Key words - mobile robots, nonlinear control, nonholonomic systems.

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SMOOTH TIME-INVARIANT CONTROL OF WHEELED MOBILE ROBOTS

Mobile robot point stabilization requirements can be achieved with complex control laws, like nonsmooth [2], time-variant [5] or hybrid feedback [8]. However, these types of control law tend to produce oscillating trajectories. Furthermore, stability proof becomes more involved, due to the discontinuous and/or time varying nature of the control law. It is well known that a suitable coordinate transformation can be very effective to the design of controllers for nonholonomic systems. Even though a general method for obtaining such transformations is not available, there are results showing that a differential driver robot can be controlled by means of a polar coordinate transformation. On the other hand, it is also known that with a suitable choice of reference axis, all models of wheeled mobile robots can be reduced to five classes. This paper proposes a set of further variable transformations that enable us to obtain control laws for all classes of wheeled mobile robot. In this approach, the Brockett limitations are avoided by applying a discontinuous transformation. Then, depending on the class of the robot, another transformation is performed, resulting in an equivalent differential drive or omnidirectional robot. The control laws are designed using Lyapunov-like analysis. While the transformed system is discontinuous in open-loop, the resulting closed-loop system becomes continuous. The resulting smooth, time-invariant control laws ensure asymptotic convergence to specified position and orientation. Real-time results obtained on a mobile robot built in our labs is also shown.

1. INTRODUCTION

As suggested by [6][7], a nonsmooth coordinate transformation followed by a smooth feedback is a promising direction on nonholonomic systems control. The main idea is to transform the system, in such a way that the transformed system becomes discontinuous, thereby avoiding the limitations imposed by Brockett conditions [3]. Unfortunately, there is no standard way to obtain such transformations. Hence, they are developed on a case-by-case basis. Furthermore, it can be shown that, with a suitable choice of reference axis, all models of wheeled mobile robots can be reduced to five classes [4].

This paper presents a general procedure for converting these five models to polar coordinates and proposes a set of further variable transformations that enable us to obtain control laws for all of them, thereby generalizing to all wheeled mobile robot classes the

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results presented in [1]. The designed feedback control law ensures asymptotic convergence to specified position and orientation.

2. MOBILE ROBOT MODEL

This paper considers wheeled mobile robots described by their kinematic models. These models describe the robot position and orientation given the angular velocities of the wheels. All model have the form

$$\dot{\xi} = {}^{0}R_{C}\Sigma(\beta_{C})\eta \tag{1}$$

where $\dot{\xi}$ is the posture coordinates $[x_c, y_c, \theta]^{T}$, ${}^{0}R_{C}$ is the rotation matrix between the robot and the reference coordinate system, $\Sigma(\beta_C)$ is a matrix whose columns constitutes a basis for the null space of the robot mobility matrix, and β_C and η are control inputs.The five possible models are shown in Table I. See [4] for details.

Table I - Wheeled mobile robots models

Class	Model	Class	Model
(3,0)	$\begin{bmatrix}\dot{x}_C\\ \dot{y}_C\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{bmatrix}$ (2)	(2,0)	$\begin{bmatrix}\dot{x}_C\\ \dot{y}_C\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$ (3)
(2,1)	$\begin{bmatrix}\dot{x}_C\\ \dot{y}_C\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}\cos(\theta+\beta_c) & 0\\ \sin(\theta+\beta_c) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$ (4)	(1,1)	$\begin{bmatrix}\dot{x}_C\\ \dot{y}_C\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}d\cos\theta\sin\beta_c\\ d\sin\theta\cos\beta_c\\ \cos\beta_c\end{bmatrix}\eta_1$ (5)
(1,2)	$\begin{bmatrix}\dot{x}_C\\ \dot{y}_C\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}L[\sin\beta_{c1}\cos(\theta+\beta_{c2})+\sin\beta_{c2}\cos(\theta+\beta_{c1})]\\ L[\sin\beta_{c1}\sin(\theta+\beta_{c2})+\sin\beta_{c2}\sin(\theta+\beta_{c1})]\\ \sin(\beta_{c2}-\beta_{c1})\end{bmatrix}\eta_1$ (6)		

3. COORDINATE TRANSFORMATION

In order to represent the model (1) in polar coordinates, define e and ϕ such that

$$e=\sqrt{x_C^2+y_C^2} \qquad \phi=\mathrm{atan2}(y_C,x_C) \qquad x_C=e\cos\phi \qquad y_C=e\sin\phi \tag{7}$$

thus, by defining $\alpha = \theta - \phi$ and $x=[e \quad \phi \quad \alpha]^{T}$ we can write the model in polar coordinates, i.e.,

$$\begin{bmatrix} \dot{e} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \Gamma(e,\phi) \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\frac{\sin\phi}{e} & \frac{\cos\phi}{e} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} \tag{8}$$

$$\dot{x} = \Gamma(e,\phi)^0 R_c \Sigma(\beta_c)\eta \tag{9}$$

4. FEEDBACK CONTROL

The control laws proposed here are based on a Lyapunov-like and Barbalat lemma analysis. Without loss of generality, it is assumed that e, ϕ and α are required to converge to zero.

4.1. CLASS (3,0)

Consider the following Lyapunov function candidate,

$$V = \frac{1}{2}\lambda e^2 + \frac{1}{2}(\alpha^2 + h\phi^2) \tag{10}$$

where λ and h are positive constants. By time differentiating (10) and replacing $\dot{e}$, $\dot{\phi}$ and $\dot{\alpha}$ from (9) we have

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \tag{11}$$

with

$$\dot{V}_1 = \lambda e \cos\alpha\, \eta_1 + h\phi \frac{\cos\alpha}{e}\eta_2 \qquad \dot{V}_2 = \alpha\left(\eta_3 - \left(\lambda e \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{e}\right)\eta_2 - \frac{\sin\alpha}{e\alpha}(\alpha - h\phi)\eta_1\right) \tag{12}$$

The partition of $\dot{V}$ is such that $\dot{V}_1$ retains the terms which are not null when α is null and $\dot{V}_2$ retains the remaining terms. Then, $\dot{V}_1$ can be made nonpositive if we assign

$$\eta_1 = -\gamma_1 e \cos\alpha \qquad \eta_2 = -\gamma_2 \phi e \cos\alpha \tag{13}$$

where γ_1 e γ_2 are positive constants. Hence, we have

$$\dot{V}_2 = \alpha\left(\eta_3 + \gamma_2\phi\cos\alpha\left(\lambda e^2 \frac{\sin\alpha}{\alpha} + \cos\alpha\right) + \gamma_1 \cos\alpha \frac{\sin\alpha}{\alpha}(\alpha - h\phi)\right) \tag{14}$$

and then, if we let

$$\eta_3 = -\gamma_3\alpha - \gamma_2\phi\cos\alpha\left(\lambda e^2\frac{\sin\alpha}{\alpha} + \cos\alpha\right) - \gamma_1\cos\alpha\frac{\sin\alpha}{\alpha}(\alpha - h\phi) \quad (15)$$

where γ_3 is a positive constants, it follows that

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -\lambda\gamma_1 e^2\cos^2\alpha - h\gamma_2\phi^2\cos^2\alpha - \gamma_3\alpha^2 \leq 0 \quad (16)$$

From (16) and from the fact that V is continuous and nonnegative, we conclude that the closed loop system is stable. Furthermore, as V is uniformly continuous and bounded, we have, by the Barbalat lemma, that $\dot{V}$ converges to zero. Therefore, (16) implies that e, ϕ and α also converge to zero.

4.2. CLASS (2,0)

Acting as above, the Lyapunov function candidate is partitioned into

$$\dot{V}_1 = \lambda e\cos\alpha\eta_1 \qquad \text{and} \qquad \dot{V}_2 = \alpha\left(\eta_2 - \frac{\sin\alpha}{e\alpha}(\alpha - h\phi)\eta_1\right) \quad (17)$$

in order to obtain

$$\eta_1 = -\gamma_1 e\cos\alpha \qquad \text{and} \qquad \eta_2 = -\gamma_2\alpha - \gamma_1\cos\alpha\frac{\sin\alpha}{\alpha}(\alpha - h\phi) \quad (18)$$

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -\lambda\gamma_1 e^2\cos^2\alpha - \gamma_2\alpha^2 \leq 0 \quad (19)$$

Thus, by employing the same arguments used for the clas (3,0), we conclude that e and α converge to zero. The convergence of ϕ to zero can be proved by considering the closed-loop system equations, i.e.,

$$\dot{e} = -\gamma_1 e\cos^2\alpha \qquad \dot{\phi} = -\gamma_1\sin\alpha\cos\alpha \qquad \dot{\alpha} = -\gamma_2\alpha + \gamma_1 h\phi\cos\alpha\frac{\sin\alpha}{\alpha} \quad (20)$$

As e and α converge to zero, from (20) can be concluded that $\dot{e}$ and $\dot{\phi}$ also converge to zero. The convergence of $\dot{\phi}$ to zero forces the convergence of $\dot{\alpha}$ to some constant value given by $\gamma_1 h\phi^*$, for some constant value ϕ^*. On the other hand, the uniform continuity of $\dot{\alpha}$, along with the convergence to zero of α, ensures, by the Barbalat lemma, that $\dot{\alpha}$ converges to zero. Hence, ϕ^* must be zero.

4.3. CLASS (2,1)

The control law for this class can be easily obtained by defining

$$u_1 = \eta_1 \cos \beta_c \qquad u_2 = \eta_1 \sin \beta_c \qquad u_3 = \eta_2 \tag{21}$$

then, the model in polar coorndinates assumes exactly the same form obtained for the model of a class (3,0) robot. Hence, by the same procedure outlined in section 4.1, we obtain

$$u_1 = -\gamma_1 e \cos\alpha \qquad u_2 = -\gamma_2 \phi e \cos\alpha \tag{22}$$

$$u_3 = -\gamma_3\alpha - \gamma_2\phi \cos\alpha\left(\lambda e^2 \frac{\sin\alpha}{\alpha} + \cos\alpha\right) - \gamma_1 \cos\alpha \frac{\sin\alpha}{\alpha}(\alpha - h\phi) \tag{23}$$

and the control inputs can be computed as

$$\eta_1 = \sqrt{u_1^2 + u_2^2} \qquad \beta_c = \text{atan2}(u_2, u_1) \qquad \eta_2 = u_3 \tag{24}$$

4.4. CLASS (1,1)

By transforming to polar coordinates, we obtain

$$\dot{e} = d \cos\alpha \sin \beta_c \eta \qquad \dot{\phi} = \frac{d \sin\alpha}{e} \sin \beta_c \eta \qquad \dot{\alpha} = \left(\cos \beta_c - \frac{d \sin\alpha}{e} \sin \beta_c\right)\eta \tag{25}$$

where η and β_c are the input variables. With the following definitions

$$u_1 = \eta \sin \beta_c \qquad u_2 = \eta \cos \beta_c \tag{26}$$

the model (25) becomes

$$\begin{bmatrix} \dot{e} \\ \dot{\phi} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} d\cos\alpha & 0 \\ \frac{d\sin\alpha}{e} & 0 \\ -\frac{d\sin\alpha}{e} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{27}$$

which, except for the constant *d*, is the same form obtained for the class (2,0). Hence, if we make

$$u_1 = -\gamma_1 d e \cos\alpha \qquad u_2 = -\gamma_2\alpha - \gamma_1 d^2 \cos\alpha \frac{\sin\alpha}{\alpha}(\alpha - h\phi) \tag{28}$$

the system is ensured to converge to the origin. Again, the control input can be computed as

$$\eta = \sqrt{u_1^2 + u_2^2} \qquad\qquad \beta_c = \mathrm{atan2}(u_1, u_2) \tag{29}$$

4.5. CLASS (1,2)

The model (9) in polar coordinates reads

$$\begin{cases} \dot{e} = L\left(\cos\alpha \sin(\beta_{c1} + \beta_{c2}) - 2\sin\alpha \sin\beta_{c1} \sin\beta_{c2}\right)\eta \\ \dot{\phi} = L\dfrac{\sin\alpha \sin(\beta_{c1} + \beta_{c2}) + 2\cos\alpha \sin\beta_{c1} \sin\beta_{c2}}{e}\eta \\ \dot{\alpha} = \left(\sin(\beta_{c1} - \beta_{c2}) - L\dfrac{\sin\alpha \sin(\beta_{c1} + \beta_{c2}) + 2\cos\alpha \sin\beta_{c1} \sin\beta_{c2}}{e}\right)\eta \end{cases} \tag{30}$$

where η, β_{c1} and β_{c2} are the control inputs. By defining

$$u_1 = \eta \sin(\beta_{c1} + \beta_{c2}) \qquad u_2 = 2\eta \sin\beta_{c1} \sin\beta_{c2} \qquad u_3 = \eta \sin(\beta_{c1} - \beta_{c2}) \tag{31}$$

the system model can be written as

$$\begin{bmatrix} \dot{e} \\ \dot{\phi} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} L\cos\alpha & -L\sin\alpha & 0 \\ L\dfrac{\sin\alpha}{e} & L\dfrac{\cos\alpha}{e} & 0 \\ -L\dfrac{\sin\alpha}{e} & -L\dfrac{\cos\alpha}{e} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{32}$$

which is very similar to the model for the class (3,0). Therefore, by the same analysis we have

$$u_1 = -\gamma_1 L e \cos\alpha \tag{33}$$

$$u_2 = -\gamma_2 L e \phi \cos\alpha \tag{34}$$

$$u_3 = -\gamma_3 \alpha - \gamma_2 L^2 \phi \cos\alpha \left(\lambda e^2 \frac{\sin\alpha}{\alpha} + \cos\alpha\right) - \gamma_1 L^2 \cos\alpha \frac{\sin\alpha}{\alpha}(\alpha - h\phi) \tag{35}$$

In order to obtain η, β_{c1} and β_{c2}, we divide (34) by the difference and the sum of (33) e (35), which gives

$$\beta_{c1} = \mathrm{atan2}(u_2, u_1 - u_3) \qquad\qquad \beta_{c2} = \mathrm{atan2}(u_2, u_1 + u_3) \tag{36}$$

By Summing up the squares of (33), (34) and (35) it is possible to obtain

$$\eta = \pm\sqrt{\frac{u_1^2 + u_2^2 + u_3^2}{2\left(\sin^2 \beta_{c1} + \sin^2 \beta_{c2}\right)}} \tag{37}$$

which can be computed once β_{c1} and β_{c2} have been computed from (36). It is important to note that this expression is not valid if $\beta_{c1}=n_1\pi$ and simultaneously $\beta_{c2}=n_2\pi$, for n_1, n_2=0, 1, 2…. However, under these conditions, the model (6) is also not valid. Indeed, when $\beta_{c1}=n_1\pi$ and $\beta_{c2}=n_2\pi$, the robot degenerates to a robot of class (2,0).

The correct sign for η can be determined from (31), by noting that $\sin(\beta_{c1}+\beta_{c2})$ and $\sin(\beta_{c1}-\beta_{c2})$ are simultaneously null only if $\beta_{c1}=n_1\pi$ and $\beta_{c2}=n_2\pi$. Furthermore, we have $\mathrm{sgn}(a/b)=\mathrm{sgn}(ab)$. Therefore, we can write

$$\mathrm{sgn}(\eta) = \begin{cases} \mathrm{sgn}(u_1 \sin(\beta_{c1} + \beta_{c2})), & \left|u_1 \sin(\beta_{c1} + \beta_{c2})\right| > \left|u_3 \sin(\beta_{c1} - \beta_{c2})\right| \\ \mathrm{sgn}(u_3 \sin(\beta_{c1} - \beta_{c2})), & \text{otherwise} \end{cases} \tag{38}$$

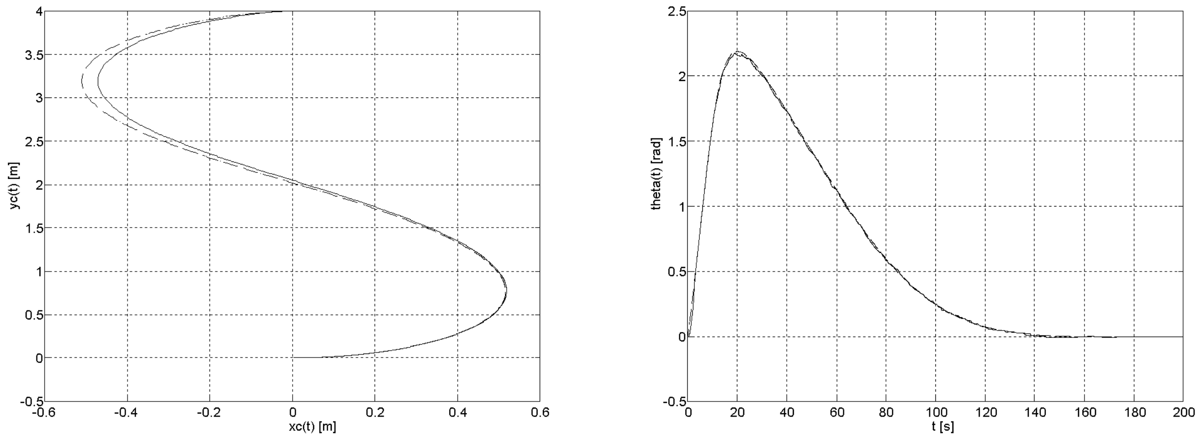
5. REAL-TIME RESULTS

So far, this paper have considered only the kinematic model of the robot, with input variables homogeneous to velocities or position. Actually, the inputs to the robot are the torques applied to rotate and orient its wheels. This does not mean, however, that the control laws obtained here can not be used in pratice. In general, for small robots, all that is needed is inner control loops ensuring that the commanded velocities and positions are effectively applied to the wheels. This can be done, for example, using conventional PID controllers.

The control law (18) was implemented in real time on a mobile robot built in our labs. This robot is a differentially actuated robot, hence belonging to class (2,0), with wheel diameter of 15cm and a wheelbase (distance between wheels) of 28.5cm. Real-time and simulation data for stabilizing the robot at origin from initial conditions $x_c(0)$=0m, $y_c(0)$=4m and $\theta(0)=0$ are shown in Fig. 2. Note that since the robot can not move sideways, this motion is a difficult one. The values γ_1=0.05, γ_2=0.1 and h=1.35 give us a good convergence rate, while avoiding actuator saturation.

7. CONCLUSION

The method presented in this paper ensures that the robot trajectory converges to a point, without excessive oscillations. Furthermore, the developed control laws are continuous and smooth. While the dynamic forces acting on the robot were neglected, real time data shows that, for small robots, an conventional PID inner loop can lead to good results. For larger robots, where the dynamic forces are significant, an adaptive controller is being developed to replace the PID controller.



(a) (b)

Fig. 2 - Real-time (continuous) and simulated (dashed) stabilization of a robot: a) spatial trajectory b) orientation.

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