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A robust state-space approach to stable predictive control strategies

J.A. Rossiter M.J.Rice
Mathematical Sciences
Loughborough University
LE11 3TU. UK
J.A.Rossiter@Lboro.ac.uk

B. Kouvaritakis
Engineering Science
Oxford University
OX1 3PJ. UK
Basil.Kouvaritakis@eng.ox.ac.uk

Abstract: A means of extending the computational and numerical advantages of the stable GPC approach to predictive strategies employing state-space models is presented.

1 Introduction

In recent years, researchers into predictive control have concentrated on algorithms with guaranteed stability. Many early results were based on a *dead-beat* form of terminal constraint (e.g. [1], [2], [3], [5], [4]); later work deployed less restrictive terminal constraints by allowing either or both the input and output predictions to have an infinite settling time, e.g. [6], [9], [8]. It has been shown [7] that Stable Generalised Predictive Control (SGPC) [4] has significant computational and numerical advantages. Here the SGPC philsophy is extended to state-space models with significant improvements in numerical conditioning for the strategies of [5], [6] and [9]. [1]

2 Background and notation

Let the m^{th} order model be

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k \qquad \mathbf{y}_k = C\mathbf{x}_k \tag{1}$$

$\mathbf{x} \in R^m$, $\mathbf{u} \in R^l$, $\mathbf{y} \in R^p$. Define three types of terminal constraint (from [5], [6], [9]):

$$\mathbf{y}_i = 0, \quad i > N \qquad \mathbf{u}_i = 0, \quad i \geq N \tag{2}$$
$$\mathbf{y}_i \to 0, \quad i \to \infty \qquad \mathbf{u}_i = 0, \quad i \geq N \tag{3}$$
$$\mathbf{y}_i \to 0, \quad i \to \infty \qquad \mathbf{u}_i = -K_q\mathbf{x}_i, \quad i \geq N \tag{4}$$

where K_q is an LQ optimal controller. The key element in the SGPC philosophy is to satisfy eqns.(2-4) and also retain an explicit handle on the degrees of freedom within $\mathbf{x}_k, \mathbf{u}_k$,. Furthermore replacing open-loop by closed-loop prediction equations avoids the ill-conditioning problems associated with open-loop predictions of unstable systems. Define a nominal performance index as:

$$J = \sum_{i=1}^{\infty}\{\|\mathbf{y}_i\|_2^2 + \lambda\|\mathbf{u}_{i-1}\|_2^2\} \tag{5}$$

A predictive control is defined by minimizing J with respect to $\mathbf{u}_k, k = 0, 1, \ldots$ subject to appropriate terminal constraints (e.g. 2-4). The terminal constraints of (2-4) must be satisfied exactly; numerical inaccuracies will remove the guarantee of stability.

3 Input/output predictions

[1]Saturation constraints and non-zero set-points are omitted here for brevity, but the results presented apply equally well

The prediction equations are derived by simulating model (1) forward in time:

$$\left[\frac{\mathbf{Y}_1}{\mathbf{Y}_2}\right] = \left[\frac{P_1}{P_2}\right]\mathbf{x}_0 + \left[\frac{H_1}{H_2}\right]\mathbf{U}; \quad \mathbf{U} = \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_{N-1} \end{bmatrix}$$

$$\left[\frac{\mathbf{Y}_1}{\mathbf{Y}_2}\right] = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \hline \mathbf{y}_{N+1} \\ \vdots \\ \mathbf{y}_{N+m} \end{bmatrix}; \quad \left[\frac{P_1}{P_2}\right] = \begin{bmatrix} CA \\ \vdots \\ \hline CA^{N+1} \\ \vdots \\ CA^{N+m} \end{bmatrix}$$

$$\left[\frac{H_1}{H_2}\right] = \begin{bmatrix} CB & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \\ \hline CA^NB & CA^{N-1}B & \ldots & CAB \\ \vdots & \vdots & \vdots & \vdots \\ CA^{N+m-1}B & CA^{N+m-2}B & \ldots & CA^mB \end{bmatrix} \tag{6}$$

The optimal control laws from minimisation of J subject to (2,3,4) are given, without proof.

Terminal constraints (2): These reduce to

$$P_2\mathbf{x}_0 + H_2\mathbf{U} = 0 \tag{7}$$

The use of an m-vector of lagrange multipliers $\boldsymbol{\mu}$ gives an additonal equation

$$0 = L_1\mathbf{U} + H_1^TP_1\mathbf{x}_0 + H_2^T\boldsymbol{\mu}; \quad L_1 = H_1^TH_1 + \lambda I \tag{8}$$

Terminal constraints (3): These reduce to

$$\begin{array}{c} F^T\mathbf{x}_N = 0; \quad \mathbf{x}_N = [P_3\mathbf{x}_0 + H_3\mathbf{U}] \\ H_3 = [A^{N-1}B, A^{N-2}B, \ldots, B]; \quad P_3 = A^N \end{array} \tag{9}$$

for an appropriate choice of F that forces $\mathbf{x}_N$ into the stable manifold only ([6]). Then it is easy to to find S_2 such that

$$\begin{array}{rcl} J & = & \|P_1\mathbf{x}_0 + H_1\mathbf{U}\|_2^2 + \lambda\mathbf{U}^T\mathbf{U} \\ & & +[P_3\mathbf{x}_0 + H_3\mathbf{U}]^TS_2[P_3\mathbf{x}_0 + H_3\mathbf{U}] \end{array} \tag{10}$$

The control law is given by eqn.(9) and

$$\begin{array}{rcl} 0 & = & L_2\mathbf{U} + [H_1^TP_1 + H_3^TS_2P_3]\mathbf{x}_0 + H_3^TF^T\boldsymbol{\mu} \\ L_2 & = & [H_1^TH_1 + \lambda I + H_3^TS_2H_3] \end{array} \tag{11}$$

Terminal constraints (4): $\mathbf{Y}_1$ is as above, but

$$\mathbf{x}_{i+1} = [A - BK_q]\mathbf{x}_i; \quad \mathbf{u}_i = -K_q\mathbf{x}_i; \quad i \geq N \tag{12}$$

where $\mathbf{y}_i = C\mathbf{x}_i$ and K_q is the optimal LQ controller minimising J of eqn.(5). Hence it is easy to compute [9] S_3 such that

$$\begin{array}{rcl} J & = & \|P_1\mathbf{x}_0 + H_1\mathbf{U}\|_2^2 + \lambda\mathbf{U}^T\mathbf{U} \\ & & +[P_3\mathbf{x}_0^T + H_3\mathbf{U}]^T S_3 [P_3\mathbf{x}_0^T + H_3\mathbf{U}] \end{array} \quad (13)$$

The control law is given as:

$$\begin{array}{c} \mathbf{U} = -M\mathbf{x}_0 \\ L_3 = [H_1^T H_1 + \lambda I + H_3 S_3 H_3] \\ M = L_3^{-1}[H_1^T P_1 + H_3^T S_3 P_3] \end{array} \quad (14)$$

Weakness of simple approaches: For unstable A the numerical conditioning of (7,8) and (11,9) and (14) can be poor because $H_i, P_i, L_i, i = 1,2,3$ all involve A^j, $j = 0,1,...,N+m$; this can lead to numerical inaccuracies.

4 SGPC approach

SGPC uses prestabilisation of the model so that matrices A^i with large elements do not arise.

Theorem 1: *Consider inputs of the form*

$$\begin{array}{ll} \mathbf{u}_i = -K_d\mathbf{x}_i + \mathbf{t}_i & i = 0,...,N-m-1 \\ \mathbf{u}_i = -K_d\mathbf{x}_i & i \geq N-m \end{array} \quad (15)$$

where K_d is a dead-beat state feedback. Terminal constraints (2) are satisfied for any $\mathbf{t}_i, i = 0,...,N-m-1$.

Proof: $[A - BK_d]^m = 0$, hence $\mathbf{x}_{N+i} = 0$. □

The implied prediction equations arising from using (15) where $\Phi = A - BK_d$ are

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = P_4\mathbf{x}_0 + H_4\mathbf{T}; \mathbf{T} = \begin{bmatrix} \mathbf{t}_0 \\ \vdots \\ \mathbf{t}_{N-m-1} \end{bmatrix}; P_4 = \begin{bmatrix} C\Phi \\ \vdots \\ C\Phi^N \end{bmatrix}$$
$$H_4 = \begin{bmatrix} CB & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \\ C\Phi^{N-1}B & C\Phi^{N-2}B & \dots & C\Phi^{m-1}B \end{bmatrix} \quad (16)$$

$$\mathbf{U} = M_4\mathbf{x}_0 + N_4\mathbf{T}; \quad M_4 = \begin{bmatrix} -K_d \\ \vdots \\ -K_d\Phi^{N-1} \end{bmatrix}$$
$$N_4 = \begin{bmatrix} I & 0 & \dots & 0 \\ -K_dB & I & \dots & 0 \\ \vdots & \vdots & & \\ -K_d\Phi^{N-2}B & -K_d\Phi^{N-3}B & \dots & -K\Phi^{m-2}B \end{bmatrix} \quad (17)$$

The performance index J is minimised by

$$\mathbf{T} = -[H_4^T H_4 + \lambda N_4^T N_4]^{-1}[H_4^T P_4 + \lambda N_4^T M_4]\mathbf{x}_0 \quad (18)$$

Combining this with eqn.(15) gives $\mathbf{U}$.

Remark 1. *The prediction equations for terminal constraints (3,4)) are similar to (16,17) except that $\Phi = A - BK_q$ and the vector $\mathbf{T}$ has N terms.*

Remark 2. $\mathbf{T}$ *constitute the degrees of freedom. It is easy to show (space does not permit) that there is a one-to-one relationship between* $\mathbf{U}$ *and* $\mathbf{T}$.

Remark 3. *Selecting the inputs to be of the form*

$$\begin{array}{ll} \mathbf{u}_i = -K_q\mathbf{x}_i + \mathbf{t}_i & i = 0,...,N-1 \\ \mathbf{u}_i = -K_2\mathbf{x}_i & i \geq N \end{array} \quad (19)$$

is equivalent to terminal constraints (3, 4) where K_2 is zero or K_q respectively.

5 Examples

The optimal control laws of eqns.(8, 7), (14) (for brevity we omit (9, 11)) are computed as $\mathbf{u}_0 = -\hat{K}\mathbf{x}_0$ for various N. Both approaches should give identical $\hat{K}$. Let the example be

$$A = \begin{bmatrix} 2.6000 & -0.0500 & -0.5000 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$C = [\ 1 \quad -2.2 \quad 1.12\ ]; \quad \lambda = 1 \quad (20)$$

Note that as $N \to \infty$, $\hat{K} \to K_q$.

$$K_q = [3.2630 \quad -0.3788 \quad -0.6226 \quad 2.1279] \quad (21)$$

	Section 3		Section 4		
N	14	15	14	15	100
$\hat{K}$	-0.5108 0.0008 0.1305 -0.4129	591.6965 -59.2203 -118.3106 394.3967	3.4001 -0.3903 -0.6517 2.1944	3.3755 -0.3882 -0.6464 2.1824	3.2630 -0.3788 -0.6226 2.1279

Table 1. $\hat{K}$ for terminal constraints (2)

	Section 3			Section 4	
N	8	13	17	8	17
$\hat{K}$	3.2630 -0.3788 -0.6226 2.1279	3.2619 -0.3786 -0.6230 2.1279	22.0189 -2.8117 -5.5989 15.5815	3.2630 -0.3788 -0.6226 2.1279	3.2630 -0.3788 -0.6226 2.1279

Table 2. $\hat{K}$ for terminal constraints (3)

The algorithm of section 4 has no problems and computes $\hat{K}$ correctly even for large N whereas the approach of section 3 quickly goes numerically ill-conditionned and produces spurious results for $N > 10$.

5.1 Conclusions: Many formulations for algorithms with guaranteed stability are not numerically stable for large prediction horizons. An alternative numerically robust algorithmic formulation for computing the identical implied state-space control laws has been derived. The key idea is prestabilisation of the plant before the predictions used in a predictive control law are computed.

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