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New results in the global stabilization of nonlinear systems via measurement feedback with application to nonholonomic systems

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Abstract

We study the problem of globally stabilizing through smooth time–varying measurement feedback a wide class of time–varying uncertain nonlinear systems, consisting of a *linear nominal time–varying system* perturbed by nonlinear terms, model uncertainties and disturbances. The nominal time–varying system is both controllable and observable. Both the uncertainties and nonlinearities are supposed to have a *lower triangular structure*. We propose a step–by step design, based on splitting the system into n one–dimensional interconnected systems Σ_j, $j = 1, \ldots, n$; assuming that for each disconnected system Σ_j there exists a smooth time–varying measurement feedback stabilizing controller $\mathcal{C}_j$ which achieves for the closed–loop system $\Sigma_j \circ \mathcal{C}_j$, $j = 1, \ldots, n$, some stability properties, we give conditions under which the interconnection of $\Sigma_j \circ \mathcal{C}_j$, $j = 1, \ldots, n$, maintains the same stability properties of the disconnected systems. In general, uniform global asymptotic (not exponential) stability can be obtained. We apply these results to nonholonomic systems with uncertainties in lower triangular form.

1 Introduction

Nonholonomic systems are Lagrange systems with linear constraints which are not integrable. The dynamics and stability of these systems have been studied in a number of papers ([4], [5]). These systems have a distinctive feature: by the Brockett's necessary condition on feedback stabilization ([6]) these systems cannot be asymptotically stabilized by any continuous time invariant feedback controller. This obstruction can be overcome by resorting to *discontinuous* or *smooth time varying controllers*. Coron ([7]) has investigated theoretical results on the existence of these non conventional controllers while constructive results have been developed in [8], [9], [10], [11], [12], [13], [14], [15], to cite a few. In all these papers the following class of nonholonomic systems in chained form has been considered

$$\begin{aligned} \dot{x}_1 &= x_2 u_2 \\ \dot{x}_2 &= x_3 u_2 \\ \vdots &= \vdots \\ \dot{x}_n &= u_1 \\ \dot{x}_{n+1} &= u_2 \end{aligned} \tag{1}$$

assuming that all the states are available for feedback: exponential global convergence to the origin can be achieved by using *discontinuous controllers*, while only asymptotic convergence can be achieved by using *smooth time–varying controllers*. Many nonlinear mechanical systems with nonholonomic constraints on velocities can be transformed (locally or globally) into a system of the form (1). Recently, a more general class of systems which take into account the presence of uncertainties in (1) has been studied in [16] and [17], more specifically

$$\begin{aligned} \dot{x}_1 &= x_2 u_2 + \psi_1(x,t) \\ \dot{x}_2 &= x_3 u_2 + \psi_2(x,t) \\ \vdots &= \vdots \\ \dot{x}_n &= u_1 + \psi_n(x,t) \\ \dot{x}_{n+1} &= u_2 \end{aligned} \tag{2}$$

with $|\psi_j(x,t)| \leq \phi_j(x_1)$ for all $j = 1, \ldots, n$, and for all x and t, with $x = \mathrm{col}(x_1, \ldots, x_{n+1})$ and $\phi_j(0) = 0$. The importance of robustifying the control schemes available for systems (1) has been pointed out by Morin in [12] on the bilinear model of a mobile robot with small angle measurement error

$$\begin{aligned} \dot{x}_l &= \Big(1 - \frac{\epsilon^2}{2}\Big)\nu \\ \dot{y}_l &= \theta_l \nu + \epsilon \nu \\ \dot{\theta}_l &= \omega \end{aligned} \tag{3}$$

When $\epsilon = 0$ one obtains a system of the form (1). As explained in [12], if one tries to globally stabilize (3) with $\epsilon = 0$, the fact that ϵ may be nonzero could induce instability of the closed–loop system. On the other hand, (3) can be brought into the form (2) after a coordinate change. The results contained in [16] and [17] give a robustification tool for designing robust discontinuous controllers which globally exponentially stabilize (3). Moreover, in [17]it is also studied the case in which only the measure of x_1 is available.

Note that the linearization of the nominal system of (2)

$$\begin{aligned} \dot{x}_1 &= x_2 u_2 \\ \dot{x}_2 &= x_3 u_2 \\ \vdots &= \vdots \\ \dot{x}_n &= u_1 \\ \dot{x}_{n+1} &= u_2 \end{aligned} \tag{4}$$

around the origin is not controllable neither observable. However, it is easy to convince that the *time–varying* linearization of (4) around a reference trajectory $(x_d(t), u_d(t))$, with $x_{dj}(t) = 0$ for $j = 1, \dots, n$, $u_{d1}(t) = 0$ and for suitable $x_{d,n+1}$ and $u_{2d}(t)$), is controllable and observable. Asymptotic convergence to the origin can be achieved if (2), after having redefined the coordinates x_{n+1} and the input u_2, respectively, as $w = x_{n+1} - x_{n+1,d}(t)$ and $v_2 = u_2 - u_{2d}(t)$, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2 u_{2d}(t) + \psi_1(x,t) + x_2 v_2 \\ \dot{x}_2 &= x_3 u_{2d}(t) + \psi_2(x,t) + x_3 v_2 \\ \vdots &= \vdots \\ \dot{x}_n &= u_1 + \psi_n(x,t) \\ \dot{w} &= v_2 \end{aligned} \tag{5}$$

can be globally stabilized via smooth time–varying measurement feedback and, in addition, $x_d(t) \to 0$ as $t \to \infty$.

In this paper we assume that only a measure of x_1 is available, possibly corrupted by noise, and that $|\psi_j(x,t)| \le \sum_{i=1}^{j} \phi_i(x_1) x_i$ for all $j = 1, \dots, n$, and for all x and t. Note that, with respect to [16] and [17], we allow for the presence of measurement noise and for terms in $\psi_j(x,t)$ containing also unmeausured state variables.

The basic ingredients of our analysis are *filtered Lyapunov functions* and *dissipation inequalities*. *Filtered Lyapunov functions* have been already used for output feedback stabilization in [3] and are based on generating the positive definite symmetric matrix P of a quadratic Lyapunov function $\|x\|_P^2$ through a nonlinear filter. On the other hand, dissipation inequalities have been extensively used in the solution of $\mathcal{H}_\infty$ control problems ([**?**], [18]).

Relying on these key tools, we consider (5) as the interconnection of n one–dimensional systems Σ_j, $j = 1, \dots, n$. Our strategy consists essentially of two steps: first, design simple controllers $\mathcal{C}_j$ for each one–dimensional disconnected system Σ_j and, secondly, tune the controller parameters so that the overall system $\Sigma_j \circ \mathcal{C}_j$, $j = 1, \dots, n$, has the same stability properties of the disconnected systems. This requires to define for each system Σ_j, $j = 1, \dots, n$, a *virtual measurement* μ_j, which in general is not available for feedback but it is crucial in the state–feedback design. Also we point out a simple procedure for the synthesis of smooth Lyapunov functions for the closed–loop system. As opposite to the discontinuous time varying controllers proposed in [16], [17] and [8], we obtain *smooth time–varying controllers* and (uniform) *global exponential convergence* to the reference trajectory $(x_d(t), u_d(t))$. Global exponential convergence to the origin cannot be ensured in general: as an example, one can take $x_{d,n+1}(t) = \dfrac{\sin(t+1)^2}{(t+1)^2}$ so that asymptotic convergence to the origin goes as $\dfrac{1}{(t+1)^2}$.

2 One dimensional systems

Let us consider systems (6) satisfying

$$\begin{aligned} \dot{x} &= \ell(t) u + \Psi_1(x,t) \\ \mu &= x + \Psi_2(x,t) \end{aligned} \tag{6}$$

with

$$\Psi_1^2(x,t) \le a_{11}(x) x^2 \tag{7}$$

$$\Psi_2^2(x,t) \le \alpha x^2 \tag{8}$$

with $\alpha < 1$, *polynomial* a_{11} (this assumption simplify the analisys and can be removed), and $\ell(\cdot)$ a smooth function such that

$$\begin{aligned} &(\ell(t) = 0) \Rightarrow (|\ell^{(1)}(t)| > r > 0) \\ &\ell^2(t) + (\ell^{(1)})^2(t) + |\ell(t)\ell^{(2)}(t)| \le \ell_0 \\ &\ell(t)\ell^{(2)}(t) \ge -q_0(t)\ell(t)\ell^{(1)}(t) - q_1(\ell(t), t) \end{aligned} \tag{9}$$

for some smooth bounded and nonnegative functions $q_0(\cdot)$ and $q_1(\ell(\cdot), \cdot)$ such that $d_1(\ell(t), t) = 0$ whenever $\ell(t) = 0$ (by rescaling, we can assume $\ell_0 = 1$). For example,

$$\ell(t) = 2\Big(-\frac{\sin(t+1)^2}{(t+1)^3} + \frac{\cos(t+1)^2}{t+1}\Big) \tag{10}$$

It is easy to establish the existence of a $\xi, \lambda > 0$ such that

$$\lambda|\mu| \le |x| \le \xi|\mu| \tag{11}$$

Let

$$W(x, \sigma, P, t) = V_{SF}(x,t) + V_m(x, \sigma, P) \tag{12}$$

where

$$V_{SF}(x,t) = \epsilon x^2 (c - \frac{2}{N\pi}\arctan(\ell(t)\ell^{(1)}(t)\kappa(x))$$
$$V_m(x,\sigma,P) = P(x-\sigma)^2 \qquad (13)$$

for some $c \geq 3$ and $N \geq 1$ and $\kappa(x)$ is a smooth positive function such that $\kappa(\lambda\mu) \leq \kappa(x) \leq \kappa(\xi\mu)$ for all μ and x in the set (11) and $\left|\frac{\partial\kappa}{\partial x}(x)x\right| \leq N\kappa(x)$ for all x and for some $N > 0$ (note that any polynomial $k(x)$ such that $k(x) = k(-x)$ has the above property).

Let $L, M > 0$ be such that

$$|d_1(x,t)x - d_1(\sigma,t)\sigma| \leq L|x-\sigma|$$
$$|d_1(x,t)x - d_1(\sigma,t)\sigma - d_1(x-\sigma,t)(x-\sigma)| \leq M|x-\sigma|$$

for all x, σ and t, where

$$d_1(x,t) = c - \frac{2}{\pi}\Big(\arctan(\ell(t)\ell^{(1)}(t)\kappa(x)) - \frac{\ell^{(1)}(t)\ell(t)}{1+\ell^2(t)(\ell^{(1)})^2(t)\kappa^2(x)}\frac{x}{N}\frac{\partial\kappa}{\partial x}(x)\Big) \qquad (14)$$

Note that L and M indeed exist since by (**??**) $d_1(x,t)$ and $\ell(t)\ell^{(1)}(t)$ are bounded and $\frac{\partial\kappa}{\partial x}(x)x$ and $\kappa(x)$ are smooth functions. Moreover, by rescaling ϵ and $\ell(t)$ one can always assume $L = M = 1$. Also let

$$L_0 = \frac{2}{r^2}\Big[\|q_0\|_\infty + \frac{2N}{\pi}\Big(2\alpha(c+5) + (c+2)^2 + 3\Big)\Big]$$

and $\kappa_0 > 0$ be such that $\frac{\kappa(\xi\mu)}{\kappa(\lambda\mu)} \leq \kappa_0$.

Proposition 2.1 *If $\beta_m > \frac{2\alpha}{1-\alpha} + 1$ and $E_0(\mu)$, $\gamma(\mu)$ and $\kappa(x)$ are such that $\frac{E_0(\mu)}{\gamma^2(\mu)} \geq a_{11}(x)$ for all μ and x in the set (11) and $\kappa(\lambda\mu) = \frac{L_0 E_0(\mu)\epsilon}{\gamma^4(\mu)}$, then the controller*

$$\begin{aligned}
v &= -\frac{\ell(t)\epsilon d_1(\sigma,t)E_0(\mu)}{\gamma^4(\mu)}\sigma \cdot \\
&\quad \cdot \begin{cases} \dfrac{m(\mu,t)+\sqrt{m^2(\mu,t)+\ell^2(t)}}{\ell^2(t)} & \text{if} \ell(t) \neq 0 \\ -\dfrac{1}{2m(\mu,t)} & \text{if} \ell(t) = 0 \end{cases} \\
\dot\sigma &= \frac{E_0(\mu)}{\gamma^2(\mu)}\frac{\epsilon d_1(\sigma,t)}{\gamma^2(\mu)}\sigma + v + \frac{\epsilon^2 E_0(\mu)}{P\gamma^4(\mu)L_1(\mu,t)}(\mu-\sigma) \\
\dot P &= \Big(-\frac{2(d_1(\sigma,t)+1)P\epsilon}{\gamma^2(\mu)} - \frac{P^2}{\gamma^2(\mu)} \\
&+ \frac{2(c+4)\epsilon^2}{\gamma^2(\mu)}\Big)\frac{E_0(\mu)}{\gamma^2(\mu)}, \qquad P(t_0) = \epsilon
\end{aligned} \qquad (15)$$

with

$$L_1(\mu,t) = \frac{1-\alpha\Big(\frac{2}{\beta_m-1}+1\Big)}{S(\mu,t)}$$
$$S(\mu,t) = 2(c+4) + 2 + \Big(\frac{2}{\beta_m-1}+1\Big)\cdot$$
$$\cdot\Big((c+2)^2 + 3 + n_{2M}(\mu,t)\Big)$$
$$m(\mu,t) = 3 + (c+2)^2 + \frac{\alpha}{L_1(\mu,t)} + \frac{2n_{2M}(\mu,t)}{N\pi}$$
$$n_{2M}(\mu,t) = \|q_0\|_\infty + q_1(\ell(t),t)\kappa_0$$
$$-\frac{(\ell^{(1)})^2 L_0}{1+\ell^2(t)(\ell^{(1)})^2(t)\kappa^2(\xi\mu)} \qquad (16)$$

is such that along the trajectories of (6)–(15)

$$\dot W \leq -x^2 Q_{SF}(\mu) - (x-\sigma)^2 Q_m(\mu) \qquad (17)$$

with

$$Q_{SF}(\mu) = \frac{E_0(\mu)\epsilon^2}{\gamma^4(\mu)}$$
$$Q_m(\mu) = \frac{\epsilon^2}{R_1(\mu)}\frac{1}{\beta_m-1} + \frac{E_0(\mu)\epsilon^2}{\gamma^4(\mu)} \qquad (18)$$

It can be shown that the expression of v in proposition 2.1 is a smooth function of their argument.

(A canonical parametrization of E_0 and γ). Let us consider systems (6) satisfying

$$\Psi_1^2(x,t) \leq h(\nu)x^2 \qquad (19)$$
$$\Psi_2^2(x,t) \leq \alpha x^2 \qquad (20)$$

with $\alpha < 1$, smooth function h and ν is some measurement such that the trajectories of

$$\dot\varphi = \max\{0, \dot h(\nu)\}\max\{0, \frac{-\varphi + h(\nu) + 1}{\sqrt{1+(-\varphi+h(\nu))^2}}\}$$

locally exist and are unique. Without loss of generality, we can assume that $h(\nu) \geq 1$. Let

$$\begin{aligned}
\gamma^2(\varphi) &= \epsilon(\varphi) = \frac{\epsilon_0}{\varphi^{2i}} \\
E_0(\varphi) &= e_0\gamma^2(\varphi)\varphi
\end{aligned} \qquad (21)$$

with $i = 1, 2, \ldots$, for some $e_0 \geq 1$, $\epsilon_0 > 0$. Moreover

$$W(x,\sigma,\varphi,t) = V_{SF}(x,\varphi,t) + V_m(x,\sigma,P) \qquad (22)$$

where

$$V_{SF}(x,\varphi,t) = \epsilon(\varphi)x^2(c - \frac{2}{\pi}\arctan(\ell(t)\ell^{(1)}(t)\kappa(\varphi))$$
$$V_m(x,\sigma,P) = P(x-\sigma)^2 \qquad (23)$$

for some $c \geq 3$ and $N \geq 1$, and

$$\kappa(\varphi) = L_0 e_0 \varphi \qquad (24)$$

with

$$L_0 = \frac{2}{r^2}\Big[\|q_0\|_\infty + \frac{2}{\pi}\Big(2\alpha(c+5)+(c+2)^2+3\Big)\Big] \quad (25)$$

Also, let

$$d_2(\varphi,t) = c - \frac{2}{\pi}\arctan(\ell(t)\ell^{(1)}(t)\kappa(\varphi))$$

Proposition 2.2 *The controller*

$$\begin{aligned}
v &= -\ell(t)d_2(\varphi,t)e_0\varphi\sigma\cdot \\
&\cdot\begin{cases} \dfrac{m(\varphi,t)+\sqrt{m^2(\varphi,t)+\ell^2(t)}}{\ell^2(t)} & \text{if}\ell(t)\neq 0 \\ -\dfrac{1}{2m(\varphi,t)} & \text{if}\ell(t)=0 \end{cases} \\
\dot\sigma &= e_0\varphi d_2(\varphi,t)\sigma + v + \frac{\epsilon(\varphi)e_0\varphi}{PL_1(\varphi,t)}(\mu-\sigma) \\
\dot P &= \Big(-2d_2(\varphi,t)P - \frac{P^2}{\epsilon(\varphi)} + 2(c+4)\Big)e_0\varphi,\ P(t_0)=\epsilon \\
\dot\varphi &= \max\{0,\dot h(\nu)\}\max\{0,\frac{-\varphi+h(\nu)+1}{\sqrt{1+(-\varphi+h(\nu))^2}}\} \quad (26)
\end{aligned}$$

with $\varphi(t_0)=h(\nu(t_0))$ *and*

$$\begin{aligned}
&L_1(\varphi,t) = \frac{1-\alpha\Big(\dfrac{2}{\beta_m-1}+1\Big)}{S(\varphi,t)} \\
&S(\varphi,t) = 2(c+4)+2+\Big(\frac{2}{\beta_m-1}+1\Big)\cdot \\
&\cdot\Big((c+2)^2+3+n_{2M}(\varphi,t)\Big) \\
&m(\varphi,t) = 3+(c+2)^2+\frac{\alpha}{L_1(\varphi,t)}+\frac{2n_{2M}(\varphi,t)}{\pi} \\
&n_{2M}(\varphi,t) = \|q_0\|_\infty + q_1(\ell(t),t) \\
&-\frac{(\ell^{(1)})^2L_0}{1+\ell^2(t)(\ell^{(1)})^2(t)\varphi^2} \qquad (27)
\end{aligned}$$

is such that, along the trajectories of (6)–(15), (17) holds true with

$$\begin{aligned}
Q_{SF}(\mu) &= E_0(\varphi) \\
Q_m(\mu,t) &= \frac{\epsilon^2(\varphi)}{R_1(\varphi,t)}\frac{1}{\beta_m-1}+E_0(\varphi) \qquad (28)
\end{aligned}$$

3 Two dimensional systems

Let us consider the following system

$$\begin{aligned}
\dot x_1 &= \ell_1(t)x_2 + \Psi_{11}(x,v,\nu,d,t) \\
\mu_1 &= x_1 + \Psi_{12}(x,v,\nu,d,t) \\
\dot x_2 &= \ell_2(t)u + \Psi_{21}(x,v,\nu,d,t) \\
\mu_2 &= x_2 + \Psi_{22}(x,v,\nu,d,t) \qquad (29)
\end{aligned}$$

with

$$\begin{aligned}
&\Psi_{11}^2(x,v,\nu,d,t) \le a_{111}(x_1)x_1^2 &(30)\\
&\Psi_{12}^2(x,v,\nu,d,t) \le \alpha_1 x_1^2 &(31)\\
&\Psi_{21}^2(x,v,\nu,d,t) \le a_{211}(x_1)x_1^2 + a_{212}(x_1)x_2^2 &(32)\\
&\Psi_{22}^2(x,v,\nu,d,t) \le a_{221}(x_1)x_1^2 + \alpha_2 x_2^2 &(33)
\end{aligned}$$

$\ell_j(t)$ satisfying (9), $\alpha_j < 1$, $j=1,2$, and for some *polynomial* functions a_{ij} (this assumption simplify the analisys and can be removed). We also assume that

$$\dot\mu_1 = b_0(x,v,\nu,d,t)x_2 + b_1(x,v,\nu,d,t) \quad (34)$$

with

$$\begin{aligned}
b_1^2(x,v,\nu,d,t) &\le n_1(x_1)x_1^2 \\
b_0^2(x,v,\nu,d,t) &\le n_0 \qquad (35)
\end{aligned}$$

for some *polynomial* functions n_1(this assumption simplify the analisys and can be removed).

It is easy to establish the existence of $\xi_1,\lambda_1,\xi_2>0$ and smooth $\varrho_2(\cdot)$ such that

$$\begin{aligned}
&\lambda_1|\mu_1| \le |x_1| \le \xi_1|\mu_1| \\
&|x_2| \le \xi_2|\mu_2| + \varrho_2(\mu_1) \qquad (36)
\end{aligned}$$

Find polynomial $h(\mu_1)$ such that

$$\begin{aligned}
&h(\mu_1) \ge \max\Big\{a_{111}(x_1), n_1(x_1), a_{211}(x_1), \\
&a_{212}(x_1), a_{221}(x_1)\Big\} \qquad (37)
\end{aligned}$$

over the set (36), and let φ be generated through

$$\dot\varphi = \max\{0,\dot h(\mu_1)\}\max\{0,\frac{-\varphi+h(\mu_1)+1}{\sqrt{1+(-\varphi+h(\mu_1))^2}}\}$$

with $\varphi(t_0)=h(\mu_1(t_0))$. Note that to implement the above filter one needs the knowledge of $\dot\mu_1$ (this assumption can be removed in some signififcant cases: see the end of this section).

Let $L_1, M_1 > 0$ be such that

$$\begin{aligned}
&|d_{11}(\zeta_1,t)\zeta_1 - d_{11}(\sigma_1,t)\sigma_1| \le L_1|\zeta_1-\sigma_1| \\
&|d_{11}(\zeta_1,t)\zeta_1 - d_{11}(\sigma_1,t)\sigma_1 \\
&-d_{11}(\zeta_1-\sigma_1,t)(\zeta_1-\sigma_1)| \le M_1|\zeta_1-\sigma_1|
\end{aligned}$$

for all ζ_1,σ_1 and t, where

$$\begin{aligned}
d_{11}(\zeta_1,t) &= c_1 - \frac{2}{N_1\pi}\Big(\arctan(\ell_1(t)\ell_1^{(1)}(t)\kappa_1(\zeta_1)) \\
&-\frac{\ell_1^{(1)}(t)\ell_1(t)\zeta_1\dfrac{\partial\kappa_1}{\partial\zeta_1}(\zeta_1)}{1+\ell_1^2(t)(\ell_1^{(1)})^2(t)\kappa_1^2(\zeta_1)}\Big) \qquad (38)
\end{aligned}$$

(w.l.o.g. $L_1=M_1=1$) and $\kappa_1(\zeta_1)$ is a polynomial positive function such that

$$\kappa_1(\lambda_1\mu_1) \le \kappa_1(\zeta_1) \le \kappa(\xi_1\mu_1) \quad (39)$$

for all μ_1 and ζ_1 in the set (36) and $\left|\frac{\partial \kappa_1}{\partial \zeta_1}(\zeta_1)\zeta_1\right| \leq N_1\kappa_1(\zeta_1)$ for all ζ_1 and for some $N_1 > 0$. Also, let

$$L_{j0} = \frac{2}{r_j^2}\Big[\|q_{j0}\|_\infty + \frac{2N_j}{\pi}\Big(2\alpha(c_j+5)+(c_j+2)^2+3\Big)\Big]$$

(with $N_2 = 1$) and $\beta_{mj} > \frac{2\alpha_j}{1-\alpha_j}+1$. Set

$$\begin{aligned}\gamma_1^2 &= \epsilon_1 = \epsilon_{10}\\ \gamma_2^2(\varphi) &= \epsilon_2(\varphi) = \frac{\epsilon_{20}}{\varphi^2}\\ E_{j0}(\varphi) &= e_{j0}\gamma_j^2(\varphi)\varphi,\ j=1,2 \qquad (40)\end{aligned}$$

with $e_{j0} > 1$ and $\epsilon_j < 1$, and set $\zeta_1 = x_1$ and $\zeta_2 = x_2 - v$, where v is as in proposition 2.1, with $d_1, E_0, \sigma, \ell, \epsilon, \gamma, \mu$ replaced by $d_{11}, E_{10}, \sigma_1, \ell_1, \epsilon_1, \gamma_1, \mu_1$.

Finally, let $\zeta = \mathrm{col}(\zeta_1,\zeta_2)$, $\sigma = \mathrm{col}(\sigma_1,\sigma_2)$, $P = \mathrm{col}(P_1,P_2)$ and

$$W(\zeta,\sigma,P,t) = \sum_{j=1}^{2} V_{SF,j}(\zeta_j,\varphi,t) + V_{mj}(\zeta_j,\sigma_j,P_j) \qquad (41)$$

where

$$\begin{aligned}&V_{SF,1}(\zeta_1,\varphi,t)\\ &= \epsilon_1 x^2(c - \frac{2}{N_1\pi}\arctan(\ell_1(t)\ell_1^{(1)}(t)\kappa_1(\zeta_1))\\ &V_{SF,2}(\zeta_2,\varphi,t)\\ &= \epsilon_2(\varphi)x^2(c - \frac{2}{\pi}\arctan(\ell_2(t)\ell_2^{(1)}(t)\kappa_2(\varphi))\\ &V_{mj}(\zeta_j,\sigma_j,P_j) = P_j(\zeta_j-\sigma_j)^2 \qquad (42)\end{aligned}$$

for some $c_j \geq 3$ and $N_1 \geq 1$.

Proposition 3.1 *For some choice of $e_{j0} > 1$ and $\epsilon_{j0} < 1$, $j = 1, 2$, there exists a controller such that along the trajectories of the closed–loop system resulting from (29)*

$$\dot W \leq -\sum_{j=1}^{2}\Big(\zeta_j^2 Q_{SF,j}(\mu) - (\zeta_j-\sigma_j)^2 Q_{mj}(\mu)\Big) \qquad (43)$$

with

$$\begin{aligned}Q_{SF,j}(\mu) &= \frac{e_{j0}h(\mu)\epsilon_{j0}}{\varphi^{2(j-1)}}\\ Q_{mj}(\mu) &= \Big(\frac{4+\frac{\alpha_j}{L_j}}{\beta_{mj}-1}+1\Big)\frac{e_{j0}h(\mu)\epsilon_{j0}}{\varphi^{2(j-1)}} \qquad (44)\end{aligned}$$

In what follows we give some general remarks on the structure of the controller (details can be found in the paper [21]). Let us consider

$$\left\{\begin{array}{rcl}\dot x_1 &=& A_1x_1 + B_{12}x_2 + \Psi_1(x,t)\\ \mu_1 &=& C_{12}x_1 + \Psi_1(x,t)\\ \dot x_2 &=& A_2x_2 + B_{22}u + \Psi_2(x,u,t)\\ \mu_2 &=& C_{22}x_2 + \Psi_2(x,u,t)\end{array}\right. \qquad (45)$$

with state $x = \mathrm{col}(x_1,x_2)$, control input u, measurement $\mathrm{col}(x_1,x_2)$ and uncertainties $\Psi = \mathrm{col}(\Psi_1,\Psi_2)$.

We will assume that both systems

$$\Sigma_1\colon\left\{\begin{array}{rcl}\dot x_1 &=& A_1x_1 + B_{12}x_2 + \Psi_1(x,t)\\ \mu_1 &=& C_{12}x_1 + \Psi_1(x,t)\end{array}\right. \qquad (46)$$

with state x_1, control x_2, measurement μ_1 and uncertainties Ψ_1, and

$$\Sigma_2\colon\left\{\begin{array}{rcl}\dot x_2 &=& A_2x_2 + B_{22}u + \Psi_2(x,u,t)\\ \mu_2 &=& C_{22}x_2 + \Psi_2(x,u,t)\end{array}\right. \qquad (47)$$

with state x_2, control u, measurement μ_2 and uncertainties Ψ_2, satisfy the conditions of proposition (2.1) (or (3.1)) and denote by $\mathcal{C}_1(\sigma_1,\varphi,x_2,\mu_1)$ and $\mathcal{C}_2(\sigma_2,\varphi,u,\mu_2)$ the controllers obtained from (15) (or (26) for (46) and (47), respectively.

A candidate controller for the overall system is obtained by taking together $\mathcal{C}_1(\sigma_1,\varphi,\widetilde x_2,\mu_1,t)$ and $\mathcal{C}_2(\sigma_2,\varphi,u,\widetilde\mu_2,t)$, with $\widetilde\mu_2 = \mu_2 - C_{22}\widetilde x_2$.

The case of n–dimensional systems can be treated as for (29). Also, the case for which only μ_1 and $\dot\mu_1$ can be treated as in [2] by using filtered Lyapunov functions to recover the information of the *virtual measurement* μ_j, $j = 2,\ldots,n$: this significantly generalizes the results contained in [17]. We stress the fact that, while in [17] *discontinuous feedback* is used, in this paper we find *smooth time varying controller*. It can be also proven that if the measure of x_1 is corrupted by noise and its derivative is available for feedback, global asymptotic stability can be achieved by smooth time varying measurement feedback.

Remark 3.1 (*Global asymptotic stabilization of nonholonomic systems in chained form with uncertainties*). Consider the system

$$\begin{array}{rcl}\dot x_1 &=& x_2v + \psi_1(x,t)\\ \vdots &=& \vdots\\ \dot x_{n-1} &=& x_nv + \psi_{n-1}(x,t)\\ \dot x_n &=& u + \psi_n(x,t)\\ \dot x_{n+1} &=& v\\ \mu_1 &=& x_1\\ \mu_{n+1} &=& x_{n+1}\end{array} \qquad (48)$$

where $x = \mathrm{col}(x_1,\ldots,x_{n+1})$. If $\widetilde\mu_2 = \mu_2 - \int \ell(t)dt$ and $v = -\epsilon\arctan(x_{n+1} - \int\ell(t)dt) + \ell(t)$, for some $\epsilon > 0$, and $\ell(t)$ as in (10). Note that $\ell(t)$ satisfies (9) and $\lim_{t\to\infty}\int \ell(t)dt = \lim_{t\to\infty}\frac{\sin(t+1)^2}{(t+1)^2} = 0$. We rewrite

(48) as follows

$$\begin{aligned}
\dot{x}_1 &= x_2\ell(t) + \psi_1(x,t) - x_2\epsilon\arctan(w)\\
\vdots &= \vdots\\
\dot{x}_{n-1} &= x_n\ell(t) + \psi_{n-1}(x,t) - x_n\epsilon\arctan(w)\\
\dot{x}_n &= u + \psi_n(x,t)\\
\dot{w} &= -\epsilon\arctan(w)\\
\mu_1 &= x_1
\end{aligned} \tag{49}$$

where $w = x_{n+1} - \int \ell(t)dt$ (the *arctan* can be replaced by a linear function and it has been used only for simplifying computations).

Assume that

$$\psi_j^2(x,t) \le \sum_{i=1}^{j} a_{j1i}(x_1)x_i^2, \qquad j = 1,\ldots,n \tag{50}$$

Apply our stabilization result via smooth time varying measurement feedback to the system

$$\begin{aligned}
\dot{x}_1 &= x_2\ell(t) + \Psi_1(x,t)\\
\vdots &= \vdots\\
\dot{x}_{n-1} &= x_n\ell(t) + \Psi_{n-1}(x,t)\\
\dot{x}_n &= u + \Psi_n(x,t)\\
\mu_1 &= x_1
\end{aligned} \tag{51}$$

with $\Psi_j(x,w,t) = \psi_j(x,t) - x_{j+1}\epsilon\arctan(w)$ for $j = 1,\ldots,n-1$ and $\Psi_n(x,w,t) = \psi_n(x,t)$, by suitably selecting ϵ. Global asymptotic stabilization of (48) follows from our results and the fact that $\lim_{t\to\infty}\int \ell(t)dt = 0$. Note that, while the convergence to the trajectory $\dfrac{\sin(t+1)^2}{(t+1)^2}$ is exponential, the convergence to the origin is not (of order $\dfrac{1}{t^2}$).

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