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Feedback Control of a Nonholonomic Wheeled Cart in Cartesian Space

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Abstract

A preliminary study of the problem, thus far little treated in existing Robotics literature, of mobile robots' feedback control is presented. The robot considered here is a two-wheel driven nonholonomic cart. Despite the controllability of the system, pure state feedback stabilization of the cart's configuration around a given terminal configuration is not possible. However, feedback stabilization of the position of any cart's point remains possible. Extension to the problem of trajectory tracking in Cartesian space is then considered, and we show that stabilization of the cart's configuration around the configuration of a virtual reference cart becomes possible as long as the reference cart keeps moving. Several simple control laws are proposed and illustrating simulation results are given. Connections with the path planning problem are pointed out.

1 Introduction

In the last years the interest of the Robotics community for the mobile robots' path planning problem has grown rapidly in relation to the development of artificial intelligence and computational geometry. Let us cite [1]-[6] among many other contributions.

The connections of path planning with control theory, and more precisely with open-loop control aspects, have been pointed out in several studies [4]-[6]. On the other hand, and unlike what can be observed from the literature on robot manipulators, feedback control issues seem to have motivated very few studies, and conference communications like [7] and [8] appear to be exceptions which confirm the rule. Although the control requirements for mobile robots and robot manipulators are usually not of the same nature, we feel that more attention could be paid to the feedback control aspects, considering their importance in practice. For instance, the planning of trajectories is only of use when low level control loops are present to ensure good tracking and local recovery from deviations provoked by unmodeled perturbations. This automatic recovery feature is essential to avoid having to reconsider the control objectives permanently, which may not be very effective, and to lighten the task of higher planning levels. It is thus important to have a clear idea of the possibilities offered by simple feedback loops so as to be able to fully exploit them. This article may be viewed as a preliminary study where these issues are formulated and treated with the automatic control point of view.

2 Cart's kinematic equations

To simplify, we consider that the cart schematized in **Fig.1** is made of three rigid bodies (the cart's platform and the two driving wheels) and that it moves on a horizontal ground. In order to derive a kinematic model of the cart, we make the rolling-without-slipping assumption according to which the contact point of each wheel with the ground has zero velocity. Also, in order to avoid the complications that would result from having to introduce the cart's dynamic equations, we will consider that the wheels' angular velocities $\dot{q}_1$ and $\dot{q}_2$ can be taken as control variables.

From this point on, we will use a parametrization of the system, obtained by expressing the cart's position coordinates in the basis of the frame F_M linked to the cart (see **Fig.2**) rather than in the basis of the fixed frame F_0.

More precisely, let:

- N denote a point linked to the cart, located at a distance $|d|$ from the wheels' axis and such that:

$$\vec{MN} = d\vec{i_M} \tag{1}$$

- x and y the coordinates of the vector $\vec{NO}$ in the basis of the mobile frame F_M
- θ the angle which characterizes the cart's orientation with respect to the frame F_0
- the configuration vector:

$$X = [q_1, q_2, x, y, \theta]^T \tag{2}$$

- the auxiliary control vector:

$$U = [v, \dot{\theta}]^T \tag{3}$$

  where v is the cart's **translational velocity** along the $\vec{i_M}$ axis.

The control vector U is related to the wheel velocities by:

$$U = D\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \quad ; \quad D = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ \frac{r}{2R} & -\frac{r}{2R} \end{bmatrix} \tag{4}$$

Since the matrix D is nonsingular it is equivalent to work with the control variables $(v, \dot{\theta})$ or $(\dot{q}_1, \dot{q}_2)$.

We then obtain the following state equation, the derivation of which is easy and left to the interested reader (see [14] also):

$$\dot{X} = B(X)U \tag{5}$$

with:

$$B(X) = \begin{bmatrix} & D^{-1} & \\ -1 & & y \\ 0 & & -(d+x) \\ 0 & & 1 \end{bmatrix} \tag{6}$$

Since the variables θ, q_1 and q_2 satisfy the following relation:

$$\theta = \frac{r}{2R}(q_1 - q_2) + constant \tag{7}$$

they are not independent and a minimal state representation is obtained by dropping, for example, the first or the last equation of the system 5.

The cart's **nonholonomy** appears through the fact that the two nonlinear equations involving the velocities $\dot{x}$ and $\dot{y}$ are not completely integrable.

3 Controllability and state-feedback stabilization

From nonlinear control theory, a system such as 5 has the property of being *controllable* when any state X can be reached under the action of piecewise constant inputs $U(t)$ [9,10,11].

When considering the reduced state vector $X^T = (x, y, \theta)$, it is rather straightforward to show that the wheeled cart modeled by equation 5 is, in the previous sense, controllable in both position and orientation [6,14].

However, in the case of nonlinear systems, it does matter to differentiate the notion of controllability from the notion of feedback stabilization on which the remainder of this article will focus.

Indeed, **an important difference with the (invariant) linear case is that the controllability of a nonlinear system does not imply the existence of a stabilizing feedback control, i.e. a control law $U(X)$ (smooth function of the state X) able to make the state converge to 0 whatever the initial state.** While the notions of controllability and feedback stabilizability are strongly related in the linear case, this is no longer true in the nonlinear case. Controllability is necessary to feedback stabilizability, but the reverse does not hold.

On the other hand, the advantages that can be expected from using a stabilizing feedback control are well known in linear control theory: insensitivity with respect to initial conditions (since the control expression does not explicitly depend on the initial conditions), robustness with respect to perturbations acting on the system and with respect to modeling errors... For these reasons, all controlled physical systems are equipped with feedback control loops. For a large part, control design consists of evaluating the possibilities of both open loop control (much related to trajectory planification in the case of mobile robots) and feedback control so as to find the right balance when mixing those two components.

In the next sections the study focusses on the feedback control aspects of the problem.

4 Feedback stabilization

4.1 Existence of stabilizing feedbacks

Let us consider the following reduced state vector:

$$X = [x, y, \theta]^T \tag{8}$$

The system's equations may still be written in the form

$$\dot{X} = B(X)U \tag{9}$$

with, this time:

$$B(X) = \begin{bmatrix} -1 & y \\ 0 & -(d+x) \\ 0 & 1 \end{bmatrix} \tag{10}$$

We would like to know if there exists a state dependent differentiable control $U(X)$ able to drive X to 0.

A classical result of control theory is that controllability of the **pseudo-linearized (p.l.)** system associated with the nonlinear system 9 at the stationary point ($X_f = 0, U = 0$) is **sufficient** to ensure the existence of a locally stabilizing feedback. When this condition is satisfied, any feedback of the form $U = KX$ which stabilizes the p.l. system also stabilizes (at least locally) the nonlinear system. Unfortunately, in the case of the wheeled cart, the p.l. system is not controllable because the matrix $B(0)$ is not square. We are thus confronted to a case where linearization results in a loss of controllability: although the nonlinear system is controllable, the corresponding p.l. system is not. This is in fact a consequence of the cart's nonholonomy, as explained in [14].

Another result is that if the p.l. system is not stabilizable, i.e. when its non-controllable modes are unstable, then there **does not exist** a stabilizing feedback for the nonlinear system. But, again, this result does not apply to the cart because the non-controllable modes of the p.l. system are marginally stable (poles on the imaginary axis).

Therefore, the only cases for which it is not possible to conclude immediately, as for the existence of stabilizing feedbacks, are the **critical** cases for which the p.l. system is neither controllable, nor strictly unstable in closed loop. Obviously, all nonlinear systems of the form 9 are critical cases whenever the $(m \times n)$ matrix $B(X)$ is rectangular with $n < m$.

However, a theorem proposed by Brockett [12] states that a condition **necessary** to the existence of stabilizing feedbacks is that the function $f(X, U) = B(X)U$ must be onto an open set of $\mathbb{R}^m$ containing 0. But it is readily verifiable that this is not the case here. Since this condition is not satisfied, **there does not exist a stabilizing C^1 feedback control $U(X)$ for the two-wheel driven cart.**

4.2 Stabilization of a cart's point

Active stabilization of the cart's orientation may not be necessary in all cases and we may now try to determine whether there exists a feedback capable of stabilizing the cart's position only, independently of the orientation's behaviour.

To do so, we are led to consider the reduced system:

$$\dot{X} = B(X)U \tag{11}$$

with:

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad , \quad B(X) = \begin{bmatrix} -1 & y \\ 0 & -(d+x) \end{bmatrix} \tag{12}$$

Let us distinguish two cases, according to whether the cart's point to be controlled is located on the wheels' axis or not.

case 1: $d = 0$

In this case, one easily verifies that the corresponding p.l. system is still not controllable. This lack of controllability results simply because the cart cannot move along the wheels' axis instantaneously. We are thus again confronted with the case of a critical system. However, Brockett's theorem no longer allows us to conclude the non-existence of a stabilizing feedback because the function $f(X, U) = B(X)U$ is onto an open set of $\mathbb{R}^2$ containing 0, in this case.

In fact, a conclusion can be reached because it is simple to show, by using an adequate Lyapunov function and Lasalle's theorem ([13], p. 58), that the following control:

$$U = \begin{bmatrix} v \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} k_1 x \\ k_2 y \end{bmatrix} \quad : k_1 > 0, \; k_2 \neq 0 \tag{13}$$

is globally stabilizing [14].

Remarks:

- The control gains k_1 and k_2 do not have to be constant.
- When these gains are chosen constant, the convergence of X to zero is not exponential in general (a consequence of the non-controllability of the p.l. system).

A simulation of the motion of the cart controlled by 13 is represented in **Fig.3**. This simulation shows how the cart's trajectory wraps around the origin and suggests the non-convergence of the angle θ in the general case. For this reason, the control 13 is probably of limited practical interest. Nevertheless, the study of this case remains conceptually interesting because it pointed out an asymptotical convergence result which could not have been obtained by working on the p.l. system only.

case 2: $d \neq 0$

In this case we try to control the cart's point N located at a distance $|d|$ from the wheels' axis.

This time the p.l. system is controllable because the control matrix $B(0)$ is now nonsingular. Moreover, any stabilizing linear feedback for the p.l. system is also **exponentially** stabilizing, at least locally (i.e. when the point N is close to the point O initially), for the nonlinear system.

It is easy to verify that such a control is, for example:

$$U = \begin{bmatrix} k_1 x \\ sign(d) k_2 y \end{bmatrix} \quad ; k_1 > 0, \; k_2 > 0 \tag{14}$$

It is in fact possible to show that this control is globally exponentially stabilizing [14]: whatever the initial cart's position, $\|X\|$ converges exponentially to 0 with a rate at least equal to $inf(k_1, k_2|d|)$. Moreover, the exponential convergence of y to 0 induces, from the control expression, the one of $\dot{\theta}$ and thus the convergence of the angle θ to some limit value.

A simulation of this control is represented in **Fig.4** which is to be compared to **Fig.3** so as to visualize the interest of choosing $d \neq 0$.

Of course, the control 14 is not the only one to possess stabilizing properties. Other possibilities, and generalization to the tracking case where the point O moves on a preplanned trajectory, are treated in [14].

5 Trajectory tracking

We now consider the problem of regulating both the cart's position and orientation around the position/orientation of an ideal "virtual" **reference cart**, the trajectory of which is predetermined and parametrized by the variable t. When the reference cart is motionless, this problem falls back on the one treated in section 4.1 where we saw that there did not exist a stabilizing feedback in this case. We are going to see that the mere fact of making the reference cart move leads to somewhat different results.

5.1 Equations of the system

The controlled cart's point, distant of $|d|$ from the wheels' axis, is denoted as before as N. The corresponding point on the reference cart is denoted as N_r. The coordinates of the vector $\vec{NN_r}$ in the basis of the mobile frame F_M tied to the controlled cart are x and y. We also define the orientation error:

$$\tilde{\theta} = \theta - \theta_r \tag{15}$$

where θ and θ_r characterize the controlled cart's orientation and the reference cart's orientation respectively.

The state vector that we wish to regulate to zero now is:

$$X = [x, y, \tilde{\theta}]^T \tag{16}$$

Let $v_r(t)$ denote the reference cart's translational velocity. In this case the cart's equations are [14]:

$$\dot{X} = A(X, t)X + B(X)U \tag{17}$$

with:

$$A(X,t) = \begin{bmatrix} 0 & \dot{\theta}_r(t) & h_1(\tilde{\theta},t) \\ -\dot{\theta}_r(t) & 0 & h_2(\tilde{\theta},t) \\ 0 & 0 & 0 \end{bmatrix} \quad (18)$$

$$B(X) = \begin{bmatrix} -1 & y \\ 0 & -(d+x) \\ 0 & 1 \end{bmatrix} \quad (19)$$

$$U = \begin{bmatrix} \tilde{v} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} v - v_r \\ \dot{\theta} - \dot{\theta}_r \end{bmatrix} \quad (20)$$

$$h_1(\tilde{\theta},t) = v_r(t)\frac{\cos\tilde{\theta} - 1}{\tilde{\theta}} + d\dot{\theta}_r(t)\frac{\sin\tilde{\theta}}{\tilde{\theta}} \quad (21)$$

$$h_2(\tilde{\theta},t) = d\dot{\theta}_r(t)\frac{\cos\tilde{\theta} - 1}{\tilde{\theta}} - v_r(t)\frac{\sin\tilde{\theta}}{\tilde{\theta}} \quad (22)$$

Comparison of equations 9 and 17 shows that the reference cart tracking problem differs from the initial regulation problem by the fact that the system's state matrix is no longer null.

5.2 Control

5.2.1 Control design based on the p.l. system

The equation of the p.l. system around the ideal configuration ($X = 0$, $U = 0$) is:

$$\dot{X} = A(0,t)X + B(0)U \quad (23)$$

Let us consider the case where $\dot{\theta}_r$ and v_r are constant. We then obtain a linear invariant system and easily verify that the associated controllability matrix is full rank if and only if $\dot{\theta}_r$ and v_r are not both equal to zero.

As pointed out in section 4.1, the p.l. system is not controllable when the reference cart is motionless ($v_r = \dot{\theta}_r = 0$). However, the important new fact is that this system becomes controllable as soon as the reference cart starts moving. It is also worth noticing that this is true whatever the value of d, i.e whatever the choice of the controlled point N on the cart.

Let us now assume that either v_r or $\dot{\theta}_r$ is different from zero. Since the p.l. system is controllable in this case, we know that any linear control:

$$U = KX \quad (24)$$

with a gain matrix K chosen so that the closed-loop system's state matrix $A(0) + B(0)K$ is "stable" (eigenvalues in the left half complex plane), stabilizes the nonlinear system 17 locally.

Any classical linear control technique may then be applied to calculate the matrix K: pole placement, quadratic criterion minimization, etc.

For example, when $d = 0$, $v_r = constant$ and $\dot{\theta}_r = 0$, the following control:

$$\begin{cases} \tilde{v} = x \\ \dot{\tilde{\theta}} = sign(v_r)y - \sqrt{1 + 2|v_r|}\tilde{\theta} \end{cases} \quad (25)$$

minimizes the quadratic cost: $J = \int_0^{+\infty}(X^TX + U^TU)dt$ (proof in [14]).

A simulation of this control in **Fig.5** shows the convergence of the controlled cart towards the reference cart when the configurations of the two carts do not coincide initially.

5.2.2 A set of nonlinear controls

In order to be able to prove more general stability properties, we define the following set of nonlinear controls:

$$\begin{cases} \dot{\tilde{\theta}} = h_\theta(X,t) = -\frac{k_3}{k_2}\tilde{\theta} - \frac{k_1}{k_2}h_2(y + d\tilde{\theta}) + \frac{k_6}{k_2}x \\ \tilde{v} = k_3k_5x + (2k_3k_4 + h_1 + k_6)\tilde{\theta} \\ \quad +[(1-k_1)y - k_1d\tilde{\theta}](h_\theta(X,t) + \dot{\theta}_r) \end{cases} \quad (26)$$

where:

- k_1 and k_2 are positive real numbers;
- k_3, k_4, k_5 are positive scalar gains that will be assumed constant to simplify, although this is not a necessary condition. The only required condition is: $0 \leq {k_4}^2 < k_5$
- k_6 is any real scalar gain.

In order to point out the fact that the invariance of $\dot{\theta}_r$ and v_r is not necessary to the convergence of X to zero, we propose the following result, the proof of which is given in [14]:

Proposition (sufficient conditions for the convergence of X to zero):

If :

- *the functions $v_r(t)$ and $\dot{\theta}_r(t)$ are differentiable ($\forall t \geq 0$)*
- $|\dot{v}_r(t)| < K_v < +\infty \qquad (\forall t \geq 0)$
- $|\ddot{\theta}_r(t)| < K_\theta < +\infty \qquad (\forall t \geq 0)$
- *either $v_r(t)$ or $\dot{\theta}_r(t)$ does not converge to zero*

then:

$\lim_{t\to+\infty} X(t) = 0$

For practical purposes, the important fact pointed out by this proposition is the **convergence of the controlled cart towards the reference cart whenever the latter keeps moving**. This property finds applications not only in the trajectory tracking problem as such, but also in the initial problem of regulation around a given terminal configuration. Indeed, it suggests a certain number of strategies for the choice of the reference cart's trajectory in order to solve this problem in practice. For example, one can imagine having the reference cart perform a periodic translational motion centered on the configuration to be reached and maintain this motion until the configuration error becomes smaller than a certain threshold determined in advance. This possibility is illustrated by the simulation represented in **Fig.6** where the controlled cart gives the impression that it performs a parking manoeuvre. Notice that this manoeuvre does not rely on an elaborate planification of the reference cart's trajectory.

The control 26 used in this simulation was obtained by making the following choices:

$$\begin{cases} d = 0 \\ k_1 = 1, \ k_4 = 0, \ k_6 = 0 \end{cases} \tag{27}$$

One easily verifies that this control may also be written:

$$\begin{cases} v = v_r \cos\tilde{\theta} + g_1 x & ; \ g_1 > 0 \\ \dot{\theta} = \dot{\theta}_r - g_3\tilde{\theta} + g_2 v_r \frac{\sin\tilde{\theta}}{\tilde{\theta}} y & ; \ g_2 > 0, \ g_3 > 0 \end{cases} \tag{28}$$

In **Fig.7**, the nonlinear control 28 is again simulated with $v_r = constant$ and $\dot{\theta}_r = 0$ in this case. The comparison of **Fig.7** and **Fig.5** shows how the convergence towards the reference cart is modified by using the nonlinear control 28 instead of a linear version, when the initial orientation error is not small.

It may also be noticed that the nonlinear control 28 is always defined, while the linear optimal control 25 is not defined when the reference cart is motionless ($v_r = \dot{\theta}_r = 0$), i.e. when the p.l. system is not controllable.

6 Concluding comments

In this study, a few feedback control principles for nonholonomic mobile robots were pointed out and applied to the two-wheel driven cart.

We saw that despite the weak controllability (or accessibility) of the cart, there was no differentiable pure state-feedback able to stabilize the cart in both position and orientation around a given terminal configuration. However, it is possible to stabilize two of the configuration variables ((x, y) or (x, θ)).

When the controlling of a cart's point position (control of x and y) is the objective, it is preferable to choose this point away from the wheels' axis in order to ensure acceptable convergence properties.

The extension of the initial regulation problem to the trajectory tracking problem in Cartesian space allowed us to show that, via the introduction of a "virtual" reference cart the trajectory of which is determined at a higher control level, feedback stabilization in both position and orientation becomes possible as long as the reference cart keeps moving. This fact does not contradict the aforementionned non-existence result because the control laws considered in the trajectory tracking case now depend also on the external variable t which is used to parametrize the reference cart's motion.

Although subtle in appearance, the distinction between the two cases is conceptually important because it concretely meddles with the difficult and only partially understood problem of linking the controllability property of nonlinear systems (which is closely related to the path planning problem, and thus to open-loop control) to the various feedback control possibilities. In the context of mobile robots, it points out some peculiarities of the problem of controlling nonholonomic systems, such as the possibility of controlling (and stabilizing) configuration variables in a number exceeding the number of the robot's actuators. A consequence of this possibility, which does not exist in the case of holonomic robot manipulators (for which the notions of controllability and feedback stabilization are undistinguishable in the configuration space), is that the control approaches developped for robot manipulators apply only partially to the case of mobile robots. Complementary studies are necessary to fully explore this possibility. In particular, it is no longer so easy to decouple the path planning problem from the feedback control problem, as already illustrated in this study. The mere choice of the control strategy to follow is not easily decided because there are many possible alternatives.

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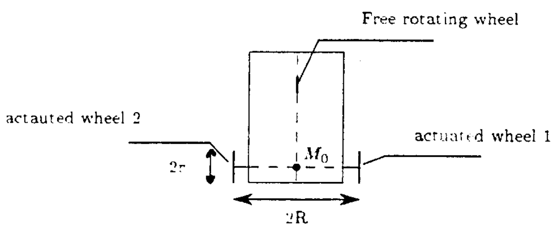

Fig.1 : Above view of the cart

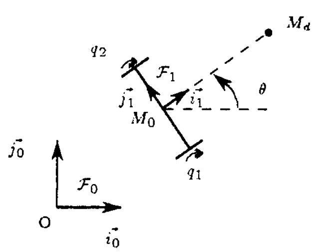

Fig.2 : Cart's parameters

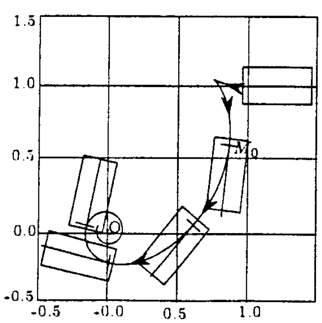

Fig.3 : Stabilisation of the point M_0

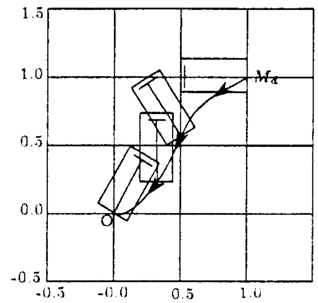

Fig.4 : Stabilisation of the point M_d ($d \neq 0$)

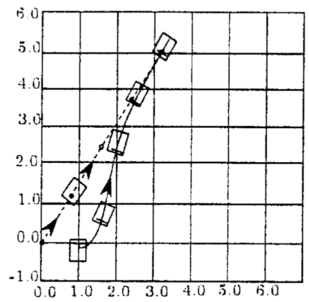

Fig.5 : Tracking of a reference cart

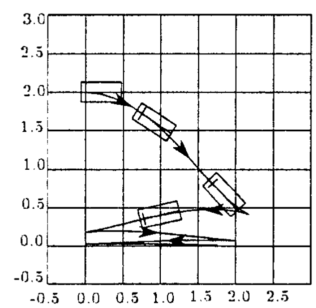

Fig 6 : Tracking of a reference cart

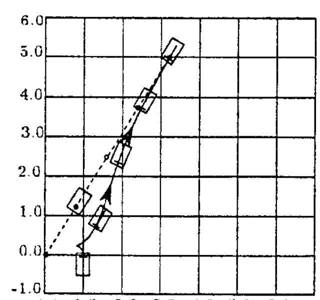

Fig.7 : Tracking of a reference cart