Automatica, Vol. 28, No. 2, pp. 383–387, 1992
Printed in Great Britain.

0005–1098/92 $5.00 + 0.00
Pergamon Press plc

Brief Paper

Synthesis of an Optimal Control Law for Path Tracking in Mobile Robots*

A. HEMAMI,† M. G. MEHRABI‡ and R. M. H. CHENG

Key Words—Optimal control; path tracking; mobile robots.

Abstract—An optimal control law is derived for path tracking control of a class of wheeled mobile robots (WMR) or automated guided vehicles with front steering wheel. Optimality is based on minimizing a quadratic performance index in the position and orientation errors and the steering angle. The existence of such a controller which is in the form of a linear feedback of the two errors (position and orientation) formally proves why such a system can be stabilized about a trajectory by using position and orientation error feedback to determine the command to the steering angle. (This has been widely used recently.) The effect of the controller is illustrated by simulation results for various initial conditions in an experimental vehicle.

Introduction

Wheeled mobile robots (WMR) and automated guided vehicles (AGV) have been used in automated factories for material handling. They can also be used in other areas such as nuclear and explosive handling, security and hospital services. These vehicles require some type of guide path to follow, such as reflective tape, paint or buried wire. Alternatively, they can use a route map stored in their computer memory. Although AGVs have been in use for a long time, literature survey shows that their dynamics and control have not been fully investigated, whereas an analysis of their kinematics and dynamics are essential for proper control of their motion. In a recent study (Muir and Neuman, 1987a, b), the kinematics of the vehicle was developed based firstly on a kinematic model for each wheel, and a condition was proposed to determine whether ideal rolling is possible. The same problem was partly investigated by Alexander and Maddocks (1989), but with a different approach. Saha and Angeles (1989) used the concept of orthogonal complement of the matrix of nonholonomic constraints to develop the dynamic equations of these systems. Grattinger and Krogh (1989) categorized the constraints that limit the class of executable trajectories for a WMR into path constraints, kinematic constraints and dynamic constraints.

The general concepts for their control can be found in Borenstein and Koren (1985), and their tracking stability is discussed in Larcombe (1981), Julliere *et al.* (1983). It is now well-known that a proportional controller, based on feedback of the offset (position error) and orientation error, as a command to the steering wheel, gives satisfactory results. However, not enough explanation on the performance of the controlled system can be observed, nor is an analysis given for calculation of the proportional gains, which are usually determined by trial and error for each particular system. Kanayama *et al.* (1990) recently formulated the tracking problem in a systematic way; by introducing a Liapunov function, they determined the structure of a controller to stabilize the system and bring the position and orientation error to zero, for plane motion of the vehicle based on geometry of motion. In a different approach, which is based on the dynamic model for plane motion of vehicle, Hemami *et al.* (1990) introduced a nonlinear controller with superior performance in path tracking for this class of vehicle, compared to a conventional controller based on a linear feedback of the position and orientation errors (Julliere *et al.*, 1983; Larcombe, 1981).

In this paper based on a dynamic model for the plane motion of a three-wheeled cart with front steering wheel, an optimal controller is sought such that a performance criterion consisting of the squared errors in position and orientation is minimized. In this way the feedback gains will be explicitly determined in terms of physical specifications of any vehicle. The results are implemented for simulation studies of a vehicle and the simulation results are presented.

Dynamic model of the system

Studies on dynamics of wheeled vehicles have been carried out by a number of researchers (Dugoff *et al.*, 1970; Wong, 1978). Three-dimensional dynamics can be considered, but for steering control purposes a simplified model defining the plane motion of such a system is sufficient. The latter is employed in this work.

In Fig. 1 the tangent to the trajectory at a general point A is taken as the momentary X-axis and the normal to that is the Y-axis. U and W are the vehicle-attached longitudinal and lateral coordinates with their origin at point C, the vehicle mass center. The angle between CU and AX is the orientation error ε_θ and the distance CA is the position error ε_d, or the offset. Figure 1 indicates, also, dimensions and the definition of some other variables. Furthermore, it illustrates that if there is no error in either the orientation or the position, then the two sets of coordinates will coincide, and $\dot{y} = v_w$. Figure 2 shows the new position of the vehicle after a small period of time, when it has rotated around point P by an angle ψ. Point P is the instantaneous centre of rotation which is determined by the intersection of a line perpendicular to the rear tires and a line perpendicular to the front tire. The location of this point depends on the steering angle δ. The nonslip condition for this class of vehicle implies that the rear wheels have no lateral velocity; in this sense, point Q the middle point on the rear axle has no lateral velocity. The velocities of all other points on the vehicle can be assumed to consist of the forward velocity of point Q and a rotation about an axis through point Q with the angular velocity ω. If the velocity components of the mass centre in the forward and lateral directions are denoted by v_u and v_w, then the new position of the vehicle can be considered as being obtained by a forward translation with

* Received 4 October 1990; revised 7 May 1991; revised 10 August 1991; received in final form 3 July 1991. The original version of this paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor M. G. Rodd under the direction of Editor H. Austin Spang III.

† Canadian Centre for Automation and Robotics in Mining (CCARM), École Polytechnique de Montréal, C.P. 6079, Succ. "A" Montréal, Québec, Canada, H3C 3A7. Author to whom all correspondence should be addressed.

‡ Department of Mechanical Engineering, Concordia University, Montreal, Québec, Canada, H3G 1M8.

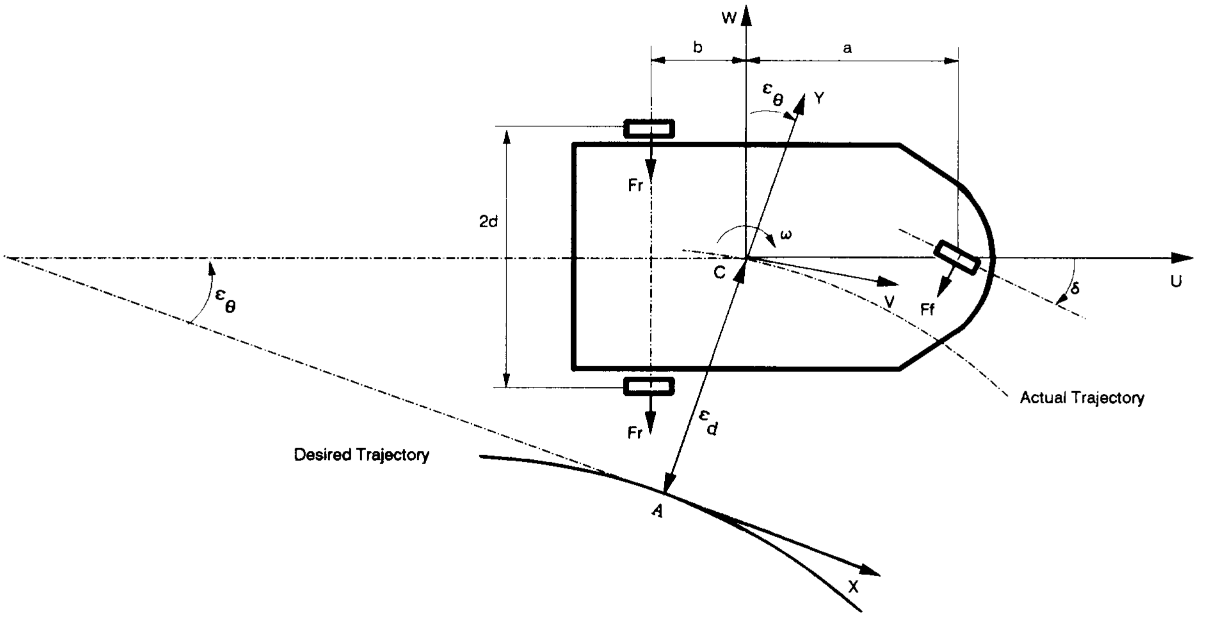

FIG. 1. Definition of various parameters and dimensions for a vehicle and its trajectory.

velocity v_u, a lateral translation with velocity v_w and a rotation by an angle ψ.

From coordinate transformation relationships, Fig. 2 reveals that

$$\dot{\varepsilon}_d = \dot{y} = v_w \cos\psi + v_u \sin\psi \tag{1}$$

where ψ is the angular displacement, that is,

$$\dot{\psi} = \omega. \tag{2}$$

For small ψ ($\sin\psi = \psi$, $\cos\psi = 1$), equation (1) can be simplified to

$$\dot{y} = v_w + v_u\psi. \tag{3}$$

The side forces on the front and rear tires are proportional to side-slip angles (the angle between a plane normal to the axle of a tire and the velocity vector of the point of the vehicle frame where that tire is attached) for small values of these angles (Wong, 1978); that is,

$$F_f = C_f\beta_f \tag{4}$$

$$F_r = C_r\beta_r \tag{5}$$

where F_f and F_r are the front wheel and rear wheel lateral forces, β_f and β_r are the side-slip angles, and C_f and C_r are the cornering stiffness of the front and rear wheels, respectively (see Appendix 1).

For the steering wheel in the front, the approximate

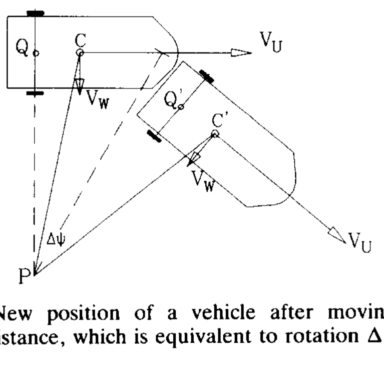

FIG. 2. New position of a vehicle after moving a short distance, which is equivalent to rotation $\Delta\psi$.

expressions for β_f and β_r can be written as (Wong, 1978):

$$\beta_f = \delta - \frac{v_w + a\omega}{v_u} \tag{6}$$

$$\beta_r = \frac{b\omega - v_w}{v_u} \tag{7}$$

where δ represents the steering angle (Fig. 1).

Writing the mass-force relationship for lateral motion of the vehicle leads to (assuming forward velocity to be constant):

$$C_f\beta_f + 2C_r\beta_r = m\ddot{y} = m\dot{v}_w + m\omega v_u. \tag{8}$$

Also, taking moments about the centre of mass gives

$$aC_f\beta_f - bC_r\beta_r = I\dot{\omega} \tag{9}$$

where a and b are constant as shown in Fig. 1, and I is the moment of inertia.

From equations (2) to (9) and after some manipulation the linearized equations of the system in the state space form can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\delta \tag{10}$$

where

$$\mathbf{x}^{\mathrm{T}} = [\varepsilon_d, v_w, \omega, \varepsilon_\theta] \tag{11}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & v_u \\ 0 & \dfrac{-C_f - 2C_r}{mv_u} & \dfrac{2bC_r - aC_f}{mv_u} - v_u & 0 \\ 0 & \dfrac{2bC_r - aC_f}{Iv_u} & \dfrac{-2b^2C_r - a^2C_f}{Iv_u} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

$$\mathbf{b} = \begin{bmatrix} 0 & \dfrac{C_f}{m} & \dfrac{aC_f}{I} & 0 \end{bmatrix}^{\mathrm{T}}. \tag{13}$$

This system has two open-loop poles (eigenvalues of matrix A) at origin, and the other two are in the form of:

$$\lambda_{2,3} = \tfrac{1}{2}[\alpha \pm \sqrt{\beta^2 + 4\gamma}] \tag{14}$$

where

$$\alpha = \frac{-1}{mIv_u}[C_f(I + ma^2) + 2(I + mb^2)C_r] \tag{15}$$

which is always negative,

$$\beta = \frac{(a^2m - I)C_f + 2(mb^2 - I)C_r}{mIv_u} \quad (16)$$

and

$$\gamma = \frac{(2bC_r - aC_f)^2}{mIv_u^2} - \frac{(2bC_r - aC_f)}{I}. \quad (17)$$

Typical values of α, β and γ for physical systems show that these two poles are far from origin and in the left half of the complex plane; that is, the system possesses two fast (stable) modes and two slow modes. The fast modes correspond to the lateral and angular velocities of the vehicle (v_w and ω), whereas the slow modes correspond to the first and fourth states; that is, the position offset and the orientation error, ε_d and ε_θ.

Synthesis of an optimal controller

For the system defined by equations (10)–(13), an optimal controller to minimize the performance index

$$J = \int_0^\infty (q_1\varepsilon_d^2 + q_2\varepsilon_\theta^2 + R\delta^2)\,dt \quad (18)$$

is sought. q_1, q_2, and R are positive scalars that play as weighting factors for the two errors, ε_d and ε_θ, and the control effort represented by the steering angle δ. However, for such a system with slow and fast modes, the effect of fast modes disappear almost instantly. For this class of system, an optimal control based on only the slow subsystem will suffice, as the results of this study show.

In order to decouple the slow and fast modes, a linear transformation of the state in the form of:

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}, \quad (19)$$

where $\mathbf{P}$ is the matrix of eigenvectors of $\mathbf{A}$, changes equation (10) into:

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{b}\delta = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{b}'\,\delta \quad (20)$$

where $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is in Jordan canonical form because of the repeated eigenvalues at the origin.

Interestingly, the two column vectors of $\mathbf{P}$ corresponding to slow modes can be in the simple form of:

$$\mathbf{s}_1 = [v_u \quad 0 \quad 0 \quad 0]^T$$

and

$$\mathbf{s}_4 = [0 \quad 0 \quad 0 \quad 1]^T$$

where v_u is the forward velocity of the vehicle.

The subsystem associated with slow modes has the following form:

$$\dot{\mathbf{z}}_{1,4} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}_{1,4} + \begin{bmatrix} b'_1 \\ b'_4 \end{bmatrix} \delta. \quad (21)$$

The unique optimal controller for a linear system in the form of

$$\dot{\mathbf{z}}(t) = \mathbf{C}\mathbf{z}(t) + \mathbf{D}\mathbf{u}(t) \quad (22)$$

where $\mathbf{z}(t)$ is the state vector and $\mathbf{u}(t)$ is the input vector, to minimize the cost function (18) is given by Athans and Falb (1966):

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{D}^T\mathbf{K}\mathbf{z}(t) \quad (23)$$

where $\mathbf{K}$ is a constant positive definite matrix which results from solving the algebraic matrix Ricatti equation:

$$-\mathbf{K}\mathbf{C} - \mathbf{C}^T\mathbf{K} + \mathbf{K}\mathbf{D}\mathbf{R}^{-1}\mathbf{D}^T\mathbf{K} - \mathbf{Q} = 0 \quad (24)$$

where $\mathbf{Q}$ is a diagonal matrix of compatible dimension of the weighting factors for each state.

In the next section numerical computations are performed to synthesize an optimal controller for an AGV. The results of simulation for implementation of this controller are also presented.

TABLE 1. THE VEHICLE'S CHARACTERISTICS EMPLOYED FOR SIMULATION

Symbol	Name	Value
a	Distance of c.g. to the front wheel axis	0.36 m
b	Distance of c.g. to the rear wheel axis	0.03 m
I	Yaw moment of inertia	14.6 kg/m^2
C_f, C_r	Cornering stiffness	6220 N/rad
m	Mass	124.4 kg
d	Half distance between two rear wheels	0.4 m

Illustrative example and simulation results

As for observing the effect of the controller designed based on the analysis in this paper, the specifications of an automated guided vehicle (Cheng and Huang, 1990) is used. These data are given in Table 1.

For this system, when the forward velocity $v_u = 0.4$ m/sec, the open loop plant matrix is:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0.4 \\ 0 & -375 & -37.5 & 0 \\ 0 & -319.5 & -140 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (25)$$

which has the following eigenvalues:

$$\begin{aligned} \lambda_{1,4} &= 0 \\ \lambda_2 &= -96.5 \\ \lambda_3 &= -418.5. \end{aligned} \quad (26)$$

The input vector has the values:

$$\mathbf{b} = [0 \quad 50 \quad 153.4 \quad 0]^T \quad (27)$$

and with the following choice of eigenvectors

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.0014 & -0.0016 & 0 \\ 0 & -0.1348 & 0.6571 & 0 \\ 0 & 0.9908 & 0.7538 & 0 \\ 0 & -0.01 & -0.0018 & 1 \end{bmatrix} \quad (28)$$

the transformed system according to equation (19) is:

$$\begin{aligned} z_1 &= 2.5x_1 + 0.0088x_2 - 0.0023x_3 \\ z_2 &= -1.0015x_2 + 0.873x_3 \\ z_3 &= 1.3164x_2 + 0.1791x_3 \\ z_4 &= -0.0076x_2 + 0.0091x_3 + x_4. \end{aligned} \quad (29)$$

As it is seen from equation (29) with good approximation

$$\begin{aligned} z_1 &\simeq 2.5x_1 = \frac{1}{v_u}\varepsilon_d \\ z_4 &\simeq x_4 = \varepsilon_\theta. \end{aligned} \quad (30)$$

The relations in equation (30) are general and are true for every system (that is, the other terms in z_1 and z_4 are always negligible).

The subsystems that result from linear transformation (19) are in the form of:

$$\dot{\mathbf{z}}_{1,4} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{z}_{1,4} + \begin{bmatrix} 0.08 \\ 1 \end{bmatrix} \delta \quad (31)$$

$$\dot{\mathbf{z}}_{2,3} = \begin{bmatrix} -96.5 & 0 \\ 0 & -418.5 \end{bmatrix} \mathbf{z}_{2,3} + \begin{bmatrix} 83.9 \\ 93.3 \end{bmatrix} \delta. \quad (32)$$

(The transformed plant matrix is in Jordan Canonical form because of repeated eigenvalue $\lambda_{1,4} = 0$.)

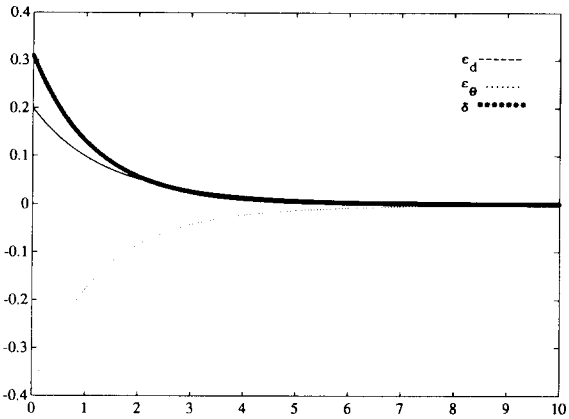

FIG. 3. Initial offset = 0.2 m, initial orientation error = $\pi/8$.

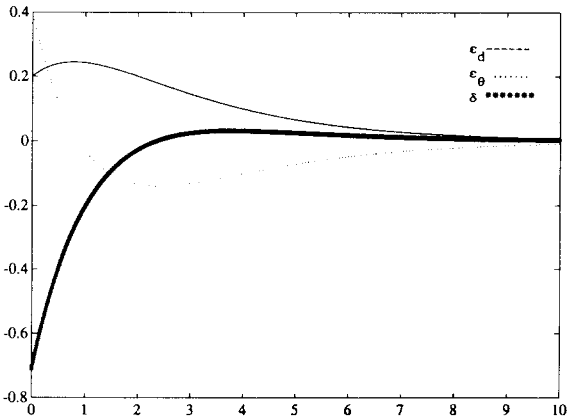

FIG. 4. Initial offset = 0.2 m, initial orientation error = $-\pi/8$.

The design of an optimal controller for the subsystem in equation (31) for unity weighting factors for ε_d, ε_θ, and δ in equation (18), while ε_d and ε_θ are related to z_1 and z_4 according to equation (30), leads to the following matrix $\mathbf{K}$ in Ricatti equation (24):

$$\mathbf{K} = \begin{bmatrix} 0.52 & 0.36 \\ 0.36 & 1.3 \end{bmatrix} \tag{33}$$

which gives rise to the control law:

$$\delta = \varepsilon_d + 1.3\varepsilon_\theta. \tag{34}$$

The result of simulation studies for various initial ε_d and ε_θ after the implementation of control law (34) are illustrated in Figs 3–6.

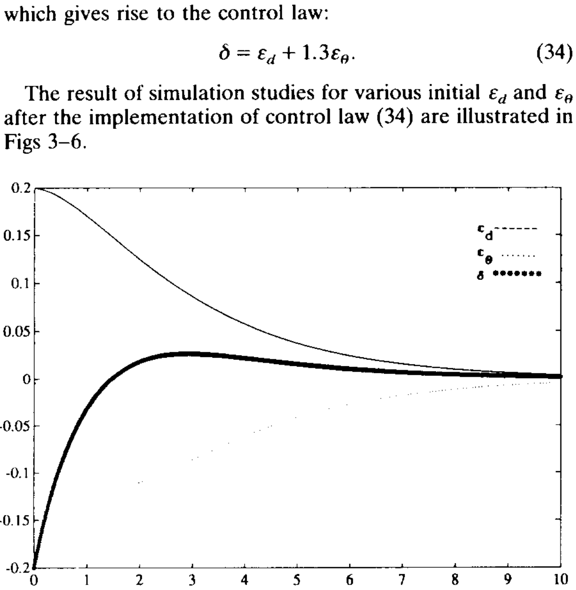

FIG. 5. Initial offset = 0.2 m, initial orientation error = 0.

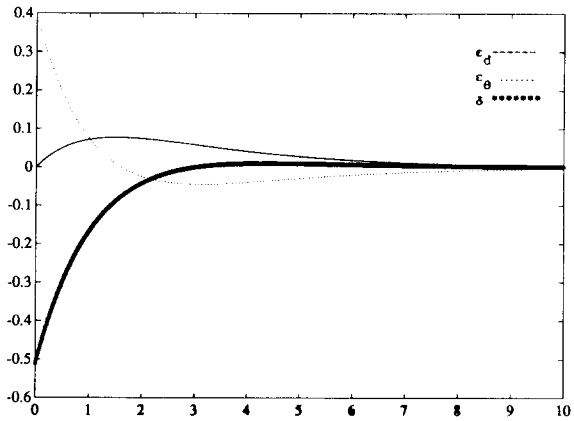

FIG. 6. Initial offset = 0, initial orientation error = $\pi/8$.

Conclusions

Studied here is the synthesis of an optimal control law for three-wheeled front steering vehicles. The synthesis is based on a linearized plane motion dynamic model of the vehicle and the minimization of a quadratic performance index incorporating the two measurable errors of the vehicle in path tracking: the offset and the orientation error. The effect of such a linear quadratic optimal controller on the controlled system is demonstrated by simulating an automated guided vehicle on a digital computer, illustrating the performance of the system in tracking the zero error condition when started from several non-zero initial conditions.

References

Alexander, J. C. and J. H. Maddocks (1989). On the kinematics of wheeled mobile robots. *Int. J. Robotic Res.*, **8,** 15–27.

Athans, M. and P. L. Falb (1966). *Optimal Control.* McGraw-Hill, New York.

Borenstein, J. and Y. Koren (1985). A mobile platform for nursing robots. *IEEE Trans. Ind. Elect.*, **IE-32,** 158–165.

Cheng, R. M. H. and M. Huang (1990). Dynamic modelling of an AGV system. Internal Report, Centre for Industrial Control, Concordia University, Montreal.

Dugoff, H., P. S. Fancher and L. Segel (1970). An analysis of tire traction properties and their influence on vehicle dynamic performance. *SAE Trans.*, 1219–1243.

Grattinger, T. J. and B. H. Krogh (1989). Evaluation and time-scaling of trajectories for wheeled mobile robots. *ASME J. Dynam. Syst. Meas. Control*, **111,** 222–231.

Hemami, A., M. G. Mehrabi and R. M. H. Cheng (1990). A new control strategy for tracking in mobile robots and AGVs, *IEEE Int. Conf. on Robotics and Automation*, Cincinnati, OH, 1122–1127.

Julliere, M., L. Marce and H. Place (1983). A guidance system for a mobile robot. *Proc. 13th ISIR/Robot 7 Conf.*, Chicago, IL 13.58–13.68.

Kanayama, Y., Y. Kimura, F. Miyazaki and T. Noguchi (1990). A stable tracking control method for an autonomous mobile robot. *IEEE Int. Conf. on Robotics and Automation*, Cincinnati, OH, 384–389.

Larcombe, M. H. E. (1981). Tracking stability of wire guided vehicles, *Proc. 1st Int. Conf. on Automated Guided Vehicle Systems*, Stratford-upon-Avon, UK, pp. 137–144.

Muir, P. F. and C. P. Neuman, (1987a). Kinematic modelling of wheeled mobile robots. *J. Robotic Syst.*, **4,** 281–340.

Muir, P. F. and C. P. Neuman (1987b). Kinematic modelling for feedback control of an omnidirectional mobile robot. *IEEE Int. Conf. on Robotics and Automation*, 1772–1778.

Saha, S. K. and J. Angeles (1989). Kinematics and dynamics of a three-wheeled 2-DOF AGV. *IEEE Int. Conf. on Robotics and Automation*, Scottsdale, AZ, 1572–1577.

Wong, J. Y. (1978). *Theory of Ground Vehicles.* Wiley, New York.

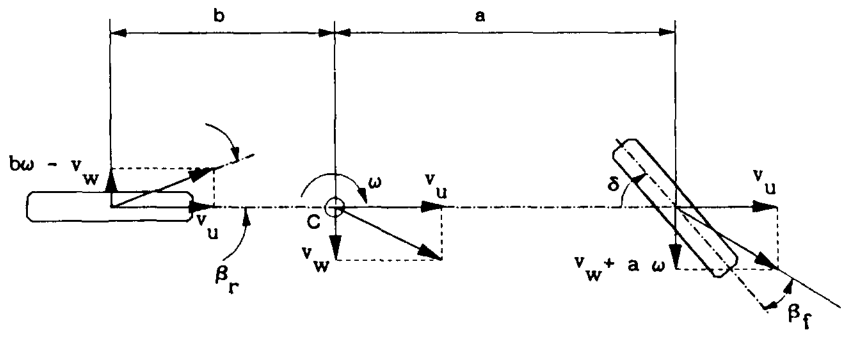

FIG. 7. Definition of side-slip angles β_f and β_r.

Appendix 1. *Side-slip angles*

The side-slip angle of a tire is defined as the angle between the plane of the tire and the velocity vector of tire center (β_f and β_r in Fig. 7). Referring to Fig. 7, the following expressions can be written for β_f and β_r (these angles are usually small and trigonometric approximation can be used as follows):

$$\beta_f = \delta - \tan^{-1}\frac{v_w + a\omega}{v_u} \cong \delta - \frac{v_w + a\omega}{v_u} \tag{A1}$$

$$\beta_r = \tan^{-1}\frac{b\omega - v_w}{v_u} \cong \frac{b\omega - v_w}{v_u}. \tag{A2}$$

The lateral tire forces acting on the front and rear wheels are functions of the corresponding side-slip angle and cornering stiffness. The expressions for lateral forces for the front and rear wheels can be written as Wong (1978):

$$F_f = C_f\beta_f \tag{A3}$$

$$F_r = C_r\beta_r \tag{A4}$$

where C_f and C_r are the cornering stiffness of the front and rear tires, respectively.