4. LAPLACE TRANSFORMS

Topics:
- Laplace transforms
- Using tables to do Laplace transforms
- Using the s-domain to find outputs
- Solving Partial Fractions

Objectives:
- To be able to find time responses of linear systems using Laplace transforms.

4.1 INTRODUCTION

Laplace transforms provide a method for representing and analyzing linear systems using algebraic methods. In systems that begin undeflected and at rest the Laplace 's' can directly replace the d/dt operator in differential equations. It is a superset of the phasor representation in that it has both a complex part, for the steady state response, but also a real part, representing the transient part. As with the other representations the Laplace s is related to the rate of change in the system.

$$D = s \qquad \text{(if the initial conditions/derivatives are all zero at t=0s)}$$
$$s = \sigma + j\omega$$

Figure 4.1 The Laplace s

The basic definition of the Laplace transform is shown in Figure 4.2. The normal convention is to show the function of time with a lower case letter, while the same function in the s-domain is shown in upper case. Another useful observation is that the transform starts at t=0s. Examples of the application of the transform are shown in Figure 4.3 for a step function and in Figure 4.4 for a first order derivative.

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

where,

$f(t)$ = the function in terms of time t

$F(s)$ = the function in terms of the Laplace s

Figure 4.2 The Laplace transform

Aside: Proof of the step function transform.

For f(t) = 5,

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} 5e^{-st} dt = -\frac{5}{s} e^{-st} \Big|_0^{\infty} = \left[-\frac{5}{s} e^{-s\infty}\right] - \left[-\frac{5e^{-s0}}{s}\right] = \frac{5}{s}$$

Figure 4.3 Proof of the step function transform

Aside: Proof of the first order derivative transform

Given the derivative of a function g(t)=df(t)/dt,

$$G(s) = L[g(t)] = L\left[\frac{d}{dt}f(t)\right] = \int_0^{\infty}(\mathrm{d/dt})f(t)e^{-st}dt$$

we can use integration by parts to go backwards,

$$\int_a^b u\,dv = uv\big|_a^b - \boxed{\int_a^b v\,du}$$

$$\int_0^{\infty}(\mathrm{d/dt})f(t)e^{-st}dt$$

therefore,

$$du = df(t) \qquad v = e^{-st}$$

$$u = f(t) \qquad dv = -se^{-st}dt$$

$$\therefore \int_0^{\infty} f(t)(-s)e^{-st}dt = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty}(\mathrm{d/dt})f(t)e^{-st}dt$$

$$\therefore \int_0^{\infty}(\mathrm{d/dt})f(t)e^{-st}dt = [f(t)e^{-\infty s} - f(t)e^{-0s}] + s\int_0^{\infty} f(t)e^{-st}dt$$

$$\boxed{\therefore L\left[\frac{d}{dt}f(t)\right] = -f(0) + sL[f(t)]}$$

Figure 4.4 Proof of the first order derivative transform

The previous proofs were presented to establish the theoretical basis for this method, however tables of values will be presented in a later section for the most popular transforms.

4.2 APPLYING LAPLACE TRANSFORMS

The process of applying Laplace transforms to analyze a linear system involves the basic steps listed below.

1. Convert the system transfer function, or differential equation, to the s-domain by replacing ’D’ with ’s’. (Note: If any of the initial conditions are non-zero these must be also be added.)
2. Convert the input function(s) to the s-domain using the transform tables.
3. Algebraically combine the input and transfer function to find an output function.
4. Use partial fractions to reduce the output function to simpler components.
5. Convert the output equation back to the time-domain using the tables.

4.2.1 A Few Transform Tables

Laplace transform tables are shown in Figure 4.5, Figure 4.7 and Figure 4.8. These are commonly used when analyzing systems with Laplace transforms. The transforms shown in Figure 4.5 are general properties normally used for manipulating equations, and for converting them to/from the s-domain.

TIME DOMAIN	FREQUENCY DOMAIN
$f(t)$	$f(s)$
$Kf(t)$	$KL[f(t)]$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sL[f(t)] - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2 L[f(t)] - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n L[f(t)] - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - \frac{d^n f(0^-)}{dt^n}$
$\int_0^t f(t) dt$	$\frac{L[f(t)]}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as} L[f(t)]$
$e^{-at} f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n f(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^{\infty} f(u) du$

Figure 4.5 Laplace transform tables

$$L[\ddot{x} + 7\dot{x} + 8x = 9] =$$

where,

$$\ddot{x}(0) = 1$$
$$\dot{x}(0) = 2$$
$$x(0) = 3$$

Figure 4.6 Drill Problem: Converting a differential equation to s-domain

The Laplace transform tables shown in Figure 4.7 and Figure 4.8 are normally used for converting to/from the time/s-domain.

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2 + \omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
$t^2 e^{-at}$		$\frac{2!}{(s+a)^3}$

Figure 4.7 Laplace transform tables (continued)

TIME DOMAIN	FREQUENCY DOMAIN
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\dfrac{C-aB}{\omega}\right)\sin\omega t\right]$	$\dfrac{Bs+C}{(s+a)^2+\omega^2}$
$2\vert A\vert e^{-\alpha t}\cos(\beta t+\theta)$	$\dfrac{A}{s+\alpha-\beta j}+\dfrac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\vert A\vert e^{-\alpha t}\cos(\beta t+\theta)$	$\dfrac{A}{(s+\alpha-\beta j)^2}+\dfrac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\dfrac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\dfrac{s+c}{(s+a)(s+b)}$
$\dfrac{e^{-at}-e^{-bt}}{b-a}$	$\dfrac{1}{(s+a)(s+b)}$

Figure 4.8 Laplace transform tables (continued)

$$f(t) = 5\sin(5t+8)$$

$$f(s) = L[f(t)] =$$

Figure 4.9 Drill Problem: Converting from the time to s-domain

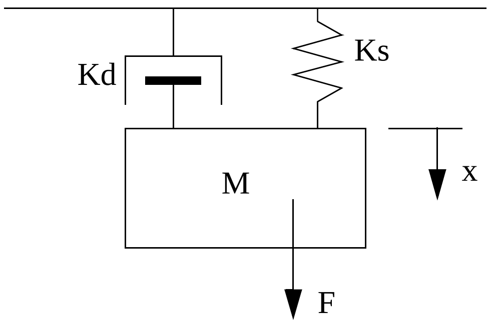
$$f(s) = \frac{5}{s} + \frac{6}{s+7}$$

$$f(t) = L^{-1}[f(s)] =$$

Figure 4.10 Drill Problem: Converting from the s-d to time domain

4.3 MODELING TRANSFER FUNCTIONS IN THE s-DOMAIN

In previous chapters differential equations, and then transfer functions, were derived for mechanical and electrical systems. These can be converted to the s-domain, as shown in the mass-spring-damper example in Figure 4.11. In this case we assume the system starts undeflected and at rest, so the ’D’ operator may be directly replaced with the Laplace ’s’. If the system did not start at rest and undeflected, the ’D’ operator would be replaced with a more complex expression that includes the initial conditions.



$$F = MD^2x + K_dDx + K_sx$$

$$\frac{F(t)}{x(t)} = MD^2 + K_dD + K_s$$

$$L\left[\frac{F(t)}{x(t)}\right] = \frac{F(s)}{x(s)} = Ms^2 + K_ds + K_s$$

ASIDE: Here ’D’ is simply replaced with ’s’. Although this is very convenient, it is only valid if the initial conditions are zero, otherwise the more complex form, shown below, must be used.

$$L\left[\frac{d^nf(t)}{dt^n}\right] = s^nL[f(t)] - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - \ldots - \frac{d^nf(0^-)}{dt^n}$$

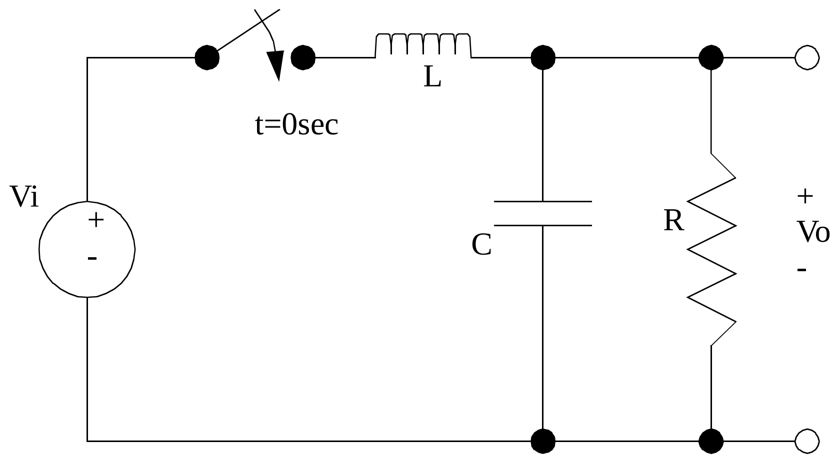
Figure 4.11 A mass-spring-damper example

Impedances in the s-domain are shown in Figure 4.12. As before these assume that the system starts undeflected and at rest.

Device	Time domain	s-domain	Impedance
Resistor	$V(t) = RI(t)$	$V(s) = RI(s)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$V(s) = \left(\frac{1}{C}\right)\frac{I(s)}{s}$	$Z = \frac{1}{sC}$
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$V(s) = LsI(s)$	$Z = Ls$

Figure 4.12 Impedances of electrical components

Figure 4.13 shows an example of circuit analysis using Laplace transforms. The circuit is analyzed as a voltage divider, using the impedances of the devices. The switch that closes at t=0s ensures that the circuit starts at rest. The calculation result is a transfer function.



Treat the circuit as a voltage divider,

$$V_o = \frac{Vi\left(\frac{1}{DC+\frac{1}{R}}\right)}{DL + \left(\frac{1}{DC+\frac{1}{R}}\right)} = \frac{V_i\left(\frac{R}{1+DCR}\right)}{DLR + \left(\frac{R}{1+DCR}\right)} = V_i\left(\frac{R}{D^2R^2LC+DLR+R}\right)$$

$$\frac{V_o}{V_i} = \left(\frac{R}{s^2R^2LC+sLR+R}\right)$$

Figure 4.13 A circuit example

At this point two transfer functions have been derived. To state the obvious, these relate an output and an input. To find an output response, an input is needed.

4.4 FINDING OUTPUT EQUATIONS

An input to a system is normally expressed as a function of time that can be converted to the s-domain. An example of this conversion for a step function is shown in Figure 4.14.

Apply a constant force of A, starting at time t=0 sec.
(*Note: a force applied instantly is impossible but assumed)

F(t) = 0 for t < 0
= A for t >= 0

Perform Laplace transform using tables

$$F(s) = L[F(t)] = \frac{A}{s}$$

Figure 4.14 An input function

In the previous section we converted differential equations, for systems, to transfer functions in the s-domain. These transfer functions are a ratio of output divided by input. If the transfer function is multiplied by the input function, both in the s-domain, the result is the system output in the s-domain.

Given,

$$\frac{x(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$F(s) = \frac{A}{s}$$

Therefore,

$$x(s) = \left(\frac{x(s)}{F(s)}\right)F(s) = \left(\frac{1}{Ms^2 + K_d s + K_s}\right)\frac{A}{s}$$

Assume,

$$K_d = 3000\frac{Ns}{m}$$

$$K_s = 2000\frac{N}{m}$$

$$M = 1000kg$$

$$A = 1000N$$

$$\therefore x(s) = \frac{1}{(s^2 + 3s + 2)s}$$

Figure 4.15 A transfer function multiplied by the input function

Output functions normally have complex forms that are not found directly in transform tables. It is often necessary to simplify the output function before it can be converted back to the time domain. Partial fraction methods allow the functions to be broken into smaller, simpler components. The previous example in Figure 4.15 is continued in Figure 4.16 using a partial fraction expansion. In this example the roots of the third order denominator polynomial, are calculated. These provide three partial fraction terms. The residues (numerators) of the partial fraction terms must still be calculated. The example shows a method for finding resides by multiplying the output function by a root term, and then finding the limit as s approaches the root.

$$x(s) = \frac{1}{(s^2+3s+2)s} = \frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \lim_{s \to 0}\left[s\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

$$B = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = -1$$

$$C = \lim_{s \to -2}\left[(s+2)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

Aside: the short cut above can reduce time for simple partial fraction expansions. A simple proof for finding 'B' above is given in this box.

$$\frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$(s+1)\left[\frac{1}{(s+1)(s+2)s}\right] = (s+1)\left[\frac{A}{s}\right] + (s+1)\left[\frac{B}{s+1}\right] + (s+1)\left[\frac{C}{s+2}\right]$$

$$\frac{1}{(s+2)s} = (s+1)\left[\frac{A}{s}\right] + B + (s+1)\left[\frac{C}{s+2}\right]$$

$$\lim_{s \to -1}\left[\frac{1}{(s+2)s}\right] = \lim_{s \to -1}\left[(s+1)\left[\frac{A}{s}\right]\right] + \lim_{s \to -1} B + \lim_{s \to -1}\left[(s+1)\left[\frac{C}{s+2}\right]\right]$$

$$\lim_{s \to -1}\left[\frac{1}{(s+2)s}\right] = \lim_{s \to -1} B = B$$

$$x(s) = \frac{1}{(s^2+3s+2)s} = \frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}$$

Figure 4.16 Partial fractions to reduce an output function

After simplification with partial fraction expansion, the output function is easily converted back to a function of time as shown in Figure 4.17.

$$x(t) = L^{-1}[x(s)] = L^{-1}\left[\frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}\right]$$

$$x(t) = L^{-1}\left[\frac{0.5}{s}\right] + L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{0.5}{s+2}\right]$$

$$x(t) = [0.5] + [(-1)e^{-t}] + [(0.5)e^{-2t}]$$

$$x(t) = 0.5 - e^{-t} + 0.5e^{-2t}$$

Figure 4.17 Partial fractions to reduce an output function (continued)

4.5 INVERSE TRANSFORMS AND PARTIAL FRACTIONS

The flowchart in Figure 4.18 shows the general procedure for converting a function from the s-domain to a function of time. In some cases the function is simple enough to immediately use a transfer function table. Otherwise, partial fraction expansion is normally used to reduce the complexity of the function.

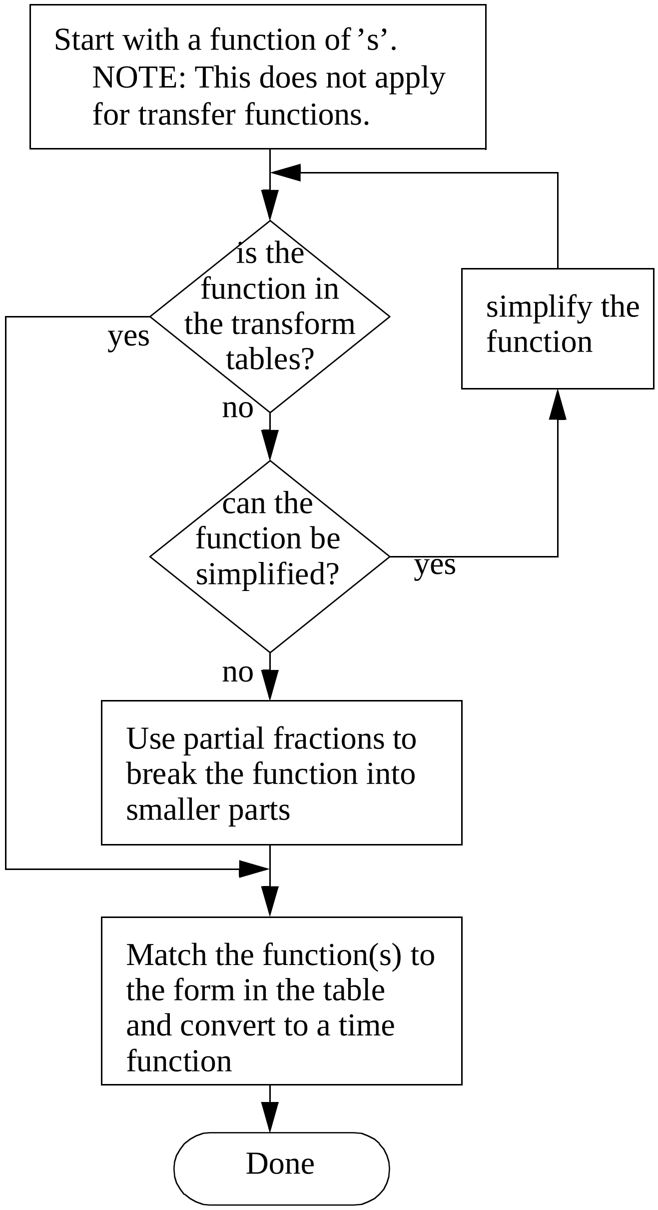


Figure 4.18 The methodology for doing an inverse transform of an output function

Figure 4.19 shows the basic procedure for partial fraction expansion. In cases where the numerator is greater than the denominator, the overall order of the expression can be reduced by long division. After this the denominator can be reduced from a polynomial to multiplied roots. Calculators or computers are normally used when the order of the polynomial is greater than second order. This results in a number of terms with unknown resides that can be found using a limit or algebra based technique.

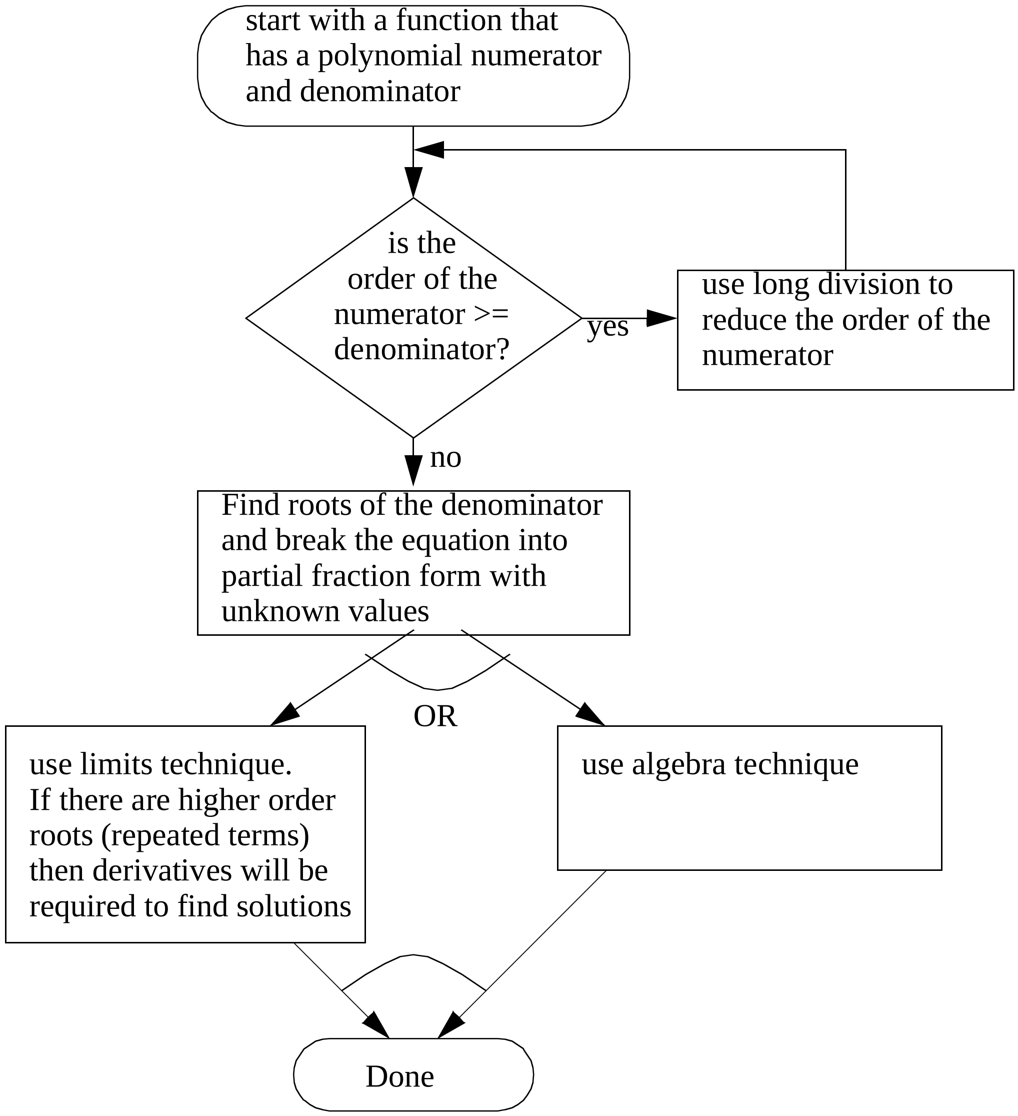


Figure 4.19 The methodology for solving partial fractions

Figure 4.20 shows an example where the order of the numerator is greater than the denominator. Long division of the numerator is used to reduce the order of the term until it is low enough to apply partial fraction techniques. This method is used infrequently because this type of output function normally occurs in systems with extremely fast response rates that are infeasible in practice.

$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r|l} & 5s + 3 \\ \hline s^2 + 4 & 5s^3 + 3s^2 + 8s + 6 \\ & 5s^3 + 20s \\ \hline & \quad 3s^2 - 12s + 6 \\ & \quad 3s^2 + 12 \\ \hline & \qquad -12s - 6 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \qquad \text{solve} \qquad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$

Figure 4.20 Partial fractions when the numerator is larger than the denominator

Partial fraction expansion of a third order polynomial is shown in Figure 4.21. The s-squared term requires special treatment. Here it produces partial two partial fraction terms divided by s and s-squared. This pattern is used whenever there is a root to an exponent.

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}[-(s+1)^{-2}] = -1$$

Figure 4.21 A partial fraction example

Figure 4.22 shows another example with a root to an exponent. In this case each of the repeated roots is given with the highest order exponent, down to the lowest order exponent. The reader will note that the order of the denominator is fifth order, so the resulting partial fraction expansion has five first order terms.

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

Figure 4.22 Partial fractions with repeated roots

Algebra techniques are a reasonable alternative for finding partial fraction residues. The example in Figure 4.23 extends the example begun in Figure 4.22. The equivalent forms are simplified algebraically, until the point where an inverse matrix solution is used to find the residues.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3}$$

$$= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \\ 5 \\ 10 \\ 15 \end{bmatrix}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 4.23 Solving partial fractions algebraically

For contrast, the example in Figure 4.23 is redone in Figure 4.24 using the limit techniques. In this case the use of repeated roots required the differentiation of the output function. In these cases the algebra techniques become more attractive, despite the need to solve simultaneous equations.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$A = \lim_{s \to 0}\left[\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{5}{(s+1)^3}\right] = 5$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{(s+1)^3}\right)\right] = \lim_{s \to 0}\left[\frac{5(-3)}{(s+1)^4}\right] = -15$$

$$C = \lim_{s \to -1}\left[\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{5}{s^2}\right] = 5$$

$$D = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{-2(5)}{s^3}\right] = 10$$

$$E = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{30}{s^4}\right] = 15$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Figure 4.24 Solving partial fractions with limits

An inductive proof for the limit method of solving partial fractions is shown in Figure 4.25.

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$\lim_{s \to -1}\left[\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}\right]$$

$$\lim_{s \to -1}\left[(s+1)^3\left(\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}\right)\right]$$

$$\lim_{s \to -1}\left[\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right]$$

For C, evaluate now,

$$\frac{5}{(-1)^2} = \frac{A(-1+1)^3}{(-1)^2} + \frac{B(-1+1)^3}{-1} + C + D(-1+1) + E(-1+1)^2$$

$$\frac{5}{(-1)^2} = \frac{A(0)^3}{(-1)^2} + \frac{B(0)^3}{-1} + C + D(0) + E(0)^2 \qquad \boxed{C = 5}$$

For D, differentiate once, then evaluate

$$\lim_{s \to -1}\left[\frac{d}{dt}\left(\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right)\right]$$

$$\lim_{s \to -1}\left[\frac{-2(5)}{s^3} = A\left(-\frac{2(s+1)^3}{s^3} + \frac{3(s+1)^2}{s^2}\right) + B\left(-\frac{(s+1)^3}{s^2} + \frac{3(s+1)^2}{s}\right) + D + 2E(s+1)\right]$$

$$\frac{-2(5)}{(-1)^3} = \boxed{D = 10}$$

For E, differentiate twice, then evaluate (the terms for A and B will be ignored to save space, but these will drop out anyway).

$$\lim_{s \to -1}\left[\left(\frac{d}{dt}\right)^2\left(\frac{5}{s^2} = \frac{A(s+1)^3}{s^2} + \frac{B(s+1)^3}{s} + C + D(s+1) + E(s+1)^2\right)\right]$$

$$\lim_{s \to -1}\left[\left(\frac{d}{dt}\right)\left(\frac{-2(5)}{s^3} = A(\ldots) + B(\ldots) + D + 2E(s+1)\right)\right]$$

$$\lim_{s \to -1}\left[\frac{-3(-2(5))}{s^4} = A(\ldots) + B(\ldots) + 2E\right]$$

$$\frac{-3(-2(5))}{(-1)^4} = A(0) + B(0) + 2E \qquad \boxed{E = 15}$$

Figure 4.25 A proof of the need for differentiation for repeated roots

4.6 EXAMPLES

4.6.1 Mass-Spring-Damper Vibration

A mass-spring-damper system is shown in Figure 4.26 with a sinusoidal input.

Given,

$$\therefore \frac{x(s)}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{K_d}{M}s + \frac{K_s}{M}}$$

Component values are,

$$M = 1kg \qquad K_s = 2\frac{N}{m} \qquad K_d = 0.5\frac{Ns}{m}$$

The sinusoidal input is converted to the s-domain,

$$F(t) = 5\cos(6t)N$$

$$\therefore F(s) = \frac{5s}{s^2 + 6^2}$$

This can be combined with the transfer function to obtain the output function,

$$x(s) = F(s)\left(\frac{x(s)}{F(s)}\right) = \left(\frac{5s}{s^2 + 6^2}\right)\left(\frac{\frac{1}{1}}{s^2 + 0.5s + 2}\right)$$

$$\therefore x(s) = \frac{5s}{(s^2 + 36)(s^2 + 0.5s + 2)}$$

$$\therefore x(s) = \frac{A}{s + 6j} + \frac{B}{s - 6j} + \frac{C}{s - 0.5 + 1.39j} + \frac{D}{s - 0.5 - 1.39j}$$

Figure 4.26 A mass-spring-damper example

The residues for the partial fraction in Figure 4.26 are calculated and converted to a function of time in Figure 4.27. In this case the roots of the denominator are complex, so the result has a sinusoidal component.

$$A = \lim_{s \to -6j}\left[\frac{(s+6j)(5s)}{(s-6j)(s^2+36)(s^2+0.5s+2)}\right] = \frac{-30j}{(-12j)(36-3j+2)}$$

$$\therefore A = \frac{-30j}{-432j-36-24j} = \frac{30j}{36+456j} \times \frac{36-456j}{36-456j} = \frac{13680+1080j}{209,232} = 0.0654 + 0.00516j$$

--Continue on to find B, C, D same way

$$\therefore x(s) = \frac{0.0654+0.00516j}{s+6j} + \frac{0.0654-0.00516j}{s-6j} + \dots$$

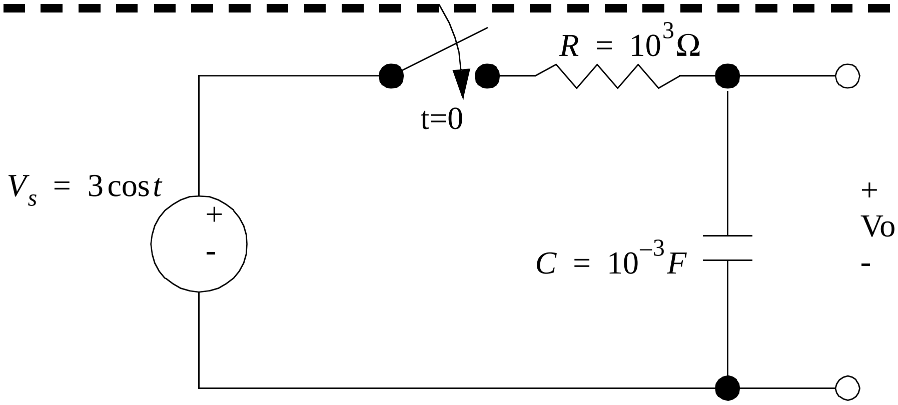
Do inverse Laplace transform

$$x(t) = 2\sqrt{0.0654^2 + 0.00516^2}e^{-0t}\cos\left(0.00516t + \operatorname{atan}\left(-\frac{0.00516}{0.0654}\right)\right) + \dots$$

Figure 4.27 A mass-spring-damper example (continued)

4.6.2 Circuits

It is not necessary to develop a transfer functions for a system. The equation for the voltage divider is shown in Figure 4.28. Impedance values and the input voltage are converted to the s-domain and written in the equation. The resulting output function is manipulated into partial fraction form and the residues calculated. An inverse Laplace transform is used to convert the equation into a function of time using the tables.



As normal, relate the source voltage to the output voltage using component values in the s-domain.

$$V_o = V_s\left(\frac{Z_C}{Z_R + Z_C}\right) \qquad V_s(s) = \frac{3s^2}{s^2 + 1} \qquad Z_R = R \qquad Z_C = \frac{1}{sC}$$

Next, equations are combined. The numerator of resulting output function must be reduced by long division.

$$V_o = \frac{3s^2}{s^2 + 1}\left(\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right) = \frac{3s^2}{(s^2 + 1)(1 + sRC)} = \frac{3s^2}{(s^2 + 1)(s10^3 10^{-3} + 1)}$$

The output function can be converted to a partial fraction form and the residues calculated.

$$V_o = \frac{3s^2}{(s^2 + 1)(s + 1)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 1} = \frac{As^2 + As + Bs + B + Cs^2 + C}{(s^2 + 1)(s + 1)}$$

$$V_o = \frac{3s^2}{(s^2 + 1)(s + 1)} = \frac{s^2(A + C) + s(A + B) + (B + C)}{(s^2 + 1)(s + 1)}$$

$$B + C = 0 \qquad \therefore C = -B$$

$$A + B = 0 \qquad \therefore A = -B$$

$$A + C = 3 \qquad \therefore -B - B = 3 \qquad \therefore B = -1.5 \qquad \therefore A = 1.5 \qquad \therefore C = 1.5$$

$$V_o = \frac{1.5s - 1.5}{s^2 + 1} + \frac{1.5}{s + 1}$$

Figure 4.28 A circuit example

The output function can be converted to a function of time using the transform tables, as shown below.

$$V_o(t) = L^{-1}[V_o(s)] = L^{-1}\left[\frac{1.5s - 1.5}{s^2 + 1} + \frac{1.5}{s + 1}\right] = L^{-1}\left[\frac{1.5s - 1.5}{s^2 + 1}\right] + L^{-1}\left[\frac{1.5}{s + 1}\right]$$

$$\therefore V_o(t) = 1.5L^{-1}\left[\frac{s}{s^2 + 1}\right] - 1.5L^{-1}\left[\frac{1}{s^2 + 1}\right] + 1.5e^{-t}$$

$$\therefore V_o(t) = 1.5\cos t - 1.5\sin t + 1.5e^{-t}$$

$$\therefore V_o(t) = \sqrt{1.5^2 + 1.5^2}\cos\left(t + \operatorname{atan}\left(\frac{-1.5}{1.5}\right)\right) + 1.5e^{-t}$$

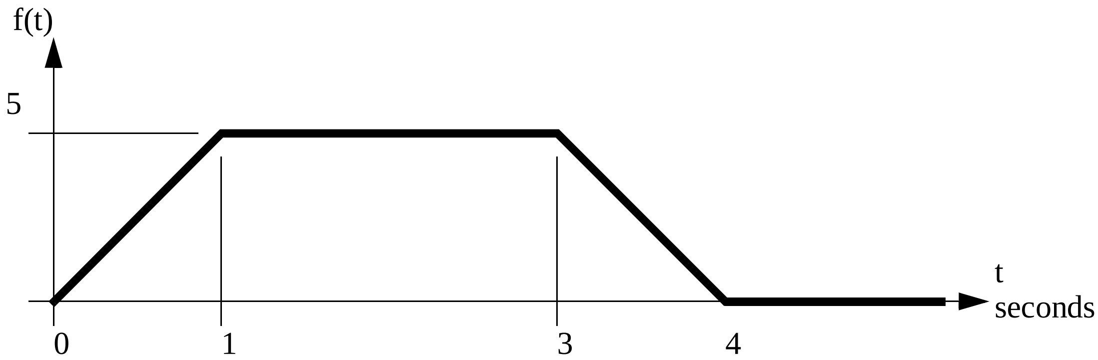
$$\therefore V_o(t) = 2.121\cos\left(t - \frac{\pi}{4}\right) + 1.5e^{-t}$$

Figure 4.29 A circuit example (continued)

4.7 ADVANCED TOPICS

4.7.1 Input Functions

In some cases a system input function is comprised of many different functions, as shown in Figure 4.30. The step function can be used to switch function on and off to create a piecewise function. This is easily converted to the s-domain using the e-to-the-s functions.



$$f(t) = 5tu(t) - 5(t-1)u(t-1) - 5(t-3)u(t-3) + 5(t-4)u(t-4)$$

$$f(s) = \frac{5}{s^2} - \frac{5e^{-s}}{s^2} - \frac{5e^{-3s}}{s^2} + \frac{5e^{-4s}}{s^2}$$

Figure 4.30 Switching on and off function parts

4.7.2 Initial and Final Value Theorems

The initial and final values an output function can be calculated using the theorems shown in Figure 4.31.

$$\boxed{x(t \to \infty) = \lim_{s \to 0} [sx(s)]}$$ Final value theorem

$$\therefore x(t \to \infty) = \lim_{s \to 0}\left[\frac{1s}{(s^2 + 3s + 2)s}\right] = \lim_{s \to 0}\left[\frac{1}{s^2 + 3s + 2}\right] = \frac{1}{(0)^2 + 3(0) + 2} = \frac{1}{2}$$

$$\boxed{x(t \to 0) = \lim_{s \to \infty} [sx(s)]}$$ Initial value theorem

$$\therefore x(t \to 0) = \lim_{s \to \infty}\left[\frac{1(s)}{(s^2 + 3s + 2)s}\right] = \frac{1}{((\infty)^2 + 3(\infty) + 2)} = \frac{1}{\infty} = 0$$

Figure 4.31 Final and initial values theorems

4.7.3 Impulse Response

• If we look at an input signal (force here) we can break it into very small segments in time. As the time becomes small we call it an impulse function.

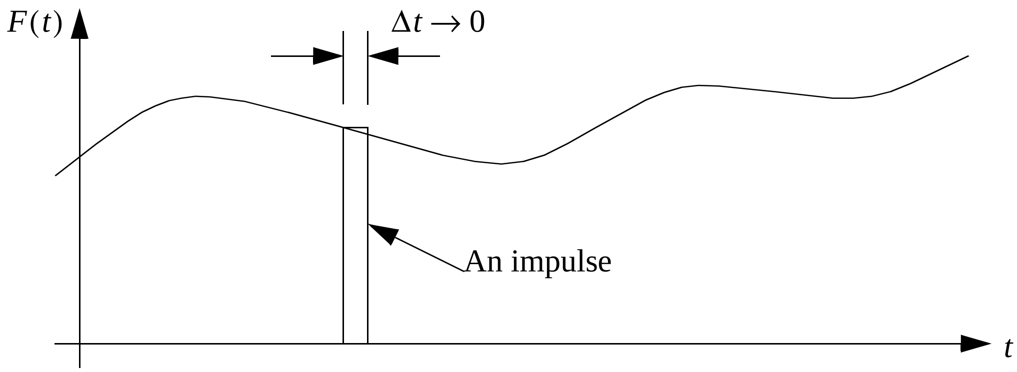


Figure 4.32 An impulse as a brief duration pulse

• If we put an impulse into a system the output will be an impulse response.

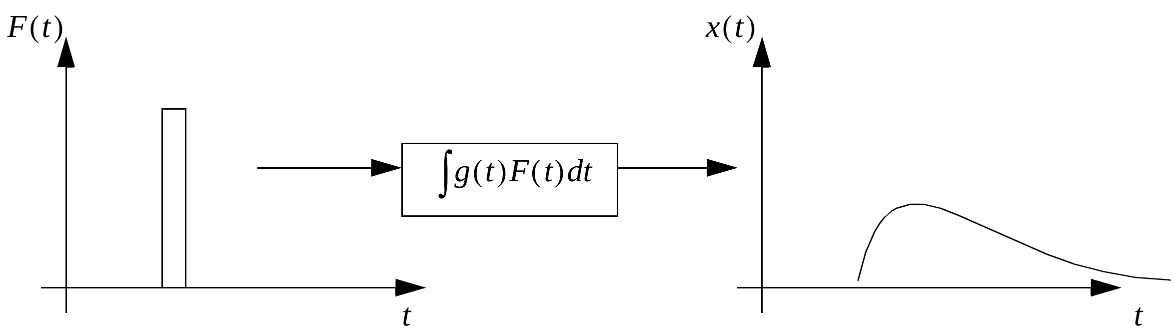
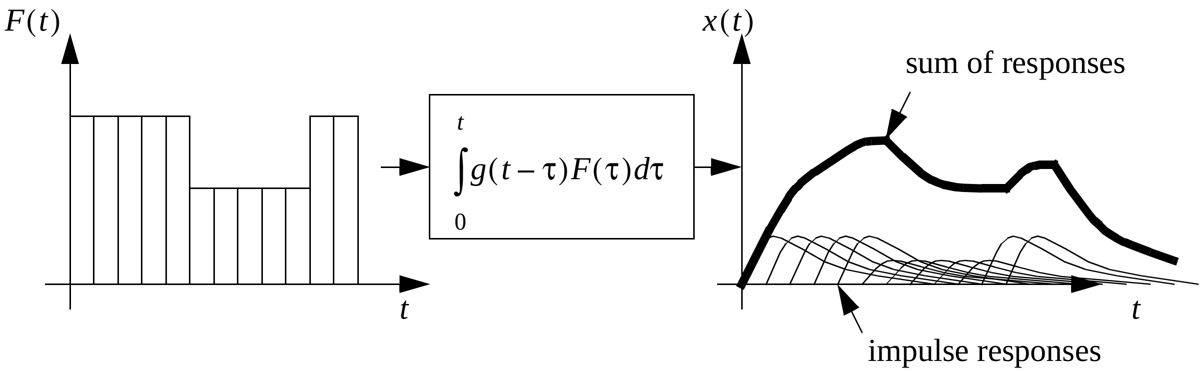


Figure 4.33 Response of the system to a single pulse

• If we add all of the impulse responses together we will get a total system response. This operation is called convolution.



The convolution integral

$$c(t) = \int_0^t g(t-\tau)r(\tau)d\tau$$

Figure 4.34 A set of pulses for a system gives summed responses to give the output

• The convolution integral can be difficult to deal with because of the time shift. But, the Laplace transform for the convolution integral turns it into a simple multiplication.

$$c(t) = \int_0^t g(t-\tau)r(\tau)d\tau$$

$$C(s) = G(s)R(s)$$

Figure 4.35 The convolution integral

4.8 A MAP OF TECHNIQUES FOR LAPLACE ANALYSIS

• The following map is to be used to organize the various topics covered in the course.

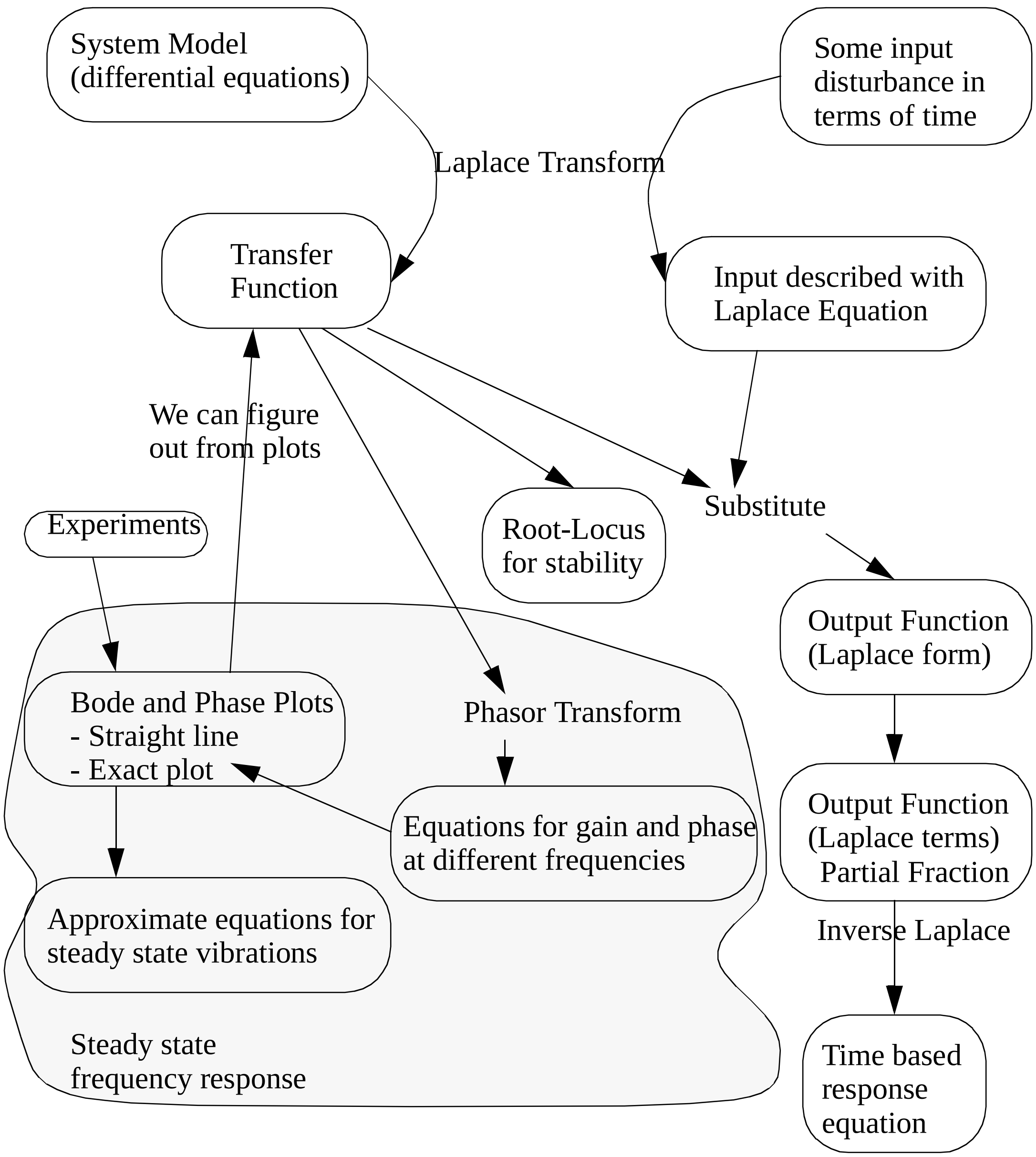


Figure 4.36 A map of Laplace analysis techniques

4.9 SUMMARY

- Transfer and input functions can be converted to the s-domain
- Output functions can be calculated using input and transfer functions
- Output functions can be converted back to the time domain using partial fractions.

4.10 PRACTICE PROBLEMS

1. Convert the following functions from time to laplace functions.

a) $L[5]$

b) $L[e^{-3t}]$

c) $L[5e^{-3t}]$

d) $L[5te^{-3t}]$

e) $L[5t]$

f) $L[4t^2]$

g) $L[\cos(5t)]$

h) $L[\cos(5t+1)]$

i) $L[5e^{-3t}\cos(5t)]$

j) $L[5e^{-3t}\cos(5t+1)]$

k) $L[\sin(5t)]$

l) $L[\sinh(3t)]$

m) $L[t^2\sin(2t)]$

n) $L\left[\frac{d}{dt}t^2e^{-3t}\right]$

o) $L[\ddot{x}+5\dot{x}+3x], \dot{x}(0) = 8, x(0) = 7$

p) $L\left[\frac{d}{dt}\sin(6t)\right]$

q) $L\left[\left(\frac{d}{dt}\right)^3 t^2\right]$

r) $L\left[\int_0^t y dt\right]$

s) $L[3t^3(t-1)+e^{-5t}]$

t) $L[u(t-1)-u(t-2)]$

u) $L[e^{-2t}u(t-2)]$

v) $L[e^{-(t-3)}u(t-1)]$

w) $L[5e^{-3t}+u(t-1)-u(t-2)]$

x) $L[\cos(7t+2)+e^{t-3}]$

y) $L[3(t-1)+e^{-(t+1)}]$

z) $L[6e^{-2.7t}\cos(9.2t+3)]$

aa)

2. Convert the following functions below from the laplace to time domains.

a) $L^{-1}\left[\frac{1}{s+1}\right]$

b) $L^{-1}\left[\frac{5}{s+1}\right]$

c) $L^{-1}\left[\frac{6}{s^2}\right]$

d) $L^{-1}\left[\frac{6}{s^3}\right]$

e) $L^{-1}\left[\frac{s+2}{(s+3)(s+4)}\right]$

f) $L^{-1}\left[\frac{6}{s^2+5s+6}\right]$

g) $L^{-1}\left[\frac{6}{4s^2+20s+24}\right]$

h) $L^{-1}\left[\frac{6}{s^2+6}\right]$

i) $L^{-1}\left[\frac{5}{s}(1-e^{-4.5s})\right]$

j) $L^{-1}\left[\frac{4+3j}{s+1-2j}+\frac{4-3j}{s+1+2j}\right]$

k) $L^{-1}\left[\frac{6}{s^4}+\frac{6}{s^2+9}\right]$

3. Convert the following functions below from the laplace to time domains using partial fractions.

a) $L^{-1}\left[\frac{s+2}{(s+3)(s+4)}\right]$

b) $L^{-1}[\ ]$

c) $L^{-1}[\ ]$

d) $L^{-1}[\ ]$

e) $L^{-1}\left[\frac{6}{s^2+5s}\right]$

f) $L^{-1}\left[\frac{9s^2+6s+3}{s^3+5s^2+4s+6}\right]$

g) $L^{-1}\left[\frac{s^3+9s^2+6s+3}{s^3+5s^2+4s+6}\right]$

h) $L^{-1}\left[\frac{9s+4}{(s+3)^3}\right]$

i) $L^{-1}\left[\frac{9s+4}{s^3(s+3)^3}\right]$

j) $L^{-1}\left[\frac{s^2+2s+1}{s^2+3s+2}\right]$

k) $L^{-1}\left[\frac{s^2+3s+5}{6s^2+6}\right]$

l) $L^{-1}\left[\frac{s^2+2s+3}{s^2+2s+1}\right]$

4. Convert the following differential equations to transfer functions.

a) $5x'' + 6x' + 2x = 5F$

b) $y' + 8y = 3x$

c) $y' - y + 5x = 0$

5. Given the following input functions and transfer functions, find the response in time.

	Transfer Function	Input
a)	$\frac{x(s)}{F(s)} = \frac{s+2}{(s+3)(s+4)}\left(\frac{m}{N}\right)$	$F(t) = 5N$
b)	$\frac{x(s)}{F(s)} = \frac{s+2}{(s+3)(s+4)}\left(\frac{m}{N}\right)$	$x(t) = 5m$

6. Do the following conversions as indicated.

a) $L[5e^{-4t}\cos(3t+2)] =$

b) $L[e^{-2t} + 5t(u(t-2) - u(t))] =$

c) $L\left[\left(\frac{d}{dt}\right)^3 y + 2\left(\frac{d}{dt}\right)y + y\right] =$ where at t=0 $y_0 = 1$

$y_0' = 2$

$y_0'' = 3$

$y_0''' = 4$

d) $L^{-1}\left[\frac{1+j}{s+3+4j} + \frac{1-j}{s+3-4j}\right] =$

e) $L^{-1}\left[s + \frac{1}{s+2} + \frac{3}{s^2+4s+40}\right] =$

7. Convert the output function to functions of time.

a) $\frac{s^3 + 4s^2 + 4s + 4}{s^3 + 4s}$

b) $\frac{s^2 + 4}{s^4 + 10s^3 + 35s^2 + 50s + 24}$

8. Solve the differential equation using Laplace transforms. Assume the system starts undeflected and at rest.

$$\dddot{\theta} + 40\ddot{\theta} + 20\dot{\theta} + 2\theta = 4$$

4.11 PRACTICE PROBLEM SOLUTIONS

1.

a) $\frac{5}{s}$

b) $\frac{1}{s+3}$

c) $\frac{5}{s+3}$

d) $\frac{5}{(s+3)^2}$

e) $\frac{5}{s^2}$

f) $\frac{8}{s^3}$

g) $\frac{s}{s^2+25}$

h) $\frac{s\cos 1 - 5\sin 1}{s^2+25}$

i) $\frac{5(s+3)}{(s+3)^2+5^2}$

j) $\frac{2\sqrt{10}\angle 1}{s+3-5j} + \frac{2\sqrt{10}\angle -1}{s+3+5j}$

k) $\frac{5}{s^2+25}$

l) $\frac{0.5}{s-3} - \frac{0.5}{s+3}$

m) $\frac{-4}{(s^2+4)^2} - \frac{16s^2}{(s^2+4)^3}$

n) $\frac{2s}{(s+3)^3}$

o) $(s^2x - 7s - 8) + 5(sx - 7) + 3x$

p) $\frac{6s}{s^2+36}$

q) 2

r) $\frac{y}{s}$

s) $\frac{72}{s^5} - \frac{18}{s^4} + \frac{1}{s+5}$

t) $\frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

u) e^{-2s-2}

w) $\frac{5}{s+3} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$

x) $\frac{\cos(2)s - \sin(2)7}{s^2+49} + \frac{e^{-3}}{s-1} = \frac{-0.416s - 6.37}{s^2+49} + \frac{e^{-3}}{s-1}$

y) $\frac{3}{s^2} - \frac{3}{s} + \frac{e^{-1}}{s+1}$

z) $\frac{3\angle 3}{s+2.7-9.2j} + \frac{3\angle -3}{s+2.7+9.2j}$

aa)

2.

a) e^{-t}

b) $5e^{-t}$

c) $6t$

d) $3t^2$

e) $-e^{-3t} + 2e^{-4t}$

f) $6e^{-2t} - 6e^{-3t}$

g) $0.5e^{-2t} - 0.5e^{-3t}$

h) $\sqrt{6}\sin(\sqrt{6}t)$

i) $5 - 5u(t - 4.5)$

j) $2(5)e^{-(1)t}\cos\left(2t + \operatorname{atan}\left(-\frac{3}{4}\right)\right)$

k) $t^3 + 2\sin(3t)$

3.

a) $-e^{-3t} + 2e^{-4t}$

b)

c)

d)

e) $1.2 - 1.2e^{-5t}$

f)

g)

h)

i)

j) $\delta(t) - e^{-2t}$

k) $\frac{\delta(t)}{6} + 0.834\cos(t + 0.927)$

l) $\delta(t) + 2te^{-t}$

4.

a) $\frac{x}{F} = \frac{5}{5s^2 + 6s + 2}$

b) $\frac{y}{x} = \frac{3}{s + 8}$

c) $\frac{y}{x} = \frac{-5}{s - 1}$

5.

a) $\frac{5}{6} + \frac{5}{3}e^{-3t} - \frac{5}{2}e^{-4t}$

b)

6.

a)

$$L[5e^{-4t}\cos(3t+2)] = L[2|A|e^{-\alpha t}\cos(\beta t+\theta)] \qquad \begin{matrix}\alpha = 4 & \beta = 3\\ |A| = 2.5 & \theta = 2\end{matrix}$$

$$A = 2.5\cos 2 + 2.5j\sin 2 = -1.040 + 2.273j$$

$$\frac{A}{s+\alpha-\beta j} + \frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j} = \frac{-1.040+2.273j}{s+4-3j} + \frac{-1.040-2.273j}{s+4+3j}$$

b)

$$L[e^{-2t} + 5t(u(t-2)-u(t))] = L[e^{-2t}] + L[5tu(t-2)] - L[5tu(t)]$$

$$= \frac{1}{s+2} + 5L[tu(t-2)] - \frac{5}{s^2} = \frac{1}{s+2} + 5L[(t-2)u(t-2) + 2u(t-2)] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + 5L[(t-2)u(t-2)] + 10L[u(t-2)] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + 5e^{-2s}L[t] + 10e^{-2s}L[1] - \frac{5}{s^2}$$

$$= \frac{1}{s+2} + \frac{5e^{-2s}}{s^2} + \frac{10e^{-2s}}{s} - \frac{5}{s^2}$$

c)

$$\left(\frac{d}{dt}\right)^3 y = s^3y + 1s^2 + 2s^1 + 3s^0$$

$$\left(\frac{d}{dt}\right)y = s^1y + s^0 1$$

$$L\left[\left(\frac{d}{dt}\right)^3 y + 2\left(\frac{d}{dt}\right)y + y\right] = (s^3y + 1s^2 + 2s + 3) + (sy + 1) + (y)$$

$$= y(s^3 + s + 1) + (s^2 + 2s + 4)$$

d)

$$L^{-1}\left[\frac{1+j}{s+3+4j} + \frac{1-j}{s+3-4j}\right] = L^{-1}\left[\frac{A}{s+\alpha-\beta j} + \frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}\right]$$

$$|A| = \sqrt{1^2+1^2} = 1.141 \qquad \theta = \operatorname{atan}\left(\frac{-1}{1}\right) = -\frac{\pi}{4} \qquad \alpha = 3 \qquad \beta = 4$$

$$= 2|A|e^{-\alpha t}\cos(\beta t+\theta) = 2.282e^{-3t}\cos\left(4t - \frac{\pi}{4}\right)$$

e)
$$L^{-1}\left[s + \frac{1}{s+2} + \frac{3}{s^2+4s+40}\right] = L[s] + L\left[\frac{1}{s+2}\right] + L\left[\frac{3}{s^2+4s+40}\right]$$
$$= \frac{d}{dt}\delta(t) + e^{-2t} + L\left[\frac{3}{(s+2)^2+36}\right] = \frac{d}{dt}\delta(t) + e^{-2t} + 0.5L\left[\frac{6}{(s+2)^2+36}\right]$$
$$= \frac{d}{dt}\delta(t) + e^{-2t} + 0.5e^{-2t}\sin(6t)$$

7.

a)
$$\frac{s^3+4s^2+4s+4}{s^3+4s}$$

$$s^3+4s \,\overline{\big)\, s^3+4s^2+4s+4}^{\;1}$$
$$\underline{-(s^3+4s)}$$
$$4s^2+4$$

$$= 1 + \frac{4s^2+4}{s^3+4s} = 1 + \frac{A}{s} + \frac{Bs+C}{s^2+4} = 1 + \frac{s^2(A+B)+s(C)+(4A)}{s^3+4s}$$

$A = 1$
$C = 0$
$B = 3$

$$= 1 + \frac{1}{s} + \frac{3s}{s^2+4}$$

$$= \delta(t) + 1 + 3\cos(2t)$$

b)
$$\frac{s^2+4}{s^4+10s^3+35s^2+50s+24} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} + \frac{D}{s+4}$$

$$A = \lim_{s\to-1}\left(\frac{s^2+4}{(s+2)(s+3)(s+4)}\right) = \frac{5}{6}$$

$$B = \lim_{s\to-2}\left(\frac{s^2+4}{(s+1)(s+3)(s+4)}\right) = \frac{8}{-2}$$

$$C = \lim_{s\to-3}\left(\frac{s^2+4}{(s+1)(s+2)(s+4)}\right) = \frac{13}{2}$$

$$D = \lim_{s\to-4}\left(\frac{s^2+4}{(s+1)(s+2)(s+3)}\right) = \frac{20}{-6}$$

$$\frac{5}{6}e^{-t} - 4e^{-2t} + \frac{13}{2}e^{-3t} - \frac{10}{3}e^{-4t}$$

8.

$$\theta(t) = -66 \cdot 10^{-6} e^{-39.50t} - 3.216e^{0.1383t} + 1.216e^{-0.3368t} + 2.00$$

4.12 ASSIGNMENT PROBLEMS

1. Convert the output function below Y(s) to the time domain Y(t).

$$Y(s) = \frac{5}{s} + \frac{12}{s^2 + 4} + \frac{3}{s + 2 - 3j} + \frac{3}{s + 2 + 3j}$$

2. Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

$$G(s) = \frac{4}{s + 2} = \frac{Y(s)}{X(s)} \qquad X(t) = 20 \quad \text{When t >= 0}$$

3. Prove the following relationships.

a) $L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

b) $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

c) $L[e^{-at}f(t)] = F(s + a)$

d) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

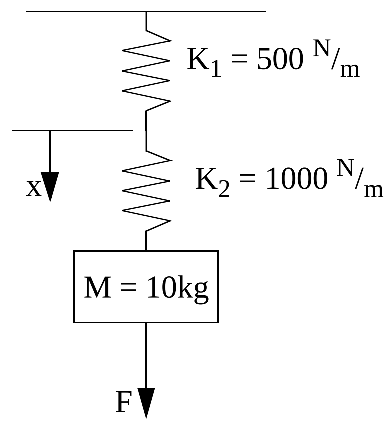
e) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

f) $L[tf(t)] = -\frac{d}{dt}F(s)$

4. Given a mass supported by a spring and damper, find the displacement of the supported mass over time if it is released from neutral at t=0sec, and gravity pulls it downward.
   a) find the transfer function x/F in the s-domain.
   b) find the input function F in the time and s-domains.
   c) find the position output as a function of ‘s’.
   d) do the inverse Laplace transform to find the position as a function of time for Ks = 10N/m, Kd = 5Ns/m, M=10kg.

5. The applied force ‘F’ is the input to the system, and the output is the displacement ‘x’.

a) find the transfer function.



b) What is the steady state response for an applied force F(t) = 10cos(t + 1) N ?
c) Give the transfer function if ‘x’ is the input.
d) Find x(t), given F(t) = 10N for t >= 0 seconds using Laplace methods.

6. The following differential equation is supplied, with initial conditions.

$$y'' + y' + 7y = F \qquad y(0) = 1 \qquad y'(0) = 0$$

$$F(t) = 10 \qquad t > 0$$

a) Solve the differential equation using calculus techniques.
b) Write the equation in state variable form and solve it numerically.
c) Find the frequency response (gain and phase) for the transfer function using the phasor transform. Roughly sketch the bode plots.
d) Convert the differential equation to the Laplace domain, including initial conditions. Solve to find the time response.

4.13 REFERENCES

Irwin, J.D., and Graf, E.R., Industrial Noise and Vibration Control, Prentice Hall Publishers, 1979.

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