Feedback Control of Nonholonomic Mobile Robots

Dr. ing. thesis

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Contents

Preface

This thesis is submitted for the Doktor ingeniør degree at the Norwegian Institute of Technology (NTH). The research was carried out in the period from January 1990 to January 1993. Professor Olav Egeland, Department of Engineering Cybernetics, was my supervisor, and Professor Carlos Canudas de Wit, Laboratoire d'Automatique de Grenoble, was my advisor. During this period, I benefitted from a one-year stay in Professor Carlos Canudas de Wit's group in Grenoble in 1991.

The background for my work on nonholonomic mobile robots was that the feedback control of such systems appeared to be a new and challenging field in robotics involving advanced nonlinear theory applied to physical systems.

I would like to thank my supervisor Professor Olav Egeland for his good advice, valuable comments, and for being a source of inspiration and encouragement. I am indebted to Professor Carlos Canudas de Wit for introducing me to the problem of controlling nonholonomic systems and the good cooperation with him. I also thank him for welcoming me to Grenoble and introducing me to many researchers. I acknowledge valuable discussions with Morten Dalsmo during his diploma work concerning the use of piecewise analytic feedback laws. I am very grateful to Ola-Erik Fjellstad for proofreading this thesis and providing valuable comments and to Professor Andrew Kantar for his editing work.

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Summary

This doctoral thesis contains new theoretical results on the feedback control of nonholonomic mobile robots like carts (unicycles) and cars with n trailers. A review of previous work on the controllability and open- and closed-loop control of nonholonomic mobile robots is also provided, and unresolved problems concerning feedback control are discussed. The following problems have been solved in this work:

- **Exponential convergence of a cart:** A piecewise analytic feedback law has been developed for the kinematic model of a 2-dof cart which gives *global*, *exponential* convergence to any given position and orientation. The main idea is to introduce a set of curves that pass through the actual position with the tangent corresponding to the desired orientation. The curves chosen in this thesis are circles. The feedback law proposed here makes the cart converge to the desired configuration asymptotically along one of these circles. This feedback law introduces discontinuity surfaces. The discontinuities are discussed, and it is shown that they are not attractive. Design guidelines and simulation results are also presented.

- **Path following:** The piecewise smooth feedback law has been extended to make the cart follow a path which is composed of arcs of circles and straight lines. This approach allows stopping and reversing phases. The method is illustrated by simulations.

- **Conversion of the kinematics of a car with n trailers into a chained form:** A new kinematic model has been derived for a car with n trailers. This model has been (locally) converted into a chained form suitable for control. This implies that such a trailer system can be locally controlled by using existing open- or closed-loop control strategies for chained systems.

- **Exponential stabilization of chained systems:** A feedback law has been proposed which makes a chained system exponentially stable in a sense which is defined in this thesis. In addition to being a function of the state, the feedback law depends on time and a parameter which switches at discrete instants of time. The convergence is analyzed theoretically and simulation results are presented. Using this controller design technique for chained systems, several types of nonholonomic mobile robots, like a car with n-trailers, can be locally stabilized with exponential convergence.

A lemma on exponential convergence of a class of time-variant systems has also been proved and used in the analysis of the cart and the chained systems. The concept of nonholonomic systems is presented and defined in this context as a mechanical system with nonintegrable constraints. A summary of important definitions and results for nonholonomic systems on controllability and stabilizability is also provided.

Chapter 1

Introduction

Wheeled mobile robots are becoming increasingly important in industry as a means of transport, inspection, and operation because of their efficiency and flexibility. In addition, mobile robots are useful for intervention in hostile environments. The motion of a wheeled mobile robot will, in general, be subject to nonholonomic constraints due to the rolling constraints of the wheels, which render a motion perpendicular to the wheels impossible. These nonholonomic constraints give rise to highly nonlinear mathematical models of the mobile robots, and the control problem is not trivial although the full state is measured. Feedback control of nonholonomic mobile robots is, therefore, a challenging problem which combines nonlinear control theory and differential geometry.

1.1 Previous Work

In this section a review of previous work on the control of nonholonomic mobile robots is provided to summarize and to distinguish between existing strategies. Topics include results on controllability, open-loop strategies, and closed-loop strategies. For an introduction to the concept of nonholonomic systems, see *Appendix A*.

The control strategies of mobile robots can be divided into open-loop and closed-loop (feedback) strategies. In open-loop control, the inputs to the mobile robots (velocities or torques) are calculated beforehand from the knowledge of the initial and end position and of the desired path between them in the case of path following. This strategy cannot compensate for disturbances and model errors. Closed-loop strategies, however, may give the required compensation, since the inputs are functions of the actual state of the system and not only of the initial and the end point. Therefore, disturbances and errors causing deviations from the predicted state are compensated by the use of the inputs.

Results on the controllability of nonholonomic mobile robots are presented in Subsection 1.1.1. Open-loop and closed-loop control strategies are summarized in Subsections 1.1.2 and 1.1.3, respectively. Unresolved problems are presented in Subsection 1.1.4.

1.1.1 Controllability of Nonholonomic Mobile Robots

Controllability of wheeled nonholonomic mobile robots has been studied from different points of view depending on the physical system, the model, and the constraints considered. For a mathematical presentation of controllability, see *Appendix B*.

The problem of nonholonomic motion planning was introduced by Laumond (1986) who proved that a car-like robot with one nonholonomic constraint is controllable, even when the steering angle is limited. The result of controllability of mobile robots was extended by Barraquand & Latombe (1989) to any linear equality constraint. Unlike Laumond (1986), no nonlinear inequality constraint was considered. In addition to showing that the car-like robot is controllable in position and orientation, Samson & Ait-Abderrahim (1990*b*) also showed that the robot is controllable in the angular position of one of the wheels.

Controllability of a system with two nonholonomic constraints was studied by Laumond & Siméon (1989). They proved that trajectories between two configurations exist for a nonholonomic mobile robot towing one trailer if both configurations are in the same connected component of the free configuration space. A kinematic model for a mobile robot towing n trailers was presented by Laumond (1991). Controllability of the proposed model was shown with four different models of the pulling car. Although two of these four models were presented as dynamic models, only the kinematic versions were used to show controllability.

Controllability of more general dynamic systems was discussed by Bloch, Reyhanoglu & McClamroch (1991). A controlled nonholonomic Caplygin system was shown to be strongly accessible and small time locally controllable at any equilibrium point. The relation between controllability for kinematic models and controllability for dynamic models was studied by Sussmann (1991). He showed that if a control-linear driftless system is completely controllable, then its dynamic extension is completely controllable as well.

Instead of showing controllability directly from the models of the mobile robots, transformed models can be used. Murray & Sastry (1991) proved controllability of two-chained systems. By converting the kinematic models of an automobile and of a car with one trailer into a chained form, they showed that these systems are controllable.

1.1.2 Open-loop Strategies

For controllable, nonholonomic, mobile robots there are feasible paths in the free configuration space connecting an initial configuration with a final configuration. Open-loop strategies seek to find such a path with several criteria, like collision-avoidance, shortest path, minimum control effort, and minimal number of maneuvers with different kinds of constraints and robot models.

Laumond (1987) proposed an algorithm, based on the work on controllability (Laumond 1986), for planning maneuver- and collision-free paths for a nonholonomic circular robot whose turning radius was lower bounded. However, this algorithm fails whenever all free paths require one or more maneuvers. The planner proposed by Barraquand & Latombe (1989) also deals with cases that require maneuvers. This was the first implemented planner capable of finding a collision-free path with minimal number of maneuvers. It consisted of discretizing the configuration space and applying a best-first search strategy by using the number of maneuvers as the cost function. Simulation results were presented for car-like and trailer-like robots with limited steering wheel angle.

The shortest path with a lower bound on its radius between two oriented points in the plane was studied by Dubins (1957) in the maneuver- and obstacle-free case. Such a path is composed of at most three segments which are arcs of circles or straight lines. Reeds & Shepp (1990) extended this work to include maneuverability. In this case the shortest path is composed of at most five segments which are arcs of circles or straight lines. In the case of obstacles Laumond, Taïx & Jacobs (1990) proposed a planner for a car-like robot based on a global/local approach. This study considered potential fields combined with a new metric in the configuration space $R^2 \times S^1$. The metric was the shortest path in the absence of obstacles. These studies on shortest path resulted in paths with a discontinuous curvature. Smooth paths were obtained locally by Kanayama & Hartman (1989) who considered the derivative of the path curvature as a cost function.

The preceding heuristic or specific approaches were devoted to path planning for mobile robots with low dimensional state spaces. Control theoretic approaches using differential geometry tools have been explored to motion planning of more general, controllable, nonholonomic systems on the form

$$\dot{x} = f_1(x)u_1 + \cdots + f_m(x)u_m$$

where $\dim(x) = n > m$.

Sinusoidal inputs were used by Murray & Sastry (1990) to steer a class of controllable nonholonomic systems on a special triangular form. This form allowed Fourier series techniques to be used to analyze the motion resulting from inputs consisting of sines and cosines at related frequencies. This approach was used to steer a local model of a front-wheel-drive car to a given position and orientation. The algorithm steers first the x-coordinate and the steering wheel angle ϕ to their desired values. Then the periodic input drives the orientation θ to the desired value while bringing x and ϕ back to their desired values. The periodic input then brings the y-coordinate to its desired value and returns the other three states to their desired values. The strategy with sinusoidal inputs can be used on systems on chained form. Sufficient conditions for determining if a system can be converted into chained form and a constructive algorithm for this conversion were proposed by Murray & Sastry (1991). This result was then used to convert the kinematic models of a car and of a car with one trailer into chained forms. By using sinusoidal inputs, these systems could then be steered to any configuration in the state space. This algorithm failed when additional trailers were added to the system. A generalization of the use of sinusoidal inputs to generate motion at a given level of Lie brackets of the input vectors was given by Lafferriere & Sussmann (1991) and Lafferriere (1991) for systems without drift. They introduced an extended system with additional input vectors corresponding to higher order Lie brackets of the original system. Lie algebraic techniques were then used to generate nonholonomic motions. This approach yields exact solutions for nilpotent and nilpotentizable systems and approximate solutions for general systems without drift. Examples of this approach were shown for kinematic models of a car, a front-wheel-drive cart, and a front-wheel-drive cart with one trailer. An example of this method for steering systems with drift was given by Lafferriere (1991). He studied the model of a knife edge which corresponds to a dynamic extension of the kinematic model of a car. This open-loop strategy consisting of a 3-move control was essentially the same as the one presented by Bloch, McClamroch & Reyhanoglu (1990) for the same system.

These open-loop strategies do not compensate for disturbances and model errors. Therefore, closed-loop strategies have been an active field of research.

1.1.3 Closed-loop Strategies

In closed-loop strategies the control input is a function of the state to compensate for disturbances and errors.

Typically, for nonholonomic mobile robots there is no smooth (i.e. C^∞) pure state feedback law that makes a given configuration asymptotically stable, in spite of the controllability property. This results from Brockett's Theorem (Brockett 1983) and was further discussed by Bloch & McClamroch (1989) and d'Andrea-Novel, Bastin & Campion (1991).

Because of the lack of stabilizing smooth state feedback laws, works on closed-loop strategies for path following and tracking have assumed non-zero reference motion. Local tracking was studied for a cart (Kanayama, Nilipour & Lelm 1988) and for a car with a front steering wheel (Nelson & Cox 1988). These studies introduced local error coordinate systems relating the reference and current postures for vehicle control. The control laws were linear (PID and P, respectively) based on a pseudo-linearized system. These approaches were further explored by Kanayama, Kimura, Miyazaki & Noguchi (1990) and Samson & Ait-Abderrahim (1990*b*) who used non-linear control laws based on a Lyapunov analysis for the tracking control of a cart where non-zero reference velocity was assumed. Implementation aspects and experimental results were presented by Kanayama & Hartman (1989); whereas, Samson & Ait-Abderrahim (1990*b*) discussed theoretical controllability and stabilizability issues and proved global stability. Tracking of dynamic models of a cart with actuator limitations was studied by Canudas de Wit & Samson (1991) who proposed a scheme for adjusting the cart's advancement velocity and the control gains. In these previous closed-loop tracking strategies only a *cart* model was considered. Tracking of a trailer-like mobile robot was studied by Sampei, Tamura, Itoh & Nakamichi (1990). They proposed a controller for local, asymptotic convergence towards a desired straight line by using time scale transformation and exact linearization.

Because of the assumption of non-zero reference motion, stabilization about a constant configuration cannot be treated as a special case by these tracking strategies.

Since there is no smooth pure state feedback law that stabilizes the mobile robots considered about a constant configuration, other kinds of closed-loop strategies have been studied. Stabilizing control laws have mainly been studied from two points of view:

- Discontinuous feedback laws
- Time-varying feedback laws

A discontinuous feedback law was proposed for the dynamic model in the knife-edge example (Bloch et al. 1990). The knife-edge example represents a dynamic extension of the kinematic model of a cart. This feedback law is an extension of the the open-loop strategy presented in the same paper. The strategy consists of constructing a set of nested submanifolds which are made invariant through a sequence of control functions from these submanifolds. Because of the dynamic model this feedback approach makes the knife edge reach the origin with zero orientation in finite time in the case of no disturbances. Although the origin is reached in finite time, stability in the sense of Lyapunov was not proven. Two examples of discontinuous stabilization were presented by Messager (1990): Brockett's example (Brockett 1983) and a rigid body steering using a single gas jet along two principal axes. Similar to mobile robots, these systems are not stabilizable by smooth pure feedback. The feedback approach is based on a partition of the state space into subsets and consists of an algorithmic sequence of first determining the actual subset and then applying the corresponding feedback law during a fixed time interval.

The use of time-varying feedback laws is another approach to stabilize a nonholonomic systems about a constant configuration. This approach was first studied by Samson (1991*b*) for the stabilization of a cart. Stabilizing control laws were found for both velocity and torque inputs. This approach was further developed for a car-like mobile robot with a steering wheel (Samson 1991*a*). Constructive approaches were presented by Samson & Ait-Abderrahim (1990*a*) and Pomet (1992). Coron (1991) studied the existence of stabilizing time-varying feedback laws for more general nonholonomic systems. The design methods by Pomet (1992) were extended by Coron & Pomet (1992) to the more general situation given by Coron (1991). Gurvits & Li (1992) presented an algorithm for computing time-periodic feedback solutions for nonholonomic motion planning. They studied the extended system using Lie bracket completion vectors like in the open-loop strategy of Lafferriere & Sussmann (1991). This feedback algorithm was based on multi-scaling averaging techniques and highly-oscillatory inputs. All these smooth time-varying approaches have revealed rather slow convergence rates. This is consistent with the statement that smooth time-periodic feedback cannot be *exponentially* stabilizing (Gurvits

1992), (Gurvits & Li 1992). To improve the performance, Pomet, Thuilot, Bastin & Campion (1992) proposed a hybrid strategy for the stabilization of a cart model. This strategy used a time-invariant feedback control outside a neighborhood of the origin to obtain a satisfactory convergence to this neighborhood and a time-varying control inside the neighborhood to obtain asymptotic convergence to the origin. However, the convergence inside the neighborhood still remained slow. Murray, Walsh & Sastry (1992) obtained local exponential convergence to a neighborhood of the origin for a system equivalent to the kinematic model of a cart. The asymptotic behavior was obtained by letting the control law be time-varying. The exponential convergence to the neighborhood was obtained by letting the control law be non-smooth at the origin.

1.1.4 Unresolved Problems from Previous Research

Controllability of general nonlinear systems has been well studied, particularly for the controllability of nonholonomic mobile robots. Constructive procedures to prove controllability have been used to find open-loop sequences to steer simple mobile robots like carts and car-like robots with one trailer. However, the problem of finding exact steering algorithms for more complex mobile robots with additional trailers has not yet been solved.

In the case of a cart with one nonholonomic constraint and upper bounded curvature, the shortest path between two oriented points in the plane has been shown to be composed of arcs of circles and straight lines. The shortest path for a car-like mobile robot with two nonholonomic constraints has to the author's knowledge, however, not yet been found.

Closed-loop strategies for mobile robots are a more recent field of research than open-loop strategies and many unresolved problems have been left for this thesis and for future research. One problem is to find a control law to improve the convergence of a mobile robot to a given configuration. A car-like robot with one trailer can be stabilized by converting this system into chained form and then using stabilizing feedback laws for such chained forms. However, a conversion into a chained form when additional trailers are added has not been found in previous work. Therefore, the stabilization of car-like robots with n trailers is another problem. A third problem is to find an *exponentially* stabilizing feedback law for a chained form to improve the convergence of the mobile robots that can be converted into such a form.

Several closed-loop strategies have been presented for tracking under the assumption that the reference cart does not stop. Tracking or path following with stopping and reversing phases and convergence to the end configuration, is thus a fourth problem that has been left unresolved.

1.2 Contributions of this Thesis

In this thesis, new feedback control strategies with improved convergence properties are developed for kinematic models of nonholonomic mobile robots like a 2-dof cart and a car with n trailers. The main contributions are:

- **Exponential convergence of a cart to a given configuration:** A piecewise analytic feedback law is proposed for the kinematic model of a mobile robot with no lower bound on its turning radius. The convergence to the desired configuration is global and *exponential* which is a significant improvement compared to other feedback strategies like time-varying smooth feedback laws. The feedback law for exponential convergence to a given configuration is developed in Chapter 3 and Section 4.2. This presentation is based on Canudas de Wit & Sørdalen (1991) and Canudas de Wit & Sørdalen (1992*b*). This feedback approach has been applied to other systems:
  - Attitude control of a rigid body with a nonholonomic constraint (Sørdalen, Egeland & Canudas de Wit 1992).
  - Global exponential convergence of an nonholonomic underwater vehicle (Sørdalen, Dalsmo & Egeland 1993).
  - Asymptotic stabilization of a 2-dimensional system with only one input (Canudas de Wit & Sørdalen 1992*a*).
  - Global exponential convergence of a 3-dimensional chained system (Canudas de Wit & Sørdalen 1993).
- **Path following with stopping and reversing phases:** The piecewise analytic feedback law for exponential convergence is extended to make the cart globally follow a path composed of arcs of circles and straight lines. This approach allows stopping and reversing phases as opposed to previous tracking strategies where non-zero reference motion has been assumed. The convergence to the end configuration is exponential. The presentation in Chapter 4 is based on Sørdalen & Canudas de Wit (1992*a*) (finalist for the Philips Prize awarded annually for young authors), Sørdalen & Canudas de Wit (1992*b*), and Sørdalen & Canudas de Wit (1993).

- **Conversion of the kinematics of a car with n trailers into a chained Form:** A kinematic model of a car with n trailers is developed and converted into a chained form by a change of coordinates and an invertible feedback transformation of the inputs. This conversion holds locally in the orientations and globally in the position. The problem of developing such a conversion has been posed in the literature, but it has not been solved in previous work. By using this conversion, a car with n trailers can be locally controlled by existing open- or closed-loop strategies for a chained system. The presentation in Chapter 5 is based on Sørdalen (1993*b*) and Sørdalen (1993*c*).

- **Exponential Stabilization of Chained Systems:** A feedback law is proposed to globally *exponentially* stabilize, in a sense defined here, a chained nonholonomic system with two inputs as opposed to previous approaches where exponential convergence was not achieved. The system may have any nonholonomic degree. All nonholonomic systems that can be converted into such a chained form can then be exponentially stabilized. For example, a car with n trailers can be (locally) exponentially stabilized about a given position and a given orientation of the car and the trailers by using this control law combined with the conversion from Chapter 5. The presentation in Chapter 6 is based on Sørdalen & Egeland (1993). This work has also been presented at an invited seminar at Berkeley, December 21th 1992.

In addition, a useful lemma on the exponential convergence of a class of time-variant systems is proved.

It is hoped that this thesis contributes to a better understanding of nonholonomic mobile robots and how to control them in order to achieve exponential convergence and perform path-following.

1.3 Outline of the Thesis

The thesis is organized as follows.

- **Chapter 2** A lemma on exponential convergence for a class of time-variant systems is presented.

- **Chapter 3:** A piecewise analytic feedback law is developed for the kinematic model of a cart resulting in *global*, *exponential* convergence to the origin with zero orientation.

- **Chapter 4:** The piecewise analytic feedback law from Chapter 3 is extended to make the cart follow a path composed of arcs of circles and straight lines where stopping and reversing phases are possible.
- **Chapter 5:** A kinematic model of a car with n trailers is developed and converted into a chained form suitable for control.
- **Chapter 6:** A stabilizing feedback control law for a chained form is developed yielding global *exponential* convergence to the origin.
- **Appendix A:** The concept of nonholonomic systems is presented.
- **Appendix B:** Theoretical preliminaries on the controllability and stabilizability of nonlinear systems with emphasis on nonholonomic systems are presented.

Chapter 2

A Lemma on Exponential Convergence

The following lemma is useful for establishing exponential convergence for a class of time-variant systems. It is used in the analysis of the exponential convergence of the orientation of a cart in Chapter 3 and in the stability analysis of chained systems in Chapter 6. This lemma is, to the author's knowledge, an original contribution.

Lemma 2.1 *Consider the nonlinear, time-variant system*

$$\dot{x} = -a(x,t)x + d(x,t)\,, \quad t \geq t_0 \tag{2.1}$$

under the following assumptions:

- *There exists a solution $x(t)$ of (2.1).*
- *$a(x,t)$ has the property that for all $x(t)$*

  $$|\int_{t_0}^{t}(a(x(\tau),\tau) - \lambda)d\tau| \leq P, \quad \forall t \geq t_0 \tag{2.2}$$

  where λ and P are positive constants.
- *The signal $d(x,t)$ is bounded for all $x(t)$ by*

  $$|d(x(t),t)| \leq De^{-\gamma(t-t_0)}\,, \quad \forall t \geq t_0 \tag{2.3}$$

  for some positive constants D and γ.

Then

$$\forall \varepsilon > 0, \quad |x(t)| \leq c\,(|x(t_0)| + D)e^{-(\beta-\varepsilon)(t-t_0)}$$

where

$$\beta = \min\{\lambda, \gamma\} > 0, \qquad c = \max\{e^P, \frac{e^{2P}}{\varepsilon}\}$$

Proof: We denote

$$F(t) = \int_{t_0}^{t} a(x(\tau), \tau) d\tau$$

where $x(t)$ is a solution of (2.1). Multiplying (2.1) with $e^F(t)$ gives

$$\frac{d}{dt}(x(t)e^{F(t)}) = d(x(t), t)e^{F(t)}$$

This implies

$$x(t)e^{F(t)} = x(t_0) + \int_{t_0}^{t} e^{F(\tau)} d(x(\tau), \tau) d\tau$$

Dividing by $e^F(t)$ then gives the following (implicit) expression for $x(t)$

$$x(t) = e^{-F(t)} \left(x(t_0) + \int_{t_0}^{t} e^{F(\tau)} d(x(\tau), \tau) d\tau \right) \tag{2.4}$$

Property (2.2) implies that

$$|F(t) - \lambda(t - t_0)| \leq P \tag{2.5}$$

which is equivalent to

$$-P + \lambda(t - t_0) \leq F(t) \leq P + \lambda(t - t_0) \tag{2.6}$$

By using (2.6) and (2.3) we get

$$|x(t)| \leq e^P e^{-\lambda(t-t_0)} \left(|x(t_0)| + D \int_{t_0}^{t} e^P e^{(\lambda-\gamma)(\tau-t_0)} d\tau \right)$$

In the case that $\lambda = \gamma$ we find

$$|x(t)| \leq e^P |x(t_0)| e^{-\lambda(t-t_0)} + De^{2P}(t - t_0)e^{-\lambda(t-t_0)}$$

In the case that $\lambda \neq \gamma$ we find

$$|x(t)| \leq e^P |x(t_0)| e^{-\lambda(t-t_0)} + \frac{De^{2P}}{\lambda - \gamma}(e^{-\gamma(t-t_0)} - e^{-\lambda(t-t_0)})$$

Since

$$(1 - e^{-\alpha t})/\alpha < t, \quad \forall \alpha > 0, \ \forall t \geq 0$$

then

$$|x(t)| \leq e^P |x(t_0)| e^{-\lambda(t-t_0)} + De^{2P}(t-t_0)e^{-\beta(t-t_0)} \tag{2.7}$$

for all $\lambda > 0$ and $\gamma > 0$ where $\beta = \min\{\lambda, \gamma\}$. Eq. (2.7) implies that

$$|x(t)| \leq (\rho(t-t_0)+\sigma)e^{-\beta(t-t_0)} = (\rho(t-t_0)+\sigma)e^{-\varepsilon(t-t_0)}e^{-(\beta-\varepsilon)(t-t_0)} \tag{2.8}$$

where

$$\rho = De^{2P}, \qquad \sigma = e^P|x(t_0)| \tag{2.9}$$

since $e^{-(\lambda-\beta)(t-t_0)} \leq 1$ for $t \geq t_0$. By comparison we find that

$$(\rho\, t + \sigma)e^{-\varepsilon t} \leq k \qquad \forall t \geq 0 \tag{2.10}$$

if

$$k = \begin{cases} \frac{\rho}{\varepsilon e} e^{\frac{\varepsilon\sigma}{\rho}} \ , & \rho > \varepsilon\sigma \\ \sigma \ , & \rho \leq \varepsilon\sigma \end{cases}$$

Since

$$e^{\frac{\varepsilon\sigma}{\rho}} < e$$

when $\rho > \varepsilon\sigma$, Eq. (2.10) will also be satisfied with the following choice of k:

$$k = \begin{cases} \frac{\rho}{\varepsilon} \ , & \rho > \varepsilon\sigma \\ \sigma \ , & \rho \leq \varepsilon\sigma \end{cases} \tag{2.11}$$

Then, from (2.8), (2.9), (2.10) and (2.11) we get that

$$|x(t)| \leq ke^{-(\beta-\varepsilon)(t-t_0)} \leq (\frac{\rho}{\varepsilon} + \sigma)e^{-(\beta-\varepsilon)(t-t_0)} \leq c(|x(t_0)| + D)e^{-(\beta-\varepsilon)(t-t_0)}$$

where $\beta = \min\{\lambda, \gamma\}$ and $c = \max\{e^P, \frac{e^{2P}}{\varepsilon}\}$.

□

This lemma implies that a solution $x(t)$ of (2.1) converges exponentially to zero if $a(x,t)$ and $d(x,t)$ have the properties (2.2) and (2.3).

Remark 1: By choosing $\varepsilon = \beta$ we see that

$$\max_{t \geq t_0} |x(t)| \leq (|x(t_0)| + D) \max\{e^P, \frac{e^{2P}}{\beta}\}$$

where $\beta = \min\{\lambda, \gamma\} > 0$.

Remark 2: If $a(x,t)$ and $d(x,t)$ are continuous in x and t, then there exists at least one solution of (2.1), (Miller & Michel 1982), Theorem 2.3.

Chapter 3

Exponential Convergence of a Mobile Robot with a Nonholonomic Constraint

3.1 Introduction

Feedback control of mobile robots is important to compensate for disturbances and model errors. From the discussion of previous work in Section 1.1 we know that such systems are typically controllable but not stabilizable by a smooth pure feedback law. Sussmann (1979) proved that if a real analytic control system is completely controllable, then for every point p in the state space there exists a *piecewise* analytic feedback control that steers every state into p. Therefore, we seek for a piecewise analytic feedback law to make the nonholonomic mobile robot considered here *globally* converge to the origin in the xy-plane with zero orientation. Another motivation for seeking a piecewise analytic feedback law is to obtain better convergence than with the time-varying smooth feedback laws developed for mobile robots.

In this chapter, a kinematic model of a 2-dof cart is presented. A piecewise analytic feedback law is proposed which yields *global*, *exponential* convergence of the cart to the origin with zero orientation. The basic idea is to let the position of the cart converge to the origin along circles that pass through the origin with the tangents parallel to the x-axis. The control law introduces two discontinuity surfaces. A discussion of the resulting motion in the neighborhood of these discontinuity surfaces is provided. Design guidelines and simulation results are also presented.

This presentation is based on Canudas de Wit & Sørdalen (1992*b*). More details are, however, added and some expressions are modified, notably the bound on $b_1(\alpha, \beta)$ in Lemma 3.2. The function sinc$(\cdot)$ is also introduced to make $\theta_d(x, y)$ well-defined everywhere. The well-known result that the cart is completely controllable but not stabilizable by a smooth static state feedback law is also shown. The discussion of the discontinuities is extended and the design guidelines are modified. Finally, more simulation results are presented to study robustness with respect to measurement noise, model errors and dynamic extension, and to compare with a time-varying feedback law.

3.2 Coordinate Transformation

A mobile robot in this context is a cart with two driving wheels. The kinematic model of the cart is given by

$$\begin{aligned} \dot{x} &= \cos\theta(v_1 + v_2)/2 = \cos\theta v \\ \dot{y} &= \sin\theta(v_1 + v_2)/2 = \sin\theta v \\ \dot{\theta} &= (v_1 - v_2)/(2c_r) = \omega \end{aligned} \tag{3.1}$$

where the state of the system (3.1),

$$q = [x, y, \theta]^T \tag{3.2}$$

is the position of the wheel axis center (x, y) and the cart orientation θ with respect to the x-axis. The configuration manifold is $M = \mathbb{R}^2 \times S^1$. The distance between the point (x, y) and each of the wheel locations is c_r. The velocities v_1 and v_2 are the tangent velocities of each wheel at its center of rotation, (i.e. motor velocities time wheel radius). We assume that the tangent velocity v and the angular velocity ω can be regarded as the inputs to the system. They are related to the wheel velocities in the following manner:

$$u = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c_r} & -\frac{1}{2c_r} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3.3}$$

The nonholonomic constraint due to the non-sliding condition is given by

$$[\sin\theta, -\cos\theta, 0]\dot{q} = 0$$

We can show the well-known result that the cart (3.1) is completely controllable but not stabilizable by a smooth static state feedback law by rewriting the system (3.1) in the following form:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 \tag{3.4}$$

where

$$g_1(q) = [\cos\theta, \sin\theta, 0]^T, \qquad g_2(q) = [0,\ 0,\ 1]^T$$

and $u_1 = v$ and $u_2 = \omega$.

g_1 and g_2 are analytic vector fields. The Lie bracket of g_1 and g_2 is, Definition B.9,

$$[g_1, g_2](q) = \frac{\partial g_2(q)}{\partial q} g_1(q) - \frac{\partial g_1(q)}{\partial q} g_2(q) = [-\sin\theta, \cos\theta, 0]^T$$

We find from Definition B.15 that the accessibility distribution C of (3.4) is given by

$$C(q) = \mathrm{span}\{g_1, g_2, [g_1, g_2]\}(q) = \mathbb{R}^3$$

since $[g_1, g_2]$ is linearly independent of g_1 and g_2. This means that

$$\forall q \in M, \quad \dim C(q) = 3$$

From Corollary B.1 we find that system (3.4) is locally accessible.

Since $\mathbb{R}^2 \times S^1$ is connected and there is no drift in this system, Theorem B.2 implies that system (3.4), or equivalently (3.1), is completely controllable.

Despite this controllability property, the cart cannot be made *globally* asymptotically stable about the origin by a smooth static state feedback law since $M = \mathbb{R}^2 \times S^1$ is not contractible, Theorem B.5. From Theorem B.6 it follows that (3.1) cannot even be locally stabilized by a smooth static state feedback since the image of the map

$$f(q, u) = g_1 u_1 + g_2 u_2 : V_{q_0} \times \mathbb{R}^2 \to \mathbb{R}^3 \tag{3.5}$$

does not contain a neighborhood of $[0, 0, 0]^T \in \mathbb{R}^3$ for all neighborhoods V_{q_0} of q_0. Taking $q_0 = [0, 0, 0]^T$ we see that $[0, \varepsilon, 0]^T$ is not in the image of (3.5) if V_{q_0} is for instance restricted to $\mathbb{R}^2 \times (-\pi/2 + \varepsilon, \pi/2 - \varepsilon)$ where $\varepsilon \in (0, \pi/2)$.

The problem is how to find a feedback control law, $u(q)$, so that the closed-loop system

$$\dot{q} = G(q)u(q) = f(q), \quad G(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{3.6}$$

converges for any initial condition, $q(0)$, to an equilibrium point in $\mathcal{O}$,

$$\mathcal{O} = \{q \mid (x, y, \theta) = (0, 0, 2\pi n)\}$$

where $n \in \{0, \pm 1, \pm 2, \ldots\}$. The set $\mathcal{O}$ then represents a constant configuration on the configuration manifold $\mathbb{R}^2 \times S^1$. Consider the circle family $\mathcal{P}$,

$$\mathcal{P} = \{(x, y) | \; x^2 + (y - r)^2 = r^2\} \tag{3.7}$$

as the set of circles in the xy-plane with radius $r = r(x, y)$. They pass through the origin and (x, y) and are centered on the y-axis with $\frac{\partial y}{\partial x} = 0$ in the origin. Let θ_d be the angle of the tangent of $\mathcal{P}$ at (x, y), defined by

$$\theta_d(x, y) = \begin{cases} 2\arctan(y/x) = 2\arcsin \frac{y\,\mathrm{sgn}(x,y)}{\sqrt{x^2+y^2}} & ; \; (x, y) \neq (0, 0) \\ 0 & ; \; (x, y) = (0, 0) \end{cases} \tag{3.8}$$

where

$$\mathrm{sgn}(x, y) = \begin{cases} 1 \; ; \; x > 0 \;\text{ or }\; (x = 0, \; y < 0) \\ -1 \; ; \; x < 0 \;\text{ or }\; (x = 0, \; y \geq 0) \end{cases} \tag{3.9}$$

θ_d is taken by convention to belong to $[-\pi, \pi)$. Hence θ_d has discontinuities on the y-axis with respect to x. The discontinuity surface is defined by

$$\mathcal{D} = \{(x, y, \theta) \mid x = 0, \; y \neq 0\} \tag{3.10}$$

In view of these definitions, we introduce the following variables:

$$a(x, y) \triangleq r\theta_d = \frac{\sqrt{x^2 + y^2}}{\mathrm{sinc}\frac{\theta_d}{2}} \mathrm{sgn}(x, y) \tag{3.11}$$

$$\alpha(x, y, \theta) \triangleq e - 2\pi n(e) \,, \qquad e = \theta - \theta_d(x, y) \tag{3.12}$$

where

$$\mathrm{sinc}\,\zeta \triangleq \frac{\sin\zeta}{\zeta} \;\in C^\infty, \;\; \zeta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \mathrm{sinc}\zeta \in [\frac{2}{\pi}, 1]$$

and $r = r(x, y)$ is given by (3.7).

$a(x, y)$ is the arc length from the origin to (x, y) along a circle which is centered on the y-axis and passes through these two points. $a(x, y)$ has the same sign as x if $x \neq 0$ and the opposite of y if $x = 0$. The angle $\alpha \in [-\pi, \pi)$ is the orientation error of the cart with respect to the desired orientation θ_d. α is a periodic and piecewise continuous function with respect to e. $n(\cdot)$ takes values in $\{0, \pm 1, \pm 2, \ldots\}$ so that α belongs to $[-\pi, \pi)$. An illustration of these definitions is shown in *Figure 3.1.*

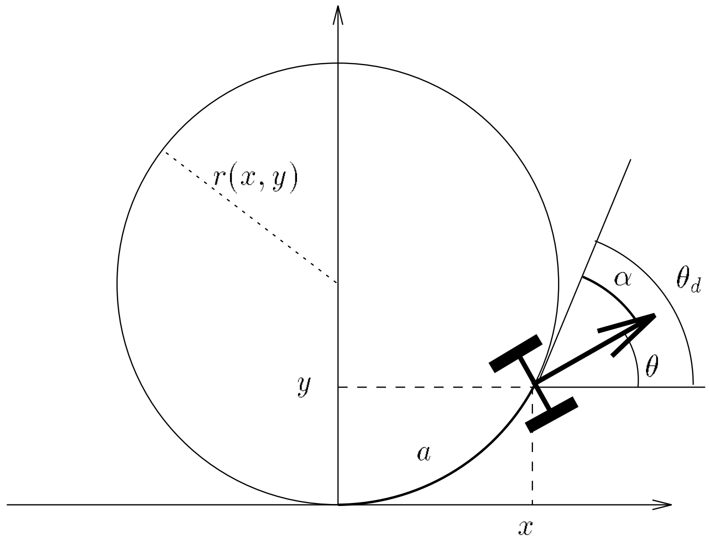

Figure 3.1: Illustration of the variables a, α and θ_d.

$\mathcal{E}$ is the set of states q where $\alpha(q)$ is discontinuous, i.e.,

$$\begin{aligned}\mathcal{E} &= \{(x,y,\theta) \mid \alpha(x,y,\theta) = -\pi\} \\ &= \{(x,y,\theta) \mid \theta = \theta_d(x,y) + 2\pi n - \pi\}, \quad n \in \{0, \pm 1, \pm 2, \ldots\}\end{aligned} \tag{3.13}$$

We see that $\mathcal{E}$ consists of several "parallel" discontinuity surfaces since $\theta \in \mathbb{R}$. There is one surface $\mathcal{E}^n$ for each $n \in \{0, \pm 1, \pm 2, \ldots\}$. Discontinuities in $a(x,y)$ only take place in $\mathcal{D}$.

Let the mapping $F(\cdot) : \mathbb{R}^3 \to \mathbb{R} \times [-\pi, \pi)$ be given by the definitions of $a(x,y)$ and $\alpha(x,y,\theta)$, (3.11)-(3.12). We denote

$$z = [a,\ \alpha]^T$$

The mapping $z = F(q)$ maps the state space coordinates, $q \in \mathbb{R}^3$, into the two-dimensional set, $\mathbb{R} \times [-\pi, \pi)$:

$$z = F(q) = \begin{bmatrix} a(x,y) \\ \alpha(x,y,\theta) \end{bmatrix} \in \mathbb{R} \times [-\pi, \pi) \tag{3.14}$$

This transformation has several useful properties. These are listed in the following lemma.

Lemma 3.1 *The mapping* $F(\cdot) : \mathbb{R}^3 \to \mathbb{R} \times (-\pi, \pi]$ *has the following properties:*

1. $F(q) = 0 \;\Leftrightarrow q \in \mathcal{O}$

2. $a^2(q), \alpha^2(q), \|F(q)\|^2$ *are continuous when traversing* $\mathcal{D}$ *and* $\mathcal{E}$*.*

3. $\|[x,y]^T\| \leq |a| \leq \frac{\pi}{2}\|[x,y]^T\|$

where $\|\cdot\|$ *denotes the Euclidian norm.*

Proof: See *Appendix 3.A*

□

3.3 Control Design and Convergence Analysis

In this section a piecewise analytic feedback law is proposed and the convergence of the closed-loop system is analyzed. First, the convergence analysis is performed in an open continuous subspace, and then the analysis is extended to the whole state space including discontinuities.

3.3.1 Dynamics in the Ψ-set

Let us first consider the case where $q \in \Psi$ where Ψ is defined as the open set $\Psi = \mathbb{R}^3 - (\mathcal{D}_0 \cup \mathcal{E}_0)$ where $\mathcal{D}_0 = \mathcal{D} \cup \{q \mid x = y = 0\}$ and $\mathcal{E}_0 = \mathcal{E} \cup \{q \mid x = y = 0\}$. The discontinuity surfaces $\mathcal{D}$ and $\mathcal{E}$ are defined in (3.10) and (3.13). Thus, the hypersurface $\mathcal{D}_0$ consists of all the configurations where $x = 0$. The set Ψ consists of the open connected sets Ψ_{de},

$$\Psi = \cup_{de}\Psi_{de}, \quad d \in \{0,1\},\; e \in \{0, \pm 1, \pm 2, \ldots\}$$

where the sets Ψ_{de} are separated by the surfaces $\mathcal{D}_0$ and $\mathcal{E}_0$. The index d is defined such that $d = 0$ corresponds to $x < 0$, which implies that $a(x,y) < 0$, and $d = 1$ corresponds to $x > 0$, which implies that $a(x,y) > 0$. The index e is defined such that $q = [x, y, \theta]^T \in \Psi_{de}$ is between the surfaces $\mathcal{E}_0^{e-1}$ and $\mathcal{E}_0^{e}$ meaning that, (3.13),

$$\theta_d(x,y) + 2\pi(e-1) - \pi < \theta < \theta_d(x,y) + 2\pi e - \pi$$

In Ψ we have,

$$\dot{z} = \frac{\partial F}{\partial q}\dot{q} = J(q)\dot{q} \;\;; \quad J(q) \in \mathbb{R}^{2\times 3} \tag{3.15}$$

where $z = [a, \alpha]^T$ and $q = [x, y, \theta]^T$. The matrix $J(q)$ is given by

$$J(q) = \begin{bmatrix} \frac{\theta_d}{\beta} - 1 & \frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta} & 0 \\ \frac{2\beta}{(1+\beta^2)x} & -\frac{2}{(1+\beta^2)x} & 1 \end{bmatrix} \tag{3.16}$$

where $\beta = y/x$. Note from the definitions of a and α (3.11)-(3.12) that $\frac{\partial a}{\partial q}$ is defined everywhere in $\mathbb{R}^3 - \mathcal{D}_0 \triangleq M_a$ and $\frac{\partial \alpha}{\partial q}$ is defined everywhere in $\mathbb{R}^3 - \mathcal{E}_0 \triangleq M_\alpha$.

From Eqs. (3.6) and (3.15) we get

$$\dot{z} = J(q)G(q)u = B(q)u; \qquad B = \begin{bmatrix} b_1 & 0 \\ b_2 & 1 \end{bmatrix} \tag{3.17}$$

with

$$b_1 = b_1(q) = \cos\theta(\frac{\theta_d}{\beta} - 1) + \sin\theta \left(\frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta}\right) \tag{3.18}$$

$$b_2 = b_2(q) = \cos\theta \frac{2\beta}{(1+\beta^2)x} - \sin\theta \frac{2}{(1+\beta^2)x} \tag{3.19}$$

By noting that $\cos\theta = \cos(\alpha + \theta_d)$, $\sin\theta = \sin(\alpha + \theta_d)$ and $\cos\theta_d = \frac{1-\beta^2}{1+\beta^2}$, $\sin\theta_d = \frac{2\beta}{1+\beta^2}$ where $\beta = \tan(\theta_d/2)$, we can rewrite b_1 as:

$$b_1(\alpha, x, y) = \cos\alpha + B(x, y)\sin\alpha \tag{3.20}$$

where

$$\begin{aligned} B(x, y) &= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d \left(\frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta}\right) \\ &= \frac{\sin(\theta_d/2)\cos(\theta_d/2) - \theta_d/2}{\sin^2(\theta_d/2)} \end{aligned} \tag{3.21}$$

where $\theta_d = \theta_d(x, y)$, (3.8).

Lemma 3.2 *The functions $b_1(q)$ and $b_2(q)$ have the following properties for any $q = [x, y, \theta]^T$*

1. $b_{min}(\alpha) \leq b_1(\alpha, x, y) \leq b_{max}(\alpha)$
2. $b_1(\alpha, x, y)$ *is continuous in* α.
3. $|b_1(\alpha, x, y) - 1| \leq |\alpha|\zeta, \quad \zeta \triangleq 1 + \frac{\pi}{2}$

4. $| b_2(q)a(q) | \leq \pi$

where

$$b_{min}(\alpha) = \cos\alpha - \frac{\pi}{2}|\sin\alpha|$$
$$b_{max}(\alpha) = \cos\alpha + \frac{\pi}{2}|\sin\alpha|$$

and $\alpha(q)$ is defined in (3.12).

Proof: see *Appendix 3.B.*

□

The properties in Lemma 3.2 will be useful in establishing the exponential convergence of the state to the desired configuration. We propose the following feedback law, with $\gamma > 0$ and $k > 0$:

$$v = -\gamma b_1 a \tag{3.22}$$
$$\omega = -b_2 v - k\alpha \tag{3.23}$$

This feedback law gives the following differential equations:

$$\dot{a} = b_1 v = -\gamma b_1^2 a \tag{3.24}$$
$$\dot{\alpha} = b_2 v + \omega = -k\alpha \tag{3.25}$$

Eq. (3.24) is defined $\forall q \in M_a$ and (3.25) is defined $\forall q \in M_\alpha$. Eqs. (3.24)-(3.25) have the following solutions for $a(t)$ and $\alpha(t)$:

$$\begin{aligned} a(t) &= a(0)\exp(-\gamma\kappa(t)) \\ \alpha(t) &= \alpha(0)\exp(-kt) \end{aligned} \tag{3.26}$$

where

$$\kappa(t) = \int_0^t b_1^2(q(\tau))d\tau \tag{3.27}$$

From these equations we have:

$$\|z(t)\|^2 \leq \|z(0)\|^2 \exp(-2\eta(t)) \tag{3.28}$$

where

$$\eta(t) = \min(\gamma\kappa(t), kt) \qquad \forall t \geq 0 \tag{3.29}$$

which provides upper bounds on the norm of $z(t) = [a(t), \alpha(t)]^T$ in the continuous set Ψ. The following subsection extends the boundedness of $z(t)$ to the region including the discontinuities by showing that the discontinuity surfaces $\mathcal{D}$ and $\mathcal{E}$ are not attractive nor invariant.

3.3.2 Dynamic Behavior on the Surfaces $\mathcal{D}$ and $\mathcal{E}$

A solution of the closed-loop system (3.6) with the input $u(q)$ given by (3.22)-(3.23) is defined similarly as by André & Seibert (1960) for more general piecewise continuous differential equations:

Definition 3.1 *We call a (single valued) vector function $q(t)$, defined in an interval I, a* **solution of the system (3.6)** *if it satisfies the following conditions:*

1. *It is continuous throughout I.*
2. *For every $t \in I$, such that $q(t) \in \Psi_{de}$, the function $q(t)$ is differentiable and satisfies Eq. (3.6).*
3. *The set of values $t \in I$ for which $q(t) \in \mathcal{D} \cup \mathcal{E}$ holds, has no cluster in I.*

André & Seibert (1960) classified the normal switching points in three principal types: *transition points* (i.e. points at which a solution traverses the switching space), *end-points* (at which two solutions "end"), and *starting points* (at which two solutions start).

Motion (or impossibility of motion) on the discontinuity surfaces $\mathcal{D}$ and $\mathcal{E}$ can be investigated by analyzing the direction of the vector field $f(q)$ in (3.6) in the neighborhood of the discontinuities. Let us first consider the behavior on the surface $\mathcal{D}$.

Lemma 3.3 *Let the feedback law be given by (3.22)-(3.23). The solutions $q(t)$ of the closed-loop system (3.6)*

$$\dot{q} = f(q), \qquad q = [x, y, \theta]^T$$

where $q(t_d) \in \mathcal{D}$ for some $t_d \geq 0$ satisfies $q(t) \in \Psi$ for all $t \in (t_d, t_d + \delta)$ for a $\delta > 0$.

Proof: We analyze the vector field $f(q)$ in the neighborhood of $\mathcal{D}$:

$$f^+(q) = \lim_{x \to 0^+} f(q) = \begin{bmatrix} \cos\theta v^+ \\ \sin\theta v^+ \\ \omega^+ \end{bmatrix}$$

$$f^-(q) = \lim_{x \to 0^-} f(q) = \begin{bmatrix} \cos\theta v^- \\ \sin\theta v^- \\ \omega^- \end{bmatrix}$$

where,

$$v^+ = -\gamma(-\cos\theta + \frac{\pi}{2}\sin\theta)\frac{\pi}{2}|y| \tag{3.30}$$

$$v^- = \gamma(-\cos\theta - \frac{\pi}{2}\sin\theta)\frac{\pi}{2}|y| \tag{3.31}$$

$$\omega^+ = -\cos\theta\gamma\pi(-\cos\theta + \frac{\pi}{2}\sin\theta)\mathrm{sgn}(y) - k(\theta - \pi - 2\pi n) \tag{3.32}$$

$$\omega^- = \cos\theta\gamma\pi(-\cos\theta - \frac{\pi}{2}\sin\theta)\mathrm{sgn}(y) - k(\theta - \pi - 2\pi n) \tag{3.33}$$

The control law (3.22)-(3.23) and the righthand side $f(q)$ are defined everywhere, notably in $\mathcal{D}$ where $f(q) = f^+(q)$ if $y < 0$ and $f(q) = f^-(q)$ if $y > 0$. The normal to the discontinuity surface $\mathcal{D}$ pointing in positive x-direction is

$$n = [1 \;\; 0 \;\; 0]^T$$

The projections of the vector fields $f^+(q)$ and $f^-(q)$ along the normal n are given by

$$n^T f^+(q) = \cos\theta v^+ = -\gamma\cos\theta(-\cos\theta + \frac{\pi}{2}\sin\theta)\frac{\pi}{2}|y| \tag{3.34}$$

$$n^T f^-(q) = \cos\theta v^- = -\gamma\cos\theta(\cos\theta + \frac{\pi}{2}\sin\theta)\frac{\pi}{2}|y| \tag{3.35}$$

We find that

$$(n^T f^+(q))(n^T f^-(q)) = (\gamma\frac{\pi}{2}y\cos\theta)^2(\frac{\pi^2}{4}\sin^2\theta - \cos^2\theta)$$

These projections are headed in opposite directions at points $q \in \mathcal{D}$ where

$$(n^T f^+(q))(n^T f^-(q)) < 0 \;\;\Leftrightarrow\;\; \sin^2\theta < \frac{1}{\frac{\pi^2}{4} + 1} \triangleq \xi^2 \tag{3.36}$$

It is straightforward to calculate from (3.34)-(3.35) that

$$\forall q \in \mathcal{D} \qquad \sin^2\theta < \xi^2 \;\Rightarrow\; n^T f^+(q) > 0, \;\; n^T f^-(q) < 0$$

which means that the vector field $f(q)$ has a component which points away from the discontinuity surface $\mathcal{D}$. This implies that the points $q \in \mathcal{D}$ where $\sin^2\theta < \xi^2$ are starting points, i.e. $q(t) \in \Psi$ for all $t \in (t_d, t_d + \delta)$ for a $\delta > 0$ where $q(t_d) \in \mathcal{D}$. André & Seibert (1960) stated that a solution exists for such starting points for $t \geq 0$.

We find from (3.34)-(3.35) that $\forall q \in \mathcal{D}$

$$\sin^2\theta > \xi^2 \;\Rightarrow\; n^T f^+(q), \; n^T f^-(q) > 0, \;\text{ or }\; n^T f^+(q), \; n^T f^-(q) < 0$$

This implies that the points $q \in \mathcal{D}$ where $\sin^2\theta > \xi^2$ are transition points, i.e. these points attract the solutions from one side of the discontinuity and repel them at the other side via a transition through $q \in \mathcal{D}$. André & Seibert (1960) stated that the existence and uniqueness theorem holds in the strict sense at such transition points.

We find from (3.34)-(3.35) that

$$(n^T f^+(q))(n^T f^-(q)) = 0 \quad \Leftrightarrow \quad \sin^2\theta = \xi^2 \quad \text{or} \quad \cos\theta = 0 \tag{3.37}$$

These situations are identified in *Figure 3.2*. If $\cos\theta = 0$ then $n^T f^+(q) =$

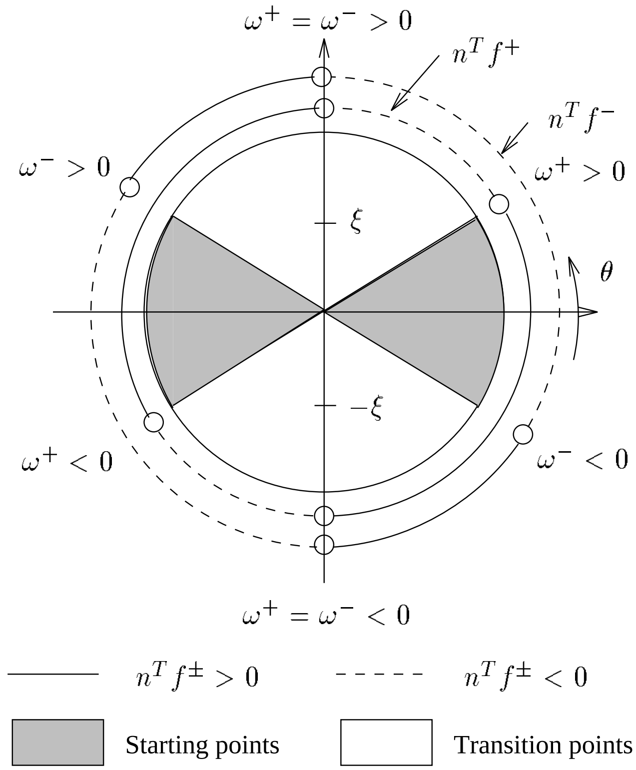

Figure 3.2: Illustration of the type of the switching points in $\mathcal{D}$ dependent on the orientation θ of the cart.

$n^T f^-(q) = 0$ but $\omega^+(q) = \omega^-(q) \neq 0$. Since $\dot\theta = \omega \neq 0$, we cannot have that $\cos\theta = 0$ in a finite time interval and the system will leave $\mathcal{D}$. (The points $q \in \mathcal{D}$ where $\cos\theta = 0$ are of LR-type according to the notation of André & Seibert (1960) at which the existence and uniqueness theorem holds in the strict sense.)

If $\sin^2\theta = \xi^2$ then either $n^T f^-(q) < 0$, $n^T f^+(q) = 0$ and $\omega^+(q) \neq 0$, or $n^T f^+(q) > 0$, $n^T f^-(q) = 0$ and $\omega^-(q) \neq 0$, (3.32)-(3.33) and (3.34)-(3.35).

Therefore, since $\dot{\theta} = \omega$ and $\omega^{\pm} \neq 0$ if $f^{\pm}(q)n = 0$, the cart cannot stay in $\mathcal{D}$ in a finite time interval with $\sin^2\theta = \xi^2$. We can therefore conclude that none of the points in $\mathcal{D}$ are end-points and $q(t) \in \Psi$ for all $t \in (t_d, t_d + \delta)$ for a $\delta > 0$ where $q(t_d) \in \mathcal{D}$.

□

We have shown that the trajectories $q(t)$ cannot stay in $\mathcal{D}$. However, it should be noticed that $\mathcal{D}$ can be traversed. This is not the case for the set $\mathcal{E}$ which is shown to consist of repelling discontinuity surfaces by the following lemma.

Lemma 3.4 *Let the feedback law be given by (3.22)-(3.23). All points $q \in \mathcal{E}$ are normal starting points for the closed-loop system*

$$\dot{q} = f(q), \qquad q = [x, y, \theta]^T$$

i.e. solutions $q(t)$ exist for $t \geq 0$ if $q(0) \in \mathcal{E}$ and $q(t) \in \Psi$ for all $t \in (0, \delta)$ for a $\delta > 0$.

Proof: To prove this, we need only to show that for any $q \in \mathcal{E}$, the projection of the vector field $f(q)$ on the normal of $\mathcal{E}$ points outwards from both sides of the surface. In other words, the inner products of $f(q)$ and the outpointing normal at each side of the discontinuity surface are strictly positive. Let $s_e(q) = 0$ denote the discontinuity surface $\mathcal{E}^e$,

$$s_e(q) = \theta - \theta_d(x, y) - 2\pi e - \pi, \quad e \in \{0, \pm 1, \pm 2, \ldots\}$$

Then the normal to $s_e(q) = 0$ is

$$n(q) = m(q)/\|m(q)\|$$

where

$$m(q) = \frac{\partial s_e(q)}{\partial q} = \begin{bmatrix} \frac{2y}{x^2+y^2} \\ -\frac{2x}{x^2+y^2} \\ 1 \end{bmatrix} = \begin{bmatrix} J_{21} \\ J_{22} \\ J_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{r} \\ -\frac{x}{yr} \\ 1 \end{bmatrix}$$

We find that

$$\|m(q)\| = \frac{4(x^2 + y^2) + 1}{(x^2 + y^2)^2}$$

We define for $q \in \mathcal{E}^e$:

$$f^+(q) \triangleq \lim_{s_e \to 0^+} f(q) = \begin{bmatrix} -\cos\theta_d \gamma r \theta_d \\ -\sin\theta_d \gamma r \theta_d \\ -\gamma\theta_d + k\pi \end{bmatrix}$$

$$f^-(q) \triangleq \lim_{s_e \to 0^-} f(q) = \begin{bmatrix} -\cos\theta_d \gamma r \theta_d \\ -\sin\theta_d \gamma r \theta_d \\ -\gamma\theta_d - k\pi \end{bmatrix}$$

where we have used the control law (3.22)-(3.23), Eqs. (3.19) and (3.20), and the definitions of $\alpha(q)$ from (3.12), $a(q)$ from (3.11), and $\theta_d(q)$ from (3.8). We note that $f^+(q)$ and $f^-(q)$ are independent of the surface $\mathcal{E}^e$. The analysis is thus equivalent for all $\mathcal{E}^e \subset \mathcal{E}$. The control law (3.22)-(3.23) and the righthand side $f(q)$ are defined for all $q \in \mathcal{E}$ where $f(q) = f^+(q)$ since $\alpha = -\pi$ in $\mathcal{E}$. Then we have for all q in a neighborhood of $\mathcal{E}^e$

$$\begin{aligned} n^T f^+(q) &= \quad k\pi/\|m\| > 0 \\ n^T f^-(q) &= -k\pi/\|m\| < 0 \end{aligned}$$

This means that in the neighborhood of every $\mathcal{E}^e \subset \mathcal{E}$, or when $q(0) \in \mathcal{E}$, the vector field, $f(q)$, always has a component driving the system away from $\mathcal{E}$, i.e. $q(t) \in \Psi$ for all $t \in (0, \delta)$ for a $\delta > 0$.

□

(The points $q \in \mathcal{E}$ are normal starting points of AA-type according to the notation of André & Seibert (1960). They stated that a solution exists for such starting points for $t \geq 0$.) Since the discontinuity surfaces $\mathcal{E}^e \subset \mathcal{E}$ are repelling, the solution $\alpha(t)$ of (3.25) exists for all $t \geq 0$. The solution is given by (3.26),

$$\alpha(t) = \alpha(0)\exp(-kt)$$

Since $|\alpha(0)| \leq \pi$, the orientation error $\alpha(t)$ therefore satisfies

$$|\sin \alpha(t)| \leq \xi, \quad \forall t \geq T = \frac{1}{k} \ln \frac{\arcsin \xi}{\pi} \tag{3.38}$$

where ξ is defined in (3.36). On the discontinuity surface $\mathcal{D}$ where $\theta_d = -\pi$, we thus have from the definition of α, (3.12),

$$|\sin \theta(t)| = |\sin(\alpha(t) + \theta_d)| = |\sin(\alpha(t) - \pi)| = |\sin \alpha(t)| \leq \xi, \quad t \geq T$$

Consequently, the discontinuity surface $\mathcal{D}$ becomes repelling after a finite time T as illustrated in *Figure 3.2.*

3.3.3 Dynamics in the Complete Configuration Space

In the set Ψ the righthand side of (3.6) is continuous and a solution of (3.6) is ensured for all time t where $q(t) \in \Psi$. Lemmas 3.1, 3.3 and 3.4 allow us to extend the properties of the dynamics of the closed-loop system to the complete configuration space including the discontinuities.

The following lemma summarizes these results.

Lemma 3.5 *Let the feedback law be given by (3.22)-(3.23). For any* $q = [x, y, \theta]^T \in \mathbb{R}^3$, $z = [a, \alpha]^T \in \mathbb{R} \times (-\pi, \pi]$, *and* $\forall t \geq 0$, *we have:*

$$\begin{aligned} \|z(t)\| &\leq \|z(0)\| e^{-\eta(t)} \\ |a(t)| &\leq |a(0)| e^{-\gamma\kappa(t)} \\ |\alpha(t)| &= |\alpha(0)| e^{-kt} \end{aligned}$$

where $\eta(t)$ *and* $\kappa(t)$ *are defined by (3.29) and (3.27).*

The following theorem establishes the main result.

Theorem 3.1 *Consider the system given by (3.17) with the feedback law (3.22)-(3.23). For all constants* $\varepsilon \in (0, 1)$ *there exists a positive constant* σ *such that the norm of* $z(t)$ *satisfies,*

$$\|z(t)\| \leq \sigma \|z(0)\| e^{-\eta t}, \quad \forall t \geq 0$$

where

$$\eta = \min(\gamma(1-\varepsilon), k)$$

with arbitrary, positive controller parameters γ *and* k.

Proof: Lemma 3.2 gives upper and lower bounds on $b_1(\alpha, \beta)$ and shows that when α approaches zero, $b_1(\alpha, \beta)$ continuously tends towards one. From Lemma 3.2 we find that

$$b_1^2(\alpha, \beta) \geq \begin{cases} (1 - \zeta|\alpha|)^2 & , \zeta|\alpha| \leq 1 \\ 0 & , \zeta|\alpha| > 1 \end{cases}$$

Lemma 3.5 shows that $\alpha(t)$ decreases exponentially to zero. Therefore, with ε given there is a constant T_ε so that

$$b_1^2(\alpha(t), \beta(t)) \geq (1-\varepsilon)^2, \ \ \forall t \geq T_\varepsilon$$

where

$$T_\varepsilon = \begin{cases} \frac{1}{k} \ln \frac{\zeta|\alpha(0)|}{\varepsilon} & , |\alpha(0)| \geq \frac{\varepsilon}{\zeta} \\ 0 & , |\alpha(0)| < \frac{\varepsilon}{\zeta} \end{cases} \tag{3.39}$$

From Lemma 3.5 where $\kappa(t) \geq 0$, $\forall t \geq 0$, we have for all $t \geq 0$,

$$\begin{aligned} |a(t)| &\leq |a(0)| e^{-\gamma\kappa(t)} \\ &\leq |a(0)| e^{-\gamma \int_{T_\varepsilon}^t b_1^2(\tau) d\tau} \\ &\leq |a(0)| e^{-\gamma \int_{T_\varepsilon}^t (1-\varepsilon)^2 d\tau} \end{aligned}$$

$$
\begin{aligned}
&= |a(0)|e^{-\gamma(1-\varepsilon)^2(t-T_\varepsilon)} &(3.40)\\
&= |a(0)|e^{-\gamma(1-\varepsilon)^2 t}e^{\gamma(1-\varepsilon)^2 T_\varepsilon}\\
&\leq |a(0)|\left(\frac{\zeta|\alpha(0)|}{\varepsilon}\right)^{\frac{\gamma(1-\varepsilon)^2}{k}} e^{-\gamma(1-\varepsilon)^2 t}\\
&\leq |a(0)|\sigma e^{-\gamma(1-\varepsilon)^2 t} &(3.41)
\end{aligned}
$$

where

$$\sigma = \left(\frac{\zeta\pi}{\varepsilon}\right)^{\frac{\gamma(1-\varepsilon)^2}{k}} \tag{3.42}$$

since $|\alpha(0)| \leq \pi$. The constant ζ is given by Property 3 of $b_1(q)$ in Lemma 3.2.

This implies that the 1-norm of $z(t)$ is exponentially bounded by

$$
\begin{aligned}
\|z(t)\| &= |a(t)| + |\alpha(t)|\\
&\leq \sigma|a(0)|e^{-\gamma(1-\varepsilon)^2 t} + |\alpha(0)|e^{-kt}\\
&\leq \sigma\|z(0)\|e^{-\eta t}
\end{aligned}
$$

where

$$\eta = \min(\gamma(1-\varepsilon), k)$$

□

It can now be established that exponential convergence of $z(t)$ to zero implies exponential convergence of the q-trajectories to any of the members of $\mathcal{O}$.

Theorem 3.2 *Consider the system given by (3.1) with the feedback law (3.22)-(3.23). For any initial condition $q(0) \in \mathbb{R}^3$, the solutions $q(t)$ of the closed-loop system (3.6)*

$$\dot{q} = f(q), \quad t \geq 0, \qquad q = [x, y, \theta]^T$$

exponentially converge to an element in $\mathcal{O} = \{q \mid (x, y, \theta) = (0, 0, 2\pi n)\}$ where $n \in \{0, \pm 1, \ldots\}$.

Proof: Note from property 3 in Lemma 3.1 that the distance from (x, y) to the origin is upper bounded by the arc length $|a|$,

$$\|[x(t), y(t)]^T\| \leq |a(t)|, \quad \forall t \geq 0$$

Eq. (3.41) implies that

$$\|[x(t), y(t)]^T\| \leq |a(0)|\sigma e^{-\gamma(1-\varepsilon)^2 t}$$

where σ is given by (3.42). Therefore, the norm of $[x, y]^T$ converges exponentially to zero. It remains to be shown that the cart's orientation, θ, converges to a point in $\mathcal{O}$. To this end, we recall from (3.12) that θ can be written as a function of α as follows:

$$\theta(t) = \alpha(t) + \theta_d(t) + 2\pi n$$

where the variable n increments when the discontinuity surface $\mathcal{D}$ is traversed in the positive x-direction and decrements when $\mathcal{D}$ is traversed in the opposite direction. Since $\alpha(t)$ tends exponentially to zero, the behavior of $\theta(t)$ will be determined by the behavior of $\theta_d(t)$. The desired orientation θ_d is defined in (3.8). From this definition we have for $(x, y) \neq (0, 0)$:

$$\sin\frac{\theta_d}{2} = \frac{y\,\mathrm{sgn}(x,y)}{\sqrt{x^2+y^2}}, \quad \cos\frac{\theta_d}{2} = \frac{|x|}{\sqrt{x^2+y^2}}, \quad \tan\frac{\theta_d}{2} = \frac{y}{x} \tag{3.43}$$

Differentiation and Eq. (3.1) give

$$\frac{1}{2}\dot{\theta}_d = \cos^2\frac{\theta_d}{2}\frac{d}{dt}\frac{y}{x} = \cos^2\frac{\theta_d}{2}\frac{1}{x^2}(x\sin\theta - y\cos\theta)v$$

From (3.43) we get

$$\frac{1}{2}\dot{\theta}_d = \frac{1}{x^2+y^2}(x\sin\theta - y\cos\theta)v$$

The control law (3.22) and Eqs. (3.11)-(3.12) and (3.43) imply

$$\begin{aligned}
\frac{1}{2}\dot{\theta}_d &= \frac{1}{x^2+y^2}(-x\sin\theta + y\cos\theta)\gamma b_1\frac{\sqrt{x^2+y^2}}{\mathrm{sinc}\frac{\theta_d}{2}}\mathrm{sgn}(x,y)\\
&= \left(-\sin\theta\cos\frac{\theta_d}{2} + \cos\theta\sin\frac{\theta_d}{2}\right)\frac{\gamma b_1}{\mathrm{sinc}\frac{\theta_d}{2}}\\
&= \sin(\frac{\theta_d}{2} - \theta)\frac{\gamma b_1}{\mathrm{sinc}\frac{\theta_d}{2}}\\
&= -\sin(\frac{\theta_d}{2} + \alpha)\frac{\gamma b_1}{\mathrm{sinc}\frac{\theta_d}{2}}\\
&= -\left(\sin\frac{\theta_d}{2}\cos\alpha + \cos\frac{\theta_d}{2}\sin\alpha\right)\frac{\gamma b_1(\alpha,\theta_d)}{\sin\frac{\theta_d}{2}}\frac{\theta_d}{2}\\
&= -\gamma\cos\alpha\; b_1(\alpha,\theta_d)\frac{\theta_d}{2} - \cos\frac{\theta_d}{2}\frac{\gamma b_1}{\mathrm{sinc}\frac{\theta_d}{2}}\sin\alpha
\end{aligned}$$

From (3.26) we have that $\alpha(t) = \alpha(0)\exp(-kt)$. This implies that

$$\dot{\theta}_d = -c(\theta_d, t)\theta_d + d(\theta_d, t) \tag{3.44}$$

where

$$\begin{aligned} c(\theta_d, t) &= \gamma \cos\alpha(t) b_1(\alpha(t), \theta_d) \\ d(\theta_d, t) &= -2\cos\frac{\theta_d}{2}\frac{\gamma b_1(\alpha(t), \theta_d)}{\operatorname{sinc}\frac{\theta_d}{2}}\sin\alpha(t) \end{aligned}$$

where $b_1(\alpha(t), \theta_d)$ is given from (3.20) and (3.21). By using Property 1 of Lemma 3.2 we can show that $c(\theta_d, t)$ and $d(\theta_d, t)$ satisfy

$$\begin{aligned} \left|\int_T^t (c(\theta_d(\tau), \tau) - \gamma) d\tau\right| &\leq P = \gamma\left(\frac{4}{\pi} + \frac{\pi}{2}\right)\frac{\pi}{k}, \quad \forall t \geq T \\ |d(\theta_d, t)| &\leq De^{-k(t-T)}, \quad \forall t \geq T, \quad D = 2\gamma\pi\sqrt{1 + \frac{\pi^2}{4}} \end{aligned}$$

for all possible solutions $\theta_d(t)$, $t \geq T$, where T is a finite time such that $\mathcal{D}$ is repelling for all $t \geq T$, (3.38). For $t \geq T$ the discontinuity surfaces are not traversed so that there exists a solution $\theta_d(t)$ of (3.44) for $t \geq T$. From Lemma 2.1 we can hence conclude that

$$\forall \varepsilon > 0, \quad |\theta_d(t)| \leq \varsigma\left(|\theta_d(T)| + D\right)e^{-(\kappa-\varepsilon)(t-T)}$$

where

$$\kappa = \min\{\gamma, k\} \quad \varsigma = \max\{e^P, e^{2P}/\varepsilon\}$$

Therefore, θ_d converges exponentially to zero. The definition of α, (3.12), gives

$$|\theta - 2\pi n| \leq |\alpha(t)| + |\theta_d(t)| \leq \pi e^{-kt} + \varsigma\left(|\theta_d(T)| + D\right)e^{-(\kappa-\varepsilon)(t-T)}, \quad \forall t \geq T$$

and the exponential convergence of $q(t)$ to an element in $\mathcal{O}$ is proven.

□

This theorem implies that the control law (3.22)-(3.23) makes the origin attractive, Definition B.20, with exponential convergence. We can further state that:

Corollary 3.1 *The control inputs $v(q)$ and $\omega(q)$ remain bounded for any $q \in \mathbb{R}^3$.*

Proof: Boundedness of v and ω follows from the properties 1 and 4 of b_1 and $b_2 a$ listed in Lemma 3.2, and the fact that $a(q)$ and $\alpha(q)$ are bounded for all $q \in \mathbb{R}^3$. It should also be observed that both the inputs v and ω tend toward zero as time goes to infinity.

$\square$

Design guidelines for choosing the controller parameters can be established from Lemma 3.5 and Theorem 3.1.

3.3.4 Design Guidelines

A suitable design specification may be to give time intervals, T_a and T_α, as the time needed to decrease $a(t)$ and $\alpha(t)$ respectively from their initial values, $a(0)$ and $\alpha(0)$, to some specified values, $a(T_a)$ and $\alpha(T_\alpha)$, relative to $a(0)$ and $\alpha(0)$, i.e.

$$|\frac{\alpha(T_\alpha)}{\alpha(0)}| \leq n_\alpha < 1 \quad \forall t \geq T_\alpha \quad \text{and} \quad |\frac{a(T_a)}{a(0)}| \leq n_a < 1 \quad \forall t \geq T_a$$

where n_α and n_a describe the desired reductions. From these quantities it is straightforward to calculate the corresponding value of k as:

$$k = -\frac{1}{T_\alpha} \ln n_\alpha$$

Similarly, we find from (3.40)

$$\gamma \geq -\frac{1}{(1-\varepsilon)^2 (T_a - T_\varepsilon)} \ln n_a$$

where T_ε is given by (3.39) as a function of ε and $|\alpha(0)|$. We then choose

$$\gamma = \min_{\varepsilon \in (0,1)} -\frac{1}{(1-\varepsilon)^2 (T_a - T_\varepsilon)} \ln n_a \tag{3.45}$$

where T_ε is given by the worst case, i.e. when $|\alpha(0)| = \pi$. For example, $T_a = 5$, $T_\alpha = 1$, $n_\alpha = n_a = 0.01$ give

$$k = 4.6, \qquad \gamma = 1.3$$

by numerical minimization of (3.45). The optimum is found for $\varepsilon = 0.28$.

3.4 Simulations

Simulations were done by using the SIMNON package, (Elmqvist, Åström, Schönthal & Wittenmark 1990), and MATLAB. The constants k and γ were chosen to 4.6 and 1.3, respectively. *Figure 3.3* shows the resulting paths in the xy-plane for several initial conditions corresponding to different points on the unit circle with an initial orientation angle $\theta(0) = \pi/2$. We see

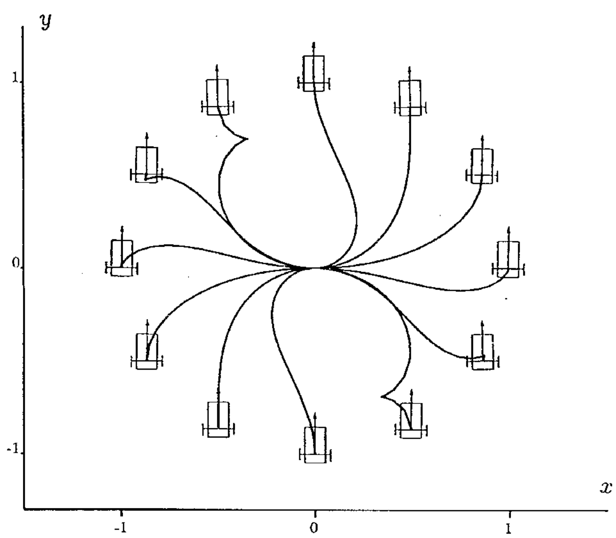

Figure 3.3: Resulting paths when the cart is initially on the unit circle in the xy-plane with $\theta(0) = \pi/2$.

from these phase trajectories that the cart converges to the origin with the desired orientation by asymptotically tracking one of the circles in the path family, $\mathcal{P}$. *Figure 3.4* shows the resulting path in the xy-plane for $q(0) = [0, 1, 0]^T$. Note that the cart starts at the discontinuity surface $\mathcal{D}$ and asymptotically converges to the origin. The shape of the path will vary with variations in the control gains, k and γ.

Several simulations were done with a traversing of the discontinuity surface $\mathcal{D}$ which consists of the y-axis except $y = 0$. The initial condition was $q(0) = [-0.05, 1, \pi/2]^T$. Euler's method was used for numerical integration with time-step 0.005. A nominal simulation was done and compared to simulations with measurement noise, model errors, torque inputs, and time-varying feedback law.

3.4.1 Nominal Simulation

Figure 3.5 shows the resulting paths in the xy-plane in the case of a kinematic model and no disturbances or errors. We see that the cart traverses the discontinuity surface only once before reaching the origin with the desired orientation. *Figure 3.6* shows the corresponding time histories of $x(t)$, $y(t)$ and $\theta(t)$, illustrating exponential convergence to 0. *Figure 3.7* illustrates the discontinuity in the inputs v and ω when the y-axis is crossed. We note that the inputs are bounded and chattering is avoided.

3.4.2 Measurement Noise

A simulation was done with measurement noise. A normal distributed noise with mean 0 and standard deviation 0.05 was added to the x- and y-positions and to the orientation θ. *Figure 3.8* shows the resulting path outside a circle about the origin with radius equal to the standard deviation, 0.05. We see that outside this domain the resulting path is essentially equal to the case without noise, *Figure 3.5*. The resulting path inside the domain is presented in *Figure 3.9*. This simulation was run for $t \in [0, 10]$. This simulation study indicates that the feedback law is not significantly sensible to measurement noise outside a domain about the origin in the xy-plane where the size of the domain is given by the uncertainty of the position measurement. An explanation for this can be that the control law for ω makes the orientation error α exponentially stable and thus less sensible for noise in the orientation measurement. The desired orientation $\theta_d(x, y)$ and the arc length $a(x, y)$ are not sensible to variations in (x, y) far away from the origin and the discontinuity surface $\mathcal{D}$. This can be shown from the definition of $\theta_d(x, y)$, (3.8), and $a(x, y)$, (3.11). We denote

$$\tilde{\theta}_d = \theta_d(x + \triangle x, y + \triangle y) - \theta_d(x, y), \quad \tilde{a} = a(x + \triangle x, y + \triangle y) - a(x, y)$$

By a first order Taylor expansion we then find from (3.8)

$$\tilde{\theta}_d \approx -J_{21}\triangle x - J_{22}\triangle y = -\frac{2y}{x^2 + y^2}\triangle x + \frac{2x}{x^2 + y^2}\triangle y$$

We see that far away from the origin, $x^2 + y^2$ dominates the variations $\triangle x$ and $\triangle y$ and $\tilde{\theta}_d$ is small implying that the variations of the resulting path are small. However, as the cart converges to the origin asymptotically along a circle, as seen from the simulations, y tends toward zero faster than x and we get

$$\tilde{\theta}_d \approx \frac{2}{x}\triangle y$$

which implies that $\tilde{\theta}_d$ becomes significant when $\triangle y$ is of the same order as x. This variation of θ_d affects the variation of θ because of the exponentially convergent control law for ω. This is illustrated in *Figure 3.9* where large variations in the orientation θ are observed in the neighborhood of the origin.

By a first order Taylor expansion we find from (3.11) and (3.16)

$$\tilde{a} \approx J_{11}\triangle x + J_{12}\triangle y = (\frac{\theta_d}{\beta} - 1)\triangle x + \left(\frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta}\right)\triangle y$$

We can see from the definition of θ_d, (3.8), that

$$|J_{11}| \leq 1$$

From (3.18) we see that $J_{12} = b_1(x, y, \pi/2)$, i.e. $\theta = \pi/2$. From Lemma 3.2 we can show that

$$\max b_1 \leq \sqrt{1 + \pi^2/4}$$

which implies that $|J_{12}| \leq \sqrt{1 + \pi^2/4}$. Therefore, the variations in $a(x, y)$ are of the same order as the variations in x and y everywhere outside the discontinuity surfaces. Since the feedback law for v depends linearly on $a(x, y)$, the feedback law is not significantly affected by small variations in x and y.

3.4.3 Model Error

Figure 3.10 shows the resulting path when the wheels are unbalanced, i.e. the real gains for the wheel velocities v_1 and v_2 are different from the gains in the model (3.1).

The applied input $u^a = [v^a, \omega^a]^T$ is therefore modified with respect to the calculated $u^c = [v^c, \omega^c]^T$ from the control law. From (3.3) we have that

$$\begin{bmatrix} v_1^c \\ v_2^c \end{bmatrix} = M^{-1} \begin{bmatrix} v^c \\ \omega^c \end{bmatrix}, \qquad M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c_r} & -\frac{1}{2c_r} \end{bmatrix}$$

where v_1^c and v_2^c are the wheel velocities resulting from the control law. We assume that the applied wheel velocities $[v_1^a, v_2^a]^T$ are modified due to a model error according to

$$\begin{bmatrix} v_1^a \\ v_2^a \end{bmatrix} = \Lambda \begin{bmatrix} v_1^c \\ v_2^c \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The corresponding applied input $u^a = [v^a, \omega^a]^T$ hence becomes

$$\begin{bmatrix} v^a \\ \omega^a \end{bmatrix} = M \begin{bmatrix} v_1^a \\ v_2^a \end{bmatrix} = M\Lambda M^{-1} \begin{bmatrix} v^c \\ \omega^c \end{bmatrix}$$

In the simulation the following parameters were chosen: $\lambda_1 = 1.2$, $\lambda_2 = 1$, and $c_r = 1$. These values give

$$\begin{bmatrix} v^a \\ \omega^a \end{bmatrix} = \begin{bmatrix} 1.1 & 0.1 \\ 0.1 & 1.1 \end{bmatrix} \begin{bmatrix} v^c \\ \omega^c \end{bmatrix}$$

Since the matrix $M\Lambda M^{-1}$ is non-singular for all $\lambda_1, \lambda_2, c_r > 0$, the only possible equilibrium point for (3.1) with $[u, \omega]^T = [u^a, \omega^a]^T$ is the equilibrium point with $[u, \omega]^T = [u^c, \omega^c]^T$, which is the origin in $\mathbb{R}^2 \times S^1$. We see from *Figure 3.10* that the cart converges to $(x, y) = (0, 0)$ along essentially the same type of path as in the case of no model errors, *Figure 3.5.* This model error causes, however, an "overshoot" in the exponential convergence of the orientation error $\alpha(t)$ as illustrated in *Figure 3.11.* $\alpha(t)$ converges to zero as the cart converges to the origin showing that the orientation θ converges to zero in the origin with the chosen model error. Convergence with other types of model errors or parameters is, however, not analyzed.

3.4.4 Torque Inputs

A simulation was done in the case of torque inputs (u_1, u_2), i.e. the dynamic extended model was given by

$$\begin{aligned} \dot{x} &= \cos\theta v_r \\ \dot{y} &= \sin\theta v_r \\ \dot{\theta} &= \omega_r \\ \dot{v}_r &= u_1 \\ \dot{w}_r &= u_2 \end{aligned}$$

The velocities $[v, \omega]^T$ from the control law (3.22)-(3.23) were used as references for the state variables v_r and ω_r. A simple feedback law was investigated:

$$u_1 = -k_1(v_r - v) \tag{3.46}$$
$$u_2 = -k_2(\omega_r - \omega) \tag{3.47}$$

In the simulation, the controller parameters were chosen to $k_1 = k_2 = 10$. The resulting path in the xy-plane is presented in *Figure 3.12* showing convergence to the origin. The orientation error $\alpha(t)$ is plotted in *Figure 3.13* showing a little "overshoot" in the exponential convergence to zero. The actual velocities v_r and ω_r are presented in *Figure 3.14*. We see that the velocities are continuous and tend toward zero. A theoretical analysis of the convergence in the case of torque inputs is quite complicated and is not done. This simulation just indicates that a feedback law like (3.46)-(3.47) controls the real velocities, v_r and ω_r, sufficiently close to the desired velocities, v and ω from the control law for the kinematic model, to obtain convergence of the state $q = [x, y, \theta]^T$ to the origin.

3.4.5 Time-varying Smooth Feedback

Finally, the behavior of the proposed piecewise analytic feedback law was compared to the behavior of a time-varying smooth feedback law. The time-varying smooth feedback law was the one proposed by Pomet et al. (1992):

$$\begin{aligned}
u_1 &= -y \sin\theta - x\cos\theta \\
u_2 &= -a\cos\frac{\theta}{a}\sin\frac{\theta}{a} \\
&\quad + a\lambda\cos^2\frac{\theta}{a}[-y(\sin t - \cos t) - (y\sin\theta + x\cos\theta)\cos t\sin\theta]
\end{aligned}$$

for the cart model

$$\begin{aligned}
\dot{x} &= \cos\theta u_1 \\
\dot{y} &= \sin\theta u_1 \\
\dot{\theta} &= u_2
\end{aligned}$$

The parameter λ was taken to 40 as in their simulation, and $a > 2$ was taken to 2.1. The resulting path in the xy-plane with $q(0) = [-0.05, 1, \pi/2]^T$ is shown in *Figure 3.15*. Compare this figure with the corresponding path from the piecewise analytic control law in *Figure 3.5*. Note the odd behavior and the significant number of cusps in the time-varying case. The slow convergence of the time-varying smooth feedback law is illustrated in *Figure 3.16* where the time history of the orientation θ is shown together with the orientation resulting from the piecewise analytic controller, *Figure 3.6*. We see that the convergence of the time-varying smooth controller is much slower than the convergence in the piecewise analytic case.

3.5 Conclusions

A piecewise analytic controller has been proposed for the kinematic model of a mobile robot with two degrees of freedom. The particularity of this discontinuous controller is that infinite high frequency components and the well-known problem of "chattering" are avoided. The cart *exponentially* converges to the origin with zero orientation for any initial condition. This is achieved by letting the motion of the cart converge to one of the circles which pass through the origin and are centered on the y-axis. The circles were chosen because they introduce new variables that are geometrically meaningful. However, other types of paths may also be possible. Common for these paths is that they must pass through the origin with the desired derivative so that the desired orientation is asymptotically reached. By letting the control law be discontinuous, the problem of interpreting the scalars θ and $\theta + 2n\pi \in \mathbb{R}$ as the same configuration in S^1 is avoided as opposed to continuous approaches.

Simulations showed that this piecewise analytic feedback law resulted in exponential convergence of the cart to the origin with zero orientation. The resulting paths are much more natural than in the case of a time-varying feedback law and the convergence is considerably faster. Simulations also indicated that this piecewise analytic feedback law works in the case of measurement noise and some model errors. An extension to torque inputs instead of velocity inputs was studied by a simulation. The simulation indicated an exponential convergence to the desired configuration. The conditions for achieving this convergence with torque inputs is, however, still an open problem.

3.A Appendix: Proof of Lemma 3.1

1. From (3.11)-(3.12) and (3.8) we see that if $(x, y, \theta) = (0, 0, 2\pi n)$ then $\theta_d(0, 0) = 0$ and

$$a(x, y) = 0, \qquad \alpha(x, y, \theta) = 0$$

Similarly, from (3.11) and (3.8) we see that if $a(x, y) = 0$ then $(x, y) = (0, 0)$ which implies that $\theta_d = 0$. Since $\alpha = 0$ and $\theta_d = 0$ then $\theta = 2\pi n$, (3.12).

2. The discontinuity surface $\mathcal{D}$ is traversed when the state $q = [x, y, \theta]$ passes through a configuration $q_\mathcal{D} = [0, y_\mathcal{D}, \theta_\mathcal{D}]^T \in \mathcal{D}$ for some $y_\mathcal{D} \neq 0$. From (3.8) we see that

$$\lim_{q \to q_\mathcal{D}} \theta_d(x, y) = \pm\pi$$

which implies that

$$\lim_{q \to q_{\mathcal{D}}} \operatorname{sinc} \frac{\theta_d}{2} = \frac{2}{\pi}$$

Eq. (3.11) then implies that

$$\lim_{q \to q_{\mathcal{D}}} a^2(x, y) = (\frac{\pi}{2} y_{\mathcal{D}}^2)^2$$

From the definition of $\operatorname{sgn}(x, y)$, (3.9), we have that $a^2(0, y_{\mathcal{D}}) = (\frac{\pi}{2} y_{\mathcal{D}}^2)^2$ and the continuity of $a^2(x, y)$ when traversing $\mathcal{D}$ then follows. Traversing $\mathcal{E}$ does not affect the continuity of $a^2(x, y)$ since $a(x, y)$ only is discontinuous in $\mathcal{D}$. The orientation error $\alpha(q)$ becomes discontinuous if $\mathcal{E}$ is traversed. Let $q_{\mathcal{E}}$ denote a configuration in $\mathcal{E}$. Then we see from (3.12) and (3.13) that

$$\lim_{q \to q_{\mathcal{E}}} \alpha(q) = \pm \pi$$

and $\alpha(q_{\mathcal{E}}) = -\pi$. The continuity of $\alpha^2(q)$ when traversing $\mathcal{E}$ then follows. When traversing $\mathcal{D}$, the desired orientation $\theta_d(x, y)$ becomes discontinuous,

$$\lim_{q \to q_{\mathcal{D}}} \theta_d(x, y) = \pm \pi$$

Eq. (3.12) then implies

$$\lim_{q \to q_{\mathcal{D}}} \alpha(q) = \theta_{\mathcal{D}} \mp \pi + 2\pi\, n(\theta_{\mathcal{D}} \mp \pi) = \begin{cases} \theta_{\mathcal{D}} + \pi \ , & \theta_{\mathcal{D}} \in [-\pi, 0) \\ \theta_{\mathcal{D}} - \pi \ , & \theta_{\mathcal{D}} \in [0, \pi) \end{cases}$$

Therefore, the limit $\lim_{q \to q_{\mathcal{D}}} \alpha(q)$ is well-defined for all $q_{\mathcal{D}} \in \mathcal{D}$. From the definition of $\alpha(q)$, $\alpha(q_{\mathcal{D}})$ is equal to this limit. The continuity of $\alpha(q)$ and consequently of $\alpha^2(q)$ when traversing $\mathcal{D}$ then follows. The continuity of $\|F(q)\|^2 = a^2(x, y) + \alpha^2(q)$ readily follows.

3. Since $\operatorname{sinc} \frac{\theta_d}{2} \in [\frac{2}{\pi}, 1]$ it follows from (3.11) that

$$\|[x, y]^T\| = \sqrt{x^2 + y^2} \leq |a(x, y)| \leq \frac{\pi}{2} \|[x, y]^T\| = \frac{\pi}{2} \sqrt{x^2 + y^2}$$

$\square$

3.B Appendix: Proof of Lemma 3.2

1. From Equation (3.20) we have, with $\beta = \frac{y}{x}$:

$$b_1(\alpha, \beta) = \cos \alpha + B(\beta) \sin \alpha \tag{3.48}$$

where

$$
\begin{aligned}
\theta_d &= \theta_d(\beta) = 2\arctan(\beta) \\
B(\beta) &= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d(\frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta}) \\
&= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d(\theta_d\frac{\beta^2 - 1}{2\beta} + 1)\frac{1}{\beta} \\
&= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d(1 - \frac{\theta_d}{\tan\theta_d})\frac{1}{\beta} \\
&= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d\frac{1}{\beta} - \frac{\cos^2\theta_d}{\sin\theta_d}\frac{\theta_d}{\beta} \\
&= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d\frac{1}{\beta} + \sin\theta_d\frac{\theta_d}{\beta} - \frac{1}{\sin\theta_d}\frac{\theta_d}{\beta} \\
&= \sin\theta_d + \cos\theta_d\frac{1}{\beta} - \frac{1}{\sin\theta_d}\frac{\theta_d}{\beta} \\
&= \frac{2\beta}{1+\beta} + \frac{1-\beta^2}{1+\beta^2}\frac{1}{\beta} - \frac{1+\beta^2}{2\beta}\frac{2\arctan\beta}{\beta} \\
&= \frac{1}{\beta} - (1 + \frac{1}{\beta^2})\arctan\beta
\end{aligned}
$$

Here, we have used the fact that

$$
\cos\theta_d = \frac{1-\beta^2}{1+\beta^2}, \quad \sin\theta_d = \frac{2\beta}{1+\beta^2}, \quad \tan\theta_d = \frac{2\beta}{1-\beta^2}
$$

In order to find the maximum and minimum values of $B(\beta)$, we analyze the derivative, $B'(\beta)$:

$$
\begin{aligned}
B'(\beta) &= -\frac{1}{\beta^2} + \frac{2}{\beta^3}\arctan\beta - (1 + \frac{1}{\beta^2})\frac{1}{1+\beta^2} \\
&= \frac{2}{\beta^2}(\frac{\arctan\beta}{\beta} - 1) \; < \; 0 \qquad \forall\beta
\end{aligned}
$$

We note that $B'(\beta)$ is continuous in $\beta = 0$, if we define $B'(0) = -\frac{2}{3}$. Since $B'(\beta)$ is negative for all $\beta \in R$, we find that

$$
-\frac{\pi}{2} = \lim_{\beta\to\infty} B(\beta) \leq B(\beta) \leq \lim_{\beta\to-\infty} B(\beta) = \frac{\pi}{2}
$$

Therefore we get:

$$
\cos\alpha - \frac{\pi}{2}|\sin\alpha| \leq b_1(\alpha,\beta) \leq \cos\alpha + \frac{\pi}{2}|\sin\alpha|
$$

2. From **1** we have that

$$b_1(\alpha, \beta) = \cos\alpha + B(\beta)\sin\alpha$$

where $B(\beta)$ is bounded. Since $\cos\alpha$ and $\sin\alpha$ are continuous in α, and $B(\beta)$ is bounded, it is clear that $b_1(\alpha, \beta)$ is also continuous in α.

3. From **2**, we have that

$$b_1(\alpha, \beta) = \cos\alpha + B(\beta)\sin\alpha$$

where $|B(\beta)| \le \frac{\pi}{2}$. This equivalent to

$$b_1(\alpha, \beta) - 1 = \cos\alpha - 1 + B(\beta)\sin\alpha$$

This implies

$$\begin{aligned} |b_1(\alpha, \beta) - 1| &= |\cos\alpha - 1 + B(\beta)\sin\alpha| \\ &\le |\cos\alpha - 1| + |B(\beta)\sin\alpha| \\ &\le |\cos\alpha - 1| + \frac{\pi}{2}|\sin\alpha| \\ &\le |\alpha| + \frac{\pi}{2}|\alpha| = \zeta|\alpha| \end{aligned}$$

4. From the definition of $a(x, y)$, (3.11), we have:

$$a = r\theta_d = \frac{\sqrt{x^2 + y^2}}{\operatorname{sinc}\frac{\theta_d}{2}}\operatorname{sgn}(x, y)$$

From the definition of θ_d (3.8) we have

$$\frac{y\operatorname{sgn}(x, y)}{\sqrt{x^2 + y^2}} = \sin\frac{\theta_d}{2}, \qquad \frac{x\operatorname{sgn}(x, y)}{\sqrt{x^2 + y^2}} = \cos\frac{\theta_d}{2}$$

From the definition of b_2, (3.19), we then get:

$$\begin{aligned} b_2 a &= \left(\cos\theta\frac{2y}{x^2 + y^2} - \sin\theta\frac{2x}{x^2 + y^2}\right) a(x, y) \\ &= (\cos\theta\sin\frac{\theta_d}{2} - \sin\theta\cos\frac{\theta_d}{2})\frac{2}{\operatorname{sinc}\frac{\theta_d}{2}} \\ &= \sin(\frac{\theta_d}{2} - \theta)\frac{2}{\operatorname{sinc}\frac{\theta_d}{2}} \end{aligned}$$

Since maximum of $\sin(\frac{\theta_d}{2} - \theta)$ is 1 and minimum of $\operatorname{sinc}\frac{\theta_d}{2}$ is $\frac{2}{\pi}$ we have

$$|b_2 a| \le \pi$$

□

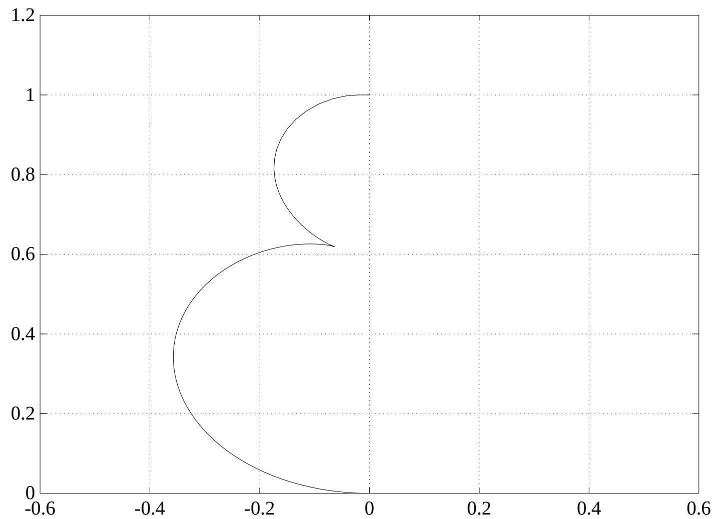

Figure 3.4: The resulting path in the xy-plane with $q(0) = [0, 1, 0]^T$.

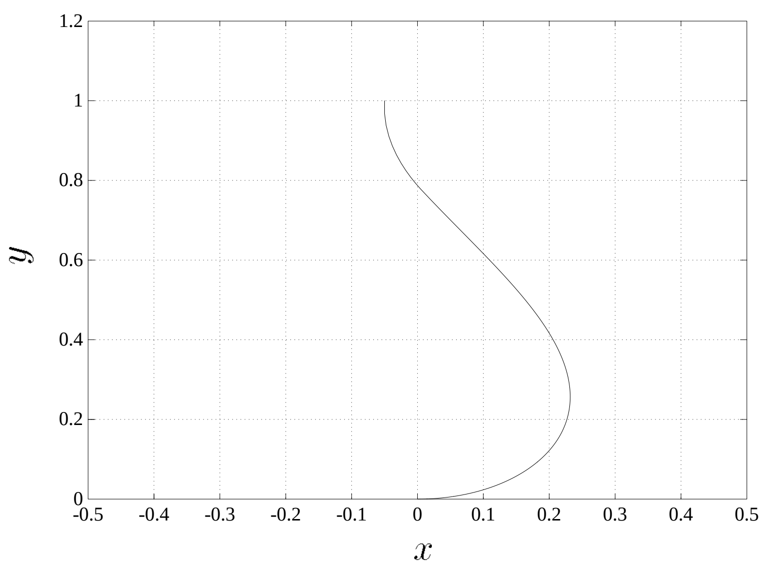

Figure 3.5: A path crossing the discontinuity surface, the y-axis, $q(0) = [-0.05, 1, \pi/2]^T$.

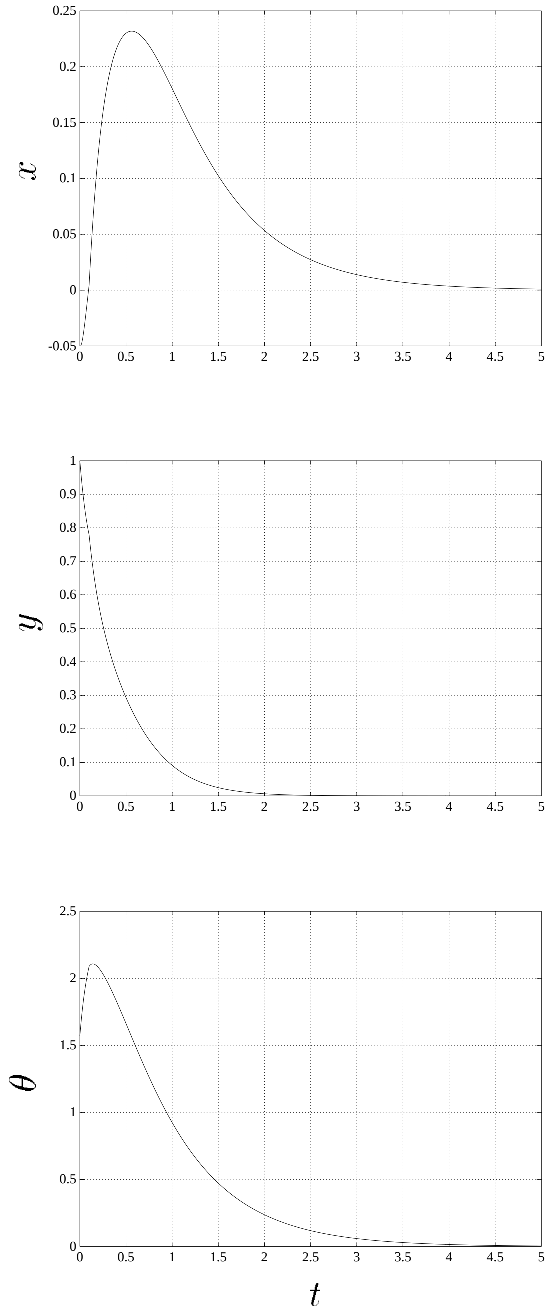

Figure 3.6: Timeplots of the position, $x(t)$ and $y(t)$, and the orientation, $\theta(t)$, for a path crossing the y-axis, see *Figure 3.5.*

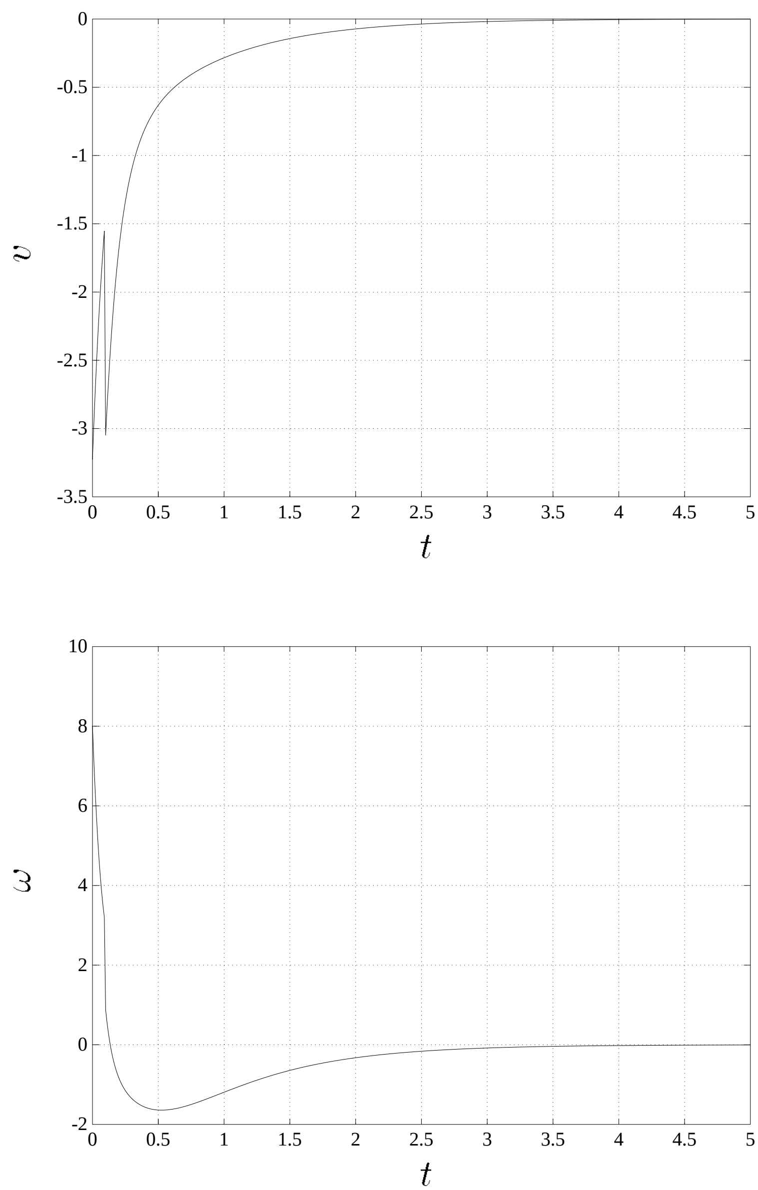

Figure 3.7: Timeplots of the inputs, $v(t)$ (tangential velocity) and $\omega(t)$ (angular velocity), for a path crossing the y-axis, see *Figure 3.5*. $v(t)$ and $\omega(t)$ become discontinuous when $x = 0$, $y \neq 0$.

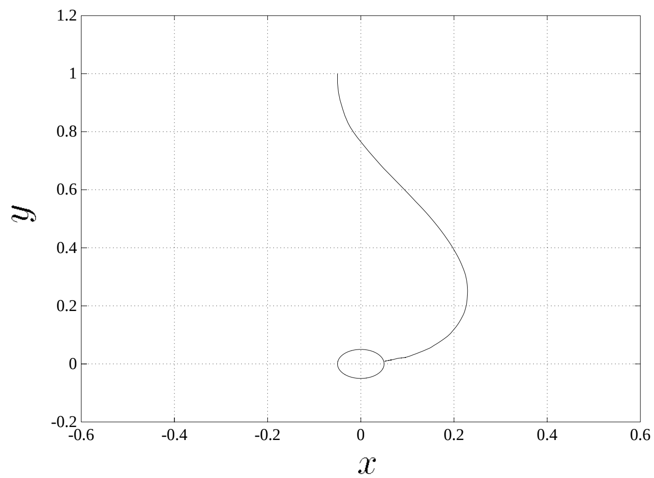

Figure 3.8: A path crossing the discontinuity surface, the y-axis, with normal distributed measurement noise at $q = [x, y, \theta]^T$. The standard deviation of the uncorrelated noise is 0.05. The initial state is $q(0) = [-0.05, 1, \pi/2]^T$.

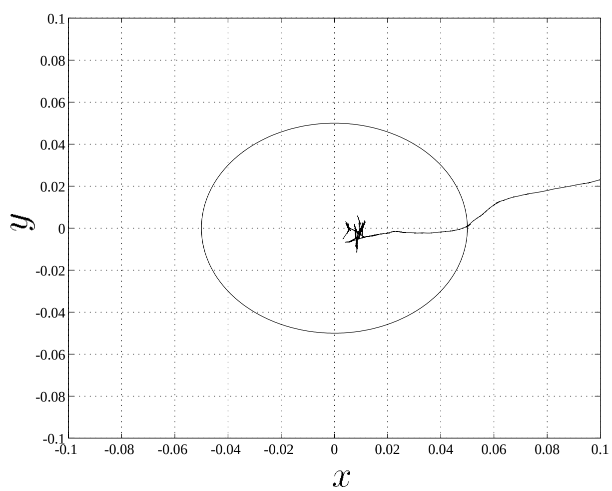

Figure 3.9: The resulting path in the neighborhood of the origin with normal distributed measurement noise at $q = [x, y, \theta]^T$. The standard deviation of the uncorrelated noise is 0.05.

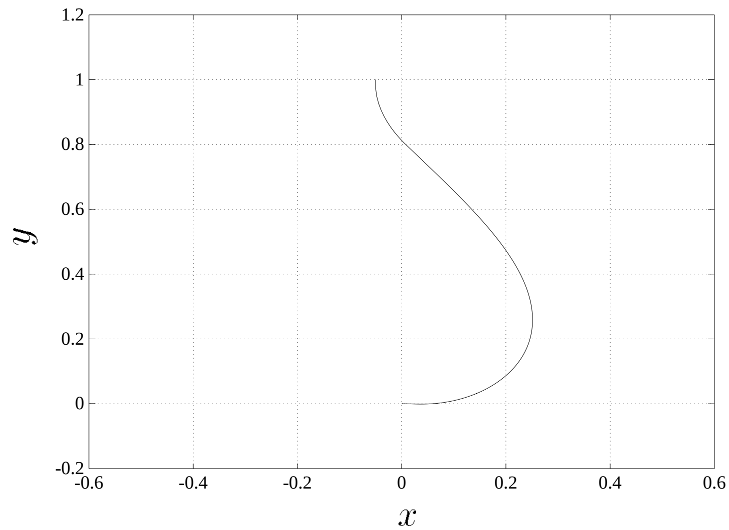

Figure 3.10: A path crossing the discontinuity surface, the y-axis, for a cart with unbalanced wheels. The initial state is $q(0) = [-0.05, 1, \pi/2]^T$.

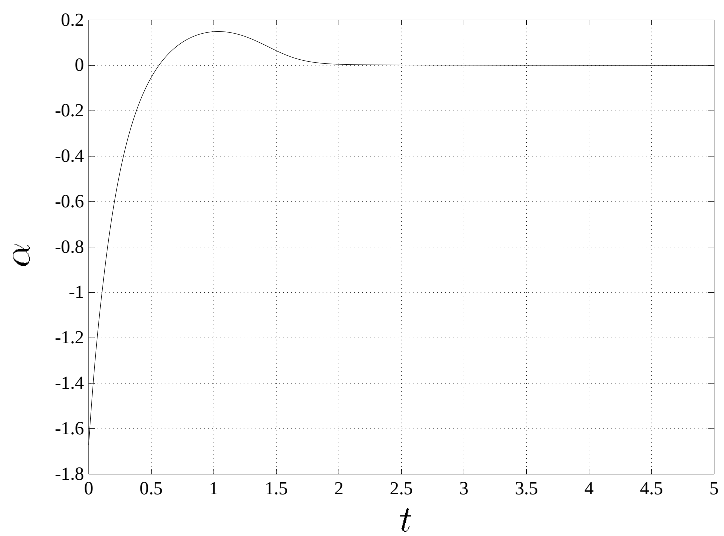

Figure 3.11: The orientation error $\alpha(t)$ resulting from unbalanced wheels corresponding to the path in *Figure 3.10*.

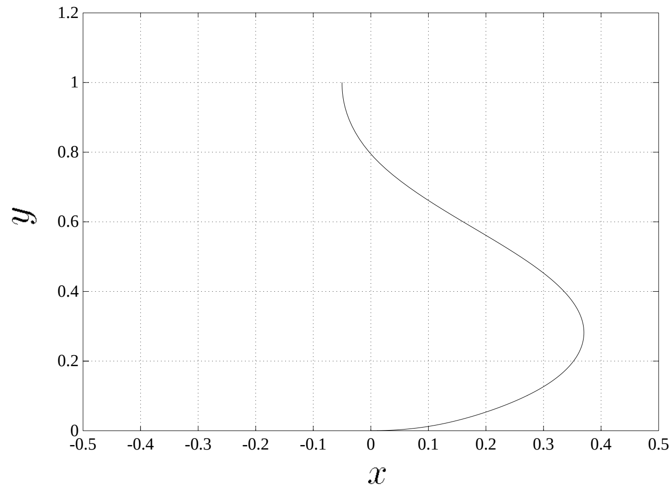

Figure 3.12: The resulting path in the xy-plane when torque inputs are applied. The initial state was $q(0) = [-0.05, 1, \pi/2]^T$ and the discontinuity surface $\mathcal{D}$ is traversed.

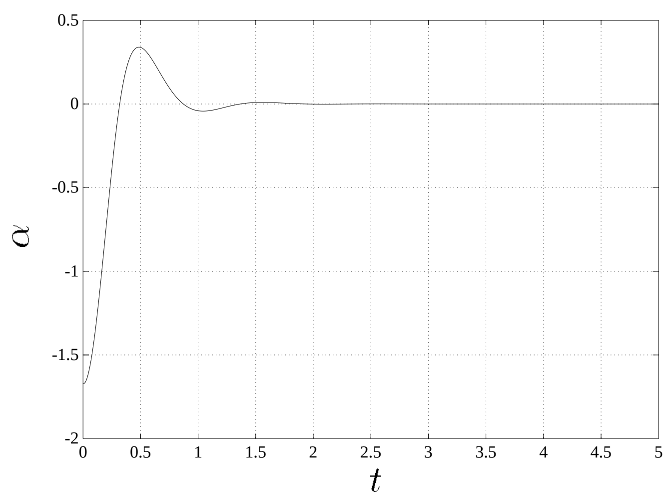

Figure 3.13: The orientation error $\alpha(t)$ when torque inputs are applied corresponding to the path in *Figure 3.12*.

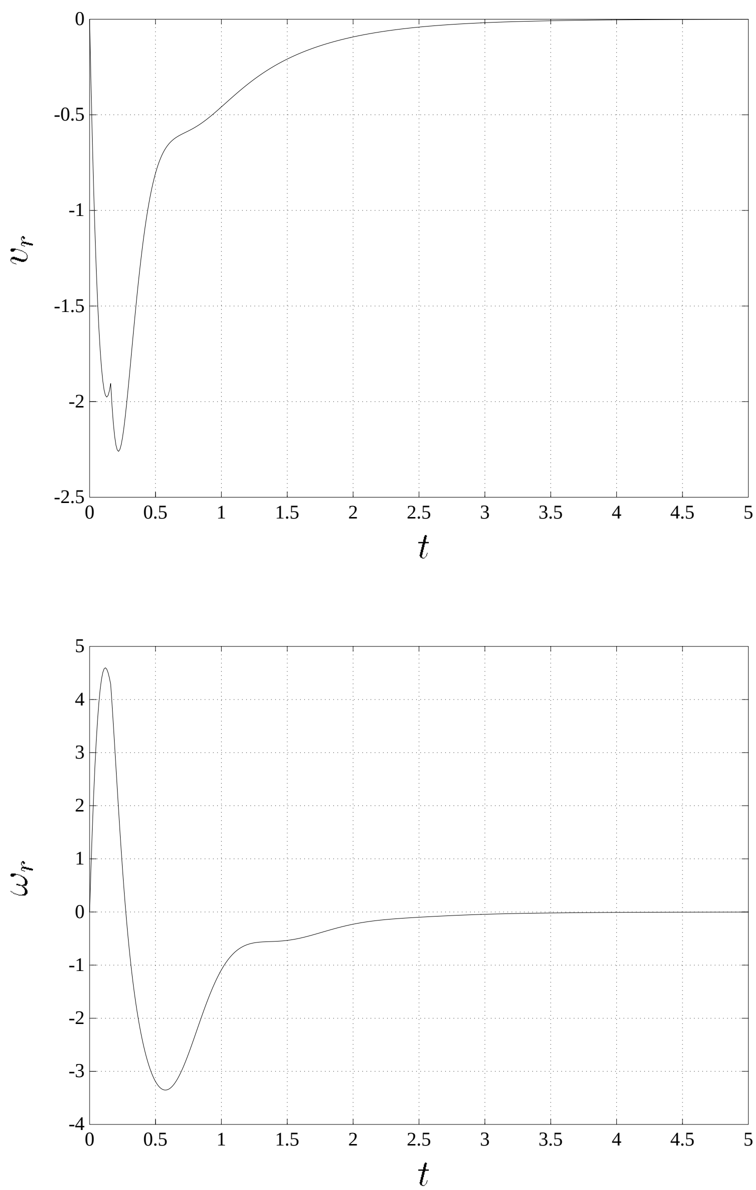

Figure 3.14: Timeplots of the actual velocities $v_r(t)$ and $\omega_r(t)$ when torque inputs are applied corresponding to the path in *Figure 3.12*.

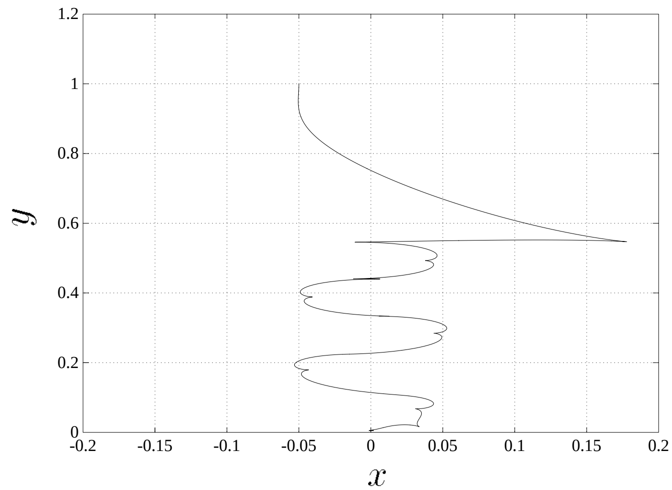

Figure 3.15: The resulting path in the xy-plane when a time-varying smooth feedback law is used. The initial state is $q(0) = [-0.05, 1, \pi/2]^T$.

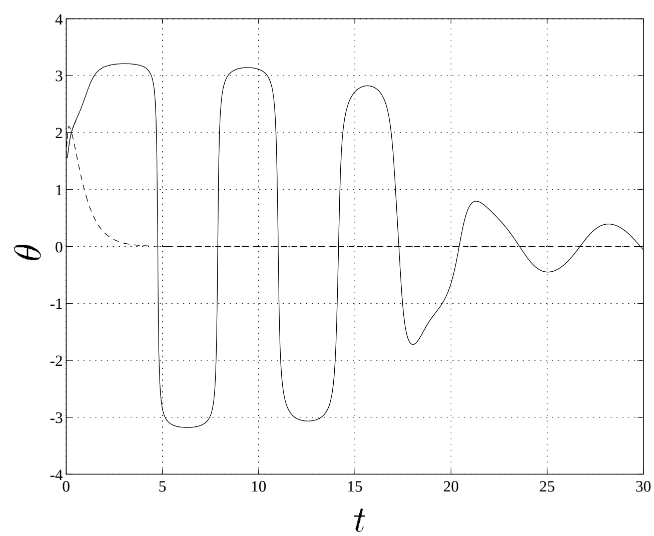

Figure 3.16: The orientation $\theta(t)$ of the cart when a time-varying feedback law is used corresponding to the path in *Figure 3.15*. The dashed line is the orientation from the piecewise analytic controller, *Figure 3.6*.

Chapter 4

Path Following

4.1 Introduction

Paths can be planned between desired configurations based on several criteria and constraints like collision avoidance and shortest path. The problem of making mobile vehicles follow such a desired path has mainly been studied by an introduction of a reference vehicle to be tracked, (Kanayama et al. 1988), (Nelson & Cox 1988), (Kanayama et al. 1990), (Samson & Ait-Abderrahim 1990*b*) (see Section 1.1). The basic assumption for these strategies has been that the advancement velocity of the reference mobile robot does not converge to zero, hence excluding stopping phases.

In this chapter, a control law is presented to make a cart follow a path composed of straight lines and arcs of circles rather than tracking a reference vehicle. Such a path is a result of several path planners and is easily represented as a sequence of positions and orientations. The shortest path between two configurations with the curvature upper bounded is also composed of straight lines and arcs of circles, in the case of no obstacles, (Dubins 1957) and (Reeds & Shepp 1990).

The global control law is an extension of the approach in Chapter 3 to make the cart move towards any configuration asymptotically along a circle segment. The desired path is then followed with arbitrary accuracy and the convergence to the terminal configuration is exponential. This presentation is essentially based on Sørdalen & Canudas de Wit (1992*a*), Sørdalen & Canudas de Wit (1992*b*), and Sørdalen & Canudas de Wit (1993).

4.2 Exponential Convergence to Arbitrary Configuration

The kinematics of a cart with two driving wheels is given by (3.1):

$$\dot{q} = G(q)\begin{bmatrix} v \\ \omega \end{bmatrix}, \quad G(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{4.1}$$

where the state of the system (4.1), $q = [x, y, \theta]^T$, is the position of the wheel axis center, (x, y), and the cart orientation, θ, with respect to the x-axis. We assume that the tangent velocity v and the angular velocity ω can be regarded as the inputs to the system, i.e. $u = [v, \omega]^T$.

Let $q_r = [x_r,\ y_r,\ \theta_r]^T$ be a reference configuration in the configuration space. The control problem addressed in this section consists of designing a control law, $u(q, q_r)$, so that the closed-loop system $\dot{q} = G(q)u(q, q_r) = f(q, q_r)$ converges for any initial condition, $q(0)$, to an equilibrium point in $\mathcal{Q}$,

$$\mathcal{Q} = \{q \mid (x, y, \theta) = (x_r, y_r, \theta_r + 2\pi n)\}, \quad n \in \{0, \pm 1, \pm 2, \ldots\}$$

which represents a constant configuration in the configuration space $\mathbb{R}^2 \times S^1$. Note that all points in $\mathcal{Q}$ are equivalent in terms of positioning and orienting the cart.

The exponential convergence to a given configuration will be obtained by using the piecewise analytic feedback law in Chapter 3 combined with a coordinate transformation. We introduce the error vector $q_e(t)$ (Kanayama et al. 1988) and (Nelson & Cox 1988),

$$q_e = [x_e, y_e, \theta_e]^T = T(\theta_r)(q - q_r) \tag{4.2}$$

$$T(\theta_r) = \begin{bmatrix} \cos\theta_r & \sin\theta_r & 0 \\ -\sin\theta_r & \cos\theta_r & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

The time derivative of q_e is given by, (4.1),

$$\dot{q}_e = T(\theta_r)\dot{q} = T(\theta_r)G(q)u = G(q_e)u \tag{4.4}$$

The convergence of $q(t)$ to q_r is equivalent to the convergence of $q_e(t)$ to an element in $\mathcal{O}_e = \{q \mid (x_e, y_e, \theta_e) = (0, 0, 2\pi n)\}$, $n \in \{0, \pm 1, \pm 2, \ldots\}$.

Similar to Section 3.2, consider the following circle family $\mathcal{P}$,

$$\mathcal{P} = \{(x_e, y_e) \mid x_e^2 + (y_e - r)^2 = r^2\} \tag{4.5}$$

as the set of circles in the $x_e y_e$-plane with radius $r = r(x_e, y_e)$. They pass through the origin and (x_e, y_e) and are centered on the y_e-axis with $\frac{\partial y_e}{\partial x_e} = 0$ at the origin. Let θ_d be the angle of the tangent of $\mathcal{P}$ at (x_e, y_e), defined as:

$$\theta_d(x_e, y_e) = \begin{cases} 2\arctan(y_e/x_e) = 2\arcsin\frac{y_e\,\mathrm{sgn}(x_e, y_e)}{\sqrt{x_e^2+y_e^2}} & ; (x_e, y_e) \neq (0,0) \\ 0 & ; (x_e, y_e) = (0,0) \end{cases} \tag{4.6}$$

where $\mathrm{sgn}(\cdot)$ is defined in (3.9). θ_d is taken by convention to belong to $[-\pi, \pi)$. Hence, θ_d has discontinuities on the y_e-axis with respect to x_e. As in Section 3.2, we introduce the following change of coordinates:

$$a(x_e, y_e) = r\theta_d = \frac{\sqrt{x_e^2 + y_e^2}}{\mathrm{sinc}\frac{\theta_d}{2}}\mathrm{sgn}(x_e, y_e) \tag{4.7}$$

$$\alpha(x_e, y_e, \theta_e) = e - 2\pi n(e)\,,\ e = \theta_e - \theta_d \tag{4.8}$$

where a is the arc length and the orientation error $\alpha \in [-\pi, \pi)$ is a periodic and piecewise continuous function with respect to e. n takes values in $\{0, \pm 1, \pm 2, \ldots\}$ so that α belongs to $[-\pi, \pi)$. α is introduced so that all the elements in $\mathcal{O}_e$ are mapped into the unique point $(a, \alpha) = (0, 0)$. $a(x_e, y_e)$ defines the arc length from the origin to (x_e, y_e) along a circle which is centered on the y_e-axis and passes through these two points. The function $a(x_e, y_e)$ may be positive or negative according to the sign of x_e. When $y_e = 0$, we define $a(x_e, 0) = 0$ which makes $a(x_e, y_e)$ continuous with respect to y_e since $a(x_e, \varepsilon) \approx x_e$ when $\varepsilon \approx 0$. Discontinuities in $a(x_e, y_e)$ only take place on the y_e-axis. An illustration of these definitions is shown in *Figure 4.1*.

The mapping $F(\cdot)$, (3.14), is here a function of q_e

$$z = F(q_e); \qquad F(q_e) = \begin{bmatrix} a(x_e, y_e) \\ \alpha(x_e, y_e, \theta_e) \end{bmatrix} \tag{4.9}$$

where $F(0) = 0$.

Like in Section 3.3.1, we introduce the set

$$\Psi = \mathbb{R}^3 - (\mathcal{D}_0 \cup \mathcal{E}_0)$$

where $\mathcal{D}_0 = \mathcal{D} \cup \{q \mid x_e = y_e = 0\}$ and $\mathcal{E}_0 = \mathcal{E} \cup \{q \mid x_e = y_e = 0\}$, and

$$\begin{aligned} \mathcal{D} &= \{q \mid x_e = 0,\ y_e \neq 0\} \\ \mathcal{E} &= \{q \mid \alpha(x_e, y_e, \theta_e) = -\pi\} \\ &= \{q \mid \theta = \theta_d(x_e, y_e) + 2\pi n - \pi\}, \quad n \in \{0, \pm 1, \pm 2, \ldots\} \end{aligned}$$

In the domain, Ψ, where $F(q_e)$ is differentiable, i.e. $q_e \in \Psi$, we find $\dot{z}$ similar to Eq. (3.17)

$$\dot{z} = B(q_e)u = \begin{bmatrix} b_1(q_e) & 0 \\ b_2(q_e) & 1 \end{bmatrix} u$$

where $b_1(q_e)$ and $b_2(q_e)$ are given by (3.18)-(3.19) with $\beta = y_e/x_e$. We propose the following feedback control law which has the same structure as (3.22)-(3.23) with $\gamma > 0$ and $k > 0$:

$$v = -\gamma b_1(q_e)a \tag{4.10}$$
$$\omega = -b_2(q_e)v - k\alpha = \gamma b_1 b_2 a - k\alpha \tag{4.11}$$

This feedback law gives the following differential equations:

$$\begin{aligned} \dot{a} &= b_1 v &&= -\gamma b_1^2 a \\ \dot{\alpha} &= b_2 v + \omega &&= -k\alpha \end{aligned} \tag{4.12}$$

In the same way as in Chapter 3, with a discussion of discontinuities, we can state the exponential convergence of the q_e-trajectories to any of the members of $\mathcal{O}_e$ and the control inputs $v(q_e)$ and $\omega(q_e)$ remain bounded.

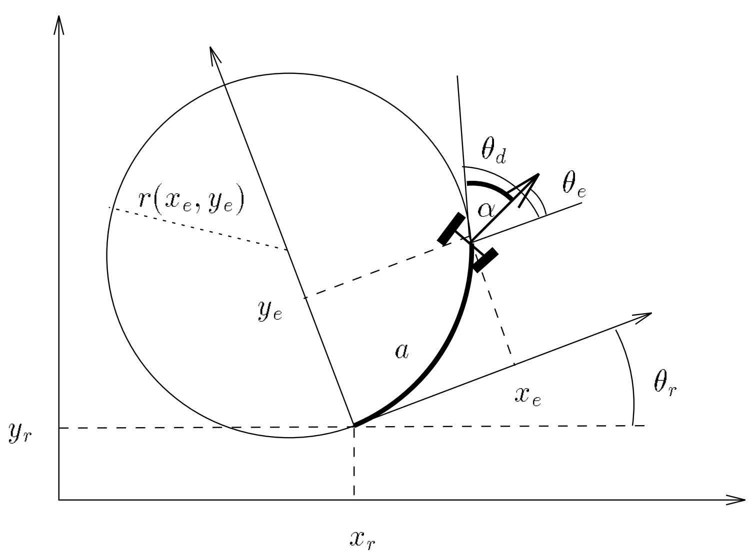

Figure 4.1: Illustration of the coordinate transformation.

4.3 Tracking a Sequence of Points

In this section we extend the results from the preceding section to track a sequence of points in the configuration space. The problem is not to regulate the cart about one point, but rather to make the cart move from one point in the state space to another by using feedback combined with some path planning. In environments with obstacles, it is customary to define a certain number of intermediate points between the initial point p_1 and final point p_n so that the obstacles can be avoided. Suppose that a path planner gives a path which is composed of arcs of circles and straight lines joined in the points $p_2,\ p_3, \ldots, p_{n-1}$ given by the sequence

$$p = (p_1, p_2, \ldots, p_n)$$

An element, p_i, is a point defined in the configuration space, i.e. given by a position (x_i, y_i) and an orientation θ_i

$$p_i = [x_i, y_i, \theta_i]^T$$

In the case of no obstacles the shortest path with constrained curvature is such a path which is composed of arcs of circles and straight lines, (Dubins 1957), (Reeds & Shepp 1990) and (Kanayama & Hartman 1989).

Note that each point, p_i, represents a local coordinate system, each with a family of circles $\mathcal{P}_i$, as defined in Eq. (4.5), associated with it. We associate each point, p_i, with a desired velocity in this point, v_i, and a maximum deviation from p_i, ε_i, before switching to p_{i+1} as the new point of attraction. We define the vector ε as the vector consisting of the maximum deviations,

$$\varepsilon = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}]^T$$

where $\varepsilon_i > 0$, and the vector V as the vector consisting of the desired velocities,

$$V = [v_1, v_2, \ldots, v_{n-1}]^T$$

We can change between forward and backward motion by separating positive and negative velocities with a zero velocity. From p_i and p_{i+1} it is easy to calculate the arc length between these two points. We denote this length by d_i which is given by

$$d_i = a(x_i^{i+1}, y_i^{i+1})$$

where a is the arc length defined by Eq. (4.7). The point (x_i^{i+1}, y_i^{i+1}) is the position (x_i, y_i) transformed to the coordinate system defined by p_{i+1}.

Thus, d_i can be positive or negative. We define the vector d as the vector consisting of these lengths,

$$d = [d_1, d_2, \ldots, d_{n-1}]^T$$

We assume that the desired path from p_i to p_{i+1} does not cross the y-axis defined by p_{i+1}. This can easily be satisfied by adding another intermediate point between p_i and p_{i+1}.

We change to p_{i+1} as the point of attraction when $|a| < \varepsilon_i$. The index function I is thus given by:

$$I = \begin{cases} 1, & \text{when } \ t = 0 \\ I+1, & \text{when } \ |a| \leq \varepsilon_I \end{cases}$$

The desired circles are made stable through a feedback law as in Section 4.2:

$$\omega = -b_2\, v - k\, \alpha \tag{4.13}$$

The orientation error, α, will be given by

$$\dot{\alpha} = -k\alpha \tag{4.14}$$

This implies that $\alpha(t)$ converges exponentially to zero implying that the motion converges towards a motion along one of the circles which pass through the actual reference point. This situation is illustrated in *Figure 4.2* where we see that the circles define a kind of funnel towards each point p_i along the desired path.

From Property 3 in Lemma 3.2 we have that b_1 will converge to 1 as $\alpha(t)$ converges to zero. This means that $\dot{a}$ converges exponentially to v, (4.12). The input v can be chosen to interpolate between the desired velocities, v_i, in many different ways. Here, a control law is chosen which gives a continuous $v(t)$ and $\dot{v}(t) = 0$ when changing from one point of attraction to another:

$$v(t^i_-) = v(t^i_+) = v_{i-1}, \ \ \dot{v}(t^i_-) = \dot{v}(t^i_+) = 0$$

where t^i is the point of time when the index function I increments from $i-1$ to i.

Since the desired path from p_{i-1} to p_i does not cross the y-axis defined by p_i, the sign of d_{i-1} is the opposite of the sign of v_{i-1} and v_i. v_{i-1} and v_i cannot have opposite signs, but one of these velocities can be zero, for instance in order to change between forward and backward motion.

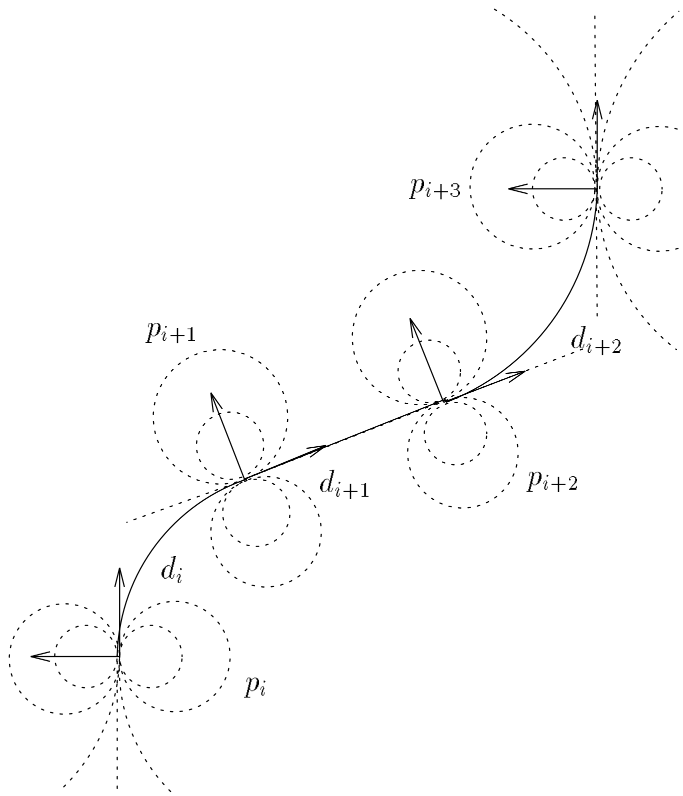

Figure 4.2: Illustration of the circles defined by the points p_i.

The tangential velocity, $v(t)$, in (4.10) is thus modified to account for the velocity requirements when $1 < i < n$ and $\tau \in [0, \tau_f]$ as:

$$v(\tau) = v_{i-1} + 3\triangle v(\frac{\tau}{\tau_f})^2 - 2\triangle v(\frac{\tau}{\tau_f})^3 \tag{4.15}$$

where

$$\tau_f = -\frac{2d_{i-1}}{v_{i-1} + v_i}, \quad \triangle v = v_i - v_{i-1} \tag{4.16}$$

τ is the time variable which is reset to zero when the index function I increments. τ_f is the time it takes to reach p_i along the desired circle if the cart is exactly at p_{i-1} at $\tau = 0$, if the input $v(\tau)$ is given by (4.15). Because of possible disturbances or model errors, the time needed to reach p_i may be different from τ_f. If the time needed is shorter than τ_f, then the index function I increments when the cart reaches the neighborhood of p_i and τ is reset to zero. Note that because of the exponential convergence of the orientation to the angle $\theta_d(x_e, y_e)$ given by the tangent of the actual circle, the cart has to pass through the neighborhood of p_i. Therefore, the cart cannot "overshoot" this neighborhood without switching to p_{i+1} as the new point of attraction. If the time needed is longer than τ_f, then the control law has to assure that the cart reaches the neighborhood of the point p_i. To this end, we let the velocity be constant, equal to v_i during the time $\mathcal{T}$ after t_f. If the cart has not reached the neighborhood of p_i after time $\tau_f + \mathcal{T}$, then we let the cart converge to p_i with a feedback law as in Section 4.2:

$$v(\tau) = \begin{cases} v_i & , \tau \in (\tau_f, \tau_f + \mathcal{T}] \\ -\gamma b_1 a & , \tau > \tau_f + \mathcal{T} \end{cases} \tag{4.17}$$

This ensures that the cart will reach the neighborhood of p_i in finite time.

When $i = 1$, v is given by a feedback law in order to converge to the starting point p_1,

$$v(t) = -\gamma b_1 a(1 - e^{-\lambda t}) \tag{4.18}$$

where γ and λ are positive constants. This results in $v(0) = 0$. It is natural to choose the starting velocity of the path, v_0, equal to zero. In that case, $v(t)$ will converge to this velocity. The closed-loop equation is:

$$\dot{a} = -\gamma b_1^2 a(1 - e^{-\lambda t}) \tag{4.19}$$

From (4.14) and Property 3 in Lemma 3.2 we have that b_1 converges to 1. This makes $\gamma b_1^2(1 - e^{-\lambda t}) > \epsilon$ for all $t > T$, where T is some finite time and $\epsilon > 0$. This implies that $a(t)$ converges to zero and $|a(t^1)| < \varepsilon_1$ for some finite t^1.

At the end of the desired path, $i = n$, we want the cart to exponentially converge to p_n. In order to make $\dot{v}(t)$ continuous, we propose the following feedback law:

$$v = -b_1\gamma a(1 - \delta e^{-\lambda\tau}) \tag{4.20}$$

where γ, λ, and δ are positive constants. In order to have $v(0) = v_{n-1}$ and $\dot{v}(0) = 0$ when $b_1 = 1$, we choose

$$\delta = 1 + \frac{v_{n-1}}{d_{n-1}\gamma},\ \lambda = \frac{v_{n-1}^2}{(d_{n-1}\gamma + v_{n-1})d_{n-1}},\ \gamma > -\frac{v_{n-1}}{d_{n-1}}$$

Note that $\frac{v_{n-1}}{d_{n-1}} < 0$ by assumption so that $\delta < 1$. This gives the closed loop equation:

$$\dot{a} = -b_1^2\gamma a(1 - \delta e^{-\lambda\tau}) \tag{4.21}$$

As in the case $i = 1$, a will exponentially converge to zero. If $b_1 = 1$, then the solution of (4.21) is

$$a(t) = a(0)e^{-\{\gamma t + \frac{\gamma\delta}{\lambda}(1 - e^{-\lambda t})\}} \tag{4.22}$$

More generally, v_i may be a function of time and the arc length a in order to let the cart stand still at the point p_{i-1} for a certain time. If we want the cart to stand still at p_{i-1} during the time T, we can choose $v_{i-1} = 0$ and v_i as

$$v_i = \begin{cases} 0,\ t < t^i + T \\ \nu,\ t \geq t^i + T \end{cases}$$

where t^i is the point of time when $|a|$ becomes less than ε_{i-1} and ν is the velocity with which we want to leave p_{i-1}. This velocity can be positive or negative, and may be a function of time. Hence, with *this* approach it is possible to follow a path where the cart can stop, drive forwards or backwards by defining p and V accordingly. It remains to be shown that the inputs remain bounded.

Theorem 4.1 *The control law defined by (4.15) and (4.13) implies that the inputs $v(t)$ and $\omega(t)$ remain bounded.*

Proof: b_1 is bounded from Property 3 in Lemma 3.2. From (4.19) and (4.21), we have that a remains bounded. With v_{i-1} (and v_i) finite, (4.15) and (4.17) show that $v(t)$ will remain bounded. Since by definition α is bounded, (4.8), w will remain bounded if $b_2 v$ is bounded. We find with $\beta = y_e/x_e$:

$$\begin{aligned} |b_2(q_e)v| &= |\cos\theta_e \frac{2\beta}{(1+\beta^2)x_e} - \sin\theta_e \frac{2}{(1+\beta^2)x_e}||v| \leq \frac{2}{x_e^2+y_e^2}\sqrt{x_e^2+y_e^2}|v| \\ &= \frac{2|v|}{\sqrt{x_e^2+y_e^2}} \ \leq \ \frac{4|v|}{\pi|a|} \ \leq \ \frac{4\max\{|v_{i-1}|,|v_i|\}}{\pi\min\{\varepsilon_i\}} \end{aligned}$$

$\square$

Here, we have used the fact that $|a| \leq \frac{\pi}{2}\sqrt{x_e^2+y_e^2}$, Lemma 3.1, and that the point of attraction changes from p_{i-1} to p_i when $|a| = \varepsilon_i$ so that $|a| \geq \min\{\varepsilon_i\} > 0$. From (4.15) we have that $|v| \leq \max\{|v_{i-1}|, |v_i|\}$. This proof shows that $|b_2(q_e)v|$ can be arbitrarily bounded by choosing ε_i, v_{i-1} and v_i appropriately.

We have shown that the proposed control law makes the cart reach the neighborhood of every point p_i where the neighborhood is given by ε_i. The exponential convergence of $\alpha(t)$ to zero can be used to show that the cart stays in the neighborhood of the desired nominal circle segment between the intermediate points p_i and p_{i+1}. The motion of the cart at a point (x_e, y_e) can be decomposed in a motion tangential to the actual circle representing the desired motion, and in a motion perpendicular to the desired motion resulting in a deviation from the actual circle. Let the deviation of the cart from the nominal circle segment in the xy-plane be denoted ρ. The sign of ρ is defined so that a positive orientation error α results in a positive derivative $\dot{\rho}$ in forward motion. We introduce the integral

$$I_\rho = \int_0^t v(\tau) \sin \alpha(\tau) d\tau + \varepsilon_i$$

where $v(\tau)$ is the velocity of the cart, ε_i is the maximal deviation from the last point of attraction p_i, and $\alpha(\tau)$ is the orientation error of the cart with respect to the tangent of the actual circle. The time t is set to zero when the cart reaches the neighborhood of p_i and p_{i+1} becomes the new point of attraction. The integral I_ρ represents a distance from the nominal circle segment along a path perpendicular to the family of circles $\mathcal{P}$. The circles in the family $\mathcal{P}$ become denser as the cart moves towards the point p_{i+1}. This situation is illustrated in *Figure 4.3*. Because of this property of $\mathcal{P}$ we

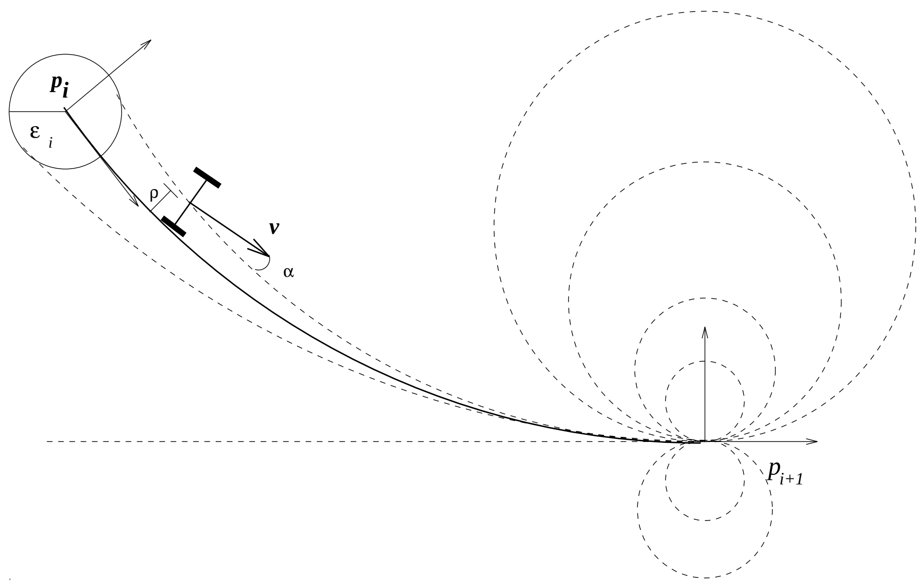

Figure 4.3: An illustration of the motion of the cart in the neighborhood of the nominal path between p_i and p_{i+1}.

see from a geometrical point of view that

$$|\rho(t)| \leq |I_\rho| \leq |\int_0^t v(\tau) \sin \alpha(\tau) d\tau| + \varepsilon_i \tag{4.23}$$

From (4.14) we find $\alpha(t) = \alpha(0)e^{-kt}$. We define $v_{mi} = \max\{v_i, v_{i+1}\}$. Eq.

(4.23) thus implies

$$
\begin{aligned}
|\rho(t)| \leq v_{mi}|\int_0^t \sin(\alpha(0)e^{-k\tau})d\tau| + \varepsilon_i \ \leq \ v_{mi}|\int_0^t \alpha(0)e^{-k\tau}d\tau| + \varepsilon_i \\
\leq \frac{v_{mi}}{k}|\alpha(0)e^{-k\tau} - \alpha(0)| + \varepsilon_i \leq \frac{2v_{mi}}{k}|\alpha(0)| + \varepsilon_i
\end{aligned}
$$

We see that by choosing k, $v_{mi} = \max\{v_i, v_{i+1}\}$ and ε_i appropriately, the resulting path of the cart will stay arbitrarily close to the nominal path between the points p_i and p_{i+1}.

As an example of this approach, we want the cart to pass through a corridor with a corner. The points p_i were taken to

$$
\begin{aligned}
p_1 = [1, 1, \tfrac{\pi}{2}]^T,\ p_2 = [1, 2, \tfrac{\pi}{2}]^T,\quad p_3 = [2, 3, 0]^T \\
p_4 = [4, 3, 0]^T,\ p_5 = [5, 2, -\tfrac{\pi}{2}]^T,\ p_6 = [5, 1, -\tfrac{\pi}{2}]^T
\end{aligned}
$$

This results in the distance vector $d = [-1,\ -\frac{\pi}{2},\ -2,\ -\frac{\pi}{2},\ -1]^T$. The velocities were chosen to $V = [0,\ 0.5,\ 0.5,\ 0.5,\ 0.5]^T$. The accuracy vector, ε, and the initial state were chosen to $\varepsilon = [0.001, 0.01, 0.01, 0.01, 0.01]^T$ and $q_0 = [0.9, 0.9, 0.5]^T$. The controller parameters k and γ were chosen to 2 and 2.

A simulation was done using the SIMNON package (Elmqvist et al. 1990). The resulting path in the xy-plane is shown in *Figure 4.4*. Initially there is a transient phase to reach p_1. Thereafter the cart follows the desired path. The time plot of α is shown in *Figure 4.5*. We see that with this approach, α remains practically equal to zero after an initial period. This is a result of the way that the points p_i are chosen so that the resulting path consists of arcs of circles and straight lines. The point (x_i, y_i) defines a specific circle in the coordinate system of p_{i+1}. The point (x_i, y_i) also defines a desired angle, $\theta_d(x_i, y_i)$, corresponding to the angle of the tangent of this circle. θ_i is chosen to this angle, $\theta_d(x_i, y_i)$. Therefore, when the cart arrives in the neighborhood of p_i and we switch to p_{i+1} as the new point of attraction, the cart already has the desired orientation, i.e. $\alpha = 0$, and the arc of the circle between p_i and p_{i+1} becomes invariant. From *Figure 4.6*, we see that the tangential velocity $v(t)$ is continuous and obtains the desired values. The angular velocity $\omega(t)$ is discontinuous when the path changes between a circle and a straight line as a result of the discontinuity in the curvature of such a path.

4.4 Conclusions

A piecewise smooth controller has been presented for a mobile robot with two degrees of freedom. The cart exponentially converges to a position in the xy-plane with a desired orientation for any initial condition. This is achieved by letting the motion of the cart converge to one of the circles which pass through the origin and are centered on the y-axis in a local coordinate system. An extension to the case of following a path composed of straight lines and arcs of circles has been proposed.

This approach allows for stopping and reversing phases. The degree of accuracy can be arbitrarily chosen. Desired velocities at certain points along the path are to be specified and can be zero, positive or negative. Therefore, stopping and reversing phases can be included by specifying the desired velocities accordingly. The convergence to the terminal configuration was exponential. Simulation results showed that the desired path was followed and the desired velocities were achieved. Different velocity profiles can be chosen by changing the structure of the control law for the tangential velocity v.

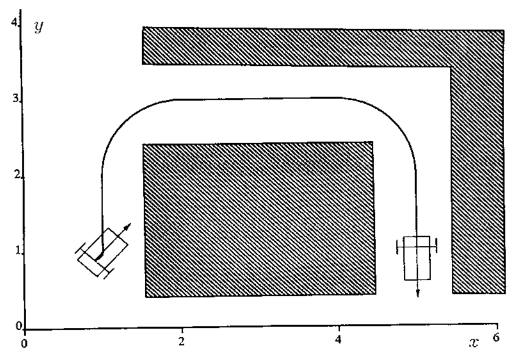

Figure 4.4: The resulting path in the xy-plane when the cart passes through a corridor. The desired path is defined by the endpoints of the straight lines and the circle segment.

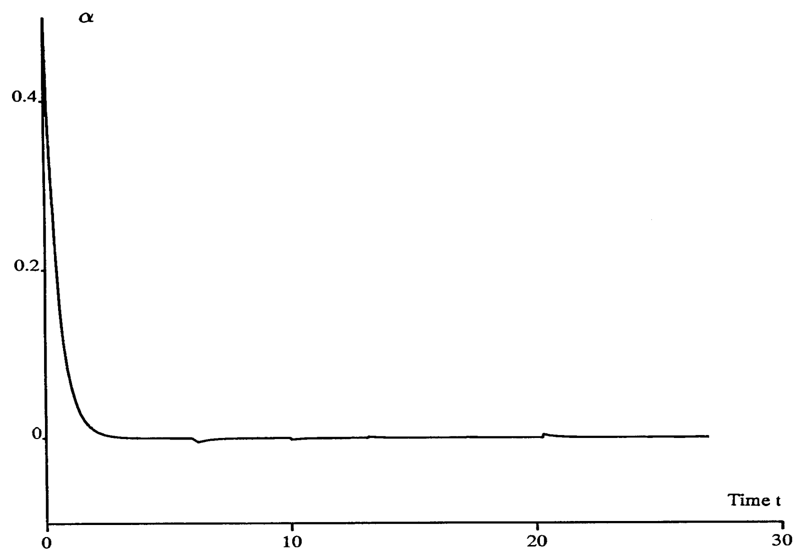

Figure 4.5: Timeplot of the orientation error, α, when the cart passes through a corridor.

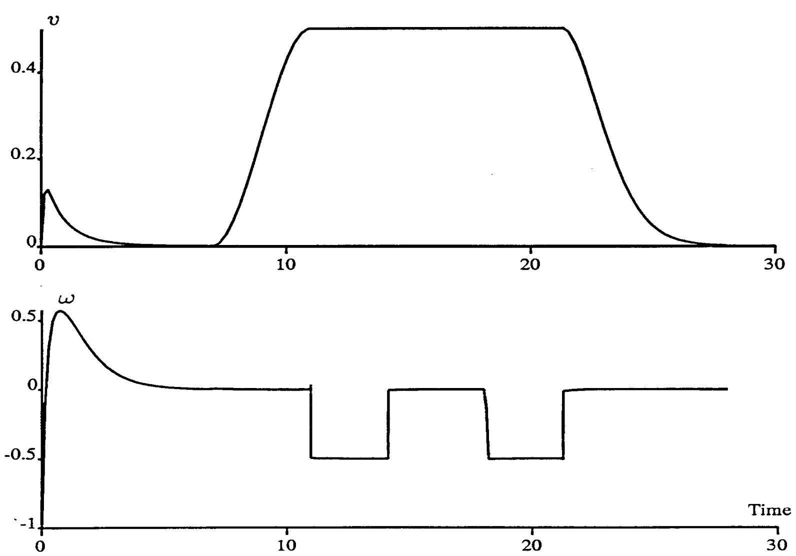

Figure 4.6: Timeplots of the tangential velocity, v, and the angular velocity, ω, when the cart passes through a corridor. The tangential velocity is continuous.

Chapter 5

Conversion of the Kinematics of a Car with n Trailers into a Chained Form

5.1 Introduction

A car with n trailers is a nonholonomic system due to the rolling constraints of the wheels. The configuration of the system is given by two position coordinates and $n+1$ angles, whereas there are only two inputs, namely one tangential velocity and one angular velocity. Thus, the system has two degrees of freedom.

A kinematic model for such a system was presented by Laumond (1991). Controllability for this model was proven, but no control law was found. Control of simpler kinematic models of cars and mobile robots have been studied from open-loop and closed-loop points of view. Closed-loop or exact open-loop strategies have, however, not been presented in previous work for a car with n trailers with two degrees of freedom where n is an arbitrary number. An interesting approach to this problem is to seek for a conversion of the kinematic model into a chained form. Then, control schemes for systems on chained form can be applied. A chained system is nilpotent, Section B.3, and has the form

$$\begin{aligned}\dot{\xi}_1 &= u_1 \\ \dot{\xi}_2 &= u_2\end{aligned}$$

$$
\begin{aligned}
\dot{\xi}_3 &= \xi_2 u_1 \\
\dot{\xi}_4 &= \xi_3 u_1 \\
&\vdots \\
\dot{\xi}_k &= \xi_{k-1} u_1
\end{aligned}
$$

An open-loop strategy to steer nonholonomic systems on a special canonical form which includes chained systems was proposed by Murray & Sastry (1990). The control strategy used sinusoid-type inputs. A general strategy for steering systems without drift was proposed by Lafferriere & Sussmann (1991) and Lafferriere (1991). This approach provides an exact solution for nilpotent or feedback nilpotentizable systems. This strategy can, therefore, also be used to find an exact open-loop solution of the Motion Planning Problem of a car with n trailers if the kinematics can be converted into a chained form.

Also closed-loop strategies have been developed for chained systems. The work on open-loop control using sinusoids was extended to asymptotic stabilization of chained systems by using time-varying feedback (Teel, Murray & Walsh 1992). The time-varying feedback law proposed by Pomet (1992) can also be used to stabilize chained systems. The problem of these smooth time-varying feedback laws is a rather slow convergence. In Chapter 6 a new stabilizing control law is proposed for chained systems having *exponential* convergence. By converting the model of a car with n trailers into a chained form, the system can, therefore, be controlled by using existing control strategies or the control law in Chapter 6.

A constructive procedure to transform a nonholonomic system with two inputs into a chained form suitable for control was given by Murray & Sastry (1991) under some assumptions on the input vectors. This was used to locally convert the kinematic model of a car pulling a single trailer into a chained form. However, the algorithm failed when additional trailers were added for the model considered.

In the work of Murray & Sastry (1991) the absolute position of the car with n trailers was given by the position of the pulling car. In this chapter, a new model structure is proposed where the absolute position of the car with n trailers is given by the position of the *rear* trailer. This formulation of the model allows for a local conversion of the kinematics of a car with n trailers into a chained form where n is an arbitrary positive number. A change of coordinates and an invertible feedback transformation of the inputs are found which convert this kinematic model into a chained form. This presentation is based on Sørdalen (1993*b*) and Sørdalen (1993*c*).

5.2 Kinematic Model

A car in this context will be represented by two driving wheels connected by an axle. A kinematic model of a car with two degrees of freedom pulling n trailers can be given by:

$$\begin{aligned}
\dot{x} &= \cos\theta_n v_n \\
\dot{y} &= \sin\theta_n v_n \\
\dot{\theta}_n &= \frac{1}{d_n}\sin(\theta_{n-1}-\theta_n)v_{n-1} \\
&\vdots \\
\dot{\theta}_i &= \frac{1}{d_i}\sin(\theta_{i-1}-\theta_i)v_{i-1} \quad i=1,\ldots,n \\
&\vdots \\
\dot{\theta}_1 &= \frac{1}{d_1}\sin(\theta_0-\theta_1)v_0 \\
\dot{\theta}_0 &= \omega
\end{aligned} \tag{5.1}$$

where (x, y) is the planar position of the center of the axle between the two wheels of the *rear* trailer. The use of the position of the rear trailer in the model as proposed here is an original contribution which is a significant improvement with respect to previous models where the position of the *first* car was determining the location of the system, (Laumond 1991) and (Murray & Sastry 1991).

θ_i is the orientation angle of trailer i with respect to the x-axis, with $i \in \{1, \ldots, n\}$. θ_0 is the orientation angle of the pulling car with respect to the x-axis. Incidentally, this model is identical to a model of a four-wheeled car pulling $n-1$ trailers where $\theta_0 - \theta_1$ is the angle of the front wheels relative to the orientation θ_1 of the four-wheeled car.

d_i is the distance from the wheels of trailer i to the wheels of trailer $i-1$, where $i \in \{2, \ldots, n\}$. d_1 is then the distance from the wheels of trailer 1 to the wheels of the car.

v_0 is the tangential velocity of the car and is an input to the system. The other input is the angular velocity of the car, ω. We denote

$$\nu = [v_0, \omega]^T$$

The tangential velocity of trailer i, v_i, is given by

$$v_i = \cos(\theta_{i-1}-\theta_i)v_{i-1} = \prod_{j=1}^{i}\cos(\theta_{j-1}-\theta_j)\ v_0 \tag{5.2}$$

where $i \in \{1, \ldots, n\}$. An illustration of these definitions is presented in *Figure 5.1*.

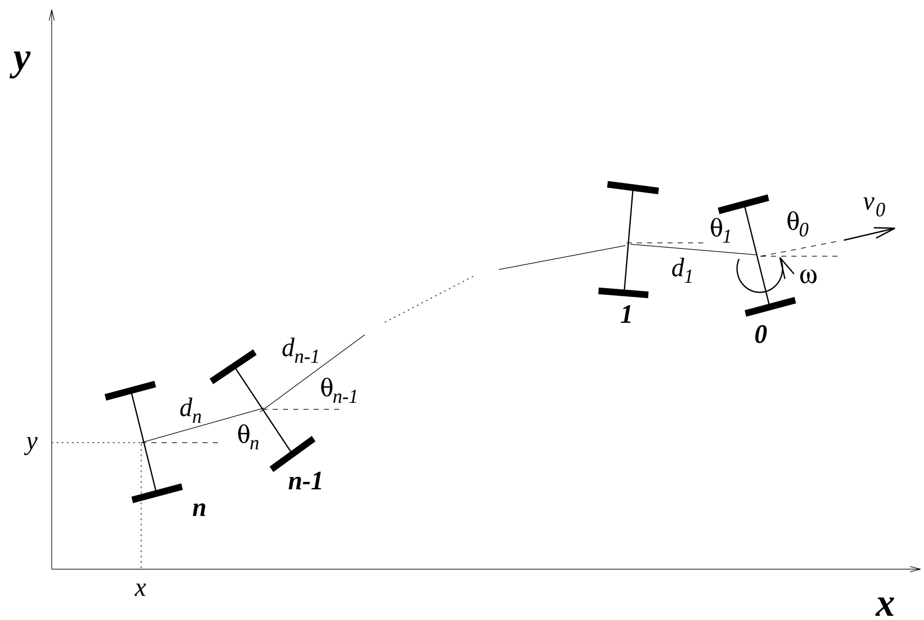

Figure 5.1: Model of a car with n trailers.

A car with n trailers (5.1) is a nonholonomic system with rolling constraints for each trailer and the car. Denote by (x_i, y_i) the planar position of trailer i where $(x_n, y_n) = (x, y)$. The planar position of the car is denoted by (x_0, y_0). The nonholonomic constraints can then be expressed as (Laumond 1991)

$$\sin\theta_i \dot{x}_i - \cos\theta_i \dot{y}_i = 0, \quad i \in \{0, 1, \ldots, n\} \tag{5.3}$$

From a geometrical consideration, *Figure 5.1*, we easily see that

$$x_i = x + \sum_{j=i+1}^{n} d_j \cos\theta_j, \quad y_i = y + \sum_{j=i+1}^{n} d_j \sin\theta_j$$

where $(x, y) = (x_n, y_n)$. Eq. (5.3) then imply

$$\sin\theta_i \dot{x} - \sin\theta_i \sum_{j=i+1}^{n} d_j \sin\theta_j \dot{\theta}_j - \cos\theta_i \dot{y} - \cos\theta_i \sum_{j=i+1}^{n} d_j \cos\theta_j \dot{\theta}_j = 0$$

which gives the following nonholonomic constraints

$$\sin\theta_i \dot{x} - \cos\theta_i \dot{y} - \sum_{j=i+1}^{n} d_j \cos(\theta_i - \theta_j)\dot{\theta}_j = 0, \quad i \in \{0, 1, \ldots, n\}$$

The system has, thus, $(n+3)-(n+1) = 2$ degrees of freedom corresponding to the two independent velocity inputs.

We will make a change of inputs under the assumption that the state $q = [x, y, \theta_n, \ldots, \theta_0]$ is in a neighborhood D of the origin where D is given by

$$\begin{aligned} (x, y) &\in \mathbb{R}^2 \\ \theta_i &\in \left(-\frac{\pi}{4} + \varepsilon, \frac{\pi}{4} - \varepsilon\right), \quad i \in \{0, \ldots, n\} \end{aligned}$$

where ε is a small constant. We introduce the transformed input v as

$$v = \cos\theta_n \; v_n = \cos\theta_n \prod_{j=1}^{n} \cos(\theta_{j-1} - \theta_j) \; v_0 \tag{5.4}$$

The transformed input is the velocity of trailer n in x-direction. This transformation from v_0 to v is nonsingular and smooth in D. The velocity of v_i from (5.2) can then be rewritten

$$v_i = \frac{1}{\cos\theta_n \prod_{j=i+1}^{n} \cos(\theta_{j-1} - \theta_j)} v = \frac{1}{p_i(\underline{\theta}_i)} v \tag{5.5}$$

for $i \in \{0, \ldots, n\}$ where

$$\underline{\theta}_i \triangleq [\theta_i, \ldots, \theta_n]^T \tag{5.6}$$

$$\begin{aligned} p_i(\underline{\theta}_i) &\triangleq \cos\theta_n \prod_{j=i+1}^{n} \cos(\theta_{j-1} - \theta_j) \\ &= \prod_{j=i}^{n} \cos(\theta_j - \theta_{j+1}) \end{aligned} \tag{5.7}$$

for $i \in \{0, \ldots, n\}$ where $\theta_{n+1} \triangleq 0$. Eq. (5.5) then gives $v = p_0(\underline{\theta}_0) v_0$.

System (5.1) can now be represented (locally) at the following form.

$$\begin{aligned} \dot{x} &= v \\ \dot{\theta}_0 &= \omega \\ \dot{\theta}_1 &= \frac{1}{d_1} \frac{\tan(\theta_0 - \theta_1)}{p_1(\underline{\theta}_1)} v \\ &\vdots \\ \dot{\theta}_i &= \frac{1}{d_i} \frac{\tan(\theta_{i-1} - \theta_i)}{p_i(\underline{\theta}_i)} v, \quad i \in \{1, \ldots, n\} \\ \dot{y} &= \tan\theta_n v \end{aligned} \tag{5.8}$$

where v is given by (5.4). We denote for $i \in \{1, \ldots, n\}$

$$f_i(\underline{\theta}_{i-1}) = \frac{1}{d_i}\frac{\tan(\theta_{i-1} - \theta_i)}{p_i(\underline{\theta}_i)} \tag{5.9}$$

$$\underline{f}_i(\underline{\theta}_{i-1}) = [f_i(\underline{\theta}_{i-1}), \ldots, f_n(\underline{\theta}_{n-1})]^T \tag{5.10}$$

This means that we can write

$$\dot{\theta}_i = f_i(\underline{\theta}_{i-1})v \tag{5.11}$$

$$\underline{\dot{\theta}}_i = \underline{f}_i(\underline{\theta}_{i-1})v \tag{5.12}$$

After a reordering of the state variables, we denote the state by the vector

$$z = [z_1, \ldots, z_{n+3}]^T = [x, \theta_0, \ldots, \theta_n, y]^T$$

which has dimension $n+3$. We note from (5.8) that this kinematic model has a special triangular structure where $\dot{z}_i$ is not a function of $z_1, \ldots, z_{i-2}$, where $i \in \{3, \ldots, n+3\}$.

5.3 Conversion into a Chained Form

In this section we will exploit the special structure of (5.8) to convert system (5.1) into a chained form. The chained form will be obtained by a constructive procedure. This is formulated in the following theorem where $\xi = [\xi_1, \ldots, \xi_{n+3}]^T$.

Theorem 5.1 *The following change of coordinates, $\xi = F(z)$, and feedback transformation, $u = G(z)\nu$, convert locally the model (5.1) of a car with n trailers into a chained form:*

$$\xi_1 = x \tag{5.13}$$

$$\xi_2 = \frac{\tan(\theta_0 - \theta_1)}{c_2(\underline{\theta}_1)} + \tau_2(\underline{\theta}_1) \tag{5.14}$$

$$\vdots$$

$$\xi_i = \frac{\tan(\theta_{i-2} - \theta_{i-1})}{c_i(\underline{\theta}_{i-1})} + \tau_i(\underline{\theta}_{i-1}) \tag{5.15}$$

$$\vdots$$

$$\xi_n = \frac{\tan(\theta_{n-2} - \theta_{n-1})}{c_n(\underline{\theta}_{n-1})} + \tau_n(\underline{\theta}_{n-1}) \tag{5.16}$$

$$\xi_{n+1} = \frac{\tan(\theta_{n-1} - \theta_n)}{d_n \cos^3 \theta_n} \quad (5.17)$$

$$\xi_{n+2} = \tan \theta_n \quad (5.18)$$

$$\xi_{n+3} = y \quad (5.19)$$

where

$$c_i(\underline{\theta}_{i-1}) = \prod_{j=i}^{n+1} \cos^{j-i+3}(\theta_{j-1} - \theta_j) d_{n+i-j}$$

$$= p_{i-1}^2(\underline{\theta}_{i-1}) \prod_{j=i-1}^{n} d_j p_j(\underline{\theta}_j) \quad (5.20)$$

$$\tau_i(\underline{\theta}_{i-1}) = \frac{\partial \xi_{i+1}}{\partial \underline{\theta}_i} \underline{f}_i(\underline{\theta}_{i-1}) \quad (5.21)$$

with $i \in \{2, \ldots, n\}$.

The transformation of the inputs, $u = G(z)\nu$, *is given by*

$$u_1 = p_0(\underline{\theta}_0) v_0 \quad (5.22)$$

$$u_2 = \frac{1}{\cos^2(\theta_0 - \theta_1) c_2(\underline{\theta}_1)} \omega + \frac{\partial \xi_2}{\partial \underline{\theta}_1} \underline{f}_1(\underline{\theta}_0) p_0(\underline{\theta}_0) v_0 \quad (5.23)$$

where $p_i(\underline{\theta}_i)$ *is given by (5.7).*

Proof: We are free to choose

$$\xi_{n+3} = y \quad (5.24)$$

From (5.8), differentiation with respect to time gives

$$\dot{\xi}_{n+3} = \dot{y} = \tan \theta_n v \quad (5.25)$$

We get $\dot{\xi}_{n+3} = \xi_{n+2} v$ by choosing

$$\xi_{n+2} = \tan \theta_n \quad (5.26)$$

From (5.8), differentiation gives

$$\dot{\xi}_{n+2} = \frac{1}{\cos^2 \theta_n} \dot{\theta}_n = \frac{\tan(\theta_{n-1} - \theta_n)}{d_n \cos^3 \theta_n} v \quad (5.27)$$

We get $\dot{\xi}_{n+2} = \xi_{n+1} v$ by choosing

$$\xi_{n+1} = \frac{\tan(\theta_{n-1} - \theta_n)}{d_n \cos^3 \theta_n} \quad (5.28)$$

We will show by induction that $\dot{\xi}_{i+1} = \xi_i v$ by choosing

$$\xi_i = \frac{\tan(\theta_{i-2} - \theta_{i-1})}{c_i(\underline{\theta}_{i-1})} + \tau_i(\underline{\theta}_i) \tag{5.29}$$

for $i \in \{2, \ldots, n\}$ where

$$c_i(\underline{\theta}_{i-1}) = p_{i-1}^2(\underline{\theta}_{i-1}) \prod_{j=i-1}^{n} d_j p_j(\underline{\theta}_j) \tag{5.30}$$

$$\tau_i(\underline{\theta}_{i-1}) = \frac{\partial \xi_{i+1}}{\partial \underline{\theta}_i} \underline{f}_i(\theta_{i-1}) \tag{5.31}$$

where $i \in \{2, \ldots, n\}$. This means that $\xi_i = \xi_i(\underline{\theta}_{i-2})$. Assume that (5.29) is satisfied for $i = m$. Eqs. (5.7), (5.8), (5.12), (5.30) and (5.31) imply

$$\begin{aligned}
\dot{\xi}_m &= \frac{\partial \xi_m}{\partial \theta_{m-2}} \dot{\theta}_{m-2} + \frac{\partial \xi_m}{\partial \underline{\theta}_{m-1}} \dot{\underline{\theta}}_{m-1} \\
&= \frac{1}{\cos^2(\theta_{m-2} - \theta_{m-1}) c_m(\underline{\theta}_{m-1})} \cdot \\
&\quad \frac{1}{d_{m-2}} \frac{\tan(\theta_{m-3} - \theta_{m-2})}{p_{m-2}(\underline{\theta}_{m-2})} v \\
&\quad + \frac{\partial \xi_m}{\partial \underline{\theta}_{m-1}} \underline{f}_{m-1}(\underline{\theta}_{m-2}) v \\
&= \left(\frac{\tan(\theta_{m-3} - \theta_{m-2})}{c_{m-1}(\underline{\theta}_{m-2})} + \tau_{m-1}(\underline{\theta}_{m-2})\right) v
\end{aligned}$$

Note from (5.7) and (5.20) that

$$\begin{aligned}
c_{m-1}(\underline{\theta}_{m-2}) &= \cos^2(\theta_{m-2} - \theta_{m-1}) c_m(\underline{\theta}_{m-1}) d_{m-2} p_{m-2}(\underline{\theta}_{m-2}) \\
&= \cos^2(\theta_{m-2} - \theta_{m-1}) p_{m-1}^2 \left(\prod_{j=m-1}^{n} d_j p_j\right) d_{m-2} p_{m-2}(\underline{\theta}_{m-2}) \\
&= p_{m-2}^2(\underline{\theta}_{m-2}) \prod_{j=m-2}^{n} d_j p_j(\underline{\theta}_j)
\end{aligned}$$

We have thus shown that if ξ_i is given by (5.29) for $i = m$ then

$$\dot{\xi}_m = \xi_{m-1} v$$

by choosing ξ_{m-1} as in (5.29) with $i = m - 1$. It remains to be shown that if ξ_n is given by (5.29) with $i = n$ and ξ_{n+1} is given by (5.28) then

$$\dot{\xi}_{n+1} = \xi_n v$$

We find from (5.28), (5.8), (5.7), (5.12), (5.30) and (5.31) that

$$\begin{aligned}
\dot{\xi}_{n+1} &= \frac{\partial \xi_{n+1}}{\partial \theta_{n-1}}\dot{\theta}_{n-1} + \frac{\partial \xi_{n+1}}{\partial \theta_n}\dot{\theta}_n \\
&= \frac{1}{\cos^2(\theta_{n-1}-\theta_n) d_n \cos^3\theta_n} \cdot \\
&\quad \frac{1}{d_{n-1}} \frac{\tan(\theta_{n-2}-\theta_{n-1})}{\cos\theta_n \cos(\theta_{n-1}-\theta_n)} v \\
&\quad + \frac{\partial \xi_{n+1}}{\partial \theta_n} f_n(\underline{\theta}_{n-1}) v \\
&= \left(\frac{\tan(\theta_{n-2}-\theta_{n-1})}{c_n(\underline{\theta}_{n-1})} + \tau_n(\underline{\theta}_{n-1})\right) v
\end{aligned}$$

This means that

$$\dot{\xi}_{n+1} = \xi_n v$$

by choosing

$$\xi_n = \frac{\tan(\theta_{n-2}-\theta_{n-1})}{c_n(\underline{\theta}_{n-1})} + \tau_n(\underline{\theta}_{n-1}) \tag{5.32}$$

Therefore, ξ_i is given by (5.29) for all $i \in \{2, \ldots, n\}$ and the transformations (5.14)-(5.19) imply that

$$\dot{\xi}_i = \xi_{i-1} v \ , \quad \forall i \ \in \{3, \ldots, n+3\}$$

To complete the proof we have to show that

$$\dot{\xi}_2 = u_2 \ , \qquad \dot{\xi}_1 = u_1$$

We have shown that ξ_2 is given by (5.29) with $i = 2$. Differentiation gives

$$\begin{aligned}
\dot{\xi}_2 &= \frac{\partial \xi_2}{\partial \theta_0}\dot{\theta}_0 + \frac{\partial \xi_2}{\partial \underline{\theta}_1}\dot{\underline{\theta}}_1 \\
&= \frac{1}{\cos^2(\theta_0-\theta_1) c_2(\underline{\theta}_1)}\omega + \frac{\partial \xi_2}{\partial \underline{\theta}_1} \underline{f}_1(\underline{\theta}_0) v \\
&= \frac{1}{\cos^2(\theta_0-\theta_1) c_2(\underline{\theta}_1)}\omega + \frac{\partial \xi_2}{\partial \underline{\theta}_1} \underline{f}_1(\underline{\theta}_0) p_0(\underline{\theta}_0) v_0
\end{aligned}$$

since $v = p_0(\underline{\theta}_0) v_0$. This implies that the transformation (5.23) makes

$$\dot{\xi}_2 = u_2 \tag{5.33}$$

From (5.8) it follows directly that

$$\dot{\xi}_1 = u_1 \tag{5.34}$$

by choosing $\xi_1 = x$ and $u_1 = v = p_0(\underline{\theta}_0)v_0$. We can thus conclude that the transformation (5.13)-(5.19) implies that

$$\begin{aligned} \dot{\xi}_1 &= u_1 \\ \dot{\xi}_2 &= u_2 \\ \dot{\xi}_3 &= \xi_2 u_1 \\ &\vdots \\ \dot{\xi}_{n+3} &= \xi_{n+2} u_1 \end{aligned}$$

where u_1 and u_2 are given by (5.22)-(5.23).

□

Several control strategies have been presented where $\xi = [\xi_1, \ldots, \xi_{n+3}]^T$ is controlled to zero using $u = [u_1, u_2]^T$, (Murray & Sastry 1991), (Pomet 1992), (Teel et al. 1992) and (Sørdalen & Egeland 1993). In order to show that the convergence of ξ to zero implies that z converges to zero, we have to show that the transformation given in Theorem 5.1 has an inverse

$$z = F^{-1}(\xi)$$

which is continuously differentiable in a neighborhood of zero and $F^{-1}(0) = 0$. Moreover, the feedback input transformation, $u = G(z)\nu$, has to have an inverse

$$\nu = G^{-1}(z)u$$

First we note from (5.7) and (5.9) that $f_i(\underline{\theta}_{i-1})$ is smooth, i.e. C^∞, in D. This implies that $\underline{f}_i(\underline{\theta}_{i-1})$ in (5.10) is smooth as well. Then it is straightforward to show by induction from the transformations in Theorem 5.1 that $\xi_i(\underline{\theta}_{i-2})$ is smooth in D for $i = \{2, \ldots, n+2\}$ since $\xi_{n+1}(\theta_{n-1}, \theta_n)$ is smooth in D. The other transformations $\xi_1(x)$, $\xi_{n+2}(\theta_{n+2})$ and $\xi_{n+3}(y)$ are obviously smooth in D. Therefore the transformation $F(z)$ is smooth. We will then show that the Jacobian matrix

$$J = \frac{\partial F(z)}{\partial z}$$

is nonsingular for all z in D. The components of z are $[z_1, \ldots, z_{n+3}]^T = [x, \theta_0, \ldots, \theta_n, y]^T$. We see from the transformation $\xi = F(z)$ in Theorem 5.1 that

$$J_{ij} = \frac{\partial F_i(z)}{\partial z_j} = 0\ , \quad \text{if} \quad i > j$$

The matrix J is thus upper triangular. The diagonal elements are found to be

$$\begin{aligned} J_{11} &= 1 \\ J_{ii} &= \frac{\partial \xi_i}{\partial \theta_{i-2}} = \frac{1}{\cos^2(\theta_{i-2} - \theta_{i-1}) c_i(\underline{\theta}_{i-1})}, \quad i \in \{2, \ldots, n+1\} \\ J_{n+2,n+2} &= \frac{1}{\cos^2 \theta_n} \\ J_{n+3,n+3} &= 1 \end{aligned}$$

From the definition of $c_i(\underline{\theta}_{i-1})$, (5.20), we have that $J(z)$ is nonsingular for all $z \in D$. From the Inverse Function Theorem we can conclude that in a neighborhood of any $z \in D$, the transformation $\xi = F(z)$ has an inverse

$$z = F^{-1}(\xi)$$

which is smooth.

We see from $F(z)$ given in Theorem 5.1 that $F(0) = 0$ which implies that $F^{-1}(0) = 0$. Since $F^{-1}(\xi)$ is smooth we can conclude that convergence of ξ to zero implies convergence of z to zero.

From (5.7), (5.9), (5.22) and (5.23) we have that the feedback input transformation, $u = G(z)\nu$ is well defined for all $z \in D$ since

$$\frac{\partial \xi_2}{\partial \underline{\theta}_1} \underline{f}_1(\underline{\theta}_0) = \tau_1(\underline{\theta}_0)$$

is smooth. We find the inverted transformation, $\nu = G^{-1}(z)u$, from (5.21) and (5.22)-(5.23):

$$\begin{bmatrix} v_0 \\ \omega \end{bmatrix} = G^{-1}(z) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where

$$G^{-1}(z) = \begin{bmatrix} \frac{1}{p_0(\underline{\theta}_0)} & 0 \\ \cos^2(\theta_0 - \theta_1) c_2(\underline{\theta}_1) \tau_1(\underline{\theta}_0), & \cos^2(\theta_0 - \theta_1) c_2(\underline{\theta}_1) \end{bmatrix}$$

This transformation is nonsingular and smooth for all $z \in D$ which implies that a control law for $u = [u_1, u_2]^T$ can be transformed to a control law for $\nu = [v_0, \omega]^T$. Therefore, locally in D a control problem for a car with n trailers can be solved by using a control law for a chained system via the transformations $\xi = F(z)$ and $\nu = G^{-1}(z)u$.

Remark: The proof of Theorem 5.1 is in fact similar to the constructive proof of Proposition 4 in Murray & Sastry (1991) which was given in terms of Lie brackets. The proof given here, therefore, shows that the new kinematic model (5.8) satisfies the necessary conditions in Murray & Sastry (1991) for converting systems into a chained form.

5.4 Conclusions

It has been shown how a kinematic model of a car with n trailers can be converted locally into a nilpotent chained form by a change of coordinates and an invertible feedback transformation of the inputs. In order to achieve this conversion, the location of the car-trailer system was modeled with the position of the *rear* trailer.

The conversion holds when the orientation angles of the trailers are less than $\pi/4 - \varepsilon$ in magnitude where ε is an arbitrary small constant. In fact, the conversion holds if the less conservative condition $\cos(\theta_{i-1} - \theta_i) > 0$ holds for all $i \in \{1, \ldots, n\}$ and $\cos\theta_n > 0$. There is no condition on the position (x, y) of the system. Since the angels θ_i will tend towards zero as the pulling car advances along the x-axis in positive x-direction. Therefore, the local assumption on the orientations will be satisfied in finite time by driving the pulling car along the x-axis.

We have shown that the proposed conversion of the car-trailer system into a chained form makes it possible to use control strategies developed for chained systems to control a car with n trailers. In particular, a stabilizing feedback law for a chained system can be used to locally stabilize a car with n trailers.

Chapter 6

Exponential Stabilization of Chained Nonholonomic Systems

6.1 Introduction

Several nonholonomic mechanical systems can be represented by kinematic models on chained form. Murray & Sastry (1991) presented a constructive procedure to transform a nonholonomic system with two inputs to a chained form where the system must satisfy some conditions. In Chapter 5, the kinematic model of a car with n trailers was converted into a chained form. Therefore, control strategies for chained nonholonomic systems can be used for the control of a broad class of nonholonomic, mechanical systems.

Sinusoids were proposed by Murray & Sastry (1990) to steer in open-loop nonholonomic systems on a special canonical form. This work on the use of sinusoids was extended to the stabilization of chained nonholonomic systems using smooth time-varying feedback (Teel et al. 1992). The time-varying feedback law proposed by Pomet (1992) can also be used to stabilize chained systems. Time-varying smooth feedback laws for the stabilization of nonholonomic systems about a constant configuration have, however, revealed a rather slow convergence. Gurvits (1992) stated that *smooth* time-periodic feedback cannot be *exponentially* stabilizing. By letting the feedback be non-smooth and time-varying, local exponential convergence to a neighborhood of the origin was obtained by Murray et al. (1992) for a chained system where the nonholonomic degree was equal to one. However, no feedback law has been presented in previous work that *exponentially* stabilizes a chained nonholonomic system with an arbitrary nonholonomic degree about a constant configuration.

In this chapter, the problem of exponential stabilization is addressed and a new feedback approach is proposed for chained nonholonomic systems. The stabilization is achieved by letting the state feedback law depend on time and on a parameter which varies with the state at discrete instants of time. The proposed feedback law globally stabilizes the system about the origin. The resulting closed-loop system is not exponentially stable as defined by Khalil (1992) p. 168, but it is shown to have a property which will be termed $\mathcal{K}$-exponential stability.

The chapter is organized as follows: A modified definition of exponential stability is proposed in Section 6.2. The nonholonomic chained system is given in Section 6.3. The control law is presented in Section 6.4. The convergence of a part of the system is analyzed in Section 6.5. The stability of the total system is analyzed in Section 6.6. This control strategy is illustrated by a simulation example in Section 6.7 and compared to the result of a time-varying smooth approach. The conclusions are given in Section 6.8. This presentation is an extension of Sørdalen (1993*d*) and based on Sørdalen & Egeland (1993).

6.2 A Modified Definition of Exponential Stability

In this section a modified definition of exponential stability will be proposed. To this end, we need the following notion, (Hahn 1967) Def. 2.5:

Definition 6.1 *A continuous function* $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be of* **class** $\mathcal{K}$ *(or belong to class* $\mathcal{K}$*) if it is strictly increasing and* $\alpha(0) = 0$*.*

One property of class $\mathcal{K}$ functions is that, (Khalil 1992) and (Hahn 1967),

$$\alpha_1, \ \ \alpha_2 \ \in \text{class } \mathcal{K} \quad \Rightarrow \quad \alpha_1 \circ \alpha_2 \ \in \text{class } \mathcal{K} \tag{6.1}$$

We then define:

Definition 6.2 *Consider the nonlinear, time-variant system*

$$\dot{x} = f(x,t) \qquad x \in D \subset \mathbb{R}^n, \quad t \geq t_0 \tag{6.2}$$

System (6.2) is $\mathcal{K}$**-exponentially stable** *about* x_p *iff there exist a neighborhood* $\Omega_p \subset D$ *about* x_p*, a positive constant* λ*, and a function* $h(\cdot)$ *of* *class $\mathcal{K}$ such that all solutions $x(t)$ of (6.2) satisfy*

$$\forall x(t_0) \in \Omega_p \ \forall t \geq t_0, \ \ \|x(t) - x_p\| \leq h(\|x(t_0) - x_p\|)e^{-\lambda(t-t_0)} \tag{6.3}$$

where the constant λ and the neighborhood Ω_p are independent of t_0, and $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$.

If (6.3) is satisfied for $\Omega_p = D$, then System (6.2) is **globally**, **$\mathcal{K}$-exponentially stable** *about x_p.*

According to this definition, if System (6.2) is $\mathcal{K}$-exponentially stable at x_p, then it is *uniformly asymptotically stable* as defined by e.g. Khalil (1992), Def. 4.3., and in addition it has an exponential rate of convergence.

Note that in this definition of stability with exponential convergence, a general function $h(\cdot)$ of class $\mathcal{K}$ is considered as opposed to the usual definition of exponential stability where a special function of class $\mathcal{K}$ is considered, namely the linear function

$$h(\|x(t_0) - x_p\|) = k\|x(t_0) - x_p\|$$

where k is a positive constant independent of t_0 and $x(t_0)$. This means that $\mathcal{K}$-exponential stability corresponds to a weaker form of exponential stability. However, $\mathcal{K}$-exponential stability and exponential stability are equal with respect to the rate of convergence. Therefore, the notion "exponential stabilization" is used in the title of this chapter; although, only $\mathcal{K}$-exponential stability will be proved.

The need for this definition is illustrated by the following examples.

Example 1:

Let an asymptotically stable, nonlinear system be given by:

$$\dot{x}_1 = -\lambda_1 x_1 + x_2^{1/3} \tag{6.4}$$

$$\dot{x}_2 = -\lambda_2 x_2 \ , \qquad \lambda_2 > 3\lambda_1 > 0 \tag{6.5}$$

We find

$$\begin{aligned} x_2(t) &= x_2(t_0)e^{-\lambda_2(t-t_0)} \\ x_1(t) &= x_1(t_0)e^{-\lambda_1(t-t_0)} + e^{-\lambda_1(t-t_0)}\int_{t_0}^{t} e^{\lambda_1(\tau-t_0)}x_2(\tau)^{1/3}d\tau \\ &= x_1(t_0)e^{-\lambda_1(t-t_0)} + e^{-\lambda_1(t-t_0)}\int_{t_0}^{t} x_2(t_0)^{1/3}e^{-(\frac{\lambda_2}{3}-\lambda_1)(\tau-t_0)}d\tau \\ &= x_1(t_0)e^{-\lambda_1(t-t_0)} + \frac{3x_2(t_0)^{1/3}}{\lambda_2 - 3\lambda_1}[e^{-\lambda_1(t-t_0)} - e^{-\frac{\lambda_2}{3}(t-t_0)}] \end{aligned} \tag{6.6}$$

Since $\lambda_2 > 3\lambda_1$, we find that the 1-norm of $x(t) = [x_1(t),\ x_2(t)]^T$ is bounded by

$$\begin{aligned}
\|x(t)\| &= |x_1(t)| + |x_2(t)| \\
&\leq |x_1(t_0)|e^{-\lambda_1(t-t_0)} + \frac{3|x_2(t_0)|^{1/3}}{\lambda_2 - 3\lambda_1}e^{-\lambda_1(t-t_0)} + |x_2(t_0)|e^{-\lambda_2(t-t_0)} \\
&\leq \|x(t_0)\|e^{-\lambda_1(t-t_0)} + \frac{3}{\lambda_2 - 3\lambda_1}|x_2(t_0)|^{1/3}e^{-\lambda_1(t-t_0)} \\
&\leq (\|x(t_0)\| + \frac{3}{\lambda_2 - 3\lambda_1}\|x(t_0)\|^{1/3})e^{-\lambda_1(t-t_0)} \\
&\leq h(\|x(t_0)\|)e^{-\lambda_1(t-t_0)}
\end{aligned}$$

where $h(q) = q + \frac{3}{\lambda_2 - 3\lambda_1}q^{1/3}$ is of class $\mathcal{K}$. Therefore, System (6.4)-(6.5) is globally, $\mathcal{K}$-exponentially stable according to Definition 6.2. However, despite the exponential convergence and the uniform asymptotic stability, the system is not exponentially stable in the usual sense, since it does not exist as a constant κ independent on $x(t_0)$ such that

$$\|x(t)\| \leq \kappa\|x(t_0)\|e^{-\lambda(t-t_0)} \tag{6.7}$$

for all $x(t_0)$ in a neighborhood of zero, for some $\lambda > 0$. This can be shown from (6.6). By choosing $x_1(t_0) = 0$ we get

$$x_1(t) = \frac{3}{\lambda_2 - 3\lambda_1}x_2(t_0)^{1/3}[e^{-\lambda_1(t-t_0)} - e^{-\frac{\lambda_2}{3}(t-t_0)}]$$

This implies

$$|x_1(t_0+T)| = \alpha|x_2(t_0)|^{1/3} \leq \|x(t_0+T)\|$$

where $\alpha = \frac{3}{\lambda_2 - 3\lambda_1}(e^{-\lambda_1 T} - e^{-\frac{\lambda_2}{3}T})$ and T is some positive constant. If (6.7) is satisfied for the constants κ and λ then

$$\alpha|x_2(t_0)|^{1/3} \leq \kappa\|x(t_0)\|e^{-\lambda T} = \kappa|x_2(t_0)|e^{-\lambda T}$$

for all $x_2(t_0)$ in a neighborhood of zero since $x_1(t_0) = 0$. Then, κ must satisfy

$$\kappa \geq \alpha e^{\lambda T}|x_2(t_0)|^{-2/3}$$

which implies that κ tends towards infinity as $x_2(t_0)$ tends towards zero. Consequently, a constant κ independent on $x(t_0)$ satisfying (6.7) does not exist.

Example 2:

An example of a system which is *globally*, $\mathcal{K}$-exponentially stable, but not globally, exponentially stable in the usual sense, is given by

$$\dot{x} = \mathrm{sat}(-\lambda x, K), \qquad \lambda,\ K\ > 0, \qquad t \geq t_0 \tag{6.8}$$

where

$$\mathrm{sat}(q, K) \triangleq \left\{ \begin{array}{cc} q \quad , & |q| < K \\ K\mathrm{sgn}(q)\ , & |q| \geq K \end{array} \right\}$$

The solution $x(t)$ of (6.8) is given by

$$x(t) = \left\{ \begin{array}{ll} x(t_0) - \mathrm{sgn}(x(t_0))K(t - t_0), & |x(t_0)| \geq \frac{K}{\lambda},\ t_0 \leq t < T \\ \mathrm{sgn}(x(t_0))\frac{K}{\lambda}e^{-\lambda(t-T)}, & |x(t_0)| \geq \frac{K}{\lambda},\ t \geq T \\ x(t_0)e^{-\lambda(t-t_0)}, & |x(t_0)| < \frac{K}{\lambda},\ t \geq t_0 \end{array} \right.$$

where $T = t_0 + \frac{|x(t_0)|}{K} - \frac{1}{\lambda}$. This implies that the solution $x(t)$ is bounded by

$$|x(t)| \leq h(|x(t_0)|)e^{-\lambda(t-t_0)}, \quad \forall x(t_0) \in \mathbb{R}, \ \ \forall t \geq t_0 \tag{6.9}$$

where

$$h(|x(t_0)|) = \left\{ \begin{array}{ll} |x(t_0)| & ,\ |x(t_0)| \leq \frac{K}{\lambda} \\ \frac{K}{\lambda}e^{\frac{\lambda|x(t_0)|}{K} - 1} & ,\ |x(t_0)| > \frac{K}{\lambda} \end{array} \right.$$

The function $h(|x(t_0)|)$ is clearly of class $\mathcal{K}$. Since (6.9) is satisfied for all $x(t_0) \in \mathbb{R}$, then the system (6.8) is globally, $\mathcal{K}$-exponentially stable about zero.

6.3 The System

The following nonholonomic chained system is considered:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{aligned} \tag{6.10}$$

The input vectors are

$$g_1 = [1, 0, x_2, x_3, \ldots, x_{n-1}]^T\ , \quad g_2 = [0, 1, 0, 0, \ldots, 0]^T$$

Repeated Lie brackets can be defined recursively by

$$\mathrm{ad}_{g_1}^0 g_2 = g_2 \ , \quad \mathrm{ad}_{g_1}^i g_2 = [g_1, \mathrm{ad}_{g_1}^{i-1} g_2]$$

It can easily be shown that for System (6.10)

$$\mathrm{ad}_{g_1}^{k-2} g_2 = (-1)^k e_k \ , \quad k \in \{2, 3, \ldots, n\}$$

where $e_k \triangleq [\delta_{1k}, \delta_{2k}, \ldots, \delta_{nk}]^T$ where δ_{ik} is the Kronecker delta. We denote $x = [x_1, x_2, \ldots, x_n]^T$. Since

$$\forall x \in \mathbb{R}^n \ , \quad \mathrm{span}\{g_1, \mathrm{ad}_{g_1}^0 g_2, \ldots, \mathrm{ad}_{g_1}^{n-2} g_2\}(x) = \mathbb{R}^n$$

the chained system (6.10) is completely controllable and the nonholonomic degree is $n-2$. In spite of the controllability, (6.10) cannot be stabilized by a smooth, static-state feedback law as stated by Brockett's Theorem, (Brockett 1983). From Definition B.22 and Theorem B.7 we see that if

$$\|u(t)\| \leq \kappa \|x(t)\|^\sigma$$

then (6.10) is not $(\sigma, ce^{-\lambda t})$ stabilizable if $\sigma > \frac{1}{n-2}$, where $n-2$ is the nonholonomic degree of (6.10). Therefore, there is no smooth, time-varying feedback law which *exponentially* stabilizes (6.10).

The nonholonomic chained system (6.10) is illustrated by a block diagram in *Figure 6.1.* This block diagram shows that one property of the chained

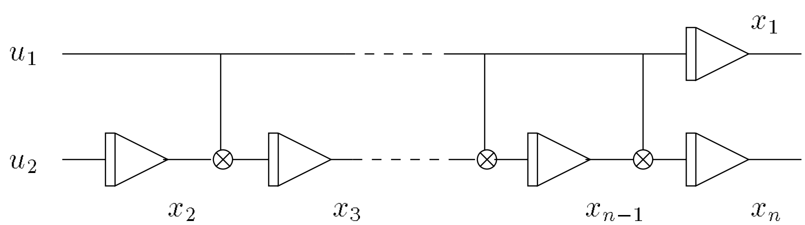

Figure 6.1: Block diagram of the nonholonomic chained system.

system (6.10) is that if the input u_1 is a time-periodic signal with a given non-zero amplitude, then the system from u_2 to x_n becomes linear and time-variant.

6.4 The Control Law

In this section, we will propose a control law to globally, $\mathcal{K}$-exponentially stabilize the nonholonomic chained system (6.10). Since there is no smooth static-state feedback law which can stabilize (6.10), we let the feedback law be time-dependent, as first proposed by Samson & Ait-Abderrahim (1990*b*) for a nonholonomic cart. To obtain exponential convergence, we let the feedback law also depend on a parameter k which is a function of the state $x(t_i)$ at discrete instants of time $t_i \in \{t_0, t_1, \ldots\}$. Motivated by the linear (time-variant) property of the system from u_2 to x_n if u_1 is a time-periodic signal with a given non-zero amplitude, we let u_1 be given by

$$u_1 = k(x(t_i))\, f(t) \tag{6.11}$$

t_i denotes the last element in the sequence $(t_0, t_1, t_2, \ldots)$ such that $t \geq t_i$. The function $f(t)$ has the properties:

P1: $f(t) \in C^\infty$

P2: $0 \leq f(t) \ \leq\ 1, \quad \forall t \geq t_0$

P3: $f(t_i) = 0, \quad t_i \in \{t_0, t_1, t_2, \ldots\}$

P4: $F_j(t) = \int_{t_k}^{t} f^{2(j-2)+1}(\tau) d\tau = \eta_j(t - t_k) + p_j(t)$
$\forall j \in \{3, \ldots, n\}, \ \forall t_k \in \{t_0, t_1, \ldots\}$

where η_j is a positive constant. The signal $p_j(t)$ satisfies $|p_j(t)| \leq P_j$ where P_j is a positive constant. This makes u_1 continuous with respect to time if $|k| < \infty$. A function satisfying these conditions is

$$f(t) = (1 - \cos \omega t)/2\ , \quad \omega = \frac{2\pi}{T} \tag{6.12}$$

where $T = t_{i+1} - t_i$ is supposed to be a constant.

We let the input u_1 be given by (6.11). The *lower part* of (6.10) will then be given by

$$\begin{aligned} \dot{x}_2 &= u_2 \\ \dot{x}_3 &= k f(t) x_2 \\ &\vdots \\ \dot{x}_n &= k f(t) x_{n-1} \end{aligned} \tag{6.13}$$

where k may switch at the time instants t_i.

We denote

$$z = [x_2, \ldots, x_n]^T$$

In the following, we derive a feedback law for u_2 which makes $z(t) = [x_2, \ldots, x_n]^T$ exponentially converge to zero under some conditions on k.

In Section 6.6 the parameter $k(x(t_i))$ will be specified to make the overall system (6.10) globally, $\mathcal{K}$-exponentially stable.

We regard (6.13) as a cascaded system where x_n can be controlled via x_{n-1} which again can be controlled via x_{n-2}, etc., see *Figure 6.1.* To derive the control law for u_2 we introduce the auxiliary variables $x_d = [x_2^d, \ldots, x_n^d]^T$ which are iteratively defined by:

$$
\begin{aligned}
x_n^d &= 0 && (6.14)\\
x_{n-1}^d &= -\frac{\lambda_n}{k} f^{2(n-2)}(t) x_n && (6.15)\\
x_{n-2}^d f(t) &= -\frac{\lambda_{n-1}}{k} f^{2(n-3)+1}(t)(x_{n-1} - x_{n-1}^d) + \frac{1}{k}\dot{x}_{n-1}^d && (6.16)\\
&\vdots\\
x_{i-1}^d f(t) &= -\frac{\lambda_i}{k} f^{2(i-2)+1}(t)(x_i - x_i^d) + \frac{1}{k}\dot{x}_i^d && (6.17)\\
&\vdots\\
x_2^d f(t) &= -\frac{\lambda_3}{k} f^3(t)(x_3 - x_3^d) + \frac{1}{k}\dot{x}_3^d && (6.18)
\end{aligned}
$$

where $\lambda_i > 0$ and $k \neq 0$ if $\|x(t_i)\| \neq 0$. If $z = [x_2, \ldots, x_n]^T = 0$ then we define $x^d(z = 0) = 0$. These auxiliary variables result from the following reasoning:

x_n^d is the desired value of x_n, i.e. $x_n^d = 0$. To obtain exponential convergence of x_n to $x_n^d = 0$ we want x_{n-1} to be equal to the auxiliary variable x_{n-1}^d which is chosen to satisfy

$$kf(t)x_{n-1}^d = -\lambda_n f^{2(n-2)+1}(t) x_n$$

This choice implies that if $x_{n-1} = x_{n-1}^d$ then

$$\dot{x}_n = -\lambda_n f^{2(n-2)+1}(t) x_n$$

according to (6.13). This defines x_{n-1}^d as given by (6.15). To obtain exponential convergence of x_{n-1} to its desired value x_{n-1}^d we want x_{n-2} to be equal to the auxiliary variable x_{n-2}^d which is chosen to satisfy

$$kf(t)x_{n-2}^d - \dot{x}_{n-1}^d = -\lambda_{n-1} f^{2(n-3)+1}(t)(x_{n-1} - x_{n-1}^d)$$

This choice implies that if $x_{n-2} = x_{n-2}^d$ then

$$\dot{\tilde{x}}_{n-1} = -\lambda_{n-1} f^{2(n-3)+1}(t)\tilde{x}_{n-1}, \qquad \tilde{x}_{n-1} \triangleq x_{n-1} - x_{n-1}^d$$

according to (6.13). This defines x_{n-2}^d as given by (6.16). The auxiliary variable x_{n-3}^d is defined similarly to make x_{n-2} exponentially converge to its desired value x_{n-2}^d, etc.

Note the exponents of $f(t)$ in these definitions of x_i^d. They are chosen sufficiently large so that when $f(t)=0$ then all the $x_i^d=0$. This structure of x^d is expressed in the following lemma which shows that $f(t)$ is a factor in $\dot{x}_i^d$.

Lemma 6.1 *Let x_i^d be defined by (6.14)-(6.18). Then, x_i^d can be expressed as a weighted sum of x_j, $j \in \{i+1,\ldots,n\}$ according to*

$$x_i^d = f^{2(i-1)}(t) \sum_{j=i+1}^{n} g_{ij} \frac{1}{k^{j-i}} x_j \,, \qquad i \in \{2,\ldots,n-1\} \tag{6.19}$$

where

$$g_{ij} = g_{ij}(f, \dot{f}, \ldots, f^{(j-i-1)}) \,, \qquad i \in \{2,\ldots,n-2\}$$

and g_{ij} is a smooth function with respect to its arguments. The functions g_{ij} are given by

$$g_{n-1,n} = -\lambda_n \tag{6.20}$$
$$g_{i-1,j} = g_{ij}(\lambda_i f^{2(i-1)} + 2(i-1)\dot{f}) + f(\dot{g}_{ij} + g_{i,j+1} f) \tag{6.21}$$
$$g_{i-1,i} = -\lambda_i + f^2 g_{i,i+1} \tag{6.22}$$
$$g_{ik} \triangleq 0 \;\; \text{if} \;\; k \le i \;\; \text{or} \;\; k = n+1 \tag{6.23}$$

where

$$\dot{g}_{ij} = \sum_{m=0}^{j-i-1} \frac{\partial g_{ij}}{\partial f^{(m)}} f^{(m+1)}$$

Proof: It is trivially seen from (6.15) that $g_{n-1,n} = -\lambda_n$. Eqs. (6.13) and (6.15) imply that

$$\dot{x}_{n-1}^d = f^{2(n-2)-1}[-2(n-2)\dot{f}\lambda_n \frac{x_n}{k} - f^2 \lambda_n x_{n-1}]$$

We can then calculate from (6.16)

$$x_{n-2}^d = f^{2(n-3)}(g_{n-2,n-1}\frac{x_{n-1}}{k} + g_{n-2,n}\frac{x_n}{k^2}) \tag{6.24}$$

where

$$g_{n-2,n-1} = -\lambda_{n-1} - \lambda_n f^2 \tag{6.25}$$
$$g_{n-2,n} = -\lambda_n(\lambda_{n-1} f^{2(n-2)} + 2(n-2)\dot{f}) \tag{6.26}$$

which are smooth functions in f and $\dot{f}$. The rest of the proof will be given by induction. Assume that for an index $i \in \{2, \ldots, n-1\}$

$$x_i^d = f^{2(i-1)}(t) \sum_{j=i+1}^{n} g_{ij} \frac{1}{k^{j-i}} x_j \tag{6.27}$$

We then find by using (6.13),

$$\begin{aligned} \dot{x}_i^d &= 2(i-1) f^{2(i-1)-1} \dot{f} \sum_{j=i+1}^{n} g_{ij} \frac{1}{k^{j-i}} x_j \\ &\quad + f^{2(i-1)} \sum_{j=i+1}^{n} [\dot{g}_{ij} \frac{1}{k^{j-i}} x_j + g_{ij} \frac{f}{k^{j-i-1}} x_{j-1}] \end{aligned} \tag{6.28}$$

The argument t of f is omitted for simplicity. From (6.14)-(6.18), (6.27) and (6.28) we get

$$\begin{aligned} x_{i-1}^d f(t) &= -\frac{\lambda_i}{k} f^{2(i-2)+1}(x_i - x_i^d) + \frac{1}{k} \dot{x}_i^d \\ &= -\frac{\lambda_i}{k} f^{2(i-2)+1}(x_i - f^{2(i-1)} \sum_{j=i+1}^{n} g_{ij} \frac{1}{k^{j-i}} x_j) \\ &\quad + f^{2(i-1)-1} \{2(i-1) \dot{f} \sum_{j=i+1}^{n} g_{ij} \frac{1}{k^{j-i+1}} x_j \\ &\quad + f \sum_{j=i+1}^{n} [\dot{g}_{ij} \frac{1}{k^{j-i+1}} x_j + g_{ij} \frac{f}{k^{j-i}} x_{j-1}]\} \\ &= f^{2(i-2)+1} \{x_i(-\frac{\lambda_i}{k} + f^2 g_{i,i+1} \frac{1}{k}) \\ &\quad + \sum_{j=i+1}^{n} \frac{x_j}{k^{j-i+1}} [(\lambda_i f^{2(i-1)} + 2(i-1)\dot{f}) g_{ij} + f(\dot{g}_{ij} + g_{i,j+1} f)]\} \\ &= f^{2(i-2)+1} \sum_{j=(i-1)+1}^{n} g_{i-1,j} \frac{1}{k^{j-(i-1)}} x_j \end{aligned} \tag{6.29}$$

This implies that

$$x_{i-1}^d = f^{2(i-2)} \sum_{j=(i-1)+1}^{n} g_{i-1,j} \frac{1}{k^{j-(i-1)}} x_j \tag{6.30}$$

where

$$\begin{aligned} g_{i-1,j} &= g_{ij}(\lambda_i f^{2(i-1)} + 2(i-1)\dot{f}) + f(\dot{g}_{ij} + g_{i,j+1} f) \\ g_{i-1,i} &= -\lambda_i + f^2 g_{i,i+1} \end{aligned}$$

$$\begin{aligned} g_{ik} &\triangleq 0 \;\text{ if }\; k \leq i \;\text{ or }\; k = n+1 \\ \dot{g}_{ij} &= \sum_{m=0}^{j-i-1} \frac{\partial g_{ij}}{\partial f^{(m)}} f^{(m+1)} \end{aligned}$$

We note that $g_{i-1,j}$ is given by g_{ij} and $g_{i,j+1}$. This iterative definition of the functions g_{ij} is initiated by $g_{n-1,n} = -\lambda_n$. Since g_{ij} and $g_{i,j+1}$ are smooth functions, this will also be the case for $g_{i-1,j}$. We also see that

$$g_{i-1,j} = g_{i-1,j}(f, \dot{f}, \ldots, f^{j-(i-1)-1})$$

Thus, if (6.27) is satisfied for $i = m$ then (6.27) is satisfied for $i = m - 1$. From (6.24) we have that (6.27) is satisfied for $i = n - 2$. By noting from (6.15) that (6.19) is satisfied for $i = n - 1$, we can conclude that (6.19) is satisfied for all $i \in \{2, \ldots, n-1\}$.

□

The definition of g_{ij} from (6.20)-(6.23) can be illustrated by the following diagram where $a \rightarrow b$ means that b depends on a:

$$\begin{array}{ccccccccc}
g_{2,3} & & \cdots & & g_{2,n-2} & & g_{2,n-1} & & g_{2,n} \\
& \nwarrow & \cdots & \nwarrow & \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\
& & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
& & & \nwarrow & \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\
& & & & g_{n-3,n-2} & & g_{n-3,n-1} & & g_{n-3,n} \\
& & & & & \nwarrow & \uparrow & \nwarrow & \uparrow \\
& & & & & & g_{n-2,n-1} & & g_{n-2,n} \\
& & & & & & & \nwarrow & \uparrow \\
& & & & & & & & g_{n-1,n}
\end{array}$$

From Chapter 5 we can think of a chained system as a local representation of a car with $n - 3$ trailers. Lemma 6.1 then expresses that with this definition of x^d we try to make the y-position of the rear trailer, x_n, exponentially converge to zero. When $f(t) = 0$ the parameter k may change sign, resulting in a cusp at the path of the rear trailer, since the sign of u_1 and k corresponds to the sign of the velocity along the x-axis of the rear trailer. By choosing the exponent of the function $f(t)$ in the definition of x_{n-1}^d sufficiently large, we try to stretch the car-trailer system at these cusps, i.e. try to make all the trailers parallel to the x-axis, which is a natural motion. This will be illustrated by the simulation example in Section 6.7.

The idea of the control law for u_2 is to make x_2 converge exponentially to x_2^d making x_3 converge exponentially to x_3^d, etc., resulting in an exponential convergence of $x_n(t)$ to zero. We, therefore, let u_2 be given by

$$u_2 = -\lambda_2(x_2 - x_2^d) + \dot{x}_2^d \tag{6.31}$$

Using Lemma 6.1 to find x_2^d and $\dot{x}_2^d$ gives

$$u_2 = (-\lambda_2 + f^3 g_{23})x_2 + f \sum_{j=3}^{n} \left(\lambda_2 f g_{2j} + 2\dot{f} g_{2j} + f \dot{g}_{2j} + f^2 g_{2,j+1}\right) \frac{1}{k^{j-2}} x_j$$

With u_1 given by (6.11), the control law is therefore given by

$$\begin{aligned} u_1 &= k(x(t_i))\, f(t) && (6.32)\\ u_2 &= \Gamma(k,t)^T z && (6.33) \end{aligned}$$

where

$$z = [x_2, \ldots, x_n]^T \in \mathbb{R}^{n-1}, \quad \Gamma(k,t) = [\Gamma_2(k,t), \ldots, \Gamma_n(k,t)]^T \in \mathbb{R}^{n-1}$$

and

$$\begin{aligned} \Gamma_2(k,t) &= -\lambda_2 + f^3 g_{2,3} \\ \Gamma_j(k,t) &= f \sum_{j=3}^{n} \left(\lambda_2 f g_{2j} + 2\dot{f} g_{2j} + f \dot{g}_{2j} + f^2 g_{2,j+1}\right) \frac{1}{k^{j-2}}, \quad j \in \{3, \ldots, n\} \end{aligned}$$

The smooth functions g_{2j} are given by (6.20)-(6.23). The parameter $k(x(t_i))$ will be chosen in Section 6.6 as a result of the following convergence analysis.

6.5 Convergence Analysis of the Lower Part

To analyze the convergence of $z = [x_2, \ldots, x_n]$ we introduce the following variables:

$$\tilde{x}_j = x_j - x_j^d, \quad i \in \{2, \ldots, n\}$$

Eqs. (6.13), (6.14)-(6.18) and the control law for u_2, (6.33) imply

$$\begin{aligned} \dot{\tilde{x}}_2 &= -\lambda_2 \tilde{x}_2 && (6.34)\\ \dot{\tilde{x}}_3 &= -\lambda_3 f^3(t)\tilde{x}_3 + k f(t)\tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_n &= -\lambda_n f^{2(n-2)+1}(t)\tilde{x}_n + k f(t)\tilde{x}_{n-1} && (6.35) \end{aligned}$$

The structure of this analysis is:

First, the convergence of the "error" variables $\tilde{x}_2, \ldots, \tilde{x}_n$ to zero will be analyzed. Then, the state variables $x_2, \ldots, x_n$ will be expressed by $\tilde{x}_2, \ldots, \tilde{x}_n$. Finally, the convergence of $z = [x_2, \ldots, x_n]^T$ to zero is analyzed under some assumptions on k. In the next section, k will be specified and global, $\mathcal{K}$-exponential stability of $x = [x_1, x_2, \ldots, x_n]^T$ will be shown.

The following lemma shows that $x_j(t)$ tends exponentially towards its reference x_j^d under certain assumptions on $k(\cdot)$.

We denote

$$\underline{q}_j = [x_2, x_3, \ldots, x_j]^T$$

Lemma 6.2 *Consider System (6.34)-(6.35). Assume that k has the properties*

- *k is constant when $f(t) \neq 0$*
- $\exists K \, \forall t_i \in \{t_0, t_1, \ldots\} \mid \quad |k(x(t_i))| \leq K$
- $\|z(t_i)\| \neq 0 \quad \Rightarrow \quad k \neq 0, \qquad t \in [t_i, t_{i+1})$

Then

$$\begin{aligned} &\forall j \in \{2, \ldots, n\} \;\; \exists \delta > 0 \;\; \forall \varepsilon_2, \ldots, \varepsilon_j \in (0, \delta) \, \exists c_j > 0 \, \forall t_k \in \{t_0, t_1, \ldots\} \\ &\qquad \forall \underline{q}_j(t_k) \in \mathbb{R}^{j-1} \quad |\tilde{x}_j(t)| \leq c_j \|\underline{q}_j(t_k)\| e^{-\gamma_j (t - t_k)}, \quad \forall t \geq t_k \end{aligned} \tag{6.36}$$

where

$$\gamma_j = \beta_j - \varepsilon_j \;>\; 0 \tag{6.37}$$

$$\beta_q = \min\{\lambda_q \eta_q, \beta_{q-1} - \varepsilon_{q-1}\}, \quad q \in \{3, \ldots, j\} \tag{6.38}$$

$$\beta_2 = \lambda_2 \tag{6.39}$$

The constants η_q are found from Property P4 of $f(t)$.

Proof: The proof will be given by induction. Assume that (6.36) is satisfied for $j = m - 1 \in \{2, \ldots, n-1\}$. Then from (6.34)-(6.35) we have that

$$\dot{\tilde{x}}_m = -\lambda_m f^{2(m-2)+1}(t) \tilde{x}_m + k f(t) \tilde{x}_{m-1} = -a(t) \tilde{x}_m + d(t)$$

where

$$\begin{aligned} a(t) &= \lambda_m f^{2(m-2)+1}(t) \\ d(t) &= k f(t) \tilde{x}_{m-1} \end{aligned}$$

From Property P4 of $f(t)$ we have that

$$|\int_{t_k}^{t}(a(\tau)-\lambda_m\eta_m)d\tau| \leq \lambda_m P_m$$

Since (6.36) is assumed to be satisfied for $j = m-1$ and since $|f(t)| \leq 1$ and $|k| \leq K$ we have

$$|d_m(t)| \leq D_m e^{-\gamma_{m-1}(t-t_k)}$$

where $D_m \stackrel{\triangle}{=} K c_{m-1}\|\underline{q}_{m-1}(t_k)\|$. From Lemma 2.1 we can then conclude that

$$\begin{aligned}\forall\varepsilon_m > 0\ \exists c_m \quad | \quad |\tilde{x}_m(t)| &\leq c_m(\|\underline{q}_{m-1}(t_k)\| + |\tilde{x}_m(t_k)|)e^{-\gamma_m(t-t_k)}\\ &= c_m\|\underline{q}_m(t_k)\|e^{-\gamma_m(t-t_k)}\end{aligned}$$

where

$$\begin{aligned}\gamma_m &= \beta_m - \varepsilon_m\\ \beta_m &= \min\{\lambda_m\eta_m, \gamma_{m-1}\}\\ c_m &= \max\{e^{\lambda_m P_m}, e^{2\lambda_m P_m}\frac{K c_{m-1}}{\varepsilon_m}\}\end{aligned}$$

The parameter $\gamma_m > 0$ if we restrict ε_i as follows:

$$0 < \varepsilon_j < \beta_j, \quad j \in \{2, \ldots, m\}$$

Consequently, there exists a constant δ such that $\varepsilon_2, \ldots, \varepsilon_n \in (0, \delta)$ implies that $\gamma_m > 0$.

We have, therefore, proved that if (6.36) is satisfied for $j = m-1$ then (6.36) is satisfied for $j = m$, for all $m \in \{3, \ldots, n\}$. The proof of the lemma is completed by showing that (6.36) is satisfied for $j = 2$. From (6.34) we find that

$$|\tilde{x}_2(t)| = |\tilde{x}_2(t_k)|e^{-\lambda_2(t-t_k)} = \|\underline{q}_2(t_k)\|e^{-\lambda_2(t-t_k)}$$

since $\tilde{x}_j(t_k) = x_j(t_k)$. Eq. (6.36) is then satisfied for $j = 2$ by choosing $c_2 = 1$. (In this case Eq. (6.36) is also satisfied for $\varepsilon_2 = 0$.)

□

Remark: Note from (6.38) that $\beta_2, \ldots, \beta_n$ can be arbitrary by choosing $\lambda_2, \ldots, \lambda_n$ appropriately which means that the exponential rate of convergence $\gamma_j = \beta_j - \varepsilon_j$ can be chosen arbitrarily large since ε_j can be chosen arbitrarily close to zero.

Lemma 6.1 can be used to show that the state $x_j(t)$ can be expressed as a weighted sum of the error variables $\tilde{x}_r$, $r \in \{j, \ldots, n\}$ as stated in the following lemma.

Lemma 6.3 *Let x_j^d be defined by (6.14)-(6.18). Then $x_j(t)$ can be expressed as*

$$x_j = \tilde{x}_j + f^{2(j-1)}(t) \sum_{r=j+1}^{n} \tilde{g}_{jr} \frac{1}{k^{r-j}} \tilde{x}_r \ , \quad j \in \{2, \ldots, n-1\} \tag{6.40}$$

where

$$\begin{aligned} \tilde{x}_r &= x_r - x_r^d \\ \tilde{g}_{jr} &= \tilde{g}_{jr}(f, \dot{f}, \ldots, f^{(r-j-1)}) \ , \qquad j \in \{2, \ldots, n-2\} \end{aligned}$$

and $\tilde{g}_{jr}$ is a smooth function with respect to its arguments. If $j = n-1$ then $\tilde{g}_{n-1,n} = -\lambda_n$.

Proof: From the definition of $\tilde{x}_{n-1}$ we have that

$$x_{n-1} = \tilde{x}_{n-1} + x_{n-1}^d \tag{6.41}$$

From (6.14) we have that $x_n^d = 0$ which means that $\tilde{x}_n = x_n$. It is trivially seen from (6.15) that

$$x_{n-1} = \tilde{x}_{n-1} + x_{n-1}^d = \tilde{x}_{n-1} + f^{2(n-2)}(-\lambda_n)\frac{1}{k}\tilde{x}_n \tag{6.42}$$

which implies that

$$\tilde{g}_{n-1,n} = g_{n-1,n} = -\lambda_n \tag{6.43}$$

We can then calculate from (6.24), Lemma 6.1, (6.41), (6.42) and (6.43)

$$\begin{aligned} x_{n-2} &= \tilde{x}_{n-2} + x_{n-2}^d \\ &= \tilde{x}_{n-2} + f^{2(n-3)}(g_{n-2,n-1}\frac{x_{n-1}}{k} + g_{n-2,n}\frac{x_n}{k^2}) \\ &= \tilde{x}_{n-2} + f^{2(n-3)}[g_{n-2,n-1}\frac{1}{k}\tilde{x}_{n-1} \\ &\quad +(g_{n-2,n} + g_{n-2,n-1}\tilde{g}_{n-1,n}f^{2(n-2)})\frac{x_n}{k^2}] \\ &= \tilde{x}_{n-2} + f^{2(n-3)}[\tilde{g}_{n-2,n-1}\frac{1}{k}\tilde{x}_{n-1} + \tilde{g}_{n-2,n}\frac{1}{k^2}x_n] \end{aligned} \tag{6.44}$$

where

$$\tilde{g}_{n-2,n-1} = g_{n-2,n-1} \tag{6.45}$$

$$\tilde{g}_{n-2,n} = g_{n-2,n} + g_{n-2,n-1}\tilde{g}_{n-1,n}f^{2(n-2)} \tag{6.46}$$

The rest of the proof will be given by induction. Assume that (6.40) is satisfied for $j+1, j+2, \ldots, n-1$, i.e.

$$x_{m+1}(t) = \tilde{x}_{m+1} + f^{2((m+1)-1)} \sum_{r=(m+1)+1}^{n} \tilde{g}_{m+1,r} \frac{1}{k^{r-(m+1)}} \tilde{x}_r \tag{6.47}$$

where $m \in \{j, j+1, \ldots, n-2\}$. We then find from Lemma 6.1

$$\begin{aligned}
x_j(t) &= \tilde{x}_j + x_j^d \\
&= \tilde{x}_j + f^{2(j-1)}(t) \sum_{r=j+1}^{n} g_{jr} \frac{1}{k^{r-j}} x_r \\
&= \tilde{x}_j + f^{2(j-1)}(t) \sum_{r=j+1}^{n} g_{jr} \frac{1}{k^{r-j}} [\tilde{x}_r + f^{2(r-1)} \sum_{m=r+1}^{n} \tilde{g}_{rm} \frac{1}{k^{m-r}} \tilde{x}_m] \\
&= \tilde{x}_j + f^{2(j-1)}(t) \sum_{r=j+1}^{n} g_{jr} [1 + f^{2(r-1)} \sum_{m=j+1}^{r-1} \tilde{g}_{mr}] \frac{1}{k^{r-j}} \tilde{x}_r \\
&= \tilde{x}_j + f^{2(j-1)}(t) \sum_{r=j+1}^{n} \tilde{g}_{jr} \frac{1}{k^{r-j}} \tilde{x}_r
\end{aligned} \tag{6.48}$$

where

$$\tilde{g}_{jr} = g_{jr} [1 + f^{2(r-1)} \sum_{m=j+1}^{r-1} \tilde{g}_{mr}] \tag{6.49}$$

Eq. (6.48) shows that if (6.40) is satisfied for $j+1, j+2, \ldots, n$, then (6.40) is satisfied for j as well. Therefore, Eqs. (6.42)-(6.43) and (6.48) imply that (6.40) is satisfied for all $j \in \{2, \ldots, n-1\}$. We see that $\tilde{g}_{jr}$ is given by $\tilde{g}_{mr}$ and g_{jr} which are known. Because of the smoothness of $f(t)$, g_{jr} and $\tilde{g}_{mr}$ for $m \in \{j+1, \ldots, r-1\}$, $\tilde{g}_{jr}$ will be smooth as well. The functions $\tilde{g}_{jr}$ are given by (6.25), (6.26), (6.43), (6.45), (6.46) and(6.49). We note that $\tilde{g}_{jr}$ has the same arguments as g_{jr}, namely $f, \dot{f}, \ldots, f^{(r-j-1)}$.

□

We can now show exponential convergence of the lower part (6.13). We denote $z = [x_2, \ldots, x_n]^T$.

Theorem 6.1 *Consider the system given by (6.13) where u_2 is given by (6.33) and $f(t)$ has the properties P1-P4. Assume that*

- *k is constant when $f(t) \neq 0$*
- $\exists K \, \forall t_i \in \{t_0, t_1, \ldots\} \mid \quad |k(x(t_i))| \leq K$

- $\|z(t_i)\| \neq 0 \quad \Rightarrow \quad k \neq 0, \qquad t \in [t_i, t_{i+1})$

and

$$|k| < K \quad \Rightarrow \quad |k| \geq \kappa_j |\tilde{x}_j(t)|^{\frac{1}{2(n-2)}}, \quad \forall\, j \in \{3, \ldots, n\} \tag{6.50}$$

where κ_j is a positive constant. Then there are a function $h_z(\cdot)$ of class $\mathcal{K}$ and a constant $\gamma_z > 0$ so that

$$\forall z(t_0) \in \mathbb{R}^{n-1} \quad \|z(t)\| \leq h_z(\|z(t_0)\|)e^{-\gamma_z(t-t_0)}, \quad \forall t \geq t_0$$

where

$$\gamma_z = \frac{\gamma_n}{2}$$

where γ_n is given from (6.37).

Proof: From Lemma 6.3 we have that

$$x_j = \tilde{x}_j + f^{2(j-1)}(t) \sum_{r=j+1}^{n} \tilde{g}_{jr} \frac{1}{k^{r-j}} \tilde{x}_r\ , \quad j \in \{2, \ldots, n-1\}$$

Since the function $\tilde{g}_{jr} = \tilde{g}_{jr}(f, \dot{f}, \ldots, f^{(r-j-1)})$ is smooth and the images of $f(t)$ and its derivatives are finite intervals in $\mathbb{R}$, we know that there is a constant G such that $\tilde{g}_{jr} \leq G$. Since $|f(t)| \leq 1$, Property P2 of $f(t)$, we get

$$|x_j(t)| \leq |\tilde{x}_j(t)| + \sum_{r=j+1}^{n} G \frac{1}{|k|^{r-j}} |\tilde{x}_r(t)|\ , \quad j \in \{2, \ldots, n-1\} \tag{6.51}$$

Since $|k| = K$ or $|k| \geq \kappa_r |\tilde{x}_r(t)|^{\frac{1}{2(n-2)}}$, $r \in \{j+1, \ldots, n\}$, (6.50), we get

$$|x_j(t)| \leq |\tilde{x}_j(t)| + G \sum_{r=j+1}^{n} \left[\frac{1}{K^{r-j}} |\tilde{x}_r(t)| + \frac{1}{\kappa_r^{r-j}} |\tilde{x}_r(t)|^{1-\frac{r-j}{2(n-2)}} \right] \tag{6.52}$$

for all $j \in \{2, \ldots, n-1\}$. From Lemma 6.2 we get, with $t_k = t_0$,

$$\begin{aligned} |x_j(t)| &\leq c_j \|\underline{q}_j(t_0)\| e^{-\gamma_j(t-t_0)} + G \sum_{r=j+1}^{n} [\frac{1}{K^{r-j}} c_r \|\underline{q}_r(t_0)\| e^{-\gamma_r(t-t_0)} \\ &\quad + (\frac{c_r}{\kappa_r^{r-j}} \|\underline{q}_r(t_0)\| e^{-\gamma_r(t-t_0)})^{1-\frac{r-j}{2(n-2)}}] \\ &\leq h_j(\|z(t_0)\|) e^{-\zeta_j(t-t_0)} \end{aligned}$$

for all $j \in \{2, \ldots, n-1\}$, since $\|z(t_0)\| \geq \|\underline{q}_r(t_0)\|$, where

$$\begin{aligned} h_j(q) &= c_j q + G \sum_{r=j+1}^{n} \left[\frac{1}{K^{r-j}} c_r q + (\frac{c_r}{\kappa_r^{r-j}} q)^{1-\frac{r-j}{2(n-2)}} \right] \\ \zeta_j &= \min_r \{\gamma_r (1 - \frac{r-j}{2(n-2)})\}, \qquad r \in \{j, \ldots, n\} \end{aligned} \tag{6.53}$$

From (6.37)-(6.39) we see that

$$0 < \gamma_n < \gamma_{n-1} < \cdots < \gamma_2 < \lambda_2$$

Eq. (6.53) then implies that

$$\zeta_j = \gamma_n (1 - \frac{n-j}{2(n-2)})$$

Note that the function $h_j(q)$ is of class $\mathcal{K}$. Since $x_n^d \stackrel{\triangle}{=} 0$ we have from Lemma 6.2

$$|x_n(t)| = |\tilde{x}_n(t)| \leq c_n \|z(t_0)\| e^{-\gamma_n (t-t_0)}$$

We define $h_n(\|z(t_0)\|) = c_n \|z(t_0)\|$ and $\zeta_n = \gamma_n$ such that from Lemma 6.2

$$|x_n(t)| = |\tilde{x}_n(t)| \leq h_n(\|z(t_0)\|) e^{-\gamma_n (t-t_0)}$$

The proof is completed by noting that

$$\begin{aligned} \|z(t)\| &= |x_2(t)| + \ldots |x_n(t)| \\ &\leq \sum_{j=2}^{n} h_j(\|z(t_0)\|) e^{-\zeta_j (t-t_0)} = h_z(\|z(t_0)\|) e^{-\gamma_z (t-t_0)} \end{aligned}$$

where

$$h_z(q) = \sum_{j=2}^{n} h_j(q) \tag{6.54}$$

$$\gamma_z = \min\{\zeta_2, \ldots, \zeta_n\} = \zeta_2 = \frac{\gamma_n}{2} \tag{6.55}$$

□

It is interesting to compare the exponent $\frac{1}{2(n-2)}$ in Assumption (6.50) with the parameter σ in the Condition (B.13),

$$\sigma > \frac{1}{\eta(x_p)}$$

where $\eta(x_p) = n - 2$ is the nonholonomic degree. This sufficient condition for the system not to be $(\sigma, ce^{-\lambda t})$ stabilizable is not satisfied if $\sigma = \frac{1}{2(n-2)}$.

6.6 Stabilization of the Total System

In this section we prove global $\mathcal{K}$-exponential stability of the total system, (6.10).

We must choose $k(x(t_i))$ so that $x = [x_1, \ldots, x_n]^T$ in (6.10) becomes $\mathcal{K}$-exponentially stable about the origin. In addition, k must satisfy the assumptions

1. k is constant when $f(t) \neq 0$
2. $\exists K \, \forall t_i \in \{t_0, t_1, \ldots\} \mid \quad |k(x(t_i))| \leq K$
3. $\|z(t_i)\| \neq 0 \quad \Rightarrow \quad k \neq 0, \qquad t \in [t_i, t_{i+1})$

and

$$|k| < K \quad \Rightarrow \quad |k| \geq \kappa_j |\tilde{x}_j(t)|^{\frac{1}{2(n-2)}}, \quad j \in \{3, \ldots, n\} \tag{6.56}$$

in Theorem 6.1.

The assumptions 1, 2, and 3 are trivially satisfied by letting k be constant in the time intervals $[t_i, t_{i+1})$, $i \in \{0, 1, \ldots\}$, and equal to

$$k = \mathrm{sat}\left(-\left[x_1(t_i) + \mathrm{sgn}(x_1(t_i)) G(\|z(t_i)\|)\right] \beta, K\right) \tag{6.57}$$

where

$$\mathrm{sat}(q, K) \triangleq \left\{ \begin{array}{cc} q \quad , & |q| < K \\ K \mathrm{sgn}(q) \ , & |q| \geq K \end{array} \right\}$$

and

$$G(\|z(t_i)\|) = \kappa \|z(t_i)\|^{\frac{1}{2(n-2)}} \tag{6.58}$$

$$\beta = 1 / \int_{t_i}^{t_{i+1}} f(\tau) d\tau \tag{6.59}$$

where κ is a positive constant and $\mathrm{sgn}(x_1(t_i))$ is defined as follows:

$$\mathrm{sgn}(x_1(t_i)) = \left\{ \begin{array}{r} 1, \ x_1(t_i) \geq 0 \\ -1, \ x_1(t_i) < 0 \end{array} \right.$$

We will now show that the assumption (6.56) is satisfied. We let t_i denote the most recent point of time in the sequence $(t_0, t_1, t_2, \ldots)$ such that $t \geq t_i$. From (6.57) we find that if $|k| < K$ then

$$|k| = |k(x(t_i))| = [|x_1(t_i)| + G(\|z(t_i)\|)]\beta \tag{6.60}$$

Combining this with the definition of $G(\cdot)$, (6.58), implies

$$|k| \geq G(\|z(t_i)\|)\beta = \beta\kappa\|z(t_i)\|^{\frac{1}{2(n-2)}} \geq \beta\kappa\|\underline{q}_j(t_i)\|^{\frac{1}{2(n-2)}}, \quad j \in \{2, \ldots, n\} \tag{6.61}$$

Lemma 6.2 implies

$$\begin{aligned} |k| &\geq \beta\kappa\left(\frac{1}{c_j}|\tilde{x}_j(t)|\right)^{\frac{1}{2(n-2)}} \\ &= \beta\kappa c_j^{\frac{-1}{2(n-2)}}|\tilde{x}_j(t)|^{\frac{1}{2(n-2)}}, \quad j \in \{3, \ldots, n\} \end{aligned} \tag{6.62}$$

Therefore, the assumption (6.56) is satisfied by defining $\kappa_j = \beta\kappa c_j^{\frac{-1}{2(n-2)}}$ and all the assumptions in Theorem 6.1 are thus satisfied.

Theorem 6.1 will now be extended to show $\mathcal{K}$-exponential stability of the total system (6.10).

Theorem 6.2 *Let the control law be given by (6.32)-(6.33) where k is given by (6.57). Then the system (6.10) is $\mathcal{K}$-exponentially stable about the origin, i.e. $\exists\delta > 0 \ \ \forall\varepsilon_2, \ldots, \varepsilon_n \in (0, \delta)$ there exist a function $h(\cdot)$ of class $\mathcal{K}$, a constant $\gamma > 0$, and a neighborhood Ω about $x = 0$ such that*

$$\forall x(t_0) \in \Omega \quad \|x(t)\| \leq h(\|x(t_0)\|)e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0 \tag{6.63}$$

where

$$\begin{aligned} \gamma &= \frac{\gamma_z}{2(n-2)} > 0 \\ \gamma_z &= \frac{\gamma_n}{2} \\ \gamma_n &= \beta_n - \varepsilon_n \\ \beta_q &= \min\{\lambda_q\eta_q, \beta_{q-1} - \varepsilon_{q-1}\}, \quad q \in \{3, \ldots, n\} \\ \beta_2 &= \lambda_2 \end{aligned}$$

The constants η_q are found from Property P4 of $f(t)$. The neighborhood Ω is given by

$$\Omega = \{x \mid \quad |x_1| < \frac{K}{2\beta}, \ \ G(h_z(\|z\|)) < \frac{K}{2\beta}\}$$

where $h_z(\cdot)$ is a function of class $\mathcal{K}$ from Theorem 6.1, and $z = [x_2, \ldots, x_n]^T$. The function $G(\cdot)$ is defined in (6.58).

Proof: By induction, we can show that if $x(t_0) \in \Omega$, then

$$k(x(t_i)) = -\left[x_1(t_i) + \operatorname{sgn}(x_1(t_i))G(\|z(t_i)\|)\right]\beta, \quad \forall t_i \in \{t_0, t_1, \ldots\} \tag{6.64}$$

which is equivalent to, (6.57),

$$|k(x(t_i))| < K, \quad \forall t_i \in \{t_0, t_1, \ldots\} \tag{6.65}$$

Note from Theorem 6.1 that

$$\|z(t)\| \leq h_z(\|z(t_0)\|), \quad \forall t \geq t_0$$

Since $G(\cdot)$ is of class $\mathcal{K}$, (6.58), this implies

$$G(\|z(t)\|) \leq G(h_z(\|z(t_0)\|)), \quad \forall t \geq t_0$$

Assume for a $t_m \in \{t_0, t_1, \ldots\}$ that

$$k(x(t_m)) = -\left[x_1(t_m) + \mathrm{sgn}(x_1(t_m))G(\|z(t_m)\|)\right]\beta$$

Integrating $\dot{x}_1 = u_1 = k(x(t_m))f(t)$ from t_m to t_{m+1} then gives

$$x_1(t_{m+1}) = -\mathrm{sgn}(x_1(t_m))G(\|z(t_m)\|) \tag{6.66}$$

which implies that

$$|x_1(t_{m+1})| = G(\|z(t_m)\|) \leq G(h_z(\|z(t_0)\|))$$

By assumption, $x(t_0) \in \Omega$ which implies that $G(\|h_z(t_0)\|) < \frac{K}{2\beta}$, and therefore

$$|x_1(t_{m+1})| < \frac{K}{2\beta}$$

Since

$$\left[|x_1(t_{m+1})| + G(\|z(t_{m+1})\|)\right]\beta < (\frac{K}{2\beta} + \frac{K}{2\beta})\beta = K$$

then

$$k(x(t_{m+1})) = -\left[x_1(t_{m+1}) + \mathrm{sgn}(x_1(t_{m+1}))G(\|z(t_{m+1})\|)\right]\beta$$

Eqs. (6.64) and (6.65) are proved by noting that since $x(t_0) \in \Omega$, then

$$\left[|x_1(t_0)| + G(\|z(t_i)\|)\right]\beta < (\frac{K}{2\beta} + \frac{K}{2\beta})\beta = K$$

which implies that

$$k(x(t_0)) = -\left[x_1(t_0) + \mathrm{sgn}(x_1(t_0))G(\|z(t_0)\|)\right]\beta$$

Integrating $\dot{x}_1 = u_1$ in (6.10) from t_i to $t < t_{i+1}$ with $u_1 = k(x(t_i))\, f(t)$ and k constant gives

$$\begin{aligned} |x_1(t)| &\le |x_1(t_i)| + |k| \int_{t_i}^{t} f(\tau) d\tau \\ &\le |x_1(t_i)| + |k| \int_{t_i}^{t_{i+1}} f(\tau) d\tau \\ &= |x_1(t_i)| + |x_1(t_{i+1}) - x_1(t_i)| \\ &\le 2|x_1(t_i)| + |x_1(t_{i+1})| \end{aligned} \tag{6.67}$$

Since $x(t_0) \in \Omega$ then $k(x(t_i))$ is given by (6.64) and we get from (6.66) and (6.67)

$$|x_1(t)| \le 2G(\|z(t_{i-1})\|) + G(\|z(t_i)\|)\ , \quad i \in \{1, 2, \ldots\} \tag{6.68}$$

From Theorem 6.1 we have that

$$\|z(t_k)\| \le h_z(\|z(t_0)\|) e^{-\gamma_z (t_k - t_0)} \tag{6.69}$$

From the definition of $G(\cdot)$, (6.58), we have

$$G(ae^{-bt}) = \kappa(ae^{-bt})^{\frac{1}{2(n-2)}} = e^{\frac{-bt}{2(n-2)}} G(a) \tag{6.70}$$

where a and b are positive constants. We denote for simplicity

$$q = h_z(\|z(t_0)\|$$

Eq. (6.70) combined with (6.68) and (6.69) implies

$$\begin{aligned} |x_1(t)| &\le 2G(qe^{-\gamma_z(t_{i-1} - t_0)}) + G(qe^{-\gamma_z(t_i - t_0)}) \\ &= 2G(q) e^{-\frac{\gamma_z}{2(n-2)}(t_{i-1} - t_0)} + G(q) e^{-\frac{\gamma_z}{2(n-2)}(t_i - t_0)} \\ &\le 3G(q) e^{-\frac{\gamma_z}{2(n-2)}(t_{i-1} - t_0)} \end{aligned}$$

By convention we have for all $i \in \{1, 2, \ldots\}$

$$t_{i-1} = t_{i+1} - 2T \ge t - 2T$$

since $t \in [t_i, t_{i+1})$. This implies

$$|x_1(t)| \le 3G(q) e^{\frac{\gamma_z 2T}{2(n-2)}} e^{\frac{-\gamma_z (t - t_0)}{2(n-2)}}\ , \quad t \ge t_1 \tag{6.71}$$

If $t \in [t_0, t_1)$ we find the following bound on $|x_1(t)|$ by integrating $\dot{x}_1 = k(x(t_0))\, f(t)$ where k is given by (6.64) and $f(t) \ge 0$:

$$\begin{aligned} |x_1(t)| &\le |x_1(t_0)| + [|x_1(t_0)| + G(\|z(t_0)\|)] \frac{\int_{t_0}^{t} f(\tau) d\tau}{\int_{t_0}^{t_1} f(\tau) d\tau} \\ &\le 2|x_1(t_0)| + G(\|z(t_0)\|) \end{aligned}$$

Since $t \in [t_0, t_1)$ this implies

$$|x_1(t)| \leq [2|x_1(t_0)| + G(\|z(t_0)\|)]e^{-\frac{\gamma_z(t-t_1)}{2(n-2)}}, \quad t \in [t_0, t_1) \tag{6.72}$$

Combining (6.71) and (6.72) and using $t_1 - t_0 = T$ implies

$$\begin{aligned} |x_1(t)| &\leq \{3G(h_z(\|z(t_0)\|))e^{\frac{\gamma_z 2T}{2(n-2)}} \\ &\quad +[2|x_1(t_0)| + G(\|z(t_0)\|)]e^{\frac{\gamma_z T}{2(n-2)}}\}e^{\frac{-\gamma_z(t-t_0)}{2(n-2)}} \\ &\leq h_1(\|x(t_0)\|)e^{\frac{-\gamma_z(t-t_0)}{2(n-2)}}, \quad t \geq t_0 \end{aligned} \tag{6.73}$$

where

$$\begin{aligned} h_1(\|x(t_0)\|) &= 3G(h_z(\|x(t_0)\|))e^{\frac{\gamma_z 2T}{2(n-2)}} \\ &\quad +[2\|x(t_0)\| + G(\|x(t_0)\|)]e^{\frac{\gamma_z T}{2(n-2)}} \end{aligned}$$

Since $G(\cdot)$ and $h_z(\cdot)$ are functions of class $\mathcal{K}$, $h_1(\cdot)$ is also of class $\mathcal{K}$, (6.1). The proof is completed by noting from Theorem 6.1 and (6.73) that

$$\begin{aligned} \|x(t)\| &= |x_1(t)| + \|z(t)\| \\ &\leq h_1(\|x(t_0)\|)e^{\frac{-\gamma_z(t-t_0)}{2(n-2)}} + h_z(\|z(t_0)\|)e^{-\gamma_z(t-t_0)} \\ &\leq [h_1(\|x(t_0)\|) + h_z(\|z(t_0)\|)]\, e^{\frac{-\gamma_z(t-t_0)}{2(n-2)}} \\ &\leq h(\|x(t_0)\|)e^{-\gamma(t-t_0)}, \quad t \geq t_0 \end{aligned} \tag{6.74}$$

where

$$\begin{aligned} h(\|x(t_0)\|) &= h_1(\|x(t_0)\|) + h_z(\|x(t_0)\|) \\ \gamma &= \frac{\gamma_z}{2(n-2)} \end{aligned} \tag{6.75}$$

where γ_z is given from Theorem 6.1.

□

Remark 1: The exponential convergence rate γ in (6.63) can be arbitrary by choosing $\lambda_2, \ldots, \lambda_n$ appropriately. For all $\lambda_2, \ldots, \lambda_n > 0$ the system is $\mathcal{K}$-exponentially stable for all time periods $T > 0$ of $f(t)$.

Remark 2: The neighborhood Ω can be chosen arbitrarily large by choosing the parameters K, κ, and β (or $T = t_{i+1} - t_i$) appropriately. Therefore, the stability is semi-global.

Remark 3: Note that since the function $f(t)$ is periodic in time with period $T = t_{i+1} - t_i$, the same analysis can be used to show $\mathcal{K}$-exponential stability about the origin at any $t_i \in \{t_0, t_1, t_2, \ldots\}$.

Remark 4: The bounds on $\|x(t)\|$ given by the function $h(\cdot)$, (6.75), and γ_z can be very conservative and do not in general provide quantitative information.

Theorem 6.2 does not prove *global*, $\mathcal{K}$-exponential stability because of the saturation function in the definition of $k(x(t_i))$, (6.57). Motivated by Example 2 in Section 6.2, we can prove global, $\mathcal{K}$-exponential stability, too.

Theorem 6.3 *Let the control law be given by (6.32)-(6.33) where k is given by (6.57). Then the system (6.10) is globally, $\mathcal{K}$-exponentially stable about the origin, i.e. $\exists \delta > 0 \;\; \forall \varepsilon_2, \ldots, \varepsilon_n \in (0, \delta)$ there exist a function $h_x(\cdot)$ of class $\mathcal{K}$ and a constant $\gamma > 0$ such that*

$$\forall x(t_0) \in \mathbb{R}^n \quad \|x(t)\| \leq h_x(\|x(t_0)\|) e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0 \tag{6.76}$$

where

$$\begin{aligned}
\gamma &= \frac{\gamma_z}{2(n-2)} > 0 \\
\gamma_z &= \frac{\gamma_n}{2} \\
\gamma_n &= \beta_n - \varepsilon_n \\
\beta_q &= \min\{\lambda_q \eta_q, \beta_{q-1} - \varepsilon_{q-1}\}, \quad q \in \{3, \ldots, n\} \\
\beta_2 &= \lambda_2
\end{aligned}$$

The constants η_q are found from Property P4 of $f(t)$ and $\lambda_2, \ldots, \lambda_n$ are arbitrary positive controller parameters.

Proof: This will be proved be showing that $z(t) = [x_2(t), \ldots, x_n(t)]^T$ and $x_1(t)$ are bounded and reach the neighborhood Ω defined in Theorem 6.2 in finite time. After this finite time, exponential convergence of $x(t)$ is ensured by Theorem 6.2. A function $h_x(\|x_(t_0)\|)$ is constructed by using expressions for the finite time and the bounds on $\|x(t)\|$ and $h(\|x(t_0)\|)$ in Theorem 6.2.

From Theorem 6.1 we have for all positive constants σ_z

$$\forall t > T_z + t_0, \quad \|z(t)\| < \sigma_z, \qquad T_z \triangleq \begin{cases} \frac{1}{\gamma_z} \ln \frac{h_z(\|x(t_0)\|)}{\sigma_z} & , \sigma_z \leq h_z(\|x(t_0)\|) \\ 0 & , \sigma_z > h_z(\|x(t_0)\|) \end{cases} \tag{6.77}$$

since $h_z(\|x(t_0)\|) \geq h_z(\|z(t_0)\|)$.

Integrating $\dot{x}_1 = u_1 = k(x(t_i))f(t)$ from t_i to t_{i+1} gives

$$|x_1(t_{i+1})| = \begin{cases} G(\|z(t_i)\|) & , |k(x(t_i))| < K \\ |\left(|x_1(t_i)| - \frac{K}{\beta}\right)| & , |k(x(t_i))| = K \end{cases} \tag{6.78}$$

Since $\dot{x}_1 = kf(t)$, we can show from the definition of k, (6.57), by using the convergence property of $z(t)$, Theorem 6.1, that

$$\begin{aligned} \max_{t \geq t_0} |x_1(t)| &\leq |x_1(t_0)| + G(h_z(\|z(t_0)\|)) \\ &\leq \|x(t_0)\| + G(h_z(\|x(t_0)\|)) \triangleq x_{1m}(\|x(t_0)\|) \end{aligned} \tag{6.79}$$

since $\|x(t_0)\| \geq \|z(t_0)\|$, and h_z and G are of class $\mathcal{K}$. We assume that σ_z is chosen small enough such that

$$\|z\| < \sigma_z \quad \Rightarrow \quad G(h_z(\|z\|)) < \frac{K}{2\beta} \tag{6.80}$$

Theorem 6.1 and (6.77) imply that for $t > T_z + t_0$, $G(h_z(\|z(t)\|)) < \frac{K}{2\beta}$. This implies that for $t > T_z + t_0$, $G(\|z(t)\|)) < \frac{K}{2\beta}$ since $h_z(\cdot)$ and $G(\cdot)$ are of class $\mathcal{K}$.

We can find a finite time $T_1 + T_z + t_0$ such that $|x_1(t_i)| < \frac{K}{2\beta}$ for all $t_i \geq T_1 + T_z + t_0$. Note from (6.64) and (6.65) that if $x(t_m) \in \Omega$, then

$$|k(x(t_i))| < K, \quad \forall t_i \in \{t_m, t_{m+1}, \ldots\}$$

Therefore, since $G(h_z(\|z(t)\|)) < \frac{K}{2\beta}$ for $t > T_z + t_0$, then having $|x_1(t_m)| < \frac{K}{2\beta}$ for a $t_m > T_z + t_0$ implies that $|k(x(t_i))| < K$ for all $t_i \in \{t_m, t_{m+1}, \ldots\}$. If $|k(x(t_m))| < K$ then $|x_1(t_{i+1})| = G(\|z(t_i)\|) \leq G(\sigma_z)$ for all $t_i \in \{t_m, t_{m+1}, \ldots\}$, (6.78) which implies that $x(t_m) \in \Omega$. Since $|k(x(t_i))| < K$ for all $t_i \in \{t_m, t_{m+1}, \ldots\}$ and $G(\|z(t)\|)) < \frac{K}{2\beta}$, then $x(t_i) \in \Omega$ for all $t_i \in \{t_m, t_{m+1}, \ldots\}$.

Denote by t_z the smallest element in the sequence $(t_0, t_1, \ldots)$ such that $t_z \geq T_z + t_0$. Hence, $G(\|z(t)\|)) < \frac{K}{2\beta}$, (6.80). If $|x_1(t_z)| \leq \frac{K}{2\beta}$ then $|k(t_z)| < K$ and Eq. (6.78) implies that

$$|x_1(t_i)| = G(\|z(t_i)\|) \leq G(\sigma_z), \quad \forall t_i \in \{t_{z+1}, t_{z+2}, \ldots\} \tag{6.81}$$

If $\frac{K}{2\beta} < |x_1(t_z)| < \frac{K}{\beta}$ then Eq. (6.78) implies that

$$|x_1(t_{z+1})| = G(\|z(t_z)\|) \leq G(\sigma_z) \quad \text{or} \quad |x_1(t_{z+1})| = |\left(|x_1(t_z)| - \frac{K}{\beta}\right)| \leq \frac{K}{2\beta}$$

Combining this with Eq. (6.81) implies that if $|x_1(t_z)| < \frac{K}{\beta}$ then

$$|x_1(t_i)| = G(\|z(t_i)\|) \leq G(\sigma_z), \quad \forall t_i \in \{t_{z+2}, t_{z+3}, \ldots\}$$

If $|x_1(t_z)| \geq \frac{K}{\beta}$, then $|k(x(t_z))| = K$ and Eq. (6.78) gives

$$|x_1(t_{z+1})| = |x_1(t_z)| - \frac{K}{\beta}$$

By continuing this sequence, we can find a finite time T_1 such that

$$|x_1(t_i)| \leq G(\sigma_z), \quad \forall t_i > T_1 + T_z + t_0$$

The finite time T_1 will be given by

$$2T + \frac{T\beta}{K}|x_1(t_z)| \leq 2T + \frac{T\beta}{K}x_{1m}(\|x(t_0)\|) \stackrel{\triangle}{=} T_1(\|x(t_0)\|)$$

where $T = t_{i+1} - t_i$ and $x_{1m}(\|x(t_0)\|)$ is given by (6.79). Therefore, we have shown that

$$\|x(t)\| = |x_1(t)| + \|z(t)\| \leq G(\sigma_z) + G(h_z(\sigma_z)), \quad \forall t > \tau(\|x(t_0)\|) + t_0$$

where $\tau(\|x(t_0)\|) \stackrel{\triangle}{=} T_1 + T_z$. We define

$$\sigma \stackrel{\triangle}{=} G(\sigma_z) + G(h_z(\sigma_z))$$

where σ_z is assumed to satisfy (6.80). We now choose σ_z, and σ, small enough such that (6.80) is satisfied and

$$\|x\| < \sigma \quad \Rightarrow \quad \|x\| \in \Omega$$

where Ω is defined in Theorem 6.2. Thus, if $\|x(t_0)\| < \sigma$, then Theorem 6.2 implies

$$\|x(t)\| \leq h(\|x(t_0)\|)e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0 \tag{6.82}$$

If $\|x(t_0)\| \geq \sigma$, then $\|x(t)\| < \sigma$ for all $t > \tau(\|x(t_0)\|) + t_0$. Theorem 6.2 then implies

$$\|x(t)\| \leq h(\|x(t_0)\|)e^{-\gamma(t-\tau(\|x(t_0)\|)-t_0)}, \quad \forall t > \tau(\|x(t_0)\|) + t_0 \tag{6.83}$$

From (6.79) and Theorem 6.2 we have

$$\|x(t)\| \leq x_{1m}(\|x(t_0)\|) + h_z(\|x(t_0)\|), \quad \forall t \geq t_0$$

since $h_z(\cdot)$ is of class $\mathcal{K}$. The bound in Eq. (6.83) can then be extended to an exponential bound for all $t \geq t_0$:

$$\|x(t)\| \leq [h(\|x(t_0)\|) + x_{1m}(\|x(t_0)\|) + h_z(\|x(t_0)\|)]e^{\gamma\tau(\|x(t_0)\|)}e^{-\gamma(t-t_0)} \tag{6.84}$$

Eq. (6.82) implies, $\forall\|x(t_0)\| < \sigma$, $\forall t \geq t_0$,

$$\|x(t)\| \leq [h(\|x(t_0)\|) + x_{1m}(\|x(t_0)\|) + h_z(\|x(t_0)\|)]e^{\gamma\tau(\|x(t_0)\|)}e^{-\gamma(t-t_0)} \tag{6.85}$$

Combining (6.84) and (6.85) gives

$$\forall\|x(t_0)\| \in \mathbb{R}^n, \qquad \|x(t)\| \leq h_x(\|x(t_0)\|)e^{-\gamma(t-t_0)}, \qquad \forall t \geq t_0$$

where

$$\begin{aligned}
h_x(\|x(t_0)\|) &\stackrel{\triangle}{=} [h(\|x(t_0)\|) + x_{1m}(\|x(t_0)\|) + h_z(\|x(t_0)\|)]e^{\gamma\tau(\|x(t_0)\|)} \\
x_{1m}(\|x(t_0)\|) &= \|x(t_0)\| + G(h_z(\|x(t_0)\|)) \\
\tau(\|x(t_0)\|) &= T_1(\|x(t_0)\|) + T_z(\|x(t_0)\|) \\
T_1(\|x(t_0)\|) &= 2T + \frac{T\beta}{K}x_{1m}(\|x(t_0)\|) \\
T_z(\|x(t_0)\|) &= \max\{\frac{1}{\gamma_z}\ln\frac{h_z(\|x(t_0)\|)}{\sigma_z}, 0\}
\end{aligned}$$

The function $h_z(\|x(t_0)\|)$ and γ_z are given from Theorem 6.1, and $h(\|x(t_0)\|)$ and γ are given from Theorem 6.2. The function $G(\cdot)$ is defined in (6.58) and the constant σ_z is chosen small enough such that

$$\|x\| < \sigma \stackrel{\triangle}{=} G(\sigma_z) + G(h_z(\sigma_z)) \quad \Rightarrow \quad \|x\| \in \Omega$$

where Ω is defined in Theorem 6.2. We see that $h_x(\cdot)$ is of class $\mathcal{K}$ since the functions $h(\cdot)$, $x_{1m}(\cdot)$, $h_z(\cdot)$ are of class $\mathcal{K}$ and $e^{\gamma\tau(\cdot)}$ is strictly increasing.

$\square$

6.7 Simulations

A simulation with $n = 4$ was done in MATLAB at a SPARC Station 1. The control law for u_1 was chosen as follows:

$$u_1 = k(x(t_i))\,f(t), \qquad f(t) = (1-\cos t)/2$$

where

$$k = \mathrm{sat}\,(-[x_1(t_i) + \mathrm{sgn}(x_1(t_i))G(\|z(t_i)\|)]\beta, K), \qquad K = 2$$

as defined in (6.57). By studying the time integrals of $f^3(t)$ and $f^5(t)$, we see that Property P4 of $f(t)$ is satisfied by choosing

$$\eta_3 = \frac{5}{11}, \quad \eta_4 = \frac{63}{256}, \quad P_3 = \frac{1}{2}, \quad P_4 = \frac{1}{2}$$

The controller parameter κ in $G(\cdot)$, (6.58), was taken to $\kappa = 3$. The constant β is given by (6.59):

$$\beta = 1/\int_0^T f(\tau)d\tau = \frac{1}{\pi}$$

where T is the time-period of the function $f(t)$, i.e. $T = 2\pi$. The instants of time t_i where $k(x(t_i))$ may switch is given by the set $\{0, 2\pi, 4\pi, 6\pi, \ldots\}$, where t_0 is taken to zero.

We find the control law for u_2 from (6.31), (6.14)-(6.18) and (6.10):

$$\begin{aligned} u_2 = &- \left(\lambda(1 + f^3 + f^5)\right) x_2 \\ &- \left(f\lambda(f\lambda + f^3\lambda + f^6\lambda + 2\dot{f} + 8f^2\dot{f})/k\right) x_3 \\ &- \left(f\lambda(f^5\lambda^2 + 4f\lambda\dot{f} + 6f^4\lambda\dot{f} + 8\dot{f}^2 + 4f\ddot{f})/k^2\right) x_4 \end{aligned}$$

Here, we have chosen

$$\lambda = \lambda_2 = \lambda_3 = \lambda_4$$

In the simulation, λ was taken to $\lambda = 1$. This expression for u_2 is easily generated by the following *Mathematica*-code:

```
u1 := k f
x3 /: Dt[x3] = x2 u1
x4 /: Dt[x4] = x3 u1
x2d:= -l f^2 (x3-x3d)/k + Dt[x3d]/(k f)
x3d:= -l f^4 x4 / k
SetAttributes[l, Constant]
SetAttributes[k, Constant]
u2=-l (x2-x2d)+Dt[x2d]
```

The initial state was chosen as follows:

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0, -0.1, 0.1, 1)$$

Euler's method was applied for the numerical integration where the time-step was taken to 0.05. In *Figure 6.2*, $z(t) = [x_2(t), x_3(t), x_4(t)]^T$ is plotted versus time showing exponential convergence to zero. The state variable $x_1(t)$, is plotted in *Figure 6.3*. We see that $x_1(t)$ converges exponentially to zero, too. Note, however, from the time-axes that the rate of convergence of $x_1(t)$ is slower than the one of $z(t)$. This coincides with the relation between γ and γ_z in Theorem 6.2 where $\gamma = \gamma_z/4$.

By using the coordinate transformation from Chapter 5, we can interpret the variables x_1 and x_4 as the x- and y-position of the midpoint of the rear axle of a four-wheeled car. The path $(x(t), y(t))$ is presented in *Figure 6.4*. We see that the motion seems natural when interpreted as a parking maneuver.

In *Figures 6.5* and 6.6 the inputs $u_1(t)$ and $u_2(t)$ are shown as functions of time. We see that they are continuous and converge exponentially to zero.

A similar simulation was done with the smooth, time-varying approach presented by Teel et al. (1992):

$$u_1 = -x_1 - \text{sat}\left((\xi_3^2 + \xi_4^2)^{\frac{1}{2}}, \varepsilon\right)^2 (\sin t - \cos t) \tag{6.86}$$

$$u_2 = -x_2 - c_1 \text{sat}(\xi_3, \varepsilon)\cos t - c_2 \text{sat}(\xi_4, \varepsilon)\cos 2t \tag{6.87}$$

where

$$\xi_3 = -x_3 + x_1 x_2, \quad \xi_4 = x_4 - x_1 x_3 + \frac{1}{2} x_1^2 x_2$$

and c_1 and c_2 are positive constants. The controller parameters were chosen as in the simulation study of Teel et al. (1992):

$$c_1 = c_2 = 2, \quad \varepsilon = 0.5$$

The resulting time-plot of $x_4(t)$ is presented in *Figure 6.7.* By comparing this plot with the corresponding plot of $x_4(t)$ from the exponentially convergent control law (*Figure 6.2*), we see that the smooth, time-varying approach converges considerably slower than the approach proposed in this chapter. (Note the time scales in these two figures.) The resulting "parking maneuver" from the smooth, time-varying approach is presented in *Figure 6.8* where the simulation was run from 0 to 100 time units. By comparing this path with the path in *Figure 6.4*, we see that the approach proposed in this chapter leads to a more appealing path when interpreted as the result of a parking maneuver. In addition, we see that the distance to the desired final configuration is much smaller with the approach presented here, after the same number of time units.

The inputs $u_1(t)$ and $u_2(t)$ from the control law (6.86)-(6.87) are presented in *Figures 6.9 and 6.10.* We see that the magnitude of the maximal values of these inputs are approximately of the same order as the inputs in *Figures 6.5 and 6.6.* However, the slow convergence of the smooth, time-varying approach is again illustrated by the slow convergence of the inputs $u_1(t)$ and $u_2(t)$.

6.8 Conclusions

A feedback control law has been proposed to globally stabilize a chained nonholonomic system with any nonholonomic degree. The resulting rate of convergence is exponential. A new definition of stability with exponential convergence, termed $\mathcal{K}$-exponential stability, has been introduced here. $\mathcal{K}$-exponential stability is a weaker form of stability than exponential stability in the usual sense but it possesses the same rate of convergence.

The feedback law depends on a time-periodic function and on a parameter which varies with the state at discrete instants of time. In the design of this control law, reference variables for the state are defined which make the total state exponentially converge to zero. The rate of convergence can be arbitrary by choosing the controller parameters appropriately. Simulation studies showed that this approach was superior to a smooth, time-varying feedback approach with respect to the rate of convergence.

The idea of introducing parameters which varies with the state at discrete instants of time seems, therefore, to be a useful approach to obtain good stability properties for nonholonomic systems. The control law has a simple structure, though the stability analysis is quite involved because of few existing mathematical tools for such systems.

A drawback of this approach is the possibility for numerical problems at digital computers as the parameter $k(x(t_i))$ converges to zero, since there are divisions by $k(x(t_i))$ in the control law for u_2. (These divisions do not cause unbounded quantities, since $k(x(t_i))$ always dominates the numerator.) Such numerical problems were, however, not observed during the simulations. Since the parameter $k(x(t_i))$ only changes at $t_i \in \{t_0, t_1, \ldots\}$, the control of x_1 may be more sensitive to disturbances and modeling imperfections than if $k = k(x(t))$, i.e. function of the state at all $t \geq t_0$. Depending on the physical application, feedback from the state only at $t_i \in \{t_0, t_1, \ldots\}$ may be sufficient to make x_1 exponentially converge to zero. Note that the state variables $z = [x_2, \ldots, x_n]^T$ are controlled by the control law $u_2 = u_2(z(t), k)$ which depends on $z(t)$ for all $t \geq t_0$.

In further work, an extension from $k = k(x(t_i))$ to $k = k(x(t))$ by redefining x^d, (6.14)-(6.18), can be studied. The structure of $k(x(t_i))$ in the present control law indicates that the use of $k = k(x(t))$ results in a non-smooth, time-varying feedback law. More mathematical tools are needed to analyze such systems. The need for the definition of $\mathcal{K}$-exponential stability for the stabilization with exponential convergence of chained systems and other nonholonomic systems should also be analyzed.

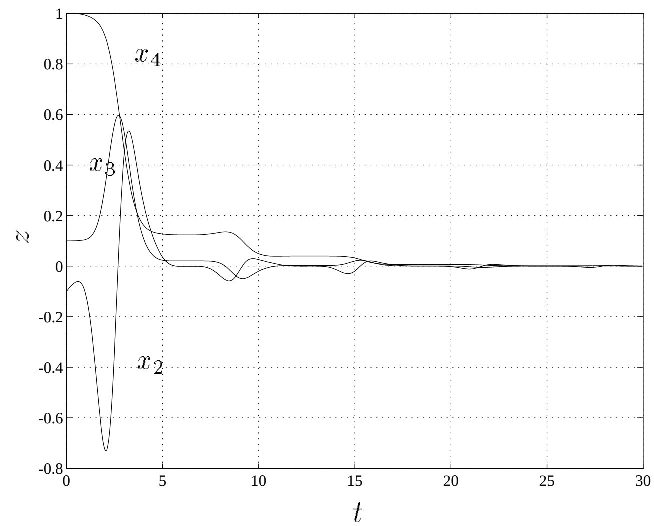

Figure 6.2: Exponential convergence of $z(t) = [x_2, x_3, x_4]^T$ to zero.

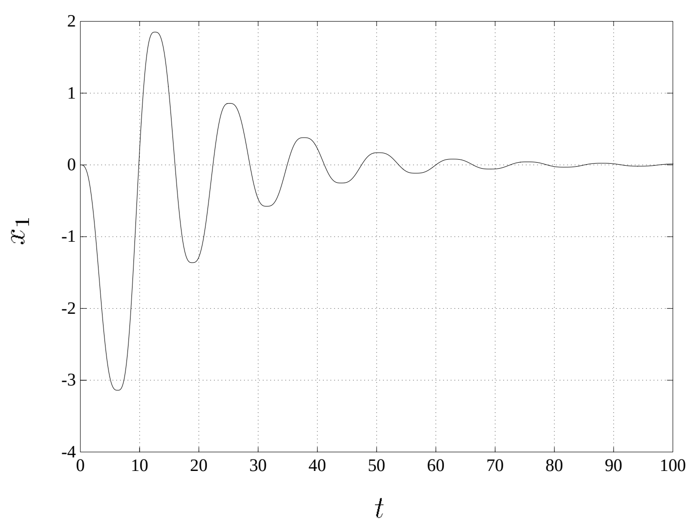

Figure 6.3: Exponential convergence of $x_1(t)$ to zero.

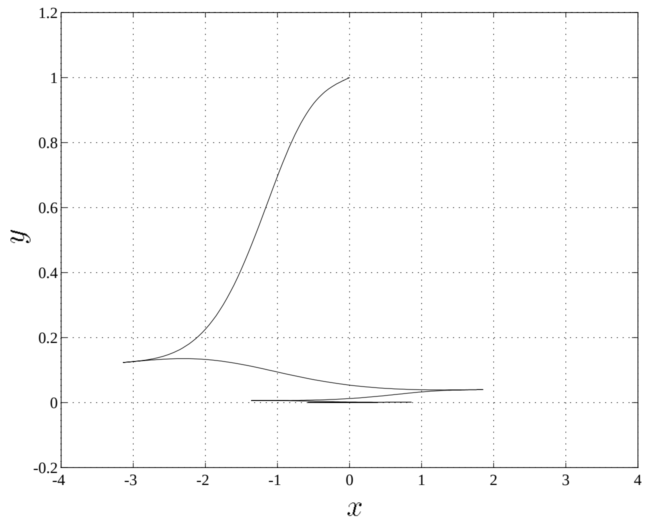

Figure 6.4: The resulting path in the xy-plane when applying the exponential convergent control law. The variables $x = x_1$ and $y = x_4$ are interpreted as the planar position of a four-wheeled car.

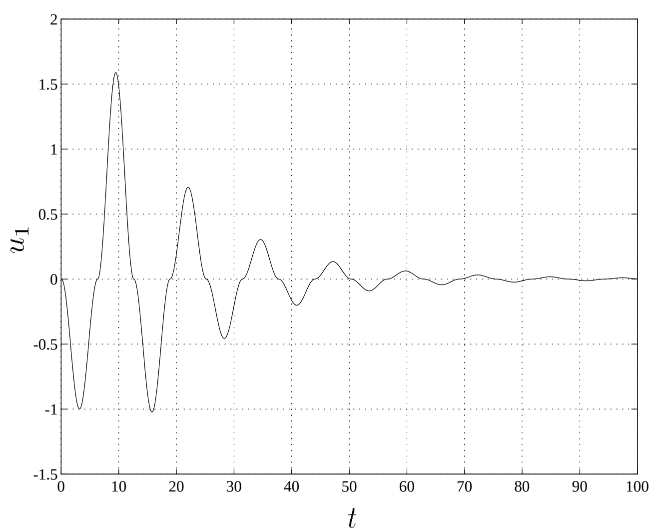

Figure 6.5: The input $u_1(t)$ from the exponentially convergent control law.

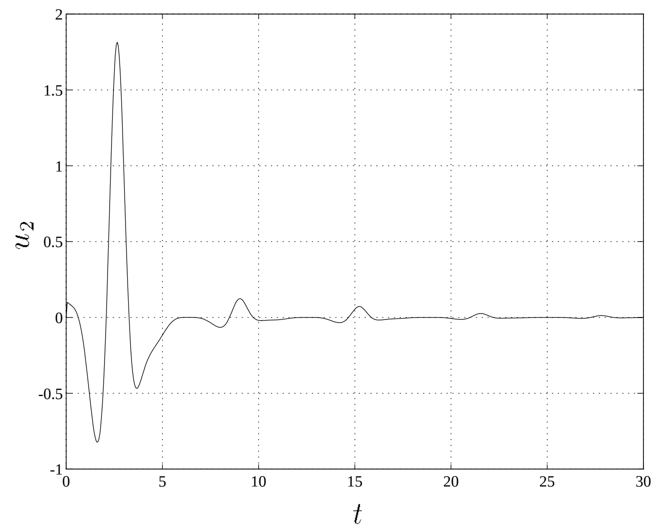

Figure 6.6: The input $u_2(t)$ from the exponentially convergent control law.

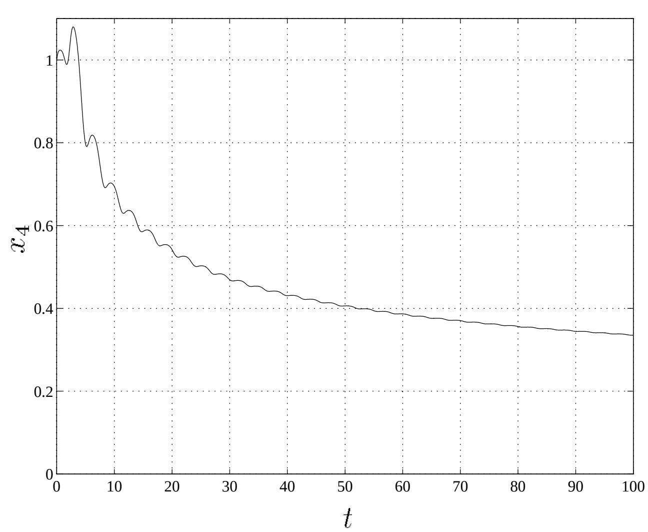

Figure 6.7: Asymptotically convergence of $x_4(t)$ to zero resulting from a smooth, time-varying feedback law.

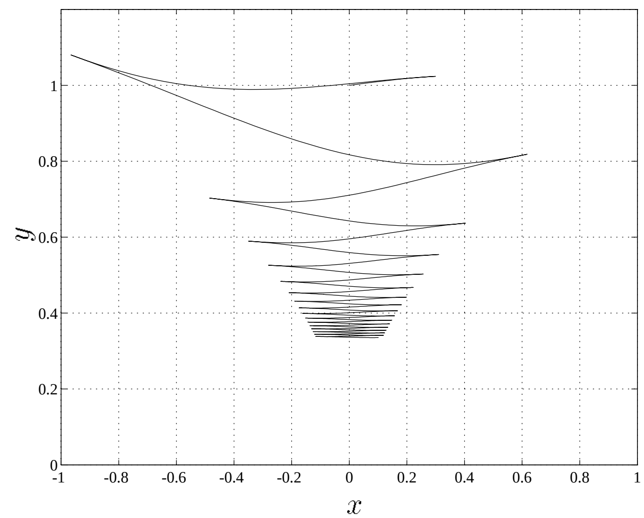

Figure 6.8: The resulting path in the xy-plane from a smooth, time-varying feedback law where $x = x_1$ and $y = x_4$ are interpreted as the planar position of a four-wheeled car.

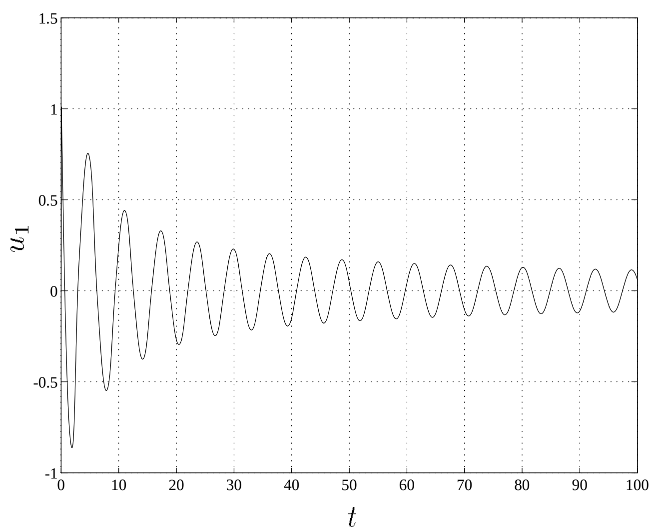

Figure 6.9: The input $u_1(t)$ from the smooth, time-varying control law.

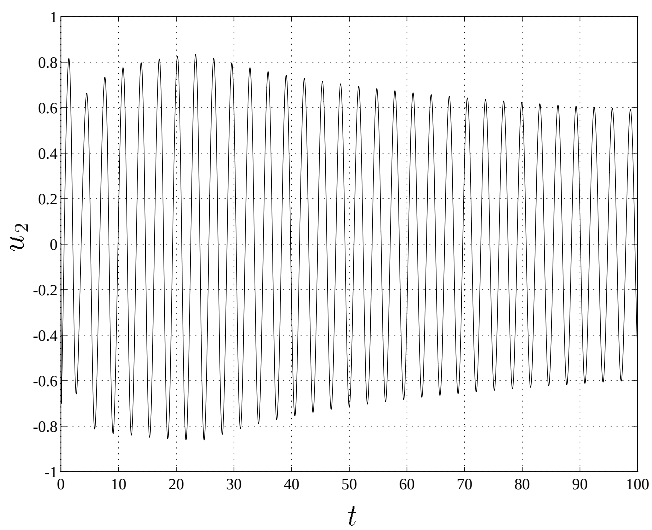

Figure 6.10: The input $u_2(t)$ from the smooth, time-varying control law.

Chapter 7

Conclusions

A piecewise analytic feedback controller has successfully been applied to obtain *global exponential* convergence to any configuration in $\mathbb{R}^2 \times S^1$ for a cart with one nonholonomic constraint. Simulations indicated that this approach is robust to measurement noise, model errors, and dynamic extension, i.e. torque inputs instead of velocity inputs. The resulting paths from this approach were intuitively more appealing than typical paths with many cusps from a smooth time-varying approach. In addition, the rate of convergence of the piecewise analytic feedback controller was superior to the convergence of the smooth, time-varying feedback controller studied. Therefore, using a piecewise analytic feedback controller seems to be a promising approach to obtain global exponential convergence for nonholonomic mobile robots. More research is needed to extend this approach to the control of other types of mobile robots.

The piecewise analytic feedback controller has been extended to make a cart (globally) follow a path composed of straight lines and arcs of circles with arbitrary accuracy. This approach allows for stopping and reversing phases. The convergence to the terminal configuration was exponential. For the path-planning, more research can be done to optimally choose the configurations that join the path segments.

A kinematic model of a car with n trailers has been developed to locally convert the kinematic model into a chained form suitable for control. The proposed conversion of the car/trailer system into a chained form makes it possible to use control strategies developed for chained systems to control a car with n trailers. In particular, a stabilizing feedback law for a chained system can be used to locally stabilize a car with n trailers. The local assumption on the orientations can be satisfied in finite time by driving the pulling car along the x-axis in a positive direction. By switching between forward motions, to obtain small orientations, and local stabilizing to intermediate points, global convergence to any position and orientation can be obtained within limited area. Developing such a point planner and switching strategy can be a field for future research.

A feedback law has been proposed to globally stabilize a chained nonholonomic system with two inputs, resulting in exponential convergence. The system may have any nonholonomic degree. The feedback law depends on a time-periodic function and on a parameter which varies with the state at discrete instants of time. By combining this control law with the conversion of the n-trailer system into a chained form, a car with n trailers can be locally stabilized with exponential convergence. A new definition of stability with exponential convergence has been introduced here. This type of stability has been called $\mathcal{K}$-exponential stability and seems to be a useful notion for chained nonholonomic systems. The need for this definition for the stabilization with exponential convergence of chained systems and other nonholonomic systems should be analyzed further in future research.

Consequently, this work shows how good convergence to a given configuration can be obtained for nonholonomic mobile robots by using

- piecewise analytic feedback law for global exponential convergence
- time-varying feedback law that also depends on the state at discrete instants of time for local stabilization with exponential convergence.

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Appendix A

Nonholonomic Systems

A mechanical system can be subject to constraints. The concept of nonholonomic constraints and nonholonomic systems is presented in the following.

Let the mechanical system be represented by n particles where $q_1, \ldots q_n$ are the position coordinates of the particles. If the constraints can be expressed as functions of the coordinates of the particles (and possibly the time) having the form

$$f_i(q_1, \ldots, q_n; t) = 0 \ , \quad i = 1, \ldots, m \tag{A.1}$$

then the constraints are said to be *holonomic*, (Goldstein 1980). An example of such a system is 2 particles connected by a rigid bar. Another example is a particle constrained to move on a given surface.

Constraints which cannot be expressed in this form are called *nonholonomic*. The nonholonomic constraints may be given as follows:

$$f_i(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t) = 0 \ , \quad i = 1, \ldots, m \tag{A.2}$$

These kinematic constraints are said to be two-sided and restrictive or limiting. They may be classified as time-dependent or time-independent, depending on whether or not they contain the time explicitly.

The constraints (A.2) are *nonintegrable* if the corresponding systems of differential equations (A.2) cannot be reduced to the form (A.1). Similarly to Neĭmark & Fufaev (1972) p. 6, and the usual notion in robotics, we define:

> A mechanical system with nonintegrable kinematic constraints is called a *nonholonomic system.*

More generally, nonintegrable differential constraints may involve higher order derivatives.

A system consisting of N particles, free from constraints has $3N$ independent coordinates or *degrees of freedom*. If there exist holonomic constraints, expressed in m equations in the form (A.1), then we may use these equations to eliminate m of the $3N$ coordinates, resulting in $3N-m$ independent generalized coordinates. The system has then $3N-m$ degrees of freedom.

In the case of nonholonomic constraints, the equations expressing the constraints cannot be used to eliminate any of the coordinates, (Goldstein 1980). For a nonholonomic system, the number of degrees of freedom is less than the number of independent generalized coordinates by the number of nonintegrable kinematic constraints, (Neĭmark & Fufaev 1972) p. 13, i.e.

$$n = 3N - m, \qquad p = n - r$$

where n is the number of independent generalized coordinates, r is the number of nonholonomic constraints, and p is the degrees of freedom which corresponds to the number of independent velocities.

Nonholonomic constraints encountered in mechanics can usually be expressed in a form which is linear in the generalized velocities

$$a_i(q_1, \ldots, q_n; t)\dot{q}_i + b_i(q_1, \ldots, q_n; t) = 0, \quad i = 1, \ldots, m$$

If $[b_1, \ldots, b_m] \equiv 0$ then the constraints are said to be homogeneous. They are independent of the time if the coefficients a_i and b_i do not contain t explicitly.

All the nonholonomic constraints studied in this thesis are of the form

$$A(q)\dot{q} = 0$$

A simple example of a nonholonomic system is a vertical disk rolling on a horizontal plane without slipping. The coordinates describing the motion may be the x- and y-coordinates indicating the position, the orientation θ and the angle of rotation ϕ, as illustrated in *Figure A.1* from Goldstein (1980). Because of the non-slipping condition, the magnitude of the velocity of the center of the disk, v, is proportional to $\dot{\phi}$,

$$v = a\dot{\phi}$$

where a is the radius of the disk. The direction of the velocity is perpendicular to the axis of the disk,

$$\begin{aligned} \dot{x} &= v \sin\theta \\ \dot{y} &= -v \cos\theta \end{aligned}$$

These conditions give the following two nonholonomic constraints:

$$\begin{aligned}\dot{x} - a \sin\theta\dot{\phi} &= 0 \\ \dot{y} + a \cos\theta\dot{\phi} &= 0\end{aligned}$$

These constraints cannot be integrated to find a holonomic constraint

$$f(x, y, \theta, \phi) = 0$$

As an example of the use of a disk, or wheel, with a nonholonomic constraint, we consider a simple planimeter of A. N. Krylov, (Neĭmark & Fufaev 1972) p. 26. It consists of a rod with a cutting wheel attached at one end and a guiding pin at the other. In order to measure a given area with this device, one must start with the guiding pin at the approximate center of the area and the guide it along the contour as shown in *Figure A.2*. Because of the nonholonomic constraint of the rolling wheel, the planimeter will be turned an angle θ at the completion of the circuit. The area inside the contour is then approximately given by

$$S \approx l^2\theta$$

where l is the distance between the guiding pin and the point of contact of the cutting wheel.

Other examples of nonholonomic systems in mechanics are wheeled mobile robots and rotating bodies like spacecrafts with a manipulator. The feedback control of wheeled nonholonomic mobile robots is studied in thesis.

Figure A.1: Vertical disk rolling on a horizontal plane (Goldstein 1980).

Figure A.2: The planimeter of A. N. Krylov, (Neĭmark & Fufaev 1972) p. 26.

Appendix B

Controllability and Stabilizability

B.1 Introduction

This chapter presents and summarizes some definitions and theorems concerning controllability and stabilizability often encountered in the study of nonholonomic systems. Typically, such systems are controllable but not stabilizable by a smooth static state feedback law as opposed to linear systems. The theory presented here is also applicable for more general nonlinear systems. The references for this study are presented in Section B.5. This presentation is based on Sørdalen (1993*a*).

B.2 Preliminaries

This section presents some elementary notions in differential geometry. For a more complete survey of this subject the reader is referred to the references in Section B.5.

Definition B.1 *Let S be a set. A* **topological structure**, *or a* **topology**, *on S is a collection T of subsets of S, called* **open sets**, *satisfying the axioms*

1. *the union of any number of open sets is open*
2. *the intersection of any finite number of open sets is open*

3. *the set S and the empty set $\emptyset$ are open*

A set S together with a topology is called a **topological space**. A **basis** for a topology T on S is a collection $B_T \subset T$ such that every open set can be written as a union of elements in B_T. A **neighborhood** of a point p of a topological space is any open set which contains p. A topological space is **Hausdorff** if any two different points p_1 and p_2 have disjoint neighborhoods. A **locally Euclidean space** X of dimension n is a topological space such that there is a homeomorphism ϕ mapping some neighborhood of p onto an open set in R^n.

Definition B.2 *A* **manifold** *M of dimension n is a topological space which is locally Euclidean of dimension n, is Hausdorff and has a countable basis.*

A **coordinate chart** on a manifold M is a pair (U, ϕ), where U is an open set of M and ϕ a homeomorphism of U onto an open set of R^n. Two coordinate charts (U, ϕ) and (V, ψ) are called **C^∞-compatible** if $U \cap V = \emptyset$, or in the case $U \cap V \neq \emptyset$, the coordinate transformation $\psi \circ \phi^{-1}$ is a diffeomorphism. A C^∞ **atlas** on a manifold M is a collection $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ of pairwise C^∞-compatible coordinate charts with the property that $\bigcup_{i \in I} U_i = M$. An atlas is **complete** if not properly contained in any other atlas.

Definition B.3 *A* **smooth** *(or C^∞)* **manifold** *is a manifold equipped with a complete C^∞ atlas.*

A function $F : U \subset \mathbb{R}^m \to \mathbb{R}$ is **analytic** if it is expandable in a power series in its arguments about each point of U. We can define analytic manifolds, analytic mappings of manifolds, and so on by assuming that the functions, mappings, etc. are analytic instead of only C^∞.

Let the set of all smooth functions in a neighborhood of p be denoted $C^\infty(p)$.

Definition B.4 *A* **tangent vector** *ν_p at p is a map*

$$\nu_p : C^\infty(p) \to \mathbb{R}$$

with the following properties for all λ, $\gamma \in C^\infty(p)$ and a, $b \in \mathbb{R}$:

- $\nu_p(a\lambda + b\gamma) = a\nu_p(\lambda) + b\nu_p(\gamma)$

- $\nu_p(\lambda\gamma) = \gamma(p)\nu_p(\lambda) + \lambda(p)\nu_p(\gamma)$

Definition B.5 *The* **tangent space** *to the manifold M at p, written T_pM, is the set of all tangent vectors at p.*

The **tangent bundle** of a smooth manifold M is defined as

$$TM = \cup_{p \in M} T_pM$$

Definition B.6 *Let M and N be smooth manifolds. Let $F : M \to N$ be a smooth mapping. The* **differential** *of F at $p \in M$ is the map*

$$F_{*p} : T_pM \to T_{F(p)}N$$

defined as follows. For $\nu_p \in T_pM$ and $\lambda \in C^\infty(F(p))$,

$$(F_{*p}(\nu_p))(\lambda) = \nu_p(\lambda \circ F)$$

Definition B.7 *Let M be a smooth manifold of dimension n. A* **vector field** *f on M is a mapping assigning to each point $p \in M$ a tangent vector $f(p)$ in T_pM.*

A smooth curve $\sigma : (t_1, t_2) \to M$ is an **integral curve** of f, if

$$\dot{\sigma}(t) = f(\sigma(t)), \quad \forall t \in (t_1, t_2) \tag{B.1}$$

where

$$\dot{\sigma}(t) = \sigma_{*t}(\frac{d}{dt})_t$$

We define the **flow** of f to be the mapping

$$\Phi_t^f : U \to M \ , \quad t \in (a, b)$$

by letting $\Phi_t^f(p)$ be the solution of the differential equation (B.1) at time t with the initial condition $\sigma(0) = p$. Here, U is a bounded set in M.

Definition B.8 *Let f be a smooth vector field on M and λ a smooth real-valued function on M. The* **Lie derivative** *of λ along f is a function $L_f\lambda : M \to R$ defined as*

$$(L_f\lambda)(p) = (f(p))(\lambda)$$

Definition B.9 *For f and g any (smooth) vector fields on M, we define a new vector field, denoted as $[f, g]$ and called the* **Lie bracket** *of f and g by setting*

$$([f,g](p))(\lambda) = (L_f L_g \lambda)(p) - (L_g L_f \lambda)(p)$$

where λ is a smooth function on M.

$[f, g]$ is skew commutative, bilinear over $\mathbb{R}$ and satisfies the Jacobi identity. If the two vector fields f and g both are defined on an open subset U of $\mathbb{R}^n$, then

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

at each x in U. Let $V^\infty(M)$ denote the set of vector fields on a manifold M, considered as a module over the ring $C^\infty(M)$ of C^∞ real valued functions on M. $V^\infty(M)$ with the product $[f, g]$ is a **Lie algebra**.

A geometrical interpretation of the Lie bracket is given by the following situation. Let a nonlinear system be given by

$$\dot{x} = f(x)u_1 + g(x)u_2 \tag{B.2}$$

where f and g are smooth vector fields on the configuration manifold. With the initial condition $x(0) = x_0$, let the input be given by

$$u(t) = \begin{cases} (1,0) & t \in [0,h) \\ (0,1) & t \in [h,2h) \\ (-1,0) & t \in [2h,3h) \\ (0,-1) & t \in [3h,4h) \end{cases} \qquad h > 0$$

From a straightforward Taylor expansion we find

$$[f,g](x_0) = \lim_{h \to 0} \frac{x(4h) - x_0}{h^2} \tag{B.3}$$

where $x(4h)$ is the solution of (B.2) at time $4h$.

Definition B.10 *A* **distribution** *D on a manifold M is a map which assigns a linear subspace $D(p)$ of the tangent space T_pM to each $p \in M$. D is called a* **smooth distribution** *if*

$$\forall p \in M \ \exists U_p \ \exists \{f_i;\ i \in I\} \mid \forall q \in U_p \ \ D(q) = \operatorname{span}\{f_i;\ i \in I\}$$

where f_i is a smooth vector field, U_p is a neighborhood of p and I is an index set.

Definition B.11 *A distribution D is called* **involutive** *if*

$$f, g \in D \Rightarrow [f, g] \in D$$

A distribution D defined on an open set U is **nonsingular** if

$$\forall p \in U \; \exists d \mid \dim D(p) = d$$

where d is an integer.

A vector field f on M is called **admissible** with respect to the distribution D if $\forall x \in M, \;\; f(x) \in D(x)$. For every point $x \in M$ we construct a chain

$$D(x) = D_1(x) \subset D_2(x) \subset \cdots \tag{B.4}$$

of linear spaces in a tangent space $T_x M$ defining $D_i(x)$ as a linear envelope of all the values of vector fields that can be represented by Lie brackets of length $\leq i$, of admissible vector fields, (Gershkovich & Vershik 1988). This means that

$$D_2 = [D_1, D_1], \ldots, D_i = [D_{i-1}, D_1]$$

The notation $[D_{i-1}, D_1]$ means the distribution defined by all Lie brackets of vector fields in D_{i-1} and vector fields in D_1. By a **growth vector** of a distribution D at a point x we mean a sequence of integers $\{n_i(x)\}$, where $n_i(x) = \dim D_i(x)$. The distribution is **regular** if

$$\forall i \, \forall x \in M \quad n_i(x) = k_i$$

where k_i is a constant.

The distribution D (B.4) is **completely nonholonomic** if for some i_0, $D_i = TM$ for all $i \geq i_0$ where TM is the tangent bundle. The smallest such i_0 is called the **nonholonomic degree** of the distribution D, (Gershkovich & Vershik 1988).

B.3 Controllability

In this section we will consider smooth nonlinear systems which are affine in control

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j \tag{B.5}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ are *local* coordinates for a smooth state space manifold M, and $u = [u_1, \ldots, u_n]^T \in U \subset \mathbb{R}^m$. The vector fields $f, g_1, \ldots, g_m$ are smooth on M. f is a **drift** vector field and g_j, $j \in \{1, \ldots, m\}$, are **input** vector fields.

Given a point $x_0 \in M$, we want to say something about the set of points which can be reached from x_0 in finite time by a suitable choice of the input functions $u_j(\cdot)$, $j \in \{1, \ldots, m\}$. This is the controllability (or reachability) problem.

For a linear, time-invariant system

$$\dot{x} = Ax + Bu, \quad u \in \mathbb{R}^m, \quad x \in \mathbb{R}^n \tag{B.6}$$

we have the well-known Kalman rank condition for controllability

$$\mathrm{rank}[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B] = n$$

(B.6) is controllable if and only if the Kalman rank condition is satisfied.

We would like to have similar conditions for controllability in some sense for nonlinear systems. In order to work out algebraic conditions for controllability, we make the following assumptions concerning the input space U and the class of admissible controls Υ:

- The input space U is such that the linear span of the set $\mathcal{F}$ of associated vector fields of the system (B.5)

$$\mathcal{F} = \{f + \sum_{j=1}^{m} g_j\, u_j \mid [u_1, \ldots, u_n]^T \ \in U\}$$

  contains the vector fields $f, g_1, \ldots, g_m$.

- Υ consists of piecewise constant functions which are piecewise continuous from the right.

These assumptions are not very restrictive. In Sontag (1990*b*), Section 3.7, and in Theorem 1 in (Sussmann 1979) it is shown that if (B.5) is controllable with u more general in U, then (B.5) is controllable with u restricted to be piecewise constant, too. This follows from standard results on the continuity of solutions of differential equations. Moreover, we assume that a unique solution of (B.5) exists in the time intervals considered. These assumptions apply in the sequel unless otherwise stated.

For nonlinear systems we need several concepts in order to describe the ability of a system to be controlled from state x_1 to state x_2, (or equally the ability of a system to reach state x_2 from x_1). This concerns the time needed and local properties of the trajectories.

The unique solution of (B.5) with initial condition $x(0) = x_0$ at time $t \geq 0$ and with input function $u(\cdot)$ is denoted $x(t, 0, x_0, u)$. We introduce the following definition.

Definition B.12 *The system (B.5) is* **completely controllable** *if*

$$\forall x_1, x_2 \in M \; \exists T < \infty \; \exists u : [0, T] \to U \mid x(T, 0, x_1, u) = x_2$$

This is called **controllability** by Nijmeijer & van der Schaft (1990). This means that every state at M can be reached from every other by using admissible inputs during a finite time interval $[0, T]$.

For (B.5), we denote $R^V(x_0, T)$ as the reachable set from x_0 at time $T > 0$, following trajectories which remain in the neighborhood V of x_0 for $t \leq T$,

$$R^V(x_0, T) = \{x \in M \mid (x(T, 0, x_0, u) = x)\&(\forall t \in [0, T] \; x(t) \in V)\}$$

where $u : [0, T] \to U$ is an admissible input. We define

$$R_T^V(x_0) = \bigcup_{\tau \leq T} R^V(x_0, \tau)$$

Definition B.13 *The system (B.5) is* **locally accessible from** x_0 *if*

$$\forall V \; \forall T > 0 \;\; R_T^V(x_0) \supset \Omega \tag{B.7}$$

where V is an open, non-empty neighborhood of x_0 and Ω is an open, non-empty subset of M. If this holds for any $x_0 \in M$ then the system is called **locally accessible**.

Locally accessible is also called **locally weakly controllable**, (Hermann & Krener 1977).

The system (B.5) is called **small-time local controllable** from x_0 if Ω in (B.7) contains x_0, (Sussmann 1987).

The advantage of local accessibility over other possible forms of controllability, is that it is suitable for a simple algebraic test. This can be done by introducing the accessibility algebra $\mathcal{C}$. The **Lie-algebra** of vector fields on M is denoted $V^\infty(M)$.

Definition B.14 *The* **accessibility algebra** $\mathcal{C}$ *for (B.5) is the smallest subalgebra of $V^\infty(M)$ that contains $f, g_1, \ldots, g_m$.*

This is also called **Control Lie Algebra**, (Isidori 1989). A typical element of $\mathcal{C}$ is a finite linear combination of elements of the form

$$[v_k, [v_{k-1}, [\ldots, [v_2, v_1] \ldots]]]$$

where $k = 0, 1, 2, \ldots$ and v_i, $i \in \{1, \ldots, k\}$, is in the set $\{f, g_1, \ldots, g_m\}$. This algebra is said to be **nilpotent** if there is an integer $k > 0$ with the property that all the Lie brackets $[v_{k+1}, [v_k, [\ldots, [v_2, v_1] \ldots]]]$ vanish, (Lafferriere & Sussmann 1991).

The vector fields which are elements in $\mathcal{C}$ span the accessibility distribution.

Definition B.15 *The* **accessibility distribution** C *of (B.5) is given by*

$$C(x) = \text{span}\,\{v(x) \mid v \in \mathcal{C}\}, \quad x \in M$$

C is the **involutive closure** of span$\{f, g_1, \ldots, g_m\}$.

We now state the following theorem:

Theorem B.1 *Consider the system (B.5). Assume that*

$$\dim\, C(x_0) = n \tag{B.8}$$

Then the system is locally accessible from x_0.

Proof: See Nijmeijer & van der Schaft (1990) p. 80 or Hermann & Krener (1977), Theorem 2.2.

□

We call (B.8) the **accessibility rank condition at** x_0. The following corollary follows from Theorem B.1:

Corollary B.1 *If the accessibility rank condition holds for all $x \in M$, then the system (B.5) is locally accessible.*

Accessibility is not the same as controllability. But in the case of no drift in the system, i.e. $f \equiv 0$, the accessibility rank condition implies controllability by Chow's Theorem, (Chow 1939), (Haynes & Hermes 1970), (Lobry 1970) Proposition 1.3.2.:

Theorem B.2 *Suppose $f \equiv 0$ in (B.5), and let $\mathcal{F}$ be symmetric, i.e. $v \in \mathcal{F} \Rightarrow -v \in \mathcal{F}$. Assume that $\forall x \in M$ $\dim C(x) = n$. Then*

1. *The system (B.5) is locally accessible.*

2. *If M is connected then (B.5) is completely controllable.*

Proof: See Nijmeijer & van der Schaft (1990) p. 83.

□

This result also holds if $\forall x \in M,\ f(x) \in \text{span}\{g_1(x), \ldots, g_m(x)\}$. The following theorem, Proposition 1 in Sussmann (1991), says that the dynamic extension also is completely controllable.

Theorem B.3 *Let $g_1, \ldots, g_m$ be C^∞ vector fields on the C^∞ manifold M. Assume that the control-linear driftless system*

$$\dot{x} = g_1(x)u_1 + \ldots + g_m(x)u_m$$

is completely controllable. Then its dynamic extension

$$\begin{aligned} \dot{x} &= g_1(x)y_1 + \ldots + g_m(x)y_m \\ \dot{y_1} &= u_1 \\ &\vdots \\ \dot{y_m} &= u_m \end{aligned}$$

with state space $M \times R^m$, is completely controllable as well.

Proof: See Sussmann (1991).

□

The system can be given rank conditions for a stronger type of controllability than local accessibility. This is called local strong accessibility:

Definition B.16 *The system (B.5) is said to be* **locally strongly accessible from** x_0 *if*

$$\forall V\ \exists \delta > 0\ \forall T \in (0, \delta]\ \exists \Omega \mid R^V(x_0, T) \supset \Omega$$

where V is a neighborhood of x_0 and Ω a non-empty open set in M.

In order to have a stronger rank condition than the accessibility rank condition, we introduce the following subalgebra and distribution:

Definition B.17 *Let $\mathcal{C}$ be the the accessibility algebra of (B.5). Define $\mathcal{C}_0$ as the smallest subalgebra which contains $g_1, \dots, g_m$ and satisfies $[f, v] \in \mathcal{C}_0$ for all $v \in \mathcal{C}_0$. Define the corresponding involutive distribution*

$$C_0(x) = \text{span}\{v(x) \mid v \in \mathcal{C}_0\}$$

$\mathcal{C}_0$ is called the **strong accessibility algebra** *and C_0 is called the* **strong accessibility distribution**.

A typical element of $\mathcal{C}_0$ is a finite linear combination of elements in the form

$$[v_k, [v_{k-1}, [\dots, [v_1, g_j] \dots]]], \quad j \in \underline{m}$$

where $k = 0, 1, 2, \dots$ and v_i, $i \in \underline{k}$, is in the set $\{f, g_1, \dots, g_m\}$.

We have the following algebraic condition for local strong accessibility:

Theorem B.4 *Consider the system (B.5). Suppose that*

$$\dim C_0(x_0) = n$$

then the system is locally strongly accessible from x_0.

Proof: See Nijmeijer & van der Schaft (1990) p. 86.

□

For analytic systems we know that local weak controllability (or local accessibility) is equivalent to that the accessibility rank condition (B.8) holds, (Hermann & Krener 1977). Furthermore, the requirement of locality, i.e. the trajectories have to lie within any neighborhood of x_0, can be omitted.

B.4 Stabilizability

In the discussion in the previous section of controllability, it was assumed that the input $u(\cdot)$ was a piecewise constant function of time, i.e. *open loop* control. To have rejection of disturbances and errors due to model errors, the input should be a *feedback* control law.

In this section we will present two theorems which give necessary conditions for when a system can be asymptotically stabilized by a pure static feedback law. A theorem on smooth control laws and the rate of convergence is also presented. First, we introduce some definitions.

Given the nonlinear system

$$\dot{x} = f(x, u) \tag{B.9}$$

where x is the state in the state space manifold M; u is the input in a space U; f is a smooth vector field, we let the solution of (B.9) after time T with input u and initial condition x_0 at time 0 be denoted $x(T, 0, x_0, u)$.

Definition B.18 *Let $x_1, x_2 \in M$ and assume that V is a subset of M containing both x_1 and x_2. Then, x_1 can be* **asymptotically controlled to x_2 without leaving** *V, written $x_1 \Gamma^V x_2$, if there exists some control $u \in U$ so that*

- $\lim_{t\to\infty} x(t, 0, x_1, u) = x_2$
- $\forall t \in [0, \infty),\ x(t, 0, x_1, u) \in V$

If $V = M$, we say simply that x_1 can be **asymptotically controlled** to x_2, written $x_1 \Gamma x_0$.

Definition B.19 *Let x_0 be an equilibrium point of the system (B.9). Then (B.9) is*

- **Locally asymptotically controllable to** x_0 *if*

$$\forall V_{x_0} \exists W_{x_0} \forall x \in W_{x_0},\ x \Gamma^{V_{x_0}} x_0$$

- **Globally asymptotically controllable to** x_0 *if it is locally asymptotically controllable and*

$$\forall x \in M,\ x \Gamma x_0$$

For systems with no control, we say that the system is **(globally) asymptotically stable** with respect to x_0.

We can introduce a weaker concept of an asymptotically stable state by dropping the large excursion part, (Hahn 1967) p. 6:

Definition B.20 *For system (B.9) with no control, i.e. $u = 0$, x_1 is* **attractive** *if*

$$\exists \delta > 0 \mid |x_0| < \delta \Rightarrow \lim_{t\to\infty} x(t, 0, x_0, 0) = x_1$$

Note that the same definition applies if the input is given by a feedback law, $u = u(x)$, by redefining $f(\cdot)$ in (B.9).

Let (x_0, u_0) be an equilibrium pair for the system (B.9).

Definition B.21 *We say that the system (B.9) is* C^1 **stabilizable** *(respectively,* **smoothly stabilizable***) with respect to this equilibrium pair if there exists a function*

$$g : V_{x_0} \to U, \quad g(x_0) = u_0$$

which is continuously differentiable (respectively, smooth) defined on some neighborhood V_{x_0} *of* x_0 *for which the closed loop system*

$$\dot{x} = f(x, g(x)) \tag{B.10}$$

is locally asymptotically stable.

If $V_{x_0} = M$, and (B.10) is globally asymptotically stable, we say that the system (B.9) is **globally** C^1 (or C^∞) stabilizable.

Unlike linear systems, controllability does not imply stabilizability for nonlinear systems. We give two theorems on the necessary conditions for stabilizability.

Theorem B.5 *If a differential equation on a manifold M*

$$\dot{x} = f(x), \quad f(x_0) = 0 \tag{B.11}$$

is globally asymptotically stable with respect to x_0*, then M must be contractible.*

Proof: See Sontag (1990*a*) p. 71 or Sontag (1990*b*) p. 181.

□

Corollary B.2 *If the state space manifold M for the system (B.11) is not contractible, then (B.11) is not* C^1 *globally stabilizable (about any* x_0*).*

Proof: See Sontag (1990*b*) p. 181.

□

The result in this corollary can be extended to show that there does not exist a globally stabilizing feedback $g(\cdot)$ for which the closed loop system is just locally Lipschitz on M, if M is noncontractible.

Theorem B.5 implies that mechanical models with a noncontractible phase space, rigid body orientations, for example, give rise to systems that cannot be smoothly, or in any reasonable sense continuously, globally stabilizable, Sontag (1990*a*) p. 71.

Brockett's Stabilization Theorem is given by

Theorem B.6 *Assume that the system*

$$\dot{x} = f(x, u)$$

is locally C^1 stabilizable with respect to x_0. Then the image of the map

$$f : M \times \Upsilon \to R^n$$

contains some neighborhood of x_0 where $\dim x = n$.

Proof: See Brockett (1983) or Sontag (1990*b*) p. 182.

□

Note that a stronger result is obtained by restricting the state space manifold M to the neighborhood of x_0, V_{x_0}. This means that

$$f : V_{x_0} \times \Upsilon \to R^n$$

contains some neighborhood of x_0 where V_{x_0} is an arbitrary open set of x_0.

The same comment on extension to locally Lipschitz as for Corollary B.2 applies here.

We will present a theorem on smooth control laws and the rate of convergence. To this end, we need the following definition from Gurvits & Li (1992):

Definition B.22 *The system $\dot{x} = B(x)u$ is $(\sigma, g(t))$* **stabilizable at** *x_p iff there exists a positive κ and a neighborhood Ω_p about x_p such that for any $x_0 \in \Omega_p$ there is a control $u(t)$, $0 \le t < \infty$, satisfying the following condition*

$$\begin{aligned} \dot{x} &= B(x)u(t), \quad x(0) = x_0 \\ \|u(t)\| &\le \kappa \|x(t) - x_p\|^\sigma \end{aligned} \tag{B.12}$$

and

$$\|x(t) - x_p\| \le \|x_0 - x_p\| g(t), \quad g(t) \ge 0$$

Theorem B.7 *The system (B.12) is not $(\sigma, g(t))$ stabilizable at x_p if*

$$\int_0^\infty (g(t))^\sigma dt \leq \infty \qquad \textit{and} \qquad \sigma > \frac{1}{\eta(x_p)} \tag{B.13}$$

where $\eta(x_p)$ is the nonholonomic degree, Section B.2.

Proof: See Gurvits & Li (1992), Theorem 3.3.

□

This theorem states that a nonholonomic system, i.e. $\eta(x_p) > 1$, on the form (B.12) cannot be exponentially stabilized in the usual sense of exponential stability.

B.5 Bibliographic Notes

Section B.2 is based on the presentation of Nijmeijer & van der Schaft (1990) and Isidori (1989). The shortest formulations and the most convenient notation are used. More theory on differential geometry can be found in Bishop & Crittenden (1964) and Chillingworth (1976). The geometrical interpretation of the Lie bracket, (B.3) can be found in Nijmeijer & van der Schaft (1990) p. 77 and in Bishop & Crittenden (1964) p. 18.

Section B.3 is mainly based on Nijmeijer & van der Schaft (1990), Section 3.1 and Hermann & Krener (1977). The idea of using Lie brackets in the study of accessibility or reachability can be traced back to Chow (1939). This is further developed by Haynes & Hermes (1970), Sussmann & Jurdjevic (1972) and Lobry (1970). Hermes (1965) extended the concept of complete controllability for linear systems to nonlinear systems. The results in this section are local. For a more global treatment see Sussmann & Jurdjevic (1972). Sussmann (1987) has given a general theorem on small-time local controllability which includes several existing theorems. The definitions concerning completely nonholonomic distributions and nonholonomic degree can be found in Gershkovich & Vershik (1988).

Section B.4 is mainly based on Sontag (1990*b*). The formulation of Definition B.18 is given in a more general manner by Sontag (1990*b*). Theorem B.5 is also given in Sontag (1990*a*). Theorem B.6 was originally given by Brockett (1983). An extension of Brockett's Theorem has been found for an n-dimensional system for $n \leq 2$, (Zabczyk 1989).