R}^n$, both sets containing the origin, for all t of interest.

A4 The function $u \mapsto f(x,u)$ is convex for each $x \in \mathbb{X}$.

Item A3 represents constraints on the states and inputs. For unconstrained MPC problems $\mathbb{U} = \mathbb{R}^m$ and $\mathbb{X} = \mathbb{R}^n$. For nonlinear unconstrained MPC the states and inputs may be required to satisfy some norm bound, e.g. $u \in L_1(0,T)$

(see [4]). In any case, given an initial state $x(t_0) = x_0$ and a control trajectory $u(\cdot) \in \mathbb{U}$, the state trajectory $x^u(\cdot; x_0)$ is the curve satisfying

$$x^u(t; x_0) = x_0 + \int_{t_0}^{t} f(x^u(\tau; x_0), u(\tau))\ \mathrm{d}\tau, \quad \forall t \geq t_0, \tag{1.2}$$

when the constraints on the state are not active.

Define the cost of applying a control $u(\cdot)$ from an initial state x_0 over the infinite-time interval $[t_0, \infty)$ as

$$J_\infty(u(\cdot); x_0) = \int_{t_0}^{\infty} q(x^u(\tau; x_0), u(\tau))\ \mathrm{d}\tau \tag{1.3}$$

where the incremental cost $q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ is assumed to satisfy

- B1 $q(0, 0) = 0$ and is twice continuously differentiable (C^2).
- B2 $q(x, u) \geq c_q(||x||^2 + ||u||^2),\ \forall x \in \mathbb{X},\ u \in \mathbb{U}$ and for some $c_q > 0$.
- B3 The function $u \mapsto q(x, u)$ is convex for each $x \in \mathbb{X}$.

These assumptions imply that the quadratic approximation of q near the origin is positive definite. Assumption B2 can be relaxed to positive semi-definiteness in x if a zero-state detectability condition is satisfied [9]. Henceforth the cost $J_\infty(\cdot)$ is referred to as the *value function*. The optimal cost, called the *optimal value function*, from x_0 is

$$J^*_\infty(x_0) = \inf_{u(\cdot)} J_\infty(u(\cdot); x_0) \tag{1.4}$$

We are now interested in approximating the infinite horizon optimization problem with a finite horizon optimization. Let V be a nonnegative C^2 function and define the finite horizon value function as

$$J(u(\cdot); x_0, T) = \int_{0}^{T} q(x^u(\tau; x_0), u(\tau))\ \mathrm{d}\tau + V(x^u(T; x_0)), \tag{1.5}$$

and denote the optimal value function and control trajectory as

$$J^*(x_0, T) = \inf_{u(\cdot)} J(u(\cdot); x_0, T), \quad u^*(t; x_0, T) = \arg\inf J(u(\cdot); x_0, T),\ t \in [0, T]. \tag{1.6}$$

In equation (1.5) we set $t_0 = 0$, i.e. we can regard the current initial time as zero since $f(\cdot)$, $V(\cdot)$ and $q(\cdot)$ are time-invariant. It is clear that the finite horizon optimal control trajectory u^* depends on the current (initial) state x_0. The terminal cost function $V(\cdot)$ in equation (1.5) will be important for proving stability later. To guarantee the existence of equation (1.6) and the corresponding optimal state trajectory, denoted as $x^*(t; x_0, T)$, we make the following standing assumption.

S1 The minimum of the value functions $J^*_\infty(\cdot)$, $J^*(\cdot, T)$, $T \geq 0$, is attained.

This does not imply uniqueness of the optimal solution. Additionally we may require that $x^u(T; x_0) \in X_f$, where X_f is called the **terminal constraint set**, a compact subset of $\mathbb{X}$ containing the origin. Also assume that inside X_f a stabilizing controller κ_f exists and is employed. Existence of κ_f local to $\{0\}$ is actually guaranteed by A2. We say that a terminal equality constraint or terminal state is employed when $X_f \equiv \{0\}$ and no terminal constraint is employed when $X_f \equiv \mathbb{X}$. The general finite horizon optimal control problem of interest is now defined.

Definition 1.1. We refer to $\mathbb{P}_T(x)$ as the problem of solving equation (1.6) subject to

1. equation (1.1), given initial condition x at current time t,
2. the stated assumptions (A1-A4, B1-B3) and
3. terminal constraint X_f.

We refer to the solution $u^*(\cdot; x, T)$ of $\mathbb{P}_T(x)$ as **stabilizing** if $x \to 0$ in some finite or possibly infinite time.

The infinite horizon solution has guaranteed stability, which is now shown. The principle of optimality for the infinite horizon problem says that for *any* optimal trajectory $(x^*_\infty(\cdot; x_0), u^*_\infty(\cdot; x_0))$ originating at x_0 and for any $\delta > 0$

$$J^*_\infty(x^*_\infty(\delta; x_0)) = J^*_\infty(x_0) - \int_0^\delta q(x^*_\infty(\tau; x_0), u^*_\infty(\tau; x_0))\, \mathrm{d}\tau. \tag{1.7}$$

Assuming $\tau \mapsto u^*_\infty(\tau; x_0)$ is continuous ($J^*_\infty(\cdot)$ is C^2), the limit of equation (1.7) as $\delta \to 0^+$ exists and can be written as

$$\begin{aligned} &\dot{J}^*_\infty(x_0, u^*_\infty(0; x_0)) + q(x_0, u^*_\infty(0; x_0)) = 0, \quad \text{where} \\ &\dot{J}^*_\infty(x, u) = DJ^*_\infty(x) \cdot f(x, u) \text{ (Lie derivative in direction } f). \end{aligned} \tag{1.8}$$

Since q is positive definite, $\dot{J}^*_\infty$ is negative definite and the STATIC feedback $u = \kappa_\infty(x) \triangleq u^*_\infty(0; x_0)$ globally asymptotically stabilizes the origin. So under the given assumptions the infinite horizon optimal controllers are stabilizing and the value function serves as an appropriate Lyapunov function for establishing such stability. Except in simple cases (e.g. linear unconstrained), the solution to the infinite horizon problem is **not tractable**.

The goal of the finite horizon problem $\mathbb{P}_T$ is to achieve the desired infinite horizon properties of stability (and performance) while posing a tractable computational problem. If the terminal cost $V(\cdot)$ in equation (1.5) is exactly the infinite horizon cost-to-go $J^*_\infty(\cdot)$, the globally stabilizing benefits above are achieved. However, in nearly all constrained and/or nonlinear cases $J^*_\infty(\cdot)$ is not known and conditions on $V(\cdot)$ in the finite horizon problem have been defined to achieve local stability. As an aside, there are alternative methods that bypass the use of the value function as a Lyapunov function altogether.

Given the definitions above, we can now state the actual MPC problem.

Definition 1.2. The Model Predictice Contol or **MPC Problem** is

1. solve $\mathbb{P}_T(\cdot)$ from state x at current time t,
2. implement the optimal input trajectory $u^*(\tau; x, T)$ for $\tau \in [0, \delta]$, where $0 \leq \delta < T$,
3. repeat step 1. from state $x \leftarrow x^*(\delta; x, T)$ at current time $t \leftarrow t + \delta$ until $x \in X_f$.

Before giving any statements about stability of MPC in the next section, one point about what is trying to be proven should be made clear. There is no uncertainty incorporated in most MPC problem formulations, including the one given in Definition 1.2. Therefore, any stabilizing solution of $\mathbb{P}_T$ (if it exists) can be used *open-loop* to achieve the desired objective. The objective of MPC stability proofs is to ensure that the stability of such stabilizing solutions is not lost when implemented according to Definition 1.2. Proofs of stability of the MPC problem are thus to ensure nominal stability, unless robustness is addressed specifically. The implementation methodology outlined in Definition 1.2 is designed to handle uncertainty, e.g. the current state update in step 3. is relplaced with the actual state of the system after δ seconds. In practice, there is uncertainty, delays in computation, etc. and much success has still been realized. However, (most of) the proofs of stability still live in the uncertainty-free (and computation-free) world. The general conditions to ensure stability and an overview of various stabilizing MPC formulations in an uncertainty and computation-free environment are given in the next section.

1.2 Stability

MPC stability is well defined for various constraint and cost conditions, for linear and nonlinear (especially control affine) systems [7], [3]. The remainder of this section will detail stability in terms of the model predictive formulation.

MPC of constrained systems is nonlinear, warranting the use of Lyapunov stability theory. The value function is almost universally employed as a Lyapunov function for stability analysis for nonlinear (constrained or not) and constrained linear systems. This practice was initiated by Chen and Shaw in 1982 [1] for continuous unconstrained systems with a terminal equality constraint. The more general and noticed result came from Keerthi and Gilbert in 1988 [5], who established stability in this way for time-varying, constrained, nonlinear, discrete systems (again a terminal equality constraint is employed). Since then, work has been done to maintain stability while relaxing the terminal equality constraint to a less restrictive inequality constraint. Sznaier in 1987 CDC paper - need to find it - introduced dual mode, but went largely unnoticed. Later, Michalska and Mayne [8] designed a *dual mode* scheme for constrained, continuous, nonlinear systems. The scheme is dual in the sense that the model predictive controller is required to steer the state to a terminal constraint set, where a local stabilizing controller is employed. More recent results for unconstrained nonlinear systems successfully define requirements solely on the terminal cost function, while implicitly incurring the desired stabilizing properties of a terminal constraint set. This is achieved in [3] where a control Lyapunov function is employed as the terminal cost.

To summarize, most stability results to date incorporate conditions in terms of the aforementioned terminal cost $V(\cdot)$, terminal constraint set X_f and local controller $\kappa_f(\cdot)$. Specifically, the generalized conditions [7] are

C1 $X_f \subset \mathbb{X},\ X_f$ closed, $0 \in X_f$.

C2 $\kappa_f(x) \in \mathbb{U},\ \forall x \in X_f$.

C3 X_f is positively invariant for $\dot{x} = f(x, \kappa_f(x))$.

C4 $\left[\dot{V} + q\right](x, \kappa_f(x)) \leq 0,\ \forall x \in X_f$.

Condition C4 implies C3 if X_f is a level set of $V(\cdot)$. In any case, conditions C1-C4 ensure that

$$\left[\dot{J}^* + q\right](x, \kappa(x, T)) \leq 0,\ \forall x \in X_T, \tag{1.9}$$

where X_T is the set of states that can be steered to X_f by an admissible control in time T. The control law $\kappa(x, T)$ is thought of in two ways.

$\kappa 1$ The first is in light of the infinite horizon case. Given the current state x at time $t = 0$ and optimal trajectories $(x^*, u^*)(\tau; x, T),\ \tau \in [0, T]$, the instantaneous model predictive feedback is $\kappa(x, T) \triangleq u^*(0; x, T)$. In reference to Definition 1.2, $\delta = 0$ in this case. The finite horizon optimization has to be continuously (and instantaneously) resolved, which is obviously not implementable.

$\kappa 2$ From the perspective of implementation, the following is a standard description for model predictive feedback controllers. The finite horizon optimization is solved every $\delta > 0$ seconds ($T > \delta$) and the control trajectory $u^*(\tau; x, T),\ \tau \in [0, \delta]$ is implemented to drive the system from x at time $t = 0$ to $x^*(\delta; x, T)$ at time $t = \delta$ (no uncertainty present). The MPC control law is $\kappa(\tau; x, T) \triangleq u^*(\tau; x, T),\ \tau \in [0, \delta]$. NOTE: this feedback is open-loop for δ seconds, i.e. $u^*(\cdot; x, T)$ depends only on time and not on the state. Although the optimization is not solved continuously, instantaneous computation is still implicit. In practice, computation time is a real issue in implementing receding horizon control, specifically when the speed of the system dynamics is comparable to computational speed. Note that $\kappa(\cdot; x, T)$ is now time-varying.

In any case, equation (1.9) ensures asymptotic stability of the closed-loop system. A sketch proof of this is given in [7], where a monotonicity property of the optimal value function is utilized. The value function is assumed to be continuously differentiable in this proof and constraints may invalidate this assumption.

The sections that follow outline stability criteria in the case where terminal constraints and costs are used, separately and together, and not used.

1.2.1 Terminal state

Use of a terminal equality constraint (terminal state) at then end of each horizon trivially produces stability. Setting $X_f = \{0\}$, $V(x) = 0$ and $\kappa_f(x) = 0$, where zero control maintains the state at the origin (see A1). The functions $V(\cdot)$ and $\kappa_f(\cdot)$ need only be defined on X_f, which is $x = 0$ in this case. Conditions C1–C3 are trivial and C4 follows from B1. In practice, some tolerance on the terminal state is used; however, it is still very taxing on the on-line optimization and can often result in infeasibility. It is often not obvious, for example, how to choose a horizon time that satisfies this constraint. Further, the presence of process noise alone may result in infeasibility.

1.2.2 Terminal cost

In this case $X_f = \mathbb{R}^n$. To date, stability can only be guaranteed for unconstrained linear systems or constrained and stable linear systems.

A technique that embeds $X_f \subset \mathbb{R}^n$ via cost function properties while removing the actual terminal constraint from the on-line optimization problem [3] are not considered here. This case is included in the terminal cost and constraint set approaches, since the constraint set is still implicitly present.

1.2.3 Terminal (inequality) constraint set

The terminal constraint $x(T) \in X_f$ satisfying C1 is enforced, but no terminal cost ($V(x) \equiv 0$). The constraint is an inequality in the sense that $x(T)$ does not have to be at a particular point. The MPC controller drives the state to X_f, where a local (not necessarily linear) controller $\kappa_f(x)$ guarantees that C2–C3 are satisfied. Properties on $q(x, \kappa_f(x))$ that satisfy C4 are detailed in [7]. As mentioned, this approach was introduced by Michalska and Mayne [8] for continuous nonlinear constrained systems, where a variable horizon length was employed. Variable horizon MPC is not discussed here. Other approaches employ a fixed horizon, but according to [7], it makes more sense to employ a terminal cost when the horizon is fixed.

1.2.4 Terminal cost and (inequality) constraint set

Linear constrained systems and nonlinear constrained and unconstrained systems have been addressed with this variant of MPC. This approach is more popular as it achieves better performance when compared to terminal state or terminal constraint set formulations and handles a wider range of problems than terminal cost MPC. Refer to [7] for discussion and references for linear and nonlinear constrained systems.

One approach to the nonlinear unconstrained case is given by Jadbabaie et al [4]. The essential requirement for stability is that $V(\cdot)$ be a control Lyapunov function in the neighborhood of the origin. For reference later, the definition of control Lyapunov function is now given.

Definition 1.3. A global **control Lyapunov function** (CLF) is a C^1, proper, positive definite and radially unbounded function $\bar{V} : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$\inf_u \left[\frac{\partial \bar{V}}{\partial x} f(x, u) \right] < 0, \quad \forall x \neq 0. \tag{1.10}$$

Krstic et al [6] give more insight into CLFs. It is known that the existence of a global CLF is equivalent to the existence of a globally asymptotically stabilizing control law $u = K(x)$ which is smooth everywhere except possibly at the origin.

Jadbabaie et al show there exists a constraint set $X_f = \{x | V(x) \leq r\}$ and a local control law $\kappa_f(\cdot)$ such that the triple $(V(\cdot), X_f, \kappa_f(\cdot))$ satisfy C1-C4. A key advantage of this approach is that the *horizon is chosen* to ensure automatic satisfaction of the terminal constraint, while the constraint itself is not required in the on-line optimal control problem.

1.2.5 No terminal cost or constraints

For unconstrained nonlinear systems, it has been shown [3] that there always exists a horizon time (long enough) such that stability to the origin can be attained. Quantification of such a horizon length has not been done. Moreover there are trivial (linear) examples where if the horizon is not long enough, instability occurs.

1.2.6 Other forms of MPC

Instead of requiring a terminal state (A.) or inequality set (B.), an intermediate solution is to incorporate a terminal constraint set (type B.) that is required to contract (become smaller) over each horizon optimization. This can relax the computational strain of requiring that a terminal state be reached for each optimization, and may improve performance over the inequality set. This is referred to as contractive MPC and was developed for constrained nonlinear systems in [2]. The terminal constraint function that is required to decrease with each successive optimization is used as a Lyapunov function to prove stability. Since this Lyapunov function is decreased discretely, rather than continuously, bounds on the trajectories between optimization updates is required for stability. It is interesting to note that the proof of (nominal) exponential stability only assumes the trajectory bound mentioned, A2 and a parameterization of the set of initial states that result in a feasible MPC problem. For the objective function to be an exponentially decaying Lyapunov function with the inclusion of a terminal equality constraint, stronger assumptions are needed (Lipshitzness of MPC feedback laws and A1 with bounded derivatives for f are two).

In the contractive MPC formulation, the terminal constraint set (which now depends on x and the optimization number $k \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the nonnegative integers) is implicitly defined by the terminal constraint

$$M(x(t_{k+1})) \leq \beta M(x(t_k)), \; \beta \in [0, 1)$$

where M is a local Lyapunov function, e.g. state quadratic function.

Chapter 2

MPC: Stability Proofs

In this chapter, stability proofs of some of the formulations discussed in the previous chapter are given. Rather than giving only the proof of the general formulation in the last chapter, alternative proofs are included to give insight into the (mathematical) stabilizing mechanism of a particular MPC formulation.

2.1 Terminal State (§1.2.1)

This 20 year old proof is given (in less detail) in the original paper by Chen and Shaw [1]. The state x and input u are unconstrained, so $\mathbb{U} = \mathbb{R}^m$ and $\mathbb{X} = \mathbb{R}^n$. The cost functional to be minimized is

$$J(u(\cdot); x_0, T) = \int_0^T q(x^u(\tau; x_0), u(\tau)) \, \mathrm{d}\tau, , \tag{2.1}$$

subject to equation (1.1) and $x(0) = x_0$. As stated in §1.2.1 of the previous chapter, there is no terminal cost in this formulation ($V(x) = 0$) and it is required that $x(T) = 0$, i.e. $X_f = \{0\}$. The optimal value function is denoted

$$J^*(x_0, T) = \inf_u J(u(\cdot); x_0, T).$$

Regarding the assumptions A1-A4, the following can be stated:

A1 No assumption is stated about the $C^?$ of f but the proof uses the Hamilton-Jacobi-Bellman (HJB) equation, so we require that f be at least C^2.

A2 $f(0, 0) = 0$ is assumed but there is no need to require controllability.

A3 not enforced.

A4 not needed.

Regarding assumptions B1-B3:

B1 $q(0,0) = 0$ and use of HJB so assume twice continuously differentiable (C^2).

B2 $q(x,u) \geq c_q(||x||^2+||u||^2)$, $\forall x \in \mathbb{X},\ u \in \mathbb{U}$ and for some $c_q > 0$ is STRONG and simplifies the proof. The authors have a relaxed assumption in that q is positive semi-definite in x but this requires the extra assumption later that the solutions of $\dot{x} = f(x,0)$ are either unbounded or converge to zero (no open-loop poles on imaginary axis).

B3 q is positive-definite in u.

For the proof here, assumption B2 is enforced. This covers the added assumption in [1] that $J^*(x,T) \to \infty$ when $||x|| \to \infty$ (radially unboundedness) and that $J^*(x,T)$ is positive definite in x. The authors also assume that $J^*(x,T)$ is differentiable. The two control laws referenced in the proof are now given, where receding horizon is equivalent to model predictive:

$\boldsymbol{u}_{opt}$ The open-loop optimal control $u^*(t;x_0,T) = \arg J^*(x_0,T)$, $t \in [0,T]$.

$\boldsymbol{u}_{RH}$ The receding horizon feedback control (RHFC) $\kappa(x,T) = u^*(0;x,T)$.

The theorem of the stability of the closed-loop system under RHFC can now be stated.

Theorem 2.1. *For any fixed $T > 0$, the closed-loop system under RHFC is asymptotically stable in the large. That is, $J^*(x,T)$ is a C^1, positive definite, radially unbounded function and*

$$\frac{\mathrm{d}}{\mathrm{d}t} J^*(x,T)\Big|_{RHFC\ applied} = \left[\frac{\partial J^*(x,T)}{\partial x}\right]^T f(x,\kappa(x,T)) < 0, \quad \forall x \neq 0. \tag{2.2}$$

Proof. Consider the optimal finite (fixed) horizon and time-varying control problem

$$V^*(t,x_0,T) = \inf_{u(\cdot)} \left\{ \int_t^T q(x^u(\tau;x_0),u(\tau),\tau)\ \mathrm{d}\tau \right\},$$

where equation (1.1) may also be time-varying and $x(t) = x_0$. The HJB equation for $V^*(t,x_0,T)$ is

$$\begin{aligned} -\frac{\partial V^*(t,x^*(t;x_0,T),T)}{\partial t} &= \\ q\left(x^*(t;x_0,T),u^*(t;x_0,T)\right) + \left[\frac{\partial V^*(t,x^*(t;x_0,T),T)}{\partial x}\right]^T & f\left(x^*(t;x_0,T),u^*(t;x_0,T)\right) \\ &= \inf_{u(\cdot)} \left\{ q(x,u) + \left[\frac{\partial V^*(t,x,T)}{\partial x}\right]^T f(x,u) \right\}. \end{aligned} \tag{2.3}$$

A less confusing notation for the partial derivatives would be

$$D_1 \equiv \frac{\partial}{\partial t},\ D_2 \equiv \frac{\partial}{\partial x},$$

where it would be understood which place holder was being differentiated while holding variables in the other place holders constant. Having said this, we leave the partial derivative notation as it is, as their meaning should now be clear. Observe that $q,\ f$ are autonomous so

$$V^*(t, x_0, T)\big|_{t=0} = J^*(x_0, T) = \int_0^T q(x^*(\tau; x_0, T), u^*(\tau; x_0, T))\ \mathrm{d}\tau,\ \text{and} \tag{2.4}$$

$$\frac{\partial J^*(x_0, T)}{\partial T} = \frac{\partial}{\partial T}\left(V^*(t, x_0, T)\big|_{t=0}\right) = \frac{\partial V^*(t, x_0, T)}{\partial T}\big|_{t=0}. \tag{2.5}$$

Given that the problem is autonomous we also have that

$$V^*(t, x_0, T) = V^*(t+s, x_0, T+s),\ \forall s \in \mathbb{R}.$$

Now examine the derivative with respect to s with $t = 0$

$$\begin{aligned}
\frac{\mathrm{d}V^*(0, x_0, T)}{\mathrm{d}s} =& \frac{\mathrm{d}V^*(s, x_0, T+s)}{\mathrm{d}s} \\
0 =& \frac{\partial V^*(s, x_0, T+s)}{\partial t} + \frac{\partial V^*(s, x_0, T+s)}{\partial T}. \\
\implies \quad \frac{\partial V^*(s, x_0, T+s)}{\partial t} &= -\frac{\partial V^*(s, x_0, T+s)}{\partial T},\ \forall s \in \mathbb{R}.
\end{aligned}$$

From equation (2.5) and the last equation it follows that

$$\frac{\partial J^*(x_0, T)}{\partial T} = \frac{\partial V^*(t, x_0, T)}{\partial T}\big|_{t=0} = -\frac{\partial V^*(t, x_0, T)}{\partial t}\big|_{t=0}.$$

At time $t = 0$, equation (2.3) becomes

$$\frac{\partial J^*(x_0, T)}{\partial T} = q + \left[\frac{\partial J^*(x_0, T)}{\partial x}\right]^T f(x_0, u^*(0; x_0, T)), \tag{2.6}$$

where equation (2.4) implies that the partials with respect to x are equal at $t = 0$. Now equation (2.6) has control law $\boldsymbol{u}_{RH}$ by definition so it can be rewritten

$$\frac{\partial J^*(x_0, T)}{\partial T} = q + \left[\frac{\partial J^*(x_0, T)}{\partial x}\right]^T f(x_0, \kappa(x_0, T)).$$

Using the notation of equation (2.2)

$$\frac{\partial J^*(x, T)}{\partial T} - q = \frac{\mathrm{d}}{\mathrm{d}t} J^*(x, T)\Big|_{\text{RHFC applied}} \tag{2.7}$$

The following lemma guarantees that the right-hand side of equation (2.7) is negative definite.

Lemma 2.2. *If $T_2 > T_1 > 0$, then $J^*(x, T_1) \geq J^*(x, T_2)$ for any x.*

Proof. The optimal control $u^*(t; x, T_1)$, $t \in [0, T_1)$ gives the cost $J^*(x, T_1)$ and terminal condition $x^*(T_1; x, T_1) = 0$. Construct a new control law as follows

$$\hat{u}(t; x, T_1, T_2) = \begin{cases} u^*(t; x, T_1) & \text{for} \quad t \in [0, T_1) \\ 0 & \text{for} \quad t \in [T_1, T_2). \end{cases}$$

Given that $f(0,0) = 0$ and $q(0,0) = 0$, $\hat{u}(t; x, T_1, T_2)$ gives the cost $J^*(x, T_1)$ and the terminal condition $x^{\hat{u}}(T_2; x) = 0$. This new control law $\hat{u}$ is at best optimal over the interval $[0, T_2)$ for which it is defined. From the definition of $J^*(x, T_2)$ then it must be that $J^*(x, T_1) \geq J^*(x, T_2)$. ∎

From Lemma 2.2

$$\frac{\partial J^*(x, T)}{\partial T} \leq 0.$$

which means by equation (2.7) that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} J^*(x, T)\right|_{\text{RHFC applied}} < 0,$$

concluding the proof of Theorem 2.1. ∎

Note that Theorem 2.1 holds independent of the choice in terminal cost, i.e. a nonzero terminal cost would not invalidate the result. The strong requirement that the terminal state be exactly zero clearly has theoretic advantages for proving stability of MPC. The tradeoff is that this formulation is computationally very taxing and generally not feasible.

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