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Adaptive Motion/Force Control of Mechanical Systems with Nonholonomic Pfaffian Constraints

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Abstract

The contribution of this work relates to two subjects. In the first part of the paper we present a novel dynamic description of mechanical (Lagrangian) systems with nonholonomic (Pfaffian) constraints. This development was motivated by the need for a convenient and simple dynamic model for the controller. Essentially, the new element of this development is QR decomposition of the constraint matrix. Following this decomposition, we have proven a new dynamic property of the considered system (see Property 3). This property allows us to express the system dynamics in terms of a new reduced- order state vector. The second part of the work is concerned with development of the adaptive position/force controllers for general Lagrangian systems with Pfaffian constraints. Two cases are elaborated: (i) motion/force tracking control with known system parameters, and (ii) adaptive control in the presence of uncertainties. The adaptive control law guarantees the uniform ultimate boundness of the tracking error.

1. Introduction

The dynamics of nonlinear systems with nonintegrable differential constraints has a long research history (early works on the subject are reviewed in [7]). Interest in these types of systems is motivated by the challenging problem of their dynamic description and the broad spectrum of practical machines subjected to nonholonomic constraints. It is sufficient to say that all mechanisms which involve rolling contact between surfaces, (vehicles, walking machines, multifinger robotic hands, some cutting machines, etc.) are systems with nonholonomic constraints. Conventional methods of dynamic modelling include Lagrange multipliers and embedding constraints in the expression of the kinetic energy [8,9,10]. Both techniques, effective and sophisticated as they are from a mechanical point of view, produce somewhat cumbersome dynamics models, as far as controller design is concerned. Moreover, the underlining idea of these approaches, (using the Lagrangian coordinates selected for the unconstrained system and eliminating the constraint forces), appears to be in contradiction with the very nature of the control problem.

Control of mechanical systems with nonholonomic constraints is a subject that has generated significant interest in recent years [1,2,5,6, 11-14]. This problem has specificity which distinguish it from other control problems; in particular, the classical position tracking problem is hardly applicable to the constrained motion control. Constraints are usually resulting from contact interaction of the machine and environment. Therefore, the control of the forces of interactions is at least as important as the position control. Applying a conventional technique one can reduce the number of state variables with those which provide motion complying with constraints . Physically this implies that the constraints provide the necessary reactions. However, the magnitude of the constrained forces may be large, as a result of control efforts or position errors. The real constraints may not be able to provide such forces, or such strong interaction may be unacceptable from other requirements (for example, technological requirements). Whatever sophisticated adaptive control algorithm there may be, it is impractical unless position and force of interaction are controlled simultaneously. Simultaneous position/force control of nonholonomic systems requires a specific dynamic description of the mechanical systems which includes a reduced set of the states and forces of interaction. To the best of our knowledge, such a dynamic model for nonholonomic systems with Pfaffian constraints has not yet been reported in the literature.

In this paper we consider Lagrangian mechanical systems with general nonholonomic Pfaffian constraints (catastatic and acatostatic [8]). The developed approach is based on QR decomposition of the constraint matrix. With this decomposition we were able to define a reduced set of "inherent states" which define the system motion in conformity with the nonholonomic constraints. A new dynamic equation is derived in terms of the inherent variables and the constraint forces; the latter are expressed in terms of Lagrange multipliers. This dynamic description allows us to prove an important relationship between the coefficients of the Lagrange equation and the elements of the decomposition ("skew-symmetric property", see Property 5).

An adaptive controller design procedure presented in the second part of the paper relies essentially on the above-mentioned skew-symmetric property of the dynamic model. Our objective is to control position and constraint force simultaneously. In order to present the main ideas and approach in a simple and clear manner, we firstly consider the position/force control with known parameters (Section 3.1). Exponential stability of the developed control law has been proven for this case. In the following section the developed procedure is extended to uncertain systems. The adaptive control law does not include any parameter-estimation procedure that makes it relatively simple and convenient for practical implementation. An essential result here is Theorem 2, which proves that, with the proposed control law, the position and force tracking errors are uniformly ultimately bounded with a controllable bound.

2. Dynamic Equation with Pfaffian Constraints

In this section, we consider a mechanical system with n degree-of-freedom whose generalized coordinates are $q_1, q_2, \ldots, q_n$. The Lagrange equations describing the motion of the system are

$$D(\mathbf{q})\ddot{\mathbf{q}} + F(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}} + G(\mathbf{q}) = \tau + \tau' \quad (1)$$

where the $n \times n$ matrix $D(\mathbf{q})$ is positive definite and symmetric, and is related to the inertial properties of system [8], the vector function $F(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}}$ is in general a nonlinear function of its arguments and $\tau = [\tau_1,\ \tau_2, \ldots \tau_n]^T$.

Two simplifying properties should be noted about this dynamic structure.

Property 1: A suitable definition of $F(\mathbf{q}, \dot{\mathbf{q}})$ makes the matrix $(\dot{D} - 2F)$ skew-symmetric.

Property 2: $D(\mathbf{q})$, $F(\dot{\mathbf{q}}, \mathbf{q})$, and $G(\mathbf{q})$ are bounded in $\mathbf{q}$, and $F(\dot{\mathbf{q}}, \mathbf{q})$ is linear in $\dot{\mathbf{q}}$, therefore, there exist positive constants κ_i, $i = 1, 2, 3$, such that

$$\begin{aligned} \|D(\mathbf{q})\| &\leq \kappa_1 \\ \|F(\dot{\mathbf{q}}, \mathbf{q})\| &\leq \kappa_2 \|\dot{\mathbf{q}}\| \\ \|G(\mathbf{q})\| &\leq \kappa_3 \end{aligned} \quad (2)$$

It should be noted that in the second property the exact values of κ_i $(i = 1, 2, 3)$ are not necessary for the controller design.

Let us consider the situation where the system is subjected to an additional $p (p < n)$ independent nonholonomic Pfaffian constraints of the form [8] [9]

$$L\dot{\mathbf{q}}(t) = W(\mathbf{q}(t), t) \quad (3)$$

where $L \in R^{p \times n}$ and W are (at least one picewise differentiable) functions of the generalized coordinates and time.

We note that the Pfaffian constraint equation (3) is very general. It includes the classical nonholonomic constraints as a special case [7]. The constraints may be scleronomic or rheonomic, catastatic or acatastatic [8].

The effect of the constraints can be viewed as restricting the dynamics to the manifold Ω defined by

$$\Omega = \{(\mathbf{q},\ \dot{\mathbf{q}}) | L\dot{\mathbf{q}}(t) = W(\mathbf{q}(t), t)\}$$

When the nonholonomic constraints (3) are imposed on the mechanical systems (1), these constraints may be thought of as imposing additional constraint forces, τ_r', on the system, thereby altering the set of the equation (1) to [8]

$$D(\mathbf{q})\ddot{\mathbf{q}} + F(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{q}} + G(\mathbf{q}) = \tau_r + \tau_r', \quad r = 1, 2, \ldots, n, \quad (4)$$

where the constraint forces τ' are expressed as

$$\tau' = L^T \lambda \quad (5)$$

where $\lambda = [\lambda_1, ...\lambda_p]^T$ is the associated Lagrangian multipliers.
Performing a QR decomposition of the constraint matrix L, one obtain

$$\begin{aligned} L^T(\mathbf{q}) &= Q(\mathbf{q})R(\mathbf{q}) \\ &= [Q_1(\mathbf{q})\ Q_2(\mathbf{q})] \begin{bmatrix} R_1(\mathbf{q}) \\ 0 \end{bmatrix} \\ &= Q_1(\mathbf{q})R_1(\mathbf{q}) \end{aligned} \quad (6)$$

where $Q(\mathbf{q}) \in R^{n\times n}$, $R_1(\mathbf{q}) \in R^{p\times p}$ are nonsingular; the columns of Q_1 are orthonormal and are constructured from p linearly independent columns of L^T so that $R(Q_1) = R(L^T)$, the columns of Q_2 are chosen from the orthogonal complement of $R(Q_1)$ so that $R(Q_2) = N(L)$. Then we have

$$R^n = R(Q_1) \oplus R(Q_2).$$

Note that in this paper, $R(\cdot)$, $N(\cdot)$ denote rang and null spaces, respectively.
Based on the above decomposition, the following property could be obtained.
Property 3:

$$\begin{aligned} Q_1^T Q_1 &= I_p &(7) \\ Q_1^T Q_2 &= 0 &(8) \\ LQ_2 &= 0. &(9) \end{aligned}$$

In view of the decomposition (6), the constraint equations (3) can be rewritten as

$$Q_1^T(\mathbf{q})\dot{\mathbf{q}}(t) = R_1^{-T}(\mathbf{q})W(\mathbf{q}(t), t) \quad (10)$$

It can easily be seen that the general solution to (3) or equivalently (10) can then be expressed as

$$\begin{aligned} \dot{\mathbf{q}} &= Q_2(\mathbf{q})\dot{\mathbf{v}} + Q_1(\mathbf{q})R_1^{-T}(\mathbf{q})W(\mathbf{q}(t), t) \\ &= Q_2(\mathbf{q})\dot{\mathbf{v}} + H(\mathbf{q}, t) \end{aligned} \quad (11)$$

where $H(\mathbf{q}, t) = Q_1(\mathbf{q})R_1^{-T}(\mathbf{q})W(\mathbf{q}(t), t)$ and $\mathbf{v} \in R^{n-p}$ is the vector of new variables which we will call "inherent (generalized) velocities". It should be noted that the $(n-p)$-vector $\dot{\mathbf{v}}$ represents internal states, so that $(\mathbf{q}(t),\ \dot{\mathbf{v}})$ is sufficient to describe the constrained motion. The first part of the solution (11) is the homogeneous solution lying in $N(Q_1^T)$ and the other is a particular solution to (3).
Remark: The internal states $\dot{\mathbf{v}} \in R^{n-p}$ can be thought of as the magnitude of generalized velocity along tangent directions defined by columns of Q_2. However, $\mathbf{v} \in R^{n-p}$ may or may not be a physical quantity, depending on the form of constraint equations (3.
Differentiating (11), we obtain

$$\ddot{\mathbf{q}} = Q_2(\mathbf{q})\ddot{\mathbf{v}} + \frac{d}{dt}(Q_2(\mathbf{q}))\dot{\mathbf{v}} + \frac{d}{dt}H(\mathbf{q}, t) \quad (12)$$

Therefore, the dynamic equation (4), when satisfying the nonholohomic constraint (??), can be expressed in terms of the inherent velocities $\dot{\mathbf{v}}$ as

$$D(\mathbf{q})Q_2(\mathbf{q})\ddot{\mathbf{v}} + F_1(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{v}} + G(\mathbf{q}) + P(\mathbf{q}, \dot{\mathbf{q}}, t) = \tau + L^T\lambda \quad (13)$$

where

$$F_1(\dot{\mathbf{q}}, \mathbf{q}) = D(\mathbf{q})\frac{d}{dt}(Q_2(\mathbf{q})) + F(\dot{\mathbf{q}}, \mathbf{q})Q_2(\mathbf{q})$$

$$P(\mathbf{q}, \dot{\mathbf{q}}, t) = D(\mathbf{q})\frac{d}{dt}H(\mathbf{q}, t) + F(\dot{\mathbf{q}}, \mathbf{q})H(\mathbf{q}, t)$$

It should be noted that the reduced state space is $2n-p$ dimensional. The system is described by the n-vector of variables $\mathbf{q}$ and the $(n-p)$-vector of internal states $\dot{\mathbf{v}}$.
Remark: We note the derived equations for the system include the effects of the Pfaffian constraints. They can therefore be thought of as the new equivalent equations of motion.
By exploiting the structure of the equation (13), two properties are obtained.
Property 4: The matrix $Q_2^T D(\mathbf{q})Q_2$ is symmetric and positive definite.
Property 5: Define $D_1(\mathbf{q}) = Q_2^T D(\mathbf{q})Q_2$. Using the same definition of the matrix $F(\dot{\mathbf{q}}, \mathbf{q})$ as in Property 1, $D_1(\mathbf{q})$ and $F_1(\dot{\mathbf{v}}, \mathbf{q})$ satisfy

$$\mathbf{x}^T(\frac{d}{dt}D_1(\mathbf{q}) - 2Q_2^T F_1(\dot{\mathbf{q}}, \mathbf{q}))\mathbf{x} = 0,$$

i.e., $(\frac{d}{dt}D_1(\mathbf{q}) - 2Q_2^T F_1(\dot{\mathbf{q}}, \mathbf{q}))$ is a skew symmetric matrix.
The aforementioned properties are fundamental for designing the force/motion control law.

3. Controller Design for Motion/Force Tracking

It has been proved (see [6] [1]) that the nonholonomic systems cannot be stabilized to a single point using pure smooth state feedback. It can only be stabilized to a manifold of dimension p due to the existence of p nonholonomic constraints. The objective of stabilizing the systems to a point has been achieved by open-loop control

[4], nonsmooth feedback law [1] [11] and time-varying feedback laws [6]. However, it is fair to say that these approaches are not yet fully general and only focused on the classical (catastatic) constraints. It is worth mentioning that different control objectives may also be pursued, such as stabilization to manifolds of equilibrium points [1][2](as opposed to a single equilibrium position) or to trajectories [5].

By assuming that $(n-p)$-vector of variables $\mathbf{v}(\mathbf{q})$ is physically motivated, similar to [5], the objective of the control can be specified as: given a desired $\mathbf{v}_d$, $\dot{\mathbf{v}}_d$, and desired λ_d, determine a control law such that for any $(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \in \Omega$, then $\mathbf{v}(\mathbf{q})$, $\mathbf{q}$, and λ asymptotically converge to a manifold Ω_d defined as

$$\Omega_d = \{(\mathbf{q}, \dot{\mathbf{q}}, \lambda) | \mathbf{v} = \mathbf{v}_d,\ \dot{\mathbf{q}} = Q_2\dot{\mathbf{v}}_d + H, \lambda = \lambda_d\}.$$

3.1. Motion/Force Control with Known Parameters

This section deals with the case where the dynamic parameters are known. We define

$$\begin{aligned} \mathbf{e}_v &= \mathbf{v} - \mathbf{v}_d && (14)\\ \mathbf{e}_\lambda &= \lambda - \lambda_d && (15)\\ \dot{\mathbf{v}}_r &= \dot{\mathbf{v}}_d - \Lambda \mathbf{e}_v. && (16) \end{aligned}$$

where Λ is a positive definite matrix whose eigenvalues are strictly in the right-hand complex plane.

Before giving the control law, the following assumptions are required.

Assumption A1: The desired trajectory $\mathbf{v}_d$ is chosen such that $\mathbf{v}_d$, $\dot{\mathbf{v}}_d$, and $\ddot{\mathbf{v}}_d$ are all bounded signals.

Assumption A2: The matrix $D_1(\mathbf{q}) = Q_2^T D(\mathbf{q}) Q_2$ can be bounded above and below, i.e.,

$$\beta_m I \le D_1(\mathbf{q}) \le \beta_M I, \quad \forall \mathbf{q} \in R^n$$

where β_m and β_M are positive constants.

Remark: Since the matrix $D(\mathbf{q})$ can be bounded above and below and the choice of the matrix Q_2 depends on the form of constraint equation (3), the above assumption can always be satisfied for a class of Pfaffian constraints.

With the above in mind, a non-adaptive control law is defined as

$$\begin{aligned} \tau &= D(\mathbf{q})Q_2(\mathbf{q})(\dot{\mathbf{v}}_r - \mu \mathbf{s}) + F_1(\dot{\mathbf{q}}, \mathbf{q})\dot{\mathbf{v}}_r + G(\mathbf{q}) \\ &\quad + P(\mathbf{q}, \dot{\mathbf{q}}, t) - Q_2 \mathbf{s} - L^T \lambda_c && (17) \end{aligned}$$

where the vector $\mathbf{s}$ is defined as

$$\mathbf{s} = \dot{\mathbf{e}}_v + \Lambda \mathbf{e}_v; \qquad (18)$$

the force related variable λ_c is defined as

$$\lambda_c = \lambda_d - K_\lambda \mathbf{e}_\lambda, \qquad (19)$$

K_λ is a constant matrix of force control feedback gains and μ is a constant.

The following theorem can be stated:

Theorem 1: *In the closed-loop error system, the errors $\dot{\mathbf{e}}_v$, $\mathbf{e}_v$, and $\mathbf{e}_\lambda$ converge to zero exponentially from a given initial condition $(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \in \Omega$, i.e., $(\mathbf{q}, \dot{\mathbf{q}}, \lambda)$ converge to Ω_d exponentially.* □□□

3.2. Adaptive Control for Motion/Force Tracking

Now, assume that the parameters of the mechanical systems are not exactly known. To synthesize the adaptive controller, in addition to Assumptions A1 and A2, the following assumptions are also required.

Assumption A3:

$$\begin{aligned} \|H(\mathbf{q})\| &\le \sum_{i=1}^{3} \rho_i \|\mathbf{q}\|^{i-1} && (20)\\ \|\frac{d}{dt}Q_2(\mathbf{q})\| &\le \rho_4 \|\dot{\mathbf{q}}\| && (21)\\ \|\frac{d}{dt}H(\mathbf{q}, t)\| &\le \sum_{i=5}^{7} \rho_i \|\mathbf{q}\|^{i-5} \|\dot{\mathbf{q}}\| && (22) \end{aligned}$$

where ρ_i are some positive constants.

The following property can be obtained due to Assumption 2.

Property 6: The minimum and maximum eigenvalues of the matrix $Q_2^T Q_2$ satisfy

$$\lambda_{min}(Q_2^T Q_2) \ge \sigma_1, \quad \lambda_{max}(Q_2^T Q_2) \le \sigma_2$$

where σ_1 and σ_2 are positive constants.

With Properties 2 and 6 and Assumptions A2 and A3, the following conditions are satisfied for all $(\mathbf{q}, \dot{\mathbf{q}})$.

Property 7:

$$\begin{aligned} \|D_1(\mathbf{q})\| &\le \delta_1 && (23)\\ \|Q_2^T F_1\| &\le \delta_2 \|\dot{\mathbf{q}}\| && (24)\\ \|Q_2^T G(\mathbf{q})\| &\le \delta_3 && (25)\\ \|Q_2^T P(\mathbf{q}, \dot{\mathbf{v}}, t)\| &\le \sum_{i=4}^{6} \delta_i \|\mathbf{q}\|^{i-4} \|\dot{\mathbf{q}}\| && (26) \end{aligned}$$

where δ_i are some positive constants. It should be noted that the bounds δ_i are not needed for the adaptive controller design.

The robust adaptive control law is then synthesized as

$$\tau = -\mu Q_2 \mathbf{s} - \frac{1}{\sigma_1} Q_2[\hat{\delta}_1 ||\ddot{\mathbf{v}}_r|| + \hat{\delta}_2 ||\dot{\mathbf{q}}|| ||\dot{\mathbf{v}}_r|| + \hat{\delta}_3 + \sum_{i=4}^{6} \hat{\delta}_i ||\mathbf{q}||^{i-4} ||\dot{\mathbf{q}}||] sat(\mathbf{s}/\phi) - L^T \lambda_c \quad (27)$$

where μ is a constant; ϕ is the thickness of the boundary layer; and

$$sat(\mathbf{s}/\phi) = (sat(s_1/\phi), ..., sat(s_{n-p}/\phi))^T$$

with

$$sat(s_i/\phi) = \begin{cases} sgn(s_i) & \text{if } |s_i| > \phi, \\ s_i/\phi & \text{if } |s_i| \le \phi, \end{cases} \quad i = 1, ..., n-p.$$

The update laws for $\hat{\delta}_i$ (i=1,...,6) are defined as follows:

$$\dot{\hat{\delta}}_1 = \eta_1 ||\mathbf{s}_\phi|| ||\ddot{\mathbf{v}}_r|| \quad (28)$$

$$\dot{\hat{\delta}}_2 = \eta_2 ||\mathbf{s}_\phi|| ||\dot{\mathbf{q}}|| ||\dot{\mathbf{v}}_r|| \quad (29)$$

$$\dot{\hat{\delta}}_3 = \eta_3 ||\mathbf{s}_\phi|| \quad (30)$$

$$\dot{\hat{\delta}}_i = \eta_i ||\mathbf{s}_\phi|| ||\mathbf{q}||^{i-4} ||\dot{\mathbf{q}}||, \quad i = 4, 5, 6, \quad (31)$$

where $\eta_i > 0$ (i=1 ... 6) are constants, determining the rate of adaptation, $\mathbf{s}_\phi = (s_{\phi 1}, ..., s_{\phi n})^T = \mathbf{s} - \phi sat(\mathbf{s}/\phi)$ is a measurement of the algebraic distance of the current state to the boundary layer.
Based on the above, the following theorem can be stated.

Theorem 2: *Consider the constrained mechanical system (13) with the control laws (27), (28-31), satisfying Assumptions A1, A2, and A3. Then the following holds for any* $(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \in \Omega$:
i) $\dot{\mathbf{e}}_v$ *and* $\mathbf{e}_v$ *are uniformly ultimately bounded.*
ii) $\mathbf{e}_\lambda$ *is uniformly ultimately bounded and inversely proportional to the norm of the matrix* $(I_p + K_\lambda)$. □□□

4. Conclusion

In this paper, the issue of appropriate control of position and constraint force is firstly addressed for mechanical systems with a class of Pfaffian nonholonomic constraints. By specifying an internal state vector, a novel dynamic model, suitable for simultaneous force and motion control, is established. A non-adaptive and an adaptive control formulations are then proposed, respectively, ensuring that a system with p nonholonomic constraints can be stabilized to a p-dimensional desired manifold. However, the definition of the desired manifold depends on the specific choice of internal state vector, which is related to the form of the constraint equations. Given the internal state vector, the proposed control law provides a convenient solution for the robust force and motion control of nonholonomic systems.

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