Proceedings of the 41st IEEE
Conference on Decision and Control
Las Vegas, Nevada USA, December 2002

WeP03-3

A universal saturation controller design for mobile robots

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ABSTRACT: We propose a global time-varying universal controller to achieve stabilization and tracking simultaneously for mobile robots with saturated inputs. The controller synthesis is based on Lyapunov's direct method and backstepping technique. Numerical simulations are provided to validate the effectiveness of the proposed controller.

I. INTRODUCTION

Over the last decade, a lot of interest has been devoted to stabilization and tracking problems of nonholonomic mechanical systems including wheeled mobile robots [1, 3, 4, 5, 6, 9, 10, 11, 14, 16, 20]. The main difficulty of solving these problems is due to the fact that the motion of the systems in question to be controlled has more degrees of freedom than the number of control inputs under nonholonomic constraints. The necessary condition of Brockett's theorem [17] shows that any continuous time invariant feedback control law does not make the null solution of the wheeled mobile robots asymptotically stable in the sense of Lyapunov. Stabilization and tracking of wheeled mobile robots have thus been often treated separately from different viewpoints.

Stabilization of wheeled mobile robots has been solved by employing two main nonlinear control approaches. The first, initiated by Bloch and McClamroch [3], uses discontinuous feedback while the second employs time-varying continuous feedback, which was firstly studied by Samson [6]. Several smooth feedback control laws were proposed by Pomet [10] with slow asymptotic convergence for the case of regulation. These controllers were then improved by an alternative control scheme by M'Closkey and Murray [16].

Tracking control of wheeled mobile robots has also received considerable attention from the control community [2, 3, 4, 9, 12, 18, 21]. In these references, based on Barbalat's lemma [8] and the popular backstepping method [13] several time-varying controllers were developed to globally follow special paths such as circles and straight-lines. In [18, 21], only certain reference trajectories can be tracked such as those generated by the reference mobile robot of which forward or angular velocity cannot converge to zero.

In [2, 15], a high-gain dynamic controller was proposed to solve both stabilization and tracking based on a transformation of the mobile robot into a so-called skew form. A recent result on developing controllers for both tasks to achieve practical stability is given in [14].

Saturated stabilization and tracking of wheeled mobile robots have been rarely addressed in the literature [19, 22]. Model-based controllers were proposed in [22] by applying passivity and normalization to achieve stabilization and tracking separately. In [19], a time-varying saturated controller was proposed to solve stabilization and tracking of mobile robots. However, several tracking cases were not covered. Firstly, the reference forward velocity must be positive. Secondly, the reference angular velocity must converge to zero when the reference forward velocity tends to zero.

In this paper, we present a complete solution to the long standing open problem of simultaneous stabilization and tracking for a wheeled mobile robot. More specifically, we propose a universal saturated controller that removes assumptions required in aforementioned papers to achieve stabilization and tracking simultaneously. No switching is needed. The control development is surprisingly simple and stability analysis is based on Barbalat's lemma and Lyapunov's direct method.

Notations. The single bar $|\bullet|$ denotes the absolute value of $\bullet$, the double bar $\|\bullet\|$ denotes the Euclidean norm. A continuous function $\alpha:\Re^+ \to \Re^+$ is of class K if it is increasing and vanishes at the origin. For any $k_x > 0$, the saturation function $\sigma_{k_x}(x)$ is defined as

$$\sigma_{k_x}(x) = \begin{cases} x & \text{if} \quad |x| \le k_x, \\ k_x & \text{if} \quad x > k_x, \\ -k_x & \text{if} \quad x < -k_x \end{cases} \tag{1}$$

II. PROBLEM FORMULATION

The kinematic model of a wheeled mobile robot moving in a plane is described as [5, 7]

$$\begin{aligned} \dot{x} &= v\cos(\theta), \\ \dot{y} &= v\sin(\theta), \\ \dot{\theta} &= \omega \end{aligned} \tag{2}$$

where v and ω are the forward and angular velocities, (x, y) and θ are the position and orientation of the mobile robot in the plane, respectively.

We assume that the reference kinematic model for (2) to be tracked is given as

$$\dot{x}_d = v_d \cos(\theta_d),$$
$$\dot{y}_d = v_d \sin(\theta_d), \quad (3)$$
$$\dot{\theta}_d = \omega_d$$

where notations in (3) have similar meaning to those in (2). In this paper, we propose the first universal controller that simultaneously solves stabilization and tracking problems of wheeled mobile robots with velocity inputs subject to the following constraints:

$$\begin{cases} \sup_{t\ge 0} |v(t)| \le v_{\max}, \ \ v_{\max} > \sup_{t\ge 0} |v_d(t)|, \\ \sup_{t\ge 0} |\omega(t)| \le \omega_{\max}, \ \omega_{\max} > \sup_{t\ge 0} |\omega_d(t)| \end{cases} \quad (4)$$

and under the following assumptions:

Assumption 1. The reference trajectory, x_d and y_d, and reference velocities, v_d and ω_d are bounded and differentiable with bounded derivative $\dot{v}_d$, i.e.

$$0 \le v_{d\min} \le |v_d| \le v_{d\max} < v_{\max},$$
$$0 \le \omega_{d\min} \le |\omega_d| \le \omega_{d\max} < \omega_{\max}, \quad (5)$$
$$|\dot{v}_d| \le \dot{v}_{d\max}.$$

Assumption 2. One of the following conditions holds:

C1. $v_d = \omega_d = 0$.

C2. $\int_0^\infty \left(|v_d(t)| + |\omega_d(t)| + |\dot{\omega}_d(t)|\right) dt < \infty$.

C3. $\int_{t_0}^{t} v_d^2(\tau) d\tau \ge \delta_v (t - t_0), \qquad \delta_v > 0, \ \ \forall \ 0 \le t_0 \le t < \infty$

C4. $\int_{t_0}^{t} \omega_d^2(\tau) d\tau \ge \delta_\omega (t - t_0), \qquad \delta_\omega > 0, \ \ \forall \ 0 \le t_0 \le t < \infty$.

Remark 1. When the condition C1 holds, the tracking problem becomes regulation/stabilization one. The condition C3 covers the case of straight line tracking while the parking problem belongs to the condition C2. The circular tracking belongs to the case when the condition C4 holds.

Remark 2. In the recent work [19], stronger conditions than C3 and C4 are required. Firstly, the reference forward velocity is positive. Secondly, the reference angular velocity is not allowed to be nonzero when the reference forward velocity is zero or approaches to the origin.

We use the following position, orientation and velocity errors [5, 9] as

$$\begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_d \\ y - y_d \\ \theta - \theta_d \end{bmatrix}, \quad (6)$$
$$v_e = v - v_d, \ \omega_e = \omega - \omega_d.$$

From (2), (3) and (6) we have the tracking error dynamics

$$\dot{x}_e = v_e - v_d\left(\cos(\theta_e) - 1\right) + \omega_e y_e + \omega_d y_e,$$
$$\dot{y}_e = v_d \sin(\theta_e) - \omega_e x_e - \omega_d x_e, \quad (7)$$
$$\dot{\theta}_e = \omega_e.$$

The tracking and regulation problems of mobile robots become the one of stabilizing the system (7). In particular, we are interested in designing explicit expressions for v and ω such that $\lim_{t\to\infty} \|X_e(t)\| = 0$, for all $t_0 \ge 0$ and $X_e(t_0) \in \Re^3$ with $X_e = \left[x_e, \ y_e, \ \theta_e\right]^T$. In case of C4, it is shown that $\|X_e(t)\| \le \gamma\left(\|X_e(t_0)\|\right) e^{-\mu(t-t_1)}$, $0 \le t_0 \le t_1 \le t < \infty$ with γ being a class-K function, and μ a positive constant.

III. CONTROL DESIGN

First, we introduce the following coordinate transformation

$$z_e = \theta_e + \operatorname{asin}\left(k(t) y_e / \sqrt{1 + x_e^2 + y_e^2}\right) \quad (8)$$

with

$$k(t) = \lambda_1\left(v_d + \lambda_2(1 - \operatorname{sgn}(v_{d\min}))\cos(\lambda_3 t)\right) \quad (9)$$

where the constants λ_i, $1 \le i \le 3$ are such that $\sup_{t\ge 0} |k(t)| < 1$ and will be specified in the stability analysis in Section 4. The choice of (9) is interpreted as follows. When $v_{d\min} \ne 0$, that is, v_d satisfies the persistent excitation (PE) condition or $|v_d(t)| \ge v_{d\min} > 0$, (Condition C3) the term, $\lambda_2(1 - \operatorname{sgn}(v_{d\min}))\cos(\lambda_3 t)$, vanishes and the term $\lambda_1 v_d$ is designed to stabilize the closed loop tracking error system. When $v_{d\min} = 0$, that is, $v_d = 0$, (Condition C1), or $\lim_{t\to\infty} v_d(t) = 0$, (Condition C2), the term $\lambda_2(1 - \operatorname{sgn}(v_{d\min}))\cos(\lambda_3 t)$ is used to stabilize the closed loop tracking error system. It is seen that (8) is well defined and convergence of z_e and y_e implies convergence of θ_e.

Under the coordinate transformation (8), the tracking error dynamics (7) is rewritten as

$$\dot{x}_e = v_e - v_d\left(\pi_2/\pi_1 - 1\right) + \omega_e y_e + \omega_d y_e + p_x,$$
$$\dot{y}_e = -k v_d y_e/\pi_1 - \omega_e x_e - \omega_d x_e + p_y,$$
$$\dot{z}_e = \left(1 - k x_e/\pi_2\right)\omega_e + \pi_2^{-1}\Big(\dot{k} y_e - k^2 v_d y_e/\pi_1 - k\omega_d x_e - \quad (10)$$
$$\frac{k y_e}{\pi_1^2}\left(x_e\left(v_e - v_d\left(\frac{\pi_2}{\pi_1} - 1\right)\right) - \frac{k v_d y_e^2}{\pi_1}\right)\Big) + p_z$$

where, for notational simplicity, we use k as $k(t)$, and

$$p_x = -v_d \left(\left(\cos(z_e) - 1\right) \frac{\pi_2}{\pi_1} + \sin(z_e) \frac{k y_e}{\pi_1} \right),$$

$$p_y = v_d \left(\sin(z_e) \frac{\pi_2}{\pi_1} - \left(\cos(z_e) - 1\right) \frac{k y_e}{\pi_1} \right), \tag{11}$$

$$p_z = \frac{1}{\pi_2} \left(k p_y - \frac{k y_e \left(x_e p_x + y_e p_y \right)}{\pi_1^2} \right),$$

$$\pi_1 = \sqrt{1 + x_e^2 + y_e^2}, \ \pi_2 = \sqrt{1 + x_e^2 + (1 - k^2) y_e^2}.$$

We propose the following universal controller

$$\begin{aligned} v_e &= -k_1 \sigma_{k_x}(x_e) + k_2 \omega_d k_y y_e / \pi_1, \\ \omega_e &= \omega_{1e} + \omega_{2e} \end{aligned} \tag{12}$$

where

$$\begin{aligned} \omega_{1e} = &-\frac{1}{\pi_2 - k x_e} \left(\dot{k} y_e - \frac{k^2 v_d y_e}{\pi_1} - k \omega_d x_e - \right. \\ &\left. -\frac{k y_e}{\pi_1^2} \left(x_e \left(v_e - v_d \left(\frac{\pi_2}{\pi_1} - 1 \right) - \frac{k v_d y_e^2}{\pi_1} \right) \right) \right), \end{aligned} \tag{13}$$

$$\omega_{2e} = -\frac{\pi_2}{\pi_2 - k x_e} \left(k_3 \sigma_{k_z}(z_e) + p_z \right),$$

All positive design constants $k_i, 1 \le i \le 3$ and k_x, k_y, k_z are to be chosen in the next section. We now present our main result whose proof is given in Section 4.

Theorem 1. Under Assumptions 1 and 2, the universal saturated controller (12) solves the tracking and stabilization control problem formulated in Section 2 with a suitable choice of the design constants, k_x, k_y, k_z, λ_i. and $k_i, 1 \le i \le 3$.

Corollary 1. The tracking error dynamics (10) with the saturated controller (12) is locally exponentially stable at the origin under Assumption 1 and conditions C3 and C4 of Assumption 2.

Proof of this corollary follows directly from the proof of Theorem 1.

IV. PROOF OF THEOREM 1

Substituting (12) into (10) yields the following tracking error closed loop system

$$\begin{aligned} \dot{x}_e &= -k_1 \sigma_{k_x}(x_e) + k_2 \omega_d k_y y_e / \pi_1 - v_d \left(\pi_2 / \pi_1 - 1 \right) + \\ &\quad (\omega_{1e} + \omega_{2e} + \omega_d) y_e + p_x, \\ \dot{y}_e &= -\frac{k v_d y_e}{\pi_1} - \omega_{1e} x_e - \omega_{2e} x_e - \omega_d x_e + p_y, \\ \dot{z}_e &= -k_3 \sigma_{k_z}(z_e). \end{aligned} \tag{14}$$

We will prove Theorem 1 in the following order. Firstly we show that the control law (12) satisfies the constraint (4). Secondly we prove that (x_e, y_e, z_e) is bounded, then that the closed loop system (14) is asymptotically stable at the origin.

IV.1 *Boundedness of* v and ω

From (12) and (13), there exist design constants, λ_i, k_i, k_x, k_y and k_z, such that

$$\left| v(t) \right| \le \left| v_d \right| + k_1 k_x + k_2 \left| \omega_d \right| k_y \le v_{\max}, \tag{15}$$

$$\begin{aligned} \left| \omega(t) \right| \le \left| \omega_d \right| + \frac{1}{1 - |k|} &\left(\frac{|\dot{k}|}{\sqrt{1 - k^2}} + k^2 \left| v_d \right| + \left| k v_d \right| \right. \\ &\left| k \omega_d \right| + |k| \left(k_1 k_x + k_2 k_y \left| \omega_d \right| + \left| v_d \right| + \left| k v_d \right| \right) + \\ &\left. k_3 k_z + |k| \left(4 \left| v_d \right| + 5 |k| \right) \right) \le \omega_{\max}. \end{aligned} \tag{16}$$

We now present two technical lemmas to simplify the proof of Theorem 1.

Lemma 1. The tracking error closed loop system (14) possesses the following:

1) There exists a time instant t_1 such that

$$\begin{aligned} &\lim_{t \to \infty} z_e(t) = 0, \qquad \forall \quad 0 \le t_0 \le t < \infty, \\ &\left| z_e(t) \right| \le \left| z_e(t_1) \right| e^{-k_3 (t - t_1)}, \ \forall \ \ 0 \le t_0 \le t_1 \le t < \infty. \end{aligned} \tag{17}$$

2) There exists a nondecreasing function χ_2 of $\left\| \left(x_e(t_0), y_e(t_0), z_e(t_0) \right) \right\|$ such that

$$\left\| \left(x_e(t), y_e(t) \right) \right\| \le \chi_2, \ \forall 0 \le t_0 \le t < \infty, \ \left| x_e(t) \right| \ge k_x. \tag{18}$$

3) There exists a nondecreasing function χ_3 of $\left\| \left(x_e(t_0), y_e(t_0), z_e(t_0) \right) \right\|$ such that

$$\left\| \left(x_e(t), y_e(t) \right) \right\| \le \chi_3, \ \forall 0 \le t_0 \le t \le t_1. \tag{19}$$

Proof of Lemma 1.

1) By differentiating $V_1 = 0.5 z_e^2$ along the solution of the last equation of (14) yields (17) readily.

2) Consider the following quadratic function

$$V_2 = 0.5 \left(x_e^2 + y_e^2 \right) \tag{20}$$

whose time derivate along the solution of the first two equations of (14) satisfies

$$\begin{aligned} \dot{V}_2 = &-k_1 x_e \sigma_{k_x}(x_e) + k_2 \omega_d x_e k_y y_e / \pi_1 - \\ &v_d x_e \left(\pi_2 / \pi_1 - 1 \right) + x_e p_x - k v_d y_e^2 / \pi_1 + y_e p_y. \end{aligned} \tag{21}$$

When $\left| x_e \right| \ge k_x$, substituting (11) into (21) yields

$$\begin{aligned} \dot{V}_2 = &-\left(k_1 k_x - k_2 k_y \left| \omega_d \right| - \left| v_d \right| \right) \left| x_e \right| + \left| x_e \right| \left| v_d \right| \left(1 + |k| \right) \left| z_e \right| + \\ &\left| y_e \right| \left| v_d \right| \left(1 + |k| \right) \left| z_e \right| - \frac{k v_d y_e^2}{\pi_1}. \end{aligned} \tag{22}$$

We choose the design constants λ_i, k_i, k_x and k_y such that

$$k_1 k_x - k_2 k_y \left| \omega_d \right| - \left| v_d \right| \ge 0. \tag{23}$$

It is seen that there always exist λ_i, k_i, k_x and k_y (23) satisfies for bounded v_d and ω_d. Substituting (23) into (22) yields

$$\dot{V}_2 = \left| x_e \right| \left| v_d \right| \left(1 + |k| \right) \left| z_e \right| + \left| y_e \right| \left| v_d \right| \left(1 + |k| \right) \left| z_e \right| - k v_d y_e^2 / \pi_1 \tag{24}$$

which is further equivalent to

$$\dot{V}_2 = \left| v_d \right| \left(1 + |k| \right) \left(1 + V_2 \right) \left| z_e \right| - k v_d y_e^2 / \pi_1. \tag{25}$$

Substituting (17) into (25) then integrating both sides of (25) yields (18) readily.

3) It is easily seen from (21).

Lemma 2. The solution of the following differential inequality

$$\dot{\xi} \le \sqrt{2}\,p(t)\sqrt{\xi} + q(t)\xi + \left(\eta_1\xi + \eta_2\right)e^{-\sigma(t-t_0)} \quad (26)$$

with $\xi \ge 0,\ \eta_1 \ge 0,\ \eta_2 \ge 0,\ \sigma > 0$, $p(t) \ge 0,\ q(t) \ge 0$ and $\int_0^\infty \left(p(t)+q(t)\right)dt < \infty$ satisfies: $\xi(t) \le \pi$ (27)

where π is a nondecreasing function of $\left\|\xi(t_0)\right\|$.

Proof of Lemma 2.

It is easy to show that (26) is equivalent to

$$\begin{aligned}\dot{\overbrace{(\eta_1+\delta)\xi+\eta_2}} \le{}& \sqrt{2(\xi_1+\delta)}\ p(t)\sqrt{(\eta_1+\delta)\xi+\eta_2} + \\ & (\eta_1+\delta)\left((\eta_1+\delta)\xi+\eta_2\right)e^{-\sigma(t-t_0)} + \\ & q(t)\left((\eta_1+\delta)\xi+\eta_2\right)\end{aligned} \quad (28)$$

where δ is an arbitrarily small positive constant. We introduce the constant δ to cover the case of $\eta_1 = 0$.

Substituting

$$\kappa = \sqrt{(\eta_1+\delta)\xi+\eta_2} \quad (29)$$

into (28) yields

$$\dot{\kappa} \le \sqrt{2(\eta_1+\delta)}\ p(t) + \left((\eta_1+\delta)e^{-\sigma(t-t_0)} + q(t)\right)\kappa \quad (30)$$

which results in

$$\kappa(t) \le \kappa(t_0)e^{(\eta_1+\delta)/\sigma}\eta + \eta\sqrt{2(\eta_1+\delta)}e^{(\eta_1+\delta)/\sigma}\int_{t_0}^{t} p(\tau)d\tau \quad (31)$$

where $\eta \ge e^{\int_{t_0}^{t} q(\tau)d\tau}$. Since $\int_0^\infty \left(p(t)+q(t)\right)dt < \infty$, the right hand side of (31) is bounded. From (31) and (29), we have (27) readily.

IV.2 *Proof of Theorem 1.*

Since $\lim_{t\to\infty} z_e(t) = 0$ (see Lemma 1), we only need to show that $\lim_{t\to\infty}(x_e(t), y_e(t)) = 0$. Furthermore $(x_e(t), y_e(t))$ is bounded when $0 \le t_0 \le t \le t_1$ as proven in Lemma 1. Therefore we only consider $t_1 < t < \infty$.

To prove $\lim_{t\to\infty}(x_e(t), y_e(t)) = 0$ for $t_1 < t < \infty$, we take the following quadratic function

$$V = 0.5y_e^2 + 0.5\left(x_e - k_2\omega_d y_e\right)^2 \quad (32)$$

Differentiating (32) along the solutions of the first two equations of (14) yields

$$\begin{aligned}\dot{V} ={}& -k_1 x_e\sigma_{k_x}(x_e) - \frac{k v_d y_e^2}{\sqrt{1+x_e^2+y_e^2}} - k_2^2\omega_d^2 k_y y_e^2/\pi_1 - \\ & k_2\omega_d^2 y_e^2 + M_1 + M_2 + \Omega\end{aligned} \quad (33)$$

where for notational simplicity, we have defined

$$\begin{aligned}M_1 ={}& k_2\omega_d x_e\sigma_y(y_e) - (x_e v_d - k_2\omega_d v_d y_e)(\pi_2/\pi_1 - 1) - \\ & k_2\dot{\omega}_d x_e y_e + (k k_2\omega_d v_d x_e y_e - k k_2^2\omega_d^2 v_d y_e^2)/\pi_1 + \\ & k_2\omega_d^2 x_e^2 + k_1 k_2\omega_d y_e\sigma_x(x_e) + k_2^2\omega_d\dot{\omega}_d y_e^2 - k_2^2\omega_d^3 x_e y_e,\end{aligned}$$

$$M_2 = k_2\omega_d\left(x_e^2 - y_e^2 - k_2\omega_d x_e y_e\right)\omega_{1e}, \quad (35)$$

$$\begin{aligned}\Omega ={}& y_e p_y + x_e p_x + k_2\omega_d\omega_{2e}x_e^2 - k_2\omega_d x_e p_y - \\ & k_2\omega_d\omega_{2e}y_e^2 - k_2\omega_d y_e p_x - k_2^2\omega_d^2\omega_{2e}x_e y_e + k_2^2\omega_d^2 y_e p_y.\end{aligned} \quad (36)$$

After a lengthy by simple calculation by completing the squares and noting that

$$x_e^2 \le \frac{\chi_2}{k_x}x_e\sigma_{k_x}(x_e) \quad (37)$$

we arrive at

$$\begin{aligned}\dot{V} \le{}& -p_1(t)x_e\sigma_{k_x}(x_e) - \frac{p_{21}(t)y_e^2}{\sqrt{1+x_e^2+y_e^2}} - p_{22}(t)y_e^2 - \\ & p_{23}(t)y_e\sigma_y(y_e) + \left(\mu_1 V + \mu_2\right)e^{-k_3(t-t_1)}\end{aligned} \quad (38)$$

where

$$\begin{aligned}p_1(t) ={}& k_1 - \frac{\chi_2}{k_x}\left(\frac{k_2}{4\varepsilon}\left(2 + k_2 + |k| + \omega_d^2(4\varepsilon + |\omega_d|)\right) + \right. \\ & \frac{2k_2|\omega_d| + k_2^2\omega_d^2}{2(1-|k|)}\left(\frac{|\dot{k}|}{\sqrt{1-k^2}} + k^2|v_d| + |k||\omega_d| + \right. \\ & \left.\left.(k_1+k_2)|k| + k^2|v_d|\left(|k|+1\right)\right) - k_1 k_2,\right.\end{aligned} \quad (39)$$

$$\begin{aligned}p_{21}(t) ={}& k v_d - \left(v_d^2(k_2^2\varepsilon + 0.5k_2k^2 + |k|k_2\varepsilon\omega_d^2) + |k|k_2^2\omega_d^2|v_d|\right) - \\ & \frac{2k_2|\omega_d| + k_2^2\omega_d^2}{2(1-|k|)}\left(\frac{k^2|v_d|}{\sqrt{1-k^2}} + \left(|k|+1\right)k^2|v_d|\right),\end{aligned} \quad (40)$$

$$\begin{aligned}p_{22}(t) ={}& k_2\omega_d^2 - k_2\varepsilon\dot{\omega}_d^2 - k_1k_2\varepsilon\omega_d^2 - k^2|\omega_d\dot{\omega}_d| - \\ & 0.5k_2k^2\omega_d^2 - k_2^2\varepsilon\omega_d^2|\omega_d| - \frac{2k_2|\omega_d| + k_2^2\omega_d^2}{2(1-|k|)}\times \\ & \left(\frac{|\dot{k}|}{\sqrt{1-k^2}} + k_1|k| + |\omega_d||k|(1+k_2)\right),\end{aligned} \quad (41)$$

$$p_{23}(t) = k_2\omega_d^2\left(k_2 - \varepsilon\right) \quad (42)$$

for some $0 < \varepsilon < 1$, and μ_1 and μ_2 are nondecreasing functions of $|z_e(t_1)|$.

We are now in a position to select the design parameters, λ_i and k_i, , $1 \le i \le 3$ such that the closed loop system (14) is globally asymptotically stable at the origin. We proceed case by case according to Assumption 2. For each case, we choose a subset of λ_i and k_i. Then the design parameters belong to the subset that is contained in all of the derived subsets.

As discussed in Section 2, we first choose

$$\lambda_1 |v_d| + \lambda_2 (1 - \mathrm{sgn}(v_{d\min})) < 1, \quad \lambda_2 > 0,\ \lambda_3 > 0. \tag{43}$$

Note that this primary choice guarantees $\sup_{t\ge 0} |k(t)| < 1$ as required in Section 2.

Secondly, we choose λ_i and k_i such that

$$p_1(t) \ge p_1^* > 0 \tag{44}$$

Before going further to the choice of λ_i and k_i, let us discuss each case of Assumption 2.

1) Case of $v_d = \omega_d = 0$

In this case, by noting that

$$p_{21}(t) = p_{22}(t) = p_{23}(t) = 0 \tag{45}$$

we have

$$\begin{aligned} \dot V &\le -k_3 k_x x_e + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} && \text{if} \quad x_e > k_x, \\ \dot V &\le k_3 k_x x_e + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} && \text{if} \quad x_e < -k_x, \\ \dot V &\le -k_3 x_e + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} && \text{if} \quad |x_e| \le k_x. \end{aligned} \tag{46}$$

Therefore $\dot V \le (\mu_1 V + \mu_2) e^{-k_3(t-t_1)}$ for any x_e. It is not hard to show that there exists a nondecreasing function μ_3 of $\|(V(t_1), \mu_1, \mu_2, k_3)\|$ such that $V(t) \le \mu_3$. Integrating both sides of (46) and applying Barbalat's lemma [9] yield $\lim_{t\to\infty} x_e(t) = 0$.

To prove $\lim_{t\to\infty} y_e(t) = 0$, applying Lemma 2 in [19] to the x_e-dynamic equation yields

$$\begin{aligned} \lim_{t\to\infty} \big(& k_2 \omega_d k_y y_e / \pi_1 - v_d (\pi_2/\pi_1 - 1) + \\ & (\omega_{1e} + \omega_{2e} + \omega_d) y_e + p_x \big) = 0. \end{aligned} \tag{47}$$

Since $v_d = \omega_d = 0$, $\lim_{t\to\infty} p_x = \lim_{t\to\infty} \omega_{2e} = 0$, (47) is equivalent to

$$\lim_{t\to\infty} \omega_{1e} y_e = 0. \tag{48}$$

From (46) we have $\overbrace{V + k_3^{-1}(\mu_1\mu_3 + \mu_2) e^{-k_3(t-t_1)}}^{\bullet} \le 0$ which implies that $V + k_3^{-1}(\mu_1\mu_3 + \mu_2) e^{-k_3(t-t_1)}$ is decreasing. Since V is bounded from below by zero, V tends to a finite nonnegative constant depending on $\|(x_e(t_1), y_e(t_1), \theta_e(t_1))\|$. This implies that the limit of $|y_e(t)|$ exists and is a finite real number, l_{y_e}. If l_{y_e} was not zero, there would exist a sequence of increasing time instant $\{\tau_i\}_{i=1}^{\infty}$ with $\tau_i \to \infty$, such that both of the limits $\dot k(\tau_i)$ and $\dot k(\tau_i) y_e^2(\tau_i)$ are not zero, which is impossible because of (48). Hence l_{y_e} must be zero. Therefore we conclude from (48) that $y_e(t) \to 0$ as $t \to \infty$ for any $\lambda_i > 0, i = 2,3$ by noting that $1 - \mathrm{sgn}(v_{d\min}) = 1$ in this case. We define the subset of λ_i, k_i, k_x and k_y satisfying (23), (43) and (44) as $\Xi_1^{\lambda k}$.

2) Case of $\int_0^\infty \left(|v_d(t)| + |\omega_d(t)| + |\dot\omega_d(t)| \right) dt < \infty$

Since

$$\begin{aligned} \dot V &\le -k_3 k_x x_e + |p_{21}(t)|\sqrt{2V} + |p_{22}(t)| V + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} && \text{if } x_e > k_x, \\ \dot V &\le k_3 k_x x_e + |p_{21}(t)|\sqrt{2V} + |p_{22}(t)| V + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} && \text{if } x_e < -k_x, \\ \dot V &\le -k_3 x_e + |p_{21}(t)|\sqrt{2V} + |p_{22}(t)| V + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} && \text{if } |x_e| \le k_x, \end{aligned} \tag{49}$$

we have

$$\dot V \le |p_{21}(t)|\sqrt{2V} + |p_{22}(t)| V + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)}, \ \forall x_e \in R. \tag{50}$$

By noting that

$$\lim_{t\to\infty} p_{21}(t) = \lim_{t\to\infty} p_{22}(t) = 0, \quad p_{23}(t) \ge 0 \tag{51}$$

applying Lemma 2 to (50) yields that V is bounded. It is not hard to show from (50) that V is decreasing. Therefore, by using the same arguments as in the previous Subsection, we have $\lim_{t\to\infty} (x_e(t), y_e(t)) = 0$.

The subset of λ_i, k_i, k_x and k_y that satisfies all requirements in this case is the same as $\Xi_1^{\lambda k}$.

3) Case of $\int_{t_0}^{t} v_d^2(\tau) d\tau \ge \delta_v (t - t_0)$

In this case, we rewrite (38)

$$\begin{aligned} \dot V \le & -p_1(t) x_e \sigma_x(x_e) - \left(\frac{p_{21}(t)}{\sqrt{1+2\chi_2^2}} + p_{22}(t) \right) y_e^2 + \\ & (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} \end{aligned} \tag{52}$$

which is further yields

$$\begin{aligned} \dot V \le & -\frac{p_1^* k_x}{\chi_2 + \delta} x_e^2 - \left(\frac{p_{21}(t)}{\sqrt{1+2\chi_2^2}} + p_{22}(t) \right) y_e^2 + \\ & (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} \end{aligned} \tag{53}$$

where δ is an arbitrarily small positive constant.

Substituting (32) into (53) yields

$$\dot V \le -\rho(t) V + (\mu_1 V + \mu_2) e^{-k_3(t-t_1)} \tag{54}$$

where

$$\rho(t) = \min\left(\frac{p_1^* k_x}{\chi_2 + \delta} \frac{2}{1 + k_2^2 \omega_d^2}, \left(\frac{p_{21}(t)}{\sqrt{1+2\chi_2^2}} + p_{22}(t) \right) \frac{2}{1 + 2k_2^2 \omega_d^2} \right). \tag{55}$$

From (54), it is seen that there exist nondecreasing functions φ_1 and φ_2 of $\|(x_e(t_1), y_e(t_1), z_e(t_1))\|$ such that

$$\|(x_e(t), y_e(t))\| \le \varphi_1 e^{-\varphi_2(t-t_1)} \tag{56}$$

as long as we choose the design constants, λ_i and k_i, such that

$$\int_{t_1}^{t} \rho(\tau) d\tau \ge \varphi_2 (t - t_1), \ \varphi_2 > 0. \tag{57}$$

Since φ_2 depends on $\|(x_e(t_1), y_e(t_1), z_e(t_1))\|$, global asymptotic tracking is achieved.

We define the subset of λ_i, k_i, k_x and k_y satisfying (23), (43), (44) and (57) as $\Xi_2^{\lambda k}$.

4) Case of $\int_{t_0}^{t} \omega_d^2(\tau)d\tau \geq \delta_\omega(t-t_0)$.

Stability analysis of this case is similar to that of the case C3. However, from (52) and (56), it can be shown that there exists a finite time instance, t_2, such that $|x_e(t)| \leq k_x$. This implies that there exists a positive constant φ_2 independence of the initial condition $\|(x_e(t_0), y_e(t_0), z_e(t_0))\|$ such that (56) holds for all $t_2 \leq t < \infty$, which means that exponential tracking is achieved. The subset of λ_i, k_i, k_x and k_y that satisfies all requirements in this case is the same as $\Xi_2^{\lambda k}$.

V. CONCLUSIONS

A universal controller has been obtained in this paper to solve simultaneously regulation and tracking problems for wheeled mobile robots with saturated inputs. The proposed controller is able to globally asymptotically force the mobile robot to follow any reference trajectory generated by a suitable virtual robot. When the reference angular velocity satisfies PE condition, we achieve the exponential stability of the closed loop system at the origin after a considerable time period.

ACKNOWLEDGEMENTS

This research supported partially by a Seed Grant from the Othmer Institute for Interdisciplinary Studies and by the NSF Grants ECS-0093176 and INT-9987317. The study of the first author is supported by IPRS and UPA from the University of Western Australia.

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