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Exponential Stabilization of a Car with n Trailers

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Abstract

It is shown how a class of two-input nonholonomic systems can be converted into a chained form. This is used to convert a car with n trailers into a chained form. The position of the system is given by the location of the last trailer. A time-varying feedback control law for the stabilization of a chained form with exponential convergence to the origin is presented. A coordinate transformation and input feedback transformation is derived to obtain asymptotic stability with exponential convergence to any desired configuration, locally in the orientations of the trailers. Simulation results are presented for a car with three trailers.

1 Introduction

A car with n trailers is a nonholonomic system due to the rolling constraints of the wheels. The configuration of the system is given by two position coordinates and $n+1$ angles, whereas there are only two inputs, namely one tangential velocity and one angular velocity. Thus, the system has two degrees of freedom.

A kinematic model for such a system was presented by [4]. Controllability for this model was proven, but no feedback control law was proposed. An interesting approach to stabilize a car with n trailers about a given position and orientation is to convert the kinematic model into a chained form. Chained form, which was introduced in [5], is nilpotent and is given by

$$\dot{\xi}_1 = u_1, \quad \dot{\xi}_2 = u_2, \quad \dot{\xi}_i = \xi_{i-1} u_1 \tag{1}$$

for $i \in \{3, \ldots, m\}$ where u_1 and u_2 are the inputs.

A constructive procedure to transform a nonholonomic system with two inputs into a chained form suitable for control was given by [6] under certain conditions on the input vectors. This was used to locally convert the kinematic model of a car pulling a single trailer into a chained form. However, the algorithm failed when additional trailers were added for the model considered. A new set of coordinates was proposed by [9] to convert the car/trailer system into a chained form with any number of trailers.

Chained forms cannot be stabilized by a smooth pure feedback, [1]. Smooth, time-varying feedback was first proposed by [8] for the stabilization of a cart, which is equivalent to a three-dimensional chained form. Smooth, time-varying feedback stabilizers for a chained form of any dimension form have been presented in [7, 12]. Since these approaches are smooth, the stability cannot be exponential, [3]. Exponential convergence to the origin of a cart, i.e. a three dimensional chained form, was obtained in [2] by letting the feedback be piecewise smooth.

In [11] a new stabilizing control law was proposed for chained forms of any dimension having *exponential* convergence to the origin. Stabilization was achieved by letting the state feedback law depend on time. Exponential convergence was obtained by letting the feedback control law be a non-smooth function of the state at discrete instants of time.

In this paper, the results from [9] and [11] are extended and combined. The conversion of the kinematics of a car with n trailers into a chained form [9] is presented here by a more general theorem applicable for a class of nonholonomic systems. The stabilization of chained forms about the origin from [11] is extended to stabilization about any desired configuration of the chained form. This paper therefore shows how a car with n trailers locally can be asymptotically stabilized about any configuration with exponential convergence. This is illustrated by simulation results for a car with three trailers.

2 Kinematic Model

A car in this context will be represented by two driving wheels connected by an axle. A kinematic model of a car with two degrees of freedom pulling n trailers can be given by, [9]:

$$\begin{aligned}
\dot{\theta}_0 &= \omega \\
\dot{\theta}_i &= \frac{1}{d_i}\sin(\theta_{i-1} - \theta_i)v_{i-1} \quad i \in \{1, \ldots, n\} \\
\dot{x} &= \cos\theta_n v_n \\
\dot{y} &= \sin\theta_n v_n
\end{aligned} \tag{2}$$

Here, (x, y) is the absolute position of the center of the axle between the two wheels of the *last* trailer.

θ_i is the orientation angle of trailer i with respect to the x-axis, with $i \in \{1, \dots, n\}$. θ_0 is the orientation angle of the pulling car with respect to the x-axis.

d_i is the distance from the wheels of trailer i to the wheels of trailer $i-1$, where $i \in \{2, \dots, n\}$. d_1 is then the distance from the wheels of trailer 1 to the wheels of the car.

v_0 is the tangential velocity of the car and is an input to the system. The other input is the angular velocity of the car, ω. We denote

$$\nu = [v_0, \omega]^T$$

The tangential velocity of trailer i, v_i, is given by

$$v_i = \cos(\theta_{i-1} - \theta_i) v_{i-1} = \prod_{j=1}^{i} \cos(\theta_{j-1} - \theta_j)\, v_0 \qquad (3)$$

where $i \in \{1, \dots, n\}$. An illustration of these definitions is presented in *Figure 1*.

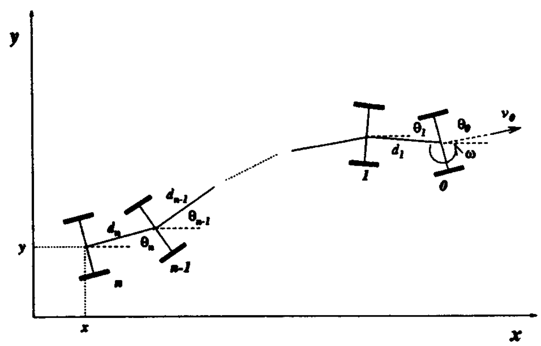

Figure 1: Model of a car with n trailers.

3 Conversion into a Chained Form

In this section, we present a theorem on the conversion of a class of nonholonomic systems into a chained form. The kinematic model (2) is then modified to belong to this class. This theorem will provide the transformation of the kinematic model of a car with n trailers into a chained form.

First, denote

$$\begin{aligned} \underline{q}_i &\triangleq [q_i, \dots, q_m]^T \\ \underline{f}_i(\underline{q}_{i-1}) &\triangleq [f_i(\underline{q}_{i-1}), \dots, f_m(\underline{q}_{m-1})]^T \end{aligned}$$

Theorem 1 *Let a driftless, two-input system be given by*

$$\begin{aligned} \dot{q}_1 &= v_1 && (4) \\ \dot{q}_2 &= v_2 && (5) \\ \dot{q}_i &= f_i(\underline{q}_{i-1}) v_1, \quad i \in \{3, \dots, m\} && (6) \end{aligned}$$

where $f_i(\cdot)$ is a smooth function. Assume that at a configuration $q = p$ on the configuration manifold

$$\frac{\partial f_i(\underline{q}_{i-1})}{\partial q_{i-1}}\Big|_{q=p} \neq 0, \quad \forall i \in \{3, \dots, m\} \qquad (7)$$

Then, a coordinate transformation $\xi = h(q)$ and an input feedback transformation $u = g(q)v$ converting (4)-(6) into the chained form (1) in a neighborhood of $q = p$ is given by

$$\begin{aligned} \xi_m &= h_n(q_m) && (8) \\ \xi_i &= h_i(\underline{q}_i) \triangleq \frac{\partial h_{i+1}(\underline{q}_{i+1})}{\partial \underline{q}_{i+1}} \underline{f}_{i+1}(\underline{q}_i) && (9) \\ \xi_1 &= h_1(q_1) \triangleq q_1 && (10) \end{aligned}$$

where $i \in \{2, \dots, m-1\}$ and $h_m(q_m)$ is any smooth function such that $\frac{\partial h_m(q_m)}{\partial q_m}\big|_{q=p} \neq 0$ and

$$\begin{aligned} u_1 &= v_1 && (11) \\ u_2 &= \frac{\partial h_2(\underline{q}_2)}{\partial \underline{q}_3} \underline{f}_3(\underline{q}_2) v_1 + \frac{\partial h_2(\underline{q}_2)}{\partial q_2} v_2 && (12) \end{aligned}$$

Proof: The proof consists of two parts. First, it will be shown that the transformations (8)–(10) and (11)–(12) result in the chained form (1). Then, it will be shown that these transformations are diffeomorphic.

Assume that ξ_i is given by (8)–(9) for all $i \in \{3, \dots, m\}$. Then, from (4)–(6) and (11) we get

$$\dot{\xi}_i = \frac{\partial h_i(\underline{q}_i)}{\partial \underline{q}_i} \dot{\underline{q}}_i = \frac{\partial h_i(\underline{q}_i)}{\partial \underline{q}_i} \underline{f}_i(\underline{q}_{i-1}) v_1 = \xi_{i-1} u_1$$

From (9), (4)–(6), and (12) we get

$$\dot{\xi}_2 = \frac{\partial h_2(\underline{q}_2)}{\partial \underline{q}_2} \dot{\underline{q}}_2 = \frac{\partial h_2(\underline{q}_2)}{\partial \underline{q}_3} \underline{f}_3(\underline{q}_2) v_1 + \frac{\partial h_2(\underline{q}_2)}{\partial q_2} v_2 = u_2$$

From (4), (10), and (11) it follows readily that $\dot{\xi}_1 = u_1$.

To show that the transformation $\xi = h(q)$ from (8)–(10) is diffeomorphic, we study the jacobian $J(q)$,

$$J(q) \triangleq \frac{\partial h(q)}{\partial q}$$

where $q = [q_1, \dots, q_n]^T$. Because of the triangular structure of $h(q)$, $J(q)$ is non-singular if and only if $J_{ii}(q) \neq 0$ for all $i \in \{1, \dots, m\}$. From (9) we have that for $i \in \{2, \dots, m-1\}$

$$\begin{aligned} h_i(\underline{q}_i) &= \frac{\partial h_{i+1}(\underline{q}_{i+1})}{\partial \underline{q}_{i+1}} \underline{f}_{i+1}(\underline{q}_i) \\ &= \frac{\partial h_{i+1}(\underline{q}_{i+1})}{\partial q_{i+1}} f_{i+1}(\underline{q}_i) + \frac{\partial h_{i+1}(\underline{q}_{i+1})}{\partial \underline{q}_{i+2}} \underline{f}_{i+2}(\underline{q}_{i+1}) \end{aligned}$$

Hence,

$$\begin{aligned} J_{ii}(\underline{q}_i) &= \frac{\partial h_i(\underline{q}_i)}{\partial q_i} = \frac{\partial h_{i+1}(\underline{q}_{i+1})}{\partial q_{i+1}} \frac{\partial f_{i+1}(\underline{q}_i)}{\partial q_i} \\ &= J_{i+1,i+1} \frac{\partial f_{i+1}(\underline{q}_i)}{\partial q_i} \end{aligned}$$

$J_{mm} = \frac{\partial h_m(q_m)}{\partial q_m}$ is assumed to be non-zero in a neighborhood of $q = p$. It then follows by induction that $J_{ii}(\underline{q}_i) \neq 0$ in a neighborhood of $q = p$ if and only if $\frac{\partial f_{i+1}(\underline{q}_i)}{\partial q_i}|_{q=p} \neq 0$ for $i \in \{2, \ldots, m-1\}$. From (10) we see that $J_{11}(q) = 1$. From The Inverse Function Theorem it then follows that $\xi = h(q)$ is diffeomorphic if (7) is satisfied. From (11)–(12) we then see that the input transformation is diffeomorfic since

$$\frac{\partial h_2(\underline{q}_2)}{\partial q_2}|_{q=p} = J_{22}(\underline{q}_2) \neq 0$$

□

This theorem assumes a more special structure of the system than the theorem in [6]. The tests and calculations are therefore simplified here. The problem may be to represent the system with the required structure. To this end, geometrical insight must be applied, if possible. We can convert the kinematic model (2) into the required form. This is obtained by an input feedback transformation in a neighborhood D of the origin where D is given by

$$\begin{aligned} (x, y) &\in \mathbf{R}^2 & (13) \\ \theta_i &\in (-\frac{\pi}{4} + \varepsilon, \frac{\pi}{4} - \varepsilon), \quad i \in \{0, \ldots, n\} & (14) \end{aligned}$$

where ε is a small constant. Introduce the transformed input v as follows:

$$v = \cos\theta_n \, v_n = \cos\theta_n \prod_{j=1}^{n} \cos(\theta_{j-1} - \theta_j) \, v_0 \qquad (15)$$

The transformed input v is the velocity of trailer n in x-direction. This transformation from v_0 to v is nonsingular and smooth in D. The velocity of v_i from (3) can then be rewritten

$$v_i = \frac{1}{\cos\theta_n \prod_{j=i+1}^{n} \cos(\theta_{j-1} - \theta_j)} v = \frac{1}{p_i(\underline{\theta}_i)} v \qquad (16)$$

for $i \in \{0, \ldots, n\}$ where

$$\begin{aligned} \underline{\theta}_i &\triangleq [\theta_i, \ldots, \theta_n]^T & (17) \\ p_i(\underline{\theta}_i) &\triangleq \prod_{j=i}^{n} \cos(\theta_j - \theta_{j+1}) & (18) \end{aligned}$$

for $i \in \{0, \ldots, n\}$ where $\theta_{n+1} \triangleq 0$. Eq. (16) then gives $v = p_0(\underline{\theta}_0) v_0$.

System (2) can now be represented (locally) at the following form

$$\begin{aligned} \dot{x} &= v \\ \dot{\theta}_0 &= \omega \\ \dot{\theta}_i &= \frac{1}{d_i} \frac{\tan(\theta_{i-1} - \theta_i)}{p_i(\underline{\theta}_i)} v, \quad i \in \{1, \ldots, n\} \qquad (19) \\ \dot{y} &= \tan\theta_n v \end{aligned}$$

where v is given by (15).

From Theorem 1 it now follows that the kinematic model of a car with n trailers is convertible into a chained form.

Corollary 1 *The kinematic model (2) is convertible into the chained form (1) with $m = n + 3$ in the subspace D (13)–(14). A coordinate transformation and an input feedback transformation is given by (8)–(10) with $q = [x, \theta_0, \ldots, \theta_n, y]^T$ and $f_i(\underline{q}_{i-1}) = \frac{1}{d_{i-2}} \frac{\tan(\theta_{i-3} - \theta_{i-2})}{p_{i-2}(\underline{\theta}_{i-2})}$ for $i \in \{3, \ldots, n+2\}$ and $f_{n+3}(\underline{q}_{n+2}) = \tan\theta_n$.*

Proof: This follows readily from (19) and Theorem 1 since for all $i \in \{3, \ldots, n+2\}$ and for all $q \in D$

$$\frac{\partial f_i(\underline{q}_{i-1})}{\partial q_{i-1}} = \frac{1}{d_{i-2}} \frac{1}{\cos(\theta_{i-3} - \theta_{i-2}) p_{i-3}(\underline{\theta}_{i-3})} \neq 0$$

and

$$\frac{\partial f_{n+3}(\underline{q}_{n+2})}{\partial q_{n+2}} = \frac{1}{\cos^2\theta_n} \neq 0$$

□

The conversion into chained form is not unique. See [13] for another conversion.

4 The Control Law

In this section, we present a control law to asymptotically stabilize the chained form (1) with exponential convergence to the origin. This is defined as $\mathcal{K}$-exponential stability in [10].

Since there is no smooth, static-state feedback law which can stabilize (2), we let the feedback law be time-dependent. To obtain exponential convergence, we let the feedback law also depend on a parameter k which is a function of the state $\xi(t_i)$ at discrete instants of time $t_i \in \{t_0, t_1, \ldots\}$. The input u_1 as chosen as follows, [11],

$$u_1 = k(\xi(t_i)) f(t) \qquad (20)$$

where $\xi = [\xi_1, \ldots, \xi_m]^T$. Here, t_i denotes the last element in the sequence $(t_0, t_1, t_2, \ldots)$ such that $t \geq t_i$. The function $f(t)$ has the following properties

P1: $f(t) \in C^\infty$

P2: $0 \leq f(t) \leq 1, \quad \forall t \geq t_0$

P3: $f(t_i) = 0, \quad t_i \in \{t_0, t_1, t_2, \ldots\}$

P4: $|\int_{t_k}^{t} [f^{2(j-2)+1}(\tau) - \eta_j] d\tau| \leq P_j$
$\forall j \in \{3, \ldots, n+3\}, \ \forall t_k \in \{t_0, t_1, \ldots\}$

where η_j and P_j are positive constants. This makes u_1 continuous with respect to time if $|k| < \infty$. A function satisfying these conditions is $f(t) = (1 - \cos \omega t)/2$ where $\omega = \frac{2\pi}{T}$ and $T = t_{i+1} - t_i$ is supposed to be a constant.

We let the input u_1 be given by (20). The *lower part* of (1) is then given by

$$\dot{\xi}_2 = u_2, \quad \dot{\xi}_i = k f(t) \xi_{i-1}, \quad i \in \{3, \ldots, m\} \tag{21}$$

where k may switch at the time instants t_i. We denote the state of the lower part by $z \triangleq [\xi_2, \ldots, \xi_{n+3}]^T$.

The feedback control law is motivated from the observation that if u_1 is constant, then (21) becomes a linear, time-invariant system which can easily be exponentially stabilized to zero by u_2. If u_1 is time-periodic, then (21) becomes a linear, time-varying system which also can be stabilized by u_2 to zero. (The path-following of a car with n trailers is therefore a simpler problem than the stabilization). By letting u_1 be time-periodic, $x_1(t)$ cannot be asymptotically stabilized. The idea is then to choose the sign and magnitude of $k(\cdot)$ from (20) and u_2 such that the whole system becomes stable and converges exponentially to zero. This is obtained by the following control law (see [10] for details)

$$\begin{aligned} u_1 &= k(\xi(t_i)) f(t) && (22) \\ u_2 &= \Gamma(k,t)^T z && (23) \end{aligned}$$

where

$$z = [\xi_2, \ldots, \xi_m]^T, \quad \Gamma(k,t) = [\Gamma_2(k,t), \ldots, \Gamma_m(k,t)]^T$$

and

$$\Gamma_2(k,t) = -\lambda_2 + f^3 g_{2,3}$$

$$\Gamma_j(k,t) = \frac{f}{k^{j-2}} \sum_{j=3}^{m} \left(\lambda_2 f g_{2,j} + 2\dot{f} g_{2,j} + f \dot{g}_{2,j} + f^2 g_{2,j+1} \right)$$

where $j \in \{3, \ldots, m\}$. The smooth functions $g_{2,j}$ are given by

$$\begin{aligned} g_{m-1,m} &= -\lambda_m \\ g_{i-1,j} &= g_{i,j}[\lambda_i f^{2(i-1)} + 2(i-1)\dot{f}] + f(\dot{g}_{i,j} + g_{i,j+1} f) \\ g_{i-1,i} &= -\lambda_i + f^2 g_{i,i+1} \\ g_{i,k} &= 0 \ \text{ if } \ k \leq i \ \text{ or } \ k = m+1 \end{aligned}$$

The parameter $k(\cdot)$ is chosen as follows

$$k = \mathrm{sat}\left(-[x_1(t_i) + \mathrm{sgn}(x_1(t_i)) G(\|z(t_i)\|)] \beta, K\right) \tag{24}$$

where

$$\mathrm{sat}(q, K) \triangleq \left\{ \begin{array}{ccc} q & , & |q| < K \\ K \mathrm{sgn}(q) & , & |q| \geq K \end{array} \right\}$$

and

$$\begin{aligned} G(\|z(t_i)\|) &\triangleq \kappa \|z(t_i)\|^{\frac{1}{2(n-2)}} && (25) \\ \beta &\triangleq 1 / \int_{t_i}^{t_{i+1}} f(\tau) d\tau && (26) \\ \mathrm{sgn}(q) &= \left\{ \begin{array}{rl} 1, & q \geq 0 \\ -1, & q < 0 \end{array} \right. && (27) \end{aligned}$$

where κ is a positive constant. This control law leads to the following theorem:

Theorem 2 *Let the control law be given by (22)-(23) where k is given by (24). System (1) is globally, $\mathcal{K}$-exponentially stable about the origin, i.e. $\exists \delta > 0$, $\forall \varepsilon_2, \ldots, \varepsilon_m \in (0, \delta)$ there exist a function $h(\cdot)$ of class $\mathcal{K}$ and a constant $\gamma > 0$ such that $\forall t \geq t_0$*

$$\forall \xi(t_0) \in \mathbb{R}^m \quad \|\xi(t)\| \leq h(\|\xi(t_0)\|) e^{-\gamma(t-t_0)} \tag{28}$$

where

$$\begin{aligned} \gamma &= \frac{\beta_m - \varepsilon_m}{4(m-2)} > 0 \\ \beta_q &= \min\{\lambda_q \eta_q, \beta_{q-1} - \varepsilon_{q-1}\}, \quad q \in \{3, \ldots, m\} \\ \beta_2 &= \lambda_2 \end{aligned}$$

The constants η_q are found from Property P4 of $f(t)$. The function $G(\cdot)$ is defined in (25).

Proof: See [10] Chapter 6. □

Therefore, the control law $u = u(\xi, k, t)$ from (22)-(23) where $\xi = h(q)$ from Theorem 1 and Corollary 1 can be used to asymptotically stabilize the car/trailer system (2) to the origin with exponential convergence.

5 Stabilization about Arbitrary Configuration

We have shown that the controller (22)-(23) and (24) makes the system (1) globally, $\mathcal{K}$-exponentially stable about the *origin*, i.e. about $[\xi_1, \ldots, \xi_m]^T = 0$. In this section we show that the same control law can be used to $\mathcal{K}$-exponentially stabilize the chained system (1) about any configuration. Indeed, we show that any strategy to control the chained system to the origin can be used to control it to any desired configuration.

Let the desired, constant configuration be given by

$$\xi^p = [\xi_1^p, \xi_2^p, \ldots, \xi_m^p]^T \in \mathbb{R}^m, \quad \dot{\xi}^p \equiv 0$$

Now, we introduce the following variables:

$$\xi_i^r \triangleq \xi_i^p + \sum_{j=2}^{i-1} \xi_j^p \frac{1}{(i-j)!} (\xi_1 - \xi_1^p)^{i-j}, \quad i \in \{1, \ldots, m\} \tag{29}$$

Here, ξ_1 is a state variable of the chained system (1) satisfying $\dot{\xi}_1 = u_1$. The vector $\xi^r = [\xi_1^r, \ldots, \xi_m^r]^T$ is thus given from (29) as a smooth function $\xi^r \triangleq \phi(\xi_1; \xi^p)$.

Lemma 1 *Let $\xi^r = [\xi_1^r, \ldots, \xi_m^r]^T$ be given by (29). Then, for $i \in \{3, \ldots, m\}$*

$$\dot{\xi}_1^r = 0, \quad \dot{\xi}_2^r = 0, \quad \dot{\xi}_i^r = \xi_{i-1}^r u_1 \tag{30}$$

Proof: From (29) we have that $\xi_1^r = \xi_1^p$ and $\xi_2^r = \xi_2^p$. Since ξ_1^p and ξ_2^p are constants, the two first equations of (30) follow readily. Eq. (30) can be proved by induction. Assume that there is an index $k \in \{3, \ldots, m-1\}$ such that $\dot{\xi}_k^r = \xi_{k-1}^r u_1$. Since $\dot{\xi}_i^p = 0$, differentiating ξ_{k+1}^r from (29) gives

$$\begin{aligned}
\dot{\xi}_{k+1}^r &= \sum_{j=2}^{k} \xi_j^p \frac{1}{(k-j)!}(\xi_1 - \xi_1^p)^{k-j} u_1 \\
&= \left[\xi_k^p + \sum_{j=2}^{k-1} \xi_j^p \frac{1}{(k-j)!}(\xi_1 - \xi_1^p)^{k-j} \right] u_1 \\
&= \xi_k^r u_1
\end{aligned}$$

Consequently, if (30) is satisfied for $i = k$ then (30) is satisfied for $i = k+1$, too. The proof is then completed by showing that $\dot{\xi}_3^r = \xi_2^r u_1$. From (29) we get

$$\dot{\xi}_3^r = \frac{d}{dt}[\xi_2^p(\xi_1 - \xi_1^p) + \xi_3^p] = \xi_2^p u_1 = \xi_2^r u_1$$

□

We now define

$$\bar{\xi}_i \triangleq \xi_i - \xi_i^r, \quad i \in \{1, \ldots, m\} \tag{31}$$

From System (1) and Lemma 1 it follows that

$$\dot{\bar{\xi}}_1^r = u_1, \quad \dot{\bar{\xi}}_2^r = u_2, \quad \dot{\bar{\xi}}_i^r = \bar{\xi}_{i-1}^r u_1 \tag{32}$$

where $i \in \{3, \ldots, m\}$. This system has the same structure as the chained system (1). A control law for (1) controlling $\xi = [\xi_1, \ldots, \xi_m]^T$ to zero can be used to control $\bar{\xi} = [\bar{\xi}_1, \ldots, \bar{\xi}_m]^T$ to zero. The coordinate transformation between ξ and $\bar{\xi}$ is then given from (29) and (31) by

$$\begin{aligned}
\bar{\xi} &= \xi - \xi^r = \xi - \phi(\xi_1; \xi^p) \triangleq \tau(\xi; \xi^p) \\
\xi &= \bar{\xi} + \xi^r = \bar{\xi} + \phi(\bar{\xi}_1 + \xi_1^p; \xi^p) \triangleq \bar{\tau}(\bar{\xi}; \xi^p)
\end{aligned}$$

where $\tau(\,\cdot\,; \xi^p)$ and $\bar{\tau}(\,\cdot\,; \xi^p)$ are smooth functions.

We can then conclude with the following theorem:

Theorem 3 *Given the system (32) where $\bar{\xi}_i$ and ξ_i^r are given by (31) and (29). A control law for (32) making $\bar{\xi} = [\bar{\xi}_1, \ldots, \bar{\xi}_m]^T$ converge to zero makes ξ converge to the desired configuration ξ^p. The convergence of ξ to ξ^p is exponential if $\bar{\xi}$ converges exponentially to zero.*

Proof: From (31) and (29) we have that for $i \in \{1, \ldots, m\}$

$$\xi_i = \xi_i^r + \bar{\xi}_i = \xi_i^p + \sum_{j=2}^{i-1} \xi_j^p \frac{1}{(i-j)!} \bar{\xi}_1^{i-j} + \bar{\xi}_i$$

Therefore, the control of $\bar{\xi}$ to zero implies the control of ξ to ξ^p. We also see that exponential convergence of $\bar{\xi}$ to zero implies exponential convergence of ξ to ξ^p. □

6 Simulations

A simulation with $n = 3$, i.e. 3 trailers, was done in MATLAB. The control law for u_1 was chosen as follows

$$u_1 = k(\xi(t_i)) f(t), \quad f(t) = (1 - \cos t)/2$$

where $\xi(t_i) = h(q(t_i))$ from Theorem 1 and Corollary 1, and

$$k = \text{sat}\left(-[\xi_1(t_i) + \text{sgn}(\xi_1(t_i)) G(\|z(t_i)\|)]\, \beta, K\right)$$

as defined in (24) where K was selected to 2.

The controller parameter κ in $G(\cdot)$, (25), was taken to $\kappa = 3$. The constant β is given by (26) and $f(t)$ as $\beta = \frac{1}{\pi}$. The instants of time t_i where $k(\xi(t_i))$ may switch is given by the set $\{0, 2\pi, 4\pi, 6\pi, \ldots\}$, where t_0 is taken to zero. We find the control law for u_2 is given by (23). The controller parameters were chosen as follows $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1$.

The model parameters were taken to $d_1 = d_2 = d_3 = 1$. The initial state was chosen as follows:

$$(x(0), \theta_0(0), \theta_1(0), \theta_2(0), \theta_3(0), y(0)) = (0, 0, 0, 0, 0.1, 1)$$

Euler's method was applied for the numerical integration with the time-step equal to 0.05.

The path $(x(t), y(t))$ is presented in *Figure 2*. We see that the motion seems appealing when interpreted as a parking maneuver of a car with 3 trailers. In *Figure 3*, $y(t)$ is plotted versus time showing exponential convergence to zero. The x-position of the rear trailer is plotted in *Figure 4*. We see that $x(t)$ converges exponentially to zero, too. Note from the time-axes, however, that the rate of convergence of $x(t)$ is slower than the one of $y(t)$.

7 Conclusions

A feedback control law has been presented to stabilize a car with n trailers about a given position and orientation where n is arbitrary. The resulting rate of convergence was exponential. This was achieved by locally converting the system into a chained form and then apply a stabilizing feedback law for chained forms. The conversion holds globally in the position of the system and locally in the orientations of the trailers.

The conversion into a chained form was obtained by a theorem for a class of nonholonomic systems. The stabilization about any configuration (locally in the orientations) was obtained by a useful coordinate transformation and input feedback transformation for chained forms. Simulation studies showed exponential convergence of a car with 3 trailers to a given position and orientation.

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References

[1] R. W. Brockett. Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman, and H. J. Sussman, editors, *Differential Geometric Control Theory*, pages 181–208. Birkhauser, 1983.

[2] C. Canudas de Wit and O. J. Sørdalen. Exponential stabilization of mobile robots with nonholonomic constraints. *IEEE Transactions on Automatic Control*, 37(11):1791–1797, 1992.

[3] L. Gurvits and Z. X. Li. Smooth time-periodic feedback solutions for nonholonomic motion planning. In Z. X. Li and J. Canny, editors, *Progress in Nonholonomic Motion Planning*. Kluwer Academic Publisher, 1992.

[4] J.-P. Laumond. Controllability of a multibody mobile robot. In *ICAR'91*, Pisa, Italy, June 1991.

[5] R. M. Murray and S. S. Sastry. Steering nonholonomic systems using sinusoids. In *IEEE CDC*, pages 2097–2101, Honolulu, Hawaii, December 1990.

[6] R. M. Murray and S. S. Sastry. Steering nonholonomic systems in chained form. In *IEEE CDC*, pages 1121–1126, Brighton, England, December 1991.

[7] J.-P. Pomet. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *System and Control Letters*, 18(2):147–158, 1992.

[8] C. Samson. Velocity and torque feedback control of a nonholonomic cart. In C. Canudas de Wit, editor, *Advanced Robot Control*, pages 125–151. Springer-Verlag, 1991.

[9] O. J. Sørdalen. Conversion of the kinematics of a car with n trailers into a chained form. In IEEE Conf. on Robotics and Automation, pages 382–387, Atlanta, Georgia, May 1993.

[10] O. J. Sørdalen. *Feedback Control of Nonholonomic Mobile Robots*. PhD thesis, The Norwegian Institute of Technology, February 1993. ITK-report 1993:5-W.

[11] O. J. Sørdalen and O. Egeland. Exponential stabilization of chained nonholonomic systems. In *European Control Conference*, Groningen, The Netherlands, June 1993.

[12] A. R. Teel, R. M. Murray, and G. Walsh. Nonholonomic control systems: From steering to stabilization with sinusoids. In *IEEE CDC*, pages 1603–1609, Tucson, Arizona, December 1992.

[13] D. Tilbury, R. Murray, and S. Sastry. Trajectory generation for the n-trailer problem using goursat normal form. Technical Report UCB/ERL M93/12, UC Berkeley, February 1993.

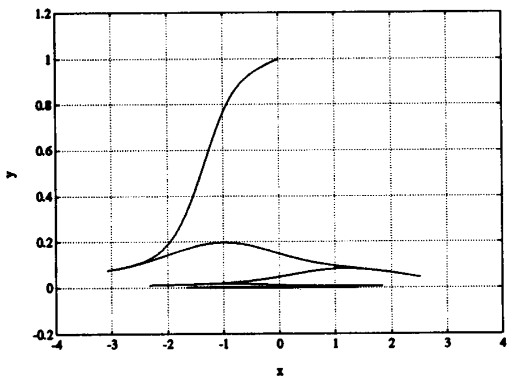

Figure 2: The resulting path of the rear trailer in the xy-plane.

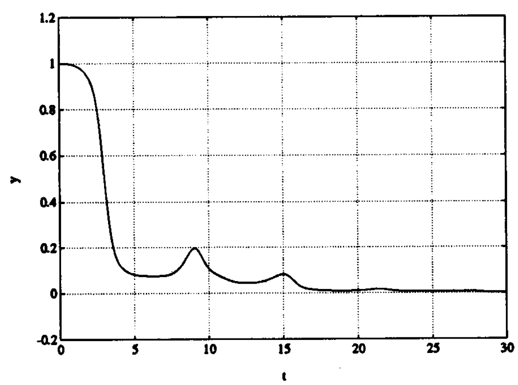

Figure 3: Exponential convergence of $y(t)$ to zero.

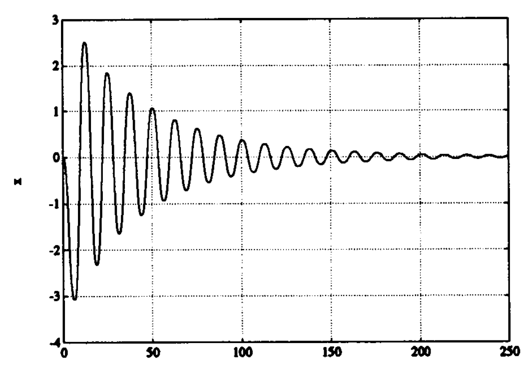

Figure 4: Exponential convergence of $x(t)$ to zero.