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A Lyapunov Approach to Exponential Stabilization of Nonholonomic Systems in Power Form

John-Morten Godhavn and Olav Egeland

***Abstract*—In this paper a continuous feedback control law with time-periodic terms is derived for the control of nonholonomic systems in power form. The control law is derived by Lyapunov design from a homogeneous Lyapunov function. Global asymptotic stability is shown by applying the principle of invariance for time-periodic systems. Exponential convergence follows since the vector fields are homogeneous of degree zero.**

I. Introduction

Systems in *power form* have in recent years been used to model the kinematic equations of nonholonomic mechanical systems (e.g., for nonholonomic wheeled vehicles, see [1]). Two types of control laws have been derived to solve this problem, namely piecewise continuous and time-varying control laws. In this paper, time-varying control will be addressed.

Asymptotic stability with exponential convergence for systems in power form has been achieved using time-varying control laws by Pomet and Samson [2], M'Closkey and Murray [3], Kolmanovsky and McClamroch [4], Sørdalen and Egeland [5], and others. In particular, M'Closkey and Murray [6] derived control laws giving ρ-exponential stability by modifying control laws from [7] and [8], where asymptotic stability was obtained. This was done by making the closed-loop vector fields homogeneous of degree zero. The control laws of [6] were derived using Lyapunov arguments, with a Lyapunov function which was the solution of an implicit function that was solved at each time sample. Also, with special relevance for the present paper, Pomet and Samson [2] derived a ρ-exponentially stable control law by Lyapunov arguments using a nonnegative function, which was a Lyapunov function for all states but one.

The main contribution of this paper is that ρ-exponential stability for systems in power form is achieved by Lyapunov design of a control law, where an explicit Lyapunov function for the complete system is used. Moreover, the Lyapunov function is closely related to the square of the homogeneous norm and can be given a relatively simple intuitive interpretation. A short version of this paper for three-dimensional (3-D) systems is given in [9].

Homogeneous feedback controllers are globally continuous and differentiable almost everywhere. Previous applications of homogeneous functions, homogeneous vector fields, and dilations are found in [10] and [11]. A short review of these terms follows.

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Definition 1 (Dilation): Let $\mathbf{x} \in R^n$ be given by $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$. A *dilation* on R^n is defined by assigning n positive rationals $(r_1 = 1 \le r_2 \le \cdots \le r_n)$, and the following map $\Delta_\lambda^r : R^n \mapsto R^n$:

$$\Delta_\lambda^r \mathbf{x} = (\lambda^{r_1} x_1, \lambda^{r_2} x_2, \cdots, \lambda^{r_n} x_n), \qquad \lambda > 0$$

where Δ_λ^r usually is written Δ_λ.

Definition 2 (Homogeneous Function): A continuous function $f : R \times R^n \mapsto R$ is *homogeneous of degree* $l \ge 0$ with respect to Δ_λ if

$$f(t, \Delta_\lambda \mathbf{x}) = \lambda^l f(t, \mathbf{x}).$$

Definition 3 (Homogeneous Vector Field): A continuous vector field $\mathbf{X}(t, \mathbf{x})$ on $R \times R^n$ is *homogeneous of degree* $m \le r_n$ with respect to Δ_λ if its ith coordinate X^i is a homogeneous function of degree $r_i + m$, i.e.,

$$X^i(t, \Delta_\lambda \mathbf{x}) = \lambda^{r_i + m} X^i(t, \mathbf{x}).$$

Definition 4 (Homogeneous Norm): A homogeneous norm ρ is a positive definite homogeneous degree-one function with respect to Δ_λ.

Definition 5 (ρ-Exponential Stability): The equilibrium $\mathbf{x} = \mathbf{0}$ of the system $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ is locally exponentially stable with respect to the homogeneous norm ρ, if for some neighborhood U of the origin there exist two strictly positive numbers, c_1 and c_2, such that

$$\rho(\mathbf{x}(t)) \le c_1 \rho(\mathbf{x}(0)) e^{-c_2 t}, \qquad \forall t \ge 0, \qquad \forall \mathbf{x}(0) \in U.$$

This is called ρ-exponential stability to distinguish it from the usual definition of exponential stability.

The following lemma will be used.

Lemma 1 (From Murray and M'Closkey [10, Lemma 3.4]): Let $\mathbf{X}(t, \mathbf{x})$ be a homogeneous degree-zero vector field. Then uniform asymptotic stability is equivalent to global ρ-exponential stability.

In this paper the definition of a system in power form is given, a set of control laws is derived and analyzed, and some illustrating simulations are given.

II. Control of Systems in Power Form

In this section a control law is derived from a Lyapunov function for systems in power form. The kinematic equations of various nonholonomic mechanical systems have recently been modeled by systems in power form [2].

A. Systems in Power Form

Let $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$ be the state vector, and consider the differential equations for systems in *power form*

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{x}_2 &= v_2 \\ \dot{x}_i &= \frac{1}{(i-2)!} x_2^{(i-2)} v_1, \qquad i \in \{3, \cdots, n\}. \end{aligned} \tag{1}$$

The control problem is to find control laws $v_1 = \alpha_1(t, \mathbf{x})$ and $v_2 = \alpha_2(t, \mathbf{x})$, which stabilize (1). In this paper, time-varying continuous homogeneous control laws are derived, which stabilize (1) ρ-exponentially.

Remark 1: Sufficient properties for conversion of systems $\dot{\mathbf{z}} = \mathbf{X}_1(\mathbf{z})u_1 + \mathbf{X}_2(\mathbf{z})u_2$ to power form via diffeomorphic state and input transformations are given in [12].

B. Derivation of Control Laws for Systems in Power Form

A dilation for systems in power form is given by

$$\Delta_\lambda \mathbf{x} = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \cdots, \lambda^{n-1} x_n). \tag{2}$$

A homogeneous *norm* for this system is given by

$$\rho(\mathbf{x}) = \left(x_1^c + x_2^c + x_3^{c/2} + x_4^{c/3} + \cdots + x_n^{c/(n-1)}\right)^{1/c} \tag{3}$$

where c is the least evenly divisible number of $2, 3, \cdots, (n-1)$.

Differentiation of ρ with respect to time t gives

$$\begin{aligned} \frac{d}{dt}\rho(\mathbf{x}) &= \frac{1}{\rho^{c-1}(\mathbf{x})}\left(x_1^{c-1} + \sum_{i=3}^{n} \frac{1}{(i-1)!} x_i^{\frac{c}{i-1}-1} x_2^{i-2}\right) v_1 \\ &\quad + \frac{x_2^{c-1}}{\rho^{c-1}(\mathbf{x})} v_2 \\ &= q_1 v_1 + q_2 v_2 \end{aligned} \tag{4}$$

where $q_1 = q_1(\mathbf{x})$ and $q_2 = q_2(\mathbf{x})$ as defined above are bounded homogeneous degree-zero functions.

A vector $\mathbf{z} = [z_3, z_4, \cdots, z_n]^T$ is defined by

$$\mathbf{z} = \left[\frac{x_3}{\rho(\mathbf{x})}, \frac{x_4}{\rho^2(\mathbf{x})}, \cdots, \frac{x_n}{\rho^{n-2}(\mathbf{x})}\right]^T \tag{5}$$

where all $z_i(\mathbf{x}) = \frac{x_i}{\rho^{i-2}(\mathbf{x})}$, $i \in \{3, \cdots, n\}$ are homogeneous degree-one functions. Note that $(z_i(\mathbf{x})/\rho(\mathbf{x}))$ are bounded homogeneous degree-zero functions.

Differentiation of z_i gives

$$\begin{aligned} \frac{d}{dt} z_i(\mathbf{x}) &= \left(\frac{1}{(i-2)!}\frac{x_2^{i-2}}{\rho^{i-2}(\mathbf{x})} - (i-2)\frac{x_i}{\rho^{i-1}(\mathbf{x})} q_1(\mathbf{x})\right) v_1 \\ &\quad - (i-2)\frac{x_i}{\rho^{i-1}(\mathbf{x})} q_2(\mathbf{x}) v_2 \\ &= p_{1i} v_1 + p_{2i} v_2 \end{aligned} \tag{6}$$

which in vector form is written $d\mathbf{z}/dt = \mathbf{p}_1 v_1 + \mathbf{p}_2 v_2$, where $\mathbf{p}_1 = \mathbf{p}_1(\mathbf{x}) = [p_{13}(\mathbf{x}), \cdots, p_{1n}(\mathbf{x})]^T$ and $\mathbf{p}_2 = \mathbf{p}_2(\mathbf{x}) = [p_{23}(\mathbf{x}), \cdots, p_{2n}(\mathbf{x})]^T$. The functions $p_{ji}(\mathbf{x})$ defined above, where $j \in \{1, 2\}$ and $i \in \{3, \cdots, n\}$ are all bounded and of homogeneous degree zero.

Remark 2: The vector $\mathbf{z}(\mathbf{x})$ is defined $\forall \mathbf{x} \in R^n$ and is differentiable $\forall \mathbf{x} \neq \mathbf{0}$.

To derive a control law by Lyapunov design, a state transformation $\mathbf{x} \mapsto \tilde{\mathbf{x}}$ is introduced, where

$$\tilde{\mathbf{x}} = [x_1, \tilde{x}_2, x_3, \cdots, x_n]^T \tag{7}$$

$$\tilde{x}_2 = x_2 - \mathbf{z}^T(\mathbf{x})\boldsymbol{\beta}(t) \tag{8}$$

$$\boldsymbol{\beta}(t) = h[\sin t, \sin 2t, \sin 3t, \cdots, \sin(n-2)t]^T. \tag{9}$$

h is a scalar design parameter.

Remark 3: Several choices of $\boldsymbol{\beta}(t)$ are possible. However, $\boldsymbol{\beta}$ must be an $(n-2)$-vector consisting of linearly independent time-periodic functions $\beta_i \in \mathbf{C}^1$ with given bound on amplitude. The functions β_i must be linearly independent in the sense that $\sum_{i=3}^n a_i\beta_i(t) = 0$ $\forall t \geq 0$ implies that all the constants $a_i = 0$, $i \in \{3, \cdots, n\}$. It is not necessary that $\boldsymbol{\beta}$ is trigonometric. In this paper, $\boldsymbol{\beta}$ is selected according to (9) to simplify the proof.

The transformed system satisfies (1), where $\dot{x}_2$ is replaced with

$$\dot{\tilde{x}}_2 = -\mathbf{p}_1^T\boldsymbol{\beta} v_1 + \left(1 - \mathbf{p}_2^T\boldsymbol{\beta}\right) v_2 - \mathbf{z}^T\dot{\boldsymbol{\beta}}. \tag{10}$$

The homogeneous norm defined by (3) with respect to $\tilde{\mathbf{x}}$ is given by

$$\rho(\tilde{\mathbf{x}}) = \left(x_1^c + \tilde{x}_2^c + x_3^{c/2} + x_4^{c/3} + \cdots + x_n^{c/(n-1)}\right)^{1/c}. \tag{11}$$

Consider the positive definite time-periodic function V given by

$$\begin{aligned} V(t, \mathbf{x}) &= \frac{1}{2}\rho^2(\tilde{\mathbf{x}}) \\ &= \frac{1}{2}\left(x_1^c + \tilde{x}_2^c + x_3^{c/2} + \cdots + x_n^{c/(n-1)}\right)^{2/c}. \end{aligned} \tag{12}$$

V is positive definite and decrescent and a function of homogeneous degree two with respect to $\Delta_\lambda \mathbf{x}$. Note that V is a Lyapunov function candidate also with respect to $\tilde{\mathbf{x}}$.

Remark 4: The Lyapunov function V is the square of the homogeneous norm of the transformed state. Note that the state transformation $\mathbf{x} \mapsto \tilde{\mathbf{x}}$ is simply a time-varying offset of x_2 by $\mathbf{z}^T\boldsymbol{\beta}$ and that convergence in $\mathbf{x}$ and $\tilde{\mathbf{x}}$ is equivalent since $\mathbf{x} = \mathbf{0} \Leftrightarrow \tilde{\mathbf{x}} = \mathbf{0}$. Thus, the transformation corresponds to letting the origin of the transformed system oscillate along the x_2 axis around the origin of the original system. The structure of systems in power form implies that controllability is lost when $x_2 = 0$, since then $\dot{x}_i = 0$ $\forall i \in \{3, \cdots, n\}$. V can be thought of as appearing by periodically "shaking" the potential function defined by the square of the homogeneous norm. Lyapunov design with (12) obviously leads to a time-varying control law, and this control law turns out to give the desired result. The control laws will be designed so that x_2 tracks $\mathbf{z}^T\boldsymbol{\beta}$, i.e., convergence of $\tilde{x}_2 = x_2 - \mathbf{z}^T\boldsymbol{\beta}$ to zero.

The control laws $v_1 = \alpha_1(t, \mathbf{x})$ and $v_2 = \alpha_2(t, \mathbf{x})$ given by

$$\alpha_1(t, \mathbf{x}) = -\gamma_1 \frac{1}{\rho^{c-2}(\tilde{\mathbf{x}})}\left(x_1^{c-1} + \sum_{i=3}^{n}\frac{1}{(i-1)!} x_i^{\frac{c}{i-1}-1} x_2^{i-2}\right) \tag{13}$$

$$\alpha_2(t, \mathbf{x}) = \frac{1}{1 - \mathbf{p}_2^T\boldsymbol{\beta}}\left(-\gamma_2\tilde{x}_2 + \mathbf{z}^T\dot{\boldsymbol{\beta}} + \mathbf{p}_1^T\boldsymbol{\beta}\alpha_1\right) \tag{14}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are controller parameters, give

$$\dot{V}(t, \mathbf{x}) = -\gamma_1 \frac{\rho^{2(c-1)}(\mathbf{x})}{\rho^{2(c-2)}(\tilde{\mathbf{x}})} q_1^2(\mathbf{x}) - \gamma_2 \frac{\tilde{x}_2^c}{\rho^{c-2}(\tilde{\mathbf{x}})} \leq 0. \tag{15}$$

Remark 5: Note that the control law $\alpha_1(t, \mathbf{x})$ also can be written as

$$\alpha_1(t, \mathbf{x}) = -\gamma_1 \frac{\rho^{c-1}(\mathbf{x})}{\rho^{c-2}(\tilde{\mathbf{x}})} q_1(\mathbf{x}).$$

Remark 6: Exponential convergence of $\tilde{x}_2(t) = \tilde{x}_2(0)e^{-\gamma_2 t}$ follows from (10) and (14):

$$\dot{\tilde{x}}_2 = -\gamma_2 \tilde{x}_2. \tag{16}$$

Remark 7: To ensure an analytic control law α_2 given by (14), the amplitude h of the time-varying function $\boldsymbol{\beta}$ must be bounded above. A conservative bound for h is derived in the Appendix for a given dimension n.

Proposition 1: The vector field $\mathbf{X}(t, \mathbf{x})$ given by

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}) = \begin{bmatrix} \alpha_1(t, \mathbf{x}) \\ \alpha_2(t, \mathbf{x}) \\ x_2\alpha_1(t, \mathbf{x}) \\ \vdots \\ \frac{1}{(n-2)!} x_2^{n-2}\alpha_1(t, \mathbf{x}) \end{bmatrix}$$

is homogeneous of degree zero with respect to the dilation $\Delta_\lambda \mathbf{x}$.

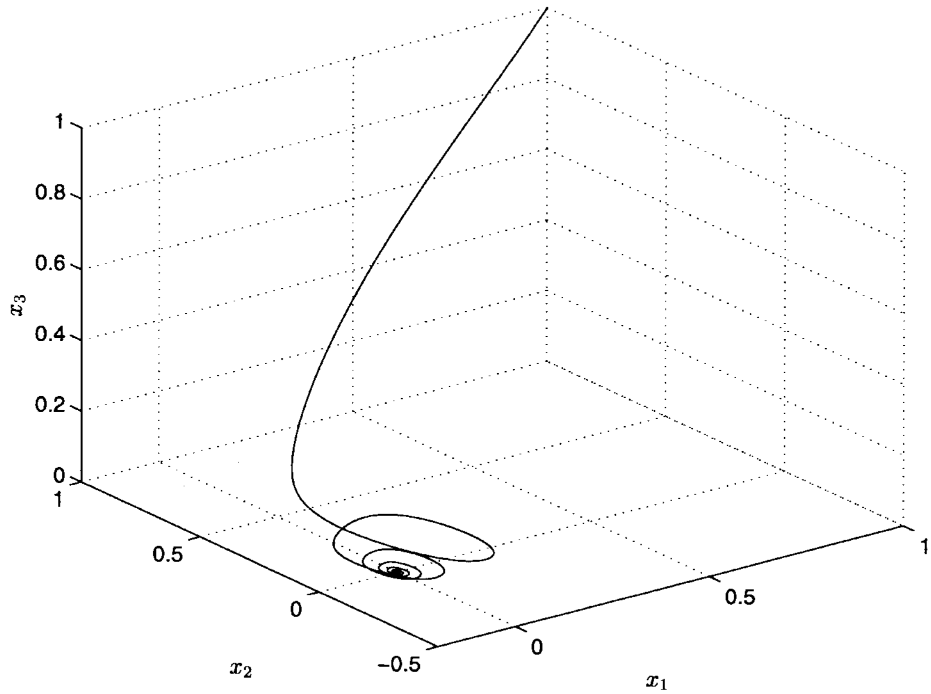

Fig. 1. A trajectory for the 3-D power form. The trajectory spirals around the x_3-axis in a smooth motion toward the origin. The initial state was $\mathbf{x} = [1, 1, 1]^T$. The parameters used in the simulation were $\gamma_1 = \gamma_2 = h = 1$.

Proof: This follows from direct calculation:

$$\mathbf{X}(t, \Delta_\lambda \mathbf{x}) = \begin{bmatrix} \alpha_1(t, \Delta_\lambda \mathbf{x}) \\ \alpha_2(t, \Delta_\lambda \mathbf{x}) \\ \lambda x_2 \alpha_1(t, \Delta_\lambda \mathbf{x}) \\ \vdots \\ \frac{1}{(n-2)!}\lambda^{n-2} x_2^{n-2} \alpha_1(t, \Delta_\lambda \mathbf{x}) \end{bmatrix} = \operatorname{diag}(\lambda, \lambda, \lambda^2, \cdots, \lambda^{n-1})\mathbf{X}(t, \mathbf{x})$$

since both α_1 and α_2 are homogeneous degree-one functions. □

Theorem 1 (Main Result): The control laws α_1 and α_2 given by (13) and (14) give ρ-exponential stability by Definition 5 of systems in power form (1).

Proof: The proof of stability is based on the Krasovskii–LaSalle theorem (e.g., [13, Th. 5.3.79]) for time-periodic systems.

Consider the function $V(t, \mathbf{x})/\rho^2(\mathbf{x})$, which is homogeneous of degree zero, time-periodic, and well defined $\forall \mathbf{x} \neq 0$. Hence, it is a strictly positive continuous function that lives on a compact set. From this it can be concluded that there exist two constant scalars $0 < k_1 \leq k_2$ so that V is bounded above and below by

$$k_1 \rho^2(\mathbf{x}) \leq V(t, \mathbf{x}) \leq k_2 \rho^2(\mathbf{x}). \tag{17}$$

Hence, V is positive definite, decrescent, and radially unbounded.

The set $M \subset R^n$, where $\dot{V} = 0$, is given by

$$\begin{aligned} M = \{\mathbf{x} \in R^n; \exists t \geq 0, \\ x_2 - \mathbf{z}^T(\mathbf{x})\boldsymbol{\beta}(t) = 0, q_1(\mathbf{x}) = 0\}. \end{aligned} \tag{18}$$

Note that whenever $\mathbf{x} \in M$, then x_1 and x_i, $i \in \{3, \cdots, n\}$, are constants. This is seen from (1), (4), (6), and (13), which imply $v_1 = \alpha_1(t, \mathbf{x}) = -\gamma_1 \frac{\rho^{c-1}(\mathbf{x})}{\rho^{c-2}(\mathbf{x})} q_1(\mathbf{x}) = 0 \ \forall \mathbf{x} \in M$.

Assume now that M contains a nonzero solution $\mathbf{x} \neq \mathbf{0}$, where x_1, x_i, $i \in \{3, \cdots, n\}$ are all constants and $x_2(t)$ is time-varying. The case $\mathbf{z} = \mathbf{0}$ implies that both $x_1 = x_2 = 0$ in M, so the case to consider is when $\mathbf{z} \neq \mathbf{0}$.

In the Appendix it is shown that if x_2 is a constant, then $\mathbf{z} = \mathbf{0}$, i.e., x_2 is not a constant in M when $\mathbf{z} \neq \mathbf{0}$.

Further, $\forall \mathbf{x} \in M$

$$\begin{aligned} 0 &= q_1(\mathbf{x})\rho^{c-1}(\mathbf{x}) \\ &= x_1^{c-1} + \sum_{i=3}^{n} \frac{1}{(i-1)!} x_i^{\frac{c}{i-1}-1} x_2^{i-2} \\ &= a + \sum_{i=3}^{n} b_i x_2^{i-2} \end{aligned} \tag{19}$$

where a and b_i are constants given by

$$a = x_1^{c-1} \tag{20}$$

$$b_i = \frac{1}{(i-1)!} x_i^{\frac{c}{i-1}-1}. \tag{21}$$

The time-varying function $x_2(t)$ is given as a sum of trigonometric functions of different order (linearly independent with respect to time t) on M by

$$x_2(t) = \mathbf{z}^T \boldsymbol{\beta}(t) = \sum_{i=3}^{n} z_i \beta_i(t) = h \sum_{i=3}^{n} z_i \sin(i-2)t. \tag{22}$$

It follows that the elements of the set $\{x_2(t), x_2^2(t), \cdots, x_2^{n-2}(t)\}$ are linearly independent functions $t \mapsto x_2^{i-2}(t)$, $i \in \{3, \cdots, n\}$. This can be shown by writing the trigonometric functions as complex exponentials and using the fact that exponentials of different order are linearly independent. The linear independence of the elements of the set $\{x_2(t), x_2^2(t), \cdots, x_2^{n-2}(t)\}$ implies that the only solution to (19) is $b_i = 0$, $\forall i \in \{3, \cdots, n\}$. From the definition of b_i (21), it follows that

$$b_i = 0 \Leftrightarrow x_i = 0, \qquad \forall i \in \{3, \cdots, n\} \tag{23}$$

and hence $x_i = 0 \ \forall i \in \{3, \cdots, n\}$. Then $\mathbf{z} = \mathbf{0} \Rightarrow x_1 = x_2 = 0$ by the definition of the invariant set M (18), and hence $\mathbf{x} = \mathbf{0}$. This is a contradiction to the assumption $\mathbf{x} \neq \mathbf{0}$, and hence the zero-solution $\mathbf{x} = \mathbf{0}$ is the only possible trajectory in the invariant set M.

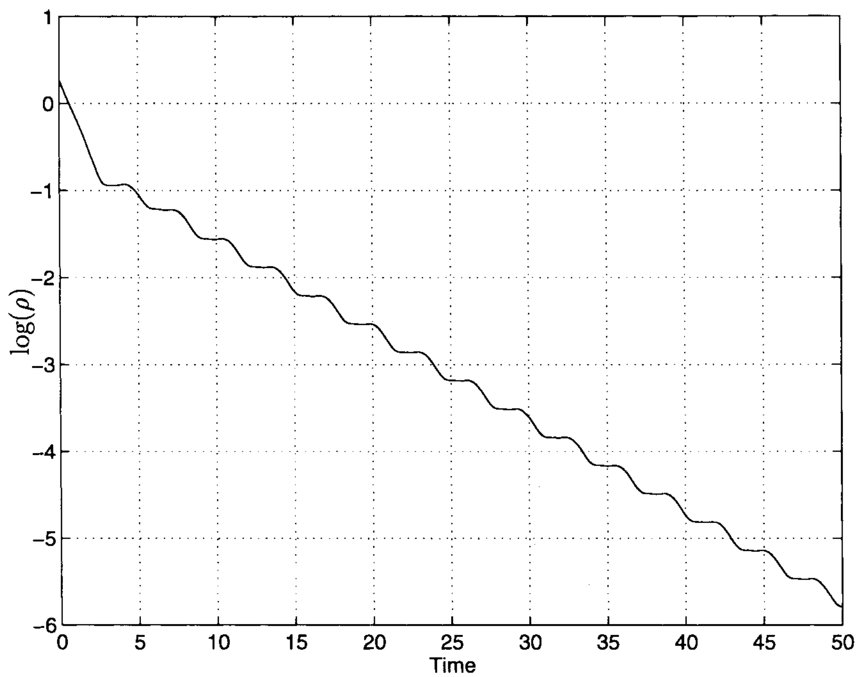

Fig. 2. The logarithm of the homogeneous norm $\rho(\mathbf{x}) = (x_1^4 + x_2^4 + x_3^2)^{1/4}$ is bounded above by a function which decreases linearly. This illustrates the exponential convergence of the system.

Global uniform asymptotic stability of the origin is established by the principle of invariance, since the trivial trajectory is the only trajectory contained in M. ρ-exponential stability follows by Lemma 1, since $\mathbf{X}(t, \mathbf{x})$ is a homogeneous degree-zero vector field. This concludes the proof of Theorem 1. □

Remark 8: In this proof it is clearly demonstrated why the time-periodic terms appearing in the control law make the system ρ-exponentially stable. The property that the periodic functions are linearly independent with respect to time is used directly.

Remark 9: An extension of the results to the dynamic case, i.e., with integrators added to the inputs, can be made by applying a method proposed in [14].

Appendix

Proposition 2: If the amplitude h of the time-periodic signals $\beta_i(t)$ is selected so that $\forall i \in \{3, \cdots, n\}$

$$|\beta_i(t)| \leq h < \sqrt{\frac{6}{(n-1)(n-2)(2n-3)}} \tag{24}$$

then the denominator $(1 - \mathbf{p}_2^T \boldsymbol{\beta})$ of the control law (14) is strictly positive $\forall t \geq 0$ and $\forall \mathbf{x} \in R^n$.

Proof: The following conservative computation gives the result:

$$\begin{aligned} 1 - \mathbf{p}_2^T(\mathbf{x})\boldsymbol{\beta}(t) &\geq 1 - \|\boldsymbol{\beta}(t)\| \cdot \|\mathbf{p}_2(\mathbf{x})\| \\ &\geq 1 - h\sqrt{\sum_{i=3}^{n} \left(-(i-2)\frac{x_i}{\rho^{i-1}(\mathbf{x})} \frac{x_2^{c-1}}{\rho^{c-1}(\mathbf{x})} \right)^2} \\ &\geq 1 - h\sqrt{\sum_{i=3}^{n} (i-2)^2} \\ &= 1 - h\sqrt{\frac{(n-1)(n-2)(2n-3)}{6}}. \end{aligned} \tag{25}$$

In the computation above, (6), (3), and (4) were used. □

Proposition 3: If x_2 is a constant and $\mathbf{x} \in M$, then $\mathbf{z} = \mathbf{0}$. Hence, if $\mathbf{z} \neq \mathbf{0}$, then $x_2(t)$ is a time-varying function.

Proof: Assume that x_2 is a constant. This, together with the fact that $\forall \mathbf{x} \in M$, x_1 and x_i, $\forall i \in \{3, \cdots, n\}$ are constants, implies that $\rho(\mathbf{x})$ is constant and that $\mathbf{z}$ is a constant vector. Further

$$\text{constant} = x_2 = \sum_{i=3}^{n} z_i \beta_i(t) = h \sum_{i=3}^{n} z_i \sin(i-2)t, \qquad \forall t \geq 0 \tag{26}$$

has only the trivial solution that all constants $z_i = 0$, $\forall i \in \{3, \cdots, n\}$, since $\beta_i(t)$ are linearly independent functions (trigonometric functions of different order). The result follows. □

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A Dual Formulation of Mixed μ and on the Losslessness of (D, G) Scaling

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***Abstract*—This paper studies the mixed structured singular value, μ, and the well-known (D, G)-scaling upper bound, ν. A dual characterization of μ and ν is derived, which intimately links the two values. Using the duals it is shown that ν is guaranteed to be lossless (i.e., equal to μ) if and only if $2(m_r + m_c) + m_C \leq 3$, where $m_r, m_c,$ and m_C are the numbers of repeated real scalar blocks, repeated complex scalar blocks, and full complex blocks, respectively. The losslessness result further leads to a variation of the well-known Kalman–Yakubovich–Popov lemma and Lyapunov inequalities.**

***Index Terms*—Duality, Kalman–Yakubovich–Popov lemma, linear matrix inequalities, mixed structured singular values.**

I. Introduction

In two adjoining papers, Doyle [1] and Safonov [2] coined the structured singular value as a tool to test for robust stability of closed-loop systems. The D-scaling upper bound introduced in the very first paper on structured singular values [1] is to date still the most widely used upper bound of the structured singular value. As claimed in [3], D-scaling for complex structures with full blocks is in practice close to the actual structured singular value (or μ for short), and for several nontrivial complex structures the D-scaling upper bound is proved to be lossless [1], [3].

Progress in the theory of mixed μ has been slower. Mixed real/complex μ is an extension of μ that allows the structure to consist of real and complex parts. Such mixed structures arise, for example, if robust stability is to be tested with respect to parametric uncertainties. In Fan *et al.* [4], an upper bound for mixed μ is presented, but, unlike its pure complex counterpart, this upper bound—which we call ν—can be far from the actual mixed μ [5].

So far not much is known about losslessness of ν for mixed structures. Fan *et al.* [4] have shown that ν is lossless if there is one *nonrepeated* real scalar and one full complex block. Young [6] showed that ν is lossless for rank-one matrices.

In this paper we show that the upper bound ν of mixed μ is lossless iff $2(m_r+m_c)+m_C \leq 3$, where m_r, m_c, and m_C are the numbers of repeated real scalar blocks, repeated complex scalar blocks, and full complex blocks, respectively. These losslessness results come as a by-product of a dual formulation of μ and ν that we derive. The duality theory forms the bulk of this paper and is of independent interest. It is partly based on Rantzer's proof of the Kalman–Yakubovich–Popov (KYP) lemma [7], which in turn is influenced by [3].

Section II introduces notation and a few well established results. In Section III the dual characterizations of μ and ν are derived. As an example of the use of these dual results, we show that $\mu(M) = \nu(M)$ if M has rank one (the proof is a substantial simplification compared to that in Young [6]). The dual characterizations are applied in Section IV to prove the losslessness of ν for the mentioned structures. In Section V we give examples that show that for all other structures ν is not lossless. Section VI is about a variation of the KYP lemma and Lyapunov inequalities.

II. Notation and (D, G) Scaling

The norm $\|T\|$ of a matrix $T \in C^{m\times n}$ is in this paper the spectral norm. The Euclidean norm of T is denoted as $\|T\|_2$. T^{H} is the complex conjugate transpose of T, and $\mathrm{He}\, T$ is the Hermitian part of T defined as

$$\mathrm{He}\, T = \frac{1}{2}(T + T^{\mathrm{H}}).$$

Given a subset X of $C^{n\times n}$, the *(mixed) structured singular value* of $M \in C^{n\times n}$ is denoted by $\mu_X(M)$ and is defined as

$$\mu_X(M) = \frac{1}{\inf\{\|\Delta\| : I - \Delta M \text{ is singular and } \Delta \in X\}}.$$

$\mu_X(M)$ is set to zero if $I - \Delta M$ is nonsingular for every $\Delta \in X$. Obviously $\mu_X(M)$ depends on the "structure" X. Whenever $\mu_X(M)$ is used it is implicitly assumed that some structure X is given. Invariably, X is assumed block-diagonal of the form

$$\begin{aligned} X = \mathrm{diag}(&RI_{k_1}, \cdots, RI_{k_{m_r}},\ CI_{l_1}, \cdots, CI_{l_{m_c}},\\ &C^{f_1\times f_1}, \cdots, C^{f_{m_C}\times f_{m_C}}) \end{aligned} \tag{1}$$

where $m_r, m_c,$ and m_C are the number of repeated real scalar blocks, repeated complex scalar blocks, and full complex blocks, respectively.

A. (D, G) Scaling

Let H^q denote the set of $q \times q$ Hermitian matrices, and denote its subset of positive definite elements by P^q. Given the structure X of (1), the sets $\mathcal{D}_X$ and $\mathcal{G}_X$ are defined as

$$\begin{aligned} \mathcal{D}_X = \mathrm{diag}(&P^{k_1}, \cdots, P^{k_{m_r}},\ P^{l_1}, \cdots, P^{l_{m_c}},\\ &P I_{f_1}, \cdots, P I_{f_{m_C}})\\ \mathcal{G}_X = \mathrm{diag}(&H^{k_1}, \cdots, H^{k_{m_r}}\ 0_{l_1\times l_1}, \cdots, 0_{l_{m_c}\times l_{m_c}},\\ &0_{f_1\times f_1}, \cdots, 0_{f_{m_C}\times f_{m_C}}). \end{aligned}$$

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