MODEL-BASED PREDICTIVE CONTROL

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Glossary

ANCBI: Acronym for time-invariant dynamic linear systems which are *asymptotically null-controllable by bounded* (arbitrarily small) *inputs*.
ARE: Acronym for *algebraic Riccati equation* relevant to the Linear-Quadratic optimal control.
CG: Acronym for a nonlinear device, called *command governor* or reference governor, which is added to a pre-compensated control system so as to satisfy prescribed constraints on the system variables.
COLOC: Acronym for constrained open-loop optimal control problem underlying any model-based predictive control algorithm.
Feasible control sequence: A sequence of controls satisfying control, state and terminal-state constraints in a COLOC problem.
H_∞ **control**: A min-max control problem whereby the control law minimizes an operator gain from the disturbance l_2-norm to the performance-vector l_2-norm.
LQ: Acronym for an optimal regulation of the *Linear-Quadratic* type where the plant to be regulated is a dynamic linear system and the performance index to be minimized quadratic.
LQG: Acronym for an optimal regulation of the *Linear-Quadratic-Gaussian* type where in addition to the features of LQ regulation the plant to be regulated is a dynamic stochastic linear system with state and disturbances jointly Gaussian distributed.
MBPC: Acronym for *model-based predictive control*.
PCG: Acronym for *predictive* command governor, *viz.* a CG designed on the grounds of conceptual tools of MBPC.
Receding horizon control: A control strategy according to which the effective control action applied to the plant at the current time is the first control of a string of possible controls solving a COLOC problem over a time-horizon beginning from the current time onwards.

Summary

MBPC is a feedback-control methodology suitable to enforce efficently hard constraints on the variables of the controlled system. It is shown that the method hinges upon a constrained open-loop optimal control problem along with the adoption of the so-called receding-horizon control strategy. In the important case of time-invariant linear saturated ANCBI systems, MBPC algorithms can be devised with the property of ensuring global feasibility/stability. Considerations on how to deal with disturbances and model uncertainties are also given. A presentation of a simplified form of MBPC, *viz*. the PCG, is finally discussed.

1. Introduction

Model-Based Predictive Control (MBPC) is conceptually a natural method for generating feedback control actions for linear and nonlinear plants subject to pointwise-in-time input and/or state-related constraints. A human being, for instance, while driving a vehicle, generates steering-wheel commands by forecasting or *predicting* over a finite time-horizon the (possible) vehicle state-evolutions on the basis of vehicle current state and dynamics, and a *virtual* or potential steering-wheel command sequence. Then, one, among such sequences, is sorted out, which fulfills safety constraints and meets performance requirements. Only a short initial portion of such a sequence is applied by the driver to the steering-wheel, while its remaining part is discarded. After such an initial portion is applied, the driver repeats the whole operation by restarting predictions over a moved-ahead or *receded* time-horizon from the updated vehicle state as determined by the applied command. MBPC complies with the same logical scheme: the control sequence is computed by solving on-line, over a finite control horizon, an open-loop optimal control problem, given the plant dynamical model and current state. Though this computation hinges upon an open-loop control problem, MBPC yields a feedback-control action. Indeed, similarly to the driver behaviour, in a discrete-time setting only the first control of the open-loop control sequence is applied to the plant, and, according to the *receding-horizon control* philosophy, the whole optimization cycle is repeated at the subsequent time-instant based on the new plant-state. Because it involves a control horizon made up by only a finite number of time-steps, MBPC can be often calculated on-line by existing optimization routines so as to minimize a performance index in the presence of hard constraints on the time evolutions of input and/or state. MBPC ability of handling constraints is of paramount importance whenever constraints are part of the control design specifications. In fact, constraints are typically present in applications, as they stem from actuators saturations and/or physical, safety or economical requirements. Despite the importance of constraints, there is a shortage of control methods for handling them effectively. The main reason for the interest of control engineers in MBPC is therefore its ability to systematically and effectively handle hard constraints. An important observation in this connection is that, in contrast to MBPC, in feedback-control systems of more traditional type, *e.g.*, LQG or H_∞ control, constraints are indirectly enforced, by imposing, whenever possible, a conservative behaviour at a performance-degradation expense. Other instances where MBPC can be advantageously used comprise unconstrained plants for which off-line computation of a control law is a difficult task as compared with on-line computations via receding-horizon control.
The presentation of MBPC given hereafter aims at enlightnening the main features of the approach, related well-established feasibility/stability constructive arguments, and current open problems. Consideration will be also given to the *command governor*, a specific control architecture of practical interest, which, though introduced independently of MBPC, in its recent developments has taken advantage of using conceptual tools of predictive control. For more specialized topics, the reader

is referred to the three article level contributions dealing with MBPC, *viz.*: **6.43.16.1 "MBPC for linear systems"**; **6.43.16.2 "MBPC for nonlinear systems"**; and **6.43.15.4 "Adaptive predictive control"**.

The presentation is organized as follows: Sect. 2 sets up the general ingredients of the constrained open-loop optimal control problem underlying any MBPC scheme. Sect. 3 describes the earliest and simplest form of a stabilizing MBPC algorithm. Sect. 4 introduces a convenient form of a set-membership (ellipsoidal) terminal state-constraint devised so as to improve in terms of feasibility the algorithm of Sect. 3. Sect. 5 extends the scheme of Sect. 4 by considering a state-dependent ellipsoidal constraint which allows one to get *global* feasibility/stability whenever such a property is achievable in principle. Sect. 6 describes how to deal with constant disturbances and nonzero set-points, as well as model uncertainties of polytopic type. Sect. 7 describes predictive reference governors. In Sect. 8 a brief assessment of the current status of MBPC concludes the contribution.

2. The constrained open-loop optimal control (COLOC) problem

In MBPC, the system to be controlled (plant) is usually represented by an ordinary differential equation. However, as the control is normally piecewise constant, the plant is most of the times described in terms of a difference equation

$$x(k+1) = \varphi(x(k), u(k)) \tag{1}$$

$$y(k) = \eta(x(k)) \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the state at time k, $u(k) \in \mathbb{R}^m$ the input, $y(k) \in \mathbb{R}^p$ a state-related vector connected to performance requirements (see (6) below), φ is assumed to be continuous at the origin with $(\varphi(0_X, 0_U) = 0_X)$ and $\eta(0_X) = 0_Y$. The plant input and state sequences are required to satisfy the constraints

$$u(k) \in U \tag{3}$$

$$x(k) \in X \tag{4}$$

where, usually, U is a convex and compact subset of $\mathbb{R}^m$, and X is a convex and closed subset of $\mathbb{R}^n$, both sets containing the origin in their interior. For the event (x,t) (*viz.*, for state x at time t), the cost is defined by

$$J(x,t,\mathbf{u}) = \sum_{k=t}^{t+N-1} l(x(k), u(k)) + L(x(t+N)) \tag{5}$$

where $\mathbf{u} := \{u(t), u(t+1), \ldots, u(t+N-1)\}$ and $x(k) = x^{\mathbf{u}}(k;(x,t))$, the latter notation denoting the state at time k resulting from state x at time $t \le k$ and a control sequence $\mathbf{u}$. The terminal time $t+N$ increases with time t and, consequently, is referred to as a *receding horizon*. Various choices for the instantaneous loss l and the terminal loss L are in principle possible. However, according to the usual MBPC choice, hereafter the loss functions will be taken to be quadratic

$$\left. \begin{array}{l} l(x(k), u(k)) := \|y(k)\|^2_{\psi_y} + \|u(k)\|^2_{\psi_u} \\ L(x(t+N)) := \|x(t+N)\|^2_{\psi_N} \end{array} \right\} \tag{6}$$

where $\|v\|^2_{\psi} := v'\psi v$, the prime denotes transpose, $\psi_y = \psi_y' > 0$, $\psi_u = \psi_u' > 0$ and $\psi_N = \psi_N' \ge 0$. In general, a terminal-state constraint

$$x(t+N) \in X_N \tag{7}$$

is also imposed.
At the event (x,t), the COLOC problem $\mathcal{P}(x,t)$ is to find, provided it exist, the optimal (*virtual*) control sequence

$$\mathbf{u}^\circ(x,t) := \{u^\circ(t;(x,t)), u^\circ(t+1;(x,t)), \dots, u^\circ(t+N-1;(x,t))\} \tag{8}$$

which minimizes $J(x,t,\mathbf{u})$ subject to the control, state and terminal-state constraints, and yields the value function

$$V(x,t) := J(x,t,\mathbf{u}^\circ(x,t)) \tag{9}$$

According to the receding-horizon mode of operation, only the first control $u^\circ(t;(x,t))$ is applied to the plant input at time t. In such a way, a feedback-control action is obtained

$$u(t) = c(x,t) := u^\circ(t;(x,t)) \tag{10}$$

Since $\varphi(\cdot,\cdot)$, $\eta(\cdot)$, $l(\cdot,\cdot)$ and $L(\cdot)$ are time-invariant, problems $\mathcal{P}(x,t)$ are time-invariant in that $V(x,t) = V(x,0)$ and $c(x,t) = c(x,0)$. Consequently, it suffices, at each event (x,t), to solve $\mathcal{P}_N(x) := \mathcal{P}(x,0)$. Problem $\mathcal{P}_N(x)$ is therefore as follows

$$\mathcal{P}_N(x): \quad V_N(x) = \min_{\mathbf{u}} \{J_N(x,\mathbf{u}) | \mathbf{u} \in \mathcal{U}_N(x)\} \tag{11}$$

$$J_N(x,\mathbf{u}) := \sum_{k=0}^{N-1} l(x(k),u(k)) + L(x(N)) \tag{12}$$

where $\mathbf{u} = \{u(0), u(1), \dots, u(N-1)\}$, $x(k) = x^{\mathbf{u}}(k;(x,0))$ and $\mathcal{U}_N(x)$ is the set of *feasible* control sequences, *viz.* sequences satisfying the control, state and terminal-state constraints. Because N is finite, the minimum exists provided that $\varphi(\cdot,\cdot)$ and $h(\cdot)$ are continuous, U compact, X and X_N closed, and $\mathcal{U}_N(x)$ non-empty. At the event (x,t), $\mathcal{P}_N(x)$ is solved yielding the optimizing (virtual) control sequence

$$\mathbf{u}^\circ(x) = \{u^\circ(0;x), u^\circ(1;x), \dots, u^\circ(N-1;x)\} \tag{13}$$

the optimal (virtual) state trajectory

$$\mathbf{x}^\circ(x) = \{x^\circ(0;x) = x, x^\circ(1;x), \dots, x^\circ(N;x)\} \tag{14}$$

and the value function

$$V_N(x) = J_N(x,\mathbf{u}^\circ(x)) \tag{15}$$

The first control in the optimizing sequence $\mathbf{u}^\circ(x)$ is applied to the plant input at the time t, and the MBPC action results

$$c_N(x) = u^\circ(0;x) \tag{16}$$

It is to be underlined that MBPC computes numerically on-line at the event (x,t) the optimal control action (16) rather than computing off-line the optimal control law $c_N(\cdot)$. It would be more convenient to explicitly compute off-line, once for all, $c_N(\cdot)$ via dynamic programming. As this is usually very hard of even impossible, MBPC computes at the event (x,t) the optimal control action $c_N(x)$ rather than pre-computing the optimal control law $c_N(\cdot)$.

3. Zero terminal-state MBPC

This can be regarded as the earliest and, conceptually, the simplest form of MBPC which guarantees stability to the controlled system, whenever feasibility is satisfied. Here, $X_N = \{0_X\}$ and, hence, the terminal-state constraint is the equality constraint

$$x(N) = 0_X \tag{17}$$

Given the plant state x, here $\mathcal{U}_N(x)$, the set of feasible control sequences, is the set of sequences which drive the plant initial state x to the zero-state in N steps with no constraint violation. Assume that $\mathcal{U}_N(x)$ is non-empty, and satisfaction of all other conditions for the existence of the optimizing control sequence $\mathbf{u}^\circ(x)$ in (13). Then, asymptotic stability of the plant fed by the MBPC action can be easily proved by the following direct argument. Let $\bar{\mathbf{u}}(x)$ be the control sequence obtained from $\mathbf{u}^\circ(x)$ by deleting its first control and inserting the zero-control 0_U in its final position

$$\bar{\mathbf{u}}(x) := \{u^\circ(1;x), u^\circ(2;x), \ldots, u^\circ(N-1;x), 0_U\} \tag{18}$$

This is a control sequence again of length N which is feasible for $x^\circ(1;x) = \varphi(x, u^\circ(0;x))$. In fact, $x^\circ(1;x)$ is driven by $\bar{\mathbf{u}}(x)$ to 0_X in $N-1$ steps, and held at 0_X at the N-th step because

$$\varphi(0_X, 0_U) = 0_X$$

Specifically, the state trajectory over N steps resulting from the initial state $x^\circ(1;x)$ and the control sequence $\bar{\mathbf{u}}(x)$ is

$$\bar{\mathbf{x}}(x) := \{x^\circ(1;x), \ldots, x^\circ(N-1;x), 0_X, 0_X\} \tag{19}$$

Moreover,

$$\begin{aligned} J_N(x^\circ(1;x), \bar{\mathbf{u}}(x)) &= J_N(x, \mathbf{u}^\circ(x)) - l(x, u^\circ(0;x)) \\ &= V_N(x) - \|y(0)\|^2_{\psi_y} - \|u(0)\|^2_{\psi_x} \end{aligned} \tag{20}$$

where $y(0) = \eta(x)$ is the initial plant output and $u(0) = c_N(x) = u^\circ(0;x)$ is the effective input supplied at time 0 by MBPC to the plant. Then, if $x(1) = \varphi(x, u(0))$ denotes the *effective* plant state at time 1 in the MBPC-controlled system, according to (11) and (20) we get

$$V_N(x(1)) \le V_N(x(0)) - \|y(0)\|^2_{\psi_y} - \|u(0)\|^2_{\psi_u}$$

Going to the generic time k, $k = 0, 1, \ldots$, for the closed-loop system we have

$$V_N(x(k)) - V_N(x(k+1)) \ge \|y(k)\|^2_{\psi_y} + \|u(k)\|^2_{\psi_u} \tag{21}$$

where $u(k)$, $x(k)$ and $y(k)$ denote, respectively, the effective plant input, state and output at time k in the MBPC-controlled system. Eq. (21) shows that $\{V_N(x(k))\}_{k=0}^{\infty}$ is a monotonically non-increasing sequence. Hence, being $V_N(x(k))$ nonnegative, as $k \to \infty$ it converges to $V_N(x(\infty))$, $0 \le V_N(x(\infty)) \le V_N(x(0))$. Consequently, summing both sides of (21) from $k = 0$ to $k = \infty$, we get

$$\infty > V(x(0)) - V(x(\infty)) \ge \sum_{k=0}^{\infty} \left[\|y(k)\|^2_{\psi_y} + \|u(k)\|^2_{\psi_u} \right] \tag{22}$$

This, in turn, implies as $\psi_y > 0$ and $\psi_u > 0$

$$\lim_{k\to\infty} y(k) = 0_Y \quad \text{and} \quad \lim_{k\to\infty} u(k) = 0_U \tag{23}$$

Then, under a detectability condition on (1) and (2), for $\forall x \in \mathcal{U}_N(x) \neq \emptyset$ one can conclude asymptotic stability of the closed-loop system: stability that can be seen to be of exponential type if the plant is linear, *viz.*

$$\left.\begin{aligned} x(k+1) &= \Phi x(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}\right\} \tag{24}$$

The foregoing stability proof hinges upon the following two crucial points:

$$\mathcal{U}_N(x) \neq \emptyset \tag{25}$$

$$\mathbf{u}^\circ(x) \in \mathcal{U}_N(x) \Rightarrow \bar{\mathbf{u}}(x) \in \mathcal{U}_N(x^\circ(1;x)) \tag{26}$$

Now, feasibility condition (25) can be lost in zero terminal-state MBPC, because of the need of driving the plant state to 0_X in a finite time. If the initial state is far from 0_X, this brings about the use of large plants inputs which can violate possible input saturation constraints. Various approaches can be adopted so as to enlarge the set of admissible states (states such that (25) is fulfilled). *E.g.*, given a plant state $x \neq 0_X$ and an integer N_1 such that $\mathcal{U}_{N_1}(x) = \emptyset$, one can always find, if the plant is controllable and only input saturation constraints are present, a possibly large but finite integer N_2, $N_2 > N_1$, so as to make $\mathcal{U}_{N_2}(x)$ non-empty. The disadvantage with this approach is that N_2 can be too large, and, hence, the associated COLOC problem $\mathcal{P}_{N_2}(x)$ too complex for an on-line solution for the available computing power and sampling time.
Hereafter, forms of MBPC will be presented which are particularly tailored for both solving the feasibility problem and yielding a highly performing closed-loop system. In so doing, for the sake of simplicity, we shall address linear plants of the form (22) in the presence of only input-saturation constraints. The reason for the latter choice is that, while hard bounds on the manipulated variables are typically dictated by physical constraints, e.g., limited power of actuators, state-related constraints are frequently of the ”soft-type”, and, hence, can be properly addressed by penalizing additional state-related variables in the cost.

4. Set-membership terminal constraint

Here, instead of using a terminal constraint consisting in an *equality* as (17), a *set-membership* terminal constraint $x(N) \in X_N$ is considered in the COLOC problem $\mathcal{P}_N(x)$. Specifically, we shall assume that the plant is linear in the form (24) with

$$(H, \Phi, G) \quad \text{stabilizable and detectable} \tag{27}$$

It is also assumed that the plant state is unconstrained, *i.e.*, $X = \mathbb{R}$ in (4), and the plant input is subject to the saturation constraint (3) with

$$U = \left\{ u \in \mathbb{R}^m : -u^- \leq u \leq u^+ \right\} \tag{28}$$

where $u^- := \left[u_1^-, u_2^-, \ldots, u_m^-\right]' \in \mathbb{R}^m$, and $u^+ := \left[u_1^+, u_2^+, \ldots, u_m^+\right]' \in \mathbb{R}^m$, with $u_i^-, u_i^+ > 0$, $i \in \underline{m} := \{1, 2, \ldots, m\}$, and the vector inequalities in (28) have to be interpreted in a component-wise sense. Moreover, the cost is as follows

$$\begin{aligned} J_N(x, \mathbf{u}) &:= \sum_{k=0}^{N-1} l_{\underline{\rho}}(y(k), u(k)) + L_\rho(x(N)) & (29) \\ l_{\underline{\rho}}(y, u) &:= \|y\|_{\psi_y}^2 + \underline{\rho}\, \|u\|_{\psi_u}^2 & (30) \\ L_\rho(x) &:= \|x\|_{P_\rho}^2 & (31) \end{aligned}$$

and the terminal-state constraint is the set-membership (ellipsoidal) constraint (7) where

$$\begin{aligned} X_N &= \mathcal{E}(\rho) \\ &:= \left\{ x \in \mathbb{R}^n : \|x\|^2_{\overline{P}_\rho} \leq \underline{\lambda}(\psi_u) v^2 \right\} \end{aligned} \tag{32}$$

In (29)-(32), $\underline{\rho}$ and ρ are two positive real numbers such that

$$0 < \underline{\rho} \leq \rho \tag{33}$$

$\underline{\lambda}(\psi_n)$ denotes the minimum eigenvalue of ψ_u, $v := \min_{i \in \underline{m}} \{u_i^-, u_i^+\}$

$$\overline{P}_\rho := P_\rho / \rho \tag{34}$$

and P_ρ is the unique semipositive-definite solution of the following algebraic Riccati equation (ARE)

$$P_\rho = \Phi'_\rho P_\rho \Phi_\rho + \rho F'_\rho \psi_u F_\rho + H' \psi_y H \tag{35}$$

$$\Phi_\rho := \Phi + G F_\rho \tag{36}$$

$$F_\rho := -\left(\rho \psi_u + G' P_\rho G\right)^{-1} G' P_\rho \Phi \tag{37}$$

The COLOC problem (27)-(32) is a special case of a quadratically constrained quadratic programming problem which can be converted in a linear matrix inequality (LMI) problem and handled efficiently via semi-definite programming solvers. A key feature of COLOC (27)-(37) is that $x \in \mathcal{E}(\rho)$ implies $F_\rho x \in U$, and that (32) is an *invariant set* for the state of plant (24) under state-feedback controls $u(k) = F_\rho x(k)$, *viz.*

$$x \in \mathcal{E}(\rho) \;\Rightarrow\; \Phi_\rho x \in \mathcal{E}(\rho) \tag{38}$$

In other terms, constraint $x(N) \in \mathcal{E}(\rho)$, which is enforced in COLOC (27)-(37), ensures that a sequence **u** made up by N controls drives in N steps the plant state into $\mathcal{E}(\rho)$ where thereafter it can be held by state-feedback controls with feedback-gain F_ρ. Feasibility $(\mathcal{U}_N(x) \neq \emptyset)$ is ensured for all states x such that

$$x \in \mathcal{E}(\rho) \tag{39}$$

It can be seen that inclusion (39) is satisfied whenever

$$\|x\|^2 \leq \underline{\lambda}(\psi_u) v^2 / \overline{\lambda}(\overline{P}_\rho) \tag{40}$$

where $\overline{\lambda}(\overline{P}_\rho)$ denotes the maximum eigenvalue of $\overline{P}_\rho$. Further, for any initial state satisfying (39), the MBPC action satisfies the input saturation constraints at all times and makes the closed-loop system exponentially stable. The reader is referred to the Appendix for a detailed proof of these statements. We underline that condition (33) is a requirement for stability of the closed-loop system. The other interesting point is that the smaller $\overline{P}_\rho$ the larger the set of (admissible) states satisfying (39). This is an important consideration in view of MBPC schemes where ρ-varying ellipsoidal constraints are adopted so as to maximize feasibility. One of such schemes is the subject of next section.

5. Time-varying ellipsoidal terminal constraint

We have seen that (39) is a sufficient condition for feasibility of the MBPC scheme with COLOC (27)-(32). In this connection, an important property is the fact that the solution P_ρ of the ARE (35)-(37) and its normalized version $\overline{P}_\rho = P_\rho / \rho$ have the following monotonicity properties

$$\rho_2 > \rho_1 \Rightarrow \begin{cases} P_{\rho_2} \geq P_{\rho_1} \\ \overline{P}_{\rho_2} \leq \overline{P}_{\rho_1} \end{cases} \tag{41}$$

In particular, taking into account (32), from (41) it follows

$$\rho_2 > \rho_1 \Rightarrow \mathcal{E}(\rho_2) \supset \mathcal{E}(\rho_1) \tag{42}$$

and that $\lim_{\rho \to \infty} \overline{P}_\rho = \overline{P}_\infty \geq 0$. $\overline{P}_\infty$ can be shown to coincide with the unique solution $\overline{P}_\rho$ of ARE (35)-(37) with $\rho = \infty$ such that $\Phi_\infty := \Phi + GF_\infty$ has eigenvalues on the closed unit disk. Consequently, if $\mathcal{E}(\infty) := \lim_{\rho\to\infty} \mathcal{E}(\rho)$, we see that $x \in \text{Int}\, \mathcal{E}(\infty)$ implies that there exist ρ, $\underline{\rho} \leq \rho < \infty$, such that COLOC (27)-(37) is feasible.

An important class of systems consists of the linear time-invariant dynamical systems whose states can be globally asymptotically steered to 0_X by arbitrarily small controls. In the literature, these systems are referred to as ANCBI (Asymptotically Null Controllable by Bounded Inputs) systems. A necessary and sufficient condition for a discrete-time time-invariant linear system to be ANCBI is that it be stabilizable and have eigenvalues of modulus less than or equal to one, irrespective of their multiplicity. ANCBI systems are very relevant to process control applications in that they encompass neutrally stable systems with chains of integrators of arbitrary complexity. Input-saturated ANCBI systems are all and the only systems for which it make sense to look for MBPC schemes with *global* feasibility and stability properties. Indeed, such schemes can be constructively devised on the grounds of (42) and the fact that for ANCBI systems

$$\overline{P}_\infty = 0_{nxn} \tag{43}$$

Hence, for ANCBI systems $\mathcal{E}(\infty) = \mathbb{R}^n$. Therefore, for any ANCBI system, given an arbitrary $x \in \mathbb{R}^n$, we can find ρ, $\underline{\rho} \leq \rho < \infty$, under which COLOC (27)-(37) is feasible and the resulting closed-loop system exponentially stable. Now, for reasons clarified in the next section which deals with step-wise disturbances, it is in practice important to handle unpredictable state jumps. Such jumps are responsible for the fact that at a time t the plant state $x(t)$ need not coincide with the one predicted in accordance with (24). Under such circumstances, in order to ensure feasibility, the ρ parameter in (27)-(37) has to be made state-dependent. In order to describe a possible answer to the problem, assume that at the event $(x(t-1), t-1)$ the weight ρ was set equal to $\rho(t-1)$, and COLOC (27)-(37) with $\rho = \rho(t-1)$ generated the N (virtual) optimal controls

$$\mathbf{u}^\circ(x(t-1), t-1) := \{u^\circ(t-1|t-1)), u^\circ(t|t-1)), \ldots, u^\circ(t+N-2|t-1))\} \tag{44}$$

where simpler notation than in (8) is adopted. Let now $(x(t), t)$ be the next event. In order to select $\rho(t)$, define

$$\bar{x}(t+N-1|t) := \Phi^{N-1}x(t) + \sum_{k=0}^{N-2} \Phi^{N-k-2} G u^\circ(t+k|t-1) \tag{45}$$

Then, set

$$\rho(t) := \min\left\{\rho : \rho \geq \underline{\rho}, \|\bar{x}(t+N|t)\|^2 \leq \underline{\lambda}(\psi_u)v^2/\overline{\lambda}(\overline{P}_\rho)\right\} \tag{46}$$

Here, $\bar{x}(t+N|t)$ can be seen to coincide with the state response at time $t+N-1$ from $(x(t), t)$ to the input sequence $\{u^\circ(t|t-1)), \ldots, u^\circ(t+N-2|t-1))\}$. Consequently, $\rho(t)$ as in (46) ensures that the mentioned control sequence drives $x(t)$ to $\mathcal{E}(\rho(t))$ in $N-1$ steps. It follows that COLOC (27)-(37) is solvable at the current event $(x(t), t)$ provided that ρ is changed into the $x(t)$-dependent weight $\rho(t)$ as in (46). The procedure can be initialized from the event $(x(0), 0)$ by taking

$$\rho(0) := \min\left\{\rho : \rho \geq \underline{\rho}, \|x(0)\|^2 \leq \underline{\lambda}(\psi_u)v^2/\overline{\lambda}(\overline{P}_\rho)\right\} \tag{47}$$

Under the choice ([46](#)) and in the ideal case (absence of unpredictable state jumps), we obtain global feasibility and global asymptotic (exponential, for all initial states in a compact set) stability. Further, $\rho(t)$ turns out to be monotonically nonincreasing and reaches its lower bound $\underline{\rho}$ in a finite time. It is important to point out that these properties do not depend crucially on (46): *e.g.* it is enough that $\rho(t)$ be chosen via the use of a look-up table listing only a discrete set of $\overline{\lambda}(\overline{P}_\rho)$ with ρ-values in an interval having $\underline{\rho}$ as its minimum value, and that the value of ρ be chosen at each time t as the minimum value in the mentioned set under which (40) is satisfied. A useful remark here is that the positive real $\underline{\rho}$, the control horizon N, and the weights ψ_y and ψ_u are design knobs. Their choice has to be typically made in accordance with the desired performance of the MBPC-controlled system in its linear dynamic range, *viz.* performance with inactive constraints. Such a performance coincides with the one of the LQ regulated plant, because, as shown in the Appendix, the MBPC action equals the one given by the LQ optimal feedback-control law as soon as $\rho(t)$ reaches $\underline{\rho}$.

Example 1 - Consider the following single-input single-output (continuous-time) triple integrator plant, $(d^3/dt^3)y = u$, with u constrained between ± 1 and a sampling interval of 0.1 s. No linear controller can globally stabilize this system. Let $x = [y \;\; \dot{y} \;\; \ddot{y}]' = [3 \;\; -1 \;\; 3]'$ be the initial state. We apply to the discretized plant the predictive controller based on COLOC (27)-(37) equipped with the state-dependent ρ-selection rule (46), for three different values of the control horizon $N = 0, 4, 20$. In Fig. 1, the relevant responses from the initial non-zero state are reported. In particular, for $N = 0$, the behavior of the controlled plant looks very conservative in that only after a very long transient the output approaches zero. However, one can appreciate that, as N increases, the regulation performance of the MBPC-controlled system improves.
Some advantages can be advocated for the MBPC with state-dependent ellipsoidal terminal constraint relative to alternative MBPC schemes, *e.g.*, the ones which select on-line the control horizon N so as to possibly avoid unfeasibility. In fact, feasibility and regulation results are achieved by the scheme of this section irrespective of the control horizon N. This can be freely chosen off-line by trading off performance, which improves as N increases, with computational burden, which also increases with N. Further, a prequantifiable numerical burden results for the underlying on-line COLOC problem, irrespective of the current state.

6. Models, disturbances and robustness

The discussion so far has focussed on the ideal case where the plant model is exact and no disturbance is present. Within this context, the state model (24) encompasses also input-output or external representations of plants in the form of difference equations such as

$$y(t) + A_1 y(t-1) + \cdots + A_{n_A} y(t-n_A) = B_1 u(t-1) + \cdots + B_{n_B} u(t-n_B) \tag{48}$$

This can be seen by transforming (48) in the form (24), *e.g.*, via the introduction of the (nonminimal) state

$$x(t) := \left[y'(t)\; y'(t-1) \ldots y'(t-n_A+1)\; u'(t-1) \ldots u'(t-n_B+1) \right]' \tag{49}$$

Another issue which the reader should be aware of is that if step-wise disturbances are present, some manipulations are required in order to arrive at a noiseless state-space model like (22). Consider in fact a plant with *physical* model

$$\begin{aligned} \underline{x}(t+1) &= \underline{\Phi x}(t) + \underline{Gu}(t) + \xi \\ y(t) &= \underline{Hx}(t) + \zeta - r \end{aligned} \tag{50}$$

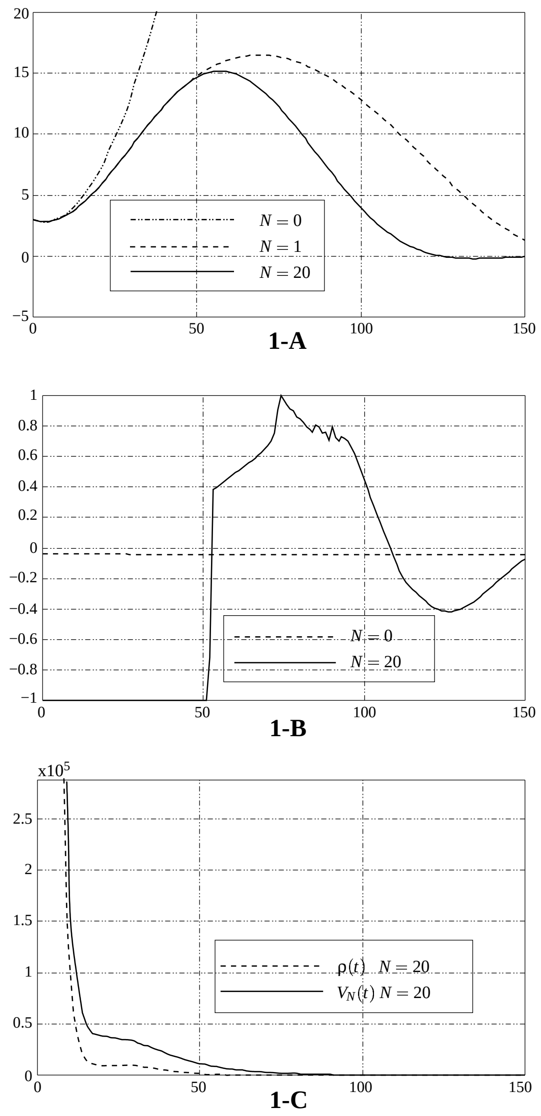

Figure 1: Simulation results for Example 1 and different control horizons N. 1-A: output $y(t)$. 1-B: input $u(t)$ with $|u| \leq 1$. 1-C: value $V_N(t) = V_N(x(t))$ and $\rho(t)$.

where $\underline{x}(t)$ is the state, $\underline{u}(t)$ the manipulable input, $y(t)$ the tracking error or output to be regulated to 0_Y , r the set-point or step-wise constant reference, and ξ and ζ step-wise disturbances on the state and, respectively, the output. Over time-intervals in which disturbances and set-point are constant, one can see that (50) can be put in the (ideal) form (24), provided that the following relationships hold

$$\begin{aligned} x(t) &:= \begin{bmatrix} \delta\underline{x}'(t) & y'(t) \end{bmatrix} && (51) \\ \delta\underline{x}(t) &:= \underline{x}(t) - \underline{x}(t-1) && (52) \\ u(t) &:= \delta\underline{u}(t) \\ &:= \underline{u}(t) - \underline{u}(t-1) && (53) \end{aligned}$$

Under (51)-(53), (24) becomes the so-called *incremental model* associated with the physical model(50). It can be found that detectability of $(\underline{H}, \underline{\Phi})$ implies detectability of (24), provided that (51)-(53) hold. Further, if the plant is square, *viz.*, dim y = dim u, and the transfer matrix of (50) from $\underline{u}$ to y evaluated at $z = 1$ is nonsingular, the property that (50) is stabilizable (ANCBI) via the input $\underline{u}$ implies that (24), with (51)-(53), is stabilizable (ANCBI) via the input u. The state evolutions of the physical plant can be exactly predicted via the incremental model except at those times where variations (steps) of disturbances and/or set-point take place. Such variations cause, in fact, instantaneous jumps of the x-state. This discussion indicates that the MBPC technique of Sect. 5, which embodies a time-varying ellipsoidal terminal constraint, can be applied *"mutatis mutandis"* to cover the practical important case of plants affected by piece-wise constant disturbances and set-points. In so doing, the MBPC action turns out to be of Proportional-Integral type and designed so as to (globally, in case of ANCBI plants) fulfill saturation constraints on the input-increments $\delta \underline{u}(t)$. Extensions of the MBPC scheme with a time-varying ellipsoidal terminal constraint are available which can jointly handle input and input-increment saturation constraints, as required in some applications.

So far our discussion has been restricted to the ideal case wherein the model used for prediction is exact. It is important to extend MBPC to cases where the plant model is uncertain. Though several ways for describing model uncertainty are possible, our considerations will be focussed hereafter on a polytopic description of model uncertainty. Such a description fits well for the practical case of a possibly time-varying plant with uncertain dynamics. Consider, in place of model (24), the following p-parameterized family of discrete-time linear dynamic systems:

$$\begin{aligned} x(k+1) &= \Phi(p(k))x(k) + G(p(k))u(k) \\ y(k) &= Hx(k) \end{aligned} \tag{54}$$

where $p \in \mathbb{R}^l$ is an unknown and possibly time-varying parameter. The system matrices $\Phi(p)$ and $G(p)$ belong to the polytopic matrix family

$$\Sigma(\mathcal{P}) := \left\{ (\Phi(p), G(p)) = \sum_{i=1}^{l} p_i(\Phi_i, G_i), \forall p \in \mathcal{P} \right\} \tag{55}$$

where (Φ_i, G_i) denotes the i-th vertex of the polytope $\Sigma(\mathcal{P})$, *viz.* $(\Phi_i, G_i) \in \text{Vert}\{\Sigma(\mathcal{P})\}, \forall i \in l := \{1, 2, \ldots l\}$. The parameter vector $p = [p_1, \ldots, p_l]'$ is in the unit simplex

$$\mathcal{P} := \left\{ p \in \mathbb{R}^l : \sum_{i=1}^{l} p_i = 1, p_i \geq 0 \right\} \tag{56}$$

In (54), $p(k)$ can take on any value in $\mathcal{P}$ and the time-evolution of $p(\cdot)$ is allowed to be any conceivable trajectory in $\mathcal{P}$. This means that the model-based k-step ahead state predictions from the initial event $(x, 0)$ given a control sequence $\mathbf{u}$ are subsets $\mathcal{Z}^{\mathbf{u}}(k; x)$ of $\mathbb{R}^n$, *e.g.*

$$\mathcal{Z}^{\mathbf{u}}(2; x) = \{z \in \mathbb{R}^n; z = \Phi(p(1)) [\Phi(p(0))x + G(p(0))u(0)] + G(p(1))u(1), p(0), p(1) \in \mathcal{P}\}$$

Such sets are closed and convex polyhedra with vertices in Vert $\mathcal{Z}^{\mathbf{u}}(k; x)$ and the maximum of any convex function defined on $\mathcal{Z}^{\mathbf{u}}(k; x)$ is attained at some vector in Vert $\mathcal{Z}^{\mathbf{u}}(k; x)$. An extension of the MBPC technique of Sect. 5 to the present context can be based upon the following points. The optimization criterion to be considered is now of *min-max* type

$$\begin{aligned} J_N(x, \mathbf{u}) = \min_{\mathbf{u}} \Bigg[\sum_{k=0}^{N-1} & \max \left\{ l(x(k), u(k)); x(k) \in \text{Vert } \mathcal{Z}^{\mathbf{u}}(k; x) \right\} + \\ & \max \left\{ \|x(N)\|^2_{Q(x)}; x(N) \in \text{Vert } \mathcal{Z}^{\mathbf{u}}(N; x) \right\} \Bigg] \end{aligned} \tag{57}$$

under the control saturation constraints (3) and (28), and the extended terminal state constraint

$$\text{Vert } \mathcal{Z}^{\mathbf{u}}(N; x) \in \overline{\mathcal{E}}(Q, \sigma) \subset \hat{X} \tag{58}$$

where:

$$l(x, u) := \|x\|^2_{H'\psi_y H} + \|u\|^2_{\psi_u}, \quad \psi_y = \psi_y' > 0, \quad \psi_u = \psi_u' > 0;$$

the pair (F,Q), $Q = Q' \geq 0$ satisfies the inequalities

$$Q \geq \Phi'_{F,i} Q \Phi_{F,i} + F' \psi_u F + \psi_x, \quad \forall i \in \underline{l} \tag{59}$$

$$\Phi'_{F,i} := \Phi_i + G_i F \tag{60}$$

$$\overline{\mathcal{E}}(Q,\sigma) := \left\{ x \in \mathbb{R}^n : \|x\|_Q^2 \leq \sigma \right\} \tag{61}$$

and $\hat{X}$ is the subset of all states $z \in \mathbb{R}^n$ such that the state-feedback Fz fulfills the saturation constraints

$$\hat{X} := \{ z \in \mathbb{R}^n ; \; Fz \in \mathbb{U} \} \tag{62}$$

It is a positive fact that, if it exists, a triplet (F,Q,σ) satisfying (59) and $x \in \overline{\mathcal{E}}(Q,\sigma) \subset \hat{X}$ can be efficiently found by solving an LMI (Linear Matrix Inequality) problem. The existence of such a triplet ensures feasibility of COLOC (54)-(62) and of the minimax control sequence $\mathbf{u}^\circ(x) = \{(u^\circ(0;x), u^\circ(1;x), \ldots, u^\circ(N-1;x))\}$. If, when the current plant state is x, the plant is fed by the control $u^\circ(0;x)$, feasibility is ensured at all future times and asymptotic stability results. It is to be underlined that the positive real σ here plays a similar role as ρ in Sect. 5. Further, similarly to ρ, σ is chosen on-line so as to handle possible state jumps in such a way that feasibility be possibly held.
Example 2 that follows exhibits simulation results obtained by using an MBPC algorithm based on COLOC (54)-(62) in order to control a plant with polytopic model uncertainties as in (55).

Example 2 - The plant consists of a two-mass spring system as shown in Fig. 2. Using Euler's

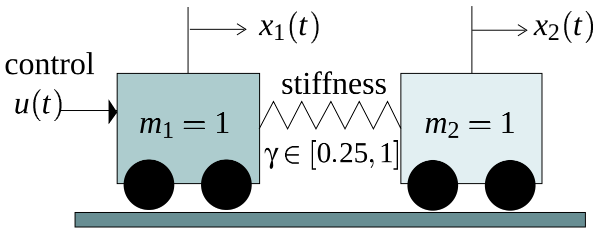

Figure 2: Two-mass spring system.

first-order approximation for the derivative and a sampling time of 0.1 s., the following discrete-time state-space representation is obtained from the continuous-time equations of the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1\gamma/m_1 & 0.1\gamma/m_1 & 1 & 0 \\ 0.1\gamma/m_2 & -0.1\gamma/m_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.1/m_1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = x_2(t)$$

Here, x_1 and x_2 are the positions of body 1 and 2 of masses $m_1 = m_2 = 1$, and x_3 and x_4 are, respectively, their velocities. The parameter γ denotes the spring stiffness constant. All entities are expressed in MKS units. The control force u acts on m_1. The spring constant is uncertain in the interval $0.25 \leq \gamma \leq 1$. The problem is to find a state-feedback control u capable of moving body 2 with an acceptable step response, under the input saturation constraint $|u(t)| \leq 0.1$ and the stated (possibly time-varying) uncertainty on γ. All simulations are referred to an actual plant with a constant value of γ, $\gamma = 1$, and $\psi_u = \psi_y = 1$. In Table 1, the numerical complexity of an implementation of COLOC (54)-(62) is reported for three different values of N, $N = \{0,1,2\}$. Fig 3-5 show, respectively, the output, input and cost V_N as a function of time. In this example, a significative reduction of conservativeness is achieved by just having one free control move $(N = 1)$.

	$N=0$	$N=1$	$N=1$
No. of flops	5455	11181	19445

Table 1: Computational complexity (average over 400 steps) as a function of the control horizon N

In particular observe in Fig. 4 that for $N=1,2$ the initial controls touch the constraint boundary (0.1), whereas for $N=0$ they stay far from the input saturation levels.

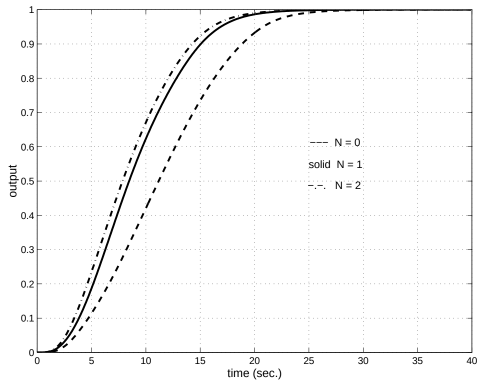

Figure 3: Output for $N=\{0,1,2\}$.

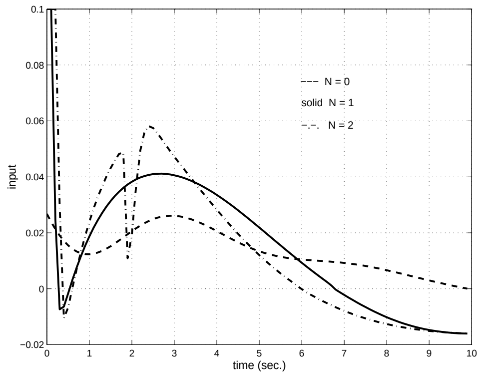

Figure 4: Control input $|u| \leq 0.1$ for $N=\{0,1,2\}$ (initial 10 s.).

The presence of persistent unpredictable state-disturbances $n(t)$ in (24), *viz.*

$$\begin{aligned} x(k+1) &= \Phi x(k) + Gu(k) + w(k) \\ y(k) &= Hx(k) \end{aligned}$$

$w(t) \in W$, W compact, strongly complicates MBPC, even in the case of full information on the current state. The related existing approaches are of the min-max type and typically entail large computational loads. A related issue, which has received some attention, is how to embody suitable state-observers in MBPC schemes in the case of partial state information on the current state. Receding horizon state-observers have also been considered, particularly when hard constraints on the state have to be imposed.

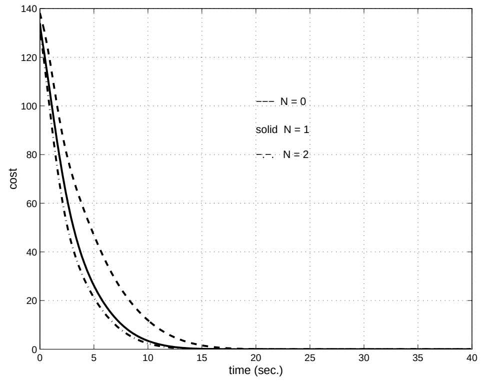

Figure 5: V_N for $N = \{0, 1, 2\}$.

7. Predictive Command Governors

A *Command Governor* (CG), or reference governor, is a nonlinear device which is added to a primal compensated control system. The latter is designed, independently of the CG, so as to perform satisfactorily in the absence of constraints. Whenever necessary, the CG modifies the set-point entering the primal control system so as to possibly avoid violation of prescribed constraints. In such a way the CG action is finalized to let the control system operate linearly within a wider dynamic range than that which would result with non CG. The resulting control system takes a special simplified structure with respect to standard MBPC schemes which on the other hand are computationally very demanding. In practice, CG's are attractive in industrial applications wherein the primal compensated system includes typically PID-like controllers, and the constraints must only be taken care of by adding peripheral units which, as with CG's, do not modify the primal compensated system, and do not require a massive amount of flops per sampling time. CG's can be designed on the grounds of conceptual tools of MBPC: whenever this happens, they are referred to as *predictive* CG's (PCG). Similarly to MBPC, the action of a PCG is based on the system current state, desired set-point and prescribed constraints. It selects at any time a modified value for the set-point from a family of simple *virtual* sequences, *e.g.* constant sequences of infinite duration. The selection is carried out by typically solving a simple quadratic programming problem, and, according to the receding horizon control philosophy, the instantaneously modified value of the set-point sent to the primal system equals the value of the first sample of the selected sequence. Thanks to its simple special structure, it is relatively easy to establish analytically convergence properties of pre-compensated systems whose set-point is managed by a PCG, and how to take care of the presence of persistent unpredictable disturbances, partial state information and uncertainties on the pre-compensated system model. Along with their moderate computational burden, these are the main advantages of PCG over general MBPC schemes. This is typically paid for by a reduced feasibility and performance, as the PCG approach can be rightly looked at as a sub-optimal version of a general MBPC scheme.

8. Conclusive Remarks

MBPC is a powerful methodology for solving challenging control problems, particularly the ones where prescribed point-wise-in-time input and/or state constraints have to be satisfied. When MBPC was first advocated for process control, existing techniques for control design, such as LQ control, were not extensively used in industry mainly for their inadequacy to cope with problems raised by constraints, nonlinearities and uncertainty. Nowadays it is recognized that, as shown in this contribution, properties of LQ regulation can be conveniently exploited in MBPC schemes, at least in the ideal case, *viz.* when there are no unpredictable disturbances and no model uncertainties. When one moves out the ideal case, MBPC becomes computationally extremely demanding, particularly in the case of unpredictable disturbances. MBPC schemes with guaranteed feasibility/stability properties tend therefore to provide *conceptual* controllers which work well in principle but are too complex to be used in practice. In fact, because of its heavy computational load, use of MBPC is yet limited, largely in simplified ad-hoc forms, to slow processes, such as the petro-chemical and process industries, where economic considerations require operating points close to the boundary of the set of operating points fulfilling the prescribed constraints. This has been, and it continues to be, in any case an intensive area for MBPC applications that has made MBPC a successful multi-million dollar industry. An exception to the high complexity of MBPC is the predictive command governor of Sect. 7: an example of a suboptimal MBPC. It is to be expected that in order to make MBPC viable for different application areas, other forms of suboptimal MBPC have to be developed, *e.g.* schemes where feasibility, rather than optimality, plays a major role.

APPENDIX

In order to show that (39) implies feasibility of COLOC (27)-(37), consider the standard (unconstrained) Linear-Quadratic (LQ) regulation problem for the linear plant (24) with initial event $(x,0)$ and quadratic cost

$$J_\infty(x,\mathbf{u}) = \sum_{k=0}^{\infty} l_\rho(y(k),u(k)) \tag{63}$$

As well known, under (27) the optimal control sequence minimizing J_∞ can be expressed in terms of the state-feedback

$$u^{LQ}(k;x) = F_\rho x^{LQ}(k;x)$$

with $x^{LQ}(0;x) = x$, $x^{LQ}(k+1;x) = \Phi_\rho x^{LQ}(k;x)$, $\Phi_\rho := \Phi + GF_\rho$, and F_ρ the LQ feedback-gain (37). Under these circumstances, it is known that

$$\begin{aligned} V_\infty(x) &= \min_{\mathbf{u}} J_\infty(x,\mathbf{u}) \\ &= \sum_{k=0}^{\infty} l_\rho(y^{LQ}(k;x), F_\rho x^{LQ}(k;x)) = \|x\|^2_{P_\rho} \end{aligned} \tag{64}$$

with P_ρ solution of (35)-(37). We prove next that under (39) each control in the LQ-optimal control sequence $\mathbf{u}^{LQ} := \left\{F_\rho x^{LQ}(k;x)\right\}_{k=0}^{\infty}$ satisfies the saturation constraints. In fact, for every k,

$$\rho\underline{\lambda}(\psi_u)\left\|u^{LQ}(k;x)\right\|^2 \le \rho\left\|u^{LQ}(k;x)\right\|^2_{\psi_u} \le \|x\|^2_{P_\rho} \tag{65}$$

Hence, from (39) and (65) it follows that $\left\|u^{LQ}(k;x)\right\| \le v$. This shows that, under (39), the whole LQ-optimal control sequence satisfies the constraints. Hence, the LQ-optimal control sequence belongs to $\mathcal{U}_N(x)$, and, consequently, COLOC (27)-(32) is feasible. Said in different words, feasibility follows from the fact that the ellipsoidal set $\mathcal{E}(\rho)$ is a *positive-invariant* set for the LQ-controlled plant, *viz.*

$$x^{LQ}(k;x) \in \mathcal{E}(\rho) \Rightarrow x^{LQ}(k+1;x) \in \mathcal{E}(\rho) \tag{66}$$

In fact, (35) implies that $\Phi_\rho' P_\rho \Phi_\rho - P_\rho \le 0$.
In order to show exponential stability under feasibility, we resort to a method of proof similar to

the one used in the stability proof of the zero terminal-state MBPC. To this end, consider that, as a consequence of (64), the value of the cost (29) for a control sequence of length N $\mathbf{u}_N := \{u(0), u(1), \dots, u(N-1)\}$ coincides with the value taken on by the following modified cost

$$J_{N;\infty}(x, \mathbf{u}_N \otimes \mathbf{u}^{LQ}) = \sum_{k=0}^{N-1} l_{\underline{\rho}}(y(k), u(k)) + \sum_{k=N}^{\infty} l_{\rho}(y(k), u(k)) \tag{67}$$

for the control sequence of infinite length

$$\mathbf{u}_N \otimes \mathbf{u}^{LQ} = \begin{cases} u(k) & k = 0, 1, \dots, N-1 \\ F_\rho \Phi_\rho^{k-N} x^u(N, x) & k = N, N+1, \dots \end{cases} \tag{68}$$

where $x^u(N, x)$ denotes the state response at time N of plant (24) from event $(x, 0)$ to the input sequence u_N . Assume now feasibility, *viz.*, $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbf{u}^\circ(x) := \{u^\circ(0;x), u^\circ(1;x), \dots u^\circ(N-1;x)\}$ be the optimal control sequence for COLOC (28)-(32), and $x^\circ(N, x)$ the related state response at time N. Let

$$\bar{\mathbf{u}}(x) := \{u^\circ(1, x), u^\circ(2, x), \dots, u^\circ(N-1, x), F_\rho x^\circ(N, x)\}$$

We see that $\mathbf{u}^\circ(x) \in \mathcal{U}_N(x)$ implies $\bar{\mathbf{u}}(x) \in \mathcal{U}_N(x^\circ(1;x))$ because of (65). Moreover,

$$J_N(x^\circ(1;x), \bar{\mathbf{u}}(x)) = J_{N;\infty}(x^\circ(1;x), \bar{\mathbf{u}}(x) \otimes \mathbf{u}^{LQ}(x^\circ(N+1, x))) =$$
$$J_{N;\infty}(x, \mathbf{u}^\circ \otimes \mathbf{u}^{LQ}(x^\circ(N, x)) - \|y(0)\|^2_{\Psi_y} - \underline{\rho}\|u(0)\|^2_{\Psi_y} - (\rho - \underline{\rho}) \left\|F_\rho x^\circ(N, x)\right\|^2_{\Psi_u}$$

Taking into account (33), we get

$$V_N(x(1)) \leq V_N(x(0)) - \|y(0)\|^2_{\Psi_y} - \|u(0)\|^2_{\Psi_u}$$

where $y(0) = Hx$ and $u(0) = u^\circ(0;x)$. The remaining part of the proof follows the same lines as the one from (21) throughout (23) for the zero terminal-state MBPC.

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