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Efficient robust predictive control

B. Kouvaritakis
Dept. Engineering Science, Parks Road, Oxford. OX1 3PJ, UK
Email: Basil.Kouvaritakis@eng.ox.ac.uk

J.A. Rossiter J.Schuurmans
Department of Mathematical Sciences,
Loughborough University, Leicestershire. LE11 3TU, UK
Email: J.A.Rossiter@lboro.ac.uk

Abstract

Invariant sets can be used in conjunction with known control laws to guarantee convergence of LTV or uncertain systems with constraints. However, invariant sets are often associated with fixed control laws and hence can be limited in size by constraints. Here it is shown how to utilise degrees of freedom in the transient predictions in order to enlarge the invariant set and so widen the applicability of the convergence proof, for a given control law. The same degrees of freedom allow for computationally efficient and systematic performance optimisation.

1 Introduction

Predictive control [3] is a popular strategy because it systematically accounts for performance and constraints. Moreover, it is possible to give stability results for linear systems, even during online constraint handling (e.g. [9], [8], [13]). Researchers are now concentrating more on non-linear and uncertain systems where rigourous results and computationally efficient algorithms are more difficult to find. One early result guaranteeing convergence for non-linear and uncertain systems uses the concept of a dual mode controller [5], that is a control law which takes a different form depending on how close the system state is to its desired value; this paper will make use of that result. Dual mode schemes rely on a terminal condition that once the system state enters a region near the origin, a fixed control law is sufficient for convergence and constraint satisfaction. A recent study [6] followed this approach further and defined an appropriate invariant set (a set such that once the state enters, it will remain within). An algorithm was proposed for selecting a control law which gave an invariant set containing the current state consequently giving a guarantee of stability even for uncertain linear time varying (LTV) systems. However, although this control scheme allowed the control law to vary from sample to sample, it assumed a fixed linear control law in the predictions; this limits performance and requires a significant on-line computational burden which limits its applicability. The scheme is also limited to those states lying inside invariants sets for fixed linear control laws.

Performance can be improved considerably by allowing the controller to be nonlinear in the predictions. One way of achieving this is by allowing the first N future moves to become degrees of freedom. In [12] the condition that the control law becomes fixed and the state enter the invariant set is delayed until a few samples into the future. However since the system considered is uncertain the predictions over the transients are not single valued but rather are contained in polytopic sets and for anything other than small horizons the number of vertices becomes unmanageable. Here we overcome this problem by using invariant sets to capture the system prediction, even in the transient part where the control law is not fixed. It is shown that the appropriate invariant set can be significantly larger than that obtained with a fixed linear law (e.g. [6]) and moreover the online computational burden can be reduced dramatically. The particular predictive control strategy of ([7], [10], [11]) is best suited to these developments and is used here. In this approach the predictive control law optimises closed-loop predictions rather than open-loop predictions; the predictions are optimised by varying a free variable which is a loop input. The advantage being that apart from giving better numerical conditionning to the computations, it also allows the nominal closed-loop to be 'optimally designed' in some sense, say with regard to performance and robustness. When there are no predicted constraint violations, the extra d.o.f. can be set to zero.

The paper is structured as follows. Section 2 gives some backgrund and notation. In section 3 it is demonstrated how d.o.f. in transients can enlarge invariant sets. Section 4 illustrates the advantages and discusses how to augment the robustness yet further. The paper finishes with a conclusion.

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2 Notation and problem description

This paper adopts state-space models but the results easily transcribe to transfer function models. Consider a polytopic set of state space systems:

$$\begin{array}{c} \mathbf{x}_{k+1} = A(k)\mathbf{x}_k + B(k)\mathbf{u}_k, \quad \mathbf{y}_k = C\mathbf{x}_k, \quad \mathbf{x} \in \mathcal{R}^m \\ [A(k),\ B(k)] \in \Omega, \quad \Omega = \mathrm{Co}\{[A_1, B_1], ...[A_n, B_n]\} \end{array} \tag{1}$$

(Co{..}) means convex hull). The polytopic set can be taken to define an uncertain LTI (linear time invariant) or an uncertain LTV system. Predicted performance is measured by the cost function

$$J = \sum_{i=0}^{\infty} \mathbf{x}_{k+i} Q \mathbf{x}_{k+i} + \mathbf{u}_i^T R \mathbf{u}_i \tag{2}$$

where $\mathbf{x}_{k+i+1},\ \mathbf{u}_{k+i},\ i \geq 0$ are predicted values.

2.1 Ellipsoidal invariant sets

This paper considers ellipsoidal invariant sets only because polytopic invariant sets (e.g. [4]) do not lend themselves to analysis. An ellipsoidal set $\mathcal{E}_x$ is defined in terms of positive definite matrix Q_x as follows:

$$\mathcal{E}_x = \{\mathbf{x} | \mathbf{x}^T Q_x^{-1} \mathbf{x} \leq 1\} \tag{3}$$

Select a fixed state feedback control law which stabilises all models in the set (1) and define the corresponding closed loop system as follows:

$$\mathbf{u}_k = K\mathbf{x}_k; \quad \mathbf{x}_{k+1} = [A(k) - B(k)K]\mathbf{x}_k = \Phi(k)\mathbf{x}_k \tag{4}$$

Then $\mathcal{E}_x$ is an invariant set for $\mathbf{x}_{k+1} = \Phi(k)\mathbf{x}_k$ if

$$\mathbf{x}_k \in \mathcal{E}_x \ \Rightarrow \ \mathbf{x}_{k+1} \in \mathcal{E}_x \tag{5}$$

It has been shown (e.g. [1]) that $\mathcal{E}_x$ is invariant for model set (1) if the following conditions hold:

$$Q_x^{-1} - \Phi_i^T Q_x^{-1} \Phi_i > 0, \ \ i = 1, ..., n; \ \Phi_i = A_i - B_i K \tag{6}$$

Remark 2.1 *There is no guarantee that an invariant set defined by (6) exists for a model set (4). We assume that at least one pair K, Q_x exist.*

Let the system constraints be

$$|(\mathbf{u}_k)_j| \leq d_j, \ \ \forall j; \qquad |(\mathbf{x}_{k+1})_i| \leq \overline{x}_i, \ \ \forall i \tag{7}$$

Using control law $\mathbf{u}_k = K\mathbf{x}_k$ and letting $\mathbf{K}_j^T$ be the jth row of K, the input limits can be rewritten as

$$|\mathbf{K}_j^T \mathbf{x}| \leq d_j, \ \ \forall\ j \tag{8}$$

Hence $\mathbf{x} \in \mathcal{E}_x$ (eqn.3), implies the following

$$|\mathbf{K}_i^T \mathbf{x}| = |\mathbf{K}_i^T Q_x^{\frac{1}{2}} Q_x^{-\frac{1}{2}} \mathbf{x}| \leq \|\mathbf{K}_i^T Q_x^{\frac{1}{2}}\| \ \|Q_x^{-\frac{1}{2}} \mathbf{x}\| \leq \|\mathbf{K}_i^T Q_x^{\frac{1}{2}}\| \tag{9}$$

Therefore input constraints are satisfied if:

$$\|\mathbf{K}_i^T Q_x^{\frac{1}{2}}\| \leq d_i \ \ \Rightarrow \ \ d_i^2 - \mathbf{K}_i Q_x \mathbf{K}_i^T \geq 0, \ \ i = 1, 2, ... \tag{10}$$

Remark 2.2 *Given model set (1) and Q_x, K such that both (6, 10) are satisfied, define $\mathcal{E}_x$ as in eqn.(3). Then $\mathbf{x}_k \in \mathcal{E}_x$ implies that $\lim_{i\to\infty} \mathbf{x}_{k+i} = 0$.*

Several authors, (e.g.[6]), used the above results to design a robust stabilising control strategy. K is varied to ensure that $\mathbf{x}$ is inside an invariant set. Further degrees of freedom in the choice of K are taken up by minimising an upper bound w.r.t. to model set Ω on the predicted performance of eqn.(2). The two main weaknesses of this work are: (i) the on-line computational burden is prohibitive as a new K has to be selected at each time instant and (ii) as K is assumed fixed over the prediction horizon, only K is available to alter the shape/size of the invariant set. However, it is possible to use predictive control strategies, which are not restricted to a fixed control law in transients, combined with an invariant set to: (i) reduce the computational burden enormously; (ii) increase the size of the invariant sets and (iii) gain more flexibility for optimising robust performance.

3 Using future control moves to enlarge the invariant set

Enhance the closed-loop (implied by the stabilising state-feedback of eqn.(4)) with an additional term $\mathbf{c}_k$:

$$\mathbf{u}_k = K\mathbf{x}_k + \mathbf{c}_k; \quad \mathbf{c}_{k+n_c+i} = 0, \ \ i \geq 0 \tag{11}$$

with $\mathbf{c}_k, i = 0, 1, ..., n_c - 1$ a free variable. This gives the closed-loop state-space systems as

$$\mathbf{x}_{k+1} = \Phi(k)\mathbf{x}_k + B(k)\mathbf{c}_k \tag{12}$$

Write (12) as an autonomous system ($\mathbf{z} \in \mathcal{R}^{m+n_c}$)

$$\begin{array}{c} \mathbf{z}_{k+1} = \Psi(k)\mathbf{z}_k; \ \ \mathbf{z} = \left[\begin{array}{c} \mathbf{x} \\ \mathbf{f} \end{array}\right]; \ \mathbf{f}_{k+i} = M\mathbf{f}_{k+i-1}; \\ \Psi(k) = \left[\begin{array}{c|cccc} \Phi(k) & B(k) & \mathbf{O} & \ldots & \mathbf{O} \\ \hline 0 & & & M & \end{array}\right]; \ \ \mathbf{O} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right] \\ \mathbf{f}_k = \left[\begin{array}{c} \mathbf{c}_k \\ \mathbf{c}_{k+1} \\ \vdots \\ \mathbf{c}_{k+n_c-1} \end{array}\right]; \ \ M = \left[\begin{array}{cccc} 0_{n_c} & I_{n_c} & \ldots & 0_{n_c} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{n_c} & 0_{n_c} & \ldots & I_{n_c} \\ 0_{n_c} & 0_{n_c} & \ldots & 0_{n_c} \end{array}\right] \end{array} \tag{13}$$

0_{n_c} is an $n_c \times n_c$ matrix of zeros. Define the invariant space in $\mathbf{z}$ as

$$\mathcal{E}_z = \{\mathbf{z} | \mathbf{z}^T Q_z^{-1} \mathbf{z} \leq 1\} \tag{14}$$

Note that $\mathcal{E}_x \subseteq \mathcal{E}_z$ as $\mathbf{f} = 0$ implies $\mathcal{E}_z = \mathcal{E}_x$.

The aim now is to enlarge the size of the invariant space in $\mathcal{R}^n$ (that is x-space) by utilising the freedom in $\mathbf{c}$. More specifically, $\mathcal{E}_z$ allows for values of $\mathbf{x}$ not in $\mathcal{E}_x$, so that stability is extended to a wider region. That this is so can be illustrated easily. Define $\mathcal{E}_{xz}$ to be the subspace of $\mathcal{E}_z$ projected onto $\mathcal{R}^m$. Then

$$\mathcal{E}_x \subseteq \mathcal{E}_{xz} \subseteq \mathcal{E}_z \tag{15}$$

For example consider figure 1 where $\mathbf{x}$ and $\mathbf{f}$ are both one dimensional so that $\mathcal{E}_x \in \mathcal{R}^1$, $\mathcal{E}_{xz} \in \mathcal{R}^1$ and are lines on the x-axis. $\mathcal{E}_z \in \mathcal{R}^2$ is a 2-dimensional ellipse. Clearly $\mathcal{E}_{xz}$ is far longer than $\mathcal{E}_x$.

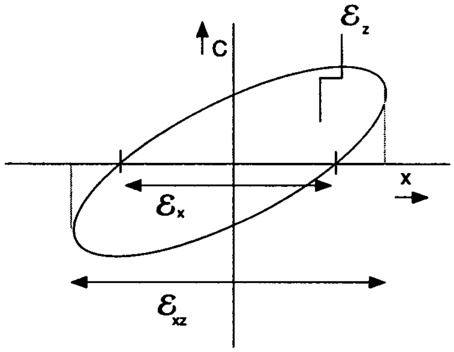

Figure 1. Illustration of invariant sets $\mathcal{E}_x, \mathcal{E}_{xz}, \mathcal{E}_z$.

The following lemma gives an analytical formula for the projection of an ellipsoid defined in $\mathcal{R}^{m+n_c}$ into $\mathcal{R}^m$.

Lemma 3.1 *Let the ellipsoid $\mathcal{E}_z$ in $\mathcal{R}^{m+n_c}$ be defined as $\mathcal{E}_z = \{\mathbf{z}|\mathbf{z}^T Q_z^{-1}\mathbf{z} \leq 1\}$. Then, its projection into $\mathcal{R}^m$ is given by $\mathcal{E}_{xz} = \{\mathbf{x}|\mathbf{x}^T Q_{xz}^{-1}\mathbf{x} \leq 1\}$ with T defined by $\mathbf{x} = T\mathbf{z}$ and $Q_{xz} = TQ_zT^T$.*

Proof: Omitted due to space restrictions.

3.1 Maximum volume of invariant set $\mathcal{E}_{xz}$

The aim then is to maximise the volume of $\mathcal{E}_{xz}$; this is a relatively simple LMI problem.

Theorem 3.1 *Maximization of the volume of the ellipsoid $\mathcal{E}_{xz}$ can be achieved by minimizing the function $\log\det(TQ_zT^T)^{-1}$, which is a convex function.*

Proof: Lemma (3.1) has given a condition for the mapping of ellipsiod $\mathcal{E}_z$ into the $\mathcal{E}_{xz}$. The volume of ellipsoid $\mathcal{E}_{xz}$ is proportional to $\det(Q_{xz})$ or equivalently $\det(TQ_zT^T)$ from which the proof is obvious. □

The conditions for $\mathcal{E}_z$ of eqn.(14) to be positive invariant for the system (13) are needed. This is implied by the following LMIs

$$\Psi_i^T Q_z^{-1} \Psi_i - Q_z^{-1} \leq 0, \quad i = 1, ..., n \tag{16}$$

Alternatively, multiplying left and right by Q_z,

$$\begin{bmatrix} Q_z & Q_z\Psi_i^T \\ \Psi_i Q_z & Q \end{bmatrix} \geq 0, \quad i = 1, ..., n \tag{17}$$

Hence an algorithm maximising the volume of $\mathcal{E}_{xz}$ is:

Algorithm 3.1 *Optimise* $\log\det(TQ_zT^T)^{-1}$ *s.t. LMIs (17,10) (rewritten in terms of Ψ, Q_z).*

3.2 Examples of invariant sets for various n_c, K

For fixed K, the size of the invariant space $\mathcal{E}_{xz}$ can be increased significantly with n_c for fixed K. Two examples will be given. The first illustrates the increase in $\mathcal{E}_{xz}$ as n_c increases. The second contrasts the invariant set sizes for different nominal choices of K and how $\mathbf{c}$ can be used to allow more highly tuned K.

3.3 Example 1

Consider the system described by $A(k) = Co\{A_1, A_2\}$

$$A_1 = \begin{bmatrix} 1 & .1 \\ 0 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1.2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ .0787 \end{bmatrix}, \tag{18}$$

with $C = [1\ 0]$, $|u(k)| \leq 2$. This system can be stabilized by the controller $\mathbf{u} = K\mathbf{x}$ with $K = [-8.48\ -5.73]$; this was obtained using the algorithm of [6] with an initial state $\mathbf{x}_0 = [0.38\ 0]^T$. Fig. 2 shows how the invariant set $\mathcal{E}_{xz}$ varies for $n_c = 0,\ 5,\ 10$.

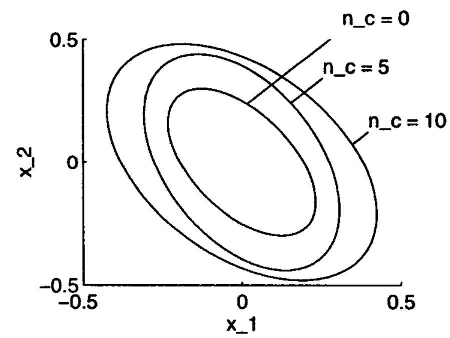

Figure 2. Positive invariant sets for $n_c = 0$, 5 and 10.

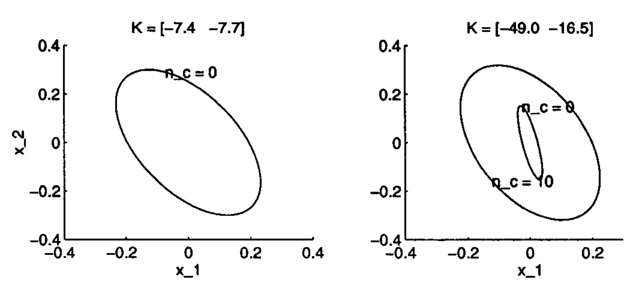

Figure 3. Illustration of invariant sets

Next consider the case where the nominal system is assumed to be given by $x(k+1) = A_2x(k) + Bu(k)$. The quadratic performance criterion (2) is defined with $Q = C^TC$ and $R = 10^{-4}$. This gives a highly tuned control law $K = [-49, \ -16.5]^T$. If R is increased to the suboptimal value of 10^{-2}, then performance is poorer; the corresponding K is $K = [-7.4, \ -7.7]^T$. Figure 3. shows positive invariant sets $\mathcal{E}_x$ for both controllers and $\mathcal{E}_{xz}$ for the optimal controller.

For the 'optimal' control law, the invariant set $\mathcal{E}_x$ is smaller than for the suboptimal control law. However, use of $\mathbf{f}$ and $\mathcal{E}_{xz}$ has ensured the effective invariant set has recovered to the volume of the detuned case. So use of $\mathbf{f}$ allows a more highly tuned control law to be used for a given region $\mathcal{R}^m$.

4 Optimising performance for predictive controllers using invariant sets

The examples in section (3.2) illustrated that $\mathcal{E}_{xz}$ gets larger as n_c increased. Given that the space used in $\mathcal{R}^m$ is known and finite, say $\mathcal{E}_r$, it would seem reasonable to select K to give as good performance as possible so long as $\mathcal{E}_r \in \mathcal{E}_{xz}$. This philosophy is in contrast to [6] for instance where K was used to vary the size of $\mathcal{E}_x$ to ensure that $\mathbf{x} \in \mathcal{E}_x$. As the requirement for $\mathbf{x} \in \mathcal{E}_{xz}$ is now handled by appropriate selection of $\mathbf{f}$, the techniques of this paper offer more flexibility in the choice of K. Moreover, the nominal control law can be be designed to give robust performance.

Algorithm (3.1) maximises the volume of the invariant $\mathbf{x}$-space $\mathcal{E}_{xz}$. The implication is that if $\mathbf{x} \in \mathcal{E}_{xz}$, then there exists a vector $\mathbf{f}$ such that $\mathbf{z} \in \mathcal{E}_z$. Once a state is inside an invariant set, convergence to the origin is guaranteed with the control law (11). However, this assumes that the update of $\mathbf{f}$ follows that of eqn.(13). Clearly there may exist other values of $\mathbf{f}$ such that $\mathbf{z} \in \mathcal{E}_z$ and this flexibility could be used to enhance performance. Under the condition that K is selected to give optimal performance (e.g. minimising cost function (2) over model set Ω in the absence of constraints), the most appropriate choice of $\mathbf{f}$ is the one minimising $\mathbf{f}^T\mathbf{f}$, [10], [13]; that is ideally $\mathbf{f} = 0$. Hence the following algorithm is proposed.

Algorithm 4.1 Offline design: *Design of K is performed to give optimal robust performance in the absence of constraints, but under the constraint that $\mathcal{E}_x \neq \emptyset$. Q_{xz} is then determined using algorithm (3.1) with an appropriate n_c.*

Online tasks: *At each sampling instant perform the minimisation*

$$\min_{\mathbf{f}} \quad J = \mathbf{f}^T\mathbf{f} \quad \text{s.t.} \quad \mathbf{z}^T Q_z^{-1} \mathbf{z} \leq 1 \qquad (19)$$

Of the optimising $\mathbf{f}$, implement the first element $\mathbf{c}_k$ in control law (11) and move onto the next sample.

The optimisation of (19) is trivial and hence omitted.

Theorem 4.1 *The control law of algorithm (4.1) is robustly stabilising.*

Proof: By assumption (see eqn.(13)) the $\mathbf{f}$ selected now can also be used at the next sampling instant, bar the first element. Therefore $\mathbf{f}^T\mathbf{f}$ is Lyapunov. Once $\mathbf{f} = 0$, the control law $\mathbf{u} = K\mathbf{x}$ is stabilising and avoids constraints violations. □

4.1 Simulation examples

In this section the advantages of algorithm (4.1) will be illustrated. These are twofold:

1. **Computational.** The on-line optimisation is trivial, especially when this is compared to that required in [6].
2. **Performance.** Performance can be improved substantially because the choice of K is not constrained to those which ensure $\mathbf{x} \in \mathcal{E}_x$.

The only on-line computation is the optimisation of (19) which is trivial - in fact a univariate search problem. Other proposals such as that in [6] require the solution on line of large LMI problems w.r.t several d.o.f. and hence the online computation is several orders of magnitude more demanding.

Improvements in performance are illustrated here by way of the same example used in section 3. A comparsion is made between the algorithm of [6] which varies K to achieve invariance and the algorithm of this paper which fixes K (via an offline design) and selects only $\mathbf{c}$ online. Here $n_c = 10$ and the K corresponding to $R = 10^{-4}$ is used. Figure 4 shows the simulations of the nominal LTI system for the initial state $\mathbf{x}_0 = [0.1 \ 0]^T$. Figure 5 shows simulations of the LTV system controlled by the same algorithms. At each time step of the simulation, the system matrices were varied within their set such that the performance index at that time step was maximized.

4.2 Improving the robust performance further

There are various approaches to robust design, this section discusses briefly two strategies. In [6], the objective is set as minimising an upper bound on a quadratic performance index (2). The nominal controller K in some sense minimises the worst case, however nominal performance maybe poor. Another strategy is to design the controller solely for the nominal case. This gives better nominal performance but may lack robustness. An alternative proposed recently ([7], [11]) is to augment the nominal optimal controller K with the well known Youla parameter [14] to give a robust design. The Youla parameter by enhancing robustness, can be used to enlarge an invariant set. For completeness an illustration of the efficacy of a Youla parameter in the context of invariant sets is given next.

Take the example system (18) with A_1, B_1 the nominal and a control weighting $R = 0.1$. In this case the nominal LQ controller K does not give an invariant set for the whole system set ! However if the controller is augmented with a 1st order Youla parameter, an invariant set exists again, see Fig. 6.

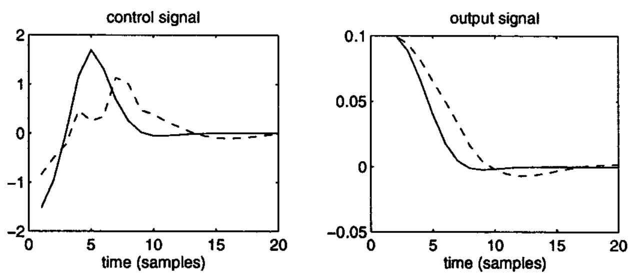

Figure 4. Simulation with nominal system controlled by; (i) algorithm of eqn.(19) (solid line) and (ii) Kothare's algorithm (dashed line).

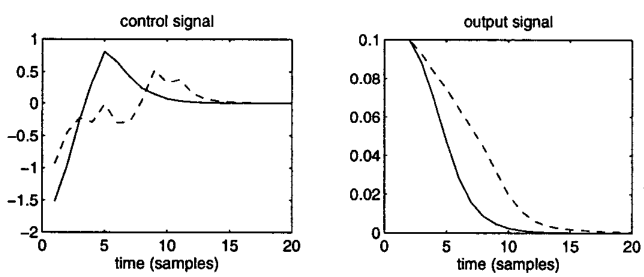

Figure 5. Simulation with LTV system controlled by; (i) algorithm of eqn.(19) (solid line) and (ii) Kothare's algorithm (dashed line).

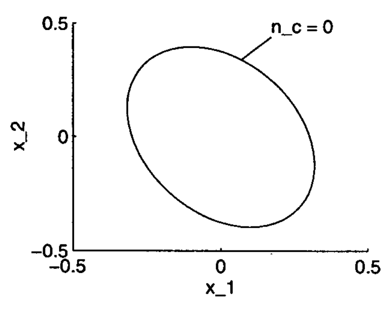

Figure 6. Invariant sets with controller augmented by Youla parameter

5 Conclusion

There are few stability results for predictive control in the presence of constraints where the plant is uncertain; one notable result used invariant sets. Here it has been shown how to enlarge the invariant set and therefore enlarge the region of applicability of predictive control. The proposed technique is computationally far less demanding and yet also allows for improving performance. Finally the scheme is able to make effective use of the Youla parameterisation of robust stabilising controllers to improve the results yet further.

Acknowledgements

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