# Optimal Infinite-Horizon Control and the Stabilization of Linear Discrete-Time Systems: State-Control Constraints and Nonquadratic Cost Functions

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*Abstract*—**Stability results are given for a class of feedback systems arising from the regulation of time-invariant, discrete-time linear systems using optimal infinite-horizon control laws. The class is characterized by joint constraints on the state and the control and a general nonlinear cost function. It is shown that weak conditions on the cost function and the constraints are sufficient to guarantee asymptotic stability of the optimal feedback systems. Prior results, which concern the linear quadratic regulator problem, are included as a special case. The proofs make no use of discrete-time Riccati equations and linearity of the feedback law, hence, they are intrinsically different from past proofs.**

## I. Introduction

Consider the following linear, discrete-time system with a joint constraint on state and control:

$$x_{k+1}=Ax_k+Bu_k,\ y_k=Cx_k+Du_k, \qquad k\geq 0, \tag{1.1}$$

$$(x_k,\ u_k)\in Z\subset R^{n+m},\ y_k\in R^l, \qquad k\geq 0,\ x_0=a. \tag{1.2}$$

Let $h:R^{l+m}\rightarrow R$ be a nonnegative function. Our problem is to find a control law, $u=\eta(x)$, which generates, through (1.1), a sequence $\{u_k\}_{k\geq 0}$ that minimizes

$$J=\sum_{k=0}^{\infty} h(y_k,\ u_k) \tag{1.3}$$

subject to (1.1), (1.2).

Let the following special conditions hold: a) $h(y, u)=(y'Qy+u'Ru)$, where prime denotes transpose and $Q$ and $R$ are symmetric positive definite matrices; b) the pairs $(A, B)$ and $C, A)$ are, respectively, stabilizable and detectable; and c) $Z=R^{n+m}$. Then (1.1)–(1.3) becomes a linear quadratic regulator problem (LQRP) and it is known [4] that the optimal control law is linear and that the resulting feedback system is exponentially stable.

In this note we extend the stability results for the LQRP to the wider class of problems described by (1.1)–(1.3). Under weak conditions on $h$ and $Z$ we show that the optimal feedback system is asymptotically stable. Also, we give conditions which guarantee exponential stability of the optimal system. The known stability results for the LQRP are a trivial specialization of our results. Our proofs are based on a standard Lyapunov stability theorem. Specialized to the LQRP, they do not utilize the discrete-time Riccati equation or the linearity of the feedback law. Hence, they appear to involve concepts which are more basic than those used in past proofs [1].

We conclude this section with some notations and definitions. Given a vector $x\in R^p$ and a matrix $M\in R^{q\times p}$, let $\|x\|=\sqrt{x'x}$ and $\|M\|=\max\{\|Mx\|:\|x\|=1\}$. For $x\in R^p$, $y\in R^q$, $(x, y)$ denotes the joint vector $[x', y']'\in R^{p+q}$. $R_+$ is the set of nonnegative reals. For $\epsilon\geq 0$, $N(\epsilon)=\{x:\|x\|\leq\epsilon\}$. A function $W:R_+\rightarrow R_+$ is said to belong to *class* $T_0$ if: a) it is continuous and nonnegative; b) $W(s)=0\Leftrightarrow s=0$. $W$ is in *class* $T_+$ if $W\in T_0$ and is nondecreasing. The function $W$ is in *class* $T_\infty$ if $W\in T_+$ and $W(s)\rightarrow\infty$ when $s\rightarrow\infty$.

Manuscript received March 15, 1985; revised September 3, 1985. Paper recommended by Associate Editor, M. G. Safanov.

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IEEE Log Number 8406894.

## II. Main Results

In this section we formulate more precisely the infinite-horizon optimal control problem and state theorems concerning the existence and stability of the corresponding optimal feedback system. Proofs are deferred to Section III.

For $a\in R^n$ let $P(a)$ denote the problem of minimizing the cost $J$ in (1.3) subject to (1.1) and (1.2). A sequence $\{(x_k,\ y_k,\ u_k)\}_{k\geq 0}$ is *admissible* to $P(a)$ if it satisfies (1.1) and (1.2). To state our results, a variety of assumptions will be needed.

*Assumption A.1:* $Z$ is closed and $(0, 0)\in$ interior $Z$.

*Assumption A.2:* $h:R^{l+m}\rightarrow R$ is lower semicontinuous.

*Assumption A.3:* There exists a function $H_1\in T_\infty$ such that

$$H_1(\|(y,\ u)\|)\leq h(y,\ u), \qquad (y,\ u)\in R^{l+m}. \tag{2.1}$$

*Assumption A.4:* There exist positive numbers $p_1$, $p_2$, $q_1$, and $q_2$ such that

$$h(y,\ u)\leq p_1\|(y,\ u)\|^{q_1}+p_2\|(y,\ u)\|^{q_2}, \qquad (y,\ u)\in R^{l+m}. \tag{2.2}$$

*Assumption A.5:* There exist positive numbers $p_3$, $p_4$, and $q$ such that

$$p_3(\|(y,\ u)\|)^q\leq h(y,\ u)\leq p_4\|(y,\ u)\|^q, \qquad (y,\ u)\in R^{l+m}. \tag{2.3}$$

*Remark 2.1:* Assumption A.4 is a very weak condition. Assumption A.5 implies both A.3 and A.4 and is relatively strong.

*Remark 2.2:* Assumption A.3 is implied by another condition which may be easier to verify. Specifically, assume that there exists a function $\bar{h}:R^{l+m}\rightarrow R$ such that $\bar{h}(y, u)\leq h(y, u)$, $(y, u)\in R^{l+m}$, and the following conditions hold: a) $\bar{h}$ is continuous and nonnegative; b) $\bar{h}(y, u)=0$ if and only if $(y, u)=(0, 0)$; c) $\bar{h}(y, u)\rightarrow\infty$ whenever $\|(y, u)\|\rightarrow\infty$. To verify Assumption A.3, define $H_1(s)=\inf\{\bar{h}(y, u): \|(y, u)\|\geq s\}$.

Let

$$X=\{a: P(a)\ \text{has an admissible sequence for which}\ J\ \text{is finite}\}. \tag{2.4}$$

The following theorem concerns properties of $X$ and the existence of an optimal control. The properties of $X$ follow from simple arguments; the existence is a direct consequence of results in [3].

*Theorem 2.1:* Suppose $(A, B)$ is stabilizable and Assumptions A.1–A.4 hold. Then: (i) $X$ is nonempty and $0\in$ interior $X$; ii) $Z=R^{n+m}$ implies $X=R^n$; iii) $a\in X$ implies that $P(a)$ has a solution. □

Assume that the hypotheses of the theorem are satisfied so that, for $a\in X$ there is an optimal control sequence, $\{\bar{u}_k(a)\}_{k\geq 0}$, for $P(a)$. Let $\eta:X\rightarrow R^m$ be defined by $\eta(a)=\bar{u}_0(a)$. Then by the principle of optimality $\eta$ is an optimal control law and an optimal feedback system is given by

$$x_{k+1}=F(x_k)=Ax_k+B\eta(x_k), \qquad x_k\in X,\ k\geq 0. \tag{2.5}$$

Clearly, $F:X\rightarrow X$. Also, by Assumptions A.1, A.3, and A.4, $\eta(0)=0$, $F(0)=0$. For state constrained systems like (2.5), definitions of stability are obvious modifications of the usual stability definitions [2], [5]. Let $x_k^*(a)$, $k\geq 0$ denote the solution of (2.5) given $x_0=a\in X$.

*Theorem 2.2:* Suppose $(A, B)$ and $(C, A)$ are, respectively, stabilizable and observable, and A.1–A.4 hold. Then for (2.5): i) for all $a\in X$, $\lim_{k\rightarrow\infty} x_k^*(a)=0$; ii) $x=0$ is the only equilibrium state and it is asymptotically stable. □

*Remark 2.3:* The requirements A.2 and $H_1\in T_\infty$ in A.3 are used in the proof of the theorem only to guarantee existence of an optimal solution. If the existence can be verified by some other assumption, such as those described in [3], then it is sufficient to require $H_1\in T_+$.

The results of this theorem can be viewed as a weak form of global stability. Under stronger hypotheses true global stability properties are obtained.

*Theorem 2.3:* Suppose $(A, B)$ and $(C, A)$ are, respectively, stabilizable and observable and $Z=R^{n+m}$. Then for (2.5): i) A.2–A.4 imply that $x=0$ is asymptotically stable in the large; ii) Assumptions A.2, A.5 imply that $x=0$ is exponentially stable, and that there exists a constant $\rho>0$ such that $\|\eta(x)\|\leq\rho\|x\|$, $x\in R^n$. □

Observability of $(C, A)$ is needed in Theorems 2.2, 2.3. Detectability suffices provided Assumption A.1 is replaced by the following stronger assumption which allows only output-control constraints.

*Assumption A.1':* $Z$ is expressed by $Z = \{(x, u): y = Cx + Du, (y, u) \in W\}$, where $W \subset R^{l+m}$ is closed and $(0, 0) \in$ interior $W$.

*Theorem 2.4:* The conclusions of Theorems 2.2 and 2.3 hold if their hypotheses are modified as follows: observability of $(C, A)$ is replaced by detectability of $(C, A)$ and Assumption A.1 is replaced by Assumption A.1'. □

It is easy to see that Assumptions A.2 and A.5 (with $q = 2$) are satisfied for the LQRP of Section I. Thus, Theorem 2.1 implies the existence of an optimal feedback law and Theorem 2.4 shows that the corresponding optimal feedback system is exponentially stable.

## III. Proofs

To prove Theorem 2.1, we need the following lemma.

*Lemma 3.1:* Suppose $(A, B)$ is a stabilizable pair and that Assumption A.4 holds. Then, for every $a \in R^n$, there exists a sequence triple $\{(\tilde{x}_k(a), \tilde{y}_k(a), \tilde{u}_k(a))\}_{k\geq 0}$ which satisfies (1.1), $\tilde{x}_0(a) = a$, and

$$\|(\tilde{x}_k(a), \tilde{u}_k(a))\| \leq \mu_c \|a\|, \qquad k \geq 0, \tag{3.1}$$

$$\tilde{V}(a) = \sum_{k=0}^{\infty} h(\tilde{y}_k(a), \tilde{u}_k(a)) \leq \tilde{\phi}(\|a\|) \tag{3.2}$$

where $\mu_c > 0$ and $\tilde{\phi} \in T_\infty$. If Assumption A.4 is replaced by the stronger Assumption A.5, then there exists $p > 0$ such that (3.2) holds with

$$\phi(s) = ps^q. \qquad \square \tag{3.3}$$

*Proof:* Since $(A, B)$ is stabilizable, there exists a $K \in R^{m\times n}$ such that

$$\|(A + BK)^k\| \leq \alpha(\beta)^k, \qquad k \geq 0, \tag{3.4}$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$. For $A \in R^n$, let $\{(\tilde{x}_k(a), \tilde{y}_k(a), \tilde{u}_k(a))\}_{k\geq 0}$ be the solution of (1.1) together with $\tilde{x}_0(a) = a$ and $u_k = Kx_k$. Then (3.4) implies (3.1) with $\mu_c = \alpha\|[I_n\ K']'\|$, where $I_n \in R^{n\times n}$ is the identity matrix. Let $\lambda = \|[(C + DK)'K']\|$. Then by Assumption A.4 and (3.4)

$$\tilde{V}(a) \leq \frac{p_1(\lambda\alpha)^{q_1}}{1-\beta^{q_1}}\|a\|^{q_1} + \frac{p_2(\lambda\alpha)^{q_2}}{1-\beta^{q_2}}\|a\|^{q_2} \triangleq \tilde{\phi}(\|a\|). \tag{3.5}$$

Clearly, $\tilde{\phi} \in T_\infty$. Similarly, Assumption A.5 implies (3.3). □

*Proof of Theorem 2.1:* By Assumption A.1 there exists $\epsilon_1 > 0$ such that $N(\epsilon_1) \subset Z$. Let $\epsilon_2 = \epsilon_1/\mu_c$. Then, for all $a \in N(\epsilon_2)$, inequalities (3.1) and (3.2) imply $\{(\tilde{x}_k(a), \tilde{y}_k(a), \tilde{u}_k(a))\}_{k\geq 0}$ is admissible for $P(a)$ and has a finite cost. Thus, $N(\epsilon_2) \subset X$, which proves part i). Part ii) follows immediately since $Z = R^{n+m}$ implies that $\epsilon_1$, and hence $\epsilon_2$ can be arbitrarily large. By Assumption A.3, $h(y, u) \geq H_1(\|u\|)$, $k \geq 0$, $(y, u) \in R^{l+m}$. Existence of a solution to $P(a)$, $a \in X$ then follows from Assumptions A.1, A.2, and results in [3]. Specifically, see Theorem 1 and Condition $(d_3)$ of Theorem 2. □

Stability properties of (2.5) will be established by using the following theorem, which is mainly an extension of a Lyapunov stability theorem [2], [5] to constrained systems. We omit its proof since it is similar to the proof in [2], [5].

*Theorem 3.2:* Consider the system $x_{k+1} = F(x_k)$, $x_k \in X$, $k \geq 0$, where $0 \in X$, $F: X \to X$ and $F(0) = 0$. Let $x_k^*(a)$, $k \geq 0$, denote its solution given $x_0 = a$. Suppose there exist: $V: X \to R$, $\alpha \in T_+$, $\beta \in T_+$, $\gamma \in T_0$, $\lambda > 0$ and a positive integer $M$ such that the following conditions hold: a) $V(a) \leq \beta(\|a\|)$, $a \in X \cap N(\lambda)$; b) $\alpha(\|a\|) \leq V(a)$, $V(a) - V(x_M^*(a)) \geq \gamma(\|a\|)$, $V(a) - V(x_1^*(a)) \geq 0$, $a \in X$. Then: i) $x = 0$ is asymptotically stable; ii) $\lambda = \infty$, $\alpha \in T_\infty$ imply $x = 0$ is asymptotically stable in the large; iii) $\lambda = \infty$, $\alpha(s) = \alpha_1 s^q$, $\beta(s) = \beta_1 s^q$, $\gamma(s) = \gamma_1 s^q$ ($\alpha_1$, $\beta_1$, $\gamma_1$ and $q$ positive) imply $x = 0$ is exponentially stable. □

To prove Theorem 2.2 we also need the following lemma.

*Lemma 3.3:* Suppose $(C, A)$ is an observable pair. Then there exists $\mu_0 > 0$ such that, for every $i \geq 0$ and sequence triple $\{(x_k, y_k, u_k)\}_{k\geq i}$ satisfying (1.1) for $k \geq i$, we have

$$\sum_{k=i}^{i+n-1} \|(y_k, u_k)\| \geq \mu_0\|x_i\|. \qquad \square \tag{3.6}$$

*Proof:* Since the system (1.1) is time-invariant, we can, without loss of generality, let $i = 0$. For $0 \leq k \leq n - 1$, $y_k = CA^k a + L_k U$, where $U = (u_0, u_1, \cdots, u_{n-1})$ and $L_k = [C A^{k-1} B \cdots CB\ D\ 0 \cdots]$. Hence,

$$\sum_{k=0}^{n-1} \|y_k\|^2 \geq a'E'Ea + 2a'TU, \tag{3.7}$$

where $E' = [C'(CA)' \cdots (CA^{n-1})']$ and $T = \Sigma_{k=0}^{n-1} (CA^k)'L_k$. Since $(C, A)$ is observable, $E$ is full rank, and hence $E'E$ is positive definite. Let $\mu_1$ denote the smallest eigenvalue of $E'E$. Then, from (3.7), $\Sigma_{k=0}^{n-1} \|y_k\|^2 \geq \mu_1\|a\|^2 - 2\|T\|a\|U\|$. For $0 \leq \delta \leq 1$, this implies

$$\left(\sum_{k=0}^{n-1} \|(y_k, u_k)\|\right)^2 \geq \sum_{k=0}^{n-1} \|(y_k, u_k\|^2 \geq \delta \sum_{k=0}^{n-1} \|y_k\|^2 + \|U\|^2$$
$$\geq (\delta\mu_1 - \delta^2\|T\|^2)\|a\|^2. \tag{3.8}$$

Choose $\delta = \min(1, \mu_1/2\|T\|^2)$ to get (3.6) with $\mu_0 = \sqrt{\delta\mu_1/2}$. □

*Proof of Theorem 2.2:* For $a \in X$ let $\{(x_k^*(a), y_k^*(a), u_k^*(a))\}_{k\geq 0}$ denote the solution of (1.1) together with $x_0^*(a) = a$ and $u_k^*(a) = \eta(x_k^*(a))$, $k \geq 0$. This is consistent with the definition of $x_k^*(a)$, $k \geq 0$ as the solution of (2.5). Let $V(a)$ denote the optimal cost of $P(a)$. By optimality,

$$V(a) = \sum_{k=0}^{\infty} h(y_k^*(a), u_k^*(a)), \qquad a \in X. \tag{3.9}$$

To prove i) let $a \in X$. Clearly $V(a) < \infty$. Hence, by Assumption A.3, $H_1(\|(y_k^*(a), u_k^*(a))\|) \to 0$, $k \to \infty$. Since $H_1 \in T_\infty$, $\|(y_k^*(a), u_k^*(a))\| \to 0$, $k \to \infty$. It follows from Lemma 3.3 that $|x_k^*(a)| \to 0$, $k \to \infty$. This also implies that $x = 0$ is the only equilibrium point of (2.5).

To prove asymptotic stability of $x = 0$, we use part i) of Theorem 3.2. Let $\lambda = \epsilon_2$, where $\epsilon_2$ is the constant defined in the proof of Theorem 2.1. For $a \in N(\lambda)$, the sequence $\{(\tilde{x}_k(a), \tilde{y}_k(a), \tilde{u}_k(a))\}_{k\geq 0}$ of Lemma 3.1 is feasible for $P(a)$, and hence, by optimality and (3.2)

$$V(a) \leq \tilde{V}(a) \leq \tilde{\phi}(\|a\|), \qquad a \in N(\lambda). \tag{3.10}$$

For $a \in X$, we get using Assumption A.3 and (3.6),

$$\sum_{k=0}^{n-1} h(y_k^*(a), u_k^*(a)) \geq H_1(\max_{0\leq k\leq n-1} \|(y_k^*(a), u_k^*(a))\|)$$
$$\geq H_1\left((1/n)\sum_{k=0}^{n-1} \|(y_k^*(a), u_k^*(a))\|\right)$$
$$\geq H_1(\mu_0\|a\|/n) \triangleq \bar{\phi}(\|a\|). \tag{3.11}$$

Clearly $\bar{\phi} \in T_\infty$. Then by Assumption A.3 and (3.11),

$$V(a) \geq V(a) - V(x_n^*(a)) = \sum_{k=0}^{n-1} h(y_k^*(a), u_k^*(a)) \geq \bar{\phi}(\|a\|), \qquad a \in X \tag{3.12}$$

$$V(a) - V(x_1^*(a)) = h(y_0^*(a), u_0^*(a)) \geq 0, \qquad a \in X. \tag{3.13}$$

Asymptotic stability of the origin follows from (3.10), (3.12), (3.13), and part i) of Theorem 3.2 with $\alpha = \bar{\phi}$, $\beta = \tilde{\phi}$, $\gamma = \bar{\phi}$ and $M = n$. □

*Proof of Theorem 2.3:* The proof of part i) is essentially the same as the proof of part ii) of Theorem 2.2. It is only necessary to let $\lambda = \infty$; then part ii) of Theorem 3.2 implies that the origin is asymptotically stable in the large. Part ii) is proved as follows. Assumption A.5 implies that A.3 holds with $H_1(s) = p_3 s^q$, and hence we get from (3.11), $\bar{\phi}(s) = \bar{p}s^q$, where $\bar{p} = p_3(\mu_0/n)^q$. Part iii) of Theorem 3.2 together with (3.3), (3.10), (3.12), and (3.13), then proves exponential stability of the origin. The required bound on $\eta(x)$ follows from (3.9), (2.3), (3.10), (3.3) and by letting $\rho = (\bar{p}/p_1)^{1/q}$. □

*Proof of Theorem 2.4:* Since $(C, A)$ is detectable, we can, by a change of coordinates in (1.1) and without loss of generality, assume that

$$x=\begin{pmatrix}\bar{x}\\ \hat{x}\end{pmatrix},\; A=\begin{pmatrix}\bar{A} & 0\\ A_0 & \hat{A}\end{pmatrix},\; B=\begin{pmatrix}\bar{B}\\ \hat{B}\end{pmatrix},\; C=(\bar{C},\ 0), \tag{3.14}$$

where $\bar{x} \in R^{\bar{n}}$, $\hat{x} \in R^{\hat{n}}$, $\bar{n} + \hat{n} = n$, $(\bar{C}, \bar{A})$ is observable, and $\hat{A}$ is stable. Because of Assumption A.1$'$, and the structure of (3.14), $P(a)$ is equivalent to the problem $\bar{P}(\bar{a})$ of minimizing $J$ in (1.3) subject to

$$\bar{x}_{k+1}=\bar{A}\bar{x}_k+\bar{B}u_k,\; y_k=\bar{C}\bar{x}_k+Du_k, \tag{3.15}$$

$$(y_k,\ u_k) \in W,\; k\geq 0,\; \bar{x}_0=\bar{a}, \tag{3.16}$$

where $a = (\bar{a}, \hat{a})$. Hence, the optimal control, $\{\bar{u}_k(a)\}_{k\geq 0}$, depends only on $\bar{a}$. Let

$$\bar{X}=\{\bar{a} : \bar{P}(\bar{a}) \text{ has an admissible sequence for which } J \text{ is finite}\} \tag{3.17}$$

and $u = \bar{\eta}(\bar{x})$, $\bar{x} \in \bar{X}$ be an optimal control law. Clearly, $X = \bar{X} \times R^{\hat{n}}$ and the optimal feedback system is

$$\bar{x}_{k+1}=\bar{A}\bar{x}_k+\bar{B}\bar{\eta}(\bar{x}_k),\; \bar{x}_k \in \bar{X}, \tag{3.18}$$

$$\hat{x}_{k+1}=\hat{A}\hat{x}_k+A_0\bar{x}_k+\hat{B}\bar{\eta}(\bar{x}_k), \qquad k\geq 0. \tag{3.19}$$

Given $a = (\bar{a}, \hat{a})$, $\bar{a} \in \bar{X}$, let $(\bar{x}_k^*(\bar{a}), \hat{x}_k^*(a))$, $k \geq 0$ be the solution of (3.18), (3.19) with $\bar{x}_0^*(\bar{a}) = \bar{a}$ and $\hat{x}_0^*(a) = \hat{a}$.

By applying (3.10) to the subsystem (3.18) and using Assumption A.3, we get

$$H_1(\|\bar{\eta}(\bar{a})\|)\leq \bar{V}(\bar{a})\leq \tilde{V}(\bar{a})\leq \tilde{\phi}(\|\bar{a}\|),\; \bar{a} \in \bar{X} \cap N(\bar{\lambda}) \tag{3.20}$$

where all the variables and functions associated with (3.18) are indicated by an overhead bar. The completion of the proof is straightforward, but rather tedious. So we only indicate the ideas involved. Stability of (3.18) follows from the observability of $(\bar{C}, \bar{A})$ and the application of Theorems 2.2 and 2.3. Standard arguments usually employed in proving bounded input-bounded state stability of linear systems can then be used together with the stability of $\hat{A}$ and the bound (3.20) to prove the stability of (3.19). □

## V. Conclusions

In this note we have presented stability theorems for optimal infinite-horizon control laws. While it is usually difficult to characterize $X$ in (2.4) and compute the optimal infinite-horizon control law $\eta$, the theory shows that it may be worthwhile to investigate these issues. One approach is to use moving-horizon control laws. We have obtained some interesting results in this direction and will report on them in a later paper.

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# Numerical Simulation of Trajectory Prescribed Path Control Problems by the Backward Differentiation Formulas

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*Abstract*—**The equations which describe trajectory prescribed path control problems naturally form nonlinear semiexplicit, differential-algebraic systems with index greater than one. It is known that not all fully implicit systems may be solved stably by the $k$-step backward differentiation formulas, yet these methods do produce convergent numerical solutions to some semiexplicit systems. In this note numerical results are presented for the simplest backward differentiation formula when applied to an index three, semiexplicit system concerning the reentry of the space shuttle. The application of this numerical method to a realistic problem illustrates some unresolved implementation difficulties.**

## I. Introduction

In trajectory prescribed path control (TPPC) problems [1], a vehicle flying in space along a trajectory, constrained by a set of path equations, is modeled by a system of differential-algebraic equations (DAE's). These systems involve a set of $l$ state equations corresponding to the equations of motion of the vehicle,

$$\dot{y}=F(t,\ y,\ u) \tag{1a}$$

and $m$ auxiliary path constraints

$$0=H(t,\ y,\ u) \tag{1b}$$

where $y$ and $F$ have dimension $l$, $u$ and $H$ have dimension $m$, and $m \leq l$. Traditionally, the $y$ variables are referred to as state variables, while the $u$ variables are considered the control (algebraic) variables. The state variables describe the position, velocity, and possibly the mass of the vehicle. The control variables are typically angle of attack ($\alpha$) and/or bank angle ($\beta$) because they effectively determine the magnitude and direction of the aerodynamic forces acting on the vehicle. Often, the path constraints in a TPPC problem are functions only of the state variables. In particular, the TPPC problems of interest in this note can be written as either

$$\dot{y}=F(t,\ y,\ u) \tag{2a}$$

$$0=H(t,\ y) \tag{2b}$$

where the matrix product $(\partial H/\partial y)(\partial F/\partial u)$ is nonsingular for all $t$ of interest, or as

$$\dot{v}=F(t,\ v,\ w,\ u) \tag{3a}$$

$$\dot{w}=G(t,\ v,\ w) \tag{3b}$$

$$0=H(t,\ w) \tag{3c}$$

where $v$ and $F$ have dimension $p$, $w$ and $G$ have dimension $q$, $u$ and $H$ have dimension $m$, and the matrix product $(\partial H/\partial w)(\partial G/\partial v)(\partial F/\partial u)$ is nonsingular for all $t$ of interest.

Before the advent of the differential-algebraic integrator [5], it was necessary to either reduce the DAE system to a set of explicit ordinary equations (ODE's) by a reduction technique, or to approximate the control by constant values (determined from an appropriate control law) defined on small intervals of time. However, it may be effectively impossible in practice to reduce the system to a set of ODE's. In the other case, a discontinuous control will introduce noise into the simulation, often

Manuscript received January 7, 1985; revised September 23, 1985. Paper recommended by Associated Editor, T. L. Johnson.
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IEEE Log Number 8406893.