Optimization Issues in Nonlinear Model Predictive Control

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Nonlinear Model

Continuous Time:

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= g(x)\end{aligned}$$

Discrete Time:

$$\begin{aligned}x_{k+1} &= F(x_k,u_k)\\ y_k &= G(x_k)\end{aligned}$$

with the constraints

$$x, x_k \epsilon \mathbf{X} \qquad \text{and} \qquad u, u_k \epsilon \mathbf{U}$$

We also require the states and inputs to be deviation variables such that the steady states occur at the origin.

Linearized Model

We can linearize our model about the steady state (origin) to yield the following forms:

Continuous Time:

$$\dot{x} = Ax + Bu$$

Where

$$A = \frac{\partial f}{\partial x}(\mathbf{0}, \mathbf{0}) \qquad \text{and} \qquad B = \frac{\partial f}{\partial u}(\mathbf{0}, \mathbf{0})$$

Discrete Time:

This model can be discretized to the form

$$x_{k+1} = \widetilde{A} x_k + \widetilde{B} u_k$$

using a discretization time Δt.

Model Predictive Control Formulation

Infinite Horizon:

$$\begin{aligned}\min_{u^N} \Phi_k \quad = \quad & \sum_{j=0}^{\infty} (y_{k+j}^T Q y_{k+j} + u_{k+j}^T R u_{k+j} + \\ & \Delta u_{k+j}^T S \Delta u_{k+j})\end{aligned}$$

Finite Horizon:

$$\begin{aligned}\min_{u^N} \Phi_k \quad = \quad & \sum_{j=0}^{N} (y_{k+j}^T Q y_{k+j} + u_{k+j}^T R u_{k+j} + \\ & \Delta u_{k+j}^T S \Delta u_{k+j})\end{aligned}$$

with the additional constraint

$$x_N = 0$$

Challenges of Standard MPC Scheme

- Computation of a global optimum for the cost function is not always fast or reliable.
- The endpoint stability constraint complicates the optimization problem.
- Poor initial guesses may prevent optimization algorithm from making progress.

Proposed Solution: Terminal Region

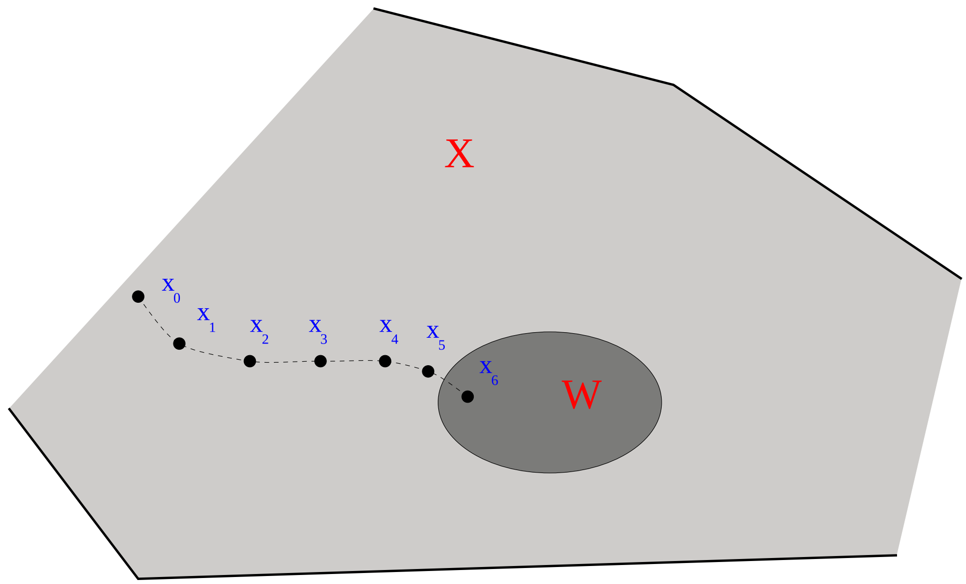


Figure 1: Feasible predicted state horizon for suboptimal approach (horizon length = 6).

- Calculate the region around the origin that can be stabilized by a linear control law. Instead of requiring the final state to be at the origin, we require the final state to be in this region, **W**.

Proposed Solution: Suboptimal MPC

- Rather than finding the optimal sequence of control moves, we settle for a feasible sequence of moves that decrease the cost function from its prior value.

- Most of the computation to determine the endpoint region and the initial control sequence can be performed off-line to increase speed of performance.

Suboptimal MPC:

$$\begin{aligned}\min_{u^N} \Phi_k &= \sum_{j=0}^{N-1} (y_{k+j}^T Q y_{k+j} + u_{k+j}^T R u_{k+j} + \\ &\quad \Delta u_{k+j}^T S \Delta u_{k+j}) + x_N^T P x_N\end{aligned}$$

with the additional constraint

$$x_N \in W_\alpha$$

Calculation of the W Region

We require that we improve at all points x in our region **W** under the linear control law Kx by at least one half the linear stage cost.

The desired neighborhood is defined as:

$$W_\alpha := \{x | x^T P x \leq \alpha\}$$

where

$$W_\alpha \subset \mathbf{X}, \quad KW_\alpha \subset \mathbf{U}.$$

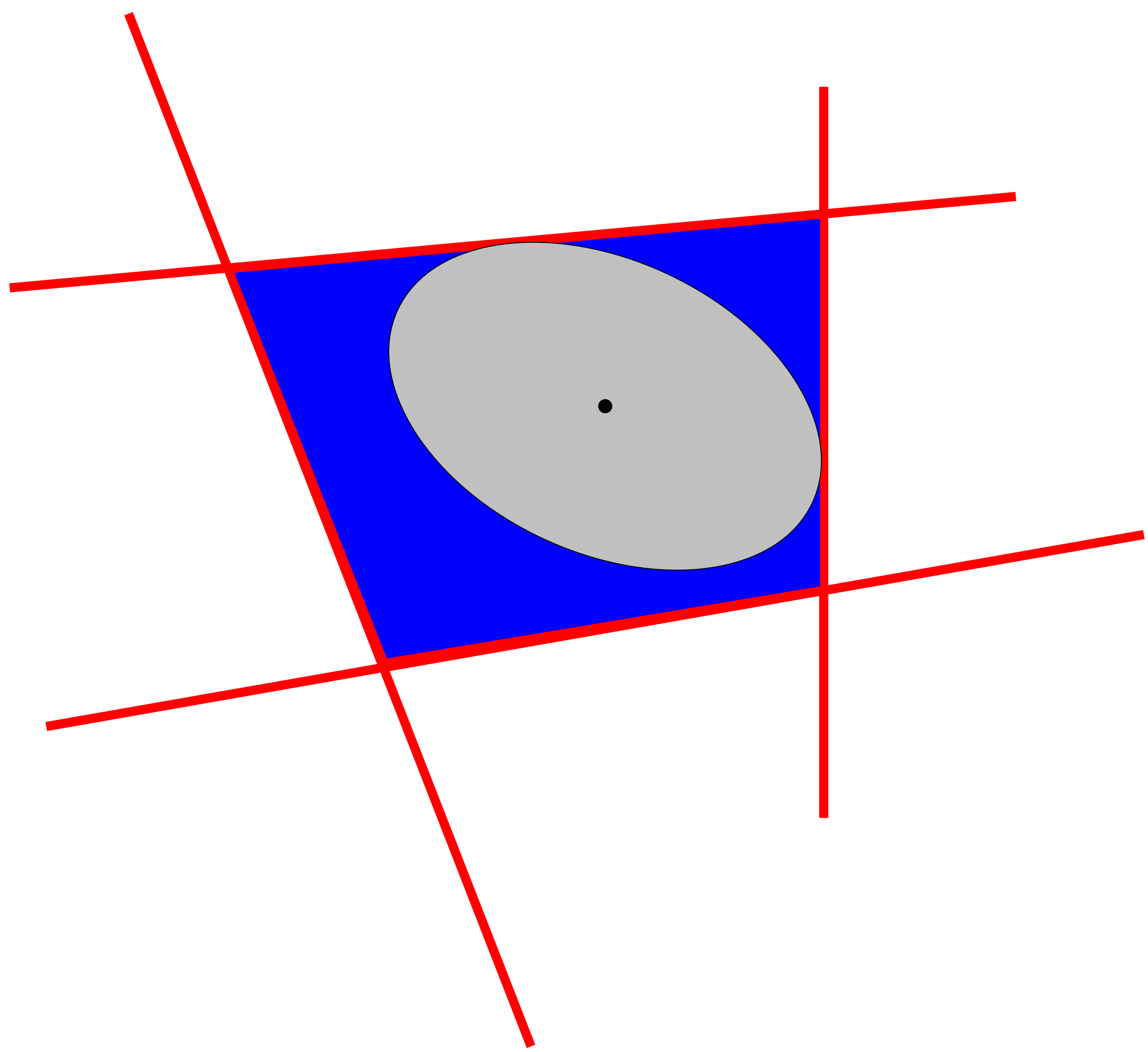
Define

$$V(x) = (1/2) x^T P x \qquad \text{and} \qquad F_K(x) = F(x, Kx)$$

We choose α such that

$$\max_{x \epsilon W_\alpha} \{V(F_K(x)) - V(x) + (1/4) x^T [Q + K^T R K] x\} \leq 0$$

The Largest Possible Starting α_0



The largest ellipse $x^T P x \leq \alpha_0$ that satisfies the constraints and is contained entirely within the feasible region is calculated.

Reformulating the Optimization Problem

We reformulate the optimization problem to locate the largest α for which the inequality is satisfied everywhere in W_α.

We therefore seek the critical value of α, i.e., the smallest α for which the left side is equal to zero.

This is done by:

- first removing the trivial solution at the origin,
- squaring the function, to make it a minimization,
- and rescaling the problem to drive solutions to the center.

Reformulating the Optimization Problem

We now choose the initial guess $x_0 \in W_{\alpha_0}$ and define the new α by:

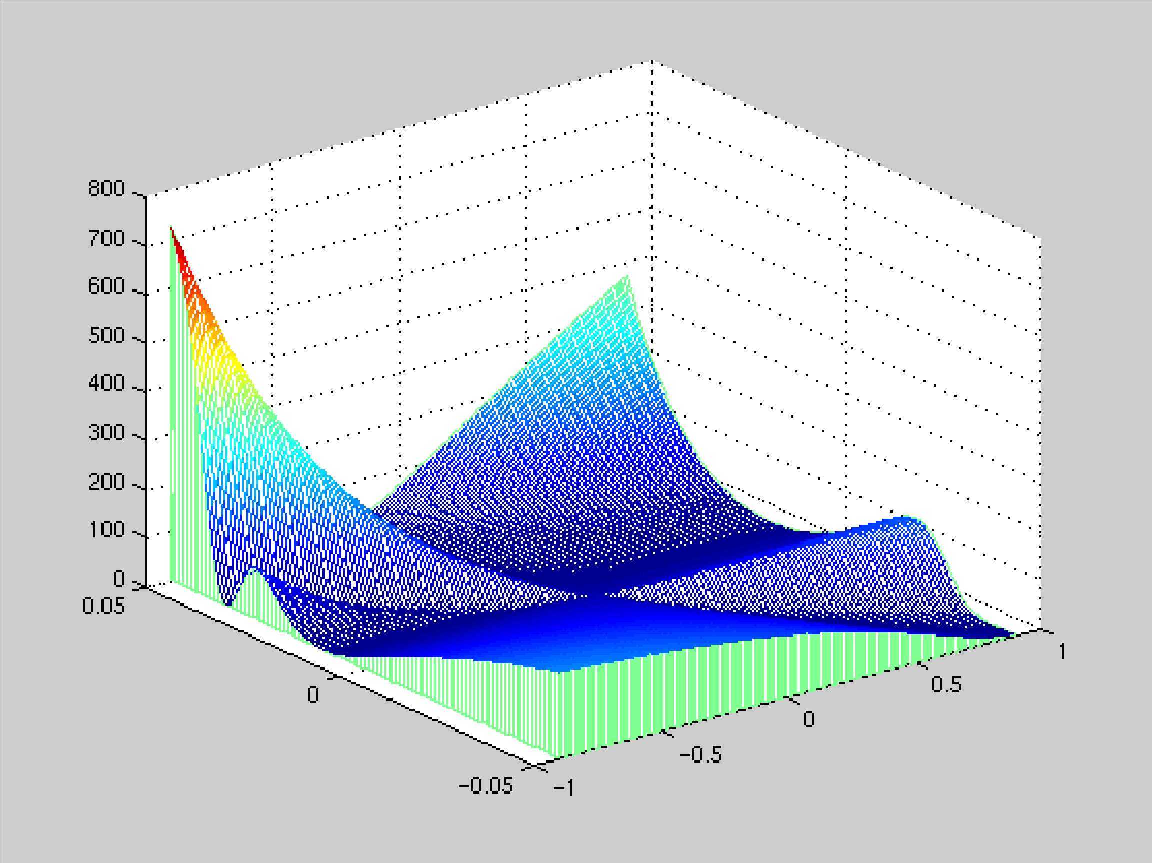
$$\overline{x} = \arg\min \quad \left(\frac{V(F_K(x)) - V(x) + (1/4)x^T[Q+K^TRK]x}{(1/2)x^T[Q+K^TRK]x}\right)^2 \mu + x^TPx$$

$$\alpha = \overline{x}^T P \overline{x}$$

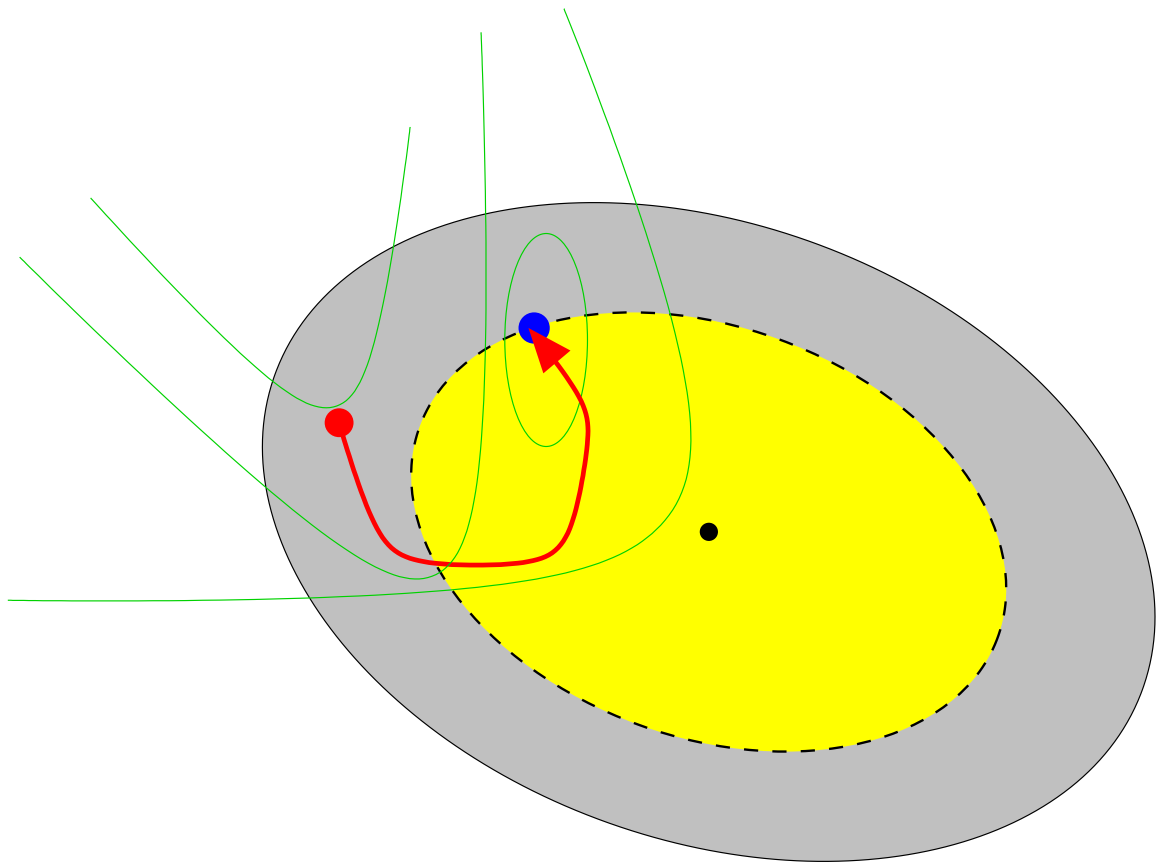
where μ is “large.”

If μ is chosen appropriately, $\overline{x}$ will be a local critical value of α. Now, the smallest of these values must be found by a global optimization scheme.

Geometric Character of the Optimization Problem

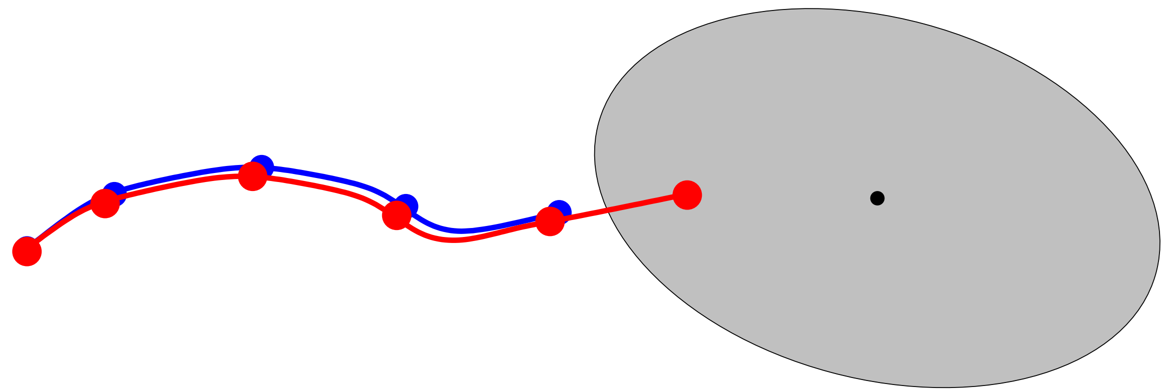


Decreasing to the Final α



A random point is selected within the ellipse. It is used as a starting point for optimization, yielding a local minimum. If this point is within a smaller ellipse, then we shrink α. We then repeat the process.

Initial Feasible Control



- Guess a horizon length N and an initial set of control moves.

- Repeat until converged:
  - Generate states with nonlinear process model.
  - Approximate model with time varying linear model.
  - Minimize objective function subject to linear model. (QP)
  - Perform line search in direction of solution.

- If final state is in terminal region, end. Otherwise, repeat with a longer horizon, using initial guess from this attempt and Kx for the additional controls.

Closed Loop Operation

At each sample time:

- The objective function is decreased by performing a linesearch in the SQP direction.
- The first control is injected and the horizon is shifted one move.
- The final move is guessed to be Kx_{N-1}.
- The process repeats.

Sample Problem

We adapt the simple model presented by Henson and Seborg [1] for a continuously stirred tank reactor (CSTR) undergoing reaction $A \rightarrow B$ at an unstable steady state:

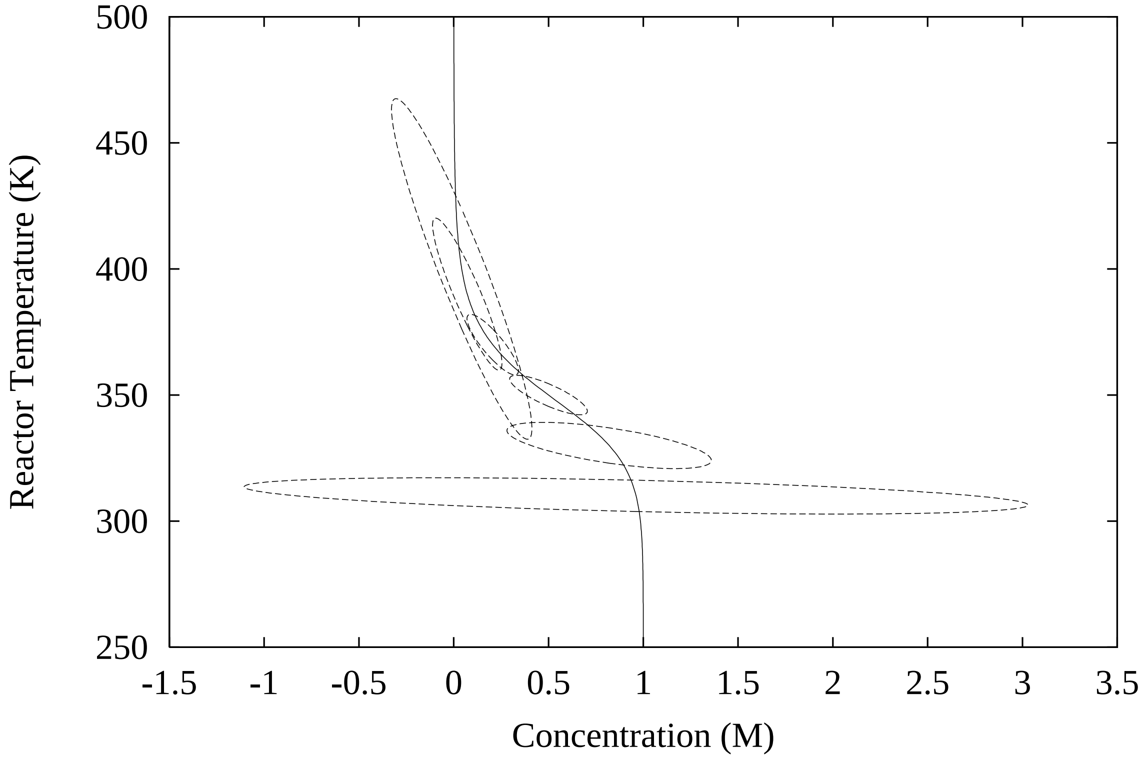
$$\begin{aligned}
\dot{C}_A &= \frac{q}{V}(C_{Af} - C_A) - k_0 \text{exp}\left(-\frac{E}{RT}\right) C_A \\
\dot{T} &= \frac{q}{V}(T_f - T) + \frac{(-\Delta H)}{\rho C_p} k_0 \text{exp}\left(-\frac{E}{RT}\right) C_A \\
&\quad + \frac{UA}{V \rho C_p}(T_c - T)
\end{aligned}$$

The steady state operating conditions are as follows:

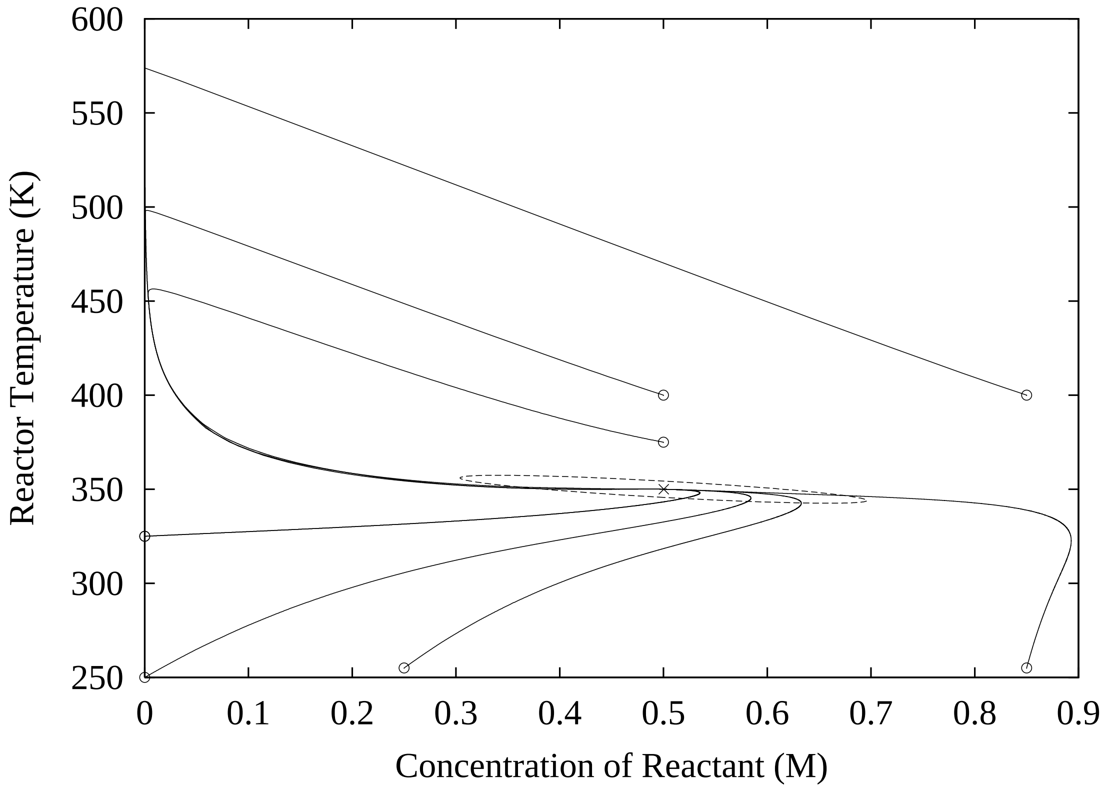
$$\begin{aligned}
T &= 350K \\
C_A &= 0.5M \\
T_c &= 300K
\end{aligned}$$

[1] M. A. Henson and D. E. Seborg. *Nonlinear Process Control.* Prentice Hall PTR, Upper Saddle River, New Jersey, 1997.

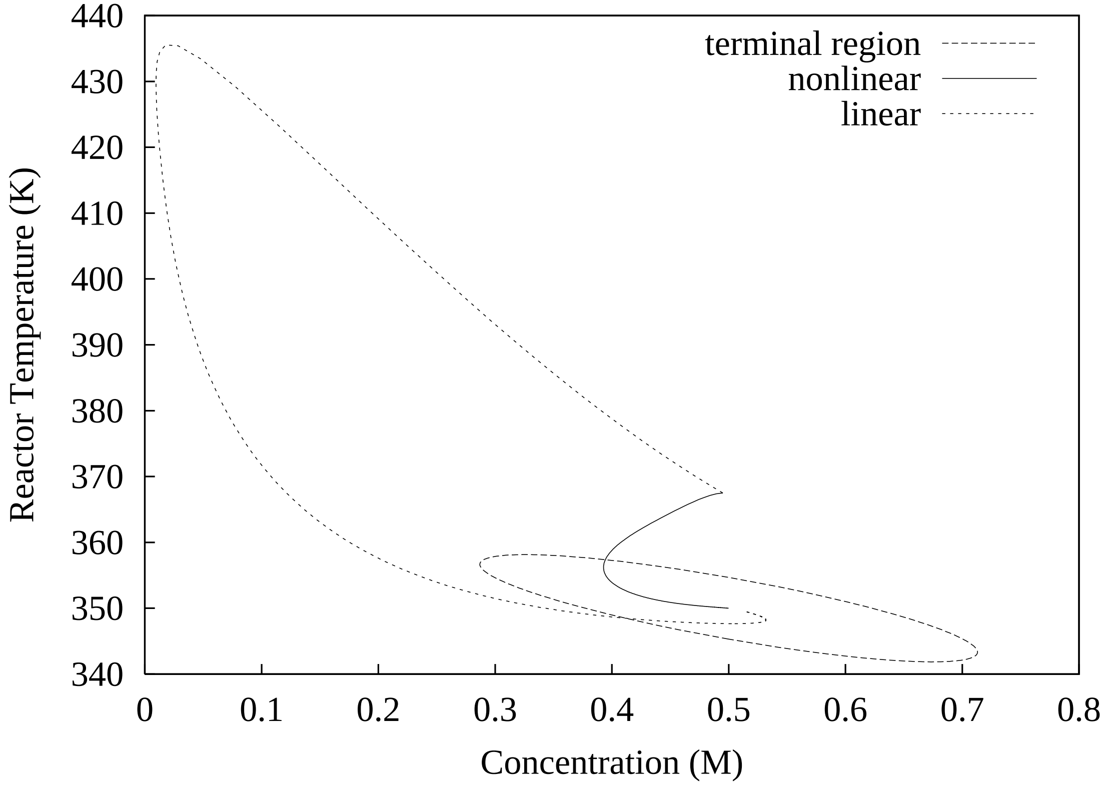
W Regions Around Other Steady States



Controlling to a Steady State



Linear Vs. Nonlinear Control



Conclusions

- Established an implementable algorithmic framework for calculating the terminal region.
- Developed a numerically tractable optimization approach to reach the terminal region.
- Demonstrated control performance superior to linear control.

Future Work

- Handle integrated disturbance models for offset-free control.
- Incorporate state estimation as part of the feedback.
- Analyze terminal constraint enforcement as a constraint vs. horizon extension.

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