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Newton's Law and Nonholonomic Systems

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Abstract

In this paper we consider nonholonomic control systems on Riemannian manifolds. Such systems evolve on sub-bundles of tangent bundles, defined by the nonholonomic constraints. This paper promotes the view of such systems as the restriction to the nonholonomic sub-bundle of "Newton-law" type problems on the entire tangent bundle, defined by, in general, non-Riemannian connections. These connections should be related to specific geometric properties of the nonholonomic system. We define a particular class of connections based on special symmetry properties, and demonstrate the validity of the construction through three examples.

1 Introduction

We suppose that M^n, $\langle \cdot, \cdot \rangle$ is a Riemannian manifold with symmetric Riemannian connection ∇ and covariant derivative $D/\partial t$. We sometimes denote the metric by g. We may describe a class of nonholonomic control systems on M, driven by external forces, by following the works of Bloch and Crouch ([1, 2, 3]). These are generally defined by systems of equations:

$$\frac{D^2 q}{\partial t^2} = \sum_{i=1}^{m} \lambda_i W_i + F; \quad q \in M, \omega_i(\dot{q}) \equiv 0, \quad 1 \leq i \leq m, \tag{1}$$

where F is an arbitrary external force field, modeled as a vector field on M, and ω_i, $1 \leq i \leq m$, are m independent constraint forms on M satisfying

$$\omega_i(X) = \langle W_i, X \rangle; \quad \forall X \in \Gamma(TM)$$

for m vector fields W_i, $1 \leq i \leq m$ on M, where $\Gamma(V)$ denotes the space of sections of the vector bundle V over M. The system (1) is nonholonomic precisely when the distribution $N \subset TM$ defined by

$$N_p = \{X_p \in \Gamma(TM); \quad \omega_i(X) \equiv 0, \quad 1 \leq i \leq m\}, \quad p \in M, \tag{2}$$

is not integrable. This paper sketches results obtained in the forthcoming, Bloch and Crouch [8].

Let $a_{ij} = \omega_i(W_j)$, $1 \leq i,\ j \leq m$. The independence of the forms ω_i implies that the matrix $[a_{ij}]_{1 \leq i,j \leq m}$ is invertible on the whole of M. By differentiating the constraints we may eliminate the multipliers λ_i and obtain an equivalent formulation of the flow described by (1) in the form:

$$\begin{aligned} \frac{D^2 q}{\partial t^2} + \sum_{i,j} W_k a_{kj}^{-1} \frac{D\omega_j(\dot{q})}{\partial t} &= F - \sum_{k,j} W_k a_{kj}^{-1} \omega_j(F) \\ \omega_i(\dot{q}) &= 0, \quad 1 \leq i \leq m. \end{aligned} \tag{3}$$

Much effort has gone into understanding or rationalizing this system of equations. We briefly review the approach of Vershik and Gershkovich [4] and Vershik and Fadeev [5]. For any $X \in \Gamma(TM)$ we set

$$\pi_N(X) = X - \sum_{k,i} W_k a_{ki}^{-1} \omega_i(X).$$

It is evident that π_N is the orthogonal projection onto the subbundle $N \subset TM$, and that we may write (3) in the form

$$\pi_N\left(\frac{D^2 q}{\partial t^2}\right) = \pi_N(F), \quad \dot{q} \in N. \tag{4}$$

Definition 1 If $\overline{\nabla}$ is any connection on M, with corresponding covariant derivative $\overline{D}/\partial t$, then we define the following second-order system:

$$\frac{\overline{D}^2 q}{\partial t^2} = \overline{F}, \quad q \in M \tag{5}$$

to be a "Newton Law" system on M, with external forces modeled by the vector field $\overline{F}$ on M.

Thus the nonholonomic system (1) may be viewed as simply the projection (4) of the Newton Law system $D^2q/\partial t^2 = F$, onto the subbundle N. We may however define a new connection ∇' on M by setting

$$\nabla'_X Y = \nabla_X Y + \sum_{k,i=1}^{m} W_i a_{ik}^{-1}(\nabla_X \omega_k)(Y); \tag{6}$$

$$X, Y \in \Gamma(TM),$$

which, in turn, defines a covariant derivative $D'/\partial t$. Clearly from (6) ∇' has the property

$$\omega_i(\nabla'_X Y) = \omega_i(\nabla_X Y) + (\nabla_X \omega_i)(Y) = X(\omega_i(Y)).$$

Thus if $X, Y \in \Gamma(N)$, $\nabla'_X Y \in \Gamma(N)$, and so $\nabla'|_N$ defines a connection on the subbundle N. This system (1) may be written in the form

$$\frac{D'^2 q}{\partial t^2} = \pi_N(F), \quad \dot{q} \in N, \tag{7}$$

and view this system as a Newton Law system on N. This is the perspective of Vershik and Fadeev [5].

We take a different perspective in this paper and note that system (7) defines a perfectly good Newton Law system on all of TM:

$$\frac{D'^2 q}{\partial t^2} = \pi_N(F), \quad q \in N. \tag{8}$$

Definition 2 Consider a Newton Law system defined on M, employing a connection $\overline{\nabla}$ and external force $\overline{F}$ of the form

$$\frac{\overline{D}^2 q}{\partial t^2} = \overline{F}, \quad q \in M. \tag{9}$$

We say that the system has the restriction property if it restricts to the subbundle N and on N it coincides with the nonholonomic system (1).

It follows that system (8) has the restriction property. However, ∇' seems to be an arbitrary choice and, motivated by the work of Vershik, we ask if there are not more natural choices? In particular, one may ask if there is another metric G on M, with corresponding Riemannian connection ∇^G, such that the associated Newton-Law system defined by ∇^G has the restriction property?

We give a partial answer to this question in section 4, by defining in section 2 a class of connections on M, under certain symmetry assumptions on the nonholonomic system, which do define Newton Law systems with the required restriction property. In section 3 we detail a particular class of systems for which the structure introduced in section 2 has a particularly simple form. In section 5 we briefly discuss related integrability questions and give examples in section 6.

2 Connections and Bundle Maps

We introduce another vector bundle V over M, isomorphic to TM; $\pi : V \to M$, with $V_q = \pi^{-1}(q)$ and $\dim(V_q) = n$ for all $q \in M$. We assume that V comes equipped with a connection ∇^V which defines a covariant derivative $D^v/\partial t$ on V. We let $A : TM \to V$ be a vector bundle isomorphism so that

$$A_q : T_q M \to V_q$$

is a nonsingular vector space isomorphism for each $q \in M$. We are interested in how the Newton Law system (9) defined by a connection $\overline{\nabla}$ on M behaves under the bundle map $v = A_q(\dot{q})$. To differentiate this expression we introduce a connection on the bundle over M, $L(TM; V)$ of bundle maps from TM to V. The connections $\overline{\nabla}$ and ∇^V allow us to define the induced connection $\overline{\overline{\nabla}}$ by setting

$$(\overline{\overline{\nabla}}_X A)(Y) \triangleq \nabla^V_X(AY) - A(\overline{\nabla}_X Y), \tag{10}$$

$$A \in \Gamma(L(TM; V)), X, V \in \Gamma(TM).$$

(We use the notation $\overline{\overline{\nabla}}$ since ∇^V is viewed as a fixed connection, while we manipulate $\overline{\nabla}$ to obtain desired results.)

Differentiating the constraint $v = A_q(\dot{q})$ and applying the Newton Law (9) we obtain

$$\frac{D^V v}{\partial t} = \frac{\overline{\overline{D}} A_q}{\partial t}(A_q^{-1} v) + A_q(\overline{F}), \dot{q} = A_q^{-1}(v). \tag{11}$$

We naturally ask the question, "Can we choose the connection $\overline{\nabla}$ so that the term

$$\frac{\overline{\overline{D}} A_q}{\partial t}(A_q^{-1} v) = \frac{\overline{\overline{D}} A_q}{\partial t}(\dot{q}) \tag{12}$$

vanishes identically?". We make another definition

Definition 3 $A \in \Gamma(L(TM; V))$ is Killing with respect to $\overline{\overline{\nabla}}$ if

$$(\overline{\overline{\nabla}}_X A)(Y) + (\overline{\overline{\nabla}}_Y A)(X) \equiv 0, \quad X, Y \in \Gamma(TM). \tag{13}$$

We note that (13) is equivalent to $(\overline{\overline{\nabla}}_X A)(X) \equiv 0$ for all $X \in \Gamma(TM)$. Thus the question above becomes, "Can we choose $\overline{\nabla}$ so that A is Killing?" It turns out that there is an elegant solution to this question.

Theorem 1 *The unique symmetric connection ∇^A on M, so that $A \in \Gamma(L(TM;V))$ is Killing with respect to ∇^A is given by*

$$\nabla^A_X Y = \nabla_X Y + \frac{1}{2} A^{-1}((\nabla_X A)(Y) + (\nabla_Y A)(X)); \quad (14)$$

$$X, Y \in \Gamma(TM).$$

Since we are interested primarily in nonholonomic systems (1), it is useful to place these results in the context of a Newton Law system which has the restriction property. In this case we simply require that the term vanishes on N. In this case the transformed system (11), will have the form

$$\frac{D^v v}{\partial t} = A_q(\pi_N(F)), \quad \dot{q} \in A_q^{-1}(v), \quad \dot{q} \in N.$$

Definition 4 $A \in \Gamma(L(TM;V))$ is Killing on N with respect to $\overline{\nabla}$ if

$$(\overline{\nabla}_X A)(Y) + (\overline{\nabla}_Y A)(X) \equiv 0; \quad X, Y \in \Gamma(N). \quad (15)$$

Theorem 2 *Let $\nabla^{(A,S)}$ be a symmetric connection on M for which $A \in \Gamma(L(TM, V))$ is Killing on N with respect to $\nabla^{(A,S)}$. Then for $X, Y \in \Gamma(TM)$*

$$\begin{aligned} \nabla^{(A,S)}_X Y &= \nabla_X Y + \frac{1}{2} A^{-1}((\nabla_X A)(Y) \quad (16) \\ &\quad + (\nabla_Y A)(X)) + S(X,Y) \end{aligned}$$

for some symmetric two tensor S, such that $S|_N \equiv 0$.

Corresponding to the connection $\nabla^{(A,S)}$ on M defined in (16) there exists a corresponding covariant differentiation $D^{(A,S)}/\partial t$. From (16) we have

$$\frac{D^{(A,S)^2} q}{\partial t^2} = \frac{D^2 q}{\partial t^2} + A^{-1}\left(\frac{DA}{\partial t}\right)(\dot{q}) + S(\dot{q}, \dot{q}).$$

We may now ask the question, when does the corresponding Newton Law system

$$\frac{D^{(A,S)^2} q}{\partial t^2} = \pi_N(F), \quad q \in M, \quad (17)$$

have the restriction property?

Theorem 3 *Given a vector bundle V, connection ∇^V, symmetric tensor S, $S|_N \equiv 0$, then a necessary and sufficient condition for the Newton Law system (17) to have the restriction property is that $A \circ \pi_N$ is Killing on N with respect to ∇.*

3 A Special Class of Nonholonomic Systems

Given a vector bundle V, connection ∇^V, we wish to find examples of bundle maps $A : TM \to V$ so that $A \circ \pi_N$ is Killing on N with respect to ∇.

Definition 5 A one form ν on M is said to be a Killing form with respect to a connection $\overline{\nabla}$ on M if

$$(\overline{\nabla}_X \nu)(X) \equiv 0, \quad X \in \Gamma(TM),$$

and similarly for a Killing form on $N \subset TM$.

If $\nu(x) = \langle V, X \rangle$ for a vector field V on M, then V is Killing if and only if V is Killing in the classical sense. Killing vector fields and forms are important in the theory of nonholonomic systems.

Lemma 1 *If ν is a Killing form with respect to $\overline{\nabla}$, then $\nu(\dot{q})$ is a constant of motion for the Newton Law system $\overline{D}^2 q/\partial t^2 = 0$.*

Lemma 2 (Arnold [10]) *If ν is a Killing form on N with respect to ∇ such that $\nu(N^\perp) = 0$ then $\nu(\dot{q})$ is a constant of motion for the nonholonomic system (1), with $F \equiv 0$.*

We introduce a special class of systems through a set of assumptions.

Assumption 1

(i) M is parallelizable

(ii) In the nonholonomoic system (1) we may complete the set $\{\omega_1, \ldots, \omega_m\}$ to a basis of $\Gamma(T^*M)$ by the set $\{\nu_1, \ldots, \nu_{n-m}\}$ of one forms ν_k that are Killing on N with respect to ∇ and $\nu_k(N^\perp) \equiv 0$, $1 \leq k \leq n - m$.

Defining V_k, $1 \leq k \leq n - m$, by $\nu_k(X) = \langle V_k, X \rangle$, $X \in \Gamma(TM)$, and setting $(AW_k = \hat{W}_k, 1 \leq k \leq m)$ $AV_k = \hat{V}_k$, $1 \leq k \leq n - m$, $b_{kj} = \nu_k(V_j)$, $1 \leq k$, $j \leq n - m$. Assumption 1 ensures that the matrix $[b_{kj}]$ is invertible on M. It follows that

$$A = \sum_{jk} \hat{W}_j a_{jk}^{-1} \omega_k + \sum \hat{V}_j b_{jk}^{-1} \nu_k.$$

Lemma 3 *Under assumption 1, $A \circ \pi_N$ is Killing on N with respect to ∇ if and only if*

$$\nabla^V_X \left(\sum_j \hat{V}_j b_{jk}^{-1} \right) \equiv 0, \quad 1 \leq k \leq n - m, \quad X \in \Gamma(N). \quad (18)$$

To make condition (18) more attractive, we make a further assumption.

Assumption 2 Under assumption 1, set $V = TM$, $\hat{V}_k = \sum_M V_m b_{mk}$, $\hat{W}_k = \sum_m W_m a_m k$, so that

$$A = \sum_m V_m \nu_m + \sum_m W_m \omega_m, \tag{19}$$

and set

$$\nabla^V V_k = 0, \quad 1 \leq k \leq n-m, \quad \nabla^V W_k = 0, \quad 1 \leq k \leq m.$$

Corollary 1 *Under assumptions 1 and 2, $A \circ \pi_N$ is always Killing on N with respect to ∇.*

Corollary 2 *Under assumptions 1 and 2, for any symmetric two tensor S, $S|_N \equiv 0$, the Newton Law system on M,*

$$\frac{D^{(A,S)^2} q}{\partial t^2} = \pi_N(F), \quad q \in M, \tag{20}$$

has the restriction property. The corresponding nonholonomic system may be rewritten in the form:

$$\frac{D^v v}{\partial t} = \sum_i V_i \nu_i(F); \quad \dot{q} = A_q^{-1}(v), \quad v, \dot{q} \in N.$$

4 Riemannian Connections Compatible with Bundle Maps

In this section we consider the question posed in the introduction, but in the following form: "Does there exist a metric G on M such that $\nabla^G = \nabla^{(A,S)}$ for some bundle map $A : TM \to V$, and symmetric tensor S?". We first note that we may write the unique Riemannian connection ∇^G on M, corresponding to G in the form

$$\begin{aligned} G(z, \nabla_X^G Y) &= G(z, \nabla_X Y) + \frac{1}{2}\{(\nabla_Y G)(X, Z) \quad (21) \\ &\quad +(\nabla_X G)(Y, Z) - (\nabla_2 G)(X, Y)\}, \\ &\quad X, Y, Z \in \Gamma(TM). \end{aligned}$$

We may compare this expression directly with the one for ∇^A in (14), and deduce that $\nabla^G = \nabla^A$ if and only if

$$G(Z, A^{-1}((\nabla_X A)(Y) + (\nabla_Y A)(X)))$$
$$= (\nabla_Y G)(X, Z) + (\nabla_X G)(Y, Z) - (\nabla_Z G)(X, Y).$$

We may simplify this noting the symmetry in X and Y.

Lemma 4 $\nabla^G = G^A$ *if and only if*

$$\begin{aligned} G(Z, A^{-1}(\nabla_X A)(X)) &= (\nabla_X G)(X, Z) \quad (22) \\ &\quad -\frac{1}{2}(\nabla_Z G)(X, X); \\ &\quad X, Z \in \Gamma(TM). \end{aligned}$$

Ascertaining solutions A and G of (22) is a hard problem in general, and is the basis of solving the general problem posed in the introduction. However, we may simplify the situation by choosing a particular class of matrices G.

Theorem 4 *For the metric $G(X, Y) = H(AX, AY)$, $O = \nabla^V H$, where H is a metric on V, then $\nabla^G = \nabla^{(A,S)}$ where S is the symmetric tensor defined by*

$$H(AZ, AS(X, Y)) = \frac{1}{2} H(AX, (\nabla_Y A)(Z) \tag{23}$$
$$-(\nabla_Z A)(Y)) + \frac{1}{2} H(AY, (\nabla_X A)(Z) - (\nabla_Z A)(X))$$

Applying this result in the situation of Theorem 3 we obtain the following

Corollary 3 *The Newton Law system on M with G given by $G(X, Y) = H(AX, AY)$, $\nabla^V H = 0$,*

$$\frac{D^{G^2} q}{\partial t^2} = \frac{D^{(A,S)^2} q}{\partial t^2} = \pi_N(F), \quad q \in M$$

has the restriction property if and only if

$$\begin{array}{lll} (i) & A \circ \pi_N \text{ is Killing on } N \text{ with respect to } \nabla, & (24) \\ (ii) & S|_N \equiv 0, \quad S \text{ defined by (23).} & \end{array}$$

We now examine condition (24)(ii) in the presence of assumptions 1 and 2. Let $\{\hat{\omega}_1, \ldots, \hat{\omega}_m, \hat{\nu}_1, \ldots, \hat{\nu}_m\}$ be a dual frame for $\{W_1, \ldots, W_m, V_1, \ldots, V_{n-m}\}$ and choose the metric H by setting

$$H = \frac{1}{2} \sum_k \hat{\omega}_k \otimes \hat{\omega}_k + \frac{1}{2} \sum_k \hat{\nu}_k \otimes \hat{\nu}_k. \tag{25}$$

It is easily checked that if $V = TM$ and ∇^V is defined as in assumption 2, then $\nabla^V H = 0$. We may now simplify the expression (23) for S to obtain

$$\begin{aligned} &\sum_m \nu_m(z) \nu_m(S(X, Y)) + \sum_m \omega_m(z) \omega_m(S(X, Y)) \\ = \; &\frac{1}{2} \sum_m \nu_m(X) d\nu_m(Y, Z) + \omega_m(X) d\omega_m(Y, Z) \\ &+ \frac{1}{2} \sum_m \nu_m(Y) d\nu_m(X, Z) + \omega_m(Y) d\omega_m(X, Z). \end{aligned}$$

Lemma 5 *If assumptions 1 and 2 hold, and if the tensor S is defined by (23), then $S|_N \equiv 0$ if and only if*

$$d\nu_k(X, \cdot) \equiv 0; \quad 1 \leq k \leq n - m, \quad X \in \Gamma(N). \tag{26}$$

It turns out that in all of the examples in section 6 this condition is never satisfied. But this negative result does not exclude the possibility that there are solutions of $\nabla^G = \nabla^{(A,S)}$ for some metric G on M and pairs (A, S) satisfying (24). The work by A. Lewis [12, 13] develops some related ideas to those expressed here, but no answer to the above question is offered.

5 Integrability of Nonholonomic Systems

Consider the situation studied above, where we are given V, ∇^V and the map A, and corresponding Newton Law system on M,

$$\frac{D^{A^2} q}{\partial t^2} = \overline{F}, \quad q \in M.$$

By Theorem 1 we may rewrite this system in the form

$$\frac{D^v v}{\partial t} = A_q(\overline{F}), \quad \dot{q} = A_q^{-1}(v), \quad (q,v) \in V. \tag{27}$$

In general, this global transformation of coordinates does nothing to simplify the integration of the system equations. In the case of a nonholonomic system (1), Theorem 3 provides a means of viewing it as a restriction to N of the Newton Law system (20). We may apply the same transformation $v = A_q(\dot{q})$ to the Newton Law system and obtain a system in the form (27). If in addition assumptions 1 and 2 hold true the nonholonomic system reduces to the form

$$\begin{aligned} \dot{v}_i &= \nu_i(\overline{F}), \quad 1 \le i \le n-m, \\ \dot{w}_i &= 0, \quad 1 \le i \le m \\ \dot{q} &= \sum_{k,j} V_j b_{jk}^{-1} v_k \end{aligned} \tag{28}$$

where $v \in V$ has been expressed in the form

$$v = \sum_{i=1}^{n-m} v_i V_i(q) + \sum_{i=1}^{m} w_i W_i(q).$$

The corresponding system based on "kinematics" alone is now simply expressed as

$$\dot{q} = \sum_{j,k} V_j b_{jk}^{-1} \nu_i(\overline{F}).$$

The reader should compare this discussion with the less succinct discussion in [2]. Note that the process of reducing the full nonholonomic dynamics to the kinematics is a different reduction from those detailed in Bloch and Crouch [1] and Bloch, Krishnaprasad, Marsden and Murray [9].

6 Examples

Example 1 (Rolling Penny, Bloch and Crouch [2]).

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ -\cos\phi \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -\sin\phi \\ 0 \end{pmatrix}$$
$$+\frac{u_1}{2} \begin{pmatrix} \cos\phi \\ \sin\phi \\ 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with the nonholonomic constraints: $\dot{x} = \cos\phi\dot{\theta}$, $\dot{y} = \sin\phi\dot{\theta}$, and Euclidean metric structure.

Thus,

$$\omega_1 = dx - \cos\phi d\theta, \qquad \omega_2 = dy - \sin\phi d\theta$$
$$\nu_2 = d\phi, \qquad \nu_1 = d\theta + \cos\phi dx + \sin\phi dy$$

Although W_1, W_2, V_1 and V_2 are not mutually orthogonal, W_1 and W_2 are orthogonal to V_1 and V_2. Clearly, ν_2 is Killing. We calculate $(D\nu_1/\partial t)(\dot{q})$.

$$\begin{aligned} \frac{D\nu_1}{\partial t}(\dot{q}) &= -\sin\phi\dot{\phi}\dot{x} + \cos\phi\dot{\phi}\dot{y} \\ &= \dot{\phi}(\dot{y}\cos\phi - \dot{x}\sin\phi) \\ &= \dot{\phi}(\cos\phi\omega_2(\dot{q}) - \sin\phi\omega_1(\dot{q}). \end{aligned}$$

Thus ν_1 is not Killing, but it is Killing when restricted to $N = \{X; \omega_1(X) = \omega_2(X) = 0\}$. Thus the example does satisfy assumption (1).

We also note that if we set $\eta_1 = \sin\phi d\phi$ and $\eta_2 = \cos\phi d\phi$, then

$$(\nabla_X \nu_1)(Y) = \eta_2(X)\omega_2(Y) - \eta_1(X)\omega_1(Y)$$

and

$$\begin{aligned} d\nu_1(X,Y) &= (\nabla_X \nu_1)(Y) - (\nabla_Y \nu_1)(X) \\ &= (\eta_2 \wedge \omega_2 - \eta_1 \wedge \omega_1)(X,Y). \end{aligned}$$

In particular, condition (26) is not satisfied.

Example 2 (Bates-Sniatycki example [6]).

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \lambda \begin{pmatrix} -y \\ 0 \\ 1 \end{pmatrix} + \frac{u_1}{\sqrt{1+y^2}} \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

with the nonholonomic constraint $\dot{z} = y\dot{x}$ and Euclidean metric structure.

Thus

$$\begin{aligned} \omega &= dz - ydx, \\ \nu_1 &= \frac{1}{\sqrt{1+y^2}}(dx + ydz), \\ \nu_2 &= dy. \end{aligned}$$

Thus, W, V_1 and V_2 form an orthogonal set. Clearly ν_2 is Killing. We calculate $(D\nu_1/\partial t)(\dot{q})$.

$$\begin{aligned} \frac{D\nu_1}{\partial t}(\dot{q}) &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+y^2}}\right)\dot{x} + \frac{d}{dt}\left(\frac{y}{\sqrt{1+y^2}}\right)\dot{z} \\ &= \frac{\dot{y}\omega(\dot{q})}{(1+y^2)^{3/2}}. \end{aligned}$$

Thus ν_1 is not Killing, but it is Killing when restricted to $N = \{X; \omega(X) = O\}$. Thus the example does satisfy assumption (1).

We also note that if we set $\eta = (dy/(1+y^2)^{3/2})$ then $(\nabla_X \nu_1)(Y) = \eta(X)\omega(Y)$, so

$$d\nu_1(X,Y) = (\nabla_X \nu_1)(Y) - (\nabla_Y \nu_1)(X) = (\eta \wedge \omega)(X,Y).$$

In particular, condition (26) is not satisfied.

Example 3 (Rolling Ball, Bloch and Crouch [2]).

$$\begin{aligned} J\dot{\nu} &= S(\nu)J\nu + \lambda_1 Pe_1 + \lambda_2 Pe_2, \qquad \nu \in \mathbf{R}^3 \\ \dot{P} &= S(\nu)P \qquad P \in SO(3) \\ m\ddot{x} &= \lambda_2 + u_1 \qquad x, y \in \mathbf{R} \\ m\ddot{y} &= -\lambda_1 + u_2 \end{aligned}$$

with the nonholonomic constraints

$$e_2^T P^T \nu + \dot{x} = 0, \qquad e_1^T P^T \nu - \dot{y} = 0$$

and metric structure

$$\begin{aligned} &\langle(\nu_A, \dot{x}_A, \dot{y}_A), (\nu_B, \dot{x}_B, \dot{y}_B)\rangle \\ &= \frac{1}{2}\nu_A^T J \nu_B + \frac{1}{2} m(\dot{x}_A \dot{x}_B + \dot{y}_A \dot{y}_B). \end{aligned}$$

This system is ten dimensional, evolving on $T(\mathbf{R}^2 \times SO(3))$. We let $a^T \partial/\partial\nu$ ($a^T d\nu$) the right invariant vector field (one form) on $SO(3)$ with generator a. We may obtain

$$\begin{aligned} \omega_1 &= (Pe_1)^T d\nu - dy, \qquad \omega_2 = (Pe_2)^T d\nu + dx \\ \nu_1 &= (JPe_1)^T d\nu + m dy, \quad \nu_2 = (JPe_2)^T d\nu - m dx \\ \nu_3 &= (JPe_3)^T d\nu. \end{aligned}$$

V_3 corresponds to the fact that along the motion we have that $e_3^T P^T J\nu$ is a constant. In general, we could insert another torque, u, exerted about Pe_3, by adding a term uPe_3 to the first equation. Although W_1, W_2, V_1, V_2, V_3 is not an orthogonal set relative to the metric, we do have that $\{W_1, W_2\}$ is orthogonal to $\{V_1, V_2, V_3\}$, from which it is easy to see that they form a spanning set for the tangent spaces $T_q(\mathbf{R}^2 \times SO(3))$.

It was demonstrated in Bloch and Crouch [1] that $(Pe_k)^T(\partial/\partial\nu)$ are Killing vector fields, relative to the metric structure so V_1, V_2, V_3 are indeed Killing, even without restricting to N. It is also interesting to integrate the system equations using the easily verified identities:

$$\begin{aligned} \frac{d}{dt}\nu_1(\dot{q}) &= u_2, \qquad \frac{d}{dt}\nu_2(\dot{q}) = -u_1 \\ \frac{d}{dt}\nu_3(\dot{q}) &= \omega_2(\dot{q}) = \omega_1(\dot{q}) \equiv 0. \end{aligned}$$

Setting $\mathbf{J} = J + m(Pe_1 e_1^T P^T + Pe_2 e_2^T P^T)$ and $e_3^T P^T J\nu = d (= \text{const})$ we obtain the reduced equations (28) in the form:

$$\begin{aligned} \dot{P} &= S(\mathbf{J}^{-1}P(e_1 a_2 + e_2(-a_1) + e_3 d))P \\ \dot{x} &= -e_2^T P^T \mathbf{J}^{-1} P(e_1 a_2 + e_2(-a_1) + e_3 d) \\ \dot{y} &= e_1^T P^T \mathbf{J}^{-1} P(e_1 a_2 + e_1(-a_1) + e_3 d) \\ \dot{a}_1 &= u_1, \qquad \dot{a}_2 = u_2. \end{aligned}$$

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