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Nonlinear Predictive Control for Setpoint Families

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Abstract

Varying product specifications and load changes require that operating points have to be changed frequently during the operation of many processes. Hence controllers have to guarantee stability for all setpoints and allow smooth transfer between setpoints. We propose to combine pseudolinearization and quasi-infinite horizon nonlinear predictive control for the solution of this problem. The pseudolinearization is used to obtain a closed expression for the controller parameters as a function of the setpoints, such that for every setpoint stability of the closed loop is guaranteed.

1 Introduction

For many control problems it is required to stabilize not only one single setpoint. In practice it does vary, for example due to changing product specifications or load changes. Suitable controllers should lead to satisfactory behavior of the closed loop around all setpoints while allowing for smooth transfer between them. In this paper we suggest an extension to the so-called quasi-infinite horizon nonlinear predictive control (QIH-NMPC) scheme [4, 3, 6], such that a whole family of operating points can be stabilized.

In recent years many advances in NMPC for the stabilization of one fixed operating point have been made (see for example [5, 1] for a review) and several NMPC strategies that guarantee stability have been proposed. For all these schemes certain controller parameters need to be determined, before stability for the considered setpoint can be guaranteed. In principle one can use NMPC algorithms that guarantee stability for a single setpoint to stabilize any setpoint of a setpoint family. Yet if such schemes are used, the parameters of the NMPC controller have to be either precalculated off–line for a limited number of possible setpoints or have to be calculated on–line to account for the changing setpoint conditions. Clearly this is very unsatisfying and not practical if a large (infinite) number of operating points is considered.

We adopt results from pseudolinearization [10, 12] to derive a model predictive control strategy based on the so called quasi–infinite horizon NMPC (QIH-NMPC) concept [3]. The resulting controller does stabilize the whole setpoint family without increasing the on-line computational load.

The combination of feedback-linearization techniques with predictive control has been proposed before [8, 9, 2]. Compared to [8, 9] we do only use feedback linearization to obtain suitable controller parameters off-line. We do *not* use it in a hierarchical control structure, where the MPC controller is used to stabilize an inner loop, namely the feedback linearized system as in [8, 9]. In [2] a similar approach as in this paper utilizing exact input to state linearization instead of pseudolinearization is proposed. There no varying setpoints are considered and the goal is mainly to reduce the necessary computation time. The use of pseudolinearization does also allow to consider a much richer class of systems then the use of exact input to state linearization.

The paper is organized as follows. In Section 2 the considered system class and the control problem is introduced. Section 3 presents several NMPC schemes for the stabilization of a setpoint family. The main attention is laid on the combination of pseudo–linearization and QIH-NMPC and the resulting stability properties are discussed.

For simplicity, if not otherwise stated, all functions and derivatives that appear in the sequel are assumed to be sufficiently smooth.

2 Problem Setup and Preliminaries

In this paper the control of multi–input nonlinear systems of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{f}(\mathbf{0},\mathbf{0}) = \mathbf{0}$ is considered. The input $\mathbf{u}$ and the state $\mathbf{x}$ are required to stay in predefined sets $\mathbf{U}$ and $\mathbf{X}$: $\mathbf{u}(t) \in \mathbf{U} \subseteq \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbf{X} \subseteq \mathbb{R}^n$, $\forall t \geq 0$. We consider the stabilization of (1) around a family of (constant) operating points given by:

$$\mathbf{f}(\mathbf{x}_s(\gamma), \mathbf{u}_s(\gamma)) = \mathbf{0}, \quad \forall \gamma \in \Gamma. \tag{2}$$

Here $\gamma \in \Gamma$ is the parameterization variable for the setpoints. We assume, as standard in pseudolinearization, that $\gamma \in \mathbb{R}^m$ and that $\Gamma \subseteq \mathbb{R}^m$ is open. Typically the operating point family is parametrized in terms of the input or state components [11]. To guarantee admissibility of the considered operating points and solutions of (1): we require that:

A1 $\mathbf{U} \subseteq \mathbb{R}^m$ is compact and convex, $\mathbf{X} \subseteq \mathbb{R}^n$ is connected and $(\mathbf{x}_s(\gamma), \mathbf{u}_s(\gamma)) \in \mathbf{X} \times \mathbf{U}, \ \forall \gamma \in \Gamma$.

A2 System (1) has an unique, continuous solution for any initial condition $\mathbf{x}(0) \in \mathbf{X}$ and any piecewise continuous and right–continuous input function $\mathbf{u}(\cdot)$ with $\mathbf{u}(t) \in \mathbf{U}$.

2.1 Pseudolinearization

We will need the so called parameterized linearization family [11] of (1), which is the linearization of (1) about its constant operating point family (2):

$$\frac{d}{dt}\Delta\mathbf{x} = A(\gamma)\Delta\mathbf{x} + B(\gamma)\Delta\mathbf{u} \tag{3}$$

with $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_s(\gamma)$, $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_s(\gamma)$ and

$$A(\gamma) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_s(\gamma), \mathbf{u}_s(\gamma)), B(\gamma) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_s(\gamma), \mathbf{u}_s(\gamma)).$$

Additionally to the assumptions made so far, we need for the pseudolinearization, that $\forall \gamma \in \Gamma$:

A3 $\operatorname{rank}\left(\frac{\partial \mathbf{x}_s}{\partial \gamma}(\gamma)\right) = m$ and $\operatorname{rank}(B(\gamma)) = m$.

A4 The linearization family (3) is controllable, $\operatorname{rank}\left[B(\gamma), A(\gamma)B(\gamma), \dots, A^{n-1}(\gamma)B(\gamma)\right] = n$.

A5 The controllability indices of (3) are constant.

The aim of pseudolinearization [10] is to find an invertible state variable change $\mathbf{z} = \Phi(\mathbf{x})$, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and a state feedback law $\mathbf{u} = \mathbf{k}(\mathbf{x}, \mathbf{v})$, $\mathbf{k} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, such that the transformed and controlled closed loop system has the same (constant) linearization (A_z, B_z) independent of γ when linearized about any setpoint within the operating point family $(\mathbf{z}_s(\gamma), \mathbf{v}_s(\gamma))$:

$$\frac{d}{dt}\Delta\mathbf{z} = A_z\Delta\mathbf{z} + B_z\Delta\mathbf{v}, \quad \gamma \in \Gamma, \tag{4}$$

with $\Delta\mathbf{z} = \mathbf{z} - \mathbf{z}_s(\gamma)$, $\Delta\mathbf{v} = \mathbf{v} - \mathbf{v}_s(\gamma)$, $\mathbf{z}_s(\gamma) = \Phi(\mathbf{x}_s(\gamma))$ and A_z, B_z given by:

$$A_z = \begin{bmatrix} A_{z1} & & 0 \\ & \ddots & \\ 0 & & A_{zm} \end{bmatrix}, \ A_{zj} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{k_j \times k_j} \tag{5}$$

$$B_z = \begin{bmatrix} B_{z1} & & 0 \\ & \ddots & \\ 0 & & B_{zm} \end{bmatrix}, \ B_{zj} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{k_j}. \tag{6}$$

Here $k_1, \dots, k_m$ are the controllability indices. In what follows we will assume, that:

A6 An invertible transformation $\mathbf{z} = \Phi(\mathbf{x})$ and a feedback $\mathbf{u} = \mathbf{k}(\mathbf{x}, \mathbf{v})$ with $\frac{\partial \Phi}{\partial \mathbf{x}}(\mathbf{x}(\gamma))$ and $\frac{\partial \mathbf{k}}{\partial \mathbf{v}}(\mathbf{x}(\gamma), \mathbf{v}(\gamma))$ non–singular for all $\gamma \in \Gamma$ is known, such that the pseudolinearized system has the constant linearization (4) about (3).

We thus only consider systems for which a pseudolinearization can be obtained.

Remark 2.1 *Note, that the necessary conditions for the existence of a pseudolinearization transformation are much less restrictive than the ones required for exact input to state linearization. However the computation of the pseudolinearization transformation and feedback may be still rather cumbersome.*

3 NMPC for Setpoint Families

In this section we will show how NMPC can be utilized for the stabilization of (1) around all fixed setpoints in the setpoint family. In the framework of predictive control, at each "sampling" time an open loop optimal control problem with the measured state $\mathbf{x}(t)$ of the plant as initial condition is solved. We will base our considerations on the following, rather general open–loop optimal control problem, that has to be solved at every sampling instance:

Problem 1: *Solve*

$$\min_{\bar{\mathbf{u}}(\cdot)} J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); \gamma) \tag{7}$$

subject to:

$$\dot{\bar{\mathbf{x}}}(\tau) = \mathbf{f}(\bar{\mathbf{x}}(\tau), \bar{\mathbf{u}}(\tau)), \quad \bar{\mathbf{x}}(t; \mathbf{x}(t), t) = \mathbf{x}(t) \tag{8a}$$

$$\bar{\mathbf{u}}(\tau) \in \mathbf{U}, \quad \tau \in [t, T_p] \tag{8b}$$

$$\bar{\mathbf{x}}(\tau; \mathbf{x}(t), t, \bar{\mathbf{u}}) \in \mathbf{X}, \quad \tau \in [t, T_p] \tag{8c}$$

$$\bar{\mathbf{x}}(t + T_p; \mathbf{x}(t), t) \in \Omega(\gamma) \tag{8d}$$

with the (finite) horizon cost function:

$$J(\mathbf{x}(t), \bar{\mathbf{u}}(\cdot); \gamma) := \int_t^{t+T_p} F(\bar{\mathbf{x}}(\tau; \mathbf{x}(t), t, \bar{\mathbf{u}}), \bar{\mathbf{u}}(\tau); \gamma) d\tau + E(\bar{\mathbf{x}}(t + T_p; x(t), t, \bar{\mathbf{u}}); \gamma). \tag{9}$$

Internal controller variables are denoted by a bar, $\bar{\mathbf{x}}(\cdot; \mathbf{x}(t), t, \bar{\mathbf{u}})$ is the solution of (8a) driven by the input $\bar{\mathbf{u}}(\cdot) : [t, T_p] \to \mathbf{U}$ with initial condition $\mathbf{x}(t)$ at time t up to the prediction horizon T_p. $F(\cdot)$ is the so called *stage cost.* Since we want to stabilize constant operating points parameterized by γ, $F(\cdot)$ has to satisfy:

A7 $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \ni (\mathbf{x}, \mathbf{u}, \gamma) \to \mathbb{R}$ is smooth and satisfies for $\gamma \in \Gamma$: 1. $F(\mathbf{x}_s(\gamma), \mathbf{u}_s(\gamma), \gamma) = 0$, 2. $F(\mathbf{x}, \mathbf{u}, \gamma) > 0$, $\forall \ (\mathbf{x}, \mathbf{u}) \neq (\mathbf{x}_s(\gamma), \mathbf{u}_s(\gamma))$.

The term $E(\bar{\mathbf{x}}(t + T_p); \gamma)$ is often called *terminal penalty term,* while $\Omega(\gamma)$ denotes the so called *terminal region.* The system input during the sampling time δ is given by the optimal input $\bar{\mathbf{u}}^\star(\cdot; \mathbf{x}(t), t, T_p) : [t, t+T_p] \to \mathbf{U}$ which is the solution of the open loop optimization Problem 1 at time t: $\mathbf{u}^\star(\tau) = \bar{\mathbf{u}}^\star(\tau; \mathbf{x}(t), t, T_p)$, $\tau \in [t, t + \delta)$.

The stability of the closed loop system does, as in the case of one setpoint, depend on the proper choice for the controller parameters T_P, E, F.

One way to achieve stability for a *single fixed* setpoint, is the use of an infinite prediction horizon $T_p = \infty$. It is rather easy to see, that this does also lead to stability in the case of a fixed setpoint of a setpoint family, if $F(\cdot)$ is chosen according to A7. This immediately follows from the results for one fixed setpoint. The main disadvantage is, that an open-loop optimization problem over an infinite horizon has to be solved. A second possibility to guarantee stability is the use of a finite horizon T_p together with a ("zero") state terminal constraint:

$$\bar{\mathbf{x}}(t+T_p;\mathbf{x}(t),t)=\mathbf{x}_s(\gamma).$$

The drawback of this is, that feasibility of the resulting open-loop optimization problem often requires the use of a rather large control horizon T_p [7].

3.1 QIH-NMPC for Setpoint Families

One way to reduce feasibility and solvability problems in the case of a single setpoint is the use of the so called quasi–infinite horizon NMPC strategy [4, 6]. The idea behind QIH-NMPC [3, 4, 6] is to approximate an infinite horizon cost functional that is known to lead to closed loop stability by a finite horizon cost functional utilizing a terminal penalty term and terminal region constraint (equations (9) and (8d) in Problem 1).

In the following we will show how QIH-NMPC can be modified for the stabilization of (1) around all considered, fixed setpoints. The problem is to determine "one" terminal region $\Omega(\gamma)$ and "one" terminal penalty term $E(\cdot;\gamma)$ (as an explicit function of γ), such that stability can be guaranteed for all fixed $\gamma\in\Gamma$.

In a first step we consider the system (8a) and the cost function (9) after having applied the pseudolinearization feedback and state transformation:

$$\dot{\bar{\mathbf{z}}}(\tau)=\mathbf{f}_z(\bar{\mathbf{z}}(\tau),\bar{\mathbf{v}}(\tau)),\quad \bar{\mathbf{z}}(t;\mathbf{z}(t),t)=\mathbf{z}(t) \tag{10}$$

$$J_z(\mathbf{z}(t),\bar{\mathbf{v}}(\cdot);\gamma):=\int_t^{t+T_p}F_z(\bar{\mathbf{z}}(\tau;\mathbf{z}(t),t,\bar{\mathbf{v}}),\bar{\mathbf{v}}(\tau);\gamma)d\tau + E_z(\bar{\mathbf{z}}(t+T_p;\mathbf{z}(t),t,\bar{\mathbf{v}});\gamma).$$

Here $\mathbf{f}_z$, F_z and E_z are given by:

$$\begin{aligned}
\mathbf{f}_z(\cdot)&=\frac{\partial\Phi}{\partial\mathbf{x}}(\Phi^{-1}(\bar{\mathbf{z}}))\mathbf{f}\left(\Phi^{-1}(\bar{\mathbf{z}}),\mathbf{k}(\Phi^{-1}(\bar{\mathbf{z}}),\bar{\mathbf{v}})\right),\\
F_z(\cdot)&=F(\Phi^{-1}(\bar{\mathbf{z}}),\mathbf{k}(\Phi^{-1}(\bar{\mathbf{z}}),\bar{\mathbf{v}});\gamma),\\
E_z(\cdot)&=E(\Phi^{-1}(\bar{\mathbf{z}});\gamma).
\end{aligned}$$

The main advantage of this transformation is, that the linearization of (10) around all operating points $(\mathbf{z}_s(\gamma),\mathbf{v}_s(\gamma))$, $\gamma\in\Gamma$ is constant (compare Section 2.1). This allows us to utilize the same techniques as for standard QIH-NMPC [4] in the case of a setpoint family. That is a local *linear* feedback and a corresponding, positive invariant region of attraction are utilized to obtain a the desired terminal region and terminal penalty term. Since the linearization of (8a) around every setpoint is the same, we can always use the same linear control law.

What remains is to find a terminal region Ω_z such that this linear controller stabilizes the nonlinear system in Ω_z and renders Ω_z positive invariant for *every* setpoint . For stability we also need, that the resulting terminal penalty term is an upper bound for the infinite horizon cost. Since the determination of a terminal region with the methods in [4] is based on a quadratic stage cost function, we have to assume that:

A8 There exists a set Ω_1 around the operating point family $(\mathbf{z}_s(\gamma),\mathbf{v}_s(\gamma))$ and two positive definite matrices $Q>0\in\mathbb{R}^{n\times n}$, $R>0\in\mathbb{R}^{m\times m}$, such that the nonlinear stage cost $F_z(\mathbf{z},\mathbf{v};\gamma)$ can be bounded from above by a quadratic stage cost: $F_z(\mathbf{z},\mathbf{v};\gamma)\leq\Delta\mathbf{z}^TQ\Delta\mathbf{z}+\Delta\mathbf{v}^TR\Delta\mathbf{v}$ in Ω_1.

Once Assumptions A1–A8 are satisfied, we can utilize the following procedure for the calculation of a terminal region of the form $\Omega_z(\gamma)=\{\mathbf{z}|\Delta\mathbf{z}^TP\Delta\mathbf{z}\leq\alpha\}$ and a quadratic penalty term $E_z(\cdot)=\Delta\mathbf{z}^TP\Delta\mathbf{z}$ as in the case of one setpoint [4]:

Step 1 Find a locally stabilizing linear state feedback $\Delta\mathbf{v}=K_z\Delta\mathbf{z}$, for the constant linearization family (4).

Step 2 Choose a constant $\kappa\in[0,-\lambda_{\max}(A_{zK}))$, $A_{zK}=A_z+B_zK_z$. Solve the Lyapunov equation $(A_{zK}+\kappa I)^TP+P(A_{zK}+\kappa I)=-(Q+K_z^TRK_z)$ for the positive definite and symmetric terminal weight P.

Step 3 Find the largest possible α_2 such that $\Omega_2\subseteq\Omega_1$ holds for all $\gamma\in\Gamma$, with $\Omega_2:=\{\mathbf{z}\in\mathbb{R}^n|\Delta\mathbf{z}^TP\Delta\mathbf{z}\leq\alpha_2\}$.

Step 4 Find the largest possible $\alpha_3\leq\alpha_2$ such that for $\mathbf{v}=K_z\Delta\mathbf{z}+\mathbf{v}_s(\gamma)$ the state and input constraints are satisfied in the set $\Omega_3:=\{\mathbf{z}\in\mathbb{R}^n|\Delta\mathbf{z}^TP\Delta\mathbf{z}\leq\alpha_3\},\ \forall\gamma\in\Gamma$.

Step 5 Determine a $\Omega_z=\{\mathbf{z}\in\mathbb{R}^n|\Delta\mathbf{z}^TP\Delta\mathbf{z}\leq\alpha\}$ via iterations of

$$\max_{\mathbf{z}\in\Omega_3,\gamma\in\Gamma}\{\Delta\mathbf{z}^TP\phi(\Delta\mathbf{z},\gamma)-\kappa\Delta\mathbf{z}^TP\Delta\mathbf{z}\} \tag{11}$$

$$\text{subject to: }\ \Delta\mathbf{z}^TP\Delta\mathbf{z}\leq\alpha$$

by reducing α from α_3, until the optimal value of (11) is non-positive. $\phi(\Delta\mathbf{z},\gamma):=\mathbf{f}_z(\mathbf{z}_s(\gamma)+\Delta\mathbf{z},\mathbf{v}_s(\gamma)+K_z\Delta\mathbf{z})-A_{zK}\Delta\mathbf{z}$ is the difference between the linearized and the nonlinear system with the input $\mathbf{v}=K_z\Delta\mathbf{z}+\mathbf{v}_s(\gamma)$ at the operating point γ.

Remark 3.1 *In Step 1 a local linear controller is found, which stabilizes the constant linearization family. This implies, that there exists for every operating point a region, in which the linear controller does stabilize the nonlinear system. In Step 2 the terminal weighting matrix P is calculated. It remains to establish the region in which a quadratic Lyapunov function with P as weighting matrix can guarantee stability for the system controlled by the "linear" control law. This is done in Step 5. In Steps 3 and 4 regions are calculated, such that*

Assumption A8 and the input and state constraints for the nonlinear system controlled by the linear controller are satisfied. Note that the existence of these regions is implied by the fact that the state transformation $\Phi(\mathbf{x})$ and the feedback $k(\mathbf{x},\mathbf{v})$ are continuous and do possess continuous derivatives.

Remark 3.2 *In Step 4 we check, that inside of the terminal region the transformed input and state constraints are satisfied. These are given by:*

$$\mathbf{k}\left(\Phi^{-1}(\bar{\mathbf{z}}),\bar{\mathbf{v}}\right)\in\mathbf{U},\quad \Phi^{-1}\left(\bar{\mathbf{z}}\right)\in\mathbf{X}. \tag{12}$$

Note that both constraints do involve the transformed state $\bar{\mathbf{z}}$ and the input $\bar{\mathbf{v}}$ and represent "coupled" nonlinear constraints.

Assumption A8 can be rather restrictive. The main problem is, that this assumption can only be checked once the pseudolinearizing transformation is known. We will see in the proof of Theorem 3.4, that Assumption A8 can be replaced by the following, less restrictive assumption:

A9 There exists a set Ω_1 around the operating point family $(\mathbf{z}_s(\gamma),\mathbf{v}_s(\gamma))$ and two positive definite matrices $Q>0\in\mathbb{R}^{n\times n}$, $R>0\in\mathbb{R}^{m\times m}$, such that the "controlled" nonlinear stage cost $F_z(\mathbf{z},K_z\Delta\mathbf{z}+\mathbf{v}_s(\gamma);\gamma)$ can be bounded from above as follows: $F_z(\mathbf{z},K_z\Delta\mathbf{z}+\mathbf{v}_s(\gamma);\gamma)\le\Delta\mathbf{z}^T\left(Q+K_z^TRK_z\right)\Delta\mathbf{z}$ in Ω_3.

Remark 3.3 *In Procedure 1 "one" terminal region and "one" local controller for all setpoints is determined. It is also possible to obtain for every setpoint a different terminal region and local control law. However then one has the problem that these values have to be obtained and stored for all considered setpoints.*

3.2 Stability of the Closed Loop

The terminal penalty term and the terminal region as computed by the procedure above are given in the original coordinates by:

$$\Omega(\gamma)=\{\bar{\mathbf{x}}\in\mathbb{R}^n|\Delta\Phi(\bar{\mathbf{x}};\gamma)^TP\Delta\Phi(\bar{\mathbf{x}};\gamma)\le\alpha\},$$
$$E(\cdot;\gamma)=\Delta\Phi(\bar{\mathbf{x}};\gamma)^TP\Delta\Phi(\bar{\mathbf{x}};\gamma).$$

with $\Delta\Phi(\bar{\mathbf{x}};\gamma)=\Phi(\bar{\mathbf{x}})-\Phi(\bar{\mathbf{x}}_s(\gamma)$. With these, the following stability theorem for *any fixed* $\gamma\in\Gamma$ holds:

Theorem 3.4 *Suppose that a) Assumptions A1–A7, A9 are satisfied; b) the operating point is fixed, i.e. $\gamma=const.$, $\gamma\in\Gamma$; c) the terminal region and terminal penalty term are calculated according to Procedure 1; d) the open–loop optimal control Problem 2 is feasible at $t=0$. Then the nominal closed–loop system is asymptotically stable with respect to the constant operating point $(\mathbf{x}_s(\gamma),\mathbf{u}_s(\gamma))$ for small enough sampling times δ and the region of attraction contains the set of states for which the optimization problem described in d) has a solution.*

Proof: The proof is split up in four parts. In the first part it is shown, that feasibility of the optimization problem at one sampling time implies feasibility at the next sampling time. In the second part it is shown, that inside the terminal region $\Omega(\gamma)$

$$\frac{dE}{dt}(\mathbf{x};\gamma)\le -F(\mathbf{x},\mathbf{k}(\mathbf{x},K_z(\Phi(\mathbf{x})-\mathbf{v}_s(\gamma)) + \mathbf{v}_s(\gamma));\gamma) \tag{13}$$

holds. Integration on both sides establishes that:

$$E(\bar{\mathbf{x}}(t+T_p;\mathbf{x}(t),t,\bar{\mathbf{u}});\gamma) \ge \int_{t+T_p}^{\infty}F(\bar{\mathbf{x}}(\tau;\mathbf{x}(t),t,\bar{\mathbf{u}}),\bar{\mathbf{u}}(\tau);\gamma)d\tau$$

In the third part it is shown that the sequence of optimal cost functions $J^\star(\mathbf{x}(t);T_p,\gamma)$ (the value function), i.e. the solution of Problem 2 for every sampling instance, is not increasing. This fact allows in the fourth step to establish asymptotic stability with respect to the considered setpoint. For this the value function $J^\star(\bar{\mathbf{x}}(t);T_p,\gamma)$ is used as a "Lyapunov function". Utilizing continuity of the value function at the origin asymptotic stability can be established. Furthermore due to the decrease of the value function it is established, that the region of attraction consists of all points for which Problem 2 has a solution.

Since the proof follows in major parts [2, 4] we do only show part 2 of the proof, that differs most and refer the reader for the remaining parts to [4, 2].

To show (13) we assume that inside of the terminal region all constraints are satisfied. This is assured by Steps 3 and 4 of Procedure 1. We use $\Delta\mathbf{z}^TP\Delta\mathbf{z}$ as a Lyapunov function candidate for the nonlinear system

$$\dot{\mathbf{z}}(\tau)=\mathbf{f}_z(\mathbf{z}(\tau),\mathbf{v}(\tau)) \tag{14}$$

controlled by the linear state feedback

$$\mathbf{v}=K_z\Delta\mathbf{z}+\mathbf{v}_s(\gamma) \tag{15}$$

with $\Delta\mathbf{z}=(\mathbf{z}-\mathbf{z}_s(\gamma))$ around the operating point $\mathbf{z}_s(\gamma)$. Differentiation of $\Delta\mathbf{z}^TP\Delta\mathbf{z}$ along (14) with (15) does lead to:

$$\frac{d}{dt}\left(\Delta\mathbf{z}^TP\Delta\mathbf{z}\right)=\Delta\mathbf{z}^T\left(A_{zK}^TP+PA_{Kz}\right)\Delta\mathbf{z} + 2\Delta\mathbf{z}^TP\phi(\Delta\mathbf{z},\gamma)$$

with $\phi(\Delta\mathbf{z},\gamma):=\mathbf{f}_z(\mathbf{z}_s(\gamma)+\Delta\mathbf{z},\mathbf{v}_s(\gamma)+K_z\Delta\mathbf{z})-A_{zK}\Delta\mathbf{z}$. Since in Step 5 of Procedure 1 the terminal region Ω_z was chosen such that $\Delta\mathbf{z}^TP\phi(\Delta\mathbf{z},(\gamma))-\kappa\Delta\mathbf{z}^TP\Delta\mathbf{z}\le 0$, we obtain

$$\frac{d}{dt}(\Delta\mathbf{z}^TP\Delta\mathbf{z})\le\Delta\mathbf{z}^T(A_{zK}+\kappa I)^TP+P(A_{zK}+\kappa I)\Delta\mathbf{z}.$$

This does lead with Step 2 to

$$\frac{d}{dt}\left(\Delta\mathbf{z}^T P\Delta\mathbf{z}\right) \leq -\Delta\mathbf{z}^T(Q+K_z^T R K_z)\Delta\mathbf{z}^T.$$

It follows, that $\Delta\mathbf{z}^T P\Delta\mathbf{z}$ is in fact a valid Lyapunov function for (14) under the feedback (15) and that (15) renders $\Omega_z(\gamma)$ positive invariant for the system (14). Utilizing Assumption A9 we obtain

$$\frac{d}{dt}\left(\Delta\mathbf{z}^T P\Delta\mathbf{z}\right) \leq -F_z(\mathbf{z},\mathbf{v}_s(\gamma)+K_z\Delta\mathbf{z},\gamma),$$

which corresponds to (13) in original coordinates. Note also, that integration of (13) from $t+T_p$ to ∞ does establish the fact, that $E(\cdot)$ is an upper bound for $\int_{t+T_p}^{\infty} F(\mathbf{x}(\tau;\mathbf{x}(t),t,\mathbf{u}),\mathbf{u}(\tau);\gamma)d\tau$. $\square$

The main advantage of the proposed scheme is, that if a pseudolinearization transformation and feedback can be computed, a closed expression for the terminal region and the terminal penalty term can be given, such that stability can be guaranteed for all considered setpoints. No additional on-line computation is necessary. However, as in the case of the exact feedback-linearization it is in general not an easy task to obtain such a pseudolinearization transformation and feedback. Additionally it is difficult to decide a priori weather Assumption A8 can be satisfied. This does depend on the pseudolinearization transformation and on the cost functional used.

Remark 3.5 *We only consider the case of a constant operating point ($\gamma = const \in \Gamma$) which can be chosen from a set of possible operating points $(\mathbf{x}_s(\gamma),\mathbf{u}_s(\gamma))$, $\gamma \in \Gamma$. For feasibility it would be also possible to consider slowly varying setpoints, provided that the terminal region around the new setpoint could be reached in time T_p from the current state. If for example the new setpoint is chosen such, that last predicted final state $\bar{\mathbf{x}}(t+T_p;\mathbf{x}(t),t)$ is inside of the terminal region belonging to the new steady state: $\bar{\mathbf{x}}(t+T_p;\mathbf{x}(t),t) \in \Omega(\gamma_{new})$ then feasibility of the new setpoint can be guaranteed.*

The application of the proposed NMPC scheme to a process control example, namely the control of a continuous stirred tank reactor can be found in [7].

4 Conclusions

Over the past decade there has been significant progress in the area of nonlinear model predictive control for the stabilization around *one fixed* steady state. However, for practical applications the restriction to one a priori known steady state is too stringent. In the present paper an NMPC approach is presented that allows to stabilize a broad class of constrained nonlinear systems using *one* predictive controller with fixed controller parameters such that stability is achieved in the neighborhood of (almost) *arbitrary* steady states. This method is based on the quasi-infinite horizon nonlinear model predictive control concept that has been extended using ideas from pseudolinearization. The main advantage of this scheme is, that a closed expression for the terminal region and terminal penalty term used in QIH-NMPC in dependence of the setpoint can be obtained, such that stability for the whole setpoint family can be guaranteed. The major drawback is, that a pseudolinearization transformation and feedback is in general not easy to obtain and might involve lengthy calculation. The proposed method can be seen as a possible first step in direction to the (setpoint) tracking problem for nonlinear systems.

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References

[1] F. Allgöwer, T. A. Badgwell, J. S. Qin, J. B. Rawlings, and S. J. Wright. Nonlinear predictive control and moving horizon estimation – An introductory overview. In P. M. Frank, editor, *Advances in Control, Highlights of ECC'99*, pages 391–449. Springer, 1999.

[2] H. Chen. *Stability and Robustness Considerations in Nonlinear Model Predictive Control.* Fortschr.-Ber. VDI Reihe 8 Nr. 674. VDI Verlag, Düsseldorf, 1997.

[3] H. Chen and F. Allgöwer. A quasi-infinite horizon predictive control scheme for constrained nonlinear systems. In *Proc. 16th Chinese Control Conference*, pages 309–316, Qindao, 1996.

[4] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10):1205–1218, 1998.

[5] G. De Nicolao, L. Magni, and R. Scattolini. Stability and robustness of nonlinear receding horizon control. In F. Allgöwer and A. Zheng, editors, *Nonlinear Predictive Control.* Birkhäuser, 2000.

[6] R. Findeisen. Suboptimal nonlinear model predictive control. Master's thesis, University of Wisconsin–Madison, 1997.

[7] R. Findeisen and F. Allgöwer. A nonlinear model predictive control scheme for the stabilization of setpoint families. To appear in: Journal A, 2000.

[8] M.A. Henson and M.J. Kurtz. Input-output linearization of constraint nonlinear processes. AIChE Annual Meeting, San Francisco, CA, 1994.

[9] V. Nevistić and M. Morari. Constrained control of feedback-linearizable systems. In *Proc. 3rd European Control Conference ECC'95*, pages 1726–1731, Rome, 1995.

[10] C. Reboulet and C. Champetier. A new method for linearizing non-linear systems: The pseudolinearization. *Int. J. Contr.*, 40(4):631–638, 1984.

[11] J. Wang and W. J. Rugh. Feedback linearization families for nonlinear systems. *IEEE Trans. Automat. Contr.*, AC-32:935–940, 1987.

[12] J. Wang and W. J. Rugh. On the pseudo-linearization problem for nonlinear systems. *Syst. Contr. Lett.*, 12:161–167, 1989.