Contrôle de Systèmes Euler-Lagrange par retour de sortie

On Output Feedback Control of Euler-Lagrange Systems

Thèse de Doctorat
PhD Thesis

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To Lena,
with
$r = 1 - sin(\theta)$

Préambule

Ce mémoire de recherche est présenté comme une formalité partielle pour l'obtention du titre de “Docteur”, de l'Université de Tecnologie de Compiègne. Ce manuscrit est un rapport sur le travail de recherche que j'ai réalisé sous la direction de M. Romeo Ortega, dans le Département d'Informatique, au sein du laboratoire HEUDIASYC URA CNRS 817, dirigé par M. Rogelio Lozano. La période de cette recherche comprend à partir de Septembre 1993 à Octobre 1996. Ce travail, ainsi que celui que j'ai réalisé pendant mon stage de DEA, a été subventionné par CONACyT, Mexique ; et SFERE, France, dans le cadre du XVI programme d'échange CONACyT-CEFI.

Ce mémoire porte le label “Européen”, ce qui implique que ce document doit être écrit dans deux langues européennes différentes. En ce qui concerne ce document, il comporte une première partie en français, qui présente mes résultats théoriques principaux, sans les preuves. La seconde partie, étant écrite en anglais et avec plus de détail, inclut les preuves théoriques des propositions ainsi que des résultats de simulation. Chaque partie, peut être lue indépendamment de l'autre.

Ce travail a bénéficié de la collaboration de différents chercheurs avec qui, grâce à mon encadreur, j'ai eu l'occasion de travailler pendant la période de mes recherches. Ainsi, les résultats contenus dans les premiers deux chapitres sont le produit d'une collaboration avec M. Rafael Kelly, et M. Laurent Praly. Aussi, les résultats sur le contrôle de position, présentés dans le quatrième chapitre, ont été possibles grâce à la collaboration de M. Kelly et M. Victor Santibañez. Dans la dernière année de ma recherche, j'ai eu l'opportunité de visiter M. Hendrik Nijmeijer à l'Université de Twente, avec qui j'ai travaillé et obtenu le résultat sur le contrôle de suivi de trajectoire, contenu dans le chapitre **4**. Finalement, l'instruction de Mlle. Elena Panteley, a fait possible l'obtention des résultats du chapitre **5**. La plupart du travail en collaboration avec Mlle. Panteley, a été effectué pendant plusieurs visites à l'Institut de Problèmes de Génie Mécanique de l'Académie de Sciences de la Russie. Tout ce travail a été dirigé par mon encadreur, Prof. Romeo Ortega.

Preamble

The present work is presented in partial fulfillment of the requirements for the degree of Dr. ("Docteur") at the Université de Tecnologie de Compiègne. The manuscript is a documentation of the research work that I have carried out under the supervision of Prof. Romeo Ortega at the Department of Information Sciences, within the lab. HEUDIASYC URA CNRS 817, held by Prof. Rogelio Lozano. My research period extends from March 1993 to October 1996. This work, as well as the work I have done towards the Masters degree, was supported by a scholarship furnished by CONACyT, Mexico and SFERE, France within the XVI exchange CONACyT–CEFI program.

The present manuscript carries the label "European", what implies that this thesis must be written in two different European languages. In my case, this document contains a first part written in French, which presents the main contributions without proofs. Then, a second part written in English consists of a full and detailed description of the main contributions to the control of Lagrangian systems, and the proofs.

This work has benefited from the collaboration with different researchers that, thanks to my advisor, I had the opportunity to work with during the last years of my research period. Thus, the results contained in chapters **6** and **7** are product of the collaboration with Dr. Rafael Kelly, and Prof. Laurent Praly. Also, the results on set-point control presented in the chapter **9** wouldn't have been possible without the collaboration with Dr. Kelly and Mr. Victor Santibañez. In the last year of my research, I had the opportunity to visit Prof. Hendrik Nijmeijer at the University of Twente with whom I worked towards the results on bounded tracking control, contained also in chapter **9**. Finally, the instruction of Ms. Elena Panteley, made possible the results of chapter **10**, on constrained manipulators. Most of the collaboration with Ms. Panteley was carried out during repeated visits of myself to the Institute of Problems of Mechanical Engineering of the Academy of Sciences of Russia. All this work was led by my supervisor, Prof. Romeo Ortega.

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I have no words to express my gratitude to my advisor, Romeo Ortega. Through the last three years, Romeo has been a true master for me. He proposed me a promising research subject and motivated me to work on challenging problems. He came down to my level in order to *collaborate* with me, he was a real partner. He was always ready to discuss new ideas or to answer the most naive questions. All this happened in a very friendly atmosphere. For all these things I have no words to say “thank you” but I think that the only way to correspond to all what I have received from him, will be doing the same for my future students, and to honour Romeo with my future research. I will do my best in order to keep on the track where he has placed me.

My deep gratitude goes to Rogelio Lozano who has been a key part of my starting. I owe to him, first, my incorporation to the UTC. but more important, I have received form Rogelio many valuable advises in order to find a place in the complex “system” which is the Control Community. Rogelio has been like a “godfather” to me.

How to forget Rafael Kelly who was my robotics teacher in the last term of undergraduate school in ITESM, Mexico, back in late 1991. With his course on robot control, Rafael introduced me to the world of Lyapunov stability and robotics. Through my PhD research period I had the opportunity to collaborate with him and hence to continue learning from him.

Next, I want to express my gratitude to all the other members of my dissertation committee: Hendrik Nijmeijer, Laurent Praly, Claude Samson, and Patrizio Tomei. Their only presence in my defense has been a honour for me. Nevertheless, I had much feedback from Laurent Praly and Hendrik Nijmeijer before my dissertation. I am very grateful to them for their dedication and interest they showed in my work.

A mi familia...

Gracias a mis padres y a mi querida hermana, Maria, por su incondicional apoyo en todas mis necedades y obseciones, a lo largo de mi corta vida. Confieso que al momento de mi partida a Francia yo *tampoco* comprendia el significado de las palabras “investigacion” ni “doctorado” (ahora... mas o menos!). Aun en la duda de “cuando se me ocurrira comenzar a trabajar” recibi todo el apoyo moral (y a veces economico) de mi familia. Gracias a mis padres y mi Mari, por sus valiosas ensenanzas.

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Merci Tarek pour ta naïveté et ta noblesse... Sinon, avec qui d’autre, mes histoires montées auraient pu marcher?. Nous nous en sommes bien marrés (nous, les autres!).

Merci Manu pour ta générosité et ouverture d’esprit. Tes gestea généreux de me conduire et/ou chercher à Charles de Gaule, quelque part sur l’A1, à la Gare de Compiègne... etc. Ton ouverture d’esprit pour ne pas me tuer à coups de pieds pour te téléphoner à 6:00am pour te demander quel jour on est!.

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Et puis, merci énormément à toute l’équipe technique du GI. Longue est la liste de gens qui m’ont supporté pendant mes 4 ans. Je voudrais donc remercier à Nathalie, Dominique, Jacqueline, Corinne, Brigitte, David, et tous les autres, leur preuve de patience envers un mexicain désorienté et pas toujours aimable.

Finalement, je devrais ajouter un mot sur les “reproches” (bien gentilles) que m’ont été faites sur mes éclats de rire!. Je dois dire en ma défense que c’est bien de votre faute puisque c’est bien vous qui avez cultivé ma bonne humeur pendant environs 3 ans!

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Introduction

Le concept de *système* peut être expliqué comme une partie de la réalité, laquelle en est séparée et a des connexions avec son environnement. Ces connexions reflètent le *comportement* externe du système comme il est perçu par un observateur. Les systèmes sont appelés "dynamiques", s'il sont sujets à des changements temporels et réciproquement, s'ils exercent une influence quelconque sur leur environnement.

Le concept de *systèmes de contrôle*, est lié au contrôle de la relation existante entre le système et son environnement. Un but typique des chercheurs dans la théorie de contrôle, est la conception de lois de commande qui soient applicables a une *classe de systèmes*, laquelle est définie sur la base des caractéristiques invariantes des systèmes. Cela a une motivation pratique : celle d'associer une *interprétation physique* à la classe de systèmes en question, de manière à concevoir des contrôleurs qui exploitent leurs propriétés physiques.

Dans cette thèse, on considère une grande classe de systèmes physiques : les systèmes Euler-Lagrange (EL). Le principe Lagrangien constitue une manière compacte et invariante d'impliquer les équations mécaniques du mouvement. L'intérêt de la formulation Lagrangienne est que cette possibilité n'est pas réservée exclusivement à la mécanique ; dans presque tous les domaines de la physique variationelle, des principes variationels peuvent être utilisés pour exprimer les équations du mouvement, soient les équations de Newton, ou celles de Maxwell ou de Schrödinger. On verra que, quand un principe variationel est utilisé comme la base de formulation, tous les domaines exhiberont une analogie structurelle.

Motivation et problèmes ouverts

Notre développement est inspiré des systèmes mécaniques, qui appartiennent à la classe EL. Notre première motivation est le théorème de Josephe Louis La Grange, qui établit :

"Les équilibres stables d'un système mécanique libre, sont déterminés par les minima de sa fonction d'énergie potentielle.".

Ce "simple" théorème qui a été démontré par la suite par Dirichlet, a motivé l'étude du problème de stabilisation par retour d'état, des systèmes Lagrangiens. Un groupe important parmi la classe EL, et qui a motivé notre recherche est celui des robots manipulateurs. La commande par retour d'état des robots manipulateurs a été étudiée pendant des années ; parmi de plusieurs livres écris sur la dynamique et la commande des robots manipulateurs, on peut citer par exemple (Spong and Vidyasagar, 1989), et (Samson et al., 1991) où les auteurs ont traité l'approche dite "tâche fonctionnelle". Parmi des textes plus récents on compte (Sciavicco and Siciliano, 1996), et (Bastin et al., 1996) qui donne une aperçue sur les résultats les plus récents et plus significatifs dans la théorie de commande des robots manipulateurs.

Ainsi, la commande des robots est une de nos motivations principales pour considérer le contrôle par retour de sortie des systèmes Lagrangiens :

- La commande par retour d'état a l'inconvénient, en pratique, que les dispositifs utilisés pour mesurer la vitesse, sont contaminés avec du bruit. D'ailleurs, la différentiation numérique ne parait pas adéquate pour estimer de vitesses très rapides ou très lentes (Bélanger, 1992). D'où l'intérêt des chercheurs, à concevoir des lois de commande par retour de position, pour les robots manipulateurs.

- Comme il sera éclairci par la suite, l'interconnexion de deux systèmes Lagrangiens est aussi un système EL. Cette propriété fondamentale, ainsi que le principe de superposition, nous permet de considérer la stabilisation des *interconnexions* de systèmes EL, comme il a été considéré pour les "robots plus moteurs", dans (Canudas de Wit et al., 1993; Panteley and Ortega, 1997), ou les convertisseurs CD-à-CD (Sira-Ramirez et al., 1996).

- Un des problèmes les plus attrayants dans la théorie de commande des robots manipulateurs est celui de contrôle de trajectoire par retour de position. Il faut remarquer que ce problème a été ouvert pendant des années. Un deuxième problème qui a motivé notre recherche est celui du contrôle de position sans mesure de vitesse et sans connaissance de la charge.

Contributions de cette thèse

Dans ce travail, on adresse de divers problèmes de contrôle par retour de sortie (de position), des systèmes EL : contrôle de position, contrôle de trajectoire, contrôle à entrées bornées et contrôle de force des robots manipulateurs. Nos contributions dans chaque domaine sont brièvement décrites ci-dessous. On rappelle au lecteur cependant, que cette thèse est écrite en anglais et en français ; la première partie, en français, comprend uniquement les résultats, les plus significatifs, sans les preuves. Pour une lecture plus approfondie, le lecteur est invité à lire la deuxième partie de ce document.

Il faut aussi mentionner que, dans cette thèse, on a un intérêt particulier sur les robots manipulateurs ainsi, quelques résultats de simulation seront présentés pour illustrer la performance de nos lois de commande, sur des robots manipulateurs. Aussi, le dernier chapitre est dédié à la commande de manipulateurs sous contraintes holonomes.

Chapitre 1. Ce chapitre contient un bref descriptif des systèmes Lagrangiens. On soulignera quelques propriétés fondamentales qui sont à la base des résultats présentés dans les chapitres suivants.

Chapitre 2. On traite le contrôle de position par retour de sortie. Notre première contribution dans ce domaine est la définition de la classe de contrôleurs Lagrangiens. Autrement dit, on propose une classe de contrôleurs qui possèdent des caractéristiques semblables à celles du système EL que l'on veut contrôler. La deuxième contribution de ce chapitre est un contrôleur de position où l'on suppose que la connaissance de la gravité est incertaine. On exploitera les propriétés de passivité du système Lagrangien pour concevoir une loi de commande a double intégrateur, appelée PI^{2}D.

Chapitre 3. Ce troisième chapitre concerne le problème ouvert de contrôle de trajectoire sans mesure de vitesse. Notre contribution est, à notre connaissance, le premier contrôleur assurant la stabilité *globale* asymptotique pour systèmes Lagrangiens d'un dégrée de liberté.

Chapitre 4. Les contrôleurs basés sur des observateurs parus dans la littérature, qui assurent la stabilité semiglobale asymptotique en boucle fermée, ont besoin en générale, de hauts gains. Cependant, il est souhaitable dans la pratique, de concevoir des lois de commande qui produisent des entrées de contrôle bornées. La première contribution présentée dans ce chapitre est une classe de contrôleurs Lagrangiens qui donnent des entrées de contrôle bornées. Deuxièmement, on présente le premier contrôleur de suivi de trajectoire par retour de position, à des entrées de contrôle bornées.

Chapitre 5. Ce chapitre est consacré à l'application des résultats présentés dans les chapitres **1**–**4**, aux robots manipulateurs en interaction avec leur environnement. En particulier, on propose un contrôleur de type PI^{2}D pour le contrôle de force/position d'un robot en contact avec un milieu élastique. Aussi, on présente quelques résultats de contrôle de position et trajectoire des manipulateurs soumis à des contraintes holonomes. Il faut remarquer que très peu de résultats ont été publiés sur le contrôle de manipulateurs en interaction avec leur milieu, sans mesure de vitesse.

Introduction

The concept of *system* may be explained as a part of reality which is separated from and has some connections with its environment. These connections reflect the external *behaviour* of a system perceived by some observer. Systems are referred to as "dynamical" if they are subject to temporal changes caused by environmental changes and conversely, if they exert some influence on their environment.

The concept of *systems control* has to do with the control of the relation between a system and its environment. A typical goal of control theory researchers is the design of control laws applicable to a *class of systems* which is defined upon systems' invariants. The practical motivation to this, is to attach a *physical interpretation* to the class of systems in question, so as to design controllers which exploit their physical properties.

In this thesis we consider a wide class of physical systems, the Euler-Lagrange (EL) systems. The Lagrangian principle forms a compact invariant way of implying the mechanical equations of motion. The interest of the Lagrangian formulation is that this possibility is not reserved for mechanics only; in almost every field of variational physics, variational principles can be used to express the equations of motion, whether they be Newton's equations, Maxwell's equations or Schrödinger equation. We will see that, whenever a variational principle is used as the basis of the formulation, all such fields will exhibit at least to some degree, a structural analogy.

Motivation and open problems

Our development is inspired upon mechanical systems which belong to the EL-class and our first motivation is the famous theorem of Joseph Louis La Grange, that states:

"The stable equilibria of a free mechanical system are determined by the minima of its potential energy function".

This "simple" theorem which was later proved by Dirichlet, has motivated researchers to study the state-feedback stabilization problem of EL systems. An important group of mechanical systems which lies in the EL-class and has motivated our research, are the robot manipulators. The state feedback control of robot manipulators has been studied for many years now, among the many books that have been written about robot dynamics and control we can cite for instance (Spong and Vidyasagar, 1989), and (Samson et al., 1991) where the task functional approach is broached. More recent texts are (Sciavicco and Siciliano, 1996), and (Bastin et al., 1996) which give an overview of the most recent and significant results in robot control theory.

Thus, robot control constitutes one of our main motivations to consider the output feedback control of EL systems:

- State-feedback control is stymied in practice, by the well known fact that the devices used to measure velocities are contaminated with noise. Moreover, numerical differentiation appears not to be adequate for very fast or slow velocities (Bélanger, 1992). This fact has motivated robotics researchers to look for control laws not needing velocity measurements.
- As it will become clear later, the interconnection of two EL systems yield an EL system. This fundamental property, together with the superposition principle allows us to consider the stabilization of

interconnection of different EL systems as for instance, control of robots + motors (Canudas de Wit et al., 1993), DC-to-DC converters (Sira-Ramirez et al., 1996).

- Output feedback control of robot manipulators: In particular we are interested in the long standing open problem of *global* output (position) feedback tracking control of robot manipulators. Another interesting problem that motivated this work was the *position* feedback set-point control of robot manipulators with unknown gravity knowledge.

Contributions and outline of this thesis

We address in this thesis several output (position) feedback control problems of EL systems such as set-point control, trajectory tracking control, control under input constraints (i.e. via bounded inputs) and force control of robot manipulators. Our contributions in each direction is briefly described below, classified by chapters.

It is worth mentioning that throughout the whole manuscript we put particular emphasis on robotic manipulators, thus the simulation results and applications of our results are illustrated through these systems. Also, we dedicate the last chapter to the motion control of robot manipulators under holonomic constraints.

Chapter 61. We present an introduction to Lagrangian mechanics, borrowed from (Goldstein, 1974). We then define the subclass of Euler-Lagrange systems in which we focus our interest throughout this thesis, underlining some fundamental properties to our results.

Chapter 7. We deal with the output feedback regulation problem. Our first contribution is a new controller design methodology which exploits the physical properties of Euler Lagrange (EL) systems such as *passivity*. We define a class of controllers whose dynamics is derived from the Euler-Lagrange equations and that we have called EL controllers. The efficiency of these controllers hinges upon the ability of shaping the plants potential energy in order to ensure that the unique global minimum is the desired equilibrium point, and injecting a suitable damping to insure global asymptotic stability. Our second contribution is the extension of this result to the case of unknown potential energy. In particular, we propose the $\mathrm{PI^2D}$ controller which is based on a traditional PID scheme with a second integrator. Then, in order to avoid the use of velocity measurements, it uses a linear filter. The class of EL controllers was introduced in (Ortega et al., 1995c), a particular application of this approach to the flexible joint robots was originally presented in (Ortega et al., 1995a). This methodology groups together several previous apparently unrelated results. The $\mathrm{PI^2D}$ controller was proposed in (Ortega et al., 1995b) and as far as we know, together with (Colbaugh and Glass, 1995), it is the first solution to the position feedback regulation problem with uncertain gravity knowledge.

Chapter 8. Concerning the tracking control problem, our main contribution has been the *first* position feedback controller for one degree of freedom EL systems ensuring *global* asymptotic stability. This result, which was originally presented in (Loria, 1996) is based on a previous semiglobally stable controller which uses a linear filter in order to avoid velocity measurements, previously reported in (Loria and Ortega, 1995).

Chapter 9. Observer based controllers reported in the literature ensuring semiglobal asymptotic stability often require high controller and observer gains to increase the domain of attraction. Nonetheless, it is of practical interest to design controllers yielding *bounded* control inputs. In this domain our contribution is twofold: to solve the set-point control problem we define a subclass of EL controllers which insure global asymptotic stability and use saturation functions to yield bounded inputs; second, we propose as far as we know, the *first* semiglobal position feedback tracking controller with bounded inputs, hence proving that controller *high gains are not necessary* to increase the domain of attraction. The saturated EL controllers approach was originally proposed in (Loria et al., 1996) while the saturated tracking controller was presented in (Loria and Nijmeijer, 1995).

Chapter 10. In this chapter we focus our study on one particular but important group of EL systems: robot manipulators in interaction with their environment. The problem of force/position and force/tracking control without velocity measurements has been poorly studied in the literature. In this work, we address both problems considering two different kinds of interaction environments: infinitely stiff and elastic. Our contributions in this domain extend the results of previous chapters to the constrained motion case. Firstly, we consider the force/regulation problem of a robot manipulator interacting with an elastic environment and extend the results of the PI^{2}D in this direction. This result was published in (Loria and Ortega, 1996). Secondly, we have used the reduced order model of constrained manipulators of (Panteley and Stotsky, 1993a) in order to extend the results on set point control under input constraints (Panteley et al., 1996) and our previous results on semiglobal tracking control (Loria and Panteley, 1996).

Notation

Common mathematical symbols.

$\mathbb{R}$	Field of real numbers.
$\mathbb{R}^n$	Linear space of real vectors of dimension n.
$\mathbb{R}^{n\times m}$	Ring of matrices with n rows and m columns and elements in $\mathbb{R}$.
$\mathbb{R}_{\geq 0}$	Field of nonnegative real numbers.
I_n	The identity matrix of dimension n.
I_t	The set $[t_0,\infty)$, $t_0 \in \mathbb{R}_{\geq 0}$.
t	Time, $t \in \mathbb{R}_{\geq 0}$ or $t \in I_t$.
$\bar{\Gamma}$	Closure of the set Γ.
$\mathring{\Gamma}$	Stands for the interior of Γ.
$\partial\Gamma \triangleq \bar{\Gamma}\setminus\Gamma$	The boundary of Γ.
$\|x\|$	The Euclidean norm of $x \in \mathbb{R}^{n\times m}$.
$\mathcal{L}_2^n$	Space of n-dimensional square integrable functions.
$\mathcal{L}_{2e}^n$	Extended space of n-dimensional square integrable functions.
$\langle\cdot\|\cdot\rangle$	Inner product in $\mathcal{L}_2^n$.
$\langle\cdot\|\cdot\rangle_T$	Inner product in $\mathcal{L}_{2e}^n$.
$\|\cdot\|_{2T}$	The $\mathcal{L}_2$ norm.
$\rightarrow$	Mapping from a domain into a range.
$\mapsto$	Mapping of two elements into their image.
$\triangleq$	"defined as".
$\frac{dz}{d\xi}$	Derivative of $z = f(\xi)$.
$\frac{d}{dt}(\cdot) = \dot{(\cdot)}$	Total time derivative.
$p \triangleq \frac{d}{dt}$	Derivative operator.
$\frac{\partial}{\partial\xi}$	Partial derivative with respect to ξ.
$\underline{n}$	The set of integers $[1,\ldots,n]$.

Matrices.

K_p	Frequently used to denote a "proportional" error gain in a controller.
K_d	Used to denote a "derivative" gain.
K_i	Stands for an "integral gain".
k_m	Smallest eigenvalue of matrix K. Correspondingly k_{p_m} for K_p.
k_M	Largest eigenvalue.
$(\cdot)^\top$	Transpose operator.

Euler-Lagrange systems.

q	Vector of generalized positions.
$\dot{q}$	Vector of generalized velocities.
q_d	Vector of desired references.
$\tilde{q}$	Position error, that is, $\tilde{q} \triangleq q - q_d$.
ϑ	Output of a dynamic extension, e.g. output of the "dirty derivatives" filter.
$T(q, \dot{q})$	Kinetic energy.
$V(q)$	Potential energy.
$\mathcal{F}(\dot{q})$	Rayleigh's dissipation function.
$D(q)$	Inertia matrix.
$C(q, \dot{q})$	Coriolis and centrifugal forces matrix.
$g(q)$	Potential forces vector.
Q	External generalized forces.
u, u_p	Control inputs.

In particular, in chapters **7** and **9** we make the following distinction in the notation, between Euler-Lagrange *plants* and *controllers*.

q_p	Generalized positions vector of an EL *plant*.
$T_p(q_p, \dot{q}_p)$	Plant's kinetic energy.
$V_p(q_p)$	Plant's potential energy.
$\mathcal{F}_p(\dot{q}_p)$	Plant's Rayleigh dissipation function.
$\mathcal{M}_p$	Matrix which maps the inputs to an EL plant's coordinates.
q_c	Generalized positions of an EL *controller*.
$T_c(q_c, \dot{q}_c)$	Controller's kinetic energy.
$V_c(q_c, q_p)$	Controller's potential energy.
$\mathcal{F}_c(\dot{q}_c)$	Controller's Rayleigh dissipation function.

Numbering of sections, equations, theorems, etc.

In this document, the sections are numbered in each chapter as 1, 2, etc. The subsection numbers are relative to the section which they belong to, that is, 1.1, 1.2, etc. The sub-subsections are numbered according to the number of the subsections but they are not listed in the table of contents. When citing a section or subsection belonging to a different chapter, we use the **bold** font for the chapter number. Thus, the reader should not be confused for instance, between section 2 of chapter 1, cited **1**.2 in chapter 3, from subsection 2 of section 1, cited in chapter 3, as 1.2.

The equation numbers are relative to the corresponding chapter, for instance, equation (3.56), is the 56th equation of chapter 3. Theorems, propositions, lemmas, definitions, and remarks have the same counter, which is relative to the chapter number. For instance, remark 3.2 follows proposition 3.1 in the third chapter.

Figure counters are also relative to the chapter number only.

Partie I

Contrôle de Systèmes Euler-Lagrange par retour de sortie

Chapitre 1

Définition et propriétés des systèmes Euler-Lagrange

Dans ce premier chapitre on définit une classe de systèmes Lagrangiens, en soulignant des propriétés fondamentales qui seront utilisées dans les chapitres qui suivent.

1 La formulation Lagrangienne

Les systèmes Euler-Lagrange sont caractérisés par les paramètres EL

$$\{T(q,\dot{q}), V(q), \mathcal{F}(\dot{q}), M\} \tag{1.1}$$

où $q \in \mathbb{R}^n$ sont les coordonnées généralisées, et n correspond au nombre de dégrées de liberté du système. Dans cette thèse on s'intéresse aux systèmes dont la fonction d'énergie cinétique

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} \tag{1.2}$$

où la matrice d'inertie $D(q)$ satisfait $D(q) = D^\top(q) > 0$. On suppose que la fonction d'énergie potentielle $V(q)$ est bornée par en dessous, c'est à dire qu'il existe un $c \in \mathbb{R}$ tel que $V(q) > c$ pour tout $q \in \mathbb{R}^n$. $\mathcal{F}(\dot{q})$ est la fonction de dissipation de Rayleigh (voir par exemple (Nijmeijer and van der Schaft, 1990)). Le dernier paramètre, M, est une matrice de rang plein qui applique les entrées aux coordonnées généralisées.

Par convenance, on divise le vecteur $q = \operatorname{col}[q_p\ q_c]$ où l'on appelle q_p, les coordonnées non amorties et q_c, les coordonnées amorties. Selon cette notation, on peut distinguer entre deux classes de systèmes :

1.1 Définition. *Un système EL avec paramètres (1.1) est dit* totalement amorti *si* $(\alpha > 0)$

$$\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \geq \alpha \|\dot{q}\|^2 . \tag{1.3}$$

Un système EL est sous-amorti *si*

$$\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \geq \alpha \|\dot{q}_c\|^2 . \tag{1.4}$$

On fera aussi une distinction entre deux autres classes de systèmes EL en fonction de la structure de la matrice M :

1.2 Définition. *Un système EL est dit* totalement-agis *si $M = I_n$. Aussi, on dira qu'un système est* sous-agi *si*

$$M = [0 \mid I_m]^\top, \qquad m < n. \tag{1.5}$$

Dans ce cas-là, q peut être divisé en coordonnées agies et non agies[1] *respectivement*

$$q_1 \triangleq M^{\perp} q = \begin{bmatrix} I_{n-m} & | & 0 \end{bmatrix} q, \quad q_2 \triangleq Mq \tag{1.6}$$

1.3 Remarque. On suppose dans cette thèse que les variables agies sont mesurables et les coordonnées non agies sont les coordonnées contrôlées.

En utilisant les paramètres Lagrangiens, les systèmes EL sont définis par les équations d'Euler-Lagrange :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = Q \tag{1.7}$$

où $\mathcal{L} \triangleq T(q,\dot{q}) - V(q)$ est appelé "Lagrangien", et $Q \in \mathbb{R}^n$, est le vecteur de forces généralisées. Dans cette thèse on suppose que Q est composé uniquement de forces potentielles $Mu \in \mathbb{R}^n$, et de forces dissipatives, $-\frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}$; invariantes dans le temps. C'est à dire,

$$Q = Mu - \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}. \tag{1.8}$$

Les équations (1.7) sont équivalentes à (Spong and Vidyasagar, 1989)

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} = Mu \tag{1.9}$$

où la matrice $C(q,\dot{q})$ est la matrice de forces centrifuges et de Coriolis. Son kj-ème élément est donné par

$$C_{kj} = \sum_{i}^{n} c_{ijk}(q)\dot{q}_i. \tag{1.10}$$

où

$$c_{ijk} \triangleq \frac{1}{2}\left(\frac{\partial d_{ik}(q)}{\partial q_j} + \frac{\partial d_{jk}(q)}{\partial q_i} - \frac{\partial d_{ij}(q)}{\partial q_k}\right)$$

sont appelés symboles de Christoffel.

Dans ce travail on concentre notre attention sur les systèmes qui possèdent les propriétés suivantes :

P 1.1 La matrice $D(q)$ est positive définie et la matrice $N = \dot{D}(q) - 2C(q,\dot{q})$ est anti-symmétrique. D'ailleurs, il existe des constantes positives d_m et d_M tel que

$$d_m I < D(q) < d_M I \tag{1.11}$$

P 1.2 La matrice $C(x,y)$ est bornée en x. En outre, il est évidente de (1.10), que $C(x,y)$ est linéaire en y, donc pour tout $z \in \mathbb{R}^n$:

$$C(x,y)z = C(x,z)y \tag{1.12}$$

$$C(x,y) \leq k_c \|y\|, \quad k_c > 0. \tag{1.13}$$

P 1.3 Il existe des constantes positives k_g et k_v telles que

$$k_g \geq \sup_{q \in \mathbb{R}^n} \left\| \frac{\partial^2 V(q)}{\partial q^2} \right\|, \quad \forall\, q \in \mathbb{R}^n \tag{1.14}$$

$$k_v \geq \sup_{q \in \mathbb{R}^n} \left\| \frac{\partial V(q)}{\partial q} \right\|, \quad \forall\, q \in \mathbb{R}^n \tag{1.15}$$

En effet, la propriété d'anti-symmétrie de la matrice $\dot{D}(q) - 2C(q,\dot{q})$, ainsi que (1.12) sont des conséquence directes de la définition de $C(q,\dot{q})$. Aussi, l'inégalité (1.13) suit en utilisant (1.11) et la définition des symboles de Christoffel. Voir par exemple (Spong and Vidyasagar, 1989; Ortega and Spong, 1989; Stepanenko and Yuan, 1992).

1.4 Remarque. Il est important de remarquer que les robots manipulateurs possèdent la propriété **P1.3**.

[1] C'est à dire des coordonnées généralisées dont la ligne correspondante dans la matrice M, contient des '1' et des '0' respectivement.

2 Propriétés des systèmes Euler-Lagrange

Dans cette section on présente quelques propriétés fondamentales des systèmes EL, lesquelles on a classifiées en propriétés “de stabilité entrée-sortie” et propriétés “de stabilité de Lyapunov”.

2.1 Propriétés de stabilité entrée-sortie

Il est connu (voir par exemple (van der Schaft, 1996)) que les systèmes EL ont quelques propriétés de dissipation d'énergie. En particulier, on a le suivant :

1.5 Proposition. *(Passivité) Un système EL définit un* opérateur passif *des entrées u aux vitesses généralisées (agies) $M^\top \dot{q}$. C'est à dire, il existe un $\beta \in I\!R$ tel que*

$$\langle u \mid M^\top \dot{q} \rangle \ \geq \beta \tag{1.16}$$

pour tout $u \in \mathcal{L}_{2e}^m$. D'ailleurs, cet opérateur est strictement passif à la sortie *si la fonction de dissipation de Rayleigh définit un opérateur strictement passif à la sortie. Dans ce cas,*

$$\langle u \mid M^\top \dot{q} \rangle \ \geq \alpha \|M^\top \dot{q}\|_2^2 + \beta \tag{1.17}$$

pour un $\alpha > 0$, $\beta \in I\!R$ et pour tout $u \in \mathcal{L}_{2e}^m$. □

1.6 Remarque. La proposition 1.5 ci-dessus, est importante étant donné qu'elle affirme que l'opérateur $u \mapsto M^\top \dot{q}$ peut être rendu strictement passif à la sortie, même si l'énergie n'est pas dissipée “dans toutes les directions”. En effet, il suffit d'assurer que $\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \geq \alpha \|M^\top \dot{q}\|^2$. Cette caractéristique sera exploitée dans le chapitre 2, pour prouver la stabilité globale asymptotique par injection *partielle* d'amortissement.

2.2 Propriétés de stabilité de Lyapunov

Dans cette section, on mentionne d'autres propriétés des systèmes Euler-Lagrange, qui sont liées directement à la stabilité dans le sens de Lyapunov. Par souci de clarté, on distinguera deux classes de systèmes : Totalement amortis et sous-amortis.

2.2.1 Systèmes totalement amortis

La proposition ci-dessous établit des conditions suffisantes pour assurer la *stabilité interne* des systèmes complètement amortis.

1.7 Proposition. (GAS avec amortissement complet) *Les équilibres d'un système EL complètement amorti, en état libre, c'est à dire, avec $u = 0$, sont $(q, \dot{q}) = (\bar{q}, 0)$ où $\bar{q}$ est une solution de*

$$\frac{\partial V(q)}{\partial q} = 0. \tag{1.18}$$

L'équilibre est unique et stable *s'il est un point minimum global et unique de la fonction d'énergie potentielle $V(q)$, et V est radialement non bornée. Par exemple, si V satisfait les conditions du lemme B.7. En outre, cet équilibre est global asymptotiquement stable si la fonction de dissipation est* strictement passive à l'entrée, *par exemple, une fonction de dissipation de Rayleigh.* □

2.2.2 Systèmes sous-amortis

Dans la proposition ci-dessous, on montre que les systèmes avec un amortissement partielle sont aussi asymptotiquement stables si : la matrice d'inertie présente une certaine structure diagonale par blocs et la dissipation est adéquatement propagée à travers toutes les coordonnées généralisées.

Pour distinguer entre les coordonnées *amorties* et les *non amortis*, on introduit la partition de q suivante :

$$q_c \triangleq [0 \mid I_{n_c}]q, \quad q_p \triangleq [I_{n_p} \mid 0]q, \quad n = n_p + n_c,$$

1.8 Proposition. (GAS avec amortissement partiel) *L'équilibre d'un système EL sous-amorti, libre ($u = 0$), est GAS si la fonction d'énergie potentielle est radialement non bornée avec un minimum global et unique et si :*

(i) $D(q) := \begin{bmatrix} D_p(q_p) & 0 \\ 0 & D_c(q_c) \end{bmatrix}$, *où* $D_c(q_c) \in I\!R^{n_c \times n_c}$,

(ii) $\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \geq \alpha \|\dot{q}_c\|^2$ *pour un* $\alpha > 0$,

(iii) Pour chaque q_c, *la fonction* $\frac{\partial V(q)}{\partial q_c} = 0$ *a uniquement des zéros isolés dans* q_p. □

1.9 Remarque. Il est important de remarquer que la condition sur la structure diagonale par blocs, de la matrice d'inertie, est intéressante pour découpler les parties amorties et non amorties du système, par rapport a l'énergie cinétique.

Grosso-modo, les systèmes sous amortis peuvent être regardés comme l'interconnexion de deux sous-systèmes : l'un totalement amorti, l'autre sans amortissement ; les coordonnées généralisées étant respectivement, q_c et q_p. Les deux systèmes sont couplés par les forces qui se dérivent de l'énergie potentielle, de façon telle que la dissipation d'énergie se propage des coordonnées amorties aux coordonnées non amorties. Ainsi, une propriété fondamentale des systèmes Lagrangiens est :

P1.4 L'interconnexion de deux systèmes EL $\Sigma_p : \{T_p(q_p, \dot{q}_p),\ V_p(q_p),\ \mathcal{F}_p(\dot{q}_p),\ M_p\}$ et $\Sigma_c\,\{T_c(q_c, \dot{q}_c),\ V_c(q_c, q_p),\ \mathcal{F}_c(\dot{q}_c)\ \}$ est un système EL $\Sigma = \Sigma_p + \Sigma_c$, c'est à dire

$$\Sigma : \{T(q, \dot{q}),\ V(q),\ \mathcal{F}(\dot{q})\}$$

où

$$T(q, \dot{q}) = T_c(q_c, \dot{q}_c) + T_p(q_p, \dot{q}_p) \qquad V(q) = V_c(q_c, q_p) + V_p(q_p), \tag{1.19}$$

$$\mathcal{F} = \mathcal{F}_c(\dot{q}_c) + \mathcal{F}_p(\dot{q}_p). \tag{1.20}$$

L'importance de la propriété **P1.4** est que les propriétés de deux systèmes Lagrangiens, décrites dans ce chapitre, sont conservées lorsque ces systèmes sont interconnectés. Ainsi, on verra que la proposition 1.8 et l'invariance des propriétés des systèmes Lagrangiens lors d'une interconnexion, sont fondamentales pour la stabilisation par retour de sortie, développée dans les chapitres **2** et **4**.

3 Conclusions

Dans ce chapitre on a présenté quelques propriétés fondamentales que l'on utilisera par la suite, dans nos contributions. Ces propriétés peuvent être résumées en :

1. Les systèmes EL définissent des opérateurs passifs.
2. Les systèmes EL sont caractérisés par ses paramètres EL, à savoir : Énergie cinétique, énergie potentielle, fonction de dissipation, matrice d'entrées généralisées.
3. Les équilibres stables d'un système EL sont déterminés par les minima de sa fonction d'énergie potentielle.
4. Les systèmes EL sont asymptotiquement stables s'ils ont un amortissement adéquat.
5. L'interconnexion de deux systèmes EL constitue un système Lagrangien. Les paramètres des systèmes EL sont simplement l'addition des paramètres de chaque sous-systèmes.

Chapitre 2

Contrôle de position

Le problème de contrôle de position est formulé comme : Trouver un contrôleur pour le procédé EL Σ_p : $\{T_p(q_p, \dot{q}_p),\ V_p(q_p),\ \mathcal{F}_p(\dot{q}_p),\ M_p\}$, tel que l'équilibre souhaité $q_p = q_{pd}$, soit globale asymptotiquement stable.

Pour résoudre ce problème rappelons nous des propriétés fondamentales des systèmes EL, que ln vient de souligner dans le chapitre précédent. On a vu que les équilibres d'un système Lagrangien Σ_p sont déterminés par les minima de $V_p(q_p)$, d'ailleurs l'équilibre est unique et stable s'il est un minimum global et unique de la fonction d'énergie potentielle. On a vu aussi qu'une condition nécessaire pour assurer la stabilité asymptotique est la présence d'un certain amortissement.

Une autre propriété clé est que l'interconnexion de deux systèmes Lagrangiens résulte en un système EL.

En nous appuyant sur ces propriétés, dans ce chapitre on présente deux contributions au problème de contrôle de position par retour de sortie : premièrement, on définit une classe de contrôleurs que l'on appelle *Contrôleurs EL*. Les contrôleurs EL *façonnent* l'énergie potentielle du procédé, de manière à placer un minimum global et unique à l'équilibre souhaité, $q = q_d$. En outre, les contrôleurs EL injectent l'amortissement adéquat pour assurer la stabilité asymptotique.

Deuxièmement, on étend ces résultats pour les systèmes totalement amortis, au cas où l'énergie potentielle du procédé n'est pas connu avec précision. En particulier, on propose le contrôleur PI^2D dont la caractéristique principale est l'utilisation d'un double intégrateur pour compenser l'effet de forces potentielles inconnues, tout en exploitant la passivité du système.

1 Introduction

1.1 Motivation : Façonnement d'énergie plus injection d'amortissement

Les propriétés de stabilité entrée-sortie et de stabilité interne des systèmes EL, a motivé le développement de la méthodologie appelée *façonnement d'énergie plus injection d'amortissement*, et qui est basée sur la passivité. Cette technique a été proposée par (Takegaki and Arimoto, 1981). Comme son nom l'indique, cette méthodologie, cherche à façonner l'énergie potentielle du procédé, via un contrôleur passif, de manière à ce que la "nouvelle" fonction d'énergie ait un minimum global et unique. Par souci de clarté, on illustre cette méthodologie avec un simple pendule, néanmoins, cette technique est applicable à des systèmes de n dégrées de liberté.

L'énergie totale (cinétique + potentielle) du pendule est

$$H = \underbrace{\frac{1}{2}ml^2\dot{q}^2}_{T(\dot{q})} + \underbrace{mgl(1-\cos(q))}_{V(q)} \tag{2.1}$$

où $q \in \mathbb{R}$, cependant, puisque la position du pendule est répétée tous les 2π radians, par souci de simplicité on limitera notre analyse en supposant que $\{q : 0 \leq q < 2\pi\}$. En utilisant les équations Lagrangiennes, on peut dériver facilement l'équation dynamique du pendule :

$$ml^2\ddot{q} + g(q) = u \tag{2.2}$$

où u est la force externe généralisée, $g(q)$ est la force dérivée de l'énergie potentielle, c'est à dire :

$$g(q) \triangleq \frac{\partial V(q)}{\partial q} = mgl\sin(q) \tag{2.3}$$

et g est l'accélération de la gravité. Donc, avec un abus de notation, on appellera $g(q)$, force gravitationnelle.

Les équilibres du système libre (2.2) (c.à.d. avec $u \equiv 0$), sont les points critiques de la fonction d'énergie potentielle, c'est à dire, les solutions de

$$\frac{\partial V(q)}{\partial q} = mgl\sin(q) = 0,$$

ou bien $q = 0$ et $q = \pi$. En prenant la seconde dérivée partielle de $V(q)$ par rapport à q, on trouve :

$$\frac{\partial^2 V(q)}{\partial q^2} = mgl\cos(q)$$

qui est positif pour $q = 0$ et négative pour $q = \pi$. On conclut donc que l'origine correspond à un minimum de la fonction d'énergie potentielle, en contraste, $q = \pi$ en est un point maximal.

Il est bien connu (voir par exemple (Arnold, 1989; van der Schaft, 1996)) que le minimum de l'énergie potentielle correspond à un équilibre stable, et le maximum, à un équilibre instable de (2.1).

Notre problème de contrôle est celui de stabiliser le pendule à un autre équilibre que l'origine, par exemple, $q = q_d$. Sachant que le minimum de la fonction d'énergie potentielle correspond à un minimum de l'énergie potentielle, une solution assez attractive, est celle de changer l'énergie potentielle $V(q)$ de manière à ce que le point d'équilibre souhaité, soit le nouveau minimum global et unique. Suivant cette idée, proposons la nouvelle énergie potentielle *souhaité* (Takegaki and Arimoto, 1981) :

$$V_d(q) = V(q) - V(q_d) + \frac{1}{2}k_p\tilde{q}^2 - g(q_d)\tilde{q}. \tag{2.4}$$

En utilisant le lemme B.7, on peut prouver que (2.4) a un minimum global et unique en $q = q_d$ pourvu que $k_p > k_g$, où k_g est définie par (1.14) (voir p.e. (Tomei, 1991b)). Ainsi, la première partie de la loi de commande est

$$u = \frac{\partial}{\partial q}\left(V(q) - V_d(q)\right) = -k_p\tilde{q} + g(q_d), \tag{2.5}$$

à laquelle on ajoute l'amortissement nécessaire, $-k_d\dot{q}$, pour assurer la stabilité asymptotique. Ainsi, on retrouve la loi de commande proposée par [1] (Takegaki and Arimoto, 1981) :

$$u = -k_p\tilde{q} - k_d\dot{q} + g(q_d). \tag{2.6}$$

Le gain proportionnel k_p peut être regardé comme la constante d'élasticité d'un ressort linéaire, lequel soutient le pendule à la position souhaité, q_d. Ainsi, ensemble avec le terme de compensation $g(q_d)$, le gain proportionnel change l'équilibre stable de l'origine à la position souhaitée (façonnement d'énergie). Le gain dérivatif k_d représente un amortisseur visqueux qui introduit de la friction, pour atteindre la stabilité asymptotique.

Noter aussi que le système en boucle fermée (2.2), (2.6) est en effet un système EL totalement amorti, avec énergie potentielle $V_d(q)$ définie par (2.4) et énergie cinétique $T(\dot{q})$. Le terme $-k_d\dot{q}$ correspond à des forces dissipatives qui peuvent être dérivées d'une fonction de Rayleigh : $\frac{1}{2}k_d\dot{q}^2$. En outre, la fonction d'énergie potentielle $V_d(q)$ a un minimum global et unique en $q = q_d$, et la fonction de Rayleigh définit un opérateur strictement passif à l'entrée si $k_d > 0$, donc, d'après la proposition 1.7, l'équilibre $q = q_d$ est globale asymptotiquement stable.

Bien que cette technique s'appuie sur les propriétés physiques du système, elle a deux inconvénients importants dans les applications pratiques : (1) on requiert des mesures de vitesse pour injecter l'amortissement, (2) on suppose connaître l'énergie potentielle de manière précise. Ainsi, dans ce chapitre on adresse les problèmes suivantes : (1) contrôle de position sans mesure de vitesse, et (2) sans connaissance exacte de l'énergie potentielle.

[1] Voir (Tomei, 1991b) pour une extension de ce résultat, pour les robots à articulations flexibles

2 Stabilisation des systèmes EL par retour de sortie

2.1 Formulation du problème

Dans cette section, on considère uniquement des systèmes sans amortissement interne, c.à.d., dont les paramètres peuvent être

$$\{T_p(q_p, \dot{q}_p), V_p(q_p), M_p\}$$

où l'indexe $_p$ correspond au "procédé". Le modèle dynamique a la forme (1.9), que l'on récrit ci-dessous par convenance :

$$D_p(q_p)\ddot{q}_p + C(q_p, \dot{q}_p)\dot{q}_p + g(q_p) = u_p \tag{2.7}$$

où $u_p = M_p u \in \mathbb{R}^n$ est le vecteur des entrées de contrôle.

Problème de stabilisation globale par retour de sortie. Considérer le système EL (2.7) où q_p est divisé en $q_p = \text{col}[q_{p_1},\ q_{p_2}]$, $q_{p_2} = M_p q_p$. Supposer que q_{p_2} sont les *sorties mesurables* et q_{p_1} les *sorties contrôlées* avec valeur souhaitée *constante*. Trouver une loi de commande $q_{p_2} \mapsto u_p$ qui rende le système en boucle fermée, globalement asymptotiquement stable à l'équilibre souhaité.

Pour résoudre ce problème, on utilisera les propriétés de stabilité entrée-sortie et de stabilité interne, montrées dans le chapitre **1**. On définira une classe de contrôleurs qui modifient l'énergie potentielle et les propriétés de dissipation du procédé, tout en préservant sa structure Lagrangienne.

2.2 Injection d'amortissement par retour dynamique de sortie

Motivés par la technique de façonnement d'énergie plus injection d'amortissement, proposée par Takegaki et Arimoto, ainsi que par les propriétés décrites dans le premier chapitre, on considère ici des *contrôleurs EL* avec coordonnées généralisées $q_c \in \mathbb{R}^{n_c}$ et paramètres EL $\{T_c(q_c, \dot{q}_c), V_c(q_c, q_{p_2}), \mathcal{F}_c(\dot{q}_c), 0\}$. C'est à dire, les équations dynamiques du contrôleur sont

$$D_c(q_c)\ddot{q}_c + \dot{D}_c(q_c)\dot{q}_c - \frac{\partial T_c(q_c, \dot{q}_c)}{\partial q_c} + \frac{\partial V_c(q_c, q_{p_2})}{\partial q_c} + \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} = 0. \tag{2.8}$$

Noter que l'énergie potentielle du contrôleur dépend uniquement de la sortie mesurable q_{p_2}, donc q_{p_2} entre dans le contrôleur par le terme $\frac{\partial V_c(q_c, q_{p_2})}{\partial q_c}$. D'un autre côté, l'interconnexion du procédé avec le contrôleur est donnée par

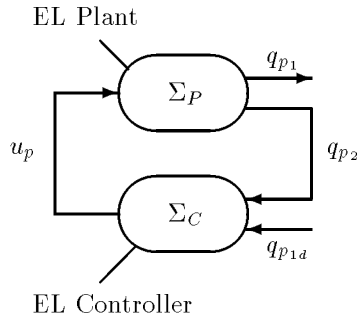
$$u_p = -\frac{\partial V_c(q_c, q_{p_2})}{\partial q_{p_2}}. \tag{2.9}$$

De cette manière, on conclut de **P1.4** que le système en boucle fermée est caractérisé par les paramètres EL $\{T(q, \dot{q}), V(q), \mathcal{F}(\dot{q}), 0\}$, où

$$T(q, \dot{q}) \triangleq T_p(q_p, \dot{q}_p) + T_c(q_c, \dot{q}_c),\ V(q) \triangleq V_p(q_p) + V_c(q_c, q_{p_2}),\ \mathcal{F}(\dot{q}) \triangleq \mathcal{F}_p(\dot{q}_p) + \mathcal{F}_c(\dot{q}_c)$$

et $q = \text{col}[q_p,\ q_c]$. Il est claire de la nouvelle définition de $\mathcal{F}(\dot{q})$, que l'extension dynamique injecte (à travers le contrôleur) l'amortissement nécessaire si $\mathcal{F}_c$ est une fonction de dissipation de Rayleigh.

Le système résultant est montré dans la figure ci-dessous, où $\Sigma_p : u_p \mapsto q_{p_2}$ est un opérateur défini par les équations dynamiques (2.7), et l'opérateur $\Sigma_c : q_{p_2} \mapsto u_p$ est défini par (2.8), (2.9).



Système EL en boucle fermée.

2.3 La classe des contrôleurs EL

On sait du chapitre précédent que, pour atteindre la stabilité globale asymptotique, $V(q)$ doit posséder un minimum global et unique à l'équilibre souhaité, et $\mathcal{F}(\dot{q})$ doit satisfaire (1.4). Ces conditions sont résumées dans la proposition ci-dessous, dont la preuve suit directement des calculs de la section précédente et de la proposition 1.8.

2.1 Proposition. (Stabilisation par retour de sortie) *Considérer un* procédé EL *(2.7) avec paramètres* $\{T_p(q_p, \dot{q}_p), V_p(q_p), \mathcal{F}_p(\dot{q}_p), M_p\}$. *Un* contrôleur EL *(2.8), (2.9) avec paramètres EL* $\{T_c(q_c, \dot{q}_c), V_c(q_c, q_{p_2}), \mathcal{F}_c(\dot{q}_c), 0\}$, *où*

$$\dot{q}_c^\top \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} \geq \alpha \|\dot{q}_c\|^2, \qquad \alpha > 0,$$

résout le problème de stabilisation globale par retour de sortie *si*

(i) (Façonnement d'énergie)
$\frac{\partial V(q)}{\partial q} = 0$ *admet une solution constante* $\bar{q}$ *telle que* $q_{p1d} = [I_{n_{p1}} \mid 0]\bar{q}$, *et* $q = \bar{q}$ *est un minimum global et unique de* $V(q)$, *par exemple si* $\frac{\partial^2 V(q)}{\partial q^2} > I_n \varepsilon > 0, \;\; \varepsilon > 0 \;\; \forall\, q \in I\!R^n$

(ii) (Propagation de la dissipation)
Pour toute trajectoire telle que $q_c \equiv const$ *et* $\frac{\partial V_c(q_c, q_{p_2})}{\partial q_c} \equiv 0$, *on a que* $q_p \equiv const.$

□

2.2 Remarque. La condition 2.1.ii garantie que, lorsque les coordonnées amorties q_c sont constantes (i.e. $\dot{q}_c \equiv 0$), aussi q_p = constante. Noter que cette condition a l'air de la condition de détectabilité, nécessaire pour relier la stabilité asymptotique avec la stabilité entrée-sortie (Hill and Moylan, 1976; Byrnes and Martin, 1995; van der Schaft, 1996). En effet, cette condition peut être remplacée par la supposition (plus forte) de détectabilité état-zéro, en définissant la sortie $\dot{q}_{p_2}$.

3 Régulation avec connaissance incertaine de la gravité

Pour résoudre le problème de stabilisation de systèmes EL par retour de sortie, on a introduit le concept de contrôleurs Lagrangiens, basés sur la méthodologie de façonnement d'énergie plus injection d'amortissement. Un inconvénient évident de cette technique est que, l'énergie potentielle du système doit être précisément connue. Dans cette section, on propose notre contrôleur PI^{2}D, pour résoudre le problème de stabilisation par retour de sortie, avec connaissance incertaine des forces potentielles.

3.1 Le contrôleur PI²D

On considère des systèmes avec modèle dynamique (2.7) où $M_p = I_{n_p}$ c.à.d., on suppose que toutes les coordonnées sont agies. Il faut remarquer que le contrôleur n'appartient pas à la classe de systèmes EL donc, par souci de simplicité, dans cette section on n'utilise pas l'indexe $_p$.

2.3 Proposition. *Considérer le modèle dynamique (2.7) avec $u \in I\!R^n$, en boucle fermée avec la loi de commande PI²D :*

$$\left\{ \begin{array}{rcl} u &=& -K_P\tilde{q} + \nu - K_D\vartheta \\ \dot{\nu} &=& -K_I(\tilde{q} - \vartheta), \quad \nu(0) = \nu_0 \in I\!R^n \end{array} \right. \tag{2.10}$$

$$\left\{ \begin{array}{rcl} \dot{q}_c &=& -A(q_c + Bq_p) \\ \vartheta &=& q_c + Bq_p \end{array} \right. \tag{2.11}$$

Soit K_P, K_I, K_D, A, B des matrices diagonales positives définies tels que

$$B > \frac{4d_M}{d_m} I_n \tag{2.12}$$

$$K_P > (4k_g + 1)I_n \tag{2.13}$$

où k_g est défini par (1.14).

Sous ces conditions, on peut toujours trouver un gain intégral (suffisamment petit) K_I tel que l'équilibre $x \triangleq col[\tilde{q},\ \dot{q},\ \vartheta,\ \nu - g(q_d)] = 0$ soit asymptotiquement stable avec un domaine d'attraction :

$$\left\{ x \in I\!R^{4n} : \|x\| < c_3 \right\} \tag{2.14}$$

où $\lim_{b_m \to \infty} c_3 = \infty$. En d'autres mots, pour n'importe quelle condition initiale (arbitrairement large) $\|x(0)\|$, il existe des gains de contrôle qui assurent : $\lim_{t\to\infty} \|x(t)\| = 0$. □

Noter que la loi de commande (2.10) peut être récrite comme

$$u(t) = -K_P\tilde{q} - K_I \int_0^t \tilde{q}d\tau - K_D \text{ diag}\left\{ \frac{b_i p}{p + a_i} \right\} q - K_I \int_0^t \text{ diag}\left\{ \frac{b_i p}{p + a_i} \right\} q d\tau. \tag{2.15}$$

Les trois premiers termes du côté droit de l'égalité, constituent respectivement, le gain d'action proportionnelle, intégrale et dérivative (filtrée), tandis que le dernier terme motive le nom de PI²D. On a pourtant gardé la notation “D” pour le gain dérivatif puisque dans les application pratiques, les régulateurs PID sont souvent implantés en introduisant ce filtre.

2.4 Remarque. Il est important de mentionner que, lors de l'implantation du contrôleur PI²D, on doit fixer les conditions initiales du filtre de différentiation approximative, très soigneusement : D'après la proposition 2.3, le gain B doit être choisi suffisamment grand pour admettre des conditions initiales $\vartheta(0)$ et $\tilde{q}(0)$ grands. Néanmoins, la condition initiale $\vartheta(0)$ dépend du gain B (voir éq. (2.11) donc, plus grand est B, plus grand sera $\vartheta(0)$. Une façon simple de sortir de cette boule est celle de fixer les conditions initiales $q_c(0) = \vartheta(0) - Bq(0)$ pour certains $\vartheta(0)$, $q(0)$ et B donnés.

3.1.1 Idée centrale de la preuve

La preuve de stabilité semiglobale asymptotique de la proposition (2.3) peut être établie en utilisant la seconde méthode de Lyapunov à l'aide de la fonction candidate $V = \sum_{i=1}^4 V_i$ où

$$V_1(\tilde{q}, \dot{q}, \vartheta) = \frac{1}{2}\dot{q}^\top D\dot{q} + U_g(q) + \frac{1}{2}(q - \delta)^\top K_p'(q - \delta) + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta + c_1 \tag{2.16}$$

$$V_2(z) = \frac{1}{2}z^\top K_I^{-1} z \tag{2.17}$$

$$V_3(\tilde{q}, \dot{q}, \vartheta) = \dot{q}^\top D(q)(\tilde{q} - \vartheta). \tag{2.18}$$

En utilisant les propriétés **P1.1** – **P1.3**, il n'est pas difficile de vérifier que la dérivé temporelle de Lyapunov est localement semi-définie négative avec un domaine d'attraction

$$||x|| \leq c_3 \triangleq \frac{1}{2k_c}\left[\frac{1}{2}b_m d_m - d_M\right]\sqrt{\frac{\alpha_1}{\alpha_2}},$$

où les constantes α_1, α_2 dépendent des bornes sur les paramètres du contrôleur. La stabilité asymptotique suit en utilisant le théorème de Krasovskii-LaSalle. Finalement, on peut prouver que le cocient $(\alpha_1/\alpha_2) \to \infty$ lorsque $b_m \to \infty$.

3.2 Du point de vue de la passivité

Les termes croisés dans V_3 sont particulièrement importants pour notre analyse de stabilité. Plusieurs auteurs ont utilisé cette astuce auparavant pour résoudre de problèmes divers, (Arimoto and Miyazaki, 1986; Wen and Bayard, 1988; Koditschek, 1989). A manière d'exemple on peut citer la méthode (bien connue) de Slotine et Li (Slotine and Li, 1988) : en effet, le changement de variable $s = \dot{q} + \lambda\tilde{q}$ proposé, entraîne des termes croisés dans la fonction de type Lyapunov, en particulier $V = \frac{1}{2}s^\top D(q)s$. Cette fonction, communément utilisée pour prouver la stabilité asymptotique, contient en effet les termes croisés : $\lambda\tilde{q}^\top D(q)\dot{q}$. Dans cette section, on souligne l'importance des termes croisés du point de vue de la passivité, ou plus précisément, *passification*.

Dans la figure ci-dessous, on montre la boucle fermée de la loi de commande PI^{2}D avec un système EL.

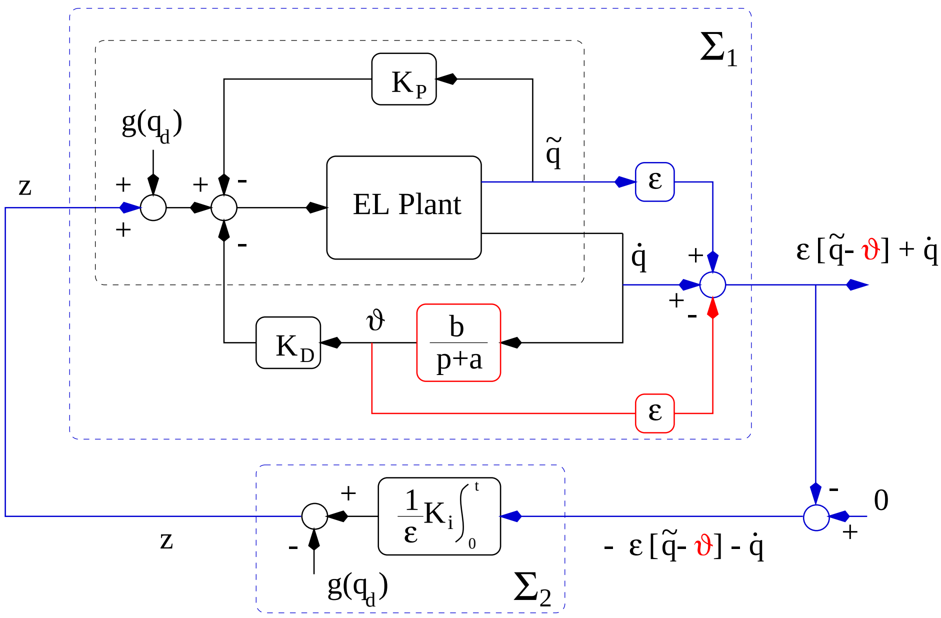


Figure 2.1: Contrôleur PI^{2}D

Pour ce système on peut prouver la proposition suivante :

2.5 Proposition. *Pour le système montré dans la fig. 2.1, défini par la boucle fermée de (2.7) avec (2.10), (2.11) il existe des gains de contrôle K_P, K_D, K_I, A and B tels que*

(i) *Le système Σ_1 définit un opérateur* Localement Strictement Passif à la Sortie *(LOSP) : $z \mapsto \varepsilon[\tilde{q} - \vartheta] + \dot{q}$ avec fonction d'emmagasinage*

$$\phi(\tilde{q}, \dot{q}, \vartheta) \triangleq V_1(\tilde{q}, \dot{q}) + \varepsilon V_3(\tilde{q}, \dot{q}, \vartheta)$$

(ii) *Le système Σ_2 définit un opérateur passif : $-\varepsilon[\tilde{q} - \vartheta] - \dot{q} \mapsto z$ avec fonction d'emmagasinage $V_2(z)$.*

□

4 Conclusions

Dans ce chapitre, on a adressé deux problèmes de contrôle : (1) régulation par retour de sortie et (2) régulation par retour de sortie sans connaissance exacte des forces potentielles.

En ce qui concerne le premier problème, on a donné des conditions suffisantes pour la stabilisation globale asymptotique des systèmes EL, sans mesure de vitesse. On a définit la classes de contrôleurs EL. Ces contrôleurs, sont conçus sur la base de la méthodologie de façonnement d'énergie et injection d'amortissement.

Un inconvénient de notre approche est que, pour façonner l'énergie potentielle, il est supposé que l'on en a une connaissance exacte. Or, ce cas ne se présente pas en générale dans les application pratiques. Pour résoudre ce problème, on a proposé le contrôleur PI^2D.

Notre approche est basée aussi sur les propriétés de passivité du système que l'on veut contrôler. Les caractéristiques principales sont : 1) Elle assure la stabilité asymptotique *semiglobale* de l'équilibre désiré. 2) On suppose disponibles uniquement des mesures de position. 3) La seule connaissance nécessaire à priori, pour l'implantation du contrôleur, sont des bornes supérieures sur certains paramètres du procédé. Notre contrôleur est linéaire.

Chapitre 3

Contrôle de suivi de trajectoire

1 Introduction

La solution au problème de contrôle de suivi de trajectoire par retour *d'état*, des systèmes Lagrangiens complètement agis (en particulier, les robots manipulateurs à articulations rigides) est connu depuis plusieurs années. Pour une révision de la littérature, voir (Ortega and Spong, 1989; Wen and Bayard, 1988). Cependant, un inconvénient de ces résultats est l'utilisation de mesures de vitesses.

Tout comme dans le problème de régulation, une approche alternative souvent adoptée dans la littérature, est la conception d'un observateur qui utilise des mesures de position pour reconstruire le signal de la vitesse. Alors, le contrôleur est réalisé en remplaçant la mesure de vitesse par son estimée. A la porte de notre connaissance, le premier résultat dans cette direction revient à (Nicosia and Tomei, 1990), qui ont utilisé un observateur non linéaire qui reproduit la dynamique complète du robot, dans un schéma de type "computed torque" plus compensation de la gravité. Les auteurs ont prouvé la stabilité asymptotique de la boucle fermée, sous la condition que les gains de l'observateur satisfassent quelques bornes inférieures, sur les paramètres du robot et sur la norme de l'erreur de suivi de trajectoire. Voir également (Canudas de Wit et al., 1990).

(Berghuis et al., 1992a) ont proposé un contrôleur "computed torque" avec un observateur linéaire. Cette approche exploite le propriété de linéarisation par retour, du contrôleur "computed torque" et donne une technique efficace pour le réglage des gains. Plus tard, en utilisant cette même idée, (Berghuis and Nijmeijer, 1993b) ont présenté une procédure systématique qui exploite la propriété de passivité des robots manipulateurs, dans la conception des systèmes contrôleur-observateur, afin de résoudre les problèmes de régulation et suivi de trajectoire par retour de position. La stabilité semiglobale asymptotique a été démontrée pour de gains suffisamment grands.

Dernièrement, dans (Loria and Ortega, 1995) on a utilisé le filtre linéaire de différentiation approximative, avec un contrôleur de type "computed torque", proposé auparavant par (Wen and Bayard, 1988). Dans cet article, on a prouvé la stabilité semiglobale asymptotique de la boucle fermée, en montrant donc, que le domaine d'attraction peut être arbitrairement élargi en grandissant les gains du filtre. Quelques résultats plus récents et plus forts, qui adressent le même problème sont par exemple : (Lim et al., 1994; Nicosia and Tomei, 1994), et (Nicosia and Tomei, 1995). Lim *et al* ont proposé le premier contrôleur adaptatif pour des robots à articulations flexibles, sans mesure de vitesse. Simultanément, (Nicosia and Tomei, 1994) ont proposé un contrôleur basé sur un observateur, globale asymptotiquement stable, en utilisant uniquement des mesures des bras (positions et vitesses). Plus tard, (Nicosia and Tomei, 1995) ont étendu ce résultat en utilisant uniquement la mesure de position du bras ; cependant, dans ce dernier article les auteurs ont prouvé uniquement la stabilité semiglobale asymptotique.

A notre connaissance, le premier contrôleur globale asymptotiquement stable pour des systèmes EL d'un degré de liberté, a été présenté très récemment dans (Loria, 1996). Notre approche est basée sur une structure de type "computed torque" plus PD, et un retour dynamique de sortie inspirée du filtre linéaire de différentiation approximative. La stabilité globale asymptotique de la boucle fermée est assurée pourvu que les gains du contrôleur et de l'extension dynamique, satisfassent quelques bornes inférieures qui dépendent de la trajectoire de référence souhaitée. Malheureusement, notre approche ne s'applique qu'a des systèmes d'un seul degré de liberté, rien ne pouvant être dit pour des systèmes de n degrés de liberté. Cependant, à la meilleure porté de notre connaissance, on ne peut pas prouver la stabilité *globale* pour aucun des résultats reportés auparavant dans la littérature, même en considérant des systèmes d'un seul degré de liberté.

Il faut remarquer, que (Burkov, 1995b) a montré, en utilisant des techniques de perturbations singulières, la stabilité globale asymptotique d'un système Lagrangien de n degrés de liberté, en boucle fermée avec un contrôleur linéaire, par retour dynamique de position. L'inconvénient de ce résultat est qu'il n'est pas possible d'établir des bornes explicites pour les gains du contrôleur. Burkov a donc montré "l'existence" dans contrôleur globale asymptotique sans mesure de vitesse.

Dans ce chapitre, on présente d'abord le résultat reporté dans (Loria and Ortega, 1995), à manière de motivation pour notre résultat principal, le contrôleur global asymptotiquement stable, pour des systèmes d'un degré de liberté. Dans une dernière section on entame une discussion sur l'amélioration de notre résultat (Loria, 1996) par rapport à celui de (Loria and Ortega, 1995).

2 Contrôle semiglobal de suivi de trajectoire

Considérer le problème suivant :

Contrôle de trajectoire par retour de position, des robots rigides. (RR/RP) Pour le système

$$D(q_{p_1})\ddot{q}_{p_1} + C(q_{p_1}, \dot{q}_{p_1})\dot{q}_{p_1} + g(q_{p_1}) = u_p \tag{3.1}$$

supposer que, seulement la position du bras est disponible comme mesure. Sous ces conditions, définir une loi de commande dont les gains peuvent dépendre des conditions initiales, qui assure

$$\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} (q(t) - q_d(t)) = 0 \tag{3.2}$$

pour tout $q_d \in \mathcal{C}^4$, $\|q_d(t)\|$, $\|\dot{q}_d(t)\|$, $\|\ddot{q}_d(t)\| < B_d$.

Une solution au problème posé ci-dessus est présenté par la suite.

3.1 Proposition. *Considérer le système (3.1) en boucle fermée avec la loi de commande*

$$u = D(q)\ddot{q}_d + C(q, \dot{q}_d)\dot{q}_d + g(q) - K_p\tilde{q}_1 - K_d\vartheta \tag{3.3}$$

$$\left\{ \begin{array}{rcl} \dot{q}_c & = & -A(q_c + B\tilde{q}) \\ \vartheta & = & q_c + B\tilde{q} \end{array} \right. \tag{3.4}$$

où

$$K_p,\ K_d > 0 \tag{3.5}$$

sont des matrices diagonales, $a_i > 0$ et b_i satisfont

$$b_i > \frac{3d_M}{d_m}. \tag{3.6}$$

Alors, pour n'importe quelle condition initiale bornée, $x_0 = col[\tilde{q}(0), \dot{\tilde{q}}(0), \vartheta(0)]$, il existe des gains du contrôleur (3.3), (3.4) suffisamment grands, telle que (3.2) est satisfaite avec le domaine d'attraction

$$\left\{x \in I\!R^{3n} : \|x\| < c_1\right\} \tag{3.7}$$

où $\lim_{b_m\to\infty} c_1 = \infty$. □

3.2 Remarque. Il faut remarquer que (3.3) correspond au contrôleur (4.6) proposé par (Wen and Bayard, 1988) où l'on a simplement remplacé le signal de vitesse par la dérivé approximée (3.4). D'ailleurs, dans le cas d'une référence invariante dans le temps, $\dot{q}_d(t) = 0$, la loi de commande (3.3) se réduit au contrôleur EL, globale asymptotiquement stable, de (Kelly et al., 1994a).

3 Contrôle GAS de suivi de trajectoire des systèmes EL d'un degré de liberté

Motivés par le résultat contenu dans la proposition 3.1, et l'idée des termes croisés, on a cherché une manière d'élargir "automatiquement" le domaine d'attraction, au même taux de croissance que celui des variables d'état. Le résultat de cette recherche est présenté dans cette section : un contrôleur de type "computed torque", plus une extension dynamique non linéaire inspirée du filtre linéaire (3.4). Afin d'élargir automatiquement, le domaine d'attraction, on introduit des fonctions trigonométriques hyperboliques dans le contrôleur et l'extension dynamique[1].

3.1 Formulation du problème et sa solution

Considérer le problème

Contrôle de suivi de trajectoire par retour de sortie.
Considérer le modèle dynamique d'un système EL d'un degré de liberté :

$$d(q)\ddot{q} + c(q)\dot{q}^2 + g(q) = u \tag{3.8}$$

où $q \in \mathbb{R}$ est la position généralisée, $d(q)$ est le terme d'inertie, qui satisfait $0 < d_m \leq d(q) \leq d_M$, le terme des forces centrifuges et de Coriolis : $c(q) \triangleq \frac{1}{2}\frac{\partial}{\partial q}\{d(q)\}$ satisfait $k_c > |c(q)|$, $\forall\, q \in \mathbb{R}$, et $g(q)$ est le terme de la force potentielle.

Sous ces conditions, en considérant uniquement des mesures de position, *n'importe quelles* conditions initiales $\mathrm{col}[q(0),\ \dot{q}(0)]$, et n'importe quelle référence continûment différentiable, $q_d(t) \in \mathcal{C}^2$ telle que $|\dot{q}_d| < \beta_d$; trouver une loi de commande $u(q, q_d)$ telle que

$$\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} [q(t) - q_d(t)] = 0. \tag{3.9}$$

Une solution au problème posé ci-dessus est donnée par la suite. Celui-ci est le résultat principal de ce chapitre.

3.3 Proposition. *Considérer le système (3.8) en boucle fermée avec*[2]

$$\begin{aligned} u &= -k_p\ \tanh(\tilde{q}) - k_d\ \cosh(\vartheta)\ \sinh(\vartheta) + g(q) + c(q)\dot{q}_d^2 + d(q)\ddot{q}_d && (3.10)\\ \dot{q}_c &= -a\ \sinh(q_c + b\tilde{q}) && (3.11)\\ \vartheta &= q_c + b\tilde{q} && (3.12)\end{aligned}$$

où k_p, k_d, a, et b sont des constantes positives telles que

$$b > \frac{6k_c + 3d_M}{d_m} \tag{3.13}$$

$$\frac{3k_c\beta_d}{bd_m} < \min\left\{\frac{ak_pk_d}{(k_p+k_d)^2 b},\ \frac{k_d d_m}{3ad_M^2},\ \frac{a}{4b},\ \left(\frac{k_p}{6d_M}\right)^{1/2},\ \left(\frac{k_d}{3d_M b}\right)^{1/2}\right\} \tag{3.14}$$

alors, le problème de contrôle de suivi de trajectoire par retour de sortie est résolu. En particulier, les conditions ci-dessus son satisfaites pour n'importe quelle b choisi conformément à (3.13) et

$$\frac{3d_m^2}{d_M^3}b \geq k_p = k_d = 3a > \frac{36k_c\beta_d}{d_m}. \tag{3.15}$$

$\square$

[1] Pour un rappel sur les fonctions trigonométriques hyperboliques, voir l'Annexe D.
[2] Pour un rappel sur les fonctions trigonométriques hyperboliques, voir l'annexe D.

3.1.1 Idée principale de la preuve de stabilité

Tout d'abord, on écrit les équations de la boucle fermée (3.10) – (3.12), (3.8) comme

$$d(q)\ddot{\tilde{q}} + [c(q)\dot{q} + c(q)\dot{q}_d]\dot{\tilde{q}} + k_p \tanh(\tilde{q}) + k_d \cosh(\vartheta)\sinh(\vartheta) = 0 \tag{3.16}$$

$$\dot{\vartheta} = -a \sinh(\vartheta) + b\dot{\tilde{q}}. \tag{3.17}$$

Considérer maintenant la fonction candidate de Lyapunov :

$$V(\tilde{q}, \dot{\tilde{q}}, \vartheta) = \frac{1}{2}d(q)\dot{\tilde{q}}^2 + k_p \ln|\cosh(\tilde{q})| + \frac{k_d}{2b}\sinh(\vartheta)^2 + \varepsilon[\tanh(\tilde{q}) - \sinh(\vartheta)]\dot{\tilde{q}} \tag{3.18}$$

où $\varepsilon > 0$ est une constante choisie suffisamment petite pour assurer la positivité de $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$. La dérivée temporelle de $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ le long des trajectoires de la boucle fermée (3.16), (3.17), est bornée par

$$\dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta) \le -\frac{\varepsilon}{3}\lambda_1\dot{\tilde{q}}^2 - \lambda_2 \tanh(\tilde{q})^2 - \lambda_3 \sinh(\vartheta)^2 \tag{3.19}$$

où λ_2 et λ_3 sont des constantes positives pour des valeurs de K_p, K_d, a et b suffisamment grands et

$$\lambda_1 = [bd_m \cosh(\vartheta) - 3k_c(|\sinh(\vartheta)| + |\tanh(\tilde{q})|)]. \tag{3.20}$$

Par exemple, on peut choisir ces paramètres en accord avec (3.15). Noter aussi, que λ_1 est positif puisque le terme $|\tanh(\tilde{q})| < 1$ et $\cosh(\vartheta) > |\sinh(\vartheta)|$.

Ainsi, on peut montrer que $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ est définie positive et que sa dérive temporelle est définie négative dans toutes les variables d'état. On peut donc montrer que V est une fonction de Lyapunov. En utilisant la seconde méthode de Lyapunov, on peut affirmer que le système (3.16), (3.17) est *globale asymptotiquement stable.*

4 Discussion

De la proposition 3.1, on sait que le système en boucle fermée (3.3) – (3.4), (3.8) est semiglobale asymptotiquement stable. Dans le cas particulier d'un système EL d'un degré de liberté ce résultat peut être obtenu en utilisant la fonction candidate de Lyapunov

$$V_{ch}(\tilde{q}, \dot{\tilde{q}}, \vartheta) = \frac{1}{2}d(q)\dot{\tilde{q}}^2 + \frac{1}{2}k_p\tilde{q}^2 + \frac{k_d}{2b}\vartheta^2 + \varepsilon[\tilde{q} - \vartheta]d(q)\dot{\tilde{q}}. \tag{3.21}$$

où les termes croisés $\varepsilon[\tilde{q} - \vartheta]d(q)\dot{\tilde{q}}$ sont introduits dans (3.21) afin d'obtenir les termes quadratiques $-\varepsilon bd_m\dot{\tilde{q}}^2$ and $-\varepsilon k_p\tilde{q}^2$, dans la dérivée temporelle de la fonction candidate de Lyapunov :

$$\dot{V}_{ch}(\tilde{q}, \dot{\tilde{q}}, \vartheta) \le -\frac{\varepsilon}{3}[bd_m - 3k_c(|\vartheta| + |\tilde{q}|) - 3d_M]\dot{\tilde{q}}^2 - \frac{1}{3}k_p\tilde{q}^2 - \frac{k_d a}{3b}\vartheta^2. \tag{3.22}$$

Ces termes nous sont fondamentaux pour affirmer la stabilité asymptotique. Cependant, le prix á payer pour ces termes quadratiques négatifs, est les termes cubiques positifs : $\varepsilon 3k_c(|\vartheta| + |\tilde{q}|)\dot{\tilde{q}}^2$. Grosso modo, cela est ce qui nous amène à la définition d'un domaine d'attraction et par conséquent, à la stabilité locale asymptotique ou dans le meilleur des cas, semiglobale. La difficulté pour borner ces termes est une faiblesse commune de plusieurs résultats dans la littérature. Voir par exemple (Loria and Ortega, 1995; Nicosia and Tomei, 1990; Berghuis and Nijmeijer, 1993b; Canudas de Wit et al., 1990).

Pour surmonter ce problème, on a proposé l'utilisation des fonctions trigonométriques hyperboliques dans la fonction candidate de Lyapunov et donc aussi, dans le contrôleur. Cette astuce nous donne une dérivée temporelle de la fonction candidate de Lyapunov, qui est négative définie.

Il est important de remarquer la différence entre le premier terme sur le côté droit de (3.22), et λ_1 dans (3.20). Noter que dans (3.21), le terme constant bd_m doit borner les valeurs absolues des variables d'état : $|\vartheta|$ et $|\tilde{q}|$. En revanche, l'utilisation dans (3.18), des fonctions trigonométriques hyperboliques nous aide : à saturer le terme $k_c \tanh(\tilde{q})$ (en comparaison avec $k_c|\tilde{q}|$ dans l'éq. 3.22), et à avoir le terme $bd_m \cosh(\vartheta)$ qui domine toujours le terme $k_c \sinh(\vartheta)$. Tout cela, en dépit du gain *constant*, b, aussi dans (3.4) que dans (3.12).

3.4 Remarque. L'astuce d'introduire des termes croisés dans la fonction candidate de Lyapunov, n'est pas nouveau. Cependant, l'utilisation des fonctions trigonométriques hyperboliques est à notre connaissance une nouvelle approche qui pourrait aider à résoudre d'autres problèmes difficiles, tel que le contrôle de position sans connaissance exacte des forces potentielles, et sans mesure de vitesse.

4.1 Le cas de n degrés de liberté

Le contrôle de suivi de trajectoire des systèmes de n degré de liberté, par retour de sortie, mérite une attention particulière. Du point de vu d'une analyse basée sur les méthodes de Lyapunov, on a vu jusqu'ici que, l'utilisation de termes croisés et des fonctions trigonométriques hyperboliques, nous permettent d'avoir une dérivée de la fonction de Lyapunov, qui est négative définie. Ceci nous a permis d'affirmer la stabilité globale asymptotique. Malheureusement, cette approche ne s'applique pas aux systèmes de $n > 1$ degrés de liberté. Dans cette section, on explique la raison.

Considérer le système de n degrés de liberté

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

où $q \in \mathbb{R}^n$, en boucle fermée avec le contrôleur

$$\begin{aligned} u &= -k_p \operatorname{Tanh}(\tilde{q}) - k_d \operatorname{Cosh}(\vartheta) \operatorname{Sinh}(\vartheta) + g(q) + C(q,\dot{q}_d)\dot{q}_d + D(q)\ddot{q}_d && (3.23)\\ \dot{q}_c &= -a \operatorname{Sinh}(q_c + b\tilde{q}) && (3.24)\\ \vartheta &= q_c + b\tilde{q} && (3.25) \end{aligned}$$

où k_p, k_d, a, et b sont constantes positives. On définit $\operatorname{Tanh}(\tilde{q}) \triangleq \operatorname{col}[\tanh(\tilde{q}_1), \ldots, \tanh(\tilde{q}_n)]$, $\operatorname{Sinh}(\tilde{q}) \triangleq \operatorname{col}[\sinh(\tilde{q}_1), \ldots, \sinh(\tilde{q}_n)]$, and $\operatorname{Cosh}(\tilde{q}) \triangleq \operatorname{diag}\{[\cosh(\tilde{q}_1), \ldots, \cosh(\tilde{q}_n)]\}$. Le système de l'erreur est :

$$\begin{aligned} &D(q)\ddot{\tilde{q}} + [C(q,\dot{q}) + C(q,\dot{q}_d)]\dot{\tilde{q}} + k_p \operatorname{Tanh}(\tilde{q}) + k_d \operatorname{Cosh}(\vartheta) \operatorname{Sinh}(\vartheta) = 0, && (3.26)\\ &\dot{\vartheta} = -a \operatorname{Sinh}(\vartheta) + b\dot{\tilde{q}}. && (3.27) \end{aligned}$$

Proposons maintenant la fonction candidate de Lyapunov $V_n(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ similaire à (3.18) :

$$V_n(\tilde{q}, \dot{\tilde{q}}, \vartheta) = \frac{1}{2}\dot{\tilde{q}}^\top D(q)\dot{\tilde{q}}^\top + k_p \sum_i \ln|\cosh(\tilde{q}_i)| + \frac{k_d}{2b}\|\operatorname{Sinh}(\vartheta)\|^2 + \varepsilon[\operatorname{Tanh}(\tilde{q}) - \operatorname{Sinh}(\vartheta)]^\top \dot{\tilde{q}} \qquad (3.28)$$

où $\varepsilon > 0$ est suffisamment petit pour assurer la positivité de V_n. Il peut être montré aussi que la dérivée temporelle de V_n, le long des trajectoires de (3.26), (3.27), est bornée par

$$\dot{V}_n(\tilde{q}, \dot{\tilde{q}}, \vartheta) \leq -W_n(\tilde{q}, \dot{\tilde{q}}, \vartheta) - \varepsilon\dot{\tilde{q}}^\top [\frac{b}{6}I_n - D^{-1}C(q, \operatorname{Tanh}(\vartheta))] \operatorname{Cosh}(\vartheta)\dot{\tilde{q}} \qquad (3.29)$$

où $W_n(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ est négative définie. Remarquer le dernier terme de (3.29) ; d'un côté, on sait de l'annexe D que, $\operatorname{Cosh}(\vartheta) > I_n > 0$, $\forall\vartheta \in \mathbb{R}^n$, et d'un autre côté, puisque $|\operatorname{Tanh}(\vartheta)|_i < 1$, $\forall\vartheta \in \mathbb{R}^n$, on a que la matrice $[\frac{b}{3}I - D^{-1}C(q, \operatorname{Tanh}(\vartheta)) - C^\top(q, \operatorname{Tanh}(\vartheta))D^{-1}]$ peut être une matrice positive définie pour un b suffisamment large. Ainsi, la matrice contenue dans le dernier terme de (3.29) peut être regardée comme le produit de deux matrices définies positives.

Or, bien que le produit de deux matrices positives définies, XY, pourrait être positive définie pour deux matrices *spécifiques* $X > 0$, $Y > 0$, (Lancaster and Tismenetsky, 1985), malheureusement, dans notre cas ces matrices varient dans le temps, en conséquence rien ne peut être affirmé.

5 Conclusions

Dans ce chapitre, on a présenté une solution alternative au problème de suivi de trajectoire de robots à articulations rigides, *sans mesure de vitesse*. Notre première contribution est la preuve que, un contrôleur du type "computed-torque" avec le filtre linéaire de différentiation approximative, rend la boucle fermée semiglobale asymptotiquement stable.

On a étendu ce résultat au contrôle *global asymptotique* avec seule mesure de position, des systèmes EL d'un dégrée de liberté. Notre approche exploite les propriétés des fonctions trigonométriques hyperboliques. Pour éviter la mesure de vitesse, on a proposé une extension dynamique non linéaire, inspirée du filtre de différentiation approximative. Malheureusement, cette approche ne s'applique pas aux systèmes de n dégrées de liberté.

On croit que la contribution théorique de cette approche pourrait contribuer à la solution d'autres problèmes difficiles tels que le problème de régulation global par retour de position, avec connaissance incertaine de l'énergie potentielle.

Un inconvénient de nos approches est que, pour assurer la stabilité (semi-) globale, de très hauts gains sont nécessaires. En effet, notre contrôleur global pour systèmes de un dégrée de liberté, délivre des entrées de contrôle qui croissent exponentiellement par rapport à l'erreur de suivi. Il s'avère donc attirant, de concevoir des lois de commandes saturées. Ces problèmes sont traités dans le chapitre suivant.

Chapitre 4

Contrôle sous contraintes à l'entrée

1 Introduction

Dans le chapitre **2**, on a identifié une classe de systèmes Euler-Lagrange, lesquels, peuvent être stabilisés globale asymptotiquement par retour dynamique de sortie. La conception du contrôleur, qui exploite la passivité des systèmes EL, est basée sur le fait que la fonction d'emmagasinage pour des systèmes EL interconnectés, est simplement la somme des fonctions d'emmagasinage respectives. Ces propriétés de base sont exprimées en termes des principes tels que le "façonnement d'énergie" et l'injection de l'amortissement nécessaire. En particulier, on a considéré le cas où le contrôleur est aussi un système Lagrangien, donc passif. De cette manière, le système en boucle fermée est un système EL avec une énergie totale et une fonction de dissipation, égales à la somme des fonctions d'énergie potentielle et de dissipation, du contrôleur et du système à contrôler.

Dans le chapitre **3**, on a présenté quelques solutions au problème de suivi trajectoire par retour de position. Cependant, un inconvénient de ces résultats qui paraît évident, est que de grands gains sont nécessaires pour assurer la stabilité semiglobale asymptotique de la boucle fermée. D'ailleurs, la structure du contrôleur GAS pour des systèmes d'un degré de liberté est telle que la magnitude de l'entrée de contrôle grandit exponentiellement avec l'erreur de suivi de trajectoire.

Du point de vue théorique, on a vu dans les chapitres précédents que divers problèmes de contrôle par retour dynamique de sortie peuvent être résolus avec le filtre linéaire de différentiation approximative. Ce filtre, peut en effet, être regardé comme un observateur linéaire (Berghuis and Nijmeijer, 1993a). Néanmoins, cet observateur peut introduire des valeurs très grandes, de l'estimée de la vitesse, sur une certaine période de temps. Ce fait entraîne deux mauvaises conséquences : premièrement, comme il a été dit dans le chapitre **2**, plus grand est le gain de l'observateur, plus grandes seront les conditions initiales, donc on entre dans une boucle. Deuxièmement, sur la période de temps où l'estimée de la vitesse est trop grande, cette estimée n'a aucun intérêt donc, elle ne doit être prise en compte. Une manière simple de surmonter le premier problème est celle de définir soigneusement les conditions initiales du filtre. Une autre approche pour résoudre les deux problèmes est l'utilisation des saturations. Cette mesure a été prise d'abord par (Esfandiari and Khalil, 1992). Ceci a motivé les chercheurs à étudier le problème de contrôle de suivi de trajectoire par retour dynamique de sortie, avec des *saturations*. Un article très intéressant est (Teel and Praly, 1995) où les auteurs donnent divers outils pour la stabilisation semiglobal par retour dynamique de sortie avec saturations ; parmi eux, il faut remarquer le filtre de différentiation approximative. Voir également (Teel, 1992; Saberi et al., 1996; Coron et al., 1995; Sontag and Sussman, 1994) et les références qui y sont contenues, pour citer quelques exemples.

Dans ce chapitre, on s'intéresse à la commande saturée par retour de sortie, d'une classe de systèmes non linéaires : les systèmes EL complètement agis. Concernant le problème du contrôle de position, une des premières contributions revient à (Cai and Song, 1993), qui ont proposé un contrôleur PD saturé plus compensation de la gravité. Plus tard, (Kelly et al., 1994b) ont étendu ces résultats en utilisant la compensation pré-calculée de la gravité. Cependant, ces deux résultats utilisent des mesures de vitesses. Quelques résultats plus récents qui utilisent uniquement des mesures de position ont été présentés dans (Burkov, 1995c) et (Loria et al., 1996). Burkov (1995) a proposé un contrôleur qui utilise compensation exacte de la gravité, pendant que nous avons introduit une sous-classe de contrôleurs EL (qui contient celui de Burkov), en étendant ainsi, les résultats reportés dans (Ortega et al., 1995c), au cas des systèmes EL complètement agis et soumis à des contraintes à l'entrée.

Très récemment, (Burkov and Freidovich, 1995) ont étendu leurs propres résultats, et ceux reportés dans (Loria et al., 1996) au cas des systèmes Lagrangiens sous-agis. A la meilleure porté de notre connaissance, celui-ci est le premier résultat dans cette direction.

Cependant, tous les résultats mentionnés ci-dessus s'appliquent uniquement au problème de régulation. A notre connaissance, la première solution au problème de suivi de trajectoire par retour de position, et à des entrées bornées, a été présentée dans (Loria and Nijmeijer, 1995). Dans cet article, on propose une loi de commande par retour dynamique de sortie. L'extension dynamique proposée, est basée sur la structure du filtre linéaire de différentiation approximée, utilisée précédemment dans (Loria and Ortega, 1995). Il faut remarquer aussi que l'on a prouvé qu'il n'est pas nécessaire d'introduire de hauts gains pour élargir le domaine d'attraction ; ceci nous permet de respecter les contraintes à l'entrée.

Ce chapitre comprend les résultats reportés dans (Loria et al., 1996) et dans (Loria and Nijmeijer, 1995). On considère uniquement des systèmes EL complètement agis, et qui possèdent les propriétés **P1.1**–**P1.2**.

2 Régulation sans mesure de vitesse avec des saturations

2.1 Formulation du problème

Dans cette section, on considère des systèmes EL possédant la propriété **P1.3**. Sous cette condition, on considère le problème suivant

Problème de régulation globale par retour de sortie. Pour le système (1.7) avec $u_p \in \mathrm{IR}^n$ et $\mathcal{F}_p(\dot{q}_p) \equiv 0$, c'est à dire,

$$D(q_p)\ddot{q}_p + C(q_p,\dot{q}_p)\dot{q}_p + g(q_p) = u_p, \tag{4.1}$$

supposer que, uniquement des mesures de position généralisées, q_p, sont disponibles et que les entrées du système sont restreintes par :

$$|u_{p_i}| \le u_{p_i}^{\max} \qquad \forall i \in \underline{n} \tag{4.2}$$

alors, trouver un contrôleur par retour de sortie qui rende le système en boucle fermée, globale asymptotiquement stable, c'est à dire, un contrôleur par retour de sortie, tel que

$$\lim_{t\to\infty} \tilde{q}_p \triangleq \lim_{t\to\infty} [q_p - q_{pd}] = 0, \tag{4.3}$$

où q_{pd} est le vecteur de positions souhaités.

Basés sur les résultats du deuxième chapitre, dans cette section on définit une famille de contrôleurs Lagrangiens à des entrées bornes. On rappelle que notre motivation pour considérer des contrôleurs EL est la suivante : Puisque l'interconnexion de deux systèmes EL, est un système EL, et le comportement dynamique d'un système EL est complètement caractérisé par par ses paramètres EL, il suffit de choisir les paramètres EL souhaités en boucle fermée, pour en dériver les paramètres EL du contrôleur. En particulier, en ce qui concerne le problème de régulation, il suffit de choisir $V(q)$ ayant un minimum global et unique à l'équilibre souhaité et de choisir $\mathcal{F}_c(\dot{q}_c)$ de manière à assurer l'amortissement nécessaire pour la stabilité asymptotique.

A ce point-là, on rappelle le lecteur que, en accord avec la méthodologie des contrôleurs EL introduite au chapitre **2**, l'entrée de contrôle est définie par

$$u_p = -\frac{\partial V_c(q_c, q_p)}{\partial q_p};$$

d'ici, on peut déduire que la contrainte sur l'entrée, établie par (4.2), entraîne des restrictions de croisement sur $V_c(q)$. D'ailleurs, puisque $V_c(q)$ doit être conçue de manière qu'elle domine la fonction d'énergie potentielle du procédé, $V_p(q_p)$, il est nécessaire d'imposer des restrictions sur le taux de croissance de $V_p(q_p)$. Ainsi, on considère dans ce chapitre une sous-classe de procédés EL totalement agis, dont la fonction d'énergie potentielle satisfait (1.15). Cette condition restreint le taux d'accroissement de $V_p(q_p)$ à l'ordre $\mathcal{O}(\|q_p\|^2)$ pour tout q_p dans une boule B_β, et à un ordre $\mathcal{O}(\|q_p\|)$ en dehors de B_β.

Naturellement des conditions similaires sont imposées sur $V_c(q_c, q_p)$. Comme il le deviendra plus clair par la suite, afin de satisfaire cette contrainte, on peut choisir par exemple

$$V_c(q_c, q_p) = \int_0^{\tilde{q}_p} \mathrm{sat}(x)dx + \dots$$

où la fonction de saturation $\mathrm{sat}(x)$ est définie par

4.1 Définition. *Une fonction de saturation $sat(x) : I\!R \to I\!R$ est une fonction $\mathcal{C}^2$, strictement croissante, qui en outre satisfait*

1. $sat(0) = 0$,
2. $|\, sat(x)| < 1$,
3. $\frac{\partial^2\, sat(x)}{\partial x^2} \neq 0 \quad \forall\, x \neq 0 \in I\!R$.

Des fonction de saturation définies par 4.1 satisfont les propriétés suivantes :

P 4.1 $\int_0^{\tilde{q}_{p_i}} \mathrm{sat}(x)dx \geq \frac{1}{2}\,\mathrm{sat}(\tilde{q}_{p_i})\tilde{q}_{p_i}, \quad \tilde{q}_{p_i} \in \mathrm{I\!R}$.

P 4.2 Il existe $\varepsilon > 0$ tel que

$$\mathrm{sat}(\tilde{q}_{p_i})\tilde{q}_{p_i} \geq \frac{\mathrm{sat}(\varepsilon)}{\varepsilon}\tilde{q}_{p_i}^2 \quad |\tilde{q}_{p_i}| < \varepsilon, \tag{4.4}$$

$$\mathrm{sat}(\tilde{q}_{p_i})\tilde{q}_{p_i} \geq \mathrm{sat}(\varepsilon)|\tilde{q}_{p_i}| \quad |\tilde{q}_{p_i}| \geq \varepsilon. \tag{4.5}$$

Par exemple, on peut choisir $\mathrm{sat}(x) := \tanh(\omega x)$, $\omega > 0$, comme il a été proposé par (Cai and Song, 1993; Kelly et al., 1994b).

2.2 Une famille de contrôleurs GAS, saturés

Puisque l'on traite avec des systèmes complètement agis, la manière la plus simple de “dominer” l'énergie potentielle du procédé, est celle d'annuler $V_p(q_p)$ et d'imposer une énergie souhaitée en boucle fermée. Néanmoins, cela entraîne des problèmes de robustesse ; on favorise donc, une solution qui ne soit pas basée sur cette annulation. Il est intéressant de remarquer que si l'on utilise un contrôleur qui n'annule pas le vecteur de forces potentielles, la contrainte sur le taux de croisement de $V_p(q_p)$, lui est imposée uniquement à la position souhaitée. Le prix à payer dans ce cas, est que l'on a besoin d'utiliser des gains plus importants dans $V_c(q_c, q_p)$ pour dominer l'effet de $V_p(q_p)$. Par conséquent, on requiert des conditions plus sévères sur la borne supérieure, $u_{p_i}^{\max}$.

4.2 Proposition. (Contrôleur avec annulation des forces potentielles)
Supposer que la fonction d'énergie potentielle du procédé, satisfait

$$\sup_{q_p \in I\!R^n} \left| \left(\frac{\partial V_p(q_p)}{\partial q_p} \right)_i \right| < u_{p_i}^{\max}, \quad i \in \underline{n} \tag{4.6}$$

où $(\cdot)_i$ est le $i-$ème composant du vecteur. Sous ces conditions, il existe un contrôleur EL par retour dynamique de sortie, de la forme (2.8), (2.9) qui assure que les contraintes (4.2) sont satisfaites, et fait

$$(\dot{q}_p, q_p, \dot{q}_c, q_c) = (0, q_{pd}, 0, q_{cd}), \tag{4.7}$$

avec q_{cd} une constante quelconque, un point d'équilibre GAS du système en boucle fermée. $\square$

2.2.1 Un contrôleur avec annulation des forces potentielles (Burkov, 1995c)

Considérer le contrôleur EL caractérisé par

$$\begin{aligned}
T_c(q_c, \dot{q}_c) &= 0, \quad \mathcal{F}_c(\dot{q}_c) = \frac{1}{2}\|\dot{q}_c\|^2 &(4.8)\\
V_c(q_c, q_p) &= V_{c_1}(q_c) + V_{c_2}(q_c, q_p) - V_p(q_p) &(4.9)\\
V_{c_1}(q_c) &= \frac{1}{2} q_c^\top K_1 q_c &(4.10)\\
V_{c_2}(q_c, q_p) &= \sum_{i=1}^{n} k_{2_i} \int_0^{(q_{c_i} - \tilde{q}_{p_i})} \mathrm{sat}(x_i) dx_i &(4.11)
\end{aligned}$$

où $\tilde{q}_{p_i} := q_{p_i} - q_{pd_i}$, $k_{1_i}, k_{2_i} > 0$, $K_1 :=$ diag$\{k_{1_i}\}$. En utilisant les équations de Lagrange, on peut dériver la dynamique du contrôleur :

$$\begin{aligned}
\dot{q}_{c_i} &= -k_{1_i} q_{c_i} - k_{2_i}\, \mathrm{sat}(q_{c_i} - \tilde{q}_{p_i}) &(4.12)\\
u_{p_i} &= k_{2_i}\, \mathrm{sat}(q_{c_i} - \tilde{q}_{p_i}) + \left(\frac{\partial V_p(q_p)}{\partial q_p}\right)_i &(4.13)
\end{aligned}$$

qui correspond à celui proposé par Burkov.

4.3 Proposition. (Contrôleur sans annulation des forces potentielles)
Supposer que, le gradient de l'énergie potentielle évalué à la position souhaitée, satisfait l'inégalité

$$\left|\left(\frac{\partial V_p}{\partial q_p}(q_{pd})\right)_i\right| \leq k_{g_i}^{\max}, \quad i \in \underline{n} \tag{4.14}$$

où $k_{g_i}^{\max} < u_{p_i}^{\max}$, et son Hessian satisfait (1.15). Sous ces conditions, il existe un contrôleur EL global asymptotiquement stable qui n'annule pas les forces potentielles, et qui satisfait les contraintes à l'entrée (4.2) pourvu que $u_{p_i}^{\max}$ soit suffisamment grand. ▢

2.2.2 Un contrôleur sans annulation des forces potentielles

Dans ce cas-là, les paramètres EL du contrôleur peuvent être choisis comme

$$\begin{aligned}
T_c(q_c, \dot{q}_c) &= 0, \quad \mathcal{F}_c(\dot{q}_c) = \frac{1}{2}\dot{q}_c^\top K_2 B^{-1} A^{-1} \dot{q}_c &(4.15)\\
V_c(q_c, q_p) &= V_{c_2}(q_c, q_p) - q_p^\top \frac{\partial V_p}{\partial q_p}(q_{pd}) &(4.16)\\
V_{c_2}(q_c, q_p) &= \sum_{i=1}^{n} \left(\frac{k_{2_i}}{b_i} \int_0^{(q_{c_i} + b_i q_{p_i})} \mathrm{sat}(x_i) dx_i + k_{3_i} \int_0^{\tilde{q}_{p_i}} \mathrm{sat}(x_i) dx_i\right) &(4.17)
\end{aligned}$$

où $A :=$ diag$\{a_i\}$, $B :=$ diag$\{b_i\}$, $K_2 :=$ diag$\{k_{2_i}\} > 0$, et $k_{3_i} > 0$ *est suffisamment grand.* Ce choix donne le contrôleur :

$$\begin{aligned}
\dot{q}_{c_i} &= -a_i\, \mathrm{sat}(q_{c_i} + b_i q_{p_i})\\
u_{p_i} &= -k_{2_i}\, \mathrm{sat}(q_{c_i} + b_i q_{p_i}) - k_{3_i}\, \mathrm{sat}(\tilde{q}_{p_i}) + \left(\frac{\partial V_p}{\partial q_p}(q_{pd})\right)_i. &(4.18)
\end{aligned}$$

Les propositions ci-dessus, caractérisent en termes énergétiques, une classe de contrôleurs par retour dynamique de sortie, globale asymptotiquement stables avec des entrées bornées.

4.4 Remarque. Une caractéristique clé du contrôleur de la section 2.2.2 est que, pour renforcer sa robustesse, on a évité des annulations explicites des dynamiques du procédé. Comme il a été mentionné auparavant, le prix à payer est que la fonction d'énergie potentielle ne croisse pas plus vite que linéairement, ce qui est exprimé dans (1.15). Aussi, de gains plus élevés doivent être injectés à la boucle à travers k_{3_i}. Cette condition sur k_{3_i} viens du fait que, afin d'imposer un minimum souhaité á la fonction d'énergie potentielle en boucle fermée, maintenant on doit dominer (et non pas annuler) l'énergie potentielle.

3 Contrôle de suivi de trajectoire

Dans cette dernière section on présente, à notre meilleure connaissance, le *premier* contrôleur saturé, de suivi de trajectoire, pour les systèmes EL complètement agis. Par simplicité on choisit $\mathrm{sat}(x) := \tanh(x)$. Aussi, on rappelle au lecteur que l'indexe $_p$ utilisé dans la section précédente a été introduit pour distinguer les coordonnées généralisées des plantes EL, de celles des contrôleurs EL. Par souci de simplicité, dans cette section on omet cet indexe.

Problème de contrôle de suivi de trajectoire par retour de sortie. Pour le système (4.1) supposer que, uniquement des mesures de position sont disponibles. Supposer également que les entrées de contrôle au procédé son soumises aux contraintes (4.2). Sous ces conditions, concevoir un contrôleur par retour dynamique de sortie qui fasse le système en boucle fermée, semiglobale asymptotiquement stable, c'est à dire, un contrôleur par retour de sortie dont les paramètres peuvent dépendre de conditions initiales, tel que

$$\lim_{t\to\infty} \tilde{q}(t) \triangleq \lim_{t\to\infty} [q(t) - q_d(t)] = 0 \tag{4.19}$$

où la trajectoire souhaitée $q_d(t) \in \mathcal{C}^2$ satisfait $\|\ddot{q}_d(t)\|$, $\|\dot{q}_d(t)\|$, $\|q_d(t)\| \le B_d$.

4.5 Proposition. *Considérer le système EL (4.1) en boucle fermée avec la loi de commande*

$$\begin{aligned} u &= -K_p \;\mathrm{Tanh}(\tilde{q}) - K_d \;\mathrm{Tanh}(\vartheta) + D(q)\ddot{q}_d + C(q,\dot{q}_d)\dot{q}_d + g(q) && (4.20)\\ \dot{q}_c &= -A \;\mathrm{Tanh}(q_c + B\tilde{q}) && (4.21)\\ \vartheta &= q_c + B\tilde{q}, && (4.22) \end{aligned}$$

où A, B, K_p et K_d sont des matrices diagonales positives définies. Soit $I_t \triangleq [t_0, \infty)$, $t_0 \ge 0$ et $x(t) \triangleq col[\tilde{q}(t), \dot{\tilde{q}}(t), \vartheta(t)]$. Soit $\eta > 0$ et B_η une boule à rayon η sur $(0,\ 0,\ 0)$ dans l'espace $(\tilde{q},\ \dot{\tilde{q}},\ \vartheta)$. Sous ces conditions, il existe toujours $A > 0$ et $B > 0$ suffisamment grands tels que, pour n'importe quelle condition initiale $(t_0, x_0) \in I_t \times B_\eta$, la solution $x(t)$ du système en boucle fermée reste dans la boule B_η pour tout $t \ge t_0$. D'ailleurs, pour des k_{p_m}, k_{d_m}, donnés et n'importe quel $\eta > 0$ arbitrairement grand, le système en boucle fermée est uniformément asymptotiquement stable. En outre, les contraintes (4.2) sont satisfaites si

$$u^{\max} > B_d(d_M + k_c B_d) + k_v \tag{4.23}$$

□

4 Conclusions

Dans ce chapitre, on a étendu nos résultats précédents sur le contrôle des systèmes EL par retour de sortie, au cas d'entrées bornées. En ce qui concerne le contrôle de position, on a étendu les résultats de (Kelly et al., 1994b) sur le contrôle de robots manipulateurs par retour de sortie, à entrées saturées. En particulier, on a définit une sous-classe de contrôleurs Lagrangiens qui assurent la stabilité asymptotique globale avec des entrées saturées.

On a aussi résolu le problème de contrôle de suivi de trajectoire des systèmes EL par retour dynamique de position avec des contraintes à l'entrée. Notre approche assure la stabilité semiglobale asymptotique de la boucle fermée. Il est important de remarquer que, notre proposition garantie que l'on peut élargir le domaine d'attraction en accroissant uniquement les gains de l'extension dynamique. Ainsi, pour toute condition initiale, et des gains de contrôle suffisamment petits – tels que les contraintes d'entrées soient satisfaites –, on peut toujours trouver des gains de l'extension dynamique, suffisamment grands pour assurer la stabilité asymptotique du système.

A la meilleure porté de notre connaissance, ce résultat est le premier dans son genre.

Chapitre 5

Contrôle de robots manipulateurs en interaction avec leur environnement

Dans les chapitres précédents on a présenté plusieurs résultats sur le contrôle des systèmes EL, par retour de sortie. Comme il a été mentionné auparavant, les robots manipulateurs sont un exemple important de systèmes Lagrangiens. Ce chapitre est consacré au problème particulier du contrôle des manipulateurs en interaction avec leur environnement. Les résultats de ce chapitre sont une application directe des résultats des chapitres précédents.

1 Résumé

Au cours de la dernière décennie, de différentes approches pour le contrôle de manipulateurs en interaction avec leur environnement, ont été proposées dans la littérature. Celles-ci peuvent être classifiées en fonction de l'objectif de contrôle et du modèle utilisé ; par exemple, (Whitney, 1977) a identifié 6 approches différentes qui peuvent être divisées à leur tour, en deux groupes : Les unes cherchent à contrôler simultanément la position du manipulateur et la force exercée, de manière non conflictuelle. Les autres, cherchent à contrôler la relation entre la position du manipulateur et la force d'interaction avec l'environnement.

Un exemple assez populaire, appartenant au premier groupe, est la commande hybride. Cette approche, introduite par (Raibert and Craig, 1981), cherche à contrôler la force exercée dans les directions contraintes par l'environnement, et la position dans les autres. Plus tard, cette approche a été étendue à la commande hybride dynamique, par (Yoshikawa, 1986) et à une formulation de l'espace d'opération, par (Khatib, 1987).

Quelques exemples du deuxième groupe sont la "commande de rigidité" (Salisbury, 1980), "compliant control" (Spong and Vidyasagar, 1989; Waibel and Kazerooni, 1991) et contrôle d'impédance (Hogan, 1985; Dawson et al., 1993). Plus récemment, (Chiaverini and Sciavicco, 1993) ont introduit la *commande parallèle*, qui consiste à contrôler la position et la force, en donnant une priorité au contrôle de force, par moyen d'un intégrateur, ceci en contraste avec la matrice de sélection utilisée dans la commande hybride. Un texte récent qui décrit les approches mentionnées ci-dessus, est (Sciavicco and Siciliano, 1996).

D'un autre côté, en ce qui concerne le modèle utilisé, deux types de modèles de contact sont utilisés principalement, en dépendant du type d'interaction avec l'environnement : *élastique* ou *infiniment rigide*.

Dans le premier cas, on considère que l'interaction avec l'environnement est élastique, la constante d'élasticité étant directement proportionnelle à la déformation de l'environnement. Lorsque l'on considère un environnement infiniment rigide, à partir de (Yoshikawa, 1986; McClamroch, 1986; Koivo and Kankaanrantaes, 1988), l'interaction est modelée par des contraintes holonomes (algébriques) qui sont imposées au manipulateur. Malheureusement, ces équations sont singulières donc, pour surmonter cette difficulté plusieurs techniques pour dériver des *modèles réduits* ont été proposées, soit basées sur la projection des équations dynamiques du manipulateur, soit sur une variété décrite par les contraintes algébriques.

Une approche alternative est basée sur le *principe d'orthogonalisation* (Arimoto et al., 1993). La caractéristique principale de cette approche est l'introduction d'une matrice qui projète les signaux de vitesse et de position, sur un plan tangent à la surface de contrainte (dans l'espace des articulations), afin de distinguer les signaux de force et de position.

En ce qui concerne le contrôle de force, plusieurs résultats ont été proposés pour la stabilisation de robots manipulateurs. Voir parmi autres : (McClamroch and Wang, 1988; Carelli and Kelly, 1991; Panteley and Stotsky, 1993a; Arimoto et al., 1992). Le problème de régulation a été considéré par exemple par (Wang and McClamroch, 1993; Arimoto, 1994b), qui ont utilisé un modèle basé sur un environnement rigide, et par (Chiaverini and Sciavicco, 1993; Siciliano and Villani, 1993) qui considèrent que le manipulateur interagit avec un environnement élastique. Il faut remarquer que (Wang and McClamroch, 1993) ont prouvé la stabilité *locale* asymptotique d'un contrôleur hybride, du type PD avec compensation de la gravité et des forces de contacte, évaluées à la référence souhaitée. Ce résultat, qui étend de forme naturelle la contribution fondamentale de (Takegaki and Arimoto, 1981), est basé sur le fait que la propriété de passivité du robot est conservée, même en contacte avec son environnement.

Cependant, un grand inconvénient de tous les résultats cités ci-dessus et la nécessité de mesures de vitesse. Il est donc souhaitable de concevoir des lois de commande dans lesquelles on utilise uniquement des mesures de position. A la meilleure porte de notre connaissance ce problème a été considéré premièrement par (Huang and Tseng, 1991) et (Panteley and Stotsky, 1993b). Huang et Tseng ont étudié le problème de conception d'observateurs pour robots avec contraintes, en boucle ouverte. Plus récemment (Panteley and Stotsky, 1993b) ont proposé un observateur non linéaire, et ils ont prouvé pour la première fois, la stabilité asymptotique locale du système en boucle fermée.

Un deuxième inconvénient important qui concerne les approches basées sur des modèles d'environnements rigides, est que la plupart d'entre eux, sont conçus en supposant que l'équation algébrique des contraintes est soluble *globalement*. Il est souvent supposé, soit que le Jacobien des contraintes n'est pas singulier, (McClamroch and Wang, 1988; Caiti and Cannata, 1994; Yao et al., 1992; Carelli and Kelly, 1991), soit que la trajectoire actuelle du manipulateur est contenu dans un voisinage de la référence souhaitée (Wang and McClamroch, 1993; Arimoto et al., 1993). Pourtant, il faut remarquer que ces suppositions sont très restrictives ; par exemple, le Jacobien peut présenter des singularités même dans des cas simples comme celui d'un manipulateur de deux dégrées de liberté dont l'outil est restreint par un plan.

Dans ce dernier chapitre, on étend les résultats des chapitres précédents pour un exemple important de systèmes EL : les robots manipulateurs. Nos contributions dans ce domaine, sont des solutions à plusieurs problèmes de contrôle des manipulateurs par retour de sortie, en interaction avec leur environnement. On a étudié les deux types d'environnement principaux mentionnés ci-dessus.

Premièrement, on considère que le manipulateur interagit avec un environnement élastique. Le problème de concevoir un régulateur de position/force qui n'utilise ni la *connaissance exacte* du vecteur de force gravitationnelle, ni la *mesure de vitesse*, a été résolu dans (Loria and Ortega, 1996). Dans cet article on a utilisé le modèle du robot défini dans l'espace des tâches, proposé par (Khatib, 1987). On a supposé, comme dans (Chiaverini and Sciavicco, 1993) que l'outil exerce une force normale à l'environnement élastique, dont on connaît la constante. Notre résultat principale est la preuve que, sous ces conditions, il est possible de concevoir un régulateur *semiglobal asymptotiquement stable*. Le contrôleur proposé est du type PI^{2}D. Ainsi, il étend les résultats proposés dans la section 2.3 au cas du mouvement restreint, en observant que le problème de contrôle de force/position peut être reformulé comme un problème de contrôle de position pure, lorsque l'on considère un environnement élastique. Une deuxième contribution dans cette direction et la preuve qu'un contrôleur de type PID avec un gain intégral normalisé, est *globale* asymptotiquement stable.

Deuxièmement, on considère que l'outil est en interaction avec un environnement rigide. On utilise le modèle proposé par (Panteley and Stotsky, 1993a), qui est définit dans une variété déterminée par les équations des contraintes. On considère qu'il existe un sous-ensemble de l'espace des coordonnées où il est raisonnable de supposer que le contact n'est pas perdu pendant le mouvement. En outre, si les trajectoires généralisées commencent et restent dans ce sous-ensemble, il suit du théorème de la fonction implicite globale (Sandberg, 1981), que le Jacobien est non singulier. En contraste avec d'autres résultats qui utilisent des modèles semblables, on *ne suppose pas* que les coordonnées généralisées restent dans ce sous-ensemble, tout le temps. Or, on le prouve.

Sous ces conditions, on propose premièrement, un contrôleur qui assure que, pour toute condition initiale strictement contenue dans le sous-ensemble “de régularité”, les trajectoires généralisées restent à l'intérieur de celui-ci. Sur cette base, on prouve la stabilité asymptotique de la boucle fermée. On prouve finalement que notre loi de commande satisfait une borne supérieure. Ce résultat, est le premier dans son genre est il a été reporté dans (Panteley et al., 1996). Il étend le concept de contrôleurs EL bornés, présenté dans le chapitre 4.

Finalement, le troisième résultat de ce chapitre est un contrôleur de trajectoire/force qui rend la boucle fermée, uniformément asymptotiquement stable. Ce résultat est une extension directe des travaux publiés dans (Panteley et al., 1996) et (Loria and Ortega, 1995). En contraste avec (Panteley and Stotsky, 1993b), notre contrôleur utilise le filtre linéaire de différentiation approximative, à la place d'un observateur non linéaire. Étant donné la popularité de ce filtre, dans les applications pratiques, l'intérêt de notre contrôleur est évident. Ce contrôleur a été proposé dans (Loria and Panteley, 1996).

Conclusions

On a étudié dans cette thèse, plusieurs problèmes de contrôle des systèmes Lagrangiens : contrôle de position, contrôle de suivi de trajectoire, contrôle à entrées bornées.

Pour résoudre le problème de régulation, on a présenté le concept de *Contrôleurs EL* qui, comme leur nom le suggère, ils sont des systèmes Lagrangiens. En s'appuyant sur le fait que l'interconnexion de deux systèmes Lagrangiens donne comme résultat un système Lagrangien, notre classe de contrôleurs exploite leurs propriétés physiques. Le but de contrôle est donc : premièrement, de façonner l'énergie potentielle de manière à placer un minimum global et unique à l'équilibre souhaité ; et deuxièmement, d'injecter un amortissement adéquat pour atteindre la stabilité asymptotique. Cette méthodologie est inspiré du connu théorème de Josephe L. La Grange, qui établit que les équilibres stables d'un système mécanique libre, sont déterminés par les minima de la fonction d'énergie potentielle. Dans la littérature moderne, notre travail, reprend les idées de Takegaki et Arimoto, bien connues dans la littérature de la robotique.

Le problème de suivi de trajectoire par retour de sortie, des robots manipulateurs est connu déjà depuis quelques années. Toutes les approches reportées dans la littérature jusqu'à l'année dernière, assure uniquement la stabilité asymptotique semiglobale. Même pour les systèmes d'un dégrée de liberté, aucun contrôleur assurent la stabilité asymptotique *globale* avait été publié. Dans ce travail, nous avons présenté le premier contrôleur pour systèmes EL d'un degré de liberté, globale asymptotiquement stable. Malheureusement, rien ne peut être rigoureusement démontré pour systèmes de n degrés de liberté.

On a étendu nos résultats sur le contrôle de position et de suivi de trajectoire au cas où l'on a des contraintes à l'entrée. On a défini une sous-classe de contrôleurs EL dont l'entrée de contrôle est bornée. En ce qui concerne le problème de suivi de trajectoire, on a proposé un contrôleur par retour d'état dynamique, qui assure la stabilité semiglobale asymptotique. On a donc montré qu'il n'est pas nécessaire d'utiliser de hauts gains pour élargir le domaine d'attraction. L'inconvénient de ces résultats est qu'ils s'appliquent uniquement á des systèmes complètement agis.

Recherche future

- Au début de cette recherche, cela fait 3 années, plusieurs questions été ouvertes. D'un intérêt particulier, Prof. Ortega m'a à résoudre le problème *global* de suivi de trajectoire par retour de position. Aujourd'hui, nos résultats pour des systèmes d'un dégrée de liberté marquent un pas en avant dans cette direction, néanmoins ce problème reste ouvert.

- Le problème de contrôle avec restrictions à l'entrée, me défi également à étendre les résultats obtenus, au cas des systèmes sous-agis.

- Dans le chapitre **4** on traite le problème de contrôle des systèmes Lagrangiens avec saturations. Or, des auteurs tels que Andrew Teel, Laurent Praly, Ali Saberi, (parmi autres) ont fait plusieurs contributions importantes à la commande de systèmes non linéaires avec saturations. Ainsi, une autre question intéressante qui se pose est celle de l'extension de nos résultats à la commande bornée de systèmes non linéaires (non autonomes).

- Dans cette thèse, presque toutes nos contributions s'appuient sur la supposition que les paramètres du procédé sont connus avec certitude. Uniquement le contrôleur PI^{2}D tient compte de l'incertitude dans les paramètres. Il serait donc intéressant d'étendre nos contributions en concevant des versions "adaptatives", spécialement pour les problèmes de suivi de trajectoire.

- A la fin de cette période de 3 années, Prof. Nijmeijer a attiré mon attention vers les systèmes chaotiques, en particulier, on a commencé par étudier l'équation de Duffing. Ce qui est intéressant à remarquer dans cette équation, est qu'elle modèle un système mécanique, perturbé par un signal périodique. Motivé par les articles récents (Nijmeijer and Berghuis, 1995), qui appliquent des idées utilisées auparavant dans la commande des robots manipulateurs, et de (Nijmeijer and Mareels, 1996), qui applique la théorie standard des observateurs non linéaires aux systèmes chaotiques, on croit que les résultats présentés dans cette thèse pourraient apporter des idées nouvelles au contrôle de certains systèmes chaotiques.

Part II

On Output Feedback Control of Euler-Lagrange Systems

Chapter 6

What is an Euler-Lagrange System?

The answer to this question is quite simple: An Euler-Lagrange system (in short EL) is a system whose motion is governed by the Euler-Lagrange equations. The logical question which arises next is What are the EL equations?. The answer to this question being far from simple is the subject of this first chapter. Even though these concepts are well known, in order to make this thesis self-contained we recall in sections 1 through 3, how the EL equations are derived from both, D'Alembert's and Hamilton's principles. These sections heavily borrow from (Goldstein, 1974). In section 4 we define the subclass of EL systems which we deal with throughout this thesis. Section 5 provides some applications of the EL equations and in particular, we derive the model of a two degrees of freedom planar manipulator, which we use in further chapters for simulation experiments. Finally, in section 6 we show some fundamental properties of EL systems that we will use throughout this document.

1 From Cartesian to generalized coordinates

Let r be the radius of a particle from some given origin and v its vector velocity:

$$v = \frac{dr}{dt}.$$

The linear momentum p of the particle is defined by the product of the particle mass and its velocity:

$$p = mv.$$

In consequence of interaction with external objects and fields, the particle may experience forces of various types, e.g. gravitational or electro-dynamic; after Newton's Second Law, the vector sum of these forces exerted on the particle is the total force F:

$$F = \frac{dp}{dt}.$$

When generalizing these ideas to a system of many particles we must distinguish between the external forces acting on the particles due to sources outside the system and internal forces among the particles of the system. Thus the equation of motion of the ith particle is

$$\sum_j F_{ji} + F_i^{(e)} = \dot{p}_i \tag{6.1}$$

where $F_i^{(e)}$ stands for the external forces and F_{ji} is the internal force on the ith particle due to the jth particle (hence $F_{ii} = 0$). After Newton's Law of Action and Reaction the sum $F_{ji} + F_{ij}$ is zero for all different i and j, hence the sum of all forces acting on the system yields

$$\frac{d^2}{dt^2}\sum_i m_i r_i = \sum_i F_i^{(e)}. \tag{6.2}$$

Equation (6.2) describes the dynamics of a free *unconstrained* system, nevertheless it may be necessary to take into account the constraints that limit the motion of the system.

Constraints can be classified in various ways. If the constraints can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$\Psi(r_1, r_2, \ldots, t) = 0,$$

then the constraints are called holonomic. Constraints which can not be expressed in this form are non holonomic.

Constraints introduce two types of difficulties in the solution of mechanical systems: First, the coordinates r_i are no longer independent, since they are connected by the constraints equations; hence the equations of motion (6.1) are not independent. Second, the forces of constraint are not known a priori.

In the case of holonomic constraints, the first difficulty is overcome by the introduction of *generalized coordinates*.

So far we have been thinking implicitly in terms of Cartesian coordinates. A system of N particles, free from constraints has $3N$ independent coordinates or *degrees of freedom*. If there exist k holonomic constraints then the system can be defined by $3N-k$ equations, hence the system has $3N-k$ degrees of freedom. This elimination of the dependent coordinates can be expressed by the introduction of new $3N-k$ independent variables $q_1, q_2, \ldots, q_{3N-k}$ in terms of which the old coordinates $r_1, r_2, \ldots, r_N$ are expressed by

$$\begin{aligned} r_1 &= r_1(q_1, q_2, \ldots, q_{3N-k}, t) \\ &\vdots \\ r_N &= r_N(q_1, q_2, \ldots, q_{3N-k}, t) \end{aligned}$$

containing the constraints in them implicitly. These are transformation equations from the set of (q_i) variables to the (r_i) set. It is always assumed that one can also transform back from the (r_i) set to the (q_i) set.

Usually the generalized coordinates q_i, unlike the Cartesian coordinates, will not divide into convenient groups of 3 that can be associated together to form vectors. For instance let us take the example of a particle constrained to move in the surface of a sphere, the two angles expressing position on the sphere, say latitude and longitude, are obvious possible generalized coordinates. Or in the example of a double pendulum with revolute joints moving in a plane, satisfactory generalized coordinates are the two joint angles.

Generalized coordinates in the sense other than Cartesian, are often used in systems without constraints. For instance in the problem of a particle moving in an external central force field – $V = V(r)$ – there is no constraint involved, but it is clearly more convenient to use spherical polar coordinates than Cartesian coordinates. One must not however, think of generalized coordinates in terms of conventional position coordinates. All sorts of quantities may be impressed to serve as generalized coordinates. Thus the amplitude in a Fourier expansion of r_j may be used as generalized coordinates, or we may find it convenient to employ quantities with dimensions of energy or angular momentum.

Finally to surmount the 2nd difficulty introduced by the constraints, namely that the forces of constraint are not known a priori, it is desirable to formulate the mechanics so that the forces of constraint disappear. We must then deal only with the known applied forces. This idea is developed in next section.

2 From D'Alembert's principle to Lagrange's equations

A virtual infinitesimal displacement of a system refers to a change in the configuration system as a result of any arbitrary infinitesimal change of coordinates δr_i, consistent with the forces and constraints imposed on the system at the given instant t.

Suppose that the system is in equilibrium, i.e., the total force of each particle vanishes, then clearly the dot product $F_i \cdot \delta r_i$, which is the virtual work of the force F_i in the displacement δr_i also vanishes. The sum over all particles also vanishes:

$$\sum_i F_i \cdot \delta r_i = 0. \tag{6.3}$$

Decompose F_i into the applied $F_i^{(a)}$ and the force of constraint f_i,

$$F_i = F_i^{(a)} + f_i$$

then (6.3) becomes,

$$\sum_i F_i^{(a)} \cdot \delta r_i + f_i \cdot \delta r_i = 0$$

now, supposing that the virtual work due to the forces of constraints is zero, we therefore have the condition for the equilibrium of a system is that the virtual work of applied forces vanishes

$$\sum_i F_i^{(a)} \cdot \delta r_i = 0. \tag{6.4}$$

Equation (6.4) is often called the *principle of virtual work*. However this equation deals only with systems which are in equilibrium and we want a condition that involves the general motion of a system. To obtain such a principle consider the equation of motion

$$F_i = \dot{p}_i$$

which can be written as

$$F_i - \dot{p}_i = 0$$

This equation states that the particles in the system will be in equilibrium under the force equal to the actual force plus a "reverse effective force" $-\dot{p}_i$. Thus we can write:

$$(F_i - \dot{p}_i) \cdot \delta r_i = 0,$$

considering again that the virtual work due to the force of constraint vanishes we obtain

$$\sum_i (F_i^{(a)} - \dot{p}_i) \cdot \delta r_i = 0. \tag{6.5}$$

This is *D'Alembert's principle.*

In (6.5) the virtual displacements are not independent due to the presence of constraints so it's not possible to conclude that each coefficient $F_i^{(a)} - \dot{p}_i$ individually equals zero. In order to apply such reasoning, we shall transform this principle into an expression that involves virtual displacements of the generalized coordinates which are independent of each other (in the case of holonomic constraints), so that the coefficients of the δq_i can be set separately equal to zero.

First, since the forces of constraint no longer appear we can drop the superscript $^{(a)}$. We recall that

$$r_i = r_i(q_1, q_2, \ldots, q_n, t)$$

so v_i is expressed in terms of the $\dot{q}_k$ by

$$v_i = \frac{dr_i}{dt} = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \tag{6.6}$$

similarly, the virtual displacement δr_i is connected to the displacement δq_i by

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j}\delta q_j. \tag{6.7}$$

It is worth noticing that there are no variations of time δt considered here since a virtual displacement considers only displacements of the coordinates.

In terms of the generalized coordinates the virtual work of the force F_i becomes

$$\sum_i F_i \cdot \delta r_i = \sum_j Q_j \delta q_j \tag{6.8}$$

where

$$Q_j \triangleq \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} \tag{6.9}$$

are called the components of *generalized forces*. It is worth remarking that Q_j does not have necessarily dimensions of force, just as q_j need not have dimensions of length; however, $Q_j \delta q_j$ must always have dimensions of work.

We next turn to the second term involved in (6.5) which may be written as

$$\sum_i \dot{p}_i \cdot \delta r_i = \sum_i m_i \ddot{r}_i \cdot \delta r_i,$$

expressing δr_i by (6.7), this becomes

$$\sum_{i,j} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j .$$

Consider now the relation

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt} \left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i \cdot \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right\}, \tag{6.10}$$

changing the order of differentiation with respect to t and q_j in the last term of (6.10) we get using (6.6)

$$\frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) = \sum_k \frac{\partial^2 r_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 r_i}{\partial q_j \partial t} = \frac{\partial \dot{r}_i}{\partial q_j} = \frac{\partial v_i}{\partial q_j}.$$

Further from (6.6) we see that

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}, \tag{6.11}$$

substituting this equality in (6.10) it yields

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left\{ \frac{d}{dt} \left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_j} \right\},$$

while the second term on the left hand side of (6.5) can be expanded into

$$\sum_j \left\{ \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right) - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} \delta q_j = 0.$$

Identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the system's *kinetic energy*, T, D'Alembert's principle becomes

$$\sum_j \left[\left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0 \tag{6.12}$$

considering that each coordinate q_j is independent, any virtual displacement δq_j is independent of δq_k and therefore the only way for (6.12) to hold is that the separate coefficients vanish, i.e.,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \qquad \forall\, j \in \underline{n}. \tag{6.13}$$

Equations (6.13) are often referred to as Lagrange's equations but this designation is frequently reserved for the form (6.13) when the forces are derivable from a scalar potential function V:

$$F_i = -\nabla_i V$$

in which case, the generalized forces are written

$$Q_j = -\frac{\partial V}{\partial q_j}. \tag{6.14}$$

Then equations (6.13) can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0, \tag{6.15}$$

and considering that V only depends on the generalized positions, we obtain

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \qquad \forall\, j \in \underline{n} \tag{6.16}$$

where $\mathcal{L} \triangleq T - V$ is called the *Lagrangian*. In the sequel, we will refer to equations (6.16) as the *Lagrange's* equations

It should be remarked that if not all the forces acting on the system are derivable from a potential then the Lagrange's equations can always be written in the form:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \qquad \forall\, j \in \underline{n} \tag{6.17}$$

where $\mathcal{L}$ contains the potential of the conservative forces, i.e. the potential of those forces which do not depend explicitly on time, and Q_j are forces which are not necessarily derived from a potential. Such a situation often occurs when there are frictional forces acting into the system. This kind of forces can be simply thought of proportional to the velocity of the particle so that its x component has the form[1]

$$F_{f_x} = -k_x v_x.$$

Frictional forces of this type may be derived in terms of a function $\mathcal{F}$ called *Rayleigh's dissipation function* defined as

$$\mathcal{F} \triangleq \frac{1}{2}\sum_i (k_x v_{x_i}^2 + k_y v_{y_i}^2 + k_z v_{z_i}^2), \tag{6.18}$$

where the summation is over all particles of the system. From this definition, it is clear that the forces of friction

$$F_f = -\nabla_v \mathcal{F}.$$

A physical interpretation of the dissipation function can be given in terms of the work done by the system against friction

$$dW_f = -F v dt = (k_x v_{x_i}^2 + k_y v_{y_i}^2 + k_z v_{z_i}^2)dt.$$

Hence $2\mathcal{F}$ is the rate of energy dissipation due to friction. The component of the generalized forces resulting from the force of friction is given by

$$Q_j = \sum_j F_{f_i} \cdot \frac{\partial r_i}{\partial q_j} = -\sum \nabla_v \mathcal{F} \cdot \frac{\partial r_i}{\partial q_j} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

where we have used (6.11). Thus we can write the Lagrange's equations as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0. \tag{6.19}$$

Equations (6.19) model the dynamics of a system affected only by dissipative forces, however we may also consider systems affected by other external inputs, $u \in \mathbb{R}^m$ $(m \leq n)$:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = u. \tag{6.20}$$

[1]For a new dynamic model of friction effects see (Canudas de Wit et al., 1995).

3 From Hamilton's principle to Lagrange's equations

We have shown how to derive the Lagrangian equations from a differential principle, in particular, from D'Alembert's principle. Nevertheless it is also possible to obtain Lagrange's equations from a principle that considers the entire motion of the system between two times t_1 and t_2 and small virtual variations of the entire motion from the actual motion. Such a principle is called "integral principle".

First, let us clarify the meaning of "motion of the system between times t_1 and t_2". The instantaneous configuration of the system is described by the values of the generalized coordinates $(q_1, \ldots, q_n)$ and corresponds to a particular point in the Cartesian hyperspace where the q's form the n coordinate axes. As time goes on, the state of the system changes and the system point changes its position in the configuration space, hence tracing out a curve which describes the "path of motion of the system" from the configuration $(q_1, \ldots, q_n, t_1)$ to the configuration $(q_1, \ldots, q_n, t_2)$. Thus the "motion of the system" as used above, refers to the motion of the system in the *configuration space.*

The integral Hamilton's principle describes the motion of a system for which all the forces (except the force of constraint) can be derived from a generalized scalar potential that may be a function of the coordinates, velocities and time.

In this work we consider only systems whose potential function depends explicitly on position coordinates only. For this kind of systems the Hamilton's principle states that the motion of the system from time t_1 to time t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} \mathcal{L} dt \tag{6.21}$$

where $\mathcal{L}$ is the Lagrangian, has a stationary value for the correct path of motion. That is, among all possible paths the system will travel from its configuration at time t_1 to its configuration at time t_2 along the path for which the integral (6.21) is stationary.

In summary, Hamilton's principle says that the motion is such that the variation of the integral I for fixed t_1 and t_2 is zero:

$$\delta I = \delta \int_{t_1}^{t_2} \mathcal{L}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) dt = 0. \tag{6.22}$$

When the system's constraints are holonomic it can be shown that Hamilton's principle is a necessary and sufficient condition for Lagrange's equations (6.16). See (Goldstein, 1974; Whitaker, 1937).

The proof of the sufficient part uses the techniques of the calculus of variations. Consider the problem of finding the curve for which the integral value is stationary, firstly in the one-dimensional form: let $f(y, \dot{y}, x)$ be a function defined on the path $y = y(x)$ between two values x_1 and x_2, where $\dot{y}$ is the derivative of y with respect to x. We wish to find a particular path $y(x)$ such that the integral

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x) dx \tag{6.23}$$

has a stationary value referring to the paths differing infinitesimally from the correct function $y(x)$. Such a set of paths might be denoted by $y(x, \alpha)$ where $y(x, 0)$ represents the correct path. For any such family of curves, J is also a function of α:

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x, \alpha), \dot{y}(x, \alpha), x) dx$$

and the condition for obtaining a stationary point is

$$\left(\frac{dJ}{d\alpha} \right)_{\alpha=0} = 0. \tag{6.24}$$

In (Goldstein, 1974) it is shown that (6.24) is equivalent to

$$\delta J = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}} \right\} \delta y \; dx = 0 \tag{6.25}$$

requiring that $y(x)$ satisfy the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0.$$

This problem is generalized to the case where f is a function of many independent variables $y_i(x)$. In such case (6.25) becomes

$$\delta J = \sum_{i=1}^{n} \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} \right\} \delta y_i dx = 0, \qquad \forall\, i \in \underline{n}$$

and since the y_i variables are independent, the variations δy_i are independent, hence the condition that δJ is zero requires that the coefficients of the δy_i vanish separately, i.e.,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} = 0, \qquad \forall\, i \in \underline{n} \tag{6.26}$$

Equations (6.26) are known as the *Euler-Lagrange* differential equations.

When considering the integral of Hamilton's principle I as in (6.21) in this problem of the calculus of variations, then y_i corresponds to q_i, x to the time variable t and $f(y_i, \dot{y}_i, x)$ to the Lagrangian $L(q_i, \dot{q}_i, t)$, hence the Euler-Lagrange equations become the Lagrange's equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0, \qquad \forall\, i \in \underline{n}$$

When derived from Hamilton's principle, the Lagrangian equations are often referred to in the literature as Euler-Lagrange equations.

In this manner the Lagrangian and Hamilton's principle together form a compact invariant way of implying the mechanical equations of motion. The interest of the Lagrangian formulation is that this possibility is not reserved for mechanics only; in almost every field of variational physics, variational principles can be used to express the equations of motion, whether they be Newton's equations, Maxwell's equations or Schrödinger equation. Consequently when a variational principle is used as the basis of the formulation, all such fields will exhibit at least to some degree, a structural analogy.

4 A subclass of Euler Lagrange systems

We define in this section the subclass of EL systems, which we focus on in the present thesis. Our class is characterized by a list of properties defined upon a "Lagrangian formulation". For an excellent reference on the definition of *Hamiltonian* systems see (Crouch et al., 1995).

4.1 The Lagrangian formulation

From the discussion of previous sections, it comes natural to characterize Euler-Lagrange systems by the EL parameters:

$$\{T(q, \dot{q}), V(q), \mathcal{F}(\dot{q}), M\} \tag{6.27}$$

where $q \in \mathbb{R}^n$ are the generalized coordinates and n corresponds to the number of degrees of freedom of the system. In this work we focus our attention in EL systems whose kinetic energy function is of the form

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} \tag{6.28}$$

where the inertia matrix $D(q)$ satisfies $D(q) = D^\top(q) > 0$. Next, $V(q)$ represents the potential energy which is assumed to be bounded from below that is, there exists a $c \in \mathbb{R}$ such that $V(q) > c$ for all $q \in \mathbb{R}^n$, $\mathcal{F}(\dot{q})$ is the Rayleigh's dissipation function (6.18) and the last EL parameter, M, is a full column rank matrix mapping the external inputs to the generalized coordinates.

In this thesis we assume that the external forces, $Q \in \mathbb{R}^n$, are composed only of potential forces[2] $Mu \in \mathbb{R}^n$ and dissipative forces $-\frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}$, hence

$$Q = Mu - \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}. \tag{6.29}$$

We distinguish two classes of EL systems according to the structure of matrix M:

6.1 Definition. *An EL system is* fully-actuated *if* $M = I_n$. *Also, we say that an EL system is* underactuated *if*

$$M = [0 \mid I_m]^\top, \qquad m < n. \tag{6.30}$$

In this case, q can be partitioned into non-actuated and actuated components[3] respectively

$$q_1 \triangleq M^\perp q = \left[\begin{array}{c|c} I_{n-m} & 0 \end{array}\right] q, \quad q_2 \triangleq Mq. \tag{6.31}$$

We find it convenient at this point, to partition the vector q as $q = \mathrm{col}[q_p \; q_c]$ where we call q_p, the undamped coordinates and q_c, the damped ones. With this notation we can distinguish two classes of systems:

6.2 Definition. *An EL system with parameters (6.27) is said to be a* fully damped *EL system if* $(\alpha > 0)$

$$\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \geq \alpha \left\|\dot{q}\right\|^2. \tag{6.32}$$

An EL system is underdamped *if*

$$\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \geq \alpha \|\dot{q}_c\|^2. \tag{6.33}$$

6.3 Remark. When designing a controller for an EL plant, we will assume that the actuated variables are available for measurement, whereas the non-actuated ones correspond to those variables that we want to control (regulated coordinates).

Now we are ready to derive a dynamic model for a subclass of EL systems with quadratic kinetic energy and potential energy which depends only on the generalized position q. Let us evaluate the left hand side of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = Q. \tag{6.34}$$

First, we express the kinetic energy as

$$T(q,\dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j \tag{6.35}$$

and we define the vector

$$g_k(q) \triangleq \frac{\partial V(q)}{\partial q_k}, \tag{6.36}$$

hence the kth Lagrangian equation is written

$$\begin{aligned} &\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}\right)\frac{1}{2}\sum_{i,j=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j + \frac{\partial V(q)}{\partial q_k} = \\ &\quad = \frac{d}{dt}\sum_{i=1}^{n} d_{ik}(q)\dot{q}_i - \frac{1}{2}\sum_{i,k=1}^{n}\frac{\partial d_{ij}(q)}{\partial q_k}\dot{q}_i\dot{q}_j + g_k(q) \end{aligned}$$

[2] Forces derived from a time-invariant potential $V(q)$.

[3] That is, generalized coordinates whose corresponding row in the input matrix contains a zero (resp. nonzero) entry M.

$$
\begin{aligned}
&= \sum_{i=1}^{n} d_{ik}(q)\ddot{q}_i + \sum_{i,j=1}^{n} \left(\frac{\partial d_{ik}(q)}{\partial q_j} - \frac{d_{i,j}(q)}{q_k} \right) \dot{q}_i \dot{q}_j + g_k(q) \\
&= \sum_{i=1}^{n} d_{ik}(q)\ddot{q}_i + \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial d_{ik}(q)}{\partial q_j} + \frac{\partial d_{jk}(q)}{\partial q_i} - \frac{\partial d_{ij}(q)}{\partial q_k} \right) \dot{q}_i \dot{q}_j + g_k(q) \\
&= \sum_{i=1}^{n} d_{ik}(q)\ddot{q}_i + \sum_{i,j=1}^{n} c_{ijk}(q) \dot{q}_i \dot{q}_j + g_k(q)
\end{aligned}
$$

where

$$
c_{ijk}(q) \triangleq \frac{1}{2} \left(\frac{\partial d_{ik}(q)}{\partial q_j} + \frac{\partial d_{jk}(q)}{\partial q_i} - \frac{\partial d_{ij}(q)}{\partial q_k} \right)
$$

are the so called Christoffel symbols of the first kind. Then the Lagrangian equations can be written in the equivalent form (Spong and Vidyasagar, 1989)

$$
D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) + \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} = Mu \tag{6.37}
$$

where the matrix $C(q,\dot{q})$ is called the "Coriolis and centrifugal forces" matrix and its kjth entry is given by

$$
C_{kj}(q,\dot{q}) = \sum_{i}^{n} c_{ijk}(q)\dot{q}_i. \tag{6.38}
$$

In this work we focus our attention on those systems for which the following properties hold:

P 6.1 The matrix $D(q)$ is positive definite, and the matrix $N = \dot{D}(q) - 2C(q,\dot{q})$ is skew symmetric. Moreover, there exist some positive constants d_m and d_M such that

$$
d_m I < D(q) < d_M I \tag{6.39}
$$

P 6.2 There exists some positive constants k_g and k_v such that

$$
k_g \geq \sup_{q \in \mathbb{R}^n} \left\| \frac{\partial^2 V(q)}{\partial q^2} \right\|, \quad \forall\, q \in \mathbb{R}^n \tag{6.40}
$$

$$
k_v \geq \sup_{q \in \mathbb{R}^n} \left\| \frac{\partial V(q)}{\partial q} \right\|, \quad \forall\, q \in \mathbb{R}^n \tag{6.41}
$$

P 6.3 The matrix $C(x,y)$ is bounded in x. Moreover, it is easy to see from (6.38) that $C(x,y)$ is linear in y, then for all $z \in \mathbb{R}^n$

$$
C(x,y)z = C(x,z)y \tag{6.42}
$$

$$
C(x,y) \leq k_c \|y\|, \quad k_c > 0. \tag{6.43}
$$

As a matter of fact, the skew-symmetry property of $\dot{D}(q) - 2C(q,\dot{q})$, as well as (6.42) are direct consequences of the definition of $C(q,\dot{q})$. Also, inequality (6.43) follows using (6.39) and the definition of Christoffel symbols. See for instance (Spong and Vidyasagar, 1989; Ortega and Spong, 1989; Stepanenko and Yuan, 1992).

6.1 Remarque. It is worth remarking that the subclass of EL plants satisfying property **P6.2** includes the robot manipulators (Tomei, 1991b).

4.2 The Hamiltonian formulation

In the Lagrangian formulation a system with n degrees of freedom possesses n 2nd order equations of motion of the form (6.17). As the equations are of second order, $2n$ initial conditions must be specified.

In the Hamiltonian formulation we seek to describe the motion in terms of first order differential equations. Since the number of initial conditions must be $2n$ then there must be $2n$ differential equations in terms of $2n$ independent variables. A choice of these variables is the n generalized coordinates q_i and the generalized conjugate momenta p_i:

$$p_i = \frac{\partial \mathcal{L}(q_j, \dot{q}_j)}{\partial \dot{q}_j} \tag{6.44}$$

where $\mathcal{L}$ satisfies the Legendre condition, then the quantities (q,p) are known as the *canonical variables.*

Treated strictly as a mathematical problem, the transition from the Lagrangian to the Hamiltonian formulation corresponds to the change of variables from $(q,\dot{q})$ to (q,p). Such transformation is provided by the Legendre transformation of the so called Hamiltonian $H(q,p)$:

$$H(q,p) = \dot{q}_i p_i - \mathcal{L}(q,\dot{q}). \tag{6.45}$$

Considered as a function of q, and p only, the differential of H is given by

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \tag{6.46}$$

and using (6.45) we can write

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i \tag{6.47}$$

where the terms in $d\dot{q}_i$ cancel by virtue of (6.44) and from Lagrange's equations it follows that

$$\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i$$

hence we get

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i$$

comparing these equations to (6.46) we have that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{6.48}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}. \tag{6.49}$$

Equations (6.48) – (6.49) are known as the *canonical equations of Hamilton* and they constitute the desired set of $2n$ first order differential equations replacing Lagrange's equations.

It is important to remark that in the case of physical systems whose generalized coordinates do not depend explicitly on time and are affected only by forces derivable from a conservative potential V, the Hamiltonian (6.45) is exactly the total energy of the system:

$$H = T + V = E.$$

It has been mentioned that Lagrange's equations can be derived from a variational principle, namely from Hamilton's Principle. The advantage of this derivation is that Hamilton's principle can be applied to other problems than those arising in mechanics, hence it is desirable as well, to derive the Hamilton's equations from a variational principle.

With this purpose, let us apply the Hamilton Principle in the following way: consider the integrand in the action integral (6.22) defined as a function of (q,p) only:

$$\delta I = \int_{t_1}^{t_2} [p_j \dot{q}_j - H(q,p)]dt = 0. \tag{6.50}$$

Equation (6.50) is often called the *modified Hamilton's principle.* For this principle the Euler-Lagrange equations (6.26) are written

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{q}_j} - \frac{\partial f}{\partial q_j} = 0, \qquad j \in \underline{n} \tag{6.51}$$

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{p}_j} - \frac{\partial f}{\partial p_j} = 0, \qquad j \in \underline{n} \tag{6.52}$$

$$\tag{6.53}$$

where we defined $f \triangleq p_j\dot{q}_j - H(q,p)$ hence we can write (6.51) – (6.52) directly as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}$$

which are the Hamiltonian equations of motion for systems affected only by forces derivable from a conservative potential.

5 Some applications of the Lagrangian formulation

We have shown that, for systems where one can define a Lagrangian, i.e. holonomic systems with applied forces derivable from an ordinary or generalized potential and workless constraints, there is a very convenient way of setting up the equations of motion. One has simply to write T and V in generalized coordinates and use (6.17) to derive the equations of motion. In this section we give some examples of dynamical systems whose equations of motion can be derived, using the EL equations, from their corresponding EL parameters.

5.1 Robot manipulators

Throughout this thesis we put particular interest into the robot manipulators as an example of Lagrangian systems. In this section we present the n degrees of freedom manipulator model that we use in the sequel, to illustrate our contributions in simulations. For the sake of illustration we derive firstly, the kinetic and potential energy of a two degrees of freedom planar manipulator.

Example 1: Two-revolute-joint planar manipulator

Consider the manipulator depicted in figure 6.1.

In order to derive the EL parameters of this system, we first notice that the kinetic energy is composed of mainly two parts: $T \triangleq T_v + T_w$ where we denote

$$T_v = \frac{1}{2}\left(m_1\|v_1\|^2 + m_2\|v_2\|^2\right), \tag{6.54}$$

the kinetic energy due to the linear velocities, v_1 and v_2, of the centres of mass of each link, and

$$T_w = \frac{1}{2}\left(I_1\|\omega_1\|^2 + I_2\|\omega_2\|^2\right), \tag{6.55}$$

is the part of the kinetic energy due to the angular velocities, ω_1 and ω_2.

We would like to find an expression of the kinetic energy as a function of the generalized coordinates q and the generalized velocities $\dot{q}$, i.e., $T \triangleq T(q,\dot{q})$. For this, we write the linear and angular velocities as

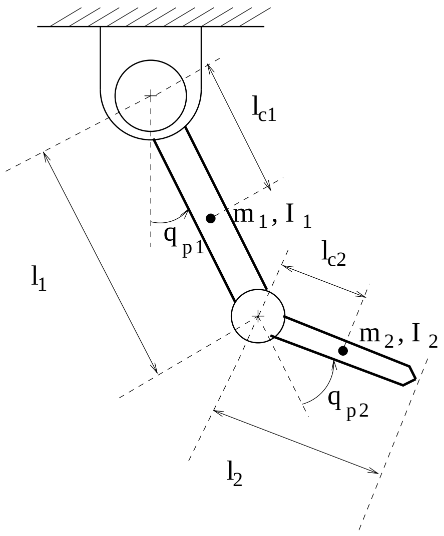


Figure 6.1: Two-revolute-joint planar manipulator

functions of q and $\dot{q}$ (Spong and Vidyasagar, 1989):

$$v_1 = \begin{bmatrix} -l_{c_1}\sin(q_1) & 0 \\ l_{c_1}\cos(q_1) & 0 \end{bmatrix} \dot{q} \tag{6.56}$$

$$v_2 = \begin{bmatrix} -l_1\sin(q_1) - l_{c_2}\sin(q_1+q_2) & -l_{c_2}\sin(q_1+q_2) \\ l_1\cos(q_1) + l_{c_2}\cos(q_1+q_2) & l_{c_2}\cos(q_1+q_2) \end{bmatrix} \dot{q} \tag{6.57}$$

$$\omega_1 = \dot{q}_1 \bar{k} \tag{6.58}$$

$$\omega_2 = (\dot{q}_2 + \dot{q}_1)\bar{k} \tag{6.59}$$

where $\bar{k}$ is the unitary vector in the direction perpendicular to this page. Substituting (6.56) – (6.59) in (6.54), (6.55) we obtain after some calculations

$$T(q,\dot{q}) = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^\top \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{6.60}$$

where

$$\begin{aligned} d_{11} &= m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2}\cos(q_2)) + I_1 + I_2 \\ d_{12} = d_{21} &= m_2(l_{c_2}^2 + l_1 l_{c_2}\cos(q_2)) + I_2 \\ d_{22} &= m_2 l_{c_2}^2 + I_2. \end{aligned}$$

Next, the potential energy is $V = V_1 + V_2$, where V_1 and V_2 are respectively the potential energy of the first and second link, given by

$$\begin{aligned} V_1 &= m_1 \mathrm{g} l_{c_1}\cos(q_1) \\ V_2 &= m_2 \mathrm{g}(l_1 + l_2 - l_1\cos(q_1) - l_2\cos(q_1+q_2)). \end{aligned}$$

We find it convenient to this point, to introduce the model of two degrees of freedom manipulator of (Kruise, 1990; Berghuis, 1993) that we use in this thesis in order to illustrate our contributions in simulations. For this robot, the EL parameters are (with zero payload)

$$T(q_p,\dot{q}_p) \triangleq \begin{bmatrix} \dot{q}_{p_1} \\ \dot{q}_{p_2} \end{bmatrix}^\top \begin{bmatrix} 1.02\cos(q_{p_2}) + 8.77 & 0.76 + 0.51\cos(q_{p_2}) \\ 0.76 + 0.51\cos(q_{p_2}) & 0.62 \end{bmatrix} \begin{bmatrix} \dot{q}_{p_1} \\ \dot{q}_{p_2} \end{bmatrix} \tag{6.61}$$

$$V_p(q_p) \triangleq 9.81(7.6\sin(q_{p_1}) + 0.63\cos(q_{p_1} + q_{p_2})) \tag{6.62}$$

$$\mathcal{F}_p(\dot{q}_p) = 0 \tag{6.63}$$

Moreover the bounds mentioned in properties **P6.1** – **P6.2** for this particular case are (assuming a payload of $m_l \leq 2$[kg])

$$d_m = 1,\ d_M = 25,\ k_c = 6,\ k_g < 10,\ k_v = 135. \tag{6.64}$$

Example 2: Flexible joints robots

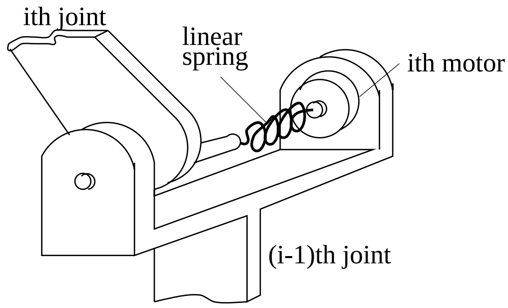


Figure 6.2: Ideal model of a flexible joint.

Some robotic manipulators present flexibility in the joints, for instance, manipulators where harmonic drives, elastic bands, or motors with long shafts are used. The joint flexibility phenomenon can be modeled by a linear spring (Marino and Nicosia, 1984; Marino and Nicosia, 1985; Spong, 1987; Burkov and Zaremba, 1987) as it is illustrated in figure 6.2.

Flexible joint robots are underactuated EL systems with generalized coordinates $q_p \triangleq \mathrm{col}[q_{p_1}, q_{p_2}]$, $q_{p_1}, q_{p_2} \in \mathbb{R}^{\frac{n_p}{2}}$ being the link and motor shaft angles respectively. The control variables are the torques at the shafts, thus $m_p = \frac{n_p}{2}$ and $M_p \triangleq [0 \mid I_{m_p}]^\top$.

The *kinetic and potential energies* of a flexible joint robot are given by[4]

$$T_p(q_{p_1}, \dot{q}_p) \triangleq \frac{1}{2}\dot{q}_p^\top D_p(q_{p_1})\dot{q}_p,\ V_p(q_p) \triangleq \frac{1}{2}q_p^\top \mathcal{K}_p q_p + V_g(q_{p_1}) \tag{6.65}$$

where

$$\mathcal{K}_p \triangleq \begin{bmatrix} K & -K \\ -K & K \end{bmatrix},\ D_p(q_{p_1}) \triangleq \begin{bmatrix} D_{11}(q_{p_1}) & D_{12}(q_{p_1}) \\ D_{12}^\top(q_{p_1}) & J \end{bmatrix} \tag{6.66}$$

with $D_{12}(q_{p_1})$ of the form

$$D_{12}(q_{p_1}) = \begin{bmatrix} 0 & d_{12}(q_{p_{1,1}}) & d_{13}(q_{p_{1,1}}, q_{p_{1,2}}) & \cdots & d_{1m_p}(q_{p_{1,1}}, q_{p_{1,m_p-1}}) \\ 0 & 0 & d_{23}(q_{p_{1,2}}) & \cdots & d_{2m_p}(q_{p_{1,2}}, q_{p_{1,m_p-1}}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{6.67}$$

and $q_{p_{1,j}}$, for $j \in [1..m_p]$, is the j-th component of the vector q_{p_1}. $D_p(q_{p_1}) = D_p^\top(q_{p_1}) > 0$ is the robot inertia matrix, $J \in \mathbb{R}^{m_p \times m_p}$ is a diagonal matrix of actuator inertias reflected to the link side, K is a diagonal matrix containing the joint stiffness coefficients, and $V_g(q_{p_1})$ is the potential energy due to the gravitational forces.

Assuming zero internal damping, that is, $\mathcal{F}_p(\dot{q}_p) = 0$ we get the *dynamic equations* of the flexible joint robot:

$$D_p(q_{p_1})\ddot{q}_p + C_p(q_{p_1}, \dot{q}_p)\dot{q}_p + g_p(q_{p_1}) + \mathcal{K}_p q_p = M_p u \tag{6.68}$$

[4] For further details on this model see, e.g., (Tomei, 1991b). Notice that the model we consider here contains, as a particular case, the model of (Spong, 1987; Burkov and Zaremba, 1987) where $D_p(q_{p_1})$ is assumed to be block diagonal.

where $g_p(q_{p_1}) \triangleq [g_{p1}^\top(q_{p_1}), 0]^\top = \frac{\partial V_g(q_{p_1})}{\partial q_{p_1}}$. Defined by the Christoffel symbols of the first kind, the Coriolis matrix has the structure:

$$C_p(q_{p_1}, \dot{q}_p) \triangleq \begin{bmatrix} C_{11}(q_{p_1}, \dot{q}_{p_1}) + C'_{11}(q_{p_1}, \dot{q}_{p_2}) & C_{12}(q_{p_1}, \dot{q}_{p_1}) \\ C_{21}(q_{p_1}, \dot{q}_{p_1}) & 0 \end{bmatrix}, \tag{6.69}$$

$$(C_{21})_{i,j}(q_{p_1}, \dot{q}_p) \triangleq \frac{1}{2}\left[\dot{q}_{p_1}^\top \frac{\partial (D_{12})_{j,i}}{\partial q_{p_1}} + \frac{\partial (D_{12})^i}{\partial \dot{q}_{p_1}} \dot{q}_{p_1}\right] \tag{6.70}$$

where $(\cdot)_{i,j}$, $(\cdot)^i$ denote the (i,j)-th term and i-th row of a matrix respectively. When the angular part of the kinetic energy of each rotor can be considered due only to its own rotation then we obtain the simplified model of (Spong, 1987)

$$\begin{cases} D_l(q_{p_1})\ddot{q}_{p_1} + C(q_{p_1}, \dot{q}_{p_1})\dot{q}_{p_1} + g(q_{p_1}) = K(q_{p_2} - q_{p_1}) \\ J\ddot{q}_{p_2} + K(q_{p_2} - q_{p_1}) = u_p. \end{cases} \tag{6.71}$$

In the case where flexibility is negligible ($K \to \infty$) it is shown in (Spong, 1987) that the model (6.71) reduces to

$$D(q_{p_1})\ddot{q}_{p_1} + C(q_{p_1}, \dot{q}_{p_1})\dot{q}_{p_1} + g(q_{p_1}) = u_p \tag{6.72}$$

where $D(q_{p_1}) \triangleq D_l(q_{p_1}) + J$.

5.2 A single dynamic model for electric and mechanic systems

Let us now consider a system for which the EL parameters are (Goldstein, 1974)

$$T(\dot{q}) = \frac{1}{2}\sum_j L_j \dot{q}_j^2 + \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k, \tag{6.73}$$

$$V(q) = -\sum_j \frac{q_j^2}{2C_j}, \tag{6.74}$$

$$F(\dot{q}) = \frac{1}{2}\sum_j R_j \dot{q}_j^2, \tag{6.75}$$

$$M = I_3. \tag{6.76}$$

Suppose that this system is subject to some external time varying forces $Q_j = E_j(t)$ then the Lagrangian equations (6.19) are written

$$L_j\ddot{q}_j + \sum_k M_{jk}\ddot{q}_k + R_j\dot{q}_j + \frac{q_j}{C_j} = E(t), \qquad j \neq k. \tag{6.77}$$

These dynamic equations can be interpreted in at least two ways: they can model either a coupled RC network or a mass spring system with dampers.

Example 3: Coupled RC networks

One can say that q's are charges, the L_j's self-inductances, the M_{jk}'s mutual inductances, the R_j's resistances, the C_j's capacities and the $E(t)$ external voltage sources. Then the equations (6.77) describe a system of mutually coupled networks. For instance for $j = 1, 2, 3$ we would have a system like illustrated in figure 6.3.

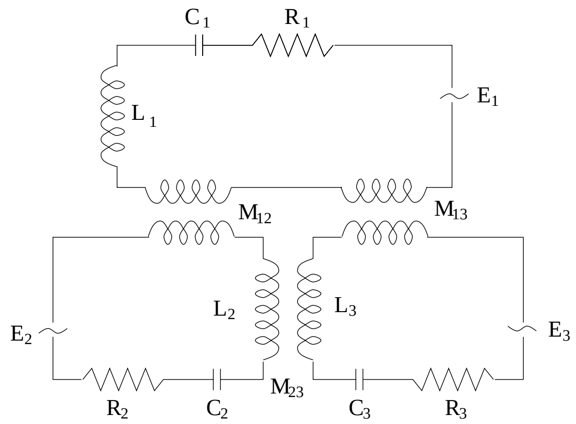


Figure 6.3: A system of coupled circuits to which the Lagrangian formulation can be applied. (Goldstein 1980)

Example 4: Mass spring system with damper

On the other hand, it is also seen that T defined by (6.73) is a homogeneous quadratic function of the generalized velocities. The coefficients L_j and M_{jk} in (6.73)–(6.75) can also been seen as masses, hence they can be inertial terms. The potential energy can be thought of the corresponding to a set of springs which satisfy Hooke's law,

$$F = -kx$$

with $k_j = i/C_j$. The dissipation function F corresponds to the existence of dissipative or viscous forces proportional to the generalized velocities.

Finally the matrix $M = I_3$ indicates in both cases that the system is fully actuated, hence that the external forces $E(t)$ affect each generalized coordinate q_j.

6.4 Remark. See also (Sira-Ramirez et al., 1996) for a Lagrangian approach to modeling of switch-regulated DC-to-DC power converters.

6 Properties of Euler-Lagrange systems

In this section we will present some fundamental properties of EL systems[5] which, for clarity of exposition we have classified into "input-output" and "internal-stability" properties.

6.1 Input–Output properties

It is well known now that EL systems have some nice *energy dissipation properties*. In particular we have the following:

6.5 Proposition. *(Passivity)*[6] *An EL system defines* a passive operator *from the inputs u to the actuated generalized velocities $M^\top \dot{q}$. That is, there exists $\beta \in I\!R$ such that*

$$\langle u \mid M^\top \dot{q} \rangle \ \geq \beta \tag{6.78}$$

for all $u \in \mathcal{L}_{2e}^m$. Further, this property is strengthened to output strict passivity *(OSP) if the Rayleigh's dissipation function defines an input strictly passive (ISP) operator. In this case if*

$$\langle u \mid M^\top \dot{q} \rangle \ \geq \alpha \|M^\top \dot{q}\|_2^2 + \beta \tag{6.79}$$

for some $\alpha > 0$, $\beta \in I\!R$ and all $u \in \mathcal{L}_{2e}^m$. □

[5] In the following, by "EL systems" we mean the particular subclass of EL systems defined in section 4.1

[6] See appendix A for definitions

Proof. The property can be easily established taking the time derivative of the Lagrangian function $\mathcal{L}(q,\dot q)$

$$\frac{d\mathcal{L}}{dt} = \sum_j \frac{\partial \mathcal{L}}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial \mathcal{L}}{\partial \dot q_j}\frac{d\dot q_j}{dt} \tag{6.80}$$

and using the EL equations (6.17) to write

$$\frac{\partial \mathcal{L}}{\partial q_j} = \frac{d\mathcal{L}}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_j}\right) - Q_j,$$

so (6.80) can be rewritten as

$$\frac{d\mathcal{L}}{dt} = \sum_j \frac{\partial \mathcal{L}}{\partial \dot q_j}\frac{d\dot q_j}{dt} + \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_j}\right)\dot q_j - \sum_j \dot q_j Q_j$$

then, reordering the terms above and using (6.29), it follows that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot q)}{\partial \dot q}\dot q - \mathcal{L}(q,\dot q)\right) = \dot q^\top \left(Mu - \frac{\partial \mathcal{F}(\dot q)}{\partial \dot q}\right).$$

Now, notice that the term in parenthesis on the left hand side coincides with the systems total energy $H(q,\dot q) \triangleq T(q,\dot q) + V(q)$, integrating from 0 to t we establish the key energy balance equation

$$\underbrace{H[q(t),\dot q(t)] - H[q(0),\dot q(0)]}_{stored\ energy} + \underbrace{\int_0^t \dot q^\top \frac{\partial \mathcal{F}(\dot q)}{\partial \dot q} ds}_{dissipated} = \underbrace{\int_0^t \dot q^\top M u ds}_{supplied}. \tag{6.81}$$

In words, the energy balance equation (6.81) states that a passive system cannot dissipate more energy than it is supplied. Now, we use definition A.4 and observe that, since $V(q)$ is bounded from below and the Rayleigh dissipation function meets (6.33) there exists a $\beta \in \mathbb{R}$ such that (6.79) holds. If no dissipation is present into the system (i.e. $\mathcal{F}(\dot q) = 0$) notice that (6.78) is also satisfied. ■

Proposition 6.5 above is important in the sense that it states that the operator $u \mapsto M^\top \dot q$ may be output strictly passive even if energy is *not dissipated "in all directions"*. Namely, it is enough to insure $\dot q^\top \frac{\partial \mathcal{F}(\dot q)}{\partial \dot q} \geq \alpha \|M^\top \dot q\|^2$. This feature will be exploited in the next chapter to achieve partial damping injection for global asymptotic stabilization.

6.2 Lyapunov stability properties

In this section, we stress other properties of Euler-Lagrange systems which are related to the stability in the sense of Lyapunov. For the sake of clarity, we distinguish two classes of EL systems: Fully-damped and underdamped systems.

6.2.1 Fully-Damped systems

The proposition below establishes conditions for *internal stability* of fully damped EL systems.

6.6 Proposition. (GAS with full damping) *The equilibria of a fully damped free EL system i.e., with $u = 0$ are $(q,\dot q) = (\bar q, 0)$ where $\bar q$ is the solution of*

$$\frac{\partial V(q)}{\partial q} = 0. \tag{6.82}$$

The equilibrium is unique and stable *if it is a global and unique minimum of the potential energy function $V(q)$ and V is proper, for instance if V satisfies the conditions of lemma B.7. Further, this equilibrium is globally asymptotically (GAS) stable if the Rayleigh dissipation function is* input strictly passive. □

Proof. The existence of the equilibrium follows immediately writing the EL equations with $u = 0$ as

$$D(q)\ddot{q} + \dot{D}(q)\dot{q} - \frac{\partial T(q,\dot{q})}{\partial q} + \frac{\partial V(q)}{\partial q} + \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} = 0$$

which as shown in section 4.1, can be written in the equivalent form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V(q)}{\partial q} + \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} = 0 \tag{6.83}$$

noting that $\frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}|_{\dot{q}=0} = 0$ it follows that the equilibria $\bar{q}$ are the solutions of (6.82).

The proof of stability can be established invoking Lyapunov's second method (Arnold, 1989): taking the time derivative of the Lyapunov function candidate $\mathcal{V} = H(q,\dot{q}) - V(\bar{q})$, it yields using **P6.1**

$$\dot{\mathcal{V}} = -\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}.$$

Since $\mathcal{F}(\dot{q})$ is a Rayleigh dissipation function then $\dot{\mathcal{V}} \le 0$, hence the equilibrium $q = \bar{q}$ is stable. Moreover, since the Rayleigh dissipation function defines an ISP operator $\dot{q} \mapsto \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}$ then there exists $\alpha > 0$ such that

$$\dot{\mathcal{V}} \le -\alpha \|\dot{q}\|^2,$$

asymptotic stability follows invoking Krasovskii-LaSalle's principle and observing that $\dot{\mathcal{V}} \equiv 0$ if and only if $\dot{q} = 0$, then from (6.83) it follows that the largest invariant set is the unique equilibrium $q = \bar{q}$. ∎

6.2.2 Underdamped Systems

As far as we know, the first paper which establishes sufficient conditions for asymptotic stability of underdamped *Lagrangian* systems is 35 years old (Pozharitskii, 1961). In the proposition below we show that *global* asymptotic stability of a unique equilibrium point can still be insured even when energy is not dissipated "in all directions" provided the inertia matrix $D(q)$ has a certain block diagonal structure, and the dissipation is suitably propagated.

Thus, to distinguish the *damped* and *undamped* coordinates we introduce the following partition of q:

$$q_c \triangleq [0 \mid I_{n_c}]q, \quad q_p \triangleq [I_{n_p} \mid 0]q, \quad n = n_p + n_c,$$

for these systems we enunciate the following

6.7 Proposition. (GAS with partial damping) *The equilibrium $(\dot{q},q) = (0,\bar{q})$ of a free ($u = 0$) underdamped EL system is GAS if the potential energy function is proper and has a global and unique minimum at $q = \bar{q}$, and if*

(i) $D(q) \triangleq \begin{bmatrix} D_p(q_p) & 0 \\ 0 & D_c(q_c) \end{bmatrix}$, *where* $D_c(q_c) \in I\!R^{n_c \times n_c}$,

(ii) $\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \ge \alpha \|\dot{q}_c\|^2$ *for some* $\alpha > 0$,

(iii) For each q_c, the function $\frac{\partial V(q)}{\partial q_c} = 0$ has only isolated zeros *in q_p.* □

Proof. We proceed as in proposition 6.6 writing the EL equations (6.34) with $u = 0$ by exploiting the block diagonal structure of $D(q)$, in the form

$$D_p(q_p)\ddot{q}_p + C_p(q_p,\dot{q}_p)\dot{q}_p + \frac{\partial V(q)}{\partial q_p} = 0 \tag{6.84}$$

$$D_c(q_c)\ddot{q}_c + C_c(q_c,\dot{q}_c)\dot{q}_c + \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}_c} + \frac{\partial V(q)}{\partial q_c} = 0 \tag{6.85}$$

where $C_c(q_c,\dot{q}_c)$, $C_p(q_p,\dot{q}_p)$ are suitably defined matrices. The equilibria are determined by the critical points of the potential energy function $V(q)$, that is, the solutions of $\frac{\partial V(q)}{\partial q} = 0$.

The stability proof is carried out using Lyapunov's second method: taking the time derivative of the Lyapunov function candidate $\mathcal{V} := H(q, \dot{q}) - V(\bar{q})$ which, using **P6.1** yields

$$\dot{\mathcal{V}} = -\dot{q}^\top \frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}}.$$

Moreover, using (ii) we get

$$\dot{\mathcal{V}} = -\alpha \left\|\dot{q}_c\right\|^2.$$

Clearly, $\dot{\mathcal{V}} \equiv 0$ if and only if $\dot{q}_c = 0$ (equivalently if and only if $q_c = const$). Next, from (6.85), it follows from (iii) that q_p is also constant. Since $q = \bar{q}$ is the unique equilibrium of (6.84), (6.85) asymptotic stability follows invoking Krasovskii-LaSalle's invariance principle. ∎

6.3 Discussion

The results established in propositions 6.5–6.7 are essential to some of the contributions presented in the sequel. Therefore we find it convenient to comment further the proofs, in terms of some recent results on internal stability of passive systems.

6.8 Corollary. (Corollary 3.4. (Byrnes and Martin, 1995).) [7] *Consider a lossless system (6.34) with a positive definite and proper storage function $\mathcal{V}$. The output feedback law*

$$u = -ky, \quad k > 0, \tag{6.86}$$

where $y = h(q, \dot{q})$, renders the origin of the closed loop system, globally asymptotically stable if and only if (6.34) is zero-state observable.

Let us show now the relation between proposition 6.7 and corollary 6.8. For this, we proceed first to show that an underdamped EL system is zero-state observable from the output $y = \frac{\partial \mathcal{F}}{\partial \dot{q}_c}$ and the state $x = \text{col}[q_p - \bar{q}_p,\ q_c - \bar{q}_c,\ \dot{q}_p,\ \dot{q}_c]$, under the conditions of proposition 6.7.

First, notice that condition (6.7.ii) implies that, $y = 0$ is equivalent to $\dot{q}_c = 0$, hence to $q_c = const$. This fact, together with the second part of condition (6.7.iii), implies that $q_p = const$ (that is, $\dot{q}_p = 0$). Next, considering condition (6.7.i), we get that the equilibria of (6.34) are the solutions of $\frac{\partial V(q)}{\partial q} = 0$, that is, the critical points of $V(q)$. The hypothesis that $V(q)$ has a global and unique minimum at $q = \bar{q}$ implies that $q = \bar{q}$ is the unique solution of $\frac{\partial V(q)}{\partial q} = 0$. Thus the whole state $x = 0$, if $y = 0$.

Second, notice that we can use condition (6.7.i) to rewrite equations (6.34) in the form

$$D_p(q_p)\ddot{q}_p + C_p(q_p, \dot{q}_p)\dot{q}_p + \frac{\partial V(q)}{\partial q_p} = 0 \tag{6.87}$$

$$D_c(q_c)\ddot{q}_c + C_c(q_c, \dot{q}_c)\dot{q}_c + \frac{1}{2}\frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}_c} + \frac{\partial V(q)}{\partial q_c} = -\frac{1}{2}\frac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}_c}. \tag{6.88}$$

From the discussion above, it is easy to see that system(6.87)–(6.88) is zero-state observable. Then, if we consider the term on the right hand side of (6.88) as a control input, global asymptotic stability of the origin follows immediately using corollary 6.8 with $k = 1/2$.

It is worth mentioning that after Byrnes and Martyn, the damping in the system (6.84)–(6.85) is called *pervasive*.

One more interesting result that it is worth mentioning is the following (van der Schaft, 1996): Consider the feedback system of figure 6.4.

[7] See also Theorem 3.2 of (Byrnes et al., 1991)

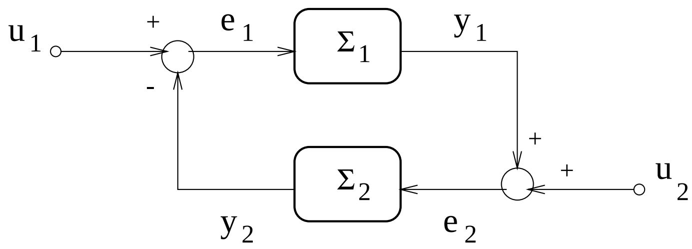


Figure 6.4: Feedback interconnection of passive systems

Assume that systems

$$\Sigma_i : \left\{ \begin{array}{l} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i \\ y_i = h_i(x_i) \end{array} \right. \quad , \qquad i = 1, 2$$

are passive or output strictly passive, with storage functions $H_1(x_1)$ and $H_2(x_2)$ respectively, i.e.

$$H_1(x_1(t) - H_1(x_1(0)) + \int_0^t \|y_1(s)\|^2 \leq \int_0^t u_1(s)y_1(s)ds \tag{6.89}$$

$$H_2(x_2(t) - H_2(x_2(0)) + \int_0^t \|y_2(s)\|^2 \leq \int_0^t u_2(s)y_2(s)ds \tag{6.90}$$

then consider the following claim.

6.9 Proposition. (Proposition 3.2.5 of A. J. van der Schaft 1996)

(i) Suppose Σ_1 and Σ_2 are passive (respectively output strictly passive), then the feedback interconnected system (Σ_1, Σ_2) of figure 6.4, defines a passive (respectively output strictly passive) operator $(e_1, e_2) \mapsto (y_1, y_2)$.

(ii) Suppose that H_1 and H_2, satisfying (6.89)–(6.90) have strict local minima in $x_1^\star$, respectively $x_2^\star$, then $(x_1^\star, x_2^\star)$ is a stable equilibrium of the feedback system, with $e_1 = e_2 = 0$.

(iii) Suppose that Σ_1 and Σ_2 are output strictly passive and zero-state detectable, and that H_1 and H_2, satisfying (6.89)–(6.90), and proper, have a global and unique minimum in $x_1^\star = 0$, respectively $x_2^\star = 0$ then $(0, 0)$ is a globally asymptotically stable equilibrium of the feedback system (Σ_1, Σ_2) with $e_1 = e_2 = 0$.

□

The proposition above gives sufficient conditions for GAS of the feedback interconnection of passive systems. Among these, we are particularly interested in Euler-Lagrange systems, hence consider the feedback interconnection of the EL systems $\Sigma_p : \{T_p(q_p, \dot{q}_p),\ V_p(q_p),\ \mathcal{F}_p(\dot{q}_p),\ M_p\}$ and $\Sigma_c\ \{T_c(q_c, \dot{q}_c),\ V_c(q_c, q_p),\ \mathcal{F}_c(\dot{q}_c)\ \}$. Let $x_1 \triangleq \mathrm{col}[\tilde{q}_p,\ \dot{q}_p]$, $u_1 = 0$, $y_1 \triangleq \frac{\partial \mathcal{F}_p(\dot{q}_p)}{\partial \dot{q}_p}$, and $x_2 \triangleq \mathrm{col}[q_c - \bar{q}_c,\ \dot{q}_c]$, $u_2 = 0$, $y_2 \triangleq \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c}$, that is, the system shown in fig. 6.4.

Under the conditions of proposition 6.7, it can be proven that both systems are zero-state detectable and that Σ_c is OSP. Furthermore, if we impose that the function $\mathcal{F}_p(\dot{q}_p)$ define a ISP operator then Σ_p is also OSP. From the hypotheses of proposition 6.7 we have that the origin is a global and unique minimum of the storage functions $H_1(q_p, \dot{q}_p) - H_1(\bar{q}_p, 0)$, and $H_2(q_p, \dot{q}_p) - H_2(\bar{q}_p, 0)$. Then GAS can be proven using proposition 6.9.

One more key result contained in proposition 6.9 is that the Lyapunov stability and IO properties of passive systems can be preserved under feedback interconnection. As a matter of fact, the following property, with respect to the EL systems is also true.

P 6.4 The feedback interconnection of two EL systems $\Sigma_p : \{T_p(q_p, \dot{q}_p),\ V_p(q_p),\ \mathcal{F}_p(\dot{q}_p),\ M_p\}$ and $\Sigma_c\ \{T_c(q_c, \dot{q}_c),\ V_c(q_c, q_p),\ \mathcal{F}_c(\dot{q}_c)\ \}$, as shown in figure 6.4, yields an EL system $\Sigma = \Sigma_p + \Sigma_c$, that is

$$\Sigma : \{T(q, \dot{q}),\ V(q),\ \mathcal{F}(\dot{q})\}$$

where

$$T(q,\dot{q}) = T_c(q_c,\dot{q}_c) + T_p(q_p,\dot{q}_p) \qquad V(q) = V_c(q_c,q_p) + V_p(q_p), \tag{6.91}$$

$$\mathcal{F} = \mathcal{F}_c(\dot{q}_c) + \mathcal{F}_p(\dot{q}_p). \tag{6.92}$$

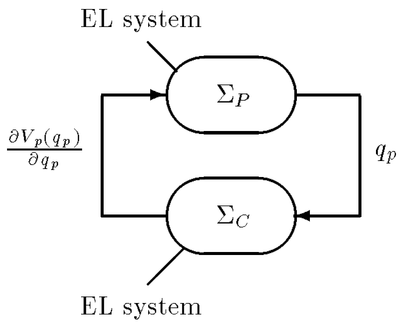


Figure 6.5: **P6.4**. Feedback interconnection of two EL systems.

The result contained in proposition 6.7 and the invariance of properties of Lagrangian systems under feedback interconnection are fundamental for the output feedback stabilization problem which we address in chapters **7** and **9**.

7 Concluding remarks

We have made in this chapter, a recall on the important concept of Lagrangian systems. For this class of systems we underlined some fundamental properties that will be used in the sequel in our control design. These properties can be summarized as follows:

1. EL systems define passive operators.
2. EL systems are characterized by their EL parameters: Kinetic energy, potential energy, Rayleigh dissipation function and inputs matrix.
3. The stable equilibria of an EL system are determined by the minima of its potential energy.
4. EL systems are asymptotically stable if they have a suitable damping.
5. The interconnection of two EL systems yields an EL system. The EL parameters of the resulting system are simply the addition of those of both subsystems.

Chapter 7

Set point control

In the previous chapter we underlined several fundamental properties of EL systems. We saw that the equilibria of an EL plant Σ_p are determined by the minima of $V_p(q_p)$, moreover the equilibrium is unique and stable if it is a global and unique minimum of the potential energy function. We also saw that a necessary condition for asymptotic stability is that suitable damping is present into the system.

Another key property is that the feedback interconnection of two EL systems yields a Lagrangian system.

The set-point control problem is simply formulated as follows: Find an output feedback controller for the EL plant $\Sigma_p : \{T_p(q_p, \dot{q}_p),\ V_p(q_p),\ \mathcal{F}_p(\dot{q}_p),\ M_p\}$ such that the desired equilibrium $q_p = q_{pd}$ be globally asymptotically stable.

Based on the passivity properties of EL plants, we present in this chapter two contributions to the output feedback set point control problem: firstly, we exploit these nice properties of EL systems in order to solve the problem of global output feedback stabilization. In particular we will define a class of controllers which achieve our control aim and we have called *EL controllers*. The EL controllers shape the potential energy of the plant, so as to have a global and unique minimum at the desired position, $q = q_d$. Moreover, they inject a suitable damping to attain asymptotic stability.

Secondly, we extend these results (for the fully actuated systems) to the case when the potential energy is not exactly known. In particular, we present the PI^2D controller whose main feature is that it uses a double integrator to compensate for the unknown potential forces. Even though this double integrator destroys the Lagrangian structure of the controller, the PI^2D controller exploits the passivity of the Lagrangian plant.

1 Introduction

1.1 Motivation: Energy shaping plus damping injection

The input-output and internal stability properties of EL systems motivated the development of the *passivity-based energy shaping plus damping injection* controller design methodology launched in the seminal paper (Takegaki and Arimoto, 1981). As its name suggests, this methodology aims at shaping the potential energy of the plant via a passive controller in such a way that the "new" energy function has a global and unique minimum in the sense of definition (B.3). For the sake of clarity we illustrate this methodology with the simple pendulum shown in figure 7.1, nevertheless, this technique is suitable for n degrees of freedom systems.

The total (kinetic + potential) energy of the simple pendulum is

$$H = \underbrace{\frac{1}{2}ml^2\dot{q}^2}_{T(\dot{q})} + \underbrace{mgl(1-\cos(q))}_{V(q)} \tag{7.1}$$

where $q \in \mathbb{R}$, nevertheless, since the position of the pendulum is repeated every 2π radians, for the sake of simplicity we restrict our analysis to the set $\{q : 0 \le q < 2\pi\}$. Using the Lagrangian equations we can easily derive the dynamics

$$ml^2\ddot{q} + g(q) = u \tag{7.2}$$

where u is the external generalized force, $g(q)$ is the force derived from the potential energy, that is,

$$g(q) \triangleq \frac{\partial V(q)}{\partial q} = mgl\sin(q) \tag{7.3}$$

and g is the gravity acceleration. Hence we call $g(q)$ the gravitational force.

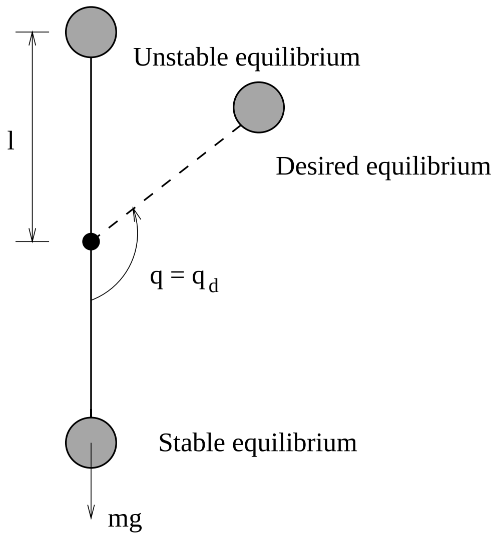


Figure 7.1: Simple pendulum

Now let us calculate the equilibria of the unforced system (7.2) (i.e. with $u \equiv 0$), they are the critical points of the potential energy function, that is, the solutions of the equation:

$$\frac{\partial V(q)}{\partial q} = mgl\sin(q) = 0,$$

that is $q = 0$ and $q = \pi$. Next, taking the second partial derivative of $V(q)$ with respect to q it yields

$$\frac{\partial^2 V(q)}{\partial q^2} = mgl\cos(q),$$

which is positive for $q = 0$ and negative for $q = \pi$. We then conclude that the origin corresponds to a minimum of the potential energy function, while $q = \pi$ is a local maximum.

It is well known now that a minimum of the potential energy corresponds to a stable equilibrium, nevertheless, for the sake of completeness we cite below a theorem borrowed from (Arnold, 1989) which establishes this fact. See also lemma 3.2.1 of (van der Schaft, 1996).

7.1 Theorem. *If the point $\bar{q}$ is a strict local minimum of the potential energy $V(q)$, then the equilibrium $q = \bar{q}$ is stable in the sense of Lyapunov.* □

Proof. Let $V(\bar{q}) = h$ then there exists $\varepsilon > 0$ and a neighbourhood of $\bar{q}$, $\{q : V(q) \leq h + \varepsilon\}$. Furthermore, there exists a neighbourhood of $\dot{q} = 0$, $q = \bar{q}$ in the phase space $(\dot{q}, q)$, defined as $B_\varepsilon \triangleq \{\dot{q}, q : H(\dot{q}, q) \leq h+\varepsilon\}$. By the law of conservation of energy, the region B_ε is invariant in the phase flow therefore, for initial conditions $(\dot{q}(0), q(0))$ close enough to $(0, \bar{q})$, every trajectory $(\dot{q}(t), q(t))$ is close to $(0, \bar{q})$. ■

In contrast, notice that the fact that $q = \pi$ correspond to a local maximum of the potential energy, contradicts the existence of $\varepsilon > 0$ and an invariant neighbourhood B_ε defined as above, hence the maximum of the potential energy corresponds to a unstable equilibrium.

Our design problem is stabilizing the pendulum at a different equilibrium other than the origin, in particular $q = q_d$. Since we know that a minimum of the potential energy corresponds to a stable equilibrium point, an appealing solution is to shape the potential energy $V(q)$ such that the desired position be the global and unique minimum of the "new" potential energy function. With this idea let us propose as *desired* potential energy (Takegaki and Arimoto, 1981)

$$V_d(q) = \frac{1}{2}k_p\tilde{q}^2 \tag{7.4}$$

where $k_p > 0$ and $\tilde{q} \triangleq q - q_d$. It is trivial to see that the global and unique minimum of $V_d(q)$ is the origin $\tilde{q} = 0$.

So far we have proposed a desired potential for our system, the problem now is how to shape the original potential of the pendulum so that the resulting potential be the desired one. This is the role of the control input u, which may be defined as

$$u = \frac{\partial}{\partial q}\left(V(q) - V_d(q)\right) = g(q) - k_p\tilde{q}. \tag{7.5}$$

We then deduce that it suffices to propose a desired potential energy function and a globally stabilizing control law is easily derived from it. Finally from proposition (6.6) we know that the origin is asymptotically stable if adequate damping is present, hence we write the control law

$$u = g(q) - k_p\tilde{q} - k_d\dot{q}$$

where $k_d > 0$, and using proposition (6.6) asymptotic stability follows.

The control law above has the drawback that besides the computational charge that it represents to compute on line the term $g(q)$, it is widely believed that compensating with a constant term instead of cancelling the nonlinear term $g(q)$ might enhance the robustness of the system vis-a-vis parametric uncertainties[1].

Thus, consider the desired potential energy function (Takegaki and Arimoto, 1981)

$$V_d(q) = V(q) - V(q_d) + \frac{1}{2}k_p\tilde{q}^2 - g(q_d)\tilde{q}. \tag{7.6}$$

Using lemma B.7 it can be proven (Tomei, 1991b) that (7.6) has a global and unique minimum at $q = q_d$ provided $k_p > k_g$ where k_g is defined by (6.40). The first part of the control law is given by

$$u = \frac{\partial}{\partial q}\left(V(q) - V_d(q)\right) = -k_p\tilde{q} + g(q_d) \tag{7.7}$$

and adding a suitable damping we find the well known PD + precompensated gravity[2] controller of (Takegaki and Arimoto, 1981):

$$u = -k_p\tilde{q} - k_d\dot{q} + g(q_d). \tag{7.8}$$

A nice physical interpretation of the controller (7.8) as an application of the energy shaping plus damping injection methodology is illustrated in figure 7.2. The proportional gain k_p can be regarded as the stiffness constant of a linear spring, which holds the pendulum at the desired position q_d. Thus, together with the compensation term, $g(q_d)$, the proportional gain k_p changes the natural stable equilibrium to the desired position (energy shaping). The derivative gain k_d represents a damper which introduces viscous friction to attain asymptotic stability.

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[1] Although this is very hard to prove in general, interested readers are referred to (Ki-Chul et al., 1996; Brogliato et al., 1996) where this claim is experimentally proven.

[2] See also (Tomei, 1991b) for an extension of this result to flexible joint robots.

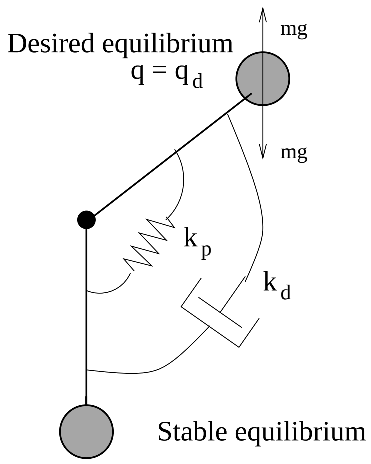


Figure 7.2: Physical interpretation of a PD plus gravity compensation controller

Notice that the closed loop system (7.2), (7.8) is indeed a fully damped Euler-Lagrange system with potential energy $V_d(q)$ defined by (7.6) and kinetic energy $T(\dot{q})$ while the term $-k_d\dot{q}$ corresponds to dissipative forces which can be derived from the Rayleigh function $\frac{1}{2}k_d\dot{q}^2$. Moreover, the potential energy function $V_d(q)$ has a global and unique minimum at $q = q_d$ and the Rayleigh dissipation function defines an ISP operator if $k_d > 0$, therefore, by virtue of proposition 6.6 the equilibrium point $q = q_d$ is globally asymptotically stable.

Even though this technique is physically appealing and based upon the properties of EL systems described in chapter **6**, its utilization is stymied in some applications mainly by two facts: (1) measurement of the generalized velocities, $\dot{q}$, is typically required to add the damping, (2) the potential energy, $V(q)$, is supposed to be exactly known.

In this chapter we address both problems: (1) set point control without velocity measurements and (2) with uncertain knowledge of the potential energy. In order to put our contributions in perspective we first briefly review the literature.

1.2 Brief literature review

The first problem motivated robotics researchers to look for regulators which avoid velocity measurements. Among the first results in this direction we can cite: (Burkov, 1995a; Berghuis and Nijmeijer, 1993a; Kelly, 1993b; Ailon and Ortega, 1993; Kelly et al., 1994a). It must be remarked that the last two results hold as well for flexible joint robots (underdamped EL systems). These first results extended, to the output feedback case, the controllers of (Arimoto and Miyazaki, 1986) and (Tomei, 1991b).

The controllers proposed by (Burkov, 1995a; Berghuis and Nijmeijer, 1993a; Kelly, 1993b) are all based upon the incorporation of a similar n-order linear filter where n is the number of degrees of freedom of the manipulator, however, these results appeared independently. Ailon and Ortega (1993) proposed a controller based on a $2n$ order filter. Later, the result of (Kelly, 1993b) was extended for flexible joint robots in (Kelly et al., 1994a). Finding a common feature to all these controllers and extending these results to the more general frame of Euler-Lagrange systems, motivated us to look for a new methodology for position feedback regulation. This research culminated in the definition of the *Euler-Lagrange controllers* for flexible joint robots presented in (Ortega et al., 1995a), and for a wider class of EL systems, (Ortega et al., 1995c). Other interesting results in the aim of our research are (Samson, 1987; Abesser and Katzschmann, 1994).

Concerning the problem that the potential energy is supposed to be known exactly, it is well known that the physical structure of robot manipulators, and in particular its passivity properties, can be suitably exploited to design robustly performant simple (proportional integral derivative (PID)-like) controllers of the form (Arimoto and Miyazaki, 1986)

$$u = -K_P\tilde{q} - v_1 - K_D\dot{q} \tag{7.9}$$

$$\dot{v}_1 \quad = \quad -K_I \tilde{q} \tag{7.10}$$

where $\tilde{q} \triangleq q - q_d$, q_d is the desired link position which is assumed to be *constant*, K_P, K_D are positive definite diagonal matrices.

In (Wen and Murphy, 1990) it is shown that this PID controller is *locally* asymptotically stable provided K_P, K_I and K_D satisfy some complex relationships. Explicit tuning rules for these gains may be found in (Kelly, 1995).

Adaptive control is an alternative approach to remove the assumption of known gravity forces[3]. In this case we select a parameterization of $g(q)$ as $g(q) = \Phi(q)\theta$, where $\Phi(q)$ is a measurable regressor and θ is a vector of unknown constant parameters. A certainty equivalent PD plus adaptive gravity compensation controller is obtained by replacing in (7.10):

$$v_1 \quad = \quad -\Phi(q)\hat{\theta} \tag{7.11}$$

$$\dot{\hat{\theta}} \quad = \quad -\Phi^\top(q)\left[\gamma\dot{q} + 2\frac{\tilde{q}}{1+2\left\|\tilde{q}\right\|^2}\right] \tag{7.12}$$

where γ is a suitable positive constant. In (Tomei, 1991a) it is shown that if the parameters are updated with the normalized estimator (7.12) then *global convergence* follows. The normalization, first introduced by (Koditschek, 1988), is needed to attain globality – the unnormalized case being only semiglobally stable –. The establishment of this result also uses the Lyapunov function based on the total energy plus the cross term mentioned above, but this time normalized in order to insure $\dot{V}$ is negative semi-definite. An improvement upon this result is due to Kelly (Kelly, 1993a) where it is proven that global convergence is still preserved if in the estimator we *replace the regressor* $\Phi(q)$ by $\Phi(q_d)$. This result is fundamental for the solution of the *output feedback* problem of interest here, since in this case $\Phi(q_d)$ being constant allows us to integrate the velocity term in the parameter update law and to incorporate it as a proportional feedback in the control law (7.11). More recently, (Arimoto, 1994a) proposed a nonlinear PID to prove global asymptotic stability by saturating the proportional term. See also (Nakayama and Arimoto, 1996).

Thus, one can identify several solutions to both posed problems *separately* that is, on one hand the controllers which do not need measurement of generalized velocities need exact a priori knowledge of the potential energy, and on the other hand, the different approaches to regulation with uncertain potential energy knowledge needed the measurement of the generalized velocities.

To the best of our knowledge the first solutions to the problem of designing an asymptotically stable regulator that does not require the *exact knowledge* of $g(q_d)$ nor the *measurement of speed* appeared independently in (Ortega et al., 1995b) and (Colbaugh and Glass, 1995). The contribution of (Ortega et al., 1995b), the $\mathrm{PI^2D}$ controller, is a semiglobally stable control law that solves this problem. Colbaugh and Glass obtained a similar result for robot manipulators considering the dynamics of DC actuators.

An obvious drawback of all above mentioned approaches is that they can be applied only to rigid-joint robots. As far as we know the only results where joint flexibility is considered are (Ailon, 1996) and (Burkov et al., 1996). In the first reference, the author makes use of the global contraction mapping theorem to prove semi-global ultimate boundedness of the solutions. In the second reference Burkov proposes a common PID controller, the novelty is the use of singular perturbation techniques in order to prove that "there exists" a sufficiently small integral gain such that the closed loop system is globally asymptotically stable. Unfortunately, no specific bounds for the integral gain are given. Both approaches are based on the assumption that velocity measurements are available.

2 Output feedback stabilization of EL systems

2.1 Problem formulation

We consider in this section EL plants with no internal damping that is, we consider plants with EL parameters

$$\{T_p(q_p, \dot{q}_p), V_p(q_p), M_p\}$$

[3] See also (De Luca and Panzieri, 1993) for a learning-based scheme.

where the subindex $_p$ stands for "plant". The dynamic model has the form (6.37) which for convenience of presentation we write below

$$D_p(q_p)\ddot{q}_p + C(q_p, \dot{q}_p)\dot{q}_p + g(q_p) = u_p \tag{7.13}$$

where $u_p = M_p u \in \mathbb{R}^n$ is the vector of the control inputs to the plant. In particular, the problem we study in this section is formulated as follows:

Output feedback global stabilization problem. Consider the EL system (7.13) where q_p is partitioned as $q_p = \text{col}[q_{p_1},\ q_{p_2}]$, $q_{p_2} = M_p q_p$. Assume that the *measurable outputs* are q_{p_2} and *regulated outputs* are q_{p_1} with *constant* desired value q_{p1d}. Then, design a controller $q_{p_2} \mapsto u_p$ that makes the closed loop system *globally asymptotically stable* (GAS) at an equilibrium point $\bar{q}$ such that $\bar{q}_{p_1} = q_{p_{1d}}$.

We now use the input–output and internal stability properties established in chapter **6** to solve the output feedback global stabilization problem for our defined subclass of EL systems (see section **6**.4). Hence we define a class of controllers which, preserving the EL structure, suitably modifies the potential energy and dissipation properties of the EL plant.

2.2 Damping injection via dynamic extension

Motivated by the energy shaping plus damping injection technique of Takegaki and Arimoto, and the properties of Lagrangian systems described in the previous chapter, we will consider here *EL controllers* with generalized coordinates $q_c \in \mathbb{R}^{n_c}$ and EL parameters[4] $\{T_c(q_c, \dot{q}_c), V_c(q_c, q_{p_2}), \mathcal{F}_c(\dot{q}_c), 0\}$. That is, the controller dynamics is given by

$$D_c(q_c)\ddot{q}_c + \dot{D}_c(q_c)\dot{q}_c - \frac{\partial T_c(q_c, \dot{q}_c)}{\partial q_c} + \frac{\partial V_c(q_c, q_{p_2})}{\partial q_c} + \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} = 0. \tag{7.14}$$

Notice that the potential energy of the controller depends on the measurable output q_{p_2}, therefore q_{p_2} enters into the controller via the term $\frac{\partial V_c(q_c, q_{p_2})}{\partial q_c}$. On the other hand, the *feedback interconnection* between plant and controller is established by

$$u_p = -\frac{\partial V_c(q_c, q_{p_2})}{\partial q_{p_2}}. \tag{7.15}$$

In this way, it follows from **P6.4** that the closed-loop behaviour is characterized by the Euler-Lagrange parameters $\{T(q, \dot{q}), V(q), \mathcal{F}(\dot{q}), 0\}$ where

$$T(q, \dot{q}) \triangleq T_p(q_p, \dot{q}_p) + T_c(q_c, \dot{q}_c),\ V(q) \triangleq V_p(q_p) + V_c(q_c, q_{p_2}),\ \mathcal{F}(\dot{q}) \triangleq \mathcal{F}_p(\dot{q}_p) + \mathcal{F}_c(\dot{q}_c)$$

and $q = \text{col}[q_p,\ q_c]$ as defined before. From the new definition of $\mathcal{F}(\dot{q})$ it is clear that the dynamic extension we just introduced injects damping (through the controller) if $\mathcal{F}_c$ is a Rayleigh dissipation function.

The resulting feedback system is depicted in figure 7.3, where $\Sigma_p : u_p \mapsto q_{p_2}$ is an operator defined by the dynamic equations (7.13) and operator $\Sigma_c : q_{p_2} \mapsto u_p$ is defined by (7.14), (7.15).

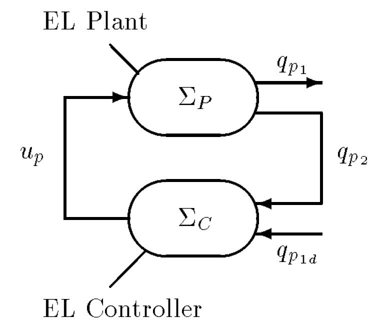


Figure 7.3: EL Closed loop system.

[4] Since we are dealing here with a regulation and not a tracking problem there are no external inputs to the controller, which explains our choice of 0 as the "input matrix" M_c.

2.3 The class of EL controllers

From the results presented in the previous chapter we see that to attain the GAS objective $V(q)$ must have a global and unique minimum at the desired equilibrium, $q = q_d$, and $\mathcal{F}(\dot{q})$ must satisfy (6.33). These conditions are summarized in the proposition below whose proof follows trivially from the derivations of previous section and proposition 6.6.

7.2 Proposition. (Output feedback stabilization)
Consider an EL plant *(7.13) with EL parameters* $\{T_p(q_p, \dot{q}_p), V_p(q_p), \mathcal{F}_p(\dot{q}_p), M_p\}$. *An* EL controller *(7.14), (7.15) with EL parameters* $\{T_c(q_c, \dot{q}_c), V_c(q_c, q_{p_2}), \mathcal{F}_c(\dot{q}_c), 0\}$, *where*

$$\dot{q}_c^\top \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} \geq \alpha \|\dot{q}_c\|^2$$

for some $\alpha > 0$, *solves the* global output feedback stabilization *problem if*

(i) (Energy shaping)
$\frac{\partial V(q)}{\partial q} = 0$ *admits a constant solution* $\bar{q}$ *such that* $q_{p1d} = [I_{n_{p1}} \mid 0]\bar{q}$, *and* $q = \bar{q}$ *is a global and unique minimum of* $V(q)$, *and* V *is proper. For instance if* $\frac{\partial^2 V(q)}{\partial q^2} > I_n \varepsilon > 0, \;\; \varepsilon > 0 \;\; \forall\, q \in I\!R^n$

(ii) (Dissipation propagation)
For each trajectory such that $q_c \equiv const$ *and* $\frac{\partial V_c(q_c, q_{p_2})}{\partial q_c} = 0$, *we have that* $q_p \equiv const$.

□

Proof. The proof follows from proposition 6.7 observing that the closed loop system is an underdamped system with damped coordinates q_c and undamped coordinates q_p. Notice that condition *(iii)* of proposition 6.7 implies that $\dot{q}_p \neq 0$ and $\frac{\partial V_c}{\partial q_c} = 0$ cannot happen simultaneously, hence condition *(ii)* of proposition 7.2 is a consequence of 6.7.*(iii)*. ■

From proposition 7.2 it is clear that the kinetic energy of the controller plays no role on the stabilization task since only the plant's *potential* energy is shaped. Furthermore, the conditions on the Rayleigh dissipation function of proposition 7.2 are satisfied with $\mathcal{F}_c(\dot{q}_c) = \frac{1}{2}\dot{q}_c^\top R_c \dot{q}_c$, where $R_c = R_c^\top > 0$. Thus, with this choice of $\mathcal{F}_c(\dot{q}_c)$ and making an obvious abuse of terminology, we will also refer to controllers with kinetic energy

$$T_c(\dot{q}_c, q_c) = 0$$

as EL controllers. This is to enlarge the class of suitable output feedback globally asymptotically stabilizing controllers and actually, to obtain simpler solutions. In particular in the following section we investigate the stability of a controller with dynamics

$$\dot{q}_c = -R_c^{-1} \frac{\partial V_c(q_c, q_{p_2})}{\partial q_c} \tag{7.16}$$

$$u_p = -\frac{\partial V_c(q_c, q_{p_2})}{\partial q_{p_2}} \tag{7.17}$$

in closed loop with a flexible joint robot.

The action of the controller (7.16), (7.17) has the following nice *passivity interpretation*. First, notice that the EL system (7.13) in closed loop with the control signal (7.15) defines a passive operator $\frac{\partial V_c(q_{p_2}, q_c)}{\partial q_{p_2}} \mapsto -\dot{q}_{p_2}$ with storage function $T_p(\dot{q}_p, q_p) + V_p(q_p)$. On the other hand, the controller (7.14) defines a passive operator $\dot{q}_{p_2} \mapsto \frac{\partial V_c(q_{p_2}, q_c)}{\partial q_{p_2}}$ with storage function $T_c(\dot{q}_c, q_c) + V_c(q_{p_2}, q_c)$. These properties follow, of course, from the passivity of EL systems established in proposition 6.5.

3 Example: stabilization of flexible joint robots

We have proven that (for general EL systems) damping injection is still possible with *only output feedback* provided we can design a passive (EL) controller which satisfies a *dissipation propagation* condition. In this section we apply this general result to identify a *class of passive controllers* solving the output feedback global stabilization problem for flexible joint robots. This provides a unified framework to compare different schemes via analysis of their energy dissipation properties. Further, we show that as particular cases of this class we can obtain the (apparently unrelated) controllers of (Kelly et al., 1994a) and (Ailon and Ortega, 1993). Other efforts aimed in the direction of our research have been reported in (Lanari and Wen, 1992; Berghuis and Nijmeijer, 1993b; Arimoto, 1994b; Battiloti and Lanari, 1995).

We use in this section, the dynamic model (6.68). We recall the reader that in our notation, q_{p_1} stands for the vector of link positions and q_{p_2} for the vector of motor shafts angles. The control variables are the torques at the shafts. We are interested in the set-point control of the link angles to a constant desired value q_{p1d}, and we assume that only the motor shaft angles, q_{p_2} are available for measurement.

3.1 First EL controller: Ailon and Ortega (1993)

Assume that we would like that the flexible joint robot behaved as an EL system characterized by the EL parameters:

$$\begin{aligned} T(q,\dot q) &= \frac{1}{2}\dot q_p^\top D(q_p)\dot q_p + \frac{1}{2}\left\|\dot q_c\right\|^2, \quad \mathcal{F}(\dot q) = \frac{1}{2}\dot q_c^\top R_c \dot q_c, && (7.18)\\ V(q) &= \frac{1}{2}q_p^\top \mathcal{K}_p q_p + V_g(q_{p_1}) + \\ &\quad +\frac{1}{2}\{(q_c - q_{p_2})^\top K_2 (q_c - q_{p_2}) + (q_c-\delta)^\top K_1 (q_c - \delta)\} && (7.19)\end{aligned}$$

with K_1, K_2, R_c symmetric and positive definite matrices and δ, a constant vector to be defined below in such a way that V have a global and unique minimum at the desired equilibrium. Knowing that the feedback interconnection of two EL systems yields an EL system consider the EL controller characterized by

$$\begin{aligned} T_c(q_c,\dot q_c) &= \frac{1}{2}\left\|\dot q_c\right\|^2, \quad \mathcal{F}_c(\dot q_c) = \frac{1}{2}\dot q_c^\top R_c \dot q_c, && (7.20)\\ V_c(q_c, q_{p2}) &= \frac{1}{2}\{(q_c - q_{p_2})^\top K_2 (q_c - q_{p_2}) + (q_c-\delta)^\top K_1 (q_c - \delta)\}. && (7.21)\end{aligned}$$

Using the Euler–Lagrange equations (6.34), we easily derive the controller dynamics

$$\ddot q_c + R_c \dot q_c + K_1(q_c - \delta) + K_2(q_c - q_{p_2}) = 0. \qquad (7.22)$$

Next, using (7.15)) we derive the feedback interconnection

$$u_p = -\frac{\partial V_c(q_c, q_{p2})}{\partial q_{p_2}} = K_2(q_c - q_{p_2}) \qquad (7.23)$$

Next, in order to inject a suitable damping the matrix R_c is chosen positive definite, hence (6.33) is satisfied with $\alpha = r_{c_m}$. Henceforth we verify conditions (i) and (ii) of proposition 7.2.

(*i*) (*Energy Shaping*)
First, we need to insure that our EL controller (7.22), (7.23) shapes the closed loop system's potential energy to make it have a global and unique minimum at the desired equilibrium point. A simple way to do this, is to verify the conditions of lemma B.7 hence, we first calculate $\frac{\partial V(q)}{\partial q} = 0$

$$\begin{bmatrix} K & -K & 0 \\ -K & K+K_2 & -K_2 \\ 0 & -K_2 & K_1+K_2 \end{bmatrix}\begin{bmatrix} q_{p_1} \\ q_{p_2} \\ q_c \end{bmatrix} + \begin{bmatrix} g_{p1}(q_{p_1}) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ K_1\delta \end{bmatrix}. \qquad (7.24)$$

We see that by defining the constant

$$\delta \triangleq q_{p1d} + (K^{-1} + K_1^{-1} + K_2^{-1}) g_{p1}(q_{p1d}) \qquad (7.25)$$

we assure that (7.24) has a solution of the required form

$$\bar{q} = \begin{bmatrix} \bar{q}_{p_1} \\ \bar{q}_{p_2} \\ \bar{q}_c \end{bmatrix} = \begin{bmatrix} q_{p_{1d}} \\ q_{p_{1d}} + K^{-1} g_{p1}(q_{p_{1d}}) \\ q_{p_{1d}} + (K^{-1} + K_2^{-1}) g_{p1}(q_{p_{1d}}) \end{bmatrix}. \tag{7.26}$$

Notice that as a matter of fact, equations (7.22), (7.23) and (7.25) define the controller of (Ailon and Ortega, 1993).

Now, to enforce $V(q)$ to have a global and unique minimum at $q = q_d$ we verify that $V(q)$ is indeed strictly convex, for this, we verify the condition

$$\frac{\partial^2 V(q)}{\partial q^2} = \begin{bmatrix} K + \frac{\partial g_{p1}(q_{p_1})}{\partial q_{p_1}} & -K & 0 \\ -K & K + K_2 & -K_2 \\ 0 & -K_2 & K_1 + K_2 \end{bmatrix} \geq I_n \varepsilon > 0 \tag{7.27}$$

which happens to hold if $k_{a_m} > k_g$ where we recall that (according to our notation) k_{a_m} is the minimum eigenvalue of K_a:

$$K_a \triangleq \begin{bmatrix} K & -K & 0 \\ -K & K + K_2 & -K_2 \\ 0 & -K_2 & K_1 + K_2 \end{bmatrix},$$

and k_g is defined by (6.40). Ailon and Ortega (1993) observed that K_a accepts the congruence transformation

$$\begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ I & I & I \end{bmatrix} K_a \begin{bmatrix} I & I & I \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} K & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & K_2 \end{bmatrix},$$

therefore $k_{a_m} > k_g$ if and only if

$$\text{block-diag}\,\{K,\ K_1,\ K_2\} > k_g \begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ I & I & I \end{bmatrix} \begin{bmatrix} I & I & I \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix} = k_g \begin{bmatrix} I & I & I \\ I & 2I & 2I \\ I & 2I & 3I \end{bmatrix}.$$

On the other hand, it can be shown that there exists a permutation matrix $P \in \mathbb{R}^{3n \times 3n}$ such that

$$P \begin{bmatrix} I & I & I \\ I & 2I & 2I \\ I & 2I & 3I \end{bmatrix} P^\top = \text{block-diag}\,\{E\}$$

where

$$E \triangleq \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Thus, recalling that permutation matrices are orthogonal, the required condition on K_1, K_2, K reduces to $K_1, K_2, K > I_n e_M k_g$.

(ii) (Dissipation propagation)
Notice that

$$\frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} + \frac{\partial V_c(q_c, q_{p2})}{\partial q_c} = R_c \dot{q}_c + K_1(q_c - \delta) + K_2(q_c - q_{p_2}) \tag{7.28}$$

then setting $q_c = const$ and equalizing the right hand side of (7.28) to zero, it follows that $q_{p_2} = const$. The proof is completed as done by (Tomei, 1991b) observing the upper triangular structure of $D_{12}(q_{p_1})$ (see eq. 6.67) in order to conclude that q_{p_1} is also constant: Since q_{p_2} is constant then we can write the last n differential equations of (6.68) using (6.66) and (6.69) as

$$D_{12}^\top(q_{p_1})\ddot{q}_{p_1} + C_{21}(q_{p_1}, \dot{q}_{p_1})\dot{q}_{p_1} - K q_{p_1} = -K q_{p_2} - K_2(q_c - q_{p_2}) + g_{p_1}(q_{p1d}) = \text{constant}. \tag{7.29}$$

Considering (6.70) and (6.67), the first equation of (7.29) becomes

$$q_{p_{1,1}} = \text{constant}, \tag{7.30}$$

substituting (7.30) into the second equation of (7.29) we get $q_{p_{1,2}} = \text{constant}$. Proceeding in the same way till the nth differential equation of (7.29), we can conclude that q_{p_1} is constant. Since we proved that $q = \bar{q}$ is the only equilibrium point, the control goal has been achieved.

3.2 Second controller: Kelly *et al* (1994a)

The EL controller that we derive in this section, corresponds to a linear filter of order n which is well known as the approximate differentiation filter or "dirty derivative".

Suppose that we wish that the EL parameters of the closed loop were

$$\begin{aligned} T(q,\dot q) &= \frac{1}{2}\dot q_p^\top D(q_p)\dot q_p, \quad \mathcal{F}(\dot q) = \frac{1}{2}\dot q_c^\top K_2 B^{-1}A^{-1}\dot q_c, & (7.31)\\ V(q) &= \frac{1}{2}\{(q_c + Bq_{p_2})^\top K_2 B^{-1}(q_c + Bq_{p_2}) + (q_{p_2} - \delta)^\top K_1 (q_{p_2} - \delta)\} + \\ &\quad + \frac{1}{2} q_p^\top \mathcal{K}_p q_p + V_g(q_{p_1}) & (7.32) \end{aligned}$$

with K_1, K_2, A, B diagonal and positive definite matrices and δ a constant vector to be defined below. Then consider an *EL controller* with EL parameters

$$T_c(q_c,\dot q_c) = 0, \quad \mathcal{F}_c(\dot q_c) = \frac{1}{2}\dot q_c^\top K_2 B^{-1} A^{-1}\dot q_c$$

$$V_c(q_c, q_{p_2}) = \frac{1}{2}(q_c + Bq_{p_2})^\top K_2 B^{-1}(q_c + Bq_{p_2}) + \frac{1}{2}(q_{p_2} - \delta)^\top K_1 (q_{p_2} - \delta),$$

that is, we will try to "dominate" the robot's potential energy with a quadratic function. As in the previous section, using (6.34) and (7.15) we derive the controller dynamic equations

$$\begin{aligned} \dot q_c &= -A(q_c + Bq_{p_2}) & (7.33)\\ u_p &= -K_1(q_{p_2} - \delta) - K_2(q_c + Bq_{p_2}) & (7.34) \end{aligned}$$

From the definition of $\mathcal{F}_c(\dot q_c)$, it is clear that $\dot q_c^\top \frac{\partial \mathcal{F}_c(\dot q_c)}{\partial \dot q_c} \geq \alpha \|\dot q_c\|^2$ with $\alpha = r_{c_m}$.

Now, we determine the values of K_1, K_2 and δ such that the rest of the conditions of proposition 7.2 hold.

(i) (*Energy shaping*)
To verify this condition first notice that setting $\frac{\partial V(q)}{\partial q}|_{\bar q} = 0$ yields

$$\begin{bmatrix} K & -K & 0 \\ -K & K + K_1 + K_2 & K_2 \\ 0 & K_2 & K_2 \end{bmatrix} \begin{bmatrix} \bar q_{p_1} \\ \bar q_{p_2} \\ \bar q_c \end{bmatrix} + \begin{bmatrix} g(\bar q_{p_1}) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ K_1\delta \\ 0 \end{bmatrix}$$

which has a (unique) solution of the required form $\bar q = [q_{p1d}^\top, *, *]^\top$ with

$$\delta = q_{p1d} + (K^{-1} + K_1^{-1}) g_{p1}(q_{p1d}).$$

The second part of this condition is met if

$$\frac{\partial^2 V(q)}{\partial q^2} = \begin{bmatrix} K + \frac{\partial g_{p1}(q_{p_1})}{\partial q_{p_1}} & -K & 0 \\ -K & K + K_1 + K_2 & K_2 \\ 0 & K_2 & K_2 \end{bmatrix} \geq I_n\varepsilon > 0, \quad \varepsilon > 0 \ \ \forall\, q \in \mathbb{R}^n.$$

To satisfy this requirement we partition the two diagonal sub-blocks of the Hessian matrix as

$$Q_1 \triangleq \begin{bmatrix} K + \frac{\partial g_{p1}(q_{p_1})}{\partial q_{p_1}} & -K \\ -K & K + \frac{1}{2}K_1 \end{bmatrix}, \quad Q_2 \triangleq \begin{bmatrix} \frac{1}{2}K_1 + K_2 & K_2 \\ K_2 & K_2 \end{bmatrix}$$

and look for conditions that ensure that both are bounded from below by some matrix $I_{n_p}\varepsilon$, $\varepsilon > 0$ for all $q \in \mathbb{R}^n$. Submatrix Q_1 is positive definite if $k_{a_m} > k_g$ where we have redefined

$$K_a \triangleq \begin{bmatrix} K & -K \\ -K & K + \frac{1}{2}K_1 \end{bmatrix}$$

then we can proceed as before observing that K_a satisfies the congruence transformation (Ailon and Ortega, 1993)

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} K_a \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix},$$

on the other hand we can prove that there exists a permutation matrix $P \in \mathbb{R}^{n_p \times n_p}$ such that

$$P \begin{bmatrix} I & I \\ I & 2I \end{bmatrix} P^\top = \text{block-diag}\{E\}$$

where

$$E \triangleq \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Thus, the condition on K, K_1 reduces to $K_1, K > e_m k_g I_{m_p}$. Following the same procedure, it can be also shown that submatrix Q_2 is positive definite for all $K_1, K_2 > 0$.

(*ii*) (*Dissipation propagation*)
Finally, this condition is verified as follows: we set the right hand side of

$$\frac{\partial V_c(q_c, q_{p_2})}{\partial q_c} = K_2(q_c + Bq_{p_2})$$

to zero and consider $q_c \equiv const$, we then get that $q_{p_2} \equiv const$. The proof is completed as before exploiting the special triangular structure of the robot inertia matrix in order to conclude that also q_{p_1} is constant.

■

In this section we have only derived the controllers of (Ailon and Ortega, 1993) and (Kelly et al., 1994a) using the methodology described in section 2 nevertheless some other related works, for instance (Berghuis and Nijmeijer, 1993a), also fit in the proposed framework, as a matter of fact the latter can be viewed as a special case of (Kelly et al., 1994a) with $A = B = L$ where L is the observer gain in (Berghuis and Nijmeijer, 1993a).

7.3 Remark. It is also worth remarking that the EL controller of (Kelly et al., 1994a), (7.33) with output $\vartheta = (q_c + Bq_p)$, corresponds to the well known dirty derivatives (or approximate differentiation) filter with transfer function

$$\vartheta = \ \text{diag}\left\{\frac{b_i p}{p + a_i}\right\} q_p, \qquad i \in \underline{n}$$

3.3 Simulation results

In this section we illustrate through simulations the performance of both EL controllers presented above. We have used the two degrees of freedom simplified model of (Berghuis and Nijmeijer, 1993a) described in section **6**.5.1 for which we have considered a joint stiffness of $K = 3500 I_2$. The constant reference to be followed is $q_{p1d} = [\frac{\pi}{4}, \frac{\pi}{4}]^\top$ with zero initial conditions. Figure 7.4 shows the result of the controller described in (Kelly et al., 1994a); in this case we have set the gains to $K_1 = \text{diag}([5000, 6000])$, $K_2 = \text{diag}([4000, 5000])$, $a_i = 30$ and $b_i = 10$. In figure 7.5 we illustrate the response to the same reference using the controller of (Ailon and Ortega, 1993). We have used the same model as before and the controller gains were set in such a way to have a transient approximately similar in time to that of the first controller. These values are $K_1 = K_2 = \text{diag}([8500, 8500])$, $R_c = \text{diag}([3500, 3500])$. Gains can be tuned to have a smoother but slower transient response.

In this brief section we have limited ourselves to illustrate the behaviour of the controllers contained in section 3, further simulation results comparing the performance of (Tomei, 1991b) and (Kelly et al., 1994a) can be found in the latter reference. Other simulations concerning robustness of the controllers in (Kelly et al., 1994a) and (Ailon and Ortega, 1993) to uncertainty on the robots potential energy were reported in (Loria, 1993).

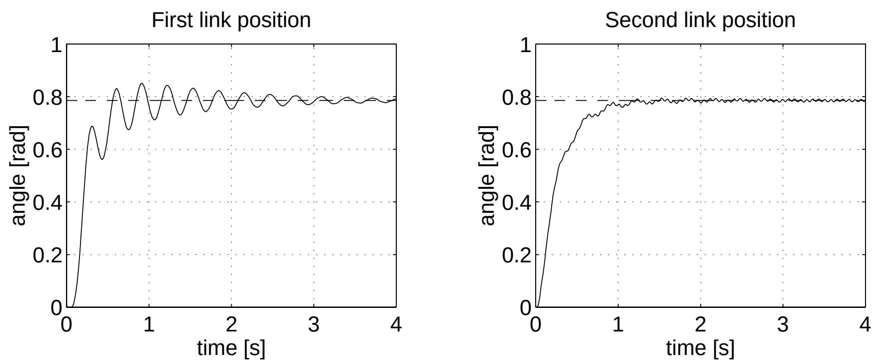


Figure 7.4: EL Controller of Kelly *et al* (1994a)

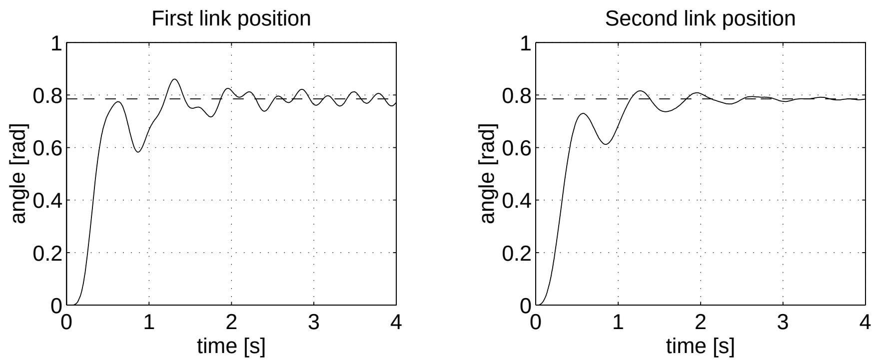


Figure 7.5: EL Controller of Ailon and Ortega (1993)

4 Regulation with uncertain knowledge of the potential energy

In previous sections we introduced the energy shaping plus partial damping injection methodology to solve the output feedback regulation problem. As it was previously discussed, an obvious drawback of this approach is that it assumes that the potential forces are accurately known, which in practice rarely happens. In this section we propose our so called PI^{2}D passivity based controller, which solves the output feedback set-point control problem with uncertain gravity knowledge.

4.1 The PI^{2}D controller

We consider the dynamic model (7.13) where $u_p \in \mathbb{R}^{n_p}$ that is, all generalized coordinates are considered to be actuated. As will become clear later, even though the PI^{2}D controller is based on the passivity properties of Lagrangian systems, it is not an EL controller therefore, for the sake of simplicity in this section we drop index $_p$.

7.4 Proposition. *Consider the dynamic model (7.13):*

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u,$$

with $u \in \mathbb{R}^n$ in closed loop with the PI^{2}D control law

$$\begin{aligned} u &= -K_P\tilde{q} + \nu - K_D\vartheta & (7.35)\\ \dot{\nu} &= -K_I(\tilde{q} - \vartheta), \quad \nu(0) = \nu_0 \in \mathbb{R}^n & (7.36)\\ \dot{q}_c &= -A(q_c + Bq) & (7.37)\\ \vartheta &= q_c + Bq & (7.38) \end{aligned}$$

Let K_P, K_I, K_D, $A \stackrel{\triangle}{=} diag\{a_i\}$, $B \stackrel{\triangle}{=} diag\{b_i\}$ be positive definite diagonal matrices with

$$B > \frac{4d_M}{d_m} I \quad (7.39)$$

$$K_P > (4k_g + 1)I \quad (7.40)$$

where k_g is defined by (6.40).

Under these conditions, we can always find a (sufficiently small) integral gain K_I such that the equilibrium $x \stackrel{\triangle}{=} col[\tilde{q},\ \dot{q},\ \vartheta,\ \nu - g(q_d)] = 0$ is asymptotically stable with a domain of attraction including

$$\{x \in I\!R^{4n} : \|x\| < c_3\} \quad (7.41)$$

where $\lim_{b_m \to \infty} c_3 = \infty$. In other words, given any (possibly arbitrarily large) initial condition $\|x(0)\|$, there exist controller gains that ensure $\lim_{t\to\infty} \|x(t)\| = 0$. □

Proposition 7.4 establishes the *semiglobal stability* of our controller, in the sense that the domain of attraction can be arbitrarily enlarged with a suitable choice of the gains, namely by increasing b_m. However, as shown in the proof of proposition 7.4 (cf. subsection 4.2), the stability conditions impose an order relationship between K_I and B such that we must correspondingly decrease K_I.

7.5 Remark. Notice that control law (7.35) can be alternatively written as

$$u(t) = -K_P\tilde{q} - K_I \int_0^t \tilde{q} d\tau - K_D \ \text{diag}\left\{\frac{b_i p}{p + a_i}\right\} q - K_I \int_0^t \ \text{diag}\left\{\frac{b_i p}{p + a_i}\right\} q d\tau. \quad (7.42)$$

The first three right hand side terms constitute the proportional, integral and filtered derivative actions, while the presence of the fourth right hand term motivates the name PI^{2}D. We have kept the notation D for the derivative term because in all practical applications of PID regulators this is implemented incorporating a filter.

4.2 Stability proof

The proof relies on *classical Lyapunov theory* and is divided in five parts. First, we define a suitable error equation for the closed loop system, whose (unique) equilibrium is at the desired value. Then, we propose a Lyapunov function candidate. Third, we prove that under the conditions of proposition 7.4 the proposed function qualifies as a Lyapunov function, and establish the asymptotic stability of the equilibrium invoking Krasovskii-LaSalle's invariance principle. Then, we define the domain of attraction, and finally prove that it can be arbitrarily enlarged.

4.2.1 Error Equation

First, let us define

$$K_p' \stackrel{\triangle}{=} K_P - \frac{1}{\varepsilon} K_I$$

where, for a given K_I, $\varepsilon > 0$ is chosen such that

$$I > \frac{1}{\varepsilon} K_I \quad (7.43)$$

thus insuring $K_p' > 4k_g I$ as required. Then, partition the controller's proportional term as

$$K_P\tilde{q} = K_p'(q - \delta) + \frac{1}{\varepsilon} K_I \tilde{q} + g(q_d)$$

where we have defined $\delta \stackrel{\triangle}{=} q_d + (K_p')^{-1} g(q_d)$. Replacing (7.35) in (7.13), using the equation above, and after some simple manipulations, we get the error equation

$$\begin{aligned} &D\ddot{q} + C\dot{q} + g(q) + K_p'(q - \delta) + K_d\vartheta = -\tfrac{1}{\varepsilon} K_I \tilde{q} + \tilde{\nu} \\ &\dot{\vartheta} = -A\vartheta + B\dot{q} \\ &\dot{\tilde{\nu}} = -K_I(\tilde{q} - \vartheta) \end{aligned}$$

where $\tilde{\nu} \triangleq \nu - g(q_d)$. The notation $\tilde{\nu}$ being motivated by the fact that with $\nu = g(q_d)$ in (7.35) we recover the GAS controller of (Kelly, 1993b), studied in the previous section. To simplify the derivation of the Lyapunov function we find it convenient to introduce the (linear) change of coordinates

$$z \triangleq -\frac{1}{\varepsilon}K_I\tilde{q} + \tilde{\nu}$$

which yields the new *error equations*

$$D\ddot{q} + C\dot{q} + g(q) + K_p'(q-\delta) + K_d\vartheta = z \tag{7.44}$$
$$\dot{\vartheta} = -A\vartheta + B\dot{q} \tag{7.45}$$
$$\dot{z} = -K_I(\tilde{q} + \frac{1}{\varepsilon}\dot{q} - \vartheta) \tag{7.46}$$

It is easy to see that, with the state vector $x' \triangleq \operatorname{col}[\tilde{q},\ \dot{q},\ \vartheta,\ z]$, an equilibrium point of (7.44) is the origin $x' = 0$.

4.2.2 Lyapunov Function Candidate

We will now construct a Lyapunov function for (7.44) – (7.46) whose derivate is negative definite in $(\tilde{q},\ \dot{q},\ \vartheta)$. To this end, remark that (7.44), (7.45) with $z \equiv 0$ are the closed loop equations of (Kelly, 1993b) which is known to be GAS with Lyapunov function

$$V_1(\tilde{q},\dot{q},\vartheta) \;=\; \frac{1}{2}\dot{q}^\top D\dot{q} + U_g(q) + \frac{1}{2}(q-\delta)^\top K_p'(q-\delta) + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta + c_1 \tag{7.47}$$

where c_1 is added to the systems total energy to enforce $(\tilde{q},\dot{q},\vartheta) = (0,0,0)$ to be a global and unique minimum of V_1. Taking the derivative of V_1 we get

$$\dot{V}_1 = -\vartheta^\top K_d B^{-1}A\vartheta + \dot{q}^\top z.$$

To cancel the second right hand term we add to V_1 the function $V_2(z) \triangleq \frac{\varepsilon}{2}z^\top K_I^{-1}z$ to get now

$$\dot{V}_1 + \dot{V}_2 = -\vartheta^\top K_d B^{-1}A\vartheta - \varepsilon z^\top(\tilde{q} - \vartheta).$$

At this point, to remove the cross products above and to enforce the desired negative definiteness of the derivative, we propose to add some cross terms to the Lyapunov function. First observe that with $V_3(\tilde{q},\dot{q}) \triangleq \tilde{q}^\top D\dot{q}$ we have

$$\dot{V}_3 = \tilde{q}^\top[C^\top(q,\dot{q})\dot{q} - g(q) + g(q_d) - K_p'\tilde{q} - K_D\vartheta + z] + \dot{q}^\top D\dot{q}$$

where we have used $\dot{D}(q,\dot{q}) = C(q,\dot{q}) + C^\top(q,\dot{q})$ to get the first right hand side term. Notice the presence of the term $\tilde{q}^\top z$. Using the bounds

$$\begin{aligned} \tilde{q}^\top C(q,\dot{q})\dot{q} \;&\le\; k_c\|\tilde{q}\|\|\dot{q}\|^2 \\ -\tilde{q}^\top\left[g(q) - g(q_d)\right] \;&\le\; k_g\|\tilde{q}\|^2 \end{aligned}$$

and some simple inequalities we get

$$\dot{V}_3 \;\le\; k_c\|\tilde{q}\|\|\dot{q}\|^2 - k'_{p_M}\|\tilde{q}\|^2 + k_{d_M}\|\tilde{q}\|\|\vartheta\| + k_g\|\tilde{q}\|^2 + \tilde{q}^\top z + d_M\|\dot{q}\|^2.$$

Proceeding analogously with the term $V_4(\tilde{q},\dot{q},\vartheta) \triangleq -\vartheta^\top D\dot{q}$, and adding them all together we get our *Lyapunov function candidate*[5] as

$$V \triangleq V_1 + V_2 + \varepsilon(V_3 + V_4), \qquad\qquad \varepsilon > 0 \tag{7.48}$$

[5] This function has been proposed in (Loria and Ortega, 1995) to study the output feedback tracking control problem.

which after some straightforward calculations and a suitable partition yields

$$\begin{aligned}\dot V &\le -\frac{\varepsilon}{2}k'_{p_m}\|\tilde q\|^2 - \frac{\varepsilon}{4}b_m d_m\|\dot q\|^2 - \frac{k_{d_m}a_m}{4b_M}\|\vartheta\|^2 - \varepsilon\left[\frac{1}{4}b_m d_m - d_M - k_c\|\vartheta\| - k_c\|\tilde q\|\right]\|\dot q\|^2 \\ &-\frac{\varepsilon}{2}\left\{\begin{bmatrix}\|\tilde q\| \\ \|\vartheta\|\end{bmatrix}^\top Q_1 \begin{bmatrix}\|\tilde q\| \\ \|\vartheta\|\end{bmatrix} - \begin{bmatrix}\|\vartheta\| \\ \|\dot q\|\end{bmatrix}^\top Q_2 \begin{bmatrix}\|\vartheta\| \\ \|\dot q\|\end{bmatrix}\right\} - \left[\frac{k_{d_m}a_m}{4b_M} - \varepsilon k_{d_m}\right]\|\vartheta\|^2\end{aligned} \tag{7.49}$$

where we defined

$$Q_1 \triangleq \begin{bmatrix} k'_{p_m} - 2k_g & -k'_{p_M} - k_{d_m} - k_g \\ -k'_{p_M} - k_{d_m} - k_g & \frac{k_{d_m}a_m}{2\varepsilon b_M}\end{bmatrix}, \quad Q_2 \triangleq \begin{bmatrix}\frac{k_{d_m}a_m}{2\varepsilon b_M} & -a_M d_M \\ -a_M d_M & b_m d_m\end{bmatrix}.$$

4.2.3 Asymptotic Stability

We will now give sufficient conditions for *positive definiteness* of V. To this end, we will partition V as $V = W_1 + W_2 + W_3 + W_4$ where

$$W_1 = \frac{1}{8}\dot q^\top D\dot q + \frac{1}{8}\tilde q^\top K'_p\tilde q + \varepsilon\tilde q^\top D\dot q, \tag{7.50}$$

$$W_2 = \frac{1}{8}\tilde q^\top K'_p\tilde q + U_g + \tilde q^\top K'_p(q_d - \delta) + \frac{1}{2}(q_d-\delta)^\top K'_p(q_d-\delta) + c_1, \tag{7.51}$$

$$W_3 = \frac{1}{8}\dot q^\top D\dot q + \frac{1}{4}\vartheta^\top K_d B^{-1}\vartheta - \varepsilon\vartheta^\top D\dot q, \tag{7.52}$$

$$W_4 = \frac{1}{4}\tilde q^\top K'_p\tilde q + \frac{\varepsilon}{2}\tilde z^\top K_I^{-1}\tilde z^\top + \frac{1}{4}\vartheta^\top K_d B^{-1}\vartheta + \frac{1}{4}\dot q^\top D\dot q. \tag{7.53}$$

Under the conditions of proposition 7.4, W_2 is positive definite (Kelly et al., 1994a). W_1 is positive definite if

$$\frac{1}{2}\sqrt{\frac{k'_{p_m}}{d_M}} > \varepsilon \tag{7.54}$$

while

$$2\sqrt{\frac{k_{d_m}}{b_M d_M}} > \varepsilon \tag{7.55}$$

insures W_3 to be positive definite. Thus, V is positive definite for ε sufficiently small, which entails, via (7.43), K_I sufficiently small.

We derive now sufficient conditions for $\dot V$ to be negative semidefinite in $(\tilde q,\ \dot q,\ \vartheta)$. If

$$\frac{(k'_{p_m} - 2k_g)k_{d_m}a_m}{2b_M\left[k'_{p_M} + k_{d_M} + k_g\right]^2} > \varepsilon \tag{7.56}$$

we have $Q_1 > 0$. In a similar way, (for all $\frac{b_m}{b_M} < \infty$), we have that $Q_2 > 0$ if

$$\frac{k_{d_m}a_m d_m}{2[a_m d_M]^2} > \varepsilon. \tag{7.57}$$

The third right hand term of (7.49) is negative if[6]

$$\frac{1}{2k_c}\left[\frac{1}{4}b_m d_m - d_M\right] > \|x\| \tag{7.58}$$

where the left hand side is positive due to (7.39). Finally, the last term in (7.49) is negative if

$$\frac{k_{d_m}a_m}{4b_M k_{d_M}} > \varepsilon \tag{7.59}$$

[6] Observe that we give this condition in terms of the original state x, instead of x'. This in order to derive the domain of attraction (and prove the semiglobal stability claim that requires ε arbitrarily small) in the coordinates x.

while (7.56), (7.57) and (7.59) are satisfied for ε sufficiently small. Therefore, $\dot{V}$ is locally negative semidefinite and the equilibrium is stable in the sense of Lyapunov. *Asymptotic stability* follows immediately invoking Krasovskii-LaSalle's invariance principle.

4.2.4 Domain of attraction

To define the *domain of attraction* we will first find some positive constants α_1, α_2 such that

$$\alpha_1 ||x||^2 \leq V(x) \leq \alpha_2 ||x||^2. \tag{7.60}$$

Notice that

$$V \geq W_4 \geq \frac{1}{4}\left[k'_{p_m}||\tilde{q}||^2 + \frac{k_{d_m}}{b_M}||\vartheta||^2\right] + \frac{1}{4}\left[d_m||\dot{q}||^2 + \frac{2\varepsilon}{k_{i_M}}||z||^2\right].$$

To obtain the lower bound in terms of x we need the following inequality

$$\left[1 - \frac{k_{i_M}}{\varepsilon}\right]||\tilde{v}||^2 + \left[\frac{k_{i_m}^2}{\varepsilon^2} - \frac{k_{i_M}}{\varepsilon}\right]||\tilde{q}||^2 \leq ||z||^2$$

which leads to

$$V \;\; \geq \;\; \tfrac{1}{4}\left\{\left[k'_{p_m} + 2\tfrac{k_{i_m}^2}{\varepsilon k_{i_M}} - 2\right]||\tilde{q}||^2 + d_m||\dot{q}||^2\right\} + \tfrac{1}{4}\left\{\tfrac{kdm}{b_M}||\vartheta||^2 + 2\left[\tfrac{\varepsilon}{k_{i_M}} - 1\right]||\tilde{v}||^2\right\} \tag{7.61}$$

so we define α_1 as

$$\alpha_1 \;\triangleq\; \frac{1}{4}\min\left\{k'_{p_m} + 2\left[\frac{k_{i_m}^2}{\varepsilon k_{i_M}} - 1\right],\; \frac{k_{d_m}}{b_M}, d_m,\; 2\left[\frac{\varepsilon}{k_{i_M}} - 1\right]\right\}.$$

In a similar manner, an upperbound on (7.48) is

$$\begin{aligned} V \;\; \leq \;\; & \left\{\frac{1}{4}\left[k'_{p_m} + 2k_g\right] + \frac{1}{2}\left[\frac{\varepsilon}{2}d_M + k'_{p_M}\right]\right\}||\tilde{q}||^2 + \left[(\varepsilon + \frac{1}{2})d_M\right]||\dot{q}||^2 \\ & + \frac{1}{2}\left[\varepsilon d_M + \frac{k_{d_m}}{b_M}\right]||\vartheta||^2 + \frac{\varepsilon}{2k_{i_m}}||z||^2. \end{aligned}$$

Now using (7.43) we have

$$||z||^2 \leq \left[1 + \frac{k_{i_M}}{\varepsilon}\right]\left[||\tilde{\nu}||^2 + \frac{k_{i_M}}{\varepsilon}||\tilde{q}||^2\right] \leq 2[||\tilde{q}||^2 + ||\tilde{\nu}||^2]$$

so we define

$$\alpha_2 \;\triangleq\; \max\left\{\left[(\varepsilon + \tfrac{1}{2})d_M\right], \tfrac{1}{2}\left[\varepsilon d_M + \tfrac{k_{d_M}}{b_m}\right], \tfrac{1}{4}\left[k'_{p_m} + 2k_g\right] + \tfrac{1}{2}\left[\tfrac{\varepsilon}{2}d_M + k'_{p_M}\right] + \tfrac{\varepsilon}{k_{i_m}}\right\}. \tag{7.62}$$

From (7.60) and (7.58) we conclude that the domain of attraction contains the set

$$||x|| \;\leq\; c_3 \triangleq \frac{1}{2k_c}\left[\frac{1}{2}b_m d_m - d_M\right]\sqrt{\frac{\alpha_1}{\alpha_2}}. \tag{7.63}$$

4.2.5 Semiglobal Stability

To establish *semiglobal stability* we must prove that, with a suitable choice of the controller gains, we can arbitrarily enlarge the domain of attraction. To this end, we propose to increase b_m and b_M at the same rate. The key question here is whether this can be done without violating the order relationships between B and ε imposed by the stability conditions (7.55), (7.56) and (7.59). The order relationship due to (7.55) is[7] $\varepsilon(b_m) = \mathcal{O}(1/\sqrt{b_m})$, while that of (7.56) and (7.59) is $\varepsilon(b_m) = \mathcal{O}(1/b_m)$. The latter being implied by the former for ε sufficiently small.

[7] A function of μ is denoted $\mathcal{O}(\mu^k)$ when for all $\mu \in [0, \mu_*]$ its norm is less than $c\mu^k$, where $c > 0$, $\mu_* > 0$ and k are some constants.

On the other hand, for b_m sufficiently large we can always find $\varepsilon > 0$ so that $\alpha_1 = c_4/b_m$ and $\alpha_2 = c_5$, where c_4, c_5 are constants independent of B. Replacing this in (7.63) we get

$$\lim_{b_m \to \infty} c_3^2 = \lim_{b_m \to \infty} c_6 \sqrt{b_m}$$

where c_6 is also independent of B. This proves that there exists $\varepsilon > 0$ such that 1) the stability conditions, are satisfied, i.e., verifying $\varepsilon = \mathcal{O}(1/\sqrt{b_m})$; 2) the domain of attraction is arbitrarily enlarged, that is, $\lim_{b_m \to \infty} c_3 = \infty$.

The proof is completed choosing, for the given ε, $k_{i_M} = \mathcal{O}(\varepsilon)$ and such that (7.43) holds.

∎

7.6 Remark. It is important to remark that when implementing the PI^{2}D controller, one should be careful with setting the initial conditions of the dirty derivatives filter (7.37)–(7.38): from the calculations above, the initial conditions $\vartheta(0)$ should be small enough to guarantee asymptotic stability. Also, in order to enlarge the domain of attraction, we proceed to increase b_m. Nevertheless, notice from (7.38) that the initial condition $\vartheta(0) = q_c(0) + Bq(0)$, hence the larger b_m is, the larger $\vartheta(0)$ may also be. A simple way to overcome this problem, is to set $q_c(0) = \vartheta(0) - Bq(0)$ for some fixed $\vartheta(0)$, $q(0)$ and B.

7.7 Remark. When velocity is *measurable* semiglobal stability of the classical PID controller follows as a corollary of our result. This can be established following the same steps of our proof with the Lyapunov function (Kelly, 1995)

$$V_1 + V_2 + \varepsilon V_3 + \frac{\varepsilon}{2}\tilde{q} K_D \tilde{q} - \frac{1}{2} g(q_d)^\top (K_p')^{-1} g(q_d) - U_g(q_d)$$

4.3 A passivity interpretation

Instrumental for our stability analysis has been the addition to the system's total energy of the *cross terms* $V_3 = \tilde{q}^\top D\dot{q}$ and $V_4 = \vartheta^\top D\dot{\tilde{q}}$ in the Lyapunov function (7.48). Several authors have used this kind of terms to solve various robotics problems, e.g., (Arimoto and Miyazaki, 1986; Wen and Bayard, 1988; Koditschek, 1989). It is worth mentioning that in the well known approach of (Slotine and Li, 1988), where the change of variable $s = \dot{q} + \lambda\tilde{q}$ was introduced, the Lyapunov-like function $V = \frac{1}{2}s^\top D(q)s$ used to prove GAS, contains also cross terms, namely $\lambda\tilde{q}^\top D(q)\dot{q}$. In this section we explain the significance of this modification in terms of passivity, or more precisely *passifiability*,[8] properties.

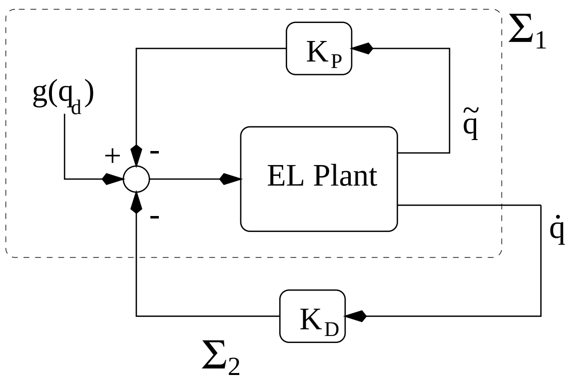


Figure 7.6: PD plus gravity compensation

[8] Notation for the possibility of rendering a system passive via feedback—a problem first studied in (Rodriguez and Ortega, 1990)–, also referred as feedback equivalence to a passive system in (Byrnes et al., 1991).

To this end, we find it convenient to first briefly review, from a passivity perspective, the schemes mentioned in section 1.2. For this, consider the system depicted in fig. 7.6. From proposition 6.5 we know that the plant's total energy function $T(q,\dot{q}) + V(q)$ qualifies as a storage function (Willems, 1972) for the supply rate $w(u,\dot{q}) = u^\top \dot{q}$. From this property *output strict passifiability* of the map $u \mapsto \dot{q}$ follows taking $u = -K_D\dot{q} + u_1$, with u_1 the input that shapes the potential energy.

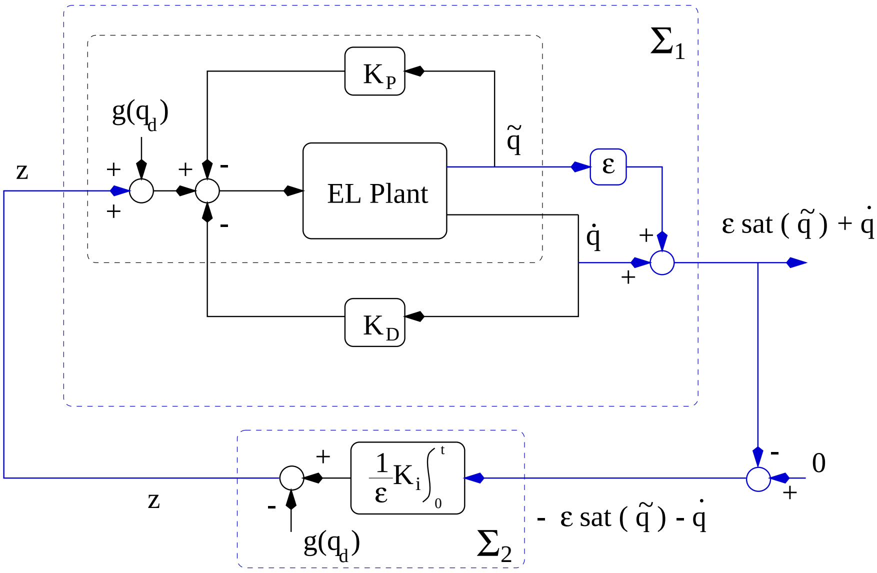


Figure 7.7: Nonlinear PID Controller (Arimoto 1994a)

Other applications –including the present study of PI controllers–, require a passifiability property of a map with q, instead of $\dot{q}$, at the output. This can be accomplished with a storage function that includes cross terms. Very recently, (Arimoto, 1994a) showed by using a saturation function $\text{sat}(\cdot)$ that both maps $\Sigma_1 : -z \mapsto \varepsilon\ \text{sat}(\tilde{q}) + \dot{q}$ and $\Sigma_2 : -\varepsilon\ \text{sat}(\tilde{q}) - \dot{q} \mapsto z$ (cf. figure 7.7) are passive blocks, hence the closed loop system is also passive.

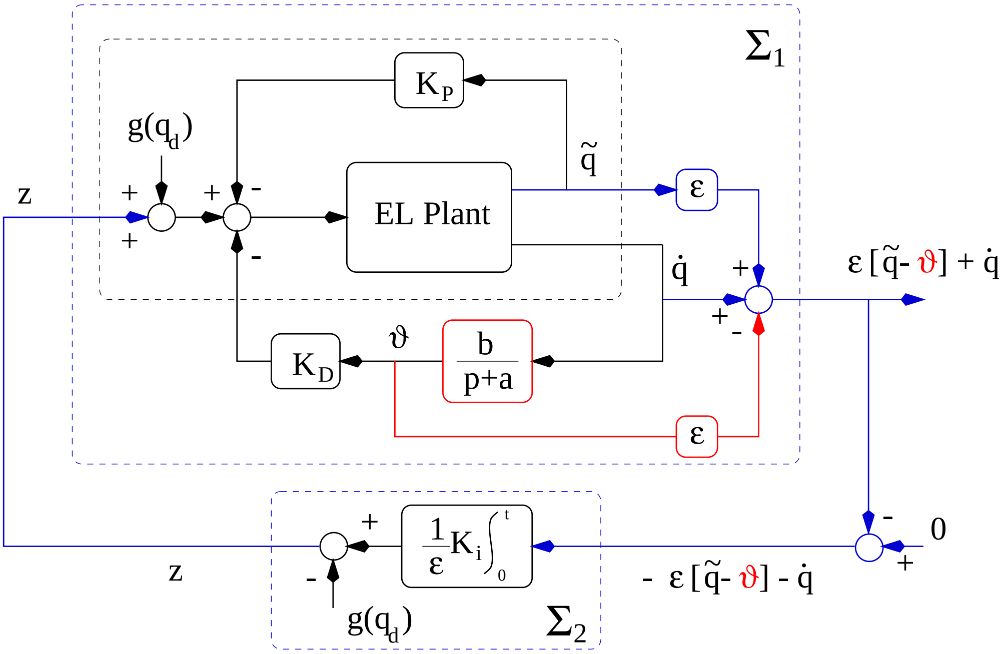


Figure 7.8: PI^{2}D Controller

As a matter of fact it can be proven that Σ_1 is OSP. Unfortunately, in our particular problem –where the controller structure is *restricted*–, the "trick" of using a saturation function does not lead us to OSP, actually, the output strict passifiability property we can establish is only *local*. That is, the property holds only for inputs that restrict the operator state to remain within a compact subset of the state space (Pota and Moylan, 1993). Nonetheless, this compact subset can be arbitrarily enlarged with high gains, this explains the semiglobal, instead of global, nature of our result.

These ideas are contained in the proposition below.

7.8 Proposition. *For the system depicted in fig. 7.8 defined by (7.44)-(7.46), there exist some control gains* K_P, K_D, K_I, A *and* B *such that*

(i) System Σ_1*: (7.44), (7.45) defines a* Locally Output Strictly Passive *(LOSP) operator* $z \mapsto \varepsilon[\tilde{q} - \vartheta] + \dot{q}$ *with storage function*

$$\phi(\tilde{q}, \dot{q}, \vartheta) \triangleq V_1(\tilde{q}, \dot{q}) + \varepsilon[V_3(\tilde{q}, \dot{q}) + V_4(\vartheta, \dot{q})].$$

(ii) System Σ_2*: (7.46) defines a passive operator* $-\varepsilon[\tilde{q} - \vartheta] - \dot{q} \mapsto z$ *with storage function* $V_2(z)$.

□

Proof.

(i): The time derivative of $\phi_2(\tilde{q}, \dot{q}, \vartheta)$ along the trajectories of (7.44), (7.45) can be written in the compact form

$$\dot{\phi}_2(\tilde{q}, \dot{q}, \vartheta) = \dot{V} + z^\top[\varepsilon(\tilde{q} - \vartheta) + \dot{q}].$$

From (7.49), for all $\tilde{q}, \vartheta, \dot{q}$ satisfying (7.56) – (7.59) we have that

$$\dot{\phi}_2(\tilde{q}, \dot{q}, \vartheta) \leq -\gamma_1 ||\tilde{q}||^2 - \gamma_2 ||\dot{q}||^2 - \gamma_3 ||\vartheta||^2 + z^\top[\varepsilon(\tilde{q} - \vartheta) + \dot{q}] \tag{7.64}$$

where we defined

$$\gamma_1 \triangleq \frac{\varepsilon b_m d_m}{2}, \quad \gamma_2 \triangleq \frac{\varepsilon k'_{p_m}}{2}, \quad \gamma_3 \triangleq \frac{k_{d_m} a_m}{4 b_M}. \tag{7.65}$$

Also notice that

$$||\varepsilon[\tilde{q}-\vartheta]+\dot{q}||^2 \leq \varepsilon^2(||\tilde{q}||^2+2||\tilde{q}||||\vartheta||+||\vartheta||^2)+2\varepsilon||\dot{q}||(||\tilde{q}||+||\vartheta||)+||\dot{q}||^2$$

from this, we get after some straightforward calculations that

$$\gamma_1||\dot{q}||^2+\gamma_2||\tilde{q}||^2+\gamma_3||\vartheta||^2 \geq ||\varepsilon[\tilde{q}-\vartheta]+\dot{q}||^2, \tag{7.66}$$

if $\gamma_1 \geq 2\varepsilon+1$ and $\gamma_2,\ \gamma_3 \geq 2\varepsilon^2+\varepsilon$. This clearly imposes new conditions on the controller gains:

$$\frac{8}{b_m d_m - 8} < \varepsilon \tag{7.67}$$

$$\frac{k_{p_m}-2}{4} > \varepsilon \tag{7.68}$$

$$\frac{k_{d_m} a_m}{4 b_M} > 2\varepsilon^2+\varepsilon. \tag{7.69}$$

It was proven before that b_m can be chosen arbitrarily large, hence the condition imposed by inequality (7.67) is met for sufficiently large b_m while (7.68) holds for a sufficiently small ε and (7.69) holds for sufficiently large a_m. Integrating (7.64) from 0 to T, on both sides of the inequality and using (7.66) we get

$$\phi_2(T)-\phi_2(0) \leq -||\varepsilon[\tilde{q}-\vartheta]+\dot{q}||_{2T}^2+\langle z \mid [\varepsilon(\tilde{q}-\vartheta)+\dot{q}]\rangle_T$$

since $\phi_2(\tilde{q},\dot{q},\vartheta)$ is positive definite for all $T \geq 0$,

$$\langle z \mid [\varepsilon(\tilde{q}-\vartheta)+\dot{q}]\rangle_T \geq ||\varepsilon[\tilde{q}-\vartheta]+\dot{q}||_{2T}^2-\beta_2$$

hence completing the proof.

(i): Straightforward. (Σ_2 is a simple integrator). ∎

7.9 Remark. Notice that inequality (7.64) only holds locally in the domain of attraction defined by (7.41). This is what leads us to *local* output strict passivity.

4.4 Simulation results

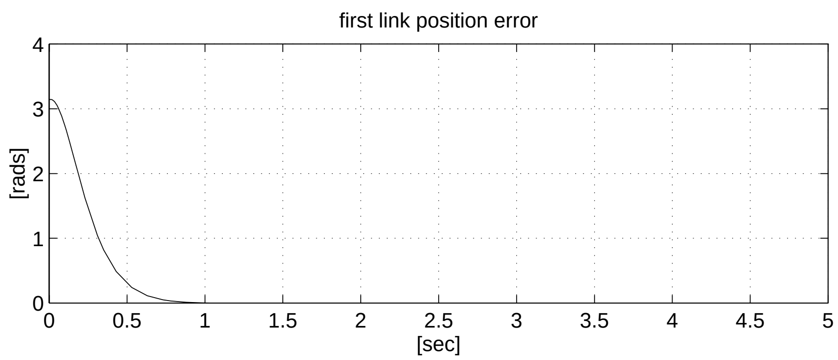


Figure 7.9: PI^{2}D control. First link position error

In this section we illustrate in simulation the performance of our algorithm in the two-rigid-link robot manipulator model of (Berghuis and Nijmeijer, 1993a) described in section **6**.5.1. For the sake of comparison we tested the performance of a simple PID controller with velocity measurements. We assumed

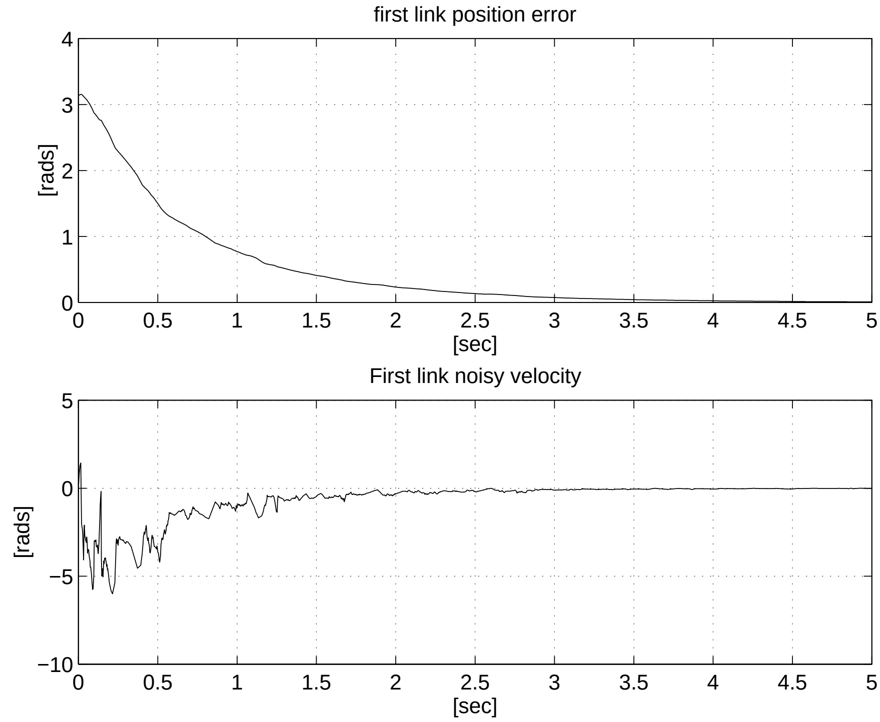


Figure 7.10: PID control. First link position error and noisy velocity measurement

that the measurements are affected by a random noise of 10% which obeys a uniform distribution. In both cases, the controller parameters were set to $K_P \triangleq \text{diag}\{[400,\ 400])\}$, $K_I \triangleq \text{diag}\{[0.004,\ 0.005]\}$, $K_D \triangleq \text{diag}\{[300,\ 300]\}$, $A \triangleq \text{diag}\{[3000,\ 3000]\}$ and $B \triangleq \text{diag}\{[1000,\ 1000]\}$. We started from zero initial conditions to achieve the constant reference $q_d \triangleq \text{col}[-\pi;\ -\pi]$.

In figure 7.9 we show the first link response using our PI^2D controller. In figure 7.10 we show the first link response using a common PID controller as well as the noisy velocity measurements corresponding to the same link. As a criterion of comparison, we have evaluated the integral square position error, (ISE) i.e. $J = \int_0^5 (\tilde{q}_p)dt$. For the PI$^2D$ controller, this calculation resulted in $J = 1.59$ while for the ordinary PID control, $J = 3.67$.

5 Concluding remarks

We have addressed in this chapter, two main control problems of EL systems: (1) output feedback regulation and (2) output feedback regulation with uncertain potential energy knowledge.

Concerning the first problem, we have given sufficient conditions for output feedback global stabilization of EL systems. The controller, which we choose to be also an EL system, is designed using the energy shaping plus damping injection ideas of the passivity-based approach.

Our contribution to this problem is the proof that damping injection *without* generalized velocity measurement is possible via the inclusion of a *dynamic extension* provided the system satisfies a *dissipation propagation* condition.

We have illustrated this technique with the case study of the flexible joints robots. In particular we showed that several apparently unrelated controllers appeared independently in the literature happen to belong to the EL class. We illustrated in simulations the performance of the two EL controllers (Ailon and Ortega, 1993; Kelly et al., 1994a).

It is clear that choosing the controller in the EL class is restrictive. For instance, the recent globally stabilizing linear controller of (Battiloti and Lanari, 1995) which measures link position does not seem to fit in the present framework.

Also, an obvious drawback of the EL controllers is that they require velocity measurements. For this, we have extended our results on output feedback set-point control, for the fully actuated EL systems, to the case when the potential energy knowledge is uncertain.

Our scheme relies on the passivity properties of the EL plant. The main features of this approach are: 1) It insures *semiglobal* stability of the desired equilibrium point; 2) only measurement of joint position is needed; 3) the only prior knowledge required for its implementation are *upperbounds* on the parameters of the gravity forces and the inertia and Coriolis matrices. Thus overcoming the important drawback of lack of robustness of all schemes that rely on exact *open-loop precompensation* of the gravity forces; 4) the controller is a linear time invariant $\mathrm{PI}^2\mathrm{D}$; 5) the stability analysis relies on basic Lyapunov theory with the Lyapunov function motivated from energy considerations. This result holds only for a reduced subclass of fully actuated EL systems.

Chapter 8

Trajectory tracking control

1 Introduction and brief literature review

The solution to the *state* feedback tracking control problem of fully damped Euler-Lagrange systems (in particular, rigid-robot manipulators) has been known from many years now – for a literature review, see e.g. (Ortega and Spong, 1989; Wen and Bayard, 1988) –, see also (Lozano and Brogliato, 1992a; Brogliato and Lozano, 1996) which was the first adaptive controller for flexible joint robots. Nevertheless, a drawback of many of the available results in the literature is that they require the measurement of joint *velocities*.

This fact has challenged researchers to solve the *global* output feedback control problem of robot manipulators. This problem has been open for many years now.

As in the regulation control problem, an alternative approach that has been widely considered in the literature is to design an observer that makes use of position information to reconstruct the velocity signal. Then, the controller is implemented replacing the velocity measurement by its estimate. As far as we know, the first result in this direction is due to (Nicosia and Tomei, 1990), who have used a nonlinear observer that reproduces the whole robot dynamics, in a PD plus gravity compensation scheme. The authors prove the equilibrium is *locally* asymptotically stable provided the observer gain satisfies some lower bound determined by the robot parameters and the trajectories error norms. See also (Canudas de Wit et al., 1990).

(Berghuis et al., 1992a) proposed a linear observer-computed torque scheme which exploits the feedback linearizing property of the computed torque scheme providing an efficient tuning technique. Later, using the same tuning idea (Berghuis and Nijmeijer, 1993b) presented a systematic procedure that exploits the passivity properties of robot manipulators into the design of controller-observer systems to solve both the position and tracking control problems. Local asymptotic stability was proved for sufficiently high gains.

Lately in (Loria and Ortega, 1995), based on a computed torque plus PD-like controller first appeared in (Wen and Bayard, 1988), we added the n-th order "approximative differentiation filter" studied in the previous chapter, to eliminate the necessity of velocity measurements. In that paper we proved semiglobal asymptotic stability of the closed loop system hence showing that the domain of attraction can be arbitrarily enlarged by increasing the filter gain. Some more recent and stronger results addressing the same problem are for instance: (Lim et al., 1994; Nicosia and Tomei, 1994), and (Nicosia and Tomei, 1995). Lim *et al* proposed the first adaptive controller for flexible joint robots by using only position measurements. Simultaneously, (Nicosia and Tomei, 1994) proposed a globally asymptotically stable observer-based controller needing only link (position and *velocity*) measurements and later in (Nicosia and Tomei, 1995) they extended this result to link position feedback.

Most recently, in (Loria, 1996), we presented as far as we know, the *first* smooth controller which renders the *one degree of freedom* (dof) EL system *globally asymptotically stable*. Our approach relies on a computed torque plus PD structure and a nonlinear dynamic extension based on the *linear* approximate differentiation filter. GAS is ensured provided the controller and filter gains satisfy some lower bound depending on the system parameters and the reference trajectory norm. Unfortunately, the performance of our approach can be ensured only for one dof systems and nothing can be claimed for the general multivariable case. On the other hand, as far as we know it is not possible to prove *global* asymptotic stability for none of the reported results which are semiglobally asymptotically stable, even for the simpler case of one dof systems.

Nonetheless, it must be underlined that (Burkov, 1995b) showed recently by using singular perturbation techniques, that a computed torque like controller plus a linear observer is capable of making a rigid joint robot track a trajectory starting from *any* initial conditions. The main drawback of this result is that no explicit bounds for the gains can be given. Thus, the author proves in an elegant way, the *existence* of an output feedback tracking controller that ensures *global* asymptotic stability.

During the typing process of this document I came aware of the interesting results contained in A.A.J. Lefeber's master's thesis (Lefeber, 1996). Of particular interest is the *composite control* approach to the global output feedback tracking control problem. The idea of this approach is simple: to apply a global output feedback *set point* control law (such as any of those mentioned in the previous chapter) from the initial time t_0 till some "switching time" t_s, at which it is supposed that the trajectories are contained in some pre-specified bounded set. At time t_s one switches to a local output feedback tracking control law (such as any among those mentioned above). The obvious drawback of this idea is that the controller is no longer smooth, furthermore, the switching time may depend on bounds on the *unmeasured* variables. The results contained in (Lefeber, 1996) concern the *existence* of the time instant t_s such that the closed loop system is globally asymptotically stable.

Thus, the composite control approach poses a new challenging theoretical problem and establishes a new path towards the solution of the longstanding open problem of global output feedback control of robot manipulators.

This chapter is organized as follows: in section 2 we present the results reported in (Loria and Ortega, 1995), in order to motivate the main result of this chapter, presented in section 3, the first globally asymptotically stabilizing controller for one degree of freedom Euler Lagrange systems without velocity measurements. In section 4 we broach a discussion on the improvement of the result reported in (Loria, 1996) with respect to (Loria and Ortega, 1995) and finally we report some simulation results in section 5, before concluding with some important remarks.

2 Semiglobal tracking control of robot manipulators

We present in this section two distinct but modest results: first, we show that by simply adding an n order filter (n being the number of degrees of freedom), in the rigid case velocity measurements are no longer needed thus obviating the necessity of observers. Second, we show that by applying a similar control law to flexible joints robots, the calculation of jerk can be removed. We prove in both cases, that for any set of initial conditions it is always possible to find a controller such that the closed loop system tends exponentially to a unique equilibrium point.

2.1 Rigid-joint robot manipulators

Consider the following problem:

Semiglobal Output Feedback Tracking Control of Rigid Joints Robots. (OF/RR) For the system

$$D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + g(q_1) = u \tag{8.1}$$

assume that *only link positions* is available for measurement. Under these conditions, define an internally stable (smooth) control law (whose gains may depend on the systems initial conditions) that ensures

$$\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} (q(t) - q_d(t)) = 0 \tag{8.2}$$

for all $q_d \in \mathcal{C}^2$, $\|q_d(t)\|$, $\|\dot{q}_d(t)\|$, $\|\ddot{q}_d(t)\| < B_d$.

A solution to the above-stated problem is presented below.

8.1 Proposition. *Consider the model (8.1) in closed loop with the control law*

$$
\begin{aligned}
u &= D(q)\ddot{q}_d + C(q,\dot{q}_d)\dot{q}_d + g(q) - K_p\tilde{q} - K_d\vartheta && (8.3)\\
\dot{q}_c &= -A(q_c + B\tilde{q}) && (8.4)\\
\vartheta &= q_c + B\tilde{q} && (8.5)
\end{aligned}
$$

where

$$K_p,\ K_d > 0 \tag{8.6}$$

are diagonal, $A > 0$ and $B > 0$ satisfies

$$b_m > \frac{3d_M}{d_m}. \tag{8.7}$$

Then, for any (bounded) initial condition $x_0 =$ col$[\tilde{q}(0), \dot{\tilde{q}}(0), \vartheta(0)]$ some sufficiently large (bounded) gains of the controller (8.3), (8.5) always exist such that (8.2) holds with a domain of attraction including

$$\left\{x \in I\!R^{3n} : \|x\| < c_1\right\} \tag{8.8}$$

where $\lim_{b_m \to \infty} c_1 = \infty$. Furthermore, $q(t) \to q_d(t)$ as $t \to \infty$, exponentially fast. □

Notice that (8.3) is the controller (4.6) proposed in (Wen and Bayard, 1988) where we have simply replaced the measurement of velocity in the last hand term by its approximate derivative (8.5). Furthermore, in the case of an invariant reference $\dot{q}_d(t) = 0$, control law (8.3) reduces to the EL controller of (Kelly et al., 1994a) which has been proven in section **7**.3.2 to be *globally asymptotically stable.*

2.1.1 Stability proof

The proof relies on *classical Lyapunov theory* and is divided in four parts. First, we define a suitable error equation for the closed loop system, whose unique equilibrium is at the desired value. Then, we propose a Lyapunov function candidate. Third, we prove that under the conditions of the theorem the proposed function qualifies as a Lyapunov function, and establish the exponential stability of the equilibrium invoking Lyapunov's second method. Finally we define the domain of attraction.

Error equation

Adding (8.3) to (8.1) we obtain

$$D\ddot{\tilde{q}} + C(q,\dot{q})\dot{q} - C(q,\dot{q}_d)\dot{q}_d + K_p\tilde{q} + K_d\vartheta = 0,$$

next adding and subtracting the term $C(q,\dot{q})\dot{q}_d$ we get by virtue of property **P6.3**

$$D\ddot{\tilde{q}} + C(q,\dot{q})\dot{\tilde{q}} - [C(q,\dot{q}_d) - C(q,\dot{q})]\dot{\tilde{q}} + K_p\tilde{q} + K_d\vartheta = 0.$$

Let us define $C_d \triangleq C(q,\dot{q}_d)$; using (8.5) we write the error equations

$$\left\{\begin{array}{l} D\ddot{\tilde{q}} + (C + C_d)\dot{\tilde{q}} + K_p\tilde{q} + K_d\vartheta = 0 \\ \\ \dot{\vartheta} = -A\vartheta + B\dot{\tilde{q}}. \end{array}\right. \tag{8.9}$$

Notice that a solution of (8.9) is col$[\tilde{q},\ \dot{\tilde{q}},\ \vartheta] =$ col$[0\ 0\ 0]$. We will now construct a Lyapunov function for (8.9) who has a global minimum at the origin and whose time derivate is negative definite in the state $x \triangleq$ col$[\tilde{q},\ \dot{\tilde{q}},\ \vartheta]$.

Lyapunov function candidate

Consider the time-varying function

$$V(t,x) = \frac{1}{2}\dot{\tilde{q}}^\top D\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^\top K_p\tilde{q} + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta + \varepsilon\tilde{q}^\top D\dot{\tilde{q}} - \varepsilon\vartheta^\top D\dot{\tilde{q}}. \tag{8.10}$$

We will now give sufficient conditions in order to guarantee *positive definiteness* of V. To simplify the proof, we will partition V as $V = W_1 + W_2$ where

$$W_1 = \frac{1}{4}\dot{\tilde{q}}^\top D\dot{\tilde{q}} + \frac{1}{4}\tilde{q}^\top K_p\tilde{q} + \frac{1}{4}\vartheta^\top K_d B^{-1}\vartheta + \varepsilon\tilde{q}^\top D\dot{\tilde{q}} - \varepsilon\vartheta^\top D\dot{\tilde{q}} \tag{8.11}$$

$$W_2 = \frac{1}{4}\dot{\tilde{q}}^\top D\dot{\tilde{q}} + \frac{1}{4}\tilde{q}^\top K_p\tilde{q} + \frac{1}{4}\vartheta^\top K_d B^{-1}\vartheta. \tag{8.12}$$

Notice that (8.11) can be written in matrix form as

$$W_1 = \frac{1}{4}\begin{bmatrix}\tilde{q}\\ \dot{\tilde{q}}\end{bmatrix}^\top \underbrace{\begin{bmatrix}K_p & 2\varepsilon D\\ 2\varepsilon D & \frac{1}{2}D\end{bmatrix}}_{P_1}\begin{bmatrix}\tilde{q}\\ \dot{\tilde{q}}\end{bmatrix} + \frac{1}{4}\begin{bmatrix}\dot{\tilde{q}}\\ \vartheta\end{bmatrix}^\top \underbrace{\begin{bmatrix}\frac{1}{2}D & -2\varepsilon D\\ -2\varepsilon D & K_D B^{-1}\end{bmatrix}}_{P_2}\begin{bmatrix}\dot{\tilde{q}}\\ \vartheta\end{bmatrix}, \tag{8.13}$$

from the definition of K_p, P_1 is positive definite if

$$\frac{1}{2}\sqrt{\frac{k_{p_m}}{2d_M}} > \varepsilon \tag{8.14}$$

similarly, from the definitions of K_d and B, P_2 is positive definite if

$$\frac{1}{2}\sqrt{\frac{k_{d_m}}{2b_M d_M}} > \varepsilon \tag{8.15}$$

while W_2 is trivially positive definite for any positive gain matrices.

Furthermore, by virtue of (8.14) and (8.15) and the nature of D, W_1 is strictly convex, hence radially unbounded.

In a similar way, from (8.6), and (8.7) W_2 is trivially strictly convex.

Lyapunov function derivative and exponential stability

In this subsection we show that the time derivative of (8.10) along the trajectories of (8.9) is *locally* negative definite in the whole state x, hence that $x(t)$ tends to zero exponentially. Using properties **P6.1** and **P6.3** we have that

$$\begin{aligned}\dot{V} \leq\ & -\vartheta^\top K_d B^{-1}A\vartheta + \dot{\tilde{q}}C_d\dot{\tilde{q}} + \varepsilon[\dot{\tilde{q}}^\top D\dot{\tilde{q}} + \tilde{q}^\top C(q,\dot{q})\dot{\tilde{q}} - \tilde{q}^\top K_p\tilde{q} - \tilde{q}^\top K_d\vartheta + \\ & +(\tilde{q}-\vartheta)^\top[C_d - C_d^\top]\dot{\tilde{q}} + \vartheta^\top A^\top D\dot{\tilde{q}} - \dot{\tilde{q}}^\top B^\top D\dot{\tilde{q}} - \vartheta^\top C(q,\dot{q})\dot{\tilde{q}} + \vartheta^\top K_p\tilde{q} + \\ & +\vartheta^\top K_d\vartheta]\end{aligned} \tag{8.16}$$

and after some simple boundings we write (8.16) in the form

$$\begin{aligned}\dot{V} \leq\ & -\frac{\varepsilon}{2}\begin{bmatrix}\|\tilde{q}\|\\ \|\vartheta\|\end{bmatrix}^\top \overbrace{\begin{bmatrix}k_{p_m} & -k_{p_M} - k_{d_M}\\ -k_{p_m} - k_{d_M} & \frac{1}{2\varepsilon}k_{d_m}b_M^{-1}a_m\end{bmatrix}}^{Q_1}\begin{bmatrix}\|\tilde{q}\|\\ \|\vartheta\|\end{bmatrix}\\ & -\frac{\varepsilon}{2}\begin{bmatrix}\|\vartheta\|\\ \|\dot{\tilde{q}}\|\end{bmatrix}^\top \overbrace{\begin{bmatrix}\frac{1}{2\varepsilon}k_{d_m}b_M^{-1}a_m & -2k_cB_d - a_Md_M\\ -2k_cB_d - a_Md_M & \frac{1}{3}b_md_m\end{bmatrix}}^{Q_2}\begin{bmatrix}\|\vartheta\|\\ \|\dot{\tilde{q}}\|\end{bmatrix}\end{aligned}$$

$$
-\frac{\varepsilon}{2}\begin{bmatrix} \|\tilde{q}\| \\ \|\dot{\tilde{q}}\| \end{bmatrix}^{\top} \overbrace{\begin{bmatrix} k_{p_m} & -2k_cB_d \\ -2k_cB_d & \frac{1}{3}b_md_m \end{bmatrix}}^{Q_3} \begin{bmatrix} \|\tilde{q}\| \\ \|\dot{\tilde{q}}\| \end{bmatrix}
$$

$$
-\left[\overbrace{\left[\frac{\varepsilon}{3}b_md_m - k_cB_d\right]}^{\lambda_1} + \varepsilon\overbrace{\left[\frac{1}{3}b_md_m - d_M - k_c(\|\vartheta\| + \|\tilde{q}\|)\right]}^{\lambda_2}\right]\|\dot{\tilde{q}}\|^2
$$

$$
-\overbrace{\left[\frac{1}{2}k_{d_m}b_M^{-1}a_m - \varepsilon k_{d_M}\right]}^{\lambda_3}\|\vartheta\|^2. \tag{8.17}
$$

We derive now sufficient conditions for $\dot{V}$ to be *locally* negative definite. First, considering (8.6) and (8.7), matrix Q_1 is positive definite if

$$
\frac{k_{p_m}k_{d_m}a_m}{2b_M\left[k_{p_M} + k_{d_M}\right]^2} > \varepsilon. \tag{8.18}
$$

In a similar way, $Q_2 > 0$ and $Q_3 > 0$ respectively if

$$
\frac{k_{d_m}a_mb_md_m}{6b_M[2k_cB_da_Md_M]^2} > \varepsilon, \qquad \frac{b_md_mk_{p_m}}{12k_c^2B_d^2} > \varepsilon. \tag{8.19}
$$

Notice that the positivity of constant λ_1 imposes a lower bound on ε, i.e.,

$$
\frac{3k_cB_d}{b_md_m} \leq \varepsilon \tag{8.20}
$$

while $\lambda_2 > 0$ if

$$
\frac{1}{2k_c}\left[b_md_m - 3d_M\right] > \|x\|, \tag{8.21}
$$

hence, condition (8.7). Finally, λ_3 is positive if

$$
\frac{k_{d_m}a_m}{2k_{d_M}b_M} > \varepsilon. \tag{8.22}
$$

Inequalities (8.14), (8.15), (8.18), (8.19) and (8.22) are satisfied for ε sufficiently small, while (8.20) is satisfied for a B sufficiently large. Thus it is always possible to find some controller gains depending on initial conditions and the desired trajectories to insure that all the above written inequalities hold. Therefore, (8.16) is *locally* negative definite and the equilibrium is exponentially stable in the sense of Lyapunov.

8.2 Remark. Notice that in contrast to (Berghuis et al., 1992b) the constant ε is not used in the controller but only for the stability proof.

Domain of attraction

In this section we define the domain of attraction and we prove that it can be enlarged by increasing the controller gains. For this, we will first find some positive constants α_1, α_2 such that

$$
\alpha_1\|x\|^2 \leq V(t,x) \leq \alpha_2\|x\|^2. \tag{8.23}
$$

Notice that from (8.14) and (8.15)

$$
V \;\geq\; W_2 \;\geq\; \frac{1}{4}\left[k_{p_m}\|\tilde{q}\|^2 + \frac{k_{d_m}}{b_M}\|\vartheta\|^2 + d_m\|\dot{q}\|^2\right]
$$

so we define α_1 as

$$
\alpha_1 \;\triangleq\; \frac{1}{4}\min\{k_{p_m},\ \frac{k_{d_M}}{b_M},\ d_m\}.
$$

In a similar manner, an upperbound on (8.10) is

$$V \leq \left[\frac{\varepsilon}{2}d_M + \frac{1}{2}k_{p_M}\right]||\tilde{q}||^2 + \left[(\varepsilon + \frac{1}{2})d_M\right]||\dot{\tilde{q}}||^2 + \frac{1}{2}\left[\varepsilon d_M + \frac{k_{d_M}}{b_m}\right]||\vartheta||^2$$

so we define

$$\alpha_2 \triangleq \max\left\{\left[\frac{\varepsilon}{2}d_M + \frac{1}{2}k_{p_M}\right], \left[(\varepsilon + \frac{1}{2})d_M\right], \frac{1}{2}\left[\varepsilon d_M + \frac{k_{d_M}}{b_m})\right]\right\}.$$

From (8.21) and (8.23) we conclude that the domain of attraction contains the set

$$||x|| \leq c_1 \triangleq \frac{1}{2k_c}[b_m d_m - 3d_M]\sqrt{\frac{\alpha_1}{\alpha_2}}.$$

Semiglobal stability

To establish *semiglobal stability* we must prove that, with a suitable choice of the controller gains, we can arbitrarily enlarge the domain of attraction. For this, we propose to increase b_m. Thus, the proof is completed checking that $\lim_{b_m\to\infty} c_1 = \infty$.

To this end, notice that

$$\lim_{b_m\to\infty} \alpha_1 = c_3\frac{1}{b_M}, \quad \lim_{b_m\to\infty} \alpha_2 = c_4$$

where c_3, c_4 are constants independent of B. Consequently,

$$\lim_{b_m\to\infty} c_1 = \lim_{b_m\to\infty} c_5\frac{b_m}{\sqrt{b_M}} = \infty$$

where c_3 is also independent of B and to get the last identity we have used the fact that $\frac{b_M}{b_m}$ is bounded.

■

2.2 Flexible joint robots

Consider the following problem:

Semiglobal State Feedback Tracking Control of Flexible Joints Robots (SF/FR) For the system

$$\begin{cases} D_l(q_1)\ddot{q}_1 + C(q_1,\dot{q}_1)\dot{q}_1 + g(q_1) = K(q_2 - q_1) \\ J\ddot{q}_2 + K(q_2 - q_1) = u. \end{cases} \tag{8.24}$$

and assume full state available for measurement. Then, find an internally stable control law (whose gains may depend on the systems initial conditions) that, without requirement of calculating $q_1^{(3)}$, insures (8.2) for all $q_{1d} \in \mathcal{C}^4$, $||q_{1d}(t)||$, $||\dot{q}_{1d}(t)||$, $||\ddot{q}_{1d}(t)|| < B_d$

for which we propose:

8.3 Proposition. *Consider the system (8.24) in closed loop with the control law*

$$u = J\ddot{q}_{2d} + K(q_{2d} - q_{1d}) - K_{P2}\tilde{q}_2 - K_{D2}\vartheta_2 \tag{8.25}$$
$$\dot{q}_{c_i} = -A_i(q_{c_i} + B_i\tilde{q}_i) \tag{8.26}$$
$$\vartheta_i = q_{c_i} + B_i\tilde{q}_i, \qquad i = 1,2. \tag{8.27}$$

where we define

$$q_{2d} \triangleq K^{-1}[D(q_1)\ddot{q}_{1d} + C(q_1,\dot{q}_{1d})\dot{q}_{1d} + g(q_1) - K_{P1}\tilde{q}_1 - K_{D1}\vartheta_1] + q_{1d},$$

matrices

$$A_i, B_i, K_{P_i},\ K_{D_i} > 0 \tag{8.28}$$

are diagonal and

$$b_m > \frac{3\lambda_{\max}(\mathcal{D})}{\lambda_{\min}(\mathcal{D})}, \quad \mathcal{D} \triangleq blockdiag[D_l \ \ J]\,. \tag{8.29}$$

Then, for any (bounded) initial conditions $x_0 = col[\tilde{q}(0), \dot{\tilde{q}}(0), \vartheta(0)]$ *where* $\tilde{q} \triangleq col[\tilde{q}_1, \tilde{q}_2]$ *and* $\vartheta \triangleq col[\vartheta_1, \vartheta_2]$, *some sufficiently large (bounded) gains for the controller (8.25) always exist such that (8.2) holds. Furthermore, a domain of attraction is defined by*

$$\left\{x \in I\!R^{6n} : \|x\| < c_2\right\} \tag{8.30}$$

where $\lim_{b_{im}\to\infty} c_2 = \infty$. *Furthermore,* $q_1(t) \to q_{1d}(t)$ *as* $t \to \infty$, *exponentially fast.* □

Notice that the calculation of u in (8.25) requires $\ddot{q}_{2d}$ however, in contrast with other solutions to this problem, our controller does not require the calculation of $q_1^{(3)}$. This stems from the use of $\dot{q}_{1d}$ instead of $\dot{q}_1$ in the second right hand term of q_{2d}, and the use of the filter. The second derivative of q_{2d} still needs link acceleration and velocity. Yet, only link velocity is considered to be available for measurement and acceleration can be computed using the first equation of (8.24).

In a similar way as in the previous proof, the error equation of (8.24) and (8.25) is

$$\begin{cases} \mathcal{D}\ddot{\tilde{q}} + (\mathcal{C} + \mathcal{C}_d)\dot{\tilde{q}} + \mathcal{K}_{\mathcal{P}}\tilde{q} + \mathcal{K}_{\mathcal{D}}\vartheta = 0 \\ \dot{\vartheta} = -\mathcal{A}\vartheta + \mathcal{B}\dot{\tilde{q}} \end{cases} \tag{8.31}$$

where for simplicity we have omitted the arguments and we defined $\mathcal{A} \triangleq \mathrm{blockdiag}\{A_1,\ A_2\}$, $\mathcal{B} \triangleq \mathrm{blockdiag}\{B_1,\ B_2\}$, $\mathcal{K}_{\mathcal{D}} \triangleq \mathrm{blockdiag}\{K_{D1},\ K_{D2}\}$, and

$$\mathcal{C} = \begin{bmatrix} C(q_1, \dot{q}_1) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C}_d = \begin{bmatrix} C(q_1, \dot{q}_{1d}) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{K}_{\mathcal{P}} = \begin{bmatrix} K_{P1} + K & -K \\ -K & K_{P2} + K \end{bmatrix}.$$

Notice that properties **P6.1** – **P6.3** hold as well for $\mathcal{D}$, $\mathcal{C}$, and $\mathcal{C}_d$. It is obvious that $\mathcal{K}_{\mathcal{P}} > 0$ if $K_{P1} > 0$ and $K_{P2} > 0$. Notice that as before, a solution of (8.31) is $\tilde{x} = \mathrm{col}[\tilde{q},\ \dot{\tilde{q}},\ \vartheta] = \mathrm{col}[0,\ 0,\ 0]$. Now, consider the Lyapunov candidate function

$$V(x(t), t) = \frac{1}{2}\dot{\tilde{q}}^\top \mathcal{D}\dot{\tilde{q}} + \frac{1}{2}\tilde{q}^\top \mathcal{K}_{\mathcal{P}}\tilde{q} + \frac{1}{2}\vartheta^\top \mathcal{K}_{\mathcal{D}} B^{-1}\vartheta + \varepsilon\tilde{q}^\top \mathcal{D}\dot{\tilde{q}} - \varepsilon\vartheta^\top \mathcal{D}\dot{\tilde{q}} \tag{8.32}$$

and note that (8.31) is similar to (8.9) as well as (8.32) is similar to (8.10). The main difference between this equations is that $\mathcal{K}_{\mathcal{P}}$ is no longer diagonal, yet, it is positive definite so all the conditions (8.14), (8.15), (8.18), (8.19), (8.20) and (8.22), apply *mutatis mutandis* to this case. The rest of the proof is based on similar arguments to those used in the proof of proposition 8.1.

■

8.4 Remark. Notice that, what hinders achieving global stability in proposition 8.1 is the presence of the cubic terms $\varepsilon k_c \left\|\dot{\tilde{q}}\right\|^2 (\|\tilde{q}\| + \|\vartheta\|)$ that appear in the Lyapunov function derivative entailed by the quadratic velocity dependence of the Coriolis and centrifugal forces term (cf. eq. 8.17). Roughly speaking, this leads us to the definition of a basin of attraction which at best can be infinitely enlarged by an infinite filter gain.

3 GAS tracking control of one-degree-of-freedom EL systems

Motivated by the result of proposition 8.1 and the technique of cross terms, we have searched for a way of "automatically" enlarging the domain of attraction at the same growing rate of the state variables. The result of this research is presented in this section, a computed torque-like controller plus a nonlinear dynamic extension based on the *linear* approximate differentiation filter (8.5). In order to automatically enlarge the domain of attraction we introduce hyperbolic trigonometric functions in both controller and filter. In particular we use $\sinh(\vartheta)$ where ϑ is the output of the linear approximative differentiation filter. By this means, we get a negative radially unbounded cubic term in the Lyapunov function derivative which dominates the Coriolis (non dissipative) forces term. Interestingly enough, we achieve this in spite of keeping the controller and filter gains *constant*.

3.1 Problem formulation and its solution

Consider the

Output feedback tracking control problem.
Consider the one degree of freedom EL system dynamic model

$$d(q)\ddot{q} + c(q)\dot{q}^2 + g(q) = u \tag{8.33}$$

where $q \in \mathbb{R}$ is the generalized position, $d(q)$ is the inertia term which satisfies $0 < d_m \leq d(q) \leq d_M$, the Coriolis and centrifugal forces term $c(q) \stackrel{\triangle}{=} \frac{1}{2}\frac{\partial}{\partial q}\{d(q)\}$ satisfies $k_c > |c(q)|$, $\forall\ q \in \mathbb{R}$ and $g(q)$ is the gravitational force term.

Under these conditions, considering only position measurements, *any* initial condition $\operatorname{col}[q(0),\ \dot{q}(0)]$ and any desired twice continuously differentiable reference trajectory, i.e. $q_d(t) \in \mathcal{C}^2$ such that $|\dot{q}_d| < \beta_d$, find some control input u such that

$$\lim_{t\to\infty} \tilde{q}(t) = \lim_{t\to\infty} [q(t) - q_d(t)] = 0. \tag{8.34}$$

8.5 Remark. It is worth mentioning that this model contains as a particular case, the simple pendulum, for an example of a physical system modeled by (8.33) see (Li and Horowitz, 1996).

Before presenting our main result we want to bring to the attention of the reader the difficulty of the output feedback tracking control problem for the *one degree of freedom* system (8.33). For this, let us cite the interesting negative result of (Mazenc et al., 1994) which establishes that the zero solution of the *autonomous* system:

$$\ddot{x} + \dot{x}^n = u \tag{8.35}$$

where $x \in \mathbb{R}$ is taken as output, *cannot* be globally asymptotically stabilized by output feedback for $n > 2$.

It is important to remark that, in (Mazenc et al., 1994) the authors provide a very simple stability proof for a dynamic output feedback assuming $n = 2$, that is, a particular case of system (8.33). Nonetheless, the problem solved in that reference is the set point control problem, hence the authors can use LaSalle's invariance principle. In this section, we are interested in the *tracking* control problem, hence we deal with a *non autonomous* nonlinear system. Our main result is stated below.

8.6 Proposition. *Consider the system (8.33) in closed loop with*[1]

$$\begin{aligned} u &= -k_p\ \tanh(\tilde{q}) - k_d\ \cosh(\vartheta)\ \sinh(\vartheta) + g(q) + c(q)\dot{q}_d^2 + d(q)\ddot{q}_d && (8.36)\\ \dot{q}_c &= -a\ \sinh(q_c + b\tilde{q}) && (8.37)\\ \vartheta &= q_c + b\tilde{q} && (8.38)\end{aligned}$$

where k_p, k_d, a, and b are positive constants such that

$$b > \frac{6k_c + 3d_M}{d_m} \tag{8.39}$$

[1] For a recall on hyperbolic trigonometric functions see appendix D.

$$\frac{3k_c\beta_d}{bd_m} < \min\left\{\frac{ak_pk_d}{(k_p+k_d)^2 b}, \frac{k_d d_m}{3ad_M^2}, \frac{a}{4b}, \left(\frac{k_p}{6d_M}\right)^{1/2}, \left(\frac{k_d}{3d_M b}\right)^{1/2}\right\} \tag{8.40}$$

then the output feedback tracking control problem is solved. In particular, the conditions above are met for any b satisfying (8.39) and

$$\frac{3d_m^2}{d_M^3} b \geq k_p = k_d = 3a > \frac{36 k_c \beta_d}{d_m} \tag{8.41}$$

□

3.2 Stability proof

The proof is based on Lyapunov stability theory. For the sake of clarity it is divided in the following steps: first we derive a "suitable" closed loop equation; second, we propose a Lyapunov function candidate and show that it is positive definite and radially unbounded; third, we show that its derivative is *globally* negative definite under the conditions of proposition 8.6; finally, we invoke the second Lyapunov method to claim *global* asymptotic stability.

3.2.1 Error equation

The first error equation follows directly by substituting (8.36) in (8.33) and adding and subtracting the term $c(q)\dot{q}\dot{q}_d$ to get

$$d(q)\ddot{\tilde{q}} + [c(q)\dot{q} + c(q)\dot{q}_d]\dot{\tilde{q}} + k_p \tanh(\tilde{q}) + k_d \cosh(\vartheta)\sinh(\vartheta) = 0, \tag{8.42}$$

second, differentiating (8.38) and using (8.37) we obtain

$$\dot{\vartheta} = -a \sinh(\vartheta) + b\dot{\tilde{q}}. \tag{8.43}$$

Next, we will study the stability of the error system (8.42), (8.43). For this we propose below a Lyapunov function candidate.

3.2.2 Lyapunov function candidate

Consider the function

$$V(\tilde{q}, \dot{\tilde{q}}, \vartheta) = \frac{1}{2} d(q)\dot{\tilde{q}}^2 + k_p \ln|\cosh(\tilde{q})| + \frac{k_d}{2b}\sinh(\vartheta)^2 + \varepsilon[\tanh(\tilde{q}) - \sinh(\vartheta)] d(q)\dot{\tilde{q}} \tag{8.44}$$

where $\varepsilon > 0$ is sufficiently small. To prove that (8.44) is positive definite and radially unbounded in $\tilde{q}$, $\dot{\tilde{q}}$ and ϑ, we define the auxiliary functions

$$W_1(\dot{\tilde{q}}, \tilde{q}) \triangleq \frac{1}{6} d(q)\dot{\tilde{q}}^2 + k_p' \tanh^2(\tilde{q}) + \varepsilon d(q)\tanh(\tilde{q})\dot{\tilde{q}}$$

$$W_2(\dot{\tilde{q}}, \vartheta) \triangleq \frac{1}{6} d(q)\dot{\tilde{q}}^2 + \frac{k_d}{2b}\sinh(\vartheta)^2 - \varepsilon d(q)\sinh(\vartheta)\dot{\tilde{q}}$$

where k_p' is a positive constant. Using lemma D.2, it is easy to see that if $k_p' < k_p/4$,

$$V(\tilde{q}, \dot{\tilde{q}}, \vartheta) > W_1(\dot{\tilde{q}}, \tilde{q}) + W_2(\dot{\tilde{q}}, \vartheta) + \frac{k_p}{2}\ln|\cosh(\tilde{q})|. \tag{8.45}$$

By definition (see Appendix D) the term $\frac{k_p}{2}\ln|\cosh(\tilde{q})|$ is radially unbounded and has a strict global minimum at $\tilde{q} = 0$, hence it suffices to prove that $W_1(\dot{\tilde{q}}, \tilde{q})$ and $W_2(\dot{\tilde{q}}, \vartheta)$ are both positive definite and further, that $W_2(\tilde{q}, \vartheta)$ is radially unbounded in $\dot{\tilde{q}}$ and ϑ. To show that this is the case, notice that we can write

$$W_1(\tilde{q}, \dot{\tilde{q}}) = \begin{bmatrix} \dot{\tilde{q}} \\ \tanh(\tilde{q}) \end{bmatrix}^\top \begin{bmatrix} \frac{1}{6}d(q) & \frac{\varepsilon}{2}d(q) \\ \frac{\varepsilon}{2}d(q) & k_p' \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}} \\ \tanh(\tilde{q}) \end{bmatrix}$$

$$W_2(\dot{\tilde{q}}, \vartheta) = \begin{bmatrix} \dot{\tilde{q}} \\ \sinh(\vartheta) \end{bmatrix}^\top \begin{bmatrix} \frac{1}{6}d(q) & \frac{-\varepsilon}{2}d(q) \\ \frac{-\varepsilon}{2}d(q) & \frac{k_d}{2b} \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}} \\ \sinh(\vartheta) \end{bmatrix}.$$

Clearly, $W_1(\dot{\tilde{q}}, \tilde{q})$ and $W_2(\dot{\tilde{q}}, \vartheta)$ are positive definite respectively in $(\tanh(\tilde{q}), \dot{\tilde{q}})$ and $(\sinh(\vartheta), \dot{\tilde{q}})$ if

$$\left(\frac{2k_p'}{3d_M}\right)^{1/2} > \varepsilon, \quad \left(\frac{k_d}{3d_M b}\right)^{1/2} > \varepsilon \tag{8.46}$$

which hold for a sufficiently small ε. From (8.45) we get that $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is *globally* positive definite in $(\tanh(\tilde{q}), \dot{\tilde{q}}, \sinh(\vartheta))$. From here, using properties **PD.3** and **PD.1** of appendix D, we can conclude that $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is *globally* positive definite in $(\tilde{q}, \dot{\tilde{q}}, \vartheta)$. Furthermore, from (8.46) we have that $W_2(\dot{\tilde{q}}, \vartheta)$ is quadratic in $\sinh(\vartheta)$ and $\dot{\tilde{q}}$, now since $|\sinh(\vartheta)| \to \infty \Leftrightarrow |\vartheta| \to \infty$, we conclude that W_2 is also radially unbounded in $|\vartheta|$. It follows finally that $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is positive definite and radially unbounded in $(\tilde{q}, \dot{\tilde{q}}, \vartheta)$.

3.2.3 Global asymptotic stability

In this section we show that under the conditions of proposition 8.6, $\dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is *globally* negative definite, stability follows directly by invoking the Lyapunov second method.

The time derivative of $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ along the trajectories of (8.42) and (8.43) using $\dot{d}(q) = 2c(q)\dot{q}$ is

$$\begin{aligned} \dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta) \quad = \quad & -c(q)\dot{q}_d\dot{\tilde{q}}^2 - \frac{ak_d}{b}\cosh(\vartheta)\sinh(\vartheta)^2 - \varepsilon k_p \tanh^2(\tilde{q}) - \varepsilon b d(q)\cosh(\vartheta)\dot{\tilde{q}}^2 + \\ & +\varepsilon d(q)\,\mathrm{sech}^2(\tilde{q})\dot{\tilde{q}}^2 + \varepsilon c(q)\tanh(\tilde{q})\dot{\tilde{q}}^2 - \varepsilon k_d \cosh(\vartheta)\sinh(\vartheta)\tanh(\tilde{q}) + \\ & -\varepsilon c(q)\sinh(\vartheta)\dot{\tilde{q}}^2 + \varepsilon a d(q)\sinh(\vartheta)\cosh(\vartheta)\dot{\tilde{q}} + \\ & +\varepsilon k_p \tanh(\tilde{q})\sinh(\vartheta) + \varepsilon k_d \cosh(\vartheta)\sinh(\vartheta)^2 \end{aligned} \tag{8.47}$$

now, using the properties presented in appendix D, we get

$$\begin{aligned} \dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta) \quad \le \quad & -\frac{\varepsilon}{2}\begin{bmatrix} |\tanh(\tilde{q})| \\ |\sinh(\vartheta)| \end{bmatrix}^\top Q_1 \begin{bmatrix} |\tanh(\tilde{q})| \\ |\sinh(\vartheta)| \end{bmatrix} - \\ & -\frac{\varepsilon}{2}\begin{bmatrix} |\sinh(\vartheta)| \\ |\dot{\tilde{q}}| \end{bmatrix}^\top Q_2 \begin{bmatrix} |\sinh(\vartheta)| \\ |\dot{\tilde{q}}| \end{bmatrix}\cosh(\vartheta) - \\ & -\frac{\varepsilon}{3}(\lambda_1 + \lambda_2)\dot{\tilde{q}}^2 - \frac{1}{2}\lambda_3 \sinh(\vartheta)^2 \end{aligned} \tag{8.48}$$

where

$$Q_1 = \begin{bmatrix} 2k_p & -k_p - k_d \\ -k_p - k_d & \frac{ak_d}{2\varepsilon b} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \frac{ak_d}{2\varepsilon b} & -ad_M \\ -ad_M & \frac{2bd_m}{3} \end{bmatrix};$$

$$\lambda_1 = [bd_m \cosh(\vartheta) - 3k_c(|\sinh(\vartheta)| + |\tanh(\tilde{q})|) - 3d_M],$$

$$\lambda_2 = [bd_m \cosh(\vartheta) - \frac{3}{\varepsilon}k_c\beta_d],$$

$$\lambda_3 = [\frac{ak_d}{b}\cosh(\vartheta) - 2\varepsilon k_d(\cosh(\vartheta) + 1)]$$

where we have used property **PD.6** in order to bound the term $|\varepsilon k_d \cosh(\vartheta)\sinh(\vartheta)\tanh(\tilde{q})| \le \varepsilon k_d[|\sinh(\vartheta)|^2|\tanh(\tilde{q})| + |\sinh(\vartheta)||\tanh(\tilde{q})|]$ to get Q_1, and property **PD.3** to get λ_3. We derive now sufficient conditions for $\dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ to be *globally* negative definite in $(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ then, we first prove that it is *globally* negative definite in $(\tanh(\tilde{q}), \dot{\tilde{q}}, \sinh(\vartheta))$. From the expression above we get that Q_1 and Q_2 are positive definite, respectively, if

$$\varepsilon < \frac{ak_pk_d}{(k_p + k_d)^2 b} \qquad \varepsilon < \frac{k_d d_m}{3ad_M^2} \tag{8.49}$$

next, using properties **PD.6** and **PD.3**, it is clear that the term $\lambda_1 > 0$ if

$$b > \frac{6k_c + 3d_M}{d_m} \tag{8.50}$$

while the positivity of the term λ_2 imposes a lower bound on ε, i.e.

$$\varepsilon \geq \frac{3k_c\beta_d}{bd_m} \tag{8.51}$$

finally, the term λ_3 is positive (from property **PD.2**) if $\varepsilon < a/4b$. Then $\dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is globally negative definite in ($\tanh(\tilde{q}), \dot{\tilde{q}}, \sinh(\vartheta)$). Finally, following a similar analysis as for the positivity of V, we can conclude that $\dot{V}(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is as well *globally* negative definite in $(\tilde{q}, \dot{\tilde{q}}, \vartheta)$.

We have shown that $V(\tilde{q}, \dot{\tilde{q}}, \vartheta)$ is a Lyapunov function according to the definition of (Desoer and Vidyasagar, 1975), now using the second Lyapunov method *global asymptotic stability* follows; the proof is completed observing that (8.40) implies that there always exists an $\varepsilon > 0$ such that all conditions above hold.

■

4 Discussion

In this section we point out the improvement of our new approach in relation to previous semi-global results, in particular to the result presented in section 2.

We proved that the closed loop system (8.3) – (8.5), (8.33) is semi-globally asymptotically stable, hence in the particular case of an one degree of freedom EL system the same result can be obtained using the Lyapunov function

$$V_{ch}(\tilde{q}, \dot{\tilde{q}}, \vartheta) = \frac{1}{2}d(q)\dot{\tilde{q}}^2 + \frac{1}{2}k_p\tilde{q}^2 + \frac{k_d}{2b}\vartheta^2 + \varepsilon[\tilde{q} - \vartheta]d(q)\dot{\tilde{q}}. \tag{8.52}$$

We recall that the cross terms $\varepsilon[\tilde{q} - \vartheta]d(q)\dot{\tilde{q}}$ are introduced in (8.52) in order to obtain the quadratic terms $-\varepsilon bd_m\dot{\tilde{q}}^2$ and $-\varepsilon k_p\tilde{q}^2$ in the Lyapunov function derivative:

$$\dot{V}_{ch}(\tilde{q}, \dot{\tilde{q}}, \vartheta) \leq -\frac{\varepsilon}{3}[bd_m - 3k_c(|\vartheta| + |\tilde{q}|) - 3d_M]\dot{\tilde{q}}^2 - \frac{1}{3}k_p\tilde{q}^2 - \frac{k_d a}{3b}\vartheta^2 \tag{8.53}$$

which have been fundamental to claim asymptotic stability. The price paid for these negative quadratic terms is the two *positive* cubic terms $\varepsilon 3k_c(|\vartheta| + |\tilde{q}|)\dot{\tilde{q}}^2$. The difficulty that bounding those terms presents is a common weakness of many existing results using this or other similar techniques. See for instance (Nicosia and Tomei, 1990; Berghuis and Nijmeijer, 1993b; Canudas de Wit et al., 1990).

In order to overcome this problem we have proposed in section 3 the use of hyperbolic trigonometric functions in the Lyapunov function and in the controller. In this manner, we get a *negative definite* Lyapunov function derivative. In particular it is important to remark the difference between the first right hand side term of (8.53) and λ_1 in (8.48). In (8.52) the constant term bd_m must upperbound the absolute values of the state variables $|\vartheta|$ and $|\tilde{q}|$. In contrast to this, the introduction of hyperbolic trigonometric functions in (8.48) helps us first, to saturate the term $k_c \tanh(\tilde{q})$ (in contrast with $k_c|\tilde{q}|$ in eq. 8.53) and second, to get the radially unbounded term $bd_m \cosh(\vartheta)$ which always upperbounds the term $k_c \sinh(\vartheta)$ despite the *constant* filter gain b in (8.5) as well as in (8.38).

Notice that we encountered a similar problem in section **7**.4 when we addressed the problem of output feedback regulation with uncertain knowledge of the potential energy (cf. eq. 7.49). In this respect, it is worth recalling that (Arimoto, 1994a) used a saturation function as we use $\tanh(\tilde{q})$ in a Lyapunov function candidate with cross terms to solve the problem of *state* feedback regulation with uncertain knowledge of the potential energy. Nevertheless, when velocity measurements are supposed unavailable, even though this "trick" can be efficiently applied to bound the "position dependent" cubic term $c(q)\tanh(\tilde{q})\dot{\tilde{q}}^2$, it is not possible to use $\tanh(\vartheta)$ in a similar way. Notice that using the saturated cross term $\varepsilon \tanh(\vartheta)d(q)\dot{\tilde{q}}$ will result in the term $-\varepsilon bd_m \operatorname{sech}^2(\vartheta)\dot{\tilde{q}}^2$ in λ_1 which vanishes when $\vartheta \to \infty$ and in consequence (8.51) can be satisfied only locally. In contrast, using $\sinh(\vartheta)$ we get a negative radially unbounded term in λ_1. This makes the difference between *semiglobal* and *global* asymptotic stability.

The trick of introducing cross terms in the Lyapunov function is not new (Koditschek, 1989; Whitcomb et al., 1991), however the use of hyperbolic trigonometric functions in both the controller and the Lyapunov function is, as far as we know, a new approach which we believe can contribute to the solution of other interesting problems such as the set-point control without velocity measurements with uncertain gravity knowledge.

4.1 The n degrees of freedom case

Particular attention deserves the case of n degrees of freedom which is still an open problem. From a Lyapunov analysis perspective, we have seen so far that by introducing some hyperbolic functions in the controller and the Lyapuov function, we can get a negative definite Lyapunov function derivative and further, global asymptotic stability for one degree of freedom systems. Unfortunately, our result fails in the general case of n degrees of freedom, in this section we intend to discuss why.

Consider the n degrees of freedom system

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

where $q \in \mathbb{R}^n$ in closed loop with the controller

$$\begin{aligned} u &= -k_p \operatorname{Tanh}(\tilde{q}) - k_d \operatorname{Cosh}(\vartheta)\operatorname{Sinh}(\vartheta) + g(q) + C(q,\dot{q}_d)\dot{q}_d + D(q)\ddot{q}_d && (8.54)\\ \dot{q}_c &= -a \operatorname{Sinh}(q_c + b\tilde{q}) && (8.55)\\ \vartheta &= q_c + b\tilde{q} && (8.56)\end{aligned}$$

where k_p, k_d, a, and b are positive constants and we define with an obvious abuse of notation $\operatorname{Tanh}(\tilde{q}) \triangleq \operatorname{col}[\tanh(\tilde{q}_1),\ldots,\tanh(\tilde{q}_n)]$, $\operatorname{Sinh}(\tilde{q}) \triangleq \operatorname{col}[\sinh(\tilde{q}_1),\ldots,\sinh(\tilde{q}_n)]$, and $\operatorname{Cosh}(\tilde{q}) \triangleq \operatorname{diag}\{[\cosh(\tilde{q}_1),\ldots,\cosh(\tilde{q}_n)]\}$. Following a similar analysis to that developed in section 2, we find that the error system is described by the n-order version of (8.42)–(8.43)

$$\begin{aligned} &D(q)\ddot{\tilde{q}} + [C(q,\dot{q}) + C(q,\dot{q}_d)]\dot{\tilde{q}} + k_p \operatorname{Tanh}(\tilde{q}) + k_d \operatorname{Cosh}(\vartheta)\operatorname{Sinh}(\vartheta) = 0, && (8.57)\\ &\dot{\vartheta} = -a \operatorname{Sinh}(\vartheta) + b\dot{\tilde{q}}. && (8.58)\end{aligned}$$

Let us propose a Lyapunov function candidate $V_n(\tilde{q},\dot{\tilde{q}},\vartheta)$ similar to (8.44):

$$V_n(\tilde{q},\dot{\tilde{q}},\vartheta) = \frac{1}{2}\dot{\tilde{q}}^\top D(q)\dot{\tilde{q}}^\top + k_p \sum_i \ln|\cosh(\tilde{q}_i)| + \frac{k_d}{2b}\|\operatorname{Sinh}(\vartheta)\|^2 + \varepsilon[\operatorname{Tanh}(\tilde{q}) - \operatorname{Sinh}(\vartheta)]^\top \dot{\tilde{q}} \qquad (8.59)$$

where as before, $\varepsilon > 0$ is sufficiently small. Following the steps of section 3.2.2, it can be shown that V_n is positive definite under similar conditions to those imposed by (8.46). Further, using the properties of appendix D, the time derivative of V_n along the trajectories of (8.57), (8.58) is bounded by

$$\dot{V}_n(\tilde{q},\dot{\tilde{q}},\vartheta) \le -W_n(\tilde{q},\dot{\tilde{q}},\vartheta) - \varepsilon\dot{\tilde{q}}^\top[\frac{b}{6}I - D^{-1}C(q, \operatorname{Tanh}(\vartheta))]\operatorname{Cosh}(\vartheta)\dot{\tilde{q}} \qquad (8.60)$$

where $W_n(\tilde{q},\dot{\tilde{q}},\vartheta)$ is negative definite. Particular attention deserves the last term of (8.60); on one hand we know from appendix D that $\operatorname{Cosh}(\vartheta) > I_n > 0$, $\forall\vartheta \in \mathbb{R}^n$, and on the other, since $|\operatorname{Tanh}(\vartheta)|_i < 1$, $\forall\vartheta \in \mathbb{R}^n$ it follows that the matrix $[\frac{b}{3}I_n - D^{-1}C(q, \operatorname{Tanh}(\vartheta)) - C^\top(q, \operatorname{Tanh}(\vartheta))D^{-1}]$ can be made positive definite for a sufficiently large b, namely $b > 6d_M k_c$. Then the matrix contained in the last term of (8.60) can be rewritten as the product of two positive definite matrices.

Even though the product of two positive definite matrices, XY, might be positive definite for two *particular* matrices $X > 0$, $Y > 0$, (Lancaster and Tismenetsky, 1985), unfortunately in our case we have two time varying matrices, hence nothing can be claimed.

5 Simulation results

Using SIMULINK™of MATLAB™ we tested our algorithm in simulations in an artificial system with the following parameters $d_m = 1$, $d_M = 2$, $k_c = 0.5$. We considered a sinusoidal reference trajectory of $2.5hz$ i.e. $q_d(t) = 0.1\cos(5\pi t)$. We started the simulation from initial condition $q(0) = 0.5$, $\dot{q}(0) = 0$, $q_c(0) = 15$. In order to meet the conditions of proposition 8.6, we selected the control gains as $k_p = 400$, $k_d = 250$, $a = 10$ and $b = 10$. In figure 8.1 we show the transient response of $q(t)$ as well as the reference.

For the sake of comparison we tested the algorithm proposed in (Loria and Ortega, 1995) which, as we mentioned before is the "linear version" of (8.36) – (8.38), this allowed us to use the *same* control parameters as defined above. The transient response produced by the linear filter based controller is depicted in figure 8.2.

In figure 8.3 we show the transient errors produced by both algorithms. Clearly, the transient response using the controller of proposition 8.6 is better than that produced by the controller of (Loria and Ortega, 1995). To quantitatively compare the tracking performance of both algorithms, we evaluated the integral of the square error $J := \int_0^5 \|q(t) - q_d(t)\|^2 dt$; for this experiment it resulted in $J = 0.20$ for the controller of (Loria and Ortega, 1995) in contrast to $J = 0.02$ for the controller of proposition 8.6. Nevertheless a practical drawback of the algorithm of proposition 8.6 is the higher control input signals yielded by the exponentially increasing term $k_d\cosh(\vartheta)\sinh(\vartheta)$. As a matter of fact, while the controller of (Loria and Ortega, 1995) applies a maximal control input of 5.19×10^3, the controller of proposition 8.6 yields a maximal control signal of 1.47×10^{19}, both maxima happen at $t = 0$.

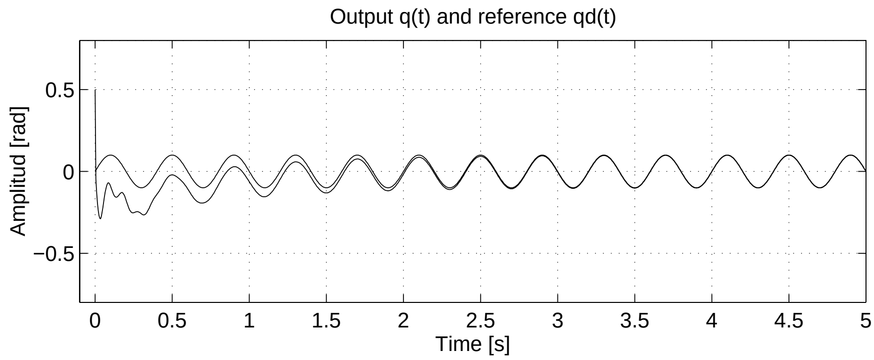


Figure 8.1: Transient response produced by the new controller of proposition 8.6.

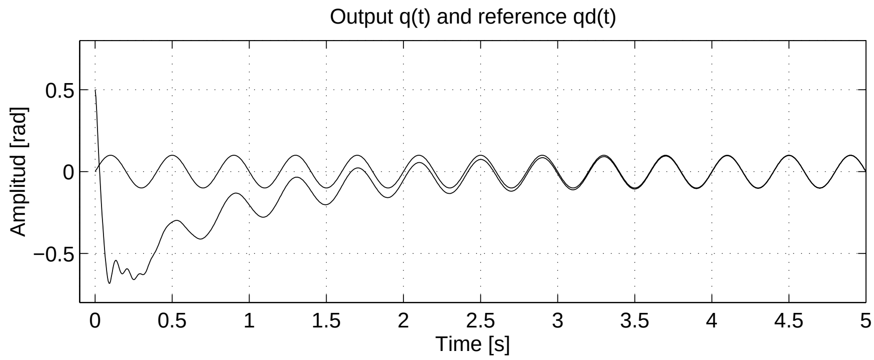


Figure 8.2: Transient response produced by the controller of Loria and Ortega (1995).

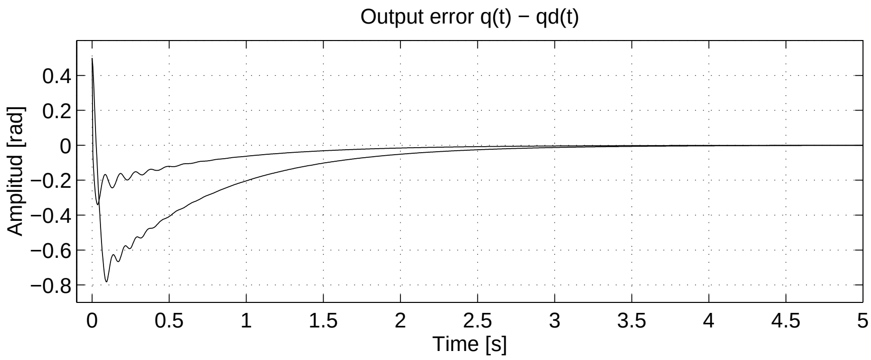


Figure 8.3: Transient errors produced by both algorithms.

6 Concluding remarks

An alternative solution to the tracking control problem for rigid joints robots *without velocity measurements* has been presented. We first proved that by simply replacing the velocity by its approximate differentiation in a computed torque based scheme it is still possible to reach semiglobal exponential stability, thus obviating the necessity of observers. In the case where flexibility in the joints cannot be neglected, we have used a similar computed torque based controller and we have proven that semiglobal exponential stability is still reachable by using the approximate derivatives. Its main feature is that it does not require calculation of jerk.

On one hand, the simplicity of this controller makes it attractive for industrial applications and on the other hand, from the theoretical point of view, even though the rate of convergence depends on the initial conditions it has been shown that it is always possible to tune the filter gain in such a way that $(x \to 0)_{t\to\infty}$ exponentially.

These results were extended to *global asymptotic stability* of one degree of freedom Euler Lagrange systems. As far as we know, this is the first global result even for one degree of freedom systems. Our approach exploits the properties of hyperbolic functions to define a “nonlinear approximate differentiation filter” which makes it possible to automatically enlarge the domain of attraction defined in our previous result (Loria and Ortega, 1995). Unfortunately, due to “technical” limitations, we cannot insure that our approach still works for the more general multivariable case.

We believe that the theoretical contribution of our approach might open a way to the solution of other difficult problems such as global output feedback regulation of robots with uncertain gravity knowledge.

Finally we illustrated in simulations that a practical drawback of our approach is the exponentially increasing terms which yield high control input signals to the system. Hence, it appears interesting to look for controllers for which it can be ensured that the control signal is bounded. This problem is studied in the next chapter.

Chapter 9

Bounded output feedback control

1 Introduction

In chapter **7** we presented two passivity based results for the output feedback set-point control problem. Our first contribution is a class of Euler-Lagrange systems that can be globally asymptotically stabilized via nonlinear dynamic *output* feedback. The controller design, relies on the fact that the storage function for feedback interconnected systems is the sum of the corresponding storage functions. This basic property is expressed in terms of the physically appealing principles of shaping the potential energy and injection of the required damping. In particular, we considered the case where the controller is also an EL system; in this way the closed loop is still an EL system with total energy and dissipation function the sum of the corresponding plant and controller total energies and dissipation functions.

In chapter **8**, we studied some solutions to the output feedback tracking control problem. An obvious drawback of these results and the references therein, that arises in practice, is that in order to guarantee local (resp. semi-global) stability, they require high gains (resp. to increase the basin of attraction). Moreover, the structure of the GAS controller of proposition 8.6 is such that the magnitude of the applied input torque increases exponentially with respect to the trajectory tracking error.

From a theoretical point of view, we have shown in previous chapters that the output feedback set-point and tracking control problems can be tackled by means of the dirty derivatives filter, which can also be regarded as a linear *high gain* observer (see (Berghuis and Nijmeijer, 1993a)). However this observer may introduce very large values of the state estimate over a short period of time. This fact entails mainly two problems: first, as we mentioned in chapter **7**, the larger is the observer gain, in order to allow a bigger domain of attraction, the larger the initial conditions may be, hence we get into a loop; second, over this short period of time, the state estimate does not make much sense and should be disregarded. As already remarked, one simple way of overcoming the "loop problem" is by adequately defining the initial conditions. Another way to solve both problems, which was adopted first in (Esfandiari and Khalil, 1992) is using saturated controls. This has motivated researchers to study the output feedback stabilization problem with saturated inputs ; see for instance (Teel and Praly, 1995) where diverse tools for semiglobal output feedback stabilization are presented, one of these, uses the dirty derivatives filter with saturations. See also (Teel, 1992; Saberi et al., 1996; Coron et al., 1995; Sontag and Sussman, 1994) and references therein to mention a few.

In this chapter we focus our attention on saturated control of a particular class of nonlinear systems, fully actuated EL systems. Concerning the set point control problem, one of the first contributions is due to (Cai and Song, 1993) who proposed a saturated PD plus gravity compensation like controller, to deal with stick-slip friction effects. Later, (Kelly et al., 1994b) extended this result using precompensated gravity, however both results use velocity measurements. Some recent extensions of these works which only use position measurements are given in (Burkov, 1995c) and (Loria et al., 1996). Burkov (1995c) proposed a PD-like controller which uses exact gravity compensation, while Loria *et al* (1996) introduced a subclass of EL controllers (which contains that of Burkov 1995c), thus extending the results of (Ortega et al., 1995c) to the case of fully actuated EL systems under input constraints. See also (Santibañez and Kelly, 1996)

Very recently, (Burkov and Freidovich, 1995) extended their own results (Burkov, 1995c) and those of (Loria et al., 1996) to the case of underdamped EL systems. As far as we know this is the first result in this direction.

In (Teel and Praly, 1995) the output feedback stabilization problem of a wider class of nonlinear systems was tackled before by using saturations, nevertheless, the *bounded control* problem was not considered in this reference.

Nevertheless, all the above mentioned results (concerning the EL systems), hold only for the set-point control problem. To the best of our knowledge the problem of output feedback *tracking* control with bounded inputs was first solved by (Loria and Nijmeijer, 1995). In that note we tackled this problem by using a nonlinear dynamic extension, based on the structure of the approximate differentiation (linear) filter used previously in (Loria and Ortega, 1995). Furthermore, interestingly enough, satisfying some input constraints we proved that the basin of attraction can be arbitrarily enlarged with high observer gains only, thus we proved semi-global asymptotic stability.

In this chapter we first present the results previously reported in (Loria et al., 1996), in particular we prove that, if the system is fully-actuated and the forces due to the potential field can be "dominated" by the constrained control signals then global output feedback regulation is still possible by incorporating some suitable saturation functions in the controller. Secondly, we present the results of (Loria and Nijmeijer, 1995) on saturated tracking control. We consider fully actuated EL plants which have the properties **P6.1**–**P6.3**.

2 Output feedback set-point control with saturations

2.1 Problem formulation

In this section we assume that property **P6.2** holds, then under this condition we deal with the

Global output feedback set-point control problem. For the system (6.34) with $M_p = I_{n_p}$ and $\mathcal{F}_p(\dot{q}_p) \equiv 0$, that is,

$$D(q_p)\ddot{q}_p + C(q_p, \dot{q}_p)\dot{q}_p + g(q_p) = u_p, \tag{9.1}$$

assume that only generalized position measurements, q_p, are available and that the manipulator inputs are constrained to

$$|u_{p_i}| \leq u_{p_i}^{\max} \qquad \forall i \in \underline{n} \tag{9.2}$$

then, find an output feedback controller which renders the closed loop system globally asymptotically stable, that is, an output feedback controller such that

$$\lim_{t\to\infty} \tilde{q}_p \triangleq \lim_{t\to\infty} [q_p - q_{pd}] = 0, \tag{9.3}$$

where q_{pd} is the desired constant position.

Based on the results of chapter **7** we define in the next section, a family of EL controllers which yield bounded control inputs. We repeat for convenience that the aim of the EL controllers is to shape the closed loop energy $V(q)$ and to inject a partial damping.It is clear that the easiest way of shaping the potential energy of the closed loop, $V(q)$, is to *cancel* the potential energy of the plant, $V_p(q_p)$, and then to impose a "new" potential energy shape. However, in the present work, we favour the passivity based controllers, hence instead of relaying on a simple cancellation we aim at *dominating* the potential energy of the plant.

To this point, we recall the reader that, according to the Lagrangian controllers methodology of chapter **7**, the control input is defined by

$$u_p = -\frac{\partial V_c(q_c, q_p)}{\partial q_p},$$

from this, we can deduce that the input constraint established by (9.2), shall entail some growth restrictions on $V_c(q)$. Now, since $V_c(q)$ should be designed in a way that it dominate $V_p(q_p)$, it is also necessary to impose some growth restrictions on $V_p(q_p)$. Thus, we consider in this chapter, a subclass of fully actuated

EL systems whose potential energy function satisfies (6.41). Loosely speaking, this condition restricts the growth rate of $V_p(q_p)$ to be $\mathcal{O}(\|q_p\|^2)$ for all q_p in some ball B_β and to $\mathcal{O}(\|q_p\|)$ outside B_β.

Naturally, similar conditions are imposed to $V_c(q_c, q_p)$. As it will become more clear later, in order to satisfy this restriction, we can choose for instance

$$V_c(q_c, q_p) = \int_0^{\tilde{q}_p} \operatorname{sat}(x)dx + V_{c_2}(q_c, q_p)$$

where $V_{c_2}(q_c, q_p)$ is a suitable defined function and we define the saturation function $\operatorname{sat}(x)$ as

9.1 Definition. *A saturation function* $sat(x) : I\!R \to I\!R$ *is a* $\mathcal{C}^2$ *strictly increasing function that satisfies*

1. $sat(0) = 0$,
2. $|\,sat(x)| < 1$,
3. $\frac{\partial^2\ sat(x)}{\partial x^2} \neq 0 \quad \forall\ x \neq 0 \in I\!R$.

Our motivation for considering saturation functions defined by 9.1 is that these functions satisfy the following properties.

P 9.1 $\int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x)dx \geq \frac{1}{2} \operatorname{sat}(\tilde{q}_{p_i})\tilde{q}_{p_i}, \quad \tilde{q}_{p_i} \in \mathrm{IR}$.

P 9.2 There exists some $\varepsilon > 0$ such that

$$\operatorname{sat}(\tilde{q}_{p_i})\tilde{q}_{p_i} \geq \frac{\operatorname{sat}(\varepsilon)}{\varepsilon}\tilde{q}_{p_i}^2 \quad |\tilde{q}_{p_i}| < \varepsilon, \tag{9.4}$$

$$\operatorname{sat}(\tilde{q}_{p_i})\tilde{q}_{p_i} \geq \operatorname{sat}(\varepsilon)|\tilde{q}_{p_i}| \quad |\tilde{q}_{p_i}| \geq \varepsilon. \tag{9.5}$$

For instance, we can take $\operatorname{sat}(x) := \tanh(\omega x)$, $\omega > 0$, as proposed in (Cai and Song, 1993; Kelly et al., 1994b).

2.2 A family of GAS saturated EL controllers

Since we are dealing with fully actuated systems, the simplest way to "dominate" the plant's potential energy is to cancel $V_p(q_p)$ and to impose a desired shape to the closed loop function. This however entails some potential robustness problems, hence we favour a solution that does not rely on this cancellation. Interestingly enough, if we use a controller that does not cancel the vector of potential forces, the growth rate restriction on $V_p(q_p)$ mentioned above is imposed only at the desired position. The price paid, however, is that in this case we need to use "high" gains in $V_c(q_c, q_p)$ to dominate $V_p(q_p)$ and this translates into stiffer requirements on the input saturation bound, $u_{p_i}^{\max}$.

9.2 Proposition. (Saturated controller with cancellation of potential forces)
Assume that the systems potential energy verifies the strict inequality

$$\sup_{q_p \in I\!R^n} \left| \left(\frac{\partial V_p(q_p)}{\partial q_p} \right)_i \right| < u_{p_i}^{\max}, \quad i \in \underline{n} \tag{9.6}$$

with $(\cdot)_i$ *the* $i-th$ *component of the vector. Under these conditions, there exists a dynamic* output *feedback EL controller* $\Sigma_c : \{T_c(q_c, \dot{q}_c), V_c(q_c, q_p), \mathcal{F}_c(\dot{q}_c)\}$ *that insures the input constraint (9.2) holds, and makes*

$$(\dot{q}_p, q_p, \dot{q}_c, q_c) = (0, q_{pd}, 0, q_{cd}) \tag{9.7}$$

with q_{cd} *some constant, a GAS equilibrium point of the closed loop system.* ▢

2.2.1 A controller with cancellation of potential forces (Burkov, 1995c)

Consider the EL controller characterized by

$$\begin{aligned} T_c(q_c, \dot{q}_c) &= 0, \quad \mathcal{F}_c(\dot{q}_c) = \frac{1}{2}\|\dot{q}_c\|^2 && (9.8) \\ V_c(q_c, q_p) &= V_{c_1}(q_c) + V_{c_2}(q_c, q_p) - V_p(q_p) && (9.9) \\ V_{c_1}(q_c) &= \frac{1}{2} q_c^\top K_1 q_c && (9.10) \\ V_{c_2}(q_c, q_p) &= \sum_{i=1}^{n} k_{2_i} \int_0^{(q_{c_i} - \tilde{q}_{p_i})} \mathrm{sat}(x_i) dx_i && (9.11) \end{aligned}$$

where $\tilde{q}_{p_i} := q_{p_i} - q_{pd_i}$, $k_{1_i}, k_{2_i} > 0$, $K_1 := \mathrm{diag}\{k_{1_i}\}$. Using Lagrange's equations we can derive the controller dynamics

$$\begin{aligned} \dot{q}_{c_i} &= -k_{1_i} q_{c_i} - k_{2_i}\, \mathrm{sat}(q_{c_i} - \tilde{q}_{p_i}) && (9.12) \\ u_{p_i} &= k_{2_i}\, \mathrm{sat}(q_{c_i} - \tilde{q}_{p_i}) + \left(\frac{\partial V_p(q_p)}{\partial q_p}\right)_i && (9.13) \end{aligned}$$

which corresponds to that proposed by Burkov.

9.3 Proposition. (Saturated controller without cancellation of potential forces)
Assume that, at the desired *reference, the gradient of the systems potential energy satisfies the inequality*

$$\left| \left(\frac{\partial V_p}{\partial q_p}(q_{pd}) \right)_i \right| \leq k_{g_i}^{\max}, \quad i \in \underline{n} \tag{9.14}$$

with $k_{g_i}^{\max} < u_{p_i}^{\max}$, and let its Hessian satisfy (6.40). Under these conditions, there exists a globally asymptotically stabilizing EL controller that does not cancel the potential forces and insures the input constraints (9.2) provided $u_{p_i}^{\max}$ is sufficiently large. □

2.2.2 A controller without cancellation of potential forces

In this case the EL parameters of the controller can be chosen as

$$\begin{aligned} T_c(q_c, \dot{q}_c) &= 0, \quad \mathcal{F}_c(\dot{q}_c) = \frac{1}{2}\dot{q}_c^\top K_2 B^{-1} A^{-1} \dot{q}_c && (9.15) \\ V_c(q_c, q_p) &= V_{c_2}(q_c, q_p) - q_p^\top \frac{\partial V_p}{\partial q_p}(q_{pd}) && (9.16) \\ V_{c_2}(q_c, q_p) &= \sum_{i=1}^{n} \left(\frac{k_{2_i}}{b_i} \int_0^{(q_{c_i} + b_i q_{p_i})} \mathrm{sat}(x_i) dx_i + k_{3_i} \int_0^{\tilde{q}_{p_i}} \mathrm{sat}(x_i) dx_i \right) && (9.17) \end{aligned}$$

where $A := \mathrm{diag}\{a_i\}$, $B := \mathrm{diag}\{b_i\}$, $K_2 := \mathrm{diag}\{k_{2_i}\} > 0$, and we select $k_{3_i} > 0$ *sufficiently large.* This choice yields the EL controller

$$\begin{aligned} \dot{q}_{c_i} &= -a_i\, \mathrm{sat}(q_{c_i} + b_i q_{p_i}) && (9.18) \\ u_{p_i} &= -k_{2_i}\, \mathrm{sat}(q_{c_i} + b_i q_{p_i}) - k_{3_i}\, \mathrm{sat}(\tilde{q}_{p_i}) + \left(\frac{\partial V_p}{\partial q_p}(q_{pd}) \right)_i && (9.19) \end{aligned}$$

The propositions above characterize, –in terms of the EL parameters $T_c(q_c, \dot{q}_c)$, $\mathcal{F}_c(\dot{q}_c)$–, a *class* of output feedback GAS controllers for EL systems with *saturated inputs.* Thus, providing an extension, to the constrained input case, of the result presented in section **7**.2.3.

A key feature of the controller given in section 2.2.2 above is that, to enhance its robustness, we *avoid* explicit cancellations of the plant dynamics. As mentioned before, the price paid for this is the requirement that the plants potential energy grow not faster than linearly; also, higher gains have to be injected into the loop through k_{3_i} (see section 2.3.2). As seen from our proposition, this imposes an additional requirement of sufficiently large input constraints for stability. The condition on k_{3_i} stems from the fact that, to impose a desired minimum point to the closed loop potential energy, now we have to dominate (and not to cancel) the systems potential energy. In this respect our controller supersedes the result of (Burkov, 1995c) which relies on exact cancellation of $V_p(q_p)$.

9.4 Remark. As a corollary of our proposition, we obtain an extension to the *output* feedback case of the result in (Kelly et al., 1994b) where a *full state* feedback solution to the problem of global regulation of rigid robots with saturated inputs was presented. It is also interesting to remark that if we write $\dot{q}_{c_i} = -a_i(q_{c_i} + b_i q_{p_i})$ instead of $\dot{q}_{c_i} = -a_i \operatorname{sat}(q_{c_i} + b_i q_{p_i})$, in (9.18) we *exactly recover* the (approximate differentiation) output feedback GAS controller of (Kelly, 1993b), see also (Kelly et al., 1994a). Our proposition then shows that by simply including the saturations we can preserve GAS even under input constraints. See also section 3 of (Teel and Praly, 1995).

2.3 Proofs

The proofs of propositions 9.2 and 9.3 are constructive. For this, we provide below the stability proofs of saturated EL controllers of sections 2.2.1 and 2.2.2.

As pointed out in section **7**.2.3 the interconnection of two EL systems, yields an EL system with potential energy $V(q) = V_c(q_c, q_p) + V_p(q_p)$, hence the proofs of both results are carried out by proving the conditions of proposition 7.2. Notice that the difficulty lies in proving that $V(q)$ has a global and unique minimum including $q_p = q_{pd}$. For this, we will use lemma B.8.

2.3.1 Proof of proposition 9.2

Notice first that the condition on the Rayleigh dissipation function of proposition 7.2 is trivially satisfied, hence we go on proving that the potential energy is adequately shaped and that the damping suitably propagates from the controller coordinates q_c to the plant coordinates q_p.

(i) (Energy shaping)

The potential energy of the closed loop system (9.1), (9.2), (9.13) is given by

$$V(q) = \frac{1}{2} q_c^\top K_1 q_c + \sum_{i=1}^{n} k_{2_i} \int_0^{(q_{c_i} - \tilde{q}_{p_i})} \operatorname{sat}(x_i) dx_i.$$

Now we use lemma B.8 to prove that $V(q)$ has a global and unique minimum at the origin[1]: $(q_c, \tilde{q}_p) = (\bar{q}_c, 0)$. The positivity condition of lemma B.8 follows from definition 9.1 and property **P9.1** while the second condition follows from equalizing $\frac{\partial V(q)}{\partial q} = 0$:

$$\begin{bmatrix} K_1 q_c + K_2 \operatorname{sat}(q_c - \tilde{q}_p) \\ -K_2 \operatorname{sat}(q_c - \tilde{q}_p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{9.20}$$

Notice that, since $\operatorname{sat}(x)$ is strictly increasing and vanishes only at $x = 0$, and K_1, K_2 are full rank, (9.20) is satisfied if and only if $(q_c, \tilde{q}_p) = (0, 0)$.

(ii) (Dissipation propagation)

This condition is easily verified by equalizing $\frac{\partial V_c(q_c, q_{p2})}{\partial q_c} = 0$:

$$K_1 q_c + K_2 \operatorname{sat}(q_c - \tilde{q}_p) = 0,$$

setting $q_c \equiv const$ and observing that effectively, $q_p \equiv const$.

The proof is completed applying the triangle inequality to (9.13), and using the fact that $|\operatorname{sat}(x)| < 1$, to get the bound

$$|(u_p)_i| < k_{2_i} + k_{3_i} + \left| \left(\frac{\partial V_p(q_p)}{\partial q_p} \right)_i \right|$$

[1] Notice that the steady state value of q_c is not important, only the final position of the plant, q_p.

thus, under assumption (9.6), we can always choose sufficiently small k_{2_i}, $k_{3_i} > 0$ such that (9.2) holds.

■

2.3.2 Proof of proposition 9.3

We provide in this section a stability proof for the controller of section 2.2.2. As in the previous proof, we verify the conditions of proposition 7.2.

We prove next that, if we take k_{2_i} sufficiently small and $\min_i \{k_{3i}\} > k_{3_i}^{\min}$, with $k_{3_i}^{\min}$ some suitably defined positive constant, then (9.7) is a GAS equilibrium point of the closed loop (9.1), (9.2), (9.19) provided[2]

$$u_{p_i}^{\max} > \left| \left(\frac{\partial V_p}{\partial q_p}(q_{pd}) \right)_i \right| + k_{3_i}, \quad i \in \underline{n}. \tag{9.21}$$

We finally show that, if in particular we take $\operatorname{sat}(x) = \tanh(x)$ then

$$k_{3_i}^{\min} \triangleq \frac{4k_v}{\tanh\left(\frac{4k_v}{k_g}\right)} \tag{9.22}$$

where k_v and k_g are given by (6.41) and (6.40) respectively.

(i) (Energy Shaping)

The closed loop potential energy is now

$$V(q) = \sum_{i=1}^{n} \left(\frac{k_{2_i}}{b_i} \int_0^{(q_{c_i}+b_i q_{p_i})} \operatorname{sat}(x_i)dx_i + k_{3_i} \int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x_i)dx_i \right) + V_p(q_p) - q_p^\top \frac{\partial V_p}{\partial q_p}(q_{pd}). \tag{9.23}$$

Hereafter we show that, if there exists a k_v as defined by (6.41), then there exists $k_{3_i}^{\min} > 0$ such that $V(q)$ has a global and unique minimum at the desired equilibrium for all $k_{3_i} \geq k_{3_i}^{\min}$.

Notice that the first right hand term of (9.23) is a nonnegative function of q_c, q_p which is zero at $q_c = -B^{-1}q_p$. Hence, to prove that (9.23) has a global and unique minimum at $(q_{pd},\ \bar{q}_c)$ it suffices to show that the last three terms have a global and unique minimum at $q_p = q_{pd}$, or equivalently, that the function $f(\tilde{q}_p) : \mathbb{R}^n \to \mathbb{R}$

$$f(\tilde{q}_p) \triangleq \sum_{i=1}^{n} \left\{ k_{3i} \int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x_i)dx_i \right\} + V_p(\tilde{q}_p + q_{pd}) - V_p(q_{pd}) - \tilde{q}_p^\top \frac{\partial V_p}{\partial q_p}(q_{pd}) \tag{9.24}$$

has a global and unique minimum at zero. We will establish the proof verifying the conditions of lemma B.8 in order to define a $k_{3_i}^{\min}$ that insures this to be the case.

Condition 1.

To prove that $f(\tilde{q}_{p_i}) > 0$ for all $\tilde{q}_p \neq 0 \in \mathbb{R}^n$ we shall prove first that there exists $\varepsilon > 0$ such that

$$\sum_{i=1}^{n} k_{3_i} \int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x)dx \geq \min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2\varepsilon} \|\tilde{q}_p\|^2 \quad \forall\ \|\tilde{q}_p\| < \varepsilon \tag{9.25}$$

$$\sum_{i=1}^{n} k_{3_i} \int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x)dx \geq \min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2} \|\tilde{q}_p\| \quad \forall\ \|\tilde{q}_p\| \geq \varepsilon. \tag{9.26}$$

Let ε be a constant that satisfies inequalities (9.4) and (9.5). For the sake of clarity we consider two cases separately:

Case 1: $\|\tilde{q}_p\| < \varepsilon$

[2] Notice that the gradient of the systems potential energy is evaluated here *at the desired* reference.

Notice that in this case we have that $|\tilde{q}_{p_i}| < \varepsilon,\ \forall\, i \in \underline{n}$, then using **P9.1** and (9.4) we get

$$\sum_{i=1}^{n} k_{3_i} \int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x)dx \geq \sum_{i=1}^{n} k_{3_i} \frac{\operatorname{sat}(\varepsilon)}{2\varepsilon} \tilde{q}_{p_i}^2 \geq \min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2\varepsilon} \|\tilde{q}_p\|^2. \tag{9.27}$$

Case 2: $\|\tilde{q}_p\| \geq \varepsilon$

Within this case we shall consider three different cases:

case a: $|\tilde{q}_{p_i}| < \varepsilon \ \ \forall i \in \underline{n}$

First notice that

$$\sum_{i=1}^{n} k_{3_i} \frac{\operatorname{sat}(\varepsilon)}{2\varepsilon} \tilde{q}_{p_i}^2 \geq \min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2\varepsilon} \|\tilde{q}_p\|^2 \geq \min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2} \|\tilde{q}_p\|.$$

Using (9.4) and **P9.1**, (9.26) follows.

case b: $|\tilde{q}_{p_i}| \geq \varepsilon \ \ \forall i \in \underline{n}$

From **P9.1** and (9.5) notice that

$$\sum_{i=1}^{n} k_{3_i} \int_0^{\tilde{q}_{p_i}} \operatorname{sat}(x)dx \geq \sum_{i=1}^{n} k_{3_i} \frac{\operatorname{sat}(\varepsilon)}{2} |\tilde{q}_{p_i}| \geq \min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2} \sum_{i=1}^{n} |\tilde{q}_{p_i}|, \tag{9.28}$$

then (9.26) easily follows observing that $\|\tilde{q}_p\| \leq \sum_{i=1}^{n} |\tilde{q}_{p_i}|$.

case c: $|\tilde{q}_{p_i}| \geq \varepsilon, \ \ |\tilde{q}_{p_j}| < \varepsilon \ \ \forall i,j \in \underline{n},\ i \neq j$

Without loss of generality we can take $i \leq n/2$ and $1 \leq j < n/2$, then a simple analysis along the lines of cases a and b, shows that (9.26) holds as well in this case.

Now we prove that, for all $\varepsilon > 0$ there exists constants $\beta_1(\varepsilon), \beta_2(\varepsilon) \in \mathbb{R}$ such that

$$V_p(q_p) - V_p(q_{pd}) - \tilde{q}_p^\top \frac{\partial V_p(q_p)}{\partial q_p}(q_{pd}) \quad \geq \quad \begin{cases} \beta_1 \|\tilde{q}_p\|^2 & \forall\ \|\tilde{q}_p\| < \varepsilon \\ \\ \beta_2 \|\tilde{q}_p\| & \forall\ \|\tilde{q}_p\| \geq \varepsilon. \end{cases} \tag{9.29}$$

On one hand, notice that using lemma B.7 it follows from (6.40) that

$$V_p(q_p) - V_p(q_{pd}) - \tilde{q}_p^\top \frac{\partial V_p(q_p)}{\partial q_p}(q_{pd}) \geq -\frac{k_g}{2} \|\tilde{q}_p\|^2, \tag{9.30}$$

on the other hand, invoking the Mean Value Theorem we have that $\exists\ \xi \in \mathbb{R}^n$ such that

$$V_p(q_{pd}) - V_p(q_p) = \left(\frac{\partial V_p(q_p)}{\partial q_p}(\xi) \right) (q_{pd} - q_p) \leq k_v \|q_{pd} - q_p\|,$$

then using (6.41) we can write

$$V_p(q_p) - V_p(q_{pd}) - \frac{\partial V_p(q_p)}{\partial q_p}(q_{pd})^\top \tilde{q}_p \geq -2k_v \|\tilde{q}_p\|. \tag{9.31}$$

Since (9.30) and (9.31) hold for all $\tilde{q}_p \in \mathbb{R}^n$, then (9.29) holds with $\beta_1 = -\frac{k_g}{2}$ and $\beta_2 = -k_v$. We then conclude from (9.25), (9.26), and (9.29) that

$$f(\tilde{q}_p) \quad \geq \quad \begin{cases} \left(\min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2\varepsilon} - \frac{k_g}{2} \right) \|\tilde{q}_p\|^2 & \forall\ \|\tilde{q}_p\| < \varepsilon \\ \\ \left(\min_i \{k_{3i}\} \frac{\operatorname{sat}(\varepsilon)}{2} - 2k_v \right) \|\tilde{q}_p\| & \forall\ \|\tilde{q}_p\| \geq \varepsilon. \end{cases} \tag{9.32}$$

From here it's easy to see that condition 1 is satisfied provided

$$\min_i \{k_{3i}\} \geq k_{3_i}^{\min} > \max\left\{ \frac{\varepsilon k_g}{\mathrm{sat}(\varepsilon)}, \frac{4k_v}{\mathrm{sat}(\varepsilon)} \right\} \tag{9.33}$$

holds with ε as in **P9.2**, k_g and k_v defined by (6.41) and (6.40) respectively.

Condition 2.
Taking the partial derivatives of $f(\tilde{q})$ we get

$$\frac{\partial f}{\partial \tilde{q}_p}(\tilde{q}_p) \;=\; K_3 \begin{bmatrix} \mathrm{sat}(\tilde{q}_{p_1}) \\ \mathrm{sat}(\tilde{q}_{p_2}) \\ \vdots \\ \mathrm{sat}(\tilde{q}_{p_n}) \end{bmatrix} + \frac{\partial V_p}{\partial \tilde{q}_p}(\tilde{q}_p) - \frac{\partial V_p}{\partial q_p}(q_{pd}). \tag{9.34}$$

Now, taking the norm and using the triangle inequality we get

$$\left\| \frac{\partial f}{\partial \tilde{q}_p}(\tilde{q}_p) \right\| \;\geq\; \left\| K_3 \begin{bmatrix} \mathrm{sat}(\tilde{q}_{p_1}) \\ \mathrm{sat}(\tilde{q}_{p_2}) \\ \vdots \\ \mathrm{sat}(\tilde{q}_{p_n}) \end{bmatrix} \right\| - \left\| \frac{\partial V_p}{\partial \tilde{q}_p}(\tilde{q}_p) - \frac{\partial V_p}{\partial q_p}(q_{pd}) \right\|. \tag{9.35}$$

On one hand, from (6.40), (6.41) and using the Mean Value Theorem we have that for all $\varepsilon > 0$

$$-\left\| \frac{\partial V_p}{\partial \tilde{q}_p}(\tilde{q}_p) - \frac{\partial V_p}{\partial q_p}(q_{pd}) \right\| \geq \begin{cases} -k_g \|\tilde{q}_p\| & \text{if } \|\tilde{q}_p\| < \varepsilon \\ -2k_v & \text{if } \|\tilde{q}_p\| \geq \varepsilon \end{cases}$$

and on the other hand since K_3 is diagonal and using **P9.2**, we obtain

$$\left\| K_3 \begin{bmatrix} \mathrm{sat}(\tilde{q}_{p_1}) \\ \mathrm{sat}(\tilde{q}_{p_2}) \\ \vdots \\ \mathrm{sat}(\tilde{q}_{p_n}) \end{bmatrix} \right\| \geq \begin{cases} \min_i \{k_{3i}\} \frac{\mathrm{sat}(\varepsilon)}{\varepsilon} \|\tilde{q}_p\| & \text{if} \|\tilde{q}_p\| < \varepsilon \\ \min_i \{k_{3i}\} \; \mathrm{sat}(\varepsilon) & \text{if} \|\tilde{q}_p\| \geq \varepsilon. \end{cases}$$

Thus, we are able to write

$$\left\| \frac{\partial f}{\partial \tilde{q}_p}(\tilde{q}_p) \right\| \geq \begin{cases} \left(\min_i \{k_{3i}\} \frac{\mathrm{sat}(\varepsilon)}{\varepsilon} - k_g \right) \|\tilde{q}_p\| & \text{if } \|\tilde{q}_p\| < \varepsilon \\ \left(\min_i \{k_{3i}\} \; \mathrm{sat}(\varepsilon) - 2k_v \right) & \text{if } \|\tilde{q}_p\| \geq \varepsilon \end{cases}$$

which happens to hold provided (9.33) is satisfied.

Notice that in the case of $\mathrm{sat}(x) = \tanh(x)$ we have that **P9.2** is true with[3] $\varepsilon := \frac{4k_v}{k_g}$. Substitution of this ε in (9.33) implies (9.22).

(ii) (Dissipation propagation)

The second condition is verified as for the previous controller, by equalizing $\frac{\partial V(q)}{\partial q_c} = 0$:

$$K_2 B^{-1} \; \mathrm{sat}(q_c + B\tilde{q}_p) = 0$$

and observing that it holds true only if $q_c = -B^{-1}\tilde{q}_p$, since K_2 is full rank, hence, $q_c \equiv \text{const}$ implies that $q_p \equiv \text{const}$.

The proof is completed applying the triangle inequality and using (9.19) to get the bound

$$|(u_p)_i| < k_{2_i} + k_{3_i} + \left| \left(\frac{\partial V_p}{\partial q_p}(q_{pd}) \right)_i \right|$$

thus under assumption (9.14), we can always choose sufficiently small $k_{2_i}, k_{3i} > 0$ such that (9.2) be satisfied.

■

[3] See (Kelly et al., 1994b)

2.4 Simulation results

Using SIMULINK™ of MATLAB™, we tested our algorithm in the two link robot arm of (Berghuis, 1993) with a desired reference $q_d = \text{col}[\pi/2,\ \pi/2]$. We have imposed the input constraint $u_{p_i}^{\max} = 320[Nm]$ in (9.2). Considering (6.64), to meet the conditions of proposition 9.3 we chose $A = \text{diag}\{100,\ 100\}$, $B = \text{diag}\{130,\ 130\}$ while the controller gains were set to $K_3 = \text{diag}\{180,\ 180\}$, $K_2 = \text{diag}\{125,\ 125\}$ according to (9.21).

Then, in order to evaluate the performance of our controller, we tested as well the one proposed in (Kelly, 1993b) with exactly the same gain values and starting from initial conditions $q_{p_0} = \text{col}[\pi/4,\ \pi/4]$, and in accordance with the previous discussions we set $q_{c_0} = \text{col}[-32.5\pi,\ -32.5\pi]$ in order to make $\vartheta_0 = 0$.

In figure 9.1 we show the transient of the first link position using the algorithm of (Kelly, 1993b), i.e. the *non* saturated controller and the control input signal yielded by this controller. In figure 9.2 we show the response of the same link driven by the saturated controller of proposition 9.3 and its control input.

On one hand, notice that the transient produced by the non saturated controller is much faster than the response using saturated controls. On the other hand, it must be remarked that the control input yielded by the linear controller fails to satisfy the input constraint; in particular the maximum absolute value of u_p is 503[Nm] for the first link. In contrast to this the saturated controller yields a control input with $|u_{p_i}|^{\max} = 211$[Nm].

Thus we verify what is not surprising: that there is a compromise between a fast transient and small control inputs.

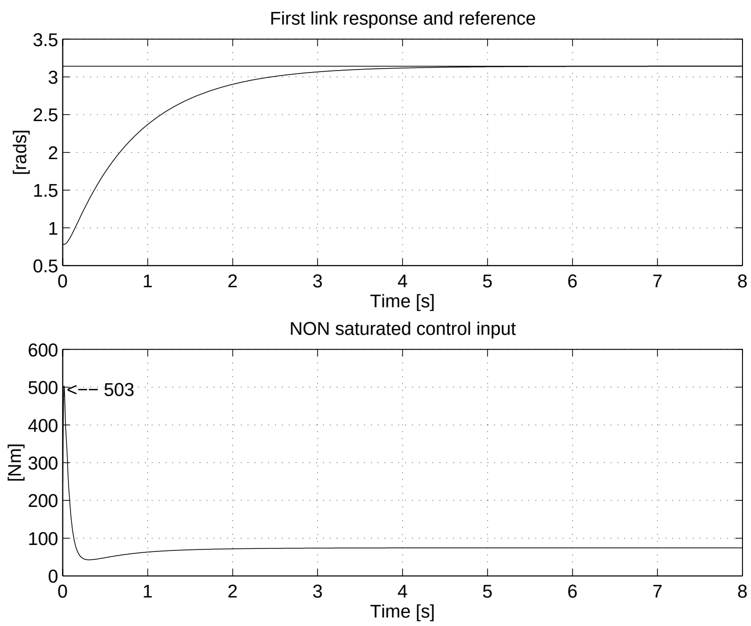


Figure 9.1: EL Controller of section **7**.2.3, (Kelly 1993b).

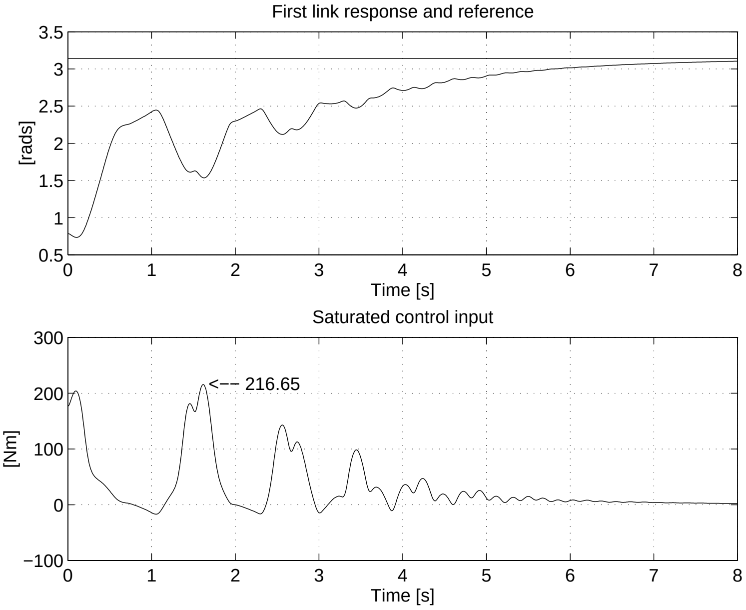


Figure 9.2: Saturated EL controller.

3 Tracking control

We present in this section, as far as we know, the *first* saturated output feedback tracking controller for fully actuated EL systems. For the sake of simplicity we limit our analysis to the particular case when $\mathrm{sat}(x) = \tanh(x)$. Also, we remind the reader that the subindex $_p$ used in last section has been introduced for distinguishing the generalized coordinates of EL plants from those of EL controllers, for the sake of simplicity, in this section we drop this subindex.

Semi-Global output feedback tracking control problem. For the system

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u, \tag{9.36}$$

assume that only position measurements are available and that the manipulator inputs are constrained to (9.2) then find an output feedback controller which renders the closed loop system semi-globally asymptotically stable, that is, an output feedback controller whose parameters may depend on the initial conditions, such that

$$\lim_{t\to\infty} \tilde{q}(t) \stackrel{\triangle}{=} \lim_{t\to\infty} [q(t) - q_d(t)] = 0 \tag{9.37}$$

where the desired trajectory $q_d(t) \in \mathcal{C}^2$ satisfies $\|\ddot{q}_d(t)\|$, $\|\dot{q}_d(t)\|$, $\|q_d(t)\| \le B_d$.

9.5 Proposition. *Consider the fully actuated EL system (9.36) in closed loop with the control law*

$$\begin{aligned} u &= -K_p\ \mathit{Tanh}(\tilde{q}) - K_d\ \mathit{Tanh}(\vartheta) + D(q)\ddot{q}_d + C(q, \dot{q}_d)\dot{q}_d + g(q) && (9.38)\\ \dot{q}_c &= -A\ \mathit{Tanh}(q_c + B\tilde{q}) && (9.39)\\ \vartheta &= q_c + B\tilde{q}, && (9.40) \end{aligned}$$

where A, B, K_p and K_d are diagonal positive definite matrices. Define $I_t \stackrel{\triangle}{=} [t_0, \infty)$, $t_0 \ge 0$ and $x(t) \stackrel{\triangle}{=} col[\tilde{q}(t), \dot{\tilde{q}}(t), \vartheta(t)]$. Let $\eta > 0$ and B_η be some ball of radius η about $(0,\ 0,\ 0)$ in the $(\tilde{q},\ \dot{\tilde{q}},\ \vartheta)$-space to be specified later. Then there always exist $A > 0$ and $B > 0$ sufficiently large such that, for any initial condition $(t_0, x_0) \in I_t \times B_\eta$, the solution $x(t)$ of the closed loop system remains in the ball B_η for all $t \ge t_0$. Moreover *for some small given k_{p_m} and any arbitrarily large $\eta > 0$, the closed loop system is uniformly asymptotically stable. Furthermore, the input constraints (9.2) are satisfied if*

$$u^{\max} > B_d(d_M + k_c B_d) + k_v \tag{9.41}$$

□

3.1 Stability Proof

The proof of the proposition above uses arguments of standard Lyapunov stability theory and is organized as follows: first we derive the error equations and propose a Lyapunov function candidate showing that it is globally positive definite, second we prove that the time derivative of this function is locally negative definite. Third we invoke theorem C.3 to show that the closed loop system is asymptotically stable. We conclude the proof by showing that it is always possible to find some constants a_i and b_i such that the domain of attraction B_η can be arbitrarily enlarged without contradicting the input constraints (9.2).

3.1.1 Error equation and Lyapunov function candidate

The error equation for the closed loop system (9.36) and (9.38) – (9.40) can be written using the properties of section **6**.4.1, as

$$D\ddot{\tilde{q}} + [C(q,\dot{q}) + C(q,\dot{q}_d)]\dot{\tilde{q}} + K_p \text{ Tanh}(\tilde{q}) + K_d \text{ Tanh}(\vartheta) = 0 \tag{9.42}$$

$$\dot{\vartheta} = -A \text{ Tanh}(\vartheta) + B\dot{\tilde{q}}. \tag{9.43}$$

Now, consider the time dependent Lyapunov function candidate

$$V(t,x) = \frac{1}{2}\dot{\tilde{q}}^\top D(q)\dot{\tilde{q}} + \sum_{i=1}^{n}\left(k_{p_i} \ln|\cosh(\tilde{q}_i)| + \frac{k_{d_i}}{b_i}\ln|\cosh(\vartheta_i)|\right) + \varepsilon\dot{\tilde{q}}^\top [\text{ Tanh}(\tilde{q}) - \text{ Tanh}(\vartheta)] \tag{9.44}$$

where ε is a (small) positive number to be defined. To prove that this function is positive definite, consider the following auxiliary functions

$$\begin{aligned} W_1(t,x) &= \frac{1}{4}\dot{\tilde{q}}^\top D(q)\dot{\tilde{q}} + \frac{1}{4}\text{ Tanh}(\tilde{q})^\top K_p \text{ Tanh}(\tilde{q}) + \varepsilon \text{ Tanh}(\tilde{q})^\top \dot{\tilde{q}} && (9.45)\\ W_2(t,x) &= \frac{1}{4}\dot{\tilde{q}}^\top D(q)\dot{\tilde{q}} + \frac{1}{4}\text{ Tanh}(\vartheta)^\top K_d B^{-1} \text{ Tanh}(\vartheta) - \varepsilon \text{ Tanh}(\vartheta)^\top \dot{\tilde{q}}, && (9.46)\end{aligned}$$

using lemma D.2

$$V(t,x) \geq W_1(t,x) + W_2(t,x) + \frac{1}{2}\sum_{i=1}^{n}\left(k_{p_i} \ln|\cosh(\tilde{q}_i)| + \frac{k_{d_i}}{b_i}\ln|\cosh(\vartheta_i)|\right). \tag{9.47}$$

Observe that since $\ln|\cosh(z)|$ is positive definite and radially unbounded in z, it is enough to show that $W_1(t,x)$ and $W_2(t,x)$ are positive definite which happens to be true if

$$\varepsilon \leq \frac{1}{2}\min\left\{(k_{p_m} d_m)^{1/2},\ \left(\frac{k_{d_m} d_m}{b_m}\right)^{1/2}\right\}. \tag{9.48}$$

Since $W_2(t,x)$ is positive definite and radially unbounded in $\dot{\tilde{q}}$, it then follows from (9.47) and (9.48) that $V(t,x)$ is positive definite and radially unbounded for all $(t,x) \in I_t \times \mathrm{IR}^{3n}$.

3.1.2 Lyapunov function candidate derivative

Taking the time derivative of (9.44) along trajectories of (9.42), (9.43) we get after some bounding and using properties **P6.1** – **P6.2**,

$$\dot{V} \;\leq\; -\frac{k_{d_m} a_m}{b_M}\|\text{ Tanh}(\vartheta)\|^2 - \frac{\varepsilon k_{p_m}}{d_M}\|\text{ Tanh}(\tilde{q})\|^2 - \varepsilon b_m \sum_{i=1}^{n}\text{ sech}^2(\vartheta_i)\dot{\tilde{q}}_i^2 +$$

$$+k_cB_d\|\dot{\tilde{q}}\|^2+\varepsilon\|\dot{\tilde{q}}\|^2\|\ \mathrm{Sech}^2(\tilde{q})\|+\varepsilon\frac{k_c}{d_m}\|\dot{\tilde{q}}\|^2\left(\|\ \mathrm{Tanh}(\tilde{q})\|+\|\ \mathrm{Tanh}(\vartheta)\|\right)+$$
$$+\varepsilon\frac{k_{p_M}+k_{d_M}}{d_m}\|\ \mathrm{Tanh}(\tilde{q})\|\|\ \mathrm{Tanh}(\vartheta)\|+\varepsilon a_M\|\dot{\tilde{q}}\|\|\ \mathrm{Sech}^2(\vartheta)\|\|\ \mathrm{Tanh}(\vartheta)\|+$$
$$+\varepsilon\frac{k_{d_M}}{d_m}\|\ \mathrm{Tanh}(\vartheta)\|^2+\varepsilon\frac{2k_cB_d}{d_m}\|\dot{\tilde{q}}\|\left(\|\ \mathrm{Tanh}(\tilde{q})\|+\|\ \mathrm{Tanh}(\vartheta)\|\right). \tag{9.49}$$

Let the ball B_η be defined as

$$B_\eta \triangleq \left\{x := \ \mathrm{col}[\tilde{q},\ \dot{\tilde{q}},\ \vartheta]\in \mathbb{R}^{3n}\ :\ V(t,x)<\eta\right\}, \tag{9.50}$$

then the closure $\bar{B}_\eta$ is defined as

$$\bar{B}_\eta \ \triangleq\ \left\{x := \ \mathrm{col}[\tilde{q},\ \dot{\tilde{q}},\ \vartheta]\in \mathbb{R}^{3n}\ :\ V(t,x)\le\eta\right\}, \tag{9.51}$$

suppose that $(t,x)\in I_t\times\bar{B}_\eta$, then we can write

$$\begin{aligned}\dot{V}\ \le\ & -\frac{1}{2}\begin{bmatrix}\|\ \mathrm{Tanh}(\tilde{q})\|\\ \|\ \mathrm{Tanh}(\vartheta)\|\end{bmatrix}^\top Q_1\begin{bmatrix}\|\ \mathrm{Tanh}(\tilde{q})\|\\ \|\ \mathrm{Tanh}(\vartheta)\|\end{bmatrix}\\ & -\varepsilon\begin{bmatrix}\|\ \mathrm{Tanh}(\tilde{q})\|\\ \|\dot{\tilde{q}}\|\end{bmatrix}^\top Q_2\begin{bmatrix}\|\ \mathrm{Tanh}(\tilde{q})\|\\ \|\dot{\tilde{q}}\|\end{bmatrix}\\ & -\begin{bmatrix}\|\dot{\tilde{q}}\|\\ \|\ \mathrm{Tanh}(\vartheta)\|\end{bmatrix}^\top Q_3\begin{bmatrix}\|\dot{\tilde{q}}\|\\ \|\ \mathrm{Tanh}(\vartheta)\|\end{bmatrix}\\ & -\gamma_1\|\ \mathrm{Tanh}(\vartheta)\|^2-\gamma_2\|\dot{\tilde{q}}\|^2.\end{aligned} \tag{9.52}$$

where

$$Q_1=\begin{bmatrix}\varepsilon k_{p_m}/d_M & -\varepsilon(k_{p_M}+k_{d_M})/d_m\\ -\varepsilon(k_{p_M}+k_{d_M})/d_m & 2k_{d_m}a_m/3b_M\end{bmatrix},$$

$$Q_2=\begin{bmatrix}k_{p_m}/2d_M & -k_cB_d/d_m\\ -k_cB_d/d_m & b_m\ \mathrm{sech}^2(\eta)/3\end{bmatrix},$$

$$Q_3=\begin{bmatrix}\varepsilon b_m\ \mathrm{sech}^2(\eta)/3 & -\varepsilon(k_cB_d/d_m+a_M/2)\\ -\varepsilon(k_cB_d/d_m+a_M/2) & k_{d_m}a_m/3b_M\end{bmatrix}$$

$$\gamma_1=\left[\frac{k_{d_m}a_m}{3b_M}-\varepsilon\frac{k_{d_M}}{d_m}\right]$$

$$\gamma_2=\left[\frac{\varepsilon}{3}b_m\ \mathrm{sech}^2(\eta)-k_cB_d-2\varepsilon k_c/d_m-\varepsilon\right]$$

The matrices Q_1, Q_3 and the constant γ_1 are positive definite if

$$\varepsilon<\min\left\{\frac{2d_m^2k_{p_m}k_{d_m}a_m}{3d_Mb_M(k_{p_M}+k_{d_M})^2},\ \frac{4d_m^2k_{d_m}a_mb_m\ \mathrm{sech}^2(\eta)}{9b_M(2k_cB_d+a_Md_m)^2},\ \frac{k_{d_m}a_md_m}{3k_{d_M}b_M}\right\} \tag{9.53}$$

while Q_2 is positive definite if

$$k_{p_m}b_m\ \mathrm{sech}^2(\eta)>\frac{6k_c^2B_d^2d_M}{d_m^2} \tag{9.54}$$

and $\gamma_2>0$ if $b_m>6$ and

$$\varepsilon>\frac{k_cB_d}{\frac{b_m}{6}\ \mathrm{sech}^2(\eta)-\frac{2k_c}{d_m}}. \tag{9.55}$$

Apparently condition (9.53) holds for any fixed control parameters and a sufficiently small ε, condition (9.54) holds for sufficiently large b_m while the last condition imposes a lower bound on ε which can be made arbitrarily small by increasing b_m, see also (9.48). This last condition seemingly may lead to a contradiction, for this we dedicate the next section to show that it is always possible to find constants a_m and b_m such that $\dot{V}(t,x)$ is negative definite $\forall\ (t,x)\in I_t\times\bar{B}_\eta$.

3.1.3 A tuning procedure

For the sake of simplicity and without loss of generality we will assume in the sequel that $A = a_M I_n$, $B = b_M I_n$, $K_p = k_{p_M} I_n$, $K_d = k_{d_M} I_n$. Under this assumption, let us define

$$b \triangleq \frac{12\alpha_1 k_c}{\operatorname{sech}^2(\eta) d_m} \tag{9.56}$$

where $\alpha_1 \gg 1$ is large enough such that (9.55) hold. With these definitions of b and α_1 notice that the inequality (9.54) holds if

$$\alpha_1 > \max\left\{\frac{k_c B_d^2 d_M}{k_p d_m}, \frac{d_m}{k_c}, 1\right\} \tag{9.57}$$

while equation (9.56) together with (9.57) implies that $b_m > 6$ as required to ensure that $\gamma_2 > 0$. On the other hand, we can define without loss of generality $k_p \triangleq \alpha_2 k_d$, $\alpha_2 > 0$ so it is easy to show that the rest of the conditions hold if

$$\frac{B_d}{(\alpha_1 - 1)} < \varepsilon \le \frac{a \operatorname{sech}^2(\eta)}{9} \min\left\{\frac{4k_d d_m}{(2k_c B_d + a d_m)^2}, \frac{d_m^2 \alpha_2}{d_M(\alpha_2+1)^2 \alpha_1 k_c}\right\}. \tag{9.58}$$

Next, notice that since there are no restrictions in the choice of k_p and k_d other than the input constraints, given condition (9.41) there always exist sufficiently small constants k_p and k_d such that (9.2) holds. Assuming that these constants are fixed in a suitable way, then the only possibility that (9.58) is satisfied, is for sufficiently large a and b; to show that this is the case we shall consider two possible situations:

case 1. We need to have

$$\frac{B_d}{(\alpha_1 - 1)} < \frac{4k_d a d_m \operatorname{sech}^2(\eta)}{9(2k_c B_d + a d_m)^2}, \tag{9.59}$$

case 2. We need to have

$$\frac{B_d}{(\alpha_1 - 1)} < \frac{a d_m^2 \alpha_2 \operatorname{sech}^2(\eta)}{9 d_M (\alpha_2+1)^2 \alpha_1 k_c}. \tag{9.60}$$

Notice that in the first case, for any fixed value of a there always exists a sufficiently large α_1 such that (9.59) holds; notice from (9.56), that this is equivalent to increase b. In the second case, notice that both terms on both sides of the inequality decrease at the same rate with respect to α_1 hence, the inequality is practically not affected for large values of α_1, therefore (9.60) is satisfied for large values of a. Since there is no contradiction in these conditions, it follows that $\dot{V}(t,x)$ is negative definite in $x' \triangleq \operatorname{col}[\operatorname{Tanh}(\tilde{q}),\ \dot{\tilde{q}},\ \operatorname{Tanh}(\vartheta)]$ for all $(t,x) \in I_t \times \bar{B}_\eta$. Finally, since $\tanh(z) = 0$ if and only if $z = 0$ it follows that $\dot{V}(t,x)$ is negative definite in $\operatorname{col}[\tilde{q}(t),\ \dot{q}(t),\ \vartheta(t)]$ for all $(t,x) \in I_t \times \bar{B}_\eta$.

So far we have shown that there always exist some constants a and b which ensure that $\dot{V}(t,x)$ is locally negative definite. Now, we recall that $V(t,x)$ is positive definite if (9.48) is satisfied; this happens to hold if, for sufficiently large b (namely $b \ge 1/\alpha_2$)

$$\frac{6k_c B_d^2}{k_d} < \frac{(\alpha_1 - 1)^2}{4\alpha_1} \operatorname{sech}^2(\eta) \tag{9.61}$$

which holds also for any fixed k_d (hence k_p), η and sufficiently large α_1.

We then conclude that $V(t,x)$ is a Lyapunov function $\forall\ (t,x) \in I_t \times \bar{B}_\eta$ according to definition of (Rouche and Mawhin, 1980).

For clarity of exposition we emphasize that due to the time varying nature of the closed loop dynamics and Lyapunov function $V(t,x)$, we are not ready to conclude (local) asymptotic stability. This issue is addressed in the following section. Nevertheless we already call $V(t,x)$ being positive definite and having $\dot{V}(t,x)$ negative definite, a Lyapunov function.

3.1.4 Semi-global asymptotic stability

In this section we prove that under the conditions established above, the closed loop system is asymptotically stable with domain of attraction B_η. Furthermore, we prove that it can be arbitrarily enlarged respecting the input constraints (9.2). For this purpose we use lemma C.1 and theorem C.3; to apply these results, we first verify the conditions of lemma C.1: notice that condition (i) is satisfied by hypothesis; second, since $V(t, x)$ is continuous in $I_t \times \mathbb{R}^{3n}$, positive definite and radially unbounded, it follows from (9.50) and (9.51) that $\forall t \in I_t$, $V(t, B_\eta) < V(t, \partial B_\eta)$ thus there exists a constant $\delta > 0$ such that (ii) and (iii) hold. Finally, in the previous section we showed that $\dot{V}(I_t \times B_\eta) \leq 0$, hence we can conclude from lemma C.1 that under conditions of proposition 9.5, the solution $x(t) \in B_\eta$ for all $t \geq t_0$.

Now, it remains to prove that the conditions of theorem C.3 are also fulfilled. For this, notice first that condition (i) is trivially satisfied from the definition of $V(t, x)$ and the assumption that $q_d(t) \in \mathcal{C}^2$. On the other hand, it was proved in the previous section that $V(t, x)$ is a Lyapunov function for all $(t, x) \in (I_t \times B_\eta)$, i.e., it is positive definite and its time derivative is negative definite, then (ii) also holds. The last condition follows directly from the definition of B_η. We conclude that the closed loop system (9.36) – (9.38) is asymptotically stable with domain of attraction, the box B_η.

Moreover since it was shown that there always exist a and b such that all the conditions of stability are fulfilled for any arbitrarily large η, semi-global asymptotic stability follows.

Finally notice that since these constants do not appear in the control law u, the input constraints are independent of them. The proof is completed by observing that since $\tanh(\cdot) < 1$, from (9.41) it is always possible to choose some sufficiently small gains K_p and K_d in (9.38) such that (9.2) hold.

■

3.2 Simulation results

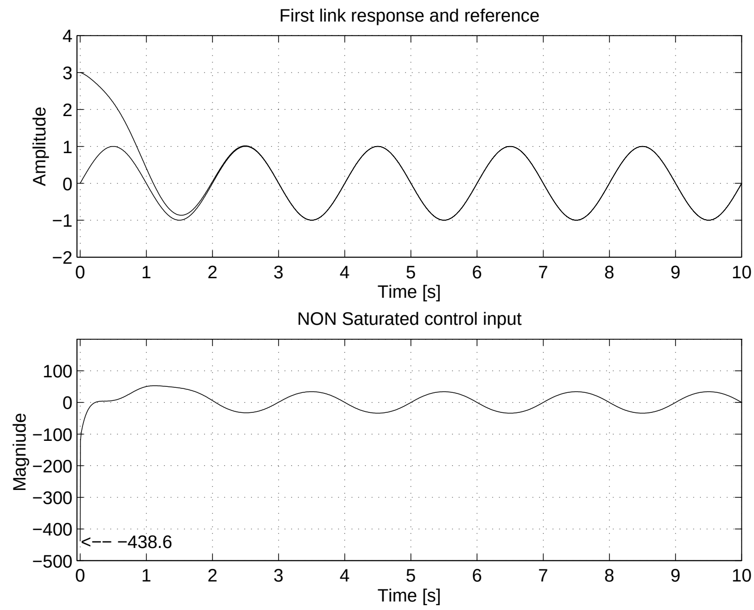


Figure 9.3: Controller of Loria and Ortega (1995)

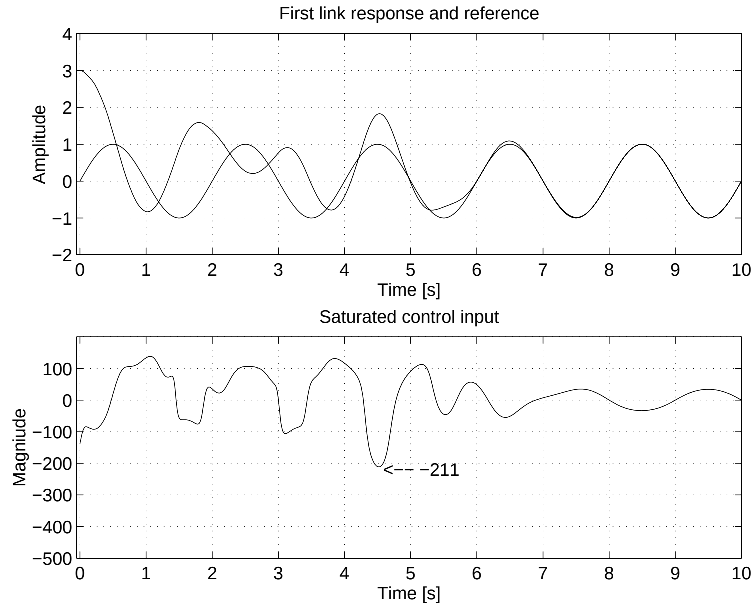


Figure 9.4: Saturated controller of proposition 9.5

Using SIMULINK™of MATLAB™, we tested our algorithm in the two link robot arm of (Kruise, 1990; Berghuis, 1993) when tracking a sinusoidal trajectory of 0.5hz i.e., $q_d(t) := \sin(\pi t - t_0)$ rads/s, hence $B_d = 14$. We imposed $u^{\max} = 250$[Nm] in (9.2). Then, to meet the conditions of proposition 9.5 we chose $A = B = \operatorname{diag}\{480000,\ 480000\}$ while the controller gains were set to to $K_p = \operatorname{diag}\{150,\ 150\}$, $K_d = \operatorname{diag}\{90,\ 90\}$ according to (9.41).

In order to evaluate the performance of our controller, we tested as well that of proposition 8.1, using the same gain values and starting from the same initial conditions: $q_p = \operatorname{col}[3,\ 0]$, $q_c = \operatorname{col}[-1440000,\ 0]$, and $t_0 = 0$. Notice that, as it was previously discussed, the initial conditions of qc are chosen in a way that the initial condition ϑ_0 be small enough.

In figure 9.3 we show the response of the first link and the control input, using the controller of 8.1, that is, without the saturation.

For this case, the maximum absolute value of the control input is 438.6[Nm], hence the input constraint is not satisfied at the beginning.

In figure 9.4 we show the response of the first link and the control input, using the controller of proposition 9.5. Notice that the maximum absolute value of the control input is 211[Nm], hence satisfying the input constraints.

In contrast, what it is not surprising is that the "big" initial value of the control input in the first case (c.f. fig. 9.3) drives the link very fast to the desired reference. In the case of the saturated controller, the transient is longer and has more oscillations till the link reaches the reference.

Thus we confirm what might be expected: there is a compromise between a fast transient or a small control input.

4 Concluding remarks

We have extended in this chapter our results on output feedback stabilization of EL systems to the practically important case of bounded control inputs. As a corollary of our work we have improved in several directions the result of (Kelly et al., 1994b) on state feedback global stabilization of robot manipulators with saturated inputs. First, we have removed the requirement of measurement of generalized velocities. In particular we have shown, that a suitably saturated approximate differentiator can be used to estimate these signals. Second, we have developed this theory for a wider class of EL systems, –that contains as a particular case robot manipulators-, and to more general saturation functions. Finally, we have identified a class of EL controllers, that achieve the global output feedback stabilization objective.

We have illustrated in simulations the practical advantages of using bounded controls. In particular we compared the performance of the controller of (Kelly, 1993b) and its "saturated" equivalent. It was shown that for a particular case with non-zero initial conditions, the linear controller of (Kelly, 1993b) failed to respect the input constraints.

We have also solved the output feedback tracking control problem of Euler-Lagrange systems under input constraints. Our scheme ensures semi-global asymptotic stability, i.e., the basin of attraction can be arbitrarily enlarged by using high observer gains, this allows to keep the controller parameters small. Hence, our result supersedes all previous solutions to the output feedback control problem in the sense that it does not need high controller gains to achieve semi-global asymptotic stability. It also improves previous results on saturated set-point control proving that it is possible to track any bounded and twice continuously differentiable trajectory.

Finally, we have compared in simulations the performance of our controller against that of (Loria and Ortega, 1995) upon which it was inspired. Simulations have showed that in a simple case when non zero initial conditions are taken, the linear controller of (Loria and Ortega, 1995) may yield inadmissible control inputs.

Chapter 10

Constrained motion control of robot manipulators

In previous chapters we have presented several results for output feedback control of EL systems. As it was mentioned before, robot manipulators are an important example of mechanical systems lying in the EL-class. In this chapter we concentrate on the particular problem of *constrained* robot motion control, or in interaction with their environment. The results of this chapter are a direct application of the results presented in previous chapters.

1 Introduction

During the last decade different approaches to control robot manipulators in interaction with their environment have been proposed in the literature which can be classified according to the control objective and the adopted model of contact force; for instance, (Whitney, 1977) identified 6 different approaches to control force which in their turn may be divided into two groups: The approaches of the first group aim at controlling simultaneously the manipulator position and exerted force in a non conflicting way. The schemes lying in the the second group focus on controlling the relationship between the manipulator position and the interaction force.

A popular example that belongs to the first group is hybrid control, introduced by (Raibert and Craig, 1981), it aims at controlling the exerted force in the directions constrained by the environment and position in the others. Later, this approach was extended to dynamic hybrid control by (Yoshikawa, 1986) and to an operational space formulation by (Khatib, 1987).

Some examples of the second group are active stiffness control (Salisbury, 1980), compliant control (Spong and Vidyasagar, 1989; Waibel and Kazerooni, 1991) and impedance control (Hogan, 1985; Dawson et al., 1993). More recently, (Chiaverini and Sciavicco, 1993) introduced the *parallel control* scheme where simultaneous force/position control is carried out giving priority to force control by means of an integrator instead of a selection matrix as it is the case of the hybrid control approach. For a recent text briefly describing the above mentioned approaches see (Sciavicco and Siciliano, 1996).

On the other hand, concerning the adopted model, mainly two types of contact force models are used based on the type of the interaction environment: *elastic* or *infinitely stiff*.

In the first case the interaction with the compliant environment is considered to be elastic so the compliant force is supposed to be proportional to the deformation of the environment. When considering an infinitely stiff environment, starting with (Yoshikawa, 1986; McClamroch, 1986; Koivo and Kankaanrantaes, 1988) the interaction is modeled by holonomic (algebraic) constraints imposed to the manipulator's motion. Unfortunately these equations are singular, then in order to cope with this difficulty, various techniques for deriving the so-called *reduced order models* have been proposed based on the projection of the dynamic robot equations onto a submanifold described by the constraints algebraic equation.

An alternative approach is based on the *principle of orthogonalization* (Arimoto et al., 1993) whose key feature is the introduction of a projection matrix that projects velocity and position error signals to a plane tangent to the constraint surface (in joint space) in order to distinguish force and position signals.

Few works take joint flexibilities into account. A more recent work on the modeling and control of constrained manipulators is (Krishnan, 1995) who offers a geometric approach which considers joint flexibility and the dynamics of direct-current actuators. See also (Wang and Soh, 1996; Lozano and Brogliato, 1992b).

Concerning the force control, several schemes have been proposed for stabilization of constrained manipulators. See among others (McClamroch and Wang, 1988; Carelli and Kelly, 1991; Panteley and Stotsky, 1993a; Arimoto et al., 1992). The regulation problem has been considered for instance by (Wang and McClamroch, 1993; Arimoto, 1994b) using an infinitely stiff environment based model and (Chiaverini and Sciavicco, 1993; Siciliano and Villani, 1993) who consider that the manipulators interacts with an elastic environment. It is worth remarking that (Wang and McClamroch, 1993) established *local* asymptotic stability of a hybrid scheme consisting of a PD regulator with compensation of the gravity and contact forces at their desired values. This result, which naturally extends to the constrained motion case the seminal contribution of (Takegaki and Arimoto, 1981) studied in chapter **7**, relies on the well known fact (Arimoto, 1994b) that the passivity property of the robot is preserved even when in contact with the environment.

Nevertheless, a major drawback of all above cited solutions is the need of velocity measurements, hence it is desirable to derive controllers where only force and position measurements are considered. To the best of our knowledge the force/position control problem without velocity measurements was first treated by (Huang and Tseng, 1991) and (Panteley and Stotsky, 1993b); Huang and Tseng studied the (open loop) observer design for constrained robots. More recently (Panteley and Stotsky, 1993b) proposed a nonlinear observer and proved for the first time, asymptotic stability of the closed loop system considering an infinitely stiff environment. Some other results in this direction are (Colbaugh and Glass, 1996; Loria and Ortega, 1996; de Queiroz et al., 1996b; de Queiroz et al., 1996a).

A second important drawback concerning those approaches based on infinitely stiff environment models, is that the majority of control schemes are designed assuming that the algebraic constraints equation is globally solvable. It is supposed either that the constraint Jacobian is non singular in the whole state space (McClamroch and Wang, 1988; Caiti and Cannata, 1994; Yao et al., 1992; Carelli and Kelly, 1991) or that the actual trajectory is contained in the neighbourhood of the desired set-point (Wang and McClamroch, 1993; Arimoto et al., 1993). However, it must be remarked that these assumptions are quite restrictive, for instance the first one can be not fulfilled even in very simple cases like for a planar two-link revolute-joint manipulator whose endpoint is constrained to a plane.

In this final chapter, we extend the results of previous chapters for an important example of EL systems: the robot manipulators. Our contribution in this domain consists on solving different force/position and force/tracking control problems by using only force and position measurements. We have studied both types of environments mentioned above.

First, the problem of designing an asymptotically stable force/position regulator that does not require the *exact knowledge* of the gravity forces nor the *measurement of speed* was first solved in (Loria and Ortega, 1996) – see also (Colbaugh and Glass, 1996) –. In that paper we carried out our design using a robot model in task space (Khatib, 1987). We supposed as (Chiaverini and Sciavicco, 1993) that the tool force is exerted in only one direction normal to an elastic environment whose stiffness constant is exactly known. The main result was the proof that under these assumptions, we can design a *semiglobally asymptotically stable* regulator consisting of a PD and two integral terms. That controller extended the results on PI^{2}D regulation of section **7**.4 to the constrained environment case observing that, under the compliant environment assumption, the force/position control problem can be reformulated as a pure position control problem with a suitable change of coordinates. A further contribution of our work was the proof that if velocity is measurable, then a simple PID with a nonlinear integral gain yields *global asymptotic stability* (GAS).

Secondly, we consider that the end-tool is interacting with an infinitely stiff environment. We use the reduced order model proposed in (Panteley and Stotsky, 1993a) which is defined in a submanifold determined by the constraints equations. It is supposed that there exists a subset of the coordinates space where it is reasonable to assume that contact is not lost during motion. Moreover, if the generalized trajectories start and remain in this subset, using the global implicit function theorem (Sandberg, 1981) it can be proven that the manipulator Jacobian is non singular. In contrast to other results using similar reduced order models we *do not* assume that the generalized coordinates remain all the time in this subset. We prove it.

Under these conditions, we first propose a controller which ensures that for every initial conditions strictly contained in this subset, the generalized positions remain in it. Based on this, asymptotic stability of the closed loop system is also proved. A drawback of the controller of (Loria and Ortega, 1996) is that it needs high controller gains, however, we know that it is of interest to design saturated control laws, therefore we achieve our control objective using bounded controls. As far as we know this result reported in (Panteley et al., 1996) is the first in its kind.

Finally, the third result contained in this chapter is an uniformly asymptotically stabilizing controller which drives the generalized trajectories of the constrained manipulator to some desired time varying trajectory, hence extending the results of (Panteley et al., 1996) to tracking/force control and those of (Loria and Ortega, 1995) to constrained motion control. In comparison to (Panteley and Stotsky, 1993b), our solution uses a *linear* filter instead of a non-linear observer making the implementation and the stability proof much simpler (Loria and Panteley, 1996).

The organization of this chapter is the following: In the next section we define and stress some important properties of the elastic environment based model, in section 3 we present our contributions considering an elastic environment, in section 4 we describe the reduced order model of (Panteley and Stotsky, 1993a), sections 5 and 6 contains our findings concerning an infinitely stiff environment and finally we conclude with some remarks to this chapter in section 7.

2 Elastic environment

We consider the robot manipulator model in the tool coordinates, $x \in \mathbb{R}^m$ (Khatib, 1987)

$$D(x)\ddot{x} + C(x,\dot{x})\dot{x} + g(x) = u - F. \tag{10.1}$$

When m is equal to the number of joints, n, and the manipulator acts in a nonsingular configuration, x constitutes a set of Lagrangian generalized coordinates in which case, $D > 0$ and assumes the role of a *true* inertia matrix (Khatib, 1987), otherwise it is only a *pseudo* inertia matrix. Throughout this work our attention is focused on non redundant robots, i.e. $m = n$. Under these assumptions, the relationship between the joints space and the task space, is given by[1]

$$\begin{aligned} D(x) &= J^{-\top}(q)D(q)J^{-1}(q) \\ C(x,\dot{x}) &= J^{-\top}(q)C(q,\dot{q}) - D\dot{J}(q)\dot{q} \\ g(x) &= J^{-\top}(q)g(q) \\ u &= J^{-\top}(q)\tau \end{aligned} \tag{10.2}$$

where q are the generalized coordinates and τ is the input force, all expressed in the joints referential frame; and finally, $J(q)$ is the manipulator Jacobian matrix.

2.1 Properties

We underline below some well known properties of the elastic environment based model useful for our controller design:

P 10.1 Considering all joints revolute, the inertia matrix D is lower and upper bounded by

$$0 < d_m I_m < D(x) \leq d_M I_m < \infty$$

where I_m stands for the $m \times m$ identity matrix.

[1] See for instance (Chiaverini and Sciavicco, 1993)

P 10.2 The matrix $N(x,\dot{x}) = \dot{D}(x) - 2C(x,\dot{x})$ is skew-symmetric. Furthermore, the matrix $C(x,\dot{x})$ is linear on $\dot{x}$ and bounded on x, hence for some $k_c > 0$

$$\|C(x,\dot{x})\| \;\leq\; k_c\|\dot{x}\|$$

P 10.3 The generalized gravitational forces vector $g(x) \triangleq \frac{\partial U_g(x)}{\partial x}$ satisfies

$$\sup_{x\in\mathbb{R}^m} \left\| \frac{\partial g(x)}{\partial x} \right\| \;\leq\; k_g \tag{10.3}$$

for a $k_g > 0$, where $U_g(x)$ is the potential energy expressed in the operational space and is supposed to be bounded from below.

2.2 Assumptions

A 10.1 We mean by 'position', both position and orientation while 'force' stands for linear force and torque, henceforth $m = 3$.

A 10.2 We consider the contact force to act only in the direction normal to the environment, for the sake of simplicity and without loss of generality we choose $F \triangleq \text{col}[0,\ 0,\ f]$. Hence, we divide the generalized coordinates x into $x \triangleq \text{col}[x_p,\ x_n]$ where $x_p \in \mathbb{R}^2$ and $x_n \in \mathbb{R}$ are the parallel and normal coordinates, i.e.

$$x_n \triangleq n_1^\top x, \quad x_p \triangleq n_2^\top x$$

where

$$n_1^\top \triangleq [0\ 0\ 1] \quad n_2^\top \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

A 10.3 We suppose the end–effector to be acting on an elastic environment with stiffness k, then the force f is defined by

$$f \triangleq k(x_n - x_{n_0}) \tag{10.4}$$

where x_{n_0} is the position in the direction of x_n where no force is exerted. Using the above defined notation we are able to write

$$F \triangleq k n_1 n_1^\top \left(x - \begin{bmatrix} 0 \\ 0 \\ x_{n0} \end{bmatrix} \right). \tag{10.5}$$

3 Force/position control in an elastic environment

Once we have described the models we use throughout this chapter, we can pass to our contributions, starting with force/regulation control of robot manipulators interacting with an elastic environment. In particular we deal with the following problems.

State feedback force/position regulation problem. For the model (10.1) assume

- The environment stiffness k is exactly known,
- $g(q)$ is unknown but k_g is known,
- Position *and velocity* are available for measurement.

Under these conditions, find a control law such that for all given desired *constant* position x_{p_d} and force f_d we have

$$\lim_{t\to\infty} x_p = x_{p_d} \tag{10.6}$$

$$\lim_{t\to\infty} f = f_d \tag{10.7}$$

for all systems initial conditions. Further, we will require that the closed loop has a GAS equilibrium.

Output feedback force/position control problem. Assume now that

- k is exactly known,
- $g(q)$ is unknown but d_m, d_M and k_g are known,
- *Only position* is available from measurement.

Then, design a controller that insures the desired equilibrium is *semiglobally asymptotically stable.*

3.1 PID and PI²D controllers

We present in this section a PID controller for the state feedback force/position control problem. As (Kelly, 1993b) and (Tomei, 1991a) we use a normalization which allows us to prove global asymptotic stability. Next, in proposition 10.2 we present a PI²D controller which guarantees semiglobal asymptotic stability of the closed loop. It extends the results presented in section **7**.4, to force/position control.

10.1 Proposition. *Consider the model (10.1) in closed loop with the control law*

$$\begin{aligned} u &= -K_p(x-x_d) - K_d\dot{x} + F_d + \nu &(10.8)\\ \dot{\nu} &= -\varepsilon K_i(x-x_d), \;\; \nu(0)=\nu_0 \in I\!R^3 &(10.9)\end{aligned}$$

where $F_d \triangleq col[0,\ 0,\ f_d]$, K_d *and* K_i *are constant positive definite diagonal matrices and, for a given* K_i, K_p *is defined as*

$$\begin{aligned} K_p &\triangleq K_p' + K_i &(10.10)\\ \varepsilon &\triangleq \frac{\varepsilon_0}{1+\|x-x_d\|} &(10.11)\end{aligned}$$

where K_p' *satisfies*

$$4k_g < k'_{p_m} \tag{10.12}$$

with k_g *as in (10.3). Under these conditions, we can always find a (sufficiently small) positive constant* ε_0, *–independent of the initial conditions–, such that the equilibrium*

$$s \triangleq col[x_p,\ f,\ \dot{x}_p,\ \dot{f},\ \nu] = col[x_{pd},\ f_d,\ 0,\ 0,\ g(x_d)]$$

is globally asymptotically stable. □

10.2 Proposition. *Consider the model (10.1) in closed loop with the PI²D control law*

$$\begin{aligned} u &= -K_p(x-x_d) - K_d\vartheta + F_d + \nu &(10.13)\\ \dot{\nu} &= -K_i(x-x_d-\vartheta), \;\; \nu(0)=\nu_0 \in I\!R^3 &(10.14)\\ \vartheta &= \mathrm{diag}\left\{\frac{b_i p}{p+a_i}\right\}x, \;\; \vartheta(0)=\vartheta_0 \in I\!R^3 &(10.15)\end{aligned}$$

where $b_i > 0$, $a_i > 0$ *and* K_p, K_i, K_d *are positive definite diagonal matrices,* K_p *satisfies*

$$K_p > (4k_g+1)I \tag{10.16}$$

and the high frequency gains b_i of the filters (10.15) are such that

$$b_i > \frac{2d_M}{d_m} \tag{10.17}$$

with k_g as in (10.3). Under these conditions, we can always find a (sufficiently small) integral gain K_i such that the equilibrium

$$s \triangleq col[x_p,\ f,\ \dot{x}_p,\ \dot{f},\ \vartheta,\ \nu] = col[x_{pd},\ f_d,\ 0,\ 0,\ 0,\ g(x_d)]$$

is asymptotically stable with a domain of attraction including the closed ball of radius c_2 in $\mathbb{R}^{12}$, where $\lim_{\min(b_i)\to\infty} c_2 = \infty$. *In other words, the desired equilibrium is semiglobally asymptotically stable.* ▢

10.3 Remark. Proposition 10.1 essentially states that GAS is insured if the integral gain is chosen sufficiently small, i.e. ε_0. This is explained noting that the convergence rate of the closed loop is determined by ε_0 thus, in a practical instance, a trade-off between performance and the size of the domain of attraction is established.

3.2 Stability proofs

In this section we carry out the proofs of propositions 10.1 and 10.2.

3.2.1 Globally asymptotically stable PID

The proof relies on *classical Lyapunov theory*, and is divided in three parts. First, we define a suitable error equation for the closed loop system, whose (global) equilibrium is at the desired value. Second, we propose a Lyapunov function candidate. Third, we prove that under the conditions of proposition 10.1 the proposed function qualifies as a Lyapunov function, and establish the global asymptotic stability of the equilibrium invoking LaSalle's invariance principle.

Error Equation

First, let us define

$$K_p'' \triangleq K_p' + k n_1 n_1^\top, \tag{10.18}$$

using (10.10) and defining $\tilde{x} \triangleq x - x_d$ we have

$$K_p \tilde{x} = K_p''(x - \delta) + K_i \tilde{x} + g(x_d) - k n_1 n_1^\top \tilde{x} \tag{10.19}$$

where $\delta \triangleq x_d + (K_p'')^{-1} g(x_d)$. Now, replacing (10.8) and (10.9) in (10.1) and using the definitions from assumption **A10.3** we get

$$\begin{cases} D\ddot{x} + C\dot{x} + g(x) + K_p''(x - \delta) + K_d \dot{x} = -K_i \tilde{x} + \tilde{\nu} \\ \dot{\tilde{\nu}} = -\varepsilon K_i \tilde{x}, \end{cases}$$

where $\tilde{\nu} \triangleq \nu - g(x_d)$, the notation $\tilde{\nu}$ being motivated by the fact that with $\nu = g(x_d)$ in (10.8) we recover the controller of (Wang and McClamroch, 1993) in tool coordinates. To simplify the derivation of the Lyapunov function we find convenient to introduce the (linear) change of coordinates

$$z = -K_i \tilde{x} + \tilde{\nu} \tag{10.20}$$

which yields the *error equation*

$$\begin{cases} D\ddot{x} + C\dot{x} + g(x) + K_p''(x - \delta) + K_d \dot{x} = z \\ \dot{z} = -\varepsilon K_i (\tilde{x} + \frac{1}{\varepsilon}\dot{x}) \end{cases} \tag{10.21}$$

It is easy to see that, with the state vector $\mathrm{col}[\tilde{x},\ \dot{x},\ z]$, the unique equilibrium point of (10.21) is the trivial one.

Lyapunov Function Candidate

We will now construct a Lyapunov function for (10.21) whose time derivate is negative definite in $\tilde{x}$, $\dot{x}$. To this end, remark that equations (10.21) with $z \equiv 0$ and $K_p = K_p''$ are similar to the closed loop equations of (Kelly, 1993b) for the *regulation problem* which is known to be GAS with Lyapunov function

$$V_1(x, \dot{x}) = \frac{1}{2}\dot{x}^\top D\dot{x} + \frac{1}{2}(x-\delta)^\top K_p''(x-\delta) + U_g(x) + c_1 \tag{10.22}$$

where $c_1 \triangleq -\frac{1}{2}g(x_d)^\top (K_p'')^{-1} g(x_d) - U_g(x_d)$ is added to the systems total energy to enforce $V_1(x_d,\ 0) = 0$. Taking the derivative of V_1 along the trajectories of (10.21) we get

$$\dot{V}_1 = -\dot{x}^\top K_d \dot{x} + \dot{x}^\top z.$$

To cancel the second right hand term we add to V_1 the function $V_2(z) \triangleq \frac{1}{2}z^\top K_i^{-1} z$ to get now

$$\dot{V}_1 + \dot{V}_2 = -\dot{x}^\top K_d \dot{x} - \varepsilon z^\top \tilde{x}.$$

At this point, to remove the cross products above and to enforce the desired negative definiteness of the time derivative, we propose to add some cross terms to the Lyapunov function. First observe that with $V_3(\tilde{x}, \dot{x}) \triangleq \tilde{x}^\top D\dot{x}$ we have

$$\dot{V}_3 = \tilde{x}^\top [C^\top(x, \dot{x})\dot{x} - g(x) + g(x_d) - K_p''\tilde{x} - K_d\dot{x} + z] + \dot{x}^\top D\dot{x}$$

where we have used property **P10.1** to get the first right hand term. Using the bound $-\tilde{x}^\top [g(x) - g(x_d)] \le k_g ||\tilde{x}||^2$, and some simple inequalities we get

$$\dot{V}_3 \le k_c ||\tilde{x}|| ||\dot{x}||^2 + k_g ||\tilde{x}||^2 - k_{p_m}'' ||\tilde{x}||^2 + \tilde{x}^\top z + d_M ||\dot{x}||^2 + k_{d_M} ||\tilde{x}|| ||\dot{x}||. \tag{10.23}$$

Adding all these terms together we get our *Lyapunov function candidate*

$$V \triangleq V_1 + V_2 + \varepsilon V_3, \tag{10.24}$$

whose time derivative using[2]

$$\dot{\varepsilon} V_3 \ \le\ \varepsilon d_M ||\dot{x}||^2 \ \le\ \varepsilon_0 d_M ||\dot{x}||^2$$

and

$$\varepsilon k_c ||\tilde{x}|| ||\dot{x}||^2 \le \varepsilon_0 ||\dot{x}||^2$$

yields, after some straightforward calculations

$$\dot{V} \le -\frac{\varepsilon}{2} \begin{bmatrix} ||\tilde{x}|| \\ ||\dot{x}|| \end{bmatrix}^\top \overbrace{\begin{bmatrix} 2[k_{p_m}'' - k_g] & -k_{d_M} \\ -k_{d_M} & \frac{1}{\varepsilon_0} k_{d_m} \end{bmatrix}}^{Q_1} \begin{bmatrix} ||\tilde{x}|| \\ ||\dot{x}|| \end{bmatrix} - \left[\frac{1}{2} k_{d_m} - \varepsilon_0 \left[k_c + 2d_M\right]\right] ||\dot{x}||^2. \tag{10.26}$$

Global Asymptotic Stability

We will now give sufficient conditions for *positive definiteness* of V. To this end, we will partition V as $V = W_1 + W_2 + W_3$ where

$$\begin{aligned} W_1 &= \frac{1}{4}\dot{x}^\top D\dot{x} + \frac{1}{8}\tilde{x}^\top K_p''\tilde{x} + \varepsilon \tilde{x}^\top D\dot{x}, & (10.27)\\ W_2 &= \frac{1}{8}\tilde{x}^\top K_p''\tilde{x} + U_g + \tilde{x}^\top K_p''(x_d - \delta) + \frac{1}{2}(x_d - \delta)^\top K_p''(x_d - \delta) + c_1, & (10.28)\\ W_3 &= \frac{1}{4}\tilde{x}^\top K_p''\tilde{x} + \frac{1}{2}z^\top K_i^{-1} z^\top + \frac{1}{4}\dot{x}^\top D\dot{x}. & (10.29) \end{aligned}$$

[2] where

$$\begin{aligned} \dot{\varepsilon} &= -\frac{\varepsilon_0}{(1+||\tilde{x}||)^2}\frac{\tilde{x}^\top \dot{x}}{||\tilde{x}||}, \\ \Rightarrow \dot{\varepsilon}\tilde{x}^\top D\dot{x} &\le \frac{\varepsilon_0}{(1+||\tilde{x}||)^2} ||\tilde{x}|| d_M ||\dot{x}||^2, \\ \Rightarrow \dot{\varepsilon}\tilde{x}^\top D\dot{x} &\le \varepsilon_0 d_M ||\dot{x}||^2. \end{aligned} \tag{10.25}$$

Under the conditions of proposition **P10.1**, W_2 is positive definite (Kelly et al., 1994a). W_1 is positive definite if

$$\frac{1}{4}\sqrt{\frac{k''_{p_m}}{d_M}} > \varepsilon_0 \geq \varepsilon. \tag{10.30}$$

Thus, V is positive definite for ε_0 sufficiently small. On the other hand, it is clear that $\dot{V}$ is *negative definite* in $\tilde{x}$, $\dot{x}$ if $k''_{p_m} > k_g$ and

$$\min\left\{\frac{2[k''_{p_m} - k_g]k_{d_m}}{\lambda^2(K_d)},\ \frac{k_{d_m}}{2\,[k_c + 2d_M]}\right\} > \varepsilon_0. \tag{10.31}$$

Inequalities (10.30) and (10.31) are satisfied for ε_0 sufficiently small. Therefore, (10.26) is negative definite and the equilibrium is stable in the sense of Lyapunov. Thus, considering assumption 2.3, GAS of the equilibrium follows immediately invoking LaSalle's invariance principle.

3.2.2 PI^{2}D controller

The proof of proposition 10.2 follows along the lines of the proof of proposition **7**.7.4, hence we will present here only a sketch of the proof.

As previously, let us first define

$$K'_p \triangleq K_p - \frac{1}{\varepsilon}K_i \tag{10.32}$$

where, for a given *constant*[3] $\varepsilon > 0$, a K_i is chosen such that

$$I > \frac{1}{\varepsilon}K_i \tag{10.33}$$

thus insuring $K'_p > 4k_g I$. Notice that the main difference in relation to the previous proof is the normalization of ε, hence considering the definitions above we can directly express the error dynamics of (10.13) – (10.15) plus (10.1) as

$$\begin{aligned}
&D\ddot{x} + C\dot{x} + g(x) + K''_p(x-\delta) + K_d\vartheta = -\tfrac{1}{\varepsilon}K_i\tilde{x} + \tilde{\nu}\\
&\dot{\vartheta} = -A\vartheta + B\dot{x}\\
&\dot{\tilde{\nu}} = -K_i(\tilde{x} - \vartheta)
\end{aligned}$$

where $A \triangleq \mathrm{diag}\{a_i\}$, $B \triangleq \mathrm{diag}\{b_i\}$ and δ as defined before. Now, redefining

$$z \triangleq -\frac{1}{\varepsilon}K_i\tilde{x} + \tilde{\nu} \tag{10.34}$$

leads us to the *error equation*

$$\left\{\begin{aligned}
&D\ddot{x} + C\dot{x} + g(x) + K''_p(x-\delta) + K_d\vartheta = z\\
&\dot{\vartheta} = -A\vartheta + B\dot{x}\\
&\dot{z} = -K_i(\tilde{x} + \tfrac{1}{\varepsilon}\dot{x} - \vartheta).
\end{aligned}\right. \tag{10.35}$$

Observe that, with the state vector $s' \triangleq \mathrm{col}[\tilde{x},\ \dot{x},\ \vartheta,\ z]$, the unique equilibrium point of (10.35) is the trivial one.

We will now construct a Lyapunov function for (10.35) whose derivate is locally negative semidefinite in $(\tilde{x},\ \dot{x},\ \vartheta)$. Let us take the Lyapunov function (10.22) and add to it the energy of the filter (10.15), to get

$$V_1(\tilde{x}, \dot{x}, \vartheta) = \frac{1}{2}\dot{x}^\top D\dot{x} + U_g(x) + \frac{1}{2}(x-\delta)^\top K''_p(x-\delta) + \frac{1}{2}\vartheta^\top K_d B^{-1}\vartheta + c_1$$

where c_1 is defined as before. Taking the derivative of V_1 we get then

$$\dot{V}_1 = -\vartheta^\top K_d B^{-1}A\vartheta + \dot{x}^\top z.$$

[3] i.e., from now on, we redefine $\varepsilon = \varepsilon_0 > 0$

To cancel the second right hand term we add to V_1 the redefined function $V_2(z) \stackrel{\triangle}{=} \frac{\varepsilon}{2} z^\top K_i^{-1} z$ to get now

$$\dot{V}_1 + \dot{V}_2 = -\vartheta^\top K_d B^{-1} A \vartheta - \varepsilon z^\top (\tilde{x} - \vartheta).$$

To remove the cross products above we add the cross term V_3 as before and the new one $V_4(x, \dot{x}, \vartheta) \stackrel{\triangle}{=} -\vartheta^\top D\dot{x}$. Summing up all of them together we get our *Lyapunov function candidate* as

$$V \stackrel{\triangle}{=} V_1 + V_2 + \varepsilon(V_3 + V_4) \tag{10.36}$$

Notice that (10.36) is similar to (7.48) as (10.35) is similar to (7.44) – (7.46) hence the proof follows *mutatis mutandis* as in subsection **7**.4.2.

3.3 Robustness properties

In the previous sections we considered exact knowledge of the stiffness constant k, this was used to avoid measuring force and to calculate an "optimal" desired position in the direction of the force, x_{n_d}. In this section we consider a more realistic case, i.e., we will assume that k is not exactly known. We show that it is still possible to make the position error converge to zero, nevertheless we can only prove that the force error remains bounded.

This is explained since we use the stiffness matrix to estimate a desired position x_{n_d} which will not be the "optimal" anymore. Formally, we introduce the notation: $x^*_{n_d}$ is defined as the optimal position which leads to zero force error as it has been proved in previous sections, $\hat{x}_{n_d}$ is the estimate of $x^*_{n_d}$, which is used in the control law. Now, the following corollary stems directly from proposition 10.1.

10.4 Corollary. *Consider the model (10.1) in closed loop with the control law*

$$\begin{aligned} u &= -K_p(x - \hat{x}_d) - K_d\dot{x} + F_d + \nu & (10.37)\\ \dot{\nu} &= -K_i\hat{\tilde{x}} \quad \nu(0) = \nu_0 \in I\!R^n & (10.38) \end{aligned}$$

Let K_p, K_i, K_d be defined like in proposition 10.2 and k_g satisfy (10.3). Under these conditions, we can always find a (sufficiently small) ε such that the equilibrium $s \stackrel{\triangle}{=}$ col$[\hat{\tilde{x}}_p,\ \dot{x}_p,\ \dot{f},\ \nu] =$ col$[0,\ 0,\ 0,\ g(x_d)]$ be globally asymptotically stable where $\hat{\tilde{x}} \stackrel{\triangle}{=} x - \hat{x}_d$ and there exists a $b_f \in I\!R$ such that $|\tilde{f}| \leq b_f$.

Proof. From previous sections we can directly write the error dynamics as

$$\left\{ \begin{array}{l} D\ddot{x} + C\dot{x} + g(x) + K_p''(x - \delta) + K_d\dot{x} = -\frac{1}{\varepsilon}K_i\hat{\tilde{x}} + \tilde{\nu} \\ \\ \dot{\tilde{\nu}} = -K_i(\hat{\tilde{x}} - \vartheta) \end{array} \right. \tag{10.39}$$

where in this case $\delta \stackrel{\triangle}{=} \hat{x}_d + (K_p'')^{-1}g(\hat{x}_d)$ and $\hat{\tilde{x}} \stackrel{\triangle}{=} x - \hat{x}_d$.

10.5 Remark. Notice that we have redefined the position error as $\hat{\tilde{x}}$, to expose the fact that $\hat{x}_d$ is only an estimate of the ideal position which yields zero force error. Nonetheless, the force is exerted only in the direction of n_1, i.e. $\hat{x}_{p_d} \equiv x_{p_d}$ and $\hat{x}_{n_d} \neq x_{n_d}$.

Under this assumptions, we propose to use the Lyapunov function

$$V(\hat{\tilde{x}}, \dot{x}, z) = \frac{1}{2}\dot{x}^\top D\dot{x} + \frac{1}{2}(x - \delta)^\top K_p''(x - \delta) + U_g(x) + \frac{\varepsilon}{2}z^\top K_i^{-1}z + \varepsilon\hat{\tilde{x}}^\top D\dot{x} + c_1 \tag{10.40}$$

where $c_1 \stackrel{\triangle}{=} \frac{1}{2}g(\hat{x}_d)^\top (K_p'')^{-1}g(\hat{x}_d) - U_g(\hat{x}_d)$. Notice that the error dynamics (10.39) and Lyapunov function (10.40) are similar to (10.21) and (10.24) respectively so the rest of the stability proof follows as in section 3.2.1.

To complete the proof, the upper bound b_f on the steady state force error can be simply derived as follows. Recalling (10.4):

$$f_d = \hat{k}(\hat{x}_{n_d} - x_{n_0}) = k(x^*_{n_d} - x_{n_0})$$

and that $f = k(x_n - x_{n_0})$, we can write

$$\begin{aligned} k(x_n - x_{n_0}) - k(x^*_{n_d} - x_n) &= k(x_n - x_{n_0}) - \hat{k}(\hat{x}_{n_d} - x_{n_0}) \\ \tilde{f} &= kx_n - \hat{k}\hat{x}_{n_d} - (k - \hat{k})x_{n_0}. \end{aligned} \tag{10.41}$$

So far, we have showed that the equilibrium containing $\hat{\tilde{x}} = 0$ is globally asymptotically stable, this implies that

$$f_\infty \triangleq \lim_{t\to\infty} f(t) = k(\hat{x}_{n_d} - x_{n_0}), \tag{10.42}$$

hence, from (10.41)

$$\tilde{f}_\infty = (k - \hat{k})(\hat{x}_{n_d} - x_{n_0}) \triangleq const \tag{10.43}$$

where *const* is some constant for which there always exists an upperbound b_f, then the thesis is proved. ∎

In corollary 10.4 we assume that velocity is available for measurement, however we saw that when this is not the case global stability can not be achieved. Nonetheless, in the following we present a result related to proposition 10.2.

10.6 Corollary. *Consider the model (10.1) in closed loop with the control law*

$$u = -K_p\hat{\tilde{x}} - K_d\vartheta + F_d + \nu \tag{10.44}$$

$$\dot{\nu} = -K_i(\hat{\tilde{x}} - \vartheta), \quad \nu(0) = \nu_0 \in I\!R^n \tag{10.45}$$

$$\vartheta = \operatorname{diag}\left\{\frac{b_i p}{p + a_i}\right\} x. \tag{10.46}$$

Let K_p, K_i, K_d, $A \triangleq \operatorname{diag}\{a_i\}$, $B \triangleq \operatorname{diag}\{b_i\}$ be defined as in proposition 10.2 and k_g satisfies (10.3). Under these conditions, we can always find a (sufficiently small) integral gain K_i such that the equilibrium $s \triangleq col[\hat{\tilde{x}}_p^\top, \dot{x}_p, \dot{f}, \vartheta, \nu] = col[0, 0, 0, 0, g(x_d)]$ is asymptotically stable and $\exists\, b_f \in I\!R$ such that $|\tilde{f}| \le b_f$. Furthermore, a domain of attraction includes

$$\left\{s \in I\!R^{4n} : ||s|| < c_2\right\} \tag{10.47}$$

where $\lim_{\underline{\lambda}(B)\to\infty} c_2 = \infty$.

Proof. Follows *mutatis mutandis* from the proof of corollary 10.4. ∎

4 Infinitely stiff environment

4.1 Preliminaries

We consider in this section the standard model of a rigid revolute joint constrained robot manipulator in the joint space (Spong and Vidyasagar, 1989)

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u - f \tag{10.48}$$

where $u \in \mathbb{R}^n$ are the applied torques and $f \in \mathbb{R}^n$ is the reaction forces vector.

In this section we assume that the manipulator's end-effector interacts with an infinitely stiff environment hence, that its motion is constrained to a smooth $(n - m)$–dimensional submanifold Φ, defined by

$$\phi(q) = 0 \tag{10.49}$$

where the function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is at least twice continuously differentiable and m is the number of constraints.

Assumption **A10.4** below, concerns the solvability of the constraints equation (10.49) and will be used in the sequel in order to derive a reduced order model of constrained motion as well as in the design of our controllers in sections 5 and 6.

A10.4 We assume that there exists an operating region $\Omega \subset \mathbb{R}^n$ defined as $\Omega \triangleq \Omega_1 \times \Omega_2$, where Ω_1 is a convex subset of $\mathbb{R}^{n-m}$, Ω_2 is an open subset of $\mathbb{R}^m$. We also assume the existence of a function $\psi : \Omega_1 \to \mathbb{R}^m$ twice continuously differentiable, such that $\phi(q^1, \psi(q^1)) = 0$ for all $q^1 \in \Omega_1$. Under these conditions, the vector q^2 can be uniquely defined by the vector q^1 such that $q^2 = \psi(q^1)$ for all $q^1 \in \Omega_1$.

Notice that under this assumption the Jacobian matrix $J(q) \triangleq \partial\phi(q)/\partial q$ can be partitioned as

$$J(q) = [J_1(q), J_2(q)]\,, \tag{10.50}$$

where $J_1(q) \triangleq \partial\phi(q)/\partial q^1$, $J_2(q) \triangleq \partial\phi(q)/\partial q^2$ and the Jacobian matrix $J_2(q)$ never degenerates in the set Ω. Necessary and sufficient conditions for global solvability have been given in (Panteley and Stotsky, 1993b), laying on results of (Sandberg, 1980; Sandberg, 1981).

Thus, without loss of generality we can assume that for all $q \in \Omega$ there exist positive constants β_1, β_2 such that

$$0 < \beta_1 \le \|J_1(q)\| \le \|J(q)\| \le \beta_2. \tag{10.51}$$

In words, assumption **A10.4** guarantees the solvability of (10.49) only in the set Ω, which is equivalent to assume that the Jacobian $J(q)$ is non singular only $\forall q \in \Omega$. This is the main difference with the other similar reduced order models used in the literature where it is supposed that $J(q)$ is full rank in the entire state space. See for instance (McClamroch and Wang, 1988; Caiti and Cannata, 1994; Yao et al., 1992; Carelli and Kelly, 1991).

A10.5 We assume that for all $q \in \Omega$ there exists a constant $\beta_3 > 0$ such that for all $i \in \underline{n}$ the following bound holds true

$$\left\| \frac{\partial J(q)}{\partial q_i} \right\| \le \beta_3. \tag{10.52}$$

Considering that the end-effector motion is constrained to the submanifold Φ, the above definition of the set Ω allows a parameterization of the generalized coordinates vector in which only $n-m$ *independent* coordinates need to be controlled (Panteley and Stotsky, 1993a; Mills and Goldenberg, 1989). Thus, without loss of generality we can choose q^1 and q^2 as independent and dependent coordinates respectively and we can express the generalized velocities vector $\dot q$ in terms of independent velocities as

$$\dot q = H(q)\dot q^1 \tag{10.53}$$

where

$$H(q) \;=\; \begin{pmatrix} I_{n-m} \\ -J_2^{-1}(q)J_1(q) \end{pmatrix}. \tag{10.54}$$

Similarly to the parameterization of the vector $\dot q$, the generalized reaction forces f which do not deliver power on admissible velocities, i.e.,

$$\dot q^\top f = 0 \tag{10.55}$$

can be parameterized by the vector of Lagrange multipliers $\lambda \in \mathbb{R}^m$ as

$$f = J^\top(q)\lambda. \tag{10.56}$$

To this point we have defined a new coordinates space, now we are ready to introduce the reduced order model that we use in sections 5 and 6.

4.2 The reduced order model of Panteley and Stotsky (1993a)

In this section we describe the reduced order model introduced by (Panteley and Stotsky, 1993a) and we stress some properties which are fundamental for further analysis. First notice that using equations (10.53) and (10.54) in (10.48) we get

$$D_*(q)\ddot q^1 + C_*(q,\dot q)\dot q^1 + g_*(q) \;=\; H^\top(q)u \tag{10.57}$$

$$\lambda = Z(q)\left(C_\lambda(q,\dot q^1)\dot q^1 + g(q) - u\right), \tag{10.58}$$

where we have defined

$$D_*(q) \triangleq H^\top(q)D(q)H(q) \tag{10.59}$$
$$C_\lambda(q,\dot q) \triangleq D(q)\dot H(q) + C(q,\dot q)H(q) \tag{10.60}$$
$$C_*(q,\dot q) \triangleq H^\top(q)C_\lambda(q,\dot q) \tag{10.61}$$
$$g_*(q) \triangleq H^\top(q)g(q) \tag{10.62}$$
$$Z(q) \triangleq \left(J(q)D^{-1}(q)J^\top(q)\right)^{-1} J(q)D^{-1}(q). \tag{10.63}$$

Next, similarly to (McClamroch and Wang, 1988) we introduce a decoupled control scheme which allows to control generalized positions and constraint forces separately, thus consider the control input u of the form

$$u = {H^+}^\top(q)u_a - J^\top(q)u_b \tag{10.64}$$

where $u_a \in \mathbb{R}^{n-m}$, $u_b \in \mathbb{R}^m$ and $H^+(q) = \left(H^\top(q)H(q)\right)^{-1} H^\top(q)$ which under assumption **A10.4**, exists and is bounded for all $q \in \Omega$.

Then the *decoupled reduced order dynamic model* for robot manipulators under holonomic constraints can be rewritten in the form (Panteley and Stotsky, 1993a)

$$D_*(q)\ddot q^1 + C_*(q,\dot q)\dot q^1 + g_*(q) = u_a \tag{10.65}$$
$$\lambda = Z(q)\left(C_\lambda(q,\dot q^1)\dot q^1 + g(q) - {H^+}^\top(q)u_a\right) + u_b \tag{10.66}$$
$$\phi(q) = 0. \tag{10.67}$$

Equation (10.65) defines the dynamics of the system in the *position direction* i.e., the dynamics of the independent coordinates, which are to be controlled. Equation (10.66) defines the dynamics in the *force direction* i.e., of the force exerted by the manipulator's end-tool when interacting with its environment. Due to the decoupled nature of this reduced order model we are able to design two separate position and force control laws u_a and u_b respectively.

4.3 Properties

We stress in this section some fundamental properties of the reduced order model (10.65) – (10.67) which will be exploited in the sequel in the design of our control algorithms.

For all $q \in \Omega$ the following properties hold.

P 10.4 The norm of the matrix $H(q)$ satisfies the inequality

$$1 \le \|H(q)\| \le 1 + \beta_2/\beta_1, \tag{10.68}$$

moreover there exist a constant $\beta_4 > 0$ such that

$$\|H(q)^+\| \le \beta_4. \tag{10.69}$$

Proof. First, notice that on one hand we have from (10.54) and (10.51) that

$$\|H(q)\| \le 1 + \|J_2^{-1}J_1\| \le 1 + \beta_2/\beta_1$$

and on the other,

$$\|H(q)\|^2 = \|H(q)^\top H(q)\| = \|I_m + J_1^{-\top}J_2^{-\top}J_2^\top J_1^\top\| \ge 1.$$

Second, by definition,

$$H^+(q) = (H^\top(q)H(q))^{-1}H^\top(q),$$

since we suppose that $q^1 \in \Omega_1$ then it follows from (10.68) that there exists a positive constant β_4 such that (10.69) holds ∎

P10.5 (Panteley and Stotsky, 1995) For all $q \in \mathbb{R}^n$, the matrix $D_*(q) = D_*^\top(q) \geq d_{m_*} I > 0$. Moreover, for all $q \in \Omega$ there exists $0 < d_{M_*} < \infty$ such that $D_*(q) \leq d_{M_*} I$.

Proof. The first part of the property holds if and only if for all $z \in \mathbb{R}^{n-m}$

$$z^\top D_*(q) z = z^\top H^\top(q) D(q) H(q) z > d_m z^\top H^\top(q) H(q) z, \tag{10.70}$$

using (10.54) we get

$$z^\top D_*(q) z \geq d_m z^\top \left(I_{n-m} + J_1^\top J_2^{-\top} J_2^{-1} J_1 \right) z > d_m z^\top z. \tag{10.71}$$

The second part can be easily proven using property **P10.4**, along the lines of (10.71) and (10.70). ∎

It is worth remarking the existence of d_{M_*} only for all $q \in \Omega$. This stems from the assumption that the Jacobian $J_2(q)$ is nonsingular only in this subset. See (10.54) and (10.59).

P10.6 (Panteley and Stotsky, 1995) A suitable choice of $C_*(q, \dot{q})$ implies that for all $q \in \mathbb{R}^n$ the matrix $\dot{D}_*(q) - C_*(q, \dot{q})$ is skew-symmetric.

Proof. From (10.59)–(10.61) we have that

$$\begin{aligned} \dot{D}_*(q) - 2C_*(q, \dot{q}) &= H^\top(q) \dot{D}(q) H(q) + 2H^\top(q) D(q) \dot{H}(q, \dot{q}) - 2H^\top(q) C(q, \dot{q}) H(q) \\ &\quad -2H^\top(q) D(q) \dot{H}(q, \dot{q}) \\ &= H^\top(q) \left(\dot{D}(q) - 2C(q, \dot{q}) \right) H(q). \end{aligned}$$

The proof is completed using **P6.1**. ∎

P10.7 (Panteley et al., 1996) There exist some positive finite constants k_{g_*}, k_{v_*} such that

$$\sup_{q \in \Omega} \left\| \frac{\partial g_*(q)}{\partial q} \right\| \leq k_{g_*} \tag{10.72}$$

$$\sup_{q \in \Omega} \| g_*(q) \| \leq k_{v_*}. \tag{10.73}$$

Proof. Consider assumption **A10.5**. Now, using (10.62), let us evaluate

$$\frac{\partial g_*(q)}{\partial q} = H^\top(q) \frac{\partial g(q)}{\partial q} + \left[\frac{\partial H^\top(q)}{\partial q_1} g(q), \ldots, \frac{\partial H^\top(q)}{\partial q_n} g(q) \right] \tag{10.74}$$

where

$$\frac{\partial H^\top(q)}{\partial q_i} = \left[0, \ -J_1^\top \frac{\partial J_2^{-\top}(q)}{\partial q_i} - \frac{\partial J_1^\top(q)}{\partial q_i} J_2^{-\top} \right].$$

Taking the norm on both sides of (10.74) and applying Schwartz rule

$$\left\| \frac{\partial g_*(q)}{\partial q} \right\| \leq \left\| H(q)^\top \frac{\partial g(q)}{\partial q} \right\| + \left\| g^\top(q) \frac{\partial H(q)}{\partial q} \right\|$$

where for the first term we can write, using **P6.2** and (10.68)

$$\left\| H^\top(q) \frac{\partial g(q)}{\partial q} \right\| \leq (1 + \beta_2/\beta_1) k_g$$

while concerning the second term we have that

$$\left\| \left[\frac{\partial H^\top(q)}{\partial q_1} g(q), \ldots, \frac{\partial H^\top(q)}{\partial q_n} g(q) \right] \right\|^2 = \sum_{i+1}^{n} \left(\left\| \frac{\partial H^\top(q)}{\partial q_i} g(q) \right\|^2 \right)$$

where

$$\left\|\frac{\partial H^\top(q)}{\partial q_i} g(q)\right\| \leq \|J_1\| \left\|\frac{\partial J_2^{-1}(q)}{\partial q_i}\right\| \|g(q)\| + \left\|\frac{\partial J_1(q)}{\partial q_i}\right\| \|J_2^{-1}\| \|g(q)\| \leq \frac{\beta_3 k_v}{\beta_1^2}(\beta_1+\beta_2)$$

and the bound $\left\|\frac{\partial J_2^{-1}(q)}{\partial q_i}\right\| \leq \beta_3/\beta_1^2$ was used. Summarizing these bounds we finally have

$$\left\|\frac{\partial g_*(q^1)}{\partial q^1}\right\| \leq \left(1+\frac{\beta_2}{\beta_1}\right)\left(k_g + \frac{\sqrt{n} k_v \beta_3}{\beta_1}\right). \tag{10.75}$$

The proof of (10.72) finishes by defining the right hand side part of (10.75) as k_{g_*}. Second, the proof of (10.73) follows directly from **P6.2** and (10.68) by defining $k_{v_*} \triangleq k_v \left(1+\frac{\beta_2}{\beta_1}\right)$. ■

P 10.8 (Loria and Panteley, 1996) The matrix $C_*(q,x)y$ is linear in x and bounded in q, that is, there exists a $k_{c_*} > 0$ such that

$$C_*(q,x)y = C_*(q,y)x \tag{10.76}$$

$$\|C_*(q,x)\| \leq k_{c*}\|x\|. \tag{10.77}$$

Proof. First notice that $C_*(q,\dot{q})$ can be written as

$$C_*(q,\dot{q}) = C_*(q, H\dot{q}^1) = C_*(q,\dot{q}^1),$$

where we recall that

$$C_*(q,\dot{q}^1)x = H^\top[D(q)\dot{H} + C(q,\dot{q})H]x. \tag{10.78}$$

For sake of clarity let us consider separately the terms on the right hand side of (10.78), starting with the second one we have that

$$H^\top C(q,\dot{q})Hx = H^\top C(q,Hx)\dot{q} = H^\top C(q,Hx)H\dot{q}^1 \tag{10.79}$$

that is, this term is linear in x. Next, concerning the first term we have

$$H^\top D(q)\dot{H}x = H^\top D(q)\begin{bmatrix} 0 \\ -(J_2^{-1}J_1)^{\cdot} \end{bmatrix} x = H^\top D(q)\begin{bmatrix} 0 \\ -(J_2^{-1}J_1)^{\cdot} x \end{bmatrix}, \tag{10.80}$$

thus we must show that $(J_2^{-1}J_1)^{\cdot}(\dot{q}^1)x = (J_2^{-1}J_1)^{\cdot}(x)\dot{q}^1$. For this, we first write

$$(J_2^{-1}J_1)^{\cdot} = -J_2^{-1}\dot{J}_2 J_2^{-1}J_1 + J_2^{-1}\dot{J}_1 = J_2^{-1}[\dot{J}_1,\ \dot{J}_2]\begin{bmatrix} I \\ -J_2^{-1}J_1 \end{bmatrix} = J_2^{-1}\dot{J}H, \tag{10.81}$$

therefore, $(J_2^{-1}J_1)^{\cdot}x = J_2^{-1}\dot{J}Hx$. Now notice that

$$J(q) = \frac{\partial\phi(q)}{\partial q} = \begin{bmatrix} \frac{\partial\phi_1}{\partial q_1} & \cdots & \frac{\partial\phi_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\phi_m}{\partial q_1} & \cdots & \frac{\partial\phi_m}{\partial q_n} \end{bmatrix}, \tag{10.82}$$

and defining

$$J^i \triangleq \left[\frac{\partial\phi_i}{\partial q_1}, \ldots, \frac{\partial\phi_i}{\partial q_n}\right] \tag{10.83}$$

we have that, for all $i \leq m$

$$\dot{J}^i = \dot{q}^\top \begin{bmatrix} \frac{\partial^2\phi_i}{\partial q_1\partial q_1} & \cdots & \frac{\partial^2\phi_i}{\partial q_n\partial q_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2\phi_i}{\partial q_1\partial q_n} & \cdots & \frac{\partial^2\phi_i}{\partial q_n\partial q_n} \end{bmatrix} \triangleq \dot{q}^\top R_i(q). \tag{10.84}$$

Using (10.53) we get

$$\dot{J} = \begin{bmatrix} \dot{q}^{1^\top} H^\top R_1(q) \\ \vdots \\ \dot{q}^{1^\top} H^\top R_m(q) \end{bmatrix} \tag{10.85}$$

and substituting in (10.81) we obtain

$$(J_2^{-1}J_1)^{\cdot}(\dot{q}^1)x = J_2^{-1} \begin{bmatrix} \dot{q}^{1^\top} H^\top R_1(q) \\ \vdots \\ \dot{q}^{1^\top} H^\top R_m(q) \end{bmatrix} Hx.$$

Since $R_i(q) \in \mathbb{R}^{n\times n}$ is symmetric for all $i \leq m$ and $(Hx) \in \mathbb{R}^n$ we can write

$$(J_2^{-1}J_1)^{\cdot}(\dot{q}^1)x = J_2^{-1} \begin{bmatrix} x^\top H^\top R_1(q) H \dot{q}^{1^\top} \\ \vdots \\ x^\top H^\top R_m(q) H \dot{q}^{1^\top} \end{bmatrix} = (J_2^{-1}J_1)^{\cdot}(x)\dot{q}^1. \tag{10.86}$$

The proof of (6.42) is completed substituting (10.86) in (10.80) hence observing that both terms on the right hand side of (10.78) are linear in x.

Next we prove that there exists a $k_{c_\star}$ such that (10.77) holds. Using (10.78) we have

$$||C_*(q,\dot{q}^1)|| \leq ||H|| \left[||D|| ||\dot{H}|| + ||C(q,\dot{q})|| \right] \tag{10.87}$$

and using properties **P10.5** and **P10.8** we get from (10.53) that

$$||C_*(q,\dot{q}^1)|| \leq (1+\beta_1/\beta_2)[d_M||\dot{H}|| + k_c||H|| ||\dot{q}^1||]. \tag{10.88}$$

From (10.54) and (10.81) we have that

$$||\dot{H}|| \leq ||J_2^{-1}|| ||\dot{J}|| ||H|| \leq ||J_2^{-1}|| \left\| \frac{\partial J(q)}{\partial q} \right\| ||\dot{q}|| ||H||^2$$

and using (10.53) (10.51) and (10.52) it follows that there exists a β_6 such that $||\dot{H}|| \leq \beta_6(1+\beta_1/\beta_2)||\dot{q}^1||$ therefore (10.77) holds with

$$k_{c_\star} \leq (1+\beta_1/\beta_2)^2[d_M\beta_6 + k_c]||\dot{q}^1||. \tag{10.89}$$

■

5 Regulation under holonomic constraints via bounded controls

Motivated by input constraints that may arise in practice, we have investigated in chapter **9**, the set-point and tracking control problems with bounded inputs. In this section we extend these results in the direction of force/regulation under holonomic constraints. In particular we consider that the manipulator interacts with an infinitely stiff environment.

5.1 Problem statement and its solution

Consider the constrained manipulator model (10.48), (10.49). Assume that only position and force measurements are available, and that the manipulator is subject to the *input constraint* (9.2), which for convenience of exposition we repeat below:

$$|u_i| \leq u_i^{\max}, \quad i \in \underline{n} \tag{10.90}$$

then, design a smooth control law such that

$$\begin{aligned} \lim_{t\to\infty} \tilde{q}(t) &\triangleq \lim_{t\to\infty} [q(t) - q_d] = 0 \\ \lim_{t\to\infty} \tilde{f}(t) &\triangleq \lim_{t\to\infty} [f(t) - f_d] = 0 \end{aligned}$$

where f_d and q_d stand for the *constant* desired values of force and position respectively which are supposed to satisfy the following restrictions:

A10.6 The desired position lies within the set Ω, i.e. $q_d \in \Omega$. Furthermore, it belongs to the interior of Ω with some distance from the boundary of the set, i.e.

$$\min_{q_b^1 \in \partial\Omega_1} \|q_d^1 - q_b^1\|^2 > \varepsilon, \tag{10.91}$$

where $\partial\Omega_1$ denotes the boundary of the set Ω_1 and ε is a small positive constant whose choice may depend on the initial conditions.

A10.7 For desired position $q_d \in \Omega$ the constraint equation $\phi(q_d) = 0$ is satisfied. Furthermore, we can express $f_d = J^\top(q_d)\lambda_d$ for all $q_d \in \Omega$.

10.7 Proposition. *Assume* **A10.4** – **A10.7** *hold and that $\|\tilde{q}(0)\| < \varepsilon$, consider the model (10.65) – (10.67) in closed loop with the control law (10.64) where for the $i-th$ coordinate we define*

$$\begin{aligned} u_{a_i} &= -k_{p_i}\ tanh(\tilde{q}_i^1) - k_{d_i}\ tanh(\vartheta_i) + g_*(q_{1_d})_i && (10.92)\\ \dot{q}_{c_i} &= -a_i\ tanh(q_{c_i} + b_i q_i^1) && (10.93)\\ \vartheta_i &= q_{c_i} + b_i q_i^1 && (10.94)\\ u_{b_i} &= \lambda_{d_i} - k_{p_i}\ tanh(\tilde{\lambda}_i) && (10.95) \end{aligned}$$

where we suppose that $k_{p_i} > k_{p_i}^{\min}$ with

$$k_{p_i}^{\min} \triangleq \frac{4k_v}{tanh\left(\frac{4k_v}{k_g}\right)}. \tag{10.96}$$

Then the equilibrium point $x = col[\tilde{q},\ \dot{q},\ \vartheta,\ \tilde{f}] = col[0,\ 0,\ 0\ ,0]$ is asymptotically stable inside the set $\Omega_x \times I\!R^n$ where

$$\Omega_x \triangleq \{x \in I\!R^{3n}\ :\ q^1 \in \Omega_1, q^2 \in \Omega_2,\ \ col[\dot{q}, \vartheta] \in I\!R^{2n}\}.$$

Furthermore, if

$$u_i^{\max} > \beta_4(k_{v_*} + k_p^{\min}) + \beta_2(\|\lambda_d\| + K_p^{\min})$$

then the input constraints (10.90) are satisfied. □

Notice that the controller in the trajectory direction, (10.92)–(10.94) corresponds to the saturated EL controller of section **9**.2.2.2.

5.2 Stability proof

The proof is based on an standard Lyapunov analysis and is organized in the following manner: First, we write the error equation and propose a Lyapunov function candidate showing that it is locally positive definite in the set Ω; next we show that its derivative is locally negative semidefinite in the same set. Third, we show that the generalized independent positions q^1 never escape from the set Ω_1 (hence $q \in \Omega\ \forall t \geq 0$). The proof is completed by invoking the LaSalle's invariance principle.

5.2.1 Error equation and Lyapunov function candidate

Notice first that since the reduced order model is decoupled into trajectory and force direction we have two error equations thus, in the trajectory direction we get from (10.65), and (10.92) – (10.94)

$$\begin{aligned} &D_*(q)\ddot{q}^1 + C_*(q,\dot{q})\dot{q}^1 + g_*(q) - g_*(q_d) + K_p\ \mathrm{Tanh}(\tilde{q}^1) + K_d\ \mathrm{Tanh}(\vartheta) = 0 && (10.97)\\ &\dot{q}_c = -A\ \mathrm{Tanh}(q_c + Bq^1) && (10.98)\\ &\vartheta = q_c + Bq^1 && (10.99) \end{aligned}$$

where we have defined $A \triangleq \text{diag}\{a_i\}$, $B \triangleq \text{diag}\{b_i\}$, $K_p \triangleq \text{diag}\{k_{p_i}\}$, $K_d \triangleq \text{diag}\{k_{d_i}\}$ and $\text{Tanh}(\cdot) \triangleq \text{col}[\,\tanh(\cdot)_i]$. Second, the force error equation is derived using (10.92) and (10.95) in (10.66) to get

$$Z(q)\sigma(q,\dot{q}) - \tilde{\lambda} - K_p\ \text{Tanh}(\tilde{\lambda}) = 0 \tag{10.100}$$

where we have defined $\sigma(q,\dot{q}) \triangleq C_\lambda(q,\dot{q}^1)\dot{q}^1 + g(q) - H^{+\top}(q)[-g_*(q_{1_d}) + K_p\ \text{Tanh}(\tilde{q}^1) + K_d\ \text{Tanh}(\vartheta)]$ and $Z(q)$ is defined by (10.63).

Now let us propose the Lyapunov function candidate for the set-point regulation part

$$\begin{aligned} V(\tilde{q}^1,\dot{q}^1,\vartheta) \ &\triangleq\ \frac{1}{2}\dot{q}^{1\top}D_*(q)\dot{q}^1 + \sum_{i=1}^{n}\left(\frac{k_{d_i}}{b_i}\ln(\,\cosh(q_{c_i}+b_i q_i^1)) + k_{p_i}\ln(\,\cosh(\tilde{q}_i^1))\right) + \\ &\quad + U_*(q) - U_*(q_d) - \tilde{q}^{1\top}g_*(q_d) \end{aligned} \tag{10.101}$$

where $U_*(q)$ is defined such that $g_*(q) \triangleq \frac{\partial U_*(q)}{\partial q}$. In section **9**.(2.2.2) it was shown that a function of the form (10.101) is positive definite provided (10.96) holds. Next, taking the time derivative of (10.101) along the trajectories (10.97) – (10.99) we get

$$\dot{V}(\tilde{q}^1,\dot{q}^1,\vartheta) = -\ \text{Tanh}(\vartheta)K_d B^{-1}A\ \text{Tanh}(\vartheta) \leq 0. \tag{10.102}$$

5.2.2 Boundedness of signals and asymptotic stability

We conclude from (10.101) and (10.102) that $K_p^{\min}\ln|\,\cosh(\tilde{q}_i^1)| \leq V(t) \leq V(0)\ \forall i \leq n$, on the other hand it is easy to prove that for any $\alpha > 0$

$$\ln|\,\cosh(\tilde{q}_i^1)| \geq \begin{cases} \text{sech}^2(\alpha)|\tilde{q}_i^1|^2 & \forall\, |\tilde{q}_i^1| < \alpha \\ \tanh(\alpha)|\tilde{q}_i^1| & \forall\, |\tilde{q}_i^1| \geq \alpha \end{cases} \tag{10.103}$$

it then follows that $|\tilde{q}_i^1| \leq \varepsilon/\sqrt{n-m}$ where

$$\varepsilon \triangleq \max\left\{\frac{V(0)}{K_p^{\min}\ \tanh(\alpha)}, \alpha\right\}\sqrt{n-m} \tag{10.104}$$

hence, $\|\tilde{q}^1\| \leq \varepsilon$. Finally using (10.104) in (10.91) we conclude that $q^1 \in \Omega_1$ for all $t \geq 0$.

In summary, we have from **A10.4** that the vector $x = \text{col}(\tilde{q},\dot{q},\vartheta)$ never leaves the set Ω_x, this allows us to invoke the LaSalle's invariance principle. For this, let us first observe that the last invariant set inside Ω_x where $\dot{V} = 0$, is $x = 0$. Next, going back to the force error equation (10.100) we conclude that, since $q^1 \in \Omega_1$ then $Z(q)$ is bounded and since $x = 0$, $\sigma(q,\dot{q}) = 0$ thus the error equation becomes simply

$$\tilde{\lambda} + K_p\ \text{Tanh}(\tilde{\lambda}) = 0$$

whose only solution clearly is $\tilde{\lambda} = 0$. Asymptotic stability follows using LaSalle's principle.

5.2.3 Boundedness of control inputs

We finally prove that under the conditions of proposition 10.7 the input constraints (10.90) are respected. For this notice that since $q^1 \in \Omega_1$ then recalling property **P10.4**

$$\|H^+(q)_i\| \leq \beta_4$$

on the other hand we get from (10.64) that for all $i \in \underline{n}$

$$|u_i| \leq \beta_4|u_{a_i}| + \beta_2|u_{b_i}|.$$

Observing (10.92) and (10.95) the input constraints are satisfied if

$$u_i^{\max} > \beta_4(k_{v_*} + K_p^{\min}) + \beta_2(\|\lambda_d\| + K_p^{\min}))$$

hence the condition of proposition 10.7.

■

6 Tracking control under holonomic constraints

We extend in this section the result of semiglobal tracking control of robot manipulators presented in chapter **8**, to the case of constrained motion control. The stability proof borrows from the proof of proposition 9.5.

6.1 Problem statement and its solution

Consider system (10.65) – (10.67). Assume that only position and force measurements are available, then design a smooth control law such that

$$\lim_{t\to\infty} \tilde{q}(t) \triangleq \lim_{t\to\infty} [q(t) - q_d(t)] = 0$$
$$\lim_{t\to\infty} \tilde{f}(t) \triangleq \lim_{t\to\infty} [f(t) - f_d(t)] = 0$$

where $f_d(t)$ and $q_d(t)$ stand for the desired values of force and trajectory respectively which are supposed to satisfy the following restrictions:

A10.8 The desired trajectory $q_d(t)$ is at least twice continuously differentiable and $\|q_d(t)\|, \|\dot{q}_d(t)\|, \|\ddot{q}_d(t)\| \leq B_d$ for all $t \geq t_0$.

A10.9 The desired trajectory lies within the set Ω, i.e. $q_d(t) \in \Omega$ for all $t \geq t_0$. Furthermore, it belongs to the interior of Ω with some distance from the boundary of the set, i.e.

$$\inf_{q_b^1 \in \partial\Omega_1} \|q_d^1(t) - q_b^1\| > \delta, \tag{10.105}$$

where δ is a small positive constant whose choice may depend on the initial conditions, the controller and robot parameters.

A10.10 For all desired trajectory $q_d(t) \in \Omega$ the constraint equation $\phi(q_d) = 0$ is satisfied. Furthermore, the desired contact force satisfies $f_d(t) = J^\top(q_d(t))\lambda_d(t)$ for all $q_d(t) \in \Omega$.

Next, we propose a solution to the above formulated problem.

10.8 Proposition. *Consider the system (10.65) – (10.67) in closed loop with the control law*

$$u_a = D_*(q)\ddot{q}_d^1 + C_*(q, \dot{q}_d)\dot{q}_d^1 + g_*(q) - K_p\tilde{q}^1 - K_d\vartheta \tag{10.106}$$
$$u_b = Z(q)H^{+^\top}(q)D_*(q)\ddot{q}_d^1 + \lambda_d - K_p\tilde{\lambda} \tag{10.107}$$

$$\begin{aligned} \dot{q}_c &= -A(q_c + B\tilde{q}^1) \\ \vartheta &= (q_c + B\tilde{q}^1) \end{aligned} \tag{10.108}$$

where K_p and K_d are diagonal positive definite and $A \triangleq diag\{a_i\}$, $B \triangleq diag\{b_i\}$ are positive definite matrices. Define $x(t) \triangleq col[\tilde{q}^1(t), \dot{\tilde{q}}^1(t), \vartheta(t)]$, then there always exist some constants $0 < \alpha_5 < \alpha_4 < \delta$ with δ as in **A10.9**, *such that for any initial condition $(t_0, x_0) \in (I_t \times \Omega_x)$ where*

$$\Omega_x \triangleq \left\{\tilde{q}^1,\ \dot{\tilde{q}}^1,\ \vartheta \in I\!R^{n-m}\ :\ \|\tilde{q}^1\|, \|\dot{\tilde{q}}^1\|, \|\vartheta\| < \alpha_5\right\} \tag{10.109}$$

the independent coordinates $q^1(t)$ remain in the set Ω_1 for all $t \geq t_0$. Moreover there always exist $k_{p_m}, a_i > 0$ and $b_i > 0$ sufficiently large to ensure that the closed loop system (10.65) – (10.67), (10.106)– (10.108) is uniformly asymptotically stable. □

6.2 Stability proof

It is worth recalling that under assumption **A10.4** it holds that the Jacobian of the constraints equation is full rank only if $q^1(t) \in \Omega_1 \; \forall \, t \geq t_0$. However we do not suppose that this happens to hold for all $t > t_0$, then we must prove that under the conditions of proposition 10.8, assumptions **A10.4**, **A10.8** – **A10.10** and for all initial conditions $x_0(t_0) \triangleq \text{col}[\tilde{q}^1(t_0), \dot{\tilde{q}}^1(t_0), \vartheta(t_0)] \in \Omega_x$, the solution $x(t)$ is always bounded, moreover we prove that $q^1(t) \in \Omega_1 \; \forall \, t \geq t_0$. This constitutes the beginning and most important part of the proof while the second concerns the uniform asymptotic stability of the error system.

6.2.1 Boundedness of solutions

A fundamental lemma to our result is lemma C.1, in order to apply it to our case we first define

$$B_\sigma \triangleq \{x \in \mathrm{I\!R}^{3(n-m)} \; : \; \|\tilde{q}^1\| < \delta; \; \dot{q}^1, \vartheta \in \mathrm{I\!R}^{n-m}\}$$

next, consider the function

$$V(t,x) = \frac{1}{2}\dot{\tilde{q}}^{1^\top} D_* \dot{\tilde{q}}^1 + \frac{1}{2}\tilde{q}^{1^\top} K_p \tilde{q}^1 + \frac{1}{2}\vartheta^\top K_d B^{-1} \vartheta + \varepsilon \tilde{q}^{1^\top} \dot{\tilde{q}}^1 - \varepsilon \vartheta^\top \dot{\tilde{q}}^1 \tag{10.110}$$

where ε is a small positive constant to be defined. Using property **P10.5** and assumption **A10.9** it is easy to see that $V(t,x)$ is a smooth function in $(I_t \times B_\sigma)$. Now, let us define

$$\begin{aligned} P_1 &\triangleq \left\{x \in \mathrm{I\!R}^{3(n-m)} \; : \; \|x\| \leq \alpha_5\right\} \\ P_2 &\triangleq \left\{x \in \mathrm{I\!R}^{3(n-m)} \; : \; \|x\| \leq \alpha_4\right\} \end{aligned} \tag{10.111}$$

where α_5 and α_4 as in proposition 10.8 . Next, we prove that that the function (10.110) meets all conditions of lemma C.2.

Condition (i)

Consider the auxiliary function

$$W_1(t,x) \triangleq \frac{1}{4}\dot{\tilde{q}}^{1^\top} D_* \dot{\tilde{q}} + \frac{1}{4}\tilde{q}^{1^\top} K_p \tilde{q} + \frac{1}{4}\vartheta^\top K_d B^{-1} \vartheta,$$

then it can be easily shown that $W_2(t,x) \triangleq V(t,x) - W_1(t,x)$ is positive definite if

$$\varepsilon < \frac{1}{\sqrt{8}} \min\left\{(k_{p_m} d_{m_*})^{1/2}, \; \left(\frac{k_{d_m} d_{m_*}}{b_M}\right)^{1/2}\right\} \tag{10.112}$$

then we have that $V(I_t \times \partial P_2) \geq W_1(I_t \times \partial P_2)$, which in its turn implies that

$$V(I_t \times \partial P_2) \geq \sup_{\substack{t \,\in\, I_t, \\ x \,\in\, \partial P_2}} W_1(t,x) = \frac{1}{4}(d_{m_*} + k_{p_m} + k_{d_m}/b_M)\alpha_4^2$$

then the first condition holds with $a \leq \frac{1}{4}(d_{m_*} + k_{p_m} + k_{d_m}/b_M)\alpha_4^2$.

Condition (ii)

First notice that, since $x \in \partial P_1$ then $\|\tilde{q}^1\| \leq \alpha_5$ and since $q_d^1(t) \in \Omega_1$ by hypothesis, it follows that $q^1(t)$ is bounded. Now, let B_{α_5, q_d^1} be the ball of radius α_5 and centered at $q_d^1(t)$, if

$$\inf_{q_b^1 \in \partial\Omega_1} \|q_d^1(t) - q_b^1\| > \alpha_5 \tag{10.113}$$

then, $B_{\alpha_5,q_d^1} \subset \Omega_1$. Since $\delta > \alpha_5$, the latter happens to hold observing (10.105). Moreover, since $\|q^1(t) - q_d^1(t)\| = \alpha_5$ it follows that $q^1(t) \in B_{\alpha_5,q_d^1(t)}$, hence $q^1(t) \in \Omega_1$.

This implies that for all $x \in \partial P_1$, there exists a d_{M_*} such that $\infty > d_{M_*} > \|D_*\|$ in accordance with property **P10.5**. Thus we can write for all $(t,x) \in (I_t \times \partial P_1)$

$$V(t,x) \leq \frac{1}{2}\left[\varepsilon + k_{p_M}\right]\|\tilde{q}^1\|^2 + \left[\varepsilon + \frac{1}{2}d_{M_*}\right]\|\dot{\tilde{q}}^1\|^2 + \frac{1}{2}\left[\varepsilon + \frac{k_{d_M}}{b_m}\right]\|\vartheta\|^2 \tag{10.114}$$

hence

$$V(I \times \partial P_1) \leq \frac{1}{2}\left(d_{M_*} + k_{p_M} + \frac{k_{d_M}}{b_m} + 4\varepsilon\right)\alpha_5^2 \tag{10.115}$$

it then follows that both conditions (a) and (b) are met if

$$\alpha_4 > \alpha_5\sqrt{\frac{2(d_{M_*} + k_{p_M} + k_{d_M}/b_m + 4\varepsilon)}{d_{m_*} + k_{p_m} + k_{d_m}/b_M}} \tag{10.116}$$

Condition (iii)

Consider the first error equation, i.e. (10.65) + (10.106), using property **P10.8** we are able to write

$$\left\{\begin{array}{l} D_*(q)\ddot{\tilde{q}}^1 + [C_*(q,\dot{q}) + C_*(q,\dot{q}_d)]\dot{\tilde{q}}^1 + K_p\tilde{q}^1 + K_d\vartheta = 0 \\ \dot{\vartheta} = -A\vartheta + B\dot{\tilde{q}}^1 \end{array}\right. \tag{10.117}$$

Now, using properties **P10.4** – **P10.6**, **P10.8** and assumptions **A10.8** – **A10.10**, we take the time derivative of (10.110) along the trajectories of (10.117) to obtain after some bounding, that for all $(t,x) \in (I_t \times P_2)$

$$\begin{aligned} \dot{V}(t,x) \leq\; & -\overbrace{\left[\frac{k_{d_m}a_m}{3b_M} - \varepsilon\frac{k_{d_M}}{d_{m_*}}\right]}^{\gamma_1}\|\vartheta\|^2 - \overbrace{\left[\frac{\varepsilon}{3}b_m - k_{c*}B_d - 2\varepsilon\alpha_4 k_{c*}/d_{m_*} - \varepsilon\right]}^{\gamma_2}\|\dot{\tilde{q}}\|^2 \\ & - \begin{bmatrix} \|\dot{\tilde{q}}\| \\ \|\vartheta\| \end{bmatrix}^T \overbrace{\begin{bmatrix} \varepsilon b_m/3 & -\varepsilon(k_{c*}B_d/d_{m_*} + a_M/2) \\ -\varepsilon(k_{c*}B_d/d_{m_*} + a_M/2) & k_{d_m}a_m/3b_M \end{bmatrix}}^{Q_3} \begin{bmatrix} \|\dot{\tilde{q}}\| \\ \|\vartheta\| \end{bmatrix} \\ & -\frac{1}{2}\begin{bmatrix} \|\tilde{q}\| \\ \|\vartheta\| \end{bmatrix}^T \overbrace{\begin{bmatrix} \varepsilon k_{p_m}/d_{M_*} & -\varepsilon(k_{p_M} + k_{d_M})/d_{m_*} \\ -\varepsilon(k_{p_M} + k_{d_M})/d_{m_*} & 2k_{d_m}a_m/3b_M \end{bmatrix}}^{Q_1} \begin{bmatrix} \|\tilde{q}\| \\ \|\vartheta\| \end{bmatrix} \\ & -\varepsilon\begin{bmatrix} \|\tilde{q}\| \\ \|\dot{\tilde{q}}\| \end{bmatrix}^T \overbrace{\begin{bmatrix} k_{p_m}/2d_{M_*} & -k_{c*}B_d/d_{m_*} \\ -k_{c*}B_d/d_{m_*} & b_m/3 \end{bmatrix}}^{Q_2} \begin{bmatrix} \|\tilde{q}\| \\ \|\dot{\tilde{q}}\| \end{bmatrix}. \end{aligned} \tag{10.118}$$

Now Q_1, Q_3 and γ_1 are positive definite if

$$\varepsilon < \min\left\{\frac{2d_{m_*}^2 k_{p_m}k_{d_m}a_m}{3d_{M_*}b_M(k_{p_M} + k_{d_M})^2},\; \frac{4d_{m_*}^2 k_{d_m}a_m b_m}{9b_M(2k_cB_d + a_M d_{m_*})^2},\; \frac{k_{d_m}a_m d_{m_*}}{3k_{d_M}b_M}\right\} \tag{10.119}$$

while Q_2 is positive definite if

$$k_{p_m}b_m > \frac{6k_{c_*}^2 B_d^2 d_{M_*}}{d_{m_*}^2} \tag{10.120}$$

and $\gamma_2 > 0$ if $b_m > 6$ and

$$\varepsilon > \frac{k_{c_*}B_d}{\frac{b_m}{6} - \frac{2k_{c_*}\alpha_4}{d_{m_*}}}. \tag{10.121}$$

Apparently, condition (10.119) holds for a sufficiently small $\varepsilon > 0$, condition (10.120) holds for sufficiently large b_m while the last condition imposes a lower bound on ε which can be made arbitrarily small by increasing b_m, see also (10.112). This last condition seemingly may lead to a contradiction, for this we dedicate the next section to show that it is always possible to find constants a_m and b_m according to the conditions of proposition 10.8 such that $\dot{V}(t,x)$ is negative definite $\forall\ (t,x) \in I_t \times P_2$.

6.2.2 A tuning procedure

For the sake of simplicity and without loss of generality we will assume in the sequel that $A = a_M I_n$, $B = b_M I_n$, $K_p = k_{p_M} I_n$, $K_d = k_{d_M} I_n$. Under this assumption, let us define

$$b \triangleq \frac{12\alpha_1\alpha_4 k_{c_*}}{d_{m*}} \tag{10.122}$$

where $\alpha_1 \gg 1$ is large enough such that (10.121) hold. With this definition of b, notice that the inequalities (10.120) and (10.121) hold if

$$\alpha_1 > \max\left\{\frac{k_{c_*} B_d^2 d_{M_*}}{\alpha_4 k_p d_{m_*}},\ \frac{d_{m_*}}{\alpha_4 k_{c_*}},\ 1\right\}, \tag{10.123}$$

moreover (10.123) together with (10.122) imply that $b > 6$, as required to ensure that $\gamma_2 > 0$. On the other hand, we can define without loss of generality $k_p \triangleq \alpha_2 k_d$, $\alpha_2 > 0$ so it is easy to show that (10.119) holds if

$$\frac{B_d}{(\alpha_1 - 1)\alpha_4} < \frac{a d_{m_*}}{9} \min\left\{\frac{8k_d}{(2k_{c_*}B_d + a d_{m_*})^2},\ \frac{d_{m_*}\alpha_2}{d_{M_*}(\alpha_2+1)^2\alpha_1\alpha_4 k_{c_*}}\right\} \tag{10.124}$$

where we have used (10.122) and (10.123). Inequality (10.124) admits the two following cases:

case 1. We need to have

$$\frac{B_d}{(\alpha_1 - 1)\alpha_4} < \frac{8k_d a d_{m_*}}{9(2k_{c_*}B_d + a d_{m_*})^2} \tag{10.125}$$

case 2. We need to have

$$\frac{B_d}{(\alpha_1 - 1)} < \frac{a d_{m_*}^2 \alpha_2}{9 d_{M_*}(\alpha_2+1)^2\alpha_1 k_{c_*}} \tag{10.126}$$

Notice that in the first case, for any fixed value of a there always exists a sufficiently large α_1 such that (10.125) holds, notice from (10.122), that this is equivalent to increase b. In the second case, notice that for any fixed a both terms on both sides of the inequality are $\mathcal{O}(\alpha_1^{-1})$ hence, the inequality (10.126) is practically not affected for large values of α_1, therefore (10.126) is satisfied for large values of a. Since there is no contradiction in these conditions, it follows that $\dot{V}(t,x)$ is negative definite for all $(t,x) \in (I_t \times P_2)$.

So far we have shown that there always exist some constants a and b which ensure that $\dot{V}(t,x)$ is locally negative definite. Now, we recall that $V(t,x)$ is positive definite if (10.112) is satisfied; this happens to hold if, for sufficiently large b (namely $b \geq 1/\alpha_2$)

$$\frac{6k_{c_*}B_d^2}{k_d\alpha_4} < \frac{(\alpha_1-1)^2}{4\alpha_1} \tag{10.127}$$

which holds also for any fixed k_d (hence k_p), and sufficiently large α_1, hence b.

We conclude that for a sufficiently large a and b satisfying (10.122) and (10.123), $\dot{V}(t,x)$ is negative definite for all $(t,x) \in (I_t \times P_2)$. It follows that condition (iii) of lemma C.1 holds as well, then $\|\tilde{q}^1(t)\| \leq \alpha_4$ for all $t \geq t_0$. Using assumption **A10.9** and observing that $\alpha_4 < \delta$, we obtain that $q^1(t) \in \Omega_1$ for all $t \geq t_0$. Finally, from **A10.4** it follows that $q(t) \in \Omega$ for all $t \geq t_0$.

6.2.3 Asymptotic stability

The proof of asymptotic stability follows directly from the results of previous section using theorem C.3

We proceed to prove that $V(t,x)$ defined by (10.110) meets all the conditions of theorem C.3. The first condition follows straightforward under assumptions **A10.8** – **A10.10** about the desired trajectory, while the second condition was already shown in the previous section, with $B_\sigma = P_2$. The third condition follows observing that, since $q^1(t) \in \Omega_1$ for all $t \geq t_0$ we are able to write using (10.114)

$$V(I_t \times P_2) \leq \frac{1}{2}\left(d_{M_*} + k_{p_M} + \frac{k_{d_M}}{b_m} + 4\varepsilon\right)\alpha_4^2. \tag{10.128}$$

In the previous section we proved that the solutions $x(t;t_0,x_0) \in P_2$ for all $t \geq t_0$ thus, using theorem 1, we conclude that the error system in the position direction, i.e. (10.117), is uniformly asymptotically stable. Now, looking at the second error equation i.e., in the force direction, we have using **P10.8** that substituting (10.107) in (10.123)

$$(K_p + I)\tilde{\lambda} = [C_\lambda(q,\dot{q}) + C_\lambda(q,\dot{q}_d)]\dot{\tilde{q}}^1 + H^{+^\top}[K_p\tilde{q}^1 + K_d\vartheta]. \tag{10.129}$$

The proof is completed by noticing that since the right hand side terms of (10.129) tend uniformly asymptotically to zero, it follows that $\tilde{\lambda}(t) \to 0$, hence $\tilde{f}(t) \to 0$ for all $t \geq t_0$ and for all $t_o \in I_t$.

7 Concluding remarks

We have applied the results presented in chapters **6** through **9** to a particular example of Lagrangian systems: the robot manipulators. In particular we have dealt with the problem of constrained motion. Our contributions in this domain of research are summarized as follows.

We investigated the force/position control problem considering an elastic environment has been visited. We presented a *globally* asymptotically stable PID controller with a normalized proportional term.

The assumption of velocity measurements was relaxed to consider only position measurements. In such case, semiglobal asymptotic stability was proved, thus defining the domain of attraction in terms of the controller gains.

In both cases, exact knowledge of the environment stiffness constant k was used to compute some desired constant position in the direction where the force is exerted, x_{nd}. In event of uncertainties on the value of k, it was shown that the force steady state error remains bounded.

We have investigated the force/position and force/tracking control problems of manipulators interacting with an infinitely rigid surface. Our approaches rely on a reduced order model whose coordinates are defined in a subset Ω of the entire coordinate space. The main property of this model is that it is ensured that, if the generalized positions start and remain in this set, the constraints Jacobian is non singular.

We provided the first solution to the practically interesting problem of output feedback force/position regulation with bounded control inputs. We considered in this case that the manipulator interacts with an infinitely stiff environment.

We have proposed a controller which makes use only of position and force measurements to keep the generalized positions bounded in the set Ω. Moreover, it was proved that the closed loop system is asymptotically stable.

Our scheme uses bounded controls, hence we have proved that it is possible to achieve asymptotic stability in the set Ω respecting some given input constraints.

Finally, concerning the problem of output feedback tracking control of manipulators under holonomic constraints, we proposed a controller which uses the output of the so called *approximate differentiation* filter instead of the velocities vector, hence only position measurements are needed.

Unlike other related results, we have not assumed that the manipulator operates away from the singularities region, but we proved that the proposed control law keeps the independent trajectories bounded in the set Ω where the solvability of the constraints equation is guaranteed. Furthermore, we proved local uniform asymptotic stability in the set Ω both the force and the tracking directions using the direct Lyapunov's Method.

Conclusions

In this thesis we have investigated several output feedback control problems for Lagrangian systems: set-point control, tracking control, bounded set-point and tracking control. Among the EL-class, special interest have deserved the robot manipulators thus, we have applied our results for Lagrangian systems to the control of constrained robots.

To tackle the set-point control problem, we have presented several different passivity based approaches. First, we presented the concept of *EL controllers*, which as their name suggests, are themselves Lagrangian systems. Then, relaying on the fact that the interconnection of Lagrangian systems yields an EL system, our class of controllers exploits the physical properties of Lagrangian plants. The control aim then becomes, firstly to shape the potential energy of the closed loop system to make the desired position, an stable equilibrium; and secondly, to inject a suitable damping to attain asymptotic stability. The efficiency of our approach hinges upon the ability of choosing a potential energy function for the closed loop system. This methodology was mainly inspired on the famous theorem of Joseph L. Lagrange that establishes that the stable equilibria of a free mechanical system are determined by the minima of the potential energy function. From the modern literature, our work borrows the ideas of Takegaki and Arimoto, well known in the robotics literature.

An obvious drawback of the EL controllers is that, in order to shape the closed loop potential energy, it is supposed that the plant's potential energy is exactly known. This problem was first solved by the PI^{2}D controller. Our approach exploits the passivity properties of the plant, by shaping the energy of the closed loop system. The PI^{2}D controller injects partial damping through a dynamic extension which consists exactly on the dirty derivatives linear filter. Then in order to compensate for the steady state error, resulting from the plant's uncertainties, a double integrator is used. This has motivated the name of PI^{2}D.

The position feedback tracking control problem of robot manipulators has been open for many years now. All approaches reported in the literature until last year, ensured only semiglobal asymptotic stability. Even for one degree of freedom (dof) Lagrangian systems, a controller ensuring *global* asymptotic stability was still unpublished. In this work, we have presented the first position feedback control law for one dof EL systems. We have identified a common weakness of many of the previously reported semiglobal results which use standard Lyapunov techniques for proving stability. The key feature of our scheme is the use of cross terms and trigonometric hyperbolic functions, in the Lyapunov function candidate. Unfortunately our approach fails in the more general case of n degrees of freedom, nonetheless, we believe that this approach may open a path to the solution of other difficult problems such as set-point control with unknown potential energy.

We have extended our results on set-point and tracking control to the interesting case when there exists input constraints. Concerning the set-point control problem, we have defined a subclass of EL controllers which yield bounded controls. We have seen that the input constraints entail some restrictions on the growth rate of the potential energy of the controller, furthermore, since the plant's potential energy must be "dominated" by that of the controller, we must impose as well, some restrictions on the growth rate of the plant's potential energy function. Concerning the tracking control problem under input constraints, we have proposed a saturated computed-torque like controller, plus a dynamic extension in order to avoid the use of velocity measurements. As far as we know, we have proven for the first time, semiglobal asymptotic stability without the necessity of high controller gains. We showed that by simply tuning the gains of the dynamic extension we can increase the domain of attraction. Unfortunately, our results only hold for fully-actuated EL systems.

We finally applied some of the results for EL systems to the control of constrained manipulators without velocity measurements. Despite the many results on position feedback control of robot manipulators in free motion, very few results have been reported concerning position feedback control of manipulators in interaction with their environment. Our contributions in this direction are direct applications of some of our previous results on bounded set-point control and tracking control.

Further research

- At the beginning of this research, 3 years ago, many questions were open. Of particular interest, Prof. Ortega challenged me to solve the *global* position feedback tracking control problem for robot manipulators. Today, our results for one degree of freedom systems, are a step forward in this direction, nevertheless this problem remains open.
- The problem of control under input constraints also challenges me to extend the results presented here, to the case of under actuated systems. A first attempt has already been made by Burkov concerning the regulation problem but to the author's best knowledge, nothing has been reported for tracking control.
- Chapter **9** deals with saturated control of EL systems. Many important contributions have been made by Andrew Teel, Laurent Praly, Ali Saberi, to mention a few. Hence, another interesting open question is to extend our results to a more general class of nonlinear (non-autonomous) systems.
- In this thesis almost all contributions are based on the assumption that the systems parameters are accuratley known. Only the PI^{2}D controller and the modest robustness results of section **10**.3.3 deal with parameters uncertainty. It would be interesting to find adaptive extensions of the contributions given in this document, specially for the most difficult problems of tracking control.
- Close to the end of this 3-years term, Prof. Nijmeijer has attracted my attention to chaotic systems, in particular we have started by studying the Duffing equation. What is interesting to remark from this equation is that it models a mechanical system perturbed by a periodic disturbance. Motivated by the recent articles (Nijmeijer and Mareels, 1996) which apply robot control ideas to control of chaos, and (Nijmeijer and Berghuis, 1995), which apply standard observer theory[4] we believe that our results for Lagrangian systems might bring some ideas to control certain chaotic systems.

[4] Unknown in the physicists' literature.

Appendix A

Passivity

A.1 Definition. ($\mathcal{L}_{2e}$-space) *We say that $f : I\!R_{\geq 0} \to I\!R$ belongs to $\mathcal{L}_{2e}$ if and only if f is locally integrable and $\int_0^T |f(t)|^2 dt < \infty$*

There exist in the literature, several definitions of passivity. In this work we adopt those of (Vidyasagar, 1993). For this, we need a fundamental tool, the inner product:

A.2 Definition. (Inner product) *Let $u, y \in \mathcal{L}_{2e}$ and $T > 0$, then the inner product is defined $\forall\ T > 0$ by*

$$\langle u \mid y \rangle_T \triangleq \int_0^T u(\tau)y(\tau)d\tau. \tag{A.1}$$

A.3 Definition. (Passivity) *An operator $H : u \mapsto y$ is passive if there exists a $\beta \in I\!R$ such that*

$$\langle u \mid y \rangle_T \geq \beta. \tag{A.2}$$

A.4 Definition. (Output Strict Passivity) *An operator $H : u \mapsto y$ is output strictly passive if there exists $\beta \in I\!R$ and $\delta_o > 0$ such that*

$$\langle u \mid y \rangle_T \geq \delta_o \, \|y\|_{2T}^2 + \beta. \tag{A.3}$$

A.5 Definition. (Input Strict Passivity) *An operator $H : u \mapsto y$ is input strictly passive if there exists $\beta \in I\!R$ and $\delta_i > 0$ such that*

$$\langle u \mid y \rangle_T \geq \delta_i \, \|y\|_{2T}^2 + \beta. \tag{A.4}$$

Passivity is a fundamental property of Input/Output stable systems, important concepts that helps to relate the Input Output stability with the stability in the sense of Lyapunov are zero-state detectability and zero-state observability (Hill and Moylan, 1976; Byrnes et al., 1991).

A.6 Definition. (Zero-state detectability)*(Byrnes et al., 1991) A state-space system $\dot{x} = f(x)$, $x \in I\!R^n$ is* zero-state detectable *from the output $y = h(x)$, if for all initial conditions $x(0) \in I\!R^n$ we have ($y(t) \equiv 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$).*

A.7 Definition. (Zero-state observability)*(Byrnes et al., 1991) A state-space system $\dot{x} = f(x)$, $x \in I\!R^n$ is* zero-state observable *from the output $y = h(x)$, if for all initial conditions $x(0) \in I\!R^n$ we have ($y(t) \equiv 0 \Rightarrow x(t) \equiv 0$).*

Appendix B

A recall on vector calculus

The definitions below concern the minima of a vector function. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a smooth function then we define the following (Mardsen and Tromba, 1988; Wismer and Chattergy, 1979).

B.1 Definition. (Critical Point) *A point* $x^* \in \mathbb{R}^n$ *is called* critical point *of* $f(x)$ *if and only if* $\frac{\partial f}{\partial x}(x^*) = 0$

B.2 Definition. (Minimum) *A point* $x^* \in \mathbb{R}^n$ *is a* local minimum *of* $f(x)$ *if there is a neighbourhood* B_δ *of* x^* *with* $0 < \delta < \infty$ *such that* $f(x) \geq f(x^*)$ *for all* $x \in B_\delta$.

B.3 Definition. (Absolute or global minimum) *A point* $x^* \in \mathbb{R}^n$ *is an* absolute or global minimum *of* $f(x)$ *if* $f(x) \geq f(x^*)$ *for all* $x \in \mathbb{R}^n$.

B.4 Definition. (Unique minimum) *A point* $x^* \in \mathbb{R}^n$ *is an* unique minimum *of* $f(x)$ *if there are no other local minima of* $f(x)$ *in* $\mathbb{R}^n$.

B.5 Definition. (Strict minimum) *A point* $x^* \in \mathbb{R}^n$ *is a* strict local minimum *of* $f(x)$ *if there exists a neighbourhood* B_δ *of* x^* *with* $0 < \delta \leq \infty$ *such that* $f(x) > f(x^*)$ *for all* $x \in B_\delta$.

B.6 Theorem. [Mean value] *Assume that* $f : \mathbb{R}^n \mapsto \mathbb{R}$ *is differentiable at each point* x *of an open set* $S \in \mathbb{R}^n$. *Let* x *and* y *be two points of* S *such that the line segment* $L(x, y) \subset S$. *Then there exists a point* z *of* $L(x, y)$ *such that*

$$f(y) - f(x) = \left(\frac{\partial f}{\partial x}(z)\right)(y - x)$$

$\square$

B.7 Lemma. (Mardsen and Tromba, 1988) *Let* $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ *and* $B_\sigma \subseteq \mathbb{R}^n$, *such that*

(i) $f(0) = 0$

(ii) $\frac{\partial f}{\partial x}(0) = 0$

(iii) $\frac{\partial^2 f}{\partial x^2} > I_n\varepsilon > 0$, $\varepsilon > 0$, *for all* $x \in B_\sigma$

then $f(x)$ *has a unique strict minimum at the origin, locally in* B_σ. *If* $B_\sigma = \mathbb{R}^n$ *then the minimum is global and unique.* $\square$

The lemma B.7 above appears too strong in some cases. Notice that a necessary condition for (iii) is that $f(x) = \mathcal{O}(\|x\|^2)$ in the ball B_σ, specially when globality is to be assured it is desirable to find milder conditions. The lemma below establishes weaker sufficient conditions for a function $f(x)$ to have a global minimum at the origin.

B.8 Lemma. (Loria et al., 1996). *Let $f(x) : I\!R^n \to I\!R$ be a $\mathcal{C}^1$ function. Assume,*

1. $f(x) > 0$, *for all* $x \in I\!R^n$, $x \neq 0$ *and* $f(0) = 0$

2. $\left\| \frac{\partial f}{\partial x}(x) \right\| > 0$, *for all* $x \neq 0 \in I\!R^n$

Then the function $f(x)$ is globally positive definite with an unique and global minimum at $x = 0$. □

Condition 1 implies that $f(x)$ is positive definite with 0, a strict global minimum. Nevertheless, it is important to remark that this condition alone does not imply the uniqueness of the minimum. Condition 2 implies that 0 is the only critical point, hence that 0 is also an unique minimum of $f(x)$.

Appendix C

Lyapunov stability theorems

In this section we cite some theorems which give sufficient conditions for local and global uniform asymptotic stability of the differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{C.1}$$

where $x(t) \in \mathbb{R}^n$ and $f : I_t \times \mathbb{R}^n \mapsto \mathbb{R}$ is continuous, and $f(t, 0) = 0$ for all $t \in I_t$. It is also assumed that (C.1) has a unique solution corresponding to each initial condition, which is the case if f is globally Lipschitzian.

We cite below some less known stability theorems borrowed from (Rouche and Mawhin, 1980) which are fundamental to the results presented in this work.

C.1 Lemma. *Let B_ρ be a ball of radius ρ and Γ be a domain of $\mathbb{R}^n$ such that $\bar{\Gamma} \in B_\rho$. Let $V : I_t \times \mathbb{R}^n \mapsto \mathbb{R}$ be of class $\mathcal{C}^1$ and δ be a constant. If*

(i) $x_0 \in \Gamma$, $t_0 \geq 0$

(ii) $V(t_0, x_0) < \delta$

(iii) $\forall\, (t, x) \in I_t \times \partial\Gamma$, $V(t, x) \geq \delta$

(iv) $\forall\, (t, x) \in I_t \times \partial\Gamma$, $\dot{V}(t, x) \leq 0$

then $\forall\, t \geq t_0$, $x(t; x_0, t_0) \in \Gamma$. □

C.2 Lemma. *Let P_1 and P_2 be two compact sets in $\mathbb{R}^n$ such that*

$$\{0\} \subset \mathring{P}_1 \subset P_1 \subset \mathring{P}_2 \subset P_2 \subset \bar{B}_\sigma \subset \mathbb{R}^n$$

Let $V : I_t \times B_\sigma \mapsto \mathbb{R}$ for all $(t, x) \mapsto V(t, x)$ be a function of class $\mathcal{C}^1$ and let δ be a constant such that

(i) $(\forall\, t \in I_t)$ and $(\forall\, x \in \partial P_2)$, $V(t, x) \geq \delta$

(ii) $(\forall\, t \in I_t)$ *and* $(\forall\, x \in \partial P_1)$, $V(t, x) < \delta$

(iii) $(\forall\, t \in I_t)$ *and* $(\forall\, x \in \mathring{P}_2 - P_1)$, $\dot{V}(t, x) \leq 0$

If $t_0 \geq 0$ and $x_0 \in P_1$, then for all $t \geq t_0$ the solution $x(t; t_0, x_0)$ is defined and belongs to P_2. □

C.3 Theorem. *Let* $V : I_t \times B_\rho \mapsto \mathbb{R}$ *be a function of class* $\mathcal{C}^1$ *and a Lyapunov function on* $I_t \times B_\sigma$*, with* $B_\sigma \subset B_{rho}$*. If*

(i) $V(t,x) \to 0$ *when* $x \to 0$ *uniformly for* $t \in I_t$

(ii) $\dot{V}(t,x)$ *is negative definite on* $I_t \times B_\sigma$

(iii) $V(t,x) \leq \delta$ *for all* $(t,x) \in I_t \times B_\sigma$*, and* $\delta \in \mathbb{R}$

then all solutions $x(t;t_0,x_0)$ *such that* $x(t;t_0,x_0) \in B_\sigma$ *for every* $t \geq t_0$ *tend to* 0 *as* $t \to \infty$ *uniformly in* t_0*,* x_0*.* □

Appendix D

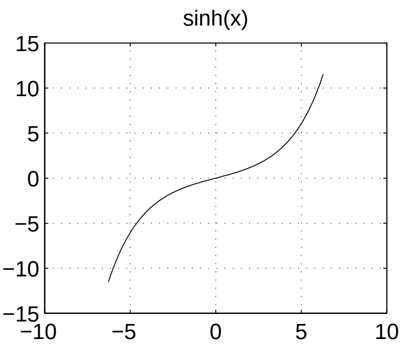
Hyperbolic trigonometric functions

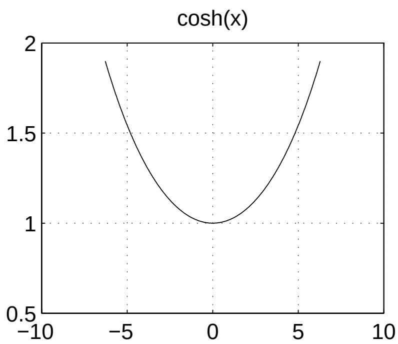
Instrumental to some of the results presented in the following chapters are the hyperbolic trigonometric functions. For the sake of clarity we recall in this section some of their properties which stem directly from their definitions.

D.1 Definition. *Hyperbolic trigonometric functions are defined for all $x \in I\!R$ by*

$$\begin{aligned}
sinh(x) &\triangleq \frac{e^{x} - e^{-x}}{2} \\
cosh(x) &\triangleq \frac{e^{x} + e^{-x}}{2} \\
tanh(x) &\triangleq \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{sinh(x)}{cosh(x)} \\
sech(x) &\triangleq \frac{2}{e^{x} + e^{-x}} = \frac{1}{cosh(x)}
\end{aligned}$$

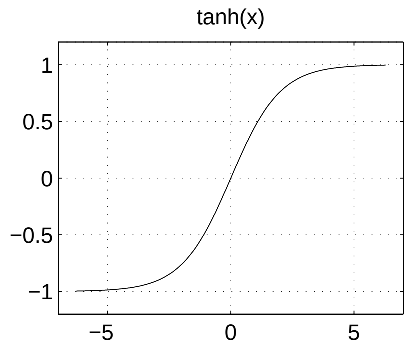
For the sake of illustration we include below the plots of the four trigonometric functions defined above.

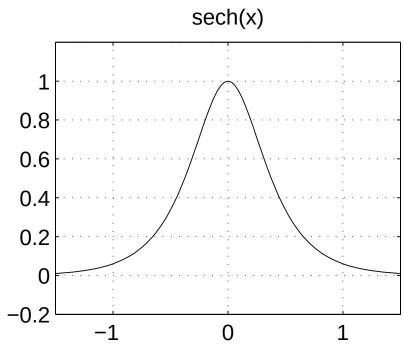




Using the definitions D.1 it is easy to see that the partial derivatives of the hyperbolic trigonometric functions are given by

$$\begin{aligned}
\frac{\partial \ \mathrm{sech}(x)}{\partial x} &= -\ \mathrm{sech}(x)\ \tanh(x) \\
\frac{\partial \ \tanh(x)}{\partial x} &= \mathrm{sech}^2(x) \\
\frac{\partial \ \ln[\ \cosh(x)]}{\partial x} &= \tanh(x).
\end{aligned}$$





Moreover, the following properties follow straightforward:

PD.1 $\sinh(x) = 0 \;\Leftrightarrow\; x = 0$, $x \sinh(x) > 0 \;\forall\; x \neq 0$ exponentially increasing.

PD.2 $\cosh(x) \geq 1$, $\forall\; x \in \mathbb{R}$, radially unbounded.

PD.3 $\tanh(x) = 0 \;\Leftrightarrow\; x = 0$, $x \tanh(x) > 0 \;\forall\; x \neq 0$, strictly increasing and $|\tanh(x)| < 1$ for all $x \in \mathbb{R}$.

PD.4 $1 \geq \operatorname{sech}(x) > 0$, $\inf\{\operatorname{sech}(x)\} = 0$.

PD.5 $\ln|\cosh(x)| = 0$ is positive definite and radially unbounded.

One more useful property which seems less evident is

PD.6 $|\sinh(x)| \leq \cosh(x) = |\cosh(x)| \leq 1 + |\sinh(x)|$.

Proof. Let us consider first, that $x \geq 0$, in this case

$$\begin{aligned} \cosh(x) - |\sinh(x)| &= \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \\ &= e^{-x} \leq 1 \;\; \forall\; x \geq 0, \end{aligned}$$

now, if $x < 0$

$$\begin{aligned} \cosh(x) - |\sinh(x)| &= \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \\ &= e^{x} \leq 1 \;\; \forall\; x \leq 0. \end{aligned}$$

In both cases we obtain that $\cosh(x) \leq 1 + |\sinh(x)|$. ■

D.2 Lemma. *Define for all $x \in \mathbb{R}^n$*

$$f(x) \triangleq ln|\,cosh(x)| - \alpha\; tanh^2(x) \tag{D.1}$$

where $cosh(x)$, $tanh(x)$ and $ln(x)$ are taken componentwise, then for all $1/2 \geq \alpha > 0$, function $f(x)$ is convex, positive! definite, and has a strict global minimum at the origin. □

Proof. Taking the first partial derivative of $f(x)$ we get

$$J(x) \triangleq \frac{\partial f(x)}{\partial x} = \tanh(x) - 2\alpha \operatorname{sech}^2(x) \tanh(x) \tag{D.2}$$

clearly $J(0) = 0$. Next, taking the partial derivative of $J(x)$ we get

$$\frac{\partial J(x)}{\partial x} = \operatorname{sech}^2(x)[1 + 4\alpha \tanh^2(x) - 2\alpha \operatorname{sech}^2(x)], \tag{D.3}$$

since $\operatorname{sech}^2(x) > 0$, $\forall\; x \in (-\infty, \infty)$, it remains to prove that $1 + 4\alpha \inf[\tanh^2(x)] \geq 2\alpha \sup[\operatorname{sech}^2(x)]$, $\forall\; x \in \mathbb{R}$ which happens to hold if $\alpha \leq 1/2$. ■

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