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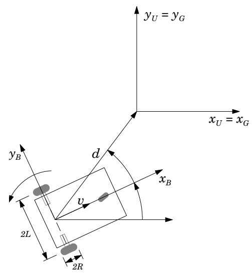

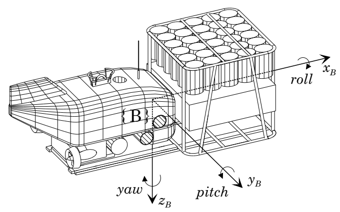

Nonlinear Motion Control of Nonholonomic and Underactuated Systems

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Abstract

This thesis addresses the problem of stabilization of nonholonomic and underactuated systems. The key motivation for this research topic stems from the fact that nonholonomic systems pose considerable challenges to control system designers since those systems cannot be stabilized by smooth, time-invariant, state-feedback control laws. Furthermore, in spite of the number of methods available for the control of nonholonomic and underactuated mechanical systems, few address important practical topics such as the explicit inclusion of dynamics in the control problem formulation and the need to cope with model parameter uncertainty and external disturbances. This thesis tackles some of theses issues, formulates and solves the related control problems, and discusses the application of the new control methodologies derived to robotic land and underwater vehicles.

The first part of the thesis focuses on the control of the so-called extended nonholonomic double integrator (ENDI), which captures the kinematics and dynamics of a class of mobile robots. A solution to the problem of global stabilization of the ENDI is given that builds on logic-based hybrid control. The methodology derived is applied to the problem of stabilizing the dynamic model of an underactuated autonomous underwater vehicle (AUV) to a desired pose.

The second part of the thesis is devoted to the problem of steering a wheeled mobile robot and an underactuated AUV to a target point with a desired orientation, in the presence of parametric modeling uncertainty. Controller design relies on non-smooth coordinate transformations in the original state space, followed by the derivation of adaptive, smooth, time invariant feedback control laws in the new coordinates. The problem of positioning and way-point tracking of an underactuated AUV in the presence of an unknown ocean current disturbance is also investigated and solved using a similar approach.

The last part of the thesis is devoted to the general problem of nonlinear system stabilization. A new controller design methodology is presented for a large class of nonlinear systems that builds on recently developed switching hybrid control techniques and classical Lyapunov based design tools.

Throughout the thesis, formal proofs of convergence of the algorithms derived are presented. Simulation results obtained with full nonlinear models of representative land and underwater robots are discussed.

Keywords: Underactuated and Nonholonomic Systems, Hybrid Control, Nonlinear and Adaptive Control, Convergence and Stability Analysis, Autonomous Wheeled Mobile Robots, Autonomous Underwater Vehicles.

Resumo

O tema principal desta tese é o controlo de sistemas subactuados e não-holonómicos. É conhecida a impossibilidade de estabilizar sistemas não holonómicos recorrendo a leis de controlo diferenciáveis e invariantes no tempo. As soluções descritas na literatura fazem apelo a controladores descontínuos, híbridos, ou até variantes no tempo. No entanto, as soluções apresentadas não se revelam eficazes em situações realistas em que: *i)* a dinâmica dos sistemas a controlar não se pode desprezar, *ii)* existem incertezas paramétricas nos modelos e *iii)* os sistemas estão sujeitos a perturbações externas. A tese propõe novas metodologias para a resolução destes problemas e contempla aplicações nos domínios da robótica terrestre e marinha.

A primeira parte da tese apresenta uma estratégia de controlo híbrido para estabilização global do denominado integrador duplo não holonómico estendido. A metodologia desenvolvida serve também de base ao desenvolvimento de um controlador híbrido para o modelo dinâmico de um veículo submarino autónomo (VSA) subactuado.

A segunda parte é dedicada ao problema da estabilização de veículos robóticos com incerteza paramétrica, nomeadamente robôs móveis terrestres e VSAs subactuados. A metodologia proposta conduz a estratégias de controlo que se baseiam numa transformação não macia de coordenadas, seguida do projecto de controladores macios e invariantes no tempo utilizando técnicas de Lyapunov. As técnicas de projecto são estendidas a fim de resolver o problema do posicionamento de VSAs subactuados sujeitos à acção de correntes marinhas desconhecidas.

A última parte da tese é dedicada ao problema geral da estabilização de sistemas não lineares. As técnicas introduzidas fazem apelo à teoria do controlo híbrido e a técnicas clássicas de Lyapunov.

Ao longo deste trabalho apresentam-se demonstrações formais de convergência dos diversos algoritmos de controlo propostos e discutem-se os resultados de simulações obtidas com modelos dos veículos estudados.

Palavras-chave: Sistemas Subactuados e não Holonómicos, Controlo Híbrido, Controlo não Linear e Adaptativo, Análise de Convergência e Estabilidade, Robôs Terrestres, Veículos Submarinos Autónomos.

To
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Contents

Chapter 1

Introduction

Nonlinear systems represent not a special case but simply everything that is not subjected to the special assumption of linearity. Scientific progress will undoubtedly occur by singling out special classes of systems subject to restrictive structural assumptions other than linearity. R. E. Kalman

This introductory chapter sets the background for the main subject of the thesis: nonlinear control of nonholonomic and underactuated systems. The chapter starts with a brief historical account of linear and nonlinear control theory. This should be viewed as quick guided tour that ends with a short summary of the new problems posed and solved in the present work. The rest of the chapter provides the practical motivation for the work carried out and revisits key nonlinear control techniques that are instrumental in dealing with practical problems of nonholonomic and underactuated vehicle control. Having introduced the proper stage for discussion, the chapter ends with the main thesis contributions and organization.

1.1 Historical perspective

Throughout the ages, the scope and the horizons of control engineering have undergone dramatic transformations. Some of the earliest ingenious control devices date back to the ancients times. Roman engineers maintained water levels for their aqueduct systems by means of floating valves that opened and closed at appropriate levels. In the Middle Ages, methods for the automatic operation of windmills were developed. The Dutch windmill of the 17th century was kept facing the wind by the action of an auxiliary vane that moved the entire upper part of the mill. The most famous example of a control device from the time of the Industrial Revolution is James Watt's flyball governor of 1769, a device that regulated steam flow to a steam engine to maintain constant engine speed despite a changing load.[1]

[1] For more background on the history of control, see the set of survey papers appearing in the IEEE Control Systems Magazine (November 1984). A good discussion of the historical developments of control is given by (Mayr, 1970), (Fuller, 1976), and (Dorf, 1980). Many other references are cited by these authors for the interested reader.

Although control has a rich history dating back to applications in the remote past, many of the accomplishments and significant theoretical contributions to the area are well rooted in the past century, with special emphasis after Word War II.

In the late 1930s classical control theory began to develop gradually. The need to design stable telephone repeater amplifiers led to the work of Black in 'regeneration theory' and to Nyquist frequency domain stability criterion, where the frequency domain approaches developed by Laplace (1749-1827), Fourier (1768-1830), Cauchy (1789-1857), and others were explored. The same stimuli led to Bode's complex frequency-domain theory and to the widely used relationship between gain and phase. Simultaneous with the development of the feedback amplifier, feedback control of industrial processes was becoming standard. The PID controller was first described by Calendar in 1936. This technology was based on extensive experimental work and relied on simple linear approximations to system dynamics. World War II (1939-1945) led to further developments in control theory because of the need for radar control and tracking and the development of servomechanisms for positioning guns.

By the late 1940s the concepts of frequency response and transfer functions were firmly established and the first analog computers were becoming available. N. B. Nichols developed his Nichols chart for the design of feedback systems and W. R. Evans in 1948 published his celebrated root-locus technique.

During the 1950s, much of the work on control was focused on the s-plane and on obtaining desirable closed loop step response characteristics. Straightforward design tools became available based on simple graphical techniques that provided great intuition to solve single loop feedback control problems. These methods became known by the generic term of *classical control*. Unfortunately, the techniques were difficult to apply to multiple inputs multiple outputs systems. This is due to the fact that looking at single-input, single-output transmission pairs one at a time does not guarantee good performance, or even closed loop stability of the original multivariable control systems. More sophisticated analysis and design techniques were thus required.

With the advent of the space age (bringing with it far more complex control problems) and stimulated by the development of digital computer technology, the 1960s witnessed the development of an alternative state space approach to control. This followed the publication of some seminal papers by Wiener, Bellman, Pontryagin, Kalman, and others on optimal control and estimation theory allowing to treat multivariable problems in a unified fashion and opening a new era in control theory, henceforth referred to as *modern control*. During this period, important new theoretical concepts and tools were introduced such as the notions of controllability, observability, and optimality in control theory, the linear quadratic regulator design method, and the Kalman-Bucy filter. Also, the importance of the ground breaking work of Lyapunov on stability of nonlinear systems was finally realized.

By the 1980s, the field of control had reached a high level of sophistication. However, with all its power and advantages, modern control was lacking in some fundamentals aspects: *i)* poor engineering intuition in striking contrast to what happens when using frequency domain techniques of classical control theory, and *ii)* limited powerful tools to analyze robustness of performance and stability in the face of poorly modeled dynamics and/or parameter uncertainty. As a contribution to overcoming these limitations, J. Doyle and others pointed out, in the early 1980s, the importance of plots of singular values versus frequency as analysis tools for robust multivariable control. This extended the work of Rosenbrock and MacFarlane, who during the 1970s had contributed to expanding the applications of classical frequency domain techniques to multivariable systems. During the 1980s, some powerful design tools emerged such as the LQG/LTR technique and H_2/H_∞ optimization paving the way for the emergence of a new control theory that blends the best features of classical and modern control techniques. A survey of this robust modern control theory is provided by (Dorato, 1987).

Over the past few years, innovative and powerful methods have been developed to tackle complex problems by casting them in the framework of Linear Matrix Inequalities (LMIs). The importance of LMIs in systems and control theory lies in the wide variety of problems that can be formulated as convex or quasi-convex optimization problems involving LMIs and then solved numerically using efficient, recently developed interior point methods. The reader interested in an overview of recent theoretical work in the field is referred to (Boyd *et al.*, 1994) and the references therein. See also (Apkarian and Gahinet, 1995; Chilali and Gahinet, 1996; Iwasaki and Skelton, 1994; Scherer *et al.*, 1997) for a lucid presentation of some of the latest developments in the field, namely in establishing a general multi objective linear controller synthesis framework using LMIs. The theory developed has reached a high degree of sophistication and extensions have been made to deal with nonlinear systems in the framework of gain-scheduled control and linear parameter varying systems (LPVs). See for example (Rugh and Shamma, 2000) and the references therein for the theory and interesting applications.

In parallel, there has been considerable research effort in the area of nonlinear control. This has been motivated by the fact that most real world problems involve nonlinear effects. Traditionally, control engineers have been concerned with local performance about selected operating points when the systems under study can be suitably approximated by linear models. A basic limitation of this approach is that it only yields stability and performance guarantees in a neighborhood of a single operating point, and is thus very restrictive for most real world systems where large deviation from nominal operating points occur. Moreover, there are many observed phenomena such as the existence of multiple equilibria or operating points, periodic variation of state variables or limit cycles, finite escape time, bifurcations, and other complex dynamic behaviours that can not be described or predicted by linear models. Nevertheless, it is important to point out that in the scope of linear systems, a vast collection of powerful methods have been developed and linear systems are the only class of models for which a reasonably complete mathematical theory of control exists so far.

Some factors make the study of nonlinear systems far more difficult than that of linear systems. A significant distinguishing property is the fact that nonlinear systems do not satisfy the so-called *superposition principle*, that is, the output response of a nonlinear system to a combination of input signals is different from the sum of its individual responses to each input. Thus, unlike the linear case, the total effect of measurement noise, external disturbances, and reference inputs cannot be simply obtained by analyzing the effect of each input separately. Nonlinear system stability analysis is also complicated by the fact that internal and input-output stability are in general not equivalent. This is also in striking contrast to linear systems, where stabilizability and detectability of a given realization are sufficient to ensure that equivalence. Thus, in general, the input-output stability properties of a nonlinear system are input dependent. An additional challenge arises from the fact that the so-called separation principle that is a workhorse of linear control system design does not hold for nonlinear systems. Thus, output feedback control system design cannot be simply done by putting together a state-feedback regulator and an observer. For general nonlinear systems, very few universal methods apply. The characterization of nonlinear systems with respect to special structures and particular properties is therefore a standard approach.

The last decades have witnessed considerable activity in nonlinear control theory both from an internal and input-output point of view (Kokotović and Arcak, 2001). The seminal work of Lyapunov was the precursor to rigorous analysis and design methods for nonlinear control. Presently, the instruments that are available for nonlinear control system design include linearization and gain-scheduling techniques, feedback linearization, circle and Popov criteria, small-gain theorems, passivity, averaging, singular perturbations, sliding mode control, Lyapunov redesign, backstepping and forwarding, adaptive control, input-to-state stability, and differential geometric approaches. For a general introduction to this area see the reference books of (Slotine and Li, 1994), (Vidyasagar, 1993), (Khalil, 1996) and recent work in (Sastry, 1999).

Beginning in the early 1970s, Brockett, Jurdjevic, Hermann, Isidori, Krener, Sussmann and others introduced the methods of differential geometry in the context of nonlinear control to express and extend notions from linear control theory such as controllability and observability. Since then, several theoretical breakthroughs led to the development of theoretical tools for the design of control laws for a large class of nonlinear control systems. Some books that describe nonlinear control in a geometric setting are for example (Isidori, 1989), (Nijmeijer and van der Schaft, 1990), and (Sontag, 1990).

Another important field of nonlinear control theory that has witnessed considerable research activity addresses the application of Lyapunov based techniques to nonlinear stabilization and adaptive identification and control. See (Krstić *et al.*, 1995) and (Sepulchre *et al.*, 1997) for a discussion of this interesting and fruitful circle of ideas.

Nonlinear control methods are increasingly being implemented in practice with great success. In fact, in the case of physical systems that are inherently nonlinear, it has proved advantageous to explicitly use the physical structure of the systems under consideration for control system design. A classical example is the computed torque method in the robotics industry.

A typical class of systems with strong nonlinear nature is the class of nonholonomic systems, that is, systems with nonintegrable constraints. The control of this kind of systems has been the subject of considerable research effort over the last few years. The reason for this trend is threefold:

i) there are a large number of mechanical systems that have non integrable constraints such as robot manipulators, mobile robots, wheeled vehicles, and space and underwater robots;

ii) there are considerable challenge in the synthesis of control laws for systems that are not transformable into linear control problems in any meaningful way and,

iii) as pointed out in a famous paper of Brockett (Brockett, 1983), nonholonomic systems cannot be stabilized by continuously differentiable, time invariant, state feedback control laws.

To overcome the limitations imposed by Brockett's celebrated result, a number of ingenious approaches have been proposed for the stabilization of nonholonomic control systems to equilibrium points. See (Kolmanovsky and McClamroch, 1995) and the references therein for a comprehensive survey of the field.

The main purpose of this dissertation is to contribute to the line of research in the field of nonlinear control theory devoted to the study of systems that cannot be rigorously controlled with linear methods. The final objective is to develop tools for the analysis and synthesis of controllers for nonlinear feedback stabilization of underactuated and nonholonomic systems. The focus of the applications is threefold: *i)* the so-called extended nonholonomic double integrator (ENDI) (a more elaborate version of the nonholonomic integrator of Brockett), *ii)* wheeled mobile robots of the unicycle-type, and *iii)* a prototype of an underactuated autonomous underwater vehicle (AUV) named SIRENE. Together with the nonholonomic constraints, the difficulty and challenging characteristic of the problems addressed are further enhanced by studying them in a more realistic scenario that explicitly includes *i)* the nonlinear dynamics (and not only the kinematics) of the mechanical nonholonomic systems under consideration, *ii)* parametric modeling uncertainty, and *iii)* the presence of external disturbances such as ocean currents for the AUV case.

To tackle these complex problems, solid nonlinear control theory and detailed knowledge of the underlying physical systems must go hand in hand. Classical Lyapunov theory combined with recent results on Lyapunov based design procedures such as backstepping and nonlinear adaptive control are some of the typical tools used in this thesis for analysis and synthesis. Also, logic-based hybrid control systems play a very important role in the development of the new control algorithms proposed.

The first part of the thesis presents a logic-based hybrid control approach to solve the global stabilization problem for the ENDI system and for a model of the SIRENE AUV. Furthermore, the problem of practical stabilization of the ENDI system under saturation constraints on the inputs and in the presence of small input additive disturbances is also solved.

The second part of the thesis is devoted to the problem of pose regulation for wheeled mobile robots and underactuated AUVs in the presence of parametric modeling uncertainty. Controller design relies on non smooth coordinate transformations in the original state space, followed by the derivation of adaptive Lyapunov-based, smooth, time invariant feedback controllers in the new coordinates. The important problem of dynamic positioning and way-point tracking of an underactuated AUV with parametric modeling uncertainty in the presence of an unknown ocean current disturbance is also posed and solved.

The last part of the thesis is dedicated to the problem of nonlinear system stabilization combining recent results on switching design of hybrid controllers and classical Lyapunov techniques. For a specific class of nonlinear plants, it is shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability.

Formal proofs of convergence of the several resulting control algorithms are derived and simulation results are presented and discussed.

The remaining of this chapter describes the practical motivation for the research presented in this thesis, presents previous related work in the general area of underactuated, nonholonomic, and hybrid system control, and summarizes the original contribution of the thesis. The objective is not to give a complete review of the work in the area reported in the literature, but to simply highlight the major developments which have been relevant in motivating the current research. At the end of this chapter, a briefly outline of the organization of the dissertation is given.

1.2 Practical motivation

At the heart of the practical problems that motivated many of the issues addressed in this thesis is the need to develop advanced marine robotic systems.

The oceans play a major role in shaping the environment and provide life support for a wide diversity of fauna and flora. Still, most of the oceans remain unexplored, and many fundamental questions aimed at understanding the mechanisms that regulate the climate, weather, and periodic phases of global warming and cooling are yet to be answered. This problem is specially important in the case of the deep sea, which probably harbors answers to many questions related to the impact of the ocean on the climate and to the origin and evolution of life on the planet.

Recently, there has been renewed interest in the development of stationary benthic[2] stations to carry out experiments on the biology, geochemistry, and physics of deep sea sediments and hydrothermal vents *in situ*, over long periods of time (Tiel and et al, 1994). However, current methods of deploying and servicing benthic laboratories are costly and require permanent support from specialized crews resident on board manned submersibles or surface ships. See, for example, (Floury and Gable, 1992) for a description of the benthic laboratory NADIA II that was first designed for re-entry of deep sea boreholes under the control of a crew stationed on board the 6000 m manned submersible *NAUTILE*.

As a contribution to overcoming some of the abovementioned problems, a European team led by IFREMER, France developed a prototype autonomous underwater shuttle vehicle named SIRENE to automatically transport and position a large range of stationary benthic laboratories on the seabed, at a desired target point, down to depths of 4000 meters. Pioneering work in this area includes the development of *ABE - Autonomous Benthic Explorer*, an autonomous underwater ve-

[2]benthic comes from the greek word *bénthos* meaning ”sea bottom”.

hicle that was designed to address the need for long term monitoring of the seafloor (Yoerger *et al.*, 1991). When compared with *ABE*, the main distinguishing feature of SIRENE is its ability to handle different benthic laboratories and to act as an underwater shuttle to position and service those laboratories automatically. The SIRENE vehicle was developed in the course of the MAST-II European project Desibel (New Methods for Deep Sea Intervention on Future Benthic Laboratories) that aimed at comparing different methods for deploying and servicing stationary benthic laboratories. The reader is referred to (Brisset *et al.*, 1995) for a general description of the project and to (Brisset, 1995) for complete technical details of the work carried out by IFREMER (FR), IST (PT), THETIS (GER), and VWS (GER). The main task of the SIRENE vehicle is to automatically transport and accurately position benthic laboratories at pre-determined target sites on the seabed. This simplifies the task of deploying dedicated instrumentation specially conceived for *in situ* measurements at the sea bottom, a technique of paramount importance in sea exploration and research.

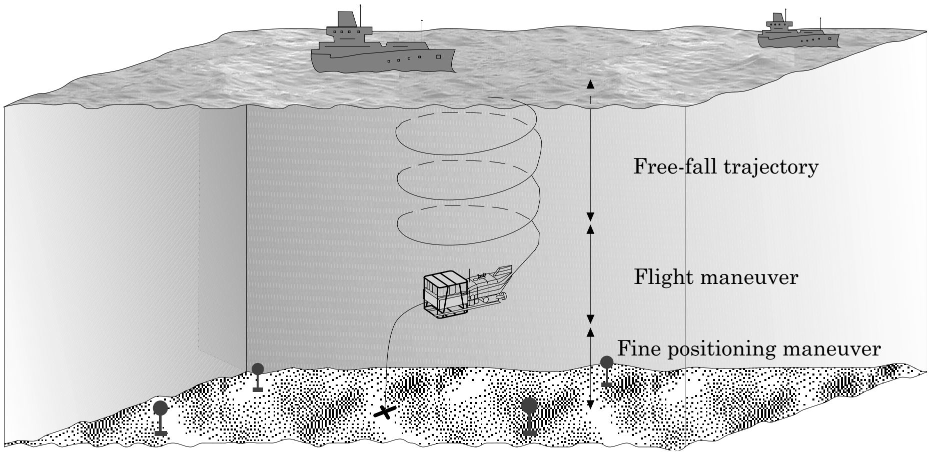

Figure 1.1: Mission scenario.

The SIRENE autonomous underwater vehicle (AUV) is equipped with two back thrusters for surge and yaw motion control in the horizontal plane and one vertical thruster for heave control. The vehicle is thus underactuated. Roll and pitch motion are left uncontrolled, since the metacentric height[3] is sufficiently large (36 cm) to provide adequate static stability. In Figure 1.1, the vehicle carries a representative benthic lab that is cubic-shaped. A typical three-phase mission scenario is illustrated. The first phase consists of selecting the precise location where the benthic lab will be deployed and marking it with acoustic beacons. SIRENE and the laboratory are then launched from a support ship. During this phase, SIRENE carries a ballast weight. The ensemble starts descending along a free-fall trajectory at a speed from 0.5 to $1\,m/s$. At approximately $150\,m$ of altitude with respect to the seabed SIRENE releases a ballast and the weight of the ensemble becomes neutral. It is then up to SIRENE to steer the benthic lab to the selected target position, deploy it with a desired heading, and return to the surface. Once deployed, the benthic laboratory executes a pre-determined plan of experiments for an extended period of time. When required, SIRENE may be instructed to service the laboratory by diving to the exact location of deployment,

[3]distance between the center of buoyancy and the center of mass.

lock onto the laboratory, and recharge its batteries using electromagnetic coupling techniques.

Steering the vehicle to the neighborhood of the selected target site is a relatively straightforward task that can be accomplished using a simple line of sight (LOS) guidance scheme combined with speed, depth, and yaw dynamic control. See (Aguiar, 1998) for complete details. However, the problem of positioning the vehicle with great accuracy at the target point with a desired heading is considerably more complex and raises some challenging questions in control system theory because the vehicle is underactuated and falls in the category of so-called nonholonomic systems. Furthermore, its dynamics are complicated due to the presence of complex hydrodynamic terms. This rules out any attempt to design a steering system for the AUV that would rely on its kinematic equations only.

The dynamic model of the SIRENE AUV plays a central role in the development that follows. The AUV is an example of an underwater robot with great potential for practical applications. At the same time, it exhibits complex dynamics that pose considerable challenges to control system designers. For these reasons, in this thesis, the AUV model was adopted as the main paradigm for nonlinear control system design. The AUV model falls in the category of nonholonomic and underactuated systems that are the subject of the following section.

1.3 Nonholonomic, underactuated and hybrid control systems: previous work

This section provides a brief overview of the general area of nonholonomic, underactuated and hybrid control systems. Key concepts are introduced and the major theoretical contributions in the area are summarized.

Over the past few years, there has been considerable interest in the study of nonholonomic and underactuated control systems. From a theoretical stand point, this interest was sparked by the realization that such systems pose challenges problems in control theory. At the same time, these systems do arise in a number of very important practical applications in the area of robotic vehicle control. Namely, in land and marine robotics when the number of actuators of a mobile vehicle is smaller that its degree of freedom (*e.g.* a wheeled robot of the unicycle type with two back steering wheels, or an autonomous underwater vehicle with no side thruster such as SIRENE). The literature on nonholonomic and underactuated systems is by now extremely vast and defies a cursory overview. The reader will find in (Kolmanovsky and McClamroch, 1995) an excellent survey that strikes a rare balance among conciseness, clarity, and rigor of analysis. See also (Wen, 1996) for a more fast paced exposition of key concepts in nonholonomic control systems. Reference (Wen, 1996) guides the reader through the different definitions of nonholonomy, starting with the most classical concepts from the field of mechanics and discusses issues related to kinematic dynamic nonholonomic control systems. What follows is a very brief introduction to this fascinating field.

1.3.1 Nonholonomic systems

A mechanical system is said to be nonholonomic if its generalized velocity satisfies an equality condition that cannot be written as an equivalent condition on the generalized position[4] (Wen, 1996). Nonholonomic equality constraints do not necessarily reduce the dimension of the space of configurations, but they do reduce the dimension of the space of possible differential motions, *i.e.*, the space of velocity directions. Nonholonomic systems typically arise in the following classes of systems (Wen, 1996):

1. *No-slip constraint.* Consider a single wheel rolling on a flat plane. The no slippage or pure rolling contact condition means that the linear velocity at the contact point is zero. This constraint is nonintegrable, *i.e.*, not reducible to a position constraint, and is therefore nonholonomic.

2. *Conservation of angular momentum.* Given a Lagrangian system, if a subset q_u of the generalized coordinates does not appear in the mass matrix $M(q)$, then they are called the cyclic coordinates. In this case, the Lagrangian equation associated with q_u is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_{u_i}}\right) = \frac{\partial L}{\partial q_{u_i}} = 0$, where L is the Lagrangian and $\dot{q}_{u_i}$ are generalized velocities. It turns out that for some cases this restriction is nonintegrable. One typical example is a free floating multibody system with no external torque.

3. *Underactuated mechanical systems.* An underactuated system is one for which the dimension of the configuration space exceeds that of the control input space. Nonintegrable conditions may then arise in terms of velocity or in terms of acceleration which cannot be integrated to a velocity condition. The latter case is called a second-order nonholonomic condition. The reader will find in (Oriolo and Nakamura, 1991) necessary and sufficient conditions for an underactuated manipulator to exhibit second-order nonholonomic, first-order nonholonomic, or holonomic constraints.

A variety of nonholonomic mathematical models have been studied in the literature. In particular, there has been a tremendous effort towards classifying nonholonomic systems and transforming them to canonical forms. As an example, the kinematic equations of several nonholonomic mechanical systems that include nonholonomic wheeled vehicles can be transformed into so-called *chained form systems*. See (Murray and Sastry, 1993), where this class of systems was defined for the first time. Interesting examples of chained form systems include tricycle-type mobile robots, cars towing several trailers, knife edge systems, vertical rolling wheels, and some nonholonomic manipulators (see, *e.g.*, (Murray and Sastry, 1993; Sørdalen, 1993; Kolmanovsky and McClamroch, 1995; Nakamura *et al.*, 2001)). Another class of systems studied in the literature can be brought into *power form* (Pomet and Samson, 1993).

A number of control strategies have been studied for various classes of nonholonomic control systems. These strategies fall into two major categories: *open loop* and *closed loop* (feedback) control. In open loop strategies, the control signal is computed beforehand, based on a priori knowledge of

[4] A formal definition of nonholonomic systems for mechanical systems is provided in chapter 2.

the initial and final configurations of the system. These strategies are often associated with motion planning problems, that is, obtaining open loop controls to steer a nonholonomic control system along a feasible path between two specified configuration points and subject to several criteria such as collision avoidance, shortest path, minimum control effort, minimal number of maneuvers, etc. Several motion planning methodologies for nonholonomic systems have been developed. They can be loosely classified into differential geometric and differential algebraic techniques, geometric phase (holonomy) methods, and control parameterization approaches. For more details, the interested reader is refereed to (Kolmanovsky and McClamroch, 1995) and the references therein. By their own nature, open loop strategies cannot compensate for disturbance and modeling errors. Thus the need to consider closed loop feedback control strategies to provide robustness against model parameter uncertainties and to yield good rejection of external disturbances.

It is well known that nonholonomic systems pose considerable challenges to closed loop feedback stabilization about a given equilibrium point since the tangent linearization of the these systems is uncontrollable. Furthermore, as pointed out in a famous paper of Brockett (Brockett, 1983), these systems cannot be stabilized by continuously differentiable (or even continuous, see extension of this theorem in (Zabczyk, 1989)), time invariant, static state feedback control laws. To overcome this basic limitation, a variety of approaches have been proposed in the literature for the stabilization of nonholonomic control systems about equilibrium points. See (Kolmanovsky and McClamroch, 1995) and the references therein for a comprehensive survey of the field. Among the proposed solutions are continuous smooth or almost smooth time-varying (periodic) controllers, discontinuous or piecewise time-invariant smooth control laws, and hybrid controllers.

The time varying feedback approach to nonholonomic system control was first studied by Samson (Samson, 1991*b*; Samson, 1991*a*; Samson, 1992) for a class of wheeled mobile robots. An overview of several nonlinear controllers for tracking, path following, and point stabilization of mobile robots is given in (Wit *et al.*, 1993). A general constructive approach for a class of controllable systems without drift was proposed by (Pomet, 1992) and (Coron and Pomet, 1992). This approach, widely known as Pomet's method, generates smooth time-periodic feedback laws and provides also closed loop Lyapunov functions. Unfortunately, as stated formally in (Gurvits and Li, 1993) and (M'Closkey and Murray, 1993*a*), *smooth time-varying controllers have the disadvantage of slow convergence and cannot achieve exponential convergence.* To solve this problem, (Coron, 1992) proved that exponential convergence rate can be achieved using a continuous state time-varying feedback law provided the control law is not smooth at the origin. Using these ideas and based on the properties associated with homogeneous systems, some authors (M'Closkey and Murray, 1993*b*; Pomet and Samson, 1993; Sørdalen and Egeland, 1995; Samson, 1995; Tell *et al.*, 1995; M'Closkey and Murray, 1997; Godhavn and Egeland, 1997; Morin and Samson, 2000; Dixon *et al.*, 2000) suggested time-periodic feedback laws that are continuous with respect to both time and state, but not differentiable at the origin of the state, thus yielding exponential convergence rates. In (M'Closkey and Murray, 1997), the theory of homogeneous systems with nonstandard dilations is used to obtain a set of constructive, sufficient conditions for extending smooth, asymptotic stabilizers to homogeneous, exponential stabilizers for a general class of driftless systems. In (Godhavn and Egeland, 1997), a local, continuous feedback control law with time-periodic terms that ρ-exponentially stabilized nonholonomic systems in power form was constructed. However, all these strategies suffer from a common problem: *the resulting closed loop system trajectories exhibit unnecessary oscillatory motions*.

Specially attractive control strategies that can in some cases overcome the complexity and lack of good performance (*e.g.*, low rates of convergence and oscillatory behaviour) that are often associated with time-varying control strategies are discontinuous control laws.

Several examples of discontinuous controllers have been reported in the literature. Among them, discontinuous time-invariant control laws that build on sliding mode control theory are worth mentioning. In (Bloch and Drakunov, 1994) a simple sliding mode feedback law is proposed to solve the stabilization problem for the so called nonholonomic integrator (Brockett, 1983). The sliding mode approach was also explored in (Guldner and Utkin, 1994) to stabilize a mobile robot with nonholonomic kinematic constraints. This type of discontinuous feedback laws force the trajectory to eventually slide along a manifold of dimension one towards the equilibrium. The disadvantage of this approach is that it may cause chattering. Other control strategies involve piecewise continuous controllers. Pioneering work in this area can be founded in (Bloch and McClamroch, 1989). Another early study is (Wit and Sørdalen, 1992), where a piecewise smooth controller was constructed to exponentially stabilize a wheeled mobile robot to a desired set point. In (Astolfi, 1998), a family of discontinuous control laws was derived to solve the problem of almost exponential stabilization for the Brockett integrator. The problem of stabilization with bounded control was also discussed and solved.

A different approach to constructing piecewise continuous controllers has been proposed in (Aicardi *et al.*, 1995) and (Astolfi, 1995). Here, the problem of exponential convergence of a nonholonomic kinematic wheeled mobile robot of unicycle type to a desired pose is solved. Control design is based on a non smooth state transformation inspired by the polar description of the kinematics of the robot. Thus, the polar representation is the key to overcoming the basic limitations imposed by Brockett's result. This is followed by the design of a smooth, Lyapunov based control law in the new coordinates. In the original coordinates, the resulting feedback law is discontinuous. In (Astolfi, 1995) it is also shown that, in contrast to what happens with other control laws, the motion of the mobile robot from any starting position to its final position is continuous, that is, no inversion of motion (velocity) occurs, thus yielding a more natural homing maneuver.

Another approach to solve the feedback stabilizing problem of nonholonomic systems involves the use of hybrid control. See for example (Bloch *et al.*, 1992; Wit and Berghuis, 1994; Hespanha, 1996). Hybrid systems are specially suited to deal with the combination of time driven and event driven dynamics. The literature on hybrid systems is extensive and discusses different modeling and analysis techniques. A more detailed presentation of this topic is deferred to Section 1.3.3. In the field of nonholonomic systems, the work in (Wit and Berghuis, 1994) yields a hybrid controller for systems in chained form that provides practical stabilization, *i.e.*, stabilization to a small neighborhood of the origin. The controller consists of two parts: a discrete-time part that practically stabilizes a subset of the system states, and a piecewise continuous-time part that steers the remaining state-components to an arbitrarily small neighborhood of zero. This constructive procedure is also applicable to dynamic extensions of systems in so-called chained form. In (Hespanha, 1996), the problem of global, exponential stabilization of the nonholonomic integrator is solved using a simple logic-based hybrid controller. This is achieved by mapping the state-space into a two dimensional closed positive quadrant space and dividing it into several overlapping regions. For each region, a feedback law is conveniently designed. The resulting hybrid controller switches among the various low-level continuous time controllers at discrete time instants. The action of the hybrid control law lends itself to a very intuitive interpretation.

Despite the large number of papers published on the stabilization of nonholonomic systems, the majority has concentrated on kinematic models of mechanical systems controlled directly by velocity inputs. Although in certain circumstances this can be acceptable, many physical systems *where forces and torques are the actual inputs* will not perform well if their dynamics are neglected. See for example (M'Closkey and Murray, 1994) where the authors extended time-varying exponential stabilizers to dynamic nonholonomic systems. The proposed control laws can be seen as consisting of a feedforward part, which drives the system along the desired trajectory when the actual and desired velocities match, and a feedback term that guarantees convergence of the velocities to their desired values. Related work that uses the idea of first designing steering velocities and then an inner velocity control loop that takes into account the dynamic behaviour of the system is described by (Fierro and Lewis, 1994). In this work, a combined kinematic/torque control law is developed for a mobile robot using a backstepping approach. Asymptotically stability is guaranteed by resorting to Lyapunov theory. See also (Morin and Samson, 1997), where an application of backstepping techniques to time-varying exponential stabilization of systems in chained form is presented. In (Astolfi, 1999), an extension of the kinematic control law described in (Astolfi, 1995) is made to incorporate the dynamics of a simple wheeled mobile robot.

It is relevant to point out that many vehicle models exhibit a drift vector field that is not in the span of the input vector fields, thus precluding the use of input transformations to bring them to driftless form. Some approaches to stabilize nonholonomic systems with drift terms have been proposed in the literature. However, most research efforts have been focused on so-called dynamic extension of drift free systems, which involves the simple addition of integrators to the velocity inputs. Thus, the problem of nonholonomic dynamical mechanical system stabilization is far from being solved.

Another problem that warrants further research is that of *controlling nonholonomic systems with modeling uncertainty*. See (Jiang and Pomet, 1996) where backstepping techniques are applied to the development of time-varying, adaptive control laws for a special class of uncertain, nonholonomic, chained systems. In (Hespanha *et al.*, 1999), the problem of parking a kinematic wheeled mobile robot of unicycle type with unknown parameters is solved using a hybrid feedback control law. The uncertainty parameters are the radius of the rear wheels and distance between them. More recently, in (Dong *et al.*, 2000), an adaptive robust controller is designed to deal with the stabilization problem of uncertainty dynamic nonholonomic chained systems. In (Valtolina and Astolfi, 2000), a simple modification of the discontinuous control law considered in (Astolfi, 1996) for chained systems is proposed. The new controller yields global asymptotic regulation, local exponential stability in the sense of Lyapunov, and local robustness against measurement errors and exogenous disturbances.

1.3.2 Underactuated AUVs

Underactuated dynamical systems have the dimension of their control vector smaller then that of the respective configuration space. Consequently, systems of this kind necessarily exhibit constraints on accelerations. See (Reyhanoglu *et al.*, 1999) for a survey of these concepts. The motivation for the study of controllers for underactuated systems, namely mobile robots is manifold and includes the following:

i) *Practical applications.* There is an increasing number of real-life underactuated mechanical systems. Mobile robots, walking robots, spacecrafts, aircraft, helicopters, missiles, surface vessels, and underwater vehicles are representative examples.

ii) *Cost reduction.* For example, for underwater vehicles that work at large depths, the inclusion of a lateral thruster is very expensive and represents large capital costs.

iii) *Weight reduction*, which can be critical for aerial vehicles.

iv) *Thruster efficiency.* Often, an otherwise fully actuated vehicle may become underactuated when its speed changes. This happens in the case of AUVs that are designed to maneuver at low speeds using thruster control only. As the forward speed increases, the efficiency of the side thruster decreases sharply, thus making it impossible to impart pure lateral motions on the vehicle.

v) *Reliability considerations.* Even for fully actuated vehicles, if one or more actuator failures occur, the system should be capable of detecting them and engaging a new control algorithm specially designed to accommodate the respective fault, and complete its mission if at all possible.

vi) *Complexity and increased challenge that this class of systems bring to the control area.* Most underactuated systems are not fully feedback linearizable and exhibit nonholonomic constraints.

It is well known that while nonholonomic mechanical systems are necessarily underactuated, the converse is not true in general. Necessary and sufficient conditions for an underactuated manipulator to exhibit second-order nonholonomic, first-order nonholonomic, or holonomic constraints are given in (Oriolo and Nakamura, 1991). See also (Wichlund *et al.*, 1995) for an extension of these results to underactuated vehicles (e.g. surface vessels, underwater vehicles, aeroplanes, and spacecraft). The work in (Wichlund *et al.*, 1995) shows that if so-called unactuated dynamics of a vehicle model contain no gravitational field component, no continuously differentiable, constant, state-feedback control law will asymptotically stabilize them to an equilibrium condition. This result brings out the importance of studying advanced control laws for underactuated systems.

The problem of steering an underactuated autonomous underwater vehicle (AUV) to a point with a desired orientation has only recently received special attention in the literature. This task raises some challenging questions in control system theory because, in addition to being underactuated, this type of vehicle exhibit complex hydrodynamic effects that must necessarily be taken into account during the controller design phase. Namely, the vehicle dynamics include sway and heave velocities that generate non-zero angles of sideslip and attack, respectively. Pioneering work in this field is reported in (Leonard, 1995), where open loop small-amplitude periodic time-varying control laws were proposed to re-position and re-orient underactuated AUVs. A piecewise smooth feedback control law that gives exponential convergence of a nonholonomic underwater vehicle to a constant desired configuration is introduced in (Egeland *et al.*, 1996). In (Pettersen and Egeland, 1996), the design of a continuous, periodic feedback control law that asymptotically stabilizes an underactuated AUV and yields exponential convergence to the origin is described. The control inputs are the force in surge and the torques in roll, pitch, and yaw. Assuming that the hydrodynamic restoring forces in roll are large enough, it is also shown how the AUV can be exponentially stabilized by the same feedback law without roll control torque.

Inspired by the work in (Morin and Samson, 1995), (Pettersen and Egeland, 1997*b*) proposed a continuous periodic time-varying exponentially stabilizing feedback law for a surface vessel with only two available actuators. The exponential stability result does not depend on the exact knowledge of the model parameters and, consequently, exponential stability is robust to model parameter uncertainty. Actuator dynamics are also included in the control design phase. Using a similar approach, a control law is proposed in (Pettersen and Egeland, 1997*a*) for attitude stabilization of an AUV with only two available control torques. In (Pettersen and Nijmeijer, 1998), a time-varying feedback control law is proposed that yields global practical stabilization and tracking for an underactuated ship using a combined integrator backstepping and averaging approach. More recently, experimental results using this approach can be found in (Pettersen and Fossen, 2000).

It is important to point out that some of the control laws developed so far for underactuated underwater vehicles do not take explicitly into account their dynamics. Contrary to what happens in the area of land robots, ignoring vehicle dynamics is totally unrealistic. Furthermore, even when the dynamics are taken into account the resulting closed loop system trajectories are often not "natural". *Thus, the control of underactuated vehicles still remains an open problem.*

1.3.3 Hybrid control systems

The last few years have witnessed increasing interest in the subject of hybrid control. Much of this interest has been motivated by applications in such diverse fields as car automation and aeronautics, real time software, communication protocols, transportation, traffic control, power distribution, robotics, and consumer electronics. At the same time, in many areas of industrial control, traditional methods of structuring large control systems are being gradually replaced by hybrid system methodologies that explicitly address the interplay between time and event driven phenomena. Modeling, analysis, control and synthesis of such systems pose a considerable number of challenging problems. In fact, while classical control theory and discrete-event system theory provide efficient tools for analysis and synthesis of time-driven and event-driven systems respectively, much work remains to be done before powerful tools for hybrid systems become available. See for example (Antsaklis and Nerode, 1998), (Morse *et al.*, 1999), (Lemmon *et al.*, 1999), and (Schaft and Schumacher, 2000) and the references therein for a comprehensive survey of the field.

In the initial phase of hybrid control systems theory, most of the discussions focused on modeling. In the early work of (Brockett, 1993*a*), a class of models was introduced as an attempt to model a range of phenomena that cut across the usual boundary between control and computer engineering. Branicky, Borkar, and Mitter (Branicky *et al.*, 1994) proposed a very general framework for hybrid control problems that encompasses several types of such hybrid phenomena. Some classical results from dynamical systems have been extended to hybrid systems. In (Ye *et al.*, 1998) a model for hybrid dynamical systems was presented that includes a very large class of systems and is suitable for qualitative analysis. The work in (Ye *et al.*, 1998) introduces also several types of Lyapunov-like stability concepts for an invariant set and establishes sufficient and necessary conditions (converse theorems) for these types of stability. The work of (Branicky, 1998) develops several tools for the analysis and synthesis of hybrid systems. Questions on the existence and uniqueness of solutions to hybrid systems are addressed in (Schaft and Schumacher, 1998) and (Lygeros *et al.*, 1999).

Hybrid controllers that combine continuous with discrete event features have been developed by a number of authors. In (Kolmanovsky and McClamroch, 1996), a hybrid controller for so called cascade systems was proposed. The methodology can be also used for stabilizing a class of nonholonomic systems, as well as for solving tracking problems. In (Lygeros *et al.*, 1996), a game-theoretic framework for designing hybrid controllers was proposed and latter applied to intelligent highway systems (Lygeros *et al.*, 1998) and air traffic control systems (Tomlin *et al.*, 1998). In another approach to hybrid control system synthesis, a family of continuous-time subsystems are switched on the basis of some supervisory control logic. A general overview of this area can be found in (Morse, 1995). This partly classical topic has regained new interest, in part due to its potential to overcome the basic limitation introduced by Brockett's celebrated result in the area of nonholonomic system control (Brockett, 1983). This is clearly shown in (Hespanha, 1996) where the problem of stabilization of the nonholonomic integrator is solved using a simple logic-based hybrid controller.

Hybrid control is a good candidate to deal with complex systems. Some methods for simulation verification and analysis have been developed in the last years, but the derivation of mathematical tools of sufficient rigor for controller synthesis is an open research issue.

1.4 Main contributions

The previous sections provided a quick overview of the field of nonholonomic and underactuated system control. In the course of the presentation, challenging theoretical problems were discussed and practical applications described. This circle of ideas sets the backdrop against which the research work reported in this thesis unfolds. The main thesis contributions are summarized below.

- **Practical stabilization of the extended nonholonomic double integrator**
  The extended nonholonomic double integrator (ENDI) model is introduced. This system captures for example the complete dynamic of a wheeled robot subject to force and torque inputs and can thus be viewed as a natural extension of the so called nonholonomic integrator described in (Brockett, 1983). Furthermore, any kinematic completely nonholonomic system with three states and two first-order dynamic control inputs can be converted to the ENDI by means of a state and control transformation. A *logic based hybrid controller is presented that solves the problem of global convergence and stabilization of the ENDI system to an arbitrarily small neighborhood of the origin.* Convergence and stability of the closed-loop hybrid system are analyzed theoretically and an application is made to the control of a wheeled mobile robot of the unicycle-type (Aguiar and Pascoal, 2000). The methodology proposed is modified to ensure *practical stabilization of the ENDI system under input saturation constraints and in the presence of small input additive disturbances* (Aguiar and Pascoal, 2002*e*).

- **Global stabilization of an underactuated autonomous underwater vehicle via logic based hybrid control**
  A feedback logic-based hybrid control law is proposed that yields *global stabilization of an underactuated autonomous underwater vehicle (AUV) in the horizontal plane to an arbitrarily small neighborhood of a target position, with a desired orientation* (Aguiar and Pascoal, 2002*d*). This is achieved by transforming the AUV dynamic model into the form of an extended double integrator plus a drift vector field, followed by the derivation of a controller that is inspired on the work of feedback hybrid control for the ENDI. It should be stressed that this is not a merely extension of the method proposed for the ENDI system described before. In fact, point stabilization of an underactuated AUV poses considerable challenges to control system analysis and design, since the models of those vehicles typically include a drift vector field that is not in the span of the input vector fields, thus precluding the use of input transformations to bring them to driftless form. To illustrate the control law developed, simulations results are presented using the model of the SIRENE AUV. In the course of proving stability of the closed loop system, *a generalized second order Gronwall-Bellman inequality is derived.*

- **Regulation of a nonholonomic dynamic wheeled mobile robot with parametric modeling uncertainty using Lyapunov functions**
  A new solution to the problem of regulating the dynamic model of a nonholonomic wheeled robot of the unicycle type to a point with a desired orientation is proposed. *A simple, discontinuous, adaptive state feedback controller is derived that yields global convergence of*

the trajectories of the closed loop system in the presence of parametric modeling uncertainty. Controller design relies on a non smooth coordinate transformation in the original state space, followed by the derivation of a smooth, time invariant control law in the new coordinates. The new control algorithm proposed, as well as the analysis of its convergence build on Lyapunov stability theory and LaSalle's invariance principle. This work was published in (Aguiar *et al.*, 2000).

- **Regulation of a nonholonomic and underactuated autonomous underwater vehicle with parametric modeling uncertainty using Lyapunov functions**
  The problem of regulating the dynamic model of an underactuated and nonholonomic AUV to a point with a desired orientation is solved. *A discontinuous, bounded, time invariant, nonlinear adaptive control law is derived that yields convergence of the trajectories of the closed loop system in the presence of parametric modeling uncertainty.* Controller design relies on a non smooth coordinate transformation in the original state space, followed by the derivation of a Lyapunov-based, smooth control law in the new coordinates. Convergence of the resulting nonlinear system trajectories is analyzed and simulations results are presented to illustrate the behaviour of the proposed control scheme (Aguiar and Pascoal, 2001).

- **Dynamic positioning and way-point tracking of underactuated AUVs in the presence of constant unknown ocean currents and parametric model uncertainty**
  The fundamental problem of positioning an underactuated AUV in the presence of unknown ocean current disturbances is considered. *A nonlinear adaptive controller is derived that yields convergence of the trajectories of the closed loop system in the presence of a constant, unknown ocean current disturbance and parametric model uncertainty.* The controller is first derived at the kinematic level, assuming that the ocean current disturbance is known. An exponential observer is then designed and convergence of the resulting closed loop system analyzed. Finally, resorting to backstepping techniques and adaptive nonlinear control Lyapunov based theory (Krstić *et al.*, 1995), the kinematic controller is extended for the dynamic case to deal with parameter uncertainty. A formal proof of convergence of the closed loop system is derived and simulation results are presented and discussed (Aguiar and Pascoal, 2002*b*). *The feedback control law derived is further extended to track a sequence of points consisting of desired positions* (x, y) *expressed in a inertial reference frame* (Aguiar and Pascoal, 2002*a*).

- **From local to global stabilization: a hybrid system approach**
  The problem of nonlinear system stabilization using hybrid control is investigated. *For a specific class of nonlinear plants, it is shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability.* The new methodology proposed for nonlinear system stabilization builds on classical Lyapunov techniques and borrows from recent results on switching design of hybrid controllers. *The resulting control laws avoid chattering and embody in themselves the interplay between the original locally stabilizing control loop and an outer control loop that comes into effect when the system trajectories deviate too much from the origin.* Formal proofs of convergence and stability of the closed loop hybrid systems are derived. The potential of the control scheme presented is illustrated with

a simple design method that yields global asymptotic stability of a particular class of single input single output nonlinear systems coupled by a suitable term is described (Aguiar and Pascoal, 2002*c*).

1.5 Organization of the dissertation

A brief outline of the content of the various chapters is presented next.

Chapter 2 introduces several examples of nonholonomic systems that will be used throughout the dissertation to illustrate theoretical results and the performance of the control systems developed. For each system, the corresponding dynamic model is described and its controllability and stabilizability properties analyzed.

Chapter 3 derives logic based hybrid control laws for the extended nonholonomic double integrator (ENDI). The problem of practical stabilization of the ENDI system under input saturation constraints and in the presence of small input additive disturbances is also posed and solved.

Chapter 4 addresses the problem of global stabilization of an underactuated AUV in the horizontal plane. A logic-based hybrid controller is proposed that yields global convergence of the vehicle to an arbitrarily small neighborhood of the origin.

Chapter 5 studies the problem of regulating the dynamics of a nonholonomic wheeled mobile robot of the unicycle type to a point with a desired orientation. A discontinuous, adaptive, state feedback controller is derived that yields global convergence of the trajectories of the closed loop system in the presence of parametric modeling uncertainty.

Chapter 6 derives a time-invariant discontinuous control law that solves the problem of regulating the dynamic model of an underactuated AUV (in the horizontal plane) to a point with a desired orientation, in the presence of parametric modeling uncertainty.

Chapter 7 proposes a solution to the problem of dynamic positioning and way-point tracking of an underactuated AUV (in the horizontal plane) in the presence of a constant, unknown, ocean current disturbance and parametric model uncertainty.

Chapter 8 investigates the problem of nonlinear system stabilization using hybrid control. For a specific class of nonlinear plants, it is shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability.

Chapter 9 summarizes the main results obtained and contains recommendations for further research.

Chapter 2

Underactuated and nonholonomic models

This chapter introduces the models of the systems that will be used throughout the dissertation to illustrate the applications of the various control techniques developed. Selected models include the celebrated nonholonomic integrator, the so-called extended nonholonomic double integrator (ENDI), a wheeled mobile robot of the unicycle-type, and an underactuated autonomous underwater vehicle (AUV). For each system, the respective dynamic model is described and its controllability and stabilizability properties analyzed. For the sake of completeness, the chapter starts by reviewing the key concepts of underactuated and nonholonomic systems, after which a brief discussion on the topic of accessibility, controllability, and stabilizability is presented.

2.1 Preliminaries

2.1.1 Underactuated and nonholonomic definitions

This section introduces the formal definitions of underactuated and nonholonomic systems, as applied to mechanical systems described by general second order equations.

Definition 2.1 (Underactuated System) *(Goldstein, 1980; Wen, 1996) Consider the affine mechanical system described by*

$$\ddot{q} = f(q, \dot{q}) + G(q)u, \tag{2.1}$$

where q is the state vector of linearly independent generalized coordinates, $f(\cdot)$ is the vector field that captures the dynamics of the system, $\dot{q}$ is the generalized velocity vector, G is the input matrix, and u is a vector of generalized inputs. System (2.1) is said to be underactuated *if the external generalized inputs are not able to command instantaneous accelerations in all directions in the configuration space. Formally stated, this occurs if $\operatorname{rank}(G) < \dim(q)$, where the dimension of q is usually defined as the number of degrees of freedom of (2.1).*

Definition 2.2 (Nonholonomic System) *(Goldstein, 1980; Wen, 1996) Consider a mechanical system described by*

$$\ddot{q} = f(q, \dot{q}, u),$$

where q is the vector of generalized coordinates, $f(\cdot)$ is the vector field representing the dynamics and u is a vector of external generalized inputs. Suppose that some constraints restrict the motion of the system. If the constraints satisfy the complete integrability property, that is, if they can be written in the form

$$h(q, t) = 0,$$

then according to standard terminology (see for example (Goldstein, 1980)), they are called holonomic*. If the constraints cannot be expressed in that fashion, then they are called* nonholonomic*. In particular, the constraints are said to be* second-order nonholonomic *if they are non-integrable on the accelerations, or* partially integrable *if they can be integrated to constraints on the velocities, that is, if they can be expressed in the form*

$$g(\dot{q}, q, t) = 0.$$

2.1.2 Controllability

The problem of characterizing local and global controllability for general nonlinear systems is extremely hard and constitutes one of the most challenging areas of nonlinear control theory. See (Lee and Markus, 1976) for a lucid presentation of early work in this area. Fundamental results on nonlinear system controllability can be traced back to a basic theorem stating that if the linearization of a nonlinear system at an equilibrium point is controllable, then the system itself is locally controllable (Lee and Markus, 1976). Over the last decades the tools available for the analysis of nonlinear system controllability have becoming increasingly powerful. Recently, a differential geometric approach to controllability analysis was adopted whereby a control system is viewed as a family of vector fields. In this set-up, valuable control theoretic information is shown to be contained in the Lie brackets of the underlying vector fields (Hermann, 1968; Sussmann and Jurdjevic, 1972). For recent work, the reader is referred to (Bullo *et al.*, 2000) where the authors provide controllability tests and motion control algorithms for underactuated mechanical control systems on Lie groups where the Lagrangian is equal to the kinetic energy of the systems under study. A good introduction to nonlinear control theory which includes many of the necessary differential geometric concepts can be found in (Isidori, 1989) and (Nijmeijer and van der Schaft, 1990).

Consider an affine nonlinear control system given by

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j, \tag{2.2}$$

where $x = [x_1, \ldots, x_n]' \in \mathbb{R}^n$ are local coordinates for a smooth manifold M (the state space manifold), $u = [u_1, \ldots, u_m]' \in U \subset \mathbb{R}^m$ are the control variables, and the mappings $f, g_1, \ldots, g_m$

are smooth vector fields on M. The vector field f is called the *drift* vector field and g_j; $j = 1, \dots, m$ are referred to as the *input* vector fields.

Given arbitrary points x_0 and x_1 in M, it is important to be able to answer very general questions about the possibility of steering x_0 to x_1 by using admissible controls over a finite time interval. Some key results are provided below.

Definition 2.3 (Controllability) *(Nijmeijer and van der Schaft, 1990) The nonlinear system (2.2) is said to be* controllable *if for any two points x_0, x_1 in M there exist a finite time $T \geq t_0 \geq 0$ and an admissible control function $u : [t_0, T] \to U$ such that the unique solution of (2.2) with initial condition $x(t_0) = x_0$ at time $t = T$ and with input function $u(\cdot)$ satisfies $x(T) = x_1$.*

Consider system (2.2), and let x_0 be an arbitrary point in M. Given a neighborhood V of x_0 in M and a time $T > 0$, let $R^V(x_0, T)$ denote the set of points that can be reached from x_0 at time T following admissible trajectories that remain in V for $t \leq T$. Stated formally,

$$R^V(x_0, T) = \big\{ x \in M : \text{there exists an admissible input } u : [t_0, T] \to U \text{ such that the evolution of (2.2) for } x(t_0) = x_0 \text{ satisfies } x(t) \in V,\ t_0 \leq t \leq T, \text{ and } x(T) = x \big\}.$$

A simple extension of this definition leads to the concept of reachable set from x_0 during the interval $[0, T]$ as follows

$$R^V(x_0, \leq T) = \bigcup_{\tau \leq T} R^V(x_0, \tau).$$

Definition 2.4 (Accessibility) *(Nijmeijer and van der Schaft, 1990) The system (2.2) is said to be* locally accessible from x_0 *if $R^V(x_0, \leq T)$ contains a non-empty open subset of M for all neighborhoods V of x_0 and all $T > 0$. If this holds for any $x_0 \in M$, then the system is called* locally accessible.

Local accessibility is easily checked by performing a simple algebraic test. This can be done by introducing the so-called accessibility algebra $\mathcal{C}$. Henceforth, the Lie-algebra[1] of vector fields on M is denoted $V^\infty(M)$.

Definition 2.5 (Accessibility algebra) *The* accessibility algebra $\mathcal{C}$ *for (2.2) is the smallest sub-algebra of $V^\infty(M)$ that contains $f, g_1, \dots, g_m$.*

The accessibility algebra $\mathcal{C}$ is also the *Control Lie Algebra*, (Isidori, 1989). A typical element of $\mathcal{C}$ is a finite linear combination of elements of the form $[v_k, [v_{k-1}, [\cdots, [v_2, v_1] \cdots]]]$, where $k = 0, 1, 2, \dots$ and v_i, $i \in \{1, \dots, k\}$ is in the set $\big\{f, g_1, \dots, g_m\big\}$. The accessibility algebra may be used to define the accessibility distribution.

[1]The Lie algebra is a vector space V together with a bilinear map $[\cdot, \cdot] : V \times V \to V$ called the Lie bracket, such that *i)* $[x, y] = -[y, x]$ (anti-symmetric) and *ii)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).
The Lie bracket of two vector fields, f and g, is defined as

$$[f, g] = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x).$$

Definition 2.6 (Accessibility distribution) *The* accessibility distribution C *of* (2.2) *is the distribution generated by the accessibility algebra* $\mathcal{C}$, i.e.,

$$C(x) = \text{span}\{v(x) : v \in \mathcal{C}\}, \quad x \in M.$$

Notice from the definition of $\mathcal{C}$ that $C(x)$ is an involutive distribution[2]. The following theorem applies:

Theorem 2.1 *The system* (2.2) *is locally accessible from* x_0 *if and only if*

$$\dim C(x_0) = n. \tag{2.3}$$

Furthermore, if condition (2.3) *(which usually referred to as* the accessibility rank condition*) holds for all* $x \in M$*, then the system is locally accessible.*

Proof: See for example (Sussmann and Jurdjevic, 1972) or (Nijmeijer and van der Schaft, 1990). □

Definition 2.7 (Local strong accessibility) *The system* (2.2) *is said to be* locally strongly accessible *from* x_0 *if for any neighborhood* V *of* x_0 *the set* $R^V(x_0, T)$ *contains a non-empty open set for any* $T > 0$ *sufficiently small. If this holds for every* x_0*, then the system is called* locally strongly accessible.

In order to have a rank condition that is stronger than the accessibility rank condition, the following subalgebra and distribution are introduced:

Definition 2.8 (Strong accessibility algebra/distribution) *Let* $\mathcal{C}$ *be the accessibility algebra of* (2.2)*. Define* $\mathcal{C}_0$ *as the smallest subalgebra which contains* $g_1, \ldots, g_m$ *and satisfies* $[f, v] \in \mathcal{C}_0$ *for all* $v \in \mathcal{C}_0$*. Define the corresponding involutive distribution*

$$C_0(x) = \text{span}\{v(x) : v \in \mathcal{C}_0\}.$$

In what follows, $\mathcal{C}_0$ *is called the* strong accessibility algebra *and* C_0 *is called the* strong accessibility distribution.

A typical element of $\mathcal{C}_0$ is a finite linear combination of elements of the form $[v_k, [v_{k-1}, [\cdots, [v_1, g_j]\cdots]]]$, $j = 1, \ldots, m$ where $k = 0, 1, 2, \ldots$ and v_i, $i \in \{1, \ldots, k\}$ is in the set $\{f, g_1, \ldots, g_m\}$. For linear systems

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i, \quad x \in \mathbb{R}^n,$$

where $b_1, \cdots, b_m$ are the columns of any input matrix B, it can be immediately checked that

$$\mathcal{C}_0 = \text{span}\{b_i, Ab_i, \cdots, A^{n-1}b_i, i = 1, \cdots, m\},$$

and therefore

$$C_0(x) = \text{Im}\big[B \vdots AB \vdots \cdots \vdots A^{n-1}B\big].$$

The following result extends Theorem 2.1.

[2] A *distribution* is the subspace generated by a collection of vector fields. An *involutive distribution* is a distribution that is closed with respect to the Lie bracket, *i.e.*, if f and g belong to a distribution Δ then $[f, g]$ also belongs to Δ.

Theorem 2.2 *The system* (2.2) *is locally strongly accessible from* x_0 *if*

$$\dim C_0(x_0) = n.$$

Proof: See (Nijmeijer and van der Schaft, 1990). □

For systems without drift, *i.e.*, $f = 0$, controllability can be ascertained through the following sufficient condition that is often referred to as Chow's theorem (Hermann, 1968):

Theorem 2.3 *The nonlinear system*

$$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i, \quad u = (u_1, \cdots, u_m) \subset U \tag{2.4}$$

is controllable if the accessibility rank condition is satisfied.

Notice that for systems with drift terms the above full rank condition only implies local accessibility. For linear time-invariant systems, the same condition reduces to the usual controllability rank condition. For nonlinear continuous-time systems, the only property that is relatively easy to completely characterize is the accessibility property. Under extra conditions such as the existence of suitable Hamiltonian structures, it is possible to prove the equivalence of controllability and accessibility. For linear systems, the two concepts are of course equivalent. It is often of interest to strengthen the definition of local controllability.

Definition 2.9 (STLC) *(Sussmann, 1987) The control system* (2.2) *is said to be* small time locally controllable (SLTC) *from* $x_0 \in M$ *if it is locally accessible from* x_0*, and* x_0 *is in the interior of* $R^V(x_0, \leq T)$ *for all* $T \geq 0$ *and each neighborhood* V *of* x_0*. If this holds for any* $x_0 \in M$*, then the system is called small time locally controllable.*

A fairly strong sufficient condition for STLC of systems of the form (2.2) is offered by Sussmann (Sussmann, 1987). A precise statement of his results is beyond the scope of this thesis. However, one can use a simpler result that can be briefly explained as follows. A Lie bracket formed of combinations of vector fields from $\{f, g_1, \cdots, g_m\}$ is *bad* if it contains an even number of each of the vector fields $g_i; i = 1, \cdots, m$ and an odd number of f. A Lie bracket is *good* if it is not bad. The *degree* of a bracket is the total number of vector fields of which it is comprised. This becomes clear with a few examples: the bracket $[[f, g_i], [f, g_j]]$ is good and of degree 4 for any $i, j \in \{1, \cdots, m\}$, and the bracket $[g_i, [f, g_i]]$ is bad and of degree 3 for any $i \in \{1, \cdots, m\}$. Let S_m denote the permutation group on m symbols. Given $\pi \in S_m$ and a Lie bracket B of vector fields from $\{f, g_1, \cdots, g_m\}$, define $\bar{\pi}(B)$ to be the bracket obtained by fixing f and sending g_i to $g_{\pi(i)}$ for $i = 1, \cdots, m$. Define

$$\beta(B) = \sum_{\pi \in S_m} \bar{\pi}(B).$$

In (Sussmann, 1987) the following sufficient condition for STLC in terms of the Lie brackets and Lie algebra generated by the vector fields $\{f, g_1, \cdots, g_m\}$ is given.

Theorem 2.4 *Suppose that an analytic control system of the form (2.2) is such that every bad bracket B has the property that $\beta(B)(x)$ is an $\mathbb{R}$-linear combination of good brackets, evaluated at x, of lower degree than B. Also suppose that (2.2) satisfies the local accessibility rank condition at x. Then (2.2) is STLC at x.*

In practice, once one finds a basis of vector fields made up of good brackets, checking STLC property amounts to checking that all bad brackets of degree not greater than the highest degree of a good bracket satisfy the hypothesis of the theorem.

2.1.3 Stabilizability

The concept of stabilizability is related to that of the existence of a feedback controller for a given system that will render the resulting closed loop system asymptotically stable about an equilibrium point. For linear systems, controllability implies stabilizability. However, this is not true for the general nonlinear case. A celebrated theorem of Brockett (Brockett, 1983) provides necessary conditions for smooth feedback stabilizability.

Theorem 2.5 (Brockett 1983) *Consider the system*

$$\dot{x} = f(x, u), \tag{2.5}$$

with $f(0,0) = 0$ and $f(\cdot,\cdot)$ continuously differentiable in a neighborhood of the origin. If (2.5) is smoothly stabilizable, i.e., if there exists a continuously differentiable function $g(x)$ such that the origin is an asymptotically stable equilibrium point of $\dot{x} = f(x, g(x))$, with stability defined in the Lyapunov sense, then the image of f must contain an open neighborhood of the origin.

For example, in the case of the driftless nonlinear system described by equation (2.4) one has $f(x, u) = f(x)u$. Thus, the range of $\{f(x, u) : (x, u) \text{ in a neighborhood of the origin}\}$ is equal to the span of the columns of $f(x)$, which we assume has dimension m (number of inputs). Because a neighborhood about the zero state is n dimensional, the necessary condition for stabilizability above is not satisfied unless $m \geq n$.

2.2 The nonholonomic integrator

In (Brockett, 1983) Brockett introduced the so called *nonholonomic integrator* system

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1, \end{aligned} \tag{2.6}$$

where $x = \big(x_1, x_2, x_3\big)' \in \mathbb{R}^3$ is the state vector and $u = \big(u_1, u_2\big)' \in \mathbb{R}^2$ is a two-dimensional input. It can be shown that any kinematic completely nonholonomic system (*e.g.*, the kinematic model of a wheeled mobile robot of the unicycle type) with three states and two inputs can be converted into the above form by a local coordinate transformation (Murray and Sastry, 1993). The nonholonomic integrator displays all basic properties of nonholonomic systems and is often quoted in the literature as a benchmark for control system design (Brockett, 1993*b*; Bloch and Drakunov, 1994; Hespanha, 1996; Astolfi, 1998; Escobar *et al.*, 1998; Morgansen and Brockett, 1999).

Consider the linearization about an equilibrium point $(x_{1_{eq}}, x_{2_{eq}}, x_{3_{eq}})$ which corresponds to the zero nominal input $u = 0$. In this case the linearized system is given by

$$\dot{x} = Ax + Bu \tag{2.7}$$

where $A = \frac{\partial f}{\partial x}(x_{eq}, u_{eq})$, $B = \frac{\partial f}{\partial u}(x_{eq}, u_{eq})$, and

$$\frac{\partial f(x,u)}{\partial x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_2 & -u_1 & 0 \end{bmatrix}, \quad \frac{\partial f(x,u)}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{bmatrix}.$$

The time-invariant linear state equation (2.7) is controllable if and only if the $n \times nm$ controllability matrix $\mathcal{C} = \left[BAB \cdots A^{n-1}B\right]$ satisfies (Rugh, 1993, Theorem 9.5)

$$\operatorname{rank}\left[BAB \cdots A^{n-1}B\right] = n.$$

Since

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -x_{2_{eq}} & x_{1_{eq}} & 0 & 0 & 0 & 0 \end{bmatrix}$$

has rank 2 for any equilibrium point $(x_{1_{eq}}, x_{2_{eq}}, x_{3_{eq}}, u_{1_{eq}}, u_{2_{eq}})$, it follows that the linear system resulting from the linearization of the nonholonomic integrator for any equilibrium point is not controllable.

However, rewriting (2.6) in the standard form (2.4), yields

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2,$$

with $g_1 = (1, 0, -x_2)'$, and $g_2 = (0, 1, x_1)'$. Computing the Lie bracket $[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1(x) - \frac{\partial g_1}{\partial x} g_2(x)$, yields $[g_1, g_2] = (0, 0, 2)'$. Clearly, it can be concluded that the nonholonomic integrator is controllable since the accessibility rank condition holds, that is, $C(x) = \operatorname{span}\{g_1, g_2, [g_1, g_2]\}$ has dimension 3 for all $x \in \mathbb{R}^3$. Notice, however that there is no time-invariant continuously differentiable control law that asymptotically stabilizes the origin. This follows from the fact that Brockett's condition is violated because no point of the form $(0, 0, \epsilon)'$, for any $\epsilon \neq 0$ is in the image of the mapping $(x, u)' \to (u_1, u_2, x_1u_2 - x_2u_1)'$.

2.3 The extended nonholonomic double integrator

The nonholonomic integrator model fails to capture the case where both the kinematics and dynamics of a wheeled robot must be taken into account. To tackle this realistic case, the nonholonomic integrator model must be extended. This is done in the next section, where it is shown that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$
\begin{aligned}
\ddot{x}_1 &= u_1, \\
\ddot{x}_2 &= u_2, \\
\dot{x}_3 &= x_1\dot{x}_2 - x_2\dot{x}_1,
\end{aligned}
\tag{2.8}
$$

where $x = \left(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2\right)' \in \mathbb{R}^5$ is the state vector and $u = \left(u_1, u_2\right)' \in \mathbb{R}^2$ is a two-dimensional control vector. In this thesis, system (2.8) will be referred to as *the extended nonholonomic double integrator* (ENDI) and will be used as a prototype system for the development of logic based hybrid controllers (see Chapter 3). It can be shown that any kinematic completely nonholonomic system with three states and two first-order dynamic control inputs can be converted to the ENDI by means of a state and control transformation. This result suggests that a fairly general class of nonholonomic systems can be stabilized by the method that will be proposed in Chapter 3.

Controllability and stabilizability properties

The ENDI system fall into the class of control affine nonlinear systems with drift described by

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

where $x \in M$, M is a smooth n-dimensional manifold, $u \in \mathbb{R}^m$ is the input vector, and the mappings $f, g_1, \ldots, g_m$ are smooth vector fields on M. The following result summarizes the controllability and stabilizability properties of the ENDI.

Theorem 2.6 *Consider the extended nonholonomic double integrator described by (2.8). Let M_e be the set of equilibrium solutions corresponding to $u = 0$, that is, $M_e = \{x \in \mathbb{R}^5 : \dot{x}_1 = \dot{x}_2 = 0\}$. Then, the ENDI system satisfies the following properties:*

1. *There is no time-invariant continuously differentiable feedback law which asymptotically stabilizes the closed loop to $x_e \in M_e$.*
2. *The ENDI system is locally strongly accessible for any $x \in \mathbb{R}^5$.*
3. *The ENDI system is small time locally controllable (STLC) at any equilibrium $x_e \in M_e$.*

Proof:

1.

The ENDI equations (2.8) can be written as $\dot{x} = f(x, u)$, where $x = (x_1, x_2, x_3, x_4, x_5)'$, $u = (u_1, u_2)'$, and the mapping $f(x, u) : \mathbb{R}^5 \times \mathbb{R}^2 \to \mathbb{R}^5$ is defined by $f(x, u) = (x_4, x_5, x_1x_5 - x_2x_4, u_1, u_2)'$. A necessary condition for the existence of a continuously differentiable asymptotically stabilizing state feedback law for system (2.8) is that the image of the mapping $f(x, u)$ contain some neighborhood of zero (see Theorem 2.5). In particular, it must contain points of the form $\epsilon = (0, 0, \gamma, 0, 0)'$, where γ is an arbitrary non null constant. Notice however that equation

$f(q,u) = \epsilon$ implies that $(x_4, x_5, x_1x_5 - x_2x_4, u_1, u_2)' = (0, 0, \gamma, 0, 0)'$ which has no solution (x, u) since γ is non-zero. Consequently, Brockett's necessary condition is not satisfied, and hence, the ENDI cannot be asymptotically stabilized to $x_e \in M_e$ using a time invariant continuously differentiable state feedback. A straightforward corollary of this theorem shows that a single equilibrium solution of (2.8) can neither be asymptotically stabilized using linear feedback nor by resorting to feedback linearization or any other control design technique that uses time-invariant smooth feedback.

2.

Rewrite system (2.8) as

$$\dot{x} = f(x) + g_1u_1 + g_2u_2,$$

where $x = (x_1, x_2, x_3, x_4, x_5)'$ and the vectors fields

$$f(x) = \begin{bmatrix} x_4 \\ x_5 \\ x_1x_5 - x_2x_4 \\ 0 \\ 0 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are real-analytic. Consider the Lie Brackets

$$[g_1, f] = \begin{bmatrix} 1 \\ 0 \\ -x_2 \\ 0 \\ 0 \end{bmatrix}, \quad [g_2, f] = \begin{bmatrix} 0 \\ 1 \\ x_1 \\ 0 \\ 0 \end{bmatrix}, \quad [g_2, [f, [g_1, f]]] = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}.$$

Observe that the vector fields g_1, g_2, $[g_1, f]$, $[g_2, f]$, $[g_2, [f, [g_1, f]]]$ span a five-dimensional space for every $x \in \mathbb{R}^5$. Thus, the strong accessibility rank condition is satisfied, and consequently from Theorem 2.2, system (2.8) is locally strongly accessible for all $x \in \mathbb{R}^5$.

3.

Notice that system (2.8) satisfies the strong accessibility rank condition and the vectors fields f, g_1, g_2 are real-analytic. From Theorem 2.4, if one can show that all bad brackets can be expressed as a linear combination of good brackets of lower degree, then it follows that the ENDI is SLTC at any equilibrium point $x_e \in M_e$. From the proof of item 2, it follows that any bracket with degree greater then 4 can be expressed as a linear combination of lower order brackets. Also, notice that all brackets in $C(x) = \text{span}\{g_1, g_2, [g_1, f], [g_2, f], [g_2, [f, [g_1, f]]]\}$ are good and that the degree of bad brackets must be odd. Hence, one only needs to check the bad brackets of order 1 and 3. The bad bracket of degree 1 is f, which vanishes at any equilibrium point. The bad brackets of order 3 are $[g_1, [f, g_1]]$ and $[g_2, [f, g_2]]$ which are identically zero vector fields. Consequently, system (2.8) satisfies the Sussmann theorem (Theorem 2.4) and is thus small time locally controllable at $x_e \in M_e$.

□

2.4 The wheeled mobile robot of the unicycle-type

This section describes the kinematic and dynamic equations of the wheeled mobile robot of the unicycle type depicted in Figure 2.1, and formulates the problem of controlling it to a point with a desired orientation. The vehicle has two identical parallel, nondeformable rear wheels which are controlled by two independent motors, and a steering front wheel. It is assumed that the plane of each wheel is perpendicular to the ground and that the contact between the wheels and the ground is pure rolling and nonslipping, *i.e.*, the velocity of the center of mass of the robot is orthogonal to the rear wheels axis[3]. It is further assumed that the masses and inertias of the wheels are negligible and that the center of mass of the mobile robot is located in the middle of the axis connecting the rear wheels. Each rear wheel is powered by a motor which generates a control torque τ_i, $i = 1, 2$.

2.4.1 Robot model

The following notation will be used in the sequel. The symbol $\{A\} := \{x_A, y_A\}$ denotes a reference frame with origin at O_A and unit vectors x_A, y_A. Let $\{U\}, \{G\}$, and $\{B\}$ be inertial, goal, and body reference frames, respectively. Assume, for simplicity of presentation, that $\{U\} = \{G\}$ and that the origin O_B of $\{B\}$ is coincident with the center of the rear wheels axle. Let $(x, y)'$ specify the position of O_B in $\{U\}$ and let θ be the parameter that describes the orientation of $\{B\}$ with respect to $\{U\}$ (i.e., the robot orientation with respect to the inertial x-axis). The kinematics and dynamics of the mobile robot are modeled by the equations

$$
\begin{aligned}
\dot{x} &= v\cos\theta, \\
\dot{y} &= v\sin\theta, \\
\dot{\theta} &= \omega, \\
m\dot{v} &= F, \\
I\dot{\omega} &= N,
\end{aligned}
\tag{2.9}
$$

where v and ω denote the linear and angular velocity of $\{B\}$ with respect to $\{U\}$, respectively. The control inputs are the force F along the vehicle axis x_B and the torque N about its vertical axes z_B. It is easy to see that

$$F = \frac{1}{R}\left(\tau_1 + \tau_2\right), \tag{2.10a}$$

$$N = \frac{L}{R}\left(\tau_1 - \tau_2\right), \tag{2.10b}$$

where R is the radius of the rear wheels and $2L$ is the length of the axis between them. The symbols m and I denote the mass and the moment of inertia of the mobile robot, respectively.

[3] By assuming that the wheels do not slide, a nonholonomic constraint on the motion of the mobile robot of the form $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$ is imposed.

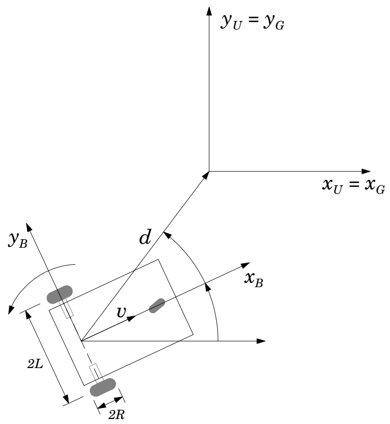

Figure 2.1: A wheeled mobile robot of the unicycle-type.

2.4.2 Problem formulation

With the notation of the previous subsection the problem considered in this thesis for the wheeled mobile robot can be formulated as follows:

Derive a feedback control for τ_1 and τ_2 to regulate $\{B\}$ to $\{G\} = \{U\}$ in the presence of uncertainty in the parameters m, I, R, and L.

In Chapter 3 this problem is solved assuming that the model parameters are known. Robustness against parameter uncertainty is addressed in Chapter 5.

2.4.3 Controllability and stabilizability results

Consider the state and control transformation defined by

$$\begin{aligned} z_1 &= \theta, \\ z_2 &= x\cos\theta + y\sin\theta, \\ z_3 &= x\sin\theta - y\cos\theta, \\ u_1 &= \frac{N}{I}, \\ u_2 &= \frac{F}{m} - \frac{N}{I}z_3 - \omega^2 z_2. \end{aligned}$$

This transformation leads to a representation of the robot dynamics in the extended power form

$$\begin{aligned} \ddot{z}_1 &= u_1, \\ \ddot{z}_2 &= u_2, \\ \dot{z}_3 &= \dot{z}_1 z_2. \end{aligned} \tag{2.11}$$

It is easy to see that by applying the additional coordinate transformation

$$\begin{aligned} x_1 &= z_1, \\ x_2 &= z_2, \\ x_3 &= -2z_3 + z_1 z_2, \end{aligned}$$

to (2.11) yields the ENDI system in equation (2.8). Consequently, since both coordinate transformations are a global diffeomorphism[4] on $\mathbb{R}^5$, the properties of controllability and stabilizability of the ENDI system hold for the wheeled mobile robot.

2.5 The autonomous underwater vehicle

This section describes the kinematic and dynamic equations of motion of an underactuated autonomous underwater vehicle (AUV). Controllability and stabilizability properties of the vehicle model are also discussed. The prototype AUV adopted in this thesis is the SIRENE vehicle depicted in Figure 2.2.

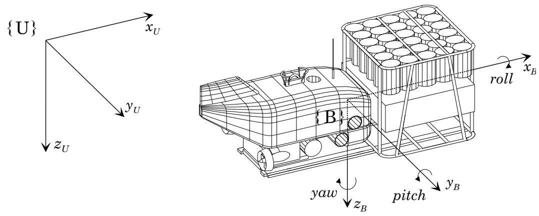

Figure 2.2: The vehicle SIRENE coupled to a benthic laboratory. Body-fixed $\{B\}$ and earth-fixed $\{U\}$ reference frames

The SIRENE AUV has an open-frame structure and is 4.0 m long, 1.6 m wide, and 1.96 m high. The vehicle has a dry weight of 4000 Kg and a maximum operating depth of 4000 m. It is equipped with two back thrusters for surge and yaw motion control in the horizontal plane and one vertical thruster for heave control. Roll and pitch motion are left uncontrolled, since the metacentric height[5] is sufficiently large (36 cm) to provide adequate static stability. The AUV has no side thruster. In the figure, the vehicle carries a representative benthic lab which is cubic-shaped, with a volume of approximately $2.3m^3$. The dynamic model of the Sirene can be found in (Aguiar and Pascoal, 1997), or in (Aguiar, 1998).

2.5.1 Vehicle modeling

General equations of motion

Following standard practice, the general kinematic and dynamic equations of motion of the AUV can be developed using a global coordinate frame $\{U\}$ and a body-fixed coordinate frame $\{B\}$, as depicted in Figure 2.2. The following notation is required (Fossen, 1994):

$\eta_1 = [x, y, z]'$ - position of the origin of $\{B\}$ measured in $\{U\}$.

$\eta_2 = [\phi, \theta, \psi]'$ - angles of roll (ϕ), pitch (θ), and yaw (ψ) that parametrize locally the orientation of $\{B\}$ with respect to $\{U\}$.

[4]A transformation $x = \Phi(x)$, with $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called a global diffeomorphism on $\mathbb{R}^n$ if satisfies the following properties: *i)* $\Phi(x)$ is invertible, *i.e.* there exists a function $\Phi^{-1}(z)$ such that $\Phi^{-1}(\Phi(x)) = x$ for all $x \in \mathbb{R}^n$, and *ii)* $\Phi(x)$ and $\Phi^{-1}(z)$ are both smooth mappings, *i.e.* have continuous partial derivatives of any order.

[5]distance between the center of buoyancy and the center of mass.

$\nu_1 =$ $[u, v, w]'$ - linear velocity of the origin of $\{B\}$ relative to $\{U\}$, expressed in $\{B\}$ (*i.e.*, body-fixed linear velocity).

$\nu_2 =$ $[p, q, r]'$ - angular velocity of $\{B\}$ relative to $\{U\}$, expressed in $\{B\}$ (*i.e.*, body-fixed angular velocity).

With this notation, the kinematics and dynamics of the vehicle can be written in compact form as

Kinematics

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} {}^U_B R(\eta_2) & 0 \\ 0 & Q(\eta_2) \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \Longleftrightarrow \dot{\eta} = J(\eta)\nu \tag{2.12}$$

Dynamics

$$M_{RB}\dot{\nu} + C_{RB}(\nu)\nu = \tau_{RB}. \tag{2.13}$$

The matrix

$${}^U_B R(\eta_2) = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

is the rotation matrix from {B} to {U} parameterized by the vector η_2 of roll, pitch, and yaw angles, and

$$Q(\eta_2) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}$$

is the matrix that relates body-fixed angular velocity with roll, pitch, and yaw rates. Notice that $Q(\lambda)$ has a singularity for $\theta = \pm\frac{\pi}{2}$. This can be avoided by using quaternions rather than Euler angles in the description of the rigid body angular motion kinematics. Nevertheless, the Euler angles representation will be adequate for representing the orientation of the SIRENE vehicle since, due to physical restrictions, it will always operate far from the singular point in θ. The vector $\nu = [u, v, w, p, q, r]^T$ consists of the body-fixed linear and angular velocity vectors, and $\tau_{RB} = [X, Y, Z, K, M, N]^T$ is the generalized vector of external forces and moments. The symbols M_{RB} and C_{RB} denote the rigid body inertia matrix and the matrix of Coriolis and Centrifugal terms, respectively. The vector τ_{RB} can further be decomposed as

$$\tau_{RB} = \tau + \tau_A + \tau_D + \tau_R, \tag{2.14}$$

where τ_R denotes the term due to buoyancy and gravity and τ_A is the added mass term. The term τ_D captures the damping and lift effects, and τ represents the forces and moments generated by the thrusters.

To be of practical use, the general equations of motion must be tuned for the vehicle in study. The main difficulty lies in computing the term τ_{RB} that arises in the dynamics equation. In the present case, this was done using both theoretical and experimental methods, and by exploring the analogy with the *Dolphin 3K* vehicle (Nomoto and Hattori, 1986). The reader will find in (Aguiar, 1998; Aguiar and Pascoal, 1997) a description of the model parameters adopted based on a series of tests that were performed in a circulating water channel at the *VWS - Versuchsanstalt fur Wasserbau und Schiffbau*, Berlin with a quarter scale model of the vehicle. In the final model, the added mass and quadratic drag terms were decomposed as $\tau_A = -M_A\dot{\nu} - C_A(\nu)\nu$ and $\tau_D = -D(\nu)\nu$ respectively, where the hydrodynamic damping matrix $D(\nu)$ is strictly positive (Fossen, 1994).

Equations of motion: compact notation

Combining equations (2.12), (2.13), and (2.14) the 6 DOF body-fixed vehicle equations of motion in the absence of ocean currents can be written in compact form as

$$M\,\dot{\nu} + C(\nu)\,\nu + D(\nu)\,\nu + g(\eta) = \tau \tag{2.15a}$$

$$\dot{\eta} = J(\eta)\,\nu \tag{2.15b}$$

where τ is the vector of actuator control forces and moments, $g(\eta) = -\tau_R$, $M = M_{RB} + M_A$, and $C(\nu) = C_{RB}(\nu) + C_A(\nu)$. It is assumed that M is constant and positive definite, and that $C(\nu)$ is skew-symmetrical, *i.e.*, $M = M^T > 0, \quad \dot{M} = 0_{6\times 6}$, and $C(\nu) = -C^T(\nu)$.

Let $\nu_c = [\nu_{c_1}', \nu_{c_2}']'$ denote the total linear and rotational fluid velocity (ocean current). If the current is nonzero, constant $\nu_{c_1} = [u_c, v_c, w_c]$ and irrotational $\nu_{c_2} = 0$, then equation (2.15a) becomes

$$M\,\dot{\nu}_r + C(\nu_r)\,\nu_r + D(\nu_r)\,\nu_r + g(\eta) = \tau, \tag{2.16}$$

where ν_r is the relative body-fluid linear velocity vector, that is,

$$\nu_r = \nu - \nu_c.$$

Simplified equations of motion: horizontal plane

In the absence of currents, the kinematic equations of motion of the vehicle in the horizontal plane can be written as

$$\dot{x} = u\cos\psi - v\sin\psi, \tag{2.17a}$$

$$\dot{y} = u\sin\psi + v\cos\psi, \tag{2.17b}$$

$$\dot{\psi} = r. \tag{2.17c}$$

Furthermore, from (2.15) and neglecting the motions in heave, roll, and pitch the simplified equations of motion for surge, sway and heading yield (Fossen, 1994)

$$m_u\dot{u} - m_v v r + d_u u = \tau_u, \tag{2.18a}$$

$$m_v\dot{v} + m_u u r + d_v v = 0, \tag{2.18b}$$

$$m_r\dot{r} - m_{uv} u v + d_r r = \tau_r, \tag{2.18c}$$

where

$$\begin{aligned} m_u &= m - X_{\dot{u}}, & d_u &= -X_u - X_{|u|u}|u|,\\ m_v &= m - Y_{\dot{v}}, & d_v &= -Y_v - Y_{|v|v}|v|,\\ m_r &= I_z - N_{\dot{r}}, & d_r &= -N_r - N_{|r|r}|r|,\\ m_{uv} &= m_u - m_v \end{aligned}$$

$m = 4000\,\text{Kg}$	$X_{\dot u} = -290\,\text{Kg}$	$X_u = -360\,\text{kg/s}$	$X_{\lvert u \rvert u} = -805\,\text{kg/m}$
$I_z = 2660\,\text{Kg m}^2$	$Y_{\dot v} = -310\,\text{Kg}$	$Y_v = -420\,\text{kg/s}$	$Y_{\lvert v \rvert v} = -1930\,\text{kg/m}$
	$N_{\dot r} = -95\,\text{Kg m}^2$	$N_r = -110\,\text{Kg m/s}$	$N_{\lvert r \rvert r} = -555\,\text{Kg m}$

Table 2.1: Table of parameters for the simplified model of the SIRENE AUV (Horizontal plane).

and τ_u and τ_r denote the external force in surge and the external torque about the z axis of the vehicle, respectively, generated by the back thrusters. In these equations, and for clarity of presentation, it was assumed that the AUV is neutrally buoyant and that the centre of buoyancy coincides with the centre of mass. The parameters of the complete model of the vehicle SIRENE can be found in (Aguiar, 1998). The parameters of the simplified model are shown in Table 2.1.

The compact form of equations (2.17)-(2.18) is given by

$$M_h\,\dot\nu_h + C_h(\nu_h)\,\nu_h + D_h(\nu_h)\,\nu_h + g_h(\eta_h) = \tau_h \tag{2.19a}$$

$$\dot\eta_h = J_h(\eta_h)\,\nu_h \tag{2.19b}$$

where $\eta_h = (x, y, \psi)'$, $\nu_h = (u, v, r)'$, $\tau_h = (\tau_u, 0, \tau_r)'$, $g_h(\eta_h) = 0_{3\times 1}$, $M_h = \text{diag}\{m_u, m_v, m_r\}$, $D_h(\nu_h) = \text{diag}\{d_u, d_v, d_r\}$, and

$$C_h(\nu_h) = \begin{bmatrix} 0 & -mr & Y_{\dot v} v \\ mr & 0 & -X_{\dot u} u \\ -Y_{\dot v} v & X_{\dot u} u & 0 \end{bmatrix}, \quad J_h(\eta_h) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the presence of ocean currents the kinematic equations (2.17) hold, with $u = u_r + u_c$ and $v = v_r + v_c$, where $(u_r, v_r)'$ is the linear velocity of the body with respect to the current. In this case the dynamics (2.18) must be modified to

$$m_u \dot u_r - m_v v_r r + d_{u_r} u_r = \tau_u, \tag{2.20a}$$

$$m_v \dot v_r + m_u u_r r + d_{v_r} v_r = 0, \tag{2.20b}$$

$$m_r \dot r - m_{uv} uv + d_r r = \tau_r, \tag{2.20c}$$

where $d_{u_r} = -X_u - X_{|u|u}|u_r|$ and $d_{v_r} = -Y_v - Y_{|v|v}|v_r|$.

The sections that follow discuss basic properties of the SIRENE AUV model in terms of holonomy, controllability, and stabilizability.

2.5.2 Test of nonholonomy for underactuated vehicles

When a vehicle is underactuated, the unactuated dynamics imply constraints on the accelerations. The work in (Oriolo and Nakamura, 1991) and (Wichlund *et al.*, 1995) provides necessary and sufficient conditions for the acceleration constraints to be second order nonholonomic, first order nonholonomic, or holonomic (see Definition 2.2).

Consider a class of systems described by

$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + g(\eta) = \begin{bmatrix} \tau \\ 0 \end{bmatrix}$$
$$\dot{\eta} = J(\eta)\nu,$$

where $\eta \in \mathbb{R}^n$, $\nu \in \mathbb{R}^n$, $\tau \in \mathbb{R}^m$, $m < n$, $\dot{M} = 0$, and M and J are nonsingular matrices. This class includes models for underactuated surfaces vessels, underwater vehicles, aeroplanes and spacecraft. Let M_u denote the last $n - m$ rows of matrix M and define $C_u(\nu)$ and $D_u(\nu)$ in a similar manner. Further, let $g_u(\eta)$ denote the $n - m$ last elements of the vector $g(\eta)$. Then the constraints implied by the unactuated dynamics may be written as

$$M_u\dot{\nu} + C_u(\nu)\nu + D_u(\nu)\nu + g_u(\eta) = 0. \tag{2.21}$$

Constraints (2.21) are said to be *partially integrable* if they can be integrated to the form

$$h_p(\nu, \eta, t) = 0,$$

where $h_p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n-m}$. If they can be further integrated to the form

$$h_T(\eta, t) = 0,$$

where $h_t : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n-m}$, then constraints (2.21) are said to be *totally integrable*. Notice that if the constraints are not partially integrable, one has a second-order nonholonomic system. If the constraints are partially but not totally integrable, the system is first-order nonholonomic, *i.e.*, it has constraints on its velocities. When the constraints are totally integrable, the system is holonomic. An underactuated vehicle will generally not be partially integrable. The following theorems gives necessary and sufficient conditions for (2.21) to be partially integrable and totally integrable.

Theorem 2.7 *The constraints (2.21) are partially integrable if and only if*

1. *g_u is constant.*
2. *$\big(C_u(\nu) + D(\nu)\big)$ is a constant matrix.*
3. *The distribution $\Omega^{\perp}$ defined by $\Omega^{\perp} = \ker\big((C_u(\nu) + D_u(\nu))J^{-1}(\eta)\big)$ is completely integrable.*

Proof: See (Wichlund *et al.*, 1995). □

Theorem 2.8 *The constraints (2.21) are totally integrable if and only if*

1. *they are partially integrable,*
2. *$C_u(\nu) + D(\nu) = 0$, and*
3. *The distribution Δ defined by $\Delta(\eta) = \ker\big(M_u J^{-1}(\eta)\big)$ is completely integrable.*

Proof: See (Wichlund *et al.*, 1995). □

The above integrability theorems imply the following proposition.

Proposition 2.1 *The autonomous underwater vehicle described by* (2.17)-(2.18) *is second order nonholonomic.*

Proof: It can be easily checked that the constraint (2.18b), which can be rewritten in the form

$$\begin{bmatrix} 0 & m - Y_{\dot{v}} & 0 \end{bmatrix} \dot{\nu}_h + \begin{bmatrix} mr & 0 & -X_{\dot{u}}u \end{bmatrix} \nu_h + \begin{bmatrix} 0 & -Y_v - Y_{|v|v}|v| & 0 \end{bmatrix} \nu_h = 0$$

violates condition 2 of Theorem 2.7 since the coriolis matrix $C_u(\nu_h) = [mr, 0, -X_{\dot{u}}u]$ and the damping matrix $D_u(\nu_h) = [0, -Y_v - Y_{|v|v}|v|, 0]$ depend on ν_h. Therefore, the constraint (2.18b) is not partially integrable, and consequently for this case underactuation implies nonholonomy. Notice that in contrast to the first order nonholonomic case (Bloch *et al.*, 1992), a second order nonholonomic constraint does not reduce the dimension of the state space. A set of three independent configuration variables and three velocity variables is required to completely specify the state of the AUV in the horizontal plane. □

2.5.3 Controllability and stabilizability results

Consider the global diffeomorphism given by the state and control coordinate transformation

$$\begin{aligned} x_1 &= \psi, \\ x_2 &= x\cos\psi + y\sin\psi, \\ x_3 &= x\sin\psi - y\cos\psi, \\ u_1 &= \frac{1}{m_r}\tau_r + \frac{m_u - m_v}{m_r}uv - \frac{d_r}{m_r}r, \\ u_2 &= \frac{m_v}{m_u}vr - \frac{d_u}{m_u}u + \frac{1}{m_u}\tau_u - u_1x_3 + vr - r^2x_2, \end{aligned} \tag{2.22}$$

that yields

$$\begin{aligned} \ddot{x}_1 &= u_1, \\ \ddot{x}_2 &= u_2, \\ \dot{x}_3 &= \dot{x}_1x_2 - v. \end{aligned}$$

Applying the additional coordinate transformation

$$\begin{aligned} z_1 &= x_1, \\ z_2 &= x_2, \\ z_3 &= -2x_3 + x_1x_2, \end{aligned} \tag{2.23}$$

finally leads to

$$\begin{aligned} \ddot{z}_1 &= u_1, \\ \ddot{z}_2 &= u_2, \\ \dot{z}_3 &= z_1\dot{z}_2 - z_2\dot{z}_1 + 2v, \end{aligned} \tag{2.24}$$

where the second order constraint (2.18b) for the sway velocity v is transformed to

$$m_v \dot{v} + m_u \left(\dot{z}_2 + \dot{z}_1 \frac{z_1 z_2 - z_3}{2} \right) \dot{z}_1 + d_v v = 0. \tag{2.25}$$

In what follows we focus on the system described by (2.24)-(2.25) to study the controllability and stabilizability properties of the AUV model of (2.17)-(2.18).

Theorem 2.9 *Consider the underactuated AUV model described by (2.17)-(2.18). Let M_e be the set of equilibrium solutions corresponding to $\tau_h = 0$, that is, $M_e = \{(x, y, \psi, u, v, r)' \in \mathbb{R}^6 : u = v = r = 0\}$. Then, the AUV model satisfies the following properties:*

1. *There is no time-invariant continuously differentiable feedback law that asymptotically stabilizes the closed loop system to $(x_e, y_e, \psi_e, 0, 0, 0)' \in M_e$.*

2. *The AUV system is locally strongly accessible for any $(x, y, \psi, u, v, r)' \in \mathbb{R}^6$.*

3. *The AUV system is small time locally controllable (STLC) at any equilibrium $(x_e, y_e, \psi_e, 0, 0, 0)' \in M_e$.*

Proof:

1.

Rewrite the equivalent system model (2.24)-(2.25) as $\dot{q} = f(q, u)$ where $q = (z_1, z_2, z_3, \dot{z}_1, \dot{z}_2, v)'$, $u = (u_1, u_2)'$ and $f(q, u) : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}^6$. A necessary condition for the existence of a continuously differentiable asymptotically stabilizing state feedback law for (2.24)-(2.25) is that the image of the mapping $f(q, u) : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}^6$ contain some neighborhood of zero (see (Brockett, 1993*b*)). Since no points of the form $(0, 0, \gamma, 0, 0, 0)$, $\gamma \neq 0$ are in the image of $f(q, u)$, it follows from Brockett's celebrated result (Brockett, 1993*b*) that (2.17)-(2.18) cannot be asymptotically stabilized to $(x_e, y_e, \psi_e, 0, 0, 0)' \in M_e$ using a time invariant continuously differentiable state feedback.

2.

Consider the system (2.24)-(2.25) rewritten as

$$\dot{q} = f(q) + g_1 u_1 + g_2 u_2, \tag{2.26}$$

where

$$f(q) = \begin{bmatrix} q_4 \\ q_5 \\ q_1 q_5 - q_2 q_4 + 2q_6 \\ 0 \\ 0 \\ -\frac{m_u}{m_v}\left(q_5 + q_4 \frac{q_1 q_2 - q_3}{2} \right) q_4 - \frac{d_v}{m_v} q_6 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The vectors fields f, g_1, g_2 are real-analytic. From the Lie brackets

$$[g_1, f] = \begin{bmatrix} 1 \\ 0 \\ -q_2 \\ 0 \\ 0 \\ -\frac{m_u}{m_v}\big(q_5 + q_4(q_1 q_2 - q_3)\big) \end{bmatrix}, \quad [g_2, f] = \begin{bmatrix} 0 \\ 1 \\ q_1 \\ 0 \\ 0 \\ -\frac{m_u}{m_v} q_4 \end{bmatrix},$$

$$[g_1, [g_2, f]] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{m_u}{m_v} \end{bmatrix}, \text{ and } \quad [[g_1, f], [g_2, f]] = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

it can be seen that the accessibility distribution C of (2.26) (see Definition 2.6) given by

$$C(q) = \operatorname{span}\big\{g_1, g_2, [g_1, f], [g_2, f], [g_1, [g_2, f]], [[g_1, f], [g_2, f]]\big\}(q),$$

has dimension 6 for every $q \in \mathbb{R}^6$. Thus, the strong accessibility rank condition is satisfied, and consequently from Theorem 2.2, it follows that system (2.26) is locally strongly accessible for all $q \in \mathbb{R}^6$.

3.

Consider system (2.26). The Sussmann condition (Sussmann, 1987) (see Theorem 2.4) for STLC amounts to requiring that the so called bad brackets associated with f, g_1, and g_2 be a linear combination of good brackets of lower degree at equilibrium. Since *i)* all brackets in $C(x)$ are good, *ii)* any bracket with degree greater then 4 can be expressed as a linear combination of lower order brackets, and *iii)* the degree of a bad bracket must be necessary odd, it follows that one only needs to check the bad brackets of order 1 and 3. The bad bracket of degree 1 is f, which vanishes at any equilibrium. The bad brackets of order 3 are $[g_1, [f, g_1]]$ and $[g_2, [f, g_2]]$ which are identically zero vector fields. Consequently, system (2.26) and thus the AUV system described by (2.17)-(2.18) are small time locally controllable at equilibrium. □

Since the system is real analytic, the above controllability results imply the existence of a piecewise analytic feedback law (Sussmann, 1979) that asymptotically stabilizes the closed loop system to a given equilibrium point. In the next chapters, inspired by this result, emphasis will be placed on designing asymptotically stabilizing discontinuous feedback laws for the system described here.

Chapter 3

Practical stabilization of the extended nonholonomic double integrator

This chapter derives hybrid control laws for the extended nonholonomic double integrator (ENDI). The ENDI system can be viewed as an extension of the so called nonholonomic integrator (Brockett, 1983). Its importance stems from the fact that it captures the dynamics and kinematics of a nonholonomic system with three states and two first-order dynamic control inputs, (*e.g.*, the dynamics of a wheeled robot subject to force and torque inputs). A new logic-based hybrid controller is proposed for the ENDI that yields global stability and convergence of the trajectories of the close loop system to an arbitrarily small neighborhood of the origin. Control system design borrows from hybrid control theory and is greatly inspired by the work of Hespanha (Hespanha, 1996) for the nonholonomic integrator. An application is made to the control of a wheeled mobile robot of the unicycle-type. The problem of practical stabilization of the ENDI system under input saturation constraints and in the presence of small input additive disturbances is also posed and solved. The main result shows that for any bounded input additive disturbances and any initial condition, the closed loop hybrid system possesses strong practical stability with bounded control input. Control system design is done by mapping the state-space into a two dimensional closed positive quadrant space and dividing it into four overlapping regions followed by the assignment of a suitable feedback law to each region. Stability and convergence proofs are presented. Simulations illustrate the performance of the controllers derived.

The remaining of this chapter is organized as follows: Section 3.1 proposes a simple piecewise smooth controller to stabilize the ENDI system. In Section 3.2 stability and convergence of the trajectories of the resulting closed loop system are analyzed. Section 3.3 extends the control law proposed to the bounded input case, and in Section 3.4 the problem of practical stability problem for the ENDI subject to input saturation constraints and in the presence of small input additive disturbances is solved. Finally, Section 3.5 contains simulation results that illustrate the performance of the control laws derived and describes an application of the hybrid control law to point stabilization of a wheeled mobile robot. Concluding remarks are given in Section 3.6.

3.1 Hybrid controller design

This section proposes a simple piecewise smooth controller to stabilize the ENDI. The key ideas involved borrow from hybrid system theory. Hybrid systems are specially suited to deal with the combination of continuous dynamics and discrete events. The literature on hybrid systems is extensive and discusses different modeling techniques (Branicky, 1998; Ye *et al.*, 1998). In this chapter, a continuous-time autonomous hybrid system Σ is defined as (Hespanha, 1996)

$$\Sigma := \begin{cases} \dot{x}(t) = f_{\sigma(t)}\big(x(t)\big), \qquad t \geq t_0 & (3.1a) \\ \sigma(t) = \phi\big(x(t), \sigma(t^-)\big) & (3.1b) \end{cases}$$

where $\sigma(t) \in \mathcal{I} \triangleq \{1, \ldots, N\}$ and $x(t) \in \mathcal{X} \triangleq \cup_{\sigma=1}^{N} \mathcal{X}_\sigma \subset \mathbb{R}^n$. Here, the differential equation (3.1a) models the continuous dynamics, where the vector fields $f_\sigma : \mathcal{X}_\sigma \times \mathbb{R}^+ \to \mathcal{X}$, $\sigma \in \mathcal{I}$ are each locally Lipschitz continuous maps from $\mathcal{X}_\sigma$ to $\mathcal{X}$. The algebraic equation (3.1b), where $\phi : \mathcal{X} \times \mathcal{I} \to \mathcal{I}$, models the state of the decision-making logic. The discrete state $\sigma(t)$ is piecewise constant. The notation t^- indicates that the discrete state is piecewise continuous from the right.

The dynamics of the system Σ can now be described as follows: starting at (x_0, i) with $x_0 \in \mathcal{R}_i \subset \mathcal{X}_i$, the continuous state trajectory $x(t)$ evolves according to $\dot{x} = f_i(x, t)$. When $\phi\big(x(\cdot), i\big)$ becomes equal to $j \neq i$, (and this could only happen when $x(\cdot)$ hits the set $\mathcal{X} \backslash \mathcal{R}_i$), the continuous dynamics switches to $\dot{x} = f_j(x, t)$, from which the process continues. As in (Hespanha, 1996), the "logical dynamics" will be determined, recursively by equation (3.1b) with $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$ where $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ from below as $\tau \to t$ and the transition function ϕ is defined by

$$\phi(x, \sigma) = \begin{cases} \sigma & \text{if } x \in \mathcal{R}_\sigma, \\ \max_{\mathcal{I}}\big\{k : x \in \mathcal{R}_k\big\} & \text{otherwise.} \end{cases} \tag{3.2}$$

The signal $\sigma(t)$ can be also generated according to the diagram in Figure 3.1.

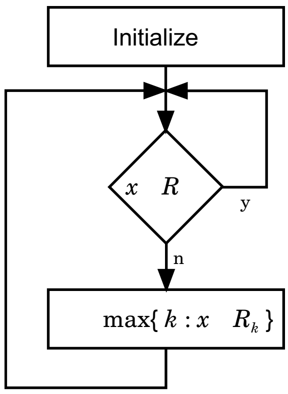

Figure 3.1: Switching Logic.

We now review the concept of stability of a hybrid system (Hespanha, 1996).

Definition 3.1 *The equilibrium point $x = 0$ of the hybrid system Σ is Lyapunov stable if for* *every $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that for every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \delta$, the solution $\{x(t), \sigma(t)\}$ satisfies $\|x(t)\| < \epsilon$, for all $t \geq t_0$. If in the above definition δ is independent of t_0, i.e., $\delta = \delta(\epsilon)$, then the origin is said to be uniformly stable.*

Consider now the ENDI system (2.8)

$$
\begin{aligned}
\ddot{x}_1 &= u_1, \\
\ddot{x}_2 &= u_2, \\
\dot{x}_3 &= x_1 \dot{x}_2 - x_2 \dot{x}_1.
\end{aligned}
$$

When the state variables x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and, consequently, x_3 will remain constant. Thus, a possible strategy to steer an initial state to the vicinity of the origin is the following (see (Hespanha, 1996) where similar ideas were applied to the control of the nonholonomic integrator):

i) first, make the state variable x_3 converge to zero while keeping x_1 and x_2 away from the axis $x_1 = x_2 = 0$;

ii) next, freeze x_3 ($\dot{x}_3 = 0$), and force x_1 and x_2 to converge to zero.

In order to derive a hybrid controller for the ENDI, it is convenient to define the function $W(x) : \mathbb{R}^5 \to \Omega \subset \mathbb{R}^2$ as

$$\omega = (\omega_1, \omega_2)' = W(x) = \left(s^2, (x_1)^2 + (x_2)^2\right)',$$

where $s = \dot{x}_3 + \lambda x_3$ and λ is a strictly positive constant. The image of W is the two-dimensional closed positive quadrant space $\Omega = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1 \geq 0, \omega_2 \geq 0\}$. This mapping has several properties, which are listed in the following lemma.

Lemma 3.1 *The mapping $W(x) : \mathbb{R}^5 \to \Omega \subset \mathbb{R}^2$ has the following properties:*

1. *$W(0) = 0$.*

2. *if w converges to zero as $t \to \infty$, then x also converges to zero as $t \to \infty$.*

3. *if $x_3(t_0) = 0$ and $\omega_1 \leq \epsilon$ for all $t \geq t_0$, then $|x_3(t)| \leq \frac{\sqrt{\epsilon}}{\lambda}$ for all $t \geq t_0$. For the case where $x_3(t_0) \neq 0$, the bound of $x_3(t)$ is given by*

$$|x_3(t)| \leq e^{-\lambda(t-t_0)} |x_3(t_0)| + \frac{\sqrt{\epsilon}}{\lambda}.$$

Divide now Ω into three overlapping regions (see Figure 3.2)

$$
\begin{aligned}
\mathcal{R}_1 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \;\wedge\; \omega_2 \leq \gamma_2\}, \\
\mathcal{R}_2 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \;\wedge\; \omega_2 \geq \gamma_1\}, \\
\mathcal{R}_3 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 \leq \epsilon_2\},
\end{aligned}
\tag{3.3}
$$

where $\epsilon_2 > \epsilon_1 > 0$ and $\gamma_2 > \gamma_1 > 0$.

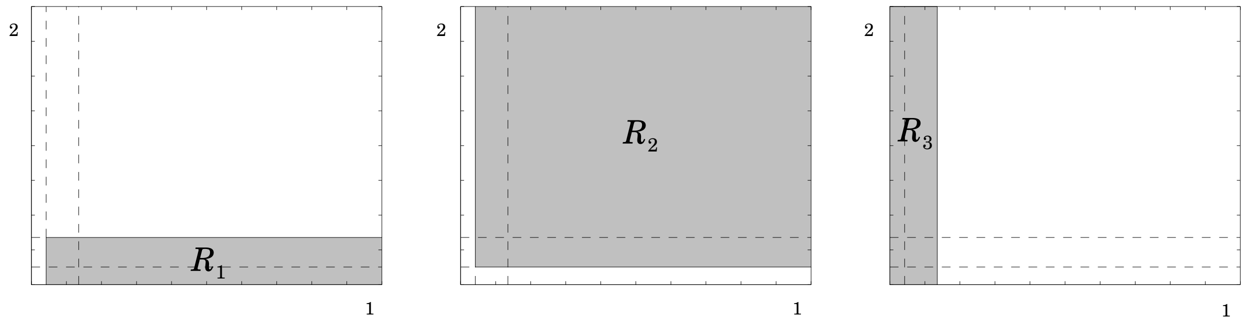

Figure 3.2: Definition of regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$.

Consider the following dynamical system as a candidate control law to steer the ENDI trajectories to a small neighborhood of the origin:

$$u = g_\sigma\left(x\right), \tag{3.4}$$

where the vector fields $g_\sigma : \mathbb{R}^5 \to \mathbb{R}^2$; $\sigma \in \mathcal{I} = \{1, 2, 3\}$ are given by

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 \\ -\lambda\dot{x}_2 + x_2 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + x_2 s \\ -\lambda\dot{x}_2 + x_2 - x_1 s \end{bmatrix}, \quad g_3(x) = \begin{bmatrix} -\lambda\dot{x}_1 - x_1 \\ -\lambda\dot{x}_2 - x_2 \end{bmatrix}, \tag{3.5}$$

and σ is a piecewise constant switching signal taking values in $\mathcal{I} = \{1, 2, 3\}$ that is determined recursively by

$$\sigma(t) = \phi\left(\omega(t), \sigma^-(t)\right), \quad \sigma^-(t_0) = \sigma_0 \in \mathcal{I} \tag{3.6}$$

where the transition function is defined according to (3.2). The control laws for each region were designed according to the following simple rule: while $\sigma = 1$, $\omega_1(t)$ must decrease or remain constant and $\omega_2(t)$ must grow without bound as $t \to \infty$; when $\sigma = 2$, $\omega_1(t)$ must decrease and reach a determined bound in finite time; and finally, when $\sigma = 3$, $\omega_1(t)$ must again remain constant and $\omega_2(t)$ must converge to zero. A sketch of a typical trajectory of W is shown in Figure 3.3. The region that is the intersection of $\mathcal{R}_2$ and $\mathcal{R}_3$ can be seen as a hysteresis region. Its aim is to avoid the possibility of infinitely fast chattering when ω_1 is near ϵ_1.

Remark 3.1 For $\sigma = 1$, if λ does not satisfies the relation

$$-\frac{1}{2}\left(1 + \sqrt{\lambda^2 + 4}\right) x_1(t_0) \neq \dot{x}_1(t_0) \ \vee \ -\frac{1}{2}\left(1 + \sqrt{\lambda^2 + 4}\right) x_2(t_0) \neq \dot{x}_2(t_0),$$

then the unstable mode of the corresponded closed loop system is not excited. In that case, $g_1(x)$ has to be modified to

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + \operatorname{sgn}(c_2)\operatorname{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - \operatorname{sgn}(c_1)\operatorname{sgn}(s) \end{bmatrix},$$

where $\operatorname{sgn}(x) = 1$ if $x \geq 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$, $c_i = \frac{\dot{x}_i(t_0) - x_i(t_0)s_1}{s_2 - s_1}$, $i = 1, 2$, and $s_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\lambda^2 + 4}$.

3.2 Stability analysis

The following theorem establishes the main result of this section.

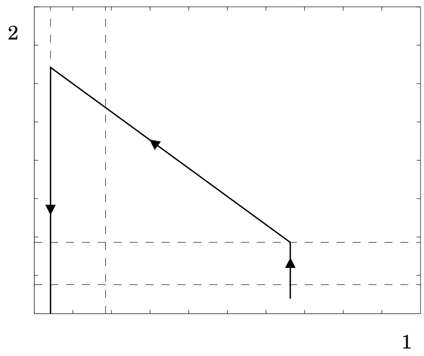

Figure 3.3: Image of a representative trajectory in the (ω_1, ω_2) plane.

Theorem 3.1 *Consider the hybrid system Σ described by (2.8), (3.4)-(3.6) and the switching logic defined in (3.2). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ. Then, the following properties hold.*

1. *Given an arbitrary pair $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ (initial condition), there exists a unique solution $\{x(t), \sigma(t)\}$ for all $t \geq t_0$ such that $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.*

2. *For any set of initial conditions $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$, there exists a finite time $T \geq t_0$ such that for $t > T$ the state variables $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$, and $\dot{x}_2(t)$ converge uniformly exponentially to zero and $\omega_1(t) \leq \epsilon_2$, where $\epsilon_2 > 0$ is a controller parameter that can be chosen arbitrarily small.*

3. *The origin $x(t) = 0$ is a Lyapunov uniformly stable equilibrium point of Σ.*

Proof: In the sequel the following notation is required: given a set $\mathcal{R} \subset \mathbb{R}^n$, its closure and boundary are denoted by $\overline{R}$ and $\partial\mathcal{R}$ respectively. $\mathcal{B}_\delta(x)$ denotes an open ball of radius $\delta > 0$ centered at x.

Existence and uniqueness

Existence and uniqueness of solutions for each subsystem

$$\dot{x} = g_i(x), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{3.7}$$

in $\mathcal{X}_i$ follows directly from two facts: *i)* each $g_i(x)$ is locally Lipschitz[1] in the domain $\mathcal{X}_i$. In fact, it can be shown that $g_i(x)$; $i = 1, 3$ is globally Lipschitz. *ii)* For $i = 2$, every solution of (3.7) lies entirely in a compact subset of $\mathcal{X}_2$. This will be shown below. Since the distance between two points in the (ω_1, ω_2)-space where consecutive switchings can occur is always nonzero, one can conclude that the hybrid system Σ has exactly one solution over $[t_0, \infty)$ for any initial condition $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$.

Convergence

The proof of convergence is based on the five claims below.

[1]Locally Lipschitz condition of g_i in x on $\mathcal{X}_i$ can be easily checked by observing that $g_i(x)$ is continuously differentiable with respect to x on the domain $\mathcal{X}_i$. Furthermore, for $\mathcal{X}_\sigma \equiv \mathbb{R}^n$, g_σ is globally Lipschitz if and only if the Jacobian matrix $[\partial g/\partial x]$ is uniformly bounded on $\mathbb{R}^n$, (Khalil, 1996).

Claim 3.1 *There exists a finite time $t_{\sigma_1} \geq t_0$ such that $\omega(t) \notin \mathcal{R}_1 \backslash \mathcal{R}_2$ for all $t \geq t_{\sigma_1}$.*

Proof. Consider first that $\omega(t_0) \in \mathcal{R}_1 \backslash \mathcal{R}_2$ and suppose by contradiction that $\omega(t)$ remains in $\mathcal{R}_1 \backslash \mathcal{R}_2$ for all $t \geq t_0$. Since $\omega(t_0) \in \mathcal{R}_1 \backslash \mathcal{R}_2$, then $\sigma(t_0) = 1$ and $\sigma(t)$ will be equal to 1 for all $t \geq t_0$. Therefore, the closed loop equation is $\dot{x} = g_1(x)$. It can be checked that for this case $\dot{\omega}_1(t) \leq 0$ for all $t \geq t_0$ and $\omega_2(t) \to \infty$ as $t \to \infty$. Consequently, ω will leave the region $\mathcal{R}_1 \backslash \mathcal{R}_2$ thus violating the assumption above. The remainder of the proof shows that $\omega(t)$ will remain outside $\mathcal{R}_1 \backslash \mathcal{R}_2$ for $t \geq t_{\sigma_1}$. Let $t_{\sigma_1} \triangleq \sup \{t \in [t_0, \infty) : \sigma(t) = 1 \wedge \omega(t) \in \partial \mathcal{R}_1 \backslash \mathcal{R}_2\}$. When $\omega(t_{\sigma_1}) \in \partial \mathcal{R}_1 \backslash \mathcal{R}_2$, $\sigma(t_{\sigma_1}) = 1$ and σ will remain constant until the next switch, which must occur after some positive time interval, say $\delta > 0$. Therefore, for $t \in [t_{\sigma_1}, t_{\sigma_1} + \delta]$ one has $\dot{\omega}_1 \leq 0$ and $\dot{\omega}_2 > 0$, which shows that the velocity vector points (non strictly) to the outside of $\mathcal{R}_1 \backslash \mathcal{R}_2$. Thus, one can conclude that $\omega(t)$ will remain outside $\mathcal{R}_1 \backslash \mathcal{R}_2$ for $t \geq t_{\sigma_1}$.

Claim 3.2 *For any $t_{\sigma_1} \geq t_0$ such that $\sigma(t_{\sigma_1}) = 1$, there exists a finite time $t_{\sigma_2} \geq t_{\sigma_1}$ such that $\sigma(t_{\sigma_2}) = 2$.*

Proof. For $\sigma = 1$, $\dot{\omega}_1 \leq 0$ and $\omega_2(t) \to \infty$ as $t \to \infty$. Thus, there exists a finite time t_{σ_2} such $\omega_2(t_{\sigma_2}) = \gamma_2$, which implies that for $t = t_{\sigma_2}$, σ switches to 2.

Claim 3.3 *For any t_{σ_2} such that $\sigma(t_{\sigma_2}) = 2$, there exists a positive time interval $\tau > 0$ such that for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$*

$$\omega_1(t_{\sigma_2} + \tau) \leq \omega_1(t_{\sigma_2}) e^{-2\gamma_1 \tau}.$$

Proof. For $\sigma(t_{\sigma_2}) = 2$, the closed loop equations of the hybrid system Σ are given by $\dot{x} = g_2(x)$. Since $g_2(x)$ is continuous in x, the solution $x(t)$ will be continuously differentiable and, consequently, there exists $\tau > 0$ such that for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$, $\omega(t)$ will be inside $\mathcal{R}_2$ and $\sigma(t) = 2$. This means that for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$,

$$\begin{aligned} \dot{\omega}_1 &= -2\omega_2 \omega_1 \\ \omega_2 &\geq \gamma_1. \end{aligned}$$

Hence, $\omega_1(t_{\sigma_2} + \tau) \leq \omega_1(t_{\sigma_2}) e^{-2\gamma_1 \tau}$.

Claim 3.4 *There exists a finite time $t_{\sigma_3} \geq t_0$ such that for all $t \geq t_{\sigma_3}$, $\omega(t) \in \mathcal{R}_3$.*

Proof. This claim will be proven in two steps. First, it will be shown that for any $\omega(t_0) \in \Omega \backslash \mathcal{R}_3$, there exists a time $t_{\sigma_3} \geq t_0$ such that $\omega(t_{\sigma_3}) \in \mathcal{R}_3$ and second, that if $\omega(t_{\sigma_3}) \in \mathcal{R}_3$ then $\omega(t) \in R_3$ for all $t \geq t_{\sigma_3}$. Consider first that $\omega(t_0) \in \Omega \backslash \mathcal{R}_3$. Then, the objective is to prove that ω_1 will reach in finite time the boundary $\omega_1 = \epsilon_2$, where σ can only takes the value 1 or 2. From Claim 3.1, it follows that after a finite time t_{σ_1}, $\omega_2(t) > \gamma_1$ (while $\omega \in \Omega \backslash \mathcal{R}_3$). Since the dynamics of ω_1 are given by

$$\dot{\omega}_1 = \begin{cases} \leq 0 & \sigma = 1, \\ -2\omega_2 \omega_1 & \sigma = 2. \end{cases}$$

from Claim 3.2 and 3.3 it can be concluded that there exists a finite time t_{σ_3} for which $\omega(t_{\sigma_3}) \in \mathcal{R}_3$. To conclude the proof, it remains to show that $\omega(t)$ will be always inside $\mathcal{R}_3$ for $t \geq t_{\sigma_3}$. This is easily proved due to the fact that when $\omega(t_{\sigma_3}) \in \partial\mathcal{R}_3$, $\sigma \in \mathcal{I}$ will remain constant for at least some positive time interval (say $\delta > 0$). Therefore for $t \in [t_{\sigma_3}, t_{\sigma_3} + \delta]$

$$\dot{\omega}_1 = \begin{cases} \leq 0 & \sigma = 1, \\ -2\omega_2\omega_1 & \sigma = 2, \\ 0 & \sigma = 3. \end{cases}$$

which shows that the velocity vector does in fact point (non strictly) to the inside of $\bar{\mathcal{R}}_3$ and consequently $\mathcal{R}_3$ is a positively invariant set.

Claim 3.5 *For $t \geq T$, $\omega_1(t) \leq \epsilon_2$ and $\omega_2(t)$ converges exponential to zero as $t \to \infty$.*

Proof. From Claim 3.4, $\omega(t) \in \mathcal{R}_3$ for all $t \geq t_{\sigma_3}$. Thus, $\omega_1(t) \leq \epsilon_2$. Moreover, there exists a finite time $T \geq t_0$ such that for all $t \geq T$ $\omega(t) \in \mathcal{R}_3 \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$. The proof of this follows *mutatis mutandi* the one given for Claim 3.4. Hence, since when $\omega(t) \in \mathcal{R}_3 \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$ it follows that $\sigma(t) = 3$, the closed loop equation are given by $\dot{x} = g_3(x)$ which shows that $\omega_2(t)$ converges exponential to zero as $t \to \infty$. This conclude the proof of Claim 3.5 and naturally item 2 of Theorem 3.1.

Stability

Given $\epsilon > 0$, let r be such that $r = \min\{\epsilon, \epsilon_1\}$. Thus if $x(t_0) \in B_r(0) \subset \mathcal{R}_3 \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$ then $x(t) \in \mathcal{R}_3 \backslash (\mathcal{R}_1 \cup \mathcal{R}_2)$ for all $t > t_0$ and $\sigma = 3$ (see Claim 3.5). Consider the Lyapunov function candidate

$$V(x) = \omega_1 + \omega_2 + \dot{x}_1^2 + \dot{x}_2^2.$$

The time derivative of V along system trajectories yields

$$\dot{V} = -\lambda(\dot{x}_1^2 + \dot{x}_2^2) \leq 0.$$

Let $c \triangleq \min_{\|x\|=r} V(x)$ and $\Omega_c \triangleq \{x \in B_r(0) : V(x) < c\}$. Since $\dot{V}(x(t)) \leq 0$, then $V(x(t)) \leq V(x_0) \leq c$ for all $t \geq t_0$, and consequently $\Omega_c \subset B_r(0)$ is a positively invariant set. Hence, since $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that $\|x\| \leq \delta \Rightarrow V(x) < c$. Thus, $B_\delta(0) \subset \Omega_c$ and

$$x_0 \in B_\delta(0) \Rightarrow x_0 \in \Omega_c \Rightarrow x(t) \in \Omega_c \Rightarrow \|x(t)\| < r \leq \epsilon.$$

Therefore the hybrid system Σ is Lyapunov uniformly stable by definition.

This completes the proof of Theorem 3.1. □

Remark 3.2 Some care is needed with the implementation of the control law proposed. In fact depending on the initial condition, it may happen that during the time interval in which $\sigma = 2$ the variable $\omega_2(t)$ grow considerably. To tackle this problem, a possible solution is to add an extra region in which $\omega_2(t)$ is forced to decrease. One example is

$$\begin{aligned} \mathcal{R}_1 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \leq \gamma_2\}, \\ \mathcal{R}_2 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \gamma_1 \leq \omega_2 \leq \gamma_4\}, \\ \mathcal{R}_3 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 \leq \epsilon_2\} \cup \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \geq \gamma_3\}, \end{aligned}$$

where $\epsilon_2 > \epsilon_1 > 0$ and $\gamma_4 > \gamma_3 > \gamma_2 > \gamma_1 > 0$. Figure 3.4 shows a sketch of a typical trajectory of W with the new modified control law.

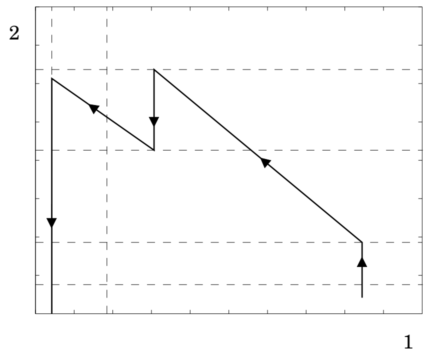

Figure 3.4: Image of a representative trajectory in the W plane with the modified regions.

3.3 Boundedness of control inputs

In this section the hybrid control law proposed is extended to the bounded input case, that is, when the ENDI system is subject to the input constraints

$$|u_1(t)| \leq \bar{u}_1,$$
$$|u_2(t)| \leq \bar{u}_2.$$

where $\bar{u}_1$ and $\bar{u}_2$ are positive constants. It will be shown that by adding one more region associated with an extra control field the control problem formulated before can still be solved. The objective of the extra region is to bring the state variables (whenever they have large amplitude values) to a region where the inputs constraints are satisfied.

Consider the following function $W_B(x) : \mathbb{R}^5 \to \Omega \subset \mathbb{R}^2$, mapping the state-space x into the two-dimensional closed positive quadrant space Ω

$$\omega = (\omega_1, \omega_2)' = W_B(x) = \left(s^2, (x_1)^2 + (x_2)^2 + (\dot{x}_1)^2 + (\dot{x}_2)^2\right)',$$

where $s = \dot{x}_3 + \lambda x_3$ and λ is a strictly positive constant. Notice that $W_B(\cdot)$ satisfies the same properties as $W(x)$ described in Lemma 3.1.

Consider also the following four overlapping regions (see Figure 3.5)

$$\mathcal{R}_1 = \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \leq \gamma_2\},$$
$$\mathcal{R}_2 = \{(\omega_1, \omega_2) \in \Omega : \omega_2 \geq \gamma_2\},$$
$$\mathcal{R}_3 = \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \gamma_1 \leq \omega_2 \leq \gamma_4\},$$
$$\mathcal{R}_4 = \{(\omega_1, \omega_2) \in \Omega : \omega_1 \leq \epsilon_2 \wedge \omega_2 \leq \gamma_5\} \cup \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \gamma_3 \leq \omega_2 \leq \gamma_5\},$$

where $\epsilon_{i+1} > \epsilon_i > 0$ for $i = 1, 2$ and $\gamma_{i+1} > \gamma_i > 0$ for $i = 1, 2, 3, 4$. The new control law is given by

$$u = g_\sigma(x), \tag{3.8}$$

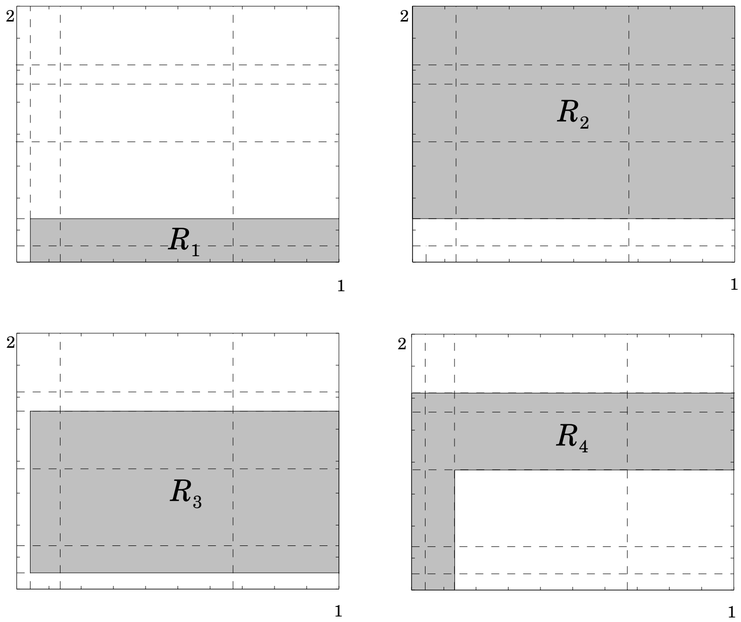

Figure 3.5: Definition of regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$.

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 \\ -\lambda\dot{x}_2 + x_2 \end{bmatrix}, \tag{3.9a}$$

$$g_2(x) = \begin{bmatrix} -k_{11}\ \mathrm{sat}(k_{12}x_1 + \dot{x}_1) - k_{12}\ \mathrm{sat}(\dot{x}_1) \\ -k_{21}\ \mathrm{sat}(k_{22}x_2 + \dot{x}_2) - k_{22}\ \mathrm{sat}(\dot{x}_2) \end{bmatrix}, \tag{3.9b}$$

$$g_3(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + k_1x_2\,\mathrm{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - k_2x_1\,\mathrm{sgn}(s) \end{bmatrix}, \tag{3.9c}$$

$$g_4(x) = \begin{bmatrix} -\lambda\dot{x}_1 - x_1 \\ -\lambda\dot{x}_2 - x_2 \end{bmatrix}, \tag{3.9d}$$

where $\sigma \in \mathcal{I} = \{1, 2, 3, 4\}$, k_i, k_{i1}, and k_{i2} are positive constants that satisfy $k_{i1} + k_{i2} = \bar{u}_i$ and $k_{i2} > k_{i1}$ for all $i = 1, 2$.

Remark 3.3 The term $\mathrm{sgn}(s)$ that appears in the vector field $g_3(x)$ does not introduce chattering since s will never change its signal while $\sigma = 3$.

Figure 3.6 shows a typical trajectory of W_B.

The following theorem can be proved.

Theorem 3.2 *Consider the hybrid system Σ_B described by (2.8), (3.8), (3.9) and the switching logic defined in (3.2). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ_B. Then, the following properties hold.*

1. *Given an arbitrary pair $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ (initial condition), there exists a unique solution $\{x(t), \sigma(t)\}$ for all $t \geq t_0$ such that $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.*

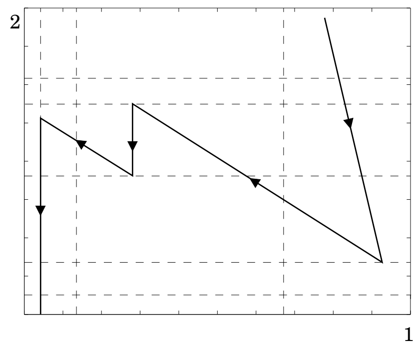

Figure 3.6: Image of a representative trajectory in the W_B plane.

2. *For any set of initial conditions* $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$, *there exists a finite time* T *such that for* $t > T$ *the state variables* $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$, *and* $\dot{x}_2(t)$ *converge uniformly exponentially to zero, and* $\omega_1(t) \leq \epsilon_2$, *where* $\epsilon_2 > 0$ *is a controller parameter that can be chosen arbitrarily small.*

3. *The origin* $x(t) = 0$ *is a Lyapunov uniformly stable equilibrium point of* Σ_B.

4. *Let* ϵ_3 *and* γ_5 *be positive constants such that* $\epsilon_3 \leq \epsilon^\star$ *and* $\gamma_5 \leq \gamma^\star$, *where the vector* $(\epsilon^\star, \gamma^\star)'$ *is an admissible or feasible parameter vector, for which the inequalities*

$$\begin{aligned} \lambda|\dot{x}_1| + |x_1| + k_1|x_2| &\leq \bar{u}_1, \\ \lambda|\dot{x}_2| + |x_2| + k_2|x_1| &\leq \bar{u}_2, \end{aligned} \tag{3.10}$$

hold for all $\omega_1 \leq \epsilon^\star$ *and* $\omega_2 \leq \gamma^\star$. *Then, for any arbitrary large and bounded continuous state initial conditions* x_0 *the control signals satisfy the constraints*

$$|u_i| \leq \bar{u}_i, \quad i = 1, 2 \tag{3.11}$$

and along the trajectories of the closed loop system Σ_B

$$\lim_{t\to\infty} u_i(t) = 0, \quad i = 1, 2.$$

Proof:

Existence and uniqueness

For each $i \in \mathcal{I}$ the vector field $g_i(x)$ is globally Lipschitz. Furthermore, the distance between two points in the (ω_1, ω_2)-space where consecutive switchings can occur is always nonzero. It now follows from classical arguments (Khalil, 1996) that the hybrid system Σ_d has exactly one solution for each initial condition $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$.

Convergence

The proof of convergence is based on the following claims.

Claim 3.6 *The equilibrium point* $x = (x, \dot{x}) = 0$ *of the system given by*

$$\ddot{x} = -k_1 \operatorname{sat}(k_2 x + \dot{x}) - k_2 \operatorname{sat}(\dot{x}), \tag{3.12}$$

where $\lambda > 0$, k_1 and k_2 are positive constants that satisfy $k_1 + k_2 = \bar{u}$ and $k_2 > k_1$, is globally asymptotically stable.

Proof. Let $y = (y_1, y_2)'$, $y_1 = k_2 x + \dot{x}$, and $y_2 = \dot{x}$. Then, system (3.12) can be rewritten as

$$\dot{y}_1 = k_2 y_2 - k_1 \text{ sat}(y_1) - k_2 \text{ sat}(y_2),$$
$$\dot{y}_2 = -k_1 \text{ sat}(y_1) - k_2 \text{ sat}(y_2).$$

Clearly, for any initial condition $y(t_0) = y_0$, there exists a finite time T such that $|y_2(t)| \leq 1$ for all $t \geq T$ since

$$y_2 \dot{y}_2 = -k_1 y_2 \text{ sat}(y_1) - k_2 y_2 \text{ sat}(y_2)$$

and, consequently, for $|y_2| \geq 1$

$$y_2 \dot{y}_2 \leq -|y_2| \, (k_2 - k_1) \; < \; 0.$$

Analyzing now the state y_1, one has

$$y_1 \dot{y}_1 = k_2 y_1 y_2 - k_1 y_1 \text{ sat}(y_1) - k_2 y_1 \text{ sat}(y_2). \tag{3.13}$$

Since after a finite time $\text{sat}(y_2)$ will be equal to y_2, expression (3.13) gives

$$y_1 \dot{y}_1 = -k_1 y_1 \text{ sat}(y_1)$$

which shows that after a finite time y_1 will enter and stay in the set $|y_1| \leq 1$. Therefore, after a finite time, the second member of equation (3.12) becomes linear for all t. Consequently, since the linear equation has two eigenvalues in $\lambda_1 = -k_1$ and $\lambda_2 = -k_2$, it follows that the equilibrium point $x = 0$ is globally uniformly asymptotically stable.

Claim 3.7 *For any $t_0 \geq 0$ such that $\sigma_0 = 2$, there exists a finite time $T \geq t_0$ such that for all $t \geq T$, $\sigma(t) \in \mathcal{I} \backslash \{2\}$.*

Proof. Consider that $\sigma_0 = 2$ and suppose by contradiction that $\sigma(t) = 2$ for all $t \geq t_0$. From Claim 3.6, there will be a finite time T such that $\omega_2 \leq \gamma_2$ and consequently $\sigma(t)$ will have to switch, which is a contradiction. Consider now that $\sigma_0 \in \mathcal{I} \backslash \{2\}$. If $\omega_2(t_0) > \gamma_5$, then σ will switch to 2 and ω_2 will decrease at least until $\omega_2 = \gamma_2$ where at that point σ will switch again to $\mathcal{I} \backslash \{2\}$. Suppose that $\omega_2(t_0) \leq \gamma_5$ (and $\sigma_0 \in \mathcal{I} \backslash \{2\}$). Then, from the definition of the regions and the transition function $\phi(\omega, \sigma)$, $\sigma(t)$ will only switch to value 2 if there exists $t_2 \geq t_0$ such that $\omega_2(t_2) > \gamma_5$. Let $t_{\gamma_5} \triangleq \sup\{t \in [t_0, \infty) : \omega_2(t) = \gamma_5\}$. Since when $\omega_2(t_{\gamma_5}) = \gamma_5$, $\sigma(t_{\gamma_5}) = 4$ and $\sigma(t)$ will remain constant for at least some positive interval (say $\delta > 0$), then for $t \in [t_{\gamma_5}, t_{\gamma_5} + \delta]$

$$\dot{\omega}_1 = 0$$
$$\dot{\omega}_2 = -2\lambda \left((\dot{x}_1)^2 + (\dot{x}_2)^2 \right) \; \leq \; 0,$$

which shows that the velocity vector points (non strictly) to the inside of the region $\omega_2(t) \leq \gamma_5$. Thus, this region is a positively invariant set and consequently $\sigma(t)$ will never switch to the value 2 again.

Claim 3.8 *The region*

$$\mathcal{R}_{inv} = \{(\omega_1, \omega_2) \in \Omega : \omega_1 \le \epsilon_3 \wedge \omega_2 \le \gamma_5\}$$

is a positively invariant set.

Proof. From Claim 3.7, if the initial condition $\omega(t_0)$ is outside $\mathcal{R}_{inv}$, then there will be a finite time T such that σ will switch to a value different from 2. At that point either $\omega(T)$ is inside $\mathcal{R}_{inv}$ or is outside with $\omega_2(T) = \gamma_2$. In the latter case, it can be seen that σ will switch to $\sigma = 3$ and $\omega(t)$ will reach the boundary of $\mathcal{R}_{inv}$ in finite time. When $\omega(t)$ is inside $\mathcal{R}_{inv}$, analysis of the velocity vector at every point of $\omega \in \partial\mathcal{R}_{inv}$ shows that it points (non strictly) to the inside of $\bar{\mathcal{R}}_{inv}$. Therefore the set $\mathcal{R}_{inv}$ is positively invariant.

Since the vector fields $g_i(x)$, $i = 1, 3, 4$ are equal to or exhibit the same properties (with respect to the direction of the velocity vector ω) to those presented in Section 3.1, the rest of the convergence proof is not presented since it is easily established using the same arguments derived the proof of Theorem 3.1.

Stability

The stability proof follows *mutatis mutandi* the one given in Section 3.2.

Boundedness

From Claim 3.8, if $\omega(t_0) \in \mathcal{R}_{inv}$, then $\omega(t) \in \mathcal{R}_{inv}$ for all $t \ge t_0$. Therefore, if $\epsilon_3 \le \epsilon^\star$ and $\gamma_5 \le \gamma^\star$ then the input signals satisfy the constraints (3.11). Notice that there exists always a feasible parameter vector $(\epsilon^\star, \gamma^\star)'$, since $(\epsilon^\star, \gamma^\star)'$ can be chosen arbitrarily small such that inequalities (3.10) hold. If $\omega(t) \notin \mathcal{R}_{inv}$, then either $\omega_2(t) > \gamma_5$ or $\omega_2(t) \le \gamma_5$. In the first case, $\sigma(t)$ can only be equal to 2, and therefore constraints (3.11) are satisfied, since $k_{i1} + k_{i2} = \bar{u}_i$ for $i = 1, 2$. In the latter case, since $\omega_2(t) \le \gamma_5$, then $\omega_2(t) \le \gamma^\star$, and it can be thus easily checked that the constraints (3.11) are also satisfied. Finally, because there is a time $t_{\sigma_4} \ge 0$ such that for all $t \ge t_{\sigma_4}$, $\sigma(t) = 4$, it can be easily derived that

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} \dot{x}_i(t) = 0 \Rightarrow \lim_{t\to\infty} u_i(t) = 0,$$

for $i = 1, 2$. □

3.4 Stability analysis under persistent disturbances

Consider now the ENDI system subject to magnitude limitations on their inputs and in the presence of small input additive disturbances, *i.e.*,

$$\begin{aligned} \ddot{x}_1(t) &= u_1(t) + d_1(t), \\ \ddot{x}_2(t) &= u_2(t) + d_2(t), \\ \dot{x}_3(t) &= x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t), \end{aligned} \tag{3.14}$$

where $x = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)' \in \mathbb{R}^5$ is a state vector and $u = (u_1, u_2)' \in \mathbb{R}^2$ is a two-dimensional control vector subject to the constraints $|u_i(t)| \le \bar{u}_i$. The disturbance vector $d = (d_1, d_2)'$ with

$d_i : [t_0, \infty) \to \mathbb{R}$ piecewise continuous in t and satisfies $|d_i(t)| \leq \delta$ for all $i = 1, 2$.

Before presenting the main result we now introduce the following definitions that extend to a hybrid control setting the usual definitions of ultimate boundedness and practical stability (La Salle and Lefschetz, 1961).

Definition 3.2 ((Global) uniform ultimate boundedness) *The solutions of the hybrid system Σ (see expression (3.1)) are said to be uniformly ultimately bounded (with bound b) if there exist positive constants b and c, and for every $\alpha \in (0, c)$, any $t_0 \in \mathbb{R}^+$ and every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \alpha$ there exists a positive constant $T = T(\alpha)$ independent of t_0 such that the continuous state solution $x(t)$ of Σ satisfies $\|x(t)\| < b$ for all $t \geq t_0 + T$. The solutions are said to be globally uniformly ultimately bounded if the above condition holds for arbitrarily large α.*

Consider now the hybrid system Σ_d defined by

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}\left(x(t), t\right) + d(x, t), \qquad t \geq t_0 & (3.15)\\ \sigma(t) &= \phi\left(x(t), \sigma(t^-)\right), & (3.16)\end{aligned}$$

where the only difference between Σ and Σ_d is the presence of a perturbation term d that for physical systems may represent input disturbances or capture unknown modeling parameters. In practice d may be an unknown function of (x, t) but satisfies the constraint

$$\|d(x, t)\| \leq \delta, \quad \forall t \geq t_0 \, \forall x \in \mathcal{X}. \tag{3.17}$$

Definition 3.3 (Practical stability) *Let δ be a positive constant and let $\mathcal{R} \subset \mathcal{X}$ and $\mathcal{R}_0$ be two sets where $\mathcal{R}$ is a closed and bounded set containing the origin and $\mathcal{R}_0$ is a subset of $\mathcal{R}$. Let S_D be the set of all perturbations d satisfying (3.17). The origin $x = 0$ is said to be* practically stable *if for each $d \in S_D$, any $t_0 \in \mathbb{R}^+$, and for every initial condition $\{x_0, \sigma_0\} \in \mathcal{R}_0 \times \mathcal{I}$, the continuous solution of Σ_d, $x(t)$ is in $\mathcal{R}$ for all $t \geq t_0$. That is, the solutions that start initially in $\mathcal{R}_0 \times \mathcal{I}$ remain thereafter in $\mathcal{R} \times \mathcal{I}$. If, in addition, each solution of the hybrid system Σ_d for each $d \in S_D$ is ultimately in $\mathcal{R}$, then one says that the hybrid system Σ (see expression (3.1)) has* strong practical stability.

To solve the problem of practical stability for Σ_d, the control law developed in Section 3.3 is used, together with a slight modification of the vectors fields $g_\sigma(x)$ as follows:

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + k_1 \operatorname{sgn}(x_2)\operatorname{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - k_2 \operatorname{sgn}(x_1)\operatorname{sgn}(s) \end{bmatrix}, \tag{3.18a}$$

$$g_2(x) = \begin{bmatrix} -k_{11}\ \operatorname{sat}(k_{12}x_1 + \dot{x}_1) - k_{12}\ \operatorname{sat}(\dot{x}_1) \\ -k_{21}\ \operatorname{sat}(k_{22}x_2 + \dot{x}_2) - k_{22}\ \operatorname{sat}(\dot{x}_2) \end{bmatrix}, \tag{3.18b}$$

$$g_3(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + k_1 h(\frac{x_2}{\delta})\operatorname{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - k_2 h(\frac{x_1}{\delta})\operatorname{sgn}(s) \end{bmatrix}, \tag{3.18c}$$

$$g_4(x) = \begin{cases} \begin{bmatrix} -\lambda\dot{x}_1 - x_1 + k_1 \operatorname{sat}(x_2)\frac{s}{\eta} \\ -\lambda\dot{x}_2 - x_2 - k_2 \operatorname{sat}(x_1)\frac{s}{\eta} \end{bmatrix}, & |s| < \eta \\ \begin{bmatrix} -\lambda\dot{x}_1 - x_1 + k_1 \operatorname{sgn}(x_2)\operatorname{sgn}(s) \\ -\lambda\dot{x}_2 - x_2 - k_2 \operatorname{sgn}(x_1)\operatorname{sgn}(s) \end{bmatrix}, & |s| \geq \eta \end{cases} \tag{3.18d}$$

where

$$h(x) = \begin{cases} x & \text{if } |x| \geq 1, \\ \operatorname{sgn}(x) & \text{otherwise,} \end{cases}$$

$\lambda > 0$, η is a positive constant that satisfies $\eta^2 < \epsilon_2$, $k_1 > \delta$, $k_2 > \delta$, and k_{i1} and k_{i2} are positive constants such that $k_{i1} + k_{i2} = \bar{u}_i$, $k_{i2} > k_{i1} + \delta$, and $k_{i1} > \delta$ for all $i = 1, 2$. Furthermore, $\epsilon_{i+1} > \epsilon_i > 0$ for $i = 1, 2$, $\gamma_{i+1} > \gamma_i > 0$ for $i = 1, 2, 3, 4$, and γ_5 satisfies

$$\beta_1^2 + \beta_2^2 \leq \gamma_5, \tag{3.19}$$

where

$$\beta_i = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\frac{\sqrt{\lambda^2+4}}{\lambda}\frac{k_i+\delta}{\theta}, \quad i = 1, 2 \tag{3.20}$$

for some positive constant $\theta < 1$, $\lambda_{max}(P) = \frac{\lambda}{4} + \frac{1}{\lambda} + \frac{\sqrt{\lambda^2+4}}{4}$, and $\lambda_{min}(P) = \frac{\lambda}{4} + \frac{1}{\lambda} - \frac{\sqrt{\lambda^2+4}}{4}$.

The following theorem establishes the main result of this section.

Theorem 3.3 *Consider the hybrid system Σ_d described by (3.14), (3.8), (3.18) and the switching logic defined in (3.2). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ_d. Then, the following properties hold.*

1. *Given an arbitrary pair $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ (initial condition), there exists a unique solution $\{x(t), \sigma(t)\}$ for all $t \geq t_0$ such that $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.*

2. *The solutions of the hybrid system Σ_d are globally uniformly ultimately bounded.*

3. *Let $|d(t)| \leq \delta$. The origin $x(t) = 0$ is practically stable relative to the positive constant δ and the two sets $\mathcal{R}$ and $\mathcal{R}_0 \subset \mathcal{R}$ defined by*

$$\mathcal{R}_0 = \{x \in \mathbb{R}^5 : \omega_1 \leq \eta^2 \wedge \omega_2 \leq \mu_1^2 + \mu_2^2\},$$
$$\mathcal{R} = \{x \in \mathbb{R}^5 : \omega_1 \leq \epsilon_2 \wedge \omega_2 \leq \beta_1^2 + \beta_2^2\},$$

where β_i, $i = 1, 2$ is given by (3.20) and

$$\mu_i = \frac{\sqrt{\lambda^2 + 4}}{\lambda} \frac{k_i + \delta}{\theta}, \quad i = 1, 2$$

for some positive constant $\theta < 1$.

Furthermore, the hybrid system composed by Σ_d without the perturbation term d possesses strong practical stability.

4. *Let $\epsilon_3 \leq \epsilon^\star$ and $\gamma_5 \leq \gamma^\star$, where the vector $(\epsilon^\star, \gamma^\star)'$ is an admissible or feasible parameter vector, for which the inequalities (3.10) and*

$$\lambda|\dot{x}_i| + |x_i| + k_i \leq \bar{u}_i, \quad i = 1, 2 \tag{3.21}$$

hold with $\omega_1 \leq \epsilon^\star$ and $\omega_2 \leq \gamma^\star$. Then, for any arbitrary large and bounded continuous state initial conditions, x_0, the control signals satisfy the constraints $|u_i| \leq \bar{u}_i$, $i = 1, 2$.

Proof:

Existence and uniqueness

Existence and uniqueness of solutions follow from classical arguments by noticing that: *i)* the number of discontinuities of the control law over any finite time interval is finite; and *ii)* between two successive discontinuities the vector field in (3.1a) is Lipschitz continuous. We now prove statement *i)*, as *ii)* is obvious.

First, it is shown that the terms $\text{sgn}(s)$, $\text{sgn}(x_1)$ and $\text{sgn}(x_2)$ in the vector fields (3.18) do not introduce an infinite number of discontinuities in a finite time interval. Starting with term $\text{sgn}(s)$, one can immediately conclude, by observing the definition of regions $\mathcal{R}_i$; $i = 1, \ldots 4$ (see Figure 3.5), that while the corresponding vector field is enabled the variable s will never change signal. For the terms $\text{sgn}(x_1)$ and $\text{sgn}(x_2)$ we will focus on the case $\sigma = 4$, since for the other cases the proof is similar. Suppose that the initial conditions are $\sigma(t_0) = 4$, $\omega(t_0) \in \mathcal{R}_4$, $|s(t_0)| > \eta$, $x_1(t_0) > 0$, $\dot{x}_1(t_0) \geq 0$, $x_2(t_0) > 0$, and $\dot{x}_2(t_0) \geq 0$. In this situation one has

$$\ddot{x}_1 = -\lambda\dot{x}_1 - x_1 + k_1 + d_1(t), \tag{3.22a}$$

$$\ddot{x}_2 = -\lambda\dot{x}_2 - x_2 - k_2 + d_2(t). \tag{3.22b}$$

Thus, for this case, the vector field g_4 is globally Lipschitz and, consequently, while equations (3.22) hold system Σ_d has a unique solution. One can also conclude that there exits a positive time interval $\tau_1 > 0$ such that $x_1(t) > 0$ and $x_2(t) > 0$ for all $t \in [t_0, t_0 + \tau_1)$, and $x_2(t_0 + \tau_1) = 0$. For $t > t_1 = t_0 + \tau_1$ the dynamics of x_1 given by (3.22a) change to

$$\ddot{x}_1 = -\lambda\dot{x}_1 - x_1 - k_1 + d_1(t) \tag{3.23}$$

and the dynamics of x_2 continue to be given by (3.22b). That is, a switch has occurred in the control signal u_1. Notice, however, that one can also conclude that while equations (3.23) and(3.22b) hold, Σ_d continues to have a unique solution. In particular, there will be a positive time interval $\tau_2 > 0$ such that $x_1(t) > 0$ and $x_2(t) < 0$ for all $t \in (t_1, t_1 + \tau_2)$.

Proceeding successively with this line of reasoning one concludes that in any finite time interval one always has a finite number of switches. From the definitions of the regions one can also observe that the distance between two points in the (ω_1, ω_2)-space where consecutive switchings can occur is always nonzero. It now follows from classical arguments (Khalil, 1996) that the hybrid system Σ_d has exactly one solution for each initial condition $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$.

Global uniform ultimate boundedness

To show global uniform ultimate boundedness of system Σ_d consider first the following claims.

Claim 3.9 *Consider the system*

$$\ddot{x} = -k_1 \text{ sat}\left(k_2 x + \dot{x}\right) - k_2 \text{ sat}\left(\dot{x}\right) + d(t),$$

where the disturbance term $d(t)$ satisfies $|d(t)| \leq \delta$ and k_1 and k_2 are positive constants that satisfy $k_1 + k_2 = \bar{u}$, $k_2 > k_1 + \delta$, and $k_1 > \delta$. Then, for any initial condition $(x(t_0), \dot{x}(t_0))' \in \mathbb{R}^2$, there exist a finite time $T \geq t_0$ and a positive constant γ given by

$$\gamma = \frac{\delta}{k_1 {k_2}^2}\sqrt{4{k_1}^2 + {k_2}^2 + 4k_1 k_2 + 4{k_1}^2{k_2}^2} \tag{3.24}$$

such that for all $t \geq T$

$$x^2(t) + \dot{x}^2(t) \leq \gamma^2. \tag{3.25}$$

Proof. Applying the same coordinate transformation that was used in the proof of Claim 3.6 of Theorem 3.2, it follows that for any initial condition $y(t_0) = y_0$ there exists a finite time T such that the state $y_2 = \dot{x}$ satisfies $|y_2| \leq 1$ for all $t \geq T$ since

$$\begin{aligned} y_2\dot{y}_2 &= -k_1 y_2 \text{ sat}(y_1) - k_2 y_2 \text{ sat}(y_2) + y_2 d \\ &\leq -|y_2|(k_2 - k_1 - \delta) \ < \ 0. \end{aligned} \tag{3.26}$$

Notice that the state $y_1 = k_2 x + \dot{x}$ satisfies

$$\begin{aligned} y_1\dot{y}_1 &= -k_1 y_1 \text{ sat}(y_1) + y_1 d \\ &\leq -|y_1| \left(k_1 \left|\text{sat}(y_1)\right| - \delta\right). \end{aligned}$$

This, coupled with the fact that $k_1 > \delta$, shows that after a finite time y_1 will enter and stay in the set $|y_1| \leq \frac{\delta}{k_1}$. It then follows that y_2 will enter and stay in the set $|y_2| \leq \frac{2\delta}{k_2}$. Condition (3.25) holds since the set given by $\mathcal{R} = \left\{(x, \dot{x}) \in \mathbb{R}^2 : |k_2 x + \dot{x}| \leq \frac{\delta}{k_1} \wedge |\dot{x}| \leq \frac{2\delta}{k_2}\right\}$ is a subset of $\Omega_\gamma = \{(x, \dot{x}) \in \mathbb{R}^2 : x^2 + \dot{x}^2 \leq \gamma^2\}$.

Claim 3.10 *Consider the system*

$$\ddot{x}_1 = -\lambda \dot{x}_1 - x_1 + k + d(t), \tag{3.27}$$

$\Leftrightarrow$

$$\dot{x} = Ax + g(x) \Leftrightarrow \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ k + d \end{bmatrix}, \tag{3.28}$$

where k and λ are positive constants and the disturbance term $d(t)$ satisfies $|d(t)| \leq \delta$. Then, there exist a finite time $T \geq t_0$, a 2×2 real symmetric positive definite matrix P, and finite positive constants γ, ξ, β, and $\theta < 1$ such that for any initial condition $x_0 = (x_1(t_0), \dot{x}_1(t_0))'$, the solution $x(t) = (x_1(t), \dot{x}_1(t))'$ of (3.28) satisfies

$$\|x(t)\| \leq \gamma e^{-\xi(t-t_0)}\|x(t_0)\|, \quad \forall t_0 \leq t < T$$

and

$$\|x(t)\| \leq \beta, \quad \forall t \geq T \tag{3.29}$$

where

$$\gamma = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}, \quad \xi = -\frac{1-\theta}{2\lambda_{max}(P)}, \quad \beta = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \frac{\sqrt{\lambda^2+4}}{\lambda} \frac{k+\delta}{\theta}. \tag{3.30}$$

Proof. Let $V(x) = x^T P x$ be a Lyapunov function candidate, where P is a real symmetric positive definite matrix that satisfies the Lyapunov equation $PA + A^T P = -I$. The derivative of $V(x)$ along the trajectories of (3.28) satisfies

$$\begin{aligned} \dot{V} &= -\|x\|^2 + 2x^T P g \\ &\leq -\|x\|^2 + \frac{\sqrt{\lambda^2+4}}{\lambda}(k+\delta)\|x\| \\ &= -(1-\theta)\|x\|^2 - \theta\|x\|^2 + \frac{\sqrt{\lambda^2+4}}{\lambda}(k+\delta)\|x\|, \quad 0 < \theta < 1 \\ &\leq -(1-\theta)\|x\|^2, \quad \forall\, \|x\| \geq \mu \end{aligned}$$

where $\mu = \frac{\sqrt{\lambda^2+4}}{\lambda}\frac{k+\delta}{\theta}$. Since

$$\lambda_{min}(P)\|x\|^2 \;\leq\; V(x) \;\leq\; \lambda_{max}(P)\|x\|^2,$$

then

$$\dot{V} \leq -\frac{1-\theta}{\lambda_{max}(P)} V, \quad \forall\, \|x\| \geq \mu$$

and consequently

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \exp\Big[-\frac{1-\theta}{2\lambda_{max}(P)}(t-t_0)\Big]\|x(t_0)\|, \quad \forall\, \|x\| \geq \mu \tag{3.31}$$

Let $\beta = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\mu$, and Ω_β a set defined by

$$\Omega_\beta = \{x \in \mathbb{R}^2 : V(x) \leq \lambda_{min}(P)\beta^2\}.$$

The set Ω_β contains the ball $B_\mu(0) = \{x \in \mathbb{R}^2 : \|x\| \leq \mu\}$ since

$$\begin{aligned} \|x\| \leq \mu = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\beta &\Rightarrow \lambda_{max}(P)\|x\|^2 \leq \lambda_{min}(P)\beta^2 \\ &\Rightarrow V(x) \leq \lambda_{min}(P)\beta^2. \end{aligned}$$

Therefore, any solution that starts inside Ω_β cannot leave it because $\dot V$ is negative on the boundary. Consequently, for a solution starting inside Ω_β, the inequality (3.29) is satisfied for all $t \geq t_0$ since

$$\begin{aligned} V(x) \leq \lambda_{min}(P)\beta^2 &\Rightarrow \lambda_{min}(P)\|x\|^2 \leq \lambda_{min}(P)\beta^2 \\ &\Rightarrow \|x\| \leq \beta. \end{aligned}$$

If the solution starts outside Ω_β, it follows from (3.31) that $\|x(t)\| \to 0$ as $t \to \infty$. Thus, there is a finite time T after which $\|x(t)\| \leq \mu$ and consequently the solution enter the set Ω_β in finite time. Once inside the set, the solution remains inside for all $t \geq T$.

Claim 3.11 *There exists a finite time $T \geq t_0$ such that for all $t \geq T$*

$$\sigma(t) = 4.$$

Proof. It can be easily checked that the vector fields $g_i(x)$ for $i \in \mathcal{I}$ have the following properties:

$\sigma = 1$: $\dot\omega_1 < 0$ if $k_1, k_2 > \delta$, and $\omega_2 \to \infty$ as $t \to \infty$.

$\sigma = 2$: From Claim 3.9, it follows that $\omega_2(t)$ reaches the boundary $\omega_2 = \gamma_2$ in finite time if its initial condition satisfies $\omega_2(t_0) \geq \gamma_2 \geq \gamma_{21}^2 + \gamma_{22}^2$, where γ_{2i}; $i = 1, 2$ is given by (3.24) by replacing k_1 and k_2 with k_{i1} and k_{i2}, respectively.

$\sigma = 3$: $\dot\omega_1 < 0$ if $k_1, k_2 > \delta$.

$\sigma = 4$: There exists a finite time $T > t_0$ such that for all $t \geq T$ one obtains $\omega_2(t) \leq \beta_1^2 + \beta_2^2$, where β_i is given by (3.20) (see Claim 3.10). If $k_1, k_2 > \delta$, then $\dot\omega_1 < 0$ while $|s| \geq \varepsilon$.

Using the arguments invoked in the proof of Theorem 3.1, and since $\eta^2 < \epsilon_2$ and $\beta_1^2 + \beta_2^2 \leq \gamma_5$, Claim 3.11 can be easily established.

It now follows, from Claim 3.10 and Claim 3.11, that for each $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ the solution of the hybrid system Σ_d is globally uniformly ultimately bounded.

Practical stability

To prove practical stability, one must show that for each initial condition $\{x_0, \sigma_0\} \in \mathcal{R}_0 \times \mathcal{I}$ and for each $d \in S_D$ (where S_D is the set of all perturbations such that $|d(t)| \leq \delta$), the continuous solution $x(t)$ of the closed loop system Σ_d must remain in $\mathcal{R}$ for all $t \geq t_0$.

From Claim 3.10, one can conclude that for all $t \geq t_0$ with $\sigma(t_0) = 4$

$$x_i^2(t_0) + \dot x_i^2(t_0) \leq \mu_i^2 \Rightarrow x_i^2(t) + \dot x_i^2(t) \leq \beta_i^2, \quad i = 1, 2$$

and, therefore $\omega_2(t) \leq \beta_1^2 + \beta_2^2$. Since $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_4$, and since for $\sigma = 4$, $\dot\omega_1 \leq 0$ for $|s| \geq \eta$, it follows that for all $t \geq t_0$ and $x_0 \in \mathcal{R}_0$, $x(t) \in \mathcal{R}$.

Since each solution of the hybrid system Σ_d for each $d \in S_D$ is ultimately in $\mathcal{R}$, it can be concluded that the hybrid system Σ possesses strong practical stability.

Boundedness of control inputs

The boundedness of the control inputs can be easily proved by adopting the main guidelines set forth in the proof of boundedness in Theorem 3.2.

This completes the proof of Theorem 3.3. □

Remark 3.4 The input constraints $\bar{u}_i$, $i = 1, 2$ impose some restrictions on the bound of the disturbance term δ. In fact, condition (3.21) implies that $k_i \leq \bar{u}_i$. Since $k_i > \delta$, then $\delta < \min_{i=1,2} \bar{u}_i$. Also, from the fact that $\gamma_5 \leq \gamma^\star$ and $\beta_1^2 + \beta_2^2 \leq \gamma_5$ it follows that $\beta_1^2 + \beta_2^2 \leq \gamma^\star$ where β_1, β_2 are functions of δ, and $\gamma^\star$ is a function of $\bar{u}_1$ and $\bar{u}_2$.

3.5 Simulation results

Two simulations results are included to illustrate the dynamic behavior of the ENDI system and the usefulness hybrid control laws described in this chapter.

The first simulation describes an application of the hybrid control law developed in Section 3.1 to stabilize a wheeled mobile robot of the unicycle type shown in Figure 2.1. See Section 2.4. The objective was to park the vehicle at position $(x, y) = (0, 0)$ with heading $\theta = 0$. Several

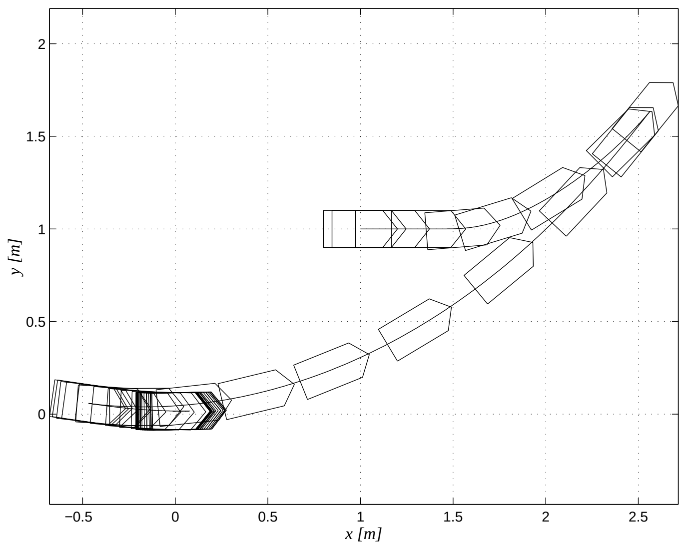

Figure 3.7: Vehicle trajectory.

computer simulations were carried out with the controller designed in Section 3.1 after applying the coordinate transformation described in Section 2.4.3. The control parameters were chosen as $\lambda = 1.0$, $\epsilon_1 = 0.001$, $\epsilon_2 = 0.2$, $\gamma_1 = 1.0$, and $\gamma_2 = 2.0$. The regions were defined according to equations (3.3). The mass and the moment of inertia of the mobile robot are unitary. Figure 3.7 shows the mobile robot trajectory and Figure 3.8 the time evolution of the states $x(t)$, $y(t)$, and $\theta(t)$ for the initial condition $x(0) = 1\,m$, $y(0) = 1\,m$, $\theta(0) = 0\,rad$, $v(0) = 0\,m/s$, and $\omega(0) = 0\,rad/s$. To better understand the performance of the hybrid control law, Figure 3.9 shows the trajectory evolution in Ω-space and Figure 3.10 displays the time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$. From the figures, one can see that while the state $\omega(t)$ is in the region $\mathcal{R}_1$, ω_2 grows in order to abandon that region and ω_1 remains constant. Then, σ switches to 2 and ω_1 starts to converge to zero until it reaches the boundary $\omega_1 = \epsilon_1$. At that moment σ switches to 3, which implies that ω_2 converges to the origin.

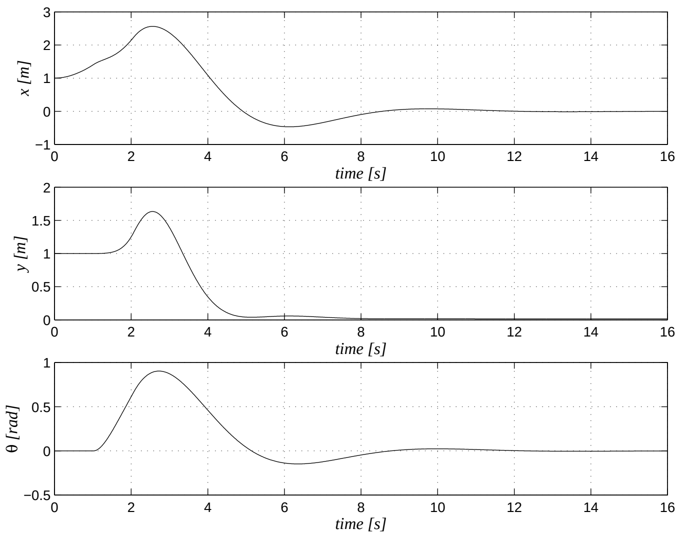

Figure 3.8: Time evolution of the position variables $x(t)$ and $y(t)$, and the orientation variable $\theta(t)$.

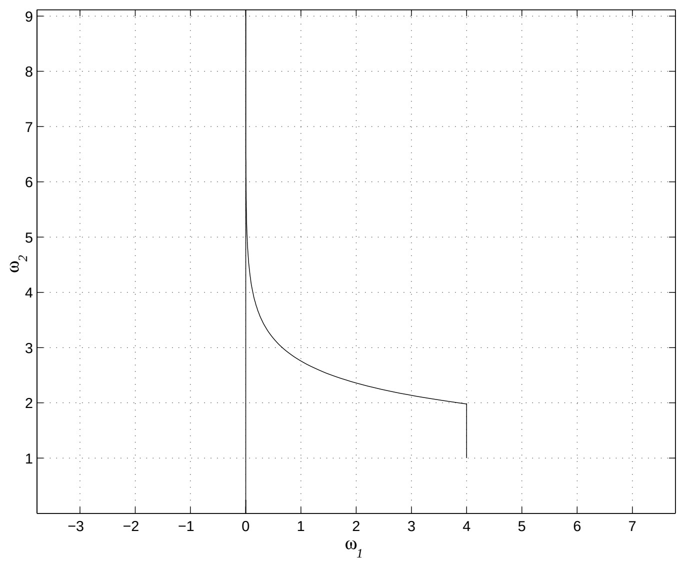

Figure 3.9: Trajectory evolution in Ω-space.

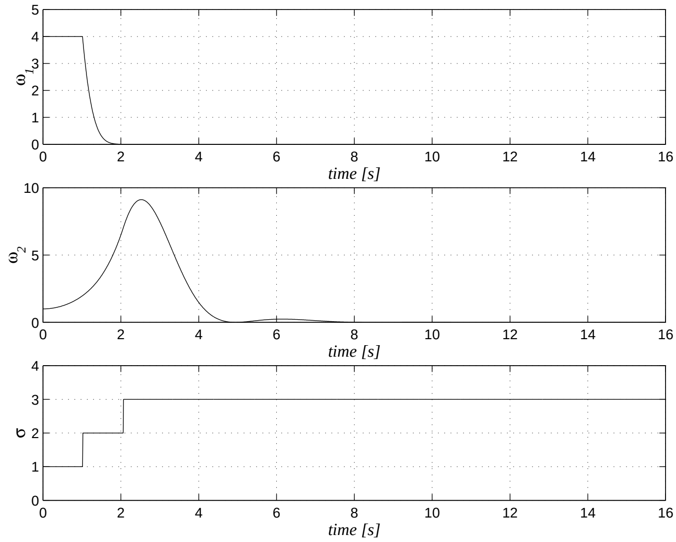

Figure 3.10: Time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$.

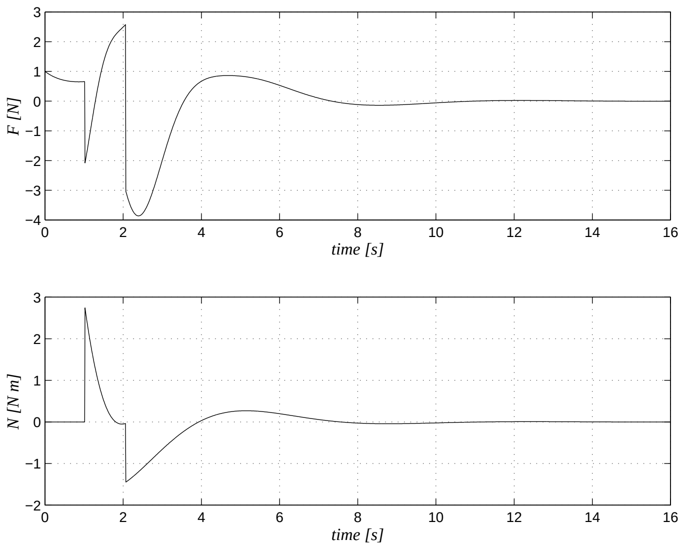

Figure 3.11: Time evolution of the control signals $F(t)$ and $N(t)$.

The second simulation illustrates the performance of the hybrid control law developed in Section 3.4 when the ENDI is subject to input saturations and input additive disturbances. The control parameters were chosen as follows: $\lambda = 1.0$, $\epsilon_1 = 0.001$, $\epsilon_2 = 0.1$, $\epsilon_3 = 0.9$, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$, $\gamma_3 = 0.35$, $\gamma_4 = 1.0$, $\gamma_5 = 1.1$, $k_1 = k_2 = 0.101$, $\eta = 0.05$, $k_{11} = k_{21} = 0.4$, $k_{12} = k_{22} = 0.6$, and $\delta = 0.1$. In the simulation, a disturbance vector $d(t) = [d_1(t), d_2(t)]'$ given by

$$d_1(t) = 0.1 \sin(t),$$
$$d_2(t) = 0.1 \sin\left(t + \frac{\pi}{2}\right),$$

was considered. Notice that γ_5 satisfies condition (3.19), where for this case $\beta_1 = \beta_2 = 0.7346$ (with $\theta = 0.99$). Notice also that for these parameters one has $\bar{u}_1 = \bar{u}_2 = 1.57$ (with $\gamma^\star = \gamma_5$ and $\theta = 0.99$).

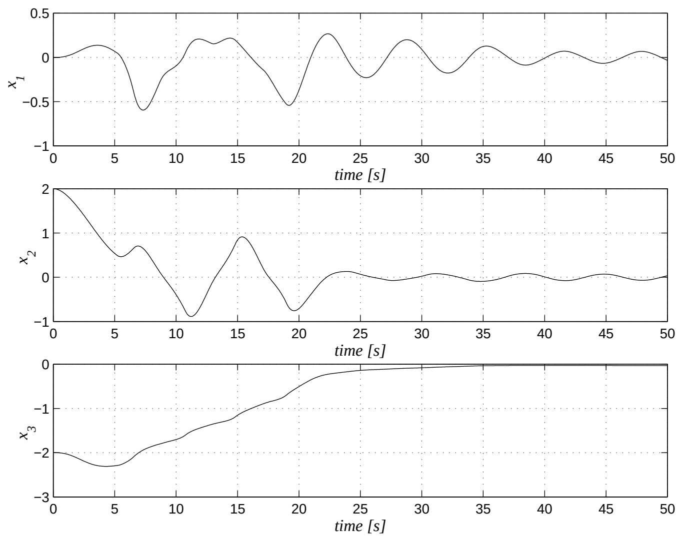

Figure 3.12: Time evolution of the state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$.

Figures 3.12-3.15 show the simulation results for the initial condition $x(0) = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2]'(0) = [0, 2\,m, -2\,m, 0, 0]'$ and $\sigma(0^-) = 2$. The state x converges to a neighborhood of the origin. Notice the oscillatory behavior of $\omega_2(t)$ that is due to input saturations.

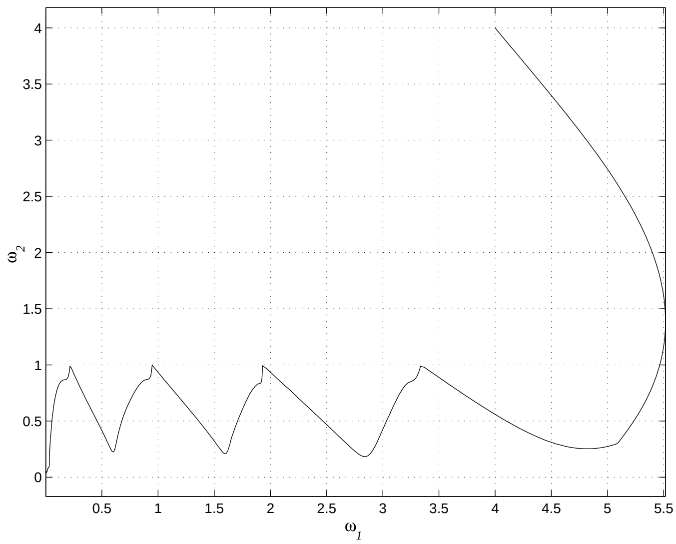

Figure 3.13: Trajectory evolution in Ω-space.

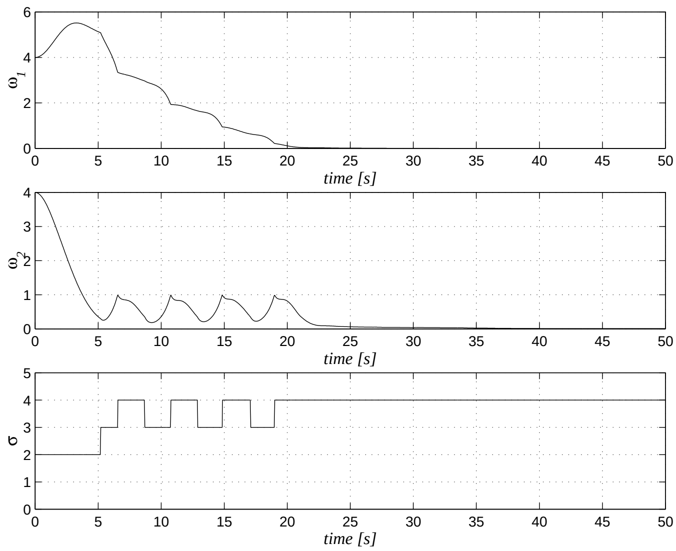

Figure 3.14: Time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$.

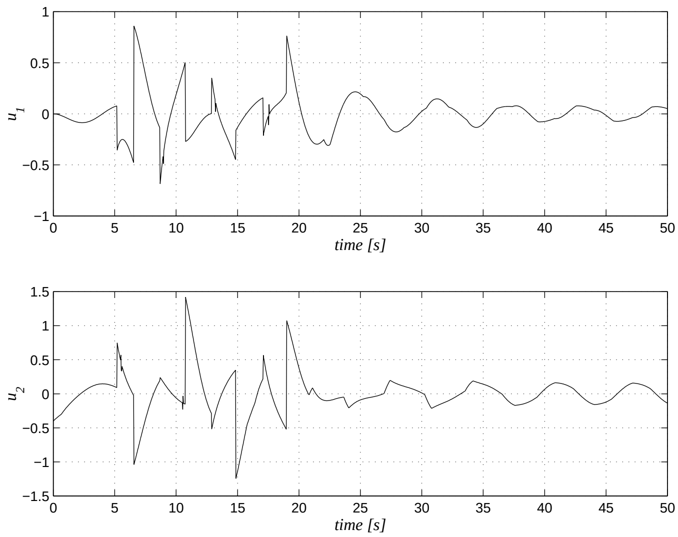

Figure 3.15: Time evolution of the control signals $u_1(t)$ and $u_2(t)$.

3.6 Concluding remarks

A hybrid control law was derived for the ENDI system that captures any kinematic completely nonholonomic model with three states and two first-order dynamic control inputs, *e.g.*, the dynamics of a wheeled robot subject to force and torque inputs. The hybrid controller yields global stability and convergence of the closed loop system to an arbitrarily small neighborhood of the origin. An extension of the controller was done to the bounded input case. It was shown that, despite saturations, the ENDI system can be stabilized for any initial condition and any arbitrarily small bound on the inputs signals. Finally, with a slightly modification of the control law, the problem of practical stabilization was solved. The main result shows that for any input additive disturbances bounded by a finite number, the closed loop hybrid system possesses strong practical stability with bounded control. Simulation results captured some of the features of the proposed control laws and illustrated their performance. An application was made to the control of a wheeled mobile robot of the unicycle-type.

An outstanding problem that has yet to be answered is that of finding out if the control strategy developed can be extended to control other mechanical nonholonomic systems, such as the important class of nonholonomic systems that include drift vector fields that there are not in the span of the input vector fields. In this case, no input transformations exist to bring them to driftless form. The answer comes in the following chapter where the stabilize problem using this strategy for the specific case of an underactuated autonomous underwater vehicle is addressed.

Chapter 4

Stabilization of an underactuated autonomous underwater vehicle via a logic-based hybrid controller

This chapter addresses the problem of stabilizing an underactuated AUV in the horizontal plane using a new technique that builds on hybrid control theory. To the best of the author knowledge, this work is the first application of hybrid control to the stabilization of underactuated marine vehicles. *A feedback logic-based hybrid control law is derived that yields global stabilization of an underactuated AUV to an arbitrarily small neighborhood of a target position with a desired orientation.* This is done by transforming the AUV dynamic model into so-called extended nonholonomic double integrator (ENDI) form plus a drift vector field, followed by the derivation a controller for that system that explores work on feedback hybrid control of the ENDI (see Chapter 3). It is worth pointing out that the technique proposed in this paper is not a simply extension of the methodology developed in the previous chapter. In fact, point stabilization of an underactuated AUV poses considerable challenges to control system designers, since the models of those vehicles typically include a drift vector field that is not in the span of the input vector fields, thus precluding the use of input transformations to bring them to driftless form. Convergence and stability of the closed loop system are analyzed. To illustrate the control law developed, simulation results are presented using the model of the SIRENE AUV. A lemma that can be interpreted as Gronwall-Bellman inequality of second order is also derived. This lemma is instrumental in proving the main theoretical result of the chapter.

The organization of this chapter is as follows: Section 4.1 derives a generalization of Gronwall-Bellman inequality. Section 4.2 proposes a piecewise smooth controller for AUV stabilization based on hybrid system theory. Section 4.3 discusses the stability of the resulting closed loop. It is shown that the closed loop system is stable and that for any initial condition the AUV converges to a small neighborhood of the desired final position with a desired orientation. The radius of the neighborhood can be chosen arbitrarily close to zero (depending only on the controller parameters). Section 4.4 contains simulation results that illustrate the performance of the proposed control strategy. Concluding remarks are presented in Section 4.5. For the sake of completeness, Section 4.6 includes a statement of the comparison lemma and Gronwall-Bellman inequality.

4.1 Technical lemma

The following lemma is useful in establishing an upper-bound on the state variable for a class of systems described by an integral inequation. This lemma, which can be interpreted as a Gronwall-Bellman inequality of second-order, is used in the analysis of convergence of the AUV to a small neighborhood of the origin.

Lemma 4.1 *Let $\mu : [t_0, T) \to \mathbb{R}$ be continuous, $\varphi : [t_0, T) \to \mathbb{R}$ continuous and nonnegative, $\nu : [t_0, T) \to \mathbb{R}$ continuous and nonnegative, and the derivative $\dot{\nu}(t)$ continuous and nonpositive. If a continuous nonnegative function $x : [t_0, T) \to \mathbb{R}$ satisfies*

$$x(t) \leq \mu(t) + \int_{t_0}^{t} \nu(t)\varphi(\tau)x(\tau)d\tau + \int_{t_0}^{t}\int_{t_0}^{\sigma} \nu(\sigma)\varphi(\tau)x(\tau)d\tau\, d\sigma$$

for $t_0 \leq t \leq T$, then on the same interval

$$x(t) \leq \mu(t) + \int_{t_0}^{t} \Phi_{11}(t,\tau)\nu(\tau)\varphi(\tau)\mu(\tau)d\tau + \int_{t_0}^{t} \Phi_{21}(t,\tau)\nu(\tau)\varphi(\tau)\mu(\tau)d\tau, \tag{4.1}$$

where $\Phi_{i,j}$ denotes the i,j-entry of the transition matrix $\Phi(t, t_0)$ defined by the Peano-Baker series

$$\begin{aligned}\Phi(t,t_0) = I &+ \int_{t_0}^{t} A(\tau_1)\, d\tau_1 + \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\, d\tau_2\, d\tau_1 \\ &+ \int_{t_0}^{t} A(\tau_1)\int_{t_0}^{\tau_1} A(\tau_2)\int_{t_0}^{\tau_2} A(\tau_3)\, d\tau_3\, d\tau_2\, d\tau_1 + \cdots\end{aligned} \tag{4.2}$$

with

$$A = \begin{bmatrix} \nu(t)\varphi(t) & \nu(t)\varphi(t) \\ 1 & 0 \end{bmatrix}.$$

Furthermore, if $\dot{\nu}(t)$ satisfies the inequality

$$\dot{\nu}(t) \geq -\lambda\nu(t), \tag{4.3}$$

for some positive constant λ, and if there exist finite constants β_1 and β_2 such that for all τ

$$\int_{\tau}^{\infty} \nu(\sigma)\varphi(\sigma)\, d\sigma \leq \beta_1, \tag{4.4a}$$

$$\int_{t_0}^{\tau} \nu(\sigma)\varphi(\sigma)\mu(\sigma)\, d\sigma \leq \beta_2, \tag{4.4b}$$

then there exists a positive constant β such that

$$x(t) \leq \beta$$

for all $t \geq t_0$.

Proof: Let

$$y_1(t) = \int_{t_0}^{t} \nu(t)\varphi(\tau)x(\tau)\, d\tau,$$
$$y_2(t) = \int_{t_0}^{t}\int_{t_0}^{\sigma} \nu(\sigma)\varphi(\tau)x(\tau)\, d\tau\, d\sigma,$$

and

$$z(t) = \mu(t) + y_1(t) + y_2(t) - x(t) \geq 0.$$

Then,

$$\begin{aligned}
\dot{y}_1(t) &= \nu(t)\varphi(t)x(t) + \dot{\nu}(t)\int_{t_0}^{t} \varphi(\tau)x(\tau)\, d\tau \\
&= \nu(t)\varphi(t)y_1(t) + \nu(t)\varphi(t)y_2(t) + \nu(t)\varphi(t)\left[\mu(t) - z(t)\right] + \dot{\nu}(t)\int_{t_0}^{t} \varphi(\tau)x(\tau)\, d\tau, \\
&\leq \nu(t)\varphi(t)y_1(t) + \nu(t)\varphi(t)y_2(t) + \nu(t)\varphi(t)\mu(t), \\
\dot{y}_2(t) &= y_1(t).
\end{aligned}$$

Consider the linear state equation

$$\dot{\mathcal{Y}}(t) = \begin{bmatrix} \nu(t)\varphi(t) & \nu(t)\varphi(t) \\ 1 & 0 \end{bmatrix}\mathcal{Y}(t) + \begin{bmatrix} \nu(t)\varphi(t)\mu(t) \\ 0 \end{bmatrix}, \quad \mathcal{Y}(t_0) = 0. \tag{4.5}$$

The solution of (4.5) can be written as

$$\mathcal{Y}(t) = \int_{t_0}^{t} \left[\begin{smallmatrix} \Phi_{11}(t,\tau)\nu(\tau)\varphi(\tau)\mu(\tau) \\ \Phi_{21}(t,\tau)\nu(\tau)\varphi(\tau)\mu(\tau) \end{smallmatrix}\right] d\tau, \quad t \geq t_0$$

where the transition matrix $\Phi(t, t_0)$ is given by the Peano-Baker series (4.2). Since $\nu(t)$ and $\varphi(t)$ are continuous functions, then the series converges absolutely and uniformly for t, $t_0 \in [-T, T]$, where $T > 0$ is arbitrary. Thus, by the comparison lemma (see (Khalil, 1996))

$$y_1(t) \leq \int_{t_0}^{t} \Phi_{11}(t,\tau)\nu(\tau)\varphi(\tau)\mu(\tau)d\tau,$$
$$y_2(t) \leq \int_{t_0}^{t} \Phi_{21}(t,\tau)\nu(\tau)\varphi(\tau)\mu(\tau)d\tau,$$

and since

$$x(t) \leq \mu(t) + y_1(t) + y_2(t), \tag{4.6}$$

inequality (4.1) holds. If $\dot{\nu}(t) \geq -\lambda\nu(t)$ for $\lambda > 0$, then one has

$$\dot{y}_1(t) \leq \left[-\lambda + \nu(t)\varphi(t)\right]y_1(t) + \nu(t)\varphi(t)y_2(t) + \nu(t)\varphi(t)\mu(t).$$

Consequently, for this case, system (4.5) can be written as

$$\dot{\mathcal{Y}}(t) = \left[A + F(t)\right]\mathcal{Y}(t) + G(t), \quad \mathcal{Y}(t_0) = 0, \tag{4.7}$$

where

$$A = \begin{bmatrix} -\lambda & 0 \\ 1 & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} \nu(t)\varphi(t) & \nu(t)\varphi(t) \\ 0 & 0 \end{bmatrix}, \quad G(t) = \begin{bmatrix} \nu(t)\varphi(t)\mu(t) \\ 0 \end{bmatrix}.$$

Since the matrix A does not have positive-real-part eigenvalues, the transition matrix $\Phi_A(t,\tau)$ of the linear state equation $\dot{\mathcal{Y}}(t) = A\,\mathcal{Y}(t)$ satisfies

$$\|\Phi_A(t,\tau)\| \leq \gamma_A$$

for all t, τ such that $t \geq \tau$, where γ_A is a finite positive constant. Recall now that the norm of a partitioned matrix with only one nonzero submatrix equals the norm of the nonzero submatrix. Then, writing $F(t)$ as a sum of two matrices, each with one nonzero element, the triangle inequality provides the inequality

$$\|F(t)\| \leq 2\,\nu(t)\varphi(t).$$

The solution of (4.7) can be written as

$$\mathcal{Y}(t) = \Phi_A(t,t_0)\mathcal{Y}(t_0) + \int_{t_0}^{t} \Phi_A(t,\tau)F(\tau)\mathcal{Y}(\tau)\,d\tau + \int_{t_0}^{t} \Phi_A(t,\tau)G(\tau)\,d\tau.$$

Therefore, taking norms, and since $\mathcal{Y}(t_0) = 0$, yields

$$\|\mathcal{Y}(t)\| \leq 2\,\gamma_A \int_{t_0}^{t} \nu(\tau)\varphi(\tau)\|\mathcal{Y}(\tau)\|\,d\tau + \gamma_A \int_{t_0}^{t} \nu(\tau)\varphi(\tau)\mu(\tau)\,d\tau.$$

Applying the Gronwall-Bellman inequality (see Section 4.6) and using 4.4 gives

$$\begin{aligned} \|\mathcal{Y}(t)\| &\leq \gamma_A \int_{t_0}^{t} \nu(\tau)\varphi(\tau)\mu(\tau)\,d\tau + 2\,\gamma_A^2 \int_{t_0}^{t} \nu(\tau)\varphi(\tau)e^{2\,\gamma_A \int_\tau^t \nu(\sigma)\varphi(\sigma)\,d\sigma} \int_{t_0}^{\tau} \nu(\sigma)\varphi(\sigma)\mu(\sigma)\,d\sigma\,d\tau \\ &\leq \gamma_A\beta_2 + 2\,\gamma_A^2\beta_1\beta_2 e^{2\gamma_A\beta_1}. \end{aligned}$$

Consequently, since $\|\mathcal{Y}(t)\|$ is bounded, from (4.6) one can conclude that $x(t)$ is bounded. $\square$

4.2 Hybrid controller

This section proposes a piecewise smooth controller to stabilize the underactuated AUV described in Section 2.5. Control system design builds on hybrid system theory. The objective is to design a feedback law for system (2.24)-(2.25) that will make the state $q = (z_1, z_2, z_3, \dot{z}_1, \dot{z}_2, v)'$ converge to an arbitrarily small neighborhood of the origin despite the drift vector field introduced by the sway velocity v that is not in the span of the input vector fields. Consequently, it is also necessary to guarantee that the sway velocity v subject to the constraint (2.25) goes to zero.

To tackle this problem, define the function $W_{AUV}(q) : \mathbb{R}^6 \to \Omega \subset \mathbb{R}^2$ as

$$\omega \triangleq (\omega_1, \omega_2)' = W_{AUV}(\cdot) = \left(s^2, \lambda_1(\lambda - \lambda_1)(z_1)^2 + (\dot{z}_1)^2\right)', \tag{4.8}$$

where $s = \dot{z}_3 + \lambda z_3$ and λ and λ_1 are positive constants that satisfy $\lambda_1 < \lambda$. The image of W_{AUV} is the two-dimensional closed positive quadrant space $\Omega = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1 \geq 0, \omega_2 \geq 0\}$. The mapping W_{AUV} has the following property: if ω_1 is bounded (say $\omega_1 \le \epsilon$) then z_3 is also bounded by

$$|z_3(t)| \le e^{-\lambda(t-t_0)}|z_3(t_0)| + \frac{\sqrt{\epsilon}}{\lambda}. \tag{4.9}$$

In particular, if $\omega_1(t)$ converges to zero, then $z_3(t)$ also converges to zero.

Consider the following three overlapping regions in Ω (see Figure 4.1) that play a key role in the definition of $\phi(q,\sigma)$ in (3.2):

$$\mathcal{R}_1 = \big\{(\omega_1,\omega_2) \in \Omega : \omega_1 > \epsilon_1 \,\wedge\, \omega_2 \le \gamma_1\big\}, \tag{4.10a}$$

$$\mathcal{R}_2 = \big\{(\omega_1,\omega_2) \in \Omega : \omega_1 > \epsilon_2 \,\wedge\, \omega_2 > 0\big\} \,\cup\, \big\{(\omega_1,\omega_2) \in \Omega : \epsilon_1 < \omega_1 \le \epsilon_2\big\}, \tag{4.10b}$$

$$\mathcal{R}_3 = \big\{(\omega_1,\omega_2) \in \Omega : \omega_1 \le \varepsilon_2\big\}, \tag{4.10c}$$

where

$$\varepsilon_2 = \begin{cases} +\infty & \sigma = 3 \wedge z_1\dot{z}_1 > 0 \wedge |z_1| < \frac{|\dot{z}_1|}{\lambda}, \\ \epsilon_2 & \text{otherwise}, \end{cases} \qquad \gamma_1(t) = \begin{cases} +\infty & |z_1(t_0)| < \frac{|\dot{z}_1(t_0)|}{\lambda}, \\ \gamma & \text{otherwise}, \end{cases}$$

and $\epsilon_2 > \epsilon_1$, ϵ_1, and γ are positive constants.

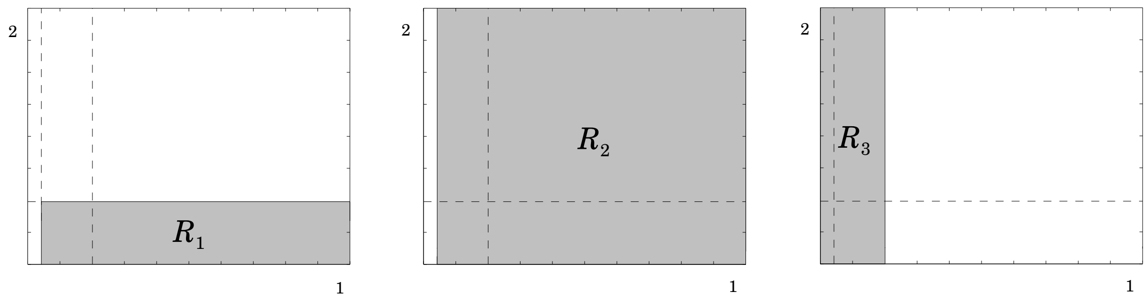

Figure 4.1: Definition of regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$.

The following control law is proposed

$$u = (u_1, u_2)' = h_\sigma(q), \tag{4.11}$$

where the vector fields $h_\sigma : \mathbb{R}^6 \to \mathbb{R}^2$, $\sigma \in \mathcal{I} = \{1,2,3\}$ are given by

$$h_1(q) = \begin{bmatrix} -\lambda\dot{z}_1 + k_1 \\ -\lambda\dot{z}_2 \end{bmatrix}, \tag{4.12a}$$

$$h_2(q) = \begin{bmatrix} -\lambda\dot{z}_1 \\ -\lambda\dot{z}_2 + k_2 \operatorname{sat}(\frac{s}{\sqrt{\epsilon_1}}) \end{bmatrix}, \tag{4.12b}$$

$$h_3(q) = \begin{bmatrix} -\lambda\dot{z}_1 - \lambda_1(\lambda-\lambda_1)z_1 \\ -\lambda\dot{z}_2 - \lambda_1(\lambda-\lambda_1)z_2 - k_3 z_1 s \end{bmatrix}, \tag{4.12c}$$

with

$$k_1 = \operatorname{sgn}\big(z_2(t_0) + \frac{\dot{z}_2(t_0)}{\lambda}\big) \operatorname{sgn}(s),$$

$$k_2 = -\operatorname{sgn}\big(z_1(t_0) + \frac{\dot{z}_1(t_0)}{\lambda}\big),$$

and k_3 a positive constant. The function $\mathrm{sgn}(\cdot)$ is defined by $\mathrm{sgn}(x) = 1$ if $x \geq 0$, and $\mathrm{sgn}(x) = -1$ if $x < 0$. The function $\mathrm{sat}(\cdot)$ is in turn defined by $\mathrm{sat}(x) = \mathrm{sgn}(x)$ if $|x| > 1$, and $\mathrm{sat}(x) = x$ if $|x| \leq 1$. The switching signal $\sigma(t)$ is piecewise constant, takes values in $\mathcal{I} = \{1, 2, 3\}$, and is determined recursively by

$$\sigma(t) = \phi\left(\omega(t), \sigma^-(t)\right), \quad \sigma^-(t_0) = \sigma_0 \in \mathcal{I} \tag{4.13}$$

where the transition function is defined according to (3.2).

The control laws for each region were designed according to the following simple rule: if $\sigma = 1$, the variable $\omega_2(t)$ must move away from zero; when $\sigma = 2$, $\omega_1(t)$ must decrease and reach a given bound in finite time and $\omega_2(t)$ must not increase; finally, when $\sigma = 3$, the variable $\omega_2(t)$ and therefore the state $z_2(t)$ must converge to zero while $\omega_1(t)$ should remain near ϵ_1 (in this case its behavior will be dictated by the drift state v). A sketch of a typical trajectory in the W_{AUV}-space is shown in Figure 4.2.

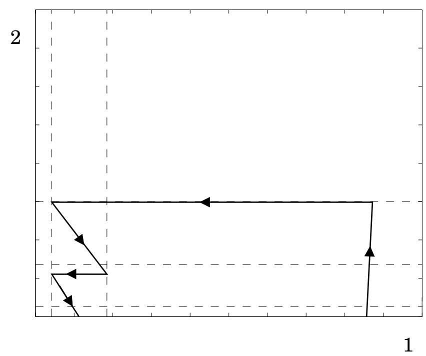

Figure 4.2: Image of a representative trajectory in the W_{AUV} - space.

4.3 Convergence and stability analysis

In this section, convergence and stability of the closed loop system is analyzed. The following is the key result of this chapter.

Theorem 4.1 *Consider the hybrid system* Σ_{AUV} *described by* (2.24)-(2.25), (4.8)-(4.13), *and* (3.2). *Let* $\{q(t), \sigma(t)\} = \{q : [t_0, \infty) \to \mathbb{R}^6, \sigma : [t_0, \infty) \to \mathcal{I}\}$ *be a solution to* Σ_{AUV}. *Then, the following properties hold.*

1. *Given an arbitrary pair* $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ *(initial condition), there exists a unique solution* $\{x(t), \sigma(t)\}$ *for all* $t \geq t_0$ *such that* $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.

2. *For any set of initial conditions* $\{q(t_0), \sigma^-(t_0)\} = \{q_0, \sigma_0\} \in \mathbb{R}^6 \times \mathcal{I}$, *there exists a finite time* T *such that for* $t > T$ *the state variables* $z_1(t)$, $\dot{z}_1(t)$, $z_2(t)$, $\dot{z}_2(t)$, *and* $v(t)$ *converge to zero, and* $\omega_1(t) \leq \epsilon_2$, *where* $\epsilon_2 > 0$ *is a controller parameter that can be chosen arbitrarily small.*

3. *The origin* $q = 0$ *is a Lyapunov uniformly stable equilibrium point of* Σ_{AUV}.

Proof:

Existence and uniqueness

Since for each $i \in \mathcal{I}$ the vector field $h_i(q)$ is continuously differentiable with respect to q, then $h_i(q)$ is locally Lipschitz in q. Moreover, since for $i = 1, 2$ the Jacobian matrix $[\partial h_i / \partial q]$ is uniformly bounded in $\mathbb{R}^6$ it follows that $h_i(q)$ is globally Lipschitz. For $i = 3$, $h_3(q)$ is not globally Lipschitz. However, it can be shown that every solution of Σ_{AUV} with $q_0 \in \mathcal{R}_3$ and $\sigma_0 = 3$ lies entirely in a compact set $S \subset \mathbb{R}^6$. A proof of this statement arises naturally in the discussion below. Notice also that the distance between two points in the (ω_1, ω_2)-space where consecutive switching may occur is always nonzero. It now follows from classical arguments (Khalil, 1996) that the hybrid system Σ_{AUV} has exactly one solution over $[t_0, \infty)$ for each initial condition $\{q_0, \sigma_0\} \in \mathbb{R}^6 \times \mathcal{I}$.

Convergence

For the sake of readability the proof of convergence will be divided into five claims.

Claim 4.1 *Given an initial time $t_0 \geq 0$, there exists a finite time $T \geq t_0$ such that*

$$\sigma(t) \in \mathcal{I} \backslash \{1\}, \quad t \geq T$$

Proof. Consider first that $\sigma_0 = 1$ at $t = t_0$, and assume by contradiction that $\sigma(t) = 1$ (and consequently that $\omega(t)$ remains in $\mathcal{R}_1$) for all $t \geq t_0$. In this case, the dynamics of z_1 satisfy

$$\ddot{z}_1 = -\lambda \dot{z}_1 + k_1,$$

and, consequently, its time evolution is given by

$$z_1(t) = \frac{1}{\lambda}\left(\dot{z}_1(t_0) - \frac{k_1}{\lambda}\right)\left[1 - e^{-\lambda(t-t_0)}\right] + \frac{k_1}{\lambda}(t - t_0) + z_1(t_0), \quad t \geq t_0.$$

Therefore, $\omega_2(t) = \lambda_1(\lambda - \lambda_1)z_1^2 + \dot{z}_1^2$ is unbounded and $\omega(t)$ leaves region $\mathcal{R}_1$, thus violating the assumption above. The remainder of the proof consist of showing that if $\sigma_0 \in \mathcal{I} \backslash \{1\}$, then σ will never switch to 1. From the definition of regions $\mathcal{R}_i$, $i = 1, 2, 3$ (see expression (4.10)) and according to the switching logic implemented for $\sigma(t)$ (see Figure 3.1) it can be easily checked that σ can only switch to 1 if there exists a finite time $\bar{t} > t_0$, such that $\omega(\bar{t}) \in \mathcal{R} = \big\{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_2(t) \wedge \omega_2 = 0\big\}$. Assume (by contradiction) that this happens. Then there exists $\tau > 0$ such that for all $t \in [\bar{t} - \tau, \bar{t})$ one has $\sigma(t) = 2$ and the dynamics of $z_1(t)$ satisfy

$$\ddot{z}_1 = -\lambda \dot{z}_1.$$

Therefore,

$$\begin{aligned}
\dot{z}_1(t) &= \dot{z}_1(\bar{t} - \tau)e^{-\lambda(t - \bar{t} + \tau)}, \\
z_1(t) &= \frac{\dot{z}_1(\bar{t} - \tau)}{\lambda}\left(1 - e^{-\lambda(t - \bar{t} + \tau)}\right) + z_1(\bar{t} - \tau).
\end{aligned}$$

This in turn shows that (since the initial condition $\omega_2(\bar{t} - \tau) \neq 0$), the variable ω_2 will never be zero at a finite time $t = \bar{t}$.

Claim 4.2 *For any $T \geq t_{\sigma_2} \geq t_0$ such that $\sigma(t) = 2$ for $t \in [t_{\sigma_2}, T)$, there exist finite constants $\bar{\gamma}_0 > 0$, $\bar{\lambda} > 0$, and $\bar{\gamma} > 0$ such that the sway velocity $v(t)$ satisfies the inequality*

$$|v(t)| \leq \bar{\gamma}_0 + \bar{\gamma}\, e^{-\bar{\lambda}(t-t_{\sigma_2})} + \int_{t_{\sigma_2}}^{t} e^{-\bar{\lambda}(t-\tau)} h(\tau)|s(\tau)|d\tau, \quad t \in [t_{\sigma_2}, T) \tag{4.14}$$

where

$$h(t) = \frac{m_u}{m_v} \frac{\dot{z}_1^2(t_{\sigma_2})}{2\lambda} e^{-2\lambda(t-t_{\sigma_2})}.$$

Moreover, if for $T = \infty$

$$\lim_{t\to\infty} h(t)|s(t)| = 0, \quad \textit{then} \quad \lim_{t\to\infty} v(t) = 0.$$

Proof. Consider the positive function

$$V = \frac{1}{2} v^2.$$

The time derivative of V along the trajectories of the closed loop system Σ is given by

$$\dot{V} = -\frac{m_u}{m_v}\left(\dot{z}_2 + \dot{z}_1 \frac{z_1 z_2 - z_3}{2}\right)\dot{z}_1 v - \frac{d_v}{m_v} v^2.$$

Using the fact that $z_3 = \frac{s-\dot{z}_3}{\lambda}$ and $\dot{z}_3 = z_1\dot{z}_2 - z_2\dot{z}_1 + 2v$, $\dot{V}$ can be written as

$$\dot{V} = -d(t)v^2 + g(t)v + h(t)vs, \tag{4.15}$$

where

$$d(t) = \frac{d_v}{m_v} + \frac{m_u}{m_v}\frac{\dot{z}_1^2}{\lambda}, \tag{4.16a}$$

$$g(t) = -\frac{m_u}{m_v}\frac{\dot{z}_1}{2\lambda}\big[2\lambda\dot{z}_2 + \dot{z}_1\left(z_1 z_2 \lambda + z_1\dot{z}_2 - z_2\dot{z}_1\right)\big], \tag{4.16b}$$

$$h(t) = \frac{m_u}{m_v}\frac{\dot{z}_1^2}{2\lambda}. \tag{4.16c}$$

Since for $\sigma = 2$

$$\ddot{z}_1 = -\lambda\dot{z}_1,$$
$$\ddot{z}_2 = -\lambda\dot{z}_2 + k_2\,\mathrm{sat}\Big(\frac{s}{\sqrt{\epsilon_1}}\Big),$$

it follows that for all $t \in [t_{\sigma_2}, T)$

$$\dot{z}_1(t) = \dot{z}_1(t_{\sigma_2})e^{-\lambda(t-t_{\sigma_2})}, \tag{4.17a}$$

$$\dot{z}_2(t) = \left[\dot{z}_2(t_{\sigma_2}) - \frac{k_2}{\lambda}\right] e^{-\lambda(t-t_{\sigma_2})} + \frac{k_2}{\lambda}, \quad \omega_1 \geq \epsilon_1 \tag{4.17b}$$

$$z_1(t) = \frac{\dot{z}_1(t_{\sigma_2})}{\lambda}\big[1 - e^{-\lambda(t-t_{\sigma_2})}\big] + z_1(t_{\sigma_2}), \tag{4.17c}$$

$$z_2(t) = \left[\dot{z}_2(t_{\sigma_2}) - \frac{k_2}{\lambda}\right]\big[1 - e^{-\lambda(t-t_{\sigma_2})}\big] + \frac{k_2}{\lambda}(t - t_{\sigma_2}) + z_2(t_{\sigma_2}), \quad \omega_1 \geq \epsilon_1. \tag{4.17d}$$

Consequently, it can be easily checked that $g(t)$ and $h(t)$ are bounded and satisfy

$$\lim_{t\to\infty} g(t) = 0,$$
$$\lim_{t\to\infty} h(t) = 0,$$

for $T = \infty$. Thus, applying norms to (4.15)

$$\dot{V} \leq -2\bar{\lambda}\, V + |g(t)|\sqrt{2V} + h(t)\sqrt{2V}|s|,$$

where $\bar{\lambda} > 0$ denotes a lower bound of $d(t)$. Performing the change of variables $W = \sqrt{V}$ to obtain a linear differential inequality, and using the fact that $\dot{W} = \frac{\dot{V}}{2\sqrt{V}}$, it follows, when $V \neq 0$, that

$$\dot{W} \leq -\bar{\lambda}\, W + \frac{\sqrt{2}}{2}|g(t)| + \frac{\sqrt{2}}{2}h(t)|s|. \tag{4.18}$$

When $V = 0$, it can be shown that the upper right-hand derivative D^+W (see Section 4.6), satisfies $D^+W \leq \frac{\sqrt{2}}{2}|g(t)| + \frac{\sqrt{2}}{2}h(t)|s|$. In fact, when $W(v(t)) = 0$, it can be seen that

$$\begin{aligned}
\frac{1}{h}\big|W(v(t+h)) - W(v(t))\big| = \frac{\sqrt{2}}{2h}|v(t+h)| \;&=\; \frac{\sqrt{2}}{2h}\left|\int_t^{t+h} f(\tau, v(\tau))\, d\tau\right| \\
&= \left|\frac{\sqrt{2}}{2}f(t,0) + \frac{\sqrt{2}}{2h}\int_t^{t+h}\big[f(\tau, v(\tau)) - f(t, v(t))\big]\, d\tau\right| \\
&\leq \frac{\sqrt{2}}{2}\,|f(t,0)| + \frac{\sqrt{2}}{2h}\int_t^{t+h}\big|f(\tau, v(\tau)) - f(t, v(t))\big|\, d\tau,
\end{aligned}$$

where

$$f(t, v(t)) = -d(t)v(t) + g(t) + h(t)s(t).$$

Since $f(t, v(t))$ is a continuous function of t, given any $\epsilon > 0$ there is $\delta > 0$ such that for all $|\tau - t| < \delta$, $\big|f(\tau, v(\tau)) - f(t, v(t))\big| < \epsilon$. Therefore, for all $h < \delta$

$$\frac{1}{h}\int_t^{t+h}\big|f(\tau, v(\tau)) - f(t, v(t))\big|\, d\tau < \epsilon,$$

and consequently

$$\lim_{h \to 0^+}\frac{1}{h}\int_t^{t+h}\big|f(\tau, v(\tau)) - f(t, v(t))\big|\, d\tau = 0.$$

It follows from property 2 of the upper right-hand derivative that $D^+W \leq \frac{\sqrt{2}}{2}|f(t,0)| \leq \frac{\sqrt{2}}{2}|g(t)| + \frac{\sqrt{2}}{2}h(t)|s|$.

Since D^+W satisfies (4.18) for all values of V, by the comparison lemma (Khalil, 1996), W satisfies the inequality

$$W(t) \leq e^{-\bar{\lambda}(t - t_{\sigma_2})}W(t_{\sigma_2}) + \frac{\sqrt{2}}{2}\int_{t_{\sigma_2}}^t e^{-\bar{\lambda}(t-\tau)}|g(\tau)|d\tau + \frac{\sqrt{2}}{2}\int_{t_{\sigma_2}}^t e^{-\bar{\lambda}(t-\tau)}h(\tau)|s|d\tau.$$

Thus, it follows that

$$\begin{aligned}
|v| &\leq e^{-\bar{\lambda}(t-t_{\sigma_2})}|v(t_{\sigma_2})| + \int_{t_{\sigma_2}}^t e^{-\bar{\lambda}(t-\tau)}|g(\tau)|d\tau + \int_{t_{\sigma_2}}^t e^{-\bar{\lambda}(t-\tau)}h(\tau)|s|d\tau \\
&\leq e^{-\bar{\lambda}(t-t_{\sigma_2})}|v(t_{\sigma_2})| + \frac{1}{\bar{\lambda}}\left[1 - e^{-\bar{\lambda}(t-t_{\sigma_2})}\right]\sup_{t \geq t_{\sigma_2}}|g(t)| + \int_{t_{\sigma_2}}^t e^{-\bar{\lambda}(t-\tau)}h(\tau)|s|d\tau,
\end{aligned} \tag{4.19}$$

and this yields (4.14), with

$$\begin{aligned}
\bar{\gamma}_0 &= \frac{m_v}{d_v}\sup_{t \geq t_{\sigma_2}}|g(t)|, \\
\bar{\gamma} &= |v(t_{\sigma_2})| - \frac{m_v}{d_v}\sup_{t \geq t_{\sigma_2}}|g(t)|.
\end{aligned}$$

Since $\lim_{t\to\infty} g(t) = 0$, it follows from (4.19) that if $\lim_{t\to\infty} h(t)|s(t)| = 0$ then

$$\lim_{t\to\infty} v(t) = 0.$$

Claim 4.3 *For any $t_{\sigma_2} \geq 0$ such that $\sigma(t_{\sigma_2}) = 2$, if $z_1(t_{\sigma_2}) \neq -\frac{\dot{z}_1(t_{\sigma_2})}{\lambda}$, then there exists a finite time $T \geq t_{\sigma_2}$ such that*

$$\omega_1(T) = \epsilon_1.$$

Proof. To prove that $\omega_1(t)$ reaches the boundary $\omega_1 = \epsilon_1$ in finite time, observe that for $\sigma = 2$ the dynamics of ω_1 are given by

$$\dot{\omega}_1 = 2s\big[z_1 k_2 + 2(\dot{v} + \lambda v)\big], \quad \omega_1 \geq \epsilon_1. \tag{4.20}$$

Since $\ddot{z}_1 = -\lambda \dot{z}_1$, then from equation (4.17c) it can be checked that there exists a finite time $\bar{t}_1 > t_{\sigma_2}$ such

$$z_1(t)k_2 = -|z_1(t)|\,\mathrm{sgn}(s)$$

for all $t \geq \bar{t}_1$. Thus, from equation (4.20) it follows that

$$\dot{\omega}_1 = -2s\big[|z_1|\,\mathrm{sgn}(s) - 2(\dot{v} + \lambda v)\big], \quad t \geq \bar{t}_1$$

Notice that by supposing that $\lim_{t\to\infty} h(t)|s(t)| = 0$ one obtains, according to Claim 3.2, that $\lim_{t\to\infty} v(t) = 0$. Notice also that $z_1(t)$ converges to a value different from zero ($z_1(t_{\sigma_2}) + \frac{\dot{z}_1(t_{\sigma_2})}{\lambda} \neq 0$). Thus, there will be a finite time $\bar{t}_2 > \bar{t}_1$ such that

$$\dot{\omega}_1(t) < 0, \quad \forall t \geq \bar{t}_2$$

Therefore, $\omega_1(t)$ will reach the boundary $\omega_1 = \epsilon_1$ in finite time. It remains to prove that $\lim_{t\to\infty} h(t)|s(t)| = 0$. Consider, for $\sigma = 2$, the dynamic of the state $s(t)$ that is given by

$$\dot{s}(t) = z_1(t)k_2 + 2\big[\dot{v}(t) + \lambda v(t)\big].$$

Integrating from t_{σ_2} to t and using (4.17c) yields

$$\begin{aligned}
s(t) &= k_2 \int_{t_{\sigma_2}}^{t} z_1(\tau)d\tau + 2\left[v(t) - v(t_{\sigma_2}) + \lambda \int_{t_{\sigma_2}}^{t} v(\tau)d\tau\right] + s(t_{\sigma_2}) \\
&= k_2 \left[\frac{\dot{z}_1(t_{\sigma_2})}{\lambda} + z_1(t_{\sigma_2})\right](t - t_{\sigma_2}) - k_2 \frac{\dot{z}_1(t_{\sigma_2})}{\lambda^2}\left[1 - e^{-\lambda(t - t_{\sigma_2})}\right] \\
&\quad + 2\left[v(t) - v(t_{\sigma_2}) + \lambda \int_{t_{\sigma_2}}^{t} v(\tau)d\tau\right] + s(t_{\sigma_2}).
\end{aligned}$$

Applying the result of Claim 3.2, it follows that

$$|s(t)| \leq \mu(t) + \int_{t_{\sigma_2}}^{t} \nu(t)\varphi(\tau)|s(\tau)|d\tau + \int_{t_{\sigma_2}}^{t}\int_{t_{\sigma_2}}^{\sigma} \nu(\sigma)\varphi(\tau)|s(\tau)|d\tau\, d\sigma,$$

where

$$\mu(t) = 2\left(\bar{\gamma}_0 + |v(t_{\sigma_2})|\right) + |s(t_{\sigma_2})| + \left[2\lambda\bar{\gamma}_0 + |k_2|\left|\frac{\dot{z}_1(t_{\sigma_2})}{\lambda} + z_1(t_{\sigma_2})\right|\right](t - t_{\sigma_2})$$
$$+ 2\bar{\gamma}e^{-\bar{\lambda}(t-t_{\sigma_2})} + 2\lambda\frac{\bar{\gamma}}{\bar{\lambda}}\left[1 - e^{-\bar{\lambda}(t-t_{\sigma_2})}\right] + \left|k_2\frac{\dot{z}_1(t_{\sigma_2})}{\lambda^2}\right|\left[1 - e^{-\lambda(t-t_{\sigma_2})}\right],$$
$$\nu(t) = 2e^{-\bar{\lambda}t}$$
$$\varphi(\tau) = e^{\bar{\lambda}\tau}h(\tau).$$

Since $\mu(t)$, $\nu(t)$, and $\varphi(t)$ satisfy (4.3)-(4.4b), then invoking Lemma 4.1 one concludes that $|s(t)|$ is bounded. Therefore, since $\lim_{t\to\infty} h(t) = 0$, it follows that

$$\lim_{t\to\infty} h(t)|s(t)| = 0,$$

and the proof of Claim 4.3 is complete.

Claim 4.4 *For any t_{σ_2} and any positive interval $\tau > 0$ such that $\sigma(t) = 2$ for $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$, if the initial conditions $\left(z_1(t_{\sigma_2}), \dot{z}_1(t_{\sigma_2})\right)'$ satisfy*

$$z_1(t_{\sigma_2})\dot{z}_1(t_{\sigma_2}) \le 0 \quad \text{and} \quad |z_1(t_{\sigma_2})| > \frac{|\dot{z}_1(t_{\sigma_2})|}{\lambda}, \tag{4.21}$$

then for all $t \in [t_{\sigma_2}, t_{\sigma_2} + \tau]$

$$\omega_2(t) \le \omega_2(t_{\sigma_2}). \tag{4.22}$$

Proof. For $\sigma = 2$, $\ddot{z}_1 = -\lambda\dot{z}_1$. Thus, the time evolution of $\dot{z}_1(t)$ and $z_1(t)$ are given by equations (4.17a) and (4.17c), respectively. Clearly, from those equations, one can see that $z_1(t)$ converges to $z_1(t_{\sigma_2}) + \frac{\dot{z}_1(t_{\sigma_2})}{\lambda}$ as $t \to \infty$ and if conditions (4.21) hold, then $|z_1(t)| \le |z_1(t_{\sigma_2})|$. Thus, since $\omega_2(t) = \lambda_1(\lambda - \lambda_1)z_1^2(t) + \dot{z}_1^2(t)$, (4.22) follows.

Claim 4.5 *There exists a finite time $T \ge t_0$ such that $\sigma(t) = 3$ and $\omega_1(t) \le \epsilon_2$ for all $t > T$. Furthermore, the state variables $z_1(t)$, $\dot{z}_1(t)$, $z_2(t)$, $\dot{z}_2(t)$, and $v(t)$ converge to zero as $t \to \infty$.*

Proof. Consider first that $\sigma(t) = 3$ for all $t \ge T$ and let define $t_{\sigma_3} = T$. In that case, the continuous dynamics can be written as

$$\ddot{z}_1 = -\lambda\dot{z}_1 - \lambda_1(\lambda - \lambda_1)z_1, \tag{4.23}$$
$$\ddot{z}_2 = -\lambda\dot{z}_2 - \lambda_1(\lambda - \lambda_1)z_2 - k_3 z_1 s, \tag{4.24}$$
$$\dot{\omega}_1 = -2k_3 z_1^2\omega_1 + 4(\dot{v} + \lambda v)s, \tag{4.25}$$
$$\dot{v} = -d(t)v + g(t) + h(t)s, \tag{4.26}$$

where $d(t)$, $g(t)$, and $h(t)$ are defined in (4.16a)-(4.16c). Clearly, from (4.23) and the fact that $\lambda_1 > 0$ and $\lambda - \lambda_1 > 0$, it follows that $(z_1, \dot{z}_1)'$ is exponentially stable. Also, from (4.24), one can conclude that $(z_2, \dot{z}_2)'$ is exponentially stable if $|s(t)|$ is bounded. Thus, one can find positive constants γ_{z_1}, γ_{z_2}, γ_h, λ_{z_1}, λ_{z_2}, λ_h, g_1, and g_2 such that

$$|z_1(t)| \leq \gamma_{z_1} e^{-\lambda_{z_1}(t-t_{\sigma_3})} \tag{4.27}$$

$$|\dot{z}_1(t)| \leq \gamma_{z_1} e^{-\lambda_{z_1}(t-t_{\sigma_3})} \tag{4.28}$$

$$h(t) \leq \gamma_h e^{-\lambda_h(t-t_{\sigma_3})}, \tag{4.29}$$

$$|z_2(t)| \leq \gamma_{z_2} e^{-\lambda_{z_2}(t-t_{\sigma_3})} + k_3 \int_{t_{\sigma_3}}^{t} \gamma_{z_2} e^{-\lambda_{z_2}(t-\tau)} |z_1(\tau)||s(\tau)|\, d\tau, \tag{4.30}$$

$$|\dot{z}_2(t)| \leq \gamma_{z_2} e^{-\lambda_{z_2}(t-t_{\sigma_3})} + k_3 \int_{t_{\sigma_3}}^{t} \gamma_{z_2} e^{-\lambda_{z_2}(t-\tau)} |z_1(\tau)||s(\tau)|\, d\tau, \tag{4.31}$$

$$|g(t)| \leq g_1|\dot{z}_2| + g_2|z_2|. \tag{4.32}$$

Observe also that $|v(t)|$ satisfies (see the derivation of equation (4.19))

$$|v| \leq e^{-\bar{\lambda}(t-t_{\sigma_3})}|v(t_{\sigma_3})| + \int_{t_{\sigma_3}}^{t} e^{-\bar{\lambda}(t-\tau)}|g(\tau)|d\tau + \int_{t_{\sigma_3}}^{t} e^{-\bar{\lambda}(t-\tau)}h(\tau)|s(\tau)|d\tau, \tag{4.33}$$

Therefore, from (4.33) it can be inferred that the sway velocity $v(t)$ converges to zero if $|s(t)|$ is bounded. To prove that $s(t)$ is bounded for $\sigma = 3$, one can use the line of reasoning that was explored in Claim 3.2 with $V = \omega_1 = s^2$. Using (4.27)-(4.33), after some algebra manipulation one gets

$$\begin{aligned}
|s(t)| \leq \mu(t) &+ \bar{k}_1 \int_{t_{\sigma_3}}^{t} h(\tau)|s(\tau)|\, d\tau \\
&+ \bar{k}_2 \int_{t_{\sigma_3}}^{t} \int_{t_{\sigma_3}}^{\tau} e^{-\lambda_{z_2}(\tau-\sigma)}|z_1(\sigma)||s(\sigma)|\, d\sigma\, d\tau \\
&+ \bar{k}_3 \int_{t_{\sigma_3}}^{t} \int_{t_{\sigma_3}}^{\tau} e^{-\bar{\lambda}(\tau-\sigma)}h(\sigma)|s(\sigma)|\, d\sigma\, d\tau \\
&+ \bar{k}_4 \int_{t_{\sigma_3}}^{t} \int_{t_{\sigma_3}}^{\tau} e^{-\bar{\lambda}(\tau-\sigma)} \int_{t_{\sigma_3}}^{\sigma} e^{-\lambda_{z_2}(\sigma-\xi)}|z_1(\xi)||s(\xi)|\, d\xi\, d\sigma\, d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\mu(t) =\ & 2\frac{\gamma_d}{\bar{\lambda}}\left[|v(t_{\sigma_3})| + \frac{g_1+g_2}{\bar{\lambda}-\lambda_{z_2}}\right]\left[1 - e^{-\bar{\lambda}(t-t_{\sigma_3})}\right] \\
& 2\frac{\gamma_{z_2}}{\lambda_{z_2}}(g_1+g_2)\left[1 + \frac{\gamma_d}{\bar{\lambda}-\lambda_{z_2}}\right]\left[1 - e^{-\lambda_{z_2}(t-t_{\sigma_3})}\right], \\
\gamma_d =\ & \lambda + \sup_{t \geq t_{\sigma_3}} d(t), \\
\bar{k}_1 =\ & 2, \\
\bar{k}_2 =\ & 2(g_1+g_2)\gamma_{z_2}k_3, \\
\bar{k}_3 =\ & 2\gamma_d, \\
\bar{k}_4 =\ & 2\gamma_d(g_1+g_2)\gamma_{z_2}.
\end{aligned}$$

The proof that $s(t)$ is bounded essentially follows the lines of the proof of Lemma 4.1. Consider the auxiliary variables

$$\begin{aligned}
y_1(t) &= \int_{t_{\sigma_3}}^{t} h(\tau)|s(\tau)|\,d\tau,\\
y_2(t) &= \int_{t_{\sigma_3}}^{t}\int_{t_{\sigma_3}}^{\tau} e^{-\lambda_2(\tau-\sigma)}|s(\sigma)|\,d\sigma\,d\tau,\\
y_3(t) &= \int_{t_{\sigma_3}}^{t}\int_{t_{\sigma_3}}^{\tau} e^{-\bar\lambda(\tau-\sigma)}h(\sigma)|s(\sigma)|\,d\sigma\,d\tau,\\
y_4(t) &= \int_{t_{\sigma_3}}^{t}\int_{t_{\sigma_3}}^{\tau} e^{-\bar\lambda(\tau-\sigma)}\int_{t_{\sigma_3}}^{\sigma} e^{-\lambda_2(\sigma-\xi)}|s(\xi)|\,d\xi\,d\sigma\,d\tau.
\end{aligned}$$

Then,

$$|s(t)| \le \mu(t) + \bar k_1 y_1(t) + \bar k_2 y_2(t) + \bar k_3 y_3(t) + \bar k_4 y_4(t). \tag{4.34}$$

Let $z(t)$ be the following positive variable

$$z(t) = \mu(t) + \bar k_1 y_1(t) + \bar k_2 y_2(t) + \bar k_3 y_3(t) + \bar k_4 y_4(t) - |s(t)| \;\ge\; 0.$$

The derivative y_i, $i = 1,2,3,4$ with respect to time yields

$$\begin{aligned}
\dot y_1 &= h(t)|s(t)|\\
&= h(t)\mu(t) + \bar k_1 h(t) y_1(t) + \bar k_2 h(t) y_2(t)\\
&\quad + \bar k_3 h(t) y_3(t) + \bar k_4 h(t) y_4(t) - h(t) z(t),\\
\dot y_2 &= y_5,\\
\dot y_3 &= y_6,\\
\dot y_4 &= y_7,
\end{aligned}$$

where

$$\begin{aligned}
y_5(t) &= \int_{t_{\sigma_3}}^{t} e^{-\lambda_2(t-\sigma)}|z_1(\sigma)||s(\sigma)|\,d\sigma,\\
y_6(t) &= \int_{t_{\sigma_3}}^{t} e^{-\bar\lambda(t-\sigma)}h(\sigma)|s(\sigma)|\,d\sigma,\\
y_7(t) &= \int_{t_{\sigma_3}}^{t} e^{-\bar\lambda(t-\sigma)}\int_{t_{\sigma_3}}^{\sigma} e^{-\lambda_2(\sigma-\xi)}|z_1(\xi)||s(\xi)|\,d\xi\,d\sigma,
\end{aligned}$$

and

$$\begin{aligned}
\dot y_5 &= -\lambda_2 y_5 + |s(t)|\\
&= -\lambda_2 y_5 + \mu(t) + \bar k_1 y_1(t) + \bar k_2 y_2(t)\\
&\quad + \bar k_3 y_3(t) + \bar k_4 y_4(t) - z(t),\\
\dot y_6 &= -\bar\lambda y_6 + h(t)|s(t)|\\
&= -\bar\lambda y_6 + h(t)\mu(t) + \bar k_1 h(t) y_1(t) + \bar k_2 h(t) y_2(t)\\
&\quad + \bar k_3 h(t) y_3(t) + \bar k_4 h(t) y_4(t) - h(t) z(t),\\
\dot y_7 &= -\bar\lambda y_7 + y_5.
\end{aligned}$$

Rearranging in a vector form, let $\mathcal{Y} = [y_1, y_2, y_3, y_4, y_5, y_6, y_7]'$, then

$$\dot{\mathcal{Y}}(t) \leq \big[A + F(t)\big]\mathcal{Y}(t) + B(t), \quad \mathcal{Y}(t_{\sigma_3}) = 0$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\bar{\lambda} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\bar{\lambda} \end{bmatrix} \quad F(t) = \begin{bmatrix} \bar{k}_1 h(t) & \bar{k}_2 h(t) & \bar{k}_3 h(t) & \bar{k}_4 h(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{k}_1 |z_1(t)| & \bar{k}_2 |z_1(t)| & \bar{k}_3 |z_1(t)| & \bar{k}_4 |z_1(t)| & 0 & 0 & 0 \\ \bar{k}_1 h(t) & \bar{k}_2 h(t) & \bar{k}_3 h(t) & \bar{k}_4 h(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B(t) = \begin{bmatrix} h(t)\mu(t) \\ 0 \\ 0 \\ 0 \\ |z_1(t)|\mu(t) \\ h(t)\mu(t) \\ 0 \end{bmatrix}$$

Consider now the linear state equation

$$\dot{\mathcal{X}}(t) = [A + F(t)]\,\mathcal{X}(t) + B(t), \quad \mathcal{X}(t_{\sigma_3}) = 0. \tag{4.35}$$

The variation of constants formula applied to (4.35) gives

$$\mathcal{X}(t) = \int_{t_{\sigma_3}}^{t} e^{A(t-\tau)} F(\tau)\mathcal{X}(\tau)\, d\tau + \int_{t_{\sigma_3}}^{t} e^{A(t-\tau)} B(\tau)\, d\tau. \tag{4.36}$$

Since matrix A does not have positive-real-part eigenvalues, then exists a positive constant γ_A such that its transition matrix satisfies $\|e^{A(t-\tau)}\| \leq \gamma_A$. Since $|\mu(t)| \leq \gamma_\mu$ and the norms of $F(t)$ and $B(t)$ satisfy

$$\begin{aligned} \|F(t)\| &\leq 2\big(\bar{k}_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4\big)\big[h(t) + |z_1(t)|\big] \\ &\leq 2\big(\bar{k}_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4\big)\big[\gamma_h e^{-\lambda_h(t-t_{\sigma_3})} + \gamma_{z_1} e^{-\lambda_{z_1}(t-t_{\sigma_3})}\big] \\ &\triangleq \gamma_F e^{-\lambda_F(t-t_{\sigma_3})}, \\ \|B(t)\| &\leq \big(2h(t) + |z_1(t)|\big)\mu(t) \\ &\leq \big(2\gamma_h e^{-\lambda_h(t-t_{\sigma_3})} + \gamma_{z_1} e^{-\lambda_{z_1}(t-t_{\sigma_3})}\big]\gamma_\mu \\ &\triangleq \gamma_B e^{-\lambda_B(t-t_{\sigma_3})}, \end{aligned}$$

then, taking norms in expression (4.36) yields

$$\|\mathcal{X}(t)\| \leq \mu_x(t) + \gamma_A\gamma_F \int_{t_{\sigma_3}}^{t} e^{-\lambda_F(\tau-t_{\sigma_3})}\|\mathcal{X}(\tau)\|\, d\tau,$$

where

$$\mu_x(t) = \frac{\gamma_A\gamma_B}{\lambda_B}\big[1 - e^{-\lambda_B(t-t_{\sigma_3})}\big].$$

Applying now Gronwall lemma (Lemma 4.3) yields

$$\begin{aligned} \|\mathcal{X}(t)\| &\leq \mu_x(t) + \gamma_A\gamma_F \int_{t_{\sigma_3}}^{t} e^{-\lambda_F(\tau-t_{\sigma_3})}\mu_x(\tau)\, e^{\gamma_A\gamma_F \int_\tau^t e^{-\lambda_F(\sigma-t_{\sigma_3})}\, d\sigma} d\tau, \\ &\leq \frac{\gamma_A\gamma_B}{\lambda_B} + \gamma_A\gamma_F \frac{\gamma_A\gamma_B}{\lambda_B} e^{\frac{\gamma_A\gamma_F}{\lambda_F}} \int_{t_{\sigma_3}}^{t} e^{-\lambda_F(\tau-t_{\sigma_3})} d\tau \\ &\leq \frac{\gamma_A\gamma_B}{\lambda_B}\big[1 + \gamma_A\gamma_F e^{\frac{\gamma_A\gamma_F}{\lambda_F}}\big] \triangleq \gamma_{\mathcal{X}}. \end{aligned}$$

Thus, since $\|\mathcal{X}(t)\|$ is bounded, then by the comparison lemma $\|\mathcal{Y}(t)\|$ is bounded. From (4.34) it can be further conclude that $|s(t)|$ is bounded. Consequently, it follows that $z_2(t)$, $\dot{z}_2(t)$, and $v(t)$ converge to zero.

To conclude the proof, it remains to show that there exists a finite time T such that for all $t \geq T$, $\sigma(t) = 3$. First, observe that one can always find positive constants $\omega_2^\star$ and $v^\star$ such that for any initial conditions $\sigma(t_{\sigma_3}) = 3$, $z_1(t_{\sigma_3})$, $\dot{z}_1(t_{\sigma_3})$, and $v(t_{\sigma_3})$ that satisfy $\omega_2(t_{\sigma_3}) \leq \omega_2^\star$ and $|v(t_{\sigma_3})| \leq v^\star$, the bound of $\omega_1(t)$ is less or equal to ϵ_2 for all $t \geq t_{\sigma_3}$. Notice also that for $\sigma = 3$, the closed loop dynamics of $z_1(t)$ are given by $\ddot{z}_1 = -\lambda z_1 - \lambda_1(\lambda - \lambda_1)z_1$. Thus, it follows that for any t_{σ_3} such that $\sigma(t_{\sigma_3}) = 3$, there exist a positive time interval $\tau > 0$ and a finite time $T \in [t_{\sigma_3}, t_{\sigma_3} + \tau]$ such that for all $t \in [T, t_{\sigma_3} + \tau]$

$$z_1(t)\dot{z}_1(t) \leq 0 \quad \text{and} \quad |z_1(t)| > \frac{|\dot{z}_1(t)|}{\lambda}. \tag{4.37}$$

From the definition of ε_2, it can be concluded that σ can not switch from 3 while condition (4.37) is not true (since $\varepsilon_2 = +\infty$). Thus, if $\sigma(t)$ switches from 3 to 2, it means that conditions (4.37) hold. Consequently, from Claim 4.4 and 4.3 it follows that $w_2(t)$ will not increase and also that, after a finite time, the signal σ will switch again to 3. Hence, there will be finite jumps between 2 and 3 until ω_2 becomes less or equal to $\omega_2^\star$. See Fig. 4.2 for a better understanding of the switching logic. From proof of Claims 4.2 and 4.3, it follows that after a finite time (say $t = T$) one has $|v(T)| \leq v^\star$, and $\omega_2(T) \leq \omega^\star$. Consequently, it can be concluded that for all $t \geq T$, $\omega_2(t) \leq \epsilon_2$, $\sigma(t) = 3$, and $v(t)$ converges to zero. This concludes the proof of Claim 4.5 and naturally item 2 of Theorem 4.1.

Stability

From the proof of Claim 4.5 it can be concluded that there exists a positive constant $\bar{r}$ such that for any $q(t_0) \in B_{\bar{r}}(0)$, $\sigma(t) = 3$ for all $t \geq t_0$. Moreover, given any $\epsilon > 0$, there exists a positive constant $r \leq \bar{r}$ such that with $q(t_0) \leq r$ it follows that $|z_3(t)| \leq \epsilon$ for all $t \geq t_0$ (see (4.9)). Also, from the closed loop system expressions for $\sigma = 3$ (see equations (4.23), (4.30), (4.31), and (4.33)), it can be easily proved that the other components of the state q, *i.e.*, z_1, $\dot{z}_1$, z_2, $\dot{z}_2$, and v are asymptotically stable. Therefore, the hybrid system Σ_{AUV} is Lyapunov uniformly stable.

This concludes the proof of Theorem 4.1. □

4.4 Simulation results

This section illustrates the performance of the proposed control scheme with a model of the SIRENE AUV described in Section 2.5.

Figures 4.3-4.8 show the simulation results for the case where the target is the origin. The initial condition is given by $\big(x(0), y(0), \theta(0),$
$u(0), v(0), r(0)\big)' = \big(2\,m, 2\,m, 0, 0, 0, 0\big)'$, or equivalently, $q_0 = \big(z_1(0), z_2(0), z_3(0), \dot{z}_1(0), \dot{z}_2(0)\big)' = (0, 2, 4, 0, 0, 0)'$, and $\sigma(0^-) = 1$. The control parameters were chosen to be $\epsilon_1 = 0.1$, $\epsilon_2 = 0.2$, $\gamma_1 = 0.6$, $\lambda = 1.0$, $\lambda_1 = 0.95$, and $k_3 = 100$.

Figure 4.3 shows the AUV trajectory in the horizontal plane while in Figure 4.4 displays the time evolution of the states $x(t)$, $y(t)$, and $\theta(t)$. The vehicle converges to a small neighborhood of the target position with a desired orientation. Figure 4.5 is a plot of the vehicle linear and angular velocities. Notice that the most aggressive motion occur during the first 20 seconds. This is clearly mirrored in the sway velocity activity over that time period. During the rest of the maneuver, as expected, the sway velocity $v(t)$ converges to zero as the trajectory of the vehicle straightens out. To better understand the action of the hybrid control law proposed, examine Figures 4.6 and 4.7. The first shows the time evolution of the state variables $z_1(t)$, $z_2(t)$, and $z_3(t)$, whereas the latter displays the time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$. It can be seen that $\omega(t)$ starts in region $\mathcal{R}_1$ and, consequently, ω_2 grows until reaches γ_1. At that moment, the signal σ switches to 2, ω_2 does not increase, and ω_1 decreases until it reaches the boundary $\omega_1 = \epsilon_1$. Then, σ switches to 3 and $z_1(t)$, $z_2(t)$, and $v(t)$ converge to zero. This implies that the state $x(t)$ and $\theta(t)$ also converge to zero. The input signals $u_1(t)$ and $u_2(t)$ are presented in Figure 4.8. Its maximal values correspond to what it was expected.

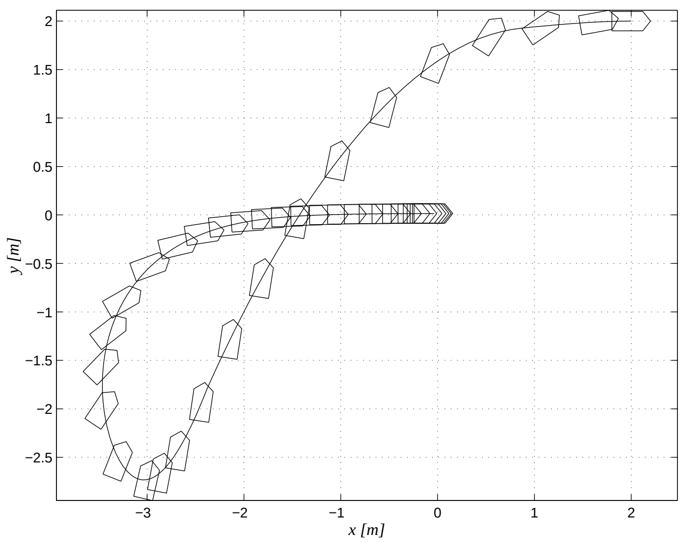

Figure 4.3: Path of the underactuated SIRENE AUV. Initial condition $(x, y, \theta, u, v, r) = (2\,m, 2\,m, 0, 0, 0, 0)$.

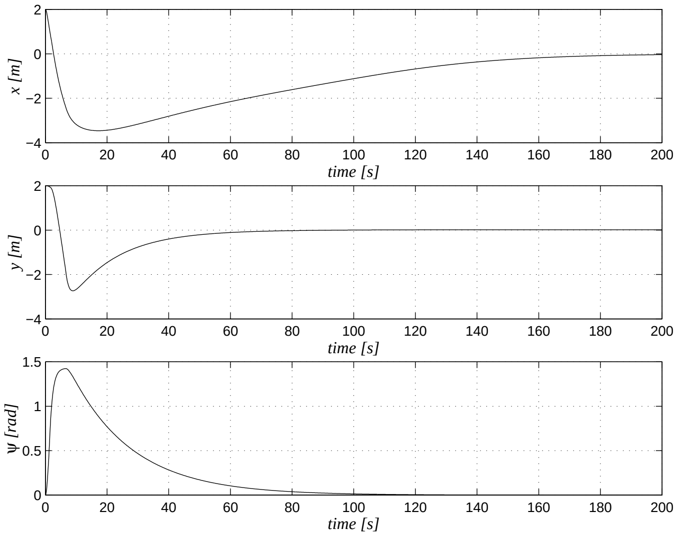

Figure 4.4: Time evolution of position variables $x(t)$ and $y(t)$ and orientation variable $\psi(t)$.

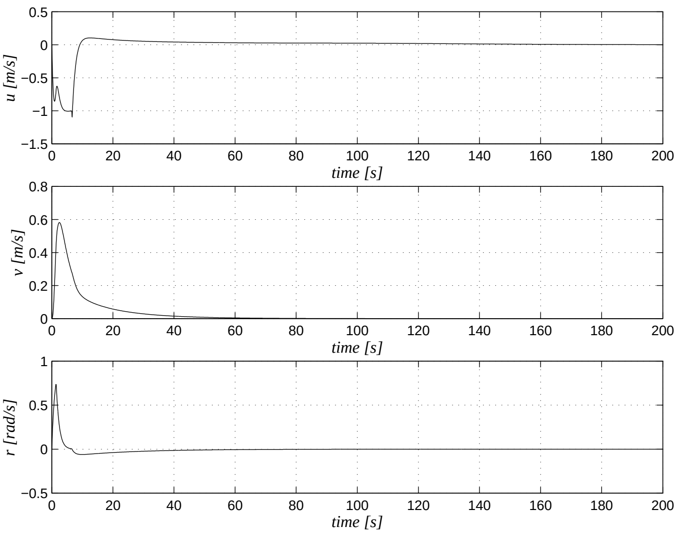

Figure 4.5: Time evolution of linear velocity in x-direction (surge) $u(t)$, linear velocity in y-direction (sway) $v(t)$, and angular velocity $r(t)$.

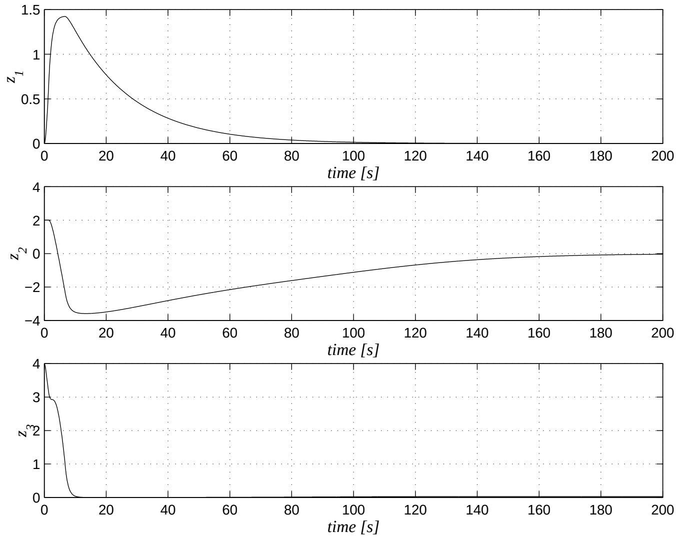

Figure 4.6: Time evolution of state variables $z_1(t)$, $z_2(t)$, and $z_3(t)$.

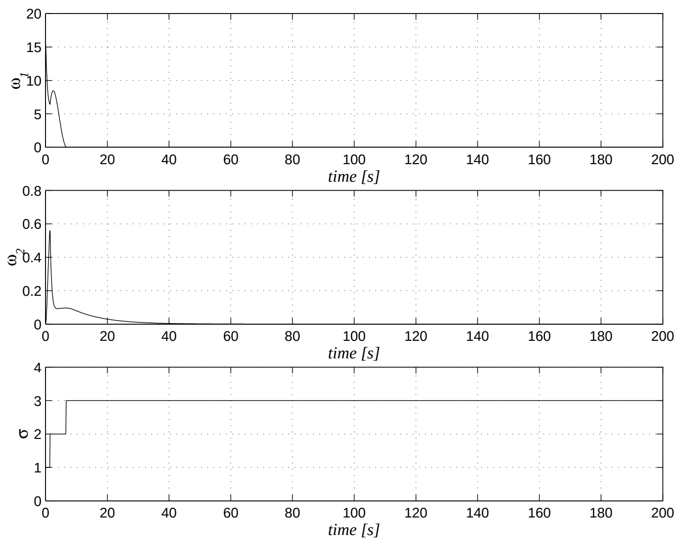

Figure 4.7: Time evolution of variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$.

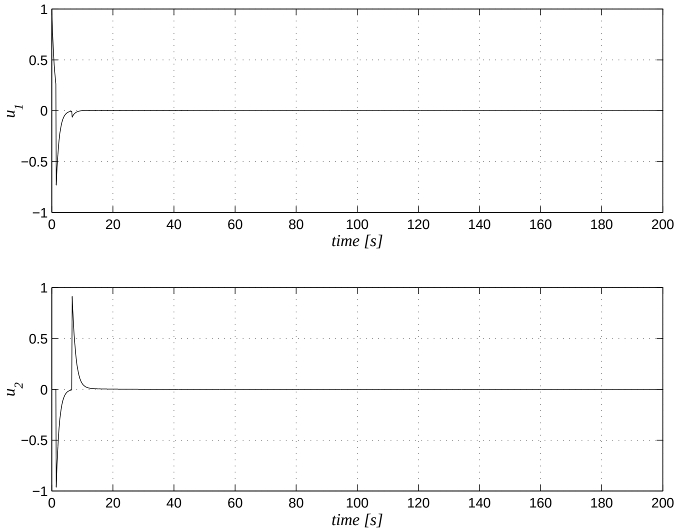

Figure 4.8: Time evolution of control signals $u_1(t)$ and $u_2(t)$.

4.5 Concluding remarks

A feedback hybrid control law has been proposed to globally stabilize an underactuated AUV in the horizontal plane to a point with a desired orientation. The controller proposed has a very simple structure and is based on hybrid systems theory. Convergence of the AUV to an arbitrarily small neighborhood of the origin and stability of the resulting closed loop system were analyzed. To illustrate the control law developed, simulations were conducted using the model of the SIRENE AUV. The simulation results show that the control objectives were achieved successfully. Notice however, that due to the existence in the control law of a coordinate transformation of the AUV model to extended nonholonomic double integrator with drift terms, the resulting path is hard to predict and may not yield a "natural maneuver". A new approach that overcomes this problem is described in the following two chapters. The first solves the stabilization problem for a nonholonomic wheeled mobile robot, while the second deals with the AUV case.

4.6 Notes

This section describes some useful lemmas that are used in this chapter.

4.6.1 Comparison principle

The comparison lemma is instrumental in analyzing the behavior of solutions of the state equation $\dot{x} = f(x,t)$. The lemma can be used in situations where the derivative of a scalar differentiable function $v(t)$ satisfies the differential inequality $\dot{v} \leq f\big(v(t),t\big)$. In this case, the lemma compares the solution of the differential inequality $\dot{v} \leq f\big(v(t),t\big)$ to the solution of the differential equation $\dot{x} = f(x,t)$. Moreover, the comparison lemma is applied even when $v(t)$ is not differentiable, but has an upper right-hand derivative $D^+v(t)$ which satisfies a differential inequality. To be more specific, the upper right-hand derivative of a continuous signal $x(t)$ is defined[1] by

$$D^+x(t) = \lim_{h\to 0^+} \sup \frac{x(t+h) - x(t)}{h},$$

and has the following properties:

1. if $x(t)$ is differentiable at t, then $D^+x(t) = \dot{x}(t)$
2. if

$$\frac{1}{h}\big|x(t+h) - x(t)\big| \le g(h,t), \quad \forall\, h \in (0,b]$$

   and

$$\lim_{h\to 0^+} g(h,t) = g_0(t),$$

   then $D^+x(t) \le g_0(t)$.

Lemma 4.2 (Comparison lemma) *Consider the scalar differential equation*

$$\dot{u} = f(u,t), \quad u(t_0) = u_0$$

where $f(u,t)$ is continuous in t and locally Lipschitz in u, for all $t \ge 0$ and all $u \in J \subset R$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \le f\big(v(t), t\big), \quad v(t_0) \le u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \le u(t)$ for all $t \in [t_0, T)$.

Proof: See for example (Khalil, 1996). □

4.6.2 Gronwall-Bellman inequality

Lemma 4.3 *Consider the continuous functions $\mu : [t_0, T) \to \mathbb{R}$, $\varphi : [t_0, T) \to \mathbb{R}^+$, and $\nu : [t_0, T) \to \mathbb{R}^+$ where φ and ν are also nonnegative and $t_0 \ge 0$. If a continuous function $x : [t_0, T) \to \mathbb{R}$ satisfies the inequality*

$$x(t) \le \mu(t) + \int_{t_0}^{t} \nu(t)\varphi(\tau)x(\tau)d\tau$$

for all $t_0 \le t \le T$, then on the same interval

$$x(t) \le \mu(t) + \int_{t_0}^{t} \nu(t)\varphi(\tau)\mu(\tau)e^{\int_\tau^t \nu(\sigma)\varphi(\sigma)\,d\sigma}d\tau.$$

In particular, if $\mu(t) \equiv \mu$ is a constant and $\nu(t) \equiv 1$, then

$$x(t) \le \mu e^{\int_{t_0}^t \varphi(\tau)\,d\tau}, \quad t_0 \le t \le T$$

[1] The limit $h \to 0^+$ means that h approaches zero from above.

Proof: Let

$$y(t) = \int_{t_0}^{t} \varphi(\tau)x(\tau)\, d\tau$$

and

$$z(t) = \mu(t) + \nu(t)y(t) - x(t) \geq 0.$$

Then, $y(t)$ is differentiable and

$$\begin{aligned}\dot{y}(t) &= \varphi(t)x(t) \\ &= \varphi(t)\mu(t) + \varphi(t)\nu(t)y(t) - \varphi(t)z(t)\end{aligned}$$

This is a scalar linear state equation with the state transition function

$$\Phi(t,\tau) = e^{\int_{\tau}^{t} \nu(\sigma)\varphi(\sigma)\, d\sigma}.$$

Since $y(t_0) = 0$, then

$$y(t) = \int_{t_0}^{t} \Phi(t,\tau)\left[\varphi(\tau)\mu(\tau) - \varphi(\tau)z(\tau)\right]\, d\tau.$$

The term

$$\int_{t_0}^{t} \Phi(t,\tau)\varphi(\tau)z(\tau)\, d\tau$$

is nonnegative. Therefore,

$$y(t) \leq \int_{t_0}^{t} \Phi(t,\tau)\varphi(\tau)\mu(\tau)\, d\tau,$$

and since $x(t) \leq \mu(t) + \nu(t)y(t)$, this completes the proof. □

Chapter 5

Regulation of a nonholonomic dynamic wheeled mobile robot with parametric modeling uncertainty using Lyapunov functions

This chapter addresses the problem of stabilizing the dynamic model of a nonholonomic wheeled mobile robot of the unicycle type to a point with a desired orientation. A simple, discontinuous, adaptive state feedback controller is derived that yields global stability and convergence of the trajectories of the closed loop system in the presence of parametric modeling uncertainty. This is achieved by resorting to a polar representation of the kinematic model of the mobile robot that is a non smooth transformation in the original state space, followed by the derivation of a smooth, time-invariant control law in the new coordinates. The new control algorithm proposed as well as the analysis of its convergence build on Lyapunov stability theory and LaSalle's invariance principle. For an introduction to the polar representation and how it can be exploited to overcome the basic limitations imposed by Brockett's result, see (Aicardi *et al.*, 1995) and (Astolfi, 1995).

The chapter is organized as follows: Section 5.1 formulates the problem of regulating a dynamic wheeled mobile robot of the unicycle type to a point with a desired orientation in the presence of model uncertainties, and presents a non-smooth polar representation of the vehicle kinematics. Section 5.2 introduces a simple Lyapunov-based adaptive strategy for vehicle control that builds on a series of candidate Lyapunov functions related to vehicle heading regulation, target distance regulation, and parameter adaptation. Section 5.3 offers a formal proof of convergence of the resulting adaptive regulation system. Section 5.4 contains simulation results that illustrate the performance of the proposed control strategy and show how it yields natural vehicle's behaviour. The chapter concludes with a summary of results and a brief review of Lyapunov stability, LaSalle's invariance principle, and the backstepping approach to regulation.

5.1 Control problem formulation. Polar representation

Consider the kinematic and dynamic equations of the wheeled mobile robot of unicycle type described in Section 2.4. The problem presented in this chapter can be formulated as follows:

Derive a feedback control for τ_1 and τ_2 to regulate $\{B\}$ to $\{G\} = \{U\}$ in the presence of uncertainty in the parameters m, I, R, and L.

In order to solve this problem, this section introduces a change of coordinates that plays a crucial role in the development that follows.

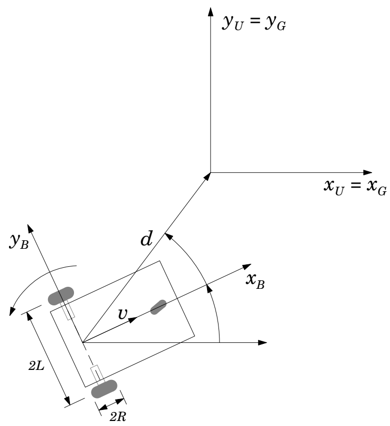

Figure 5.1: A wheeled mobile robot of the unicycle-type.

Consider the coordinate transformation (see Figure 5.1)

$$\begin{aligned} e &= \sqrt{x^2 + y^2}, \\ x &= -e\cos(\theta + \beta), \\ y &= -e\sin(\theta + \beta), \\ \theta + \beta &= \tan^{-1}\left(\frac{-y}{-x}\right), \end{aligned} \tag{5.1}$$

where d is the vector from O_B to O_U, e is the length of d, and β denotes the angle measured from x_B to d. Differentiating (5.1) with respect to time, the dynamics of the wheeled robot in the new coordinate system can be written as

$$\dot{e} = -v\cos\beta, \tag{5.2a}$$

$$\dot{\beta} = \frac{\sin\beta}{e}v - w, \tag{5.2b}$$

$$\dot{\theta} = w, \tag{5.2c}$$

$$m\dot{v} = F, \tag{5.2d}$$

$$I\dot{\omega} = N. \tag{5.2e}$$

Remark 5.1 Notice that the coordinate transformation (5.1) is only valid for non zero values of the distance error e, since for $e = 0$ the angle β is undefined. This will introduce a discontinuity in the control law that will be derived later, which will obviate the basic limitations imposed by the celebrated result of Brockett. The creation of this singular point is at the core of the design methodology adopted in this chapter, which was inspired by the work of (Astolfi, 1999).

Remark 5.2 As already pointed out, for system (5.2) the regularity assumptions needed to apply the Brockett's result do not hold and therefore asymptotic stability of the origin by means of smooth and time invariant feedback laws is not prevented.

5.2 Nonlinear controller design

This section proposes a nonlinear adaptive control law to regulate the motion of the mobile robot described by equations (2.10) and (5.2) to a point with a desired orientation, in the presence of parametric modeling uncertainty. Only the rationale for the control law proposed is introduced, a formal proof of convergence and stability being deferred to Section 5.3. For the sake of clarity, candidate Lyapunov functions are introduced recursively in a sequence of logical steps directly related to vehicle heading regulation, target distance regulation and parameter adaptation. This methodology borrows heavily from the techniques of backstepping (Krstić *et al.*, 1995). A switching term is introduced in the control law at the last stage in order to solve the indeterminacy at $e = 0$ caused by the polar representation adopted.

Step 1. (Heading regulation) Define the variables

$$\rho = \frac{v}{e},$$
$$\sigma = \beta + \theta,$$

and rewrite the equations of motion (5.2) as

$$\dot{\sigma} = \rho \sin\beta, \tag{5.3a}$$
$$\dot{\beta} = \rho \sin\beta - \omega, \tag{5.3b}$$
$$\dot{\omega} = \frac{N}{I}, \tag{5.3c}$$

and

$$\dot{e} = -\rho\cos\beta e, \tag{5.4a}$$
$$\dot{\rho} = \frac{F}{me} + \rho^2\cos\beta, \tag{5.4b}$$

where system (5.2) has been divided in two subsystems that will henceforth be referred to as the heading and distance subsystems, respectively. Consider the heading subsystem (5.3) and suppose (only at this stage) that $\rho = k_1 > 0$. Define the control Lyapunov function

$$V_1 = \frac{1}{2}k_\sigma\sigma^2 + \frac{1}{2}\beta^2,$$

and compute its time derivative along trajectories of (5.3) to obtain

$$\dot{V}_1 = \beta\left[k_1 k_\sigma \sigma \frac{\sin\beta}{\beta} + k_1 \sin\beta - \omega\right].$$

Following the nomenclature in (Krstić *et al.*, 1995) let ω be a virtual control input and

$$\alpha_1(\sigma, \beta) = k_1 k_\sigma \sigma \frac{\sin\beta}{\beta} + k_1 \sin\beta + k_2\beta, \tag{5.5}$$

$k_2 > 0$, a virtual control law. Introduce the error variable

$$z_1 = \omega - \alpha_1, \tag{5.6}$$

and compute $\dot{V}_1$ to obtain

$$\dot{V}_1 = -k_2\beta^2 - \beta z_1.$$

Step 2. (Backstepping) The function V_1 is now augmented with a quadratic term in z_1 to obtain the new candidate Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_1^2.$$

The time derivative of V_2 can be written as

$$\dot{V}_2 = -k_2\beta^2 + z_1\left[\frac{N}{I} - f_1(\cdot)\right],$$

where

$$\begin{aligned} f_1(\sigma, \beta, z_1, \rho) &= \frac{\partial\alpha}{\partial\sigma}\dot{\sigma} + \frac{\partial\alpha}{\partial\beta}\dot{\beta} - \beta \\ &= k_1 k_\sigma \dot{\sigma}\frac{\sin\beta}{\beta} + k_1 k_\sigma \sigma\dot{\beta}\frac{\beta\cos\beta - \sin\beta}{\beta^2} \\ &+ k_1\dot{\beta}\cos\beta + k_2\dot{\beta} + \beta. \end{aligned}$$

Notice that the terms $\frac{\sin\beta}{\beta}$ and $\frac{\beta\cos\beta - \sin\beta}{\beta^2}$ are well defined and continuous at zero. Using L'Hopital's rule it is easy to see that when $\beta = 0$ the first and second terms are equal to 1 and 0, respectively. Let the control law for N be chosen as

$$N = If_1(\cdot) - k_3 z_1, \quad k_3 > 0.$$

Then

$$\dot{V}_2 = -k_2\beta^2 - \frac{k_3}{I}z_1^2 \le 0,$$

that is, $\dot{V}_2$ is negative semidefinite.

Step 3. (Distance regulation) Consider now the distance subsystem (5.4). A new error variable

$$z_2 = \rho - k_1 \tag{5.7}$$

is defined and a third candidate Lyapunov function is introduced as

$$V_3 = V_2 + \frac{1}{2}z_2^2.$$

Computing its time derivative gives

$$\dot{V}_3 = -k_2\beta^2 - \frac{k_3}{I}z_1^2 + z_2\left[\frac{F}{me} + f_2(\cdot)\right],$$

where

$$f_2(\sigma, \beta, \rho) = \rho^2 \cos\beta + k_\sigma \sigma \sin\beta + \beta \sin\beta.$$

The last two terms of f_2 are due to the fact that ρ is not constant, but $\rho = k_1 + z_2$ instead. They are simply computed by replacing ρ by $k_1 + z_1$ in the expression for V_1 and propagating the corresponding terms down to $\dot{V}_3$.

Now, by choosing the control input

$$F = -mf_2(\cdot)e - k_4 z_2 e,$$

the time derivative of V_3 becomes

$$\dot{V}_3 = -k_2\beta^2 - \frac{k_3}{I} z_1^2 - \frac{k_4}{m} z_2^2,$$

that is, $\dot{V}_3$ is negative semidefinite.

Step 4. (Parameter adaptation) Suppose the values of the physical constants m, I, R, and L are not known precisely. Define the control inputs u_i, $i = 1, 2$ as $u_1 = \tau_1 - \tau_2$ and $u_2 = \tau_1 + \tau_2$. Then, from (2.10) the dynamic equations for the mobile robot can be written as

$$\begin{aligned}
\dot{\omega} &= \frac{u_1}{c_1}, \\
\dot{v} &= \frac{u_2}{c_2},
\end{aligned}$$

where $c_1 = \frac{IR}{L}$ and $c_2 = mR$ are positive unknown parameters.

Consider the augmented candidate Lyapunov function

$$V_4 = V_3 + \frac{1}{2c_1\gamma_1}\Delta c_1^2 + \frac{1}{2c_2\gamma_2}\Delta c_2^2,$$

where $\hat{c}_i$; $i = 1, 2$ are nominal value of the parameters c_i, $\Delta c_i = c_i - \hat{c}_i$ are parameter estimation errors, and $\gamma_i > 0$, $i = 1, 2$ are adaptation gains. The time derivative of V_4 can be computed to yield

$$\begin{aligned}
\dot{V}_4 = -k_2\beta^2 &+ z_1 \left[\frac{u_1}{c_1} - f_1(\cdot)\right] + z_2 \left[\frac{u_2}{c_2 e} + f_2(\cdot)\right] \\
&- \frac{\Delta c_1}{c_1\gamma_1}\dot{\hat{c}}_1 - \frac{\Delta c_2}{c_2\gamma_2}\dot{\hat{c}}_2.
\end{aligned}$$

Motivated by the choices in steps 2 and 3, choose the control laws

$$u_1 = \hat{c}_1 f_1(\cdot) - k_3 z_1, \tag{5.8a}$$

$$u_2 = -\hat{c}_2 f_2(\cdot)e - k_4 z_2 e, \tag{5.8b}$$

that make

$$\begin{aligned}
\dot{V}_4 = -k_2\beta^2 &- \frac{k_3}{c_1} z_1^2 - \frac{k_4}{c_2} z_2^2 \\
&- \frac{\Delta c_1}{c_1}\left[z_1 f_1(\cdot) + \frac{\dot{\hat{c}}_1}{\gamma_1}\right] + \frac{\Delta c_2}{c_2}\left[z_2 f_2(\cdot) - \frac{\dot{\hat{c}}_2}{\gamma_2}\right].
\end{aligned}$$

Notice in this equation how the terms containing Δc_i have been grouped together. To eliminate them, choose the parameter adaptation law as

$$\dot{\hat{c}}_1 = -\gamma_1 z_1 f_1(\cdot), \tag{5.9a}$$
$$\dot{\hat{c}}_2 = \gamma_2 z_2 f_2(\cdot), \tag{5.9b}$$

to yield

$$\dot{V}_4 = -k_2\beta^2 - \frac{k_3}{c_1}z_1^2 - \frac{k_4}{c_2}z_2^2 \leq 0. \tag{5.10}$$

Step 5. (Switching control law) So far, it has been assumed that the mobile robot will never start at or reach[1] the position $x = y = 0$ in finite time, because the polar representation (5.1) and consequently the control law described above are not defined at $e = 0$. To deal with this situation, a switching control law must be introduced at this stage. A possible solution is to make

$$u_1 = -k_{\dot{\theta}}\dot{\theta} - k_\theta\theta, \tag{5.11a}$$
$$u_2 = 0, \tag{5.11b}$$

when $e = 0$, where $k_{\dot{\theta}}$ and k_θ are positive constants. The motivation for this control law can be simply understood by noticing that it aims at rotating the vehicle in place under the action of the proportional and derivative terms $k_\theta\theta$ and $k_{\dot{\theta}}\dot{\theta}$, respectively.

The complete control law is thus given by

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{cases} (5.8), (5.9) & e \neq 0 \\ (5.11) & e = 0 \end{cases} \tag{5.12}$$

5.3 Convergence analysis

This section proves convergence to zero of the trajectories of the closed loop system consisting of equations (2.9) and (5.12). The following theorem establishes the main result.

Theorem 5.1 *Consider the closed loop nonlinear invariant system Σ described by (2.9) and (5.12). Let $\mathcal{X} : [t_0, \infty) \to \mathbb{R}^7$, $\mathcal{X}(t) = (x, y, \theta, v, w, \Delta c_1, \Delta c_2)'$; $t_0 \geq 0$ denote any solution of Σ. The following properties hold.*

1. *$\mathcal{X}(t)$ exists, is unique and defined for all $t \geq t_0$ and all $\mathcal{X}(t_0) = \mathcal{X}_0$.*
2. *$\mathcal{X}(t)$ is bounded for any $\mathcal{X}_0$.*
3. *For any initial condition $\mathcal{X}_0$, the state variables $q = (x, y, \theta, v, \omega)'$ converge to zero as $t \to \infty$.*

Proof:

Uniqueness and boundedness

Existence and uniqueness of $\mathcal{X}(t)$ is proven by showing first that for the closed loop system Σ the only switching scenario is when $e = 0$ and $v \neq 0$. It can be easily checked that the manifold

[1] In fact, it will be shown later that if the initial condition is $e(t_0) \neq 0$ than this situation will never arise.

$F = \{\mathcal{X} : x = 0 \wedge y = 0 \wedge v = 0\}$ is positively invariant. Also for $e \neq 0$, if solutions to Σ exist then (5.10) and LaSalle's (La Salle and Lefschetz, 1961) invariance principle show that $\rho(t) \to k_1$ which means that e and v must be either zero or non-zero at the same time. The only case where this is violated is when $e(t_0) = 0$ and $v(t_0) \neq 0$. In this case, from (2.9) it can be seen that for $t = t_0 + \delta$ with $\delta > 0$, $e(t_0 + \delta) \neq 0$, i.e., the control system will switch to the case $e \neq 0$. It can thus be concluded that for any initial condition $\mathcal{X}_0$ there occurs at most one switching, and the closed loop system Σ has a finite number of discontinuities. Using (Hale, 1980, Theorem 5.3, Section I.5), a unique solution to Σ exists over a maximal interval $[0, t_f)$. Now, it remains to prove that the maximum interval of existence is infinite. For $e \neq 0$, solutions of (5.3) and (5.4) exist. Also, from (5.10) one can conclude that $\sigma(t)$, $\beta(t)$, $z_1(t)$, $\rho(t)$, $z_2(t)$, $\hat{c}_1(t)$, and $\hat{c}_2(t)$ are bounded when $\mathcal{X}_0 \notin F$. Thus, the above variables are well defined on the infinite interval. Notice that $\beta(t) \to 0$ and $\rho(t) \to k_1$ as $t \to \infty$. Thus, there exists a finite time $T \geq t_0 \geq 0$ such that for all $t \geq T$, $\rho \cos\beta > 0$. From (5.4a)

$$\begin{aligned} e(t) &= e(t_0) \exp\left(- \int_{t_0}^{t} \rho(\tau) \cos\beta(\tau) d\tau \right) \\ &= e(t_0) e^{-\int_{t_0}^{T} \rho(\tau)\cos\beta(\tau) d\tau} e^{-\int_{T}^{t} \rho(\tau)\cos\beta(\tau) d\tau}, \end{aligned}$$

for all $t \geq t_0$. Therefore, it can be immediately seen that $e(t)$ is bounded and $e(t) \to 0$ as $t \to \infty$. Since $\sigma(t)$ and $\beta(t)$ are bounded, then $\theta(t)$ is bounded. Thus, the trajectory $\mathcal{X}(t)$ is bounded for all $\mathcal{X}_0 \notin F$. When $\mathcal{X}_0 \in F$ the same conclusions can be immediately drawn. Since the trajectory $\mathcal{X}(t)$ is bounded, it exists over the infinite interval, that is, $t_f = \infty$.

Convergence

If $\mathcal{X}_0 \in F$ then $\theta \to 0$ and $\omega \to 0$ as $t \to \infty$. If $\mathcal{X}_0 \notin F$, $e(t) \to 0$ which implies that $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. Moreover, since $\rho(t) \to k_1$, then $v(t) \to 0$ as $t \to \infty$. It remains to prove convergence of θ and ω. This is done by resorting to the LaSalle's invariance principle. Define $\Omega \triangleq \{\mathcal{X} : V_4(\mathcal{X}) \leq V_4(\mathcal{X}_0) = c\}$ which is a positively invariant set since $\dot{V}_4 \leq 0$. Let E be the set of all points in Ω such that $\dot{V}_4(\mathcal{X}) = 0$, that is, $E = \{\mathcal{X} \in \Omega : \beta = 0 \wedge z_1 = 0 \wedge z_2 = 0\}$. Let M be the largest invariant set contained in E. LaSalle's theorem assures that every bounded solution starting in Ω converges to M as $t \to \infty$. To characterize the set M, observe that in the set E the variables β and $\dot{\beta}$ are zero. Therefore, from (5.3b) $\omega = 0$. Notice also from (5.5) and (5.6), that if $\mathcal{X} \in E$ then $z_1 = \alpha_1 = k_1 k_\sigma \sigma$, and since $z_1 = 0$ in E it follows that $\sigma = 0$. Consequently, one can conclude that $\omega(t) \to 0$, $\sigma(t) \to 0$ and therefore $\theta(t) \to 0$ as $t \to \infty$. Thus, $\mathcal{X}(t) \to 0$ as $t \to \infty$.

□

5.4 Simulation results

This section illustrates the performance of the proposed control scheme (in the presence of parametric uncertainty) using computer simulations. The objective is to regulate the position and attitude of the robot to zero. The following parameters were adopted:

$$\begin{aligned} m &= 10.0\, Kg, & I &= 1.25\, Kg\, m^2, \\ L &= 0.5\, m, & R &= 0.1\, m. \end{aligned}$$

The control parameters were selected as

$$k_1 = 0.47, \quad k_4 = 0.5, \quad k_\sigma = 2.0,$$
$$k_2 = 0.1, \quad \gamma_1 = 0.1, \quad k_{\dot\theta} = 0.85,$$
$$k_3 = 1.0, \quad \gamma_2 = 5.0, \quad k_\theta = 0.32.$$

The initial estimates for the vehicle parameters were

$$\hat{m} = 20.0\, Kg, \quad \hat{I} = 1.6\, Kg\, m^2,$$
$$\hat{L} = 0.4\, m, \quad \hat{R} = 0.15\, m.$$

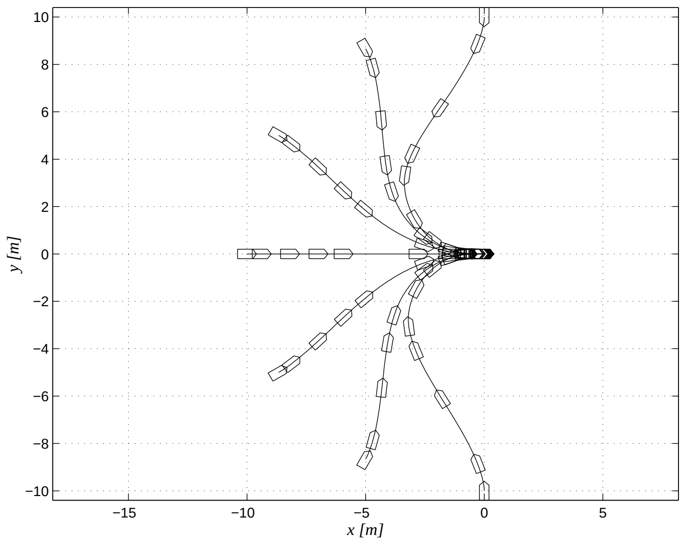

Figure 5.2: Trajectories in the xy-plane. Initial conditions: $e = 10$, $\beta = v = \omega = 0$, and $\theta = -\frac{\pi}{2}, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$.

Figure 5.2 shows the vehicle trajectories in the xy-plane for different initial conditions in θ. Figures 5.3-5.7 display the time responses of the relevant state space variables for the initial condition $q(t_0) = (x_0, y_0, \theta_0, v_0, \omega_0) = (-10\, m, -2\, m, 0, 0, 0)$. Notice how, in spite of parameter uncertainty, the mobile robot converges asymptotically to the origin with a "natural", smooth trajectory. Notice also that the estimated parameters $c_i, i = 1, 2$ are bounded, as expected. However, as Figure 5.7 shows, the estimation error Δc_1 does not converge to zero. This is due to the particular structure of the adaptive control system adopted that allows for asymptotic convergence of q with values of the estimated parameter $\hat{c}_1$ different from the "true" one.

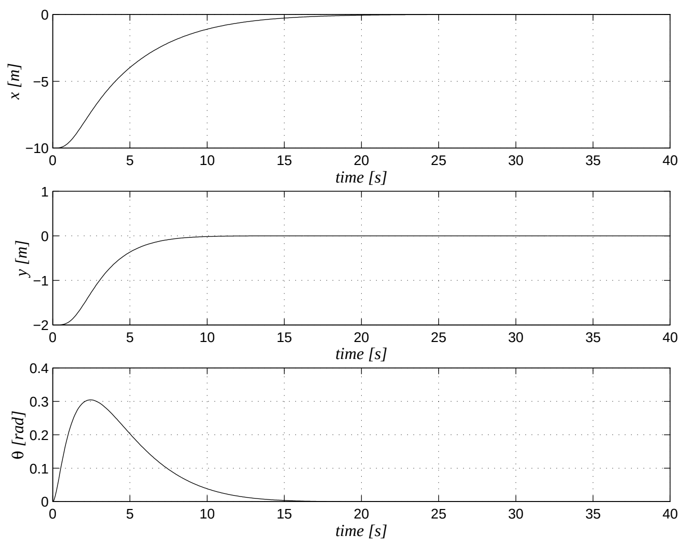

Figure 5.3: Time evolution of position variables $x(t)$, $y(t)$, and orientation variable $\theta(t)$.

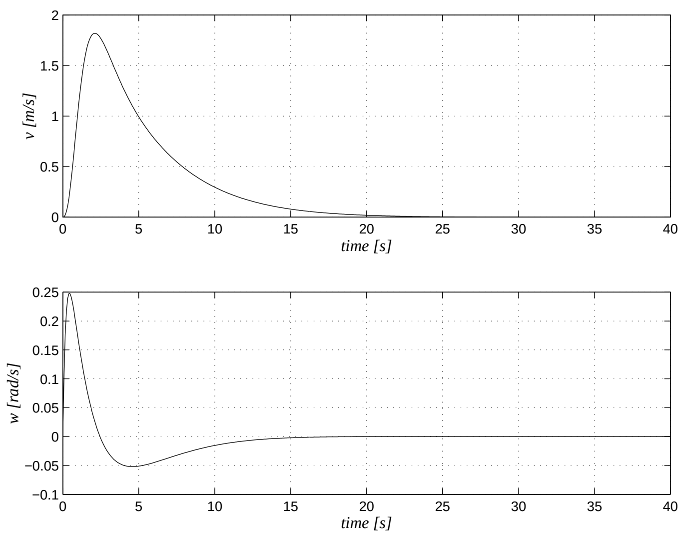

Figure 5.4: Time evolution of linear velocity $v(t)$ and angular velocity $\omega(t)$.

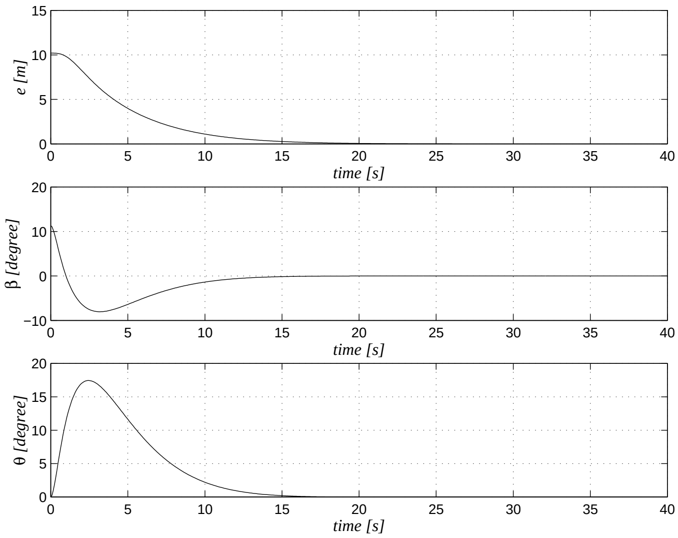

Figure 5.5: Time evolution of variables $e(t)$, $\beta(t)$, and $\theta(t)$.

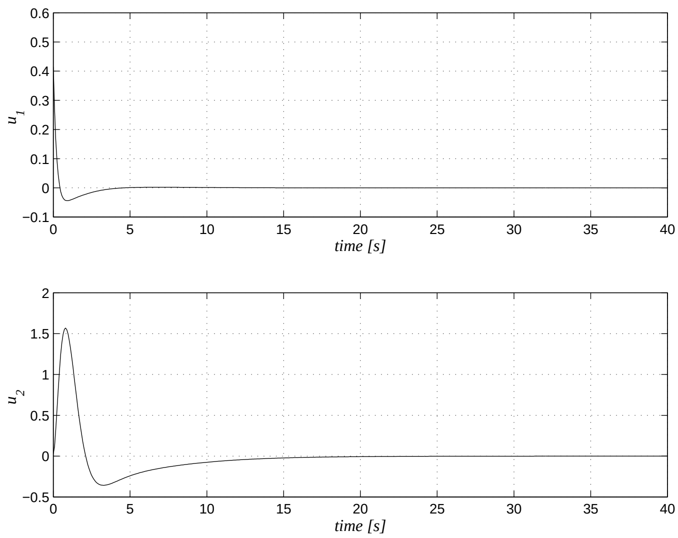

Figure 5.6: Time evolution of control signals $u_1(t)$ and $u_2(t)$.

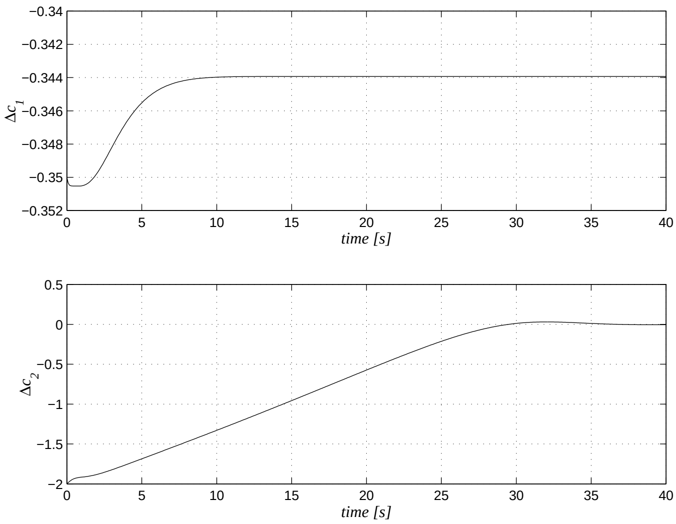

Figure 5.7: Time evolution of parameter estimation errors Δc_1 and Δc_2.

5.5 Concluding remarks

This chapter proposed a new solution to the problem of regulating the dynamic model of a nonholonomic wheeled robot of the unicycle type to a point with a desired orientation. A discontinuous, bounded, time invariant, nonlinear adaptive control law was derived that yields global convergence of the trajectories of the closed loop system in the presence of parametric modeling uncertainty. Controller design relied on a non smooth coordinate transformation in the original state space, followed by the derivation of a Lyapunov-based, smooth control law in the new coordinates. Convergence to the origin was analyzed and simulations were performed to illustrate the behaviour of the proposed control scheme. Simulation results show that the control objectives were achieved successfully and that the vehicle maneuvers exhibit an overall trend that is usually referred to as "natural"-like.

5.6 Notes

This section gives a brief review of Lyapunov stability, LaSalle's invariance principle, and the backstepping approach to stabilization.

5.6.1 Lyapunov stability

A comprehensive survey on Lyapunov stability theory is found in (Khalil, 1996). In this section, some central results and definitions of stability are reviewed.

Consider the autonomous system

$$\dot{x} = f(x), \tag{5.13}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose that the origin $x = 0$ is an equilibrium point, that is, $f(0) = 0$.

Definition 5.1 *The equilibrium point $x = 0$ of (5.13) is*

- stable *if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0$$

- unstable *if not stable.*

- asymptotic stable *if it is stable and δ can be chosen such that*

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

Theorem 5.2 (Lyapunov stability theorem) *Let $x = 0$ be an equilibrium point for (5.13) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function, such that* i) *$V(0) = 0$,* ii) *$V(x) > 0$ in $D\backslash\{0\}$, and* iii) *$\dot{V}(x) \leq 0$ in D.*

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \quad \text{in} \quad D\backslash\{0\}$$

then $x = 0$ is asymptotically stable.

Proof: See for example (Khalil, 1996). □

5.6.2 Invariance principle

This section reviews the principle of invariance of LaSalle. See (Khalil, 1996) for details.

Let $x(t)$ be a solution of (5.13). A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$. A set M is said to be a *positively invariant set* if

$$x(t_0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq t_0.$$

Lemma 5.1 *If a solution $x(t)$ of (5.13) is bounded and belongs to D for $t \geq t_0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover,*

$$x(t) \to L^+ \quad \text{as} \quad t \to \infty.$$

Theorem 5.3 (LaSalle's invariance principle) *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (5.13). Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then, every solution starting in Ω approaches M as $t \to \infty$.*

Proof: See for example (Khalil, 1996). □

5.6.3 Backstepping

This section gives a brief review of the backstepping approach to stabilization. See the book of (Krstić *et al.*, 1995) for details. This review is taken from the book of (Sastry, 1999).

Consider the problem of stabilizing a nonlinear system in so-called strict feedback form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1), \\
\dot{x}_2 &= x_3 + f_2(x_1, x_2), \\
&\vdots \\
\dot{x}_i &= x_{i+1} + f_i(x_1, x_2, \cdots, x_i), \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \cdots, x_n) + u.
\end{aligned}
$$

Note that the state equation for $\dot{x}_i$ depends only on $x_1, x_2, \cdots, x_i$ and affinely on x_{i+1}. The idea behind backstepping is to consider the state x_2 as a sort of "virtual-control" for x_1. Thus, if it is possible to make $x_2 = -x_1 - f_1(x_1)$, the x_1 state would be stabilized, as can be seen by considering the Lyapunov function $V_1 = \frac{1}{2}x_1^2$. Since x_2 is not an control input, one defines

$$
\begin{aligned}
z_1 &= x_1, \\
z_2 &= x_2 - \alpha(x_1),
\end{aligned}
$$

with $\alpha_1(x_1) = -x_1 - f_1(x_1)$. Including a partial Lyapunov function $V_1(z_1) = \frac{1}{2}z_1^2$, one obtains

$$
\begin{aligned}
\dot{z}_1 &= -z_1 + z_2, \\
\dot{z}_2 &= x_3 + f_2(x_1, x_2) - \frac{\partial \alpha_1}{\partial x_1}(x_2 + f_1(x_1)) \stackrel{\triangle}{=} x_3 + \bar{f}_2(z_1, z_2), \\
\dot{V}_1 &= -z_1^2 + z_1 z_2.
\end{aligned}
$$

Proceeding recursively, define

$$
\begin{aligned}
z_3 &= x_3 - \alpha_2(z_1, z_2), \\
V_2 &= V_1 + \frac{1}{2}z_2^2.
\end{aligned}
$$

To derive the formula for $\alpha_2(z_1, z_2)$, observe that

$$
\begin{aligned}
\dot{z}_2 &= z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2), \\
\dot{V}_2 &= -z_1^2 + z_2(z_1 + z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2).
\end{aligned}
$$

Choosing $\alpha_2(z_1, z_2) = -z_1 - z_2 - \bar{f}_2(z_1, z_2)$, yields

$$
\begin{aligned}
\dot{z}_1 &= -z_1 + z_2, \\
\dot{z}_2 &= -z_1 - z_2 + z_3, \\
\dot{V}_2 &= -z_1^2 - z_2^2 + z_2 z_3.
\end{aligned}
$$

Proceeding recursively, at the i-th step define

$$
\begin{aligned}
z_{i+1} &= x_{i+1} - \alpha_i(z_1, \cdots, z_i), \\
V_i &= \frac{1}{2}(z_1^2 + z_2^2 + \cdots + z_i^2),
\end{aligned}
$$

to get

$$
\begin{aligned}
\dot{z}_i &= z_{i+1} + \alpha_i(z_1, \cdots, z_i) + \bar{f}_i(z_1, \cdots, z_i), \\
\dot{V}_i &= -z_1^2 - \cdots - z_{i-1}^2 + z_{i-1}z_i + z_i(z_{i+1} + \alpha_i\big(z_1, \cdots, z_i) + \bar{f}_i(z_1, \cdots, z_i)\big).
\end{aligned}
$$

Using $\alpha_i\big(z_1, \cdots, z_i) = -z_{i-1} - z_i - \bar{f}_i(z_1, \cdots, z_i)$, it follows that

$$
\begin{aligned}
\dot{z}_i &= -z_{i-1} - z_i + z_{i+1}, \\
\dot{V}_i &= -z_1^2 - \cdots - z_i^2 + z_i z_{i+1}.
\end{aligned}
$$

At the last step, one obtains

$$
\dot{z}_n = \bar{f}_n(z_1, \cdots, z_n) + u.
$$

By defining

$$
u = \alpha_n(z_1, \cdots, z_n) = -z_{n-1} - z_n - \bar{f}_n(z_1, \cdots, z_n)
$$

and the composite Lyapunov function

$$
V_n = \frac{1}{2}(z_1^2 + \cdots + z_n^2)
$$

it follows that

$$
\begin{aligned}
\dot{z}_n &= -z_{n-1} - z_n, \\
\dot{V}_n &= -z_1^2 - z_2^2 - \cdots - z_n^2.
\end{aligned}
$$

The foregoing construction calculates a diffeomorphism from the x to the z coordinates. Stability is shown later by resorting to a simple quadratic Lyapunov function. Also, the dynamics are linear in the transformed z coordinates. One of the interesting flexibilities of the backstepping method is that the calculation of the α_i is done merely to produce a negative definite $\dot{V}_i$ (up to terms in z_i) rather than to cancel all system nonlinearities. This is referred as *nonlinear damping* (Krstić *et al.*, 1995).

Chapter 6

Regulation of an underactuated autonomous underwater vehicle with parametric modeling uncertainty

This chapter presents a solution to the problem of regulating the dynamic model of an underactuated autonomous underwater vehicle (AUV) in the horizontal plane to a point, with a desired orientation. A time-invariant discontinuous controller is proposed that yields convergence of the trajectories of the closed-loop system in the presence of parametric modeling uncertainty. Controller designs relies on a non smooth coordinate transformation in the original state space followed by the derivation of a Lyapunov-based, adaptive, smooth control law in the new coordinates. Convergence of the regulation system is analyzed and simulation results are presented. The control methodology adopted borrows from and extends the results presented in Chapter 5 for the stabilization of a nonholonomic wheeled mobile robot. However, as will be seen, the control law proposed and its convergence analysis are far more complex. This stems from the fact that the models of AUVs typically include a drift vector field that must be taken explicitly into account and that is not in the span of the input vector fields, thus precluding the use of input transformations to bring them to driftless form.

The chapter is organized as follows: Section 6.1 formulates the problem of regulating a dynamical model of an AUV to a point with a desired orientation in the presence of model uncertainties. A non-smooth polar representation of its kinematics is also presented. Section 6.2 introduces a simple Lyapunov-based adaptive strategy for vehicle control that builds on a series of candidate Lyapunov functions related to vehicle heading regulation and target distance regulation. Section 6.3 offers a formal proof of convergence of the resulting nonlinear regulation system. Section 6.4 describes the design of the adaptive control system by extending the results of Section 6.2-6.3 to deal with parametric uncertainty. Section 6.5 contains simulation results that illustrate the performance of the proposed control strategy and show how it yields natural vehicle's behaviour. Section 6.6 presents some concluding remarks. Finally, Section 6.7 is a short note about on the upper right-hand derivative of a function. This concept is used in the proof of the main theorem of this chapter.

6.1 Control problem formulation. Coordinate transformation

6.1.1 Problem formulation

Consider the AUV described in Section 2.5. Let $\{G\}$ be a goal reference frame and assume for simplicity of presentation that $\{G\} = \{U\}$. The problem considered in this chapter can be formulated as follows.

Given the nonholonomic underactuated autonomous underwater vehicle with kinematics and dynamics equations (2.17) *and* (2.18)*, derive a feedback control for* τ_u *and* τ_r *to regulate* $\{B\}$ *to* $\{G\}$ *in the presence of parametric model uncertainty.*

The type of parametric uncertainty considered includes the general case where all the hydrodynamic coefficients of the vehicle's dynamic model are allowed to deviate from their nominal values.

6.1.2 Coordinate transformation

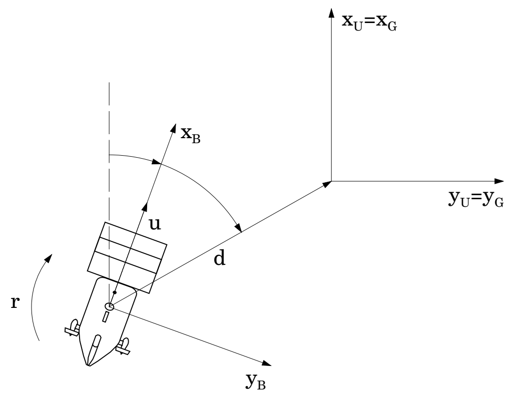

Figure 6.1: Coordinate Transformation.

Consider the coordinate transformation (see Figure 6.1)

$$e = \sqrt{x^2 + y^2}, \tag{6.1a}$$

$$x = -e\cos(\psi + \beta), \tag{6.1b}$$

$$y = -e\sin(\psi + \beta), \tag{6.1c}$$

$$\psi + \beta = \tan^{-1}\left(\frac{-y}{-x}\right), \tag{6.1d}$$

where d is the vector from O_B to O_U, e is the length of d, and β denotes the angle measured from x_B to d. Notice that in equation (6.1d) care must be taken to select the proper quadrant for β. Differentiating (6.1) with respect to time, the kinematics equations of motion of the AUV in the new coordinate system for $e \neq 0$ can be written as

$$\dot{e} = -u\cos\beta - v\sin\beta, \tag{6.2a}$$
$$\dot{\beta} = \frac{\sin\beta}{e}u - \frac{\cos\beta}{e}v - r, \tag{6.2b}$$
$$\dot{\psi} = r. \tag{6.2c}$$

Remark 6.1 Notice that the coordinate transformation (6.1) is only valid for non zero values of the distance error e, since for $e = 0$ the angle β is undefined. This will introduce a discontinuity in the control law that will be derived later, thus obviating the basic limitations imposed by the Brockett's result.

6.2 Nonlinear controller design

This section proposes a nonlinear control law to regulate the motion of the AUV described by equations (2.17) and (2.18) to a point with a desired orientation. As in the previous chapter, only the rationale for the control law proposed is introduced, a formal proof of convergence being deferred to Section 6.3. Control law design is based on previous work described in Chapter 5. For the sake of clarity, candidate Lyapunov functions are introduced recursively in a sequence of logical steps directly related to vehicle heading regulation and target distance regulation. This methodology borrows heavily from the techniques of backstepping (Krstić *et al.*, 1995). A switching term is introduced in the control law at the last stage in order to solve the indeterminacy at $e = 0$ caused by the polar representation adopted.

Step 1. (Heading regulation) Define the variables

$$\rho = \frac{u}{e}, \tag{6.3a}$$
$$\xi = \frac{v}{e}, \tag{6.3b}$$
$$\delta = \beta + \frac{1}{\gamma}\psi, \tag{6.3c}$$
$$\sigma = \beta + \psi + \int_{t_0}^{t} \xi\cos\beta d\tau + \frac{1}{\gamma}\int_{t_0}^{t} \rho\frac{\sin\beta}{\beta}\psi\, d\tau, \tag{6.3d}$$

where γ is a positive even integer constant. Rewrite the equations of motion (2.18) and (6.2) as

$$\dot{\delta} = \rho\sin\beta - \xi\cos\beta - \frac{\gamma - 1}{\gamma}r, \tag{6.4a}$$
$$\dot{\sigma} = \rho\sin\beta + \frac{1}{\gamma}\rho\frac{\sin\beta}{\beta}\psi, \tag{6.4b}$$
$$\dot{\xi} = -\frac{m_u}{m_v}\rho r - \frac{d_v}{m_v}\xi + \xi\rho\cos\beta + \xi^2\sin\beta, \tag{6.4c}$$
$$\dot{r} = \frac{1}{m_r}\left[\tau_r + m_{uv}uv - d_r r\right], \tag{6.4d}$$

and

$$\dot{e} = -\rho \cos \beta e - \xi \sin \beta e, \tag{6.5a}$$

$$\dot{\rho} = \frac{1}{m_u e} \left[\tau_u + m_v v r - d_u u \right] + \rho^2 \cos \beta + \rho \xi \sin \beta, \tag{6.5b}$$

where the final system has been divided in two subsystems that will henceforth be referred to as the heading and distance subsystems, respectively. Consider first the heading subsystem (6.4), where the control objective is to regulate the variables δ, σ, ξ, and r to zero, and ψ to a point in $\mathcal{O}_\psi = \{\psi = 2\pi n;\ n = 0, \pm 1, \pm 2, \cdots \}$. In order to do this, observe from equations (6.4a), (6.4b), and (6.4d) that r can be viewed as a virtual control input. Define the positive definite function

$$V_1 = \frac{1}{2}\delta^2 + \frac{1}{2} k_\sigma \sigma^2,$$

and compute its time derivative along the trajectories of (6.4) to obtain

$$\dot{V}_1 = \delta \left[k_\sigma \sigma \rho \frac{\sin \beta}{\beta} + \rho \sin \beta - \xi \cos \beta - \frac{\gamma - 1}{\gamma} r \right].$$

Following the methodology in (Krstić *et al.*, 1995), let r be a virtual control input and

$$\alpha(\delta, \sigma, \beta, \rho, \xi) = k_\sigma \sigma \rho \frac{\sin \beta}{\beta} + \rho \sin \beta - \xi \cos \beta + k_2 \delta, \quad k_2 > 0, \tag{6.6}$$

a virtual control law. Introduce the error variable

$$z_1 = \frac{\gamma - 1}{\gamma} r - \alpha, \tag{6.7}$$

and compute $\dot{V}_1$ to obtain

$$\dot{V}_1 = -k_2 \delta^2 - \delta z_1.$$

Step 2. (Backstepping) The function V_1 is now augmented with a quadratic term in z_1 to obtain the new candidate Lyapunov function

$$V_2 = V_1 + \frac{1}{2} z_1^2.$$

The time derivative of V_2 can be written as

$$\dot{V}_2 = -k_2 \delta^2 + z_1 \Big[\frac{1}{m_r} \frac{\gamma - 1}{\gamma} \big(\tau_r + m_{uv} u v - d_r r \big) - \dot{\alpha} - \delta \Big].$$

Notice that the terms $\frac{\sin \beta}{\beta}$ and $\frac{\beta \cos \beta - \sin \beta}{\beta^2}$ (that appears in $\dot{\alpha}$ term) are well defined and continuous at zero. Using L'Hopital's rule it is easy to see that when $\beta = 0$ the first and second terms are equal to 1 and 0, respectively. Let the control law for τ_r be chosen as

$$\tau_r = -m_{uv} u v + d_r r + m_r \frac{\gamma}{\gamma - 1} \left(\dot{\alpha} + \delta - k_3 z_1 \right), \ k_3 > 0. \tag{6.8}$$

Then

$$\dot{V}_2 = -k_2 \delta^2 - k_3 z_1^2 \ \le \ 0,$$

that is, $\dot{V}_2$ is negative semidefinite.

Step 3. (Free dynamics analysis) In this step, the dynamic motions of β and ξ in the manifold $E = \{(\delta, \sigma, z_1) \in \mathbb{R}^3 : \dot{V}_2 = 0\}$ with $\rho = k_1 > 0$ are analyzed. First, observe that because V_2 is positive definite, radially unbounded, and has a negative semidefinite derivative, it follows that δ, σ, and z_1 are globally bounded. Furthermore, LaSalle's theorem guarantees convergence of those variables to the largest invariant set M contained in E. Thus, $\delta(t) \to 0$, $z_1(t) \to 0$ as $t \to \infty$. Observe also that in the set E the variables δ and $\dot{\delta}$ are zero. Therefore, from (6.4a), (6.6), and (6.7), one has

$$k_1 k_\sigma \sigma \frac{\sin \beta}{\beta} = 0.$$

Thus, in the manifold E, the above expression leads to *i)* $\sin \beta = 0$, with $\beta \neq 0$ or *ii)* $\sigma = 0$. For the first case, it follows that in E

$$\dot{\xi} = -\Big[\frac{d_v}{m_v} - \Big(\frac{m_u}{m_v}\frac{\gamma}{\gamma - 1} + 1\Big) k_1 \cos \beta\Big]\xi.$$

Consequently, $\lim_{t \to \infty} \xi(t) = 0$ if the controller parameter k_1 is chosen such that

$$\frac{d_v}{m_v} > \Big(\frac{m_u}{m_v}\frac{\gamma}{\gamma - 1} + 1\Big) k_1.$$

For the second case, the dynamics of $\{\beta, \xi\}$ in the manifold E are given by

$$\dot{\beta} = -\frac{1}{\gamma - 1} k_1 \sin \beta + \frac{1}{\gamma - 1} \xi \cos \beta, \tag{6.9a}$$

$$\begin{aligned} \dot{\xi} = &-\Big(\frac{d_v}{m_v} - \frac{m_u}{m_v}\frac{\gamma}{\gamma - 1} k_1 \cos \beta - k_1 \cos \beta\Big)\xi \\ &+ \xi^2 \sin \beta - \frac{m_u}{m_v}\frac{\gamma}{\gamma - 1} k_1^2 \sin \beta. \end{aligned} \tag{6.9b}$$

Conclusions about the behavior of $\{\beta, \xi\}$ in the manifold E can be taken by resorting to the candidate Lyapunov function

$$V = \gamma \frac{m_u}{m_v} k_1^2 (1 - \cos \beta) + \frac{1}{2}\xi^2. \tag{6.10}$$

Computing its time derivative yields

$$\dot{V} = -\begin{bmatrix} k_1 \sqrt{\gamma \frac{m_u}{m_v}} \sin \beta \\ \xi \end{bmatrix}^T Q \begin{bmatrix} k_1 \sqrt{\gamma \frac{m_u}{m_v}} \sin \beta \\ \xi \end{bmatrix},$$

where

$$Q = \begin{bmatrix} \frac{1}{\gamma - 1} k_1 & \frac{k_1}{2(\gamma - 1)} \sqrt{\gamma \frac{m_u}{m_v}} (1 - \cos \beta) \\ \frac{k_1}{2(\gamma - 1)} \sqrt{\gamma \frac{m_u}{m_v}} (1 - \cos \beta) & \frac{d_v}{m_v} - \Big(\frac{m_u}{m_v}\frac{\gamma}{\gamma - 1} + 1\Big) k_1 \cos \beta - \xi \sin \beta \end{bmatrix}.$$

Under the assumption that ξ is bounded, *i.e.*, assuming there exists a positive number r_ξ such that

$$|\xi(t)| \leq r_\xi, \quad t \geq t_0, \tag{6.11}$$

it can be checked that Q is a positive definite matrix if the inequalities

$$\frac{1}{\gamma-1}k_1 > 0, \tag{6.12a}$$

$$\frac{d_v}{m_v} > \Big[2\frac{\gamma}{\gamma-1}\frac{m_u}{m_v}+1\Big]k_1 + r_\xi, \tag{6.12b}$$

hold. In this case,

$$\begin{aligned}
\dot{V} &\leq -\lambda_{min}(Q)\Big[\gamma\frac{m_u}{m_v}k_1^2\sin^2\beta + \xi^2\Big] \\
&\leq -\lambda_{min}(Q)\Big[\gamma\frac{m_u}{m_v}k_1^2(1+\cos\beta)(1-\cos\beta) + \xi^2\Big] \\
&\leq -\lambda_{min}(Q)\Big[\gamma\frac{m_u}{m_v}k_1^2(1+\cos\beta)(1-\cos\beta) \\
&\quad + \frac{1}{2}(1+\cos\beta)\xi^2\Big] - \lambda_{min}(Q)\Big[1-\frac{1}{2}(1+\cos\beta)\Big]\xi^2 \\
&\leq -\lambda_{min}(Q)(1+\cos\beta)\Big[\gamma\frac{m_u}{m_v}k_1^2(1-\cos\beta)+\frac{1}{2}\xi^2\Big] \\
&\leq -\lambda_{min}(Q)(1+\cos\beta)V \leq 0,
\end{aligned}$$

where $\lambda_{min}(Q)$ denotes the minimum eigenvalue of the positive matrix Q. Hence, it can be concluded that $\lim_{t\to\infty}\dot{V}(t) = 0$ which implies that $\{\sin\beta,\xi\}$ converges to zero as $t\to\infty$. Therefore, from (6.3c) and since γ takes an even integer value, it follows that $\psi \to \mathcal{O}_\psi$. To estimate a region of attraction for ξ in the manifold E, in order to validate assumption (6.11), observe that

$$\frac{1}{2}\xi^2 \leq V \leq V_0 \leq 2\gamma\frac{m_u}{m_v}k_1^2 + \frac{1}{2}\xi_0^2.$$

Thus,

$$|\xi| \leq \sqrt{4\gamma\frac{m_u}{m_v}k_1^2 + \xi_0^2} \leq r_\xi,$$

and therefore, for any $\xi(t_0) = \xi_0$ such that

$$|\xi_0| \leq \sqrt{r_\xi^2 - 4\gamma\frac{m_u}{m_v}k_1^2}, \tag{6.13}$$

on the manifold E, $\lim_{t\to\infty}\xi(t) = 0$.

Remark 6.2 In step 3 one must show convergence of $\{\sin\beta,\xi\}$ to zero in the manifold $E = \{(\delta,\sigma,z_1)\in\mathbb{R}^3 : \dot{V}_2 = 0\}$ with $\dot{\rho}=0$, and $\rho = k_1 > 0$. Notice that nothing was said about the point $\{\beta,\xi\}$. In fact, the equilibrium points $\{(\beta,\xi) = (\pi n, 0); n = 0,\pm1,\pm2,\cdots\}$ are not all stable. This can be seen by linearizing the equations (6.9) at each equilibrium point and computing the linear systems matrix eigenvalues. It can be easily concluded that the equilibrium points $\mathcal{O}_\pi = \{(\beta,\xi) = (2\pi n+\pi, 0); n = 0,\pm1,\pm2,\cdots\}$ are not stable. However, there are initial conditions in E, besides the equilibrium points themselves, such that $\{\beta,\xi\}$ converge to $\mathcal{O}_\pi$. Observe also that, if $\rho = -k_1$, then by performing first a change of variables, *e.g.* $\eta = \beta - \pi$ and following the same line of reasoning, convergence of $\{\sin\eta = -\sin\beta,\xi\}$ to zero can be concluded. In this case, the points in the set $\mathcal{O}_\pi$ are now stable equilibrium points.

Step 4. (Distance regulation) Consider now the distance subsystem (6.5) and in particularly examine equation (6.5a). Since $\{\sin\beta,\xi\}\to 0$, then, intuitively, a possible strategy to converge e to zero is the following: make the variable ρ converge to a positive value if $cos\beta \to 1$ as $t \to \infty$; for the case where $cos\beta \to -1$ (see Remark 6.2) make ρ converge to a negative value. In order to apply this strategy, a new error variable

$$z_2 = \begin{cases} \rho + k_1 & \text{if } (q, \beta, \xi)' \in \mathcal{R}_\delta, \\ \rho - k_1 & \text{otherwise}, \end{cases}$$

is defined. The region $\mathcal{R}_\delta$ is given by

$$\mathcal{R}_\delta = \left\{ q = \left(\delta, \sigma \frac{\sin\beta}{\beta}, z_1, z_2\right)', \beta, \xi : \, \|q\| \le \delta_q, |\sin\beta| \le \delta_\beta < 1, \, \cos\beta < 0, \, |\xi| \le \delta_\xi \right\},$$

where δ_q, δ_β, and δ_ξ are positive constants. These constants are chosen so that $\mathcal{R}_\delta$ is an invariant set as will be latter shown.

Consider now a third candidate Lyapunov function given by

$$V_3 = V_2 + \frac{1}{2} z_2^2.$$

Computing its time derivative gives

$$\dot{V}_3 = -k_2\delta^2 - k_3 z_1^2 + z_2 \Big[\frac{1}{m_u e} \big(\tau_u + m_v v r - d_u u\big) + \rho^2 \cos\beta + \rho\xi \sin\beta \Big].$$

Now, choosing the control input

$$\tau_u = -m_v v r + d_u u - m_u e \big[\rho^2 \cos\beta + \rho\xi \sin\beta + k_4 z_2\big], \tag{6.14}$$

the time derivative of V_3 becomes

$$\dot{V}_3 = -k_2\delta^2 - k_3 z_1^2 - k_4 z_2^2 \le 0, \tag{6.15}$$

that is, $\dot{V}_3$ is negative semidefinite.

Step 5. (Switching control law) So far, it has been assumed that the AUV will never start at or reach[1] the position $x = y = 0$ in finite time, because the polar representation (6.1) and consequently the control law described above are not defined at $e = 0$. To deal with this situation, a switching control law must be introduced at this stage. A possible solution is to make

$$\tau_u = 0, \tag{6.16a}$$

$$\tau_r = -k_r r - k_\psi \psi, \tag{6.16b}$$

where k_r and k_ψ are positive constants, when $e = 0$. The motivation for this control law can be simply understood by noticing that it aims at rotating the vehicle in place under the action of the proportional and derivative terms $k_\psi \psi$ and $k_r r$, respectively. The complete control law is thus given by

$$\tau = \begin{bmatrix} \tau_u \\ \tau_r \end{bmatrix} = \begin{cases} (6.8), (6.14) & e \neq 0 \\ (6.16) & e = 0 \end{cases} \tag{6.17}$$

[1]In fact, it will be shown later that if the initial condition is $e(t_0) \neq 0$ than this situation will never arise.

6.3 Convergence analysis

This section proves convergence to zero of the trajectories of the closed-loop system consisting of equations (2.17), (2.18), and (6.17). The following theorem establishes the main result.

Theorem 6.1 *Consider the closed-loop nonlinear invariant system Σ described by (2.17), (2.18), and (6.17). Consider also the set*

$$\mathcal{R}(\delta_u, \delta_v, k_1) = \Big\{(u, v, e) \in \mathbb{R}^3 : e > 0,\ |\frac{u}{e} - k_1| \leq \delta_u,\ |\frac{v}{e}| \leq \delta_v\Big\}.$$

Let $\mathcal{X}(t) = (x, y, \psi, u, v, r)' = \{\mathcal{X} : [t_0, \infty) \to \mathbb{R}^6\}$, $t_0 \geq 0$, be a solution to Σ. The following property holds: Given any compact neighborhood $S \subset \mathbb{R}^4$ of $(x, y, \psi, r) = (0, 0, 2\pi n, 0), n = 0, \pm1, \pm2 \cdots$, one can find sufficiently small $k_1 > 0$, $\delta_u > 0$, and $\delta_v > 0$ such that, for any initial conditions $\mathcal{X}(t_0) = \mathcal{X}_0 \in S \cup \mathcal{R}$,

1. *$\mathcal{X}(t)$ exists, is unique and defined for all $t \geq t_0$;*
2. *$\mathcal{X}(t)$ is bounded;*
3. *The solution $\mathcal{X}(t)$ converges to an equilibrium point in $\mathcal{O}$*

$$\mathcal{O} = \big\{(x, y, \psi, u, v, r)' = (0, 0, 2\pi n, 0, 0, 0)',\ n \in \mathbb{Z}\big\}$$

as $t \to \infty$.

Proof:

Uniqueness and boundedness

Existence and uniqueness of $\mathcal{X}(t)$ is proven by showing first that chattering does not occur. From (2.17), (2.18), and (6.16) it can be easily checked that the manifold $F = \{\mathcal{X} : x = 0 \wedge y = 0 \wedge u = 0 \wedge v = 0\}$ is positively invariant and consequently the system will not satisfy any switching condition. Also for $e \neq 0$, if solutions to Σ exist then (6.15) and LaSalle's invariance principle (La Salle and Lefschetz, 1961) shows that $\rho(t)$ is bounded and $|\rho(t)| \to k_1$ which means that if e is zero then u must be also zero at the same time. The only case where this is violated is when $e(t_0) = 0$ and $\{u(t_0), v(t_0)\} \neq 0$. In this case, from (2.17) it can be seen that for $t = t_0 + \delta$ with $\delta > 0$, $e(t_0 + \delta) \neq 0$, *i.e.*, the control system will switch to the case $e \neq 0$. The other switching scenario that can happen corresponds to the case where (q, β, ξ) enters $\mathcal{R}_\delta$ for the first time. It will be proved later that $\mathcal{R}_\delta$ is an invariant set. Thus, it can be concluded that for any initial condition $\mathcal{X}_0$ there occurs at most two switchings, and therefore the closed-loop system Σ has a finite number of discontinuities. Using (Hale, 1980, Theorem 5.3, Section I.5), a unique solution to Σ exists over a maximal interval $[0, t_f)$. Now, it remains to prove that the maximum interval of existence is infinite, that is, $t_f = \infty$. For $e \neq 0$, solutions of (6.4) and (5.4) exist. Also, from (6.15) one can conclude that $\delta(t)$, $\sigma(t)$, $z_1(t)$, $z_2(t)$, and $\rho(t)$ are bounded when $\mathcal{X}_0 \notin F$. Thus, the above variables are well defined on the infinite interval. To prove boundedness of $\xi(t)$, consider the positive function

$$V_\xi = \frac{1}{2}\xi^2.$$

The derivative of V_ξ along the trajectories of the system Σ is given by

$$\dot{V}_\xi = -d(t)\xi^2 + \xi^3 \sin\beta + f(t)\xi,$$

where

$$d(t) = \frac{d_v}{m_v} - \frac{m_u}{m_v}\frac{\gamma}{\gamma-1}(k_1+z_2)\cos\beta - (k_1+z_2)\cos\beta, \tag{6.18}$$

$$f(t) = -\frac{m_u}{m_v}(k_1+z_2)\frac{\gamma}{\gamma-1}\Big[k_\sigma\sigma(k_1+z_2)\frac{\sin\beta}{\beta} + (k_1+z_2)\sin\beta + k_2\delta + z_1\Big]. \tag{6.19}$$

Suppose there exist two positive constants, r_ξ and $\underline{d}$ such that for all $t \geq t_0$, $|\xi(t)| \leq r_\xi$ and $d(t) - r_\xi \geq \underline{d}$. Then,

$$\dot{V}_\xi \leq -2\underline{d}V_\xi + |f(\cdot)|\sqrt{2V_\xi}.$$

Hence, performing the change of variables $W = \sqrt{V_\xi}$ to obtain a linear differential inequality and using the fact that $\dot{W} = \frac{\dot{V}_\xi}{2\sqrt{V_\xi}}$, it follows, when $V_\xi \neq 0$, that

$$\dot{W} \leq -\underline{d}\,W + \frac{\sqrt{2}}{2}|f(\cdot)|. \tag{6.20}$$

When $V_\xi = 0$, it can be shown (see notes in Section 6.7) that the upper right-hand derivative D^+W satisfies $D^+W \leq \frac{\sqrt{2}}{2}|f(\cdot)|$ and consequently inequality (6.20) is satisfied for all values of V_ξ. Thus, by the comparison lemma [(Khalil, 1996), lemma 2.5], W satisfies the inequality

$$W(t) \leq W(t_0)e^{-\underline{d}(t-t_0)} + \frac{\sqrt{2}}{2}\int_{t_0}^{t} e^{-\underline{d}(t-\tau)}|f(\tau)|d\tau,$$

and consequently

$$\begin{aligned}
|\xi(t)| &\leq |\xi(t_0)|e^{-\underline{d}(t-t_0)} + \int_{t_0}^{t} e^{-\underline{d}(t-\tau)}|f(\tau)|d\tau \\
&\leq |\xi(t_0)|e^{-\underline{d}(t-t_0)} + \frac{1}{\underline{d}}\left[1 - e^{-\underline{d}(t-t_0)}\right]\bar{f}(t_0) \\
&\leq \max\left\{|\xi(t_0)|, \frac{1}{\underline{d}}\bar{f}(t_0)\right\},
\end{aligned} \tag{6.21}$$

where $\bar{f}(t_0) = \sup_{t\geq t_0}|f(t)|$ is finite bounded. For the inequality (6.21) to be valid the assumption considered above must be ensured. But, since $\dot{z}_2 = -k_4 z_2$, then

$$z_2(t) = \left[\rho(t_0) - k_1\right]e^{-k_4(t-t_0)}, \quad t \geq t_0$$

and therefore, from (6.18) and (6.19), it can be seen that for any compact neighborhood $S \subset \mathbb{R}^4$ of $(x, y, \psi, r) = (0, 0, 2\pi n, 0), n = 0, \pm 1, \pm 2\cdots$, one can find sufficiently small $k_1 > 0$, $\delta_u > 0$, and $\delta_v > 0$ such that, for any initial conditions $\mathcal{X}_0 \in S \cup \mathcal{R}$, the condition

$$\max\left\{|\xi(t_0)|, \frac{1}{\underline{d}}\bar{f}(t_0)\right\} \leq r_\xi \leq d(t) - \underline{d}$$

holds, thus showing that $\xi(t)$ is bounded. To prove boundedness of $e(t)$, notice that for the common case in which one has $\beta(t) \to \{\beta = 2\pi n, n = 0, \pm 1, \pm 2, \cdots\}$, $\rho(t) \to k_1$ as $t \to \infty$. Thus, there exists a finite time $T \geq t_0 \geq 0$ such that for all $t \geq T$, $\rho\cos\beta - |\xi\sin\beta| > 0$. From (6.5a)

$$\begin{aligned} e(t) &= e(t_0)\exp\Big(-\int_{t_0}^{t}\big[\rho(\tau)\cos\beta(\tau) + \xi(\tau)\sin\beta(\tau)\big]\,d\tau\Big) \\ &= e(t_0)\exp\Big(-\int_{t_0}^{T}\big[\rho(\tau)\cos\beta(\tau) + \xi(\tau)\sin\beta(\tau)\big]\,d\tau\Big) \\ &\quad \exp\Big(-\int_{T}^{t}\big[\rho(\tau)\cos\beta(\tau) + \xi(\tau)\sin\beta(\tau)\big]\,d\tau\Big), \end{aligned}$$

for all $t \geq t_0$. Therefore, it can be immediately seen that $e(t)$ is bounded and $e(t) \to 0$ as $t \to \infty$. If $\beta(t) \to \{\beta = \pi + 2\pi n, n = 0, \pm 1, \pm 2, \cdots\}$, then after a finite time $t^\star \geq t_0$, $(q(t^\star), \beta(t^\star), \xi(t^\star)) \in \mathcal{R}_\delta$ and z_2 switches to $z_2 = \rho + k_1$ taking the value $z_2(t_+^\star) = 2k_1 + z_2(t_-^\star)$. Since $\mathcal{R}_\delta$ is an invariant set (which will be proved later), there exists a finite time $T \geq t^\star$ such that for all $t \geq T$, $\rho\cos\beta - |\xi\sin\beta| > 0$. Consequently, $e(t)$ is bounded and $e(t) \to 0$ as $t \to \infty$. To prove that there exit positive constants δ_q, δ_β, and δ_ξ such that for all $t \geq t^\star$

$$\big(q(t^\star), \beta(t^\star), \xi(t^\star)\big)' \in \mathcal{R}_\delta \Rightarrow \big(q(t), \beta(t), \xi(t)\big)' \in \mathcal{R}_\delta,$$

consider the positive function (6.10), but replace β by η, that is,

$$V = \gamma\frac{m_u}{m_v}k_1^2\big(1 - \cos\eta\big) + \frac{1}{2}\xi^2,$$

where $\eta = \beta - \pi$. Computing its time derivative (not necessarily restricted to the manifold E as was done in Section 6.2, step 6.2) and following the same line of reasoning, gives

$$\dot{V} \leq -\lambda_{min}(Q)(1 + \cos\eta)V + f(\cdot), \tag{6.22}$$

where inequalities (6.12) must hold, $\lambda_{min}(Q)$ denotes the minimum eigenvalue of the positive matrix

$$Q = \begin{bmatrix} \frac{1}{\gamma-1}k_1 & \frac{k_1}{2(\gamma-1)}\sqrt{\gamma\frac{m_u}{m_v}}(\cos\eta - 1) \\ \frac{k_1}{2(\gamma-1)}\sqrt{\gamma\frac{m_u}{m_v}}(\cos\eta - 1) & \frac{d_v}{m_v} - \Big(\frac{m_u}{m_v}\frac{\gamma}{\gamma-1} + 1\Big)k_1\cos\eta + \xi\sin\eta \end{bmatrix},$$

and $f(\cdot)$ is a bounded function given by

$$\begin{aligned} f(\cdot) &= \frac{\gamma}{\gamma - 1}\frac{m_u}{m_v}k_1^2 z_2\sin^2\eta \\ &\quad + \frac{m_u}{m_v}\Big[\gamma k_1^2\sin\eta + \rho\xi\Big]\frac{\gamma}{\gamma-1}\Big[k_\sigma\sigma\rho\frac{\sin\beta}{\beta} + k_2\delta + z_1\Big] \\ &\quad - \Big[\frac{\gamma}{\gamma-1}\frac{m_u}{m_v} + 1\Big]z_2\xi^2\cos\eta \\ &\quad + \frac{m_u}{m_v}\big[z_2 - 2k_1\big]\frac{\gamma}{\gamma-1}z_2\xi\sin\eta. \end{aligned} \tag{6.23}$$

Applying the comparison lemma (Khalil, 1996) to (6.22), V satisfies the inequality

$$V(t) \leq V(t^\star)e^{-\underline{d}(t-t^\star)} + \int_{t^\star}^{t} e^{-\underline{d}(t-\tau)}|f(\tau)|d\tau.$$

Thus,

$$1 + \cos\beta \leq \frac{m_v}{m_u}\frac{1}{k_1^2\gamma}\max\Big\{V(t^\star), \frac{1}{\underline{d}}\sup_{t\geq t^\star}|f(\cdot)|\Big\}. \tag{6.24}$$

Therefore, since $\xi \to 0$ and $q \to 0$, then, from (6.23) and (6.24), it can be concluded that there exist positive constants δ_q, δ_β, and δ_ξ such that $\cos\beta < 0$ for all $t \geq t^\star$. Consequently, $\mathcal{R}_\delta$ is an invariant set.

Since $\sigma(t)$ is bounded and $\sin\beta(t) \to 0$, it follows that $\psi(t)$ is bounded. Thus, the trajectory $\mathcal{X}(t)$ is bounded for all $\mathcal{X}_0 \notin F$. When $\mathcal{X}_0 \in F$ the same conclusions can be immediately drawn. Since the trajectory $\mathcal{X}(t)$ is bounded, it exists over the infinite interval, that is, $t_f = \infty$.

Convergence

If $\mathcal{X}_0 \in F$ then $\psi \to 0$ and $r \to 0$ as $t \to \infty$. If $\mathcal{X}_0 \in (S \cup \mathcal{R})\backslash F$, then was shown $e(t) \to 0$ which implies that $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$. Moreover, since $\rho(t) \to k_1$ (or $\rho(t) \to -k_1$ if there is a time $t^\star \geq t_0$ such that $\big(q(t^\star), \beta(t^\star), \xi(t^\star)\big)' \in \mathcal{R}_\delta$), then $u(t) \to 0$ as $t \to \infty$. It remains to prove convergence of v, ψ, and r. This is done by resorting to the LaSalle's invariance principle. Define $\Omega \triangleq \{\mathcal{X} : V_3(\mathcal{X}) \leq V_3(\mathcal{X}_0) = c\}$ which is a compact and positively invariant set since $\dot{V}_3 \leq 0$. Let E be the set of all points in Ω such that $\dot{V}_3(\mathcal{X}) = 0$, that is, $E = \{\mathcal{X} \in \Omega : \delta = 0 \wedge z_1 = 0 \wedge z_2 = 0\}$. Let M be the largest invariant set contained in E. LaSalle's theorem assures that every solution starting in Ω approaches M as $t \to \infty$. Notice in this case that the set M does not reduce to the singleton 0. However, it is well known (see (Khalil, 1996, Lemma 3.1)) that any bounded solution converges to its positive limit set L^+ which must be necessarily a subset of E. To characterize the set L^+, observe that on the invariant manifold E, $\xi \to 0$ and $\sin\beta \to 0$ as was shown in step 6.2. Thus, L^+ is the origin and therefore the solution $\mathcal{X}(t)$ converges to an equilibrium point in $\mathcal{O}$. This concludes the proof of Theorem 6.1. □

Remark 6.3 The control law proposed and its convergence analysis are far more complex than those presented in chapter 5. This stems from the fact that the two following situations may occur: *i)* the initial condition can be $e_0 = 0$, and *ii)* the initial condition can be $(\beta_0, \xi_0)' \in \mathcal{O}_\pi$, or a combination of initial conditions $(\beta_0, \xi_0)'$ such that $(\beta, \xi)'$ converge to an unstable equilibrium point of $\mathcal{O}_\pi$. Another source of complexity is the drift term $\xi(t)$. It can be seen from the Ricatti equation (6.4c) that there exist initial conditions ξ_0 (with $\beta_0 \neq 0$) such that $\xi(t)$ escapes to infinity in finite time.

6.4 Adaptive nonlinear controller design

In this section the control law developed is extended to add robustness against uncertainties in the model parameters. A formal proof of convergence of the resulting adaptive regulation system is also given.

6.4.1 Control law

Consider the set of all parameters of the AUV model (2.18) concatenated in the vector

$$\Theta = \Big[m_u, m_v, m_{uv}, m_r, X_u, X_{|u|u}, N_r, N_{|r|r}, \frac{m_u}{m_v}, \frac{Y_v}{m_v}, \frac{Y_{|v|v}}{m_v}\Big]',$$

and define the parameter estimation error $\tilde{\Theta}$ as $\tilde{\Theta} = \Theta - \hat{\Theta}$, where $\hat{\Theta}$ denotes a nominal value of Θ. Consider the augmented candidate Lyapunov function

$$V_4 = \frac{1}{2}\delta^2 + \frac{1}{2}k_\sigma\sigma^2 + \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\rho}^2 + \frac{1}{2}\tilde{\xi}^2 + \frac{1}{2}\tilde{\Theta}^T P\Gamma^{-1}\tilde{\Theta}, \tag{6.25}$$

where $P = \text{diag}\Big\{\frac{1}{m_u}, \frac{1}{m_u}, \frac{1}{m_r}, \frac{1}{m_r}, \frac{1}{m_u}, \frac{1}{m_u}, \frac{1}{m_r}, \frac{1}{m_r}, 1, 1, 1\Big\}$, $\Gamma = \text{diag}\{\gamma_1, \gamma_2, ..., \gamma_{11}\}$, $\tilde{\rho} = \rho - \hat{\rho}$, and $\tilde{\xi} = \xi - \hat{\xi}$, where $\gamma_i > 0$, $i = 1, 2, ...11$ are the adaptation gains. Let the variable z_1 be now slightly modified and redefined as $z_1 = \frac{\gamma-1}{\gamma}r - \hat{\alpha}$, where

$$\hat{\alpha}(\delta, \sigma, \beta, \hat{\rho}, \hat{\xi}) = k_\sigma\sigma\hat{\rho}\frac{\sin\beta}{\beta} + \hat{\rho}\sin\beta - \hat{\xi}\cos\beta + k_2\delta. \tag{6.26}$$

Notice the inclusion of the variables $\tilde{\rho}$ and $\tilde{\xi}$ in the Lyapunov function (6.25). This was done because in the process of computing $\dot{\hat{\alpha}}$ the variables $\dot{\rho}$ and $\dot{\xi}$ must be computed and these in turn are functions of the model parameters (see equations (6.4c) and (6.5b)). The time derivative of $\dot{V}_4$ can be computed to yield

$$\begin{aligned}
\dot{V}_4 = \delta\Big[&k_\sigma\sigma\rho\frac{\sin\beta}{\beta} + \rho\sin\beta - \xi\cos\beta - \frac{\gamma-1}{\gamma}r\Big]\\
&+ z_1\Big[\frac{\gamma-1}{\gamma}\frac{1}{m_r}\big(\tau_r + m_{uv}uv - d_r r\big) - \dot{\hat{\alpha}}\Big]\\
&+ z_2\Big[\frac{1}{m_u e}\big(\tau_u + m_v vr - d_u u\big) + \rho^2\cos\beta + \rho\xi\sin\beta\Big]\\
&+ \tilde{\rho}\Big[\frac{1}{m_u e}\big(\tau_u + m_v vr - d_u u\big) + \rho^2\cos\beta + \rho\xi\sin\beta - \dot{\hat{\rho}}\Big]\\
&+ \tilde{\xi}\Big[-\frac{m_u}{m_v}\rho r - \frac{d_v}{m_v}\xi + \rho\xi\cos\beta + \xi^2\sin\beta - \dot{\hat{\xi}}\Big]\\
&- \tilde{\Theta}^T P\Gamma^{-1}\dot{\hat{\Theta}}
\end{aligned}$$

$$\begin{aligned}
= -k_2\delta^2 &+ z_1\Big[\frac{\gamma-1}{\gamma}\frac{1}{m_r}\big(\tau_r + m_{uv}uv - d_r r\big) - \dot{\hat{\alpha}} - \delta\Big]\\
&+ z_2\Big[\frac{1}{m_u e}\big(\tau_u + m_v vr - d_u u\big) + \rho^2\cos\beta + \rho\xi\sin\beta\Big]\\
&+ \tilde{\rho}\Big[\frac{1}{m_u e}\big(\tau_u + m_v vr - d_u u\big) + \rho^2\cos\beta + \rho\xi\sin\beta\\
&+ \delta k_\sigma\sigma\frac{\sin\beta}{\beta} + \delta\sin\beta - \dot{\hat{\rho}}\Big]\\
&+ \tilde{\xi}\Big[-\frac{m_u}{m_v}\rho r - \frac{d_v}{m_v}\xi + \rho\xi\cos\beta + \xi^2\sin\beta - \delta\cos\beta - \dot{\hat{\xi}}\Big]\\
&- \tilde{\Theta}^T P\Gamma^{-1}\dot{\hat{\Theta}}.
\end{aligned}$$

Motivated by the choices in steps 2 and 3, choose the control laws

$$\begin{aligned}\tau_r &= -\hat{\theta}_3 uv - \hat{\theta}_7 r - \hat{\theta}_8 |r| r + \hat{\theta}_4 \frac{\gamma}{\gamma-1}\left[\dot{\hat{\alpha}} + \delta\right] \\ &\quad - k_3 \frac{\gamma}{\gamma-1} z_1,\end{aligned} \tag{6.27a}$$

$$\begin{aligned}\tau_u &= -\hat{\theta}_2 vr - \hat{\theta}_5 u - \hat{\theta}_6 |u| u \\ &\quad - \hat{\theta}_1 e\left[\rho^2 \cos\beta + \rho\xi \sin\beta\right] - k_4\, e\, z_2,\end{aligned} \tag{6.27b}$$

and the updating laws

$$\dot{\hat{\rho}} = \delta k_\sigma \sigma \frac{\sin\beta}{\beta} + \delta \sin\beta + k_\rho \tilde{\rho}, \tag{6.28a}$$

$$\begin{aligned}\dot{\hat{\xi}} &= -\hat{\theta}_9\, \rho r + \hat{\theta}_{10}\xi + \hat{\theta}_{11}|v|\xi + \rho\xi\cos\beta \\ &\quad + \xi^2 \sin\beta - \delta\cos\beta + k_\xi \tilde{\xi},\end{aligned} \tag{6.28b}$$

that make

$$\begin{aligned}\dot{V}_4 &= -k_2\delta^2 - \frac{k_3}{m_r} z_1^2 - [z_2, \tilde{\rho}]\, Q_1\, [z_2, \tilde{\rho}]' \\ &\quad - k_\xi \tilde{\xi}^2 + \tilde{\Theta}^T P\left[Q_2 - \Gamma^{-1}\dot{\hat{\Theta}}\right],\end{aligned} \tag{6.29}$$

where $Q_1 = \begin{bmatrix} \frac{k_4}{m_u} & \frac{k_4}{2m_u} \\ \frac{k_4}{2m_u} & k_\rho \end{bmatrix}$ is a positive definite matrix if $k_\rho > \frac{k_4}{4m_u}$ and

$$\begin{aligned}Q_2 = \mathrm{diag}\Big\{&(\tilde{\rho} + z_2)(\rho^2\cos\beta + \rho\xi\sin\beta), (\tilde{\rho}+z_2)\xi r, \\ &z_1 \frac{\gamma-1}{\gamma} uv, -z_1(\dot{\hat{\alpha}} + \delta), \rho(z_2 + \tilde{\rho}), \rho(z_2+\tilde{\rho})|u|, \\ &z_1\frac{\gamma-1}{\gamma} r, z_1 \frac{\gamma-1}{\gamma}|r|r, -\tilde{\xi}\rho r, \tilde{\xi}\xi, \tilde{\xi}|v|\xi\Big\}.\end{aligned}$$

Notice in equation (6.29) how the terms containing $\tilde{\Theta}_i$ have been grouped together. To eliminate them, choose the parameter adaptation law as

$$\dot{\hat{\Theta}} = \Gamma Q_2, \tag{6.30}$$

to yield

$$\dot{V}_4 = -k_2\delta^2 - \frac{k_3}{m_r} z_1^2 - [z_2, \tilde{\rho}]\, Q_1\, [z_2, \tilde{\rho}]' - k_\xi \tilde{\xi}^2 \leq 0. \tag{6.31}$$

Thus, the complete adaptive control law is given by

$$\tau = \begin{bmatrix} \tau_u \\ \tau_r \end{bmatrix} = \begin{cases} (6.27), (6.28) & e \neq 0 \\ (6.16) & e = 0 \end{cases} \tag{6.32}$$

6.4.2 Convergence analysis

Theorem 6.2 *Consider the closed-loop nonlinear invariant system Σ_{adapt} described by (2.17), (2.18), and (6.32). Consider also the set*

$$\mathcal{R}(\delta_u, \delta_v, k_1) = \left\{(u, v, e) \in \mathbb{R}^3 : e > 0, |\frac{u}{e} - k_1| \leq \delta_u,\ |\frac{v}{e}| \leq \delta_v\right\}.$$

Let $\mathcal{X}_{adapt}(t) = (x, y, \psi, u, v, r, \tilde{\rho}, \tilde{\xi}, \tilde{\theta}')' = \{\mathcal{X}_{adapt} : [t_0, \infty) \to \mathbb{R}^{19}\}$, $t_0 \geq 0$, *denote any solution of* Σ_{adapt}. *Then the following property holds: given any compact neighborhood* $S_{adapt} \subset \mathbb{R}^{17}$ *of* $(x, y, \psi, r, \tilde{\rho}, \tilde{\xi}, \tilde{\theta}') = (0, 0, 2\pi n, 0, 0, 0, 0_{1\times 11}), n = 0, \pm 1, \pm 2 \cdots$, *one can find sufficiently small* $k_1 > 0$, $\delta_u > 0$, *and* $\delta_v > 0$ *such that, for any initial conditions* $\mathcal{X}_{adapt}(t_0) \in S_{adapt} \cup \mathcal{R}$,

1. $\mathcal{X}_{adapt}(t)$ *exists, is unique and defined for all* $t \geq t_0$;
2. $\mathcal{X}_{adapt}(t)$ *is bounded;*
3. *There exist finite constants* ρ_c, ξ_c, *and* θ_c, *such that the solution* $\mathcal{X}_{adapt}(t)$ *converges to an equilibrium point in* $\mathcal{O}_{adapt}$

$$\mathcal{O}_{adapt} = \left\{(x, y, \psi, u, v, r, \tilde{\rho}, \tilde{\xi}, \tilde{\theta}')' = (0, 0, 2\pi n, 0, 0, 0, \rho_c, \xi_c, \theta_c')', \, n \in \mathbb{Z}\right\}$$

as $t \to \infty$.

Proof: From (6.25), (6.31) and by resorting to the LaSalle's invariance principle, it can be seen that $\{\tilde{\rho}, \tilde{\xi}, \tilde{\theta}\}$ is bounded and converges to a finite constant. The rest of the proof follows *mutatis mutandi* the one given in Section 6.3. □

6.5 Simulation results

This section illustrates the performance of the proposed control scheme in the presence of parametric

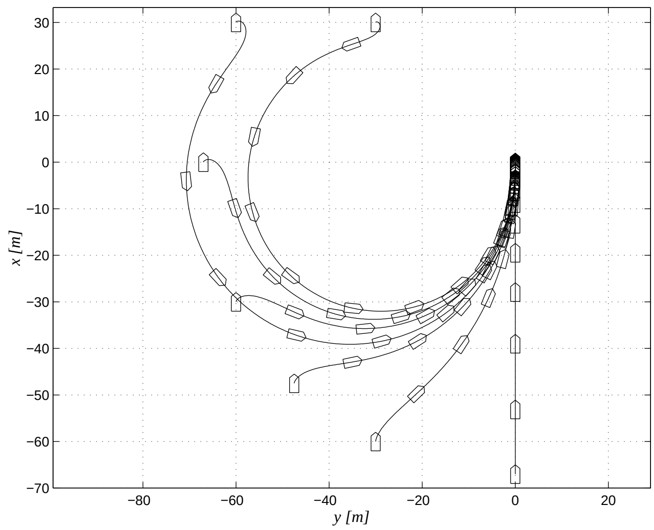

Figure 6.2: The SIRENE AUV path. The initial condition for $(\psi, u, v, r)'$ is zero.

uncertainty using computer simulations. The objective is to regulate the position and attitude of the SIRENE AUV to zero.

The control parameters were selected as following: $k_1 = 0.03$, $k_2 = 0.5$, $k_3 = 100$, $k_4 = 20$, $k_\sigma = 1$, $\gamma = 2$, $k_\rho = 10$, $k_\xi = 10$, $k_\psi = 1.6$, $k_r = 1.9$, and $\Gamma = \text{diag}(10, 10, 10, 1, 1, 2, 2, 2, 20, 20, 20)$. The initial estimates for the vehicle parameters were disturbed by 50% from their true values.

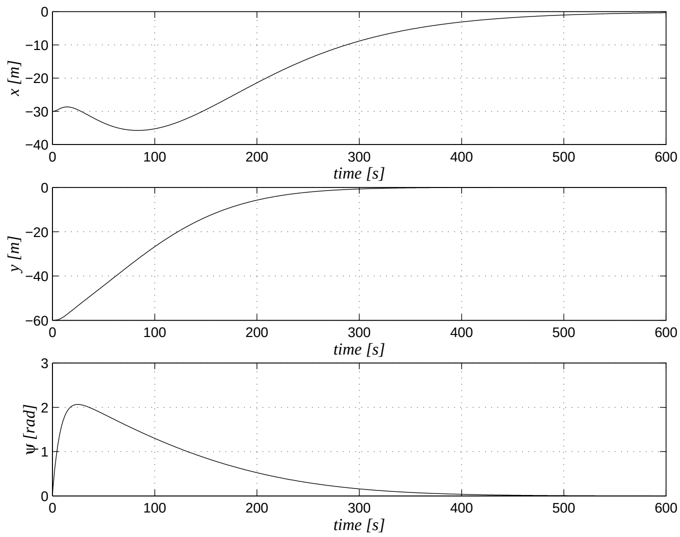

Figure 6.3: Time evolution of position variables $x(t)$ and $y(t)$ and orientation variable $\psi(t)$.

Figures 6.2-6.8 show the simulation results for the nonlinear adaptive control law (6.32). Figure 6.2 illustrates the vehicle trajectory in the xy-plane for different initial conditions in (x, y). Figures 6.3-6.8 display the time responses of the relevant state space variables for the initial condition $(x_0, y_0, \psi_0, u_0, v_0, r_0) = (-30m, -60m, 0, 0, 0, 0)$. Notice how, in spite of parameter uncertainty and the drift term (see the sway velocity activity in Figure 6.4), the vehicle converges asymptotically to the origin with a "natural", smooth trajectory. Notice also that the variables ρ, ξ and the estimated parameter $\hat{\theta}$ are bounded, as expected. However, as Figure 6.8 shows, the estimation error $\tilde{\theta}$ does not converge to zero. This is due to the particular structure of the adaptive control system adopted that allows for asymptotic convergence of $\mathcal{X}(t)$ to $\mathcal{O}$ with values of the estimated parameter $\hat{\theta}$ different from the "true" one.

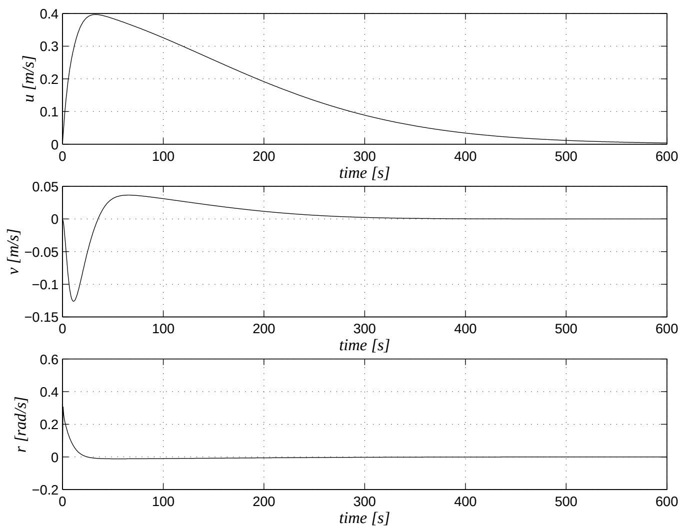

Figure 6.4: Time evolution of linear velocity in x-direction (surge) $u(t)$, linear velocity in y-direction (sway) $v(t)$, and angular velocity $r(t)$.

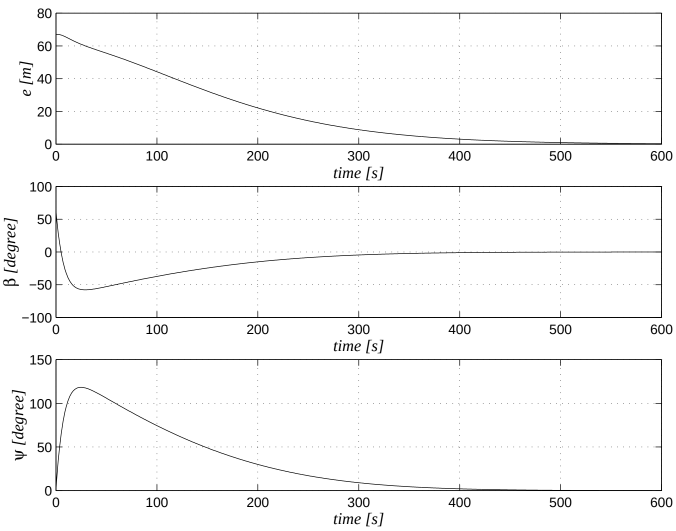

Figure 6.5: Time evolution of state variables $e(t)$, $\beta(t)$, and $\psi(t)$.

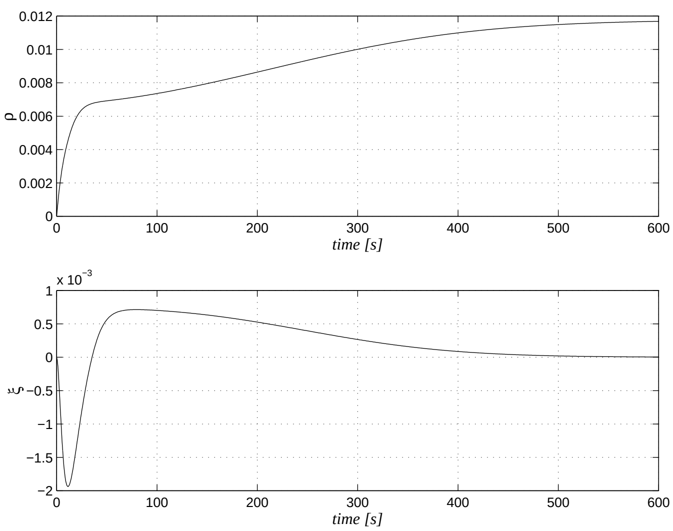

Figure 6.6: Time evolution of the variables $\rho(t)$ and $\xi(t)$.

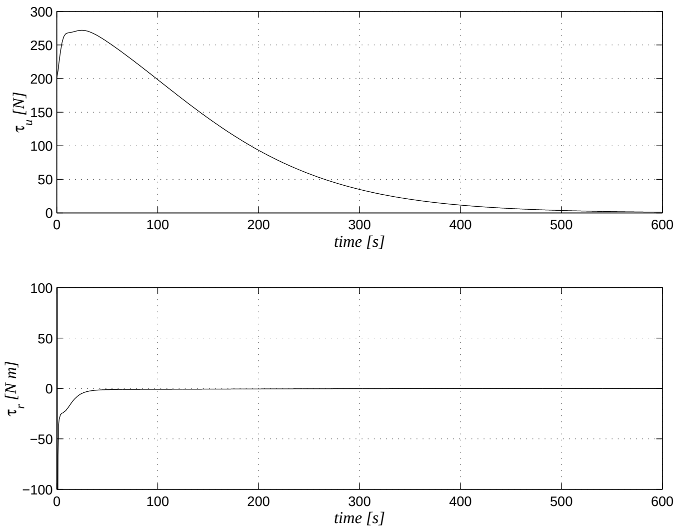

Figure 6.7: Time evolution of control signals $\tau_u(t)$ and $\tau_r(t)$.

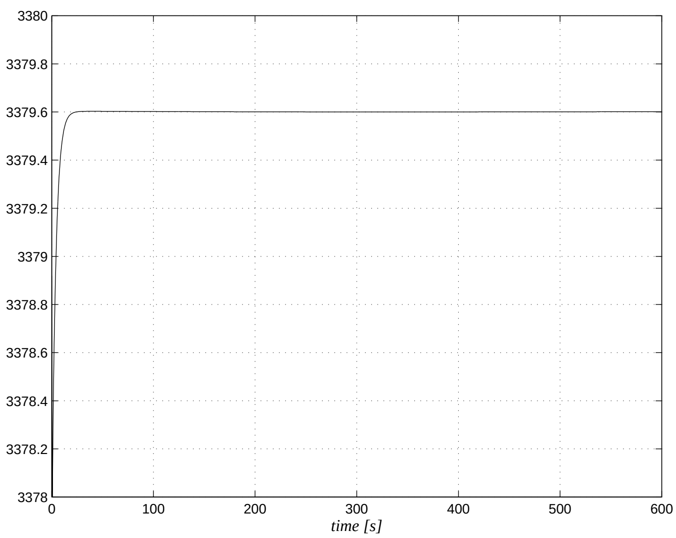

Figure 6.8: Time evolution of signal $\|\tilde{\theta}\|$.

6.6 Concluding remarks

This chapter proposed a new solution to the problem of regulating the dynamic model of an underactuated, nonholonomic AUV to a point with a desired orientation. A discontinuous, bounded, time invariant, nonlinear adaptive control law was derived that yields convergence of the trajectories of the closed loop system in the presence of parametric modeling uncertainty. Controller design relied on a non smooth coordinate transformation in the original state space, followed by the derivation of a Lyapunov-based, smooth control law in the new coordinates. Convergence of the resulting nonlinear regulation system was analyzed and simulations were performed to illustrate the behaviour of the proposed control scheme. Simulation results show that the control objectives were achieved successfully.

A typical problem that arises in the field of underwater vehicles is that of ensuring proper vehicle control in the presence of ocean currents. This important issue will be addressed in the following chapter.

6.7 Note

This section shows that the upper right-hand derivative D^+W (see inequality (6.20)) satisfies $D^+W \leq \frac{\sqrt{2}}{2}|f(\cdot)|$. Consider the following two properties (Khalil, 1996):

1. if $W(t)$ is differentiable at t, then $D^+W(t) = \dot{W}(t)$;
2. if $\frac{1}{h}|W(t+h)-W(t)| \leq g(t,h), \quad \forall h \in (0,b]$ and $\lim_{h\to 0^+} g(t,h) = g_0(t)$ then $D^+W(t) \leq g_0(t)$.

Now, when $W(\xi(t)) = 0$, it can be seen that

$$\begin{aligned}
&\frac{1}{h}\big|W(\xi(t+h)) - W(\xi(t))\big| = \frac{\sqrt{2}}{2h}|\xi(t+h)| \\
&\quad = \frac{\sqrt{2}}{2h}\left|\int_t^{t+h} F(\tau,\xi(\tau))\,d\tau\right| \\
&\quad = \left|\frac{\sqrt{2}}{2}F(t,0) + \frac{\sqrt{2}}{2h}\int_t^{t+h}\big[F(\tau,\xi(\tau)) - F(t,\xi(t))\big]\,d\tau\right| \\
&\quad \leq \frac{\sqrt{2}}{2}\,|F(t,0)| + \frac{\sqrt{2}}{2h}\int_t^{t+h}\big|F(\tau,\xi(\tau)) - F(t,\xi(t))\big|\,d\tau,
\end{aligned}$$

where

$$F(t,\xi(t)) = -d(t)\xi(t) + \xi^2(t)\sin\beta + f(t).$$

Since $F(t,\xi(t))$ is a continuous function of t, given any $\epsilon > 0$ there is $\delta > 0$ such that for all $|\tau - t| < \delta$, $\big|F(\tau,\xi(\tau)) - F(t,\xi(t))\big| < \epsilon$. Therefore, for all $h < \delta$,

$$\frac{1}{h}\int_t^{t+h}\big|F(\tau,\xi(\tau)) - F(t,\xi(t))\big|\,d\tau < \epsilon,$$

and consequently

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\big|F(\tau,\xi(\tau)) - F(t,\xi(t))\big|\,d\tau = 0.$$

Therefore, according to property 2 of the upper right-hand derivative, it can be concluded that $D^{+}W \leq \frac{\sqrt{2}}{2}|f(\cdot)|$.

Chapter 7

Dynamic positioning of an underactuated AUV in the presence of a constant unknown ocean current disturbance

In practice, an AUV must often operate in the presence of unknown ocean currents. Interestingly enough, even for the case where the current is constant, the problem of regulating an underactuated AUV to a desired point with an arbitrary desired orientation does not have a solution. In fact, if the desired orientation does not coincide with the direction of the current, normal control laws will yield one of two possible behaviors: *i)* the vehicle will diverge from the desired target position, or *ii)* the controller will keep the vehicle moving around a neighborhood of the desired position, trying insistently to steer it to the given point, and consequently inducing an oscillatory behavior.

Another problem that extends the previous one is that of designing a guidance scheme to achieve way-point tracking before the AUV stops at the final goal position. The AUV can then be made to track a predefined reference path that is specified by a sequence of way points. Way-point tracking can in principle be done in a number of ways. Most of them have a practical flavor and lack a solid theoretical background. Perhaps the most widely known is so-called line-of-sight scheme (Healey and Lienard, 1993). In this case, vehicle guidance is simply done by issuing heading reference commands to the vehicle's steering system so as to align the vehicle along the line of sight between the present position of the vehicle and the way-point to be reached. Tracking of the reference command is done via a properly designed autopilot. Notice, however, that the separation of guidance and autopilot functions may not yield stability.

Motivated by the above considerations, this chapter addresses the problem of dynamic positioning and way-point tracking of underactuated autonomous underwater vehicles (AUVs) in the presence of constant unknown ocean currents and parametric model uncertainty. A nonlinear adaptive controller is proposed that steers the AUV to track a sequence of points consisting of desired positions (x, y) in a inertial reference frame, followed by vehicle positioning at the final point. To tackle the positioning problem, the approach considered here is to drop the specification on the final desired orientation and to use this extra degree of freedom to force the vehicle to converge to the desired point. Naturally, the orientation of the vehicle at the end will be aligned with the direction of the current. The nonlinear adaptive controller proposed yields convergence of the trajectories of the closed loop system in the presence of a constant unknown ocean current disturbance and parametric model uncertainty. Controller design relies on a non smooth coordinate transformation in the original state space followed by the derivation of a Lyapunov-based, adaptive, control law in the new coordinates and an exponential observer for the ocean current disturbance. For the sake of clarity of presentation, the controller is first derived at the kinematic level, assuming that the ocean current disturbance is known. Then, an observer is designed and convergence of the resulting closed loop system is analyzed. Finally, resorting to integrator backstepping and Lyapunov techniques (Krstić *et al.*, 1995), a nonlinear adaptive controller is developed that extends the kinematic controller to the dynamic case and deals with model parameter uncertainties. Simulation results are presented and discussed.

The organization of this chapter is as follows: Section 7.1 formulates the problem of vehicle dynamic positioning and way-point tracking in the presence of a constant unknown ocean current disturbance and parametric model uncertainty. In Section 7.2, a solution to the dynamic positioning problem is proposed in terms of a nonlinear adaptive control law. Convergence of the resulting closed loop system is formally analyzed. Section 7.3 extends the strategy proposed in the previous section (to position the SIRENE AUV at the origin) to actually force the AUV to track a sequence of points consisting of desired positions (x, y) in a inertial reference frame before it converges to the finally desired point. Section 7.4 evaluates the performance of the control algorithms developed using computer simulations. Finally, Section 7.5 contains some concluding remarks.

7.1 Control problem formulation

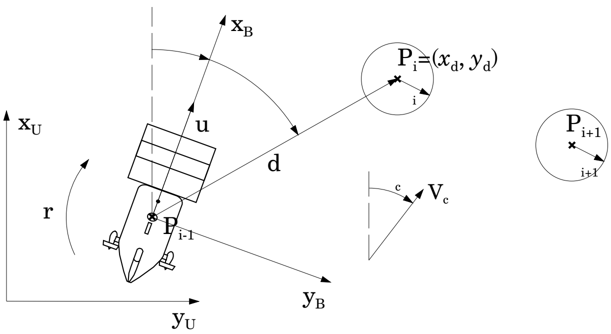

Figure 7.1: Coordinate Transformation.

Observe Figure 7.1. The problem considered in this paper can be formulated as follows:

Consider the underactuated autonomous underwater vehicle with the kinematic and dynamic equations given by (2.17) *and* (2.20)*. Let* $p = \{p_1, p_2, \ldots, p_n\}$*;* $p_i = (x_i, y_i)$*,* $i = 1, 2, \cdots, n$ *be a given sequence of points in* $\{U\}$*. Associated with each* p_i*;* $i = 1, 2, \cdots, (n-1)$ *consider the closed ball*

$N_{\epsilon_i}(p_i)$ with center p_i and radius $\epsilon_i > 0$. Derive a feedback control law for τ_u and τ_r so that the vehicle's center of mass (x, y) converges to p_n after visiting (that is, reaching) the ordered sequence of neighborhoods $N_{\epsilon_i}(p_i)$; $i = 1, 2, \cdots, (n-1)$ in the presence of a constant unknown ocean current disturbance and parametric model uncertainty.

Notice how the requirement that the neighborhoods be visited only applies to $i = 1, 2, \cdots, (n-1)$. Notice also that for the last way-point ($i = n$), because of the physics of the problem, namely the unactuated lateral dynamics and the presence of a ocean current disturbance, the vehicle's position cannot be driven to zero in an arbitrary form. The controller must recruit the control signals for the back thrusters in a way such that the trajectory described by the vehicle to the final position counteracts the effects of the ocean current disturbance and that the sway velocity relative to the water v_r be naturally driven to zero.

7.2 Nonlinear controller design for dynamic positioning

This section proposes a nonlinear adaptive control law to regulate the motion of the underactuated AUV to a given point in the presence of a constant unknown ocean current disturbance and parametric model uncertainty. The controller is first derived at the kinematic level, that is, by assuming that the control signals are the surge velocity u_r and the yaw angular velocity r. It is also assumed that the ocean current disturbance intensity V_c and its direction ϕ_c are known. This assumption will be lifted latter.

7.2.1 Coordinate transformation

Let (x_d, y_d) be a vector in $\mathbb{R}^2$. Further let d be the vector from the origin of frame $\{B\}$ to $(x_d, y_d)'$ and e its length. Denote by β the angle measured from x_B to d. Vector $(x_d, y_d)'$ plays an important role in the development of the controller for the AUV, as explained later. Figure 7.2 illustrates the particular case where $(x_d, y_d)'$ is the origin of $\{G\}$. Consider the coordinate transformation (see Figure 7.2)

$$e = \sqrt{(x - x_d)^2 + (y - y_d)^2}, \tag{7.1a}$$

$$x - x_d = -e\cos(\psi + \beta), \tag{7.1b}$$

$$y - y_d = -e\sin(\psi + \beta), \tag{7.1c}$$

$$\psi + \beta = \tan^{-1}\left(\frac{-(y - y_d)}{-(x - x_d)}\right). \tag{7.1d}$$

In equation (7.1d), care must be taken to select the proper quadrant for β. Let the ocean current disturbance be characterized by its intensity V_c and direction ϕ_c. The kinematics equations of motion of the AUV can be rewritten in the new coordinate system to yield

$$\dot{e} = -u_r\cos\beta - v_r\sin\beta - V_c\cos(\beta + \psi - \phi_c), \tag{7.2a}$$

$$\dot{\beta} = \frac{\sin\beta}{e}u_r - \frac{\cos\beta}{e}v_r - r + \frac{V_c}{e}\sin(\beta + \psi - \phi_c), \tag{7.2b}$$

$$\dot{\psi} = r. \tag{7.2c}$$

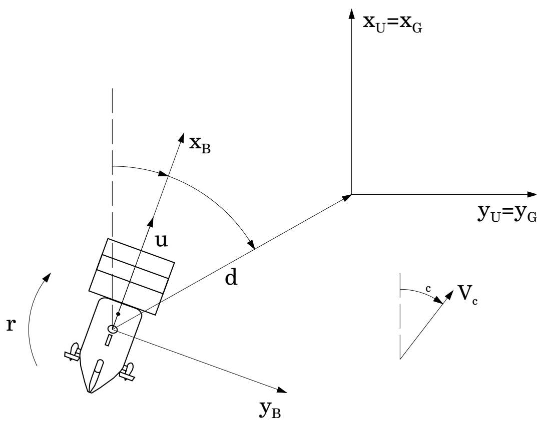

Figure 7.2: Coordinate Transformation.

Notice that the coordinate transformation (7.1) is only valid for non zero values of the variable e, since for $e = 0$ the angle β is undefined.

Throughout the section the following assumptions are considered.

Assumption 7.1 *The vehicle will not cross the singular position $e(t) = 0$ for any time $t \geq t_0$.*

Assumption 7.2 *The heading $\psi(t)$ will not converge to $\phi_c + 2\pi n, n = 0, \pm1, \pm2, \cdots$ as $t \to \infty$.*

Assumption 7.1 can be always satisfied for $t = t_0$, as will be seen later, by appropriate selection of the control parameter γ. For $t \geq t_0$, it will be proved for the kinematic case that $e(t)$ will never cross that singular position. This result strongly suggests that for the dynamic case Assumption 7.1 will also hold if the vehicle's initial conditions in velocity are small. Assumption 7.2 means that the vehicle will not converge to the other possible equilibrium configuration which corresponds to the vehicle's main axis being aligned with the ocean current. This assumption is reasonable to consider since as will be seen later, this situation correspond to an unstable equilibrium point of the closed loop system and is therefore very sensitive to perturbations.

7.2.2 Kinematic controller

At the kinematic level, the control objective consists of recruiting the linear and angular velocities u_r and r, respectively, to regulate the position (x, y) to zero. Consequently, it will be assumed that u_r and r are the control inputs. At this stage, the relevant equations of motion of the AUV are simply (7.2) and (2.20b). It is important to stress out that the dynamics of the sway velocity v must be explicitly taken into account, since the presence of this term in the kinematics equations (2.17) is not negligible (as is usually the case for wheeled mobile robots).

Returning now to the control problem, observe equations (7.2). The strategy for controller design consists basically of *i)* manipulating r to regulate β to zero (this will align x_B with vector d), and *ii)* actuating on u_r to force the vehicle position to $(x, y) = 0$. However, this must be done without driving formally e to zero, thus avoiding problems concerning the boundedness of the control inputs.

This is done by defining the coordinates x_d and y_d adequately. See the statement of the theorem below. At this stage, it is assumed that the intensity V_c and the direction ϕ_c of the ocean current disturbance are known. The following result applies.

Theorem 7.1 *Consider the nonlinear invariant system Σ_{kin} described by the AUV nonlinear model (2.17) and (2.20b) together with the control law*

$$u_r = k_1 e - k_1\gamma - V_c\cos(\psi-\phi_c), \tag{7.3a}$$

$$r = k_1\sin\beta - k_1\frac{\gamma}{e}\sin\beta + \frac{V_c}{e}\sin(\psi-\phi_c)\cos\beta - \frac{v_r}{e}\cos\beta + k_2\beta, \tag{7.3b}$$

where k_1, k_2, and γ are positive constants such that

$$\frac{d_{v_r}}{m_u} > k_1, \tag{7.4a}$$

$$k_1 > 2\frac{V_c}{\gamma}, \tag{7.4b}$$

with β and e as given in (7.1), and

$$x_d = -\gamma\cos\phi_c, \tag{7.5a}$$

$$y_d = -\gamma\sin\phi_c. \tag{7.5b}$$

Let $\mathcal{X}_{kin}(t) = (x, y, \psi, v_r)' = \{\mathcal{X}_{kin} : [t_0, \infty) \to \mathbb{R}^4\}$, $t_0 \geq 0$, be a solution of Σ_{kin}. Let assumption 7.1 and 7.2 be satisfied. Then, for any initial conditions $\mathcal{X}_{kin}(t_0) \in \mathbb{R}^4$ the control signals and the solution $\mathcal{X}_{kin}(t)$ are bounded and the position (x, y) converges to zero as $t \to \infty$.

Proof: The proof is organized as follows: First, it will be shown that β converges to zero. Then, resorting to LaSalle's invariance principle, convergence of e to γ is concluded under the assumption that the sway velocity v_r is bounded. Boundedness of v_r is proved by computing upper bounds on some relevant variables and using the comparison principle (Michel and Miller, 1977). Finally, if e converges to γ it follows from (7.1), (7.5) and Assumption 7.2 that $(x, y) \to 0$ as $t \to \infty$.

Consider the candidate Lyapunov function

$$V_{kin} = \frac{1}{2}\beta^2. \tag{7.6}$$

Computing its time derivative along the trajectories of the system Σ_{kin} gives

$$\dot{V}_{kin} = -k_2\beta^2.$$

Thus, $\beta \to 0$ as $t \to 0$. Now, consider the closed loop dynamics of e given by

$$\dot{e} = -k_1\cos\beta\, e + h_e(t) - v_r\sin\beta, \tag{7.7}$$

where $h_e(t) = k_1\gamma\cos\beta + V_c\left[\cos(\psi-\phi_c)\cos\beta - \cos(\beta+\psi-\phi_c)\right]$. Clearly, on the manifold $E = \{\mathcal{X}_{kin} : \beta = 0\}$, one can easily conclude that $e \to \gamma$ as $t \to \infty$ since $k_1 > 0$. Moreover, from (7.7), the solution $e(t)$ for $t \geq t_0$ can be expressed as

$$e(t) = \Phi_e(t, t_0)e(t_0) + \int_{t_0}^{t}\Phi_e(t,\sigma)h_e(\sigma)\,d\sigma - \int_{t_0}^{t}\Phi_e(t,\sigma)v_r\sin\beta\,d\sigma, \tag{7.8}$$

where

$$\Phi_e(t,t_0) = e^{-\int_{t_0}^{t}[k_1\cos\beta]d\sigma}. \tag{7.9}$$

Since β converges to zero, there exists a finite time $T_\beta \geq t_0 \geq 0$ such that $\cos\beta > 0$ for all $t \geq T_\beta$. Consequently,

$$\Phi_e(t,\sigma) = \begin{cases} e^{-\int_{\sigma}^{T_\beta}[k_1\cos\beta]d\tau}e^{-\int_{T_\beta}^{t}[k_1\cos\beta]d\tau} \leq \gamma_e e^{-\lambda_e(t-T_\beta)} & \text{for } \sigma < T_\beta, \\ e^{-\int_{\sigma}^{t}[k_1\cos\beta]d\tau} \leq e^{-\lambda_e(t-\sigma)} & \text{for } \sigma \geq T_\beta. \end{cases}$$

where $\gamma_e = e^{-\int_{t_0}^{T_\beta}[k_1\cos\beta]d\sigma}$ and $\lambda_e = \inf_{t\geq T_\beta} k_1\cos\beta$. Thus, if v_r is bounded, which will be proved later, it follows from (7.8) that $e(t)$ is bounded. Resorting to LaSalle's invariance principle, one can conclude that $e(t)$ converges to the largest invariant set M contained in E. Notice in this case that the set M does not reduce to the singleton 0. However, it is well known (see (Khalil, 1996, Lemma 3.1)) that any bounded solution converges to its positive limit set L^+ which must be necessarily a subset of E. To characterize the set L^+, observe that on the invariant manifold E the trajectory $(e(t)-\gamma) \to 0$ as $t \to \infty$, and therefore L^+ is the origin. This in turn implies that $e(t)$ converges to γ as $t \to \infty$.

To prove boundedness of $v_r(t)$, notice that its closed loop dynamics are given by

$$\dot{v}_r = -d(t)v_r + g(t)e + h(t), \tag{7.10}$$

where

$$\begin{aligned}
d_1(t) &= \frac{d_{v_r}}{m_v} - \frac{m_u}{m_v}k_1\cos\beta, \\
d_2(t) &= \frac{m_u}{m_v}\left[k_1\frac{\gamma}{e}\cos\beta + \frac{V_c}{e}\cos(\psi-\phi_c)\cos\beta\right], \\
d(t) &= d_1(t) + d_2(t), \\
g(t) &= -\frac{m_u}{m_v}\left[k_1^2\sin\beta + k_1k_2\beta\right], \\
h_1(t) &= -\frac{m_u}{m_v}\Big[-2k_1^2\gamma\sin\beta + k_1V_c\sin(\psi-\phi_c)\cos\beta - k_1k_2\gamma\beta \\
&\quad - V_c\cos(\psi-\phi_c)k_1\sin\beta - V_c\cos(\psi-\phi_c)k_2\beta\Big], \\
h_2(t) &= -\frac{m_u}{m_v}\Big[k_1^2\frac{\gamma^2}{e}\sin\beta - k_1\gamma\frac{V_c}{e}\sin(\psi-\phi_c)\cos\beta \\
&\quad + \frac{V_c}{e}\cos(\psi-\phi_c)k_1\gamma\sin\beta - \frac{V_c^2}{e}\cos(\psi-\phi_c)\sin(\psi-\phi_c)\cos\beta\Big], \\
h(t) &= h_1(t) + h_2(t).
\end{aligned}$$

Its solution can thus be expressed as

$$v_r(t) = \Phi_v(t,t_0)v_r(t_0) + \int_{t_0}^{t}\Phi_v(t,\sigma)g(\sigma)e(\sigma)\,d\sigma + \int_{t_0}^{t}\Phi_v(t,\sigma)h(\sigma)\,d\sigma, \tag{7.11}$$

where $\Phi_v(t,\sigma) = e^{-\int_\sigma^t d(\tau)\,d\tau}$. Notice that since k_1 and γ satisfy (7.4), then after the finite time T_β, $d(t) > 0$ for all $t \geq T_\beta$. Thus, the transition function $\Phi_v(t,\sigma)$ satisfies

$$\Phi_v(t,\sigma) = \begin{cases} e^{-\int_{\sigma}^{T_\beta}d(\tau)\,d\tau}e^{-\int_{T_\beta}^{t}d(\tau)\,d\tau} \leq \gamma_d e^{-\lambda_d(t-T_\beta)} & \text{for } \sigma < T_\beta, \\ e^{-\int_{\sigma}^{t}d(\tau)\,d\tau} \leq e^{-\lambda_d(t-\sigma)} & \text{for } \sigma \geq T_\beta, \end{cases}$$

where $\gamma_d = e^{-\int_{t_0}^{T_\beta} d(\sigma)\, d\sigma}$ and $\lambda_d = \inf_{t \geq T_\beta} d(t) > 0$. Substituting (7.8) into (7.11) gives

$$
\begin{aligned}
v_r(t) = \Phi_v(t,t_0)v_r(t_0) &+ \int_{t_0}^{t} \Phi_v(t,\sigma)h(\sigma)\, d\sigma \\
&+ \int_{t_0}^{t} \Phi_v(t,\sigma)g(\sigma)\Phi_e(\sigma,t_0)e(t_0)\, d\sigma + \int_{t_0}^{t} \Phi_v(t,\sigma)g(\sigma)\int_{t_0}^{\sigma} \Phi_e(\sigma,\tau)h_e(\tau)\, d\tau d\sigma \\
&- \int_{t_0}^{t} \Phi_v(t,\sigma)g(\sigma)\int_{t_0}^{\sigma} \Phi_e(\sigma,\tau)v_r(\tau)\sin\beta(\tau)\, d\tau d\sigma.
\end{aligned}
$$

Taking norms and computing some upper bounds, one gets

$$
\begin{aligned}
|v_r(t)| \leq \mu(t) + \gamma_d\,\gamma_e\,\bar{g} &\int_{t_0}^{T_\beta} e^{-\lambda_d(t-T_\beta)} \int_{t_0}^{\sigma} |v_r(\tau)|\, d\tau\, d\sigma \\
&+ \bar{g}\int_{T_\beta}^{t} e^{-\lambda_d(t-\sigma)}\Big[\int_{t_0}^{T_\beta} \gamma_e e^{-\lambda_e(\sigma-T_\beta)}|v_r(\tau)|\, d\tau + \int_{T_\beta}^{\sigma} e^{-\lambda_e(\sigma-\tau)}|v_r(\tau)|\, d\tau\Big]\, d\sigma,
\end{aligned}
$$

where $\mu(t)$ is given by

$$
\begin{aligned}
\mu(t) = \gamma_d e^{-\lambda_d(t-T_\beta)}|v_r(t_0)| &+ \gamma_d e^{-\lambda_d(t-T_\beta)}\bar{h} \\
&+ \frac{\bar{h}}{\lambda_d}\big[1 - e^{-\lambda_d(t-T_\beta)}\big] \\
&+ \gamma_d\,\gamma_e\,\bar{g}\,e(t_0)\,(T_\beta - t_0)\,e^{\lambda_d(t-T_\beta)} \\
&+ \gamma_e\,\bar{g}\,e(t_0)\int_{T_\beta}^{t} e^{-\lambda_d(t-\sigma)}\,e^{-\lambda_e(\sigma-T_\beta)}\, d\sigma \\
&+ \gamma_d\,\gamma_e\,\bar{g}\,\bar{h}\,e^{-\lambda_d(t-T_\beta)}\frac{(T_\beta - t_0)^2}{2} \\
&+ \bar{g}\,\bar{h}\int_{T_\beta}^{t} e^{-\lambda_d(t-\sigma)}\Big[\gamma_e(T_\beta - t_0)\,e^{-\lambda_e(\sigma-T_\beta)} + \frac{1}{\lambda_e}\big[1 - e^{-\lambda_e(\sigma-T_\beta)}\big]\Big]\, d\sigma.
\end{aligned}
$$

It can be easily checked that there exists a finite positive constant γ_μ such that $\mu(t) \leq \gamma_\mu$ for all $t \geq t_0$. The symbols $\bar{g}$ and $\bar{h}$ denote finite upper bounds of $|g(t)|$ and $|h(t)|$, respectively. Consider now the auxiliary variables

$$
\begin{aligned}
y_1(t) &= \int_{t_0}^{T_\beta} e^{-\lambda_d(t-T_\beta)} \int_{t_0}^{\sigma} |v_r(\tau)|\, d\tau\, d\sigma, \\
y_2(t) &= \int_{t_0}^{T_\beta} \gamma_e e^{-\lambda_e(t-T_\beta)}|v_r(\tau)|\, d\tau + \int_{T_\beta}^{t} e^{-\lambda_e(t-\tau)}|v_r(\tau)|\, d\tau, \\
y_3(t) &= \int_{T_\beta}^{t} e^{-\lambda_d(t-\sigma)} y_2(\sigma)\, d\sigma.
\end{aligned}
$$

Then,

$$
|v_r(t)| \leq \mu(t) + \gamma_d\,\gamma_e\,\bar{g}\,y_1(t) + \bar{g}\,y_3(t). \tag{7.12}
$$

Let

$$
z(t) = \mu(t) + \gamma_d\,\gamma_e\,\bar{g}\,y_1(t) + \bar{g}\,y_3(t) - |v_r(t)| \geq 0.
$$

The derivative of y_i, $i = 1, 2, 3$ with respect to time yields

$$\begin{aligned}
\dot{y}_1(t) &= -\lambda_d y_1(t), \\
\dot{y}_2(t) &= -\lambda_e y_2(t) + |v_r(t)| \\
&= -\lambda_e y_2(t) + \mu(t) + \gamma_d \gamma_e \bar{g}\, y_1(t) - \bar{g}\, y_3(t) - z(t), \\
\dot{y}_3(t) &= -\lambda_d y_3(t) + y_2(t).
\end{aligned}$$

Rearranging in vector form, let $\mathcal{Y}(t) = [y_1(t), y_2(t), y_3(t)]'$. Then

$$\dot{\mathcal{Y}}(t) \leq A\,\mathcal{Y}(t) + B(t), \quad \mathcal{Y}(t_0) = 0,$$

where

$$A = \begin{bmatrix} -\lambda_d & 0 & 0 \\ \gamma_d \gamma_e \bar{g} & -\lambda_e & \bar{g} \\ 0 & 1 & -\lambda_d \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ \mu(t) \\ 0 \end{bmatrix}.$$

Since all eigenvalues of the matrix A have negative real parts and $\mu(t)$ is bounded, it follows by resorting to the comparison principle that $\|\mathcal{Y}(t)\|$ is bounded. From (7.12) it can be further concluded that $|v_r(t)|$ is bounded.

To prove that (x, y) converges to zero, it remains to analyze the evolution of (ψ, v_r). Since (e, β, ψ, v_r) is bounded and $(e - \gamma, \beta)$ converges to zero as $t \to \infty$, LaSalle's theorem guarantees convergence of (ψ, v_r) to the largest invariant set M contained in $E = \{(e, \beta, \psi, v_r) \in \mathbb{R}^4, e \neq 0 : e = \gamma, \beta = 0\}$. On the manifold E, consider the closed loop dynamics of $\{\psi, v_r\}$ given by

$$\dot{\psi} = \frac{V_c}{\gamma} \sin(\psi - \phi_c) - \frac{v_r}{\gamma}, \tag{7.13a}$$

$$\dot{v}_r = -\Big[\frac{d_{v_r}}{m_v} + \frac{V_c}{\gamma}\frac{m_u}{m_v}\cos(\psi - \phi_c)\Big] v_r + \frac{m_u}{m_v}\frac{V_c^2}{\gamma}\cos(\psi - \phi_c)\sin(\psi - \phi_c), \tag{7.13b}$$

and the candidate Lyapunov function

$$V = V_c^2 \frac{m_u}{m_v}\big[1 + \cos(\psi - \phi_c)\big] + \frac{1}{2} v_r^2. \tag{7.14}$$

Computing its time derivative, gives

$$\dot{V} = -\begin{bmatrix} V_c\sqrt{\frac{m_u}{m_v}}\sin(\psi - \phi_c) \\ v_r \end{bmatrix}^T Q \begin{bmatrix} V_c\sqrt{\frac{m_u}{m_v}}\sin(\psi - \phi_c) \\ v_r \end{bmatrix},$$

where

$$Q = \begin{bmatrix} \frac{V_c}{\gamma} & -\frac{V_c}{2\gamma}\sqrt{\frac{m_u}{m_v}}\big[1 + \cos(\psi - \phi_c)\big] \\ -\frac{V_c}{2\gamma}\sqrt{\frac{m_u}{m_v}}\big[1 + \cos(\psi - \phi_c)\big] & \frac{d_{v_r}}{m_v} + \frac{V_c}{\gamma}\frac{m_u}{m_v}\cos(\psi - \phi_c) \end{bmatrix}.$$

Notice that the symmetric matrix Q is positive definite if the inequalities

$$\frac{V_c}{\gamma} > 0, \tag{7.15a}$$

$$d_{v_r} > 2\frac{V_c}{\gamma} m_u, \tag{7.15b}$$

hold. This is clearly true in view of conditions (7.4). See Remark 7.4 for the case $V_c = 0$. Therefore,

$$\begin{aligned}
\dot{V} &\leq -\lambda_{min}(Q)\left[V_c^2\frac{m_u}{m_v}\sin^2(\psi-\phi_c)+v_r^2\right]\\
&\leq -\lambda_{min}(Q)\left[V_c^2\frac{m_u}{m_v}\left[1+\cos(\psi-\phi_c)\right]\left[1-\cos(\psi-\phi_c)\right]+v_r^2\right]\\
&\leq -\lambda_{min}(Q)\left[V_c^2\frac{m_u}{m_v}\left[1+\cos(\psi-\phi_c)\right]\left[1-\cos(\psi-\phi_c)\right]+\frac{1}{2}\left[1-\cos(\psi-\phi_c)\right]v_r^2\right]\\
&\quad -\lambda_{min}(Q)\left[1-\frac{1}{2}\left[1-\cos(\psi-\phi_c)\right]\right]v_r^2\\
&\leq -\lambda_{min}(Q)\left[1-\cos(\psi-\phi_c)\right]\left[V_c^2\frac{m_u}{m_v}\left[1+\cos(\psi-\phi_c)\right]+\frac{1}{2}v_r^2\right]\\
&\leq -\lambda_{min}(Q)\left[1-\cos(\psi-\phi_c)\right]V \leq 0,
\end{aligned}$$

where $\lambda_{min}(Q)$ denotes the minimum eigenvalue of the positive matrix Q. Hence, it can be concluded that $\lim_{t\to\infty}\dot{V}(t) = 0$ which implies that $\{\sin(\psi-\phi_c), v_r\}$ converges to zero as $t\to\infty$. Thus, from (7.1b), (7.1c), (7.5), and under Assumption 7.2 one conclude that $(x,y)\to 0$ as $t\to\infty$. This concludes the proof of Theorem 7.1. □

Remark 7.1 If

$$u_r = k_1\tanh(e) - k_1\gamma - V_c\cos(\psi-\phi_c), \tag{7.16}$$

it follows directly that u_r is bounded. With this fact and using Assumption 7.1, the proof that v_r is bounded is drastically simplified by noting that replacing (7.16) and (7.3b) in (2.20b) the following expression holds: $\lim_{|v_r|\to\infty} v_r\dot{v}_r = -\infty$, which implies that v_r is bounded.

Remark 7.2 Assumption 7.1 can be always satisfied since if $e(t_0) = 0$, one can chose a different γ (see expression (7.5)) such that $e(t_0)\neq 0$. Furthermore, for any initial condition $\mathcal{X}_{kin}(t_0)\in\mathbb{R}^4; e(t_0)\neq 0$, $e(t)$ will never cross the singularity $e=0$. To show this, suppose by contradiction that $\eta = \frac{1}{e}$ is unbounded and consider the function

$$V_\eta = \frac{1}{2}\eta^2,$$

which will be unbounded by the contradiction hypothesis. Computing its time derivative yields

$$\dot{V}_\eta = k_1\cos\beta\eta^2 - a(t)\eta^3, \tag{7.17}$$

where

$$a(t) = k_1\gamma\cos\beta - v_r\sin\beta + V_c\left[\cos(\psi-\phi_c)\cos\beta - \cos(\psi-\phi_c+\beta)\right].$$

Since $\beta\to 0$, after a finite time $T\geq t_0$, the variable $a(t)$ satisfies

$$a(t) > k_1\gamma - \epsilon - v_r\sin\beta > \epsilon - |x_{v\beta}|,$$

where $k_1\gamma > \epsilon > 0$ and $x_{v\beta} = v_r\beta$.

Since

$$\dot{x}_{v\beta} = -(d_1 + d_2\eta + k_2)x_{v\beta} + g\,e\,\beta + h\,\beta,$$

then

$$x_{v\beta} = \Phi_x(t, t_0)x_{v\beta}(t_0) + \int_{t_0}^{t} \Phi_x(t, \sigma)g(\sigma)e(\sigma)h(\sigma)\beta(\sigma)\,d\sigma,$$

where $\Phi_x(t, \sigma) = e^{-\int_\sigma^t d_1 + d_2\eta + k_2}$. Clearly, $x_{v\beta}$ is bounded. Furthermore, since $\eta > 0$, $\beta \to 0$ and $\eta \to \infty$, it follows that after a finite time $T \geq t_0$, $a(t)$ will be positive for all $t \geq T$. Consequently, from (7.17) it follows that V_η is bounded, which is a contradiction. Therefore one can conclude that η is bounded.

Remark 7.3 If Assumption 7.2 does not hold, then it is possible that (x, y) will not converge to the origin but to the point $(-2\gamma\cos\phi_c, -2\gamma\sin\phi_c)$ instead. This case, to force convergence to zero, the sign of (x_d, y_d) must be changed (see equations (7.5)). Observe also that on the manifold $(e, \beta, v_r) = (\gamma, 0, 0)$, the dynamics of ψ are given by

$$\dot{\psi} = \frac{V_c}{\gamma}\sin(\psi - \phi_c),$$

which, clearly, are unstable near the equilibrium point $\psi = \phi_c$.

Remark 7.4 If $V_c = 0$, it can be easily concluded that $(e, \beta, v_r) \to (\gamma, 0, 0)$ as $t \to \infty$. From (7.2c) and (7.3b) it follows also that $\dot{\psi}$ converges to zero and ψ will converge to an equilibrium value which depends on the initial conditions. To force (x, y) to converge to zero, γ must be zero. One way to solve this problem for the general case is to choose γ as a function of V_c, *i.e.*, $\gamma = f(V_c)$, with $f(0) = 0$.

7.2.3 Observer design

In this section an observer is proposed to estimate the ocean current disturbance and convergence of the resulting closed loop system is analyzed.

Let v_{c_x} and v_{c_y} denote the components of the ocean current disturbance expressed in $\{U\}$. Then, the kinematic equations (2.17) (for position) can be rewritten as

$$\begin{aligned}\dot{x} &= u_r\cos\psi - v_r\sin\psi + v_{c_x},\\ \dot{y} &= u_r\sin\psi + v_r\cos\psi + v_{c_y},\end{aligned}$$

A simple observer for the component v_{c_x} of the current is

$$\dot{\hat{x}} = u_r\cos\psi - v_r\sin\psi + \hat{v}_{c_x} + k_{x_1}\tilde{x}, \tag{7.18a}$$

$$\dot{\hat{v}}_{c_x} = k_{x_2}\tilde{x}, \tag{7.18b}$$

where $\tilde{x} = x - \hat{x}$. Clearly, the estimate errors $\tilde{x}$ and $\tilde{v}_{c_x} = v_{c_x} - \hat{v}_{c_x}$ are asymptotically exponentially stable if all roots of the characteristic polynomial $p(s) = s^2 + k_{x_1}s + k_{x_2}$ associated with the system

$$\begin{bmatrix}\dot{\tilde{x}}\\ \dot{\tilde{v}}_{c_x}\end{bmatrix} = \begin{bmatrix}-k_{x_1} & 1\\ -k_{x_2} & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{v}_{c_x}\end{bmatrix}$$

have strictly negative real parts.

The observer for the component v_{c_y} can be written in an analogous manner, that is,

$$\dot{\hat{y}} = u_r \sin\psi + v_r \cos\psi + \hat{v}_{c_y} + k_{y_1}\tilde{y}, \tag{7.19a}$$

$$\dot{\hat{v}}_{c_y} = k_{y_2}\tilde{y}, \tag{7.19b}$$

where $\tilde{y} = y - \hat{y}$ and $\hat{v}_{c_y}$ is the estimation of v_{c_y}.

Consider now the following lemma which will be useful for establishing the convergence of the closed loop system.

Lemma 7.1 *Suppose that the ocean current disturbance is constant. Consider the observer system (7.18)-(7.19), where the gains k_{x_1}, k_{x_2}, k_{y_1}, and k_{y_2} are chosen such that the observer system is asymptotically stable. Define the variables $\hat{V}_c$ and $\hat{\phi}_c$ as the module and argument of the vector $[\hat{v}_{c_x}, \hat{v}_{c_y}]$, respectively. Consider also that $\hat{\phi}_c$ is computed to be a continuous signal with respect to time. Then, the variables $\tilde{v}_{c_x}$, $\tilde{v}_{c_y}$, $\hat{v}_{c_x}$, $\hat{v}_{c_y}$, $\hat{V}_c$, $\hat{\phi}_c$, $\tilde{V}_c = V_c - \hat{V}_c$, $\tilde{\phi}_c = \phi_c - \hat{\phi}_c$, $\dot{\tilde{V}}_c = -\dot{\hat{V}}_c$, and $\dot{\tilde{\phi}}_c = -\dot{\hat{\phi}}_c$ are bounded. Moreover, the errors $\tilde{v}_{c_x}$, $\tilde{v}_{c_y}$, $\tilde{V}_c$, $\tilde{\phi}_c$, $\dot{\tilde{V}}_c$, and $\dot{\tilde{\phi}}_c$ converge to zero as t goes to infinity.*

Theorem 7.2 *Consider the nonlinear invariant system $\Sigma_{kin+Obs}$ consisting of the nonlinear AUV model (2.17), (2.20b), the current observer (7.18)-(7.19) and the control law*

$$u_r = k_1 e - k_1\gamma - \hat{V}_c \cos(\psi - \hat{\phi}_c), \tag{7.20a}$$

$$r = k_1 \sin\beta - k_1\frac{\gamma}{e}\sin\beta + \frac{\hat{V}_c}{e}\sin(\psi - \hat{\phi}_c)\cos\beta - \frac{v_r}{e}\cos\beta + k_2\beta, \tag{7.20b}$$

where k_1, k_2, and γ are positive constants that satisfy conditions (7.4). Let variables β and e be given as in (7.1) where x_d and y_d are now redefined as

$$x_d = -\gamma\cos\hat{\phi}_c, \tag{7.21a}$$

$$y_d = -\gamma\sin\hat{\phi}_c. \tag{7.21b}$$

Let $\mathcal{X}_{kin+Obs}(t) = (x, y, \psi, v, \tilde{v}_{c_x}, \tilde{v}_{c_y})' = \{\mathcal{X}_{kin+Obs} : [t_0, \infty) \to \mathbb{R}^6\}$, $t_0 \geq 0$, be a solution to $\Sigma_{kin+Obs}$. Then, for any initial conditions $\mathcal{X}_{kin+Obs}(t_0) \in \mathbb{R}^6$ and under assumptions 7.1-7.2, the control signals and the solution $\mathcal{X}_{kin+Obs}(t)$ are bounded, and the position (x, y) converges to zero as $t \to \infty$.

Proof: First observe that x_d and y_d which were redefined in (7.21) are not constant. This will imply some modifications in the equations (7.2), which must be rewritten as

$$\dot{e} = -u_r\cos\beta - v_r\sin\beta - V_c\cos(\beta + \psi - \phi_c) - \gamma\,\dot{\hat{\phi}}_c\sin(\beta + \psi - \hat{\phi}_c), \tag{7.22a}$$

$$\dot{\beta} = \frac{\sin\beta}{e}u_r - \frac{\cos\beta}{e}v_r - r + \frac{V_c}{e}\sin(\beta + \psi - \phi_c) - \dot{\hat{\phi}}_c\frac{\gamma}{e}\cos(\beta + \psi - \hat{\phi}_c). \tag{7.22b}$$

The proof that follows resorts to LaSalle's invariance principle. From Theorem 7.1 it can be concluded that for any initial conditions $\mathcal{X}_{kin+Obs}(t_0)$ and under assumptions 7.1-7.2, on the manifold

$\{\tilde{v}_{c_x} = 0, \tilde{v}_{c_y} = 0\}$ the solution $\mathcal{X}_{kin+Obs}(t)$ is bounded and the position (x, y) vector converges to zero as $t \to \infty$. Observe also from Lemma 7.1 that $(\tilde{v}_{c_x}, \tilde{v}_{c_y}) \to 0$ as $t \to \infty$. Thus, to conclude the proof it remains to show that all off-manifold solutions are bounded. Starting with β, one has

$$\dot{\beta} = -k_2\beta - \left[\frac{\hat{V}_c}{e}\sin(\psi - \hat{\phi}_c) - \frac{V_c}{e}\sin(\psi - \phi_c)\right]\cos\beta - \left[\frac{\hat{V}_c}{e}\cos(\psi - \hat{\phi}_c) - \frac{V_c}{e}\cos(\psi - \phi_c)\right]\sin\beta.$$

Clearly it can be seen by using Lemma 7.1 that β is bounded. To prove that v_r is bounded, observe first that its dynamics, which are very similar to (7.10), are given by

$$\dot{v}_r = -\hat{d}(t)v_r + g(t)e + \hat{h}(t),$$

where

$$\begin{aligned}
\hat{d}(t) &= \frac{d_{v_r}}{m_v} - \frac{m_u}{m_v}\Big[k_1\cos\beta - k_1\frac{\gamma}{e}\cos\beta - \frac{\hat{V}_c}{e}\cos(\psi - \hat{\phi}_c)\cos\beta\Big],\\
\hat{h}(t) &= -\frac{m_u}{m_v}\Big[-2k_1^2\gamma\sin\beta + k_1\hat{V}_c\sin(\psi - \hat{\phi}_c)\cos\beta - k_1k_2\gamma\beta\\
&\quad - \hat{V}_c\cos(\psi - \hat{\phi}_c)k_1\sin\beta - \hat{V}_c\cos(\psi - \hat{\phi}_c)k_2\beta\\
&\quad + k_1^2\frac{\gamma^2}{e}\sin\beta - k_1\gamma\frac{\hat{V}_c}{e}\sin(\psi - \hat{\phi}_c)\cos\beta\\
&\quad + \frac{\hat{V}_c}{e}\cos(\psi - \hat{\phi}_c)k_1\gamma\sin\beta - \frac{\hat{V}_c^2}{e}\cos(\psi - \hat{\phi}_c)\sin(\psi - \hat{\phi}_c)\cos\beta\Big]
\end{aligned}$$

exhibit properties that are analogous to those of $d(t)$ and $h(t)$. Therefore, following the steps delineated in the proof of Theorem 7.1, it can be concluded that $v_r(t)$ is bounded.

Boundedness of e follows directly from

$$e(t) = \Phi_e(t, t_0)e(t_0) + \int_{t_0}^{t}\Phi_e(t, \sigma)\hat{h}_e(\sigma)\,d\sigma - \int_{t_0}^{t}\Phi_e(t, \sigma)v_r(\sigma)\sin\big(\beta(\sigma)\big)\,d\sigma,$$

where $\Phi_e(t, t_0)$ is the same as (7.9) and

$$\hat{h}_e(t) = k_1\gamma\cos\beta + \hat{V}_c\big[\cos(\psi - \hat{\phi}_c)\cos\beta - \cos(\beta + \psi - \hat{\phi}_c)\big] - \gamma\dot{\hat{\phi}}_c\sin(\beta + \psi - \hat{\phi}_c),$$

which satisfies $|\hat{h}(t)| \leq \bar{h}$ for some positive constant $\bar{h}$. Thus, $e(t)$ is bounded.

It has thus been shown that all off-manifold solutions are bounded and that $\{\tilde{v}_{c_x}, \tilde{v}_{c_y}\}$ converge to zero. Theorem 7.2 now follows from a straightforward application of LaSalle's invariance principle and the positive limit set lemma (Khalil, 1996, Lemma 3.1). □

7.2.4 Nonlinear dynamic controller design

This section indicates how the kinematic controller is extended to the dynamic case. This is done by resorting to backstepping techniques (Krstić *et al.*, 1995). Following this methodology, let u_r and r in equations (7.20a) and (7.20b) be virtual control inputs and α_1 and α_2 the corresponding virtual control laws. Introduce the error variables

$$\begin{aligned}
z_1 &= u_r - \alpha_1,\\
z_2 &= r - \alpha_2,
\end{aligned}$$

and consider the Lyapunov function (7.6) augmented with the quadratic terms z_1 and z_2, that is,

$$V_{dyn} = V_{kin} + \frac{1}{2}m_u z_1^2 + \frac{1}{2}m_r z_2^2. \tag{7.23}$$

The time derivative of V_{dyn} for $\hat{V}_c = V_c$ and $\hat{\phi}_c = \phi_c$ can be written as

$$\begin{aligned}\dot{V}_{dyn} &= -k_2\beta^2 - \beta z_2 + \beta\frac{z_1}{e}\sin\beta + m_u z_1\dot{z}_1 + m_r z_2\dot{z}_2 \\ &= -k_2\beta^2 + z_1\Big[\tau_u + m_v v_r r - d_{u_r}u_r - m_u\dot{\alpha}_1 + \frac{\sin\beta}{e}\beta\Big] \\ &\quad + z_2\Big[\tau_r + m_{uv}u_r v_r - d_r r - m_r\dot{\alpha}_2 - \beta\Big].\end{aligned}$$

Let the control law for τ_u and τ_r be chosen as

$$\tau_u = -m_v v_r r + d_{u_r}u_r + m_u\dot{\alpha}_1 - \frac{\sin\beta}{e}\beta - k_3 z_1, \tag{7.24a}$$

$$\tau_r = -m_{uv}u_r v_r + d_r r + m_r\dot{\alpha}_2 + \beta - k_4 z_2, \tag{7.24b}$$

where k_3 and k_4 are positive constants. Then,

$$\dot{V}_{dyn} = -k_2\beta^2 - k_3 z_1^2 - k_4 z_2^2, \tag{7.25}$$

which is negative definite.

Now, one can present the extension of Theorem 7.2 to the dynamic case.

Theorem 7.3 *Consider the nonlinear invariant system $\Sigma_{Dyn+Obs}$ described by the nonlinear AUV model (2.17) and (2.20) together with the observer (7.18)-(7.19) and the control law (7.24). Assume the control gains k_i, $i = 1, 2, 3, 4$ and the control variable γ are positive constants and satisfy conditions (7.4). Let variables β and e be given as (7.1) where x_d and y_d are defined in (7.21). Let $\mathcal{X}_{Dyn+Obs}(t) = (x, y, \psi, u, v, r, \tilde{v}_{c_x}, \tilde{v}_{c_y})' = \{\mathcal{X}_{Dyn+Obs} : [t_0, \infty) \to \mathbb{R}^8\}$, $t_0 \geq 0$, be a solution to $\Sigma_{Dyn+Obs}$. Then, for any initial conditions $\mathcal{X}_{Dyn+Obs}(t_0) \in \mathbb{R}^8$ and under assumptions 7.1 and 7.2, the control signals and the solution $\mathcal{X}_{Dyn+Obs}(t)$ are bounded, and the position (x, y) converges to zero as $t \to \infty$.*

Proof: From (7.25) it can be seen that β, z_1, and z_2 are bounded and converge to zero as $t \to \infty$. The rest of the proof is a simple application of the arguments used in the previous theorems. □

7.2.5 Adaptive nonlinear controller design

So far, it was assumed that the AUV model parameters are known precisely. This assumption is unrealistic. In this section the control law developed is extended to ensure robustness against uncertainties in the model parameters.

Consider the set of all parameters of the AUV model (2.20) concatenated in the vector

$$\Theta = \Big[m_u, m_v, m_{uv}, m_r, X_u, X_{|u|u}, N_r, N_{|r|r}, m_r\frac{m_u}{m_v}, m_r\frac{Y_v}{m_v}, m_r\frac{Y_{|v|v}}{m_v}\Big]',$$

and define the parameter estimation error $\tilde{\Theta}$ as $\tilde{\Theta} = \Theta - \hat{\Theta}$, where $\hat{\Theta}$ denotes a nominal value of Θ. Consider the augmented candidate Lyapunov function

$$V_{adp} = V_{dyn} + \frac{1}{2}\tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta}, \tag{7.26}$$

where $\Gamma = \text{diag}\{\gamma_1, \gamma_2, ..., \gamma_{11}\}$, and $\gamma_i > 0$, $i = 1, 2, ...11$ are adaptation gains and V_{dyn} is given in (7.23).

The time derivative of $\dot{V}_{adp}$ for $\hat{V}_c = V_c$ and $\hat{\phi}_c = \phi_c$ can be computed to yield

$$\begin{aligned}\dot{V}_{adp} = &-k_2\beta^2 + z_1\Big[\tau_u + m_v v_r r - d_{u_r} u_r - m_u \dot{\alpha}_1 + \frac{sin\beta}{\beta}\beta\Big]\\ &+ z_2\Big[\tau_r + m_{uv} u_r v_r - d_r r - m_r(\dot{\alpha}_{2_a} + \dot{\alpha}_{2_b}) - \beta\Big]\\ &- \tilde{\Theta}^T \Gamma^{-1} \dot{\hat{\Theta}}\end{aligned}$$

where $\alpha_{2_b} = -\frac{v_r}{e}\cos\beta$ and $\alpha_{2_a} = \alpha_2 - \alpha_{2_b}$. Motivated by the choices in the previous sections, choose the control laws

$$\tau_u = -\hat{\theta}_2 v_r r - \hat{\theta}_5 u_r - \hat{\theta}_6 |u_r| u_r + \hat{\theta}_1 \dot{\alpha}_1 - \frac{\sin\beta}{e}\beta - k_3 z_1, \tag{7.27a}$$

$$\begin{aligned}\tau_r = &-\hat{\theta}_3 u_r v_r - \hat{\theta}_7 r - \hat{\theta}_8 |r| r + \hat{\theta}_4 \dot{\alpha}_{2_a} + \hat{\theta}_9 \frac{u_r}{e} r\cos\beta\\ &+ \hat{\theta}_{10}\frac{v_r}{e}\cos\beta + \hat{\theta}_{11}|v_r|\frac{v_r}{e}\cos\beta + \hat{\theta}_4\frac{v_r}{e}\Big(\frac{\dot{\hat{e}}}{e}\cos\beta + \dot{\hat{\beta}}\sin\beta\Big) + \beta - k_4 z_2,\end{aligned} \tag{7.27b}$$

where $\hat{\theta}_i$ denotes the i-th element of vector $\hat{\Theta}$, and

$$\begin{aligned}\dot{\hat{e}} &= -u_r\cos\beta - v_r\sin\beta - \hat{V}_c\cos(\beta + \psi - \hat{\phi}_c) - \gamma\,\dot{\hat{\phi}}_c\sin(\beta + \psi - \hat{\phi}_c),\\ \dot{\hat{\beta}} &= \frac{\sin\beta}{e}u_r - \frac{\cos\beta}{e}v_r - r + \frac{\hat{V}_c}{e}\sin(\beta + \psi - \hat{\phi}_c) - \dot{\hat{\phi}}_c\frac{\gamma}{e}\cos(\beta + \psi - \hat{\phi}_c).\end{aligned}$$

Then,

$$\dot{V}_{adp} = -k_2\beta^2 - k_3 z_1^2 - k_4 z_2^2 + \tilde{\Theta}^T\left[Q - \Gamma^{-1}\dot{\hat{\Theta}}\right], \tag{7.28}$$

where Q is a diagonal matrix given by

$$\begin{aligned}Q = \text{diag}\Big\{&-\dot{\alpha}_1 z_1, z_1 v_r r, z_2 u_r v_r, -z_2\dot{\alpha}_{2_a} - z_2\frac{v_r}{e}\Big(\frac{\dot{\hat{e}}}{e}\cos\beta + \dot{\hat{\beta}}\sin\beta\Big), z_1 u_r,\\ &z_1|u_r|u_r, z_2 r, z_2|r|r, -u_r r\frac{z_2}{e}\cos\beta, \frac{v_r}{e}z_2\cos\beta, \frac{v_r}{e}|v_r|z_2\cos\beta\Big\}.\end{aligned}$$

Notice in equation (7.28) how the terms containing $\tilde{\Theta}_i$ have been grouped together. To eliminate them, choose the parameter adaptation law as

$$\dot{\hat{\Theta}} = \Gamma Q, \tag{7.29}$$

to yield

$$\dot{V}_{adp} = -k_2\beta^2 - k_3 z_1^2 - k_4 z_2^2 \leq 0. \tag{7.30}$$

The above results play an important role in the proof of the following theorem that extends Theorem 7.2 to deal with vehicle dynamics and model parameter uncertainty.

Theorem 7.4 *Consider the nonlinear invariant system Σ_{adp} consisting of the nonlinear AUV model (2.17) and (2.20), the current observer (7.18)-(7.19), and the adaptive control law (7.27), (7.29), where the adaptation gain Γ is a (11×11) diagonal positive definite matrix. Assume the control gains k_i, $i = 1, 2, 3, 4$ and the control variable γ are positive constants and satisfy conditions (7.4). Let variables β and e be given as in (7.1) where x_d and y_d are defined in (7.21).*

Let $\mathcal{X}_{adp}(t) = (x, y, \psi, u, v, r, \tilde{v}_{c_x}, \tilde{v}_{c_y}, \tilde{\Theta}')' = \{\mathcal{X}_{adp} : [t_0, \infty) \rightarrow \mathbb{R}^{19}\}$, $t_0 \geq 0$, be a solution to Σ_{adp}. Then, for any initial conditions $\mathcal{X}_{adp}(t_0) \in \mathbb{R}^{11}$ and under assumptions 7.1 and 7.2, the control signals and the solution $\mathcal{X}_{adp}(t)$ are bounded, and the position (x, y) converges to zero as $t \rightarrow \infty$.

Proof: From (7.26), (7.30) and by resorting to the LaSalle's invariance principle, it can be seen that $\tilde{\Theta}$ is bounded and converges to a finite constant. The rest of the proof follows *mutatis mutandi* the arguments introduced during the proofs in the previous sections. □

7.3 Tracking a sequence of points

This section proposes a nonlinear adaptive control law to steer the underactuated AUV through a sequence of neighborhoods $N_{\epsilon_i}(p_i)$; $i = 1, 2, \cdots, (n-1)$ in the presence of a constant unknown ocean current disturbance and parametric model uncertainty.

In what follows it is important to introduce the following notation. Let $\chi = (x, y)'$ and $\chi_d = (x_d, y_d)'$. Clearly, $e = \|\chi - \chi_d\|_2$. Notice that $e = e(i)$; $i = 1, 2, \cdots, (n-1)$, that is, the error depends on the way-point $\chi_d = p_i$ selected. Let $\mathcal{Z}_n$ be the set $\mathcal{Z}_n = \{1, 2, \ldots, n\}$. Consider the piecewise constant signal $\sigma : [t_0, \infty) \rightarrow \mathcal{Z}_n$ that is continuous from the right at every point and defined recursively by

$$\sigma = \eta(\chi, \sigma^-), \quad t \geq t_0 \tag{7.31}$$

where $\sigma^-(\tau)$ is equal to the limit from the left of $\sigma(\tau)$ as $\tau \rightarrow t$. Define the transition function $\eta : \mathbb{R}^3 \times \mathcal{Z}_n \rightarrow \mathcal{Z}_n$ as

$$\eta(\chi, i) = \begin{cases} i, & e = e(i) > \epsilon_i \\ i+1, & e = e(i) \leq \epsilon_i;\ i \neq n \\ n, & i = n. \end{cases} \tag{7.32}$$

In order to single out the last way-point as a desired target towards which the AUV should converge, $(x_d, y_d)'$ is formally defined as (see the previous section)

$$(x_d, y_d) = \begin{cases} p_\sigma & \text{if } \sigma < n, \\ p_\sigma - \gamma(\cos\phi_c, \sin\phi_c) & \text{if } \sigma = n. \end{cases} \tag{7.33}$$

7.3.1 Kinematic controller

At the kinematic level it will be assumed that u_r and r are the control inputs. At this stage, the relevant equations of motion of the AUV are simply (7.2) and (2.20b). The strategy for controller design consists basically of *i)* for $i = 1, 2, \cdots (n-1)$, fixing the surge velocity to a constant positive value U_d, *ii)* manipulating r to regulate β to zero (this will align x_B with vector d), and *iii)* for $i = n$ (the final target), actuating on u_r to force the vehicle to converge to the position p_n using the controller developed in the previous section. At this stage, it is assumed that the intensity V_c and the direction ϕ_c of the ocean current disturbance are known. The following result applies for the case where $i < n$.

Theorem 7.5 *Consider the sequence of points $\{p_1, p_2, \dots, p_{n-1}\}$ and the associated neighborhoods $\{N_{\epsilon_1}(p_1), N_{\epsilon_2}(p_2), \dots, N_{\epsilon_{n-1}}(p_{n-1})\}$. Let $\underline{\epsilon} = \min_{1 \le i < n} \epsilon_i$ and U_d, k_2, and $\underline{k}_2 > 0$ be positive constants. Consider the nonlinear system Σ_{kin} described by the AUV nonlinear model (2.17) and (2.20b) and assume that*

$$k_2 \ge \frac{U_d + V_c}{\underline{\epsilon}} + \underline{k}_2, \quad U_d > V_c, \quad \frac{d_{v_r}}{m_u} > \frac{U_d}{\underline{\epsilon}}. \tag{7.34}$$

Let the control law $u_r = \alpha_1$ and $r = \alpha_2$ be given by

$$\alpha_1 = U_d, \tag{7.35a}$$

$$\alpha_2 = k_2\beta + \frac{V_c}{e}\sin(\psi - \phi_c)\cos\beta - \frac{v_r}{e}\cos\beta \tag{7.35b}$$

with β and e as given in (7.1) where $(x_d, y_d)'$ is computed using (7.31)-(7.33).
Let $\mathcal{X}_{kin}(t) = (x, y, \psi, v_r)' = \{\mathcal{X}_{kin} : [t_0, \infty) \to \mathbb{R}^4\}$, $t_0 \ge 0$, be a solution to Σ_{kin}. Then, for any initial conditions $\mathcal{X}_{kin}(t_0) \in \mathbb{R}^4$ the control signals and the solution $\mathcal{X}_{kin}(t)$ are bounded. Furthermore, there are finite instants of time $t_1^m \le t_1^M \le t_2^m \le t_2^M, \dots, \le t_{n-1}^m \le t_{n-1}^M$ such that $\big(x(t), y(t)\big)'$ stays in $N_{\epsilon_i}(p_i)$ for $t_i^m \le t \le t_i^M$, $i = 1, 2, \dots, n-1$.

Proof: Consider the candidate Lyapunov function

$$V_{kin} = \frac{1}{2}\beta^2. \tag{7.36}$$

Computing its time derivative along the trajectories of system Σ_{kin} gives

$$\dot{V}_{kin} = -\beta^2\Big[k_2 - \frac{U_d}{e}\frac{\sin\beta}{\beta} - \frac{V_c}{e}\frac{\sin\beta}{\beta}\cos(\psi - \phi_c)\Big]$$

which is negative definite if k_2 satisfies condition (7.34). Thus, $\beta \to 0$ as $t \to \infty$. To prove that v_r is bounded consider its dynamic motion in closed loop given by

$$\dot{v}_r = -\Big[\frac{d_{v_r}}{m_v} - \frac{m_u}{m_v}\frac{U_d}{e}\cos\beta\Big]v_r - \frac{m_u}{m_v}U_d\Big[k_2\beta + \frac{V_c}{e}\cos\beta\sin(\psi - \phi_c)\Big], \tag{7.37}$$

Clearly, if condition (7.34) holds, then v_r is bounded since $\lim_{|v_r| \to \infty} v_r\dot{v}_r = -\infty$. Convergence of e is shown by observing that

$$\dot{e} = -U_d\cos\beta - v_r\sin\beta - V_c\cos(\beta + \psi - \phi_c).$$

Thus, since $\beta \to 0$, v_r is bounded and $U_d > V_c$ it follows that there exist a time $T \ge t_0$ and a finite positive constant α such that $\dot{e} < -\alpha$ for all $t > T$. Consequently, the vehicle position (x, y) reaches the neighborhood $N_{\epsilon_i}(p_i)$ of p_i in finite time. □

Notice that Theorem 7.5 only deals with the first $n - 1$ way-points. Steering to the last way-point can be done using the control structure proposed in Section 7.2.

7.3.2 Adaptive nonlinear controller design

Following the methodology described in Section 7.2.4 and Section 7.2.5, one arrives at the same control structure expressed in equations (7.27). Notice that for this case α_1 and α_2 are given by (7.35). The next result extends Theorem 7.5 to deal with vehicle dynamics, model parameter uncertainty, and constant unknown ocean current disturbance. The proof is omitted since it follows the same steps as the ones given in the previous section.

Theorem 7.6 *Consider the nonlinear invariant system Σ_{tsp} consisting of the nonlinear AUV model (2.17) and (2.20), the current observer (7.18)-(7.19), and the adaptive control law (7.35), (7.27), and (7.29) with V_c and ϕ_c replaced by their estimates $\hat{V}_c$ and $\hat{\phi}_c$. Assume that the control gains k_i, $i = 2, 3, 4$, and U_d are positive constants and satisfy conditions (7.34). Let the adaptation gain Γ be a (11×11) diagonal positive definite matrix. Let variables β and e be given as in (7.1) where $(x_d, y_d)'$ is computed using (7.31)-(7.33). Consider the sequence of points $\{p_1, p_2, \ldots, p_n\}$ and the associated neighborhoods $\{N_{\epsilon_1}(p_1), N_{\epsilon_2}(p_2), \ldots, N_{\epsilon_{n-1}}(p_{n-1})\}$. Let $\mathcal{X}_{tsp}(t) = (x, y, \psi, u, v, r, \tilde{v}_{c_x}, \tilde{v}_{c_y}, \tilde{\Theta}')' = \{\mathcal{X}_{tsp} : [t_0, \infty) \to \mathbb{R}^{19}\}$, $t_0 \geq 0$, be a solution to Σ_{tsp}. Then, for any initial conditions $\mathcal{X}_{tsp}(t_0) \in \mathbb{R}^6$ the control signals and the solution $\mathcal{X}_{tsp}(t)$ are bounded. Furthermore, there are finite instants of time $t_1^m \leq t_1^M \leq t_2^m \leq t_2^M, \ldots, \leq t_{n-1}^m \leq t_{n-1}^M$ such that $\big(x(t), y(t)\big)'$ stays in $N_{\epsilon_i}(p_i)$ for $t_i^m \leq t \leq t_i^M$, $i = 1, 2, \ldots, n-1$.*

7.4 Simulation results

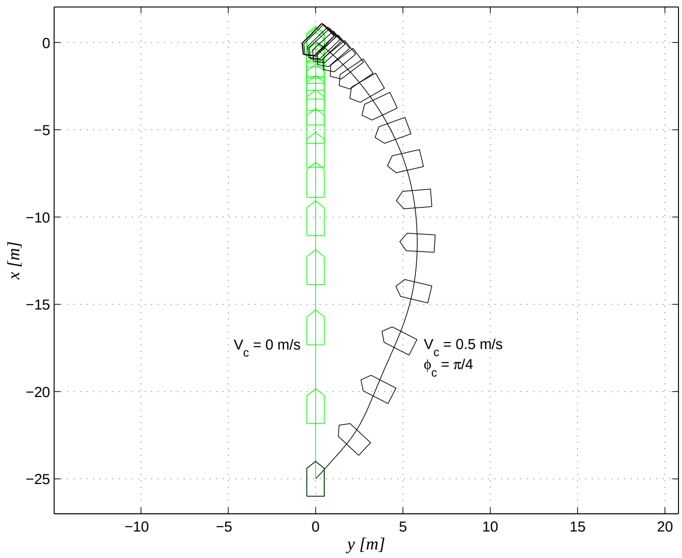

Figure 7.3: Simulation paths for the SIRENE AUV. Initial condition $(x, y, \psi, u, v, r) = (-25\,m, 0, 0, 0, 0, 0)$.

In order to illustrate the performance of the proposed control scheme for vehicle positioning in the presence of parametric uncertainty and a constant ocean current disturbance, computer simulations were carried out with a model of the SIRENE AUV. The vehicle dynamic model and its parameters nominal values can be found in Section 2.5.

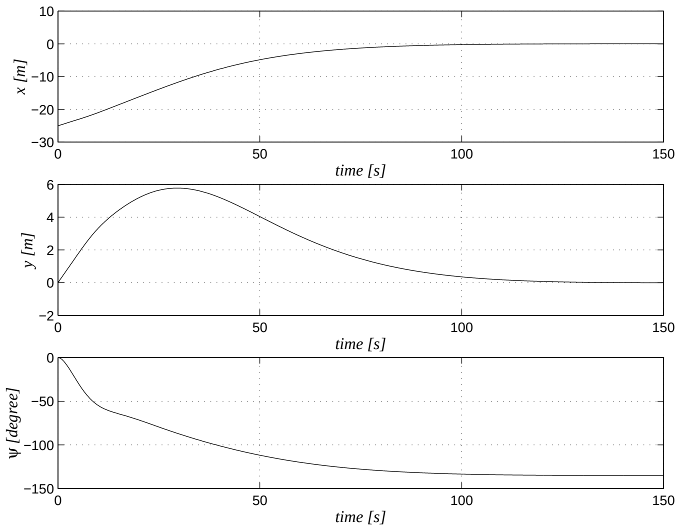

Figure 7.4: Time evolution of position variables $x(t)$ and $y(t)$ and orientation variable $\psi(t)$.

Figure 7.3 shows the resulting vehicle trajectory in the xy-plane for two simulations using the nonlinear adaptive control law (7.27), (7.29). The control parameters were selected as follows: $k_1 = 0.04$, $k_2 = 0.8$, $k_3 = 2 \times 10^3$, $k_4 = 500$, $k_{x_1} = 1.0$, $k_{x_2} = 1.0$, $k_{y_1} = 1.0$, $k_{y_2} = 1.0$, $\gamma = 15$, and $\Gamma = \mathrm{diag}(10, 10, 10, 1, 1, 2, 2, 2, 1, 0.1, .1) \times 10^3$. The parameters satisfy the constraints (7.4). The initial estimates for the vehicle parameters were disturbed by 50% from their true values. In both simulations, the initial conditions for the vehicle were $(x, y, \psi, u, v, r) = (-25\,m, 0, 0, 0, 0, 0)$. In one simulation there is no ocean current. The other simulation captures the situation where the ocean current (which is unknown from the point of view of the controller) has intensity and direction $V_c = 0.5\,m/s$, $\phi_c = \frac{\pi}{4}$ rad, respectively.

In the figure it can be seen that the vehicle converges to the desired position (the origin, in this case). Notice how in the presence of ocean current the vehicle automatically recruits the yaw angle that is required to counteract that current at the target point. Thus, at the end of the maneuver the vehicle is at the goal position and faces the current with surge velocity u_r equal to V_c. This is clearly illustrated in the Figures 7.5-7.9 that show the time responses for the case where ocean current is different from zero. Notice also form Figure 7.9 the good performance of the observer.

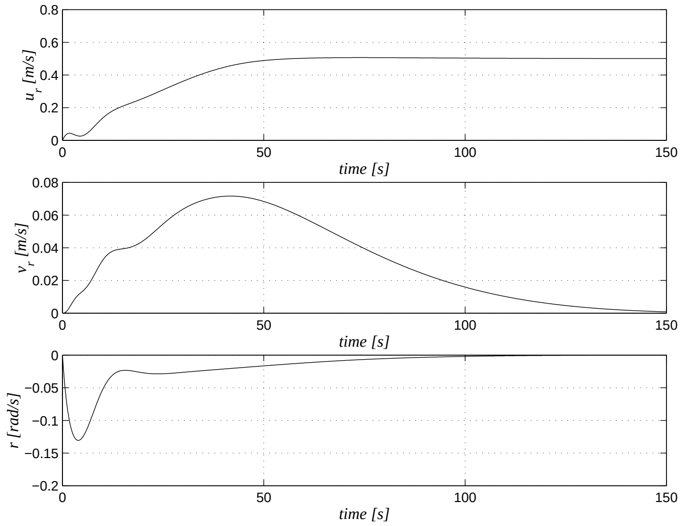

Figure 7.5: Time evolution of relative linear velocity in x-direction (surge) $u_r(t)$, relative linear velocity in y-direction (sway) $v_r(t)$, and angular velocity $r(t)$.

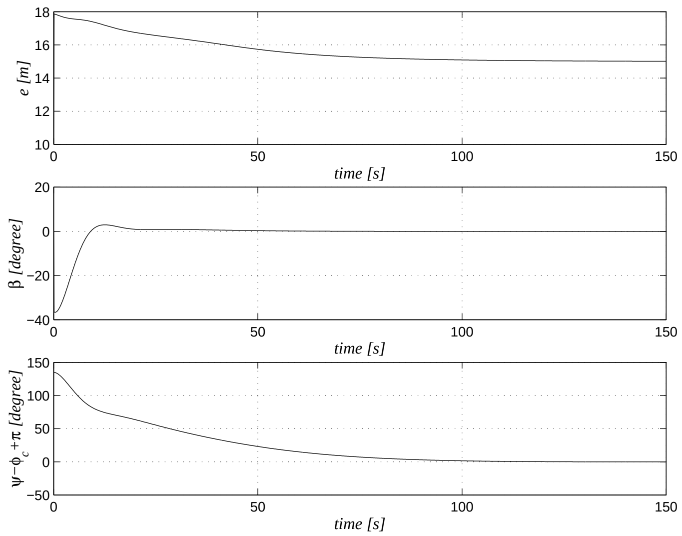

Figure 7.6: Time evolution of variables $e(t)$, $\beta(t)$ and $\psi(t) - \phi_c + \pi$.

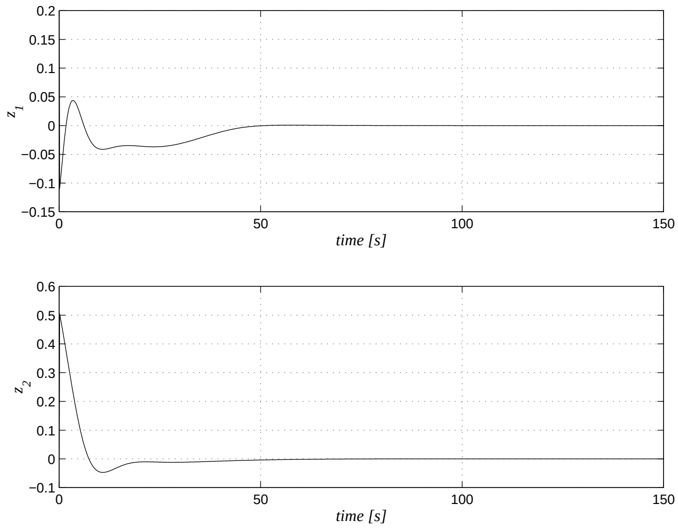

Figure 7.7: Time evolution of variables $z_1(t)$ and $z_2(t)$.

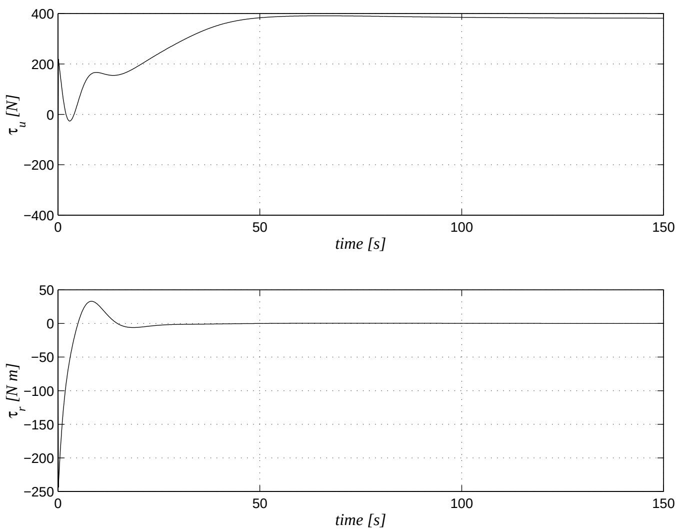

Figure 7.8: Time evolution of control signals $\tau_u(t)$ and $\tau_r(t)$.

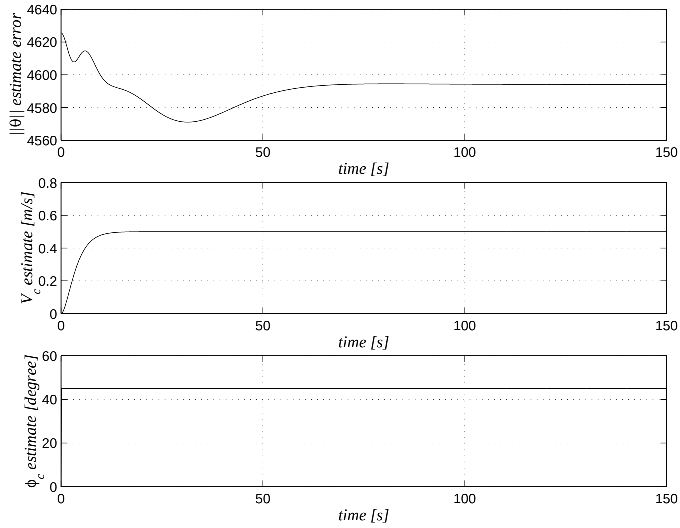

Figure 7.9: Time evolution of the signals $\|\tilde{\theta}\|$, $\hat{V}_c$, and $\hat{\phi}_c$.

Another computer simulations was carried out to illustrate the performance of the way-point tracking control algorithm derived (in the presence of parametric uncertainty and constant ocean current disturbances).

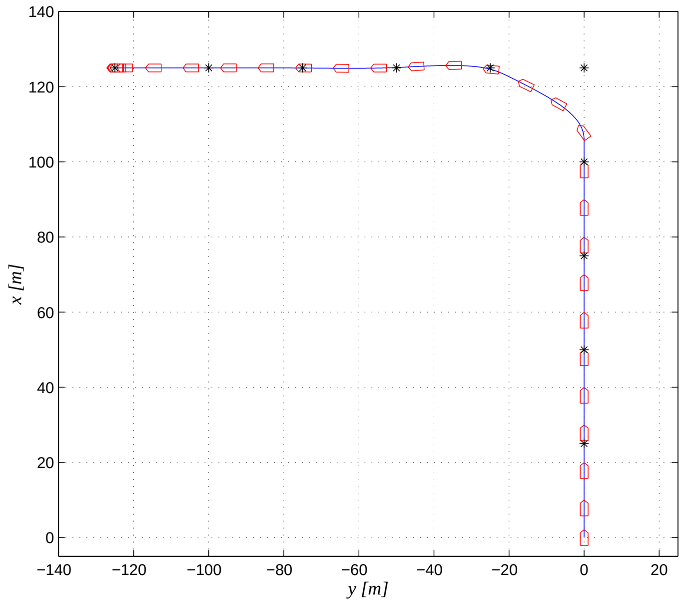

Figure 7.10: Way-point tracking with the SIRENE AUV. $U_d = 0.5\, m/s$, $V_c = \phi_c = 0$.

Figures 7.10-7.12 display the resulting vehicle trajectory in the xy-plane for three different simulations scenarios using the nonlinear adaptive control law described in Section 7.3 for $i < n$ and the controller described in Section 7.2 for $i = n$ (the last point). The control parameters (for $i < n$) were selected as following: $k_2 = 1.8$, $k_3 = 1 \times 10^3$, $k_4 = 500$, $k_{x_1} = 1.0$, $k_{x_2} = 0.25$, $k_{y_1} = 1.0$, $k_{y_2} = 0.25$, and $\Gamma = \text{diag}(10, 10, 10, 1, 1, 2, 2, 2, 1, 0.1, .1) \times 10^3$. The parameters satisfy the constraints (7.34). The initial estimates for the vehicle parameters were disturbed by 50% from their true values. The

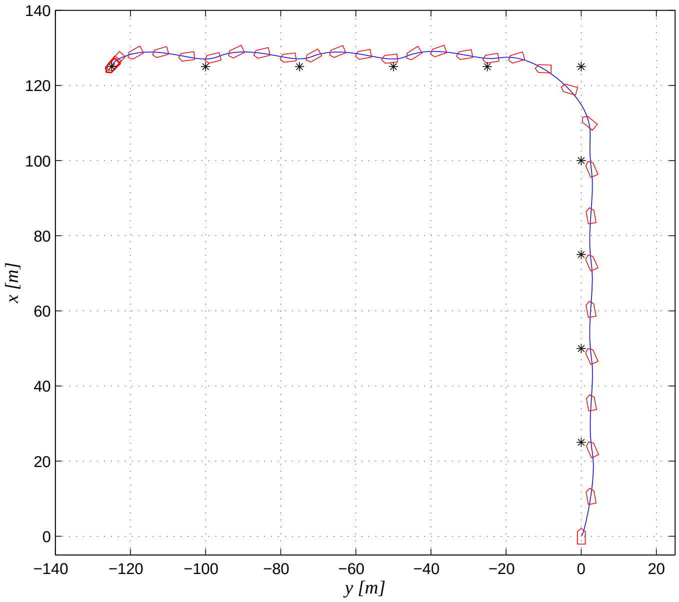

Figure 7.11: Way-point tracking with the SIRENE AUV. $U_d = 0.5\,m/s$, $V_c = 0.2\,m/s$, $\phi_c = \frac{\pi}{4}\,rad$.

sequence of points are $p = \{(25.0, 0.0), (50.0, 0.0), (75.0, 0.0), (100.0, 0.0), (125.0, 0.0), (125.0, -25.0), (125.0, -50.0), (125.0, -75.0), (125.0, -100.0), (125.0, -125.0), (125.0, -125.0)\}$. The maximum admissible deviations from p_i; $i = 1, 2, \cdots, 10$ were fixed to $\epsilon_i = 5\,m$, except for $i = 5$, where $\epsilon_5 = 20\,m$. In both simulations, the initial conditions for the vehicle were $(x, y, \psi, u, v, r) = 0$. In the first simulation (see Figure 7.10) there is no ocean current. The other two simulations capture the situation where the ocean current (which is unknown from the point of view of the controller) has intensity $V_c = 0.2\,m/s$ and direction $\phi_c = \frac{\pi}{4}$ rad, but with different values on the controller parameter U_d. See Figures 7.11 and 7.12 for $U_d = 0.5$ and $U_d = 1.0$, respectively. The figures show the influence of the ocean current on the resulting xy-trajectory. Clearly, the influence is stronger for slow forward speeds u_r. In spite of that, notice that the vehicle always reaches the sequence of neighborhoods of points $p_1, p_2, \ldots, p_{10}$ until it finally converges to the desired position $p_{11} = (125, 125)\,m$. Figures 7.13-7.15 condense the time responses of the relevant variables for the simulation with ocean current and $U_d = 0.5$.

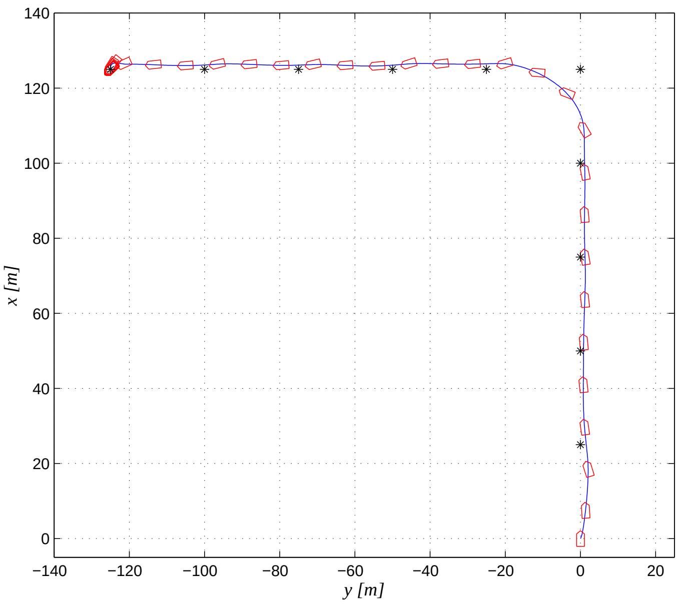

Figure 7.12: Way-point tracking with the SIRENE AUV. $U_d = 1.0\,m/s$, $V_c = 0.2\,m/s$, $\phi_c = \frac{\pi}{4}\,rad$.

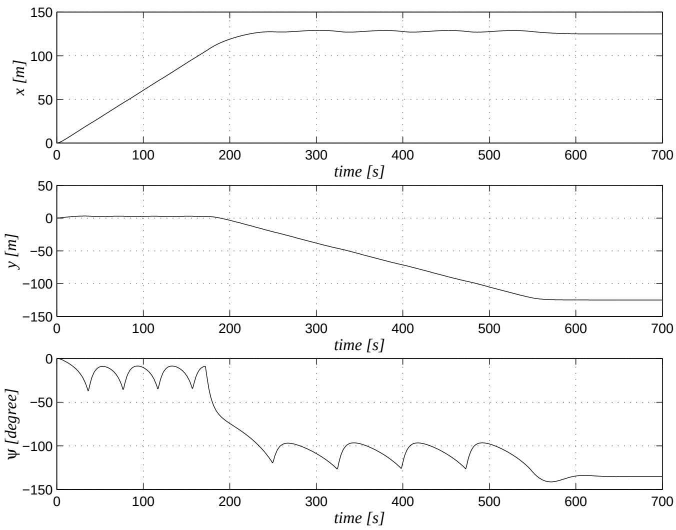

Figure 7.13: Time evolution of the position variables $x(t)$ and $y(t)$, and the orientation variable $\psi(t)$.

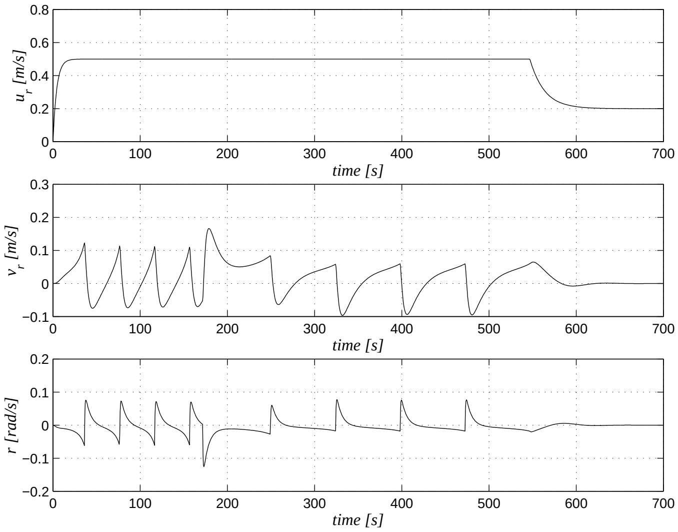

Figure 7.14: Time evolution of the relative linear velocity in x-direction (surge) $u_r(t)$, relative linear velocity in y-direction (sway) $v_r(t)$, and angular velocity $r(t)$.

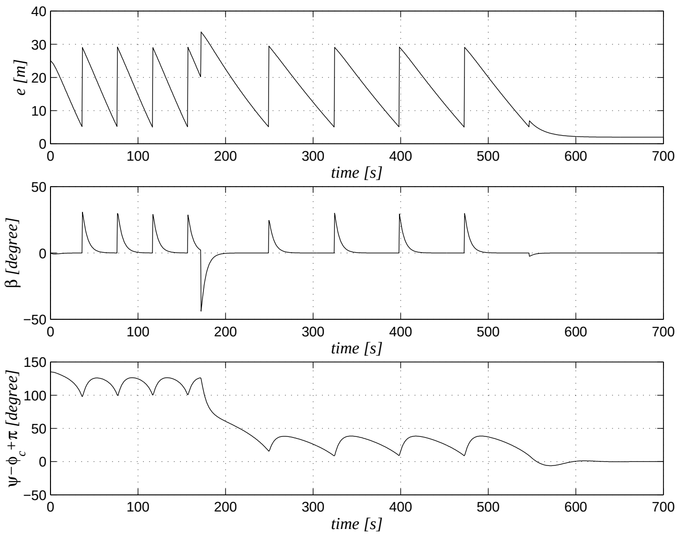

Figure 7.15: Time evolution of variables $e(t)$, $\beta(t)$, and $\psi(t) - \phi_c + \pi$.

7.5 Concluding remarks

A solution was presented to the problem of dynamic positioning and way-point tracking of an underactuated AUV (in the horizontal plane) in the presence of a constant unknown ocean current disturbance and parametric model uncertainty. Convergence of the resulting nonlinear system was analyzed and simulations were performed to illustrate the behaviour of the proposed control scheme. Simulation results show that the control objectives were achieved successfully. Future research will address the application of the new control strategy developed to the operation of a prototype marine vehicle.

As a final remark, we call attention to the fact that the control schemes derived present singularities that are extremely difficulty to avoid in the design process. Proving formally that these singularities are not reached is considerably hard. A possible solution to this problem requires blending some sound engineering judgment with mathematical rigor. From a practical point of view, a reasonable solution is to detect the proximity of a singular point and to switch to another feedback law to steer the system away from it. Care must be taken to ensure that this strategy does not lead to chattering behaviour. Notice however that this strategy may not yield stability in the usual sense. This circle of ideas is discussed in the next chapter.

Chapter 8

From local to global stabilization: a hybrid system approach

This chapter addresses the problem of nonlinear system stabilization using hybrid control. Its main contribution is the derivation of a new methodology to solve the problem of global stabilization of a specific class of nonlinear systems. Using classical Lyapunov techniques and some recent results on switching design of hybrid controllers, it is shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability. The resulting control laws avoid chattering and capture the interplay between the original locally stabilizing control loop and an outer control loop that comes into effect when the system trajectories deviate too much from the origin. Formal proofs of convergence and stability of the closed loop hybrid systems are derived. The methodology proposed is used to derive a simple control design method for global stabilization of a particular class of single input single output (SISO) nonlinear systems coupled by a given term.

The chapter is organized as follows: Section 8.1 formulates the problem of extending a locally asymptotically stabilizing control law for a given class of nonlinear systems to yield global asymptotically stability. Section 8.2 describes the general structure of the hybrid controller proposed, while Section 8.3 offers formal proofs of global asymptotic stability. An application to the control of a particular class of SISO systems coupled by a given term is described in Section 8.4. Simulation results that illustrate the performance of the control laws derived are also included. Finally, the chapter closes with some concluding remarks and recommendations for further research.

8.1 Control problem formulation

Before proceeding with the control problem formulation, consider the following definitions which will be used in the sequel.

Given $x \in \mathbb{R}^n$, let $z \in \mathbb{R}^l$; $l \leq n$ be a *subvector of* x, that is, a vector that is obtained from x by retaining l of its components. Further, let $z^c \in \mathbb{R}^{n-l}$ denote the *complementary vector of* z, that is, the vector that consists of the elements of x that are not in z. A scalar continuous function

$V(x) : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *locally positive definite with respect to* z if $V(0) = 0$, $V(x) \geq 0$ for $x \neq 0$, and $V(x) > 0$ in the set $\{x \in D : z \neq 0\}$. If $V(x)$ satisfies the weaker condition $V(x) \geq 0$ for $z \neq 0$, then it is said to be *positive semidefinite with respect to* z. A function $V(x)$ is said to be *negative definite* (respectively *negative semidefinite with respect to* z) if $-V(x)$ is positive definite (respectively positive semidefinite with respect to z). If $\dim(z) = n$, then the above definitions recover the usual concepts of positive/negative definite/semidefinite scalar continuous functions.

The class of nonlinear systems considered in this chapter is described by

$$\dot{x} = f(x, u) \tag{8.1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in its arguments, $x(t) \in \mathbb{R}^n$ is the state, and $u(t) \in \mathbb{R}^m$ is the control input. It is assumed that (8.1) satisfies the following key assumption:

Assumption 8.1 *Given* (8.1)*, there exists a feedback control law*

$$u = \alpha(x) \tag{8.2}$$

such that the origin $x = 0$ *of the closed loop system*

$$\dot{x} = f(x, \alpha(x)) \tag{8.3}$$

is an equilibrium point and system (8.3) *satisfies the following properties:*

i) There exists a subvector $z(t) \in \mathbb{R}^l$ *of* $x(t)$ *with* $l \leq n$ *and a continuously differentiable, positive semidefinite function* $V(x) : D \subset \mathbb{R}^n \to \mathbb{R}$ *defined in an open set* D *that contains the origin such that*

- *a)* $V(x)$ *is positive definite with respect to* z*,*
- *b)* $\dot{V}(x)$ *is negative definite with respect to* z*, and*
- *c)* $\|z\| \to 0 \Rightarrow V(x) \to 0$*.*

ii) For any initial condition $x(t_0) \in D$ *such that the resulting trajectory* $x(t) \in D$ *and* $z(t)$ *is bounded, there exists a positive constant* ζ *such that the solution* $z^c(t)$ *satisfies*

$$\|z^c(t)\| \leq \zeta \|z^c(t_0)\| + \sup_{t_0 \leq \tau \leq t} \|z(\tau)\|. \tag{8.4}$$

Furthermore, the set $D_0 = \{x \in D : z = 0\}$ *is positively invariant and in this set the solution* $z^c(t)$ *converges to zero as* $t \to \infty$*.*

Assumption 8.1 implies that the origin $x = 0$ is a locally asymptotically stable equilibrium point of closed loop system (8.3); the proof is a straightforward application of classical results on system stability using positive semidefinite Lyapunov functions. See for example (Khalil, 1996) and (Sepulchre *et al.*, 1997) and the results in Section 8.3.

The main problem considered in this chapter can now be briefly stated as follows:

Modify the control law (8.2) *in order to globally asymptotically stabilize* (8.1).

A solution to this problem will be obtained by resorting to hybrid control techniques, as shown in the next sections.

8.2 Hybrid controller design

This section proposes a simple piecewise smooth controller to globally asymptotically stabilize system (8.1) that borrows from hybrid system theory. Hybrid systems are specially suited to deal with the combination of continuous dynamics and discrete events. See the introduction in Chapter 1 and the model definition used in this thesis of a continuous-time autonomous hybrid system in Chapter 3.

Consider now the control problem described in Section 8.1 and define the compact set

$$\tilde{D}(\epsilon) = \left\{ x \in \mathbb{R}^n : \|z\| \leq \epsilon,\ |z_i^c| \leq \delta_i \right\},$$

for some $\epsilon > 0$ and $\delta_i > 0$, $i = 1, 2, \cdots, n - l$.

Assumption 8.2 *Given system $\dot{x} = f(x, u)$ in (8.1) and the set D in Assumption 8.1, there is a control law of the form*

$$u = \beta(x),$$

and arbitrarily large positive constants δ_i; $i = 1, 2, \cdots, n - l$ such that

i) There exists $\epsilon > 0$ such that the set $\tilde{D}(\epsilon)$ is strictly inside D, and

ii) For all $x(t_0)$ and for any arbitrarily small positive constant ϵ, a finite time $T \geq t_0$ exists such that $x(T) \in \tilde{D}(\epsilon)$.

Notice that the above conditions do not rule out the possibility that $z_i^c(t)$ grow unbounded. However, these conditions imply that $z_i^c(t)$ will not blow up in finite time.

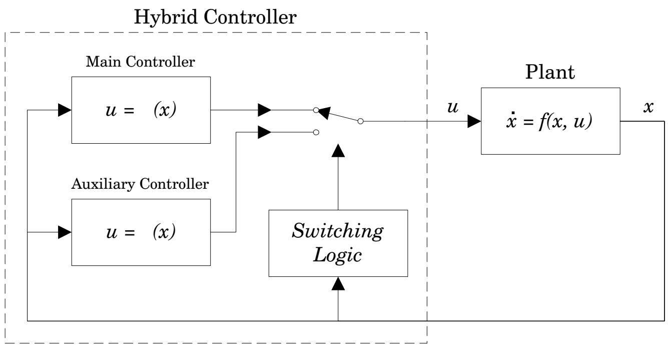

Figure 8.1: Block diagram of the hybrid system Σ.

It will be shown in the next section that the hybrid control law

$$u = g_\sigma(x), \tag{8.5}$$

with the vector fields $g_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\sigma \in \mathcal{I} = \{1, 2\}$ given by

$$g_1(x) = \alpha(x), \quad g_2(x) = \beta(x), \tag{8.6}$$

globally asymptotically stabilizes (8.1). See the block diagram in Figure 8.1, where V denotes the positive semidefinite function of Assumption 8.1. The variable σ is a piecewise constant switching signal taking values in $\mathcal{I} = \{1, 2\}$ and determines, at each instant of time, which of the feedback laws in (8.6) should be used. Its value is determined according to the switching logic S_l described by the computer diagram in Figure 8.2, where S denotes an open set that contains the origin and is strictly inside D, and $S \supseteq \tilde{D}(\epsilon)$ for some $\epsilon > 0$. The block *Initialize* sets $\gamma = +\infty$ and selects A or B as the entry point to the lower blocks of Figure 8.2 as follows: if $x_0 \in D$, $\sigma(t_0^+) = 1$ and the entry point is A; otherwise, $\sigma(t_0^+) = 2$ and the entry point is B.

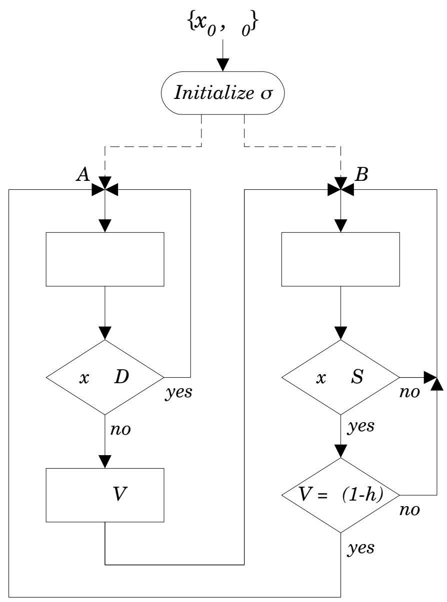

Figure 8.2: Switching logic S_l.

The rationale for the control law above can be briefly explained as follows. Assume for example that A is the starting point. If $x(t)$ remains in D for all $t \geq t_0$, then it will be later shown that $x(t)$ will, under the action of control law $\alpha(x)$, tend to 0 as $t \to \infty$. Suppose now that $x(t)$ leaves the region D at time $t = t_s$ and therefore that σ switches to 2. Let $\gamma = \lim_{t \to t_s} V(x(t))$. According to the switching logic proposed, the control law $\beta(x)$ will come into play and force the variable z to approach 0. As a consequence, x will enter $S \subset D$ and since $\|z\| \to 0 \Rightarrow V(x) \to 0$ it follows that V will reach the value $\gamma_h = \gamma(1-h)$, where $h \in (0, 1)$ is a constant variable. At that moment, σ will switch back to 1. Since $\dot{V}(x) < 0$ on the set $D_z = \{x \in D : z \neq 0\}$ it follows that the successive values of γ_h will be decreasing and z will tend to 0. Furthermore, the bound (8.4) on $z_c(t)$ guarantees that $z_c(t)$ and therefore $x(t)$ will tend to zero. The switching logic described above generates a piecewise constant signal σ that is continuous from the right everywhere. Notice how h and the construction of $S \subset D$ introduce a hysteresis in the switching logic that avoids chattering.

8.3 Stability analysis

In this section, existence and uniqueness of solutions, as well as convergence to the origin and stability of the closed loop hybrid dynamical system described in the previous section are analyzed. In order to do this, it is necessary to adopt the following definition of Lyapunov asymptotic stability for hybrid systems that borrows from the concepts introduced in (Ye *et al.*, 1998).

Definition 8.1 *The equilibrium point $x = 0$ of the hybrid system Σ of (3.1) is said to be Lyapunov stable if for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists $\delta = \delta(\epsilon) > 0$ such that for every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \delta$, the solution $\{x(t), \sigma(t)\}$ satisfies $\|x(t)\| < \epsilon$ for all $t \geq t_0$. If in addition the equilibrium point $x = 0$ is attractive* i.e., *there exists $\eta(t_0) > 0$ and, for each $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $\|x_0\| < \eta \Rightarrow \|x(t)\| < \epsilon$, for all $t \geq t_0 + T$, then the origin is said to be asymptotically stable. If these properties hold for any initial conditions, then the equilibrium point $x = 0$ is called globally asymptotically stable.*

The following theorem establishes the main result of this chapter.

Theorem 8.1 *Consider the hybrid system Σ described by (8.1), (8.5), (8.6), and the switching logic S_l defined by the computer diagram shown in Figure 8.2. Suppose that (8.1) satisfies Assumptions 8.1-8.2. Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^n, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ. Then, the following properties hold.*

i) *$\{x(t), \sigma(t)\}$ exists, is unique and defined for all $t \geq t_0$ and all $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^n \times \mathcal{I}$.*

ii) *For any set of initial conditions $\{x_0, \sigma_0\} \in \mathbb{R}^n \times \mathcal{I}$, there exists a finite time $T \geq t_0$ such that $\sigma(t) = 1$ for all $t \geq T$,* i.e., *the switching stops in finite time. Furthermore, the continuous state $x(t) \in \mathbb{R}^n$ converges to zero as $t \to \infty$.*

iii) *The origin $x(t) = 0$ is a Lyapunov globally asymptotically stable equilibrium point of Σ.*

Proof:
Existence and uniqueness
Existence and uniqueness of $\{x(t), \sigma(t)\}$ are proven by showing first that chattering does not occur. In fact, since the set $S \subset D$ is strictly contained in D and, for each discrete state, the vector fields $g_\sigma(x)$ is Lipschitz in x, it can be concluded that a trajectory that enters D cannot enter S at the same time. Conversely, if a trajectory is on the boundary of $S \subset D$, then it can not abandon immediately the set D. Therefore, there is always a positive non null time interval between switches. Standard results in (Hale, 1980, Theorem 5.3, Section I.5) show that a unique solution to Σ exists over a maximal interval $[0, t_f)$. It remains to prove that the maximum interval of existence is infinite. First, observe that while $\sigma = 1$ it follows from Assumption 8.1 that the trajectory $x(t)$ is bounded for any initial condition $x_0 \in D$. Also, for $\sigma = 2$, it can be concluded that $x(t)$ is bounded if σ switches to 1 in finite time, which will be proved later. Thus, since the trajectory $x(t)$ is bounded, it exists over the infinite interval, that is, $t_f = \infty$. This proves property *i)*.

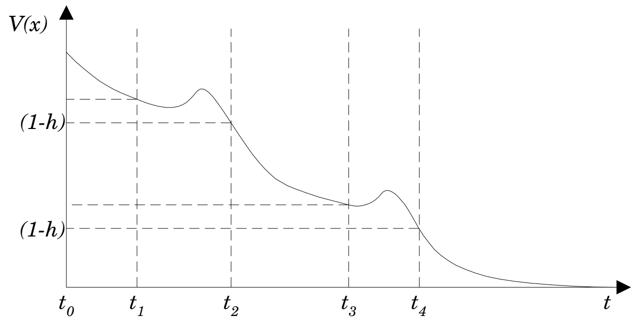

Figure 8.3: Sketch of the value of the Lyapunov function $V(x)$ as the system switches; $\sigma(t_0^+) = 1$.

Local stability

Given $\epsilon > 0$, let $r \in (0, \frac{\epsilon}{2}]$ be such that $B(z,r) = \{x \in D : \|z\| \leq r\} \subset D$. Let $\bar{c} \triangleq \min_{\|z\|=r} V(x)$. Since $V(x)$ is positive definite with respect to z, $\bar{c} > 0$. Take $c \in (0, \bar{c})$, and let $\Omega_c \triangleq \{x \in B_r : V(x) \leq c\}$. Since $\dot{V}(x(t)) \leq 0$, then

$$V(x(t)) \leq V(x_0) \leq c$$

for all $t \geq t_0$. Consequently $\Omega_c \subset B(z,r)$ is a positively invariant set, that is, for any $x_0 \in \Omega_c$, $x(t) \in \Omega_c$ for all $t \geq t_0$. Furthermore, any trajectory starting in that set is bounded with respect to z, that is, $\|z(t)\| \leq r$. Observe also that from (8.4) one has that $\|z^c(t)\| \leq \zeta \|z^c(t_0)\| + r$. Therefore, if $\delta_2 = (\frac{\epsilon}{2} - r)/\zeta$ then

$$\|z^c(t_0)\| < \delta_2 \Rightarrow \|z^c(t)\| \leq \frac{\epsilon}{2}.$$

Since $V(x)$ is continuous and $V(0) = 0$, there is $\delta_1 > 0$ such that

$$\|z\| < \delta_1 \Rightarrow V(x) < c$$

for all $t \geq t_0$. Thus,

$$x_0 \in B(z, \delta_1) = \{x \in D : \|z\| \leq \delta_1\} \subset \Omega_c$$
$$\Rightarrow x_0 \in \Omega_c \Rightarrow x(t) \in \Omega_c \Rightarrow \|z(t)\| < r \leq \frac{\epsilon}{2}.$$

Consequently, given $\epsilon > 0$ one can find $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\|x_0\| < \delta \quad \Rightarrow \|x(t)\| \leq \|z(t)\| + \|z^c(t)\| < \epsilon$$

for all $t \geq t_0$, which shows that the hybrid system Σ is locally Lyapunov stable.

Local asymptotic stability

Asymptotic stability follows by showing that $x(t) \to 0$ as $t \to \infty$. First, observe that since $V(x)$ is monotonically decreasing and bounded from bellow by zero, $V(x) \to a$ as $t \to \infty$, where a is a nonnegative constant. Using a contradiction argument it will be shown that $a = 0$. In fact, suppose that $a > 0$. By continuity of $V(x)$, there exists a $b > 0$ such that $B(z,b) \subset \Omega_a$. The limit $V(x) \to a > 0$ implies that the trajectory $x(t)$ lies outside the set $B(z,b)$ for all $t \geq t_0$. Let

$-\gamma = \max_{b\leq\|z\|\leq r} \dot{V}(x)$. Then, $-\gamma < 0$ by Assumption 8.1.i.b). Thus,

$$\begin{aligned} V(x(t)) &= V(x_0) + \int_{t_0}^{t} \dot{V}(x(\tau))\, d\tau \\ &\leq V(x_0) - \gamma(t - t_0). \end{aligned}$$

Since the right-hand side will eventually become negative, the inequality contradicts the assumption that $a > 0$. Thus, $V(x) \to 0$ as $t \to \infty$. Consequently, from the positive definiteness of $V(x)$ with respect to z, it follows that $z(t) \to 0$ as $t \to \infty$. To conclude the convergence proof, it remains to show that $z^c(t) \to\ 0$ as $t \to \infty$. Since $x(t)$ is bounded and any bounded solution of (8.3) converges to its positive limit set L^+ (also called the ω-limit set), (Khalil, 1996, Lemma 3.1), then $x(t) \to L^+$ as $t \to \infty$. To characterize the set L^+, observe that on the invariant manifold D_0 the trajectory $z^c(t) \to 0$ as $t \to \infty$ (Assumption 8.1.ii), and therefore L^+ is the origin.

Global asymptotic stability

Global asymptotic stability is proved by showing that $x(t)$ converges to zero as t goes to infinity, for any initial condition $\{x_0, \sigma_0\}$. Let the sequence $t_1 < t_2 < \cdots$ describe the times of switching between the feedback controllers g_1 and g_2 (see Figure 8.3). If $\sigma = 2$, then $u = \beta(x)$. Therefore, the closed loop satisfies the property that $z \to 0$ and, consequently, $V(x) \to 0$ as $t \to \infty$. Thus, after a finite time $\Delta > 0$, $x \in S$ and $V(x) = \gamma(1-h)$, which implies that σ will switch to 1. Hence, it can be concluded that for any $t^\star \geq t_0$ such that $\sigma(t^\star) = 2$, there exists a positive constant Δ such that $\sigma(t^\star + \Delta) = 1$. Furthermore, from the fact that the function V satisfies the nonincreasing condition

$$V(x(t_{k+1})) < V(x(t_k))$$

for every positive integer k, it follows that there will be a constant $k^\star$ such that $\sigma(t_{k^\star}) = 1$ and the surface level $V(x) = \gamma_{k^\star-1}(1-h)$, with $\gamma_{k^\star-1} = \gamma(t_{k^\star-1})$ lies inside $\Omega_c \subset D$; $c = \gamma_{k^\star-1}$, where Ω_c is the positively invariant set defined previously in the proof of local stability. Consequently, the switching stops in finite time and $x(t) \to 0$ as $t \to \infty$.

This concludes the proof of Theorem 8.1. □

8.4 An illustrative example

This section illustrates an application of the hybrid control law developed in Section 8.2. See (Isidori, 1989) for standard notation. Consider the two single-input single output (SISO) systems

$$\Sigma_i := \begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \\ y_i = h_i(x_i), \end{cases} \quad i = 1, 2$$

with $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$, f_i, and g_i are smooth vector fields on $\mathbb{R}^{n_i}$, h_i a smooth nonlinear function, and $f_i(0) = 0$, $h_i(0) = 0$ for all $i = 1, 2$. Consider also that both systems Σ_1, Σ_2 have relative degree n_1, n_2 at $x_1 = 0$, $x_2 = 0$, respectively. Then, the nonlinear coordinate transformation $\Phi_i : x_i \to \xi_i = \left(h_i, L_{f_i}h_i, \cdots, L_{f_i}^{n_i-1}h_i\right)'$; $i = 1, 2$ is a local diffeomorphism and transforms Σ_i into the "normal form" (Isidori, 1989)

$$
\begin{aligned}
\dot{\xi}_{i_1} &= \xi_{i_2}, \\
\dot{\xi}_{i_2} &= \xi_{i_3}, \\
&\vdots \\
\dot{\xi}_{i_{n_i}} &= b_i(\xi_i) + a_i(\xi_i)u_i, \\
y_i &= \xi_{i_1},
\end{aligned}
$$

where $\xi_i = (\xi_{i_1}, \cdots, \xi_{i_{n_i}})'$, $b_i = L_{f_i}^{n_i} h_i\big(\Phi_i^{-1}(\xi_i)\big)$, and $a_i(\xi_i) = L_{g_i} L_{f_i}^{n_i - 1} h_i\big(\Phi_i^{-1}(\xi_i)\big)$. Note that at $\xi_i = \Phi_i(0) = 0$, we have $a_i(0) \neq 0$, and therefore the coefficient $a_i(\xi_i)$ is nonzero for all ξ_i in a neighborhood of $\xi_i = 0$. Consequently, choosing the state feedback control law

$$
u_i = \frac{1}{a_i(\xi_i)}\big[-b_i(\xi_i) + v_i\big],
$$

yields a closed loop system that is linear and controllable from the input v_i. Thus, the closed loop system can be made to have its poles placed at the zeros of a desired polynomial $s^{n_i} + c_{i_{n_i - 1}} s^{n_i - 1} + \cdots + c_{i_0}$ by choosing

$$
v_i = -c_{i_0}\xi_{i_1} - c_{i_1}\xi_{i_2} - \cdots - c_{i_{n_i - 1}}\xi_{n_i}.
$$

Suppose now that there is a coupling term between the two SISO systems as given by

$$
\dot{x} = f(x) + g(x)u + \varphi(x, y), \tag{8.7}
$$

where $x = (x_1', x_2')'$, $f = (f_1', f_2')'$, $g = (g_1', g_2')'$, $\varphi(x, y) = (\varphi_1', \varphi_2')'$, and

$$
\begin{aligned}
\varphi_1 &= \vartheta_1(x_1)(y_2 - k), \\
\varphi_2 &= \vartheta_2(x_2)y_1,
\end{aligned}
$$

with ϑ_i; $i = 1, 2$ are smooth vector field on $\mathbb{R}^{n_i}$, k is an arbitrary non null constant, and ϑ_i satisfies the "matching condition"

$$
L_{\vartheta_i} L_{f_i}^k h_i(x_i) = 0; \quad 0 \leq k \leq n_i - 2; \quad i = 1, 2
$$

Further assume that the coordinate transformation $\xi_2 = \Phi_2(x_2)$ is a global diffeomorphism on $\mathbb{R}^{n_2}$, $\xi_1 = \Phi_1(x_1)$ is a local diffeomorphism, and that the subsystem

$$
\dot{x}_1 = f_1(x_1) + \vartheta_1(x_1)\tilde{u}; \quad \tilde{u} = y_2 - k \tag{8.8}
$$

of (8.7) that corresponds to the evolution of state x_1 with $u_1 = 0$ can be rewritten as

$$
\begin{aligned}
\dot{\xi}_{3_1} &= \xi_{3_2}, \\
\dot{\xi}_{3_2} &= \xi_{3_3}, \\
&\vdots \\
\dot{\xi}_{3_r} &= b_3(\xi_3) + a_3(\xi_3)(y_2 - k), \\
\dot{\xi}_{3_{r+1}} &= q_{r+1}(\xi_3), \\
&\vdots \\
\dot{\xi}_{n_1} &= q_{n_1}(\xi_3),
\end{aligned}
$$

where $\xi_3 = \Phi_3(x_1)$, $\Phi_3(x_1) = \big(\phi_{3_1}, \cdots, \phi_{3_{n_1}}\big)'$, $\phi_{3_1}(x_1) = h_1(x_1)$, $\phi_{3_2}(x_1) = L_{f_1} h_1(x_1)$, $\cdots$, $\phi_{3_r} = L_{f_1}^{r-1} h_1(x_1)$, and $\phi_{3_{r+1}}(x_1), \cdots, \phi_n(x_1)$ are chosen in such a way as to make $L_{\vartheta_1}\phi_{3_i} = 0$ for all $r+1 \leq i \leq n_1$ and all $x_1 \in \mathbb{R}^{n_1}$. The functions $b_3(\xi_3)$, $a_3(\xi_3)$ are given by $L_{f_1}^r h_1(x_1)$, $L_{\vartheta_1} L_{f_1}^{r-1} h_1(x_1)$, and $q_i(\xi_3) = L_{f_1}\phi_{3_i}(x_1)$ for all $r+1 \leq i \leq n_1$. Consider that the zero dynamics $\dot{\eta} = q(\bar{\xi}_3, \eta)$, with $\bar{\xi}_3 = \big(\xi_{3_1}, \cdots, \xi_{3_r}\big)'$, $\eta = \big(\xi_{r+1}, \cdots, \xi_{n_1}\big)'$, are globally asymptotically stable, and that $a_3(\xi_3) \neq 0$ for all $\xi_3 \neq 0$. Notice however, for $\xi_3 = 0$, that $a_3(0)$ can be zero.

The goal is to globally asymptotically stabilize system (8.7) at the equilibrium point $(x_1, x_2) = 0$ applying the methodology proposed in section 8.2.

8.4.1 Main controller $\alpha(x)$

A simple local controller that exponentially asymptotically stabilizes the origin and satisfies Assumption 8.1 is given by $u = \alpha(x) = \big(\alpha_1(x), \alpha_2(x)\big)'$ where

$$\begin{aligned}
\alpha_1(x) = {} & \frac{1}{L_{g_1} L_{f_1}^{n_1-1} h_1(x_1)} \big[-L_{f_1}^{n_1} h_1(x_1) - c_{1_0} h_1(x_1) \\
& - c_{1_1} L_{f_1} h_1(x_1) - \cdots - c_{1_{n_1-1}} L_{f_1}^{n_1-1} h_1(x_1) \\
& - L_{\vartheta_1} L_{f_1}^{n_1-1} h_1(x_1)(y_2 - k)\big], \\
\alpha_2(x) = {} & \frac{1}{L_{g_2} L_{f_2}^{n_2-1} h_2(x_2)} \big[-L_{f_2}^{n_2} h_2(x_2) - c_{2_0} h_2(x_2) \\
& - c_{2_1} L_{f_2} h_2(x_2) - \cdots - c_{2_{n_2-1}} L_{f_2}^{n_2-1} h_2(x_2) \\
& - L_{\vartheta_2} L_{f_2}^{n_2-1} h_2(x_2) y_1\big],
\end{aligned}$$

and the constants $c_{i_0}, c_{i_1}, \cdots, c_{i_{n_i-1}}$; $i = 1, 2$ are chosen so as to make the matrix

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_{i_0} & -c_{i_1} & -c_{i_2} & \cdots & -c_{i_{r-1}} \end{bmatrix}; \quad i = 1, 2 \tag{8.9}$$

Hurwitz.

Let $z = x_1$ and $z^c = x_2$ be the states defined in Section 8.1. A candidate Lyapunov function that satisfies the conditions of Assumption 8.1.i) can be found by solving the equation

$$P A_1 + A_1' P = -Q \tag{8.10}$$

for P, where Q is a real symmetric positive definite matrix. Let $D = \big\{x \in \mathbb{R}^{n_1+n_2} : \Phi_1(x_1)$ is invertible and $\Phi_1(x_1)$ and $\Phi_1^{-1}(\xi_1)$ are both smooth mappings$\big\}$. The quadratic function $V(x) : D \subset \mathbb{R}^{n_1+n_2} \to \mathbb{R}$,

$$V(x) = \Phi_1(z)' P \, \Phi_1(z) \tag{8.11}$$

is positive definite with respect to z, its derivative $\dot{V}(x) = -\Phi_1(z)' Q \, \Phi_1(z)$ is negative definite with respect to z, and $\|z\| \to 0 \Rightarrow V(x) \to 0$. Assumption 8.1.ii) is easily seen to be satisfied by noting that the dynamics of z^c are governed by the linear differential equation $\dot{\xi}_2 = A_2\xi_2$, with $z^c = \Phi_2^{-1}(\xi_2)$. Since A_2 is a Hurwitz matrix and Φ_2 is a global diffeomorphism on $\mathbb{R}^{n_2}$, it follows that z_c is bounded and converges to zero.

8.4.2 Auxiliary controller $\beta(x)$

In this section, a feedback control law is derived such that for any initial condition $x(t_0)$ the state $z(t)$ reaches any small neighborhood of the origin $z = 0$ in finite time. Notice that convergence to the origin is not required. Notice also that when compared to the local controller one now has an extra "degree of freedom" given by the state z^c. This vector need not converge to zero and can be viewed as another control input. In fact, by imposing $u_1 = 0$ and observing equation (8.8), one can see y_2 as a virtual control input and

$$\alpha_{3_1} = k + \frac{1}{a_3}\left[-b_3 - c_{3_0}\xi_{3_1} - c_{3_1}\xi_{3_2} - \cdots - c_{3_{r-1}}\xi_{3_r}\right]$$

as a virtual control law, where the constants $c_{3_i}; i = 0, 1, \cdots, r$ are chosen in such way that the matrix A_3 (see (8.9) for $i = 3$) is Hurwitz. Notice that if $a_3(0) = 0$, the virtual control law is not valid. However, if $x_1 = 0$ then σ must be 1 and therefore it is the local controller that is active (see Section 8.2). Introducing the error variable

$$z_1 = y_2 - \alpha_{3_1},$$

and following the backstepping approach (see the book of (Krstić *et al.*, 1995)), let V_1 be a partial Lyapunov function given by

$$V_1 = \xi_3'\, P\, \xi_3 + \frac{1}{2}z_1^2,$$

where the symmetric positive definite matrix P is the solution of the Lyapunov equation $PA_3 + A_3'P = -Q$, and Q is a real symmetric positive definite matrix. Computing the time derivative of V_1, yields

$$\dot{V}_1 = -\xi_3'\, Q\, \xi_3 + z_1\left[2\xi_3' Pb + \dot{y}_2 - \dot{\alpha}_{3_1}\right],$$

where $b = (0, \cdots, 0, 1)'$. Resorting to the coordinate transformation $\xi_2 = \Phi_2(x_2)$, it follows that

$$\dot{y}_2 = \dot{\xi}_{2_1} = \xi_{2_2}.$$

Defining $z_2 = \xi_{2_2} - \alpha_{3_2}$, where

$$\alpha_{3_2} = -2\xi_3' Pb + \dot{\alpha}_{3_1} - k_1 z_1$$

one obtains

$$\dot{V}_1 = -\xi_3'\, Q\, \xi_3 - k_1 z_1^2 + z_1 z_2.$$

Let now V_2 be the augmented Lyapunov function

$$V_2 = V_1 + \frac{1}{2}z_2^2.$$

Its derivative is given by

$$\dot{V}_2 = -\xi_3'\, Q\, \xi_3 - k_1 z_1^2 + z_2\left[z_1 + \xi_{2_3} - \dot{\alpha}_{3_2}\right].$$

Proceeding recursively, at the i-th step define

$$z_{i+1} = \xi_{2_{i+1}} - \alpha_{3_{i+1}},$$

where

$$\alpha_{3_{i+1}} = -z_{i-1} + \dot{\alpha}_{3_i} - k_i z_i$$

to obtain

$$\dot{V}_i = -\xi_3' \, Q \, \xi_3 - k_1 z_1^2 - \cdots - k_i z_i^2 - z_i z_{i+1}.$$

At the last step, one has

$$\begin{aligned}\dot{V}_{n_2} = & -\xi_3' \, Q \, \xi_3 - k_1 z_1^2 - \cdots - k_{n_2-1} z_{n_2-1}^2 \\ & + z_{n_2}\big[z_{n_2-1} + b_2(\xi_2) + a_1(\xi_2) u_2 - \dot{\alpha}_{3_{n_2}} \\ & - L_{\vartheta_2} L_{f_2}^{n_2-1} h_2(x_2) y_1\big].\end{aligned}$$

By defining

$$u_2 = \frac{1}{a_2(\xi_2)}\big[-b_2(\xi_2) - z_{n_2-1} + \dot{\alpha}_{3_{n_2}} + L_{\vartheta_2} L_{f_2}^{n_2-1} h_2(x_2) y_1\big], \tag{8.12}$$

the time derivative of the composite Lyapunov function

$$V_{n_2} = \xi_3' \, P \, \xi_3 + \frac{1}{2}(z_1^2 + \cdots + z_{n_2}^2)$$

given by

$$\dot{V}_{n_2} = -\xi_3' \, Q \, \xi_3 - k_1 z_1^2 - \cdots - k_{n_2} z_{n_2}^2$$

is negative definite. Thus, the closed loop system described by (8.7), together with $u = \beta(x) = (0, u_2)'$, where u_2 is given by (8.12) satisfies Assumption (8.2).

8.4.3 Simulation results

This section illustrates the performance of the proposed control scheme using computer simulations with the nonlinear system

$$\begin{aligned}\dot{x}_{1_1} &= x_{1_1}^2 + x_{1_2} \\ \dot{x}_{1_2} &= \cos(x_{1_1}) u_1 + p(x_1)(x_2 + 1) \\ \dot{x}_2 &= u_2\end{aligned}$$

where $p(x_1) = x_{1_1}^2 + x_{1_2}^2$, $x = (x_{1_1}, x_{1_2}, x_2)'$ and $u = (u_1, u_2)'$. The objective is to globally asymptotically stabilize the origin $x = 0$. Following the approach developed in the previous sections one has

$$\begin{aligned}\alpha(x) &= \begin{bmatrix} \frac{1}{\cos(\xi_{1_1})}(-2\xi_{1_1}\xi_{1_2} - p(x_1)(x_2+1) + v_1) \\ -c_{2_0} x_2 \end{bmatrix}, \\ \beta(x) &= \begin{bmatrix} 0 \\ -k_1 z_1 - 2P_{12}\xi_{3_1} - 2P_{22}\xi_{3_2} + \dot{\alpha}_{3_1} \end{bmatrix},\end{aligned}$$

where $\xi_{1_1} = \xi_{3_1} = x_{1_1}$, $\xi_{1_2} = \xi_{3_2} = x_{1_1}^2 + x_{1_2}$, $v_1 = -c_{1_0}\xi_{1_1} - c_{1_1}\xi_{1_2}$, $z_1 = x_2 - \alpha_{3_1}$, $\alpha_{3_1} = -1 + \frac{1}{p(x_1)}(-2\xi_{3_1}\xi_{3_2} - c_{3_0}\xi_{3_1} - c_{3_1}\xi_{3_2})$, and P_{ij} denotes the ij-element of the solution of the Lyapunov matrix $PA_3' + A_3'P = -I$, with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} 0 & 1 \\ -c_{3_0} & -c_{3_1} \end{smallmatrix}\right)$. Consider the sets

$$D = \left\{x \in \mathbb{R}^3 : |x_{1_1}| \leq \bar{x}_{1_1}\right\},$$
$$S = \left\{x \in \mathbb{R}^3 : |x_{1_1}| \leq \bar{x}_{1_1}(1 - h_{x_1})\right\},$$

with $\bar{x}_{1_1} < \frac{\pi}{2}$. Notice that with S defined as above, S is strictly contained in D.

The control parameters were selected as following: $c_{1_0} = c_{3_0} = 1$, $c_{1_1} = c_{3_1} = 2$, $c_{2_0} = 1$, $k_1 = 1$, $h = h_{x_1} = 0.1$, and $\bar{x}_{1_1} = 85\pi/180$.

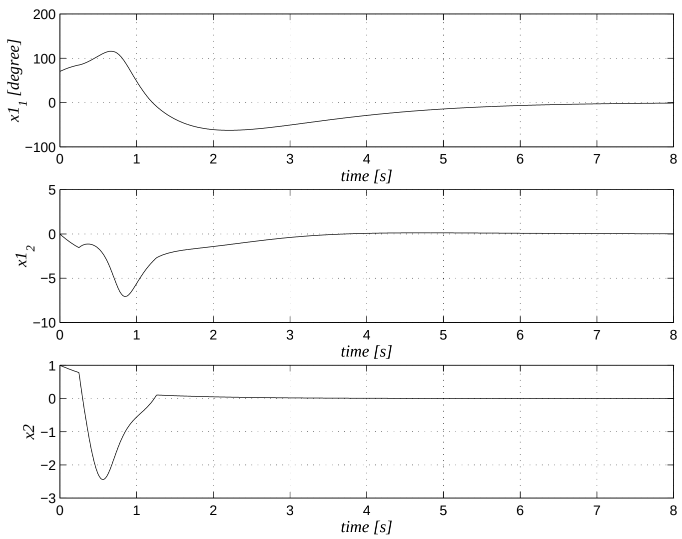

Figure 8.4: Time evolution of the state variables $\frac{180}{\pi}x_{1_1}(t)$, $x_{1_2}(t)$, and $x_2(t)$.

Figures 8.4-8.6 show the simulation results for the initial condition $x(0) = (70\pi/180, 0, 1)'$. The time evolution of the states $x_{1_1}(t)$, $x_{1_2}(t)$, and $x_2(t)$ is depicted in Figure 8.4. To better understand the performance of the hybrid control law, Figure 8.5 displays the time evolution of the Lyapunov variable $V(x) = \xi_1' P \xi_1$ and the switching signal $\sigma(t)$. Initially, σ takes the value 1. While σ is 1, one can see that V is decreasing. However, after a finite time $t \simeq 0.248\,s$, the variable x_{1_1} takes the value $\bar{x}_{1_1}$ and therefore $x(t)$ leaves the region D. As a consequence, σ switches to 2 and $z = (x_{1_1}, x_{1_2})'$ will be forced to approach 0. At $t \simeq 1.264\,s$, it can be seen that $x(t)$ is already in S and $V(x)$ reaches $\gamma(1-h)$. Therefore, σ switches again to 1, the main controller dictates the subsequent trajectory evolution. The control signals are presented in Figure 8.6.

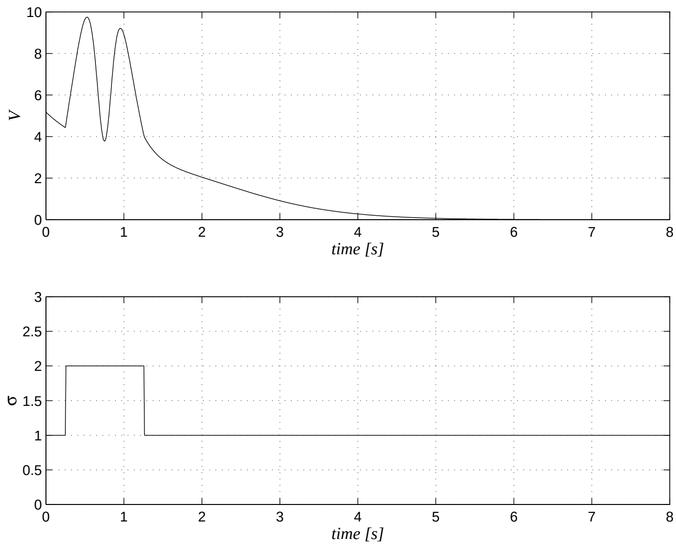

Figure 8.5: Time evolution of the Lyapunov function $V(x)$ and the switching signal $\sigma(t)$.

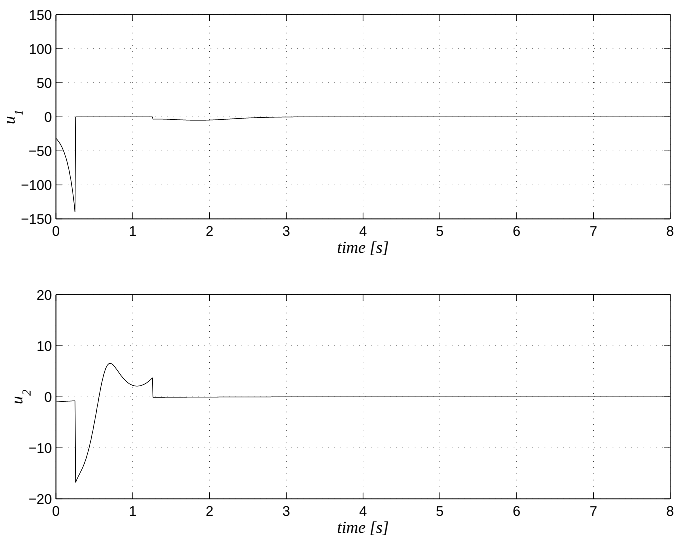

Figure 8.6: Time evolution of control signals $u_1(t)$ and $u_2(t)$.

8.5 Concluding remarks

The problem of nonlinear system stabilization using hybrid control was investigated in this chapter. For a specific class of nonlinear plants, it was shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability. The new methodology proposed for nonlinear system stabilization builds on classical Lyapunov techniques and borrows from recent results on switching design of hybrid controllers. The resulting control laws avoid chattering and embody the interplay between the original locally stabilizing control loop and an outer control loop that comes into effect when the system trajectories deviate too much from the origin. Formal proofs of convergence and stability of the closed loop hybrid systems were derived. An application was described that illustrates the potential of the control scheme presented and proposes a simple design method to globally stabilize a particular class of coupled single input single output nonlinear systems Simulation results show that the control objectives were achieved successfully.

Future research issues will address the extension of the theoretical framework to deal with systems with changing dynamics and relax some of the requirements imposed in Assumption 8.1. Another open problem that warrants further research is the control of nonlinear systems with noise and model uncertainties that arise from parameter variations or neglected dynamics.

Chapter 9

Conclusions

In this thesis new methodologies were developed for nonlinear motion control of underactuated and nonholonomic systems. The focus of the applications was threefold: *i)* the so-called extended nonholonomic double integrator (ENDI) (a more elaborate version of the nonholonomic integrator of Brockett), *ii)* wheeled mobile robots of the unicycle-type, and *iii)* the prototype of an underactuated autonomous underwater vehicle (AUV) named SIRENE. The complexity of the problems at hand can be best be realized by remarking that nonholonomic systems cannot be stabilized by smooth, time-invariant, state-feedback control laws. Together with this basic limitation, the difficulty and challenging characteristic of the problems addressed were further enhanced by studying them in a more realistic scenario that explicitly included *i)* the nonlinear dynamics (and not only the kinematics) of the mechanical nonholonomic systems under consideration, *ii)* parametric modeling uncertainty, and *iii)* the presence of external disturbances such as ocean currents for the AUV case. Formal proofs of convergence of the several resulting control algorithms were derived and simulation results were presented and discussed.

The remaining of this chapter summarizes briefly the content of the thesis, emphasizes the major original contributions, and points out some future research directions.

9.1 Summary

Chapters 1 and 2 were dedicated to setting the background for the main subject of the thesis: nonlinear motion control of nonholonomic and underactuated systems. In particular, Chapter 1 started with a brief historical account of linear and nonlinear control theory and described the practical motivation for the research presented in this thesis. Previous related work in the general area of underactuated, nonholonomic, and hybrid system control was revisited. The chapter ended with a summary of the main contributions of this thesis. In Chapter 2, several examples of nonholonomic systems that were used throughout the dissertation to illustrate theoretical results and the performance of the control systems developed were presented. For each system, the corresponding dynamic model was described and its controllability and stabilizability properties analyzed.

In Chapters 3 and 4, logic-based hybrid control laws were derived to solve the global stabilization problem for the ENDI system and for a model of the SIRENE AUV. The problem of practical stabilization of the ENDI system under input saturation constraints and in the presence of small input additive disturbances was also solved and posed. The ENDI system can be viewed as an extension of the so called nonholonomic integrator (Brockett, 1983). Its importance stems from the fact that it captures the dynamics and kinematics of a nonholonomic system with three states and two first-order dynamic control inputs. This result suggests that a fairly general class of nonholonomic systems can be treated by the method proposed in Chapter 3. Notice however that the coordinate transformation presented in the control loop may yield a path of the nonholonomic vehicle under consideration that is hard to predict and does not result into a "natural-like maneuver".

Chapters 5, 6, and 7 were devoted to the problem of pose regulation for wheeled mobile robots and underactuated AUVs in the presence of parametric modeling uncertainty. Controller design relied on non smooth coordinate transformations in the original state space, followed by the derivation of adaptive Lyapunov-based, smooth, time invariant feedback controllers in the new coordinates. The important problem of dynamic positioning and way-point tracking of an underactuated AUV with parametric modeling uncertainty in the presence of an unknown ocean current disturbance was also posed and solved. Simulations results show that, in spite of parameter uncertainty, the vehicles do not exhibit unnecessary oscillatory motions. This in striking contrast to what happens in the case of time-varying feedback.

Finally, in Chapter 8 the problem of nonlinear system stabilization was investigated by combining recent results on switching design of hybrid controllers and classical Lyapunov techniques. For a specific class of nonlinear plants, it was shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability.

9.2 Future directions

The area of nonlinear motion control of underactuated and nonholonomic systems is vast and many interesting problems that can be cast in its framework remain to be formally posed and solved. In what follows, the key results obtained in this thesis are examined critically. Related topics that warrant further research efforts are discussed. Preliminary results on some of the topics that have yet to mature into a coherent presentation are also highlighted.

Generalization of the proposed logic-based hybrid control structure to a large class of nonholonomic systems

The extended nonholonomic double integrator (ENDI) model was introduced in Chapter 2. This model captures the dynamics of a wheeled robot subject to force and torque inputs as well as that of any kinematic completely nonholonomic system with three states and two first-order dynamic inputs. Chapter 3 introduced a feedback logic-based hybrid controller that globally stabilizes the ENDI system. An open area of research is the generalization of the proposed logic-based hybrid control structure to a large class of nonholonomic systems. Namely, systems in chained form or even to dynamical extensions of chained form systems, *i.e.*, chained system appended with two integrators. For the underactuated AUV case, an open problem that warrants further research is that of finding a transformation to bring into a form similar to the ENDI in 3-D space to then generalize the hybrid controller derived in Chapter 4.

A unified approach for point-regulation, tracking control, and path-following for underactuated AUVs

The task of designing trajectory tracking and path following controllers for nonholonomic vehicles has received increasing attention in the past few years. Trajectory tracking deals with the case where a vehicle must track a time-parameterized reference. Path following refers to the problem of making a vehicle converge to a given path, without any temporal specifications. In (Pettersen and Nijmeijer, 1998) the kinematics of a surface vehicle are considered, and a time-varying control law for the surge and yaw inputs is designed such that the pose (position and orientation) error of the vehicle with respect to a reference trajectory of constant curvature is practically globally exponentially stabilized to zero. The control law developed does not require that the reference linear speed be nonzero, *i.e.*, the same control law guarantees global practical stabilization, should the reference trajectory degenerate to a constant pose. In (Jiang and Nijmeijer, 1999) a recursive technique is proposed which appears to be an extension of the currently popular integrator backstepping idea to solve the tracking problem for a class of nonholonomic chained form control systems. The authors claim that the approach presented can be also extended to treat a global path-following problem. In (Luca *et al.*, 2000), a control law which is valid for trajectory tracking as well as point stabilization tasks for kinematic wheeled mobile robots of the unicycle type is presented.

In spite of recent results however, a unified approach for point-regulation, tracking control, and path-following for underactuated AUVs or even for mobile robots remains, to the best of the author's knowledge, largely unexplored.

As a contribution towards meeting this challenging goal, Chapter 7 offered a solution to the problem of dynamic positioning of an underactuated AUV with parametric model uncertainty in the presence of a constant ocean current disturbance. It is not hard to verify that if one wants to solve a trajectory tracking problem for the AUV, the coordinate transformation given in Section 7.2.1 leads to

$$\dot{e} = -u_r \cos\beta - v_r \sin\beta - V_c \cos(\beta + \psi - \phi_c) + u_d \cos(\beta + \psi - \psi_d), \tag{9.1a}$$

$$\dot{\beta} = \frac{\sin\beta}{e} u_r - \frac{\cos\beta}{e} v_r - r + \frac{V_c}{e} \sin(\beta + \psi - \phi_c) - \frac{u_d}{e} \sin(\beta + \psi - \psi_d), \tag{9.1b}$$

$$\dot{\psi} = r. \tag{9.1c}$$

A quick comparison with equations (7.2) shows that the only difference between the two sets is the presence of the terms u_d and ψ_d that denote the desired forward velocity and steering angle, respectively. These terms are associated with the time-parameterized reference $\{x_d, y_d, \psi_d\}$ that the vehicle must track, which in this case, for a matter of convenience, is generated by a "virtual" unicycle with kinematics described by

$$\dot{x}_d = u_d \cos\psi_d,$$
$$\dot{y}_d = u_d \sin\psi_d,$$
$$\dot{\psi}_d = r_d.$$

Applying the strategy described in Chapter 7, namely, forcing e to converge to a positive constant γ, the kinematic controller for both tracking and point-regulation yields (see Section 7.2.2)

$$u_r = k_1 e - k_1\gamma - \hat{V}_c \cos(\psi - \hat{\phi}_c) + u_d \cos(\psi - \psi_d), \tag{9.2a}$$

$$r = k_1 \sin\beta - k_1 \frac{\gamma}{e} \sin\beta + \frac{\hat{V}_c}{e} \sin(\psi - \hat{\phi}_c)\cos\beta - \frac{u_d}{e}\sin(\psi - \psi_d)\cos\beta - \frac{v_r}{e}\cos\beta + k_2\beta, \tag{9.2b}$$

where k_1 and k_2 are positive constants that must satisfy condition (7.4).

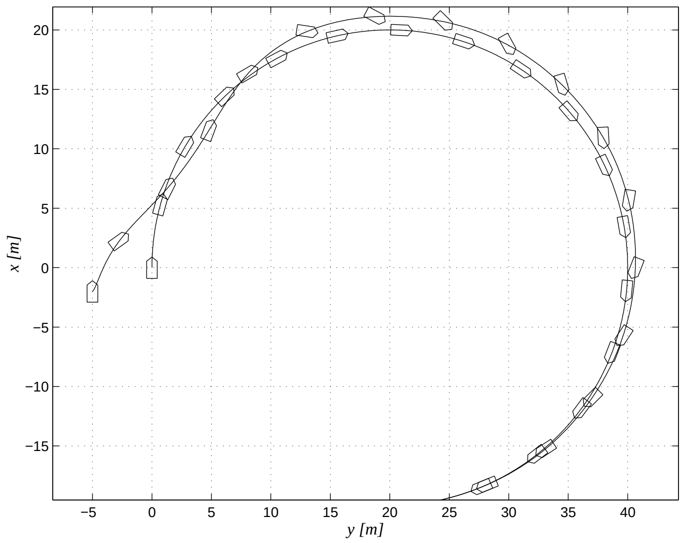

Figure 9.1: Trajectory of the SIRENE AUV in the xy-plane and reference trajectory performed by a "virtual" unicycle. Initial conditions for the AUV: $(x, y, \psi, u, v, r) = (-2\,m, -5\,m, 0, 0, 0, 0)$. An ocean current with intensity $V_c = 0.2\,m/s$ and direction $\phi_c = \frac{\pi}{4}\,rad$ is presented.

An adaptive version of the controller can be developed following the steps described in Chapter 7. Figures 9.1-9.3 show the resulting simulation results. The goal is for the underactuated AUV to track the reference generated by a "virtual" unicycle that starts with the pose $x_d(0) = y_d(0) = \psi_d(0) = 0$ and is animated with linear velocity $u_d(t) = 0.5\,m/s$, and rotational velocity $r_d(t) = 0.025\,rad/s$. The initial conditions for the AUV are $(x, y, \psi, u, v, r) = (-2\,m, -5\,m, 0, 0, 0, 0)$. An ocean current disturbance with intensity $V_c = 0.2\,m/s$ and direction $\phi_c = \frac{\pi}{4}\,rad$ is presented. The control parameters were selected as follows: $k_1 = 0.5$, $k_2 = 0.1$, $k_3 = 100$, $k_4 = 50$, $\gamma = 0.5$, and $\Gamma = \mathrm{diag}(40, 40, 40, 4, 4, 8, 8, 8, 4, 0.4, .4)$. The parameters satisfy the constraints (7.4). The initial estimates for the vehicle parameters were disturbed by 50% from their true values.

Figure 9.1 illustrate the AUV trajectory and the reference trajectory performed by a "virtual" unicycle in xy-plane. As expected, the AUV converges to a neighborhood of the virtual unicycle that is characterized by γ. Figures 9.2 and 9.3 show the time evolution of some relevant variables including the control signals τ_u and τ_r.

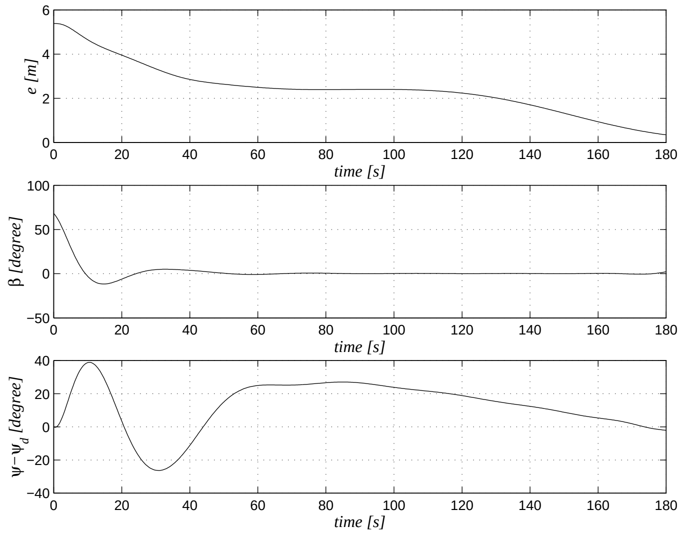

Figure 9.2: Time evolution of variables $e(t)$, $\beta(t)$, and $\psi(t) - \phi_c + \pi$.

To conclude the unified approach problem for point-regulation, tracking control, and path-following for underactuated AUVs, only the path-following problem remains to be solved. A solution to this problem can be explored by viewing the desired velocity u_d as an input control to force the AUV to converge to the path. This idea was recently explored in (Aicardi *et al.*, 2001) to solve the planar path following problem for underactuated marine vehicles. Controller design built on a polar-like kinematic model of the system. The results obtained show that the vehicle converges to and remains inside a tube centered around the desired path with a nonzero speed.

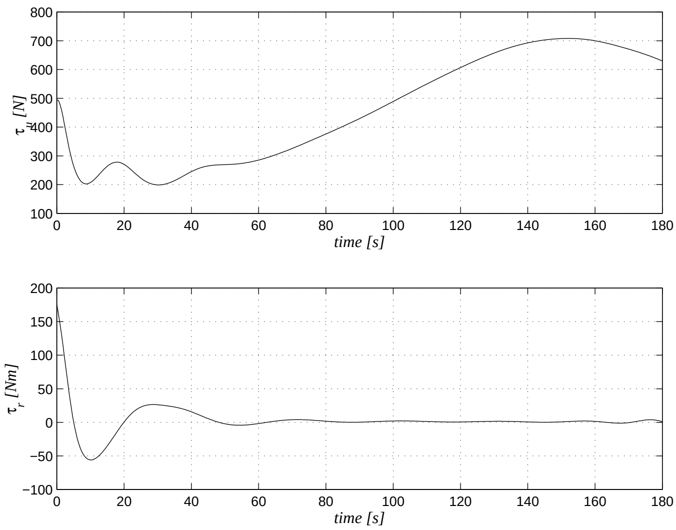

Figure 9.3: Time evolution of control signals $\tau_u(t)$ and $\tau_r(t)$.

From local to global stabilization: a hybrid system approach.

This section presents preliminary work towards the extension of the control of a nonholonomic dynamic wheeled mobile robot subject to input amplitude constraints using the control strategy described in Chapter 8. At this point, no formal proof of the most general result of global stability is available. This will be the subject of future research. It will be shown how a local stabilizing discontinuous control law can in principle be extended to yield global asymptotic stability. The design of the locally stabilizing control law explores the use of a polar representation of the kinematic model of the vehicle that is obtained through a non smooth coordinate transformation in the original state space. The discontinuity in the control law is introduced to overcome the limitations imposed by Brockett's result. See Chapter 5. Simulation results are presented and discussed. An immediately future research is the generalization of the theoretical results described in Chapter 8 to deal with this example.

Consider the wheeled mobile robot introduced in Chapter 2 and the coordinate transformation described in Section 5.1, where the angle β that denotes the angle measured from x_B to d must be in the range $\beta \in (-\pi, \pi]$. The dynamics of the wheeled robot in the new coordinate system can be written as

$$\dot{e} = -v\cos\beta, \tag{9.3a}$$

$$\dot{\beta} = \frac{\sin\beta}{e}v - w, \tag{9.3b}$$

$$\dot{\theta} = w, \tag{9.3c}$$

$$\dot{v} = u_1, \tag{9.3d}$$

$$\dot{\omega} = u_2. \tag{9.3e}$$

If the back of the robot points toward the origin, that is, $\beta \in (-\pi, -\frac{\pi}{2}] \cup (\frac{\pi}{2}, \pi]$, an alternative model can be adopted. It consists of redefining the forward direction of the robot by setting $v_a = -v$. In this case one obtains

$$\dot{e} = -v_a\cos\beta_a, \tag{9.4a}$$

$$\dot{\beta}_a = \frac{\sin\beta_a}{e}v_a - w, \tag{9.4b}$$

$$\dot{\theta} = w, \tag{9.4c}$$

where

$$\beta_a = \begin{cases} -\pi + \beta & \text{if } \beta \geq 0, \\ \pi + \beta & \text{if } \beta < 0. \end{cases}$$

In what follows a control law for the wheeled robot is derived that reflects the $\alpha(x)$, $\beta(x)$ hybrid control structure proposed in Section 8.2.

Main controller $\alpha(x)$

Consider system (9.3). System (9.4) can be dealt with in exactly the same manner.

Let $x = (e, v, \theta, \dot{\theta}, \beta)'$ be the state vector of system (9.3) and consider the local exponential control law

$$\alpha(x) = \begin{bmatrix} \frac{1}{cos\beta}\left(\sin^2\beta\,\frac{v^2}{e} - vw\sin\beta + 2\xi\omega_n\dot{e} + \omega_n^2 e\right) \\ -k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta \end{bmatrix},$$

The motivation for the structure of $\alpha(x)$ can be simply understood by dividing (9.3) in two subsystems, henceforth referred to as the distance and steering subsystems that describe the evolution of $(e, v)'$ and $(\theta, \omega, \beta)'$, respectively. Later, it will become clear that the vectors $(e, v)'$ and $(\theta, \omega, \beta)'$ play the role of z^c and z, respectively.

Distance regulation

Let $\rho = \frac{\dot{e}}{e}$. Differentiate (9.3a) and use (9.3b) to obtain

$$\ddot{e} = -v\rho\tan\beta\sin\beta - vw\sin\beta - u_1\cos\beta.$$

Further let

$$u_1 = -\frac{1}{cos\beta}\left(v\rho\tan\beta\sin\beta + vw\sin\beta + v_1\right).$$

Then,

$$\ddot{e} = v_1$$

and therefore, for $|\beta| \le \beta_{max} < \frac{\pi}{2}$, the distance error can be exponentially regulated to zero by applying the feedback control law

$$v_1 = -2\xi\omega_n\dot{e} - \omega_n^2 e,$$

where ω_n and ξ are arbitrary positive constants.

To analyze the evolution of $\rho(t)$ start by computing its derivative with respect to time, yielding

$$\dot{\rho} = -2\xi\omega_n\rho - \rho^2 - \omega_n^2. \tag{9.5}$$

If $\xi > 1$, equation (9.5) has two equilibrium points given by

$$\rho_\pm^\star = -\xi w_n \pm w_n\sqrt{\xi^2 - 1},$$

where $\rho_-^\star < \rho_+^\star < 0$, the stability of which can be investigated by resorting to the candidate Lyapunov function

$$V_\pm = \frac{1}{2}\left(\rho - \rho_\pm^\star\right)^2.$$

It is straightforward to see that $\rho_+^\star$ is an asymptotically stable equilibrium point, $\rho_-^\star$ is an unstable equilibrium point, and system (9.5) is asymptotically stable for any initial condition $\rho_0 = \rho(t_0)$ such that $\rho_0 > \rho_-^\star$. In fact, it can be seen that (9.5) is a scalar Riccati equation. Its closed-form solution for $\rho_0 > \rho_-^\star$ is given by

$$\rho(t) = \rho_a(t) + \rho_+^\star,$$

where

$$\rho_a(t) = -\frac{(\rho_0 - \rho_+^\star)2w_n\sqrt{\xi^2 - 1}}{\rho_-^\star e^{2w_n\sqrt{\xi^2-1}\,t} + \rho_0 - \rho_+^\star}.$$

Remark 9.1 Since $\rho = \frac{\dot{e}}{e}$ and $\dot{e} = -v\cos\beta$, the condition $\rho_0 > \rho_-^\star$ can be rewritten as

$$v_0 < -\frac{\rho_-^\star e_0}{\cos\beta_0}, \tag{9.6}$$

where $v_0 = v(t_0)$, $e_0 = e(t_0)$, and $\beta_0 = \beta(t_0)$.

Steering regulation

From (9.3), the steering subsystem is given by

$$\begin{aligned}\dot{\beta} &= -\rho\tan\beta - \dot{\theta},\\ \ddot{\theta} &= u_2.\end{aligned}$$

Let

$$u_2 = -k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta.$$

Performing the linearization of the closed loop system about the equilibrium point $(\theta, \dot{\theta}, \beta)' = 0$ and $\rho = \rho_+^\star$ yields $\dot{z} = Az$, where $z = (\theta, \omega, \beta)'$ and

$$A = \begin{bmatrix} 0 & 1 & 0\\ -k_\theta & -k_{\dot{\theta}} & -k_\beta\\ 0 & -1 & -\rho_+^\star \end{bmatrix}. \tag{9.7}$$

Applying the Routh-Hurwitz's stability criterion, it can be proved that the linear system is asymptotically stable if and only if

$$\begin{aligned}&k_\theta < 0,\\ &k_{\dot{\theta}} > -\rho_+^\star > 0,\\ &k_\beta < \frac{k_{\dot{\theta}}\left(\rho_+^{\star 2} + k_{\dot{\theta}}\rho_+^\star + k_\theta\right)}{k_{\dot{\theta}} + \rho_+^\star}.\end{aligned}$$

Local convergence analysis

Let $z = (\theta, \omega, \beta)'$ and $z^c = (e, v)'$ be the variables defined in Section 8.1. Consider the closed loop equations of the steering system

$$\begin{aligned}\dot{\beta} &= -\rho\tan\beta - \dot{\theta},\\ \ddot{\theta} &= -k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta,\end{aligned}$$

that can be rewritten in matrix form as

$$\dot{z} = Az + g(x),$$

where the matrix A is given in (9.7) and

$$g(x) = \begin{bmatrix} 0\\ 0\\ \rho_+^\star\beta - \rho\tan\beta \end{bmatrix}.$$

Notice that for $|\beta| \leq \beta_{max} < \frac{\pi}{2}$ and $\rho_0 > \rho_-^\star$ there exist positive constants γ_1 and γ_2 such that

$$\begin{aligned} |\beta - \tan\beta| &\leq \gamma_1|\beta|, \\ |\tan\beta| &\leq \gamma_2|\beta|, \\ |\rho(t)| &\leq |\rho_a(t)| + |\rho_+^\star| \leq |\rho_0 - \rho_+^\star| + |\rho_+^\star|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|g(x)\| &= |\rho_+^\star\beta - \rho\tan\beta| \\ &= |\rho_+^\star(\beta - \tan\beta) - \rho_a\tan\beta| \\ &\leq \left[\gamma_1|\rho_+^\star| + \gamma_2|\rho_0 - \rho_+^\star|\right]|\beta| \\ &= \bar{\gamma}\|z\| \end{aligned}$$

Since A is a Hurwitz matrix, a candidate Lyapunov function can be found by solving the Lyapunov equation

$$PA + A'P = -Q \tag{9.8}$$

for P, where Q is a real symmetric positive definite matrix. Thus, it follows that the quadratic function

$$V(x) = z'Pz \tag{9.9}$$

satisfies the conditions of Assumption 8.1.i). Clearly, $V(x)$ is positive definite with respect to z and satisfies $\|z\| \to 0 \Rightarrow V(x) \to 0$. Furthermore, the derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= -z'Qz + 2z'Pg(x) \\ &\leq -\left[\lambda_{min}(Q) - 2\lambda_{max}(P)\bar{\gamma}\right]\|z\|^2. \end{aligned}$$

Thus, choosing

$$\bar{\gamma} < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$$

ensures that $\dot{V}(x)$ is negative definite with respect to z for $\rho_0 > \rho_-^\star$ and if $|\beta| \leq \beta_{max} < \frac{\pi}{2}$.

Assumption 8.1.ii) is easily seen to be verified by noticing that the dynamics of z^c are governed by the linear differential equation $\ddot{e} + 2\xi\omega_n\dot{e} + \omega_n^2 e = 0$. Clearly, for $|\beta| \leq \beta_{max}$, the variable z_c is bounded and converges to zero.

Remark 9.2 The ratio $\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$ is maximized with the choice of $Q = I$ (see (Vidyasagar, 1993, Lemma 39)).

Remark 9.3 A simplest but crude estimate of the domain of attraction R_A for the steering system, with $\rho_0 > \rho_-^\star$, is provided by the set

$$\Omega_c = \{z \in \mathbb{R}^3 : V \leq c \text{ and } \dot{V} < 0\},$$

which represents an ellipsoidal estimate of R_A. One way of finding Ω_c is by choosing the constant c as

$$c < \min_{\|z\|=r} V,$$

where r defines the ball $B_r = \{z \in \mathbb{R}^3 : \|z\| \le r\}$ such that $\dot{V}$ is negative definite in B_r. For example, with $Q = I$, $\xi = 1.1$, $w_n = 0.9$, $k_\theta = -0.44$, $k_{\dot{\theta}} = 2.3$, and $k_\beta = -1.08$ yields $\lambda_{min}(P) = 0.24$, $\lambda_{max}(P) = 92.17$, $\rho_+^\star = -0.064$, $\rho_-^\star = -0.156$. Therefore, $\bar{\gamma} < \frac{1}{2\lambda_{max}(P)} = 0.0054$. Picking β and ρ_0 such that $\|g(x)\| \le \bar{\gamma}\|z\|$ holds, one has for example $\beta_{max} = \frac{\pi}{9}$, $r = \beta_{max}$, $\gamma_1 = 0.04$, $\gamma_2 = 1.04$, and $|\rho_0 - \rho_+^\star| = \frac{\bar{\gamma} - \gamma_1|\rho_+^\star|}{\gamma_2} = 0.0027$. Consequently, since $V = z'Pz \ge \lambda_{min}(P)\|z\|^2$, it follows that

$$c = \lambda_{min}(P)\left(\frac{\pi}{9}\right)^2 = 0.029$$

Thus for any initial condition $z_0 \in \Omega_{0.029}$ and $|\rho_0 - \rho_+^\star| \le 0.0027$ the system is asymptotically stable. Notice that even thought the estimation of the domain of attraction by means of the sets Ω_c is simple, it leads to very conservative results. The following results show how to overcome this problem.

Hybrid controller: design procedure

In this section the key ideas in the methodology proposed in Section 8.2 are applied to globally asymptotically stabilize the mobile robot. The technique used *extends the control law developed in Section 8.2 to accommodate the fact that the model of the robot changes according to the value of* β.

Auxiliary controller $\beta(x)$

Consider the feedback law

$$\beta(x) = \begin{bmatrix} -\lambda\,(v + k_1 e) + v\rho \\ -k_2\dot{\theta} - k_3\theta \end{bmatrix}, \tag{9.10}$$

where λ, $k_1 < -\rho_-^\star$, k_2, and k_3 are positive constants. Let $x = (e, v, \theta, \omega, \beta)'$ be the state vector of system (9.3). Then, the closed loop system described by (9.3) and (9.10) satisfies the property that for any initial condition, the state $z(t)$ (or $z_a(t) = [\theta, \dot{\theta}, \beta_a]'$) converges to zero as $t \to \infty$. To prove this, observe that θ and $\dot{\theta}$ converge to zero since in closed loop these two variables are governed by the linear differential equation $\ddot{\theta} = -k_2\dot{\theta} - k_3\theta$ with $k_2 > 0$ and $k_3 > 0$. The remaining of the proof follows by defining the variable $z_1 = \frac{v}{e} + k_1$. It is straightforward to see that z_1 satisfies $\dot{z}_1 = -\lambda z_1$. Thus, $z_1 \to 0$ as $t \to \infty$. Consider now the function

$$V = 1 - \cos\beta,$$

and its derivative

$$\begin{aligned}\dot{V} &= \sin\beta\left[\sin\beta(z_1 - k_1) - \dot{\theta}\right] \\ &= -k_1\sin^2\beta + \sin\beta\left[z_1\sin\beta - \dot{\theta}\right].\end{aligned}$$

Since z_1 and $\dot{\theta}$ converge to zero, it follows that $sin\beta \to 0$ as $t \to \infty$. Observe also that $\rho = -(z_1 - k_1)\cos\beta$. Therefore, $|\rho|$ is bounded, converges to k_1 as $t \to \infty$, and consequently there will be a finite time $T \ge t_0$ such that condition $\rho(t) > \rho_-^\star$ holds for all $t \ge T$. Thus, Assumption 8.2 is verified with

$$\tilde{D} = \{x \in \mathbb{R}^5 : \|z\| \le \epsilon, v < -\frac{\rho_-^\star e}{\cos\beta}\} \subset D$$

or

$$\tilde{D}_a = \{x_a : \|z_a\| \leq \epsilon, v_a < -\frac{\rho^\star_- e}{\cos\beta_a}\} \subset D_a,$$

where $\epsilon > 0$ is such that $|\beta|$ or $|\beta_a|$ is less than $\beta_{max} \leq \frac{\pi}{2}$, and $\|g(x)\| < \bar{\gamma}\epsilon$ or $\|g(x_a)\| < \bar{\gamma}\epsilon$.

Hybrid controller
Define the signals

$$V(x) = z'Pz, \qquad \dot{V}(x) = -\|z\|^2 + 2z'Pg(x),$$
$$V_a(x_a) = z_a'Pz_a, \qquad \dot{V}_a(x_a) = -\|z_a\|^2 + 2z_a'Pg(x_a),$$

where $x = (e, v, \theta, \omega, \beta)'$, $x_a = (e, v_a, \theta, \omega, \beta_a)'$, $z = (\theta, \omega, \beta)'$, $z_a = (\theta, \omega, \beta_a)'$, and the matrix P is the solution of the Lyapunov equation (9.8). Consider the sets

$$\begin{aligned}
D &= \{x : V \geq 0\,,\ \dot{V} \leq 0\,,\ |\beta| \leq \beta_{max}\} \\
S &= \{x : \|g(x)\| < \bar{\gamma}\|z\|\,,\ v < -\frac{\rho^\star_- e}{\cos\beta}\,,\ |\beta| \leq \beta_{max}(1 - h_\beta)\} \\
D_a &= \{x_a : V_a > 0\,,\ \dot{V}_a < 0\,,\ |\beta_a| \leq \beta_{max}\} \\
S_a &= \{x_a : \|g(x_a)\| < \bar{\gamma}\|z_a\|\,,\ v_a < -\frac{\rho^\star_- e}{\cos\beta_a}\,,\ |\beta_a| \leq \beta_{max}(1 - h_\beta)\}
\end{aligned}$$

where $\beta_{max} \in (0, \frac{\pi}{2})$ and the hysteresis constant $h_\beta \in (0, 1)$. Notice that with S defined as above, S is strictly contained in D.

The control law proposed requires switching among the individual control laws

$$u = g_\sigma(q), \tag{9.11}$$

where $q = [x, y, \theta, v, w]'$, and the vector fields $g_\sigma : \mathbb{R}^5 \to \mathbb{R}^2$, $\sigma \in \mathcal{I} = \{1, 2, 3, , 4, 5\}$ are given by

$$\begin{aligned}
g_1(q) &= \begin{bmatrix} \frac{1}{cos\beta}(\sin^2\beta\,\frac{v^2}{e} - vw\sin\beta + 2\xi\omega_n\dot{e} + \omega_n^2 e) \\ -k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta \end{bmatrix}, \\
g_2(q) &= \begin{bmatrix} -\lambda\,(v + k_1 e) + v\rho \\ -k_2\dot{\theta} - k_3\theta \end{bmatrix}, \\
g_3(q) &= \begin{bmatrix} \frac{1}{cos\beta_a}(\sin^2\beta_a\,\frac{v_a^2}{e} + v_a w\sin\beta_a - 2\xi\omega_n\dot{e} - \omega_n^2 e) \\ -k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta_a \end{bmatrix}, \\
g_4(q) &= \begin{bmatrix} \lambda\,(v_a + k_1 e) - v_a\rho \\ -k_2\dot{\theta} - k_3\theta \end{bmatrix}, \\
g_5(q) &= \begin{bmatrix} 0 \\ -k_2\dot{\theta} - k_3\theta \end{bmatrix}.
\end{aligned}$$

Clearly, g_1 and g_2 (respectively, g_3 and g_4) are the main and the auxiliary feedback laws for (9.3) (respectively, (9.4)).

Switching logic
The signal $\sigma(t)$ is a piecewise constant switching signal taking values in $\mathcal{I} = \{1, 2, 3, 4, 5\}$ and is

determined recursively by

$$\sigma(t) = \phi\left(q, \sigma^{-}(t)\right), \quad \sigma^{-}(t_0) = \sigma_0 \in \mathcal{I} \tag{9.12}$$

where the transition function is generated according to

$$\phi(q,\sigma) = \begin{cases} \text{diagram in Figure 9.4}, & \text{if } e \neq 0 \\ 5, & \text{if } e = 0 \end{cases}$$

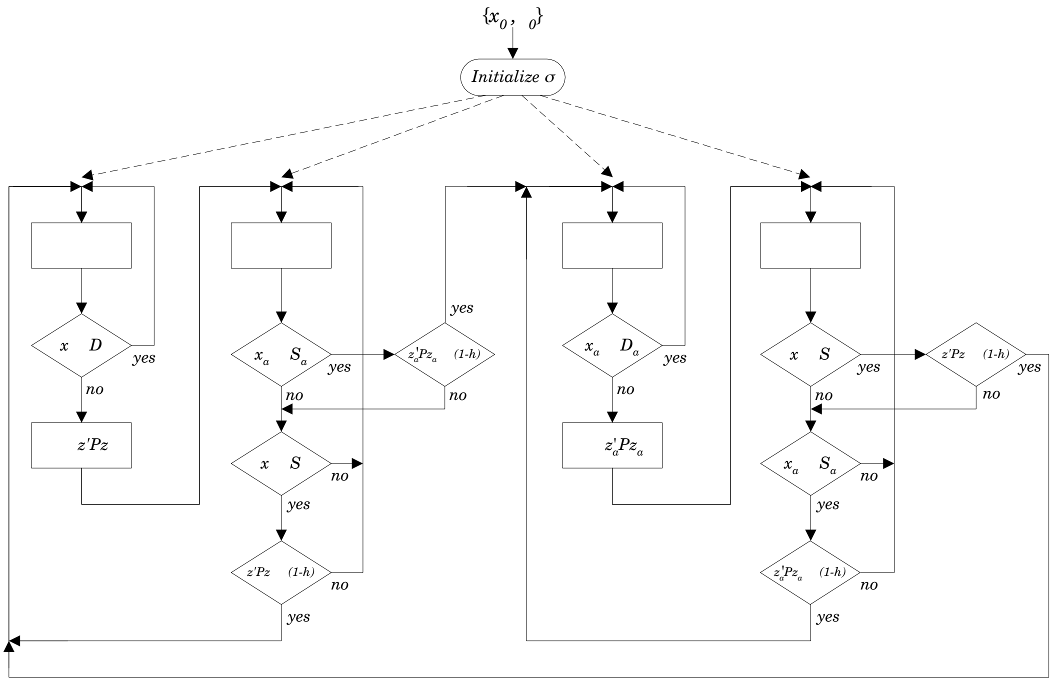

Figure 9.4: Switching logic of the hybrid controller for the mobile robot.

Boundedness of control inputs

Consider now the problem of robot stabilization subject to the input constraints

$$|F(t)| \leq F_{max},$$
$$|N(t)| \leq N_{max},$$

where F_{max} and N_{max} are positive constants.

Let the vector fields $g_\sigma : \mathbb{R}^5 \to \mathbb{R}^2$, $\sigma \in \mathcal{I} = \{1,2,3,,4,5\}$ be redefined as

$$g_1(q) = \begin{bmatrix} \operatorname{sat}\left[\frac{1}{F_{max}cos\beta}\left(\sin^2\beta\,\frac{v^2}{e} - vw\sin\beta - \rho_+^\star\operatorname{sat}(-\rho_-^\star e + \dot{e}) - \rho_-^\star\operatorname{sat}(\dot{e})\right)\right] \\ \operatorname{sat}\left[\frac{1}{N_{max}}\left(-k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta\right)\right] \end{bmatrix},$$

$$g_2(q) = \begin{bmatrix} -\lambda\operatorname{sat}(v + k_1) \\ -k_3\operatorname{sat}(k_2\theta + \dot{\theta}) - k_2\operatorname{sat}(\dot{\theta}) \end{bmatrix},$$

$$g_3(q) = \begin{bmatrix} \operatorname{sat}\left[\frac{1}{F_{max}cos\beta_a}\left(\sin^2\beta_a\,\frac{v_a^2}{e} + v_a w\sin\beta_a + \rho_+^\star\operatorname{sat}(-\rho_-^\star e + \dot{e}) + \rho_-^\star\operatorname{sat}(\dot{e})\right)\right] \\ \operatorname{sat}\left[\frac{1}{N_{max}}\left(-k_\theta\theta - k_{\dot{\theta}}\dot{\theta} - k_\beta\beta_a\right)\right] \end{bmatrix},$$

$$g_4(q) = \begin{bmatrix} -\lambda\,\mathrm{sat}\,(v_a + k_1) \\ -k_3\,\mathrm{sat}(k_2\theta + \dot{\theta}) - k_2\,\mathrm{sat}(\dot{\theta}) \end{bmatrix},$$

$$g_5(q) = \begin{bmatrix} 0 \\ -k_3\,\mathrm{sat}(k_2\theta + \dot{\theta}) - k_2\,\mathrm{sat}(\dot{\theta}) \end{bmatrix},$$

where $0 < \lambda \leq F_{max}$, $k_2 > k_3 > 0$, $k_2 + k_3 \leq N_{max}$, $-(\rho_+^\star + \rho_-^\star) \leq F_{max}$, and $sat(x) = x$, if $|x| \leq 1$, or 1 if $x > 1$, or -1 if $x < -1$.

Observe that the equilibrium point $(\theta, \omega) = 0$ of

$$\ddot{\theta} = -k_3\,\mathrm{sat}(k_2\theta + \dot{\theta}) - k_2\,\mathrm{sat}(\dot{\theta}) \tag{9.13}$$

is globally asymptotically stable. This can be proved by rewriting (9.13) as

$$\dot{y}_1 = k_2 y_2 - k_3\,\mathrm{sat}(y_1) - k_2\,\mathrm{sat}(y_2),$$
$$\dot{y}_2 = -k_3\,\mathrm{sat}(y_1) - k_2\,\mathrm{sat}(y_2),$$

where $y = (y_1, y_2)$, $y_1 = k_2\theta + \dot{\theta}$ and $y_2 = \dot{\theta}$. Clearly, for any initial condition $y(t_0) = y_0$, there exists a finite time $T \geq t_0$ such that $|y_2| \leq 1$ for all $t \geq T$, since

$$y_2\dot{y}_2 = -k_3 y_2\,\mathrm{sat}(y_1) - k_2\,\mathrm{sat}(y_2)$$

and, consequently, for $|y_2| \geq 1$

$$y_2\dot{y}_2 \leq -|y_2|(k_2 - k_3) < 0.$$

Analyzing now the state y_1, one has

$$y_1\dot{y}_1 = k_2 y_1 y_2 - k_3 y_1\ \mathrm{sat}(y_1) - k_2 y_1\ \mathrm{sat}(y_2). \tag{9.14}$$

Since after a finite time $\mathrm{sat}(y_2)$ will be equal to y_2, equation (9.14) gives

$$y_1\dot{y}_1 = -k_3 y_1\ \mathrm{sat}(y_1)$$

which shows that after a finite time y_1 will enter and stay in the set $|y_1| \leq 1$. Therefore, after a finite time the second member of equation (9.13) becomes linear for all t. Consequently, since the linear equation has two eigenvalues in $\lambda_1 = -k_3$ and $\lambda_2 = -k_2$, it follows that the equilibrium point $(\theta, \omega) = 0$ is globally uniformly asymptotically stable.

Simulation results

This section illustrates the performance of the proposed control scheme using computer simulations with a model of a wheeled robot. The objective is to regulate the position and attitude of the robot to zero.

Figure 9.5 shows the vehicle trajectories in the xy-plane for different initial positions (x_0, y_0) and with $\theta_0 = v_0 = \omega_0 = 0$. Notice how the mobile robot converges asymptotically to the origin with a "natural" smooth trajectory. The time evolution of the states $x(t)$, $y(t)$, $\theta(t)$, $v(t)$, and $\omega(t)$ for the initial condition $(x_0, y_0, \theta_0, v_0, \omega_0) = (0, -10\,m, 0, 0, 0)$ are depicted in Figures 9.6-9.7. To better understand the performance of the hybrid control law, Figure 9.8 displays the time evolution of the

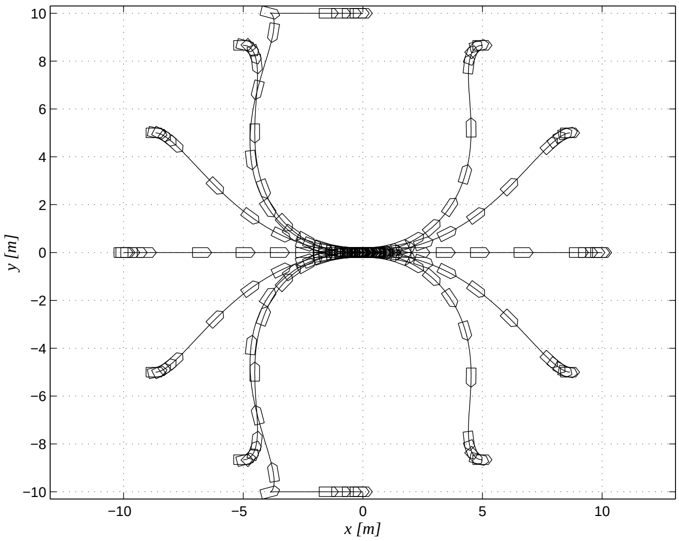

Figure 9.5: Trajectories of the wheeled mobile robot of the unicycle type in the xy-plane for different initial positions (x_0, y_0) and with $\theta_0 = v_0 = \omega_0 = 0$.

variables $e(t)$, $\beta(t)$, and $\sigma(t)$. Initially, σ takes the value 2. This happens because $\beta_0 = 90°$, which is larger than β_{max} (for this simulation, $\beta_{max} = 80°$). When the state enters S (this happens when $\beta = \beta_{max}$), σ switches to 1, after which the main controller dictates the subsequent trajectory evolution. The control signals are presented in Figure 9.9. For this simulation, the magnitude limitations are $F_{max} = 0.3\,N$ and $N_{max} = 0.2\,Nm$.

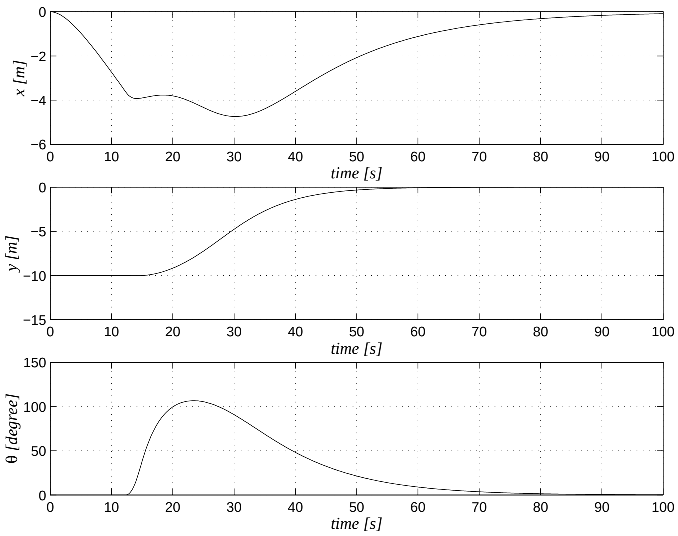

Figure 9.6: Time evolution of the position variables $x(t)$, $y(t)$, and orientation variable $\theta(t)$.

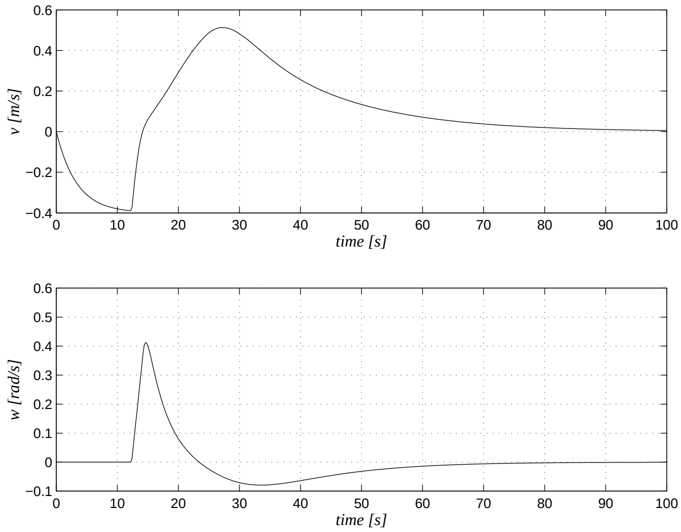

Figure 9.7: Time evolution of the linear velocity $v(t)$ and angular velocity $\omega(t)$.

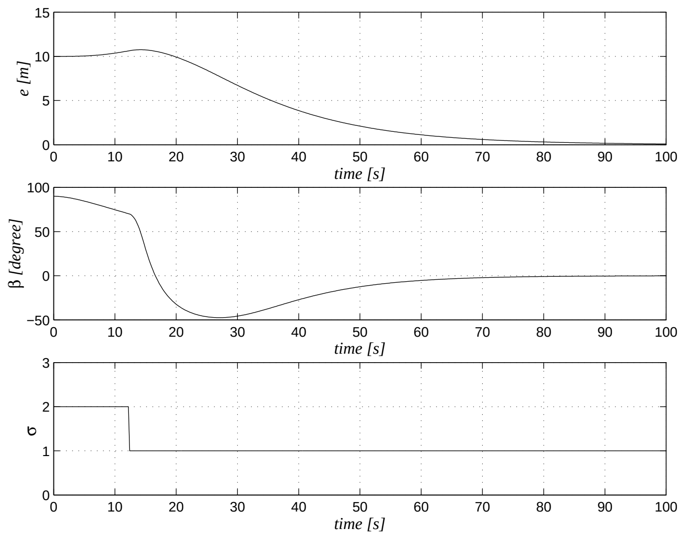

Figure 9.8: Time evolution of variables $e(t)$, $\beta(t)$, and $\sigma(t)$.

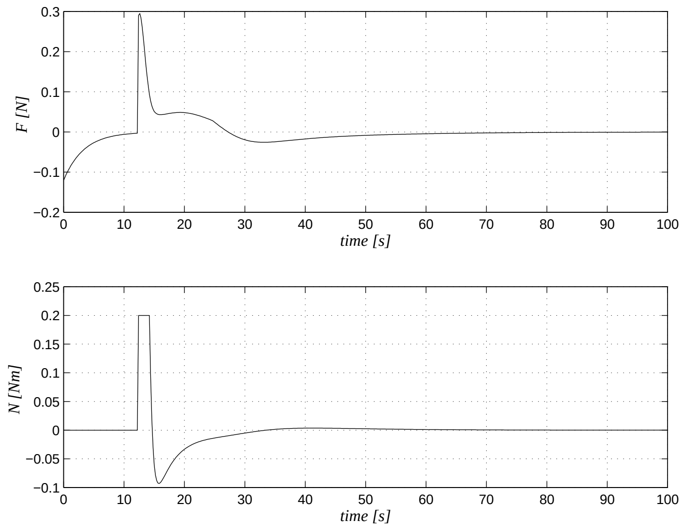

Figure 9.9: Time evolution of control signals $F(t)$ and $N(t)$.

Regulation of an underactuated and nonholonomic AUV (3D)

In Chapter 6, the regulation problem for an underactuated AUV was solved in the horizontal plane. Consider now the problem of regulating the position and attitude of an underactuated AUV in 3-dimensional space. In the case of the SIRENE AUV, the problem is not hard to solve. Since SIRENE has a vertical thruster and exhibits large restoring forces caused by small motions in roll and pitch very high, a simple strategy consists of designing separate controllers for the vertical and the horizontal planes, leaving roll and pitch passive (Aguiar and Pascoal, 1997). In this case, the horizontal controller is the one derived in Chapter 6, and the feedback controller for depth can be derived using standard nonlinear techniques such as feedback linearization.

The problem is entirely different when the configuration of the actuators is different from the one of SIRENE. Namely, when the AUV has available control force in surge and control torques in roll, pitch, and yaw, but no actuators in sway and heave. Another interesting thruster configuration arises when the AUV has no actuator in roll either.

The literature on this control problem is very scarce. In (Egeland *et al.*, 1996) and (Leonard, 1995), the control of an AUV with four control inputs was studied. However, these references only consider a kinematic, drift free model of the AUV, where the velocities are the control inputs. In (Pettersen and Egeland, 1996), the design of a continuous, periodic feedback control law that asymptotically stabilizes an underactuated AUV and yields locally exponential convergence to the origin is described. The control inputs are the force in surge and the torques in roll, pitch, and yaw. Assuming that the hydrodynamic restoring forces in roll are large enough, it is also shown how the AUV can be exponentially stabilized by the same feedback law without roll control torque. However, the resulting closed loop system trajectories exhibit undesired oscillatory motions.

Motivated by these considerations, preliminary steps were already taken towards a complete solution of the problem of regulating the pose of an AUV in 3-D space. They are briefly described below. Notice however that a formal proof of global convergence is still missing.

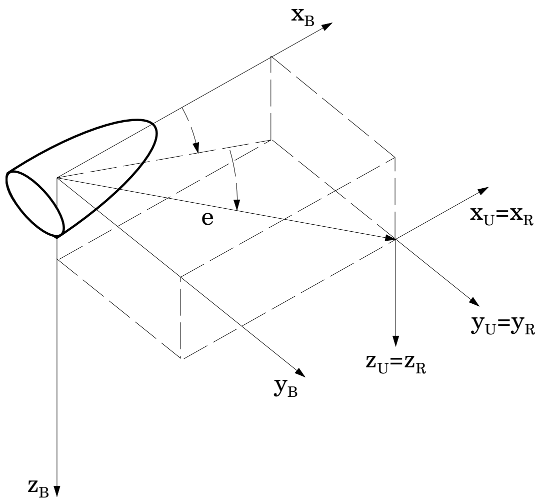

Figure 9.10: The underactuated AUV: coordinate transformation.

Consider an AUV where the inputs are the control force in surge and the control torques in pitch and yaw. Assume the roll dynamics are left passive. Then, from 2.15 (see Section 2.5) the simplified kinematics and dynamics equations of the AUV (with $\phi = p = 0$) are given by

$$\dot{u} = \frac{1}{m_u}\big(\tau_1 + m_v vr - m_w wq - d_u u\big), \tag{9.15a}$$
$$\dot{v} = \frac{1}{m_v}\big(-m_u ur - d_v v\big), \tag{9.15b}$$
$$\dot{w} = \frac{1}{m_w}\big(m_u uq - d_w w\big), \tag{9.15c}$$
$$\dot{q} = \frac{1}{m_q}\big(\tau_5 + m_{uw} uw - d_q q - Z_B B \sin\theta\big), \tag{9.15d}$$
$$\dot{r} = \frac{1}{m_r}\big(\tau_6 - m_{uv} uv - d_r r\big), \tag{9.15e}$$
$$\dot{x} = u\cos\psi\cos\theta - v\sin\psi + w\cos\psi\sin\theta, \tag{9.15f}$$
$$\dot{y} = u\sin\psi\cos\theta + v\cos\psi + w\sin\psi\sin\theta, \tag{9.15g}$$
$$\dot{z} = -u\sin\theta + w\cos\theta, \tag{9.15h}$$
$$\dot{\theta} = q, \tag{9.15i}$$
$$\dot{\psi} = \frac{1}{\cos\theta} r, \tag{9.15j}$$

where

$$\begin{array}{lclclcl}
m_u & = & m - X_{\dot{u}} & & & & \\
m_v & = & m - Y_{\dot{v}} & d_u & = & -\big(X_u + X_{u|u|}|u|\big) \\
m_w & = & m - Z_{\dot{w}} & d_v & = & -\big(Y_v + Y_{v|v|}|v|\big) \\
m_q & = & I_y - M_{\dot{q}} & d_w & = & -\big(Z_w + Z_{w|w|}|w|\big) \\
m_r & = & I_z - N_{\dot{r}} & d_q & = & -\big(M_q + M_{q|q|}|q|\big) \\
m_{uv} & = & X_{\dot{u}} - Y_{\dot{v}} & d_r & = & -\big(N_r + N_{r|r|}|r|\big) \\
m_{uw} & = & X_{\dot{u}} - Z_{\dot{w}} & & & &
\end{array}$$

Applying the coordinate transformation (see Figure 9.10),

$$
\begin{aligned}
e &= \sqrt{x^2 + y^2 + z^2}, \\
x &= -e\big[\cos\psi\cos\theta\cos\alpha\cos\beta - \sin\psi\sin\beta + \cos\psi\sin\theta\sin\alpha\sin\beta\big], \\
y &= -e\big[\sin\psi\cos\theta\cos\alpha\cos\beta + \cos\psi\sin\beta + \sin\psi\sin\theta\sin\beta\big], \\
z &= -e\big[-\sin\theta\cos\alpha\cos\beta + \cos\theta\sin\alpha\sin\beta\big], \\
\alpha &= \tan^{-1}\Big(\frac{-\cos\psi\sin\theta x - \sin\psi\sin\theta y - \cos\theta z}{cos\psi\cos\theta x - \sin\psi\cos\theta y + \sin\theta z}\Big), \\
\beta &= \sin^{-1}\Big(\frac{\sin\psi x - \cos\psi y}{e}\Big),
\end{aligned}
$$

yields

$$
\begin{aligned}
\dot{e} &= -u\cos\alpha\cos\beta - v\sin\beta - w\sin\alpha\cos\beta, && (9.16a)\\
\dot{\alpha} &= q + \frac{\sin\alpha}{e\cos\beta}u - \frac{\cos\alpha}{e\cos\beta}w + \frac{\sin(\theta-\alpha)}{\cos\theta}\frac{\sin\beta}{\cos\beta}r, && (9.16b)\\
\dot{\beta} &= -\frac{\cos(\theta-\alpha)}{\cos\theta}r + \frac{\sin\beta\cos\alpha}{e}u - \frac{\cos\beta}{e}v + \frac{\sin\beta\sin\alpha}{e}w. && (9.16c)
\end{aligned}
$$

A local control law for the vehicle is now derived.

Distance regulation
Differentiate (9.16a) to obtain

$$\ddot{e} = -\frac{\cos\alpha\cos\beta}{m_u}\Big(\tau_1 + m_v vr - m_w wq - d_u\Big) + f(\cdot),$$

where $f(\cdot)$ represents the remaining terms. Let

$$\tau_1 = -\frac{m_u}{\cos\alpha\cos\beta}\big[u_1 - f(\cdot)\big] - m_v vr + m_w wq + d_u u.$$

Then,

$$\ddot{e} = u_1.$$

Therefore, the distance error can be exponentially regulated to zero by applying the feedback control law

$$u_1 = -2\xi w_n \dot{e} - w_n^2 e,$$

where ω_n and ξ are arbitrary positive constants.

Orientation regulation
Let τ_5 and τ_6 in (9.15d)-(9.15e) be

$$\tau_5 = m_q u_2 - m_{uw} uw + d_q q + Z_B B\sin\theta$$

and

$$\tau_6 = m_r u_3 + m_{uv} uv + d_r r.$$

Then

$$\dot{q} = u_2,$$
$$\dot{r} = u_3.$$

Let u in (9.16a) be given by

$$u = -\frac{\dot{e} + v\sin\beta + w\sin\alpha\cos\beta}{\cos\alpha\cos\beta}. \tag{9.17}$$

Substituting (9.17) into (9.16b) and (9.16c), gives

$$\dot{\alpha} = q - \frac{\tan\alpha}{\cos^2\beta}\rho - \frac{\tan\alpha\tan\beta}{\cos\beta}\xi_1 - \frac{1}{\cos\alpha\cos\beta}\xi_2 + \frac{\sin(\theta-\alpha)}{\cos\theta}\tan\beta r,$$
$$\dot{\beta} = -\frac{\cos(\theta-\alpha)}{\cos\theta}r - \tan\beta\rho - \frac{1}{\cos\beta}\xi_1,$$

where $\rho = \frac{\dot{e}}{e}$, $\xi_1 = \frac{v}{e}$, and $\xi_2 = \frac{w}{e}$.

Performing the linearization of the orientation system about the equilibrium point $(\theta, q, \psi, r, \alpha, \beta)' = 0$ and $\rho = \rho_1^\star$, $\xi_1 = 0$, and $\xi_2 = 0$ yields

$$\begin{bmatrix} \dot{\theta} \\ \dot{q} \\ \dot{\psi} \\ \dot{r} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\rho_1^\star & 0 \\ 0 & 0 & 0 & -1 & 0 & -\rho_1^\star \end{bmatrix} \begin{bmatrix} \theta \\ q \\ \psi \\ r \\ \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \tag{9.18}$$

where $\rho_1^\star$ is the stable equilibrium point of the ρ dynamics. It can be seen that the linear system (9.18) is controllable. Consequently, by choosing an appropriate constant 2×6 matrix K (using for example the LQR methodology), the state feedback control law

$$u_2 = K\left[\theta, q, \psi, r, \alpha, \beta\right]'$$

achieves exponential convergence of the state $(\theta, q, \psi, r, \alpha, \beta)'$ to zero.

It can now be proved by resorting for example to the theory of interconnected systems (Khalil, 1996) that ξ_1 and ξ_2 locally converge exponentially to zero.

Figures 9.11-9.13 illustrate the performance of the local exponential convergent control law derived. In the simulation, the SIRENE AUV model was used with the following modification: the control inputs are only the force in surge and the torques in pitch and yaw. The initial condition is $x_0 = -2\,m$, $y_0 = 1\,m$, $z_0 = -1\,m$, and $\phi = \theta = \psi = u = v = w = p = q = r = 0$.

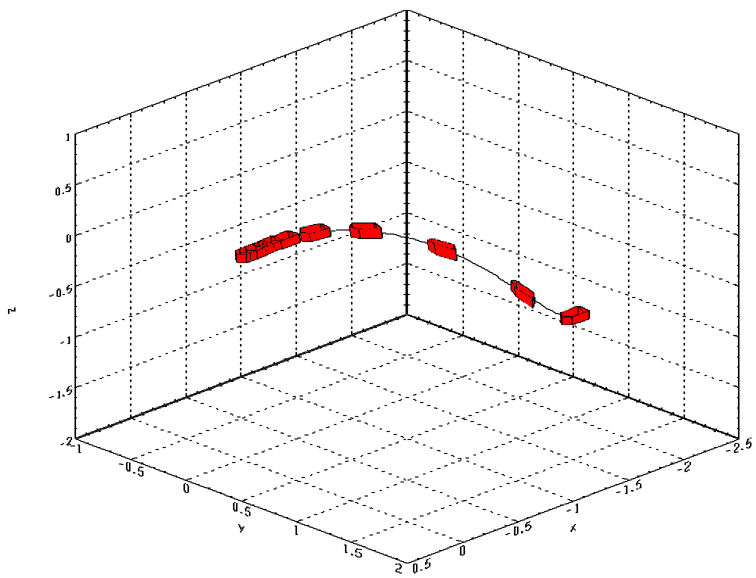

Figure 9.11: Vehicle trajectory in 3-D space. The initial condition is $x_0 = -2\,m$, $y_0 = 1\,m$, $z_0 = -1\,m$, and $\phi = \theta = \psi = u = v = w = p = q = r = 0$.

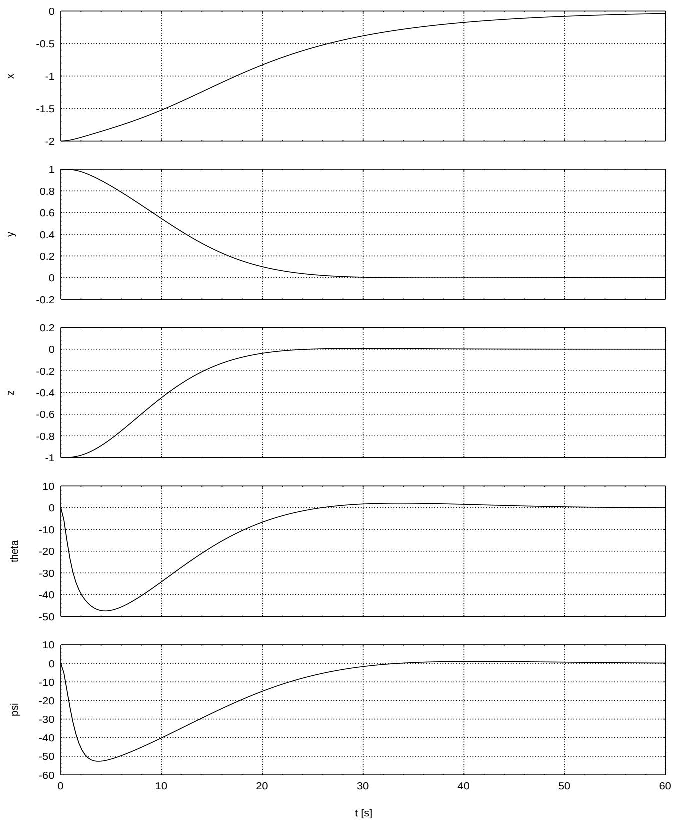

Figure 9.12: Time evolution of state variables $x(t)\,[m]$, $y(t)\,[m]$, $z(t)\,[m]$, $\theta(t)\,[degree]$, and $\psi(t)\,[degree]$.

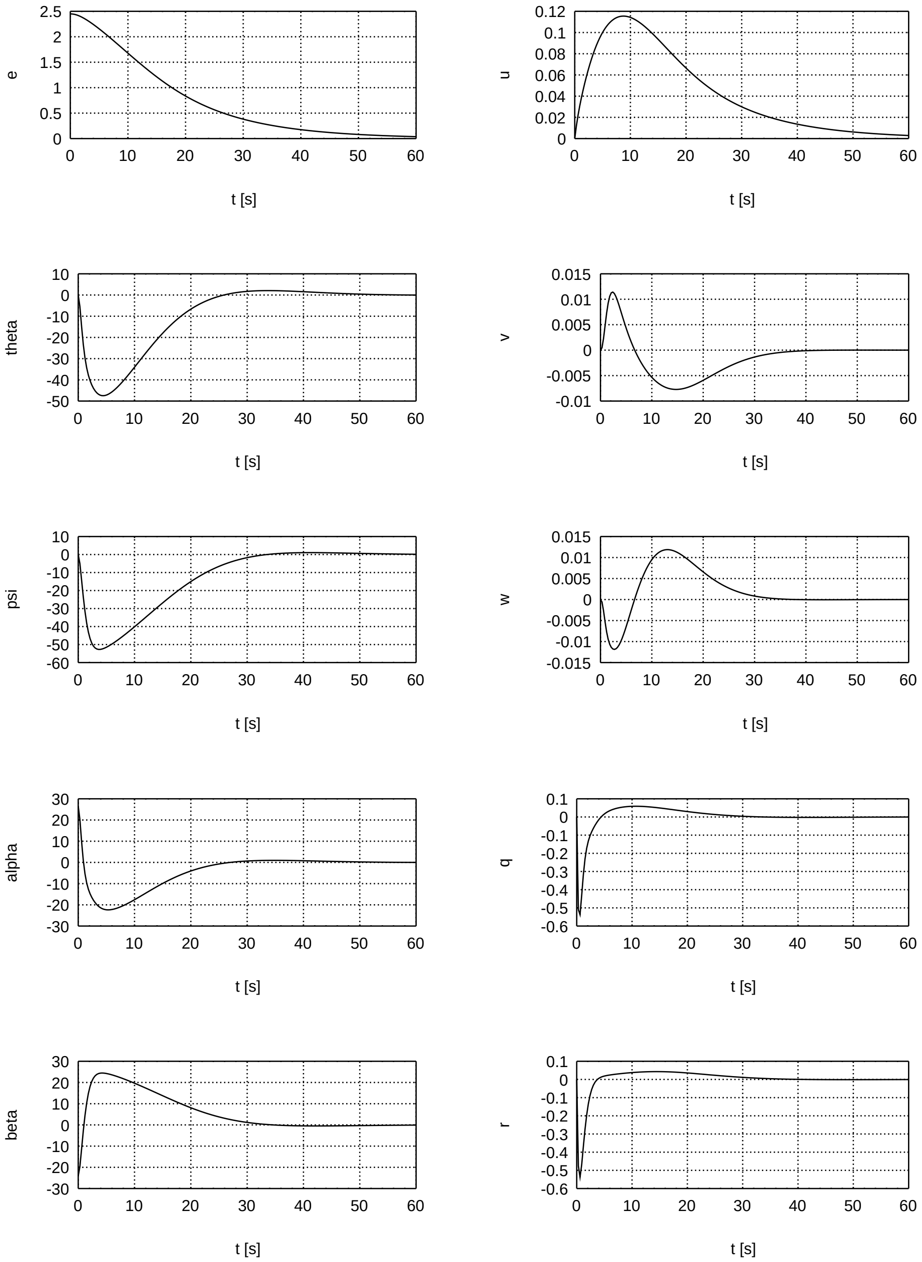

Figure 9.13: Time evolution of variables $e(t)\,[m]$, $\theta(t)\,[degree]$, $\psi(t)\,[degree]$, $\alpha(t)\,[degree]$, $\beta(t)\,[degree]$, $u(t)\,[m/s]$, $v(t)\,[m/s]$, $w(t)\,[m/s]$, $q(t)\,[rad/s]$, and $r(t)\,[rad/s]$.

Disturbance attenuation and robust control for nonholonomic systems

A promising and challenging problem that warrants further research effort is that of control system analysis and design for nonholonomic systems in the presence of exogenous signals, actuator saturation constraints, and parameter uncertainty. The problem of control under incomplete state measurements is also of great interest and touches open some deep issues in nonlinear control system analysis that addresses the explicit inclusion of observer dynamics in a control loop. Previous work in this area includes that of (Jiang and Pomet, 1996) where a backstepping based, time-varying adaptive control law for a special class of uncertain, nonholonomic, chained systems was derived. The reader is referred to (Yang and Kim, 1999) for the description of a sliding mode control law for trajectory tracking of nonholonomic wheeled mobile robots in the presence of bounded disturbances. See also (Hespanha *et al.*, 1999), where hybrid control techniques were introduced to park a kinematic wheeled mobile robot of unicycle type with parameter uncertainty. In this work the unknown parameters are the radius of the rear wheels and the distance between them. More recently, in (Dong *et al.*, 2000) an adaptive robust controller was derived to solve the stabilization problem for uncertain dynamic nonholonomic chained systems. The work of (Valtolina and Astolfi, 2000) modifies and applies the discontinuous control law considered in (Astolfi, 1996) to chained systems. The new controller assures global asymptotic regulation, local exponential stability in the sense of Lyapunov, and local robustness against measurement errors and exogenous disturbances.

In this thesis, the problem of robust control in the presence model parameter uncertainty was solved in Chapters 5 and 6 of this thesis for a wheeled robot and an AUV by resorting to adaptive control laws. External disturbances attenuation was studied in Chapter 7 for the case where the disturbance is an unknown constant ocean current acting on an AUV. The study required the analysis of the combined behaviour of a state-feedback control law and a current observer. Finally, in Chapter 3, a solution to the practical stabilization of the ENDI system under saturation constraints and input additive disturbances was proposed. These contributions are but small steps towards the goal of developing efficient tools for control system design when the nonlinear plant to be controlled is subjected to external disturbances, model uncertainty, sensor noise, and actuator constraints. Much work remains to be done in this area.

From theory to practical applications

Questions of a different flavor arise when trying to bridge the gap between theory and practice. In the spite of the tremendous effort in the field on nonlinear system control, there is still a fundamental lack of practical applications that illustrate how the theory developed can indeed be used to control of a class of systems that include wheeled vehicles and marine robots. The simulation studies carried out indicate that the types of control strategies developed are good candidates for practical applications. However, further work is required to transition from theory to practice and to actually implement the strategies developed in real vehicles. In particular, for the case of the SIRENE or similar underwater vehicles, the problem of control system re-design in the absence of full state information must be addressed and solved. Finally, an emerging line of research is that of sensor-based control of nonlinear systems, that is, control using vision and other non conventional sensors that include sonar.

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