Furthermore, $A_1 = d_1 e_1'$, $A_2 = d_2 e_2'$ and $C_1 = h_1 w_1'$ where

$$d_1 = \begin{bmatrix} 0 \\ 0.241 \\ 0 \\ 0 \end{bmatrix}; d_2 = \begin{bmatrix} 0 \\ 0 \\ 1.03 \\ 0 \end{bmatrix}; e_1 = \begin{bmatrix} 0.241 \\ -0.241 \\ -0.025 \\ 0 \end{bmatrix}; e_2 = \begin{bmatrix} 0 \\ 0.117 \\ 0 \\ -1.03 \end{bmatrix};$$

$$h_1 = \begin{bmatrix} 0.221 \\ 0 \end{bmatrix}; w_1 = \begin{bmatrix} 0 \\ -0.221 \\ 0 \\ 0 \end{bmatrix}.$$

This uncertain system was constructed so as to encompass both operating points of the synchronous turbo-generator. The weighting matrices were simply taken to be unit matrices, i.e., $R_1 = 1$, $R_2 = I$, $Q_1 = I$, and $Q_2 = I$. Using $\epsilon_1 = \epsilon_2 = 1$, Riccati equations (3.1) and (3.2) yielded the following positive definite solutions:[1]

$$P_c = \begin{bmatrix} 0.0087 & 0.0901 & -0.0040 & -0.0281 \\ 0.0901 & 5.0632 & -0.3135 & -1.5533 \\ -0.0040 & -0.3135 & 0.1780 & 0.2151 \\ -0.0281 & -1.5533 & 0.2151 & 2.0297 \end{bmatrix};$$

$$P_0 = \begin{bmatrix} 222.02 & -0.6915 & -0.0024 & -0.0462 \\ -0.6915 & 25.444 & 2.3674 & 1.8142 \\ -0.0024 & 2.3674 & 2.0536 & 0.0306 \\ -0.0462 & 1.8142 & 0.0306 & 4.5004 \end{bmatrix}.$$

It is straightforward to verify that the matrices P_c and P_o satisfy condition iii) of Theorem 3.1. Hence, the uncertain linear system (Σ_1) can be stabilized using the scheme described in the previous section. This leads to the gain matrices

$$K = [-0.9712 \quad -10.002 \quad 0.4492 \quad 3.1214]; \; L = \begin{bmatrix} 0.0003 & -0.0015 \\ 0.1810 & -0.5211 \\ -0.1915 & 5.4677 \\ -1.1471 & 0.1729 \end{bmatrix}.$$

Using these gain matrices, one may construct a dynamic compensator which will stabilize the original synchronous turbo-generator system.

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An Existence Theorem for Discrete-Time Infinite-Horizon Optimal Control Problems

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Abstract—**An existence theorem is given for a general class of deterministic, infinite-horizon, discrete-time optimal control problems. The hypotheses of the theorem are weak and can be easily verified. The objective function may be either a summation or supremum over time.**

I. INTRODUCTION

The aim of this note is to give conditions which guarantee the existence of a solution for a general class of deterministic, infinite-horizon, discrete-time optimal control problems. Except for special problems such as the linear quadratic control problem [7], most existence theorems available in the literature [2], [4], [5], [8] are for stochastic problems. Some of these theorems need restrictive assumptions and others state hypotheses which are not easily verified. Our existence theorem treats the deterministic problem directly and involves rather weak, easily verified conditions. Some of the concepts used in our approach parallel those used by Doležal [3], who has given a general existence theorem for finite horizon problems.

Compactness conditions appear in the statement of the existence theorem (Theorem 1). In a special Hilbert space setting, they may be replaced by weak compactness, extending the results in [6]. This is discussed in Section IV.

II. THE EXISTENCE THEOREM

The following notations will be used. (Z, d) denotes the metric space Z with metric d. N and $\bar{N}$ denote, respectively, the set of positive and nonnegative integers.

Consider the discrete-time system described by

$$x_{k+1} = f_k(x_k, u_k), \qquad x_0 \in T_0,$$
$$(x_k, u_k) \in W_k \subset (X, d_X) \times (U, d_U), \qquad k \in \bar{N}. \tag{2.1}$$

A sequence pair $\{(x_k, u_k)\}_{k \in \bar{N}}$ is said to be admissible if it satisfies (2.1). Problems A and B consist of minimizing, respectively, the costs

$$J_A = \sum_{k \in \bar{N}} g_k(x_k, u_k) \tag{2.2}$$

or

$$J_B = \sup_{k \in \bar{N}} g_k(x_k, u_k) \tag{2.3}$$

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over the class of all admissible sequence pairs. We now state the existence theorem.

Theorem 1: Assume $T_0 \subset X$ is compact and for each $k \in \bar{N}$ the following conditions hold: a) W_k is a closed subset of $X \times U$; b) $f_k: W_k \to X$ is continuous; c) $g_k: W_k \to R$ is lower semicontinuous and nonnegative; d) given any compact set $P \subset X$, the set $S_k(P) = \{(x, u) \in W_k | x \in P\}$ is compact in $X \times U$. Also assume: e) there exists an admissible sequence pair with a finite cost $\hat{J}_A(\hat{J}_B)$. Then problem $A(B)$ has a solution. □

By setting $g_K(x, u) = h(x)$, $W_K = T_K \times U$, and $g_k \equiv 0$, $W_k = X \times U$, $k > K$, the infinite-horizon problems become finite-horizon problems. Thus, Theorem 1 also applies to finite-horizon problems. In fact, our result is then very similar to the one of Doležal [3]. Our theorem is stronger in that it allows consideration of problem B and has a more general formulation for the constraint sets W_k; Doležal's theorem is stronger in that it applies when X and U are abstract Hausdorff topological spaces. The proof of Theorem 1 is given in Section III. Although the spaces X and U are general metric spaces, the case of greatest practical interest is $X = R^n$ and $U = R^m$. However, the proof in the general formulation is no more difficult and shows that the topological concepts needed to state and prove the existence theorem are quite elementary.

The hypotheses in Theorem 1 are not necessarily in a form which allows easy verification in applications. Some useful conditions which may replace conditions c) and d) are indicated in the following theorem and remark.

Theorem 2: Theorem 1 also holds if, for each $k \in \bar{N}$, condition d) is replaced by *any one* of the following conditions: d_1) the expression $(x, u) \in W_k$ can be written as $u \in G_k(x)$ where $G_k(\cdot)$ is a set valued function that maps elements of X into compact subsets of U, and is upper semicontinuous by set inclusion; d_2) there exists a compact set $\Omega_k \subset U$ such that $\{u \in U | (x, u) \in W_k\} \subset \Omega_k$; d_3) (U, d_U) is linear and finite dimensional and there is a function $\phi_k: U \to R$ which satisfies the following conditions: 1) $\phi_k(u) \to \infty$ whenever $d_U(u, o) \to \infty$; 2) $g_k(x, u) \geq \phi_k(u)$ whenever $(x, u) \in W_k$. □

Remark: Adding a finite number α, to J_A or J_B will not change the existence theorem. Thus, condition c) may be replaced by the following (weaker) condition: c′) $g_k: W_k \to R$ is lower semicontinuous and for problem $A(B)$ there is a sequence of real numbers $\{\alpha_k\}_{k \in \bar{N}}$ such that $\alpha = \Sigma_{k \in \bar{N}} \alpha_k (\alpha = \inf_{k \in \bar{N}} \alpha_k)$ is finite and $g_k(x, u) \geq \alpha_k$ whenever $(x, u) \in W_k$. □

An important application of c′) is to discounted cost functions [2], i.e., $g_k(x, u) = \delta^k \tilde{g}_k(x, u)$ where $0 \leq \delta < 1$ and $\tilde{g}_k$ is uniformly bounded on W_k for all k.

III. Proofs

The essence of Doležal's proof is to show that the hypotheses lead to the minimization of a lower semicontinuous function on a compact subset of a Hausdorff topological space. Existence then follows from a classical theorem. However, in the infinite-horizon case Doležal's reduction does not work and a more elaborate argument based on the following lemmas is needed.

Lemma 1: Let $\{z_k(i)\}_{k \in \bar{N}, i \in N}$ be a doubly infinite array whose elements are in a metric space (Z, d). Also, let $\{Z_k\}_{k \in \bar{N}}$ be a sequence of compact sets $Z_k \subset Z$ such that for each $k \in \bar{N}$, $z_k(i) \in Z_k$ for all $i \in N$. Then there exists an increasing subsequence $\{i_t\}_{t \in N} \subset N$, and a sequence $\{\bar{z}_k\}_{k \in \bar{N}}$ such that for each $k \in \bar{N}$, $\lim_{t \to \infty} z_k(i_t) = \bar{z}_k \in Z_k$. □

Proof: First consider the sequence $\{z_0(i)\}_{i \in N}$. Since this sequence lies in the compact set Z_0, there is an increasing subsequence $\{i_{0,t}\}_{t \in N} \subset N$, and $\bar{z}_0 \in Z_0$ such that $\lim_{t \to \infty} z_0(i_{0,t}) = \bar{z}_0$. Now, the subsequence $\{z_1(i_{0,t})\}_{t \in N}$ lies in the compact set Z_1. Hence, there is a further increasing subsequence $\{i_{1,t}\}_{t \in N} \subset \{i_{0,t}\}_{t \in N}$, and $\bar{z}_1 \in Z_1$ so that $\lim_{t \to \infty} z_1(i_{1,t}) = \bar{z}_1$. This process is repeated inductively, and we have a sequence of subsequences $\{i_{0,t}\}_{t \in N} \supset \{i_{1,t}\}_{t \in N} \supset \cdots$, forming a doubly infinite array whose elements are in N. Out of this array, choose only the diagonal elements to form a new subsequence $\{i_t\}_{t \in N}$. It is then easy to verify the lemma. □

Lemma 2: Suppose assumptions a), b), and d) of Theorem 1 hold. Then it is possible to construct a sequence of sets $\{V_k\}_{k \in \bar{N}}$ such that, for all $k \in \bar{N}$: 1) $V_k \subset W_k$; 2) V_k is compact; 3) for any admissible sequence pair, $(x_k, u_k) \in V_k$. □

Proof: Similar to [3], we use induction. Begin with $T_o \subset X$ and define

$$V_0 = \{(x, u) \in W_0 | x \in T_0\}. \tag{3.1}$$

By the compactness of T_0 and assumption d), V_0 is a compact subset of W_0. Next, consider compact sets $T_k \subset X$ and $V_k \subset W_k$. Define

$$T_{k+1} = \{x | x = f_k(y, u),\ (y, u) \in V_k\} \subset X, \tag{3.2}$$

$$V_{k+1} = \{(x, u) \in W_{k+1} |\ x \in T_{k+1}\} \subset W_{k+1}. \tag{3.3}$$

By assumptions b) and d), it is easily confirmed that T_{k+1} and V_{k+1} are compact. Thus, $\{V_k\}_{k \in \bar{N}}$ satisfies 1) and 2). Property 3) is obvious from (3.2) and (3.3). □

We now prove Theorem 1. Let I_A denote the infimum of the cost function J_A among the class of all admissible sequence pairs. By assumptions c) and e), I_A exists and $0 \leq I_A < \infty$. Thus, there exists a sequence of admissible sequence pairs $\{\{(x_k(i), u_k(i))\}_{k \in \bar{N}}\}_{i \in N}$ such that

$$I_A \leq \sum_{k \in \bar{N}} g_k(x_k(i), u_k(i)) \leq I_A + 1/i, \qquad i \in N. \tag{3.4}$$

For each $i \in N$, define $\{y_k(i)\}_{k \in \bar{N}}$ by

$$y_0(i) = 0,\ y_k(i) = \sum_{j=0}^{k-1} g_j(x_j(i), u_j(i)), \quad k \in N. \tag{3.5}$$

From (3.4), (3.5), and assumption c),

$$0 \leq y_k(i) \leq I_A + 1/i, \qquad k \in \bar{N},\ i \in N. \tag{3.6}$$

Let $\{V_k\}_{k \in \bar{N}}$ be the sequence of compact subsets of $X \times U$ constructed in Lemma 2. Then, for each $k \in \bar{N}$, $\{(y_k(i), x_k(i), u_k(i))\}_{i \in N} \subset [0, I_A + 1] \times V_k$, a compact subset of $R \times X \times U$. By Lemma 1 there is a subsequence $\{i_t\}_{t \in N} \subset N$ and a sequence $\{(\bar{y}_k, \bar{x}_k, \bar{u}_k)\}_{k \in \bar{N}}$ such that

$$\lim_{t \to \infty} (y_k(i_t),\ x_k(i_t),\ u_k(i_t)) = (\bar{y}_k,\ \bar{x}_k,\ \bar{u}_k), \tag{3.7}$$

$$(\bar{y}_k,\ \bar{x}_k,\ \bar{u}_k) \in [0,\ I_A + 1] \times V_k \subset [0,\ I_A + 1] \times W_k, \qquad k \in \bar{N}. \tag{3.8}$$

Also, by the continuity of f_k and the compactness of T_0,

$$\bar{x}_{k+1} = \lim_{t \to \infty} x_{k+1}(i_t) = \lim_{t \to \infty} f_k(x_k(i_t),\ u_k(i_t))$$
$$= f_k(\bar{x}_k,\ \bar{u}_k),\ k \in \bar{N}, \tag{3.9}$$

$$\bar{x}_0 = (\lim_{t \to \infty} x_0(i_t)) \in T_0. \tag{3.10}$$

Therefore, $\{(\bar{x}_k, \bar{u}_k)\}_{k \in \bar{N}}$ is an admissible sequence pair. We will now show that it is a solution for problem A.

First (3.6) and (3.7) imply that

$$\bar{y}_k \leq I_A, \qquad k \in \bar{N}. \tag{3.11}$$

Now let $\{\tilde{y}_k\}_{k \in \bar{N}}$ be generated by

$$\tilde{y}_0 = 0,\ \tilde{y}_k = \sum_{j=0}^{k-1} g_j(\bar{x}_j,\ \bar{u}_j); \qquad k \in N. \tag{3.12}$$

By assumption c), $\{\tilde{y}_k\}_{k \in \bar{N}}$ is a nonnegative, nondecreasing sequence of real numbers. Also by (3.5), (3.7), (3.11), and the lower semicontinuity of g_j,

$$0 \leq \tilde{y}_k \leq \sum_{j=0}^{k-1} \varliminf_{t \to \infty} g_j(x_j(i_t),\ u_j(i_t))$$
$$\leq \varliminf_{t \to \infty} y_k(i_t) = \lim_{t \to \infty} y_k(i_t) = \bar{y}_k \leq I_A. \tag{3.13}$$

Hence, $\lim_{k\to\infty} \tilde{y}_k$ exists and lies in the interval $[0, I_A]$. Note that the limit is also $\bar{J}_A$, the cost given by the admissible sequence pair $\{(\bar{x}_k, \bar{u}_k)\}_{k\in\bar{N}}$. Hence, $\bar{J}_A \geq I_A$ and the proof is complete for problem A. The detailed proof for problem B is similar and will not be given. It is only necessary to give different definitions for $\tilde{y}_k$ and $y_k(i)$. For example,

$$\tilde{y}_0 = 0,\ \tilde{y}_{k+1} = \max\ (\tilde{y}_k,\ g_k(\bar{x}_k,\ \bar{u}_k)), \qquad k \in \bar{N}. \tag{3.14}$$

We now prove Theorem 2. A proof that condition d_1) implies d) is given in [3, Prop. 6, p. 305]. We now prove that d_2) implies d). Let $P \subset X$ be a compact set and $S_k(P) = \{(x, u) \in W_k | x \in P\} \subset P \times \Omega_k$, a compact set. Then $S_k(P)$ is compact since it is closed. Thus, d) is satisfied. Finally, we prove the result of Theorem 1 for d_3). Because of condition e) of Theorem 1, it is sufficient to consider only those admissible sequence pairs with cost $J_A(J_B) \leq \hat{J}_A(\hat{J}_B)$. Since $\phi_k(u) \to \infty$ whenever $d_U(u, 0) \to \infty$, there exists $\mu_k \geq 0$ such that $d_U(u, 0) \leq \mu_k$ whenever $\phi_k(u) \leq \hat{J}_A(\hat{J}_B)$. Now let $\Omega_k = \{u | d_U(u, 0) \leq \mu_k\}$ and use condition d_2) to complete the proof.

IV. An Extension in Hilbert Spaces

For convex linear optimal control problems in Hilbert spaces, we can replace the compactness and closedness assumptions by their weak counterparts.

Theorem 3: Suppose X, U are Hilbert spaces (with inner products $\langle \cdot, \cdot \rangle_X$, $\langle \cdot, \cdot \rangle_U$), $T_0 \subset X$ is weakly compact and for each $k \in \bar{N}$ the following hold: a) W_k is weakly closed; b) $f_k(x, u) = L_k(x, u) + q_k$ where L_k is a linear bounded operator and $q_k \in X$; c) g_k is a continuous, convex, nonnegative functional; d) given any weakly compact set $P \subset X$, $S_k(P) = \{(x, u) \in W_k | x \in P\}$ is weakly compact. Also assume e) there is an admissible sequence pair with a finite cost. Then problem $A(B)$ has a solution. The existence result still holds if, for each $k \in \bar{N}$, condition d) is replaced by either of the following conditions: d_2') there is a weakly compact set $\Omega_k \subset U$ so that $\{u | (x, u) \in W_k\} \subset \Omega_k$; d_3') there is a function $\phi_k: U \to R$ which satisfies: 1) $\phi_k(u) \to \infty$ whenever $\langle u, u \rangle_U \to \infty$; 2) $g_k(x, u) \geq \phi_k(u)$ whenever $(x, u) \in W_k$. □

The proof of this theorem is along the same lines as the proof of Theorem 1, but with the following changes. All the compactness, closedness properties, and limits should be replaced by their weak forms. In proving Lemma 2 one should use the fact that the range of a linear bounded operator with a weakly compact domain is weakly compact. The arguments leading to (3.9) should be modified as follows. Let $z = \bar{x}_{k+1} - f_k(\bar{x}_k, \bar{u}_k)$. By using weak convergence and Riesz representation theorem we can show that

$$\lim_{t\to\infty} \langle x_{k-1}(i_t) - f_k(x_k(i_t),\ u_k(i_t)),\ z \rangle_X = \langle z,\ z \rangle_X.$$

Since $x_{k-1}(i_t) = f_k(x_k(i_t), u_k(i_k))$, $t \in N$, $\langle z, z \rangle_X = 0$. Thus, $z = 0$, yielding (3.9). Condition (3.13) should be derived using the following result [1, Corollary 1.8.3, p. 30]. If $F(\cdot)$ is a continuous convex functional on a Hilbert space and $\{x_n\}$ converges weakly to x, then $\underline{\lim}\ F(x_n) \geq F(x)$. In deriving d_3') one should use the fact that a bounded and closed set is weakly compact.

Theorem 3 is of some value in the optimal regulation and tracking of distributed parameter systems. The existence theorem for the discrete-time linear quadratic problem in Hilbert spaces proved by Lee *et al.* [6] is a special case of Theorem 3.

V. An Example

Consider the optimal regulator problem in which: 1) $X = R^n$, $U = R^m$; 2) J_A is the performance index. For each $k \in \bar{N}$: 3) W_k is closed and contains the origin; 4) $f_k(x, u) = A_k(x)x + B_k(x)u$, where A_k and B_k are continuous; 5) $g_k(x, u) = Q_k(x) + u'R_k(x)u$, where u' is the transpose of u, $Q_k: X \to R$ and $R_k: X \to R^{m\times m}$ are continuous, $Q_k(x) \geq 0$, $Q_k(0) = 0$, and there is a $\beta_k > 0$ such that $u'R_k(x)u \geq \beta_k u'u$ for all $(x, u) \in W_k$. Also assume: 6) $T_0 = \{x_0\}$, a compact set. Assumptions 3)–5) directly imply conditions a)–c) of Theorem 1. Let $X_0 = \{y |$ there is an admissible sequence pair $\{(x_k, u_k)\}_{k\in\bar{N}}$ and $M \geq 0$ (both depending on y) such that $x_0 = y$ and $(x_k, u_k) = 0$, $k \geq M.\}$ Assume: 7) $x_0 \in X_0$. It is easy to verify that 7) implies condition e) of Theorem 1; and, 1), 5), 7) imply condition d_3) of Theorem 2 (with $\phi_k(u) = \beta_k(u'u)$). All the hypotheses of Theorem 1 are satisfied, and hence the optimal regulator problem has a solution. The linear quadratic regulator problem is a special case of this problem.

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On the Realization of Input-Decentralized Representations for Large-Scale Systems

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Abstract—**The input-decentralization procedure proposed by Siljak and Vukcevic is analyzed in this note. It is shown that the procedure is valid on an open and dense subset of the parameter space of all large-scale systems with controllable subsystems. The examples which have appeared recently for which the proposed procedure fails are isolated points in the entire parameter space.**

I. Introduction

Recently, several authors [3] and [4] have recognized several limitations of the input decentralization procedure suggested by Siljak and Vukcevic [1], [2]. Some modifications were suggested in [4], but one problem still remained. The problem is that controllability of the original subsystems is *not* sufficient to guarantee that the subsystems in the input decentralized representation are controllable. The purpose of the current work is to investigate the class of large-scale systems (LSS) for which the input-decentralization procedure yields consistent results.

II. Main Results

Consider a system governed by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

where A is a constant $n \times n$ matrix, B is a constant $n \times m$ matrix, $x \in R^n$ is the state vector, and $u \in R^m$ is the control input. For simplicity the pair (A, B) will be used to denote (2.1). The pair (A, B) can be

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