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A Variational Approach to Optimal Nonholonomic Motion Planning

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Abstract

Nonholonomic Motion Planning (NMP) problems arise not only from the classical nonholonomic constraints, but also from symmetries and conservation laws of holonomic systems. Common to NMP problems are that *an admissible configuration space path is constrained to a given nonholonomic distribution*. Thus, NMP deals with the problem of (optimal) path finding subject to a nonholonomic distribution and also possibly to additional holonomic constraints. In this paper we first study several representative NM systems and formulate the NMP problem. Then, we use variational principles to characterize optimal solutions to these problems. Finally, we propose a simple algorithm solving a NMP problem together with simulation results.

1 Introduction

An experienced driver can easily tell that parking a car on a crowded street in Manhattan, New York, is a very challenging task. A careful pre-planning is necessary, otherwise quite a few iterations are needed, through error-and-trial method, before the car can be brought into a desired configuration. Suppose that one wants to park a car to a spot on the right hand side where there are other cars on both ends, a typical working strategy is as follows: Pull the car to the front-left position; turn the front wheel to the right and then back up until it hits the rear car; turn the wheel to the left and then drive forward (This is called a Lie-bracket operation and as a result of this operation the car has been moved to the right hand side by a certain amount); repeat this process until the car is parked. Here in this problem, there are two control inputs: *steering and driving*, while there are four variables to be controlled: two variables describing the position of the car and two more variables describing the orientation of the car and of the front wheel. For safety reasons *it is important that sliding be prevented during the parking process.* In other words, in planning trajectories for such a system rolling constraint, which is classically known as nonholonomic constraint, should be respected. This is a typical application of *nonholonomic motion planning.*

When a cat is dropped at a upside down configuration even starting from rest, the cat is usually able to land on her feet by maneuvering her upper body relative to her lower body. However, we know from mechanics that the angular momentum of the cat is conserved because these are no external torques acting. Planning trajectories for such a system which respect some given conservation laws is another application of nonholonomic motion planning.

In this paper, we will examine several important examples of Nonholonomic Motion Planning (NMP): Steering a unicycle and a system of two contacting bodies, subject to rolling constraint; attitude control of a satellite using two orthogonally attached rotors and a falling-cat, both subject to conservation of angular momentum. Then, we will characterize optimal solutions of these problems and propose an algorithm together with simulation results.

Example 1 Consider the familiar example of riding a unicycle on a plane with rolling constraint. Let $(x, y) \in \Re^2$ be the coordinates of contact relative to the plane, $\theta \in S^1$ the coordinate of contact relative to the disk and $\phi \in S^1$ the angle of the disk relative to the x-axis. Then, a configuration of the system is described by $q = (x, y, \theta, \phi)$ and the configuration space is $Q = \Re^2 \times S^1 \times S^1$. Rolling constraint implies that ([LGF91]) the velocity vector along a configuration space path has

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to satisfy the following differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \triangleq b_1(q)u_1 + b_2(q)u_2. \tag{1}$$

where $u_1 = \dot{\theta}$ is the peddling velocity and $u_2 = \dot{\phi}$ the steering velocity, both to be treated as control inputs; $b_1(q)$ and $b_2(q)$ corresponds to, respectively, the direction of peddling and steering.

Problem 1 *Given two configurations $q_0, q_f \in Q$, find a set of control inputs $u(t) \in \Re^2, t \in [0,T]$, steering the system (1) from q_0 to q_f.*

Since the $u's$ represent peddling and steering effort, it is desirable to minimize the $u's$. A choice of a cost functional would be the L_2-norm of u, i.e.,

$$\|u\|_2 = \int_0^T (u_1^2 + u_2^2)^{1/2} dt. \tag{2}$$

Thus, we have the following *optimal control problem*

Problem 2 (**Optimal Control Problem**) *Given two configurations $q_0, q_f \in Q$, find a set of control inputs $u(t) \in \Re^2, t \in [0,T]$, which minimizes the cost (2) and steers the system (1) from q_0 to q_f.*

Example 2 Consider two arbitrarily shaped objects in contact. We wish to study how to steer the system from one contact configuration to another without slippage. Applications of this study include grinding tasks by a robot manipulator, path planning for a mobile robot and dextrous manipulation with a multifingered robotic hand ([Li89]).

Let $(q_1, q_2) \in \Re^2$ be the coordinates of contact relative to object 1, $(q_3, q_4) \in \Re^2$ the coordinates of contact relative to object 2, and $q_5 \in S^1$ the angle of contact. Then, a contact configuration is specified by $q = (q_1, q_2, q_3, q_4, q_5)^T$. According to [Mon86] and [Li89], rolling constraint is equivalent to that a configuration space path $q(t) \in Q, t \in [0,T]$, satisfies a system of differential equations. For example, when object 1 is a unit ball and object 2 a flat surface, the set of equations describing the system is given by ([LC90])

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \sec q_1 \\ -\sin q_5 \\ -\cos q_5 \\ -\tan q_1 \end{bmatrix} w_x + \begin{bmatrix} -1 \\ 0 \\ -\cos q_5 \\ \sin q_5 \\ 0 \end{bmatrix} w_y. \tag{3}$$

where w_x and w_y are components of rolling velocities and is to be thought of as control input.

When object 1 is a unit ball and object 2 a ball of radius ρ, the set of equations describing rolling constraint has the form

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \end{bmatrix} = \begin{bmatrix} 0 \\ (1-\beta)\sec q_1 \\ -\beta \sin q_5 \\ -\beta \cos q_5 \sec q_3 \\ \beta \tan q_3 \cos q_5 - (1-\beta)\tan q_1 \end{bmatrix} w_x + \begin{bmatrix} -(1-\beta) \\ 0 \\ -\beta\cos q_5 \\ \beta \sin q_5 \sec q_3 \\ -\beta \tan q_3 \sin q_5 \end{bmatrix} w_y, \tag{4}$$

where $\beta = \frac{1}{1+\rho}$.

Let us define the length, or distance, of an admissible path, $q(t) \in Q, t \in [0,T]$, to be

$$d(q) = \int_0^T \langle w,\ w \rangle dt \tag{5}$$

Problem 3 *Consider the system given by (3) or (4). Let $q_0, q_f \in Q$ be two given contact configurations. Find control $w(t) \in \Re^2, t \in [0,T]$, of optimal cost, linking q_0 to q_f.*

Example 3 Attitude control of a spacecraft such as a satellite can be achieved by using two orthogonally attached rotors (or momentum exchange wheels).

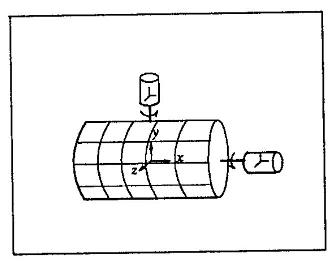

Figure 1: Attitude control of a satellite using two rotors.

Consider the system shown in Figure 1. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ be the Cayley parameters parameterizing the orientation of the satellite ([LGF91]). Then, conservation of angular momentum implies that the rate of change of α satisfies the following differential equations

$$\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\alpha}_3 \end{bmatrix} = \begin{bmatrix} (\alpha_1\alpha_2 - \alpha_3)\delta_1/2 \\ (1+\alpha_2^2)\delta_1/2 \\ (\alpha_1 + \alpha_2\alpha_3)\delta_1/2 \end{bmatrix} u_1 + \begin{bmatrix} (1+\alpha_1^2)\delta_2/2 \\ (\alpha_1\alpha_2 + \alpha_3)\delta_2/2 \\ (\alpha_1\alpha_3 - \alpha_2)\delta_2/2 \end{bmatrix} u_2 \quad (6)$$

where δ_1, δ_2 are constants and u_1, u_2 are rotor velocities thought of as control inputs.

Problem 4 *(Optimal Control Problem) Consider the system (6) with two given orientations $\alpha_0, \alpha_f \in SO(3)$, find control $u(t) \in \Re^2, t \in [0,T]$, of optimal cost, linking α_0 to α_f.*

It is interesting to note that attitude control of a falling cat can also be formulated as a problem similar to what we have discussed. See [LGF91] and [Mon89] for further details.

2 Optimal Trajectories

Problem 5 *Consider a NMP system given by (7)*

$$\dot{x} = B(x)u, \quad x_0, x_f \in \Re^n \quad (7)$$

where $x \in \Re^n$ are the configuration variables, $u \in \Re^m (m < n)$ the control inputs, and x_0, x_f the initial and final configurations. Find a set of control inputs $u(t) \in \Re^m, t \in [0,T]$, which links x_0 to x_f and minimizes the following objective function $\int_0^T \langle u, u\rangle dt$.

We assume that System (7) is controllable ([Bro76], [LC90]) and $B(x)$ has full rank so that the problem is solvable.

Theorem 1 *Define a Hamiltonian function $H(x,p)$ by*

$$H(x,p) = -\frac{1}{2}\langle p,\ B(x)B(x)^T p\rangle \quad (8)$$

where $p \in \Re^n$ is the conjugate momenta, and consider solution $(x(t), p(t)) \in \Re^{2n}$ of the Hamilton's equations $\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$ with initial condition p_0. Then the optimal control is given by $u = -B(x)^T p$ for some choice of p_0.

Several examples admitting analytic solutions of the Hamilton's equations are given in ([LGF91], [Bro81] and [MS90]).

Algorithm 2.1 *(Shooting Method)*

Input: $B(x) \in \Re^{n\times m}$, $x_0, x_f \in \Re^n$.

Step 1: *Formulate the optimal control Hamiltonian $H(x,p) = -\frac{1}{2}\langle p,\ B(x)B(x)^T p\rangle$.*

Step 2: *Pick a $p_0 \in \Re^n$ and solve the Hamilton's equation for T units of time, with initial condition (x_0, p_0). Denote the solution by $(x(t), p(t)), t \in [0,T]$.*

Step 3: *If $x(T) = x_f$, stop and set the optimal control to be $u = -B(x)^T p(t)$, otherwise, repeat step 2.*

In general there is no guarantee that a shooting algorithm will converge. In order for it to converge, a gradient vector pointing toward the final configuration needs to be incorporated.

Another formulation of an optimal NMP problem is given in ([Mon89]).

Problem 6 *Given $x_0, x_f \in \Re^n$, find a path $x(t) \in \Re^n, t \in [0,T]$, which links x_0 to x_f, satisfies the constraint*

$$A(x)\dot{x} = 0,$$

where $A(x) \in \Re^{(n-m)\times n}$ has full rank, and minimizes the cost

$$J = \int_0^T \frac{1}{2}\langle Q(x)\dot{x},\ \dot{x}\rangle dt \quad (9)$$

for $Q(x) \in \Re^{n\times n}$ a positive definite matrix.

An example of a choice of $Q(x)$ would be the inertia matrix of a falling-cat and thus (9) measures the total kinetic energy.

Theorem 2 *A trajectory $x(t) \in \Re^n, t \in [0,T]$, is an optimal solution of Problem 6 if and only if it satisfies the Hamilton's equation for a Hamiltonian function given by*

$$H(x,p) = \frac{1}{2}\langle p,\ Q^{-1}p\rangle - \frac{1}{2}\langle p,\ Q^{-1}A^*(AQ^{-1}A^*)^{-1}AQ^{-1}p\rangle. \quad (10)$$

The proof of the theorem can be found in ([Mon89], [AKN85] and [LGF91]).

We can now establish the equivalence of the *optimal control problem* with the *optimal path problem* by showing that there exist suitable choices of a positive definite matrix, $Q(x) \in \Re^{n\times n}$, and a full rank matrix, $A(x) \in \Re^{(n-m)\times n)}$, such that the *optimal path problem* becomes the *optimal control problem*.

First, because the system $\dot{x} = B(x)u$ is controllable, there exists a $n \times (n-m)$ matrix $D(x), x \in \Re^n$, such that the $n \times n$ matrix $[B(x), D(x)] \triangleq C(x)$ is nonsingular for all x. For example, we may choose for $D(x)$ the Lie-brackets of the columns of $B(x)$, i.e.,

$$D(x) = [...[b_i, b_j], ...[b_i, [b_j, b_k]], ...].$$

The matrix $C(x) \in \Re^{n\times n}$ is called a completion of $B(x)$.

Second, a curve $x(t) \in \Re^n, t \in [0,T]$, is admissible if and only if $\dot{x} \in Im(B(x))$, where $Im(B)$ denotes the image of the map $B : \Re^m \to \Re^n$. Define

$$A(x) = [0, \; I] C^{-1}, \text{ and } Q = (CC^*)^{-1}, \qquad (11)$$

where $I \in \Re^{(n-m)\times(n-m)}$ is the identity matrix.

We claim that $Ker(A) = Im(B)$ and $\langle Q\dot{x}, \dot{x}\rangle = \langle (B^\sharp)^* B^\sharp \dot{x}, \; \dot{x}\rangle$, where $B^\sharp = (B^T B)^{-1} B^T$ is the generalized inverse of B. To see this, note that if $y = Bu$, then

$$Ay = [0, I]C^{-1}y = [0, I]C^{-1}C \begin{bmatrix} u \\ 0 \end{bmatrix} = 0.$$

On the other hand, we have that $Ker(A) = C(\Re^m \oplus 0)$, which by construction is $Im(B)$. Finally, it follows from $C^{-1}y = \begin{bmatrix} u \\ 0 \end{bmatrix}$ that

$$\frac{1}{2}\langle Qy, \; y\rangle = \frac{1}{2}\langle C^{-1}y, C^{-1}y\rangle = \frac{1}{2}\langle (B^\sharp)^* B^\sharp y, y\rangle.$$

Theorem 3 *Let Q and A be given by (11). Then, the following equality holds*

$$\frac{1}{2}\langle p, Q^{-1}p\rangle - \frac{1}{2}\langle p, Q^{-1}A^*(AQ^{-1}A^*)^{-1}AQ^{-1}p\rangle = \frac{1}{2}\langle p, BB^*p\rangle. \qquad (12)$$

In other words, optimal trajectories given by solutions of the optimal control problem *are the same as that of the* optimal path problem.

The proof can be found in ([LGF91]). In fact, Theorem 3 holds for any positive definite $Q(x) \in \Re^{n\times n}$ and $A(x) \in \Re^{(n-m)\times n}$ such that $Ker(A) = Im(B)$ and

$$\langle Q\dot{x}, \; \dot{x}\rangle = \langle (B^\sharp)^* B^\sharp \dot{x}, \dot{x}\rangle.$$

3 A Basis Algorithm for Near-optimal Solutions

In this section we study near-optimal solutions of a NMP system of the form

$$\dot{x} = B(x)u, \;\; x_0, x_f \in \Re^n \qquad (13)$$

with cost functional

$$J = \int_0^T \frac{1}{2}\langle u, \; u\rangle dt.$$

Rescale time so that $T = 2\pi$ and we assume that the system is controllable so that the problem is solvable, with an optimal solution denoted by $u^*(t) \in L_2([0, 2\pi])$, where $L_2([0, 2\pi])$, or L_2 for simplicity, is the Hilbert space of measurable vector functions.

Let $\{\mathbf{e}_i(t)\}_{i=1}^\infty$ be an orthonormal basis for $L_2([0, 2\pi])$, e.g., the Fourier basis or the basis obtained from the polynominal basis using the Gram-Schmidt orthonormalization process. Then, any $u \in L_2([0, 2\pi])$ can be expressed in terms of the basis as

$$u = \sum_{i=0}^{\infty} \alpha_i \mathbf{e}_i$$

where $\alpha_i \in \Re$, the cost functional becomes

$$J = \frac{1}{2}\sum_{i=1}^{\infty} \alpha_i^2$$

and the problem is an optimization in an ∞-dimensional space. It is of course understood that searching an optimal solution in an ∞-dimensional space is not easy especially when the cost functional is not convex. Instead we will restrict our choice of control inputs to a finite dimensional space spanned by the first N basis elements, $\{\mathbf{e}_i\}_{i=1}^N$,

$$u = \sum_{i=1}^{N} \alpha_i \mathbf{e}_i \qquad (14)$$

and consider the following cost functional

$$J = \frac{1}{2}\sum_{i=1}^{N} \alpha_i^2 + \gamma \|x(2\pi) - x_f\|^2 \qquad (15)$$

where $\gamma \geq 0$ is a penalty function on the terminal conditions. Let us call the original system P and the new system with control given by (14) and cost functional by (15) $P_{N,\gamma}$, then it is shown in [LGF91]) that, under mild assumptions solutions of $P_{N,\gamma}$ converge to solution of P in a well-defined sense as N and γ go to infinity.

On the other hand, solutions of the new system $P_{N,\gamma}$ can be found using Newton's or modified Newton's algorithms. We have performed simulation studies based on this *Basis Algorithm* using the rolling disk example, the satellite example and the falling-cat example. Starting from initial condition $\alpha = 0$ the algorithm quickly converges to the optimal solution $\alpha^* \in \Re^N$ of $P_{N,\gamma}$. See [LGF91] for further details of this work.

4 Simulation Results

Example 4 The first example we used to test the *Basis Algorithm* is the rolling disk system studied in Section 1 (the θ coordinate has been eliminated because of symmetry). The initial and final configurations are

$$q_0 = \begin{bmatrix} -4 \\ -4 \\ 0 \end{bmatrix}, \quad q_f = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

and the first six basis elements were used

$$\mathbf{e}_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} \cos t \\ 0 \end{bmatrix}$$

and

$$\mathbf{e}_4 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \mathbf{e}_5 = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}, \mathbf{e}_6 = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

Thus, optimization is done in $\Re^6$. With initial condition $\alpha_0 = (0,0,0,0,0,0)^T$ Figure 2 shows plots of the optimal control inputs and Figure 3 shows plots of the optimal trajectories linking q_0 to q_f.

Example 5 The second example we used to test the *Basis Algorithm* is the satellite system with two rotors. We choose Cayley parameters $a \in \Re^3$ to parameterize the orientation of the satellite and the initial and final configurations are

$$a_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, a_f = \begin{bmatrix} 0 \\ 0 \\ 0.99924 \end{bmatrix}$$

In other words, the final configuration is a rotation from the initial configuration about the third axis without a rotor. The same basis elements as in the rolling disk example were used and plots of the optimal control and the optimal trajectories are given in Figure 4 and 5.

Example 6 The most complicated example we studied is a falling cat, modeled as two rigid bodies coupled by a ball-in-socket joint. There are three control inputs and the configuration space has dimension 6. We used Cayley parameters $a \in \Re^3$ and $b \in \Re^3$ to parameterize the orientations of the upper body and the lower body. The initial and final configurations in terms of the Cayley parameters are

$$q_0 = \begin{bmatrix} 0 - .298446 \\ 0.546302 \\ 0.546302 \\ 1 \\ -.546302 \\ -1.83 - 49 \end{bmatrix} \quad q_f = \begin{bmatrix} -5.50899 \\ -3.00959 \\ -1.83049 \\ 1.64414 \\ -3.00957 \\ 0.546302 \end{bmatrix}.$$

If we draw the falling-cat on a plane, then q_f is a rotation from q_0 about a horizontal axis by 180^0 (upside down configuration to landing configuration). We have used 21 basis elements in the simulation. These are

$$\mathbf{e}_1 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} \sin t \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} \cos t \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} \sin 2t \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{e}_5 = \begin{bmatrix} \cos 2t \\ 0 \\ 0 \end{bmatrix} \mathbf{e}_6 = \begin{bmatrix} \sin 3t \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_7 = \begin{bmatrix} \cos 3t \\ 0 \\ 0 \end{bmatrix}$$

and the remaining basis elements are obtained by permuting the rows of the above elements. Plots of the optimal control and optimal trajectories are shown in Figure 6 and 7.

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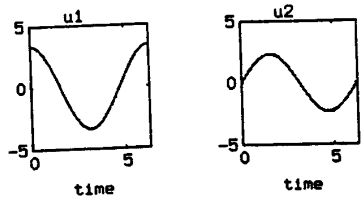

Figure 2: Optimal Control For Unicycle.

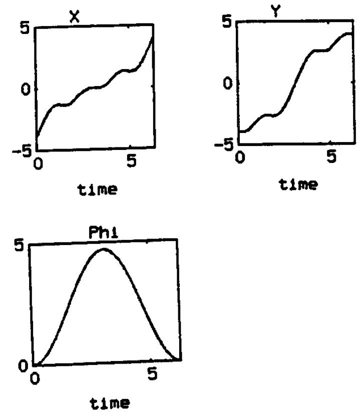

Figure 3: Optimal Trajectory For Unicycle.

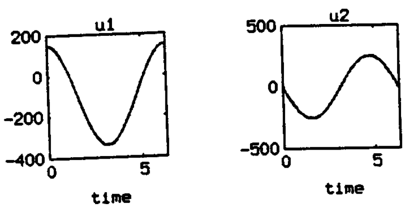

Figure 4: Optimal Control For Satellite.

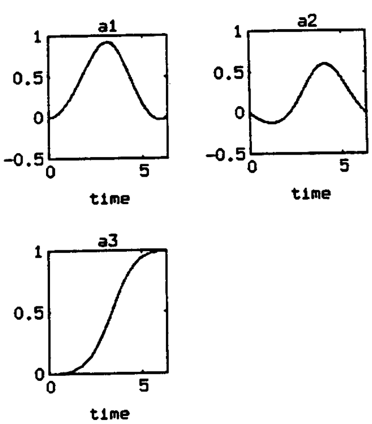

Figure 5: Optimal Trajectory For Satellite.

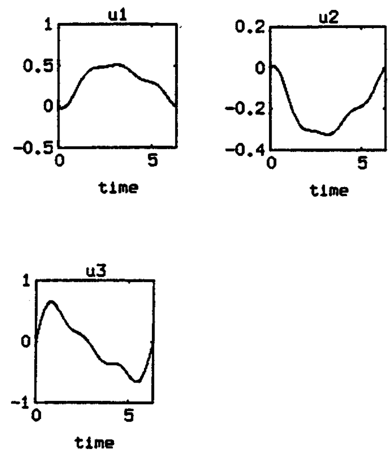

Figure 6: Optimal Control For Falling Cat.

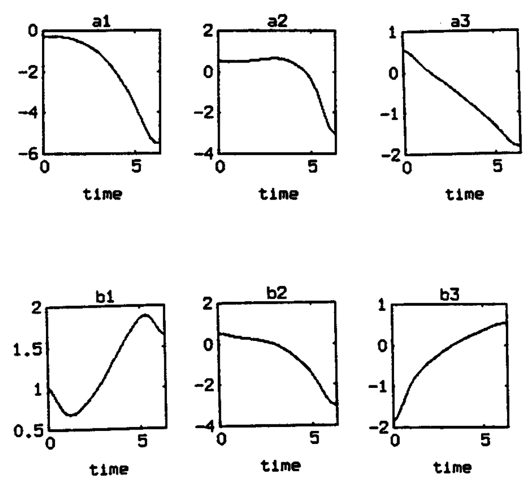

Figure 7: Optimal Trajectory For Falling Cat.