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Brief Paper

Output feedback and tracking of nonlinear systems with model predictive control☆

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Abstract

This paper presents an output feedback Receding Horizon (*RH*) control algorithm for nonlinear discrete-time systems which solves the problem of tracking exogenous signals and asymptotically rejecting disturbances generated by a properly defined exosystem. The regulator is composed by an internal model of the exosystem and a stabilizing *RH* regulator. Some robustness results are also achieved in the case of constant references.

Keywords: Model predictive control; Receding horizon control; Nonlinear control; Tracking; Output feedback

1. Introduction

Many stabilizing state-feedback control algorithms based on the Receding Horizon (*RH*) approach have recently been proposed for nonlinear systems, see the survey papers (De Nicolao, Magni, & Scattolini, 2000; Mayne, Rawlings, Rao, & Scokaert, 2000; Allgöwer, Badgwell, Qin, Rawlings, & Wright, 1999). However, stabilizing nonlinear *RH* control algorithms dealing with both the output feedback and tracking problems are still missing; at present only the method proposed in De Nicolao, Magni, and Scattolini (1997), is available, which guarantees tracking of constant references for systems described by an input/output Nonlinear AutoRegressive eXogenous (*NARX*) model. Hence, this technique is not suited to deal with all those cases in which a model of the system is a-priori known from physical considerations. Moreover, the tracking of variable signals, such as periodic trajectories, is still an open question. In the context of output feedback (without tracking), it is worth mentioning the algorithm presented in Michalska and Mayne (1995) based on a previous state-feedback control law (Michalska & Mayne, 1993) and a new moving horizon observer.

This paper presents a class of output feedback *RH* control algorithms for nonlinear discrete-time systems solving the problem of tracking exogenous signals and asymptotically rejecting disturbances generated by a properly defined exosystem. First, some results on *RH* output feedback stabilizing controllers are briefly reported (Magni, Nicolao, & Scattolini, 1998). Then, the tracking problem is considered. According to Isidori (1995), the proposed regulator is composed using an internal model of the exosystem and a stabilizing output feedback control law. Restricting attention to constant exogenous signals, it is also shown that the adopted regulator structure can guarantee robust tracking in the face of certain plant parameter perturbations.

2. Preliminaries and output feedback

The nonlinear discrete-time dynamic system is

$$x(k+1) = f(x(k), u(k)), \quad x(t) = \bar{x}, \quad k \geqslant t, \tag{1}$$

$$y(k) = h(x(k)), \tag{2}$$

where $x \in R^n$ is the state, $y \in R^m$ is the output, and $u \in R^m$ is the input. The functions $f(\cdot,\cdot)$ and $h(\cdot)$ are smooth

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(i.e. C^∞) functions of their arguments and $f(0,0)=0$, $h(0)=0$. In (1), x and u must fulfill the following constraints:

$$x(k)\in X,\quad u(k)\in U,\quad k\geqslant t, \tag{3}$$

where X and U are closed subsets of R^n and R^m, respectively, both containing the origin as an interior point.

Consider now the following *Finite-Horizon Optimal Control Problem (FHOCP)*: Minimize with respect to $u_{t,t+N-1} := [u(t)\ u(t+1)\ldots u(t+N-1)]$, $N\geqslant 1$, the cost function

$$J(\bar{x},u_{t,t+N-1},N)=\sum_{i=0}^{N-1}\psi(x(t+i),u(t+i)) + V_f(x(t+N)), \tag{4}$$

subject to (1), (3) and the terminal state constraint $x(t+N)\in X_f$ where $\psi(x,u)=x'Qx+u'Ru$, $Q>0$, $R>0$, V_f is a suitable terminal state penalty and $X_f\in R^n$ is a suitable set including the origin as an interior point.

Given an initial state $\bar{x}\in X$ the sequence $u_{t,t+N-1}$ is termed *admissible* if, when applied to system (1), $x(k)\in X$, $u(k)\in U$, $t\leqslant k<t+N-1$, and $x(t+N)\in X_f$. Associated with the *FHOCP*, the following control strategy is formulated.

Nonlinear Receding Horizon (NRH) control law: At every time instant t let $\bar{x}=x(t)$ and find an admissible control sequence $u^o_{t,t+N-1}$ solving the *FHOCP*. Then, apply the control $u(t)=u^o(\bar{x})$, where $u^o(\bar{x})$ is the first column of $u^o_{t,t+N-1}$.

The *NRH* control law is defined by the function $u=\kappa^{NRH}(x)$, where $\kappa^{NRH}(x)=u^o(x)$. Let $X_0(N)$ be the set of states $\bar{x}$ such that there exists an admissible control sequence $u^o_{t,t+N-1}$ solving the *FHOCP*.

Assumption A1. Given Q,R,N, then the terminal penalty $V_f(\cdot)$ and the set X_f are such that the closed-loop system $x(k+1)=f(x(k),\kappa^{NRH}(x(k)))$ admits the origin as an exponentially stable equilibrium (Vidyasagar, 1993) with $X_0(N)$ as the domain of attraction.

For an overview and a critical comparison of different algorithms satisfying A1, see e.g. De Nicolao et al. (2000); Mayne et al. (2000); Allgöwer et al. (1999). In order to obtain a dynamic output feedback regulator the stabilizing *RH* state-feedback control law is combined with the asymptotic state observer

$$\hat{x}(k+1)=g(\hat{x}(k),h(x(k)),u(k)),\quad \hat{x}(t)=\bar{\hat{x}} \tag{5}$$

where $\hat{x}\in R^n$ is the estimated state, $g(\cdot,\cdot,\cdot)$ is a smooth function of its arguments and $g(0,0,0)=0$.

Assumption A2. The system (1) and (2) is weakly (exponentially) detectable (Vidyasagar, 1993) and the weak (exponential) detector (5) is used.

For the design of an observer satisfying A2, see e.g. Michalska et al. (1995); Allgöwer et al. (1999); Song and Grizzle (1995). The overall dynamic regulator is then described by (5) and

$$u=\kappa^{NRH}(\hat{x}),\quad \kappa(0)=0. \tag{6}$$

Assumption A3. The control law (6) and the state observer (5) are Lipschitz continuous with respect to their arguments.

By resorting to the results given in Magni et al. (1998), the following Lemma can be proven.

Lemma 1. *Assume that* A1–A3 *hold. Then, the origin in* $\Re^n\times\Re^n$ *is an asymptotically (exponentially) stable equilibrium point of the closed-loop system* (1), (2), (5) *and* (6).

Remark 1. The Lipschitz assumption A3 is not easily relaxable, see e.g. Vidyasagar (1993) where similar requirements are introduced in the continuous time case. Anyway, under suitable assumptions on the regularity of the system and the constraints, it can be shown that the *RH* control law is Lipschitz continuous, see e.g. Ohno (1978) where the continuity of the control law is studied in the context of general finite-horizon nonlinear optimal control and Bemporad, Morari, Dua, and Pistikopoulos (2000) for the *LQR* problem with constraints.

3. Output tracking

Consider the following nonlinear, discrete-time dynamic system

$$\begin{aligned} x(k+1)&=f(x(k),u(k),w(k)),\\ e(k)&=h_e(x(k),w(k)), \end{aligned} \tag{7}$$

where $f(x,u,w)$, $h_e(x,w)$ are smooth functions with $f(0,0,0)=0$, $h_e(0,0)=0$ and $w\in R^r$ are exogenous input variables including disturbances $d(\cdot)$ to be rejected and/or references $y_{ref}(\cdot)$ to be tracked. It is assumed that $w(\cdot)$ is the solution of a (possibly nonlinear) homogeneous dynamic exosystem

$$w(k+1)=s(w(k)), \tag{8}$$

where $s(w)$ is a C^∞ function with $s(0)=0$ and the initial condition $w(0)$ belongs to some neighborhood W of the origin in R^r.

Assumption A4. The exosystem is neutrally stable (Isidori, 1995) so that all the eigenvalues of the matrix $S=(\partial s/\partial w)(0)$ are on the unit circle.

For ease of reference, some definitions and preliminary results derived from Isidori (1995), where continuous-time systems are considered, are reported.

Definition 1. Consider the smooth autonomous systems

$$x(k+1) = f(x(k)), \quad y(k) = h(x(k)), \quad x \in X \tag{9}$$

$$\tilde{x}(k+1) = \tilde{f}(\tilde{x}(k)), \quad y(k) = \tilde{h}(\tilde{x}(k)), \quad \tilde{x} \in \tilde{X} \tag{10}$$

with the same output space $Y = R^m$. Assume $f(0) = 0$, $h(0) = 0$, $\tilde{f}(0) = 0$, $\tilde{h}(0) = 0$ and let (9), (10) be denoted by $\{X, f, h\}$ and $\{\tilde{X}, \tilde{f}, \tilde{h}\}$, respectively. System $\{X, f, h\}$ is said to be immersed into system $\{\tilde{X}, \tilde{f}, \tilde{h}\}$ if there exists a C^l mapping $\tau: X \to \tilde{X}$, with $l \geqslant 1$, satisfying $\tau(0) = 0$ and $h(x) \neq h(z) \Rightarrow \tilde{h}(\tau(x)) \neq \tilde{h}(\tau(z))$, such that $\tau(f(x)) = \tilde{f}(\tau(x))$, $h(x) = \tilde{h}(\tau(x))$ for all $x \in X$.

Then, any output response generated by $\{X, f, h\}$ is also an output response of $\{\tilde{X}, \tilde{f}, \tilde{h}\}$. According to Isidori (1995), we are now in the position to state the

Output Tracking Problem (OTP): Given a nonlinear system of the form (7) and a neutrally stable exosystem (8) find a dynamic regulator

$$z(k+1) = \eta(z(k), e(k)),$$

$$u(k) = \theta(z(k)) \tag{11}$$

with z defined in a neighborhood Z of the origin in R^v such that

(S) The linearization of

$$x(k+1) = f(x(k), \theta(z(k)), 0),$$

$$z(k+1) = \eta(z(k), h_e(x(k), 0))$$

at the equilibrium $(x, z) = (0,0)$ is asymptotically stable;

(R) there exists a neighborhood $\mathscr{V} \subset X \times Z \times W$ of (0,0,0) such that, for any initial condition $(x(0), z(0), w(0)) \in \mathscr{V}$, the solution of

$$x(k+1) = f(x(k), \theta(z(k)), w(k)),$$

$$z(k+1) = \eta(z(k), h_e(x(k), w(k))),$$

$$w(k+1) = s(w(k))$$

satisfies $\lim_{k \to \infty} h_e(x(k), w(k)) = 0$.

Letting $A = \partial f(0,0,0)/\partial x$, $B = \partial f(0,0,0)/\partial u$ and $C = \partial h_e(0,0)/\partial x$, the following Lemma is the extension of the discrete-time case of Theorem 8.4.4 in Isidori (1995).

Lemma 2. *The Output Tracking Problem is solvable if and only if there exist mappings $x = \pi(w)$ and $u = c(w)$, with $\pi(0) = 0$ and $c(0) = 0$, both defined in a neighborhood $W^0 \subseteq W$ of the origin, satisfying the conditions*

$$\pi(s(w)) = f(\pi(w), c(w), w),$$

$$0 = h_e(\pi(w), w) \tag{12}$$

for all $w \in W^0$, such that the autonomous system $\{W^0, s, c\}$ through the map $\xi = \tau(w)$ is immersed into a system

$$\xi(k+1) = \varphi(\xi(k)),$$

$$u(k) = \gamma(\xi(k))$$

defined on a neighborhood Ξ^0 of the origin in R^v, in which $\varphi(0) = 0$ and $\gamma(0) = 0$, and the two matrices $\Phi = (\partial\varphi/\partial\xi)(0)$, $\Gamma = (\partial\gamma/\partial\xi)(0)$ are such that the pair $(\hat{A} + \bar{T}\bar{C}, \bar{B})$ is stabilizable for some choice of the matrix T, and the pair $(\hat{A} + \bar{B}\bar{\Gamma}, \bar{C})$ is detectable, where $\hat{A} := diag\{A, \Phi\}$, $\bar{B} := [B' \ 0]'$, $\bar{C} := [C \ 0]$, $\bar{\Gamma} := [0 \Gamma$, $\bar{T} := [0 \ T']'$.

It is worth pointing out that Lemma 2 is only a local result so that Theorem 3, below, will be a local result as well. Note that, as observed in Isidori (1995), the stabilizability of $(\hat{A} + \bar{T}\bar{C}, \bar{B})$ and the detectability of $(\hat{A} + \bar{B}\bar{\Gamma}, \bar{C})$ imply that the triplet $(\bar{A}, \bar{B}, \bar{C})$ is stabilizable and detectable where $\bar{A} := \hat{A} + \bar{T}\bar{C} + \bar{B}\bar{\Gamma}$. Conditions guaranteeing the stabilizability of the pair $(\hat{A} + \bar{T}\bar{C}, \bar{B})$ and the detectability of the pair $(\hat{A} + \bar{B}\bar{\Gamma}, \bar{C})$ are given in Isidori (1995) in terms of the original data (A, B, C) and the pair (Φ, Γ). In order to solve OTP with an NRH control law, first consider the system composed by (7) and (8) and an internal model of the exosystem:

$$x(k+1) = f(x(k), u(k), w(k)), \tag{13}$$

$$z_1(k+1) = \varphi(z_1(k)) + Te(k), \tag{14}$$

$$w(k+1) = s(w(k)), \tag{15}$$

$$e(k) = h_e(x(k), w(k)), \tag{16}$$

$$u(k) = \gamma(z_1(k)) + v(k) \tag{17}$$

where T, $\varphi(\cdot)$: $R^v \to R^v$ and $\gamma(\cdot): R^v \to R^m$ fulfill the conditions of Lemma 2. The signal $v(k)$ is the part of the input that will be used to solve the tracking problem. Let w_M and w_U be the measurable and the unmeasurable components of w so that $w(k+1) = [w_U(k+1)' \ w_M(k+1)']' = [s_U(w_U(k), w_M(k))' \ s_M(w_M(k))']'$ where s_M and s_U are suitable functions. Define the states of system (13)–(17) to be detected as $x_D = [x' \ z_1' \ w_U']'$. Then, in view of (13)–(17)

$$x_D(k+1) = f_D(x_D(k), v(k), w_M(k)),$$

$$x_D(t) = \bar{x}_D, \ k \geqslant t,$$

$$e(k) = h_D(x_D(k), w_M(k)) \tag{18}$$

where f_D, h_D, are suitable functions and $\bar{x}_D$ is defined in an obvious way. Now assume that (18) is weakly detectable and set up a weak detector for x_D:

$$\hat{x}_D(k+1) = g(\hat{x}_D(k), e(k), [v(k)' \quad w_M(k)']'). \tag{19}$$

Let also $z := [x_D' \ w_M' \ z_1^{c'}]'$, $\hat{z} := [\hat{x}_D' \quad w_M' \ z_1']$ where $z_1^c \in \Re^v$ is used to emulate the internal model with the known initial condition z_1 in order to satisfy the constraints on u. Obviously, $z \in \Re^{n+2v+r}$ and $\hat{z} \in \Re^{n+2v+r}$. In the following, Z will be a closed subset in $\Re^{n+2v+r}$ containing the origin as an interior point. Letting $\zeta(i) = [I_{n+v} \ 0_{v+r}]z(i)$, where I_n and O_n are row vectors with all the n elements equal to one and zero, respectively, in order to synthesize the feedback RH control law $v = \kappa^{NRH}(\hat{z}(k))$, minimize with respect to $v_{t,t+N-1} :=$

$[v(t) \quad v(t+1) \ldots v(t+N-1)]$, $N \geqslant 1$, the cost function

$$J(\hat{z}(k), v_{t,t+N-1}, N) = \sum_{i=t}^{t+N-1} \psi\left(\zeta(i) - \begin{bmatrix} \pi(w(i)) \\ \tau(w(i)) \end{bmatrix}, v(i)\right) + V_f(z(t+N)) \tag{20}$$

subject to (13)–(17) and

$$z_1^c(k+1) = \varphi(z_1^c(k)) + Te(k),$$

$$u^c(k) = \gamma(z_1^c(k)) + v(k)$$

with initial state $\hat{z}(t)$, to the constraints $z(i) \in Z$, $u^c(i) \in U$, $t \leqslant i < t+N-1$ and to the terminal state constraint $z(t+N) \in X_f$, where V_f is a suitable terminal penalty and $X_f \in \Re^{n+2v+r}$ is a suitable set. Note that $u(t) = u^c(t)$. Based on this finite-horizon optimization problem the following NRH state-feedback control law is obtained in the usual way:

$$v = \kappa^{NRH}(\hat{z}). \tag{21}$$

Theorem 3. *Consider system* (7) *and suppose that* A1, A3 and A4 *hold*, *system* (18) *is weakly exponentially detectable*, *and the output tracking problem is solvable. Then*, *the NRH dynamic controller* (19), (14), (17), (21) *solves OTP*.

Remark 2. Estimating w_U could seem redundant, since disturbance rejection and tracking are guaranteed by the internal model. However, the predictive performance index (20) calls for the knowledge of the future values of such states. Analogously, estimating z_1 is necessary to guarantee that the value used in the control law is equal to the one required in nominal conditions notwithstanding the presence of modelling errors or disturbances, i.e. the robustness of the regulation action.

3.1. Robust tracking for constant references

If constant references and disturbances are considered and (A, B, C) does not possess transmission zeros at point one in the complex plane, then the particular choice $\{\Xi^0, \varphi, \gamma\} = \{\Xi^0, I, I\}$ and $T = I$ solves the output tracking problem in a robust way, that is also in the presence of plant parameter perturbations. More precisely, let the perturbed plant be described by

$$\begin{aligned} x^p(k+1) &= f^p(x^p(k), u(k), w(k)) \\ &= f(x^p(k), u(k), w(k)) + \Delta f(x^p(k), u(k), w(k)), \\ e^p(k) &= h_e^p(x^p(k), w(k)) \\ &= h_e(x^p(k), w(k)) + \Delta h_e(x^p(k), w(k)), \end{aligned} \tag{22}$$

where $\Delta f(x, u, w)$ and $\Delta h_e(x, w)$ are fixed smooth functions in (x, u, w). With reference to this class of perturbations, consider the following *Robust Output Tracking Problem* (*ROTP*). Given the nonlinear system (22), find a dynamical control law (5) such that:

- Property (S) of the OTP holds with $\Delta f(x, w, u) = 0$ and $\Delta h_e(x, w) = 0$.
- (Property R^p) for each $\Delta f(x, w, u)$ and $\Delta h_e(x, w)$, such that $(x^p, z^p, w) = (0,0,0)$ is a stable equilibrium for the perturbed closed-loop system, there exists a neighborhood $\mathscr{V} \subset X \times Z \times W$ of (0,0,0) such that for each initial condition $(x^p(0), z^p(0), w(0)) \in \mathscr{V}$, the solution of

  $$x^p(k+1) = f^p(x^p(k), \theta(z^p(k)), w(k)),$$

  $$z^p(k+1) = \eta(z^p(k), h_e^p(x^p(k), w(k))),$$

  $$w(k+1) = w(k)$$

  satisfies $\lim_{k \to \infty} h_e^p(x^p(k), w(k)) = 0$

Assumption A5. For each $\Delta f(x, w, u)$ and $\Delta h_e(x, w)$, there exist mappings $x = \pi^p(w)$, $u = c^p(w)$, with $\pi^p(0) = 0$, $c^p(0) = 0$, both defined in a neighborhood $W^0 \subseteq W$ of the origin, satisfying the conditions

$$\begin{aligned} \pi^p(w) &= f^p(\pi^p(w), c^p(w), w), \\ 0 &= h_e^p(\pi^p(w), w). \end{aligned} \tag{23}$$

Assumption A6. (A, B, C) is stabilizable and does not possess transmission zeros at point one in the complex plane.

For the problem $ROTP$ the following extension of Theorem 3 can be proven.

Corollary 4. *Consider system* (7) and suppose that A1, A3, A5, A6 *hold*, *system* (18) *is weakly exponentially detectable and the output tracking problem is solvable. Then the dynamic NRH controller* (19), (14), (17), (21) with $\gamma = I$, $\varphi = I$, $T = I$ *solves the ROTP for constant exogenous signals*.

4. Example

Consider the cement milling circuit presented in Magni, Bastin, and Wertz (1999) and described by

$$\begin{aligned} 0.3\dot{y}_f &= -y_f + (1 - \alpha(l, v_s, d))\varphi(l, d), \\ \dot{l} &= -\varphi(l, d) + u_f + y_r, \\ 0.01\dot{y}_r &= -y_r + \alpha(l, v_s, d)\varphi(l, d), \end{aligned} \tag{24}$$

where y_f is the product flow rate (tons/h), l is the load in the mill (tons), y_r is the tailings flow rate (tons/h), $\varphi(l, d)$ is the output flow rate of the mill (tons/h), u_f is the feed flow rate (tons/h), v_s is the classifier speed (rpm), d represents the hardness of the material inside the mill with respect to the nominal one, $\alpha(l, v_s, d) = \varphi^{0.8} v_s^4 / (3.56 * 10^{10} + \varphi^{0.8} v^4)$, and $\varphi(l, d) = max\{0; (-0.1116dl^2 +$

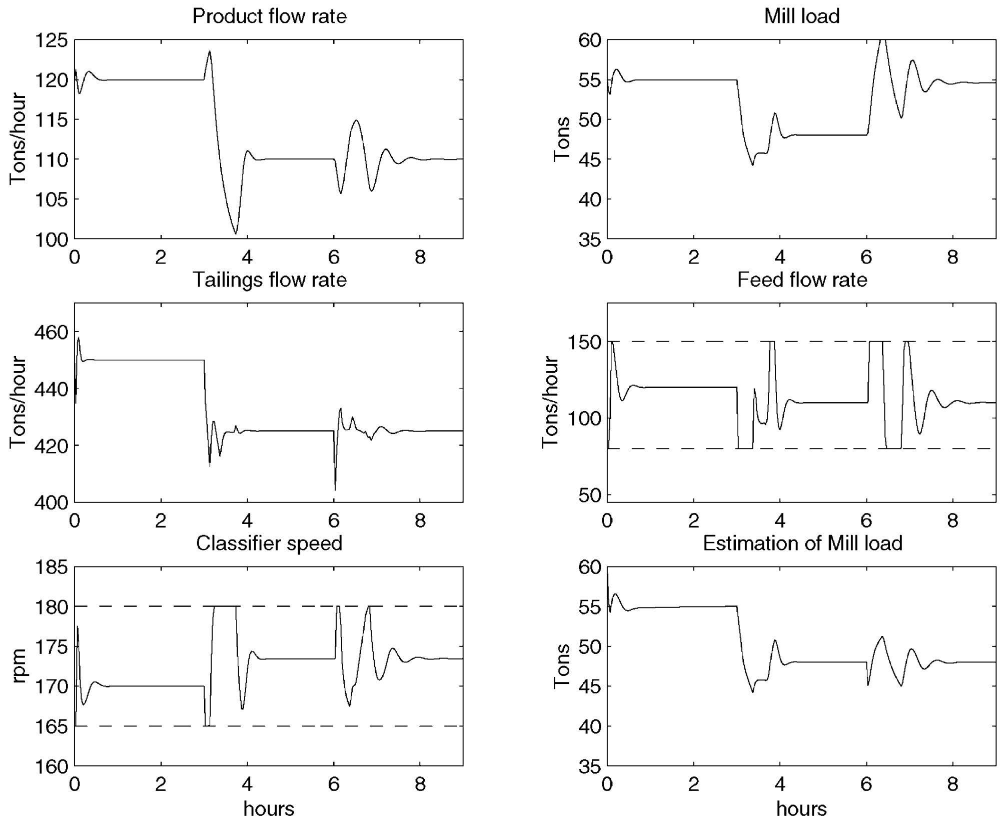

Fig. 1. Responses to step variations of the set-point from [120 450]′ to [110 425]′ after 3 h and of the hardness of the material from 1 to 1.1 after 6 h.

16.50l)}. In Magni et al. (1999) a multivariable state feedback NRH control strategy has been proposed to improve the performance and enlarge the stability region obtained with the LQ controller that is now in operation (Van Breusegen, Chen, Werbrouck, Bastin, & Wertz, 1994). However, this controller needs to measure the load l, which is a difficult task. Then, it is useful to resort to an output-feedback strategy based on the measurements of y_r and y_f. To this end, let $x = [y_f, l, y_r]'$, $u = [u_f \;\; v_s]'$, $y = [y_f, y_r]'$ and the equilibrium point $x^* = [120 \; 55 \; 450]'$, $u^* = [120,170]$. The tracking problem is to asymptotically follow any prescribed constant reference y_{ref} in a neighborhood of (x^*, u^*). System (24) has been discretized with a sampling period of 2 min using a fourth-order explicit Runge–Kutta scheme. The discretized plant so obtained takes the form (1). Then the closed-loop system (13)–(17) has been considered with $\varphi = I$, $\gamma = I$, $T = I$, $w = y_{ref}$, $e = h_e(x, w) = w - y$ and an Extended Kalman Filter has been applied to a "noisy" version of system (18) with $w_M = w$ and assuming that two jointly Gaussian and mutually independent noises $v_z(k) \sim WN(0, I)$, and $v_\xi(k) \sim WN(0, I)$ act on the state and on the output of the system, respectively. The cost function (24) has $\pi(w) = \tilde{x}$, $\tau(w) = \tilde{u}$ and $u \in \{(u_f, v_s)$: $80 \leqslant u_f \leqslant 150$, $165 \leqslant v_s \leqslant 180\}$, where $\tilde{x}$ and $\tilde{u}$ are such that $\tilde{x} = f(\tilde{x}, \tilde{u})$, $h_e(\tilde{x}, y_{ref}) = 0$. The weighting matrices are $Q = I$ and $R = 0.01{*}I$ whereas the control horizon is $N = 3$, while V_f and X_f have been chosen according to De Nicolao, Magni, and Scattolini (1998). It is rather straightforward to verify that the constraints are regular in the region were the nonlinear systems stabilized by the LQ saturated control law. Therefore, according to the results of Ohno (1978), the NRH control law in that region is Lipschitz continuous, thus satisfying A3. In Fig. 1, we present simulation results obtained starting from the nominal equilibrium point $\bar{x} = x^*$, with $z_1(0) = u^*$, $\bar{x}_D = [120 \; 60 \; 450 \; 120 \; 170]'$; after 3 h the set point changes from [120 450]′ to [110 425]′; after 6 h the hardness of the material d changes from 1 to 1.1. The last picture in Fig. 1 shows that even when a hardness change occurs, the estimation of the load in the mill tends to the nominal value.

Appendix

Proof of Theorem 3. First consider the system (13), (14), (19), (16), (17), (21) with $w = 0$. If the OTP is solvable from Lemma 2, it follows that $(\bar{A}, \bar{B})$ is stabilizable. Then, it is possible to find X_f and $V_f(\cdot)$ satisfying A1. From A1, A3, the weakly exponentially detectability of (18), and Lemma 1 it follows that condition (S) is satisfied. Moreover, by the assumption of solvability, there exist mappings $x = \pi(w)$, $u = c(w)$ and $z_1 = \tau(w)$ such that (12) are satisfied and $\tau(s(w)) = \varphi(\tau(w))$, $c(w) = \gamma(\tau(w))$. Obviously, $\kappa^{NRH}([\pi(w)' \ \tau(w)' \ w' \ \tau(w)']') = 0$ and then, in view of the weak detectability of system (18)

$$\pi(s(w)) = f(\pi(w), c(w) + 0, w),$$

$$[\pi(s(w))' \ \tau(s(w))' \ s_U(w_U, w_M)']'$$

$$= g([\pi(w)' \ \tau(w)' \ w_U']', 0, [0 \ w_M']'),$$

$$\tau(s(w)) = \varphi(\tau(w)),$$

$$0 = h_e(\pi(w), w)$$

for all $w \in W^0$. From Lemma 8.4.1 of Isidori (1995) condition (R) is satisfied.

Proof of Corollary 4. In view of A6, the pair $(\bar{A}, \bar{B})$ with $T = I$, $\Gamma = I$, $\Phi = I$ is stabilizable (De Nicolao et al., 1997). Then, it is possible to find X_f and $V_f(\cdot)$ satisfying A1. From A1, A3, the weakly exponentially detectability of (18), and Lemma 1 it follows that condition (S) is satisfied. Moreover, by hypothesis, there exist mappings $x = \pi^p(w)$, $u = c^p(w)$ such that (23) hold. Note that $\tau^p(w) = c^p(w)$ is the mapping used to obtain system $\{\Xi^0, I, I\}$ from system $\{W^0, I, c^p\}$, whereas $\pi(w)$ and $\tau(w) = c(w)$ are the mappings obtained for the nominal system. Now $\kappa^{NRH}([\pi(w)' \ \tau(w)' \ w' \ \tau^p(w)']') = 0$ and

$$\pi^p(w) = f^p(\pi^p(w), c^p(w) + 0, w),$$

$$[\pi(w)' \quad \tau(w)' \quad w_U']'$$

$$= g([\pi(w)' \quad \tau(w)' \quad w_U']', 0, [0 \quad w_M']'), \tag{25}$$

$$c^p(w) = c^p(w),$$

$$0 = h_e^p(\pi^p(w), w).$$

Therefore, $x^p = \pi^p(w)$ is by construction a center manifold for

$$x^p(k+1) = f^p(x^p(k), z_1(k) + \kappa^{NRH}(\hat{z}(k)), w(k)),$$

$$\hat{x}_D(k+1) = g(\hat{x}_D(k), h_e^p(x^p(k), w(k)),$$
$$[\kappa^{NRH}(\hat{z}(k))' \quad w_M(k)']')$$

$$z_1(k+1) = z_1(k) + h_e^p(x^p(k), w(k)),$$

$$w(k+1) = w(k).$$

Moreover, by the last equation of (25), the error satisfies $e^p(k) = h_e^p(x^p(k), w) - h_e^p(\pi^p(w), w)$. Consider $\Delta f(x, w, u)$ and $\Delta h(x, w)$, such that $(x^p, z^p, w) = (0,0,0)$ is a stable equilibrium for the perturbed closed-loop system. Then, for any sufficiently small $(x^p(0), z^p(0), w(0))$, the solution $(x^p(k), z^p(k), w)$ of the perturbed closed-loop system remains in an arbitrary small neighborhood of (0,0,0) for all $k \geqslant 0$. Using a property of center manifolds, it is deduced that there exist real numbers $a > 0$ and $b > 0$ such that $\|x^p(k) - \pi^p(w)\| \leqslant a\mathrm{e}^{-bk}\|x^p(0) - \pi^p(w)\|$ for all $k \geqslant 0$. By continuity of $h_e^p(x^p(k), w)$, $\lim_{k\to\infty} e^p(k) = 0$, i.e. the condition ($\mathrm{R}^p$) is also satisfied.

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