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A unified approach to inverse and direct kinematics for four kinds of wheeled mobile robots and its applications

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Abstract --- The development of mechanical structures for wheeled vehicles has a very long history. The forms of vehicles can be classified according to the steering and powering of the vehicle as follows: a. Ordinary car-like robots (including passenger cars, single unit trucks, single unit buses and articulated trucks); b. Dual drive robots (dual drive motors with various casters); c. Synchro drive and steering robots; d. Omnidirectional robots.

Kinematics forms the basis for feasibility analysis of the mechanism, motion feasibility, dynamics and control, trajectory planning and more complicated problem for an autonomous intelligent robot, such as path planning. This paper develops the unified kinematics, both the inverse and the direct, for the four kinds of wheeled vehicles. The application of the kinematics in path planning problem is also presented.

1. Introduction

The description of position and orientation of a manipulator is always given in the Cartesian Space whereas the execution of the motion of a manipulator is realized by actuating the joints in Joint Space. Thus, the calculation of the transformation between Cartesian Space and Joint Space, including position, velocity and acceleration, is necessary. A kinematic analysis of manipulators, always distinguishes the direct kinematics from the inverse kinematics. The direct kinematics consists of computing the position and orientation of the end-effector of the manipulator given a set of joint angles, whereas the inverse kinematics involves finding all possible sets of the joint angles for a given position and orientation of the end-effector. The establishment of the concepts of inverse and direct kinematics is to make the calculation of the translation easier. Similarly, a relationship between location description and execution of a wheeled mobile robot must be established.

For the different arrangements of the four kinds of robots, we generally define the joint variables as the powering and steering variables, while the Cartesian variables are the position of the reference point, the orientation angle and their derivatives. Essentially, the inverse and direct kinematics describes the mapping between description variables of the robot and the execution variables.

Note that the concepts of inverse and direct kinematics we used here for a wheeled mobile robot involve not only relationships among the static position, orientation, and the steering angle of the robot and so on, but also the relationship among their time derivatives. This is different from the case of a manipulator.

It is well-known that the kinematics of a manipulator forms the basis of its dynamics, control, trajectory planning and path planning. At the beginning of research on manipulators, kinematics was developed and now it is widely accepted. Similarly, kinematics for a wheeled mobile robot is of central importance for its dynamics, control, trajectory planning and path planning. Unfortunately, although research on wheeled robots has been conducted for more than twenty years, there does not exist an established kinematics for them. For lack of a complete description of the motion for a vehicle, the effects of kinematics on the trajectory generation, path planning and control problems have not been sufficiently investigated[14,15,16]. The objective of this paper is to present the unified kinematics, including the inverse and direct, for the four kinds of wheeled vehicles and show its application.

2. Mathematical model for wheeled mobile robots without kinematic constraints

2.1 Global and local reference coordinate frames

Fig. 1 is a plan view of a general robot model investigated. From the point of view of operating function, wheels used in the robot can be categorized into two types, rotatable wheels and fixed wheels. If a wheel can rotate about a vertical axle, it is defined as a rotatable wheel, otherwise, it is defined as a fixed wheel. Based on this definition, the front steered wheels of a car-like robot are rotatable wheels and its two rear wheels on the fixed axle, are fixed wheels. However, when they are described by a mathematical model, a fixed wheel may be regarded as a special rotatable wheel, so in Fig. 1, all the wheels are given in the form of rotatable wheels. A global reference coordinate frame (Oxy) is introduced to describe the motion of the robot in terms of the position of the reference point (xp,yp) and the orientation angle θ of the robot. We define a local reference coordinate frame (O1x1y1), whose origin is placed at the reference point P of the robot. We use M to represent any point in the rigid robot body when discussing the motion of the rigid robot body and also use it to represent a wheel connected to the corresponding point when discussing steering angle. The coordinates of point M are denoted by M (xm,ym) in terms of the global reference frame and M (xm1,ym1) in terms of the local reference frame. So the coordinates are constants with respect to the local reference frame.

2.2 Motion description of the rigid body

The coordinates of point M in the global frame are related to the coordinates of point M measured in the local frame by

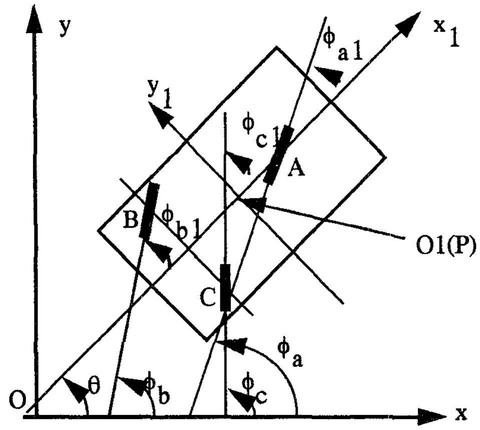

Fig. 1 Global (Oxy) and local (O1x1y1) reference coordinate frames, reference point P coincides with O1

the transformation:

$$\begin{bmatrix} x_m \\ y_m \end{bmatrix} = \begin{bmatrix} x_{m1}\cos\theta - y_{m1}\sin\theta + x_p \\ x_{m1}\sin\theta + y_{m1}\cos\theta + y_p \end{bmatrix} \tag{1}$$

Differentiating Eq. (1) with respect to time yields:

$$\begin{bmatrix} \dot{x}_m \\ \dot{y}_m \end{bmatrix} = \begin{bmatrix} (-x_{m1}\sin\theta - y_{m1}\cos\theta)\dot{\theta} + \dot{x}_p \\ (x_{m1}\cos\theta - y_{m1}\sin\theta)\dot{\theta} + \dot{y}_p \end{bmatrix} \tag{2}$$

Equations (1)-(2) give the mathematical description of any point on the robot body in terms of the reference point position, velocity and the orientation angle of the robot as well as their derivatives.

2.3 Ideal rolling condition, motion description of each wheel

The angle ϕ_m, (m=a,b,c) is defined as that between the vertical plane of the wheel M and the positive x axis, and ϕ_{m1}, between the vertical plane of the wheel M and the positive x1 axis (see Fig. 1). Their relation can be written as:

$$\phi_m = \phi_{m1} + \theta \tag{3}$$

For any types of wheeled robots, the general requirement for the mechanical design is that all the wheels connected to the rigid body should roll without any side-slip[15]. When the whole robot is considered, the following ideal rolling conditions must be satisfied:

- The direction of every wheel rolling forward or backward, whether steered or not, must coincide with the tangent to the robot body trajectory at the corresponding wheel center.
- The velocity of every wheel center point must equal to the product of wheel rotation angular velocity about its own horizontal axle and its radius.

Mathematically, the above conditions can be expressed as:

$$\tan(\phi_m) = \frac{d}{d(x_m)}(y_m) = \frac{\dot{y}_m}{\dot{x}_m} \tag{4}$$

$$v_m = r \cdot \omega_m \tag{5}$$

Where ω_m (m=a,b,c) represents the angular velocity of wheels A, B, C respectively; v_m the velocity of the center point of wheels A, B, C; and r the radius of every wheel. v_m is obtained from the vector sum of $\dot{x}_m$ and $\dot{y}_m$:

$$(v_m)^2 = (\dot{x}_m)^2 + (\dot{y}_m)^2 \tag{6}$$

From equations (3) and (4) we can obtain ϕ_{m1}, the relative angle to the robot orientation line, which is the steering angle for a steering wheel:

$$\phi_{m1} = \arctan\left(\frac{\dot{y}_m}{\dot{x}_m}\right) - \theta \tag{7}$$

3. Mathematical model for car-like robots

For a car-like robot, in order to make it move along a path, steering is necessary. Steering is normally effected by changing the heading of the front steering mechanism. In order to change its speed, the velocity from the engine must be adjusted. So two problems, one being the inverse of the other, arise. The first of these is to determine the location coordinates of the robot and their higher derivatives when the steering angle and its derivatives, and the driven velocity from the engine are given. This is referred to as the direct or forward problem. The second problem, which is referred to as the inverse problem, is, given a desired path of the reference point and its derivatives, to calculate the required steering angle and its derivatives and the driven velocities. The inverse problem is clearly relevant to the control of the robots.

3.1 General constraint equation

As defined above, when the wheels are fixed wheels, then their relative angles to the robot orientation line are constants. In this study of a car-like robot, the rear wheels B, C are fixed wheels. If we choose y1 axis parallel to the rear axle BC, then $\phi_{b1} = \phi_{c1} = 0$. Substituting Eq. (2) into (7) respectively and simplifying, we obtain:

$$x_{b1} \cdot \dot{\theta} = \dot{x}_p \sin\theta - \dot{y}_p \cos\theta \tag{8}$$

$$x_{c1} \cdot \dot{\theta} = \dot{x}_p \sin\theta - \dot{y}_p \cos\theta \tag{9}$$

As defined above, the y1 axis is parallel to the robot's rear axle, so that we may write $x_{b1} = x_{c1} = \underline{x}$. Eq. (8) and (9) become:

$$\underline{x} \cdot \dot{\theta} = \dot{x}_p \sin\theta - \dot{y}_p \cos\theta \tag{10}$$

Eq. (10) is what we need to solve for the robot's orientation angle θ when the reference point velocity components $\dot{x}_p$ and $\dot{y}_p$ are specified. Due to the fact that this equation describes the general relationship between the reference point's position relative to the rear axle, its velocity, the robot's orientation angle and its first derivative when the reference point is chosen at any point in the robot, it is called the general constraint equation.

It can be noted that y_{b1}, y_{c1} are not included in Eq. (10), which illustrates the fact that only $\underline{x}$ affects the robot's orientation angle. Eq. (10) indicates that:

- If a robot has more than one fixed horizontal axle, two constraint equations are imposed. It is obvious that the two equations conflict with one another, so, in a rigid robot, the number of the fixed horizontal axles can't exceed one.
- When different fixed wheels in the robot have the same x-coordinates in terms of the local reference frame, namely they are mounted on the same fixed horizontal axle, only one constraint is imposed.

3.2 Procedure for solving the inverse kinematics

The inverse kinematics means that given the position of a reference point on the robot and its first and second derivatives, find the needed steering angle, the driving velocity and acceleration of the middle point of the rear axle (This is due to the fact that a rear differential driving system is used for car-like robot). It can be seen that to solve the inverse kinematic problem, the key problem is to solve the general constraint equation. Once θ and $\dot{\theta}$ are known, the position and absolute velocity of any point can be determined by equations (1) and (2) and the steering angle can be solved from Eq. (7).

3.3 Direct kinematics

Given a steering angle ϕ_{a1} and a velocity $\upsilon(t)$ of the middle point Z of the rear axle as well as their derivatives, the direct kinematics consists of computing the position of the reference point P and the orientation angle of the robot, as well as their first and second derivatives. Although we can derive the direct kinematics from the inverse kinematics, it is more straightforward to use a geometrical method. As shown in Fig. 2, the velocity υ can be decomposed into the velocity components along the x and y axes. We have:

$$\dot{x}_z = \upsilon\cos\theta \tag{11}$$

$$\dot{y}_z = \upsilon\sin\theta \tag{12}$$

The rotational speed of the robot can be expressed as follows (see Fig. 2):

$$\dot{\theta} = \frac{\upsilon(t)\tan(\phi_{a1}(t))}{L} \tag{13}$$

Where L is the wheelbase of the robot. We can integrate and differentiate these equations respectively:

$$\theta(t) = \theta(0) + \int_0^t \left\{\frac{\upsilon(t)\tan(\phi_{a1}(t))}{L}\right\} dt \tag{14}$$

$$x_z(t) = x_z(0) + \int_0^t \{\upsilon(t)\cos(\theta(t))\} dt \tag{15}$$

$$y_z(t) = y_z(0) + \int_0^t \{\upsilon(t)\sin(\theta(t))\} dt \tag{16}$$

$$\ddot{\theta}(t) = \frac{\upsilon(t)}{L}\cdot(\sec(\phi_{a1}))^2\cdot\dot{\phi}_{a1} + \frac{\dot{\upsilon}(t)}{L}\tan(\phi_{a1}) \tag{17}$$

$$\ddot{x}_z(t) = \dot{\upsilon}(t)\cos(\theta) - \upsilon(t)\sin(\theta)\dot{\theta} \tag{18}$$

$$\ddot{y}_z(t) = \dot{\upsilon}(t)\sin(\theta) + \upsilon(t)\cos(\theta)\dot{\theta} \tag{19}$$

After solving equations (14) to (19), we can obtain the position of the reference point P and velocity using equations (1)-(2).

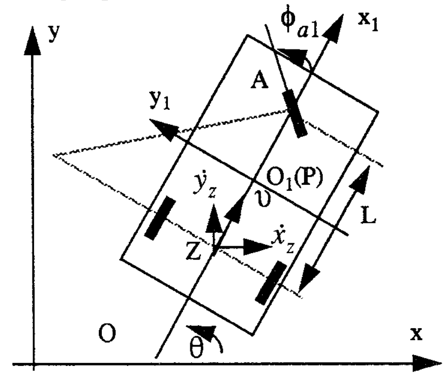

Fig. 2 Geometry for direct kinematics

3.4 Mathematical model for two or more linked vehicle units

Fig. 3 shows a linked vehicle model. Two local reference coordinate frames (O1x1y1) and (O2x2y2) are put on the two vehicle units, respectively. For convenience, the origin of the second frame coincides with the hitch point. We know that for the first vehicle unit, if the reference point O1(P1)'s position and velocity are given, then vehicle unit 1's orientation angle can be obtained by solving the general constraint differential Eq. (10), and then the position, and velocity of the hitch point O2(P2) which is a point in the vehicle unit 1 can be calculated using equations (1) and (2). This means that the position and velocity of the second vehicle unit's reference point O2 have been obtained. Consequently, by means of the identical procedure used in the calculations for vehicle unit one, the position and velocity and other interesting parameters of all the points in the second vehicle unit can be calculated. If more than two vehicle units are serially linked, the procedure for dealing with the motion problem of the complicated vehicle is analogous to the above one. In every step, a numerical method must be used to solve a simple first-order differential

Eq. (10).

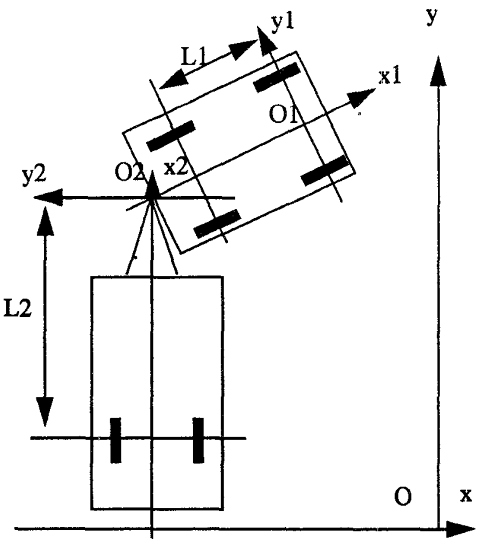

Fig. 3 A schematic for an articulated vehicle

4. Mathematical model for dual drive robots

4.1 Inverse kinematics

The steering and powering of the dual drive robot is accomplished by changing the velocity of two rear wheels B and C. Therefore, the purpose of the inverse kinematics is to solve the needed velocities υ_b and υ_c when $\dot{x}_p$ and $\dot{y}_p$ are

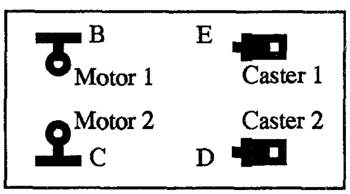

Fig. 4 Dual drive system with two front casters

given. Since the dual drive robot still has two fixed wheels B, C which are parallel, as shown in Fig. 4, there is no need to repeat the derivation for the general constraint equation and other equations describing the motion of the rigid body. In deriving the inverse kinematics of the car-like robot, we have in fact established the inverse kinematics for the dual drive robot. That is, (10), (1) and (2) can also be used to solve for the orientation angle and the driving velocities υ_b and υ_c. The only difference in analyzing the inverse kinematics from a car-like robot is that we do not need to consider the steering angle limit.

4.2 Direct kinematics

Suppose the velocities υ_b and υ_c are given, then:

$$\upsilon_b = (W + X)\dot{\theta} \qquad (20)$$

$$\upsilon_c = X\dot{\theta} \qquad (21)$$

Where X is the distance from wheel C to the centre of instantaneous rotation; W is the wheelbase. From (20) and (21), we get:

$$\dot{\theta} = (\upsilon_b - \upsilon_c)/W \qquad (22)$$

Thus, the orientation angle can be obtained by integrating the above equation:

$$\theta = \theta_0 + \int_0^t ((\upsilon_b - \upsilon_c)/W)\, dt \qquad (23)$$

The position of wheel B is:

$$x_b = x_b(0) + \int_0^t (\upsilon_b \cdot \cos\theta)\, dt \qquad (24)$$

$$y_b = y_b(0) + \int_0^t (\upsilon_b \cdot \sin\theta)\, dt \qquad (25)$$

Equations (1), (2) also apply to solve the position of the reference point and its derivative.

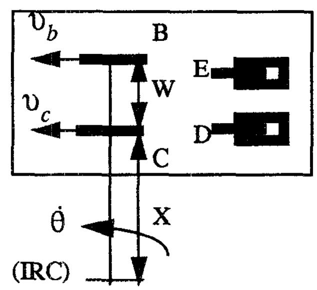

Fig. 5 Illustration of instantaneous rotation center (IRC)

4.3 Motion of the casters

The function of the casters is to balance the robot. They add no constraints on the whole robot, and their motion always follows the robot body. Each caster can be regarded as a small robot unit which has a fixed wheel. Once the motion of the robot body is solved (it may be solved by either inverse kinematics or direct kinematics in different circumstance), then the motion of the hitch point (through which point, the caster is connected to the rigid body) can be obtained. Treat the caster as a robot unit like that discussed in section 3.4, the motion of the caster can be solved using the inverse kinematics.

5. Mathematical model for synchro drive and steering robots

5.1 Inverse kinematics

As stated in[5], the synchro drive system is mechanically synchronized to each other for both steering and power. Fig. 6 illustrates the functional diagram of its steering and power. As the synchronization is accomplished by the use of chains or by gears, the mechanism requires that each wheel of A, B, C and D must have the identical steering angle and the identical velocity at any moment. Therefore, mathematically, the constraint imposed by the synchro steering is:

$$\phi_{a1} = \phi_{b1} = \phi_{c1} = \phi_{d1} = \phi(t) \tag{26}$$

and the constraint imposed by the synchro power is:

$$\dot{x}_a = \dot{x}_b = \dot{x}_c = \dot{x}_d \tag{27}$$

$$\dot{y}_a = \dot{y}_b = \dot{y}_c = \dot{y}_d \tag{28}$$

For all wheeled robots, the wheels on the rigid body should satisfy the ideal rolling conditions described by equations (4) and (5). In addition, equations (1) and (2), which describe the motion of the rigid body and Eq. (7) describing the steering angle, also apply to the synchro drive and steering system. Thus, from equations (2), (7) and (26), we have:

$$\tan(\theta+\phi) = \frac{(x_{m1}\cos\theta - y_{m1}\sin\theta)\dot{\theta} + \dot{y}_p}{(-x_{m1}\sin\theta - y_{m1}\cos\theta)\dot{\theta} + \dot{x}_p}, m = a, b, c, d \tag{29}$$

The only solution to the above set of equations is:

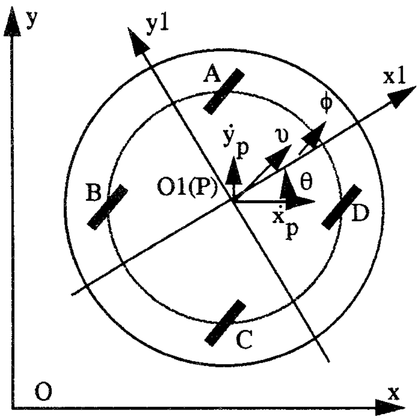

Fig. 6 Schematic for the synchro drive and steering robots

$$\dot{\theta} = 0 \Rightarrow \theta = \theta_0 \tag{30}$$

This is the general constraint equation for a synchro drive and steering robot. It is clear that the motion of the rigid body for this kind of robots is always a translation.

It is necessary and easy to verify that the motion of the synchro drive and steering robot subject to the general constraint Eq. (30) also satisfies the constraint equations (27) and (28), and they become the following one:

$$\begin{bmatrix} \dot{x}_m \\ \dot{y}_m \end{bmatrix} = \begin{bmatrix} \dot{x}_p \\ \dot{y}_p \end{bmatrix}, m = a, b, c, d \tag{31}$$

The expressions for the steering angle and the rotating angular rate of all the wheels are:

$$\phi = \arctan(\dot{y}_p/\dot{x}_p) - \theta_0 = \arctan(f'(x_p)) - \theta_0 \tag{32}$$

$$\omega = (\sqrt{\dot{x}_p^2 + \dot{y}_p^2})/r \tag{33}$$

5.2 Direct kinematics

In the following, we give the direct kinematics using the geometrical method (see Fig. 6), suppose $\phi(t)$ and $\upsilon(t)$ are given:

$$\dot{x}_p = \upsilon(t)\cos(\phi+\theta) \tag{34}$$

$$\dot{y}_p = \upsilon(t)\sin(\phi+\theta) \tag{35}$$

$$\theta = \theta_0 \tag{36}$$

$$x_p = x_p(0) + \int_0^t \{\upsilon(t)\cos(\phi+\theta)\}dt \tag{37}$$

$$y_p = y_p(0) + \int_0^t \{\upsilon(t)\sin(\phi+\theta)\}dt \tag{38}$$

$$\ddot{x}_p = \dot{\upsilon}(t)\cos(\phi+\theta) - \upsilon(t)\sin(\phi+\theta)\dot{\phi} \tag{39}$$

$$\ddot{y}_p = \dot{\upsilon}(t)\sin(\phi+\theta) + \upsilon(t)\cos(\phi+\theta)\dot{\phi} \tag{40}$$

Where $\upsilon(t)$ represents the translation velocity of the rigid body; ϕ the steering angle; P is the reference point which may be chosen at any point.

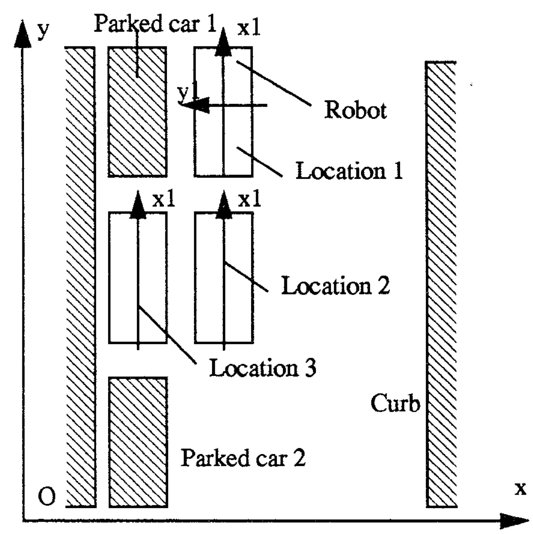

Fig. 7 Illustration of a parallel parking for four kinds of robots

6. Mathematical model for omnidirectional robots

6.1 Implication of omnidirectional robots

Before developing the kinematics of an omnidirectional robot such as The CMU Rover constructed at Carnegie Mellon University by Dr. Hans Moravec and the HERMIES-III developed at Oak Ridge National Laboratory, USA, we will discuss the meaning of omnidirectional. In the past, although a few researchers used this term, in fact nobody clarified its meaning using the expressions of the reference path and the orientation angle. For example, both the synchro drive and steering robot and the HERMIES-III were thought to be omnidirectional robots, although it is not correct to think of the synchro-drive and steering robot as an omnidirectional one [10]. We begin by considering the motivation for designing such a mobile robot with steerable wheels.

Generally speaking, the motivation to design an omnidirectional robot is to provide it with more flexible maneuverability than the conventional robots like the car-like or dual drive types. Maneuverability is still a portmanteau, meaning different things in different circumstances. In the present context, the two most important aspects of maneuverability are:

- The ability to turn sharply, and
- The ability to keep within a narrow path when turning.

The most familiar example of the manoeuvering required by a robot is parallel parking, as shown in Fig. 7. The goal is to move the robot into a space (location 3) between two parked cars 1 and 2 (they are may be two other obstacles). It is straightforward to prove using the general constraint Eq. (10) that for the robots subject to the constraint Eq. (10), it is impossible to move the robot directly from location 2 into location 3 without changing the orientation. For a car-like

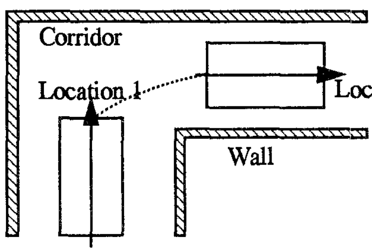

Fig.8 A robot turning around a corner

robot, the strategy adopted for such a parallel parking is to move the robot to location 2, stop, and then the following manipulations are performed: turning the wheels to left limit, backing, straightening wheels, backing, turning the wheels to the other limit, backing, and straightening the wheels (several iterations may be required by an unskilled driver). The optimum path is arc-line-arc. Of course, this path can also be traced by the dual driving robot and the needed operations are different. However, if the location 1 is occupied by obstacles, it is difficult for them to maneuver into location 3 in this way.

If the robot is synchro drive, such a requirement of maneuvering from location 2 to 3 can be met. However, in the example of turning a corner shown in Fig. 8, the car-like robot and the dual drive robot may succeed in moving from location 1 to 2 while the synchro drive one can not. These two examples of required maneuvering, which may occur very often in practice, illustrate that these kinds of robots posses their own limitations. The common feature of these three kinds of robots is that among the three parameters of x_p, y_p and θ describing the location of the robot, one is dependent on the other two by one form of constraint equation or the orientation angle is restricted. As a result, some kinds of maneuvering for them are restricted.

The purpose of designing an omnidirectional robot is to overcome these limitations. Therefore, an omnidirectional robot should be able to execute any desired path $y_p = f(x_p)$ and have any desired orientation angle θ along the path. In other words, the three parameters x_p, y_p and θ should be independent of each other. This kind of robot is called an omnidirectional robot or a free-flying robot.

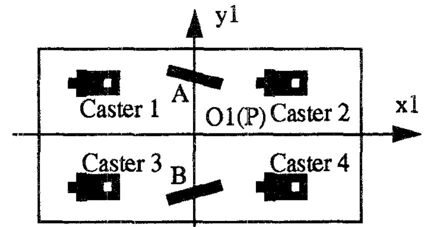

Fig. 9 Structure of the omnidirectional HERMIES-III robot

HERMIES-III, which is reported to have two drive steerable wheels A and B (see Fig. 9), functions as such a platform [11]. Another example of this omnidirectional robots is the CMU Rover[5].

6.2 Inverse kinematics

The meaning of the inverse kinematics for an omnidirectional wheeled robot is different from that of the other three kinds of robots. The difference is that the orientation angle of the robot as well as the reference point's position and their derivatives must be given to calculate the needed steering angle ϕ_{m1} and drive angular rate ϖ_m. In this case, there is no general constraint equation. Equations (1) to (7) can also be used.

6.3 Direct kinematics

In deriving the direct kinematics for an omnidirectional robot like HERMIES-III, we assume that both of wheels A and B are steered and only wheel A is driven. As shown in Fig. 10, if the steering angles ϕ_{a1} and ϕ_{b1} are given, then the position of the ICR is uniquely determined. The velocity of wheel B is a function of ϕ_{a1}, ϕ_{b1} and $\upsilon_a(t)$.

From Fig. 10, it can be seen that υ_a can be decomposed in x and y axes:

$$\dot{x}_a = \upsilon_a \cdot \cos(\phi_{a1} + \theta) \tag{41}$$

$$\dot{y}_a = \upsilon_a \cdot \sin(\phi_{a1} + \theta) \tag{42}$$

Where θ is still an unknown function that must be derived. We use the geometric method in the following (it can also be derived from the inverse kinematics using the algebraic method)

Since the directions of line AD and BD are perpendicular to wheels A and B, respectively, from Fig. 10, we have:

$$\angle ADB = \phi_{b1} - \phi_{a1} \tag{43}$$

Suppose the orientation line x1 axis is perpendicular to the line joining the two wheels A and B, then:

$$\angle ABD = 180° - \phi_{b1} \tag{44}$$

The length of AD can be obtained from the sine theorem:

$$AD = \frac{AB \cdot \sin(\angle ABD)}{\sin(\angle ADB)} = \frac{W \cdot \sin(\phi_{b1})}{\sin(\phi_{b1} - \phi_{a1})} \tag{45}$$

Where W represents the distance between the two wheels.

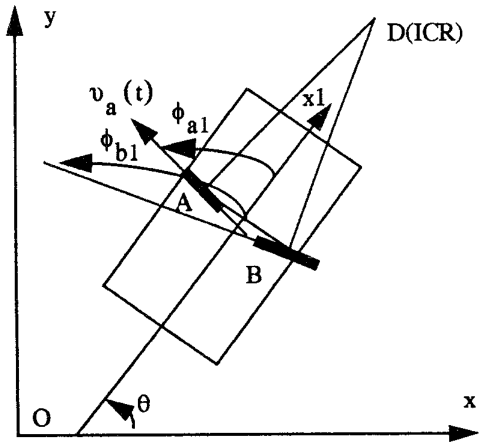

Fig. 10 A schematic for the omnidirectional robot

Thus:

$$\dot{\theta} = \frac{\upsilon_a}{AD} = \upsilon_a \cdot \frac{\sin(\phi_{b1} - \phi_{a1})}{W \cdot \sin(\phi_{b1})} \tag{46}$$

Integrating equations (46), (41) and (42) respectively gives:

$$\theta = \theta(0) + \int_0^t \left(\upsilon_a \cdot \frac{\sin(\phi_{b1} - \phi_{a1})}{W \cdot \sin(\phi_{b1})} \right) dt \tag{47}$$

$$x_a = x_a(0) + \int_0^t (\upsilon_a \cdot \cos(\phi_{a1} + \theta))\, dt \tag{48}$$

$$y_a = y_a(0) + \int_0^t (\upsilon_a \cdot \sin(\phi_{a1} + \theta))\, dt \tag{49}$$

7. Applications

As an illustrative example of applications of the kinematics to path planning problem, Let us consider the motion of the first two kinds of robots subject to the constraint equation (10), as shown in Fig. 11. The robot is required to move from the initial point A to the intersection point B of the two straight line segments AB and BC, stop at B, and then turn without moving the reference point P (we call such a motion as a pure rotation) until the orientation angle of the robot coincides with the straight line segment BC. This problem arises from the execution of the generated path using the method presented in [3]. In this method, the robot and the

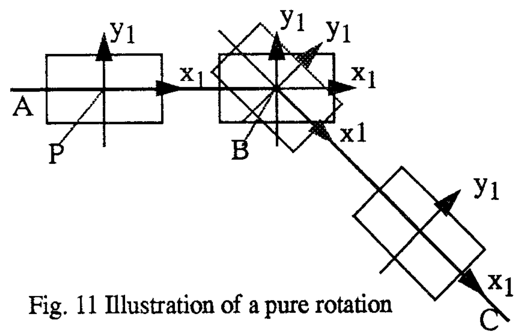

Fig. 11 Illustration of a pure rotation

obstacles are assumed to be polygons and the generated path consists of line-line segments. The reference point is chosen so that the $max_{1 \le i \le n}(d_i)$ is minimized over all the possible point on the robot as the candidate of the reference point, where d_i represents the distance of the ith vertex of the robot from the reference point (for more details, see [3]). For a rectangular robot, the reference point chosen based on this definition is located at the geometrical center. In most cases, this point is not at the rear fixed axle for a car-like or a dual drive robot. The question we are interested in here is that if the reference point is not at the rear axle, whether or not such a pure rotation can occur.

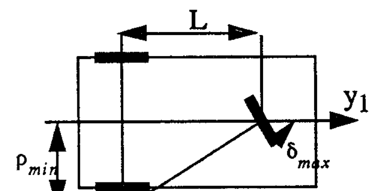

Fig. 12 The minimum distance from the middle point of the rear axle to the instantaneous rotation center

In the following, we will prove a theorem describing the condition of a pure rotation for a robot subject to the kinematic constraint equation (10).

Theorem: *For a robot subject to the kinematic constraint equation (10), only when the reference point is at the rear fixed axle, does the pure rotation occur.*

Proof: If the motion is a pure rotation, then $\dot{x}_p = 0$, $\dot{y}_p = 0$ and $\dot{\theta} \ne 0$. ($\dot{\theta} = 0$ means that the robot does not move at all). In this case, the unique solution of the general constraint equation (10), which satisfies this requirement, is $\underline{x} = 0$. That means the reference point must be at the rear fixed axle.

Note that steering angle limit is an intrinsic characteristic of the car-like robot. In this theorem, it is not taken into account. Suppose the steering angle limit is δ_{max}, then the minimum distance of the instantaneous rotation center to the middle point of the rear axle ρ_{min} can be expressed in the following form (see Fig. 12)

$$\rho_{min} = L/(\tan\delta_{max}) \tag{50}$$

From this, it can be observed that for a car-like robot which has steering angle limit (that means $\delta_{max} \neq \pm 90°$), even if the reference is at the middle of the rear axle, the motion of the pure rotation can not occur. For the dual drive robot, since steering is accomplished by changing the ratio of the velocities of the two driving wheels, only when the reference point is located at the rear axle, the pure rotation does occur. The conclusion drawn from the above analysis is that for car-like robot, the path generated by the algorithm [3] is not executable; for a dual drive robot, only when the reference point coincides with the rear axle, the path can be followed.

8. Conclusions

In this paper, a unified treatment of the kinematics for the four kinds of wheeled robots has been presented. Compared with the other available models, it has at least one of the following advantages:

1. The mathematical model is general in the sense that the reference point can be chosen at any point on the vehicle, not only on the mid-point of the rear axle (used by most of the researchers in robotics) or the front left wheel (defined in the highway and street design standards) or the vertex of the robot body (chosen by most path planning algorithms).

2. The model can apply to any dimension of rigid vehicle or vehicle combination, not only to small vehicles because it gives a transient description of the motion for a vehicle.

3. The needed steering angle can be given for tracing a specified path by the reference point, which is essential to automatically control a mobile robot by computer and useful in highway design to determine the minimum turning radius when the steering angle limit is taken into consideration.

4. The model is applicable to any kinds of paths, for example, straight line, circle, or a more general curve y=f(x).

5. The model distinguishes the inverse kinematics from the direct kinematics.

As an illustrative example, we show the applications of the kinematics established to the path planning problem. For limitation of space, we can not give more discussion of the applications.

9. Acknowledgments

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