

Model predictive control of nonlinear systems: Computational burden and stability

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Abstract: Implementation of model predictive control (MPC) for nonlinear systems requires the online solution of a nonconvex, constrained nonlinear optimisation problem. Computational delay and loss of optimality arise in the optimisation procedures. The paper presents a practical MPC scheme for nonlinear systems with guaranteed asymptotic stability. It is shown that when an initial control profile is chosen to satisfy an inequality condition in each online optimisation procedure, the nonlinear system under the proposed nonlinear MPC is asymptotically stable. The stability condition presented in the paper enables the 'fictitious' terminal control to be nonlinear, rather than only linear, thus the stability region is greatly enlarged. Furthermore it is pointed out that nominal stability is still guaranteed even though the global, or even the local, minimisation of the objective cost is not achieved within the prescribed computational time.

1 Introduction

Model predictive control (MPC) is now a recognised method in control engineering [1], and much effort has been made to extend MPC from linear systems to nonlinear systems: for example, see Allgöwer and Zheng [2]. There are, however, two main obstacles to the extension of MPC from linear systems to nonlinear systems [3–6]: one obstacle is stability, the other is the online computational burden.

Stability has been one of the main problems in MPC, ever since early MPC for linear systems was criticised for its loss of stability [7]. This problem has now been solved for linear systems in several ways: infinite horizon predictive control [8], terminal constraints [9, 10] and the fake algebraic Riccati equation [7]. For nonlinear MPC, the first interesting stability result was given by Mayne and Michalska [11]. They show that, under certain assumptions, predictive control can stabilise plants when a terminal equality constraint is embedded in the optimisation problem; that is, it is required that the terminal state is at a point in the state space. Similar results also have been reported by Genceli and Nikolaou [12] and Rawlings and Muske [13]. However, from the computational point of view, solving a nonlinear dynamic optimisation problem with equality constraints is highly computationally intensive, and in many cases is impossible to perform within a limited time. Furthermore, the feasible region of the proposed MPC is very small. To avoid this, Michalska and Mayne [14] replaced terminal equality constraints with terminal inequality constraints; that is, the allowable terminal state is a region rather than a point in the state space. To guarantee stability, a dual mode control scheme is proposed. The receding horizon control is employed

outside the terminal region and once the state arrives on the boundary of the terminal region, a local linear state feedback controller is employed to steer the state to an equilibrium.

Along these lines, a most interesting result was recently presented by Chen and Allgöwer [5]. For a wide class of nonlinear systems, a MPC scheme with a virtual *linear* stabilising control, rather than a dual mode control, is developed. Stability is achieved by involving a terminal cost which bounds the infinite horizon cost of the nonlinear system starting from the terminal region and controlled by the local linear state feedback controller. However, the local state feedback control is never implemented. This so-called quasiinfinite MPC algorithm solves online the nonlinear optimisation problem with a performance index containing the terminal cost subject to system dynamics, input constraints and the terminal inequality constraints at each sampling time. Then the receding horizon control is employed until the system arrives at the equilibrium.

Apart from the above work on the stability of nonlinear MPC, the stability of several existing nonlinear predictive control schemes has been examined by Magni and Sepulchre [15] using an inverse optimality approach. In particular, it is shown that predictive control possesses the stability margin of optimal control and it is robust in some sense. A more rigorous result for MPC was developed by Gyurkovics [16]. The receding horizon control is considered as a Bolza-type optimisation problem. Using the optimality of the value function, the stability condition is established.

Although rigorous stability results for nonlinear MPC have been established [15, 16], they are not applicable in practical implementations. A constrained nonlinear optimisation problem has to be solved online and the heavy online computational burden leads to two major problems in the implementation of nonlinear MPC. One issue is the computational delay which cannot be ignored and the other issue is that the global optimisation solution cannot be guaranteed in each optimisation cycle since, in general, it is a nonconvex, constrained nonlinear optimisation problem. Neither of these issues have been considered by Gyurkovics [16] or Magni and Sepulchre [15]. In view of the difficulty of guaranteeing the global minimisation of an

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objective cost, this requirement has been relaxed for discrete-time systems in Scokaert *et al.* [17] and for continuous-time systems in Chen and Allgöwer [5] where it is pointed out that feasibility implies stability for a particular form of MPC (with linear terminal state feedback control).

The effects of computational delay in implementation of a controller have been investigated by many authors [18, 19]. In model predictive control, several schemes taking into account computational delay have recently been proposed for linear systems [20–22]. Gomma and Owens [20] gave a delayed generalised predictive control scheme where control only depends on past measurements rather than current and past measurement. This strategy was compared with Clarke *et al.* [23] predictive control [20]. Von Wissel *et al.* [21] investigated the effect of computational delay in descriptor predictive control. It is shown that the closed-loop system remains stable as long as the delay is smaller than a specified threshold. Design of reference governors in predictive control with computation delay was addressed by Casavola and Mosca [22]. Although the computational burden in nonlinear MPC is much heavier than the linear one, only the algorithm proposed by Ronco *et al.* [24], Open-loop Intermittent Feedback Control, considers the computation delay in implementation of MPC for nonlinear systems. Similar to other schemes considering the computational delay the last control profile found by minimisation the objective cost is used until the new optimisation process is finished and a new control profile is available. It has been shown that this scheme is promising by application to several plants. In addition to computational delay, measurement delay is considered in Chen and Allgöwer [5]. Measurement delay is different from the computational delay in that control depends on current measurement.

Recently Chen *et al.* [25] proposed a new nonlinear model predictive control algorithm which can remove the computational burden and stability issues. The proposed nonlinear predictive controller is given in a closed-form and thus no online optimisation is required. The stability is also guaranteed. It is also pointed out that the well-known computed torque method or dynamic inversion control is a special case of this MPC algorithm [26]. However, this method is only applicable for nonlinear systems with stable zero dynamics and a well-defined relative degree.

This paper develops a practical MPC with guaranteed stability for general nonlinear systems. Several issues in the implementation of nonlinear MPC, including computational delay, loss of optimality in the optimisation procedure and stability, are addressed. A new stability condition is presented. The performance index in MPC is modified to take computational delay into account. At each time interval, the calculation of the control profile is divided into a two stage optimisation problem. In the first stage, a 'classic' fixed horizon MPC is solved. In the second stage, a terminal control profile within a prescribed time interval is optimised according to the stability requirements. It should be noted that the terminal control is never implemented and it is only used to generate an initial control profile in each optimisation procedure.

2 Problem formulation

Consider a nonlinear control system

$$\dot{\hat{x}}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \quad (1)$$

where $x \in R^n$ and $u \in R^m$ are the state and control vectors respectively. It is supposed that the control input is subject to the constraint

$$u(t) \in U \quad (2)$$

where $0 \in U \subset R^m$ is a compact and convex set.

To distinguish the real variables from the variables in the moving horizon time frame, hatted variables are used in the moving horizon time frame in this paper. In general, a nonlinear MPC problem can be stated as: for any state x at time t , find a continuous function $\hat{u}(\tau; x(t)): [t, t+T] \rightarrow U$, in a moving horizon time frame, T , such that the performance index

$$J_t = g(\hat{x}(t+T)) + \int_0^T \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau; x(t))^T R \hat{u}(t+\tau; x(t)) d\tau \quad (3)$$

is minimised where $Q \geq 0$, $R > 0$ and $\hat{u}(\cdot; x(t))$ explicitly indicates the control profile \hat{u} depends on the state measurement $x(t)$ at time t . It is required that $g(x)$ is a continuous, differentiable function of x , $g(0)=0$ and $g(x) > 0$ for all $0 \neq x \in R^n$. A typical choice of $g(x)$ is given by

$$g(x) = x(t)^T P x(t) \quad (4)$$

where $P \in R^{n \times n}$ is a positive definite matrix.

Then the nonlinear MPC law is determined by

$$u(t) = \hat{u}(t; x(t)) \quad (5)$$

Assumptions on the system in eqn. 1 are imposed:

A1 $f: R^n \times R^m \rightarrow R^n$ is twice continuously differentiable and $f(0,0)=0, 0 \in R^n$ is an equilibrium of the system with $u=0$.

A2 System 1 has a unique solution for any initial condition $x_0 \in R^n$ and any piece-wise continuous and right-continuous $u(\cdot): [0, \infty) \rightarrow U$;

A3 The nonzero state of the system 1 is detectable in the cost. That is, $Q^{1/2}x=0$ for all nonzero x such that $f(x,0)=0$ [27].

A4 All states, $x(t)$, are available.

To implement the MPC, a constrained nonlinear optimisation problem must be solved online. Online optimisation is a major issue in nonlinear MPC implementation [2–4]. Owing to the large computational burden, the computational delay time in MPC is significant and cannot be ignored. The optimisation process for MPC with computational delay is shown in Fig. 1.

When the state measurement $x(t)$ is available at the sampling time t , the optimal control profile $\hat{u}(\tau; x(t)), \tau \in [t, t+T]$ yielded by minimisation of the cost, eqn. 3, is not available until time $t+\delta$ where δ is the computational delay. It makes no sense to optimise the control profile $\hat{u}(\tau; x(t)), t \leq \tau \leq t+\delta$ from time t . Moreover as will be shown later, it also causes mismatching in the MPC implementation and thus degrades system performance. Hence, in this paper the performance index in eqn. 3 is modified to

$$J = g(\hat{x}(t+T)) + \int_\delta^T \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \quad (6)$$

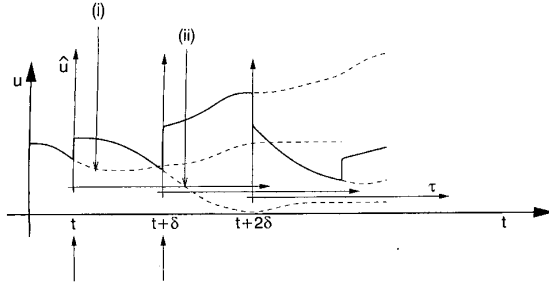


Fig. 1 Nonlinear MPC with computational delay

— implemented control profile
 --- non-implemented control profile
 (i) optimal control profile at time $t - \delta$
 (ii) optimal control profile at time t
 t = measure state $x(t)$ start time t optimisation
 $t + \delta$ = finish time t optimisation implement time t control profile measure state $x(t + \delta)$ start time $t + \delta$ optimisation

At each measurement time, the optimisation problem (OP) to be solved online can be mathematically formulated as

$$\min_{\hat{u}: [t + \delta, t + T] \rightarrow U} J \quad (7)$$

subject to

$$\begin{cases} \text{terminal state constraint } x(t + T) \in \Omega \\ \text{the system dynamics of eqn. 1} \end{cases} \quad (8)$$

The implemented control signal is determined by the equation

$$u(\tau) = \hat{u}(\tau; x(t)), \tau \in [t + \delta, t + 2\delta] \quad (9)$$

There are two main advantages in formulating the model predictive control in the form of the above OP in eqns. 7 and 8. The first advantage is that a lower computational burden exists when the optimisation occurs over the period $[t + \delta, t + T]$ compared to the period $[t, t + T]$. This reduced optimisation period occurs because the result of the optimisation is not available until time $t + \delta$ and so the optimisation cannot alter the control that is implemented in the period $[t, t + \delta]$. The second major advantage is that at time $t + \delta$ when the new control is implemented, there will be a closer match between the initial state $x(t + \delta)$ and the predicted state $\hat{x}(t + \delta)$ than with the originally measured state $x(t)$.

It should be noted that, in addition to the terminal weighting in the performance index in eqn. 6, a constraint on the terminal states is imposed in OP. As will be shown later, this is due to the stability requirement.

In MPC, one problem is to determine the computational delay off-line. This depends on the computational methods used in solving the OP. The computational issue is an active subject in nonlinear MPC and many algorithms have been proposed [3, 4, 28]. Several computational algorithms have been compared and implemented on a continuous stirred tank reactor [29]. The choice of optimisation routine is not considered in this paper. The difficulty in determining the computational delay offline lies in that the computational times for different optimisation cycles are different even for the same optimisation algorithm. It is well known that the computational time is quite different in each nonlinear optimisation procedure for different initial points and optimum. This makes it difficult to determine the length of time taken to perform the optimisation offline. However it will be shown that although the global optimal solution is not achieved and only a local minimum solution exists, or worse, the optimisation procedure

is not finished within the prescribed time period, stability can still be guaranteed in the MPC algorithm proposed in this paper. This makes it possible to determine the computational delay off-line.

Remark 1: Chen and Allgöwer [5] have considered the measurement delay issue in MPC implementation. That is, the control profile is implemented in an open-loop manner until the updating new measurement arrives. This is different from the computational delay considered in this paper. More specifically, the closed-loop control profile in Chen and Allgöwer [5] is defined as

$$u(\tau) = \hat{u}(\tau; x(t)), \tau \in [t, t + \delta_1] \quad (10)$$

where δ_1 is the measurement delay. This implies that the control depends on the current state. However the MPC algorithm with computational delay can be considered as given in eqn. 9. This implies that the control depends on the past state measurement. The measurement delay is not considered in this paper since it is much smaller than the computational time in nonlinear MPC. However if the measurement delay needs to be considered. It can be easily handled by the algorithm presented in this paper.

3 Terminal region and MPC algorithm

For the sake of simplicity, in this paper asymptotic stability of a system is taken to mean that the system has an (asymptotically) stable equilibrium at the origin.

Definition 1: The terminal region Ω in MPC is defined as a region where, once the state $\hat{x}(t + T)$, under the control $\hat{u}(\tau; x(t))$, $\tau \in [t, t + T]$ yielded by solution of the OP (eqns. 7 and 8) arrives, there exists a control $\hat{u}: [t + T, \infty) \rightarrow U$ which can steer the state to the origin.

Lemma 1: Suppose that the system in eqn. 1 satisfies Assumptions A1–A4, then, if there exists an $\alpha > 0$ such that for any $x \in \Omega(\alpha) = \{x \in R^n; g(x) \leq \alpha\}$, and if there exists control $u \in U$ satisfying

$$\frac{\partial g(x)}{\partial x} f(x, u) + x^T Q x(t) + u^T R u \leq 0. \quad (11)$$

then $\Omega(\alpha)$ is a terminal region for the nonlinear system (1) under the MPC with the performance index in eqn. 6. The proof of this is given in Appendix A.

Based on Lemma 1, a MPC algorithm with guaranteed stability (MPCGS) is proposed as follows.

MPCGS Algorithm:

1. Measure the state $x(t_0)$. No control is imposed during the time interval $[t_0, t_0 + \delta]$. Meanwhile, solving the OP in eqns. 7 and 8 at $t = t_0$ and $x(t) = x(t_0) = x_0$ yields $\hat{u}(\tau; x(t))$, $\tau \in [t + \delta, t + T]$.
2. Calculate $\hat{u}(\tau; x(t))$, $\tau \in [t + T, t + T + \delta]$, according to condition 11.
3. Implement the optimal open-loop control $\hat{u}(\tau; x(t))$ for the time interval $[t + \delta, t + 2\delta]$. Measure the state at the time $t + \delta$ and in the meantime, use the control $\hat{u}(\tau; x(t))$, $\tau \in [t + 2\delta, t + T + \delta] \rightarrow U$, defined as

$$\hat{u}(\tau; x(t + \delta))' = \begin{cases} \hat{u}(\tau; x(t)), \tau \in [t + 2\delta, t + T] \\ \hat{u}(\tau; x(t)), \tau \in [t + T, t + T + \delta] \end{cases} \quad (12)$$

as an initial solution of $\hat{u}(\tau; x(t + \delta))$ to solve the constrained optimisation problem of eqns. 7 and 8.

4. Let $t = t + \delta$ and go to Step 2.

Remark 2: When solving the OP, global minimisation or even local minimisation of the objective cost in the above

MPCGS algorithm is not required. This is important for engineering implementation. Since this problem is solved online, it is difficult to guarantee the global optimal solution, and sometimes even a local minimum cannot be achieved owing to the time limitation. One of the key features of the MPCGS algorithm is the way it determines an initial control profile for each optimisation process based on the condition in eqn. 11, which distinguishes it from other existing MPC algorithms. When the optimisation algorithm starts from this initial control profile, as will be shown later, the feasibility and stability of the proposed MPC are guaranteed.

Remark 3: In each iteration of the MPCGS, two optimisation problems need to be solved. One is the 'classic' MPC optimisation problem. The other is the optimisation problem in Step 2 to determine the terminal stabilising control. For any $x \in \Omega$, it is easy to solve this feasibility problem by a nonlinear optimisation procedure. Particularly, for an affine nonlinear system, it will be shown later that the nonlinear terminal stabilising control is given in a closed-form and on-line optimisation is not required.

Remark 4: How to determine the computational delay offline is important in the proposed algorithm. It involves a trade-off between the computational burden, robustness and the control performance. If the computation delay chosen is too small, the performance is degraded, although the stability can be guaranteed since optimisation cannot be finished at each iteration. If the computation delay is too large, the proposed MPCGS has less robustness against uncertainties and disturbances. How to analyse the robust stability of the closed-loop system with computational delay is an interesting subject and some results have been presented, for example see Chen *et al.* [30], and Chen and Gu [31]. However it is beyond the scope of this paper.

4 Stability and feasibility

Definition 2: The feasibility region refers to a set of initial state points $x(t_0)$ for which the state trajectory yielded by solving the OP terminates in the region $\Omega(\alpha)$.

Lemma 2 guarantees that if the nonlinear MPC problem is feasible at t_0 , it is also feasible for all $t > t_0$. This follows from modification of the standard argument; for example, see Genceli and Nikolaou [12], Michalska and Mayne [14], and Chen and Allgöwer [5].

Lemma 2: Consider the nonlinear system in eqn. 1 satisfying Assumptions A1–A4 with the input constraint, eqn. 2. The feasibility of the MPCGS algorithm at time t_0 implies feasibility for all $t > t_0$. The proof of this is given in Appendix B.

Theorem 1: Suppose that Assumptions A1–A4 are satisfied and the MPCGS algorithm is feasible at time $t = t_0$. The MPCGS algorithm for eqn. 1 is asymptotically stable if there exists control $u(t)$ such that the following condition is satisfied

$$\frac{\partial g(x)}{\partial x} f(x, u) + x^T Qx + u^T Ru \leq 0 \quad (13)$$

for any state x belonging to the terminal region Ω . The proof of this is given in Appendix C.

Corollary 1: When the nonlinear system considered is affine, given by

$$\dot{x}(t) = f_1(x) + g_1(x)u \quad (14)$$

the condition in eqn. 11 reduces to

$$\frac{\partial g(x)}{\partial x} f_1(x) - \frac{\partial g(x)}{\partial x} g_1(x) R^{-1} g_1(x)^T \left(\frac{\partial g(x)}{\partial x} \right)^T + x^T Qx \leq 0. \quad (15)$$

and

$$u = -R^{-1} g_1(x)^T \left(\frac{\partial g(x)}{\partial x} \right)^T \in U \quad (16)$$

The terminal region is given by

$$\Omega(\alpha) = \{x \in R^n : \text{conditions (15–16) hold for all } g(x) \leq \alpha\} \quad (17)$$

If eqn. 15 is satisfied, a control $u(t)$ which steers the state to the origin is given by eqn. 16. In particular, for a linear system

$$\dot{x} = Ax + Bu. \quad (18)$$

when $g(x)$ is chosen as eqn. 4, eqn. 15 becomes

$$A^T P + PA - PBR^{-1}B^T P + Q \leq 0. \quad (19)$$

If this condition is satisfied, the terminal region is the whole state space, that is, the MPCGS is global asymptotically stable.

Remark 5: Corollary 1 shows that for linear systems, the stability result in this paper reduces to that of the so-called fake algebraic Riccati equation [7, 32]. Hence, the stability condition for linear MPC can be considered as a special case of the stability condition presented in this paper.

Remark 6: When the terminal control is restricted to be linear and the terminal weighting item is given by eqn. 4, the stability condition for nonlinear MPCGS proposed in this paper reduces to the result in Michalska and Mayne [14]. That is, the terminal region is given in eqn. 13 by replacing $u(t)$ with $u(t) = kx(t)$ where $k \in R^{m \times n}$ is a constant matrix.

Remark 7: Before using the proposed MPCGS, the terminal region should be determined offline. To determine the maximum attraction region of the proposed MPCGS, it is required to calculate the maximum α such that eqn. 13 holds for any $x \in \Omega(\alpha)$. This can be done by making an iteration of the optimisation problem

$$\max_{x \in \Omega(\alpha)} \min_{u \in U} \left\{ \frac{\partial g(x)}{\partial x} f(x, u) + x^T Qx + u^T Ru \right\} \quad (20)$$

This is a max/min optimisation problem. The maximum α is given by increasing α such that the value in eqn. 20 is nonnegative. In general it is not trivial to solve this max/min problem. A Linear Matrix Inequalities (LMI's) approach is developed to approximately solve this problem by Chen *et al.* [33].

However for an affine nonlinear system (14), with the help of the result in Corollary 1, the maximum terminal attraction region can be achieved by solving the following optimisation problem

$$\max \alpha \quad \text{s.t. conditions (15–16) hold for all } g(x) \leq \alpha. \quad (21)$$

This is a standard semi-definite optimisation problem which has been extensively discussed: for example, see Polak *et al.* [34] for a general setting or see Michalska and Mayne [14] for MPC.

5 Simulation example

Consider the Van de Vusse reactor kinetic scheme where at an operating point, the process model is given by [35, 36]:

$$\begin{aligned}\dot{x}_1 &= -50x_1 - 10x_1^2 + (10 - x_1)u \\ \dot{x}_2 &= 50x_1 - 100x_2 - x_2u \\ y &= x_2\end{aligned}\quad (22)$$

and x_1 and x_2 are the deviation variables for the concentration of components A and B respectively. The control u is the inlet flow rate.

The control problem for this system is to control the concentration of the component B, y , by the inlet flow rate, u . The nonlinear model predictive control proposed in this paper is designed for this plant.

The performance index is chosen as eqn. 6 with g as in eqn. 4 and the weighting matrices

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}; \quad (23)$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}; \quad (24)$$

and

$$R = 0.1 \quad (25)$$

To determine the applicability and the stability region of the proposed nonlinear MPC, the terminal region needs to be determined.

Since this is an affine nonlinear system, the terminal region is determined by condition in eqn. 15 in Corollary 1. For the plant in eqn. 22 with the above weighting matrices in the performance index eqn. 6, it can be shown that eqn. 15 is satisfied for all states in the state space. The closed-loop system under the proposed nonlinear model predictive controller is globally stable.

Following eqn. 16, the terminal stabilising nonlinear controller is given by

$$\begin{aligned}u(t) &= -10[10 - x_1 - x_2]Px \\ &= -5(10 - x_1)x_1 + x_2^2\end{aligned}\quad (26)$$

The case where the terminal control is linear [5, 14] is now considered. First the nonlinear system in eqn. 22 is linearised around the original point. The linearised system is given by

$$\dot{x} = \begin{bmatrix} -50 & 0 \\ 50 & -100 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \end{bmatrix} u \quad (27)$$

Solving the LQ optimal control problem with the weighting matrices Q and R in eqn. 24–25 yields the feedback gain

$$k = [0.3148 \quad 0.3248]. \quad (28)$$

The terminal stabilising control u is given by

$$u(t) = kx(t). \quad (29)$$

The stability condition is given in eqn. 13 by replacing $u(t)$ with eqn. 29. By solving this semi-infinity optimisation problem [5, 14], the terminal region is given by

$$\Omega = \{x \in R^2 | x^T Px \leq 9.63\} \quad (30)$$

Comparing with the global stability of the nonlinear terminal control law, the above terminal region given by the linear terminal control law eqn. 29 is much smaller. Hence, the new stability condition in this paper provides a much less conservative result for nonlinear MPCGS.

The simulation results under the proposed nonlinear MPCGS are shown in Figs. 2 and 3. In the simulation, the predictive time is chosen as 0.05 hour. The computational delay is chosen as 0.01 hour (36 seconds). The 4–5th order Runge-Kutta numerical integration method with a fixed step size 0.002h (7.2s) is used for the optimisation. The whole simulation is performed in the Matlab environment with the nonlinear optimisation implemented using the Optimisation Toolbox. The initial state of the plant is measured at time $t = 0$ and MPCGS starts the optimisation procedure. Before the control profile is yielded by optimisation, no control effort is imposed on the reactor, that is, the control effort is zero. When the control profile is available after the computational delay, the control profile is implemented and new states are measured for starting the new optimisation procedure. It is difficult to determine whether or not the global minimisation of the cost is achieved in each optimisation procedure. Actually in some optimisation procedures, the optimisation process is terminated since the prescribed maximum number of evaluations is arrived at (the maximum number of evaluations in the optimisation is set to be 100 times the number of variables in the simulation). This implies that in some

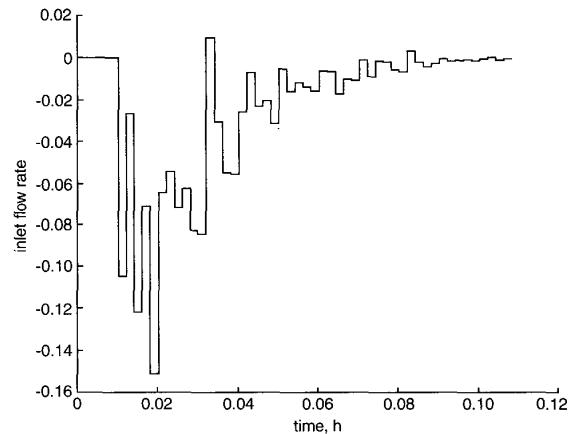


Fig. 2 Control history

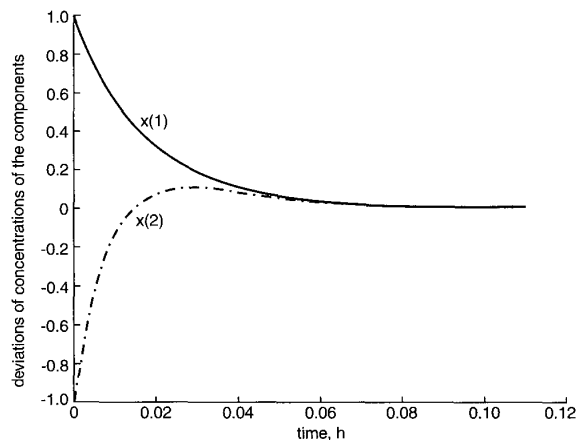


Fig. 3 Response

cases a local minimum has not been achieved. The resulting control profile is shown in Fig. 2. This may not be as smooth as expected but the stability is guaranteed by the proposed MPCGS algorithm. The response of the system is shown in Fig. 3.

6 Conclusion

A practical MPC scheme for general nonlinear systems has been presented in this paper. The main advantages and features of the proposed control scheme are that:

- the computational delay caused by online optimisation is accounted for;
- no global solution to the online nonlinear optimisation is required;
- a nonlinear terminal control can be chosen if required and hence the applicability and stability regions of the nonlinear model predictive control is enlarged;
- the feasibility of the proposed scheme implies stability when the proposed initial control profile is used in each optimisation routine.

The proposed nonlinear MPC has been illustrated by a simulation example. Compared to the linear terminal control the stability region is much enlarged due to the new stability condition derived in this paper.

Several obstacles to implementation of MPC of nonlinear systems, including loss of optimality in on-line nonlinear optimisation, computation delay due to extensive computational burden and stability issue, have been removed. As a consequence, the paper takes a step toward the development of practical MPC for general nonlinear systems with unstable zero dynamics or an ill-defined relative degree. Further investigation will focus on robustness of nonlinear MPC and reducing the online computational burden using control parametrisation techniques [37, 38].

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Appendix A Proof of Lemma 1

Suppose that there exists a control $u(t) \in U$ such that the condition in eqn. 11 is satisfied for all $x \in \Omega(\alpha)$. Then a corresponding $u(t) \in U$ can be calculated according to eqn. 11 for any $x \in \Omega(\alpha)$. Let $u(t)$, $t \in [T + t, \infty)$ be such a control profile. Let $V(x) = g(x)$ be a Lyapunov function candidate associated with the state trajectory of eqn. 1 under this control profile. It is noted that $g(x)$ is a continuous, differentiable function of x . It follows from eqn. 11 that

$$\dot{V}(x) = \frac{\partial g(x)}{\partial x} f(x, u) \leq -x^T Qx - u^T R u \leq -x^T Qx \leq 0 \quad (31)$$

The inequality in eqn. 31 implies that the Lyapunov function $V(x)$ is nonincreasing along the state trajectory. If the state of the system is within the region $\Omega(\alpha)$, it remains in this region. The inequality in eqn. 31 also implies that $\dot{V}(x) = 0$ only if $u(t) = 0$ and the state trajectory belongs to the following set

$$N = \{x(t) \in R^n : x(t)^T Qx(t) = 0, t \in R\}. \quad (32)$$

Since $Q^{1/2}x \neq 0$ for all nonzero x such that $f(x, 0) = 0$, one can prove that the only positive half trajectory contained entirely in N is $x(t) \equiv 0$ [16, 27]. Remember that $V(0) = g(0) = 0$ and $V(x) = g(x) > 0$ for all $x \neq 0$. This implies that

$$\|x(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty \quad (33)$$

Hence under the control satisfying eqn. 11, $\Omega(\alpha)$ is an attraction region of eqn. 1 with an equilibrium point $x = 0$. According to Definition 1, $\Omega(\alpha)$ is a terminal region of the MPC. QED.

Appendix B Proof of Lemma 2

When a MPC algorithm is implemented. Owing to the computational delay and no prior information about the control input, in the MPCGS proposed in this paper, no control effort is imposed in the interval $(t_0, t_0 + \delta)$. eqn. 1 becomes

$$\dot{x} = f(x, 0), \quad x(t_0) = x_0 \quad (34)$$

At time $t_0 + \delta$, the free system in eqn. 1 arrives at the state $x(t_0 + \delta)$. Since the MPCGS algorithm is feasible at time t_0 , it implies that an (open-loop) control profile $\hat{u}(\cdot; x_0) : [t_0 + \delta, t_0 + T] \rightarrow U$ which drives the state $x(t_0 + \delta)$ to arrive in the terminal set $\Omega(\alpha)$ at the terminal time $t_0 + T$ by solving the OP in eqns. 7–8 at time t_0 can be found.

The feasibility of the MPCGS at time $t_0 + \delta$ is now considered. At the time $t_0 + \delta$, the new state measurement $x(t_0 + \delta)$ is available and the optimisation problem OP with the initial state $x(t_0 + \delta)$ needs to be solved. In the MPCGS algorithm, this procedure starts from the following initial control profile $\hat{u}(\cdot, x(t_0 + \delta))' : [t_0 + 2\delta, t_0 + T + \delta] \rightarrow U$, defined as in eqn. 12. The assumption that the MPCGS is feasible at the time t_0 implies that the first part of $\hat{u}(\cdot, x(t_0 + \delta))'$ can steer the state into the terminal region at the time $t_0 + T$. The second part of $\hat{u}(\cdot, x(t_0 + \delta))'$ is calculated in Step 2 in MPCGS which satisfies eqn. 11. According to the MPCGS algorithm in Section 3, $\hat{u}(\tau, x_0)$, $\tau \in [t_0 + \delta, t_0 + T]$ is implemented in the time interval $[t_0 + \delta, t_0 + 2\delta]$, that is, $u(\tau) = \hat{u}(\tau; x_0)$, $\tau \in [t_0 + \delta, t_0 + 2\delta]$. As a result, the system state at the time $t + 2\delta$ is given by $x(t_0 + 2\delta) = \hat{x}(t_0 + 2\delta)$ under the assumption that there is no uncertainty and noise. Hence according to Lemma 1,

the terminal state $\hat{x}(t_0 + T + \delta)'$ under the control profile \hat{u}' remains in the terminal region Ω . Clearly, the optimisation problem at $t_0 + \delta$ is feasible since at least $\hat{u}(\cdot, x(t_0 + \delta))'$ is a feasible solution. Repeating the above deduction, one can prove that the feasibility of the MPCGS algorithm at the time $t = t_0$ implies feasibility for all $t > t_0$. QED

Appendix C Proof of Theorem 1

First it is shown that $x = 0$ is an equilibrium of the closed loop system under the proposed MPCGS.

When $x(t_0) = 0$ and under no control effort, it follows from $f(0, 0) = 0$ that

$$x(\tau) = 0 \text{ for } \tau \in [t_0, t_0 + \delta] \quad (35)$$

That is $x(t_0 + \delta) = 0$. It is clear that the optimal control profile which minimise the cost (6) is given by

$$\hat{u}(\tau : x(t_0)) = 0, \tau \in [t_0 + \delta, t_0 + T] \quad (36)$$

and as a result.

$$u(\tau) = 0, \text{ for } \tau \in [t_0 + \delta, t_0 + 2\delta] \quad (37)$$

and

$$x(\tau) = 0 \text{ for } \tau \in [t_0 + \delta, t_0 + 2\delta]. \quad (38)$$

Repeating the above process, one can show that $x = 0$ is an equilibrium of the closed-loop system consisting of the nonlinear system in eqn. 1 and the proposed MPCGS.

For the sake of notational simplicity, in the proof of Theorem 1, $\hat{u}(\tau; x, t)$ is replaced by $\hat{u}(\tau)$ when this will not cause confusion. Similarly, the state under the control $\hat{u}(\tau; x(t))$ in the moving horizon time frame, T , is denoted by $\hat{x}(\tau)$. Clearly, the performance index in eqn. 6 at time t depends on the initial $x(t)$ and the control profile $\hat{u}(\tau; x(t))^*$ used in the moving horizon time frame, denoted by $J(x)(t)$, $\hat{u}(\cdot, x(t))^*$ where $\hat{u}(\tau; x(t))^*$ is any control profile. Let $V(x(t + \delta))$ denote such a performance cost under the control profile $\hat{u}(\tau; x(t))$, that is.

$$\begin{aligned} V(x(t)) &= J(x(t), \hat{u}(\cdot; x(t))) \\ &= g(\hat{x}(t + T)) + \int_{\delta}^T \hat{x}(t + \tau)^T Q \hat{x}(t + \tau) \\ &\quad + \hat{u}(t + \tau)^T R \hat{u}(t + \tau) d\tau \end{aligned} \quad (39)$$

Let $V(x)$ be a candidate Lyapunov function for the closed-loop system. Similar to the proof of $x = 0$ being an equilibrium of the closed-loop system under the proposed MPCGS scheme in Lemma 2, it is trivial to show that $V(0) = 0$ since $g(0) = 0$.

It is now proved that the Lyapunov function $V(x)$ along the state trajectory under the proposed MPCGS algorithm is non-increasing. Consider the Lyapunov function value $V(x(t + \delta))$. According to MPCGS algorithm in Section 3, the control profile $\hat{u}(\tau; x(t + \delta))$, $\tau \in [t + 2\delta, t + T + \delta]$, is obtained by starting the optimisation procedure from the initial control profile u' defined in eqn. 12.

Define the performance cost in eqn. 6 for the system under this control profile $\hat{u}(\tau; x(t + \delta))'$ in eqn. 12 as

$$\begin{aligned} V(x(t + \delta))' &= g(\hat{x}(t + T + \delta))' \\ &\quad + \int_{2\delta}^{T+\delta} \hat{x}(t + \tau)' Q \hat{x}(t + \tau)' \\ &\quad + \hat{u}(t + \tau)' R \hat{u}(t + \tau)' d\tau \end{aligned} \quad (40)$$

where $\hat{x}(t + \tau)'$, $\tau \in [2\delta, T + \delta]$ denotes the system's state under the control profile $\hat{u}(\tau; x(t + \delta))'$ in eqn. 12.

Eqn. 40 can also be written as

$$\begin{aligned} V(x(t+\delta))' &= g(\hat{x}(t+T+\delta))' \\ &+ \int_{2\delta}^T \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' \\ &+ \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \\ &+ \int_T^{T-\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' \\ &+ \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (41)$$

Comparing eqn. 39 with eqn. 41 yields

$$\begin{aligned} V(x(t+\delta))' - V(x(t)) &= g(\hat{x}(t-T+\delta))' + g(\hat{x}(t+T))' \\ &+ \int_T^{T+\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' \\ &+ \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \\ &- \int_{\delta}^{2\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (42)$$

Remember that $g(x)$ is continuously differentiable with respect to x . One has

$$\begin{aligned} g(\hat{x}(t+T+\delta))' - g(\hat{x}(t+T))' &= \int_T^{T+\delta} \frac{\partial g(\hat{x}(t+\tau))}{\partial \hat{x}(t+\tau)} \hat{x}(t+\tau)' d\tau \\ &= \int_T^{T+\delta} \frac{\partial g(\hat{x}(t+\tau))}{\partial x(t+\tau)} f(\hat{x}(t+\tau)', \hat{u}(t+\tau)') d\tau \end{aligned} \quad (43)$$

Following from the definition of \hat{u}' in eqn. 12.

$$\hat{u}(t+\tau : x(t+\delta))' = \hat{u}(t+\tau : x(t)) \text{ for } \tau \in [t+\delta, t+T]. \quad (44)$$

This implies that

$$g(\hat{x}(t+T))' = g(\hat{x}(t+T)). \quad (45)$$

In the time interval $[t+T, t+T+\delta]$, the control $\hat{u}(\tau)'$ satisfies eqn. 13. Hence the following inequality holds

$$\begin{aligned} &\int_T^{T+\delta} \frac{\partial g(\hat{x}(t+\tau))}{\partial x(t+\tau)} f(\hat{x}(t+\tau)', \hat{u}(t+\tau)') d\tau \\ &\leq \int_T^{T+\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (46)$$

Invoking eqns. 45 and 46 with eqn. 43 gives

$$\begin{aligned} g(\hat{x}(t+T+\delta))' - g(\hat{x}(t+T))' &\leq \int_T^{T+\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' \\ &+ \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (47)$$

Combining the above inequality with eqn. 42 yields

$$\begin{aligned} V(x(t+\delta))' - V(x(t)) &\leq - \int_{\delta}^{2\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' \\ &+ \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (48)$$

In the proposed MPCGS algorithm, the optimisation procedure at time $t+\delta$ starts from the control profile in eqn. 12. Minimisation of the performance index in eqn. 6 yields the control profile $\hat{u}(\tau : x(t+\delta))$. The performance cost under this control profile is given by $J(x(t+\delta), \hat{u}(\cdot : x(t+\delta)))$, i.e. $V(x(t+\delta))$. The Lyapunov function $V(x(t+\delta))$ along the state trajectory satisfies

$$V(x(t+\delta)) \leq V(x(t+\delta))' \quad (49)$$

Combining eqn. 48 and 49 gives

$$\begin{aligned} V(x(t+\delta)) - V(x(t)) &\leq - \int_{\delta}^{2\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau)' \\ &+ \hat{u}(t+\tau)^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (50)$$

Remember that the control profile $\hat{u}(\cdot : x(t))$ is implemented in the time interval $[t+\delta, t+2\delta]$ until the next control profile is yielded by optimisation. That is,

$$u(t+\tau) = \hat{u}(t+\tau : x(t)), \tau \in [\delta, 2\delta] \quad (51)$$

and

$$x(t+\tau) = \hat{x}(t+\tau), \tau \in [\delta, 2\delta] \quad (52)$$

Using these equations gives

$$\begin{aligned} &\int_{t+\delta}^{t+2\delta} \hat{x}(\tau)^T Q \hat{x}(\tau)' + \hat{u}(\tau)^T R \hat{u}(\tau)' d\tau \\ &= \int_{t+\delta}^{t+2\delta} x(\tau)^T Q x(\tau)' + u(\tau)^T R u(\tau)' d\tau \end{aligned} \quad (53)$$

The inequality in eqn. 50 is equivalent to that of

$$\begin{aligned} V(x(t+\delta)) - V(x(t)) &\leq - \int_{\delta}^{2\delta} x(t+\tau)^T Q x(t+\tau)' \\ &+ u(t+\tau)^T R u(t+\tau)' d\tau \leq 0 \end{aligned} \quad (54)$$

where the latter inequality follows from $Q \geq 0$ and $R > 0$. Hence the Lyapunov function in eqn. 39 is nonincreasing along the closed-loop system's trajectory.

Repeatedly using the inequality in eqn. 54 yields

$$\begin{aligned} V(x(\infty)) - V(x(t_0)) &\leq - \int_{t_0+\delta}^{\infty} x(\tau)^T Q x(\tau)' + u(\tau)^T R u(\tau)' d\tau \\ &\leq - \int_{t_0+\delta}^{\infty} x(\tau)^T Q x(\tau)' d\tau \end{aligned} \quad (55)$$

Since $V(x)$ is monotonically non-increasing and bounded from below by zero, the integral in the right side of the inequality in eqn. 55 converges and hence exists and is bounded.

In addition, it can be shown that $x(t)$ is uniformly continuous in t on $[t_0, \infty)$ under Assumption A1 [39]. Using Baralema's Lemma, this implies that $x^T Q x \rightarrow 0$ as $t \rightarrow \infty$ [40, proof of Theorem 26.4] or [41]. When $Q > 0$ or $Q \geq 0$ and $Q^{1/2}x \neq 0$ for all nonzero x such that $f(x, 0) = 0$, it is concluded that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ [15, 27]. Hence the equilibrium point $x = 0$ of the system in eqn. 1 is asymptotically stable. QED