Proceedings of the American Control Conference
San Diego, California • June 1999

Tutorial: Model Predictive Control Technology

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Abstract

This paper provides a brief tutorial on model predictive control (MPC) theory for linear and nonlinear models. The discussion of MPC with linear models covers the topics of steady-state target calculation, infinite horizon receding horizon regulation, resolving infeasibilities caused by constraints, and state estimation and disturbance models. The section on nonlinear models briefly discusses what is desirable from a theoretical perspective and what is possible from an implementation perspective and focuses on some current efforts to bridge this gap. The paper concludes with a brief outlook for future developments in the areas of nonlinear MPC, robustness, moving horizon state estimation and MPC of hybrid systems.

1 Introduction

The purpose of this paper is to provide a reasonably accessible and self-contained tutorial exposition on model predictive control (MPC). The target audience is people with control expertise, particularly practitioners, who wish to broaden their perspective in the MPC area of control technology. This paper mainly introduces the concepts, provides a framework in which the critical issues can be expressed and analyzed, and points out how MPC allows practitioners to address the tradeoffs that must be considered in implementing a control technology. The subsequent papers and presentations by respected and experienced control practitioners provide the wider perspective on the costs and benefits of implementing MPC in a variety of industrial settings. It is hoped that this combination offers a balanced viewpoint and may stimulate further interest both in industrial application and fundamental research.

The MPC research literature is by now large, but review articles have appeared at regular intervals. We should point these out before narrowing the focus in the interest of presenting a reasonably self-contained tutorial for the non-expert. The three MPC papers presented at the CPC VI conference in 1996 are an excellent starting point [28, 33, 49]. Qin and Badgwell present comparisons of industrial MPC algorithms that practitioners may find particularly useful. Morari and Lee provide another recent review [37]. Kwon provides a very extensive list of references [26]. Moreover, several excellent books have appeared recently [38, 65, 6]. For those interested in the status of MPC for *nonlinear* plants, the proceedings of the 1998 conference [1] would be of strong interest.

2 Models

The essence of MPC is to optimize, over the manipulatable inputs, forecasts of process behavior. The forecasting is accomplished with a process model, and therefore the model is the essential element of an MPC controller. As discussed subsequently, models are not perfect forecasters, and feedback can overcome some effects of poor models, but starting with a poor process model is akin to driving a car at night without headlights; the feedback may be a bit late to be truly effective.

Linear models. Historically, the models of choice in early industrial MPC applications were time domain, input/output, step or impulse response models[54, 11, 47]. Part of the early appeal of MPC for practitioners in the process industries was undoubtedly the ease of understanding provided by this model form. It has become more common for MPC researchers, however, to discuss linear models in state-space form

$$\frac{dx}{dt} = Ax + Bu \qquad x_{j+1} = Ax_j + Bu_j \qquad (1)$$

$$y = Cx \qquad y_j = Cx_j \qquad (2)$$

in which x is the n-vector of states, y is the p-vector of (measurable) outputs and u is the m-vector of (manipulatable) inputs, t is continuous time and j is the discrete time sample number. Continuous time models may be more familiar to those with a classical control background in transfer functions, but discrete time models are very convenient for digital computer implementation. With abuse of notation, we use the same system matrices (A, B, C) for either model, but most of the subsequent discussion focuses on discrete time. The transformations from continuous time to discrete time models are available as one line commands in a language like Octave or MATLAB. Linear models in the process industries are, by their nature, empirical models, and identified from input/output data. The ideal model form for identification purposes, is perhaps best left to the experts in identification theory, but a survey of that literature indicates no disadvantage to using state-space models inside the MPC controller.

The discussion of MPC in state-space form has several advantages including easy generalization to multi-variable systems, ease of analysis of closed-loop properties, and on-line computation. Furthermore, the wealth of linear systems theory: the linear quadratic regulator theory, Kalman filtering theory, internal model principle, etc., is immediately accessible for use in MPC starting with this model form. We demonstrate the usefulness of those tools subsequently. A word of caution is also in order. Categories, frameworks and viewpoints, while indispensable for clear thinking and communication, may blind us to other possibilities. We should resist the easy temptation to formulate all control issues from an LQ, state-space framework. The tendency is to focus on those issues that are easily imported into the dominant framework while neglecting other issues, of possibly equal or greater import to practice, which are difficult to analyze, awkward, and inconvenient.

From a theoretical perspective, the significant shift in problem formulation came from the MPC practitioners who insisted on maintaining constrains, particularly input constraints in the problem formulation,

$$\frac{dx}{dt} = Ax + Bu \qquad x_{j+1} = Ax_j + Bu_j \tag{3}$$

$$y = Cx \qquad y_j = Cx_j \tag{4}$$

$$Du \le d \qquad Du_j \le d \tag{5}$$

$$Hx \le h \qquad Hx_j \le h \tag{6}$$

in which D, H are the constraint matrices and d, h are positive vectors. The constraint region boundaries are straight lines as shown in Figure 1. At this point we are assuming that $x = 0, u = 0$ is the steady state to which we are controlling the process, but we treat the more general case subsequently. Optimization over inputs subject to hard constraints leads immediately to nonlinear control, and that departure from the well understood and well tested linear control theory provided practitioners with an important, new control technology and motivated researchers to understand better this new framework. Certainly optimal control with constraints was not a new concept in the 1970s, but the implementation in a moving horizon fashion of these open-loop optimal control solutions subject to constraints at each sample time was the new twist that had not been fully investigated.

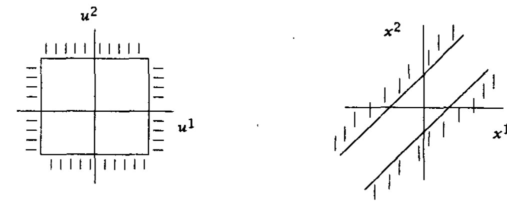

Figure 1: Example input and state constraint regions define by Equations 5–6.

Nonlinear models. The use of nonlinear models in MPC is motivated by the possibility to improve control by improving the quality of the forecasting. The basic fundamentals in any process control problem — conservation of mass, momentum and energy, considerations of phase equilibria, relationships of chemical kinetics, and properties of final products — all introduce nonlinearity into the process description. In which settings use of nonlinear models for forecasting delivers improved control performance is an open issue, however. For continuous processes maintained at nominal operating conditions and subject to small disturbances, the potential improvement would appear small. For processes operated over large regions of the state space — semi-batch reactors, frequent product grade changes, processes subject to large disturbances, for example — the advantages of nonlinear models appear larger.

Identification of nonlinear models runs the entire range from models based on fundamental principles with only parameter estimation from data, to completely empirical nonlinear models with all coefficients identified from data. We will not stray into the

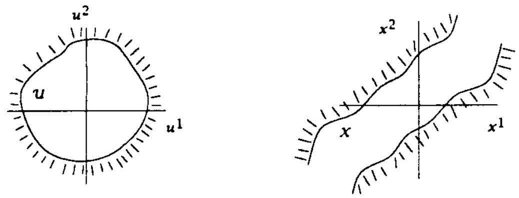

Figure 2: Example input and state constraint regions define by Equations 9–10.

issues of identification of nonlinear models, which is a large topic by itself. The interested reader may consult [44, 27] and the references therein for an entry point into this literature.

Regardless of the identification method, we represent the nonlinear model inside the MPC controller also in state space form

$$\frac{dx}{dt} = f(x, u) \qquad x_{j+1} = f(x_j, u_j) \tag{7}$$
$$y = g(x) \qquad y_j = g(x_j) \tag{8}$$
$$u \in \mathcal{U} \qquad u_j \in \mathcal{U} \tag{9}$$
$$x \in \mathcal{X} \qquad x_j \in \mathcal{X} \tag{10}$$

If the model is nonlinear, there is no advantage in keeping the constraints as linear inequalities, so we consider the constraints as membership in more general regions $\mathcal{U}, \mathcal{X}$ shown in Figure 2.

3 MPC with Linear Models

In this paper we focus on formulating MPC as an infinite horizon optimal control strategy with a quadratic performance criterion. We use the following discrete time model of the plant

$$x_{j+1} = Ax_j + B(u_j + d) \tag{11}$$
$$y_j = Cx_j + p \tag{12}$$

The affine terms d and p serve the purpose of adding integral control. They may be interpreted as modeling the effect of constant disturbances influencing the input and output, respectfully. Assuming that the state of the plant is perfectly measured, we define model predictive control as the feedback law $u_j = \rho(x_j)$ that minimizes

$$\Phi = \frac{1}{2}\sum_{j=0}^{\infty}(y_j-\bar{y})^T Q(y_j-\bar{y}) + (u_j-\bar{u})^T R(u_j-\bar{u}) + \Delta u_j^T S \Delta u_j \tag{13}$$

in which $\Delta u_j \triangleq u_j - u_{j-1}$. The matrices Q, R, and S are assumed to be symmetric positive definite. When the complete state of the plant is not measured, as is almost always the case, the addition of a state estimator is necessary (Section 3.4).

The vector $\bar{y}$ is the desired output target and $\bar{u}$ is the desired input target, assumed for simplicity to be time invariant. In many industrial implementations the desired targets are calculated as a steady-state economic optimization at the plant level. In these cases, the desired targets are normally constant between plant optimizations, which are performed at a slower time scale than the MPC controller operates. In batch and semi-batch reactor operation, on the other hand, a final time objective may be optimized instead, which produces a time-varying trajectory for the system states. Even in continuous operations, some recommend tuning MPC controllers by specifying the setpoint trajectory, often a first order response with adjustable time constant. As discussed by Bitmead et al. [6] in the context of GPC, one can pose these types of tracking problems within the LQ framework by augmenting the state of the system to describe the evolution of the reference signal and posing an LQ problem for the combined system.

For a time invariant setpoint, the steady-state aspect of the control problem is to determine appropriate values of (y_s, x_s, u_s). Ideally, $y_s = \bar{y}$ and $u_s = \bar{u}$. However, process limitations and constraints may prevent the system from reaching the desired steady state. The goal of the target calculation is to find the feasible triple (y_s, x_s, u_s) such that y_s and u_s are as close as possible to $\bar{y}$ and $\bar{u}$. We address the target calculation in Section 3.1.

To simplify the analysis and formulation, we transform (13) using deviation variables to the generic infinite horizon quadratic criterion

$$\Phi = \frac{1}{2}\sum_{j=0}^{\infty} z_j^T Q z_j + v_j^T R v_j + \Delta v_j^T S \Delta v_j. \tag{14}$$

The original criterion (13) can be recovered from (14) by making the following substitutions:

$$z_j \leftarrow y_j - Cx_s - p, \quad w_j \leftarrow x_j - x_s, \quad v_j \leftarrow u_j - u_s,$$

in which y_s, x_s and u_s are the steady states satisfying the following relation

$$x_s = Ax_s + B(u_s + d) \tag{15a}$$
$$y_s = Cx_s + p \tag{15b}$$

By using deviation variables we treat separately the steady-state and the dynamic elements of the control problem, thereby simplifying the overall analysis of the controller.

The dynamic aspect of the control problem is to control (y, x, u) to the steady-state values (y_s, x_s, u_s) in the face of constraints, which are assumed not to be active at steady state, i.e. the origin is in the strict interior of regions $\mathcal{X}, \mathcal{U}$. See [51] for a preliminary treatment of the case in which the constraints are active at the steady-state operating point. This part of the problem is discussed in Section 3.2. In particular we determine the state feedback law $v_j = \rho(w_j)$ that minimizes (14). When there are no inequality constraints, the feedback law is the linear quadratic regulator. However, with the addition of inequality constraints, there may not exist an analytic form for $\rho(w_j)$. For cases in which an analytic solution is unavailable, the feedback law is obtained by repetitively solving the open-loop optimal control problem. This strategy allows us to consider only the encountered sequence of measured states rather than the entire state space. For a further discussion, see Mayne [32].

If we consider only linear constraints on the input, input velocity, and outputs of the form

$$u_{\min} \le Du_k \le u_{\max}, \quad -\Delta_u \le \Delta u_k \le \Delta_u \tag{16}$$

$$y_{\min} \le Cx_k \le y_{\max} \tag{17}$$

we formulate the regulator as the solution of the following infinite horizon optimal control problem

$$\min_{\{w_k, v_k\}} \Phi(x_j) = \frac{1}{2} \sum_{k=0}^{\infty} z_k^T Q z_k + v_k R v_k + \Delta v_k^T S \Delta v_k \tag{18}$$

subject to the constraints

$$w_0 = x_j - x_s, \quad v_{-1} = u_{j-1} - u_s \tag{19}$$

$$w_{k+1} = Aw_k + Bv_k, \quad z_k = Cw_k \tag{20}$$

$$u_{\min} - u_s \le Dv_k \le u_{\max} - u_s \tag{21}$$

$$-\Delta_u \le \Delta v_k \le \Delta_u \tag{22}$$

$$y_{\min} - y_s \le Cw_k \le y_{\max} - y_s \tag{23}$$

If we denote

$$\{w_{k+1}^*(x_j), v_k^*(x_j)\}_{k=0}^{\infty} = \arg\min \Phi(x_j),$$

then the control law is

$$\rho(x_j) = v_0^*(x_j).$$

We address the regulation problem in Section 3.2.

Combining the solution of the target tracking problem and the constrained regulator, we define the MPC algorithm as follows:

1. Obtain an estimate of the state and disturbances $\Longrightarrow (x_j, p, d)$
2. Determine the steady-state target $\Longrightarrow (y_s, x_s, u_s)$
3. Solve the regulation problem $\Longrightarrow v_j$
4. Let $u_j = v_j + u_s$
5. Repeat for $j \leftarrow j + 1$

3.1 Target Calculation

When the number of the inputs equals the number of outputs, the solution to the unconstrained target problem is obtained using the steady-state gain matrix, assuming such a matrix exists (i.e. the system has no integrators). However for systems with unequal numbers of inputs and outputs, integrators, or inequality constraints, the target calculation is formulated as a mathematical program [40, 39]. When there are at least as many inputs as outputs, multiple combinations of inputs may yield the desired output target at steady state. For such systems, a mathematical program with a least squares objective is formulated to determine the best combinations of inputs. When the number of outputs is greater than the number of inputs, situations exist in which no combination of inputs satisfies the output target at steady state. For such cases, we formulate a mathematical program that determines the steady-state output $y_s \neq \bar{y}$ that is closest to $\bar{y}$ in a least squares sense.

Instead of solving separate problems to establish the target, we prefer to solve one problem for both situations. Through the use of an exact penalty [17], we formulate the target tracking problem as a single quadratic program that achieves the output target if possible, and relaxes the problem in a l_1/l_2^2 optimal sense if the target is infeasible. We formulate the soft constraint

$$\begin{aligned} \bar{y} - Cx_s - p &\le \eta, \\ \bar{y} - Cx_s - p &\ge -\eta, \\ \eta &\ge 0, \end{aligned} \tag{24}$$

by relaxing the constraint $Cx_s = \bar{y}$ using the slack variable η. By suitably penalizing η, we guarantee that the relaxed constraint is binding when it is feasible. We formulate the exact soft constraint by adding an l_1/l_2^2 penalty to the objective function.

The l_1/l_2^2 penalty is simply the combination of a linear penalty $q_s^T \eta$ and a quadratic penalty $\eta^T Q_s \eta$, in which the elements of q_s are strictly non-negative and Q_s is a symmetric positive definite matrix. By choosing the linear penalty sufficiently large, the soft constraint is guaranteed to be exact. A lower bound on the elements of q_s to ensure that the original hard constraints are satisfied by the solution cannot be calculated explicitly without knowing the solution to the original problem, because the lower bound depends on the optimal Lagrange multipliers for the original problem. In theory, a conservative state-dependent upper bound for these multipliers may be obtained by exploiting the Lipschitz continuity of the quadratic program [20]. However, in practice, we rarely need to guarantee that the l_1/l_2^2 penalty is exact. Rather, we use approximate values for q_s obtained by computational experience. In terms of constructing an exact penalty, the quadratic term is superfluous. However, the quadratic term adds an extra degree of freedom for tuning and is necessary to guarantee uniqueness.

We now formulate the target tracking optimization as the following quadratic program

$$\min_{x_s, u_s, \eta} \frac{1}{2} \left(\eta^T Q_s \eta + (u_s - \bar{u})^T R_s (u_s - \bar{u}) \right) + q_s^T \eta \quad (25)$$

subject to the constraints

$$\begin{bmatrix} I - A & -B & 0 \\ C & 0 & I \\ C & 0 & -I \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ \eta \end{bmatrix} \begin{Bmatrix} = \\ \geq \\ \leq \end{Bmatrix} \begin{bmatrix} Bd \\ \bar{y} - p \\ \bar{y} - p \end{bmatrix} \quad (26a)$$

$$\eta \geq 0, \quad (26b)$$

$$u_{\min} \leq D u_s \leq u_{\max}, \quad y_{\min} \leq C x_s + p \leq y_{\max}, \quad (26c)$$

in which R_s and Q_s are assumed to be symmetric positive definite.

Because x_s is not explicitly in the objective function, the question arises as to whether the solution to Equation 25 is unique. If the feasible region is nonempty, the solution exists because the quadratic program is bounded below on the feasible region. If Q_s and R_s are symmetric positive definite, y_s and u_s are uniquely determined by the solution of the quadratic program. However, without a quadratic penalty on x_s, there is no guarantee that the resulting solution for x_s is unique. Non-uniqueness in the steady-state value of x_s presents potential problems for the controller, because the origin of the regulator is not fixed at each sample time. Consider, for example, a tank in which the level is unmeasured (i.e. an unobservable integrator). The steady-state solution is to set $u_s = 0$ (i.e. balance the flows). However, any level x_s, within bounds, is an optimal alternative. Likewise, at the next time instant, a different level also would be a suitably optimal steady-state target. The resulting closed-loop performance for the system could be erratic, because the controller may constantly adjust the level of the tank, never letting the system settle to a steady state.

In order to avoid such situations, we restrict our discussion to detectable systems, and recommend redesign if a system does not meet this assumption. For detectable systems, the solution to the quadratic program is unique assuming the feasible region is nonempty. The details of the proof are given in [51]. Uniqueness is also guaranteed when only the integrators are observable. For the practitioner this condition translates into the requirement that all levels are measured. The reason we choose the stronger condition of detectability is that if good control is desired, then the unstable modes of the system should be observable. Detectability is also required to guarantee the stability of the regulator.

Empty feasible regions are a result of the inequality constraints (26c). Without the inequality constraints (26c) the feasible region is nonempty, thereby guaranteeing the existence of a feasible and unique solution under the condition of detectability. For example, the solution $(u_s, x_s, \eta) = (-d, 0, |\bar{y} - p|)$ is feasible. However, the addition of the inequality constraints (26c) presents the possibility of infeasibility. Even with well-defined constraints, $u_{\min} < u_{\max}$ and $y_{\min} < y_{\max}$, disturbances may render the feasible region empty. Since the constraints on the input usually result from physical limitations such as valve saturation, relaxing only the output constraints is one possibility to circumvent infeasibilities. Assuming that $u_{\min} \leq -d \leq u_{\max}$, the feasible region is always nonempty. However, we contend that the output constraints should not be relaxed in the target calculation. Rather, an infeasible solution, readily determined during the initial phase in the solution of the quadratic program, should be used as an indicator of a process exception. While relaxing the output constraints in the dynamic regulator is common practice [55, 18, 14, 70, 61, 59], the output constraint violations are transient. By relaxing output constraints in the target calculation on the other hand, the controller seeks a steady-state target that continuously violates the output constraints. The steady violation indicates that the controller is unable to compensate adequately for the disturbance and, therefore, should indicate a process exception.

3.2 Receding Horizon Regulator

Given the calculated steady state we formulate the regulator as the following infinite horizon optimal control problem

$$\min_{\{w_k,v_k\}} \Phi(x_j) = \frac{1}{2}\sum_{k=0}^{\infty} w_k^T C^T Q C w_k + v_k R v_k + \Delta v_k S \Delta v_k \quad (27)$$

subject to the constraints

$$w_0 = x_j - x_s, \quad v_{-1} = u_{j-1} - u_s \quad (28a)$$
$$w_{k+1} = Aw_k + Bv_k \quad (28b)$$
$$u_{\min} - u_s \le Dv_k \le u_{\max} - u_s \quad (28c)$$
$$-\Delta_u \le \Delta v_k \le \Delta_u \quad (28d)$$
$$\bar{y} - y_s \le Cw_k \le \bar{y} - y_s \quad (28e)$$

We assume that Q and R are symmetric positive definite matrices. We also assume that the origin, $(w_j, v_j) = (0, 0)$, is an element of the feasible region $\mathbb{W} \times \mathbb{V}$ [1]. If the pair (A, B) is constrained stabilizable and the pair $(A, Q^{1/2}C)$ is detectable, then $x_j = 0$ is an exponentially stable fixed point of the closed-loop system [61].

For unstable state transition matrices, the direct solution of (27)-(28) is ill-conditioned, because the system dynamics are propagated through the unstable A matrix. To improve the conditioning of the optimization, we re-parameterize the input as $v_k = Lw_k + r_k$, in which L is a linear stabilizing feedback gain for (A, B) [22, 57]. The system model becomes

$$w_{k+1} = (A + BL)w_k + Br_k \quad (29)$$

in which r_k is the new input. By initially specifying a stabilizing, potentially infeasible, trajectory, we can improve the numerical conditioning of the optimization by propagating the system dynamics through the stable $(A + BL)$ matrix.

By expanding Δv_k and substituting in for v_k, we transform (27)-(28) into the following more tractable form:

$$\min_{\{w_k,v_k\}} \Phi(x_j) = \frac{1}{2}\sum_{k=0}^{\infty} (w_k^T Q w_k + v_k^T R v_k + 2w_k^T M v_k) \quad (30)$$

[1] $\mathbb{W} = \{w \mid y_{\min} - y_s \le Cw \le y_{\max} - y_s\}$, $\mathbb{V} = \{v \mid u_{\min} \le Dv \le d_{\max}, \ -\Delta_u - u_s \le \Delta v \le \Delta_u - u_s\}$

subject to the following constraints:

$$w_0 = x_j, \quad w_{k+1} = Aw_k + Bv_k \quad (31a)$$
$$d_{\min} \le Dv_k - Gw_k \le d_{\max} \quad (31b)$$
$$y_{\min} - y_s \le Cw_k \le y_{\max} - y_s \quad (31c)$$

The original formulation (27)-(28) can be recovered from (30)-(31) by making the following substitutions into the second formulation:

$$x_j \leftarrow \begin{bmatrix} x_j - x_s \\ u_{j-1} - u_s \end{bmatrix}, \quad w_k \leftarrow \begin{bmatrix} w_k \\ v_{k-1} \end{bmatrix}, \quad v_k \leftarrow r_k$$
$$A \leftarrow \begin{bmatrix} A + BL & 0 \\ L & 0 \end{bmatrix}, \quad B \leftarrow \begin{bmatrix} B \\ I \end{bmatrix}$$
$$C \leftarrow \begin{bmatrix} C & 0 \end{bmatrix}, \quad M \leftarrow \begin{bmatrix} L^T(R+S) \\ -S \end{bmatrix}$$
$$Q \leftarrow \begin{bmatrix} C^TQC + L^T(R+S)L & -L^TS \\ -SL & S \end{bmatrix}$$
$$R \leftarrow R + S, \quad D \leftarrow \begin{bmatrix} D \\ I \end{bmatrix}, \quad G \leftarrow \begin{bmatrix} -DL & 0 \\ -L & I \end{bmatrix}$$
$$d_{\max} \leftarrow \begin{bmatrix} u_{\max} - u_s \\ \Delta_u \end{bmatrix}, \quad d_{\min} \leftarrow \begin{bmatrix} u_{\min} - u_s \\ -\Delta_u \end{bmatrix}$$

While the formulation (30)-(31) is theoretically appealing, the solution is intractable in its current form, because it is necessary to consider an infinite number of decision variables. In order to obtain a computationally tractable formulation, we reformulate the optimization in a finite dimensional decision space.

Several authors have considered this problem in various forms. In this paper, we concentrate on the constrained linear quadratic methods proposed in the literature [22, 66, 10, 61, 60]. The key concept behind these methods is to recognize that the inequality constraints remain active only for a finite number of sample steps along the prediction horizon. We demonstrate informally this concept as follows: if we assume that there exists a feasible solution to (30), (31), then the state and input trajectories $\{w_k, v_k\}_{k=0}^{\infty}$ approach the origin exponentially. Furthermore, if we assume the origin is contained within the interior of the feasible region $\mathbb{W} \times \mathbb{V}$ (we address the case in which the origin lies on the boundary of the feasible region in the next section), then there exists a positively invariant convex set [19]

$$\mathcal{O}_\infty = \{w \mid (A + BK)^j w \in \mathbb{W}_K, \quad \forall j \ge 0\} \quad (32)$$

such that the optimal unconstrained feedback law $v = Kw$ is feasible for all future time. The set $\mathbb{W}_K$

is the feasible region projected onto the state space by the linear control K (i.e. $\mathbb{W}_K = \{w|(w, Kw) \in \mathbb{W} \times \mathbb{V}\}$). Because the state and input trajectories approach the origin exponentially, there exists a finite N^* such that the state trajectory $\{w_k\}_{k=N^*}^{\infty}$ is contained in $\mathcal{O}_\infty$.

In order to guarantee that the inequality constraints (31b) are satisfied on the infinite horizon, N^* must be chosen such that $w_{N^*} \in \mathcal{O}_\infty$. Since the value of N^* depends on x_j, we need to account for the variable decision horizon length in the optimization. We formulate the variable horizon length regulator as the following optimization

$$\min_{\{w_k, v_k, N\}} \Phi(x_j) = \frac{1}{2} \sum_{k=0}^{N-1} (w_k^T Q w_k + v_k^T R v_k + 2 w_k^T M v_k) + \frac{1}{2} w_N^T \Pi w_N \quad (33)$$

subject to the constraints

$$w_0 = x_j, \quad w_{k+1} = A w_k + B v_k, \quad w_N \in \mathcal{O}_\infty \quad (34a)$$

$$d_{\min} \leq D v_k - G w_k \leq d_{\max}, \quad (34b)$$

$$y_{\min} - y_s \leq C w_k \leq y_{\max} - y_s \quad (34c)$$

The cost to go Π is determined from the discrete-time algebraic Riccati equation

$$\Pi = A^T \Pi A + Q - (A^T \Pi B + M)(R + B^T \Pi B)^{-1} (B^T \Pi A + M^T), \quad (35)$$

for which many reliable solution algorithms exist. The variable horizon formulation is similar to the dual-mode receding horizon controller [35] for nonlinear system with the linear quadratic regulator chosen as the stabilizing linear controller.

While the problem (33)-(34) is formulated on a finite horizon, the solution cannot, in general, be obtained in real-time since the problem is a mixed-integer program. Rather than try to solve directly (33)-(34), we address the problem of determining N^* from a variety of semi-implicit schemes while maintaining the quadratic programming structure in the subsequent optimizations.

Gilbert and Tan [19] show that there exist a finite number t^* such that $\mathcal{O}_{t^*}$ is equivalent to the maximal $\mathcal{O}_\infty$, in which

$$\mathcal{O}_t = \{w| \quad (A + BK)^j w \in \mathbb{W}_K, \quad \text{for } j = 0, \ldots, t\}. \quad (36)$$

They also present an algorithm for determining t^* that is formulated efficiently as a finite number of linear programs. Their method provides an easy check whether, for a fixed N, the solution to (33)-(34) is feasible (i.e. $w_N \in \mathcal{O}_\infty$). The check consists of determining whether state and input trajectories generated by unconstrained control law $v_k = K w_k$ from the initial condition w_N are feasible with respect to inequality constraints for t^* time steps in the future. If the check fails, then the optimization (33)-(34) needs to be resolved with a longer control horizon $N' > N$ since $w_N \notin \mathcal{O}_\infty$. The process is repeated until $w_{N'} \in \mathcal{O}_\infty$.

When the set of initial conditions $\{w_0\}$ is compact, Chmielewski and Manousiouthakis [10] present a method for calculating an upper bound $\bar{N}$ on N^* using bounding arguments on the optimal cost function Φ^*. Given a set $\mathbb{P} = \{x^1, \ldots, x^m\}$ of initial conditions, the optimal cost function $\Phi^*(x)$ is convex function defined on the convex hull (co) of $\mathbb{P}$. An upper bound $\bar{\Phi}(x)$ on the optimal cost $\Phi^*(x)$ for $x \in \text{co}(\mathbb{P})$ is obtained by the corresponding convex combinations of optimal cost functions $\Phi^*(x^j)$ for $x^j \in \mathbb{P}$. The upper bound on N^* is obtained by recognizing that the state trajectory w_j only remains outside of $\mathcal{O}_\infty$ for a finite number of stages. A lower bound q on the cost of $w_j^T Q w_j$ can be generated for $x_j \notin \mathcal{O}_\infty$ (see [10] for explicit details). It then follows that $N^* \leq \bar{\Phi}(x)/q$. Further refinement of the upper bound can be obtained by including the terminal stage penalty Π in the analysis.

Finally, efficient solution of the quadratic program generated by the MPC regulator is discussed in [68, 53].

3.3 Feasibility

In the implementation of model predictive control, process conditions arise where there is no solution to the optimization problem (33) that satisfies the constraints (34). Rather than declaring such situation process exceptions, we sometimes prefer a solution that enforces some of the inequality constraints while relaxing others to retain feasibility. Often the input constraints represent physical limitations such as valve saturation that cannot be violated. Output constraints, however, frequently do not represent hard physical bound. Rather, they often represent desired ranges of operations that can be violated if necessary. To avoid infeasibilities, we relax the output constraints by treating the them as "soft" constraints.

Various authors have considered formulating output constraints as soft constraints to avoid poten-

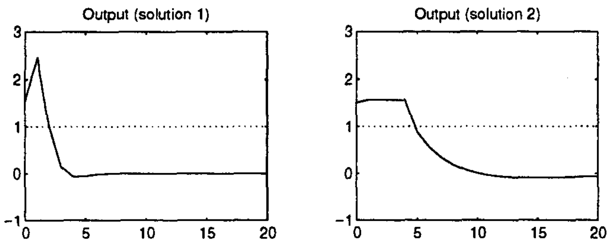

Figure 3: Two controller's resolution of output infeasibility. Output versus time. Solution 1 minimizes duration of constraint violation; Solution 2 minimizes peak size of constraint violation.

tial infeasibilities [55, 18, 14, 70, 61, 59]. We focus on the l_1/l_2 exact soft constraint strategy first advocated by de Oliveira and Biegler [14]. The attractive features of the l_1/l_2 formulation is that the quadratic programming structure is retained and the resulting solution is exact if a feasible solution exists.

Multi-objective nature of infeasibility problems. In many plants, the simultaneous minimization of the size and duration of the state constraint violations is not a conflicting objective. The optimal way to handle infeasibility is then simply to minimize both size and duration; regulator performance may then be optimized, subject to the "optimally" relaxed state constraints. Unfortunately, not all infeasibilities are as easily resolved. In some cases, such as nonminimum-phase plants, a reduction in size of violation can only be obtained at the cost of a large increase in duration of violation, and vice-versa. The optimization of constraint violations then becomes a multi-objective problem. In Figure 3 we show two different controllers' resolution of an infeasibility problem. The 2-state, SISO system model is

$$x_{k+1} = \begin{bmatrix} 1.6 & -0.64 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$
$$y_k = [-1 \quad 2] x_k$$

with constraints and initial condition

$$|y_k| \leq 1, \qquad x_0 = [1.5 \quad 1.5]'$$

Solution 1 corresponds to a controller minimizing the duration of constraint violation, which leads to a large peak violation, and solution 2 corresponds to a controller minimizing the peak constraint violation, which leads to a long duration of violation. This behavior is a system property caused by the unstable zero and cannot be avoided by clever controller design. For a given system and horizon N, the Pareto optimal size/duration curves can be plotted for different initial conditions, as in Figure 4. The user must then decide where in the size/duration plane, the plant should operate at times of infeasibility. Desired operation may lie on the Pareto optimal curve, because points below this curve cannot be attained and points above it are inferior, in the sense that they correspond to larger size and/or duration than are required.

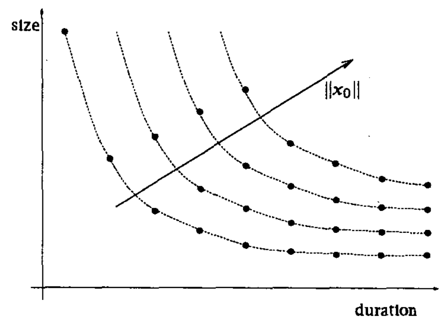

Figure 4: Pareto optimal curves for size versus duration of constraint violation as a function of initial condition, x_0.

We construct next soft output inequality constraints by introducing the slack variable ϵ_k into the optimization. We reformulate the variable horizon regulator with soft constraints as the following optimization

$$\min_{\{w_k, v_k, N\}} \Phi(x_j) = \frac{1}{2} \sum_{k=0}^{N-1} (w_k^T Q w_k + v_k^T R v_k + 2 w_k^T M v_k) + \epsilon_k^T Z \epsilon_k + z^T \epsilon_k + w_N^T \Pi w_N \quad (37)$$

subject to the constraints

$$w_0 = x_j, \quad w_{k+1} = A w_k + B v_k, \quad w_N \in \mathcal{O}_\infty \quad (38a)$$
$$d_{\min} \leq D v_k - G w_k \leq d_{\max} \quad (38b)$$
$$(y_{\min} - y_s) - \epsilon_k \leq C w_k \quad (38c)$$
$$C w_k \leq (y_{\max} - y_s) + \epsilon_k \quad (38d)$$
$$\epsilon_k \geq 0 \quad (38e)$$

We assume Z is symmetric positive definite matrix and z is a vector with positive elements chosen such that output constraints can be made exact if desired.

As a second example, consider the third-order non-

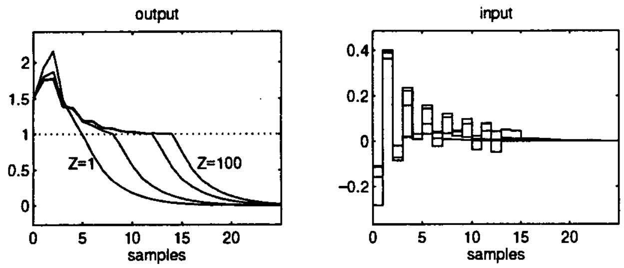

Figure 5: Least squares soft constraint solution. $Z = 1, 10, 50$ and 100. Solid lines: closed-loop, dashed lines: open-loop predictions at time 0, dotted line: output upper constraint.

minimum phase system

$$A = \begin{bmatrix} 2 & -1.45 & 0.35 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (39)$$

$$C = [-1 \quad 0 \quad 2] \quad (40)$$

for which the output displays inverse response. The controller tuning parameters are $Q = C'C$, $R = 1$ and $N = 20$. The input is unconstrained, the output is constrained between ± 1 and we perform simulations from the initial condition $x_0 = [1.5 \quad 1.5 \quad 1.5]'$. Figure 5 shows the possible trade-offs that can be achieved by adjusting the quadratic soft-constraint penalty, Z. We also see that open-loop predictions and nominal closed-loop response are in close agreement for all choices of tuning parameter.

3.4 State estimation

We now turn to reconstruction of the state from output measurements. In the model of Equation 12, the nonzero disturbances d and p are employed to give offset free control in the face of nonzero disturbances. The original industrial MPC formulations (GPC, QDMC, IDCOM) were designed for offset free control by using an integrating output disturbance model. The integrating output disturbance model is a standard device in LQG design [12, 25]. Similarly, to track nonzero targets or desired trajectories that asymptotically approach nonzero values, one augments the plant model dynamics with integrators. The disturbances may be modelled at the input, output, or some combination. These disturbance models are not used in the regulator; the disturbances are obviously uncontrollable, and are only required in the state estimator. The effects of the disturbance estimates is to shift the steady-state target of the regulator. Bitmead et al. [6] provide a nice discussion of the disturbance models popular in GPC. Lee, Morari and Garcia [29] discuss the equivalence between the original industrial MPC algorithms and different disturbance model choices. Shinskey [62] provides a nice discussion of the disadvantages of output disturbance models, in the original DMC formulation, compared to input disturbance models.

We set $d = 0$ and focus for simplicity on the output disturbance model. We augment the state of the system so the estimator produces estimates of both state, $\hat{x}$, and modelled disturbance, $\hat{p}$ with the standard Kalman filtering equations. The disturbance may be modelled by passing white noise, ξ_k, through an integrator, or by passing white noise through some other stable linear system (filter) and then through an integrator. The disturbance shaping filter enables the designer to attenuate disturbances with selected frequency content. Bitmead et al. [6] provide a tutorial discussion of these issues in the unconstrained predictive control context.

In the simplest case, the state estimator model takes the form

$$x_{j+1} = Ax_j + Bu_j + w_j \quad (41)$$

$$p_{j+1} = p_j + \xi_j \quad (42)$$

$$y_j = Cx_j + p_j + v_j \quad (43)$$

in which w_j, ξ_j, v_j are the noises driving the process, integrated disturbance, and output measurement, respectively. As shown in Figure 6, we specify $\tilde{Q}_w = R_v, \mathrm{diag}(Q_w, Q_\xi)$, which are the covariances of the zero mean, normally distributed noise terms. The optimal state estimate for this model is given by the classic Kalman filter equations [2]. As in standard LQG design, one can tune the estimator by choosing the relative magnitudes of the noises driving the state, integrated disturbance and measured output. Practitioners certainly would prefer tuning parameters more closely tied to closed-loop performance objectives, and more guidance on MPC tuning, in general, remains a valid research objective.

Assembling the components of Sections 3.1-3.4, produces the structure shown in Figure 6. This structure is certainly not the simplest available that accounts for output feedback, nonzero setpoints and disturbances, and offset free control. Nor is it the structure found in the dominant commercial vendor products. It is presented here mainly as a prototype to display a reasonably flexible means for handling these critical issues. Something similar to this structure has been implemented by industrial practitioners with success, however, and that imple-

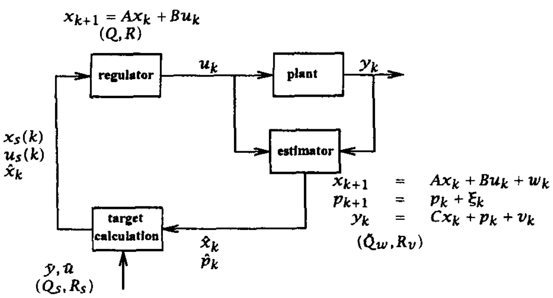

Figure 6: MPC controller consisting of: receding horizon regulator, state estimator, and target calculator.

mentation will be presented at this meeting [16].

4 MPC with Nonlinear Models

What is desirable and what is possible. From the practical side, industrial implementation of MPC with nonlinear models has already been reported [48], so it is certainly *possible.* But the implementations are largely without any established closed-loop properties, even nominal properties. A lack of supporting theory should not and does not, examining the historical record, discourage experiments in practice with promising new technologies. But if nonlinear MPC is to become widespread in the harsh environment of applications, it must eventually become reasonably reliable, predictable, efficient, and robust against on-line failure.

From the theoretical side, it would be desirable to solve in real time infinite horizon nonlinear optimal control problems of the type

$$\min_{\{x_k, u_k\}} \Phi(x_j) = \sum_{k=0}^{\infty} L(x_k, u_k) \tag{44}$$

subject to the constraints

$$x_{k+1} = f(x_k, u_k), \quad x_0 = x_j \tag{45}$$

$$u_k \in \mathcal{U}, \qquad x_k \in \mathcal{X} \tag{46}$$

Nonlinear MPC based on this optimal control problem would have the strongest provable closed-loop properties. The concomitant theoretical and computational difficulties associated with this optimal control problem, either off-line but especially on-line, are well known and formidable [33]. The current view of problem 44 is: *desirable*, but *not possible.* In the next two sections, we evince one viewpoint on the current status of bringing these two sides closer together.

4.1 State feedback

As an attempt to solve, approximately, Problem 44, it is natural to try to extend to the nonlinear case the ideas of Section 3.2. In the linear case, we define a region in state space, $\mathcal{W}$, with the following properties:

$$\mathcal{W} \subset \mathcal{X}, \qquad K\mathcal{W} \subset \mathcal{U}$$
$$x \in \mathcal{W} \implies (A + BK)x \in \mathcal{W}$$

which tells us that in $\mathcal{W}$: the state constraints are satisfied, the input constraints are satisfied under the unconstrained, linear feedback law $u = Kx$, and once a state enters $\mathcal{W}$, it remains in $\mathcal{W}$ under this control law. We can compute the cost to go for $x \in \mathcal{W}$; it is $x'\Pi x$ in which Π is given in Equation 35. For the linear case, the Gilbert and Tan algorithm provides in many cases the largest set $\mathcal{W}$ with these properties, $\mathcal{O}_\infty$.

Ingredients of open-loop, optimal control problem. In the simplest extension to the nonlinear case, consider a region $\mathcal{W}$ with the analogous properties:

$$\mathcal{W} \subset \mathcal{X}, \qquad K\mathcal{W} \subset \mathcal{U}$$
$$x \in \mathcal{W} \implies f(x, Kx) \in \mathcal{W}$$

The essential difference is that we must ensure that, under the nonlinear model, the state remains in $\mathcal{W}$ with the linear control law. Again, for this simplest version, we determine the linear control law by considering the linearization of f at the setpoint

$$A = \frac{\partial f}{\partial x}, \qquad B = \frac{\partial f}{\partial u}, \qquad C = \frac{\partial g}{\partial x}$$

For the nonlinear case, we cannot easily compute the largest region with these properties, but we can find a finite-sized region with these properties. The main assumption required is that f's partial derivatives are Lipschitz continuous [58] in order to bound the size of the nonlinear effects, and that the linearized system is controllable. Most chemical processes satisfy these assumption, and with them we can construct region $\mathcal{W}$ as shown in Figure 7. Consider the quadratic function V, and level sets of this function, $\mathcal{W}_\alpha$

$$V(x) = x'\Pi x, \qquad \mathcal{W}_\alpha = \{x | x'\Pi x \le \alpha\}$$

for α a positive scalar. Define $e(x)$ to be the difference be the state propagation under the nonlinear model and linearized model, $e(x) = f(x, Kx) - (A + BK)x$ and $e_1(x)$ to be the difference in V at these two states, $e_1(x) = V(f(x, Kx)) - V((A + BK)x)$. We can show that near the setpoint (origin)

$$\|e(x)\| \le c\,\|x\|^2, \qquad |e_1(x)| \le d\,\|x\|^3$$

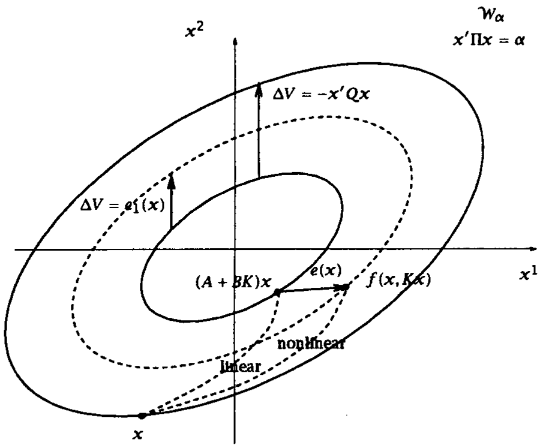

Figure 7: Region $\mathcal{W}$ level sets of $x'\Pi x$, and the effect of nonlinearity in a neighborhood of the setpoint.

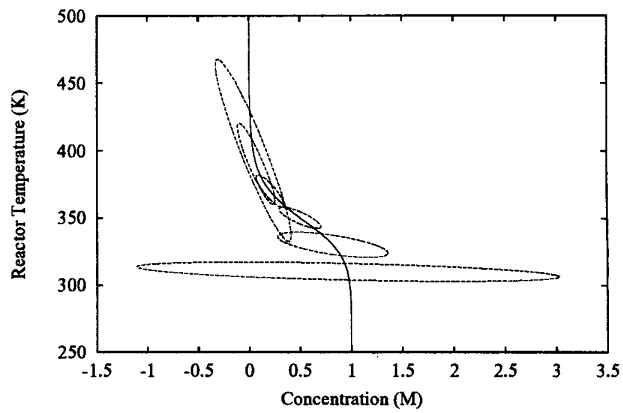

Figure 8: $\mathcal{W}$ regions surrounding the locus of steady-state operating points. Solid line: steady states. Dashed lines: $\mathcal{W}$ regions.

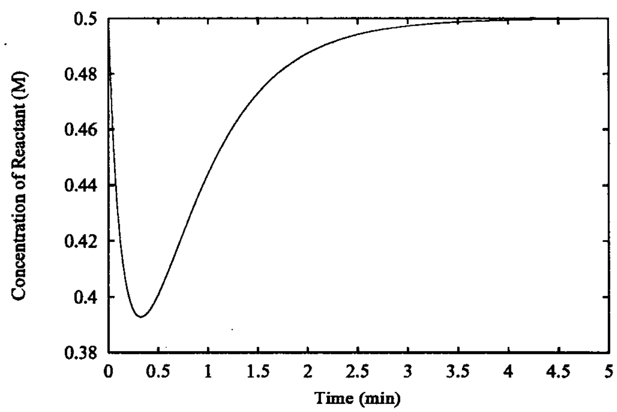

Figure 9: Closed-loop response. C_A versus t.

which bounds the effect of the nonlinearity. We can therefore find an α such that the finite horizon control law with terminal constraint and approximate cost to go penalty is stabilizing

$$\min_{\{x_k, u_k, N\}} \Phi(x_j) = \sum_{k=0}^{N-1} L(x_k, u_k) + x_N' \Pi x_N \tag{47}$$

subject to the constraints

$$x_{k+1} = f(x_k, u_k), \quad x_0 = x_j \tag{48}$$

$$u_k \in \mathcal{U}, \quad x_k \in \mathcal{X}, \quad x_N \in W_\alpha \tag{49}$$

We choose α such that

$$\max_{x \in W_\alpha} \{V(F_K(x)) - V(x) + (1/4)x^T Q x\} \leq 0 \tag{50}$$

It has also been established that global optimality in Problem 47 is not required for closed-loop stability [58]. Calculation of the $\mathcal{W}_\alpha$ region in Problem 50 remains a challenge, particularly when the target calculation and state estimation and disturbance models are added to the problem as in Sections 3.1-3.4. Under those circumstances, $\mathcal{W}_\alpha$, which depends on the current steady target, changes at each sample. It may be possible that some of this computation can be performed off-line, but resolving this computational issue remains a research challenge.

We present a brief example to illustrate these ideas. Consider the simple model presented by Henson and Seborg [21] for a continuously stirred tank reactor (CSTR) undergoing reaction A → B at an unstable steady state:

$$\frac{dC_A}{dt} = \frac{q}{V}(C_{Af} - C_A) - kC_A$$

$$\frac{dT}{dt} = \frac{q}{V}(T_f - T) + \frac{(-\Delta H)}{\rho C_p} kC_A + \frac{UA}{V\rho C_p}(T_c - T)$$

$$k = k_0 \exp(-E/RT)$$

Figure 8 displays the $\mathcal{W}$ regions computed by solving Problem 50 along the locus of steady-state operating points. For the steady-state operating point

$$T_s = 350K, \qquad C_{As} = 0.5M, \qquad T_{cs} = 300K$$

The closed-loop behavior of the states with MPC control law, Problem 47, is shown in Figures 9-10, and the manipulated variable is displayed in Figure 11. Figure 12 displays a phase-portrait of the two states

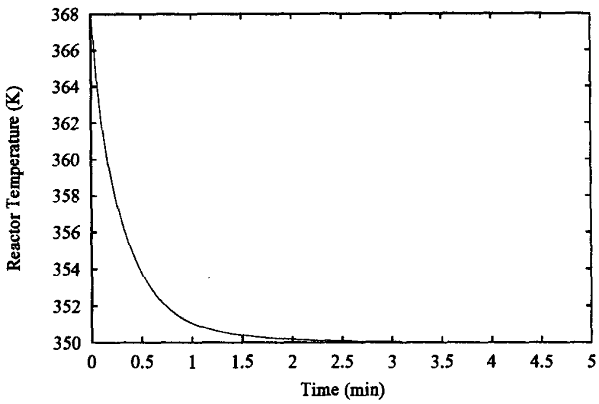

Figure 10: Closed-loop response. T versus t.

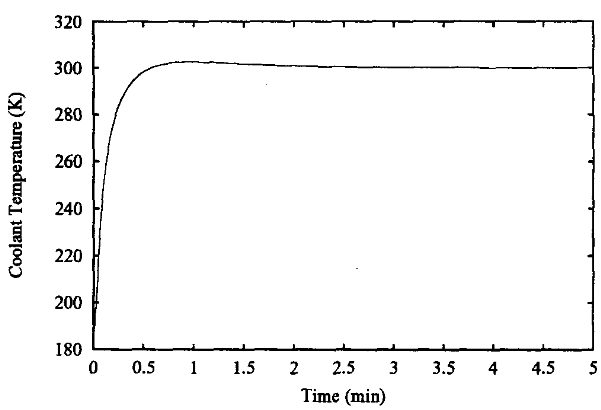

Figure 11: Manipulated variable. T_c versus t.

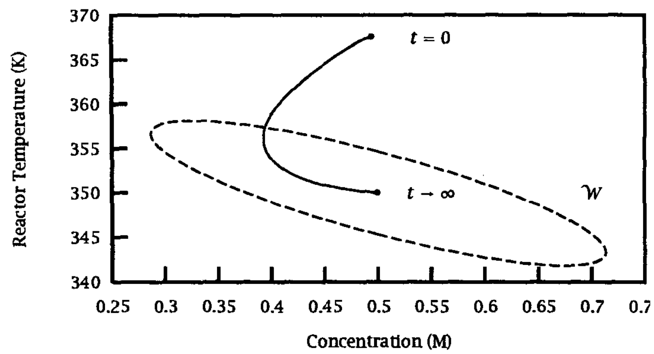

Figure 12: Closed-loop phase portrait under nonlinear MPC. Region $\mathcal{W}$ and closed-loop response T versus C_A.

converging to the setpoint and the terminal region $\mathcal{W}$.

5 Future Developments

Although this paper is intended as a tutorial, brief consideration of areas of future development may prove useful. The theory for nominal MPC with linear models and constraints is reasonably mature in that nominal properties are established, and efficient computational procedures are available. The role of constraints is reasonably well understood. Applications in the process industries are ubiquitous.

MPC with nonlinear models. In MPC for nonlinear models, the territory is much less explored. The nonconvexity of the optimal control problems presents theoretical and computational difficulties. The research covered in this tutorial on quasi-infinite horizons and suboptimal MPC provide one avenue for future development [43, 8]. Contractive MPC [45, 46, 69, 36] and exact linearization MPC [15, 24] are two other alternatives that show promise. Mayne et al. [34] and De Nicolao et al. [13] provide recent reviews of this field for further reading.

It is expected, as in the case of linear MPC of the 1970s and 1980s, that these theoretical hurdles will not impede practitioners from experimenting with nonlinear MPC [48, 31]. If the practical benefits of the nonlinear models are demonstrated, research efforts will redouble and likely produce continued improvements in the available theory and faster and more robust computational algorithms for implementation.

Robustness. Robustness to various types of uncertainty and model error is of course an active research area in MPC as well as other areas of automatic control. The difficulty that MPC introduces into the robustness question is the open-loop nature of the optimal control problem and the implicit feedback produced by the receding horizon implementation. Several robust versions of MPC have been introduced that address this issue [71, 18, 42] Lee and Yu [30] define a dynamic programming problem for the worst case cost. Badgwell [4] appends a set of robustness constraints to the open-loop problem, which ensures robustness for a finite set of plants. Kothare et al. [23] address the feedback issue by optimization over the state feedback gain rather than the open-loop control sequence subject to constraints.

The rapid development of time domain worst case controller design problems as dynamic games (see [3] for an excellent summary) has led to further proposals for robust MPC exploiting this connection to H_∞ theory [7, 9, 13]. At this early juncture, on-line computation of many of the robust MPC control laws appears to be a major hurdle for practical application.

Moving horizon estimation. The use of optimization subject to a dynamic model is the underpinning for much of state estimation theory. A moving horizon approximation to a full infinite horizon state estimation problem has been proposed by several researchers [41, 56, 67]. The theoretical properties of this framework are only now emerging [50, 52]. Again, attention should be focused on what key issues of practice can be addressed in this framework that are out of reach with previous approaches. Because moving horizon estimation with linear models produces simple, positive definite quadratic programs, on-line implementation is possible today for many process applications. The use of constraints on states or state disturbances presents intriguing opportunities, but it is not clear what applications benefit from using the extra physical knowledge in the form of constraints. Nonlinear, fundamental models coupled with moving horizon state estimation may start to play a larger role in process operations. State estimation for unmeasured product properties based on fundamental, nonlinear models may have more impact in the short term than closed-loop regulation with these models. State estimation using empirical, nonlinear models is already being used in commercial process monitoring software. Moreover, state estimation is a wide ranging technique for addressing many issues of process operations besides feedback control, such as process monitoring, fault detection and diagnosis.

MPC for hybrid systems. Essentially all processes contain discrete as well as continuous components: on/off valves, switches, logical overrides, and so forth. Slupphaug and Foss [64, 63] and Bemporad and Morari [5] have considered application of MPC in this environment. This problem class is rich with possibilities: for example, the rank ordering rather than softening of output constraints to handle infeasibility. Some of the intriguing questions at this early juncture are: how far can this framework be pushed, are implementation bottlenecks expected from the system modelling or on-line computations, what benefits can be obtained compared to traditional heuristics, and what new problem types can be tackled.

Acknowledgments

I wish to acknowledge Professor Thomas A. Badgwell for numerous helpful discussions of this material during our co-teaching of industrial MPC short courses.

The financial support of the National Science Foundation, through grant CTS-9708497, and the industrial members of the Texas-Wisconsin Modeling and Control Consortium is gratefully acknowledged.

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