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Dynamic Feedback Stabilization of Nonholonomic Systems [1]

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Abstract

Stabilization of nonholonomic systems by time-varying control laws is considered here to be a special case of time-invariant dynamic feedback. The paper presents dynamic feedback laws for stabilization of the class of drift-free controllable systems, where the first derived algebra of control vector fields span the tangent space.

Key words: Nonlinear control, nonholonomic systems, stabilization, dynamic feedback, controllability.

1 Introduction

In this paper, we present a viewpoint for finding stabilizing control laws for a class of nonlinear systems which cannot be stabilized by continuous static feedback. As a prototype in this class, consider the following system, often referred to as the nonholonomic integrator,

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 - x_2u_1,\end{aligned}$$

where $x_1, x_2, x_3, u_1, u_2 \in \mathbb{R}$. The stabilization of above system is important because it models various physical situations [7].

As shown in [3], there exists no continuous feedback law that asymptotically stabilizes the nonholonomic integrator. The topological obstruction to stabilization by a static feedback is expressible in terms of the degree of a mapping [6, 3].

A number of approaches have been proposed for the stabilization of nonholonomic systems. These approaches can be broadly classified into three categories. Stabilization by discontinuous time-invariant control laws, time-varying stabilization, and hybrid stabilization. The results given here are constructive, and intended to bring out the topological issues more clearly. Our point of view is motivated by the following consideration.

Question 1 Consider a control system $\dot{x} = f(x,u)$ $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, which cannot be stabilized by a static continuous feedback $u : \mathbb{R}^n \to \mathbb{R}^m$. Is it possible to associate with it a higher dimensional manifold $\mathbb{R}^n \times N$ and find smooth static feedback laws $u(x,\theta)$ *and* $g(x,\theta)$, $(x,\theta) \in \mathbb{R}^n \times N$ such that, for any initial condition $(x(0), \theta(0))$, the solution of

$$\begin{aligned}\dot{x} &= f(x, u(x,\theta)) &(1)\\ \dot{\theta} &= g(x,\theta), &(2)\end{aligned}$$

satisfies $\lim_{t\to\infty}(x(t),\theta(t)) \in 0 \times N$?

In finding such a system, we would achieve the goal of stabilizing the original system $\dot{x} = f(x,u)$. Thus, the problem of stabilization to a point in the original space has been transformed to the problem of stabilization to a submanifold in the enlarged state space.

Remark 1 Stabilization of nonholonomic systems by periodic time-varying feedback is a special case of the above described situation, where the dynamical system $\dot{x} = f(x, u(x,t))$, $x \in M$, a differentiable manifold, can be seen as embedded in the larger space $M \times S^1$, with

$$\dot{x} = f(x, u(x,\theta))$$

$$\dot{\theta} = 1,$$

where $\theta \in S^1$, and stabilization of x to some point x_0 in the manifold M can be thought of as stabilization of (x,θ) to the submanifold $x_0 \times S^1$ in the enlarged state space $M \times S^1$.

As shown by Coron [4], it is always possible to stabilize a driftless, controllable system by a smooth periodic time-varying control law. However, periodic time-varying control laws make two special choices in the above raised Question 1, by choosing $N = S^1$ and $g(x,\theta) = 1$. We will show that relaxing these two constraints, helps us construct simpler feedback laws for stabilization of nonholonomic systems.

This paper presents a constructive solution to the problem of feedback stabilization of the class of drift-free controllable systems where the first derived algebra of control vector fields spans the tangent space of the state space at every point (if E^0 is a subbundle of the

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tangent bundle spanned by the control fields, then the first derived algebra is given by $E^1 = E^0 + [E^0,\ E^0])$. The feedback law is derived by first solving the problem of stabilization of a generalization of the nonholonomic integrator, called the general position-area system [2]. The system is described by the equations

$$\begin{aligned} \dot{x} &= u & (3) \\ \dot{y} &= xu^T - ux^T, & (4) \end{aligned}$$

where $x,\ u$ are column vectors in $\mathbb{R}^n$ and $y \in so(n),\ n \geq 2$.

The importance this system is that it is the standard form of the class of driftless controllable systems of the form $\dot{z} = B(z)u,\ u \in \mathbb{R}^n,\ z \in \mathbb{R}^{\frac{n(n+1)}{2}}$, whose first derived algebra spans the tangent space $T\mathbb{R}^{\frac{n(n+1)}{2}}$ at any point. In [2], it was shown that a large class of systems can be transformed to the form of (3)-(4) up to a suitable order in the neighborhood of a given point such as the origin. Bloch et al. have studied the discontinuous stabilization for the position-area system [1].

The paper is organized as follows. To fix ideas, we first present the dynamic stabilization for the nonholonomic integrator. We then analyze the general position-area system in detail and present smooth feedback stabilization laws for it. Finally, we extend the stabilization results to the class of drift-free controllable systems whose first derived algebra spans the tangent space.

2 Stabilization of the nonholonomic integrator

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. We will use $O(n)$ to denote the group of n-dimensional orthogonal matrices and $so(n)$ the Lie algebra of $n \times n$ skew symmetric matrices: $y^T = -y$. The norm of a $n \times n$ real matrix A is defined as $||A|| = \sqrt{tr(AA^T)}$. If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Given the vector differential equation $\dot{x} = f(x,t)$, we say $f \in E$, if f is continuous and satisfies the Lipschitz condition.

Definition 1 Let $x \in \mathbb{R}^n$ and $\theta \in M$ a compact differentiable manifold. Following Zubov [8] we define the submanifold $N = 0 \times M$ to be a asymptotically stable submanifold of the differential equations

$$\begin{aligned} \dot{x} &= f(x,\theta,t) \quad (f \in E) \\ \dot{\theta} &= g(x,\theta,t) \quad (g \in E) \end{aligned}$$

if there exists for each $\epsilon > 0$ a number $\delta > 0$, such that the inequality $||x(0)|| < \delta$ implies that the solution $(x(t), \theta(t))$ corresponding to the initial condition $(x(0), \theta(0))$, satisfies $||x(t)|| < \delta,\ \forall t > 0$ and if there is a number $\delta_0 > 0$, such that for $||x(0)|| < \delta_0$,

$$\lim_{t\to\infty} x(t) = 0 \qquad (5)$$

If equation (5) is satisfied for all $x(0)$ we call the submanifold N to be asymptotically stable in the large.

We will motivate our stabilization approach by the following discussion. Consider the nonholonomic integrator

$$\begin{aligned} \dot{x}_1 &= u_1 & (6) \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1u_2 - x_2u_1. \end{aligned}$$

Observe that, $u_1 = -x_1$ and $u_2 = -x_2$, exponentially stabilizes x_1 and x_2. Motion in the x_3 direction is produced by generating areas in the $x_1 - x_2$ plane. Is there a natural way to stabilize x_3 ? Let us add another dimension to the above system by introducing the variable $\theta \in \mathbb{R}$. In the (x_1, x_2, x_3, θ) space consider a one-parameter family of embedded submanifolds $\{S_r\},\ r \geq 0$, defined by $S_r = \{(x_1, x_2, x_3, \theta) \in \mathbb{R}^4 : x_1 = r\ \cos\theta,\ x_2 = r\ \sin\theta\}$. Observe that if $(x_1, x_2, x_3, \theta) \in S_q,\ q > 0$, then $\dot{x}_3(t) = x_1\dot{x}_2 - x_2\dot{x}_1 = q^2\ \dot{\theta}(t)$. If we let $\dot{\theta}(t) = -x_3$, then $\dot{x}_3(t) = -q^2\ x_3$ and hence x_3 is exponentially stabilized. Thus, in (x_1, x_2, x_3, θ) space, we have identified a structure that helps us establish stability.

Our control strategy will be to design u_1 and u_2 to bring (x_1, x_2, x_3, θ) to the submanifold $S_{q(t)}$ and let $\dot{\theta} = -x_3$ to drive x_3 to zero. We make $q(t)$ go to zero as x_3 approaches zero, thereby bringing all x_1, x_2, x_3 to zero.

Theorem 1 Let $\theta\ \in \mathbb{R}$, and $(x_1, x_2, x_3) \in \mathbb{R}^3$. If $u_1 = -x_1 + x_3\cos\theta$ and $u_2 = -x_2 - x_3\sin\theta$ then, for

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1u_2 - x_2u_1 \\ \dot{\theta} &= x_3, \end{aligned}$$

the submanifold $N = \{\ (x_1, x_2, x_3, \theta) \in \mathbb{R}^4 : (x_1, x_2, x_3) = 0\}$ is asymptotically stable.

Proof: Introduce

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad (7)$$

In terms of (y_1, y_2, y_3), the above equations take the form

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -1 & -y_3 & 1 \\ y_3 & -1 & 0 \\ 0 & 0 & -y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (8)$$

$$\dot{\theta} = y_3. \qquad (9)$$

Observe that the equations for (y_1, y_2, y_3) do not depend on θ, so they can be treated as an autonomous system of equations in $\mathbb{R}^3$. Define the Lyapunov function

$$V(y_1, y_2, y_3) = (y_1 - y_3)^2 + y_2^2 + y_3^2.$$

From equation (8),

$$\dot{V}(y_1, y_2, y_3) = -2((y_1 - y_3)^2 + y_2^2 + y_2 y_3 (y_1 - y_3)). \quad (10)$$

Notice that if $\|y_3\| < 1$, then $\dot{V}(y_1, y_2, y_3) \leq 0$. Let

$$B = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : V(y_1, y_2, y_3) < 1\}.$$

Observe that $B \in \{\|y_3\| < 1\}$, therefore is a positively invariant set. Let

$$S = \{(y_1, y_2, y_3) \in B : \dot{V}(y_1, y_2, y_3) = 0\}.$$

From equation (10), if $(y_1, y_2, y_3) \in S$, then $y_1 = y_3$ *and* $y_2 = 0$. Substituting this in equation (8), we conclude that if $(y_1, y_2, y_3) \in S$, then $\dot{y}_1 = 0$, $\dot{y}_2 = y_3^2$, and $\dot{y}_3 = 0$. It follows therefore that the largest invariant set contained in S is $\{0\}$. Hence, by LaSalle's stability theorem (recall y satisfies an autonomous system), if $(y_1(0), y_2(0), y_3(0)) \in B$, then $\lim_{t\to\infty}(y_1(t), y_2(t), y_3(t)) = 0$.

Let $\Omega(\epsilon) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 < \epsilon\}$, $\epsilon > 0$. By equation (7), we have $y_1^2(t) + y_2^2(t) = x_1^2(t) + x_2^2(t)$, $y_3(t) = x_3(t)$, from which we can deduce that if $(x_1, x_2, x_3) \in \Omega(\frac{1}{3})$, then $(y_1, y_2, y_3) \in B$. This shows that if $(x_1(0), x_2(0), x_3(0)) \in \Omega(\frac{1}{3})$, then $\lim_{t\to\infty} (x_1(t), x_2(t), x_3(t)) = (0, 0, 0)$ **Q.E.D.**

We can now extend this viewpoint to find stabilizing control laws for the general position-area system.

3 General position-area system

Recall that the system is described by the following set of equations,

$$\dot{x} = u \quad (11)$$
$$\dot{y} = xu^T - ux^T, \quad (12)$$

where x and u are column vectors in $\mathbb{R}^n$ and $y \in so(n)$, $n \geq 2$. We will find smooth stabilizing control laws for the above system by embedding it in $\mathbb{R}^{n+1} \times so(n) \times O(n)$. We motivate the choice of $O(n)$ by the following discussion. To understand how the general problem has an additional level of complexity with respect to the special case of $n = 2$, we start by looking at the qualitative nature of the trajectories that need to be generated in order to stabilize y. For $n = 2$, we saw that motion in x_3 direction was produced by generating areas in the $x_1 - x_2$ plane. Now, we need to generate $n(n-1)/2$ areas ($dy_{ij} = x_i dx_j - x_j dx_i$) for stabilizing $y \in so(n)$. This can be achieved as follows. Let $e \in \mathbb{R}^n$ be a unit vector which evolves as $\dot{e}(t) = ye$. Suppose we can make $x(t) = qe(t)$ where q is a positive constant, then the norm of $y(t)$ from equation (12) evolves as

$$\frac{d\ tr(yy^T)}{dt} = -2\ q^2\ \|ye\|^2,$$

which suggests that norm of y will decrease until e begins to lie in the null space of y. Essentially, what is happening is that the vector e is generating areas in $\mathbb{R}^n$ to stabilize a subspace of yy^T. To stabilize all n orthogonal subspaces of yy^T, we will evolve n orthonormal vectors in $\mathbb{R}^n$, each of which will generate areas to stabilizes a subspace of yy^T. This is naturally achieved by introducing $\Theta \in O(n)$ such that $\dot{\Theta} = y\Theta$. The columns of Θ then form the desired orthonormal frame. We arrange matters so that we can switch between these orthonormal vectors in a smooth way. This is done using a *Selector Function* introduced in the following definition. Now by letting q gradually go to zero as y goes to zero, we can stabilize both x and y.

Definition 2 Selector Function: Let $e(t) = (e_1(t),\ e_2(t),\ \cdots,\ e_n(t)) \in \mathbb{R}^n$ be a C^1 function of time. We will call $e(t)$ a selector function of period T and strength $\epsilon > 0$, if it satisfies the following properties

- $e(t) = e(t+T)$ such that $\|e(t)\| \leq 1$ and $\|\dot{e}(t)\|$ is bounded,
- $\int_t^{t+T} \|e_i(\tau)\|\ d\tau \geq \epsilon, \forall\, t$, and $e_i(t) \cdot e_j(t) = 0$ if $i \neq j$, $i, j \in 1, \ldots, n$.

We then say that $e \in \mathcal{SF}(n, T, \epsilon)$

Lemma 1 Suppose $e(t) \in \mathbb{R}^n$ is a selector function and $y \in so(n)$, then

$$e^T(t)\ y\ \dot{e}(t) = 0.$$

Proof: If $e(t) = 0$, then the proposition is trivial. Suppose $e_i(t) \neq 0$ for some $i \in 1 \ldots n$, then by the definition of selector function, $e_j(t) = 0\ if\ i \neq j$. Differentiating $e_i(t) \cdot e_j(t) = 0$, we conclude that $e_i(t) \cdot \dot{e}_j(t) = 0$ if $i \neq j$. This shows that$e^T(t)\ y\ \dot{e}(t) = y_{ii}\ e_i(t)\dot{e}_i(t)$, but $y_{ii} = 0$ because y is skew-symmetric. **Q.E.D.**

Lemma 2 Let $e \in \mathcal{SF}(n, T, \epsilon)$, $y(t) \in so(n)$ and $\Theta \in O(n)$ such that $\dot{\Theta} = w(t)\Theta$, where $w(t) \in so(n)$. Suppose $[y(t), w(t)] = 0\ \forall t$, then

$$\int_t^{t+T} \|y\Theta e(\tau)\|\ d\tau > \epsilon \int_t^{t+T} \frac{\|y(\tau)\|}{nT} - 2\|\dot{y}(\tau)\|\ d\tau.$$

Proof: Let $z = \Theta^T y \Theta$, then it follows $\|z\| = \|y\|$ and $\|y\Theta e\| = \|ze\|$. Because $[y(t), w(t)] = 0$, it follows

$||\dot{z}(t)|| = ||\dot{y}(t)||$. Let $u_i^T = (0, \ldots, 1, \ldots 0)$ with 1 in the i^{th} position. Let $\Lambda(t) = \int_t^{t+T} ||\dot{z}(\tau)||\, d\tau$. Observe $||z(t)\, u_k|| \geq \frac{||z(t)||}{n}$ for some $k \in 1, \ldots, n$. Then,

$$\begin{aligned}
\int_t^{t+T} ||ze(\tau)||\, d\tau &\geq \int_t^{t+T} ||e_k(\tau)|| ||zu_k||\, d\tau \\
&\geq \int_t^{t+T} ||e_k(\tau)|| (||zu_k(t)|| - \Lambda) \\
&\geq \epsilon(\; ||zu_k(t)|| - \Lambda\;) && (13)\\
&\geq \epsilon(\; \frac{||z(t)||}{n} - \Lambda\;), && (14)
\end{aligned}$$

where (13) follows from the definition of selector function. Also notice that

$$\begin{aligned}
\int_t^{t+T} ||z(\tau)||\, d\tau &\leq \int_t^{t+T} (\; ||z(t)|| + \Lambda\;)\, d\tau \\
&\leq T\; (\; ||z(t)|| + \Lambda\;). && (15)
\end{aligned}$$

Combining inequalities (14)-(15) we get the desired result. Notice in particular that if $y(\tau) = y_0$ a constant, then

$$\int_t^{t+T} ||y_0 \Theta e(\tau)||\, d\tau > \frac{\epsilon}{nT} \int_t^{t+T} ||y_0||\, d\tau.$$

Q.E.D.

We now present a feedback stabilization law for the position-area system.

Theorem 2 Let $\mathcal{S}$ be a subspace of $so(n)$ and $\mathcal{P} : so(n) \to so(n)$ be a projection operator onto this subspace. Let $x \in \mathbb{R}^n$, $q \in \mathbb{R}$, $\Theta \in O(n)$, $y \in \mathcal{S}$, and $e \in \mathcal{SF}(n, T, \epsilon)$. If

$$u = -x + ||y||\Theta e + q(y\Theta e + \Theta \dot{e}),$$

then for

$$\begin{aligned}
\dot{x} &= u && (16)\\
\dot{y} &= \mathcal{P}[xu^T - ux^T] && (17)\\
\dot{q} &= -(q - ||y||) && (18)\\
\dot{\Theta} &= y\Theta && (19)
\end{aligned}$$

the submanifold $N = \{\; (x, y, q, \Theta) \in \mathbb{R}^n \times so(n) \times \mathbb{R} \times O(n) : x = 0, y = 0, q = 0\; \}$ is asymptotically stable in the large.

Proof: First notice that, for $\mathcal{S} = so(n)$ and $\mathcal{P}$ the identity operator, equations (16)-(17) reduce to the position-area system. Let $(x(t), y(t), \Theta(t), q(t))$ be the solution of equations (16)-(18) for a given initial condition $(x(0), y(0), \Theta(0), q(0))$. To simplify notation, we will often drop the time index t and just write the solution as (x, y, Θ, q). Let $p = \Theta e$, $\bar{p} = \Theta \dot{e}$, and $r = x - qp$. Then $u = -x + ||y||p + qyp + q\bar{p}$. From equations (16)-(18) we get $\dot{r} = -r$. Notice that

$$\begin{aligned}
\frac{d\; tr(y^T y)}{dt} &= 2\; tr(y^T \mathcal{P}[xu^T - ux^T]) && (20)\\
&= 2\; tr(y^T [xu^T - ux^T]) && (21)
\end{aligned}$$

where equation (21) follows from the fact $y \in \mathcal{S}$ and $\mathcal{P}$ is a projection on $\mathcal{S}$. Substituting for $x(t)$ and $u(t)$ we get

$$\begin{aligned}
\frac{d||y||^2}{dt} = -4\; \{\; & q^2 tr(p^T y y^T p) + q^2 tr(\bar{p}^T y p) \\
& + ||y|| tr(p^T y r) + \; q tr(\bar{p}^T y r) \\
& + q tr(p^T y y^T r)\}. && (22)
\end{aligned}$$

First, observe that $tr(\bar{p}^T y p) = 0$ from Lemma (1). Now using $tr(AB) < ||A|| ||B||$, from equation (22), we get

$$\begin{aligned}
\frac{d||y||^2}{dt} \leq\; & -4q^2 ||yp||^2 + \; 4\; ||q||\; ||y||\; ||yp||\; ||r|| \\
& + 4||y||\; ||yp||\; ||r|| + 4||q||\; ||\bar{p}||\; ||y||\; ||r|| && (23)\\
\frac{d||y||^2}{dt} \leq\; & -2\; (||q||\; ||yp|| - ||y||\; ||r||)^2 \; - \; 2q^2 ||yp||^2 \\
& + 2\; ||y||^2\; ||r||^2 + 4\; ||y||\; ||yp||\; ||r|| \\
& + 4\; ||q||\; ||\bar{p}||\; ||y||\; ||r|| && (24)
\end{aligned}$$

$$\frac{d||y||}{dt} \leq ||y||\; ||r||^2 \; + 2\; ||yp||\; ||r|| \; + 2\; ||q||\; ||\bar{p}||\; ||r|| \quad (25)$$

Because $||\bar{p}(t)||$ and $||p(t)||$ is bounded by the definition of selector function and $r(t) = r(0)e^{-t}$, equation (25) can be written as

$$\frac{d||y||}{dt} \leq A\; ||q||\; e^{-t} + B\; ||y|| e^{-t}. \quad (26)$$

for positive constants A and B. From equations (26) and (18), we can deduce that $||y(t)||$ is bounded and $q(t)$ is bounded. Therefore, for the given initial condition $(y(0), q(0))$, there exists $M < \infty$ such that $||y(t)|| < M$ and $||q(t)|| < M, \forall t$. Hence, we can rewrite equation (23) as

$$\frac{d||y||^2}{dt} \leq -4\; q^2\; ||yp||^2 + M_1\; e^{-t}, \quad (27)$$

for some positive constant M_1 which depends on the initial condition $(x(0), y(0), q(0))$. Defining $V(t) = ||y(t)||^2 + M_1\; e^{-t}$, observe that we have $\dot{V}(t) \leq -4\; q^2\; ||yp||^2 \leq 0$. As $V(t) \geq 0$ and non-increasing, it follows that $\lim_{t\to\infty} \dot{V}(t) = 0$, implying that $\lim_{t\to\infty} \frac{d||y(t)||}{dt} = 0$, i.e. $\lim_{t\to\infty} y(t) = y_0$ for some $y_0 \in so(n)$. Therefore, by equation (18), $\lim_{t\to\infty} q(t) = ||y_0||$. We now argue that $||y_0|| = 0$.

As $\lim_{t\to\infty} \frac{d||y(t)||}{dt} = 0$, from equation (27) we obtain $lim_{t\to\infty} ||y_0||^2\; ||y_0 p(t)||^2 = 0$. Since $p(t) = \Theta(t) e(t)$, where $e(t)$ is a selector function, we conclude from Lemma 2 that $||y_0|| = 0$. Therefore, $\lim_{t\to\infty} q(t) = 0$ and,

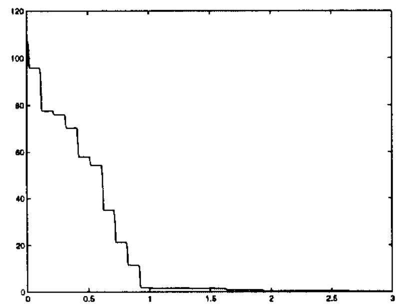

Figure 1: The panel shows the result of stabilization of the position-area system for $n = 10$, with the graphs showing the evolution of $||y||$ plotted against time.

from $x(t) = qp + ae^{-t}$, it follows $\lim_{t\to\infty} x(t) = 0$. **Q.E.D.**

A simulation of the stabilization of the general position-area system using the feedback control law given in Theorem 2 is shown in Figure 1, where the following selector function $e(t) \in \mathbb{R}^n$ with period $T = 1$ was used in the simulations

$$\begin{aligned} e_1(t) &= sin^2(\frac{\pi nt}{T}) \; ; \; 0 \le t \le \frac{T}{n} \\ &= 0 \, , \; \frac{T}{n} < t \le T \\ e_{k+1}(t) &= e_k(t - \frac{T}{n}) \, , \; k\{\in 1 \dots n\}. \end{aligned}$$

We now extend the result of Theorem 2 to the class of drift-free systems $\dot{z} = B(z)u$, which are first bracket controllable. As shown in [2], for such systems, we can choose coordinates in a neighborhood of a point, say $z = 0$, so that equations of motion take the form (33)-(34). In the following, theorem we present feedback laws that asymptotically stabilize such systems. First, we state a modification of the result due to Krasovskii. For details see [5].

Result 1 Let M be a compact differentiable manifold, $x \in \mathbb{R}^n$ and $\theta \in M$. Let

$$\begin{aligned} \dot{x} &= f(x, \theta, t) \quad (f \in E) \\ \dot{\theta} &= g(x, \theta, t) \quad (g \in E). \end{aligned}$$

The existence of a Lyapunov function $v(x, \theta, t)$ satisfying inequalities of the form

$$v < a_1 ||x||^\gamma, \quad \dot{v} < -a_2 ||x||^{\gamma+\eta} \qquad (28)$$

for sufficiently small $||x||$ is necessary and sufficient for the solution $(x(t), \theta(t))$ of the differential equation (28) to satisfy an estimate of the form

$$||x(t)||^{-\eta} - \beta ||x(0)||^{-\eta} > \alpha t \quad \forall t \ge 0. \qquad (29)$$

for small initial values $||x(0)||$. Here, a_1, a_2, η, γ, α and β are positive constants.

In case an estimate of the type (29) is satisfied, then $v \in C_1$ can be determined such that, in addition to (28), the inequality

$$||\frac{\partial v}{\partial x_i}|| < a_3 ||x||^{\gamma-1} \quad (i = 1, \dots, n; a_3 > 0, \gamma > 1) \quad (30)$$

is valid. As a result, the solutions of the modified differential equation

$$\begin{aligned} \dot{x} &= f(x, \theta, t) + h(x, \theta, t) \;\; (f + h \in E) \qquad (31) \\ \dot{\theta} &= g(x, \theta, t) \qquad (32) \end{aligned}$$

with sufficiently small initial values $||x(0)||$ also satisfy an estimate of the form (29) if $\lim_{||x||\to 0} \frac{||h(x,\theta,t)||}{||x||^{\eta+1}} = 0$.

Theorem 3 Let $\mathcal{S}$ be a subspace of $so(n)$ and $\mathcal{P} : so(n) \to so(n)$ be a projection operator onto this subspace. Let $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $q \in \mathbb{R}$, $\Theta \in O(n)$, $y \in \mathcal{S}$, and $e \in \mathcal{SF}(n, T, \epsilon)$. Let $R(x, y, u) \in \mathbb{R}^n$ and $R^1(x, y, u) \in so(n)$ have vanishing first partials with respect to x and y at the origin and be linear in u, such that $R(x, y, 0) = 0$, and $R^1(x, y, 0) = 0$. If

$$u = -x + ||y||\Theta e + q(||y||^{-\nu} y \Theta e + \Theta \dot{e}) \;\; , \; 0 < \nu < 1$$

then, for

$$\begin{aligned} \dot{x} &= u + R(x, y, u) \qquad (33) \\ \dot{y} &= \mathcal{P}[xu^T - ux^T] + R^1(x, y, u) \qquad (34) \\ \dot{q} &= -(q - ||y||) \qquad (35) \\ \dot{\Theta} &= \frac{y}{||y||^\nu} \Theta, \qquad (36) \end{aligned}$$

the submanifold $N = \{ \, (x, y, q, \Theta) \in \mathbb{R}^n \times so(n) \times \mathbb{R} \times O(n) : x = 0, y = 0, q = 0 \, \}$ is asymptotically stable.

Proof: First notice that, by definition, R and R^1 satisfy

$$\lim_{||x,y,q||\to 0} \frac{R}{||x, y, q||^{3-\nu}} = 0 \quad , \lim_{||x,y,q||\to 0} \frac{R^1}{||x, y, q||^{3-\nu}} = 0. \qquad (37)$$

We will show that, for small initial values $||x(0), y(0), q(0)|| < \delta_0$, the solutions to equations

$$\begin{aligned} \dot{x} &= u \qquad (38) \\ \dot{y} &= \mathcal{P}[xu^T - ux^T] \qquad (39) \\ \dot{q} &= -(q - ||y||) \qquad (40) \\ \dot{\Theta} &= \frac{y}{||y||^\nu} \Theta \;\; , \; 0 < \nu < 1 \qquad (41) \end{aligned}$$

satisfy for $\alpha > 0$, $\beta > 0$ and $t > 0$, an estimate of the form

$$||x(t), y(t), q(t)||^{-2+\nu} - \beta ||x(0), y(0), q(0)||^{-2+\nu} > \alpha t \qquad (42)$$

which using the result (1) and (37) proves the theorem.

Let $p = \Theta e$, and $r = x - qp$. From equations (38), (40) and (41), we get

$$\dot{r} = -r. \tag{43}$$

From equations (43) and (39), we obtain

$$\frac{d\,||y||^2}{dt} = -4\{q^2\,||y||^{-\nu}\,||yp||^2 + ||y||tr(p^T yr) + q\,||y||^{-\nu} tr(p^T yy^T r) + q\,tr(\bar{p}^T yr)\}. \tag{44}$$

$$\frac{d\,||y||}{dt} = -2q^2\,||y||^{-(1+\nu)}\,||yp||^2 + o(r,y,q) \tag{45}$$

where $o(r,y,q) < b||r,y,q||$ for some positive b. Observe that for $r=0$, equation (44) reduces to $\frac{d\,||y||^2}{dt} = -4q^2\,||y||^{-\nu}||yp||^2$. We will first show that the solutions to the system of reduced equations

$$\frac{d\,||y||^2}{dt} = -4q^2\,||y||^{-\nu}||yp||^2 \tag{46}$$

$$\dot{q} = -(q - ||y||) \tag{47}$$

$$\dot{\Theta} = \frac{y}{||y||^{\nu}}\,\Theta \tag{48}$$

satisfy, for $\alpha_1 > 0$, $\beta_1 > 0$ and $t > 0$, an estimate of the form

$$||y(t), q(t)||^{-2+\nu} - \beta_1||y(0), q(0)||^{-2+\nu} > \alpha_1 \tag{49}$$

for sufficiently small $||y(0), q(0)||$. Observe, from equation (39), that if $r = 0$,

$$||\dot{y}|| \le 2q^2\,||y||^{-\nu}||p||\,||yp||. \tag{50}$$

From equation (47), we have

$$\begin{aligned} q(t) &= q(0)e^{-t} + \int_0^t e^{-(t-\tau)}||y(\tau)||d\tau \qquad (51)\\ &\ge (q(0) - ||y(0)||)\,e^{-t} + ||y(t)|| \end{aligned}$$

where the last inequality follows from the fact that $||y(t)||$ is non-increasing function of time (equation (46)). Let $M = q(0) - ||y(0)||$, from (46) and (51), we have

$$\begin{aligned} \frac{d\,||y||^2}{dt} &\le -4\,||y||^{-\nu}\{Me^{-t} + ||y(t)||\,\}^2\,||yp||^2\\ \frac{d\,||y||}{dt} &\le -2||yp||^2||y||^{1-\nu} + 4\,||M||\,||y(0)||^{2-\nu}\,e^{-t}\\ &\quad +2\,M^2\,||y(0)||^{1-\nu}\,e^{-2t}. \qquad (52) \end{aligned}$$

We first show that solutions to

$$\frac{d\,||y||}{dt} = -||yp||^2||y||^{1-\nu} \tag{53}$$

satisfy for $\alpha_2 > 0$, $\beta_2 > 0$ and $t > 0$,

$$||y(t)||^{-2+\nu} - \beta_2||y(0)||^{-2+\nu} > \alpha_2 t. \tag{54}$$

From equation (50) and Lemma 2, it follows that

$$\int_t^{t+T} ||yp(\tau)||\,d\tau \;\ge\; \gamma\int_t^{t+T} ||y(\tau)||\,d\tau$$

for some positive constant γ, implying that

$$\int_t^{t+T} ||yp(\tau)||^2\,||y||^{1-\nu}d\tau \ge \beta\int_t^{t+T} ||y(\tau)||^{3-\nu}\,d\tau \tag{55}$$

for some positive constant β. Therefore (54) follows, and we can deduce (49) from (52) and (47). Using Result 1, there exists a Lyapunov function $v(y,q,\Theta)$ satisfying (28) and (30) for $\gamma = 2$ and $\eta = 2 - \nu$, where γ, η as defined in Result 1. Consider the Lyapunov function

$$v_1(r,y,q,\Theta) = v(y,q,\Theta) + r^2.$$

Observe from equations (43), (45), (40) and (41), v_1 satisfies estimates of the form (28), for sufficiently small $||r,y,q||$, hence by Result 1, we conclude that the assertion (42) is valid and thus the proof of the theorem follows. **Q.E.D.**

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