# 5. Nonholonomic constraint

## *Mechanics of Manipulation*

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon

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# Outline.

- An example: the unicycle.
- Integrable and nonintegrable constraints
- Vector fields and distributions
- Frobenius's theorem

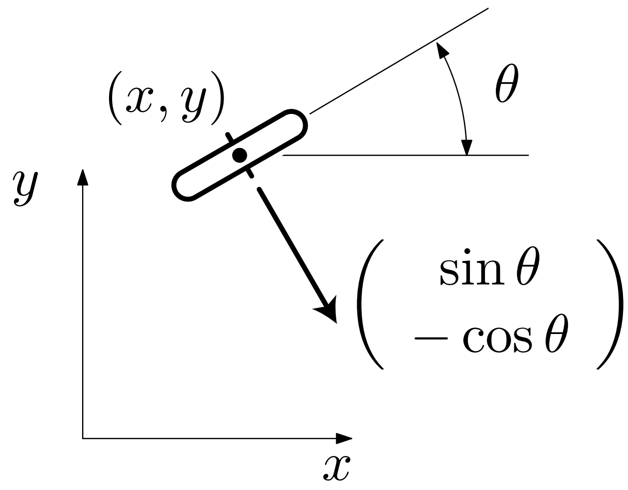
# Holonomic does not mean unconstrained!!!

- Holonomic means the constraints can be written as equations independent of $\dot{q}$
$$f(q,t) = 0$$
- A mobile robot with no constraints is holonomic.
- A mobile robot capable of arbitrary planar velocities is holonomic.
- A mobile robot capable of only translations is holonomic.

# Unicycle constraint

The unicycle cannot move sideways.
Let

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$




and let

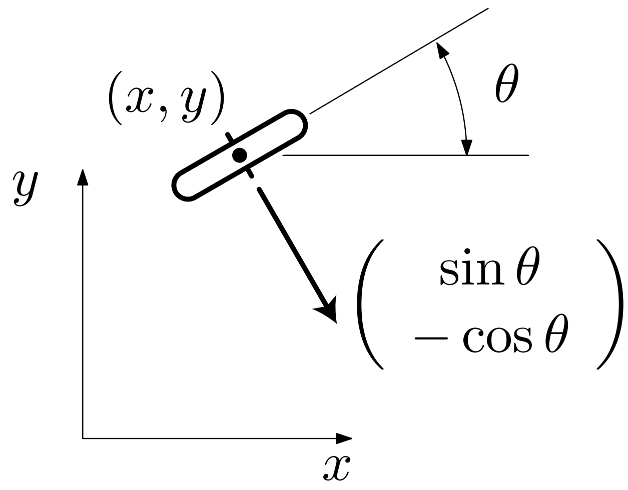
$$\mathbf{w}_1 = (\sin\theta, -\cos\theta, 0)$$

so the constraint is written $\mathbf{w}_1 \dot{\mathbf{q}} = 0$.

# Unicycle freedom

The unicycle can move in two directions, expressed by defining

$$\mathbf{g}_1(\mathbf{q}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{g}_2(\mathbf{q}) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$




and noting that the unicycle's motion can be expressed as

$$\dot{\mathbf{q}} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2$$

where $u_1$ and $u_2$ are arbitrary reals. They are the *controls*.

So, how many DOFs does the unicycle have?

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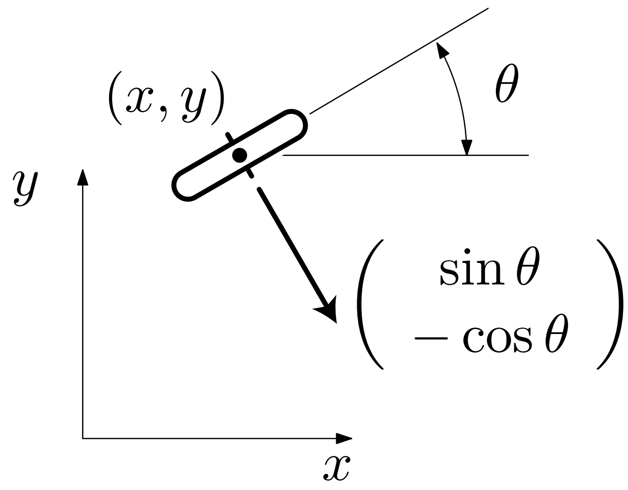


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So, how many DOFs does the unicycle have? *THREE!!!*

# Unsteered cart constraint and freedom

The unsteered cart cannot turn, and cannot move sideways. Let

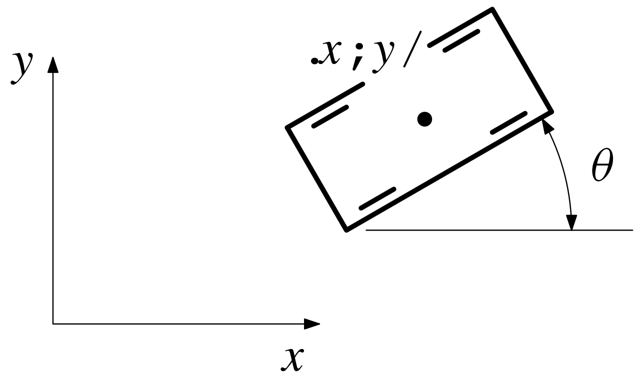
$$\mathbf{w}_1 = (\sin\theta, -\cos\theta, 0)\,,\mathbf{w}_2 = (0,0,1)$$

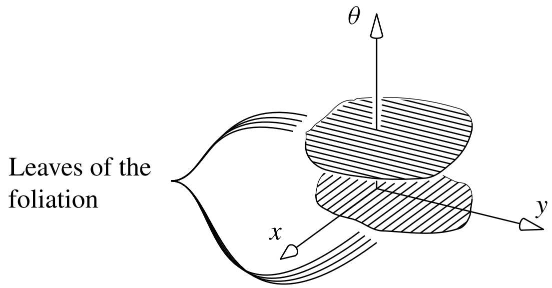
so the two constraints are written $\mathbf{w}_1\dot{\mathbf{q}} = 0$, $\mathbf{w}_2\dot{\mathbf{q}} = 0$. Expanding the products:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
$$\dot{\theta} = 0$$

These can be integrated:

$$\theta = \theta_0$$
$$(x - x_0)\sin\theta_0 - (y - y_0)\cos\theta_0 = 0$$

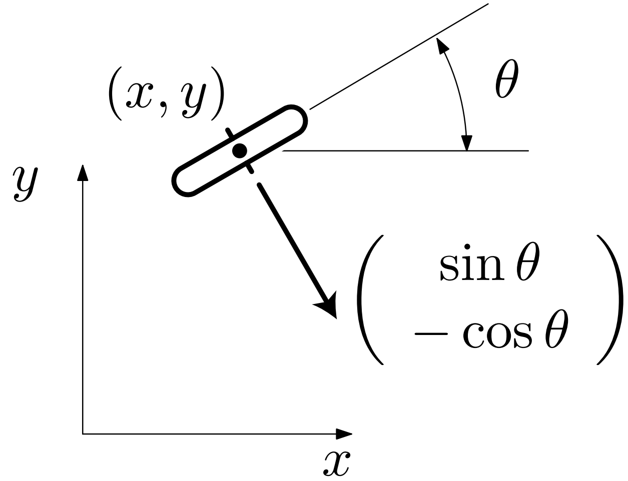


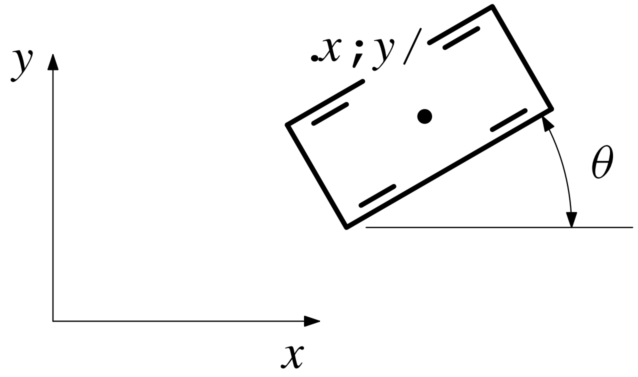



# Unicycle versus cart

- Unicycle.
  - One velocity constraint.
  - Three freedoms.
- Unsteered cart
  - Two velocity constraints.
  - *Integrable*. Equivalent to two configuration constraints.
  - One freedom.

System is nonholonomic if the constraint *cannot* be written in the form $f(q,t) = 0$.

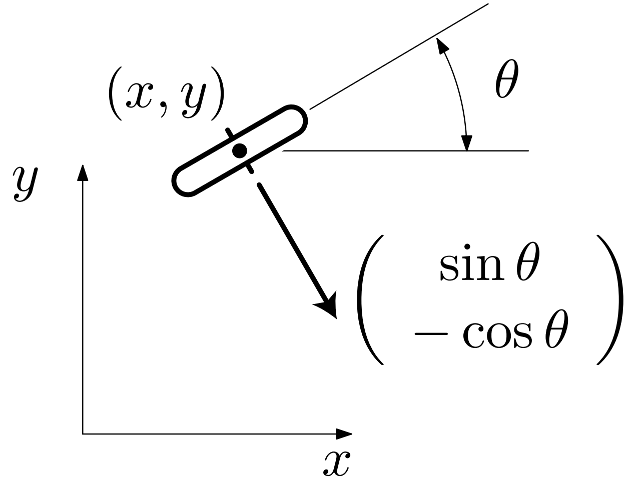


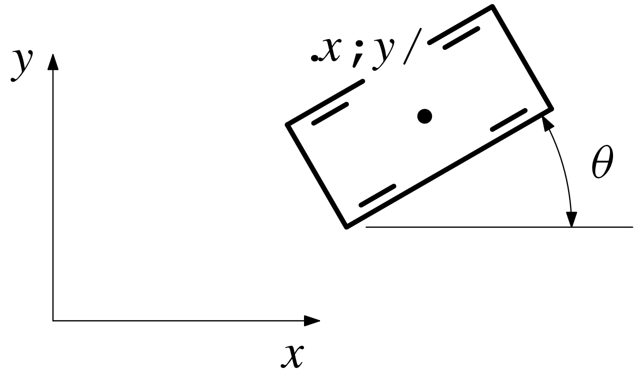



# How can you tell?

How can you tell whether a velocity constraint is integrable?

1. Try to integrate it for a while.
2. Determine whether the DOFs were reduced.
3. Lie brackets!!! (Frobenius's theorem)



# Pfaffian constraints

A set of $k$ **Pfaffian constraints** are of the form

$$\mathbf{w}_i(\mathbf{q})\dot{\mathbf{q}} = 0, i = 1 \ldots k$$

where the $\mathbf{w}_i$ are linearly independent row vectors, and $\dot{\mathbf{q}}$ is a column vector.

All the velocity constraints we have considered for the unicycle and the cart are Pfaffian.

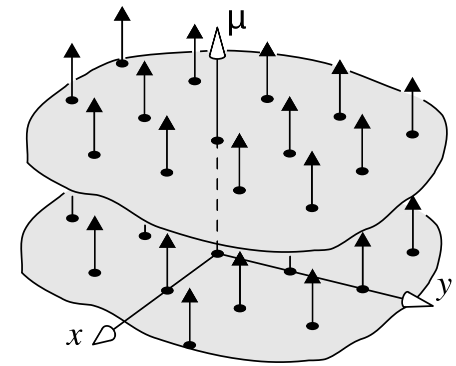
# Vector fields

A **vector field** is a smooth map

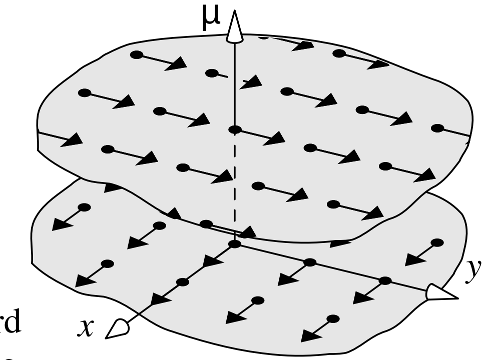
$$f(\mathbf{q}) : \mathbf{C} \mapsto \mathbf{T_q C}$$

from configurations $\mathbf{q}$ to velocity vectors $\dot{\mathbf{q}}$.

*Note:* In differential geometry "vector" sometimes means specifically "velocity vector".



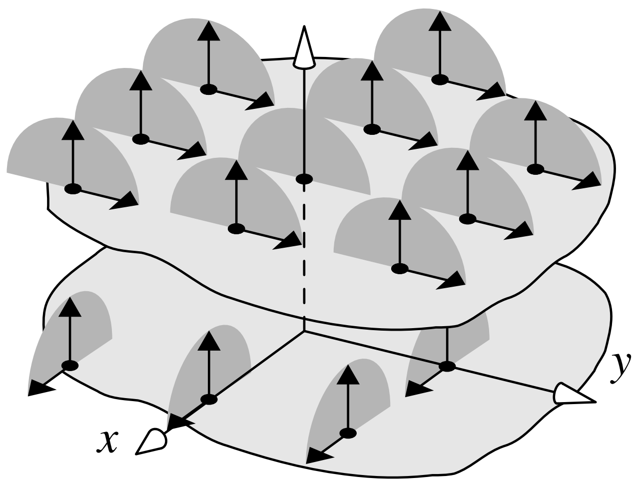

$\mathbf{g}_1$: turning




$\mathbf{g}_2$: forward rolling

# Distributions

A **distribution** is a smooth map assigning a linear subspace of $\mathbf{T_q C}$ to each configuration $\mathbf{q}$ of $\mathbf{C}$.




*Example:* The linear span of $\mathbf{g}_1$ and $\mathbf{g}_2$.

Recall that for the unicycle

$$\mathbf{q} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2$$

for $u_1, u_2 \in \mathbf{R}$. So the figure shows the feasible velocities for every $\mathbf{q}$.

(Well, it only shows a circular patch where it should show a whole plane at every $\mathbf{q}$.)

# Regular distributions and Lie brackets

A distribution is **regular** if its dimension is constant over the configuration space.

Let $\mathbf{f}$, $\mathbf{g}$ be two vector fields on $\mathbf{C}$. Define the **Lie bracket** $[\mathbf{f}, \mathbf{g}]$ to be the vector field

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\mathbf{g}$$

What is this thing written $\frac{\partial \mathbf{g}}{\partial \mathbf{q}}$ or $\frac{\partial \mathbf{f}}{\partial \mathbf{q}}$? Matrix. Each column is partial of velocity w.r.t. configuration variable.

# Lie brackets, example.

Let's take the Lie bracket $[\mathbf{g}_1, \mathbf{g}_2]$.

$$\frac{\partial \mathbf{g}_1}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix}$$

For the new vector field defined by the Lie bracket we obtain

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] = \frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\mathbf{g}_2$$

$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

# Lie brackets example continued

$$\mathbf{g}_3 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Physically, $\mathbf{g}_3$ moves sideways. It is linearly independent of $\mathbf{g}_1$ and $\mathbf{g}_2$, and it violates the constraint $\mathbf{w}_1$.

What is its physical significance? Given two vector fields $\mathbf{f}$ and $\mathbf{g}$,

1. Follow $\mathbf{f}$ for some time $\epsilon$;
2. Follow $\mathbf{g}$ for $\epsilon$;
3. Follow $-\mathbf{f}$ for $\epsilon$;
4. Follow $-\mathbf{g}$ for $\epsilon$.

In the limit as $\epsilon$ approaches zero, the result of the above motion approaches the Lie bracket $[\mathbf{f}, \mathbf{g}]$. The Lie bracket could have been called "parallel parking product".

# Involutive distribution

- A distribution is **involutive** if it is closed under Lie bracket operations.
- The **involutive closure** of a distribution $\Delta$ is the closure $\overline{\Delta}$ of the distribution under Lie bracketing.

# Frobenius's theorem

Theorem 2.8 (Frobenius's theorem):
A regular distribution is integrable if and only if it is involutive.
Proof:

> To prove that an integrable distribution is involutive, take the Taylor series expansion of the parallel parking maneuver as a function of $\epsilon$. The second order terms are Lie brackets! If the distribution is involutive, the Lie brackets must also be contained in the distribution.
>
> To prove that involutive distributions are integrable . . . $\square$

nonholonomic $\leftrightarrow$ parallel parking helps

# Rotation

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