Robust Receding Horizon Control of Constrained Nonlinear Systems

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***Abstract*—We present a method for the construction of a robust dual-mode, receding horizon controller which can be employed for a wide class of nonlinear systems with state and control constraints and model error. The controller is dual-mode. In a neighborhood of the origin, the control action is generated by a linear feedback controller designed for the linearized system. Outside this neighborhood, receding horizon control is employed. Existing receding horizon controllers for nonlinear, continuous time systems, which are guaranteed to stabilize the nonlinear system to which they are applied, require the exact solution, at every instant, of an optimal control problem with terminal *equality* constraints. These requirements are considerably relaxed in the dual-mode receding horizon controller presented in this paper. Stability is achieved by imposing a terminal *inequality*, rather than an equality, constraint. Only approximate minimization is required. A variable time horizon is permitted. Robustness is achieved by employing conservative state and stability constraint sets, thereby permitting a margin of error. The resultant dual-mode controller requires considerably less on-line computation than existing receding horizon controllers for nonlinear, constrained systems.**

I. Introduction

IN receding horizon control, the control at state x and time t is obtained by determining on-line the (open-loop) control $\hat{u}$ to solve a (finite horizon) optimal control problem over the interval $[t, t+T]$ and setting the current control equal to $\hat{u}(t)$. Repeating this calculation continuously yields a feedback control (since $\hat{u}(t)$ depends on the current state x). The finite horizon constrained optimal control problem is usually posed as that of minimizing a quadratic functional over the interval $[t, t+T]$ subject to the terminal constraint $x(t+T) = 0$ (cf. [1], [2], [4]). This strategy provides a relatively simple conceptual procedure for determining feedback control of constrained nonlinear systems when off-line computation of a nonlinear control law is prohibitively difficult.

Receding horizon control has been employed in diverse applications. However, naive application of the strategy may result in instability, and it is important to establish conditions under which receding horizon control stabilizes the nonlinear system to which it is applied. These conditions were obtained for linear time-varying systems by Kwon and Pearson [1] and Kwon, Bruckstein, and Kailath [2], for nonlinear discrete-time systems by Keerthi and Gilbert [3] and, more recently, for nonlinear, continuous-time systems by Michalska and Mayne [4]–[7]. For an early discussion of receding horizon control of nonlinear continuous-time systems see Chen and Shaw [8]. An independent and very interesting stream of research, associated with the process industry, is the literature on 'model predictive control.' This literature, which is extremely well reviewed in [9], stresses the potential of this type of control, compared with other approaches, for dealing with state and control constraints and nonlinearity. While much of this literature is concerned with the control of linear systems, there is some, for example [10], [11], devoted to nonlinear control. In model predictive control an explicit stability constraint is generally not included in the finite horizon optimal control problem, necessitating a 'cut-and try' adjustment of parameters in the cost function to obtain stability of the resultant controlled system; this approach is criticized in [12] and appears inappropriate for nonlinear systems. An important exception in the Model Predictive literature to this criticism is the recent paper by Rawlings and Muske [13]. Here stability is achieved, when the plant is stable, without a terminal equality constraint by restricting the control to a finite horizon (so that it is zero beyond the horizon) and adding a terminal cost which is the infinite horizon cost due to zero control. However, when the plant is unstable, a terminal equality constraint (on the unstable modes) must be imposed on the finite horizon problem. One of the contributions of this paper is that it shows how stability may be ensured for the linear or nonlinear case, whether the plant is stable or unstable, without requiring exact satisfaction of an equality terminal constraint (which would be computationally prohibitive in the nonlinear case).

The feedback controller in [4], [5] is conceptual in that it requires the calculation of an exact solution of a constrained optimal control problem at every instant of time. Exact minimization, and exact satisfaction of the terminal (stability) constraint, place considerable, if not impossible, demands on the on-line optimizing controller, both requiring an infinite number of iterations in the nonlinear case. It is clear that an implementable version of such a controller should allow the optimal control problem to be

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solved approximately with the terminal constraint $x(t + T) = 0$ replaced by a relaxed constraint $x(t + T) \in W$ (where W is some neighborhood of the origin). Additionally, it should stabilize the system to which it is applied even if the optimal open-loop control is updated only at a discrete set of times. In this paper, we show that the construction of a receding horizon controller which satisfies the above requirements is possible, and extend considerably our initial results [14]. Since the terminal constraint is relaxed to $x(t + T) \in W$, the receding horizon strategy loses its stabilizing properties inside W. In order to compensate for this effect, a locally stabilizing mechanism is needed. Hence, inside W, a linear stabilizing controller $u = Kx$ is employed. This enables the stability terminal equality constraint to be dispensed with; in effect, the control is set equal to Kx (rather than 0 as in [13]) beyond the horizon. Since $u = Kx$ is stabilizing, a terminal equality constraint is not necessary, even when the plant is unstable. The controller is, therefore, a linear feedback controller in W and a nonlinear, receding horizon, controller in the complement of W. The receding horizon controller is, of course, a feedback controller (it uses the measured value of the state of the system as the initial point for the optimal control problem at the discrete times when the open-loop optimal control is updated). The resultant dual-mode controller is shown to be globally stabilizing. It is able to deal with control and state constraints and a simple modification, which introduces a degree of conservatism, ensures robustness. A useful feature of the controller is that it merely requires the approximate solution of this on-line optimization problem. Moreover, once a feasible solution of the *initial* finite horizon problem is determined, feasible solutions to all *subsequent* finite horizon problems are easily obtained. This considerably reduces the amount of on-line computation required to implement the control strategy and is an important advantage of the proposed controller since computing a feasible or optimal, solution to a nonlinear optimal control problem when an initial feasible solution is *not* available can be computationally extremely expensive.

II. Dual-Mode Control Strategy

The symbol $\|\cdot\|$ denotes any vector norm in $\mathbb{R}^n$ (where the dimension n follows from the context). For an arbitrary Hermitian, positive definite matrix P, the weighted norm $\|\cdot\|_P$ is defined by $\|x\|_P^2 \triangleq x^T P x$ for all $x \in \mathbb{R}^n$, where x^T denotes the transpose of x. $B(x; \rho)$ ($\bar{B}(x; \rho)$) denotes an open (closed) ball in $\mathbb{R}^n$ with center x and radius ρ. For any Hermitian matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote, respectively, the smallest and largest eigenvalues of the matrix A. For any subset $W \subset \mathbb{R}^n$, W^c and δW denote, respectively, the complement and boundary of the set W.

We consider receding horizon control of the time-invariant nonlinear system described by

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1}$$

subject to the control constraint

$$u(t) \in \Omega \tag{2.2}$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and Ω satisfy the following:

A0) f is twice continuously differentiable; $f(0, 0) = 0$.

A1) Ω is a compact subset of $\mathbb{R}^m$ containing the origin in its interior.

We shall refer to the system being controlled (the 'plant') as the *real* system, and to the model of the system, used in computing the control, as the *model*. When model error is absent, both the real system and the model are described by (2.1). When model error is present, we use (2.1) to describe the behavior of the model, and introduce a new differential equation (4.1) to describe the dynamic behavior of the real system.

Conventionally, *receding horizon control* at state x and time t is obtained by solving the following *finite-horizon* control problem $P(x, t)$ defined by

$$P(x,t): \min\{V(x,t,u) | u \in S,\ x^u(T; x, t) = 0\} \tag{2.3}$$

where $S = S(t, T)$ is the family of piecewise continuous functions (functions continuous from the right) mapping $[t, t + T]$ into Ω, T is the specified horizon, and the performance index is

$$V(x,t,u) \triangleq (1/2)\int_t^{t+T}\left[\|x^u(s; x, t)\|_Q^2 + \|u(s)\|_R^2\right] ds, \tag{2.4}$$

where R and Q are strictly positive definite, symmetric matrices, and $x^u(\cdot; x, t)$ denotes the solution of (2.1), due to control u, with initial state x at time t. The minimizer of $P(x, t)$ will be denoted by $\hat{u}(\cdot; x, t)$ and the corresponding optimal value function is $V^0(x, t)$. With the assumption that the optimal controls $\hat{u}$ exist and are unique for every initial state $x \in \mathbb{R}^n$, the receding horizon control law $h^*: \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$h^*(x) \triangleq \hat{u}(t; x, t). \tag{2.5}$$

The basic concern in [4], [5] is proving that the *closed-loop* system

$$\dot{x}(t) = f(x(t), h^*(x(t))) \tag{2.6}$$

is globally asymptotically stable. The approach adopted is to employ $\hat{V}$ as a Lyapunov function for the closed-loop system (2.7) (under the additional assumption that h^* is continuous). This immediately reveals the necessity for the terminal constraint.

Since an exact solution to the finite horizon control problem $P(x, t)$ with a terminal equality constraint $x(t + T) = 0$ is difficult to calculate, an implementable controller should relax this constraint to an inequality constraint of the form $x(t) \in W$, where W is some neighborhood of the origin. If the terminal equality constraint is relaxed in this way, the receding horizon control loses its stabilizing properties inside W. To overcome this difficulty, we introduce a dual-mode receding horizon control strategy which uses a locally stabilizing control law inside W and a receding horizon controller outside W.

A local linear control law which stabilizes the nonlinear system in W is obtained as follows (see [6], [8]). With the assumption that $f(0,0) = 0$, the linearized system is described by

$$\dot{x}(t) = f_x(0,0)x(t) + f_u(0,0)u(t). \tag{2.7}$$

The local, linear control law h_L: $\mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$h_L(x) \triangleq Kx. \tag{2.8}$$

Provided that the linearized system is stabilizable, the matrix K is chosen so that the closed-loop system

$$\dot{x}(t) = Ax(t) \tag{2.9}$$

where $A \triangleq [f_x(0,0) + f_u(0,0)K]$ is asymptotically stable. Hence, the function V_L: $\mathbb{R}^n \to \mathbb{R}$ defined, for some positive definite, symmetric matrix P, by

$$\begin{aligned} V_L(x) &\triangleq (1/2)\int_0^{\infty}\left[\|x_L(s;x)\|_Q^2 + \|Kx_L(s;x)\|_R^2\right] ds \\ &= (1/2)\|x\|_P^2 \end{aligned} \tag{2.10}$$

where $x_L(\cdot;x)$ denotes the solution of (2.9) with initial state x at $t = 0$, takes finite values and is a Lyapunov function for (2.9). Computing the time derivative of the value function V_L along a solution of (2.9) yields

$$\dot{V}_L(x) = (1/2)x^T(A^TP + PA)x = -(1/2)x^TQ^*x \tag{2.11}$$

where $Q^* \triangleq Q + K^TRK$. System (2.11) is a local linearization of the nonlinear system

$$\dot{x}(t) = f(x(t), Kx(t)). \tag{2.12}$$

Let ϕ: $\mathbb{R}^n \to \mathbb{R}^n$ be defined by $\phi(x) \triangleq f(x, Kx) - Ax$. Clearly, ϕ satisfies $\phi(0) = 0$ and $\|\phi(x)\|_P/\|x\|_P \to 0$ as $\|x\|_P \to 0$. Computing the time derivative of the value function V_L along the trajectories of (2.12) yields

$$\begin{aligned} \dot{V}_L(x) &= (1/2)x^T(A^TP + PA)x + x^TP\phi(x) \\ &\le \|x\|_P^2[\|P\|_P(\|\phi(x)\|_P/\|x\|_P)] - (1/2)x^TQ^*x. \end{aligned} \tag{2.13}$$

Since Q^* is positive definite and $\|\phi(x)\|_P/\|x\|_P \to 0$ as $\|x\|_P \to 0$, there exist constants $\epsilon \in (0,\infty)$ and $\alpha_1 \in (0,\infty)$ such that

$$\begin{aligned} \dot{V}_L(x) &\le -\epsilon V^L(x) \qquad \text{whenever} \\ V_L(x) &= (1/2)\|x\|_P^2 \le (1/2)\alpha_1^2 \end{aligned} \tag{2.14}$$

where $\dot{V}_L$ denotes the time derivative of V_L along a solution of (2.9) or (2.12). The constant ϵ may be chosen to be $(1/2)\lambda_{\min}(Q^*)/\lambda_{\max}(P)$ or, less conservatively, as $\lambda_{\min}(P^{-1/2}Q^*P^{-1/2})$. Let $\alpha \in (0, \alpha_1)$ be such that the level set $W = W_\alpha$ defined by

$$W_\alpha \triangleq \{x \in \mathbb{R}^n | V_L(x) \le \alpha^2/2\} = \{x \mid \|x\|_P \le \alpha\} \tag{2.15}$$

satisfies $Kx \in \Omega$ for all $x \in W_\alpha$. Then, W_α is a region of attraction and an invariant set for both (2.9) and (2.12) and V_L is a Lyapunov function for both systems. Hence, any trajectory of (2.9) or (2.12), starting in W_α, remains in W_α and converges to the origin; the origin is an asymptotically stable equilibrium point of (2.9) and (2.12). Moreover, the control constraint is satisfied everywhere on such trajectories.

Let $X \subset \mathbb{R}^n$ denote the st of initial states which can be steered to W_α by a control in S (thereby satisfying the control constraint).

The following assumption ensures the existence of a linear controller with the properties specified above.

A2) The linearized system (2.11) is stabilizable.

We note in passing that, if $X \cap W_\alpha^c$ is empty, the linear controller suffices.

It is important to note that the linear controller h_L (the feedback matrix K), the positive definite matrix P, and the constant α, may be determined *off-line*. This can be done as follows. First, K is determined, using standard linear design techniques, so that the linear system $\dot{x} = Ax$, $A \triangleq f_x(0,0) + f_u(0,0)K$, is stable and has suitable regulatory properties. Then P is determined by solving the Lyapunov equation $A^TP + PA = -Q^*$. The constant ϵ may be chosen as above. Finally an α, preferably as large as possible, which satisfies the semi-infinite inequalities $x^TP\phi(x) - (1/2)x^TQ^*x \le -(\epsilon/2)\|x\|_P^2$, and $Kx \in \Omega$ for all $x \in W_\alpha = \{x \in \mathbb{R}^n \mid \|x\|_P \le \alpha\}$, is determined. Let ψ_j: $\mathbb{R}^n \to \mathbb{R}$, $j = 1,2$ be defined by:

$$\begin{aligned} \psi_1(x) &\triangleq x^TP\phi(x) - (1/2)x^TQ^*x + (\epsilon/2)\|x\|_P^2, \\ \psi_2(x) &= d(Kx, \Omega) \end{aligned} \tag{2.16}$$

where $d(u, \Omega)$ is the distance of a point u from the set Ω. The problem is, therefore, to determine α to satisfy $\psi_1(x) \le 0$ and $\psi_2(x) \le 0$ for all $x \in W_\alpha$. Since $\alpha \in \mathbb{R}$, this is a relatively simple semi-infinite feasibility problem which can be solved by existing algorithms [16], [17]. Because K maps the ellipsoid W_α into another ellipsoid $V_\alpha \triangleq \{v \in \mathbb{R}^m \mid \|v\|_{K^TPK} \le \alpha\}$, the quantity $\max\{\psi_2(x) \mid x \in W_\alpha\}$ can easily be computed if, for example, Ω is a box in $\mathbb{R}^m$. The quantity $\max\{\psi_1(x) \mid \|x\|_P \le \alpha\}$ has to be determined using global optimization; however, for small α, the function ψ_1 is concave on W_α, making global optimization relatively easy. A suitable α can be found by halving an initial guess until the inequalities $\psi_1(x) \le 0$ and $\psi_2(x) \le 0$ for all $x \in W_\alpha$ are satisfied. More difficult problems than this are encountered in the design of locally stabilizing controllers for nonlinear systems; see, for example the interesting paper [18], which describes an algorithm for determining an ellipsoidal region of attraction of maximum volume.

One candidate for the finite horizon control problem, which is defined for all $x \in X \cap W_\alpha^c$ (whose repeated solution yields a receding horizon control) is the free end-time problem defined by

$$P(x,t)\text{: } \min\{V(x,t,u,T) \mid x^u(t+T;x,t) \in W_\alpha, \; u \in S, T \ge 0\} \tag{2.17}$$

where $V: \mathbb{R}^n \times \mathbb{R} \times S \times \mathbb{R} \to \mathbb{R}$ is defined by

$$V(x,t,u,T) \triangleq (1/2)\int_t^{t+T} \left[\|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\right] ds. \quad (2.18)$$

For all $x \in W_\alpha^c$, let $\hat{u}(\cdot;x,t)$: $[t, t+\hat{T}(x)] \to \Omega$ denote the minimizing control, and $\hat{T}(x)$ the smallest, minimizing T for $P(x,t)$, for all $x \in W_\alpha^c$. Clearly, the optimal trajectory $\hat{x}(\cdot;x,t)$ crosses the boundary of W_α at time $t+\hat{T}(x)$.

For the conceptual version of the receding horizon controller, we shall assume, in addition to A0), A1), and A2), that the minimizing control horizon pair $(\hat{u}(\cdot;x,0), \hat{T}(x))$ exists for every initial state $x \in X$.

Applying the Principle of Optimality to problem $P(x,t)$ yields the following proposition.

Proposition 1: For any $x \in X \cap W_\alpha^c$, any $t > 0$, and any $\delta < \hat{T}(x)$ the solution $\hat{u}(\cdot;\hat{x}(t+\delta;x,t), t+\delta)$ to problem $P(\hat{x}(t+\delta;x,t), t+\delta)$ is defined on $[t+\delta, t+\hat{T}(\hat{x}(t+\delta;x,t))]$ and is equal to $\hat{u}(\cdot;x,t)$, the minimizer for $P(x,t)$, restricted to $[t+\delta, t+\hat{T}(x)]$. Also, $\hat{T}(\hat{x}(t+\delta;x,t)) = \hat{T}(x) - \delta$.

Proof: Suppose the contrary, that $\hat{u}(\cdot;\hat{x}(t+\delta;x,t), t+\delta)$ produces a lower cost for $P(\hat{x}(t+\delta;x,t), t+\delta)$ than (and therefore differs from) $\hat{u}(\cdot;x,t)$ in the interval $[t+\delta, t+\hat{T}(x)]$. Then, a control ν: $[t, t+\hat{T}(x)] \to \Omega$ defined by

$$\nu(s) \triangleq \begin{cases} \hat{u}(s;x,t) & \text{for all } s \in [t, t+\delta] \\ \hat{u}(s;\hat{x}(t+\delta;x,t), t+\delta) & \\ \qquad \text{for all } s \in (t+\delta, t+\hat{T}(x)] & \end{cases} \quad (2.19)$$

would yield a (strictly) lower cost for $P(x,t)$ than $\hat{u}$, and this contradicts the optimality of $\hat{u}$. That $\hat{T}(\hat{x}(t+\delta;x,t) = \hat{T}(x) - \delta$ follows from the fact that the trajectory $\hat{x}(\cdot;x,t)$ crosses the boundary of W_α at time $t+\hat{T}(x)$ so that the optimal horizon for $P(\hat{x}(t+\delta;x,t), t+\delta)$ is $(t+\hat{T}(x)) - (t+\delta)$. □

The receding horizon control law $h: \mathbb{R}^n \to \Omega$ is defined by

$$h(x) \triangleq \begin{cases} \hat{u}(t;x,t) = \hat{u}(0;x,0) & \text{for all } x \in W_\alpha^c \\ Kx & \text{for all } x \in W_\alpha. \end{cases} \quad (2.20)$$

The control law h is defined uniquely at each initial point $x \in X \cap W_\alpha^c$ for which the solution $\hat{u}$ of $P(x,t)$ is unique, and is time-invariant.

Proposition 1 leads to the following result.

Theorem 1: Suppose that assumptions A0)–A2) are satisfied, and that for all $x \in X$ and all $u \in S$ there exists a unique, absolutely continuous solution to the nonlinear system (2.1) with initial state x at $t = 0$. Suppose, additionally, that, for all $x \in X$, $\hat{u}(\cdot;x,t)$ exists and is unique so that the receding horizon law, given by (2.5), is uniquely defined. Then, in the absence of disturbances, the closed-loop system with receding horizon control

$$\dot{x}(t) = f(x(t), h(x(t))) \quad (2.21)$$

is uniform-asymptotically stable in the sense of Lyapunov [15] with a region of attraction X.

Proof: First, since $h(0) = 0$, the origin is an equilibrium state for the closed-loop system (2.20). From the definition of the receding horizon control law, (2.5), and Proposition 1 it follows next that $h(\cdot)$ is a 'closed-loop representation' of the open-loop control $u(\cdot;x,t)$. Hence, for any initial $x \in X \cap W_\alpha^c$, the trajectory of the closed-loop system (2.20) is, in the absence of disturbances, identical to $\hat{x}(\cdot;x,0)$ (generated by the open-loop finite horizon optimal control). Hence, the feedback controller h steers any initial state of the system lying in $X \cap W_\alpha^c$ to W_α and therefore to the origin (since W_α is a region of attraction for the system). The latter implies that the origin is a uniform-asymptotically stable equilibrium state of the closed-loop system (2.20) and that X is a region of attraction (and an invariant set) for the closed-loop system. □

It is worth noting that, in conventional receding horizon control (defined by (2.5)), the trajectories of the closed-loop system (2.7) and $x^{\hat{u}}(\cdot;x,0)$ [the optimal, open-loop solution to $P(x,0)$] generally differ; this has the important practical implication that feasible solutions to the current finite horizon optimal control problem cannot be obtained as restrictions of previous solutions.

It is known that if the origin is a uniform-asymptotically stable equilibrium point of (2.26) and the function $f(\cdot, h(\cdot))$: $\mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous, then this system is also totally stable and, therefore, possesses a degree of robustness [15].

In the next section, a dual-mode receding horizon controller, which merely requires approximate solutions of P, will be described.

III. Dual-Mode Receding Horizon Control

As before, let X denote the set of states which can be steered to W_α by a control in S. For any initial state $x \in \mathbb{R}^n$, the set $Z(x)$ of admissible controls and horizons for problem P, is defined by

$$Z(x) \triangleq \{u \in S, T \in (0,\infty) | x^u(T;x,0) \in \delta W_\alpha\}. \quad (3.1)$$

The admissible set $Z(x)$ can be defined equivalently (without affecting the solution to problem $P(x,t)$ or changing its computational complexity) by replacing δW_α in (3.1) by W_α. We use definition (3.1), since it assigns a unique horizon T to each admissible control u, thus simplifying our discussion.

The dual-mode receding horizon controller for system (2.1) is defined below. We assume (temporarily) that the model error is zero, so that both the model and the real system are assumed to be described by (2.1). Let $x_R(t)$ denote the state of the *real* system (i.e., the system being controlled) at time t. For all $i \in \mathbb{N}$, let $t_i \triangleq i\delta$ where δ is the 'sampling interval.' Let x_i denote the state of the real system at time t_i, i.e., $x_i = x_R(t_i)$. The motivation for the controller is to avoid the computationally very expensive task of repetitively determining an admissible control (one satisfying all the constraints, including the stability constraint) at each iteration. Suppose, therefore, one has an admissible control horizon pair (u_{i-1}, T_{i-1}) for $P(x_{i-1}, t_{i-1})$, so that u_{i-1} steers the model from state x_{i-1} to the boundary of W_α in time T_{i-1}. The control u_{i-1} is applied to the real system over the interval $[t_{i-1}, t_i]$, yielding the state x_i at time t_i. It is easily seen (since there is no model error) that the control u'_i, which is defined to be the control u_{i-1} restricted to the interval $[t_i, t_{i-1} + T_{i-1}] = [t_i, t_i + T_{i-1} - \delta]$ steers the model from x_i to the boundary of W_α in a time $T'_i \triangleq T_{i-1} - \delta$. Thus, an admissible control horizon pair (u'_i, T'_i) for $P(x_i, t_i)$ is easily obtained. See Fig. 1.

Dual-Mode Receding Horizon Controller

Data: $x_0 \in X$, $\delta \in (0, \infty)$.

Initialization: At time $t_0 = 0$, if $x_0 \in W_\alpha$, switch to local linear control, i.e., employ the linear feedback law h_L for all t such that $x_R(t) \in W_\alpha$. Else, compute a control horizon pair $(u_0, T_0) \in Z(x_0)$ for problem $P(x_0, t_0)$. Apply the control u_0 to the real system over the interval $[t_0, t_0 + \delta']$ where $\delta' = \min\{\delta, T_0\}$, so that $x_1 = x^{u_0}(t_1; x_0, t_0)$ if $T_0 \geq \delta$.

Controller:

1) At any time t, if $x_R(t) \in W_\alpha$, switch to local linear control, i.e., employ the linear feedback control law h_L for all t such that $x_R(t) \in W_\alpha$. Else:

2) At any time t_i, $i \in \mathbb{N}$:

a) Obtain an admissible control horizon pair $(u'_i, T'_i) \in Z(x_i)$ for problem $P(x_i, t_i)$ where u'_i is equal to the restriction of u_{i-1} to the interval $[t_{i-1}, t_{i-1} + T_{i-1}]$ and $T'_i = T_{i-1} - \delta$.

b) Compute an admissible control horizon pair (u_i, T_i) which is better than the preceding control horizon pair in the sense that:

$$V(x_i, t_i, u_i, T_i) \leq V(x_i, t_i, u'_i, T'_i). \tag{3.2}$$

Apply the control u_i to the real system over the interval $[t_i, t_i + \delta']$ where $\delta' = \min\{\delta, T_i\}$, (so that $x_{i+1} = x^{u_i}(t_{i+1}; x_i, t_i)$ if $T_i \geq \delta$). □

Comment 1: Once x_R enters W_α, it remains there. If Step 2) is entered, then $x_{i-1}, x_i \in W_\alpha^c$ which implies $T_{i-1} > \delta$. The control horizon pair (u'_i, T'_i), obviously satisfies the inequality (3.2). Thus, a stabilizing control horizon pair (u'_i, T'_i) can be obtained with virtually *no* computation, making Step 2)-b) of the controller unnecessary if the only objective is stability. This should be contrasted with existing receding horizon controllers which require repeated solution of an optimal control problem.

Proposition 2: Suppose that assumptions A0)–A2) are satisfied. Then:

i) For all $i \in \mathbb{N}$ such that $T_{i-1} > \delta$, there exists a control horizon pair $(u_i, T_i) \in Z(x_i)$ such that (3.2) is satisfied (and the control algorithm is well defined).

ii) There exists constant $\eta \in (0, \infty)$ such that

$$V(x_{i+1}, t_{i+1}, u_{i+1}, T_{i+1}) \leq V(x_i, t_i, u_i, T_i) - \eta \tag{3.3}$$

for all $i \in \mathbb{N}$ such that both x_i and x_{i+1} are in W_α^c.

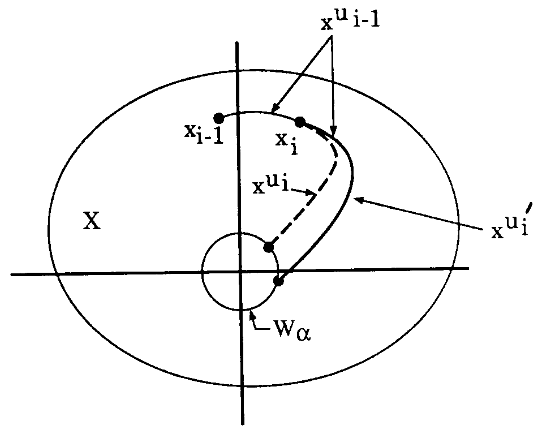

Fig. 1. Receding horizon trajectories.

Proof: Part i) follows from the definition of X (which ensures the existence of $(u_0, T_0) \in Z(x_0)$) and the fact that, for all $i \in \mathbb{N}$, $(u_i, T_i) \in Z(x_i)$ implies $(u_i, T_i - \delta) \in Z(x_{i+1})$. To prove part ii), we note that (x_{i+1}, t_{i+1}) and (x_i, t_i) are points on the same trajectory of (2.1). Since both x_i and x_{i+1} lie in W_α^c, the trajectory $x^{u_i}(\cdot; x_i, t_i)$ does not enter the set W_α during the interval $[t_i, t_{i+1}]$ (because, otherwise, x_{i+1} would also belong to W_α). Therefore, $T_i > \delta$ and

$$\begin{aligned} V(x_i, t_i, u_i, T_i) &- V(x_{i+1}, t_{i+1}, u'_{i+1}, T'_{i+1}) \\ &= (1/2)\int_{t_i}^{t_i+\delta}\left[\|x^{u_i}(s; x_i, t_i)\|_Q^2 + \|u_i(s)\|_R^2\right] ds \\ &\geq (1/2)\int_{t_i}^{t_i+\delta}\|x^{u_i}(s; x_i, t_i)\|_Q^2 \, ds \\ &\geq (1/2)\delta \inf\left\{\|x\|_Q^2 \mid x \in W_\alpha^c\right\} \\ &\geq (\delta/2)\inf\left\{\left(\|x\|_Q^2/\|x\|_P^2\right)\|x\|_P^2 \mid x \in W_\alpha^c\right\} \end{aligned} \tag{3.4}$$

for all $i \in \mathbb{N}$ such that x_i and x_{i+1} both lie in W_α^c. From Step 2)-b) of the controller, $V(x_{i+1}, t_{i+1}, u_{i+1}, T_{i+1}) \leq V(x_{i+1}, t_{i+1}, u'_{i+1}, T'_{i+1})$. Since $x \in W_\alpha^c$ implies $\|x\|_P \geq \alpha$, the desired result follows from (3.4) with $\eta \triangleq (\delta\alpha^2/2)(\lambda_{\min}(Q)/\lambda_{\max}(P))$. □

Theorem 2: Let assumptions A0)–A2) be satisfied. In the absence of disturbances, the dual-mode receding horizon controller is asymptotically stabilizing with a region of attraction X. For all $x_0 \in X$, there exists a finite time t such that $x(t) \in W_\alpha$.

Proof: Suppose that there is no such t for which $x_R(t) \in W_\alpha$. From Proposition 2, it follows that there exists a $\eta \in (0, \infty)$ such that $V(x_{i+1}, t_{i+1}, u_{i+1}, T_{i+1}) \leq V(x_i, t_i, u_i, T_i) - \eta$ for all $i \in \mathbb{N}$, which immediately implies that $V(x_i, t_i, u_i, T_i) \to -\infty$ as $i \to \infty$. However, this contradicts the fact that $V(x_i, t_i, u_i, T_i) \geq 0$ for all $i \in \mathbb{N}$. Therefore, there exists a t such that $x_R(t) \in W_\alpha$. The stabilizing property of the controller follows since, by construction, such a t is detected in Step 1) and the stabilizing local linear feedback law is applied thereafter. □

The analysis of stability does not require the actual existence of a minimizing control for problem P, merely that, at each $x \in X$, the admissible set is nonempty. Since the algorithm does not employ optimal controls, uniqueness of these is also irrelevant.

In the next section, we will show that the receding horizon controller can be modified to ensure robustness.

IV. Robust, Dual-Mode Receding Horizon Control

Receding horizon controller requires repeated solution of the open-loop optimization problem P. This solution clearly depends on the model (2.1) of the real system which is employed. Since modeling errors are always present, it is important that the receding horizon controller be robust.

To analyze robustness suppose that the *real* system is described by

$$\dot{x}(t) = f_R(x(t), u(t)), \tag{4.1}$$

and its *model* by (2.1) (i.e., by $\dot{x} = f$). Let $x_R^u(\cdot; x, t)$ denote the trajectory of the (real) system (4.1), due to control u, passing through state x at time t. As before, x^u denotes the trajectory of the model (2.1) due to the control u.

For all $\beta \in (0, \infty)$, let F_β be defined as the set $\{f_R\}$ of functions mapping $X \times \Omega$ into $\mathbb{R}^n$ and satisfying:

i) f_R is twice continuously differentiable, and

ii) $\|f_R(x, u) - f(x, u)\|_P \le \beta \|(x, u)\|_P$ for all $(x, u) \in X \times \Omega$.

We strengthen our assumptions by requiring the following.

A3) The function f is Lipschitz continuous on $X \times \Omega$.

Note that A3) is automatically satisfied if X is bounded. An immediate consequence of A3) is that, for all $\beta \in (0, \infty)$, every $f_R \in F_\beta$ is also Lipschitz continuous on $X \times \Omega$.

Our first requirement is that the local linear controller be robust. That such a controller exists is ensured by the following proposition.

Proposition 3: Let assumptions A0)–A3) be satisfied. Suppose $\beta_L \in (0, \infty)$ is given. Then there exists an $\alpha \in (0, \infty)$ and a $K \in \mathbb{R}^{m \times n}$ such that $W_\alpha = \{x \in \mathbb{R}^n \mid V_L(x) \le \alpha^2/2\}$ is an invariant region of attraction for the systems $\dot{x} = Ax$, $A \triangleq [f_x(0,0) + f_u(0,0)K]$ and $\dot{x} = f_R(x, u)$ for all $f_R \in F_{\beta_L}$ and, additionally, $h_L(x) = Kx \in \Omega$ for all $x \in W_\alpha$. □

The proof of Proposition 3 is a straightforward consequence of the Bellman–Gronwall Lemma and is not given here.

We suppose, in the sequel, that "h_L is (robustly) stabilizing in W_α" by which we mean that α, β_L and K are such that W_α has the properties specified in Proposition 3. Sufficient conditions for this are that α and β_L satisfy the inequalities:

$$\psi_1'(\beta_L, x) \triangleq \psi_1(x) + \beta_L[1 + \|K\|_P]\|x\|_P^2 \le 0 \tag{4.2}$$

and $\psi_2(x) \le 0$ for all $x \in W_\alpha$, where ψ_1 and ψ_2 are defined in (2.16). Note that robustness imposes a stronger requirement on α and a limit β_L on the permissible model error β.

Our second requirement is that the controller stabilizes system (4.1) even though it calculates the current control action using solutions of an optimal control problem defined in terms of the model (2.1). Since, for any (x, t, u), the trajectories $x_R^u(\cdot; x, t)$ and $x^u(\cdot; x, t)$ do not necessarily coincide, the dual-mode controller presented in Section III is not necessarily robust. To ensure robustness, it is necessary to replace the stability constraint $x^u(t + T; x, t) \in W_\alpha$ by a more conservative constraint. To describe, and analyze, the controller, it is helpful to introduce the control $\tilde{u}$ defined as follows. For all $z \in W_\alpha$, all $t \ge 0$, let $\tilde{u}(\cdot; z, t)$ denote the control generated by the feedback law $u = Kx$ applied to the *model* with initial condition (z, t). More precisely

$$\tilde{u}(s; z, t) \triangleq K\tilde{x}(s; z, t) \tag{4.3}$$

where $\tilde{x}(\cdot; z, t)$ is the solution of

$$\dot{x}(s) = f(x(s), Kx(s)), \qquad x(t) = z. \tag{4.4}$$

To obtain robustness, we modify our definitions of P and Z. Suppose that an upper bound $T_{\max}$ on admissible horizons is given. Let $X_{\max} \subset X$ be defined as the set of initial states of the *model* which can be steered, by a control in S, to $W_{\alpha/2}$ in a time not exceeding $T_{\max}$. Because $X_{\max} \subset X$, f and every $f_R \in F_\beta$ ($\beta \in (0, \infty)$) are Lipschitz continuous on $X_{\max} \times \Omega$. Our set of admissible control-horizon pairs is now defined by

$$Z(x) \triangleq \left\{(u, T) \in S \times [0, T_{\max}] \,\middle|\, x^u(t + T; x, t) \in \delta W_{\alpha/2}\right\}. \tag{4.5}$$

Essentially, to increase robustness, $W \triangleq W_\alpha$ is replaced by $W_{\alpha/2}$ in the modified finite horizon optimal control problem P defined by

$$P(x, t): \min \left\{V(x, t, u, T) \,\middle|\, u \in S, T \in [0, T_{\max}], x^u(t + T; x, t) \in W_{\alpha/2}\right\} \tag{4.6}$$

where V is defined in (2.4).

The robust version of the dual-mode receding horizon controller can now be stated. For all $t \ge 0$, let $x_R(t)$ denote the state of the real system (the system being controlled) at time t. For all i, let x_i denote the state of the real system at time $t_i \triangleq i\delta$, so that $x_i = x_R(i\delta)$. The motivation for the controller is, as before, to obtain, at each time t_i, an admissible control horizon pair (u_i', T_i') for $P(x_i, t_i)$ with minimal computation. Suppose, therefore, one has an admissible control horizon pair (u_{i-1}, T_{i-1}) for $P(x_{i-1}, t_{i-1})$, so that u_i steers the *model* from state x_{i-1}, along a trajectory $x^{u_{i-1}}$, to a point z_i on the boundary of $W_{\alpha/2}$ in time T_{i-1}. See Fig. 2. The

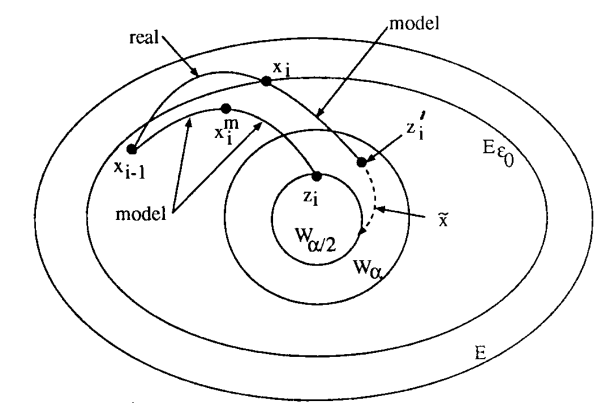

Fig. 2. Receding horizon trajectories—Robust case.

control u_{i-1} is applied to the *real* system over the interval $[t_{i-1}, t_i]$, yielding the state x_i at time t_i. In the absence of model error, x_i would be equal to x_i^m and lie on the trajectory $x^{u_{i-1}}$, and the control u_{i-1}, restricted to the interval $[t_i, t_{i-1} + T_{i-1}] = [t_i, t_i + T_i - \delta]$, would steer the model from $x_i = x_i^m$ to the point z_i on the boundary of W_α in a time $T_{i-1} - \delta$. In the presence of model error, however, x_i does not coincide with x_i^m, so the control u_{i-1}, restricted to the interval $[t_i, t_i + T_i - \delta]$ steers the *model* from x_i to a point z_i' which is not necessarily on the boundary of $W_{\alpha/2}$ (see Fig. 2). However, if α and β satisfy certain conditions, z_i' will lie in W_α. The local linear controller can then be used to generate a control $\tilde{u}$ which steers the model from z_i' to the boundary of $W_{\alpha/2}$ in a time not exceeding $\delta/2$. An admissible control horizon pair (u_i', T_i') for $P(x_i, t_i)$ can then be constructed by concatenating the restriction of u_{i-1} with the control $\tilde{u}$.

Robust Receding Horizon Controller

Data: $x_0 \in X_{\max}$, $\delta \in (0, \infty)$.

Initialization: At time $t_0 = 0$, if $x_0 \in W_\alpha$, switch to local linear control, i.e., employ the linear feedback control law h_L for all t such that $x_R(t) \in W_\alpha$. Else:

a) Calculate an admissible control horizon pair $(u_0, T_0) \in Z(x_0)$, $T_0 \le T_{\max}$ for problem $P(x_0, t_0)$.

b) Apply the control u_0 to the real system ($\dot{x} = f_R$) over the interval $[0, \delta']$ where $\delta' \triangleq \min\{\delta, T_0\}$, (so that $x_1 = x_R^{u_0}(t_1; x_0, t_0)$ if $T_0 \ge \delta$).

Controller:

1) If, at any time t, $x_R(t) \in W_\alpha$, switch to local linear control, i.e., employ the local linear controller h_L for all t such that $x_R(t) \in W_\alpha$. Else:

2) At any time t_i, $i \in \mathbb{N}$:

a) Obtain an admissible control horizon pair $(u_i', T_i') \in Z(x_i)$ as follows:

i) Compute

$$z_i' = x^{u_{i-1}}(t_i + T_{i-1} - \delta; x_i, t_i), \tag{4.7}$$

ii) Compute the least time T_i' which satisfies

$$\tilde{x}(t_i + T_i'; z_i', t_i + T_{i-1} - \delta) \in W_{\alpha/2}, \tag{4.8}$$

iii) Set

$$u_i'(s) = \begin{cases} u_{i-1}(s; x_i, t_i) & \text{when } s \in [t_i, t_i + T_{i-1} - \delta] \\ \tilde{u}(s; z_i', t_i + T_{i-1} - \delta) & \\ \qquad \text{when } s \in (t_i + T_{i-1} - \delta, t_i + T_i']. & \end{cases} \tag{4.9}$$

b) Obtain an improved control-horizon pair $(u_i, T_i) \in Z(x_i)$ satisfying

$$V(x_i, t_i, u_i, T_i) \le V(x_i, t_i, u_i', T_i'), \qquad T_i \le T_i'. \tag{4.10}$$

c) Apply the control u_i to the real system ($\dot{x} = f_R$) over the interval $[t_i, t_i + \delta']$ where $\delta' \triangleq \min\{\delta, T_i\}$ (so that $x_{i+1} = x^{u_i}(t_{i+1}; x_i, t_i)$ if $T_i \ge \delta$). □

Comment 2: Once $x_R(t)$ enters W_α, it remains there. We note again that $x_{i-1}, x_i \in W_\alpha^c$ if Step 2) is executed; this implies $T_{i-1} > \delta$. An admissible control horizon pair satisfying (4.10) can be obtained with almost no computation [see Step 2)-a)] in the controller description). If $X_{\max} \cap W_\alpha^c$ is empty, the linear controller suffices.

First of all, we have to show that the controller is well defined, i.e., that there exists a T_i' satisfying (4.8). This will be the case if $z_i' \in W_\alpha$. Second, to establish that the controller is stabilizing, we have to show that T_i' is suitably small (e.g., $T_i' \le T_i - \delta/2$), else the horizon time could increase without bound, preventing asymptotic convergence of the state to the origin. A sufficient condition for the controller to have these two properties is given in Lemma 4.

To motivate this condition, notice that the control u_{i-1} steers the model from state x_{i-1} at time t_{i-1} to state $z_i \in \delta W_{\alpha/2}$ at time $t_{i-1} + T_{i-1}$, where

$$z_i \triangleq x^{u_{i-1}}(t_{i-1} + T_{i-1}; x_{i-1}, t_{i-1}). \tag{4.11}$$

In the absence of model error, x_i lies on the trajectory $x^{u_{i-1}}$, restricted to the interval $[t_{i-1}, t_{i-1} + T_{i-1}]$, so that z_i', defined in (4.7), is equal to $z_i \in \delta W_{\alpha/2}$. When model error is present, z_i' is not necessarily equal to z_i; to quantify $z_i' - z_i$, we proceed as follows. We note that z_i is the terminal state of a trajectory of the *model* for which the control is u_{i-1} over the entire interval $[t_{i-1}, t_{i-1} + T_{i-1}]$. If $x_{i-1}, x_i \in W_\alpha^c$, then z_i' is the terminal state of a composite trajectory, for which the control is also u_{i-1} over the entire interval $[t_{i-1}, t_{i-1} + T_{i-1}]$; however, in this case, the control is applied to the real system over $[t_{i-1}, t_i]$ and to the model over $[t_i, t_{i-1} + T_{i-1}]$. Hence, the error $z_i' - z_i$ depends on both the sampling period δ and the model error which is quantified by β. To obtain a bound on this error we require the following Lemma whose proof can be found in [7].

Lemma 1: Suppose assumptions A0)–A3) are satisfied and that $f_R \in F_\beta$ for some $\beta \in (0, \infty)$. Let L be a Lipschitz constant for f and f_R. For any $\Delta \in (0, \infty)$ and any compact subset X_1 of $\mathbb{R}^n$, there exists a constant k (dependent on Δ and X_1) such that the following bounds hold:

$$\|x_R^u(t+\tau;x,t)-x\|_P \le k(\exp(L\tau)-1), \quad (4.12)$$

$$\|x^u(t+\tau;x,t)-x\|_P \le k(\exp(L\tau)-1), \quad (4.13)$$

$$\|x_R^u(t+\delta;x,t)-x^u(t+\delta;x,t)\|_P \le (\beta k/L)(\exp(L\delta)-1), \quad (4.14)$$

$$\|x^u(t+\tau;x_1,t)-x^u(t+\tau;x_2,t)\|_P \le \|x_1-x_2\|_P \exp(L\tau) \quad (4.15)$$

for all $\delta \in (0,\Delta]$, all $\tau \in [0,\infty)$ all $x, x_1, x_2 \in X_1$, and all $u \in S$. □

We can now obtain a bound on $\|z_i - z_i'\|$ where z_i and z_i' are defined in (4.11) and (4.7).

Lemma 2: Suppose assumptions A0)–A3) are satisfied and that $f_R \in F_\beta$ for some $\beta \in (0,\infty)$. Let L be a Lipschitz constant for f and f_R. If $x_{i-1} \in W_\alpha^c \cap X_{\max}$ and $(u_{i-1}, T_{i-1}) \in Z(x_{i-1})$, then:

$$d(z_i', W_{\alpha/2}) = \|z_i' - z_i\|_P \le \beta\phi(\delta),$$

where the metric d: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and the function ϕ: $\mathbb{R} \to \mathbb{R}$ are defined by $d(x,y) \triangleq \|x-y\|_P$, and

$$\phi(\delta) \triangleq (k/L)(\exp(L\delta)-1)\exp(LT_{\max}), \quad (4.16)$$

where k is defined in Lemma 1.

Proof: For all $i \in \mathbb{N}$, let x_i^m be defined as follows:

$$x_i^m \triangleq x^{u_{i-1}}(t_i; x_{i-1}, t_{i-1}). \quad (4.17)$$

Comparing the definition of x_i with that of x_i^m, we see that $x_i = x_i^m$ in the absence of model error. It follows from Lemma 1 that

$$\|x_i - x_i^m\|_P \le (\beta k/L)(\exp(L\delta)-1), \quad \text{and}$$

$$\|z_i - z_i'\|_P \le \|x_i - x_i^m\|_P \exp(L(T_{i-1}-\delta)) \le (\beta k/L)[\exp(L\delta)-1]\exp(LT_{\max}) = \beta\phi(\delta). \quad □$$

Lemma 3: Suppose $z \in W_\alpha$. Then $d(z, W_{\alpha/2}) = \gamma\alpha/2$ where $\gamma \in [0,1]$ and an upper bound on the time required by the linear control law h_L to steer the model from z to $W_{\alpha/2}$ is Δ_z where $\Delta_z \le \Delta^\gamma \le \gamma/\epsilon$ and $\Delta^\gamma \triangleq (1/\epsilon)\ln(1+\gamma)$.

Proof: Suppose $z \in W_{\alpha/2}^c$ (else $\Delta_z = 0$). The time Δ_z then satisfies $\alpha/2 \le V_L(z)\exp(-\epsilon\Delta_z) \le (1+\gamma)(\alpha/2)\exp(-\epsilon\Delta_z)$, which implies $1 \le \exp(\epsilon\Delta^\gamma)\exp(-\epsilon\Delta_z)$. Hence, $\Delta_z \le \Delta^\gamma \le \gamma/\epsilon$. □

In order to ensure that the dual-mode controller is stabilizing we impose the design condition as follows:

DC: $\Delta_{z_i'} \le \delta/2$ for all $i \in \mathbb{N}$.

We now obtain a bound on the pair (β, δ) which enables us to establish that the controller is stabilizing.

Lemma 4: Suppose $x_{i-1} \in X_{\max} \cap W_\alpha^c$.

A sufficient condition for z_i' to lie in W_α ($d(z_i', W_{\alpha/2}) \le \alpha/2$) and to satisfy the design condition $\Delta_{z_i'} \le \delta/2$ is that the model error parameter β and the sampling period δ satisfy $\beta\phi(\delta) \le \min\{\alpha\delta\epsilon/4, \alpha/2\}$.

This condition is satisfied if either:

i) $\delta \in (0, T_{\max}]$ and $\beta \le \beta_\delta$ where

$$\beta_\delta \triangleq \min\{\alpha\delta\epsilon/4, \alpha/2\}/\phi(\delta), \quad \text{or} \quad (4.18)$$

ii) $\beta \in (0, \beta_1)$ and $\delta \le \delta_\beta$, where

$$\beta_1 \triangleq \alpha\epsilon/[4k\exp(LT_{\max})],$$

$$\delta_\beta \triangleq \min\{\delta_\beta', \delta_\beta''\}, \quad (4.19)$$

and δ_β' and δ_β'' are the unique, positive solutions of $\beta\phi(\delta) = (\alpha\epsilon/4)\delta$ and $\beta\phi(\delta) = \alpha/2$, respectively.

Proof: Since $\beta\phi(\delta) \le \alpha/2$, it follows, from Lemma 2, that $d(z_i', W_{\alpha/2}) \le \alpha/2$ so that $z_i' \in W_\alpha$; defining γ implicitly by $d(z_i', W_{\alpha/2}) = \gamma(\alpha/2)$ yields $\gamma \le 1$. Similarly, since $\beta\phi(\delta) \le \alpha\delta\epsilon/4$, we obtain $d(z_i', W_{\alpha/2}) = \gamma(\alpha/2) \le \alpha\delta\epsilon/4$ which implies $\gamma \le \delta\epsilon/2$. Because $z_i' \in W_\alpha$ and $\gamma \le \delta\epsilon/2$, we deduce from Lemma 3 that $\Delta_{z_i'} \le \gamma/\epsilon \le \delta/2$.

i) Obvious.

ii) Suppose $\beta < \beta_1$ is fixed. The function ϕ is strictly convex and continuously differentiable. Since $(d/d\delta)\phi(0) = k\exp(LT_{\max})$, the equation $\beta\phi(\delta) = (\alpha\epsilon/4)\delta$ has a unique, positive solution δ_β' if, and only if $\beta k \exp(LT_{\max}) < \alpha\epsilon/4$, i.e., if and only if $\beta < \beta_1$, in which case $\beta\phi(\delta) \le (\alpha\epsilon/4)\delta$ for all $\delta \in [0, \delta_\beta']$. The equation $\beta\phi(\delta) = \alpha/2$ clearly has a unique positive solution δ_β''. Hence, $\beta\phi(\delta) \le \min\{\alpha\delta\epsilon/4, \alpha/2\}$ for all $\delta \in [0, \delta_\beta]$. □

We are now in a position to establish the robust stability of the receding horizon controller.

Theorem 3: Suppose assumptions A0)–A3) are satisfied, and that h_L is (robustly) stabilizing in W_α.

a) For any given sampling period $\delta \in (0, T_{\max}]$, there exists a $\beta_\delta > 0$ such that every closed-loop trajectory of the real system, whose model error parameter β satisfies $\beta \le \min\{\beta_\delta, \beta_L\}$ ($f_R \in F_\beta$), whose sampling period is δ, and whose initial state $x_0 \in X_{\max}$, remains in $X_{\max}$ and converges to the origin (the controller is asymptotically stabilizing with a region of attraction $X_{\max}$).

b) For any given model error $\beta < \min\{\beta_1, \beta_L\}$, there exists a $\delta_\beta > 0$ such that every closed-loop trajectory of the real system, whose model error parameter is β ($f_R \in F_\beta$), whose sampling period δ satisfies $\delta \le \delta_\beta$, and whose initial state $x_0 \in X_{\max}$, remains in $X_{\max}$ and converges to the origin (the controller is asymptotically stabilizing with a region of attraction $X_{\max}$).

Proof: Our first task is to prove that the controller is well defined, i.e., given $(u_{i-1}, T_{i-1}) \in Z(x_{i-1})$ where $x_{i-1} \in X_{\max} \cap W_\alpha^c$, then $x_i \in X_{\max}$ and, if $x_i \in W_\alpha^c$, there exists a $(u_i, T_i) \in Z(x_i)$ satisfying the conditions in Step b) of the controller. These conditions will be satisfied if, whenever $x_i \in W_\alpha^c$, there exists an admissible pair $(u_i', T_i') \in Z(x_i)$ (the controller is clearly well defined if $x_i \in W_\alpha$).

Suppose, then, that $x_{i-1} \in X_{\max} \cap W_\alpha^c$, $x_i \in W_\alpha^c$ and that $(u_{i-1}, T_{i-1}) \in Z(x_{i-1})$. Let β_δ and δ_β in assertions a) and b) of the theorem be defined by (4.18) and (4.19). It follows from Lemma 4 that, under either hypothesis of the theorem, $\Delta_{z_i'} \le \delta/2$. Hence, $T_i' \le T_{i-1} - \delta + \Delta_{z_i'} \le T_{i-1} - \delta/2$. It follows that u_i' steers the model from x_i to $W_{\alpha/2}$ in time $T_i' \le T_{i-1} \le T_{\max}$, so that $x_i \in X_{\max}$ and $(u_i', T_i') \in Z(x_i)$. Hence, the controller is well defined.

Suppose that the controller generates an infinite sequence $\{x_i\}$ in W_α^c. From the above discussion, this sequence will lie in $X_{\max}$. However, for all $i \in \mathbb{N}$ such that $x_{i-1}, x_i \in W_\alpha^c$ it follows from the discussion above that $T_i \le T_i' \le T_{i-1} - \delta/2$. Hence, there exists a finite $i_1 \in \mathbb{N}$ such that $T_{i_1} \le \delta$. This implies that $x_{i_1+1} \in W_\alpha$, a contradiction. It follows that the number of iterations in which

x_i lies outside W_α is finite so that there exists a finite integer $i_2 \leq 2T_0/\delta$ such that $x_{i_2} \in W_\alpha$. At this point, the controller switches to the linear control law h_L which steers x_{i_2} to the origin. □

V. State Constraints

Suppose now that the system being controlled is subject to the state constraint

$$x(t) \in E \qquad \text{for all } t \in [0, \infty) \tag{5.1}$$

where E is a closed subset of $\mathbb{R}^n$ defined by

$$E \triangleq \{x | g^j(x) \leq 0, j \in \boldsymbol{p}\}, \qquad \boldsymbol{p} \triangleq \{1, \cdots, p\} \tag{5.2}$$

and containing the origin in its interior. Consider first the case when there is no model error ($f = f_R$). Let X denote the set of initial states of the model which can be steered to W_α by a control u in S which satisfies both the control and state constraints, i.e., $X \subset E$. The finite horizon control problem is now defined by

$$\begin{aligned} P(x,t)\colon \min \{&V(x,t,u,T) | u \in S, T > 0, \\ &x^u(t+T; x,t) \in W_\alpha,\ x^u(s;x,t) \in E \text{ for all} \\ &s \in [t, t+T]\}. \end{aligned} \tag{5.3}$$

The set $Z(x)$ of admissible control-horizon pairs for the initial state x is now defined by

$$\begin{aligned} Z(x) \triangleq \{&(u,T) \in S \times (0,\infty) | x^u(t+T;x,t) \in \delta W_\alpha, \\ &x^u(s;x,t) \in E \quad \text{for all } s \in [t, t+T]\}. \end{aligned} \tag{5.4}$$

We assume that α, β_L and K are chosen so that the set $W \triangleq W_\alpha$, in addition to possessing the properties specified in Section III, is a subset of E ($W_\alpha \subset E$). Consequently, α must be chosen so that $\psi_j(x) \leq 0$, $j = 1,2,3$, for all $x \in W_\alpha$, where ψ_1 and ψ_2 are defined in (2.16) and $\psi_3(x) \triangleq d(x, E)$. Consequently, any trajectory of the model, the linearized system or the real system with $f_R \in F_{\beta_L}$, starting in W_α, remains in W_α and converges to the origin, simultaneously satisfying the state and control constraints. We will again refer to this property as "h_L is stabilizing in W_α". The following result is an obvious extension of Theorem 2.

Theorem 4: Suppose assumptions A0)–A3) are satisfied and h_L is stabilizing in W_α. In the absence of disturbances and model-system error, the dual-mode receding horizon controller, defined in Section III with P and Z defined as above, is asymptotically stabilizing with a region of attraction X and satisfies both the control and state constraints (2.2) and (5.1). □

Robustifying this controller is difficult due to the presence of the state constraint. For the case of control constraints only, we replace the stability constraint $x^u(t + T; x,t) \in W_\alpha$ by the more conservative constraint $x^u(t + T; x,t) \in W_{\alpha/2}$. We have to replace, in addition, the state constraint by a more conservative version. To this end, we define, for all $\epsilon > 0$, the set E_ϵ as follows:

$$E_\varepsilon \triangleq \{x | g^j(x) \leq -\epsilon, j \in \boldsymbol{p}\}. \tag{5.5}$$

We refer to E_ϵ as the ϵ state constraint. The definition of problem P is modified to incorporate the ϵ state constraint

$$\begin{aligned} P_\epsilon(x,t)\colon \min \{&V(x,t,u,T) | u \in, T \in [0, T_{\max}], \\ &x^u(s;x,t) \in E_\epsilon \text{ for all } s \in [t, t+T], \\ &x^u(t+T;x,t) \in W_{\alpha/2}\}. \end{aligned} \tag{5.6}$$

This definition involves an implicit controllability assumption which is discussed more fully in the sequel. The corresponding set of ϵ-admissible control-horizon pairs is defined by

$$\begin{aligned} Z_\epsilon(x) \triangleq \{&(u,T) | u \in S, T \in [0, T_{\max}], \\ &x^u(s;x,t) \in E_\epsilon \text{ for all } s \in [t, t+T], \\ &x^u(T;x,0) \in \delta W_{\alpha/2}\}. \end{aligned} \tag{5.7}$$

Suppose $T_{\max}$ and $\epsilon_0 > 0$ are given. Let $X_{\max}$ denote the set of initial states of the *model* which can be steered to the set $W_{\alpha/2}$ by a control in S which satisfies the ϵ_0 state constraint, i.e., $X_{\max} \subset E_{\epsilon_0}$. Since the trajectories of the real system differ from those of the model, it is necessary to employ E_{ϵ_0} in place of E in the finite horizon control problem in order to ensure that the trajectory of the real system remains in E. The phrase "h_L is stabilizing in W_α" will imply, in addition to the stabilizing properties specified in Section III, that W_α is a subset of E_{ϵ_0}, ($W_\alpha \subset E_{\epsilon_0}$) so that α and β_L are chosen to satisfy $\psi_1'(\beta_L, x) \leq 0$, $\psi_2(x) \leq 0$ and $\psi_3'(x) \leq 0$ for all $x \in W_\alpha$ where ψ_1' and ψ_2 are defined in (4.2) and (2.16) and

$$\psi_3'(x) \triangleq d(x, E_{\epsilon_0}). \tag{5.8}$$

We can now present a robust version of the receding horizon controller for control problems with state and control constraints. As before, $x_R(t)$ denotes the state of the real system (the system being controlled) at time t, and $x_i \triangleq x_R(t_i)$ for all $i \in \mathbb{N}$. The controller is similar to the previous one, having the same motivation, to reduce the computation required to obtain an admissible control horizon pair to a minimum. However, the controller has one new feature, a variable sampling period, which is introduced to obtain robustness in the presence of state constraints. Thus, if the trajectory of the real system leaves E_{ϵ_0}, the controller waits until it re-emerges, which it is guaranteed to do (provided δ and β satisfy certain conditions). See Fig. 2.

Robust Receding Horizon Controller for State and Control Constraints

Data: $x_0 \in X_{\max}$, $\Delta \in (0, \infty)$, $\epsilon_0 \in (0, \infty)$, $T_{\max} \in (0, \infty)$.

Initialization: At time $t_0 = 0$, if $x_0 \in W_\alpha$, switch to local linear control, i.e., employ the local feedback control law h_L for all t such that $x_R(t) \in W_\alpha$. Else:

a) Calculate an admissible control horizon pair $(u_0, T_0) \in Z_{\epsilon_0}(x_0)$ for problem $P_{\epsilon_0}(x_0, t_0)$.

b) Apply the control u_0 to the *real* system over the interval $[0, \delta']$ where $\delta' \triangleq \min\{\delta_0, T_0\}$, and δ_0 is the smallest $\delta \in [\Delta, T_0]$ such that $x_R^{u_0}(\delta; x_0, 0) \in E_{\epsilon_0}$.

Controller:

1) If, at any time t, $x_R(t) \in W_\alpha$, switch to local linear control, i.e., employ the linear feedback law h_L for all t such that $x_R(t) \in W_\alpha$. Else:

2) At any time t_i, $i \in \mathbb{N}$:

a) Obtain an admissible control horizon pair $(u_i', T_i') \in Z_{\epsilon_0}(x_i, t_i)$ as follows:

i) Compute

$$z_i' = x^{u_{i-1}}(t_i + T_{i-1} - \delta_{i-1}; x_i, t_i).$$

ii) Compute the least time T_i' which satisfies

$$\tilde{x}(t_i + T_i'; z_i', t_i + T_{i-1} - \delta_{i-1}) \in W_{\alpha/2}.$$

iii) Set

$$u_i'(s) = \begin{cases} u_{i-1}(s; x_i, t_i) \\ \qquad \text{when } s \in [t_i, t_i + T_{i-1} - \delta_{i-1}] \\ \tilde{u}(s; z_i', t_i + T_{i-1} - \delta_{i-1}) \\ \qquad \text{when } s \in (t_i + T_{i-1} - \delta_{i-1}, t_i + T_i']. \end{cases}$$

b) Obtain an improved control horizon pair $(u_i, T_i) \in Z_{\epsilon_0}(x_i, t_i)$ satisfying

$$V(x_i, t_i, u_i, T_i) \le V(x_i, t_i, u_i', T_i'), \qquad T_i \le T_i'.$$

c) Apply the control u_i to the real system ($\dot{x} = f_R$) over the interval $[0, \delta']$ where $\delta' = \min\{\delta_i, T_i\}$, and δ_i is the smallest $\delta \in [\Delta, T_i]$ satisfying $x_R^{u_i}(t_i + \delta; x_i, t_i) \in E_{\epsilon_0}$, so that $x_{i+1} = x_R^{u_i}(t_{i+1}; x_i, t_i)$ and $t_{i+1} = t_i + \delta_i$. □

The proof of the following result is similar to that of Theorem 3.

Theorem 5: Suppose assumptions A0)–A3) are satisfied, and that h_L is ϵ_0-stabilizing in W_α. Then, there exists a $\beta_2 > 0$ such that every closed-loop trajectory of the real system whose model error parameter β satisfies $\beta \le \beta_2$ and whose initial state $x_0 \in X_{\max}$ converges to the origin along a trajectory which satisfies both the control and state constraints, and satisfies $x_i \in X_{\max}$ for all $i \in \mathbb{N}$. □

Proof: Let $\beta_3 \le \beta_L$ be such that $x_R^u(t; x_0, 0) \in E$ for all $x_0 \in X_{\max}$, all $t \in [0, T]$, all $(u, T) \in Z_{\epsilon_0}(x, 0)$, and all $f_R \in F_{\beta_3}$; that β_3 exists follows from Lemma 1 [inequality (4.17)], the fact that $X_{\max} \subset E_{\epsilon_0}$ and the fact that $(u, T) \in Z_{\epsilon_0}(x, 0)$ ensures that $x^u(s; x, 0) \in E_{\epsilon_0}$ for all $s \in [0, T]$ and $T \le T_{\max}$. Let $\beta_2 \triangleq \min\{\beta_3, \beta_{\max}\}$, where $\beta_{\max} \triangleq \min\{\alpha \Delta \epsilon/4, \alpha/2\}/\phi(T_{\max})$ (Δ is the minimum sampling period employed in the receding horizon controller).

Suppose, that $\beta \le \beta_2$ (i.e., $f_R \in F_{\beta_2}$) and that, as in the proof of Theorem 3, $x_i \in X_{\max} \cap W_\alpha^c$. By the definition of $X_{\max}$, there exists a control u_i which steers the *model* from x_i to $z_i \in \delta W_{\alpha/2}$ along a trajectory $x^{u_i}(\cdot; x_i, t_i)$ which lies entirely in E. Since $\beta \le \beta_3$, it follows that the trajectory $x_R^{u_i}(\cdot; x_i, t_i)$ of the real system $\dot{x} = f_R$, restricted to the interval $[t_i, t_i + T_i]$, lies entirely in E. Since $\beta \le \beta_{\max}$, it follows that:

$$\beta\phi(\delta) \le \beta_{\max}\phi(T_{\max}) \le \min\{\alpha\Delta\epsilon/4, \alpha/2\} \le \min\{\alpha\delta\epsilon/4, \alpha/2\}$$

for all $\delta \in [\Delta, T_{\max}]$. Because $z_i = x^{u_i}(t_i + T_i; x_i, t_i) \in \delta W_{\alpha/2}$ where $T_i \le T_{\max}$, it follows, from Lemma 4, that $z_i' = x_R^{u_i}(t_i + T_i; x_i, t_i) \in W_\alpha$ and $\Delta_{z_i'} \le \delta/2$ for all $\delta \in [\Delta, T_{\max}]$. Because $x_R^{u_i}(t_i + T_i; x_i, t_i) \in W_\alpha$ and $W_\alpha \subset E_{\epsilon_0}$ (by assumption), there exists a $\delta_i \in [\Delta, T_i]$ such that $x_R^{u_i}(t_i + \delta_i; x_i, t_i) \in E_{\epsilon_0}$. Thus Step 2)-c) of the controller is well defined.

The rest of the proof follows as before. □

VI. Conclusion

We have presented, in this paper, dual-mode receding horizon controllers for nonlinear systems with state and control constraints, and have shown how these controllers may be robustified to cope with a degree of model error. The controllers presented here appear to have several advantages.

First if stability is to be ensured *a priori*, it has appeared necessary to incorporate a terminal equality constraint ($x(t + T) = 0$) in the on-line optimization problem. Solving a nonlinear optimal control problem with this constraint requires, in principle, an infinite number of iterations of the optimization algorithm. By incorporating dual-mode control, we are able to replace this constraint with a terminal inequality constraint and to replace exact by approximate minimization. A nonlinear optimal control problem of this type may, in principle, be solved in a finite number of iterations.

Second the dual mode feature permits the employment of variable horizon time T. This has two advantages. With fixed horizon time, the receding horizon controller faces solving optimal control problems which vary considerably as time proceeds. Consequently, the controller does not necessarily always have a feasible control (one that satisfies the control, state and stability constraints) available. Hence, as is well known, the algorithm may jam before finding a feasible control. In the algorithms presented in this paper, because variable horizons are employed, a feasible control for the current on-line optimization problem may always be constructed from the control employed at the previous iteration (obviously, a feasible control for the initial optimization problem will still have to be found). The controller merely concentrates on *improving* this feasible control. This removes the danger of jamming, which is ever present in nonlinear optimal control algorithms and reduces considerably the amount of online computation required. A second advantage is "operator transparency:" the trajectory followed by the controlled system is very close to the initial trajectory generated by the controller, differing only if there is improvement to be gained. This is not usually the case if fixed horizon times are employed.

Third it is relatively easy to robustify the dual-mode controller, preserving the features described above. We anticipate that the local linear controller, and the parameters α and ϵ are determined off-line. This presupposes that there are a finite number of set points to which it is desired to steer the system. If there is a manifold of equilibrium points, to which it is desired to steer the system, it is possible to envisage the local linear controller being replaced by a gain scheduled controller which takes over in a suitable neighborhood of this manifold. This neighborhood would replace the ellipsoid W_α employed in this paper. Again, the gain scheduled controller, and the neighborhood in which it is employed, would be determined off line. This would represent an interesting, and useful, extension to the work described here. If the local controller (local to a finite set or a manifold of equilibrium points) can be determined offline, the receding horizon controller would have the simple task of improving feasible controls, making it relatively easy to implement.

Other developments are possible. The local linear controller may be replaced by a local nonlinear controller designed, for example, using linearization techniques. The neighborhood W_α in which this controller is employed can be determined using the procedure described in [18]. Alternatively, an optimization based controller could also be employed in W_α which would then be chosen to ensure that the associated optimal control problem is convex in W_α; this would ensure that an admissible control could always be calculated in a finite number of iterations.

The parameter $T_{\max}$ should, of course, be larger than the maximum time required to steer the system to the origin from any state that will be encountered during operation of the plant. A rule of thumb for the parameter δ is that it should not exceed $T_{\max}/10$ or $T_{\max}/20$. For simplicity, we have not discussed integral control. This may be introduced in the usual way by incorporating additional integrators in the model of the system; the inputs to the integrators are set point deviations and the outputs are included in the state of the model. The receding horizon controller stabilizes the augmented system.

The control can be discretized in time, thereby reducing the optimal control problem to a finite dimensional nonlinear programming problem for which many algorithms exist. If the plant being controlled is linear, and the state and control constraints are affine inequalities, the on-line optimal control problem may be reduced to a quadratic program by discretizing the control in time and replacing W_α and $W_{\alpha/2}$ by inner simplicial approximations, i.e., by a finite set of affine inequalities.

References

[1] W. H. Kwon and A. E. Pearson, "A modified quadratic cost problem and feedback stabilization of a linear system," *IEEE Trans. Automat. Contr.*, vol. 22, pp. 838–842, 1977.

[2] W. H. Kwon, A. N. Bruckstein, and T. Kailath, "Stabilizing state-feedback design via the moving horizon method," *Int. J. Contr.*, vol. 37, 1983.

[3] S. S. Keerthi and E. G. Gilbert, "Optimal, infinite-horizon feedback laws for a general class of constrained discrete-time systems," *J. Optimiz. Theory Appl.*, vol. 57, pp. 265–293, 1988.

[4] D. Q. Mayne and H. Michalska, "Receding horizon control of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 814–824, 1990.

[5] H. Michalska and D. Q. Mayne, "Receding horizon control of nonlinear systems without differentiability of the optimal value function," *Systems Contr. Lett.*, vol. 16, pp. 123–130, 1991.

[6] H. Michalska, Ph.D. dissertation, University of London, England, 1989.

[7] H. Michalska and D. Q. Mayne, "On implementation and robustness of receding horizon control for nonlinear systems," *Tech. Rep.* IC/EE/CON/89/20, Dept. Electrical Eng. Imperial College of Science, Technology and Medicine, 1989.

[8] C. C. Chen and L. Shaw, "On receding horizon feedback control," *Automatica*, vol. 18, 1982, pp. 349–352.

[9] C. E. Garcia and M. Morari, "Model predictive control: theory and practice—A survey," *Automatica* vol. 25, no. 3, pp. 335–348, 1989.

[10] A. A. Patwadian, J. B. Rawlings, and T. F. Edgar, "Nonlinear model predictive control," *Chemical Eng. Commun.*, vol. 87, pp. 123–141, 1990.

[11] M. Agarwal and D. E. Seborg, "A multivariable nonlinear self-tuning controller," *AIChE J.*, vol. 33, p. 1397, 1987.

[12] R. R. Bitmead, M. Gevers, and V. Wertz, *Adaptive Optimal Control—The Thinking Man's GPC.* London: Prentice-Hall, Int., 1990.

[13] J. B. Rawlings and K. Muske, "The stability of constrained receding horizon control," *IEEE Trans. Automat. Contr.*, to appear.

[14] D. Q. Mayne and H. Michalska, "An implementable receding horizon controller for stabilization of nonlinear systems," in *Proc. 29th Conf. Decision Contr.*, Honolulu, HI, 1990, pp. 3396–3397.

[15] T. Yoshizawa, *Stability Theory by Lyapunov's Second Method.* Publications of the Mathematical Society of Japan, 1966.

[16] E. Polak, D. Q. Mayne, and D. Stimmler, "Control system design via semi-infinite optimization: A review," *IEEE Proc.*, Vol. 72, pp. 1777–1794, 1984.

[17] D. Q. Mayne, E. Polak, and R. Trahan, "An outer approximations algorithm for computer aided design problems," *J. Optimiz. Theory Appl.*, vol. 28, pp. 435–438, 1977.

[18] J. Hauser and M. C. Lai, "Estimating quadratic stability domains by nonsmooth optimization," in *Proc. Amer. Contr. Conf.*, Chicago, IL, pp. 571–576, 1992.

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