

# 6. Representing Rotation

## *Mechanics of Manipulation*

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# Outline.

- Generalities
- Axis-angle
- Rodrigues's formula
- Rotation matrices
- Euler angles

# Why representing rotations is hard.

- Rotations do not commute.
- The topology of spatial rotations does not permit a smooth embedding in Euclidean three space.

# Choices

- More than three numbers
  - Rotation matrices
  - Unit quaternions. (aka Euler parameters)
- Many-to-one
  - Axis times angle (matrix exponential)
- Unsmooth and many-to-one
  - Euler angles
- Unsmooth and many-to-one and more than three numbers
  - Axis-angle

# Axis-angle

Recall Euler's theorem: every spatial rotation leaves a line of fixed points: the rotation axis.

Let  $O$ ,  $\hat{\mathbf{n}}$ ,  $\theta$ , be ...

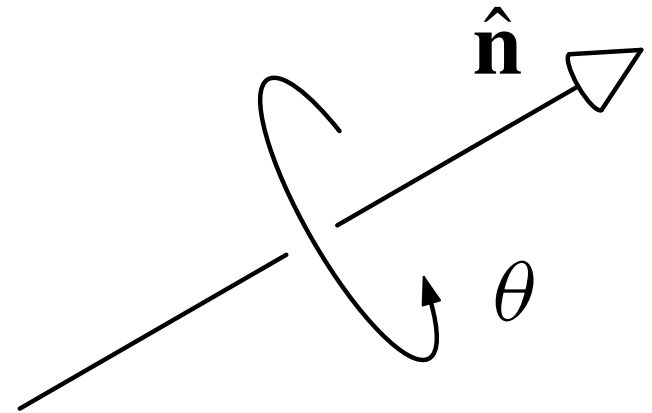
Let  $\text{rot}(\hat{\mathbf{n}}, \theta)$  be the corresponding rotation.

Many to one:

$$\text{rot}(-\hat{\mathbf{n}}, -\theta) = \text{rot}(\hat{\mathbf{n}}, \theta)$$

$$\text{rot}(\hat{\mathbf{n}}, \theta + 2k\pi) = \text{rot}(\hat{\mathbf{n}}, \theta), \text{ for any integer } k.$$

When  $\theta = 0$ , the rotation axis is indeterminate, giving an infinity-to-one mapping.



# Representation

What do we want from a representation? For a start:

- Rotate points;  
Rodrigues's formula
- Compose rotations;  
Using axis-angle? Ugh.
- (Convert to other representations.)

# Rodrigues's formula

Others derive Rodrigues's formula using rotation matrices, missing the geometrical aspects.

Given point  $\mathbf{x}$ , decompose into components parallel and perpendicular to the rotation axis

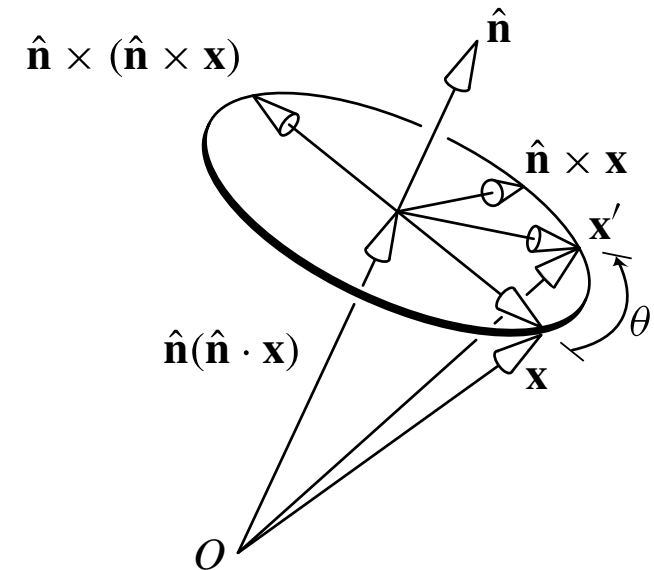
$$\mathbf{x} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Only  $\mathbf{x}_\perp$  is affected by the rotation, yielding *Rodrigues's formula*:

$$\mathbf{x}' = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \sin \theta (\hat{\mathbf{n}} \times \mathbf{x}) + \cos \theta \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

A common variation:

$$\mathbf{x}' = \mathbf{x} + (\sin \theta) \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos \theta) \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$





# Rotation matrices

Choose  $O$  on rotation axis. Choose frame  $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$ .

Let  $(\hat{\mathbf{u}}'_1, \hat{\mathbf{u}}'_2, \hat{\mathbf{u}}'_3)$  be the image of that frame.

Write the  $\hat{\mathbf{u}}'_i$  vectors in  $\hat{\mathbf{u}}_i$  coordinates, and collect them in a matrix:

$$\hat{\mathbf{u}}'_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}'_1 \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}'_1 \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}'_1 \end{pmatrix}$$

$$\hat{\mathbf{u}}'_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}'_2 \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}'_2 \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}'_2 \end{pmatrix}$$

$$\hat{\mathbf{u}}'_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}'_3 \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}'_3 \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}'_3 \end{pmatrix}$$

# So many numbers

A rotation matrix has nine numbers,  
but spatial rotations have only three degrees of freedom,  
leaving six excess numbers ...  
There are six constraints that hold among the nine numbers.

$$|\hat{\mathbf{u}}'_1| = |\hat{\mathbf{u}}'_2| = |\hat{\mathbf{u}}'_3| = 1$$
$$\hat{\mathbf{u}}'_3 = \hat{\mathbf{u}}'_1 \times \hat{\mathbf{u}}'_2$$

*i.e.* the  $\hat{\mathbf{u}}'_i$  are unit vectors forming a right-handed coordinate system.

Such matrices are called *orthonormal* or *rotation* matrices.

# Rotating a point

Let  $(x_1, x_2, x_3)$  be coordinates of  $\mathbf{x}$  in frame  $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$ .

Then  $\mathbf{x}'$  is given by the same coordinates taken in the  $(\hat{\mathbf{u}}'_1, \hat{\mathbf{u}}'_2, \hat{\mathbf{u}}'_3)$  frame:

$$\begin{aligned}\mathbf{x}' &= x_1 \hat{\mathbf{u}}'_1 + x_2 \hat{\mathbf{u}}'_2 + x_3 \hat{\mathbf{u}}'_3 \\ &= x_1 A \hat{\mathbf{u}}_1 + x_2 A \hat{\mathbf{u}}_2 + x_3 A \hat{\mathbf{u}}_3 \\ &= A(x_1 \hat{\mathbf{u}}_1 + x_2 \hat{\mathbf{u}}_2 + x_3 \hat{\mathbf{u}}_3) \\ &= A\mathbf{x}\end{aligned}$$

So rotating a point is implemented by ordinary matrix multiplication.

# Rotating a point

Let  $A$  and  $B$  be coordinate frames. Notation:

$x$  a point

$\mathbf{x}$  a geometrical vector, directed from an origin  $O$  to the point  $x$   
or, a vector of three numbers, representing  $x$  in an unspecified frame

${}^A\mathbf{x}$  a vector of three numbers, representing  $x$  in the  $A$  frame

Let  ${}^B_A R$  be the rotation matrix that rotates frame  $B$  to frame  $A$ .

Then (see previous slide)  ${}^B_A R$  represents the rotation of the point  $x$ :

$${}^B\mathbf{x}' = {}^B_A R {}^B\mathbf{x}$$

Note prescripts all match. Both points, and xform, must be written in same coordinate frame.

# Coordinate transform

There is another use for  ${}^B_A R$ :

${}^A \mathbf{x}$  and  ${}^B \mathbf{x}$  represent the same point, in frames  $A$  and  $B$  resp.

To transform from  $A$  to  $B$ :

$${}^B \mathbf{x} = {}^B_A R {}^A \mathbf{x}$$

For coord xform, matrix subscript and vector superscript “cancel”.

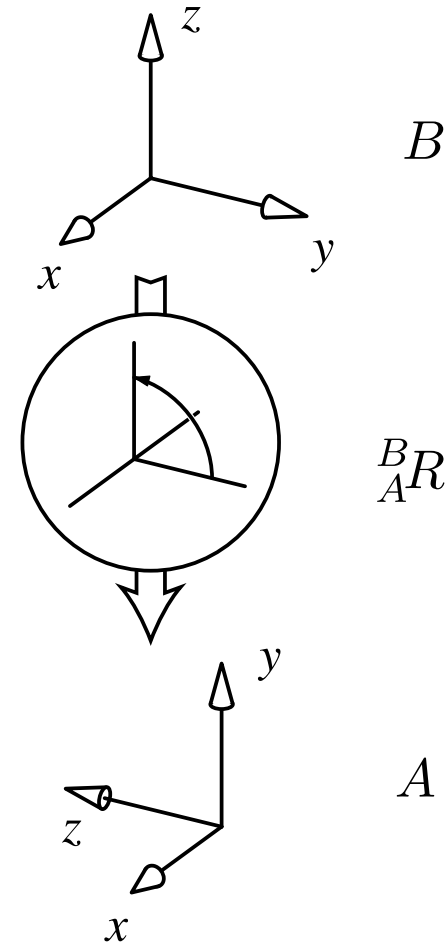
Rotation from  $B$  to  $A$  is the same as coordinate transform from  $A$  to  $B$ .

# Example rotation matrix

$$\begin{aligned} {}^B_A R &= \left( {}^B\mathbf{x}_A \mid {}^B\mathbf{y}_A \mid {}^B\mathbf{z}_A \right) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

How to remember what  ${}^B_A R$  does?  
Pick a coordinate axis and see. The  $x$  axis isn't very interesting, so try  $y$ :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



# Nice things about rotation matrices

- Composition of rotations:  $\{R_1; R_2\} = R_2 R_1$ .  
( $\{x; y\}$  means do  $x$  then do  $y$ .)
- Inverse of rotation matrix is its transpose  ${}^B_A R^{-1} = {}^A_B R = {}^B_A R^T$ .
- Coordinate xform of a rotation matrix:

$${}^B R = {}^B_A R {}^A R {}^A_B R$$

## Converting $\text{rot}(\hat{\mathbf{n}}, \theta)$ to $R$

Ugly way: define frame with  $\hat{\mathbf{z}}$  aligned with  $\hat{\mathbf{n}}$ , use coordinate xform of previous slide.

Keen way: Rodrigues's formula!

$$\mathbf{x}' = \mathbf{x} + (\sin \theta) \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos \theta) \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Define “cross product matrix”  $N$ :

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

so that

$$N\mathbf{x} = \hat{\mathbf{n}} \times \mathbf{x}$$



## ... using Rodrigues's formula ...

Substituting the cross product matrix  $N$  into Rodrigues's formula:

$$\mathbf{x}' = \mathbf{x} + (\sin \theta)N\mathbf{x} + (1 - \cos \theta)N^2\mathbf{x}$$

Factoring out  $\mathbf{x}$

$$R = I + (\sin \theta)N + (1 - \cos \theta)N^2$$

That's it! Rodrigues's formula in matrix form. If you want to you could expand it:

$$\begin{pmatrix} n_1^2 + (1 - n_1^2)c\theta & n_1n_2(1 - c\theta) - n_3s\theta & n_1n_3(1 - c\theta) + n_2s\theta \\ n_1n_2(1 - c\theta) + n_3s\theta & n_2^2 + (1 - n_2^2)c\theta & n_2n_3(1 - c\theta) - n_1s\theta \\ n_1n_3(1 - c\theta) - n_2s\theta & n_2n_3(1 - c\theta) + n_1s\theta & n_3^2 + (1 - n_3^2)c\theta \end{pmatrix}$$

where  $c\theta = \cos \theta$  and  $s\theta = \sin \theta$ . Ugly.

# Rodrigues's formula for differential rotations

Consider Rodrigues's formula for a differential rotation  $\text{rot}(\hat{\mathbf{n}}, d\theta)$ .

$$\begin{aligned}\mathbf{x}' &= (I + \sin d\theta N + (1 - \cos d\theta) N^2) \mathbf{x} \\ &= (I + d\theta N) \mathbf{x}\end{aligned}$$

so

$$\begin{aligned}d\mathbf{x} &= N \mathbf{x} d\theta \\ &= \hat{\mathbf{n}} \times \mathbf{x} d\theta\end{aligned}$$

It follows easily that differential rotations are vectors: you can scale them and add them up. We adopt the convention of representing angular velocity by the unit vector  $\hat{\mathbf{n}}$  times the angular velocity.

# Converting from $R$ to $\text{rot}(\hat{\mathbf{n}}, \theta) \dots$

Problem:  $\hat{\mathbf{n}}$  isn't defined for  $\theta = 0$ .

We will do it indirectly. Convert  $R$  to a unit quaternion (next lecture), then to axis-angle.

# Euler angles

Three numbers to describe spatial rotations. *ZYZ* convention:

$$(\alpha, \beta, \gamma) \mapsto \text{rot}(\gamma, \hat{\mathbf{z}}'') \text{rot}(\beta, \hat{\mathbf{y}}') \text{rot}(\alpha, \hat{\mathbf{z}})$$

Can we represent an arbitrary rotation?

Rotate  $\alpha$  about  $\hat{\mathbf{z}}$  until

$$\hat{\mathbf{y}}' \perp \hat{\mathbf{z}}'';$$

Rotate  $\beta$  about  $\hat{\mathbf{y}}'$  until

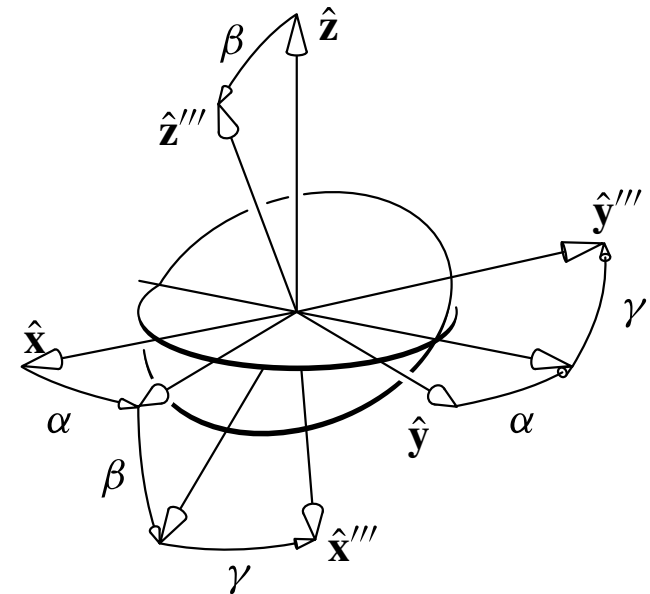
$$\hat{\mathbf{z}}'' \parallel \hat{\mathbf{z}}''';$$

Rotate  $\gamma$  about  $\hat{\mathbf{z}}''$  until

$$\hat{\mathbf{y}}'' = \hat{\mathbf{y}}'''.$$

Note two choices for  $\hat{\mathbf{y}}'$  ...

... except sometimes infinite choices.



## From $(\alpha, \beta, \gamma)$ to $R$

Expand  $\text{rot}(\alpha, \hat{\mathbf{z}}) \text{rot}(\beta, \hat{\mathbf{y}}) \text{rot}(\gamma, \hat{\mathbf{z}})$

(Why is that the right order?)

$$\begin{aligned} & \begin{pmatrix} \mathbf{c}\alpha & -\mathbf{s}\alpha & 0 \\ \mathbf{s}\alpha & \mathbf{c}\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{c}\beta & 0 & \mathbf{s}\beta \\ 0 & 1 & 0 \\ -\mathbf{s}\beta & 0 & \mathbf{c}\beta \end{pmatrix} \begin{pmatrix} \mathbf{c}\gamma & -\mathbf{s}\gamma & 0 \\ \mathbf{s}\gamma & \mathbf{c}\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{c}\alpha \mathbf{c}\beta \mathbf{c}\gamma - \mathbf{s}\alpha \mathbf{s}\gamma & -\mathbf{c}\alpha \mathbf{c}\beta \mathbf{s}\gamma - \mathbf{s}\alpha \mathbf{c}\gamma & \mathbf{c}\alpha \mathbf{s}\beta \\ \mathbf{s}\alpha \mathbf{c}\beta \mathbf{c}\gamma + \mathbf{c}\alpha \mathbf{s}\gamma & -\mathbf{s}\alpha \mathbf{c}\beta \mathbf{s}\gamma + \mathbf{c}\alpha \mathbf{c}\gamma & \mathbf{s}\alpha \mathbf{s}\beta \\ -\mathbf{s}\beta \mathbf{c}\gamma & \mathbf{s}\beta \mathbf{s}\gamma & \mathbf{c}\beta \end{pmatrix} \quad (1) \end{aligned}$$

## From $R$ to $(\alpha, \beta, \gamma)$ the ugly way

Case 1:  $r_{33} = 1$ ,  $\beta = \pi$ .  $\alpha - \gamma$  is indeterminate.

$$R = \begin{pmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & 0 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Case 2:  $r_{33} = -1$ ,  $\beta = -\pi$ .  $\alpha + \gamma$  is indeterminate.

$$R = \begin{pmatrix} -\cos(\alpha - \gamma) & -\sin(\alpha - \gamma) & 0 \\ -\sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For generic case: solve 3rd column for  $\beta$ . (Sign is free choice.)  
Solve third column for  $\alpha$  and third row for  $\gamma$ .

... but there are numerical issues ...

# From $R$ to $(\alpha, \beta, \gamma)$ the clean way

Let

$$\sigma = \alpha + \gamma$$

$$\delta = \alpha - \gamma$$

Then

$$r_{22} + r_{11} = \cos \sigma (1 + \cos \beta)$$

$$r_{22} - r_{11} = \cos \delta (1 - \cos \beta)$$

$$r_{21} + r_{12} = \sin \delta (1 - \cos \beta)$$

$$r_{21} - r_{12} = \sin \sigma (1 + \cos \beta)$$

(No special cases for  $\cos \beta = \pm 1$ ?)

Solve for  $\sigma$  and  $\delta$ , then for  $\alpha$  and  $\gamma$ , then finally

$$\beta = \tan^{-1}(r_{13} \cos \alpha + r_{23} \sin \alpha, r_{33})$$

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