From (12) we obtain

$$L = \begin{bmatrix} 0 & 0 \\ -1 & -\lambda_2\lambda_3 \\ 0 & -1-\lambda_2-\lambda_3 \end{bmatrix}.$$

Then the observer is

$$\dot{z}_1 = \lambda_1 z_1$$

$$\dot{z}_2 = \lambda_2\lambda_3 z_3 - y_1 - \lambda_2\lambda_3 y_2$$

$$\dot{z}_3 = -z_2 + (\lambda_2 + \lambda_3)z_3 - (1 + \lambda_2 + \lambda_3)y_2$$

and

$$\hat{x}_1 = y_1 + z_1$$

$$\hat{x}_2 = z_2$$

$$\hat{x}_3 = z_3.$$

This observer can be reduced to a second-order one, since $\mathrm{Re}(\lambda_1) < 0$.

IV. Conclusion

In this note, we have presented a simple method to design a full-order observer for a linear system with unknown inputs. This method reduces the design procedure of full-order observers with unknown inputs to a standard one where the inputs are known. The existence conditions are given, and it was shown that these conditions are generally adopted for unknown inputs observer problem.

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Robust Motion/Force Control of Mechanical Systems with Classical Nonholonomic Constraints

Chun-Yi Su and Yury Stepanenko

***Abstract*—The position/force control of mechanical systems subject to a set of classical nonholonomic constraints represents an important class of control problems. In this note, a reduced dynamic model, suitable for simultaneous independent motion and force control, is developed. Some properties of the dynamic model are exploited to facilitate the controller design. Based on the theory of guaranteed stability of uncertain systems, a robust control algorithm is derived, which guarantees the uniform ultimate boundedness of the tracking errors. A detailed numerical example is presented to illustrate the developed method.**

I. Introduction

The control of mechanical systems with kinematic constraints has received increasing attention and is a topic of great interest. A lot of papers have been published in recent years to deal with the control problem when the kinematic constraints are holonomic constraints [1]–[4]. In contrast, if the kinematic constraints are nonholonomic, control laws developed for holonomic constraints are not applicable; only a few papers have been proposed to address these control issues. In this note, our discussions are focused on the classical nonholomonic case, and analyses are given from the Lagrangian point of view. As for the Hamiltonian case with other forms of nonholonomic constraints, the reader may refer to [12].

It is well known that in rolling or cutting motions the kinematic constraint equations are classical nonholonomic [10], and the dynamics of such systems is well understood (see, e.g. [10]). However, the literature on control with classical nonholonomic constraints is quite recent [5], [7], [8], and the discussion mainly focuses on some special examples [11], [13]–[15]. Earlier work that deals with control of nonholonomic systems is described in [9]. Bloch and McClamroch [5], Bloch *et al.* [7], and Campion *et al.* [8] demonstrated that systems with nonholonomic constraints are always controllable, but cannot be feedback stabilized to a single point with smooth feedback. By using a decomposition transformation and nonlinear feedback, conditions for smooth asymptotic stabilization to an equilibrium manifold are established. d'Andrea-Novel *et al.* [11] and Yun *et al.* [13] showed that the system is linearizable by choosing a proper set of output equations, and then applied, respectively, their results to the control of wheeled mobile robots and multiple arms. Researchers have also offered both nonsmooth feedback laws [6], [7], [14] and time-varying feedback laws [15] for stabilizing the system to a point. However, it is fair to say that the last two approaches are not yet fully general.

The above mentioned approaches, e.g. [5], [7], and [8], indeed provide a theoretic framework which can serve as a basis for the study of mechanical systems with nonholonomic constraints; however, all of those results are based on the method of a diffeomorphism and nonlinear feedback (for details, see [16]), which requires a detailed dynamic model and may be sensitive to parametric uncertainties.

In this note, a different control approach is proposed, in which the control of the constraint force due to the existence of classical constraints is also included. By assuming complete knowledge of the constraint manifold, and recognizing that the degree of freedom

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The authors are with the Department of Mechanical Engineering, University of Victoria, Victoria, B. C. V8W 3P6, Canada.

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of mechanical systems decrease due to nonholonomic constraints, a reduced-form equation suitable for motion and force control is derived. Then, by exploiting the particular structure of the reduced dynamics, several fundamental properties are obtained to facilitate the controller design. Finally, with the specification of a desired manifold, a smooth robust control algorithm is derived, using only the measurements of joint position, velocity, and constraint force. Stability analysis shows the stabilization of the manifold in the sense that tracking errors are uniformly ultimately bounded.

II. Model of Robotic System with Classical Nonholonomic Constraints

In this section, we are concerned with mechanical systems whose configuration space is an n-dimensional simply connected manifold $\mathfrak{R}$, and whose dynamics are described, in local coordinates (termed generalized coordinates), by so called Euler–Lagrangian formulation as [8], [11]

$$D(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = \boldsymbol{f} + B(\boldsymbol{q})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{q}$ denotes the n-vector of generalized coordinates; $\boldsymbol{u}$ denotes the r-vector of generalized control input force; $\boldsymbol{f}$ denotes the n-vector of constraint forces; $D(\boldsymbol{q})$ is the $(n\times n)$ symmetric, bounded, positive definite inertia matrix; $C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ presents the n-vector of centripetal and Coriolis torques; $G(\boldsymbol{q})$ is the n-vector of gravitational torques, $B(\boldsymbol{q})$ is an $(n \times r)$ input transformation matrix.

Two simplifying properties should be noted about this dynamic structure.

Property 1: There exists a p-vector α with components depending on mechanical parameters (masses, moments of inertia, etc.), such that [17]

$$D(\boldsymbol{q})\dot{\boldsymbol{v}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\boldsymbol{v} + G(\boldsymbol{q}) = \Phi(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}, \dot{\boldsymbol{v}})\alpha \tag{2}$$

where Φ is an $n \times p$ matrix of known functions of $\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{v}, \dot{\boldsymbol{v}}$; and α is the p-vector of inertia parameters [18].

Property 2: A suitable definition of $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ makes the matrix $(\dot{D} - 2C)$ skew-symmetric [17]. In particular, this is true if the elements of $C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are defined as in [8].

$$C_{ij} = \frac{1}{2}\left[\dot{\boldsymbol{q}}^T \frac{\partial D_{ij}}{\partial \boldsymbol{q}} + \sum_{k=1}^{n}\left(\frac{\partial D_{ik}}{\partial q_j} - \frac{\partial D_{jk}}{\partial q_i}\right)\dot{q}_k\right]. \tag{3}$$

It should be noted that the first property says that the Lagrangian dynamic equation are linearly parameterizable and the second property is related to the passivity of the mechanical dynamics.

Let consider the situation where kinematic constraints are imposed, which are described by [5], [7], [8], [10], [11], and [13]

$$J(\boldsymbol{q})\dot{\boldsymbol{q}} = 0 \tag{4}$$

where $J(\boldsymbol{q})$ is an $(m \times n)$ constraint matrix which is assumed to have full rank m.

The constraint equations (4) are assumed to be classical nonholonomic constraints. Such constraints can arise in many cases, including the case when two surfaces roll against each other [10], [13]. The classical constraints are assumed not integrable. Nonintegrable constrains cannot be reduced to geometric constraints while integrable constraints are essentially geometric constraints (see [10] for the detailed explanation).

The effect of the constraints can be reviewed as restricting the dynamics to the manifold Ω defined by

$$\Omega = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) | J(\boldsymbol{q})\dot{\boldsymbol{q}} = 0\}.$$

It should be noted that since the constraints are nonintegrable, there is, in fact, no explicit restriction on the values of the variables $\boldsymbol{q}$.

When the nonholonomic constraints (4) are imposed on the mechanical systems (1), the constraint (generalized reaction) forces are given by

$$\boldsymbol{f} = J^T(\boldsymbol{q})\lambda \tag{5}$$

where $\lambda \in R^m$ is the associated Lagrangian multipliers [5], [7], [8], [10].

In the following, we denote the constraint matrix $J(\boldsymbol{q})$ as

$$J^T(\boldsymbol{q}) = [J_1(\boldsymbol{q}), \cdots, J_m(\boldsymbol{q})]$$

where $J_1, \cdots, J_m$ are smooth n-dimensional covector fields on $\mathfrak{R}$. Then, the annihilator of the codistribution spanned by the covector fields $J_1, \cdots, J_m$ is an $(n-m)$-dimensional smooth nonsingular distribution Δ on $\mathfrak{R}$. This distribution Δ is spanned by a set of $(n-m)$ smooth vector fields $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_{n-m}$:

$$\Delta = \text{span}\{\boldsymbol{r}_1(\boldsymbol{q}), \cdots, \boldsymbol{r}_{n-m}(\boldsymbol{q})\}$$

which satisfy, in local coordinates, the following relations [8]

$$R^T(\boldsymbol{q})J^T(\boldsymbol{q}) = 0 \tag{6}$$

where the full rank matrix $R(\boldsymbol{q})$ is made up of the vector function $\boldsymbol{r}_i(\boldsymbol{q})$:

$$R(\boldsymbol{q}) = [\boldsymbol{r}_1(\boldsymbol{q}), \cdots, \boldsymbol{r}_{n-m}(\boldsymbol{q})].$$

The constraints (4) and (6) imply the existence of an $(n-m)$-vector $\dot{\boldsymbol{z}}$ such that

$$\dot{\boldsymbol{q}} = R(\boldsymbol{q})\dot{\boldsymbol{z}}. \tag{7}$$

It should be noted that the $(n-m)$-vector $\dot{\boldsymbol{z}}$ represents internal states, so that $(\boldsymbol{q}, \dot{\boldsymbol{z}})$ is sufficient to describe the constrained motion.

Differentiating (7), we obtain

$$\ddot{\boldsymbol{q}} = R\ddot{\boldsymbol{z}} + \dot{R}\dot{\boldsymbol{z}}. \tag{8}$$

Therefore, the dynamic equation (1), when satisfying the nonholonomic constraint (4), can be rewritten in terms of the internal state variables $\dot{\boldsymbol{z}}$ as

$$D(\boldsymbol{q})R(\boldsymbol{q})\ddot{\boldsymbol{z}} + C_1(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{z}} + G(\boldsymbol{q}) = B(\boldsymbol{q})\boldsymbol{u} + J^T(\boldsymbol{q})\lambda \tag{9}$$

where $C_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = D(\boldsymbol{q})\dot{R}(\boldsymbol{q}) + C(\boldsymbol{q}, \dot{\boldsymbol{q}})R(\boldsymbol{q})$.

It should be noted that reduced state space is $2n-m$ dimensional. The system is described by the n-vector of variables $\boldsymbol{q}$ and the $(n-m)$-vector of variables $\dot{\boldsymbol{z}}$.

Remark: Equation (9) is suitable for control purposes and forms the basis for the subsequent development. This is because the equality constraint equation (4) are embedded into the dynamic equation, resulting in an affine nonlinear system without constraints.

By exploiting the structure of the equation (9), three properties are obtained.

Property 3: The generalized inertia matrix $R^T D(q) R$ is symmetric and positive definite.

Property 4: Define $D_1(q) = R^T D(q) R$. If $C(q, \dot{q})$ is defined as that Property 2 is verified, $(\dot{D}_1 - 2R^T C_1)$ is a skew symmetric matrix.

Proof: Directly, by using the definition of $\dot{D}_1$ and C_1 and by considering the skew symmetry of $(\dot{D} - 2C)$ in property 2.

Property 5: The dynamic structure (9) is linear in terms of the same suitably selected set of inertia parameters as used in Property 1

$$D(q)R(q)\ddot{z} + C_1(q, \dot{q})\dot{z} + G(q) = \Phi_1(q, \dot{q}, \dot{z}, \ddot{z})\alpha \tag{10}$$

where Φ_1 is an $(n \times p)$ regressor matrix; α is the p-vector of inertia parameters.

Property 5 may be easily understood by observing that the transformations do not change the linearity in terms of constant parameters α, established for model (1) by Property 1.

The aforementioned properties are fundamental for designing the robust force/motion control law.

III. Controller Design for Motion/Force Tracking

It has been proved (see [5], [7], and [8]) that the nonholonomic system (1) and (4) cannot be stabilized to a single point using smooth state feedback. It can only be stabilized to a manifold of dimension m due to the existence of m nonholonomic constraints. The objective of stabilizing these systems to a point has been achieved by nonsmooth feedback law [6], [7], [14], and time-varying feedback laws [15]. However, it is fair to say that these approaches are not yet fully general. It is worth mentioning that different control objectives may also be pursued, such as stabilization to manifolds of equilibrium points (as opposed to a single equilibrium position) or to trajectories (as long as they do not converge to a point).

By appropriately selecting a set of $(n - m)$-vector of variables $z(q)$ and $\dot{z}(q)$, the objective of the control can be specified as: given a desired z_d, $\dot{z}_d$, and desired constraint λ_d, determine a control law such that for any $(q(0), \dot{q}(0)) \in \Omega$ then $z(q)$, $\dot{q}$, and λ asymptotically converge to a manifold Ω_d specified as

$$\Omega_d = \{(q, \dot{q}, \lambda) | z(q) = z_d, \dot{q} = R(q)\dot{z}_d, \lambda = \lambda_d\}.$$

The variables $z(q)$ can be thought as $n - m$ "output equations" of the nonholonomic system. The choice of $z(q)$ is, as an example, illustrated in the next section.

Remark: If z_d, $\dot{z}_d$ are zero and λ_d free, then $\Omega_d = \{(q, \dot{q}) | z(q) = 0, \dot{q} = 0\}$ is the equilibrium manifold defined in (7). If $q = [q^1)^T, (q^2)^T]$ is a partition, where $q^1 \in R^{n-m}$, $q^2 \in R^m$, and $z(q) = q^1$, $z_d = \dot{z}_d = 0$, and λ_d free, then $\Omega_d = \{(q, \dot{q}) | q^1 = 0, \dot{q}^1 = 0, \dot{q}^2 = 0\}$ is the equilibrium manifold in [5]. If $z(q) = R^T q$, $\dot{z}_d = 0$, and λ_d free, then $\Omega_d = \{(q, \dot{q}) | R^T q = 0\}$ is the invariant set in [8]. Therefore, compared to [5], [7], and [8], we extend the stabilization problem to the tracking problem, including tracking of the contact force λ.

In the following, we define

$$e_z = z - z_d \tag{11}$$

$$e_\lambda = \lambda - \lambda_d \tag{12}$$

$$\dot{z}_r = \dot{z}_d - \Lambda e_z \tag{13}$$

where Λ is a positive definite matrix whose eigenvalues are strictly in the right-hand complex plane.

Defining α as a p-vector, containing the unknown elements in the suitably selected set of equivalent dynamic parameters, then the linear parameterizability of the dynamics (Property 5) leads to

$$D(q)R(q)\ddot{z}_r + C_1(q, \dot{q})\dot{z}_r + G(q) = \Phi_1(q, \dot{q}, \dot{z}_r, \ddot{z}_r)\alpha \tag{14}$$

where Φ_1 is the $(n \times p)$ regressor matrix.

A robust control law is defined as

$$Bu = \Phi_1(q, \dot{q}, \dot{z}_r, \ddot{z}_r)\varphi - KRs - J^T\lambda_c \tag{15}$$

$$\varphi = \begin{cases} -\rho \frac{\Phi_1^T Rs}{\|\Phi_1^T Rs\|} & \text{if } \|\Phi_1^T Rs\| > \epsilon \\ -\frac{\rho}{\epsilon}\Phi_1^T Rs & \text{if } \|\Phi_1^T Rs\| \le \epsilon \end{cases} \tag{16}$$

where Φ_1 is defined in (14); R is defined in (6); K is an $n \times n$ positive-definite matrix, $\rho \in R_+$ used in (16) is the upper bounds of inertia parameter α, i.e., $\|\alpha\| \le \rho$, which is assumed known; ϵ is an constant; the vector s, which can be thought of as a sliding surface, is defined as

$$s = \dot{e}_z + \Lambda e_z; \tag{17}$$

the force term λ_c is defined as

$$\lambda_c = \lambda_d - K_\lambda e_\lambda \tag{18}$$

where K_λ is a constant matrix of force control feedback gains.

The above controller consists of two parts. The first part provides the input torques for achieving desired "output" and internal state tracking. The second part provides the desired force tracking.

The following theorem can be stated:

Theorem: Consider the mechanical system described by (1) and (4), using the control law (15) and (16), then the following holds for any $(q(0), \dot{q}(0)) \in \Omega$:

i. e_z and $\dot{e}_z$ are uniformly ultimately bounded.
ii. e_λ is uniformly ultimately bounded and inversely proportional to the norm of the matrix $K_\lambda + I$.

Proof: Based on (17), using (9), (14), and (15), and after some calculations, the following is obtained:

$$DR\dot{s} = \Phi_1\varphi - \Phi_1\alpha - KRs - C_1 s - J^T(\lambda_c - \lambda). \tag{19}$$

According to (6), the above equation becomes

$$D_1\dot{s} = R^T\Phi_1\varphi - R^T\Phi_1\alpha - R^T KRs - R^T C_1 s. \tag{20}$$

Let us consider the generalized Lyapunov function

$$V = \frac{1}{2}s^T D_1 s \tag{21}$$

A simple calculation shows that along solutions of (20)

$$\begin{aligned}\dot{V} &= s^T(R^T\Phi_1\varphi - R^T\Phi_1\alpha - R^T KRs) + s^T\left(\frac{1}{2}\dot{D}_1 - R^T C_1\right)s \\ &= -s^T R^T KRs + s^T R^T\Phi_1(\varphi - \alpha)\end{aligned} \tag{22}$$

where we have used Property 6 to eliminate the term $s^T(1/2\dot{D}_1 - C_1)s$. Using an argument similar to [18], if $\|\Phi_1^T Rs\| > \epsilon$, the second term in (22) is

$$(\Phi_1^T Rs)^T(\varphi - \alpha) = (\Phi_1^T Rs)^T\left(-\alpha - \rho\frac{\Phi_1^T Rs}{\|\Phi_1^T Rs\|}\right) \leq \|\Phi_1^T Rs\|(\|\alpha\| - \rho) \leq 0 \tag{23}$$

from the Cauchy–Schwartz inequality and our assumption on $\|\alpha\|$. If $\|\Phi_1^T Rs\| \leq \epsilon$, we have

$$(\Phi_1^T Rs)^T(\varphi - \alpha) \leq (\Phi_1^T Rs)^T\left(\rho\frac{\Phi_1^T Rs}{\|\Phi_1^T Rs\|} + \varphi\right) = (\Phi_1^T Rs)^T\left(\rho\frac{\Phi_1^T Rs}{\|\Phi_1^T Rs\|} - \frac{\rho}{\epsilon}\Phi_1^T Rs\right). \tag{24}$$

The last term achieves a maximum value of $\epsilon/2\rho$ when $\|\Phi_1^T Rs\| = \epsilon/2$. Thus we have that

$$\dot{V} \leq -s^T R^T K R s + \epsilon/2\rho \leq -\eta\|s\|^2 + \delta \tag{25}$$

where $\eta = \lambda_{\min}(R^T K R)$, $\delta = \epsilon/2\rho$. From the above we obtain an upper bound of s as

$$\|s\| \leq \left[\frac{\lambda_2}{\lambda_1}\|s(0)\|^2 e^{-2\overline{\eta}t} + \frac{\delta}{\overline{\eta}\lambda_1}[1 - e^{-2\overline{\eta}t}]\right]^{1/2}. \tag{26}$$

where $\overline{\eta} = \eta/\lambda_2$, and λ_1 and λ_2 are positive scalars with property $\lambda_1 I \leq D_1 \leq \lambda_2 I$. Therefore, s is uniformly ultimately bounded. By standard arguments and the definition of s in (17), it can be shown that e_z, and $\dot{e}_z$ are also uniformly ultimately bounded.

Since s, e_z and $\dot{e}_z$ are bounded, it follows that $\dot{z}$, $\dot{z}_r$, and $\ddot{z}_r$ are all bounded. Therefore, all signals on the right side of (20) are bounded and we can conclude that $\dot{s}$ and therefore $\ddot{z}$ are bounded. Substituting the control (15) and (16) into reduced order dynamic model (9) yields

$$J^T(\lambda - \lambda_c) = [\Phi_1(q, \dot{q}, \dot{z}, \ddot{z})\alpha - \Phi_1(q, \dot{q}, \dot{z}_r, \ddot{z}_r)\varphi + KRs] = \sigma(q, \dot{q}, \dot{z}, \ddot{z}, \dot{z}_r, \ddot{z}_r) \tag{27}$$

where σ is a bounded function. Thus

$$J^T e_\lambda = (K_\lambda + I)^{-1}\sigma$$

and therefore the force tracking error $(f - f_d)$ are bounded and can be adjusted by changing the feedback gain K_λ. Thus, the theorem is proved. ▽

Remarks: 1) The control law is, in a simple fashion, related to the bounds of the inertia parameters α so that the parameter variations in the plant can be taken into account easily.

2) From (26), it is shown how ϵ affects the size of the ball within which $\|s\|$ is ultimately confined. If $\epsilon \to 0$, then $s \to 0$ and therefore $e_z \to 0$ and $\dot{e}_z \to 0$, $(q, \dot{q}, \lambda) \to \Omega_d$ exponentially. In such a case, φ in (15) becomes $\varphi = \mathrm{sgn}(\Phi_1^T Rs)$, which is a typical sliding mode control law. As a matter of fact, the control law (16) is just a smoothing realization of the switch function $\varphi = \mathrm{sgn}(\Phi_1^T Rs)$ so as to overcome chattering, which is undesirable in practice.

3) If the inertia parameters α is known, we can simply take $\varphi = \alpha$ in control law (15). In this case, it can be easily shown that $\dot{V} \leq -s^T R^T K R s$; therefore, $(q, \dot{q}, \lambda) \to \Omega_d$ as $t \to \infty$.

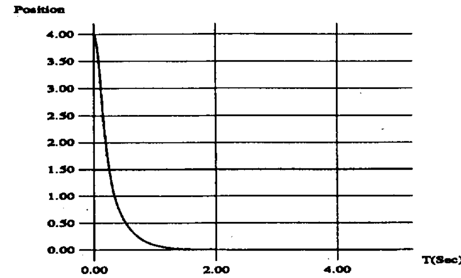

Fig. 1. Position trajectory of y.

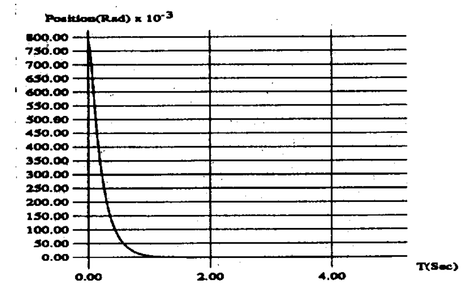

Fig. 2. Position trajectory of θ.

4) Suppose φ in control law (15) is replaced by $\hat{\alpha}$, representing estimation of α, and $\hat{\alpha}$ is updated by $\dot{\hat{\alpha}} = -\Gamma\Phi_1^T Rs$. With this algorithm, the closed loop system is globally convergent, i.e., $(q, \dot{q}, \lambda) \to \Omega_d$ as $t \to \infty$. Following the argument of [18], the question of whether to use robust control or adaptive control does not have an obvious answer. Clearly the adaptive control is easier to design and would be expected to work better if the uncertainty is large. However, it is known that adaptive control performs poorly in the presence of external disturbance and unmodeled dynamics unless the algorithm is modified. Such modification will result in a more complicated design comparable to the present robust design.

IV. Simulated Example

A simplified model of a mobile wheeled robot moving on a horizontal plane, constituted by a rigid trolley equipped with non-deformable wheels, as given in details in [8], is used to verify the validity of the control approach outlined in this note.

The dynamic model can be expressed as [8]

$$m\ddot{x} = \lambda\,\mathrm{cod}\,\theta - \frac{1}{P}(u_1 + u_2)\sin\theta$$

$$m\ddot{y} = \lambda\sin\theta + \frac{1}{P}(u_1 + u_2)\cos\theta$$

$$I_o\ddot{\theta} = \frac{L}{P}(u_1 - u_2) \tag{28}$$

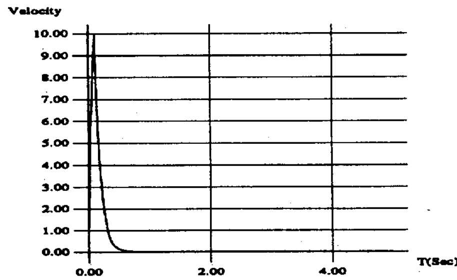

Fig. 3. Velocity of x.

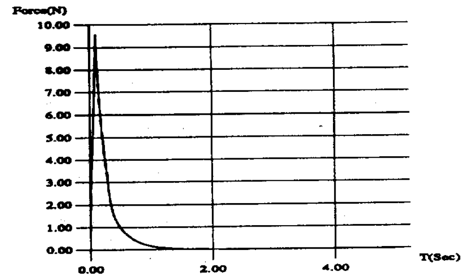

Fig. 6. Contact force error.

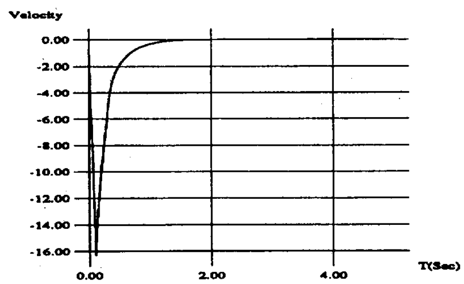

Fig. 4. Velocity of y.

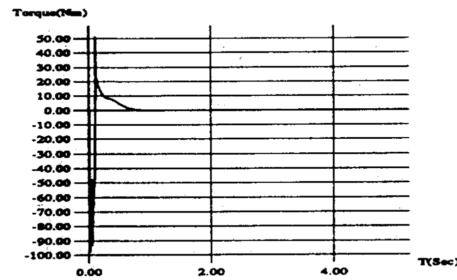

Fig. 7. Input u_1.

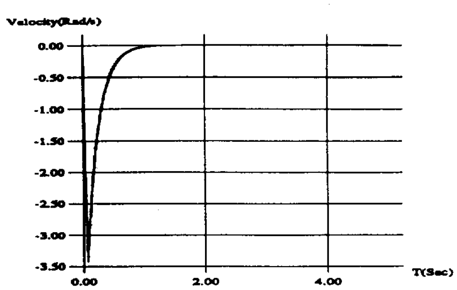

Fig. 5. Velocity of θ.

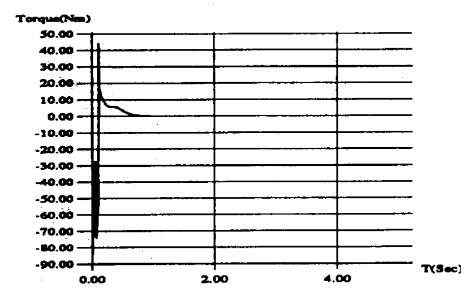

Fig. 8. Input u_2.

where x, y are coordinates in an inertial frame, θ is an orientation of the wheel with respect to the inertial frame, m is the mass of the robot, and I_o is its inertial moment around the vertical axis, P is the radius of the wheels and $2L$ the length of the axis of the front wheels, and u_i is the torques provided by the motors. For simplicity, we set $L = P = 1$.

The nonholonomic constraint is written as

$$\dot{x}\cos\theta + \dot{y}\sin\theta = 0 \tag{29}$$

The matrix $J(q)$ is therefore defined as $J(q) = [\cos\theta \;\; \sin\theta \;\; 0]$, where $q = [x \; y \; \theta]^T$. The constraint forces are $f = [\cos\theta \;\; \sin\theta \;\; 0]^T\lambda$. The "outputs" are chosen as

$$z(q) = [y,\, \theta]^T.$$

The matrix $R(q)$ defined in (6) is chosen as:

$$R = \begin{bmatrix} -\tan\theta & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so the relation $\dot{q} = R(q)\dot{z}$ is satisfied.

The desired manifold Ω_d is chosen as

$$\Omega_d = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}, \lambda) | \boldsymbol{z}(\boldsymbol{q}) = 0, \ \dot{\boldsymbol{q}} = 0, \ \lambda = 10\}.$$

The robust control law (15) with (16) is used so that $\boldsymbol{q} = [xy\theta]^T$, and λ approach Ω_d.

The unknown parameters α in (14) is chosen as $\alpha = [m, I_o]^T$, then, the regressor matrix defined in (14) can be written as

$$\Phi_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{z}_r, \ddot{z}_r) = \begin{bmatrix} -\ddot{y}_r \tan\theta - \dot{y}_r \sec^2\theta & 0 \\ \ddot{y}_r & 0 \\ 0 & \ddot{\theta}_r \end{bmatrix}.$$

The true values of m and I_c are $m = 0.5$ and $I_o = 0.5$. Thus, we choose $\rho = 1$ in the control law (15) with (16). The two tunable parameters Λ_1 and Λ_2 are chosen as $\Lambda_1 = 5$, $\Lambda_2 = 5$, and ϵ is chosen as $\epsilon = 5$. The control gain K and the force control gain K_λ are chosen as $K = \text{diag}\,(1, 1, 1)$, $K_\lambda = 0.8$.

The initial positions and velocities of robot are chosen as

$$x(0) = 0, \quad y(0) = 4, \quad \theta(0) = 45^\circ$$

$$\dot{x}(0) = 0, \quad \dot{y}(0) = 0, \quad \dot{\theta}(0) = 0.$$

Using the controller (15) with (16), the results of the simulation are shown in Figs 1–8. Fig. 1 shows the trajectory of y, Fig. 2 shows the trajectory of θ, Figs. 3–5 show the trajectories of $\dot{x}$, $\dot{y}$, $\dot{\theta}$, Fig. 6 shows the tracking error of λ, Figs. 7 and 8 show the torques exerted at the mobile robot. These results verify the validity of the proposed algorithm.

V. Conclusion

In this note, the issue of appropriate control of position and constraint force is addressed for a class of nonholonomic mechanical systems. By specifying an "output" function vector, a reduced dynamic model, suitable for simultaneous force and motion control, is established. A robust smooth control formulation is then proposed, ensuring that a system with m nonholonomic constraints can be stabilized to an m-dimensional desired manifold. However, the definition of the desired manifold depends on the specific choice of "output" function vector, which is related to the form of the constraint equations and the dynamic system. One choice is demonstrated via a simple simulation example. It should be noticed that the "output" function vector may or may not be physically motivated. Given the "output equations," the proposed control law indeed provides a convenient solution for the robust force and motion control of nonholonomic systems. A simplified mobile robot has been used to illustrate the methodology developed in this note, and simulation results are satisfactory.

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Stabilizing I–O Receding Horizon Control of CARMA Plants

L. Chisci and E. Mosca

***Abstract*—Stabilizing input–output receding horizon control (SIORHC) yields dynamic feedback compensators capable of stabilizing any stabilizable linear plant under sharp conditions. The guaranteed stabilizing property is insured by using output terminal constraints in addition to input terminal constraints. This note extends previous SIORHC results to CARMA plants.**

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The authors are with Dipartimento di Sistemi e Informatica, Universitá di Firenze, Via di Santa Marta, 3-50139 Firenze, Italy.

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