# Linear and Nonlinear Controllability Concepts: a Geometric Approach through Invariant Subspaces [1]

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**Abstract**

This paper has the following aims: (i) to provide novel dynamical interpretations of the conditions for controllability and stabilizability for linear control systems and in the process try to understand what exactly they mean *geometrically*; (ii) to re-visit the theory of canonical forms from an algebraic viewpoint; (iii) to give some refined genericity results on controllable pairs; (iv) to make precise the way in which classes of nonlinear systema really are linear in essence and to reveal some of the secrets of recursive algortithms (such as *back-stepping*.)

# 1 Introduction

Let us start by recalling that the differential–geometric approach to nonlinear control attempted to derive nonlinear analogs of the fundamental control concepts of controllability, minimality etc as generalizations of their linear counterparts. It also made an effort to derive testable conditions, again by imitating the linear ones, which are admirable in their simplicity and elegance. The nonlinear conditions arrived at shared some of the mathematical elegance (such as the controllability Lie algebra rank condition), but they stopped short of providing a clear link between test and controllability.

In this paper, we again take up the task of reasoning by analogy with the linear control system case. Our viewpoint will be very different, though. The formation of Lie brackets between vector fields will be de-emphasized in favor of arguments using more directly linear-algebraic concepts such as subspaces with special properties such as invariance under the linear flow or dissipativeness ('minimum-phaseness'?).

Linear controllability is an easily tested property through the controllability rank condition. The test applies to **all** linear systems without distinction. The meaning of the test is well understood using the explicit variation–of–constants formula and the Cayley–Hamilton theorem; the demonstration of an explicit control action to move from some initial condition to an arbitrary terminal condition is convincing. It is perhaps less clear how a controllable system can acquire arbitrary modal behavior (poles placed arbitrarily), though the controller canonical form gives an explicit algorithm.

We start by translating the Jordan form information on the state dynamics into more geometric language.

# 2 The linear theory revisited

## 2.1 Modal Structure of Linear Systems

We consider the linear control system

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in k^n$ and $u \in k^m$, where $k$ is a field. The real number field $k = \mathbb{R}$ is of interest in practice, but, as is well known, the algebraically closed complex number field $k = \mathbb{C}$ gives the cleanest form of the results. We shall present the theory for $k = \mathbb{C}$ and then give the real version, pointing out any differences. (One can think of $\mathbb{C}^n$ as the **complexification** of $\mathbb{R}^n$ obtained by forming the tensor product $\mathbb{C} \otimes \mathbb{R}^n$; this is rather abstract for us and we shall not make any use of it.)

The **Jordan decomposition** of the linear mapping represented by the matrix $A : k^n \to k^n$ (an endomorphism of $k^n$) can be translated into a modal structure of the linear system $\dot{x} = Ax$. (We knowingly commit the sin of confusing the endomorphism/linear map with its matrix representation in a basis;

this is mostly harmless, but the reader should keep in mind at the very least that, when we write $A$, we really mean the equivalence class of $A$, say $[A]$, under the action of the group of similarity transformations.) It is known that the Jordan form can then be related to the control matrix $B$ to yield the modal version of controllability and stabilizability (e.g. in Chen [4].) The main argument of this section is that this is the most geometrical view of linear controllability concepts; this is not really new. What is novel is the claim that *it is also the one that gives the most natural generalization to nonlinear systems.*

We start by recalling some familiar concepts from linear systems theory.

**Definition 1.** *A subspace $V \subset k^n$ is* **invariant** *for the linear flow $\phi(x,t) = e^{At}x$ if $x \in V$ implies $\phi(x,t) \in V$ for all $t \in \mathbb{R}$.*

Note that a subspace $V$ is invariant for a linear flow if $x \in V$ implies $Ax \in V$, so invariance is easy to check and is equivalent to invariance for the linear *map*.

Let $c_A$ and $\mu_A$ be the *characteristic and minimal polynomials* of $A$, respectively, of degrees $n$ and $m$, $n \geq m$. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the roots of $c_A$; let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be the roots of $\mu_A$. Finally, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $A$ (hence $k \leq m \leq n$, with equality if all eigenvalues are distinct.) Suppose the Jordan form of $A$ is

$$J_A = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_k \end{pmatrix} \tag{2}$$

where each $J_i$, $i = 1, \ldots, k$ corresponds to a distinct eigenvalue of $A$; there are further decompositions into Jordan blocks

$$J_{i_l} = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix} \tag{3}$$

with $\sum_l \dim J_{i_l} = \dim J_i$, for $i = 1, \ldots, k$ and $\max_l\{\dim J_{i_l}\}$ equals the multiplicity of the eigenvalue $\lambda_i$ in the minimal polynomial $\mu_A$. The number of Jordan blocks will be denoted by $K$ (so $K \geq k$.)

Now we can translate this information about the eigenstructure of the endomorphism $A$ into a collection of invariant subspaces for the linear flow $e^{At}x$. (This has the added advantage that we do not need to refer to any particular coordinate system.) First we need some more elementary definitions.

**Definition 2.** *An invariant subspace $V$ for the flow $e^{At}$ is called* **indecomposable** *if it is not the direct sum $V = V' \oplus V''$ of proper invariant subspaces $V'$ and $V''$ of $V$.*

*An invariant subspace $V$ is* **maximal** *for the linear flow if it not contained in an invariant subspace as a* proper *subspace.*

*An invariant subspace $V$ is* **minimal** *or* **irreducible** *for the flow if it does not contain an invariant subspace as a* proper *subset.*

We shall find the following definition useful; it is motivated by the fundamental concept of moduli in algebraic geometry, originally developed in the theory of classification of Riemann surfaces and elliptic functions.

**Definition 3.** *An invariant subspace $V$ is called a* **moduli of invariant subspaces** *if every subspace of $V$ is also invariant.*

Note that an invariant subspace may contain other invariant subspaces; the requirement of the definition of indecomposability is that it does not contain two such spaces that span it. In matrix terms, a decomposable subspace corresponds to a block diagonal matrix.

**Definition 4.** *A* **flag** *of subspaces of length $\nu$ is a nested sequence of subspaces*

$$\{0\} \subset V^1 \subset V^2 \subset \ldots \subset V^\nu$$

*where* $\dim V^i = i$.

We have the following basic decomposition theorem, expressed in the complex domain, because it is there that it assumes its clearest expression.

**Theorem 1.** *Any endomorphism $A$ of the vector space $\mathbb{C}^n$ corresponds to a unique (up to re-ordering) decomposition into maximal indecomposable invariant subspaces of the corresponding linear flow:*

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_K. \tag{4}$$

*Here $V_i$ is the invariant subspace corresponding to the Jordan block $J_i$.*

*Furthermore, each indecomposable invariant subspace $V_i$ contains a flag of length* $\dim J_i$.

*Finally, if $k \neq m$, there exist* **moduli** *of invariant subspaces that are maximal. The dimension of each moduli space is equal to the number of Jordan blocks corresponding to the same eigenvalue.*

**Remark:** The presence of moduli subspaces is troublesome: not only do we have a *continuum* of invariant subspaces all corresponding to a single eigenvalue, but we must also have a continuum of invariant subspace formed by taking the direct sum of an arbitrary invariant subspace with an invariant subspace in the moduli subspace (these subspaces are *decomposable*.) This makes endomorphisms with moduli *non-cyclic*, see Definition 5 below.

### 2.2 Linear Controllability and Stabilizability Conditions

In this section, we shall examine how the control matrix $B$ relates to the invariant subspace structure described above and we shall give formulations of controllability and stabilizability in terms of these relations. Classical references for these results are [18], [5] and [4], Section 5.5 (see also the book by Klamka [12] summarizing the linear controllability methods.)

We start by writing $b_1, \ldots, b_m$ for the columns of $B$. We will write $< B >$ for the range space of $B$, $\mathcal{R}(B)$. The case $m = 1$ is given separately, followed by the general case of arbitrary $m$.

The classical result on modal controllability is the following

**Proposition 1.** *In the single input case, a necessary and sufficient condition for linear controllability is that each eigenvalue of $A$ has a unique Jordan block and that the element of the $B$ vector corresponding to the last entry of each Jordan block be non-zero.*

*In the multi-input case, we require that the row vectors of $B$ corresponding to the last elements of all the Jordan blocks for the* **same** *eigenvalue be linearly independent.*

The translation into the invariant subspace form is direct and gives the simplest expression of the controllability condition:

**Theorem 2.** *Linear controllability is simply equivalent to $< B >$ not being contained in a proper invariant subpace of the linear flow.*

**Remarks:**

1. In the **single input case**, the theorem requirements are more explicitly that the characteristic and minimal polynomials of $A$ coincide, a condition equivalent to each distinct eigenvalue having a unique Jordan block (which in turn implies the absence of moduli), and that $< B >$ not be contained in any proper invariant subspace of the linear flow (see the next section for a characterization of proper invariant subspaces in terms of direct sum of indecomposable.)

2. In the presence of multiple Jordan blocks for an eigenvalue and hence of nontrivial moduli subspaces, it is easy to see that our condition implies the condition on the components of the column vectors of $B$ corresponding to the moduli subspace forming a linearly indpendent set.

## 2.3 Characterization of PISs of linear flows

It should be clear from the preceding discussion that the concept of **proper invariant subspace** (abbr. PIS) of a linear flow is crucial to our treatment of controllability questions. Note once again that we have not brought Lie brackets into the discussion: we simply examine the structure of PISs in relation to the constant control distribution $< B >$.

In the absence of moduli, a complete description of PISs mirrors almost directly the Jordan form structure of the linear map.

**Proposition 2.** *If the linear map has no moduli, then any proper invariant subspace of the linear flow is of the form*

$$V_{i_1}^{k_{i_1}} \oplus \ldots \oplus V_{i_l}^{k_{i_l}}$$

*where each summand is one of the flag subspaces of the indecomposable $V_{i_t}$, for $t \in \{1, \ldots, k\}$. In particular, every PIS is a direct sum of indecomposables.*

It is clear that the subspaces of the form given in the proposition are invariant. The point is that they are **the only ones**.

When moduli are present, there is a proliferation of PIS and this is the most natural way to see how controllability (a 'generic' property) fails.

**Proposition 3.** *If the linear flow has moduli, then any vector belongs to a proper invariant subspace of the linear flow. In particular, cyclic endomorphisms yield linear flows with no moduli.*

**Proof of Proposition 3:** Remember that we have the decomposition $\mathbb{C}^n = \oplus_i V_i$ into indecomposable invariant subspaces. In the presence of moduli –say a two-dimensional moduli of subspaces– the point is that we get a **continuum** of invariant subspaces (not indecomposable or irreducible, but proper invariant subspaces nonetheless) by taking the sum of flag subspaces $V_i^k$ with subspaces of the moduli subspace.

Pick a basis of the subspace corresponding to an eigenvalue with moduli subordinate to the decomposition into indecomposables and such that $e_2, e_3$ span a two-dimensional moduli space. In other words, the equations for the derivatives in these two coordinates are $\dot{x}_2 = \lambda x_2$ and $\dot{x}_3 = \lambda x_3$. Let us also assume, without loss of generality, that we have an equation of the form $\dot{x}_1 = \lambda x_1 + x_2$ and that the $(x_1, x_2, x_3)$–plane is invariant. Then defining $y = \alpha x_2 + \beta x_3$, for arbitrary nonzero $\alpha$ and $\beta$, we have new coordinates $(x_1, y, x_3)$ and the equations

$$\begin{array}{rcl} \dot{x}_1 & = & \lambda x_1 + \frac{(y - \beta x_3)}{\alpha} \\ \dot{y} & = & \lambda y \\ \dot{x}_3 & = & \lambda x_3; \end{array} \tag{5}$$

it follows that the $\{x_3 = 0\}$–plane (in the new coordinates) is invariant. Since the coefficients $\alpha$ and $\beta$ were arbitrary, it follows that the three-dimensional span of the basis vectors $e_1, e_2$ and $e_3$ is filled by two-dimensional invariant subspaces.

By taking sums with other flag subspaces, we get

**Claim 1.** *There exist proper invariant subspaces that fill $\mathbb{C}^n$.*

The proposition is now established, since any vector must belong to a proper invariant subspace. $\square$

### 2.4 Relation to the Kalman form

Recall that we had

$$(A, B) \text{ controllable } \iff < B > \not\subset PIS$$

We have

**Proposition 4.** *Let $U$ be the minimal invariant subspace containing $< B >$. Then*

$$U = \text{span}\,\{B, AB, \ldots, A^{n-1}B\}.$$

Now suppose $< B >$ is contained in a *proper* invariant subspace, say $U$, and suppose this PIS is minimal.

Let us suppose for simplicity that there are no moduli. Then, by Proposition 2 this PIS $U$ is a direct sum of indecomposables. We ask the question
**Question:** When is it possible to choose a complement of $U$ that is also a sum of indecomposable?

If this is possible, then we shall conclude that the Kalman form is in fact block diagonal.

**Proposition 5.** *A complement of $U$ that is a sum of indecomposables can be chosen iff the indecomposable components of the PIS $U$ are subspaces of a flag of maximal dimension.*

**Corollary 1. Kalman forms:** *Choose a subspace $U^c$ complemantary to the PIS $U$. Thus*

$$\mathbb{C}^n = U \oplus U^c.$$

*Then any matrix representation of the linear system takes the Kalman form*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \; B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \tag{6}$$

*where the pair $(A_1, B_1)$ is controllable. If it is possible to choose a PIS complement $U^c$, then $A_2 = 0$ and the Kalman form is* **diagonal**.

### 2.5 Approach through cyclicity; genericity results

There is a slightly different way of expressing the above results that is also, I think, instructive. First, a definition:

**Definition 5.** *Let $T$ be an endomorphism of a vector space $V$ ($\dim V = n$) over an arbitrary field. We say $T$ is* **cyclic** *if there exists a vector $v \in V$ such that the successive applications of $T$ to $v$ generate a basis of $V$.*

We call such a vector $v$ a **cyclic vector** or a **primitive element** of the endomorphism. This is clearly equivalent to requiring that the vectors

$$v, Tv, T^2v, \ldots, T^{n-1}v \tag{7}$$

be linearly independent.

Using this concept, it is easy to see a nice geometric characterization of linear controllability in the single input case:

**Corollary 2.** *A single input system is controllable iff $B$ is a primitive element for $A$ (in other words $A$* **is cyclic** *and $B$ is one of its primitive elements.)*

(A discussion of cyclicity is found in Wonham's nice algebraic analysis [17].) The matrix of the linear map in the basis generated by the primitive element is the **right canonical form** (RCF)

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_n \\ 1 & 0 & \cdots & 0 & -\alpha_{n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{n-2} \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 1 & -\alpha_1 \end{pmatrix} \tag{8}$$

which after a further change of basis gives the **controller canonical form** (CCF)

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & \cdots & \cdots & \cdots & -\alpha_1 \end{pmatrix}. \tag{9}$$

The $\alpha_i$ are the coefficients of the characteristic polynomial:

$$c_A(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n.$$

Note that the proposition does not rely on the algebraic closure of the field, so that the above canonical forms are valid in all fields and the vectors $v, Tv, \ldots$ are all real if $A$ is real.

**Definition 6.** *An endomorphism $T$ of a vector space $V^n$ is* **nilpotent** *if its iterates have as images subspaces of strictly decreasing dimension or, equivalently, if $T^N = 0$ for some $N \leq n$.*

**Proposition 6.** *Every cyclic nilpotent endomorphism can be expressed in the basis whose nth element is a cyclic vector by the matrix*

$$\mathfrak{N}_n = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{10}$$

These **cyclic nilpotent** matrices $\mathfrak{N}_n$ are at the core of the geometric approach to controllability. The structure of the invariant subspaces for a cyclic nilpotent flow is as simple as can be: there is a single flag of length $n$ and the control vector, denoted $\mathfrak{b}_n$, does not belong to any PIS. Also note, since $\mathfrak{N}_n^k \mathfrak{b}_n = \mathfrak{N}_n(\mathfrak{N}_n^{k-1}\mathfrak{b}_n)$, the direction of the linear vector field along the $(k-1)$-st axis is given by the $k$th basis vector.

Here is a rather unusual re-formulation of the controllability condition in terms of $\mathfrak{N}_n$

**Proposition 7.** *A single input system is controllable iff the class of feedback matrices $A + BK$ corresponding to linear state feedback includes the $n$-th cyclic nilpotent (up to equivalence.)*

In the theory of **moduli spaces** (see Mumford [14] or Newstead [16]), as well as in **singularity and bifurcation theory** (see Arnol'd [1] and [2]) the cyclic nilpotent form is the one that when perturbed gives all possible classes of nearby endomorphism and is thus is some sense *'universal.'*

We shall see below in Section 3 how the cyclic nilpotent form arises in bact-stepping.

From the above cyclicity results, one immediately recovers the classical result that controllability is a generic property for pairs $(A, B)$. In fact, we have a slightly refined version that separates the effect of the linear map $A$ from that of the control subspace $B$.

Genericity is best expressed in an algebraic framework as follows: Once the endomorphism $A$ has been selected from the set $M(n, k)$ of all endomorphisms of the vector space $k^n$, the decomposition of the vector space is fixed. Then the choices of controllable pairs $(A, B)$ corresponds to picking a vector not belonging to a proper invariant subspace of the flow of $A$. This is obviously an open dense subset of $k^n$, assuming the absense of moduli, by the characterization of invariant subspaces of Proposition 2. Provided we can convince ourselves that endomorphisms with moduli form a thin set in $M(n, k)$, we are done. In fact, it is known that even the condition of multiple eigenvalues is non-generic, let alone multiple Jordan blocks for the same eigenvalues (see Arnol'd [2].)

**Caution:** In control theory, we often meet non-generic objects according to the general theory that persist under a narrower class of perturbations, for example those preserving a chain of integrators. This does not destroy the genericity results, since the integrators will give a long Jordan or canonical block, but no moduli; the generic choice of control vector or $< B >$ still lies outside the collection of PISs.

## 2.6 Designing linear dynamics

The traditional aim of control is to first of all make a system stable, before other control requirements are met. From the point of view of *global* system dynamics, though, it makes perfect sense to convert an equilibrium of given stability index into one of a different stability index –e.g. turn a repeller into a saddle (in the case when the equilibrium is not stabilizable, for instance; a more global nonlinear object, such as a limit cycle, could attract the state even if the equilibrium cannot.)

The linear controllability theory covers this case, even though in practice one rarely meets the need to turn an unstable system into another unstable one (**unbounded control action** is an implicit, but crucial assumption.)

From now on, and in preparation for the nonlinear case, we write $D$ for the subspace spanned by the columns of $B$. Choosing our bases carefully, we can assume that $B$ takes the form $\begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

**Theorem 3.** *Suppose that the linear control system $(A, B)$ is stabilizable and that the rank of $B$ is $m$. Then there exists an $(n-m)$-dimensional subspace $D^c$ such that*

1. *$D \oplus D^c = \mathbb{R}^n$. Write $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for the decomposition of the state according to this direct sum.*
2. *The linear dynamics on $D^c$ are stable, in other words the state equation for the $x_2$ variables is*
$$\dot{x}_2 = A_{22} x_2$$
*with $A_{22}$ Hurwitz (remember $x_1 = 0$ on $D^c$.)*

Note that the subspace $D^c$ is *not* in general invariant for the linear flow, since $\dot{x}_1 = A_{12} x_2$. Thus, to summarize, the state vector field on $D^c$ is decomposed into a component in $D =< B >$ and a 'stable' part in $D^c$.

It is easy to see, however, that the subspace $D^c$ **can be made invariant through control**, since the choice $u = -A_{12} x_2$ gives $\dot{x}_1 = 0$!

The linear feedback control that stabilizes the system now follows immediately: Choose a Hurwitz $m \times m$ matrix $\tilde{A}_{11}$. The choice

$$u = (-A_{11} + \tilde{A}_{11}) x_1 - A_{12} x_2$$

yields the state dynamics

$$\begin{pmatrix} \tilde{A}_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \tag{11}$$

which is obviously Hurwitz.

The comparison with the approach using the Kalman decomposition should be clear. What we have done is to bring out the crucial role played by the existence of a subspace of appropriate dimension on which the dynamics is stable.

In the Kalman approach, the decomposition is in terms of the controllable and uncontrollable part; in our approach, we decompose into the $D$ directions and *an appropriately chosen complement*!

**Proof of Theorem 3:** (We do the single input case only, for simplicity.) If the system is controllable, the CCF gives the equations for the first $n-1$ components of the state vector

$$\begin{array}{ccc} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \cdots & \cdots & \\ \dot{x}_{n-1} & = & x_n \end{array} \tag{12}$$

Choose a monic Hurwitz polynomial of order $n-1$, say

$$p(s) = s^{n-1} + k_{n-1} s^{n-2} + \ldots + k_2 s + k_1$$

and set

$$x_n = -(k_1 x_1 + \ldots + k_{n-1} x_{n-1}).$$

This gives the state matrix

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & 0 & 1 \\ -k_1 & \cdots & \cdots & \cdots & -k_{n-1} \end{pmatrix}. \tag{13}$$

# 3 Nonlinear control of linear systems: recursive designs

The class of *nonlinear controlled systems* of the following form

$$\begin{array}{rcl} \dot{z}_1 & = & z_2 - f_1(z_1) \\ \dot{z}_2 & = & z_3 - f_2(z_1, z_2) \\ \cdots & \cdots & \\ \dot{z}_{n-1} & = & z_n - f_{n-1}(z_1, z_2, \ldots, z_{n-1}) \\ \dot{z}_n & = & u - f_n(z) \end{array} \tag{14}$$

with $f_1, f_2, \ldots, f_n$ all zero at the origin is rather special: it is possible to design 'clever' nonlinear controls to achieve stabilization through the **back-stepping recursive algorithm**. This class we shall call **control nilpotent systems** (in the literature they are known as *strict feedback systems*, see [13] or [7].)

It is shown in the present section that this class of nonlinear systems is in fact a class of **controllable linear systems** 'in disguise' –the disguise being a nonlinear change of coordinates. After this change, the form of the equations is exactly the familiar **cyclic nilpotent** one, namely

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \cdots & \cdots & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & u \end{array} \tag{15}$$

*A word of caution:* In going from the nonlinear form to the linear one, we may have to use a very large control action. One of the main advantages of the back-stepping methodology is that it avoids unnecessary cancellations and thus can use a smaller control action than the one required to cancel **all** the nonlinearities.

However, the point we intend to make here is that the class of control nilpotent nonlinear systems is really a class of *linear* systems and thus we may as well address the issue of choosing controllers from the viewpoint of the linear theory. The second aim is to use the opportunity to point out how the method can fail and how the stabilization region can fail to be the whole of the state space. Let us also point out that, in the context of our general nonlinear theory, control nilpotent systems are a subset of the class of control pairs with **constant control distribution** (in particular, it may not be possible to make sense of them in state spaces other than Euclidean ones.)

## 3.1 Nonlinear state transformations of linear systems:

Starting with the cyclic nilpotent system 15, perform a nonlinear change of coordinates as follows:

$$\begin{array}{rcl} z_1 & = & x_1 \\ z_2 - f_1(z_1) & = & x_2 \\ z_3 - f_2(z_1, z_2) - \frac{df_1}{dx_1}(z_2 - f_1(z_1)) & = & x_3 \\ z_4 - f_3(z_1, z_2, z_3) - F_3(z_1, z_2, z_3) & = & x_4 \\ \cdots & \cdots & \cdots \\ z_n - f_{n-1}(z_1, \ldots, z_{n-1}) - F_{n-1}(z_1, \ldots, z_{n-1}) & = & x_n \end{array} \tag{16}$$

where the terms $F_i$, necessary to cancel unwanted terms, become more and more complicated with increasing index, *but are still only functions of the state variables 'fed back'.* The precise form is found by writing down the equations for the derivatives of the $z_i$ coordinates. The important thing is that the Jacobian of this nonlinear transformation is lower triangular with ones along the diagonal and the map is thus a (local) diffeomorphism.

At the last step, we have that

$$\dot{z}_n = H(z_1, z_2, \ldots, z_n) + u$$

for some function $H$, since $\dot{x}_n = u$. The choice of global feedback

$$u = -H(z) + v$$

will thus give $\dot{z}_n = v$ and we have derived the 'feedback equivalence' of the linear and nonlinear cyclic nilpotent forms:

**Proposition 8.** *There exists a feedback control section and a nonlinear change of coordinates that transforms the cyclic nilpotent linear system into the strict feedback nonlinear system 14 in an open set containing the origin.*

(The open set can be all of the state space; we are just being cautious about the local diffeomorphism condition not yielding global diffeomorphism.)

This has an interesting consequence that is not obvious from a consideration of the nonlinear system alone:

**Proposition 9.** *For any strict feedback system, there is a choice of feedback control defined in some open set $U$ containing the origin such that the resulting flow has a system of nested invariant manifolds $\{0\} \subset W^1 \subset W^2 \subset \cdots \subset W^n = U$ such that $\dim W^i = i$.*

**Corollary 3.** *A strict feedback system permits a feedback control such that the equilibrium at the origin is hyperbolic of index $k$, for $k$ arbitrary.*

Thus, not only is it possible to stabilize the equilibrium of a strict feedback system, but its stability index can be assigned arbitrarily.

The conclusion follows that strict feedback systems are a rather restricted and special class of nonlinear systems; nonlinear systems with truly nonlinear behavior are not so easy to control –hence the need for a general analysis and for the theory we develop in [8] and [9].

**Example:** A simple example of a nonlinear system that is not feedback equivalent to a control-nilpotent form is the following *saddle-node equilibrium*

$$\begin{aligned} \dot{x} &= -x^2 + u \\ \dot{y} &= y + u. \end{aligned} \tag{17}$$

The change of coordinates $x = x, z = x - y$ gives the form

$$\begin{aligned} \dot{x} &= -x^2 + u \\ \dot{z} &= -x^2 - x + z \end{aligned} \tag{18}$$

and it is easily seen that, for $z < -1/4$, $\dot{z} < 0$ and so $z(t) < -1/4$ for all $t \geq 0$ and there is no way control can be used to drive the state to zero (we may say that the $z$-coordinate is a Lyapunov function in the set $\{(x, z) \; ; \; z < -1/4\}$.) However, the system is **locally** stabilizable (see [9], where the geometric reason for the failure to convert to nilpotent form is given.)

# 4 Conclusions

We have presented a geometric analysis of linear and certain nonlinear systems that emphasizes the aspect of feedback equivalence, understood as the problem of finding the class of control systems resulting from a (preliminary) choice of state feedback and a (possibly nonlinear) change of coordinates. The central 'canonical representative' for both linear and the class of strict feedback nonlinear systems is the *linear* **cyclic nilpotent form**. Furthermore, we pointed out the importance of proper invariant subspaces and gave a characterization of all PISs –pointing out the problems arising if moduli are present. This work is part of a more general theory that attempts to develop a nonlinear control theory that gives a central role to the global dynamical and geometric aspects of nonlinear control from the start ([8], [9].)

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