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Nonlinear Moving Horizon State Estimation for a Hovercraft with Continuation/GMRES Method

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Abstract

This paper proposes a fast algorithm for nonlinear moving horizon state estimation. The estimates are updated by a differential equation to trace the solution of an associated two-point boundary-value problem. A linear equation involved in the differential equation is solved by the generalized minimum residual (GMRES) method, one of the Krylov subspace methods. The proposed algorithm is applied to a hovercraft whose dynamics is highly nonlinear. The estimates are compared with the actual measurements.

1 Introduction

Nonlinear state estimation has been an important and challenging problem for many years. By now, extended Kalman filer, statistical approximation approach and extended Luenberger observer have been proposed for a nonlinear state estimation problem. However, those methods often result in poor performance when they are applied to highly nonlinear systems, since they are formulated with linearization or approximation of nonlinearities.

On the other hand, moving horizon state estimation is applied successfully to chemical industry [1], where the sampling interval is sufficiently large. The advantage of this method is that it is possible to formulate a nonlinear estimation problem without linearization. Another key feature of this method is the ability to handle constraints directly, including constraints on the system disturbances or internal states. However, it is difficult to apply this method to mechanical systems, where the sampling intervals are much shorter than those in chemical processes, since this method requires a lot of computational time for on-line optimization even if the newest computers are used.

In this paper, we propose a fast algorithm for nonlinear moving horizon state estimation, which can be executed in real time for nonlinear mechanical systems. As a related problem, C/GMRES [2] has been proposed for nonlinear receding horizon control in real time. In this method, the optimal control problem is discretized over the horizon first, and then a differential equation to update the sequence of control inputs is obtained through use of the continuation method [6]. Since that differential equation involves a large linear equation, the GMRES (Generalized Minimum RESidual) method [5] is employed to solve the linear equation. Hence we consider applying that method to moving horizon state estimation as a dual problem.

The proposed algorithm is applied to a hovercraft in order to examine the computational time. Experimental results show that nonlinear moving horizon state estimation is possible in real time for the highly nonlinear system with the proposed algorithm.

2 Problem Formulation

This section briefly summarizes the nonlinear moving horizon state estimation problem. We consider a generic nonlinear system expressed by the following state equation and measurement equation:

$$\begin{cases} \dot{x}(t) = f[x(t), u(t), w(t)] \\ y(t) = g[x(t), u(t), v(t)] \end{cases}, \tag{1}$$

where functions f, g are differentiable as many times as necessary, $x(t) \in \mathbf{R}^n$ denotes a state vector, $u(t) \in \mathbf{R}^{m_u}$ an input vector, $w(t) \in \mathbf{R}^{m_w}$ a vector of unknown disturbance, $y(t) \in \mathbf{R}^{m_y}$ a measurement vector, and $v(t) \in \mathbf{R}^{m_v}$ a vector of unknown measurement noise, respectively. For the sake of simple notation, all known quantities are put together into a vector $p(t) \in \mathbf{R}^{m_p}$; e.g., $p(t) = [y^T(t) \;\; u^T(t)]^T$. It is assumed that $v(t)$ can be represented by the state vector $x(t)$ and the known quantity $p(t)$ as

$$v(t) = h[x(t), p(t)].$$

This assumption holds trivially in the case of additive measurement noise and is not a significant restriction on the problem. If the measurement noise $v(t)$ is zero, the estimate of the state $\hat{x}(t)$ should be determined so that estimate of the measurement noise $\hat{v}(t) = h[\hat{x}(t), p(t)]$ is zero. The measurement noise $\hat{v}(t)$ corresponds to the estimation residual and should be minimized. Therefore quantities to be minimized are expressed by known quantities and estimates. In general, the estimates of the state $x^*(t)$ and unknown disturbance $w^*(t)$ at each time t are determined so as to minimize a performance index with a moving horizon:

$$J = \eta[x^*(t), p(t)] + \phi[x^*(t-T), p(t-T)] + \int_{t-T}^{t} L[x^*(t'), p(t'), w^*(t')]dt'.$$

Equality constraints are also imposed in general as:

$$C[x(t), w(t), p(t)] = 0,$$

where C is an m_c dimensional vector-valued function. An inequality constraint can be converted to an equality constraint with a dummy input (slack variable). Therefore, the present problem setup can deal with various performance indexes and constraints.

At each time t, we find the optimal trajectory over $[t-T,\ t]$ minimizing the performance index and employ it terminal value at time t as the present estimate $\hat{x}(t)$. To this end, we divide the horizon into N steps and discretize the optimal estimation problem with the backward difference as follows:

$$x_i^*(t) = x_{i+1}^*(t) - f[x_{i+1}^*(t), u_{i+1}^*(t), w_{i+1}^*(t)]\Delta\tau, \quad (2)$$

$$x_N^*(t) = \hat{x}(t), \quad (3)$$

$$C[x_i^*(t), w_i^*(t), p_i^*(t)] = 0, \quad (4)$$

$$J^* = \eta[x_N^*(t), p_N^*(t)] + \phi[x_0^*(t), p_0^*(t)] + \sum_{i=1}^{N} L[x_i^*(t), p_i^*(t), w_i^*(t)]\Delta\tau, \quad (5)$$

where the discretization step is determined by $\Delta\tau := T/N$, $x_i^*(t)$ corresponds to the state at time $t - T + i\Delta\tau$ on the optimal trajectory terminating at $\hat{x}(t)$, and $p_i^*(t)$ and $u_i^*(t)$ are given by $p(t-T+i\Delta\tau)$ and $u(t-T+i\Delta\tau)$, respectively. The horizon T of the performance index may be time-dependent in general. We assume that the initial guess $\hat{x}(0)$ is given and L, η and ϕ are differentiable as many times as necessary. The estimate of the state $\hat{x}(t)$ and estimates of disturbance $\{w_i^*(t)\}_{i=1}^N$ are optimized at each time t.

The Hamiltonian H is defined as

$$H(x, \lambda, p, w, \mu) := L(x, w, p) + \lambda^T f(x, w, p) + \mu^T C(x, w, p).$$

where $\lambda \in \mathbf{R}^n$ denotes the costate, and $\mu \in \mathbf{R}^{m_c}$ denotes the Lagrange multiplier associated with the equality constraint. The first-order necessary conditions for the optimal estimates $\hat{x}(t), \{w_i^*(t)\}_{i=1}^N$ are readily obtained as a two-point boundary-value problem by the calculus of variations as

$$H_w^T[x_i^*(t), \lambda_{i-1}^*(t), p_i^*(t), w_i^*(t), \mu_i^*(t)] = 0, \quad (6)$$

$$\lambda_i^*(t) = \lambda_{i-1}^*(t) - H_x^T[x_i^*(t), \lambda_{i-1}^*(t), p_i^*(t), w_i^*(t), \mu_i^*(t)]\Delta\tau, \quad (7)$$

$$\lambda_N^*(t) = \eta_x^T[x_N^*(t)], \quad (8)$$

$$\lambda_0^*(t) = -\phi_x^T[x_0^*(t)], \quad (9)$$

where H_x is the partial derivative of H with respect to x. The sequences of the optimal estimates $\{w_i^*(t)\}_{i=1}^N$, $\hat{x}(t)$ and the multiplier $\{\mu_i^*(t)\}_{i=1}^N$ have to satisfy Eqs. (2)–(4), (6)–(9). The solution of the discretized problem converges to the solution of the continuous problem as $N \to \infty$ under mild conditions [3].

We define a vector of the estimates and multipliers and a vector of known quantities as

$$W(t) := [w_1^{*T}(t)\ \mu_1^{*T}(t)\ w_2^{*T}(t)\ \mu_2^{*T}(t) \ldots w_N^{*T}(t)\ \mu_N^{*T}(t)\ x_N^{*T}(t)]^T \in \mathbf{R}^m,$$

$$P(t) := [p_1^{*T}(t)\ p_2^{*T}(t) \ldots p_N^{*T}(t)]^T \in \mathbf{R}^{m_p N},$$

where $m := (m_w + m_c)N + n$. We also define a projection $P_N : \mathbf{R}^m \to \mathbf{R}^n$ as

$$P_N(W) := x_N^*.$$

For given $W(t)$ and $P(t)$, $\{x_i^*(t)\}_{i=1}^N$ are calculated backward recursively by Eq. (2), and then, $\{\lambda_i^*(t)\}_{i=1}^N$ are also calculated recursively from $i = 0$ to $i = N$ by Eqs. (7) and (9). Therefore, Eqs. (4), (6) and (8) can be regarded as an equation defined as

$$F[W(t), P(t), t] := \begin{bmatrix} H_w^T[x_1^*(t), \lambda_0^*(t), p_1^*(t), w_1^*(t), \mu_1^*(t)] \\ C[x_1^*(t), w_1^*(t), p_1^*(t)] \\ \vdots \\ H_w^T[x_N^*(t), \lambda_{N-1}^*(t), p_N^*(t), w_N^*(t), w_N^*(t), \mu_N^*(t)] \\ C[x_N^*(t), w_N^*(t), p_N^*(t)] \\ \lambda_N^*(t) - \eta_{x_N}^T[x_N^*(t)] \end{bmatrix} = 0.$$

If the equation is solved with respect to $W(t)$ for the known quantities $P(t)$, then the optimal estimate $\hat{x}(t) = P_N(W(t))$ is obtained.

3 Continuation/GMRES Method

Instead of solving $F(W, P, t) = 0$ itself at each time with such an iterative method as the Newton method, we find the derivative of W with respect to time such that $F(W, P, T) = 0$ is stabilized. Namely, we determine $\dot{W}(t)$ so that

$$\dot{F}(W, P, t) = -\zeta F(W, P, t), \tag{10}$$

where ζ is a positive real number. Then, if F_W is nonsingular, the derivative of W is given by

$$\dot{W} = F_W^{-1}(-\zeta F - F_P \dot{P} - F_t), \tag{11}$$

which is integrated in real time to obtain the estimates. If $W(0)$ is chosen so that $F(W(0), P(0), 0) = 0$, then the solution $W(t)$ is traced by integrating Eq. (11), which is a kind of the continuation method [6]. We assume that the sequence of optimal estimates is a smooth function in Eq. (11). This assumption is not a practical limitation, since the actual computational process is discrete in time.

From the computational point of view, the differential equation (11) involves expensive operations, i.e., Jacobians F_W, F_P and F_t and a linear equation associated with F_W^{-1}. In order to reduce the computational cost in the Jacobians and the linear equation, we employ two devices, i.e., forward difference approximation for products of Jacobians and vectors, and the GMRES method [5] for the linear equation.

First, we approximate the products of the Jacobians and some $a_1 \in \mathbf{R}^m$, $a_2 \in \mathbf{R}^{m_p N}$ and $a_3 \in \mathbf{R}$ with the forward difference as follows:

$$\begin{aligned} & F_W(W, P, t)a_1 + F_P(W, P, t)a_2 + F_t(W, P, t)a_3 \\ & \simeq D_h F(W, P, t : a_1, a_2, a_3) \\ & := \frac{F(W + ha_1, P + ha_2, t + ha_3) - F(W, P, t)}{h}, \end{aligned}$$

where h is a positive real number. Then Eq. (10) is approximated by

$$D_h F(W, P, t : \dot{W}, \dot{P}, 1) = -\zeta F(W, P, t),$$

which is equivalent to

$$D_h F(W, P, t : \dot{W}, 0, 0) = b(W, P, \dot{P}, t), \tag{12}$$

where

$$b(W, P, \dot{P}, t) := -\zeta F(W, P, t) - D_h F(W, P, t : 0, \dot{P}, 1).$$

It should be noted that the forward difference approximation is different from the finite difference approximation of the Jacobians themselves. The forward difference approximation of the products of the Jacobians and vectors can be calculated with only one additional evaluation of the function, which requires notably less computation than approximation of the Jacobians themselves. Because of the forward difference approximation, even a sparse Jacobians is not necessary.

Second, we apply GMRES to Eq. (12) since Eq. (12) approximates a linear equation with respect to $\dot{W}$.

Algorithm 1 (FDGMRES)

$\dot{W} :=$ FDGMRES$(W, P, \dot{P}, t, h, k_{max})$

1. $r_0 := b(W, P, \dot{P}, t)$, $v_1 := r_0/||r_0||$, $\rho := ||r_0||$, $\beta := \rho, k := 0$.
2. While $k < k_{max}$, do
   - (a) $k := k + 1$
   - (b) $v_{k+1} := D_h F(W, P + h\dot{P}, t + h : v_k, 0, 0)$ for $j = 1, ..., k$
     - i. $h_{jk} := v_{k+1}^T v_j$
     - ii. $v_{k+1} := v_{k+1} - h_{jk} v_j$
   - (c) $h_{k+1,k} := ||v_{k+1}||$
   - (d) $v_{k+1} := v_{k+1}/||v_{k+1}||$
   - (e) For $e_1 = [1\ 0...0]^T \in \mathbf{R}^{k+1}$ and $H_k = (h_{ij}) \in \mathbf{R}^{(k+1)\times k}$ ($h_{ij} = 0$ for $i > j + 1$), Minimize $||\beta e_1 - h_k z^k||$ to determine $z^k \in \mathbf{R}^k$.
   - (f) $\rho := ||\beta e_1 - H_k z^k||$.
3. $\dot{W} := V_k z^k$, where $V_k = [v_1 ... v_k] \in \mathbf{R}^{m \times k}$.

GMRES is a kind of the Krylov subspace methods for such a linear equation as $Ax = b$ with a nonsymmetric matrix A. GMRES at kth iteration minimizes the residual $\rho := ||b - Ax||$ with $x \in x_0 + \mathcal{K}_k$, where x_0 is the initial guess and $\mathcal{K}_k$ denotes the Krylov subspace defined as $\mathcal{K}_k := \text{span}\{r_0, Ar_0, ..., A^{k-1} r_0\}$ with $r_0 := b - Ax_0$. GMRES also generates an orthonormal basis $\{v_j\}_{j=1}^k$ for $\mathcal{K}_k$ successively. Minimization in step 2(e) is executed efficiently through use of Givens rotations. In principle, GMRES reduces the residual monotonically and converges to the solution within the same number of iterations as the dimension of the equation. However, an important advantage of GMRES for a large linear equation is that a specified error tolerance, e.g., $\rho \le \xi ||b||$ ($\xi > 0$), can be achieved with much less iterations.

In step 2(b), the product of a matrix and a vector, $F_W(W, P, t)v_j$, is replaced with its forward difference approximation, $D_h F(W, P + h\dot{P}, t + h : v_k, 0, 0)$. It is clear that the orthonormal basis $\{v_i\}_{i=1}^k$, k vectors in $\mathbf{R}^m$, must be stored during calculations, which may require a huge amount of data storage for a large problem. Furthermore, much iteration may be impossible from the viewpoint of execution time in real-time implementation. Therefore, k_{max} should be chosen as small as possible. The state equation (1) and the costate equation (5) are evaluated over the horizon $(3 + k_{max})$ times in FDGMRES.

With $\dot{W}$ obtained approximately through use of FDGMRES, $W(t)$ is updated by integrating $\dot{W}$ in real-time. The continuation/GMRES method for real-time algorithm of nonlinear moving horizon state estimation is summarized as follows.

Algorithm 2 (C/GMRES)

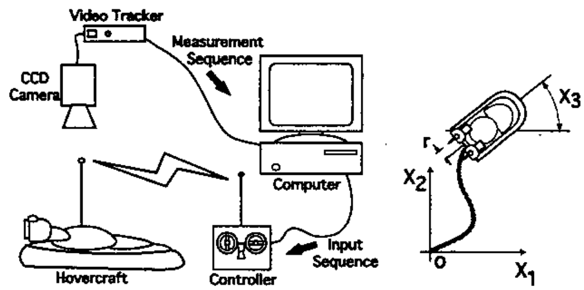
1. Let the horizon $T(t)$ be a smooth function such that $T(0) = 0$ and $T(t) \to T_f$ $(t \to \infty)$. Let $\hat{x}(0)$ be an appropriate initial guess. Let $t = 0$, measure the initial known quantities $p(0)$, and let $x_i^*(0) = \hat{x}(0)$, $\lambda_i^*(0) = \phi_x^T[\hat{x}(0)]$ $(i = 0, ..., N)$. Find $\hat{w}(0)$ and $\mu(0)$ analytically or numerically such that
$$\left\| \begin{bmatrix} H_w^T[\hat{x}(0), \phi_x^T[x(0)], \hat{w}(0), \hat{\mu}(0), p(0)] \\ C[\hat{x}(0), \hat{w}(0), p(0)] \end{bmatrix} \right\| \le \frac{\delta}{\sqrt{N}},$$
for some positive δ. Let $w_i^*(0) = \hat{w}(0)$ and $\mu_i^*(0) = \mu(0)$ $(i = 1, ..., N)$, which gives the initial condition $W(0)$ such that $\|F(W(0), P(0), 0)\| \le \delta$.
2. At time $t + \Delta t$, measure the known quantities $p(t + \Delta t)$ and let $\Delta P = P(t + \Delta t) - P(t)$, compute $\dot{W}(t)$ by $\dot{W} := \text{FDGMRES}(W, P, \Delta P/\Delta t, t, h, k_{max})$. Let $W(t + \Delta t) = W(t) + \dot{W}(t)\Delta t$. The estimate is obtained as $\hat{x}(t + \Delta t) = P_N(W(t + \Delta t))$.
3. Let $t = t + \Delta t$, and go back to Step 2.

In C/GMRES, explicit solution of $[H_w \ C^T]^T = 0$ is not necessary, since the estimate itself is a quantity to be determined numerically. Such higher order derivatives as H_{ww} and H_{wx} are also not necessary, because the linear equation for the update is solved by GMRES with the forward difference approximation. It should be noted that the iterative method is used only to solve the linear equation with respect to $\dot{W}$, and, through use of its solution, the solution of the nonlinear equation, $F(W, P, t) = 0$, is traced without any other iterative methods.

4 Experiment

4.1 System model of the hovercraft

In order to evaluate the effectiveness of the algorithm C/GMRES, we apply the algorithm to a state estimation of a hovercraft. Figure 1 shows the experimental apparatus.



(a) Experimental apparatus. (b) Hovercraft.

Figure 1: Experimental apparatus and Hovercraft.

The system of the hovercraft (Fig. 1(b)) is given by

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ -\dfrac{u_1(t) + u_2(t)}{M} \sin x_3(t) + w_1(t) \\ \dfrac{u_1(t) + u_2(t)}{M} \cos x_3(t) + w_2(t) \\ \dfrac{u_1(t) - u_2(t)}{I} r + w_3(t) \end{bmatrix},$$
$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = Cx(t) + v,$$

where ($x_1(t)$ and $x_2(t)$) is the hovercraft's position indicating the center of mass, $x_3(t)$ is the attitude angle, I, M and r are the moment of inertia, the mass and the distance between the thrusters and the center of mass, $u_1(t)$ and $u_2(t)$ are outputs of the thrusters, $w_1(t)$, $w_2(t)$ and $w_3(t)$ are unknown disturbances, and $v_1(t)$, $w_2(t)$ and $w_3(t)$ are unknown measurement noises, respectively.

The performance index with a moving horizon is chosen as:

$$J = \eta[x^*(t), p(t)] + \phi[x^*(t - T), p(t - T)] + \int_{t-T}^{t} L[x^*(t'), p(t'), w^*(t')]dt',$$

where

$$p = [y^T \ u^T \ \hat{x}^T]^T$$
$$L = \frac{1}{2}\{(y(t) - Cx^*(t))^T Q(y(t) - Cx^*(t)) + w^{*T}(t) W w^*(t)\},$$
$$\eta = 0,$$
$$\phi = \frac{1}{2}\{(\hat{x}(t - T) - x^*(t - T))^T S$$

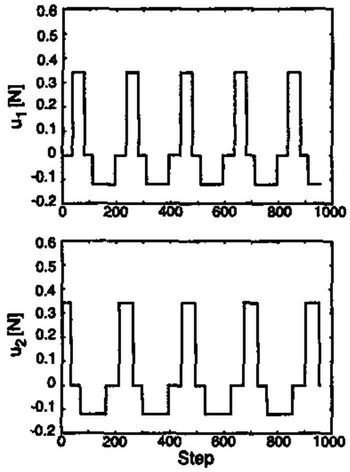


Figure 2: Sequence of inputs.

$$\cdot (\hat{x}(t-T) - x^*(t-T))\},$$

where S_f, Q and R are weighting matrices, and $\hat{x}$ is the estimate in the past which has already been obtained through optimization over $[t-2T,\ t-T]$. The final penalty ϕ is added because it is preferable that the optimal trajectory is consistent with the past estimate. Such a final penalty is often effective for numerical stability of computation.

4.2 Experimental results

State estimation is carried out for an experimental data over 8 [sec]. The physical parameters of the experiment are: the moment of inertia $I = 0.0125$ [kg $\cdot$ m^2], the mass $M = 0.86$ [kg] and the distance $r = 0.0485$ [m]. Figure 2 indicates the outputs of the two thrusters. For the moving horizon state estimation, the weighting matrices are chosen as follows:

$$\begin{aligned} S_f &= \mathrm{diag}[0.2\ 0.2\ 0.2\ 0.001\ 0.001\ 0.001], \\ W &= \mathrm{diag}[3\ 3\ 0.1], \\ Q &= \mathrm{diag}[50\ 50\ 50]. \end{aligned}$$

The program of C/GMRES is coded in C, and the computation is performed on a personal computer (CPU: PowerPC, 240 MHz) with the integration step $\Delta t = 1/120$ [sec], the number of grids $N = 15$, $k_{max} = 20$ and $\zeta = 1/\Delta t$. The length of the horizon is chosen so that $T(0) = 0$ and $T(t) \to T_f$ as $t \to \infty$, namely, $T(t) := T_f(1 - e^{-\alpha t})$ with $T_f = 0.2$ [sec] and $\alpha = 1$.

Figures 3, 4 and 5 show the results of computation by C/GMRES and measurement data. Figure 3 compares the estimates of the position and attitude with the measurements. The estimates of position and attitude almost overlaps with the measurements. Figure 4 compares the estimates of the velocities with the finite difference approximation processed by a low-pass filter. It should be noted that the estimates are smoother than corresponding measurements or filtered data in Figs. 3 and 4.

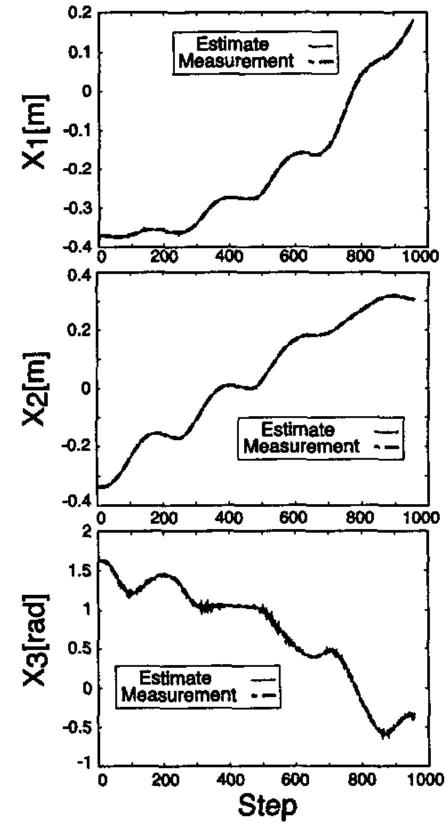


Figure 3: Estimates of position and attitude.

Figure 5 indicates the estimates of disturbances. And it implies that the smoothness assumption in Eq. (11) does not restrict the disturbance class in practice.

The total computational time for estimation is 5.55 [sec], which implies that the algorithm can be implemented in real time with a smaller sampling interval than 1/120 [sec].

5 Conclusion

This paper has proposed a real-time algorithm for nonlinear moving horizon state estimation by combining the continuation method with GMRES.

The proposed algorithm has been demonstrated in an

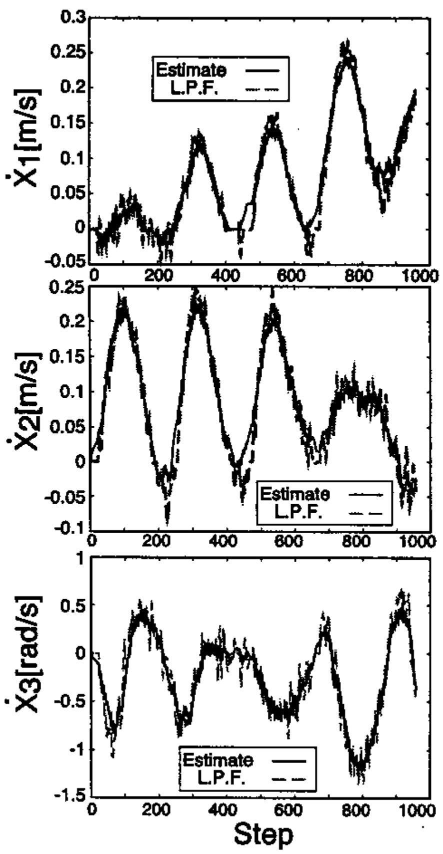


Figure 4: Estimates of velocities.

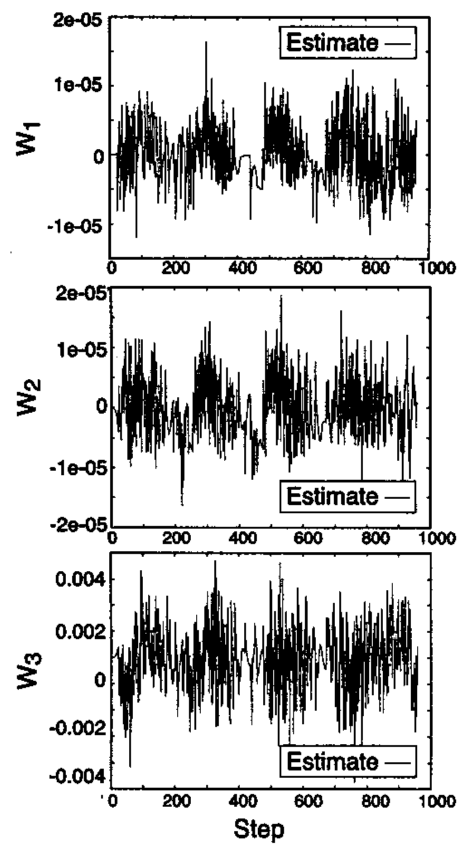


Figure 5: Estimates of disturbances.

experiment of a hovercraft whose dynamic is highly nonlinear. The experiment shows the proposed algorithm is executable in real time and that the smoothness assumption on the disturbance does not limit application of the algorithm in practice.

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