Proceedings of the 33rd
Conference on Decision and Control
Lake Buena Vista, FL - December 1994

FM-12 1:50

Practical Stabilization of Nonlinear Systems in Chained Form

C. Canudas de Wit
Laboratoire d'Automatique de Grenoble
Institut National Poytechnique de Grenoble
B.P. 46, 38402 Saint-Martin d'Hères, France

Harry Berghuis
Hollandse Signaal Apparaten B. V.
PO Box 42, 7550 GD Hengelo, The Netherlands

Henk Nijmeijer
Dept. of Applied Mathematics
University of Twente,
PO Box 217, 7500 AE Enschede, The Netherlands

Abstract

This paper presents a *hybrid controller* for the practical stabilization of general n-dimensional nonlinear systems in one-chained form. This controller consists of two parts: 1. A discrete-time part that practically stabilizes a subset of the system states, and 2. A piece-wise continuous-time part that steers the remaining state-components to an arbitrarily small neighborhood of zero. One attractive feature of the proposed control approach is that it straightforwardly allows for generalizations in the sense that integrators can be put in cascade with the control inputs without affecting the closed-loop stability properties. This yields smoother control inputs, which makes the hybrid controller particularly useful for some relevant applications like mobile robots.

Keywords - Practical stabilization, chained systems, hybrid control.

I. Introduction

In recent years there has been a lot of research interest in the stabilization of nonlinear systems in chained form. Such systems are generally obtained after a state transformation of a class of mechanical systems such as mobile robots [8], [14]. The main difficulty of systems in chained form is that it is not possible to stabilize this class of driftless nonlinear systems by smooth static-state feedback [2]. However, it turns that via other type of control strategies the stabilization problem for such systems is still solvable. For instance, [10] introduced a periodic time-varying controller to stabilize the kinematics of a two DOF mobile robot that, through a change of coordinates, can be transformed into a three-dimensional one-chain system. Also constructive procedures to design periodic time-varying controllers applicable to larger-dimensional systems with one chain have been developed, see for instance the work of [9], [12] and [5].

Another approach to the stabilization problem mentioned above was given by [3]. In particular, these authors propose a piece-wise continuous exponentially stabilizing controller for the three-dimensional one-chain system. The non-trivial extension of this result to general n-dimensional systems with one chain was studied by [13] by combining ideas of piece-wise continuous and periodic time-varying controllers.

In this paper we present a so-called *hybrid controller* for the practical stabilization of n-dimensional nonlinear systems in one-chain form. This controller consists of two parts: 1. A discrete-time and thus piece-wise constant part that practically stabilizes a subset of the system state-components, and 2. A piece-wise continuous-time part that stabilizes the remaining state-components to a set arbitrarily close to zero. One advantage of using such a hybrid approach is that it easily allows for generalizations in the sense that integrators can be put in cascade with the control inputs without affecting the closed-loop stability properties. The introduction of such integral actions is motivated by the need for smooth control inputs for some relevant applications like steering of mobile robots.

Our result relates to some extent to the multi-rate approach that was recently proposed by [7], but it has the following major differences: First, the control law presented here is for stabilization and not for open-loop steering purposes. Second, we do not discretize the full nonlinear system, but only the linear part of the dynamics, whereas the stabilization of the nonlinear part is solved with a piece-wise continuous control. Third, potential singularity problems as present in the control law of [7] are avoided.

This paper is organized as follows. Section 2 considers the third-order system having one chain, followed in Section 3 by the extension to the general n-th-order case. Next, Section 4 discusses the results that are obtained when putting integrators in cascade with the system inputs. Finally, Section 5 gives the conclusions.

II. Three-dimensional system with one chain

Consider the three-dimensional one-chain system

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & u_1(t) \\ \dot{x}_2(t) & = & u_2(t) \\ \dot{x}_3(t) & = & x_2(t)u_1(t) \end{array} \right. \tag{1}$$

with $x(0) = [x_1(0), x_2(0), x_3(0)]^T := x_0$. Assume that $u_1(t)$ is piece-wise constant during the time-intervals $I_k = [k\delta, (k+1)\delta)$, that is

$$u_1(t) = u_1(k\delta), \quad for\ all\ t \in I_k \tag{2}$$

where $k = 0, 1, 2, \cdots, \delta > 0$ arbitrary, and $u_1(k\delta)$ to be specified below. With abuse of notation, in the sequel we write k for $k\delta$. Then the system (1), (2) can be rewritten as

$$\dot{x}_1(t) = u_1(k) \tag{3}$$

$$\begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ u_1(k) & 0 \end{bmatrix} \begin{bmatrix} x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2(t) \tag{4}$$

for all $t \in I_k$. As can be seen, the subsystem (4) describes a controllable piece-wise linear time-invariant system as long as $u_1(k)$ is different from zero. This indicates that stabilization of the $z(t)$-coordinates, where

$$z(t) = [x_2(t), x_3(t)]^T \tag{5}$$

should be performed somehow faster than the stabilization of the $x_1(t)$-coordinate. Define the sign-function $s(\cdot)$ as

$$s(y) = \begin{cases} 1 & if\ y \geq 0 \\ -1 & if\ y < 0 \end{cases} \tag{6}$$

We call a function $\alpha(r) : [0, \infty) \to [0, \infty)$, a function of class K_∞ (cf. [6]) if $\alpha(r)$ is continuous, strictly increasing w.r.t. r, $\alpha(r) \to \infty$ as $r \to \infty$. Then the following lemma can be proved.

Lemma II.1: Consider the controller

$$u_1(k) = k_1(x_1(k)) + s(k_1(x_1(k)))\alpha(\|z(k)\|) \tag{7}$$

$$u_2(t) = -\mid u_1(k) \mid [k_2 x_2(t) + s(k_1(x_1(k)))k_3 x_3(t)] \tag{8}$$

where $k_1(x_1(k)), k = 0, 1, 2, \cdots$, is any asymptotically stabilizing discrete-time feedback controller for the system

$$\dot{x}_1(t) = k_1(x_1(k)), \tag{9}$$

and $k_2, k_3 > 0$ are constants, $s(\cdot)$ is the sign function as defined in (6) and $\alpha(\cdot)$ is a class K_∞ function. Then the solution of the closed-loop system (3)–(8) is globally uniformly ultimately bounded, i.e.: $\forall x(0) \in \mathbb{R}^3, b > 0, \exists N \in \mathbf{Z}^+ and \delta > 0$ such that

$$\|x(k\delta)\| \leq b \quad \forall k \geq N \tag{10}$$

where b is arbitrarily small. □

Proof Let

$$s_k = s(k_1(x_1(k))), \tag{11}$$

the closed-loop system (3)–(8) can be written as

$$\dot{x}_1(t) = k_1(x_1(k)) + s_k\alpha(\|z(k)\|) \tag{12}$$

$$\dot{z}(t) = \mid u_1(k) \mid A_k z(t) \tag{13}$$

for all $t \in I_k$, where

$$A_k = A(s_k) = \begin{bmatrix} -k_2 & -s_k k_3 \\ s_k & 0 \end{bmatrix} \tag{14}$$

The proof proceeds in two steps. First, the existence for all t of a solution is proved, and then the stability is analyzed.

skip 1.*Existence of the solution*
skip Equation (12) has as exact solution

$$x_1(t) = x_1(k) + (t - \delta k)(k_1(x_1(k)) + s_k\alpha(\|z(\delta k)\|)) \tag{15}$$

for all $t \in I_k$. In addition, (13) describes on each time-interval I_k a linear time-invariant system, which has a well-defined solution on I_k. By continuity, existence of the solution is guaranteed for all $t \in [0, \infty)$.

skip 2.*Stability analysis*
skip Consider the solution of equation (13), $\forall t \in I_k$

$$z(t) = e^{|u_1(k)|A_k(t-\delta k)}z(k) \tag{16}$$

at $t = \delta(k+1)$, and simplifying the notation we have

$$\begin{aligned} z(k+1) &= e^{\delta|u_1(k)|A_k}z(k) & (17)\\ &= T_k^{-1}e^{\delta|u_1(k)|\Lambda_k}T_k z(k) & (18) \end{aligned}$$

where $A_k = T_k^{-1}\Lambda_k T_k$ and T_k is a similarly transformation matrix and Λ_k is the Jordan matrix of A_k. It is interesting to note that Λ_k is invariant with respect to the index k, more precise it does not depends on the sign function s_k. The characteristic polynomial of the system matrix A_k is given by

$$\lambda^2 + k_2\lambda + k_3 s_k^2 = 0 \tag{19}$$

Taking into account that $s_k^2 = 1$, then

$$\lambda^2 + k_2\lambda + k_3 = 0 \tag{20}$$

This implies that A_k is Hurwitz since $k_2 > 0$ and $k_3 > 0$ and that its eigenvalues are independent of s_k. Therefore $\Lambda_k = \Lambda = diag\{\lambda_1, \lambda_2\}$, where $\lambda_1 \neq \lambda_2$ are the real roots of (20).

From (18) and (7) we have that

$$\begin{aligned} \|z(k+1)\| &\leq \kappa_0 e^{-\delta\lambda_0|u_1(k)|}\|z(k)\| \\ &= \kappa_0 e^{-\delta\lambda_0\{|k_1(x_1(k))|+\alpha(\|z(k)\|)\}}\|z(k)\| \\ &\leq \kappa_0 e^{-\delta\lambda_0\alpha(\|z(k)\|)}\|z(k)\| & (21) \end{aligned}$$

or equivalently,

$$\|z(k+1)\| - \|z(k)\| \leq \left(\kappa_0 e^{-\delta\lambda_0\alpha(\|z(k)\|)} - 1\right)\|z(k)\| \tag{22}$$

where $\kappa_0 = \sup_{s_k}\{\|T_{s_k}^{-1}\|\|T_{s_k}\|\}$ and $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. From here we have that $\|z(k)\|$ decreases as long as the term within the parenthesis in (22) remains strictly negative ,i.e.

$$\alpha(\|z(k)\|) > \frac{\ln(\kappa_0)}{\delta\lambda_0} \triangleq \varepsilon \tag{23}$$

Hence, outside of the ball B defined by

$$B = \left\{z \in R^2 \mid \|z(k)\| \leq \alpha^{-1}(\varepsilon)\right\} \tag{24}$$

we have that

$$\|z(k+1)\| - \|z(k)\| < 0 \tag{25}$$

but this does not suffice (due to the discrete-time nature of the system) to prove that B is an invariant set since due to the discrete-time nature of the system, jumps out of the ball B are still possible. Inside of B, that is when $\|z(k)\| \leq \alpha^{-1}(\varepsilon)$ the maximal possible "jump" of $\|z(k+1)\|$ can be derived from (21) as:

$$\|z(k+1)\| \leq \kappa_0\|z(k)\| \leq \kappa_0\alpha^{-1}(\varepsilon) \tag{26}$$

which defines a new ball B_{κ_0}

$$B_{\kappa_0} = \left\{z \in \mathbb{R}^2 \mid \|z(k)\| \leq \kappa_0\alpha^{-1}(\varepsilon)\right\} \tag{27}$$

which includes B since $\kappa_0 \geq 1$. Therefore all trajectories starting out of B_{κ_0} converge in finite time to B_{κ_0} and remain there, i.e. $\exists N \in \mathbf{Z}^+$, such that

$$\forall z(0) \in \mathbb{R}^2 \quad \Rightarrow \quad \|z(k)\| \leq \kappa_0 \alpha^{-1}(\varepsilon) \quad \forall k \geq N \tag{28}$$

System (12) can also be written as a I/O discrete-time operator $G(\cdot)$ with "input", $s_k\alpha(\|z(k)\|)$ and "output" $x_1(k)$, i.e.

$$x_1(k) = G(q^{-1})s_k\alpha(\|z(k)\|) \tag{29}$$

where q^{-1} is the delay operator. Let μ be define the H_∞ norm of the operator G, then it follows that

$$\begin{aligned} |x_1(k)| &\leq \|G(q^{-1})\|_\infty \, | s_k\alpha(\|z(k)\|) | \\ &\leq \mu\alpha(\|z(k)\|) \quad \forall k \in \mathbf{Z}^+ \end{aligned} \tag{30}$$

Therefore in view of the ultimate bound (28), we have that

$$\begin{aligned} |x_1(k)| &\leq \mu\alpha(\|z(k)\|) \\ &\leq \mu\alpha(\kappa_0\alpha^{-1}(\varepsilon))\varepsilon \quad \forall k \geq N \end{aligned} \tag{31}$$

Finally we get $\forall k \geq N$,

$$\begin{aligned} \|x(k)\| &\leq \|x_1(k)\| + \|z(k)\| \\ &\leq \mu\alpha(\kappa_0\alpha^{-1}(\varepsilon)) + \kappa_0\alpha^{-1}(\varepsilon) \triangleq b \end{aligned} \tag{32}$$

since $\mu < \infty$ and ε can be make arbitrarily small with a suitable choice of gains, and because $\alpha(\cdot)$ is a class K_∞ function ($\alpha^{-1}(0) = 0$), then the ultimate bound b can also be rendered arbitrarily small. Uniformness with respect to the sequence k, follows from the fact that the above bound is independent of the initial time. This completes the proof. □

Discussion

skip 1. A stabilizing discrete-time controller for the dynamics associated with $x_1(\cdot)$ is

$$k_1(x_1(k)) = \frac{1}{\delta}(a-1)x_1(k) \tag{33}$$

where $0 < a < 1$. Then (7) gives

$$x_1((k+1)\delta) = ax_1(k\delta) + \delta s_k\alpha(\|z(k)\|) \tag{34}$$

so

$$\mu = \delta \sup_\omega \left| \frac{1}{e^{-i\omega} - a} \right| < \infty \tag{35}$$

skip 2. The representation (3) describes two subsystems, the first having a piece-wise constant input function $u_1(k)$, and the second having a piece-wise continuous input function $u_2(t)$, see (7). This shows that the controller (7) can be understood as a *hybrid design*.

III. One-chain systems of n dimension

It is straightforward to extend the procedure of Section 2 to the general n-dimensional one-chain system

$$\left\{ \begin{aligned} \dot{x}_1(t) &= u_1(t) \\ \dot{x}_2(t) &= u_2(t) \\ \dot{x}_3(t) &= x_2(t)u_1(t) \\ \dot{x}_4(t) &= x_3(t)u_1(t) \\ &\vdots \\ \dot{x}_n(t) &= x_{n-1}(t)u_1(t) \end{aligned} \right. \tag{36}$$

Proceeding as before, let $u_1(t)$ be piece-wise constant according to (2), and redefine

$$z(t) = [x_2(t), x_3(t), \cdots, x_n(t)]^T \tag{37}$$

Then we have the following result.

Lemma III.1: Consider the controller

$$u_1(k) = k_1(x_1(k)) + s_k\alpha(\|z(k)\|) \tag{38}$$

$$\begin{aligned} u_2(t) = &- | u_1(k) | \, [k_2(s_k)x_2(t) + k_3(s_k)x_3(t) + \cdots \\ &+ k_n(s_k)x_n(t)] \end{aligned} \tag{39}$$

where $k_1(x_1(k))$, s_k and $\alpha(\|z(k)\|)$ are as in Lemma II.1. Assume that $k_2(\cdot), k_3(\cdot), \cdots, k_n(\cdot)$ are selected as

$$k_i(s_k) = \begin{cases} k_i & \text{if } i = even \\ k_i s_k & \text{if } i = odd \end{cases} \tag{40}$$

where $k_i > 0, i = 2, \cdots, n$, such that all the roots of the characteristic polynomial

$$\lambda^{n-1} + k_2\lambda^{n-2} + k_3\lambda^{n-3} + \cdots + k_{n-1}\lambda + k_n = 0 \tag{41}$$

have negative real part. Then the solution of the closed-loop system (36), (37), (38), (39) is uniformly ultimate bounded, with the bound $b = \mu\alpha(\kappa_0\alpha^{-1}(\varepsilon)) + \kappa_0\alpha^{-1}(\varepsilon)$, as defined before. □

Proof - The closed-loop system (36), (37), (38), (39) can be written as

$$\dot{x}_1(t) = k_1(x_1(k)) + s(k_1(x_1(k)))\alpha(\|z(k)\|) \tag{42}$$

$$\dot{z}(t) = | u_1(k) | A_k z(t) \tag{43}$$

where

$$A_k = \begin{bmatrix} -k_2(s_k) & -k_3(s_k) & \cdots & -k_{n-1}(s_k) & -k_n(s_k) \\ s_k & 0 & & 0 & 0 \\ 0 & s_k & & 0 & 0 \\ & 0 & \ddots & & \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & & s_k & 0 \end{bmatrix} \tag{44}$$

It can easily be verified that the characteristic polynomial of A_k is given by

$$\lambda^{n-1} + k_2(\cdot)\lambda^{n-2} + k_3(\cdot)s_k\lambda^{n-3} + \cdots + k_{n-1}(\cdot)s_k^{n-3}\lambda + k_n(\cdot)s_k^{n-2} = 0 \tag{45}$$

which by the choice (40) this polynomial can be rewritten as (41). This ensures that A_k is Hurwitz and, as in the three-dimensional case, the eigenvalues of A_k are rendered invariant with respect to $s(k_1(\cdot))$. Then the proof can be completed along the lines of the proof of Lemma II.1 □

IV. Extension to chained systems with cascaded integrators

In this section we consider a particular dynamic extension of the results in Sections 2 and 3. This extension consists of adding nonlinear integrators in cascade to the system inputs. The motivation for doing so is twofold. First, cascading integrators to the system inputs is one way to smooth the control signals. This is particularly suitable in a number of relevant applications where piece-wise continuous changes are too stringent. As an example, in control of mobile robots it is generally desirable to have smooth velocity changes. Second, these dynamic extensions lead to systems with drift, which significantly complicates controller design. Therefore, chained systems with cascaded integrators deserve more investigation (cf. [11]). Also in [1] the stabilization of mechanical systems with drift was analyzed.

Consider the system (1) with the following dynamical change in the input variables:

$$u_1(t) = \int_0^t \nu_1(\tau)d\tau \quad (46)$$

$$u_2(t) = \left[\int_0^t \nu_1(\tau)d\tau\right]\cdot\left[\int_0^t \nu_2(\tau)d\tau\right] \quad (47)$$

where $\nu_1(t), \nu_2(t)$ are the new inputs to the system. It is clear that if $\nu_1(\cdot)$ and $\nu_2(\cdot)$ are piece-wise continuous functions of time, then $u_1(\cdot)$ and $u_2(\cdot)$ will be continuous with respect to time. It is also important to remark that the change of inputs in (46) is not performed by straightforwardly adding integrators in cascade with the original inputs, but can be interpreted as a nonlinear dynamical extension, see the discussion at the end of this section.

Define the following coordinate transformation:

$$\begin{cases} \xi_1(t) = x_1(t) \\ \xi_2(t) = u_1(t) \\ \xi_3(t) = \int_0^t \nu_2(\tau)d\tau \\ \xi_4(t) = x_2(t) \\ \xi_5(t) = x_3(t) \end{cases} \quad (48)$$

Hence, the dynamic equations (46), (47), (48) and (1) rewrite as

$$\begin{cases} \dot\xi_1(t) = \xi_2(t) \\ \dot\xi_2(t) = \nu_1(t) \\ \dot\xi_3(t) = \nu_2(t) \\ \dot\xi_4(t) = \xi_2(t)\xi_3(t) \\ \dot\xi_5(t) = \xi_2(t)\xi_4(t) \end{cases} \quad (49)$$

Now, proceeding as in Sections 2 and 3, assume that $\nu_1(t)$ is piece-wise constant on I_k, that is

$$\nu_1(t) = \nu_1(k) \quad (50)$$

where $\nu_1(k)$ to be specified below. Introduce

$$\xi(t) = [\xi_1(t), \xi_2(t)]^T \quad (51)$$

$$z(t) = [\xi_3(t), \xi_4(t), \xi_5(t)]^T \quad (52)$$

Then the system (1), (2) can be rewritten as

$$\dot\xi(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\nu_1(k) \quad (53)$$

$$\dot z(t) = \begin{bmatrix} 0 & 0 & 0 \\ \xi_2(t) & 0 & 0 \\ 0 & \xi_2(t) & 0 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\nu_2(t) \quad (54)$$

for all $t \in I_k$.

Since $\nu_1(\cdot)$ is constant on I_k, we have that

$$\begin{aligned}\xi_2(t) &= \xi_2(k) + (t-\delta k)\nu_1(k) \\ &= c_0(k) + c_1(k)t \end{aligned} \quad (55)$$

where $c_0(k), c_1(k)$ are some constants. Therefore, $\xi_2(t)$ grows at most linearly with respect to time. This implies that (54) describes a time-varying subsystem with a linear time-dependency. The following result can be proved.

Lemma IV.1: Consider the system (49) together with the controller

$$\nu_1(k) = k_1(\xi(k)) + h(\xi(k))\alpha(\|z(k)\|) \quad (56)$$

$$\begin{aligned}\nu_2(t) = &-|\xi_2(t)|\,[k_3\xi_3(t) + s(\xi_2(t))k_4\xi_4(t) \\ &= +k_5\xi_5(t)] \end{aligned} \quad (57)$$

where $k_1(\xi(k)), k = 0, 1, 2, \cdots$, is any asymptotically stabilizing discrete-time feedback controller for the subsystem (53), i.e.

$$\dot\xi(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\xi(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} k_1(\xi(k)), \quad (58)$$

$h(\xi(k))$ given by

$$h(\xi(k)) = s(\xi_2(k) + \frac{\delta}{2}k_1(\xi(k))), \quad (59)$$

$\alpha(\cdot)$ is chosen as in Lemma II.1, and $k_3, k_4, k_5 > 0$ are constants that ensure that all the roots of the polynomial

$$\lambda^3 + k_3\lambda^2 + k_4\lambda + k_5 = 0 \quad (60)$$

have negative real part. Then the solutions of the closed-loop system (53), (54), (56), (57) are globally uniformly ultimately bounded, with the bound $b = \mu\alpha(\kappa_0\alpha^{-1}(\varepsilon)) + \kappa_0\alpha^{-1}(\varepsilon)$ arbitrarily small. □

Proof The proof essentially follows the lines of the proof of Lemma II.1. The closed-loop dynamics (53), (54), (56), (57) can be written as

$$\xi(k+1) = A_1\xi(k) + Bh(\xi(t))\alpha(\|z(k)\|) \quad (61)$$

$$\dot z(t) = |\xi_2(t)|\, A_2(s(\xi_2(t)))z(t) \quad (62)$$

for all $t \in I_k$, where, by definition of $k_1(\xi(k)), A_1$ is a stable matrix for the subsystem (61), and

$$A_2(s(\xi_2(t))) = \begin{bmatrix} -k_3 & -k_4 s(\xi_2(t)) & -k_5 \\ s(\xi_2(t)) & 0 & 0 \\ 0 & s(\xi_2(t)) & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \delta \end{bmatrix} \quad (63)$$

The characteristic polynomial of $A_2(s(\xi_2(t)))$ is given by

$$\lambda^3 + k_3\lambda^2 + k_4 s(\cdot)^2\lambda + k_5 s(\cdot)^2 = 0 \quad (64)$$

which is equivalent to (60). This implies that $A_2(\cdot)$ is Hurwitz by the choice of k_3, k_4, k_5.

Now, by virtue of (55), $s(\xi_2(t))$ will at most switch once during the time-interval I_k, say at $t_k = k\delta(1+\Delta)$, with $0 \le \Delta \le 1$. Note that (62) defines a linear time *varying* system within this time interval. Because time variations are only due to the scalar $\xi_2(t)$, the solution of the $z(t)$ can be computed explicitly, as:

$$z(t_k) = e^{A_2(s(\delta k))\int_{\delta k}^{t_k}|\xi_2(\tau)|d\tau} z(\delta k) \quad (65)$$

$$z(\delta(k+1)) = e^{A_2(s(t_k))\int_{t_k}^{\delta(k+1)}|\xi_2(\tau)|d\tau} z(t_k) \quad (66)$$

Since $A_2(\cdot)$ is stable matrix, then there exist constants $\kappa_0 > 1$ and $\lambda_0 > 0$ such that

$$\|z(t_k)\| \le \kappa_0 e^{-\lambda_0\int_{\delta k}^{t_k}|\xi_2(\tau)|d\tau}\|z(\delta k)\| \quad (67)$$

$$\|z(\delta(k+1))\| \le \kappa_0 e^{-\lambda_0\int_{t_k}^{\delta(k+1)}|\xi_2(\tau)|d\tau}\|z(t_k)\| \quad (68)$$

combining gives,

$$\begin{aligned}\|z(\delta(k+1))\| &\le \kappa_0^2 e^{-\lambda_0\left(\int_{\delta k}^{t_k}|\xi_2(\tau)|d\tau + \int_{t_k}^{\delta(k+1)}|\xi_2(\tau)|d\tau\right)}\|z(\delta k)\| \\ &= \kappa_0^2 e^{-\lambda_0\int_{\delta k}^{\delta(k+1)}|\xi_2(\tau)|d\tau}\|z(\delta k)\| \\ &\le \kappa_0^2 e^{-\lambda_0\left|\int_{\delta k}^{\delta(k+1)}\xi_2(\tau)d\tau\right|}\|z(\delta k)\| \\ &\le \kappa_0^2 e^{-\lambda_0\frac{\delta^2}{2}\alpha(\|z(\delta k)\|)}\|z(\delta k)\| \end{aligned} \quad (69)$$

where the last inequality follows from

$$\begin{aligned}\int_{\delta k}^{\delta(k+1)} \xi_2(\tau)d\tau &= \delta\xi_2(\delta k) + \frac{\delta^2}{2}\nu_1(\delta k) \\ &= \delta\xi_2(\delta k) + \frac{\delta^2}{2}(k_1(\delta k) + h(k)\alpha(\delta k)) \\ &= \delta h(\delta k)\left\{ \mid \xi_2(\delta k) + \frac{\delta}{2}k_1(\delta k) \mid + \right. \\ &\qquad \left. \frac{\delta}{2}\alpha(\delta k) \right\} \end{aligned} \tag{70}$$

which implies

$$\begin{aligned}\mid \int_{\delta k}^{\delta(k+1)} \xi_2(\tau)d\tau \mid &= \delta\left\{ \mid \xi_2(\delta k) + \frac{\delta}{2}k_1(\delta k) \mid + \frac{\delta}{2}\alpha(\delta k) \right\} \\ &\leq \frac{\delta^2}{2}\alpha(k) \end{aligned} \tag{71}$$

the rest of the proof follows as before with now ε defined as

$$\epsilon = \frac{2\ln(\kappa_0)}{\lambda_0\delta^2} \tag{72}$$

This completes the proof. □

Discussion As can be seen from (46), the new input variables $\nu_1(\cdot)$ and $\nu_2(\cdot)$ were introduced in a rather particular way. This was done in order to allow for a simple change of coordinates (48) that transforms the original system (1) into the chained form (49). It is important to note that the integrators could also have been introduced in the standard way, i.e. in cascade with the original inputs $u_1(\cdot)$ and $u_2(\cdot)$, but in that case the coordinate transformation for obtaining the chained form would become much more involved (cf. [8] and [7]).

V. Conclusions

This paper presents a novel approach to the practical stabilization of general n-dimensional nonlinear systems in one-chain form. This approach in named *hybrid* since one part of the controller is a discrete-time while the other is piece-wise continuous. The major feature of the proposed control approach is that it straightforwardly allows for generalizations in the sense that arbitrarily number of integrators can be put in cascade with the control inputs without affecting the closed-loop stability properties. This yields smooth control inputs, which makes the hybrid controller particularly useful for some relevant applications like mobile robots.

Acknowledgments
The authors would like to thank the Dutch Network on Systems and Control, who financed the stay of the first author at the University of Twente. The second author want to thank O. Sordalen for useful comments during the preparation of this paper. This work was performed while the second author was with the Electrical Engineering Dept. at the University of Twente.

References

[1] Bloch, A.M., and N.H. McClamroch (1989), *Control of mechanical systems with classical nonholonomic constraints*, Proc. IEEE Conf. on Decision and Control, Tampa, Florida, USA, pp. 201-205.

[2] Brockett, R.W. (1983), *Asymptotic stability and feedback stabilization*, in 'Differential Geometric Control Theory', eds. R.W. Brockett, R.S. Millman and H.J. Sussman, pp. 181-208, Birkhauser.

[3] Canudas de Wit, C., and O. Sordalen (1992), *Exponential stabilization of mobile robots with nonholonomic constraints*, IEEE Trans. on Automatic Control, Vol. 37, pp. 1791-1797.

[4] Corless, M. and J. Manela (1986) *Control of uncertain discrete-time systems.* Proceeding of the 1986 American Control Conference, Seattle, WA, pp. 515-520, 1986.

[5] Coron, H.-M., and J.-B. Pomet (1992), *A remark on the design of time-varying stabilizing feedback laws for controllable systems without drift*, Proc. IFAC Symp. Nonlinear Control Systems Design, Bordeaux, France, pp. 413-417.

[6] Khalil, H.K. (1992), *Nonlinear Systems*, Macmillan, New York.

[7] Monaco, S. and D. Normand-Cyrot (1992), *An introduction to motion planning under multirate digital control*, Proc. IEEE Conf. on Decision and Control, Tucson, Arizona, USA, pp.

[8] Murray, R.M., and S.S. Sastry (1991), *Steering nonholonomic systems in chained form*, Proc. IEEE Conf. on Decision and Control, Brighton, England, pp. 1121-1126.

[9] Pomet, J.-B. (1992), *Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift*, Systems and Control Letters, pp.

[10] Samson, C., and K. Ait-Abderrahim (1990), *Feedback stabilization of a nonholonomic wheeled mobile robot*, Proc. Int. Conf. on Intelligent Robots and Systems, —-

[11] Samsom, C. (1990), *Velocity and torque feedback control of a nonholonomic cart*, Int. Workshop in Adaptive and Nonlinear Control: Issues in Robotics, Grenoble, France, Nov.

[12] Samsom, C. (1993), *Mobile robot control part 2: control of chained systems and application to path following and time varying point stabilizaiton of wheeled vehicles*, Report, INRIA no. 1994, July 1993.

[13] Sordalen, O.J. (1993), *Feedback control of nonholonomic mobile robots*, PhD thesis. University of Trondheim, Norway.

[14] Sordalen, O. J.(1993), *Conversion of the kinematic of a car with n trailers into a chained form* , IEEE-Conference on Robotics and Automation, Atlanta, Georgia.

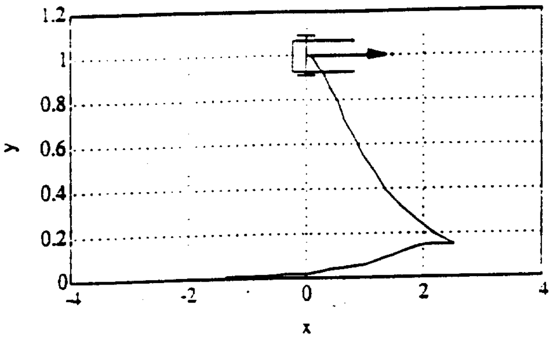

Figure 1: The resulting path in the xy-plane when a hybrid piecewise control is used

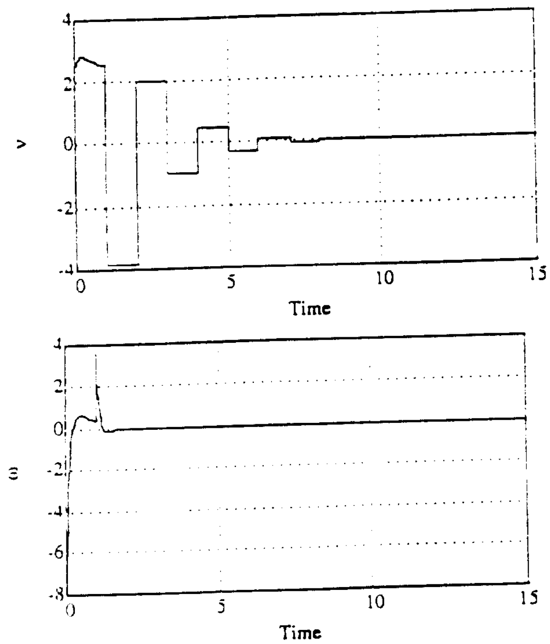

Figure 2: Time plots of the inputs, $v(t)$ and $\omega(t)$ when a hybrid piecewise control

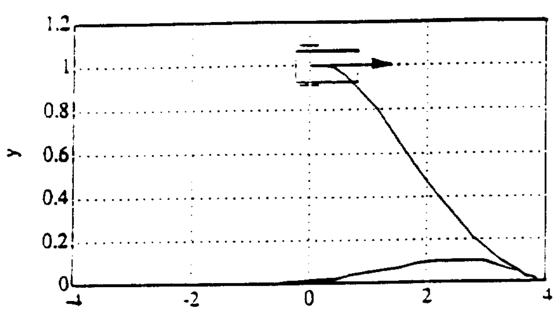

Figure 3: The resulting path in the xy-plane when the dynamic extension is used

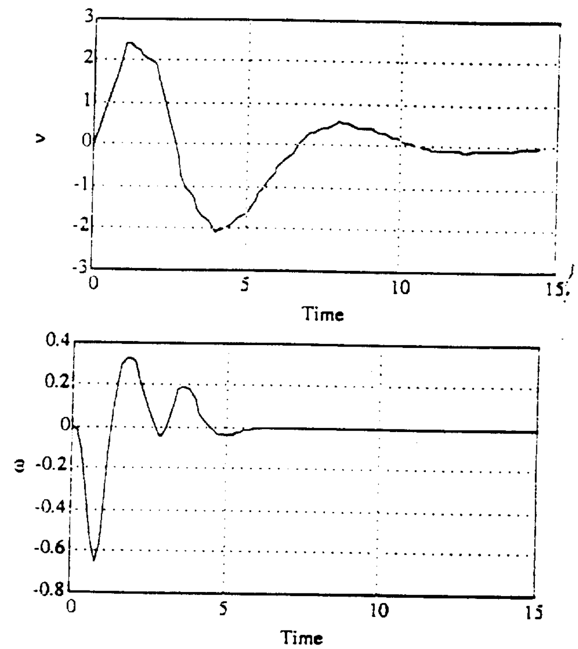

Figure 4: Time plots of the inputs, $v(t)$ and $\omega(t)$ for the case of the dynamic extension