Model Predictive Control of Nonlinear Systems: Computational Burden and Stability

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Abstract

Implementation of model predictive control for nonlinear systems requires on-line solution of a non-convex, constrained nonlinear optimisation problem. Computational delay and loss of optimality arise in optimisation procedures. This paper presents a practical Model Predictive Control (MPC) scheme for nonlinear systems with guaranteed asymptotic stability. It is shown that when an initial control profile is chosen to satisfy an inequality condition in each on-line optimisation procedure, the nonlinear system under the proposed nonlinear model predictive control is asymptotically stable. The stability condition presented in this paper enables the "fictitious" terminal control to be not only linear but also nonlinear and thus the stability region is greatly enlarged. Furthermore it is pointed out that nominal stability is still guaranteed although the global, even the local, minimisation of the objective cost is not achieved within the prescribed computational time.

1 Introduction

Since model predictive control (MPC) is now regarded as a most promising method in control engineering (Clarke, 1994), much effort has been taken to extend model based predictive control from linear systems to nonlinear systems, for example see (Allgöwer and Zheng, 1998). There are, however, two main obstacles in the extension of MPC from linear systems to nonlinear systems (Biegler and Rawlings, 1991; Mayne, 1996; Chen and Allgower, 1998; Qin and Badgwell, 1998). One obstacle is the stability issue. The other obstacle is the on-line computational burden.

Stability is one of the main problems in model predictive control since the early model predictive control for linear systems was criticised for its loss of stability (Bitmead *et al.*, 1990). This problem has now been solved in several ways, for example infinite horizon predictive control (Rossiter *et al.*, 1996), terminal constraints (Mosca and Zhang, 1992; Demircioglu and Clarke, 1992) and Fake Riccati Algebraic equation (Bitmead *et al.*, 1990). Regarding stability of nonlinear model predictive control, the first interesting result was given by Mayne and Michalska (1990). They shown that under some assumptions, the predictive control can stabilise plants when a terminal equality constraint is embedded in the optimisation problem. That is, it is required that the terminal state arrives at a point in the state space. Similar results also have been reported by (Genceli and Nikolaou, 1993; Rawlings and Muske, 1993). However from the computational points of view, to solve a nonlinear dynamic optimisation problem with equality constraints is highly computationally intensive and it is impossible to be performed within a limited time in many cases. Furthermore, the feasible region of the proposed MPC is very small. To avoid this, Michalska and Mayne (1993) replaced terminal equality constraints with terminal inequality constraints. That is, the allowable terminal state is a region rather than a point in the state space. To guarantee stability, a dual mode control scheme is proposed. In other words, the receding horizon control is employed outside the terminal region and once the state arrives on the boundary of the terminal region, a local linear state feedback controller is employed to steer the state to an equilibrium. Along this line, the most interesting result was recently presented by (Chen and Allgower, 1998). For a wide class of nonlinear systems, a model predictive control scheme with a virtual *linear* stabilising control rather than a dual mode control is developed. The stability is achieved by involving a terminal cost which bounds the infinite horizon cost of the nonlinear system starting from the terminal region and controlled by the local linear state feedback controller. However, the local state feedback control is never implemented. The so-called quasi-infinite model predictive control algorithm solves on-line the nonlinear optimisation problem with a performance index containing the terminal cost subject to system dynamics, input constraints and the terminal inequality constraints at each sampling time. Then the receding horizon control is employed until the system arrives the equilibrium.

Apart from the above works on the stability of nonlinear model predictive control, the stability of several existing nonlinear predictive control schemes has been examined by Magni and Sepulchre (1997) using inverse optimality approach. In particular, it is shown that the predictive control possesses the stability margin of optimal control and it is robust in some sense. A more rigorous result of model predictive control was developed by Gyurkovics (1998). The receding horizon control is considered as a Bolza-type optimisation problem. Using the optimality of the value function, the stability condition is established.

Although rigorous stability results for nonlinear model predictive control have been established (Gyurkovics, 1998; Magni and Sepulchre, 1997), it is not applicable in practical implementations. Since a constrained nonlinear optimisation problem has to be solved on-line, the heavy on-line computational burden causes two important issues in implementation of nonlinear model predictive control. One is the computational delay which cannot be ignored and the other is the global optimisation solution cannot be guaranteed in each optimisation procedure since, in general, it is a non-convex, constrained nonlinear optimisation problem. Both these issues have not been considered in Gyurkovics (1998), Magni and Sepulchre (1997). In view of the difficulty to guarantee the global minimisation of an objective cost, this requirement has been relaxed for discrete-time systems in Scokaert *et* *al.* (1999) and for continuous-time systems in Chen and Allgower (1998) where it was pointed out that the feasibility implies the stability for a special MPC (with linear terminal state feedback control).

The effects of computational delay in implementation of a controller have been investigated by many authors (Pierre and Pierre, 1995; Rattan, 1989). In model predictive control, recently several schemes taking into account computational delay have been proposed for linear systems (Gomma and Owens, 1998; Von Wissel *et al.*, 1997; Casavola and Mosca, 1998). Gomma and Owens (1998) gave a delayed generalised predictive control scheme where control only depends on past measurements rather than current and past measurement. This strategy was compared with Clarke et al. predictive control (Gomma and Owens, 1998). Van Kin et al.(Von Wissel *et al.*, 1997) investigated the effect of computational delay in descriptor predictive control. It is shown that the closed-loop system remains stable as long as the delay is smaller than a threshold. Reference governor in predictive control with computation delay was addressed in (Casavola and Mosca, 1998). Although the computational burden in nonlinear model predictive control is much heavier than the linear one, to the best of our knowledge, only the algorithm – Intermittent Optimal Open Loop Feedback (IOOLF) proposed by Ronco et al (Ronco *et al.*, 1999*b*; Ronco *et al.*, 1999*a*) considers the computation delay in implementation of model predictive control for nonlinear systems. Similar to other schemes considering the computational delay, in this algorithm, the last control profile yielded by minimisation of the objective cost is used until the new optimisation process is finished and the new control profile is available. It has been shown that this scheme is promising by the way of application in several plants. Apart from computational delay, sampling delay is considered in Chen and Allgower (1998). Sampling delay is different from the computational delay in that control depends on current measurement.

Recently Chen et al propose a new nonlinear model predictive control algorithm which can remove the computational burden and stability issues (Chen *et al.*, 1999*d*; Chen *et al.*, 1999*c*). The proposed nonlinear predictive controller is given in a closed-form and thus no on-line optimisation is required. The stability is also guaranteed. It is also pointed out that the well-known computed torque method or dynamic inversion control is a special case of this model predictive control algorithm (Chen *et al.*, 1999*b*). However this method is only applicable for nonlinear systems with stable zero dynamics and a well-defined relative degree.

The purpose of this paper is to develop a practical model predictive control for general nonlinear systems with guaranteed stability. Several issues in implementation of nonlinear model predictive control, including computational delay, loss of optimality in the optimisation procedure and stability, are addressed. A new stability condition is presented. The performance index in model predictive control is modified to take computational delay into account. A new practical model predictive control algorithm is presented. At each time, the calculation of the control profile is divided into a two stage optimisation problem. In the first stage, a "classic" fixed horizon MPC is solved. In the second stage, a terminal control profile within a prescribed time interval is optimised according to the stability requirement. It should be noted that the terminal control is never implemented and it is only used to generate an initial control profile in each optimisation procedure.

2 Problem formulation

Consider a nonlinear control system

$$\dot{x}(t) = f(x(t), u(t)), \; x(t_0) = x_0 \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are the state and control vectors respectively. It is supposed that the input is subject to constraint

$$u(t) \in U \tag{2}$$

where $0 \in U \subset R^m$ is a compact and convex set.

In order to distinguish the real variables from the variables in the moving horizon time frame, hatted variables are used in the moving horizon time frame in this paper. In general, a nonlinear Model Predictive Control (MPC) problem can be stated as: for any state x at time t, find a continuous function $\hat{u}(\tau; x(t))$:

$[t, t+T] \to U$, in moving horizon time frame such that the performance index

$$J_1 = g(\hat{x}(t+T)) + \int_0^T \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \qquad (3)$$

is minimised where $Q \geq 0$, $R > 0$ and $\hat{u}(\cdot; x(t))$ explicitly indicates the control profile $\hat{u}$ depends on the state measurement $x(t)$ at time t. It is assumed that $g(x)$ is a continuous, differentiable function of x, $g(0) = 0$ and $g(x) > 0$ for all $0 \neq x \in R^n$. A typical choice of $g(x)$ is given by

$$g(x) = x(t)^T P x(t) \qquad (4)$$

where $P \in R^{n \times n}$ is a positive definite matrix.

Then the nonlinear Model Predictive Control (MPC) law is determined by

$$u(t) = \hat{u}(t; x(t)) \qquad (5)$$

Assumptions on the system (1) are imposed:

A1 $f : R^n \times R^m \to R^n$ is twice continuously differentiable and $f(0,0) = 0$. $0 \in R^n$ is an equilibrium of the system with $u = 0$.

A2 System (1) has an unique solution for any initial condition $x_0 \in R^n$ and any piece-wise continuous and right-continuous $u(\cdot) : [0, \infty) \to U$;

A3 The nonzero state of the system (1) is detectable in the cost. That is, $Q^{1/2} x \neq 0$ for all nonzero x such that $f(x, 0) = 0$ (Sepulchre *et al.*, 1996);

A4 All states are available.

To implement the MPC, the constrained nonlinear optimisation problem must be solved on-line. On-line optimisation is a main issue in nonlinear MPC implementation (Allgöwer and Zheng, 1998; Mayne, 1996; Biegler and Rawlings, 1991). Due to the large computational burden, the computational delay time in MPC is significant and cannot be ignored. The optimisation process for MPC with computational delay is shown in Fig. 1.

When the state measurement $x(t)$ is available at the sampling time t, due to the computational time for the solution of the constrained nonlinear optimisation, the open loop optimal control profile $\hat{u}(\tau; x(t)), \tau \in [t, t+\delta]$ yielded by minimisation of the cost (3) is not available until at the time $t+\delta$ where δ is the computational delay. It makes no sense to optimise the control profile $\hat{u}(\tau; x(t))$, $t \leq \tau \leq t+\delta$ at time t. Moreover as will shown later, it also causes mismatching in the MPC implementation and thus degrades system performance.

Hence in this paper the performance index (3) is modified to

$$J = g(\hat{x}(t+T)) + \int_{\delta}^{T} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \tag{6}$$

In each measurement time, the optimisation problem (OP) to be solved on-line can be mathematically formulated as

$$\min_{\hat{u}:[t+\delta, t+T) \to U} J \tag{7}$$

subject to

$$\begin{cases} \text{terminal state constraint } x(t+T) \in \Omega \\ \text{the system dynamics (1)} \end{cases} \tag{8}$$

There are several advantages in formulating the model predictive control in the form of the above OP (7) and (8). The first advantage is the control profile to be optimised is over the period $[t, t+T]$ in the performance index (3) whereas the control profile to be optimised in OP (7–8) is over the shorter period $[t+\delta, t+T]$. Since the optimisation is stoped at the time $t+\delta$, the control $\hat{u}(\tau; x(t))$ over the period $[t, t+\delta)$ is useless. With the same maximum predictive time T, the computational load in $[t+\delta, t+T)$ is smaller than that in $[t, t+T)$. The second disadvantage is, more importantly, that there exists mismatching in optimisation of the original MPC performance index (3) when the computational delay cannot be ignored. The control profile in $[t+\delta, t+T]$ is optimal in the sense that the state $x(t)$ at the time t drived by the control $\hat{u}(\tau; x(t)), t \leq \tau \leq t+\delta$ arrives at the state $\hat{x}(t+\delta)$ at the time $t+\delta$. However, due to the computational delay, the last control profile is employed until the new control profile becomes available. The real state $x(t+\delta)$ which is driven by $\hat{u}(\tau; x(t-\delta)), t \leq \tau \leq t+\delta$ from the state $x(t)$ is different from $\hat{x}(t+\delta)$. Hence the control profile $\hat{u}(\tau; x(t)), \tau \in [t+\delta, t+T]$ is not optimal anymore.

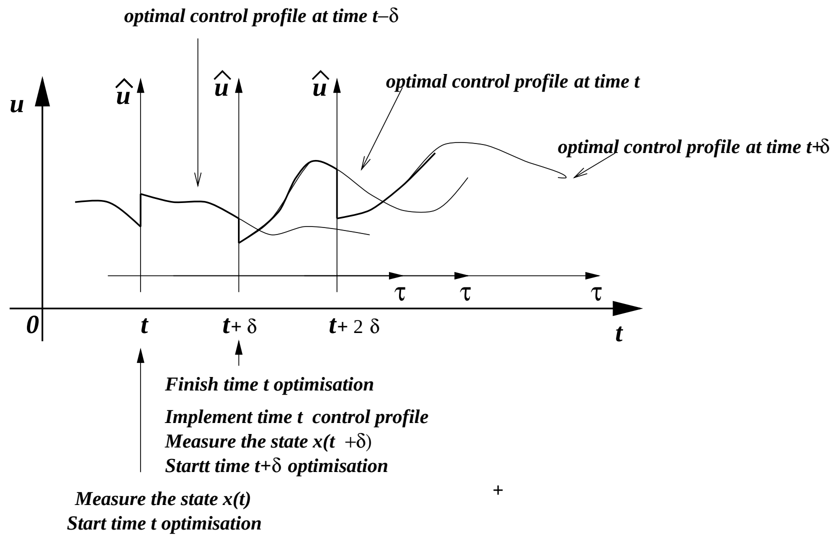

Figure 1: Nonlinear Model predictive Control with computational delay

It should be noted that, in additional to the terminal item in the performance index (6), a constraint on the terminal states is imposed in OP. As will be shown later, this is due to the stability requirement.

In design of Model Predictive Control, one problem is to determine the computational delay off-line. This depends on the computational methods used in solving the OP. The computational issue is an active subject in nonlinear model predictive control and many algorithms have been proposed (Biegler and Rawlings, 1991; Zheng, 1997; Mayne, 1996). Several computational algorithms have been compared and implemented on a continuous stirred tank reactor (Sistu *et al.*, 1993). We will not pursue this direction in this paper. The difficulty in determining the computational delay off-line lies in that the computational times for different optimisation routines are different even for the same optimisation algorithm. It is well known that the computational time is quite different in each nonlinear optimisation procedure for different initial points and optimum. This makes it difficult to determine the length of time taken to perform the optimisation off-line. However it will be shown that although the global optimal solution is not achieved and only a local minimum solution exits, or worse the optimisation procedure is not finished within the prescribed time period, stability can still be guaranteed in the MPC algorithm proposed in this paper. This makes it is possible to determine the computational delay off-line.

Remark 2.1 *Chen and Allgower (1998) have considered the sampling time issue in MPC implementation. That is, the open-loop control profile is implemented until the updating new measurement arrives. This is different from the computational delay considered in this paper. More specifically, the closed-loop control profile in Chen and Allgower (1998) is defined as*

$$u(\tau) = \hat{u}(\tau; x(t)), \tau \in [t, t + \delta_1] \tag{9}$$

where δ_1 is the sampling time. This implies that the control depends on the current state. However the MPC algorithm with computational delay can be considered as

$$u(\tau) = \hat{u}(\tau; x(t)), \tau \in [t + \delta, t + 2\delta] \tag{10}$$

This implies that the control depends on the past state measurement. The sampling time is not considered in this paper since it is much smaller than the computational time in Nonlinear MPC. However if the sampling time needs to be considered, it is easy to handle it in our algorithm.

3 Terminal Region and MPC Algorithm

For the sake of simplicity, in this paper asymptotic stability of a system means the system has an (asymptotically) stable equilibrium at the origin.

Definition 1: The terminal region Ω in MPC is defined as a region where once the terminal state $\hat{x}(t+T)$ under the control $\hat{u}(\tau; x(t))$, $\tau \in [t, t+T)$ yielded by solution of the problem OP (7–8), arrives, there exists a control $\hat{u} : [t+T, \infty) \to U$ which can steer the state to the origin.

Lemma 1: Suppose that the system (1) satisfies Assumptions (A1–A3). If there exists an $\alpha > 0$ such that for any $x \in \Omega(\alpha) = \{x \in R^n : g(x) \le \alpha\}$, there exists control $u \in U$ satisfying

$$\frac{\partial g(x)}{\partial x} f(x, u) + x^T Q x(t) + u^T R u \le 0, \tag{11}$$

then $\Omega(\alpha)$ is a terminal region for the nonlinear system (1) under the MPC with the performance index (6).

Based on Lemma 1, a MPC algorithm with Guaranteed Stability (MPCGS) is proposed as follows.

MPCGS Algorithm:

1. Measure the state $x(t_0)$. No control is imposed during the time interval $[t_0, t_0 + \delta]$. In the meanwhile, solving the OP (7) and (8) at $t = t_0$ and $x(t) = x(t_0) = x_0$ yields $\hat{u}(\tau; x(t)), \tau \in [t + \delta, t + T)$.

2. Calculate $\hat{u}(\tau; x(t))$, $\tau \in [t + T, t + T + \delta)$, according to the condition (11).

3. Implement the optimal open-loop control $\hat{u}(\tau; x(t))$ for the time interval $[t + \delta, t + 2\delta]$. Measure the state at the time $t + \delta$ and in the meanwhile, using the control $\hat{u}(\cdot; x(t))' : [t + 2\delta, t + T + \delta] \rightarrow U$, defined as

$$\hat{u}(\tau; x(t+\delta))' = \begin{cases} \hat{u}(\tau; x(t), \tau \in [t + 2\delta, t + T] \\ \hat{u}(\tau; x(t)), \tau \in [t + T, t + T + \delta] \end{cases} \tag{12}$$

   as an initial solution of $\hat{u}(\tau; x(t + \delta))$, solve the constrained optimisation problem (7) and (8).

4. Let $t = t + \delta$ and Go to Step 2.

Remark 3.1 *When solving OP, the global minimisation or even local minimisation of the objective cost in the above MPCGS algorithm is not required. This is important for engineering implementation. Since this problem is solved on-line, it is difficult to guarantee the global optimal solution, and sometimes even the local minimum cannot be achieved due to the time limitation. One of the key features of the MPCGS algorithm is the way it determines an initial control profile for each optimisation process based on the condition (11), which distinguishes it from other existing MPC algorithms. When the optimisation algorithm starts form this initial control profile, as we will show later, the feasibility and stability of the proposed MPC are guaranteed.*

Remark 3.2 *In each iteration of the MPCGS, two optimisation problems need to be solved. One is the "classic" MPC optimisation problem. The other is the optimisation problem in Step 2 to determine the terminal stabilising control. For any* $x \in \Omega$*, it is easy to solve this feasibility problem by a nonlinear optimisation procedure. Especially, for an affine nonlinear system, it will be shown later that the non-linear terminal stabilising control is given in a closed-form and on-line optimisation is not required.*

Remark 3.3 *How to determine the computational delay off-line is important in the proposed algorithm. It involves a trade-off between the computational burden, robustness and the control performance. If the computation delay is chosen too small, the performance is degraded although the stability can be guaranteed since optimisation cannot be finished in each iteration. If the computation delay is too large, the proposed MPCGS has less robustness against uncertainties and disturbances. How to analyse the robust stability of the closed-loop system with computational delay is an interesting subject and some results have been presented, for example see Chen* et al. *(1997), Chen and Gu (1991). However it is beyond the scope of this paper.*

4 Stability and Feasibility

Definition 2: The feasibility region refers to a set of initial state points $x(t_0)$ for which the state trajectory yielded by solving the OP terminates in the region $\Omega(\alpha)$.

Lemma 2 guarantees that if the nonlinear MPC problem is feasible at t_0, it is also feasible for all $t > t_0$. This follows from modification of the standard argument; for example, see Genceli and Nikolaou (1993), Michalska and Mayne (1993), Chen and Allgower (1998).

Lemma 2: Consider the nonlinear system (1) satisfying Assumptions A1-A3 with the input constraint (2). The feasibility of the MPCGS algorithm at time t_0 implies feasibility for all $t > t_0$.

Theorem 1: Suppose that Assumptions A1)–A3) are satisfied and the MPCGS algorithm is feasible at the time $t = t_0$. The MPCGS algorithm for the system (1) is asymptotically stable if there exists control $u(t)$ such that the following condition is satisfied

$$\frac{\partial g(x)}{\partial x} f(x,u) + x^T Q x + u^T R u \leq 0 \tag{13}$$

for any state x belonging to the terminal region Ω.

Corollary: When the nonlinear system considered is affine, given by

$$\dot{x}(t) = f_1(x) + g_1(x) u \tag{14}$$

condition (11) reduces to that

$$\frac{\partial g(x)}{\partial x} f_1(x) - \frac{\partial g(x)}{\partial x} g_1(x) R^{-1} g_1(x)^T \left(\frac{\partial g(x)}{\partial x} \right)^T + x^T Q x \leq 0. \tag{15}$$

and

$$u = -R^{-1} g_1(x)^T \left(\frac{\partial g(x)}{\partial x} \right)^T \in U \tag{16}$$

The terminal region is given by

$$\Omega(\alpha) = \{ x \in R^n : \text{conditions (15–16) holds for all } g(x) \leq \alpha \}$$

If condition (15) is satisfied, a control $u(t)$ which can steer the state to the origin is given by (16). In particular, for a linear system

$$\dot{x} = Ax + Bu,$$

when $g(x)$ is chosen as (4), condition (15) becomes

$$A^T P + PA - PBR^{-1} B^T P + Q \leq 0. \tag{17}$$

If this condition is satisfied, the terminal region is the whole state space, that is, the MPCGS is global asymptotically stable.

Remark 4.1 *Corollary shows that for linear systems, the stability result in this paper reduces to that of the so-called Fake Algebraic Riccati Equation (Bitmead* et al.*, 1990; Poubelle* et al.*, 1988). Hence the stability condition for linear model predictive control can be considered as a special case of our stability condition.*

Remark 4.2 *When the terminal control is restricted to be linear and the terminal weighting item is given by (4), the stability condition for nonlinear MPCGS proposed in this paper reduces to the result in Chen and Allgower (1998),Michalska and Mayne (1993). That is, the terminal region is given in (13) by replacing* $u(t)$ *with* $u(t) = kx(t)$ *where* $k \in R^{m\times n}$ *is a constant matrix.*

Remark 4.3 *Before using the proposed MPCGS, the terminal region should be determined off-line. To determine the maximum attraction region of the proposed MPCGS, it is required to calculate the maximum* α *such that (13) holds for any* $x \in \Omega(\alpha)$*. This can be done by making an iteration of the optimisation problem*

$$\max_{x\in\Omega(\alpha)} \min_{u\in U}\{\frac{\partial g(x)}{\partial x} f(x,u) + x^T Qx + u^T Ru\} \tag{18}$$

This is a maximin optimisation problem. The maximum α *is given by increasing* α *such that the value in (18) is nonnegative. In general it is not trivial to solve this maximin problem. A Linear Matrix Inequalities (LMI's) approach is developed to approximately solve this problem in (Chen* et al.*, 1999*a*).*

However for an affine nonlinear system (14), with the help of the result in the Corollary, the maximum terminal attraction region can be achieved by solving the following optimisation problem

$$\begin{aligned} &\max \alpha \\ &\textit{s.t. conditions (15–16) hold for all } g(x) \leq \alpha. \end{aligned} \tag{19}$$

This is a standard semi-definite optimisation problem which has been extensively discussed for example see Polak et al. *(1984) for general setting or see Michalska and Mayne (1993) for MPC.*

5 A simulation example

Consider the van de Vusse reactor kinetic scheme where at an operating point, the process model is given by (Mutha *et al.*, 1997; Doyle *et al.*, 1995)

$$\dot{x}_1 = -50x_1 - 10x_1^2 + (10 - x_1)u$$

$$\begin{aligned}\dot{x}_2 &= 50x_1 - 100x_2 - x_2 u \\ y &= x_2\end{aligned} \tag{20}$$

and x_1 and x_2 are the deviation variables for the concentration of component A and the concentration of the component B respectively. The control u is the inlet flow rate.

The control problem for this system is to control the concentration of the component B, y, by the inlet flow rate. The nonlinear model predictive control proposed in this paper is designed for this plant.

The performance index is chosen as (6) with the weighting matrices

$$P = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}; \tag{21}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}; \tag{22}$$

and

$$R = 0.1 \tag{23}$$

To determine the applicability and the stability region of the proposed nonlinear model predictive control, the terminal region needs to be determined.

Since this is an affine nonlinear system, the terminal region is determined by the condition (15) in the Corollary. For the plant (20) with the above weighting matrices in the performance index (6), it can be shown that this condition (15) is satisfied for all states in the state space. The closed-loop system under the proposed nonlinear model predictive controller is globally stable.

Following (16), the terminal stabilising nonlinear controller is given by

$$\begin{aligned}u(t) &= -10 \begin{bmatrix} 10 - x_1 & -x_2 \end{bmatrix} Px \\ &= -5(10 - x_1)x_1 + x_2^2\end{aligned} \tag{24}$$

Now we consider the case where the terminal control is linear (Chen and Allgower, 1998; Michalska and Mayne, 1993). First the nonlinear system (20) is linearised around the original point. The linearised system is given by

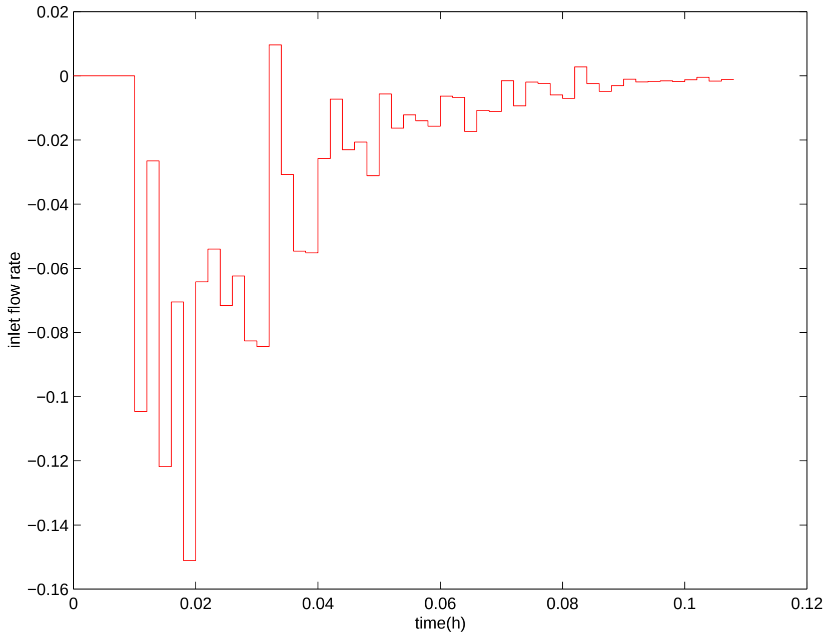

Figure 2: Control history

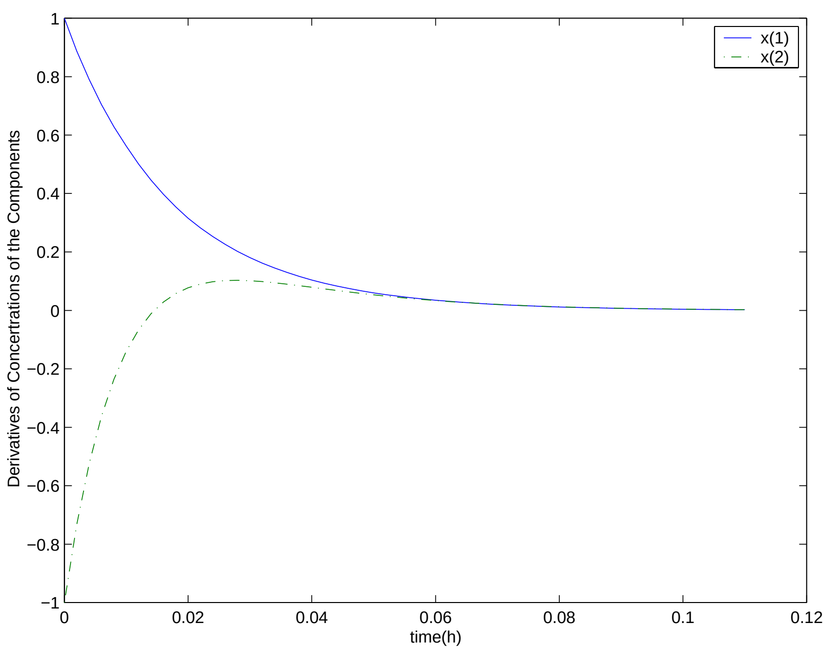

Figure 3: Response

$$\dot{x} = \begin{bmatrix} -50 & 0 \\ 50 & -100 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \end{bmatrix} u$$

Solving the LQ optimal control problem with the weighting matrices Q and R in (22–23) yields the feedback gain

$$k = \begin{bmatrix} 0.3148 & 0.3248 \end{bmatrix};$$

The terminal stabilising control u is given by

$$u(t) = kx(t), \tag{25}$$

The stability condition is given in (13) by replacing $u(t)$ with (25). By solving this semi-infinity optimisation problem (Chen and Allgower, 1998; Michalska and Mayne, 1993), the terminal region is given by

$$\Omega = \{x \in R^2 | x^T P x \leq 9.63\}$$

Comparing with the global stability of the nonlinear terminal control law, the above terminal region given by the linear terminal control law (25) is much small. Hence the new stability condition in this paper provides a much less conservative result for nonlinear MPCGS.

The simulation results under the proposing nonlinear MPCGS are shown in Fig 2 and 3. In the simulation, the predictive time is chosen as 0.05 hour. The computational delay is chosen as 0.01 hour (36 seconds). The 4-5th order Runge-Kutta numerical integration method with a fixed step size $0.001h(3.6s)$ is used for both prediction in optimisation and system model simulation. The whole simulation is performed in the Matlab environment with the nonlinear optimisation implemented by Optimisation Toolbox. The initial state of the plant is measured at time $t = 0$ and MPCGS starts the optimisation procedure. Before the control profile is yielded by optimisation, no control effort is imposed on the reactor, that is, the control effort is zero. When the control profile is available after the computational delay, the control profile is implemented and new states are measured for starting the new optimisation procedure. It's difficult to determine whether or not the global minimisation of the cost is achieved in each optimisation procedure. Actually in some optimisation procedures, the optimisation process is terminated since the prescribed maximum number of evaluations is arrived at (the maximum number for evaluations in the optimisation is set to be 100 times of the number of variables in the simulation). This implies that in some cases the local minimum is not yet achieved. The yielded control profile is shown in Figures 2 which is not smooth as expected. This may be due to loss of optimality in nonlinear optimisation but the stability is guaranteed by the proposed MPCGS algorithm.

6 Conclusion

A practical model predictive control scheme for general nonlinear systems was presented in this paper. The main advantages and features of the proposed control scheme are that

- the computational delay for on-line optimisation is accounted for;
- no global optimal solution of on-line nonlinear optimisation is required;
- a nonlinear terminal control is allowable and hence the applicability and stability regions of the nonlinear model predictive control is enlarged;
- the feasibility of the proposed scheme implies the stability when the proposed initial control profile is used in each optimisation routine.

The proposed nonlinear model predictive control was illustrated by a numerical example. Comparing with the linear terminal control, the stability region is much enlarged. This is because a new stability condition was derived in this paper. Several obstacles for implementation of model predictive control of nonlinear systems, including loss of optimality in on-line nonlinear optimisation, computation delay due to extensive computational burden and stability issue, have been removed in this paper. Therefore it takes a step toward developing practical model predictive control for general nonlinear systems with unstable zero dynamics or an ill-defined relative degree. Further investigation will focus on robustness of nonlinear model predictive control and reducing on-line computational burden using control parametrisation techniques (Gawthrop *et al.*, 1998; Vassiliadis *et al.*, 1994).

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Appendix

A: Proof of Lemma 1

Suppose that there exists a control $u(t) \in U$ such that the condition (11) is satisfied for all $x \in \Omega(\alpha)$. Then a corresponding $u(t) \in U$ can be calculated according to the condition (11) for any $x \in \Omega(\alpha)$. Let $u(t)$, $t \in [T+t, \infty)$ be such a control profile. Let $V(x) = g(x)$ be a Lyapunov function candidate associated with the state trajectory of the system (1) under this control profile. It is noted that $g(x)$ is a continuous, differentiable function of x. It follows from (11) that

$$\dot{V}(x) = \frac{\partial g(x)}{\partial x} f(x,u) \leq -x^T Q x - u^T R u \leq -x^T Q x \leq 0 \tag{26}$$

First the inequality (26) implies that the Lyapunov function $V(x)$ is non-increasing along the state trajectory. If the state of the system is within the region $\Omega(\alpha)$, it remains in this region. The inequality (26) also implies that $\dot{V}(x) = 0$ only if $u(t) = 0$ and the state trajectory belongs to the following set

$$N = \{x(t) \in R^n : x(t)^T Q x(t) = 0, t \in R\}. \tag{27}$$

Since $Q^{1/2} x \neq 0$ for all nonzero x such that $f(x, 0) = 0$, one can prove that the only positive half trajectory contained entirely in N is $x(t) \equiv 0$ (Sepulchre *et* *al.*, 1996; Gyurkovics, 1998). Remember that $V(0) = g(0) = 0$ and $V(x) = g(x) > 0$ for all $x \neq 0$. This implies that

$$\|x(t)\| \to 0 \text{ as } t \to \infty$$

Hence under the control satisfying the condition (11), $\Omega(\alpha)$ is an attraction region of the system (1) with an equilibrium point $x = 0$. According to Definition 1, $\Omega(\alpha)$ is a terminal region of the MPC. QED.

B: Proof of Lemma 2

When a model predictive control algorithm is implemented, due to the computational delay and no prior information about the control input, in the proposed MPCGS in this paper, no control effort is imposed in the interval $[t_0, t_0 + \delta)$. The system (1) becomes

$$\dot{x} = f(x, 0), \; x(t_0) = x_0 \tag{28}$$

At the time $t_0 + \delta$, the free system (1) arrives the state $x(t_0 + \delta)$. Since the MPCGS algorithm is feasible at the time t_0, it implies that we can find an (open-loop) control profile $\hat{u}(\cdot; x_0)$: $[t_0 + \delta, t_0 + T] \to U$ which can drive the state $x(t_0 + \delta)$ to arrive in the terminal set $\Omega(\alpha)$ at the terminal time $t_0 + T$ by solving the OP (7–8) at time t_0.

Now we consider the feasibility of the NGPC at the time $t_0 + \delta$. At the time $t_0 + \delta$, the new state measurement $x(t_0 + \delta)$ is available and the optimisation problem OP with the initial state $x(t_0 + \delta)$ needs to be solved. In our MPCGS algorithm, this procedure starts from the following initial control profile $\hat{u}(\cdot, x(t_0 + \delta))' : [t_0 + 2\delta, t_0 + T + \delta] \to U$, defined as in (12). The assumption that the MPCGS is feasible at the time t_0 implies that the first part of $\hat{u}(\cdot, x(t_0 + \delta)'$ can steer the state into the terminal region at the time $t_0 + T$. The second part of $\hat{u}(\cdot, x(t_0 + \delta)'$ is calculated in Step 2 in MPCGS which satisfies the condition (11). According to the MPCGS Algorithm in Section 3, $\hat{u}(\tau, x_0), \tau \in [t_0 + \delta, t_0 + T]$ is implemented in the time interval $[t_0 + \delta, t_0 + 2\delta]$, that is, $u(\tau) = \hat{u}(\tau; x_0), \tau \in [t_0 + \delta, t_0 + 2\delta]$. As a result, the system state at the time $t + 2\delta$ is given by $x(t_0 + 2\delta)= \hat{x}(t_0 + 2\delta)$

under the assumption that there is no uncertainty and noise. Hence according to Lemma 1, the terminal state $\hat{x}(t_0+T+\delta)'$ under the control profile $\hat{u}'$ remains in the terminal region Ω. Clearly, the optimisation problem at $t_0+\delta$ is feasible since at least $\hat{u}(\cdot, x(t_0+\delta)'$ is a feasible solution. Repeating the above deduction, one can prove that the feasibility of the MPCGS algorithm at the time $t = t_0$ implies feasibility for all $t > t_0$. QED

C: Proof of Theorem

First we show that $x = 0$ is an equilibrium of the closed loop system under the proposed MPCGS.

When $x(t_0) = 0$ and under no control effort, it follows from $f(0,0) = 0$ that

$$x(\tau) = 0 \text{ for } \tau \in [t_0, t_0 + \delta]$$

That is, $x(t_0 + \delta) = 0$. It is clear that the optimal control profile which minimise the cost (6) is given by

$$\hat{u}(\tau; x(t_0)) = 0, \tau \in [t_0 + \delta, t_0 + T]$$

and as a result,

$$u(\tau) = 0, \text{ for } \tau \in [t_0 + \delta, t_0 + 2\delta].$$

and

$$x(\tau) = 0 \text{ for } \tau \in [t_0 + \delta, t_0 + 2\delta].$$

Repeating the above process, one can show that $x = 0$ is an equilibrium of the closed-loop system consisting of the nonlinear system (1) and the proposed MP-CGS.

For the sake of notational simplicity, in the proof of Theorem, $\hat{u}(\tau; x(t))$ is replaced by $\hat{u}(\tau)$ when this will not cause the confusion. Similarly, the state under the control $\hat{u}(\tau; x(t))$ in the moving horizon time frame is denoted by $\hat{x}(\tau)$. Clearly, the performance index (6) in the time t depends on the initial $x(t)$ and the control profile $\hat{u}(\tau; x(t))^*$ used in the moving horizon time frame, denoted by $J(x(t), \hat{u}(\cdot; x(t))^*)$. Let $V(x(t+\delta))$ denote such a performance cost under the control profile $\hat{u}(\tau; x(t))$ achieved by solving the optimisation problem OP, that is,

$$\begin{aligned} V(x(t)) &= J(x(t), \hat{u}(\cdot; x(t)) \\ &= g(\hat{x}(t+T)) + \int_{\delta}^{T} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \end{aligned} \quad (29)$$

Let $V(x)$ be a candidate Lyapunov function for the closed-loop system. Similar to the proof of $x = 0$ being an equilibrium of the closed-loop system under the proposed MPCGS scheme in Lemma 2, it is trivial to show that $V(0) = 0$ since $g(0) = 0$.

Now we prove that the Lyapunov function $V(x)$ along the state trajectory under the proposed MPCGS algorithm is non-increasing. Consider the Lyapunov function value $V(x(t+\delta))$. According to MPCGS Algorithm in Section 3, the control profile $\hat{u}(\tau; x(t+\delta))$, $\tau \in [t+2\delta, t+T+\delta)$, is obtained by starting the optimisation procedure from the initial control profile u' defined in (12).

Define the performance cost in (6) for the system under this control profile $\hat{u}(\tau; x(t+\delta))'$ in (12) as

$$V(x(t+\delta))' = g(\hat{x}(t+T+\delta)') + \int_{2\delta}^{T+\delta} \hat{x}(t+\tau)'^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)'^T R \hat{u}(t+\tau)' d\tau \quad (30)$$

where $\hat{x}(t+\tau)', \tau \in [2\delta, T+\delta]$ denotes the system's state under the control profile $\hat{u}(\tau; x(t+\delta))'$ in (12).

The equation (30) can also be written as

$$\begin{aligned} V(x(t+\delta))' &= g(\hat{x}(t+T+\delta)') + \\ &+ \int_{2\delta}^{T} \hat{x}(t+\tau)'^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)'^T R \hat{u}(t+\tau)' d\tau + \\ &+ \int_{T}^{T+\delta} \hat{x}(t+\tau)'^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)'^T R \hat{u}(t+\tau)' d\tau \end{aligned} \quad (31)$$

Comparing (29) with (31) yields

$$\begin{aligned} V(x(t+\delta))' - V(x(t)) &= g(\hat{x}(t+T+\delta)') - g(\hat{x}(t+T)) + \\ &+ \int_{T}^{T+\delta} \hat{x}(t+\tau)'^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)'^T R \hat{u}(t+\tau)' d\tau - \\ &- \int_{\delta}^{2\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \end{aligned} \quad (32)$$

Remember that $g(x)$ is continuously differentiable with respect to x. One has

$$\begin{aligned}
& g(\hat{x}(t+T+\delta)') - g(\hat{x}(t+T)') \\
= & \int_T^{T+\delta} \frac{\partial g(\hat{x}(t+\tau)')}{\partial \hat{x}(t+\tau)} \dot{\hat{x}}(t+\tau)' d\tau \\
= & \int_T^{T+\delta} \frac{\partial g(\hat{x}(t+\tau)')}{\partial x(t+\tau)'} f(\hat{x}(t+\tau)', \hat{u}(t+\tau)') d\tau
\end{aligned} \tag{33}$$

Following from the definition of $\hat{u}'$ in (12),

$$\hat{u}(t+\tau; x(t+\delta))' = \hat{u}(t+\tau; x(t)) \text{ for } \tau \in [t+\delta, t+T].$$

It implies that

$$g(\hat{x}(t+T))' = g(\hat{x}(t+T)). \tag{34}$$

In the time interval $[t+T, t+T+\delta]$, the control $\hat{u}(\tau)'$ satisfies condition (13). Hence the following inequality holds

$$\begin{aligned}
& \int_T^{T+\delta} \frac{\partial g(\hat{x}(t+\tau)')}{\partial x(t+\tau)'} f(\hat{x}(t+\tau)', \hat{u}(t+\tau)') d\tau \\
\leq & \int_T^{T+\delta} \hat{x}(t+\tau)'^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)'^T R \hat{u}(t+\tau)' \tau
\end{aligned} \tag{35}$$

Invoking (34) and (35) with (33) gives

$$g(\hat{x}(t+T+\delta)') - g(\hat{x}(t+T)) \leq \int_T^{T+\delta} \hat{x}(t+\tau)'^T Q \hat{x}(t+\tau)' + \hat{u}(t+\tau)'^T R \hat{u}(t+\tau)' \tau \tag{36}$$

Combining the above inequality with (32) yields

$$V(x(t+\delta))' - V(x(t)) \leq -\int_\delta^{2\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \tag{37}$$

In the proposed MPCGS algorithm, the optimisation procedure at time $t+\delta$ starts from the control profile (12). Minimisation of the performance index (6) yields the control profile $\hat{u}(\tau; x(t+\delta))$. The performance cost under this control profile is given by $J(x(t+\delta), \hat{u}(\cdot; x(t+\delta)))$, ie., $V(x(t+\delta))$. The Lyapunov function $V(x(t+\delta))$ along the state trajectory satisfies

$$V(x(t+\delta)) \leq V(x(t+\delta))' \tag{38}$$

Combining (37) and (38) gives

$$V(x(t+\delta)) - V(x(t) \leq -\int_\delta^{2\delta} \hat{x}(t+\tau)^T Q \hat{x}(t+\tau) + \hat{u}(t+\tau)^T R \hat{u}(t+\tau) d\tau \tag{39}$$

Remember that the control profile $\hat{u}(\cdot; x(t))$ is implemented in the time interval $[t+\delta, t+2\delta]$ until the next control profile is yielded by optimisation. That is,

$$u(t+\tau) = \hat{u}(t+\tau; x(t)), \tau \in [\delta, 2\delta]$$

and

$$x(t+\tau) = \hat{x}(t+\tau), \tau \in [\delta, 2\delta]$$

We have

$$\int_{t+\delta}^{t+2\delta} \hat{x}(\tau)^T Q \hat{x}(\tau) + \hat{u}(\tau)^T R \hat{u}(\tau) d\tau = \int_{t+\delta}^{t+2\delta} x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) d\tau$$

The inequality (39) is equivalent to that of

$$V(x(t+\delta)) - V(x(t)) \leq -\int_{\delta}^{2\delta} x(t+\tau)^T Q x(t+\tau) + u(t+\tau)^T R u(t+\tau) d\tau \leq 0 \quad (40)$$

where the latter inequality follows from $Q \geq 0$ and $R > 0$. Hence the Lyapunov function (29) is non-increasing along the closed-loop system's trajectory.

Repeatedly using the inequality (40) yields

$$\begin{aligned} V(x(\infty)) - V(x(t_0)) &\leq -\int_{t_0+\delta}^{\infty} x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) d\tau \\ &\leq -\int_{t_0+\delta}^{\infty} x(\tau)^T Q x(\tau) d\tau. \end{aligned} \quad (41)$$

Since $V(x)$ is monotonically non-increasing and bounded from below by zero, the integral in the right side of the inequality (41) converges and hence exists and is bounded.

In addition, it can be shown that $x(t)$ is uniformly continuous in t on $[t_0, \infty)$ under Assumption A1 (Desoer and Vidyasagar, 1975). Using Baralema's Lemma, this implies that $x^T Q x \to 0$ as $t \to \infty$ (Hahn, 1967, proof of Theorem 26.4) or (Khalil, 1992). When $Q > 0$ or $Q \geq 0$ and $Q^{1/2} x \neq 0$ for all nonzero x such that $f(x, 0) = 0$, it is concluded that $x(t) \to 0$ as $t \to \infty$ (Sepulchre *et al.*, 1996; Magni and Sepulchre, 1997). Hence the equilibrium point $x = 0$ of the system (1) is asymptotically stable. QED

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