

# Linear Systems with State and Control Constraints: The Theory and Application of Maximal Output Admissible Sets

Elmer G. Gilbert, *Fellow, IEEE*, and Kok Tin Tan, *Student Member, IEEE*

**Abstract**—The initial state of an unforced linear system is output admissible with respect to a constraint set  $Y$  if the resulting output function satisfies the pointwise-in-time condition  $y(t) \in Y$ ,  $t \geq 0$ . The set of all possible such initial conditions is the maximal output admissible set  $O_\infty$ . The properties of  $O_\infty$  and its characterization are investigated. In the discrete-time case, it is generally possible to represent  $O_\infty$ , or a close approximation of it, by a finite number of functional inequalities. Practical algorithms for generating the functions are described. In the continuous-time case simple representations of the maximal output admissible set are not available; however, it is shown that the discrete-time results may be used to obtain approximate representations. Maximal output admissible sets have important applications in the analysis and design of closed-loop systems with state and control constraints. To illustrate this point, a modification of the error governor control scheme proposed by Kapasouris, Athans, and Stein [6] is presented. It works as well as their implementation but reduces the computational load on the controller by several orders of magnitude.

## I. INTRODUCTION

IN this paper, we are concerned with characterizing those initial states of an unforced linear system whose subsequent motion satisfies a specified pointwise-in-time constraint. Such characterizations have important applications. Consider the following example. A linear discrete-time time-invariant plant is given together with a linear control law

$$x(t+1) = Ax(t) + Bu(t), \quad u(t) = Kx(t). \quad (1.1)$$

In addition, there may be physical constraints on either or both the state and the control variables or on linear combinations of them. If the constraints are violated for any  $t$  serious consequences may ensue, for example, physical components may be damaged or saturation may cause a loss of closed-loop stability [1]–[3]. By an appropriate choice of matrices  $C$  and  $D$  and a set  $Y$ , all constraints of the type mentioned may be summarized by a single set inclusion

$$Cx(t) + Du(t) \in Y. \quad (1.2)$$

Typically, the set  $Y$  is convex and contains the origin. For instance, it may be a polytope or a product set of balls associated with various norms. With (1.1) and (1.2) speci-

fied, it is desired to obtain a safe set of initial conditions, i.e., a set  $Z$  such that  $x(0) \in Z$  implies (1.2) is satisfied for all integers  $t \geq 0$ .

The problem just cited, as well as many others of practical interest, can be stated equivalently as a problem involving an unforced linear system with pointwise in time *output* constraints. Specifically, we are given a triple,  $A \in \mathbb{R}^{n \times n}$ ,  $C \in \mathbb{R}^{p \times n}$ , and  $Y \subset \mathbb{R}^p$ , and are to determine if the motion of the discrete-time system

$$x(t+1) = Ax(t), \quad x(t) \in \mathbb{R}^n, \quad y(t) = Cx(t) \quad (1.3)$$

satisfies the output constraint

$$y(t) \in Y \quad (1.4)$$

for all  $t \in \mathcal{J}^+$ , where  $\mathcal{J}^+$  is the set of nonnegative integers. Problem (1.1)–(1.2) fits this format by making the assignments  $A + BK \rightarrow A$ ,  $C + DK \rightarrow C$ , and  $Y \rightarrow Y$ .

To assist our subsequent discussions, we introduce some terminology. The *state constraint set* associated with  $A$ ,  $C$ ,  $Y$  is

$$X(C, Y) = \{x \in \mathbb{R}^n : Cx \in Y\}. \quad (1.5)$$

A set  $Z \subset \mathbb{R}^n$  is *A-invariant* if  $AX \subset Z$ ; it is *A, C, Y output admissible* if  $CA^t Z \subset Y$  for all  $t \in \mathcal{J}^+$ . In the future we may omit specific reference to  $A$ ,  $C$  and  $Y$  in these and other situations, when by context it causes no confusion. The same ideas and terminology extend to the following continuous-time system and output constraint:

$$\dot{x}(t) = Ax(t), \quad x(t) \in \mathbb{R}^n, \quad y(t) = Cx(t) \in Y, \quad \forall t \in \mathbb{R}^+ \quad (1.6)$$

where  $\mathbb{R}^+$  is the set of nonnegative reals. In the definition of output admissible it is required that  $Ce^{At}Z \subset Y$  for all  $t \in \mathbb{R}^+$ .

If  $Z$  is output admissible and  $x(0) \in Z$ ,  $x(0)$  is a safe initial condition in the sense that (1.4) is satisfied for all  $t \in \mathcal{J}^+$  (discrete time) or for all  $t \in \mathbb{R}^+$  (continuous time). Output admissible sets have many important applications in the areas of stability analysis and controller design. They have appeared before in a variety of contexts, often without explicit mention. See, e.g., [3]–[13].

Much of the prior literature is based on the idea of positively invariant sets. A set  $Z \subset \mathbb{R}^n$  is *positively invariant* for a dynamical system  $S$ , if for every initial state

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The authors are with the Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140.

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$x(0) \in Z$ , the subsequent motion  $x(t)$ ,  $t > 0$  belongs to  $Z$ . If  $Z$  is positively invariant and  $Z \subset X(C, Y)$ , it is obvious that  $Z$  is output admissible. This observation has been used in several different contexts to produce output admissible sets [3], [9]–[13]. Assume, as is common in these papers, that  $S$  is the discrete-time system  $x(t+1) = Ax(t)$ ,  $A$  is asymptotically stable, and  $X$  is a polyhedron containing the origin in its interior. Use  $'$  to denote matrix transpose. Let  $V(x) = x'Px$  be a Lyapunov function for the system (1.3), generated by solving  $A'PA - P = -Q$ , where  $Q' = Q > 0$ . Then  $P' = P > 0$  and it is clear that [14] the set  $W = \{x \in \mathbb{R}^n: V(x) \leq c\}$ ,  $c > 0$ , is positively invariant. Clearly,  $c$  may be selected so that  $W \subset X$  and  $W$  is output admissible. Even if  $c$  is maximized subject to  $W \subset X$ , this approach is likely to be very conservative in the sense that much larger output admissible sets exist. The difficulty is that the ellipsoid  $W$  may not fit tightly into the polyhedron  $X$  [3]. This has led to the consideration of conditions which imply that a specified polyhedral set  $W$  is positively invariant [9]–[13]. The conditions have been applied with special assumptions on  $C$ ,  $D$ , and  $Y$  to closed-loop systems of the form (1.1), (1.2). This leads to techniques for finding  $K$  such that (1.1) is asymptotically stable and (1.2) is satisfied for all initial conditions in  $W$ . For the techniques to work the prespecified  $W$  must be sufficiently small and, as is the case with ellipsoidal  $W$ , much larger sets of safe initial conditions may exist.

It is worth noting that the approaches taken in [9]–[13] apply to discrete-time systems, but not to continuous-time systems. The reason is apparent: for discrete-time systems, positive invariance of  $Z$  is equivalent to  $A$ -invariance of  $Z$ . Thus, the positive invariance required in the various developments of the papers is expressed simply in terms of the algebraic properties of  $A$ . The situation for continuous-time systems is certainly more complex:  $Z$  is positively invariant if and only if  $Z \subset e^{At}Z$  for all  $t \in \mathbb{R}^+$ .

As a final point of interest, it is easy to give examples of output admissible sets which are not positively invariant; such sets have received little if any attention in the literature.

This paper departs from past lines of research in that it is concerned with the investigation of *maximal* output admissible sets. For the discrete-time and continuous-time systems these sets are defined, respectively, by

$$O_\infty(A, C, Y) = \{x \in \mathbb{R}^n: CA^t x \in Y \forall t \in \mathbb{J}^+\} \quad (1.7)$$

and

$$O_\infty^c(A, C, Y) = \{x \in \mathbb{R}^n: Ce^{At} x \in Y \forall t \in \mathbb{R}^+\}. \quad (1.8)$$

Our emphasis will be on the discrete-time case because of its relative simplicity and the susceptibility of  $O_\infty$  to algorithmic determination. However, we shall also examine the relationship between  $O_\infty(e^{AT}, C, Y)$  and  $O_\infty^c(A, C, Y)$  where  $T \in \mathbb{R}^+$ . As  $T \rightarrow 0$  the first set approximates the second set, a result which allows us to apply discrete-time computational techniques to continuous-time systems. References to maximal output admissible sets have appeared previously in connection with several nonlinear feedback control scheme which take into account control and state constraints; see [15] for the discrete-time case and [6]–[8] for the continuous-time case.

We have introduced  $O_\infty$  and  $O_\infty^c$  in the context of output admissibility because of the close connection between output admissibility and the way in which physical constraints are expressed. It is also possible to introduce  $O_\infty$  and  $O_\infty^c$  in the context of positively invariant sets. From (1.7) and (1.8) it may be verified that if  $O_\infty$  and  $O_\infty^c$  are not empty they are positively invariant. Thus, over the class of positively invariant subsets of  $X(C, Y)$  they are *maximal*. To our knowledge, such maximal positively invariant sets have not been studied previously.

The results of this paper are organized in the following way. Section II deals with basic issues concerning the discrete-time case:  $Y$ -dependent properties of  $O_\infty$ , the determination of  $O_\infty$  by a finite number of operations, the characterization of  $Y$  and  $O_\infty$  by functional inequalities, simplifications which occur when  $C$ ,  $A$  is unobservable and/or  $A$  is unstable. A condition for finite determinability, which is given in Section II, leads to algorithmic procedures for the computation of  $O_\infty$ . These procedures are presented along with some simple examples in Section III. In Section IV it is shown that  $O_\infty$  is finitely determined if:  $A$  is asymptotically stable, the pair  $C$ ,  $A$  is observable,  $Y$  contains the origin in its interior and is bounded. If  $A$  is only Lyapunov stable, it often follows that  $O_\infty$  is not finitely determined. However, if the only characteristic roots of  $A$  which have unit magnitude are at  $\lambda = 1$ , an approximation of  $O_\infty$  is finitely determined. These matters are taken up in Section V. Section VI treats continuous-time systems and proves the approximation result mentioned above. In Section VII some additional conditions on  $Y$ , called the Minkowski assumptions, are considered. With these assumptions some of the results of the previous sections are strengthened. In Section VIII we introduce a modification of the error governor control scheme of [6] which is based on  $O_\infty$  rather than  $O_\infty^c$ . An example application shows that it is simple, fast, and effective. Section IX contains some final remarks.

The following notations and terminology will be used. The identity matrix in  $\mathbb{R}^{n \times n}$  is  $I_n$ . Let  $\alpha \in \mathbb{R}$ ,  $r \in \mathbb{R}^+$ ,  $x \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{m \times n}$ , and  $Z, Z_1, Z_2 \subset \mathbb{R}^n$ . Then:  $x^i$  is the  $i$ th component of  $x$ ,  $\|x\| = \sqrt{(x'x)}$ ,  $\|A\| = \max\{\|Ax\|: \|x\| = 1\}$ ,  $S(r) = \{x: \|x\| \leq r\}$ ,  $\alpha Z = \{\alpha x: x \in Z\}$ ,  $-Z = (-1)Z$ ,  $Z_1 + Z_2 = \{x_1 + x_2: x_1 \in Z_1, x_2 \in Z_2\}$ .  $Z$  is *symmetric* if  $Z = -Z$ . The boundary and interior of  $Z$  are denoted, respectively, by  $\text{bd } Z$  and  $\text{int } Z$ . The product set of  $Z$  with itself  $k$  times is  $Z^k$ .

We assume hereafter that  $0 \in Y$ . This assumption is satisfied in any reasonable application and has nice consequences. In particular,  $O_\infty$  and  $O_\infty^c$  are nonempty and contain the origin.

## II. BASIC RESULTS FOR THE DISCRETE-TIME CASE

Imposing special conditions on  $Y$  often imposes corresponding conditions on  $O_\infty$ . Some implications of this type are summarized in the following theorem.

*Theorem 2.1:*

- i) Each of the following properties of  $Y$  are inherited by  $O_\infty$ : closure, convexity, symmetry.

ii) Suppose  $C, A$  is observable and  $Y$  is bounded. Then,  $O_\infty$  is bounded.

iii) Suppose  $A$  is Lyapunov stable (the characteristic roots of  $A$  satisfy the following conditions:  $|\lambda_i(A)| \leq 1$ ,  $i = 1, \dots, n$ , and  $|\lambda_i(A)| = 1$  implies  $\lambda_i(A)$  is simple) and  $0 \in \text{int } Y$ . Then,  $0 \in \text{int } O_\infty$ .

iv) Let  $\alpha \in \mathcal{R}$ . Then  $O_\infty(A, C, \alpha Y) = \alpha O_\infty(A, C, Y)$ .

*Proof:* The results in i) are easily verified from the definition of  $O_\infty$ . To prove ii) define

$$H' = [C' \ A' C' \ \dots \ (A')^{n-1} C'] \in \mathcal{R}^{n \times np}.$$

The observability assumption implies that  $H$  has rank  $n$ . Thus, if  $Hx = z$  has a solution, the solution is unique and is given by  $x = H^\dagger z$  where  $H^\dagger = (H'H)^{-1}H'$ . Define

$$O_t(A, C, Y) = \{x \in \mathcal{R}^n: CA^k x \in Y \text{ for } k = 0, \dots, t\}. \quad (2.1)$$

Clearly,  $O_\infty \subset O_{N-1} = \{x: Hx \in Y^n\} = H^\dagger Y^n$ . Since  $Y$  is bounded, so is  $H^\dagger Y^n$ . Thus, ii) follows. The assumption of Lyapunov stability in iii) implies that there exists a constant  $\gamma_1 > 0$  such that for all  $x \in \mathcal{R}^n$  and  $t \in \mathcal{J}^+$ ,  $\|CA^t x\| \leq \gamma_1 \|x\|$ . Choose  $\gamma_2 > 0$  so that  $S(\gamma_2) \subset Y$ . Then,  $\gamma_1 \|x\| \leq \gamma_2$  implies  $CA^t x \in Y$  for all  $t \in \mathcal{J}^+$ . Hence,  $S(\gamma_2/\gamma_1) \subset O_\infty$  and  $0 \in \text{int } O_\infty$ . Result iv) is obvious for  $\alpha = 0$ , so assume  $\alpha \neq 0$ . Then,  $O_\infty(A, C, \alpha Y) = \{x: CA^t \alpha^{-1} x \in Y \ \forall t \in \mathcal{J}^+\} = \{\alpha(\alpha^{-1}x): CA^t(\alpha^{-1}x) \in Y \ \forall t \in \mathcal{J}^+\} = \alpha O_\infty(A, C, Y)$ .  $\square$

*Remark 2.1:* The identity in iv) shows that  $O_\infty$  is scaled in direct proportion to the "size" of the output constraint set. There is no change in its shape.

Obviously, the set  $O_t(A, C, Y)$  satisfies the following condition:

$$O_\infty \subset O_{t_2} \subset O_{t_1} \quad \forall t_1, t_2 \in \mathcal{J}^+ \text{ such that } t_1 \leq t_2. \quad (2.2)$$

We say  $O_\infty$  is *finitely determined* if for some  $t \in \mathcal{J}^+$ ,  $O_\infty = O_t$ . Let  $t^*$  be the smallest element in  $\mathcal{J}^+$  such that  $O_\infty = O_{t^*}$ . We call  $t^*$  the *output admissibility index*. From (2.2) it follows that  $O_\infty = O_t$  for all  $t \geq t^*$ .

*Theorem 2.2:*  $O_\infty$  is finitely determined if and only if  $O_t = O_{t+1}$  for some  $t \in \mathcal{J}^+$ .

*Proof:* If  $O_\infty$  is finitely determined the equality holds for and  $t \geq t^*$ . From  $O_t = O_{t+1}$  it is easily confirmed that  $x \in O_t$  implies that  $Ax \in O_t$ ; thus,  $O_t$  is  $A$ -invariant. Also,  $O_t \subset X$ . Hence,  $O_t$  is output admissible and  $O_t \subset O_\infty$ . Property (2.2) completes the proof.  $\square$

Finitely determined maximal output admissible sets generally have a simpler structure than those which are not. Moreover, they can be obtained by finite recursive procedures. These matters, and conditions which imply finite determinability, are discussed in subsequent sections.

Next, consider the situation where  $Y$  is defined by functional inequalities:

$$Y = \{y \in \mathcal{R}^p: f_i(y) \leq 0, i = 1, \dots, s\}. \quad (2.3)$$

Under what circumstances is  $O_\infty$  defined in a similar way?

*Theorem 2.3:* Suppose  $A$  is Lyapunov stable and that for  $i = 1, \dots, s$  the functions  $f_i: \mathcal{R}^p \rightarrow \mathcal{R}$  are continuous and

satisfy  $f_i(0) \leq 0$ . Then: i) the functions  $g_i: \mathcal{R}^n \rightarrow \mathcal{R}$  given by

$$g_i(x) = \sup \{f_i(CA^t x): t \in \mathcal{J}^+\} \quad (2.4)$$

are defined, ii)  $0 \in O_\infty$  and iii)

$$O_\infty = \{x \in \mathcal{R}^n: g_i(x) \leq 0, i = 1, \dots, s\}. \quad (2.5)$$

*Proof:* Let  $x \in \mathcal{R}^n$ . Using the notation defined in the proof of Theorem 2.1, it follows that for all  $t \in \mathcal{J}^+$ ,  $CA^t x \in S(\gamma_1 \|x\|)$ . Thus, by the compactness of  $S$  and the continuity of the  $f_i$ , the sequence  $f_i(CA^t x)$ ,  $t \in \mathcal{J}^+$ , is bounded from above. Therefore, the supremum in (2.4) is defined for all  $x \in \mathcal{R}^n$ . Results ii) and iii) are a direct consequence of  $f_i(0) \leq 0$  and the definition of  $O_\infty$ .  $\square$

When  $Y$  is given by (2.3) and  $O_\infty$  is finitely determined

$$O_\infty = \{x \in \mathcal{R}^n: f_i(CA^t x) \leq 0, i \in \{1, \dots, s\}, t \in \{0, \dots, t^*\}\}. \quad (2.6)$$

Not all of the  $s \cdot (t^* + 1)$  inequalities in (2.6) may be active in the sense that there exists  $x \in O_\infty$  such that  $f_i(CA^t x) = 0$ . The active inequalities have a special structure which is given in the following theorem.

*Theorem 2.4:* Suppose  $O_\infty$  is given by (2.6). Then there exists a nonempty set of integers  $S^* \subset \{1, \dots, s\}$  and indexes  $t_i^*$ ,  $i \in S^*$ , such that i)  $t^* = \max \{t_i^*: i \in S^*\}$ , ii)

$$O_\infty = \{x \in \mathcal{R}^n: f_i(CA^t x) \leq 0, i \in S^*, t \in \{0, \dots, t_i^*\}\} \quad (2.7)$$

iii) for all  $i \in S^*$  and  $t \in \{0, \dots, t_i^*\}$  there exists and  $x \in O_\infty$  such that  $f_i(CA^t x) = 0$ .

*Proof:* Let  $S^* = \{i: \exists t \in \{0, \dots, t^*\} \text{ and } x \in O_\infty \text{ such that } f_i(CA^t x) = 0\}$ . If  $i \notin S^*$  it follows that  $f_i$  is inactive in (2.6), i.e.,  $f_i(CA^t x) < 0$  for all  $t \in \{0, \dots, t^*\}$  and  $x \in O_\infty$ . For  $i \in S^*$ , define  $t_i^* = \max \{t \in \{0, \dots, t^*\}: \exists x \in O_\infty \text{ such that } f_i(CA^t x) = 0\}$ . Then it is obvious that (2.7) holds. Result iii) follows from (2.7) and the definition of  $t_i^*$ . Clearly, there exists  $x \in O_\infty$  such that  $f_i(CA^{t_i^*} x) = 0$ . This implies  $f_i(CA^{t_i^* - t} x(t)) = 0$ ,  $t = 0, \dots, t_i^*$ , where  $x(t) = A^t x$ . By (1.7),  $x(t) \in O_\infty$  for all  $t \in \mathcal{J}^+$ .  $\square$

*Remark 2.2:* Suppose there exists an  $i \in \{1, \dots, s\}$  such that  $f_i(Cx) < 0$  for all  $x \in X(C, Y)$ . Then it is obvious that  $i \notin S^*$ . Even when such trivial situations are excluded, it is possible that  $S^* \neq \{1, \dots, s\}$ . Consider, for instance, the example:  $n = p = 1$ ,  $A = [-0.8]$ ,  $C = [1]$ ,  $f_1 = y - 1$ ,  $f_2 = -y - 2$  then  $S^* = \{1\}$  and  $t_1^* = t^* = 1$ .

*Remark 2.3:* Result iii) states that all the inequalities in (2.7) are active. This does not exclude the possibility that some of the inequalities may be redundant in the sense that they can be removed from (2.7) without destroying the characterization of  $O_\infty$ . For example, redundant inequalities in (2.3) lead to redundant inequalities in (2.7). If such redundant inequalities are excluded, it is still possible to give examples where there are redundant inequalities in (2.7). However, these examples appear to be structurally unstable, i.e., small changes in  $A, C$  or the  $f_i$  eliminate redundant

inequalities. Thus, from a practical point of view it appears that (2.7) offers the most economical representation of  $O_\infty$ .

*Remark 2.4:* Because of Theorem 2.4, (2.4) and (2.5) can be replaced by

$$g_i(x) = \max \{f_i(CA^t x) : t \in \{0, \dots, t_i^*\}\} \quad (2.8)$$

and

$$O_\infty = \{x \in \mathcal{R}^n : g_i(x) \leq 0, i \in S^*\}. \quad (2.9)$$

These simplification in (2.4) and (2.5) are valuable in control applications where it is necessary to have an efficient computational procedure for testing whether or not  $x \in O_\infty$ .

A change in state coordinates for the system (1.3) produces a simple transformation of  $O_\infty$ . To see this let  $U \in \mathcal{R}^{n \times n}$  be the nonsingular matrix which describes the change in coordinates. Then it can be verified from (1.7) that

$$O_\infty(A, C, Y) = UO_\infty(\tilde{A}, \tilde{C}, Y) \quad \text{where } \tilde{C} = CU, \tilde{A} = U^{-1}AU. \quad (2.10)$$

Because of this relationship, we shall feel free in our subsequent derivations to express the pair  $C, A$  in whatever coordinate system seems most natural. This path is acceptable for both theoretical developments and the computation of  $O_\infty$ .

The most obvious application of this idea is to unobservable systems. Choose the coordinate system in the usual way [16] so that  $C, A$  has the form

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix} \quad (2.11)$$

where  $C_1 \in \mathcal{R}^{p \times m}$ ,  $A_1 \in \mathcal{R}^{m \times m}$ ,  $A_2 \in \mathcal{R}^{(n-m) \times (n-m)}$ ,  $A_3 \in \mathcal{R}^{(n-m) \times m}$  and the pair  $C_1, A_1$  is observable. The application of (1.7) then shows that

$$O_\infty(A, C, Y) = O_\infty(A_1, C_1, Y) \times \mathcal{R}^{n-m}. \quad (2.12)$$

Thus,  $O_\infty$  is a cylinder set whose determination depends only on the triple  $A_1, C_1, Y$ . Stated in a coordinate-free way,  $O_\infty$  is a cylinder set which has infinite extent in those directions which belong to the unobservable subspace. Another consequence of (2.12) is the necessity of the observability condition in result ii) of Theorem 2.1

Finally, we consider systems (1.3) which are observable but have *divergent* motions. Define  $L = \{x \in \mathcal{R}^n : \text{the sequence } A^t x, t \in \mathcal{J}^+, \text{ is bounded}\}$ . Clearly,  $L$  is a linear space. It is determined by the direct sum of the invariant subspaces of  $A$  which are associated with those eigenvalues that satisfy  $|\lambda_i(A)| < 1$ , together with the span of the eigenvectors of  $A$  which correspond to  $|\lambda_i(A)| = 1$ . We will not dwell on the details, but it is possible to determine a basis for  $L$  numerically from the real Schur form [17] of  $A$ . Let  $m = \text{dimension of } L$ . Choose a basis for  $\mathcal{R}^n$  in the following way: let the first  $m$  basis vectors be a basis for  $L$ , let the remaining  $n - m$  basis vectors be a basis for the orthogonal complement of  $L$ . In this basis it is easy to see that the pair  $C, A$  has the form

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix} \quad (2.13)$$

where  $A_1 \in \mathcal{R}^{m \times m}$  is Lyapunov stable and  $CA^t x$  diverges on  $\mathcal{J}^+$  unless the last  $n - m$  components of  $x$  are zero. Now assume  $Y$  is *bounded*. Then it is obvious that

$$O_\infty(A, C, Y) = O_\infty(A_1, C_1, Y) \times \{0\}. \quad (2.14)$$

A coordinate-free interpretation of (2.14) is that  $O_\infty \subset L$ .

*Remark 2.5:* With respect to the determination of  $O_\infty$  the consequences of the preceding two paragraphs are clear. It is possible to successively eliminate from the representation of  $C, A$  the "unobservable" and "divergent motion" subspaces. Hence there is no loss of generality in restricting our attention to systems which are observable and (if  $Y$  is bounded) Lyapunov stable. This observation is important because several of our key results depend on the assumptions of observability and Lyapunov stability.

### III. THE ALGORITHMIC DETERMINATION OF $O_\infty$

Theorem 2.2 suggests the following conceptual algorithm for determining  $t^*$  and  $O_\infty$ .

*Algorithm 3.1:*

*Step 1:* Set  $t = 0$ .

*Step 2:* If  $O_{t+1} = O_t$ , stop. Set  $t^* = t$  and  $O_\infty = O_t$ . If  $O_{t+1} \neq O_t$ , continue.

*Step 3:* Replace  $t$  by  $t + 1$  and return to Step 2.  $\square$

Clearly, the algorithm will produce  $t^*$  and  $O_\infty$  if and only if  $O_\infty$  is finitely determined. There appears to be no finite algorithmic procedure for showing that  $O_\infty$  is not finitely determined. Fortunately, as subsequent developments show, it is often possible to resolve the issue of finite determination by other means. Algorithm 3.1 is not practical because it does not describe how the test  $O_{t+1} = O_t$  is implemented. The difficulty can be overcome if  $Y$  is defined by (2.3) and the hypotheses of Theorem 2.3 are satisfied, for then Step 2 leads to a set of mathematical programming problems.

*Algorithm 3.2:*

*Step 1:* Set  $t = 0$ .

*Step 2:* Solve the following optimization problems for  $i = 1, \dots, s$ :

$$\text{maximize } J_i(x) = f_i(CA^{t+1}x) \quad (3.1)$$

subject to the constraints

$$f_j(CA^k x) \leq 0, \quad j = 1, \dots, s, k = 0, \dots, t. \quad (3.2)$$

Let  $J_i^*$  be the maximum value of  $J_i(x)$ . If  $J_i^* \leq 0$  for  $i = 1, \dots, s$ , stop. Set  $t^* = t$  and define  $O_\infty$  by (2.6). Otherwise, continue.

*Step 3:* Replace  $t$  by  $t + 1$  and return to Step 2.  $\square$

*Remark 3.1:* After Algorithm 3.2 has terminated, it is possible to obtain  $S^*$  and the indexes  $t_i^*$  by solving a sequence of mathematical programming problems. Let  $J_{it}^* = \max f_i(CA^t x)$  subject to the constraints  $f_j(CA^k x) \leq 0, j = 1, \dots, s, k = 0, \dots, t^*$ . For each  $i \in \{1, \dots, s\}$  determine  $\tilde{t} \in \mathcal{J}^+$  so that  $J_{it}^* < 0$  for  $t = \tilde{t} + 1, \dots, t^*$  and  $J_{it} = 0$ . If this is possible,  $i \in S^*$  and  $t_i^* = \tilde{t}$ . If  $J_{it} < 0$  for  $t = 0, \dots, t^*, i \notin S^*$ .

The success of Algorithm 3.2 depends on the existence of effective algorithms for solving the rather large mathematical

programming problems which arise. This presents some difficulty because global optima are needed. Even when the  $f_i$ ,  $i = 1, \dots, s$ , are convex, the difficulty remains because the programming problems require the *maximum* of a convex function subject to convex constraints. When  $Y$  is a polyhedron, the difficulty disappears. Then, the programming problems are linear and efficient computer codes for obtaining the global maxima abound.

**Remark 3.2:** Suppose the optimization algorithm used in Step 2 is not guaranteed to provide a global minimum. Algorithm 3.2 may still produce a useful result. If it has been determined that for some  $x$  and  $t$  in Step 2 that (3.2) is satisfied and  $J_i(x) \leq 0$ ,  $i = 1, \dots, s$ , then  $O_t = O_\infty$ . Of course, there is not guarantee that Algorithm 3.2 will stop when  $O_\infty$  is finitely determined. Moreover, if Algorithm 3.2 does stop, it is possible that  $t$  may be greater than  $t^*$ .

Using the above ideas it is possible to characterize  $O_\infty$  for complex finitely determined systems. In this section, we are content to apply them to some simple examples. Our purpose is to illustrate how various  $A$ ,  $C$ , and  $Y$  affect both the determination of  $O_\infty$  and the properties of  $O_\infty$ .

The data for the examples, along with  $t^*$  and the characteristic roots of  $A$  are summarized in Table I. Examples 3.3–3.10 are particularly simple and it is possible to obtain  $t^*$  and  $O_\infty$  by hand calculations. In Example 3.1, Algorithm 3.2 was implemented by linear programming. Example 3.2 requires the maximization of quadratic functions subject to quadratic inequality constraints; these nonlinear programming problems were solved numerically using the optimization program VMCON [18]. Fig. 1 shows  $O_\infty$  for Examples 3.1 and 3.2. In each example,  $O_\infty$  is determined by functional inequalities of the form  $f_1(CA^t x) \leq 0$ . The dots on the boundary of  $O_\infty$  show where two inequalities are simultaneously active; the number adjacent to the intervening arcs show the value of  $t$  for which the corresponding inequality is active. Note that  $A$  is asymptotically stable in Examples 3.1–3.6 and only Lyapunov stable in Examples 3.7–3.10.

Example 3.1 is taken from [11]. The double shaded region shows the  $A$ -invariant subset of  $X$  obtained in that paper. The maximal output admissible set  $O_\infty$  is obviously much larger. Example 3.2 illustrates the fact that  $Y$  need not be a polyhedron as in the prior literature devoted to the determination of  $A$ -invariant subsets of  $X$ . The boundary segments of  $O_\infty$  are sections taken from three ellipses. Examples 3.3–3.6 show that  $O_\infty$  may or may not be finitely determined when  $Y$  is unbounded or contains the origin in its boundary. In Example 3.7,  $A$  has its characteristic roots on the unit circle. Because  $CA^t$  is periodic in  $t$  with period  $t = 4$ , it is easy to see that  $O_\infty$  is finitely determined. In Example 3.8, the characteristic roots are again on the unit circle but  $CA^t$  never repeats itself for  $t \in \mathcal{J}^+$ . In fact, for  $t \in \mathcal{J}^+$ ,  $\|CA^t\| = 1$  and  $CA^t$  takes on essentially all directions. Thus,  $O_\infty$  is not finitely determined and  $O_\infty = S(1)$ . In Examples 3.9 and 3.10,  $A$  has a characteristic root at  $\lambda = 1$ . Again, both possibilities with respect to finite determination exist. In Example 3.10, it is easy to see that  $O_\infty = \{x: |[1 \ 1]x| \leq 1, |[1 \ 0]x| \leq 1\}$ , a very simple set despite the fact that  $O_\infty$  is not finitely determined.

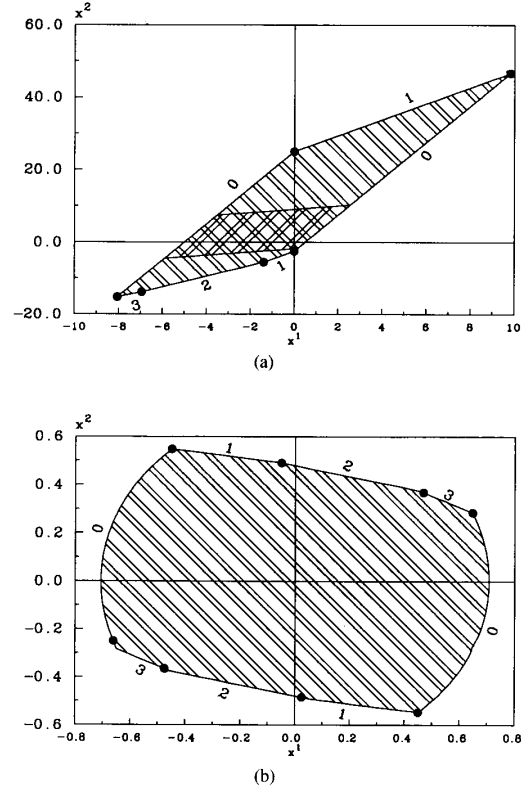


Fig. 1. (a)  $O_\infty$  for Example 3.1. (b)  $O_\infty$  for Example 3.2.

#### IV. CONDITIONS FOR FINITE DETERMINATION OF $O_\infty$

It is desirable to have simple conditions which assure the finite determination of  $O_\infty$ . Our main result in this direction is the following theorem.

**Theorem 4.1:** Suppose the following assumptions hold: i)  $A$  is asymptotically stable ( $|\lambda_i(A)| < 1$ ,  $i = 1, \dots, n$ ), ii) the pair  $C, A$  is observable, iii)  $Y$  is bounded, iv)  $0 \in \text{int } Y$ . Then,  $O_\infty$  is finitely determined.

**Proof:** It is apparent from ii) and iii) and the proof of Theorem 2.1, that there exists an  $r > 0$  such that  $O_t \subset S(r)$ ,  $t \in \mathcal{J}^+$ ,  $t \geq n - 1$ . Moreover, by i),  $CA^t \rightarrow 0$  as  $t \rightarrow +\infty$ . Thus, by iv), there exists a  $k \in \mathcal{J}^+$ ,  $k \geq n - 1$ , such that  $CA^{k+1}S(r) \subset Y$ . Hence,  $CA^{k+1}O_k \subset Y$ . This result and (2.1) imply that  $O_{k+1} = O_k$  and by Theorem 2.2 the proof is complete.  $\square$

**Remark 4.1:** The  $k$  in the proof is an upper bound for  $t^*$ . There is a simple test for obtaining it. Choose  $\gamma$  so that  $S(\gamma) \subset Y$ . Then,  $\|CA^{k+1}\| \leq r^{-1}\gamma$  implies  $CA^{k+1}S(r) \subset Y$  and  $t^* \leq k$ .

While the conditions in the theorem are sufficient for finite determinability, Examples 3.3, 3.5, 3.7, and 3.9 show that they are not necessary. On the other hand, Examples 3.4, 3.6, 3.8, and 3.10 illustrate difficulties in sharpening the sufficient conditions. As has been noted in Remark 2.2, Assumption ii) is not really a limitation. It is possible by making additional assumptions on  $C$  and  $A$  to eliminate assumptions iii) and iv). We will not pursue the investigation

TABLE I  
DATA FOR EXAMPLES

Example	$A$	$C$	$Y$	$\lambda_i$	$t^*$
3.1	$\begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.5 \end{bmatrix}$	$\begin{bmatrix} -1.0 & 0.2 \end{bmatrix}$	$[-0.5, 5.0]$	0.6, 0.3	3
3.2	$\begin{bmatrix} 0.6 & 1.0 \\ 0.0 & 0.76 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & -1.0 \end{bmatrix}$	$S(1)$	0.76, 0.6	3
3.3	$\begin{bmatrix} 0.5 & 1.0 \\ 0.0 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.0 \\ 1.0 & 1.0 \end{bmatrix}$	$[-1.0, +\infty)$ $\times (-\infty, 1.0]$	0.5, 0.3	2
3.4	$\begin{bmatrix} 0.5 & 1.0 \\ 0.0 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 0.0 \end{bmatrix}$	$[-1.0, +\infty)$	0.5, 0.3	$\infty$
3.5	$\begin{bmatrix} 0.6 & -0.1 \\ 0.1 & -0.3 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[0.0, 1.0]$	0.58, -0.28	1
3.6	$\begin{bmatrix} 0.5 & 1.0 \\ 0.0 & 0.2 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[0.0, 1.0]$	0.5, 0.2	$\infty$
3.7	$\begin{bmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[-1.0, 1.0]$	$e^{\pm j(\pi/4)}$	3
3.8	$\begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[-1.0, 1.0]$	$e^{\pm j1}$	$\infty$
3.9	$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \\ 1.0 & 0.0 \end{bmatrix}$	$[-1.0, 1.0]^2$	1.0, 0.5	0
3.10	$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	$[-1.0, 1.0]$	1.0, 0.5	$\infty$
4.1	$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 1.0 \\ 0.0 & 0.0 & 0.6 \end{bmatrix}$	$\begin{bmatrix} 1.0 & 1.0 & 1.0 \end{bmatrix}$	$[-1.0, 1.0]$	1.0, 0.6, 0.5	$\infty$

of such assumptions here. Instead, because it has greater practical interest, we will investigate the relaxation of i).

In particular, we consider systems where  $A$  is Lyapunov stable. By Remark 2.3 this represents no loss of generality when  $Y$  is bounded. The most common situation, and the one which we treat here, is where the only characteristic roots of  $A$  that are on the unit circle are at  $\lambda = 1$ . The Lyapunov stability of  $A$  implies that these roots are all simple. Thus, by doing a block diagonalization of  $A$ , where each block corresponds to a distinct eigenvalue of  $A$  [16], it is possible to find a choice of coordinates which puts  $A$ , and consequently  $C$ , into the form

$$C = \begin{bmatrix} C_L & C_s \end{bmatrix}, \quad A = \begin{bmatrix} I_d & 0 \\ 0 & A_s \end{bmatrix}. \quad (4.1)$$

Here, the partitioning of  $C$  and  $A$  is dimensionally consistent,  $d$  is the number of characteristic roots at  $\lambda = 1$ , and  $A_s \in \mathcal{R}^{(n-d) \times (n-d)}$  is asymptotically stable. The representation (4.1) simplifies our developments.

**Theorem 4.2:** Suppose  $A, C$  has the form (4.1) and  $Y$  is closed. Define

$$\hat{C} = \begin{bmatrix} C_L & 0 \\ C_L & C_s \end{bmatrix} \in \mathcal{R}^{2p \times n},$$

$$X_L = \{x: \begin{bmatrix} C_L & 0 \end{bmatrix} x \in Y\} \subset \mathcal{R}^n. \quad (4.2)$$

Then: i)  $O_\infty \subset X_L$ , ii)  $O_\infty(A, C, Y) = O_\infty(A, \hat{C}, Y \times Y)$

**Proof:** Result i) implies ii) because if it holds, the additional condition  $\begin{bmatrix} C_L & 0 \end{bmatrix} x \in Y$ , which is contained in the augmented triple  $A, \hat{C}, Y \times Y$ , is satisfied automatically. Suppose, contrary to i), there exists an  $x \in O_\infty(A, C, Y)$  such

that  $C_L x \notin Y$ . Since  $Y$  is closed, this implies the existence of an  $r > 0$  such that for all  $w \in S(r)$ ,  $C_L x + w \notin Y$ . Because  $x \in O_\infty(A, C, Y)$ , it follows that  $y(t) = CA^t x = \begin{bmatrix} C_L & 0 \end{bmatrix} x + \begin{bmatrix} 0 & C_s A_s^t \end{bmatrix} x \in Y$  for all  $t \in \mathcal{J}^+$ . From the asymptotic stability of  $A_s$  there exists a  $k \in \mathcal{J}^+$  such that  $\| \begin{bmatrix} 0 & C_s A_s^k \end{bmatrix} x \| < r$ . Thus,  $y(k) \notin Y$  and this contradiction completes the proof.  $\square$

Consider the application of the theorem to Example 3.10. The matrices  $C$  and  $A$  already have the form (4.1) with  $d = 1$ , so no transformation of coordinates is necessary. It is easily confirmed that while  $O_\infty(A, C, Y)$  is not finitely determined,  $O_\infty(A, \hat{C}, Y \times Y)$  is. Unfortunately, the trick of replacing  $A, C, Y$  by  $A, \hat{C}, Y \times Y$  does not work generally. See Example 4.1 in Table I. Again, this system satisfies (4.1) with  $d = 1$ . But now neither  $O_\infty(A, C, Y)$  nor  $O_\infty(A, \hat{C}, Y \times Y)$  is finitely determined.

To see why, examine Fig. 2, which shows  $x^2$ - $x^3$  sections of  $O_\infty(A, C, Y)$  taken for several values of  $x^1 \in [0, 1]$ . Because  $O_\infty \subset \{x: |x^1| \leq 1\} = X_L$  and is symmetric, there is no need to consider values of  $x^1$  outside  $[0, 1]$ . Each of the sections is a polygon. At  $x^1 = 1$ , the section of  $O_\infty(A, C, Y)$  is a polygon with only 6 vertices; but as  $x^1$  approaches 1 from below, the number of vertices in the polygon near the origin increases. Note the differences in scale for Fig. 2(a) and 2(b). A detailed analysis shows that if  $x^1$  is sufficiently close to 1, the number of vertices may be arbitrarily large. Thus, Example 4.1 is vastly more complicated than Example 3.10. The characterization of  $O_\infty(A, C, Y)$  requires the intersection of an infinite number of half spaces and there is no way that  $O_\infty(A, C, Y)$  can be finitely determined.

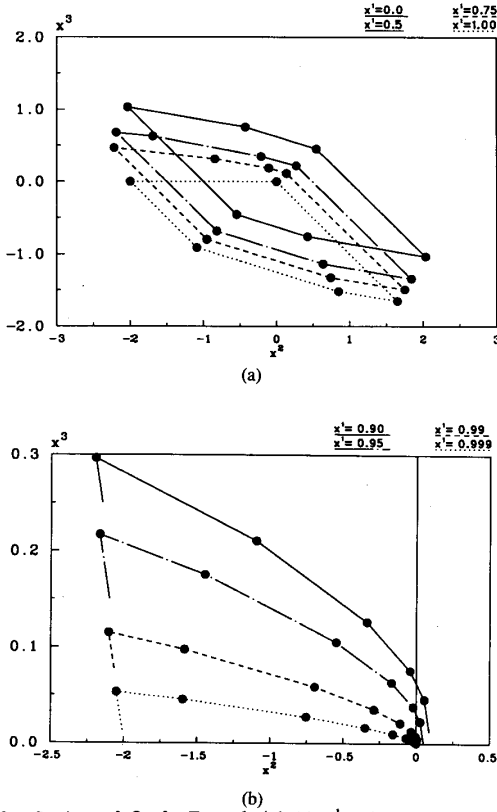


Fig. 2. Sections of  $O_\infty$  for Example 4.1. (a)  $x^1 = 0.0, 0.5, 0.75, 1.0$ . (b)  $x^1 = 0.9, 0.95, 0.99, 0.999$ .

The sections displayed in Fig. 2 do show an interesting property. The number of vertexes becomes very large only when  $x^1$  is very close to 1. This suggests that  $O_\infty(A, C, Y)$  can be approximated closely by  $O_\infty(A, C, Y) \cap \{x: |x^1| \leq 1 - \epsilon\} = O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$  where  $Y(\epsilon) = [-1 + \epsilon, 1 - \epsilon]$  and  $0 < \epsilon$  is small. As will now be seen, the approximation  $O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$  has the advantage that it is finitely determined.

#### V. THE APPROXIMATION OF $O_\infty$ FOR LYAPUNOV STABLE SYSTEMS

Assume that  $Y$  is given by (2.3). Then there is a natural choice for  $Y(\epsilon)$

$$Y(\epsilon) = \{y: f_i(y) \leq -\epsilon, i = 1, \dots, s\}. \quad (5.1)$$

Define

$$\epsilon_0 = -\max \{f_i(0): i = 1, \dots, s\}. \quad (5.2)$$

**Theorem 5.1:** Suppose  $C, A$ , and  $\hat{C}$  are given by (4.1) and (4.2),  $C, A$  is observable and  $A_S$  is asymptotically stable. Assume: i) the functions  $f_i: \mathcal{R}^p \rightarrow \mathcal{R}, i = 1, \dots, s$ , are continuous, ii)  $\epsilon_0 > 0$ , iii)  $Y = Y(0)$  is bounded. Then, for each  $\epsilon \in (0, \epsilon_0]$ ,  $O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$  is finitely determined. Furthermore

$$O_\infty(A, C, Y(\epsilon)) \subset O_\infty(A, \hat{C}, Y(\epsilon) \times Y) \subset O_\infty(A, C, Y). \quad (5.3)$$

*Proof:* Note that

$$\begin{aligned} O_t &= O_t(A, \hat{C}, Y(\epsilon) \times Y) \\ &= \{x: f_i([C_L \ 0]x) \leq -\epsilon, \\ &\quad f_i([C_L \ 0]x + [0 \ C_S A_S^k]x) \leq 0, \\ &\quad i = 1, \dots, s, k = 0, \dots, t\}. \end{aligned} \quad (5.4)$$

Clearly,  $O_t$  is nonempty. As in the proof of Theorem 4.1 there exists an  $r > 0$  such that  $O_{n-1} \subset S(r)$ . Suppose  $x \in O_{n-1}$ . Then  $\|x\| \leq r$  and  $f_i([C_L \ 0]x) \leq -\epsilon$ . Since  $A_S^t \rightarrow 0$  as  $t \rightarrow \infty$  and  $f_i$  is continuous, there exists a  $k \in \mathcal{J}^+$ , which is independent of  $x \in O_{n-1}$  and  $i = 1, \dots, s$ , such that  $f_i([C_L \ 0]x + [0 \ C_S A_S^{k+1}]x) \leq 0$ . Thus,  $\hat{C}A^{k+1}O_{n-1} \subset Y(\epsilon) \times Y$ . Consequently,  $\hat{C}A^{k+1}O_k \subset Y(\epsilon) \times Y$  and, by the reasoning used in the proof of Theorem 4.1,  $O_\infty$  is finitely determined. The right inclusion of (5.3) follows from part ii) of Theorem 4.2 and  $Y(\epsilon) \subset Y$ . Again by Theorem 4.2,  $O_\infty(A, C, Y(\epsilon)) = O_\infty(A, \hat{C}, Y(\epsilon) \times Y(\epsilon))$  and the left inclusion is obvious.  $\square$

As Example 4.1 shows, the output admissibility index for  $A, \hat{C}, Y(\epsilon) \times Y$  may increase without bound as  $\epsilon \rightarrow 0$ . The inclusions of (5.3) provide a way of judging whether or not for a given  $\epsilon$  the approximation of  $O_\infty(A, C, Y)$  by  $O_\infty(A, \hat{C}, Y(\epsilon) \times Y)$  is sufficiently good. They state, in effect, that the approximation is no worse than what would happen if the constraint set  $Y$  were replaced by  $Y(\epsilon)$ .

In most applications, an acceptable choice for  $\epsilon$  is clear. This is certainly true in Example 4.1. Table II shows how  $t^*$  changes with  $\epsilon$ . For  $\epsilon = 0.05$ , there is a reasonable compromise between  $t^*$  and the accuracy of the approximation. There are several ways to measure this accuracy. By (5.3) the reduction in  $O_\infty$  is no worse than what would be obtained by replacing  $Y = [-1, +1]$  by the slightly tighter constraint set  $Y = [-0.95, +0.95]$  and using the corresponding infinitely determined  $O_\infty$ . Alternatively, by applying result iv) of Theorem 2.1, it follows that  $0.95O_\infty(A, C, Y) \subset O_\infty(A, \hat{C}, Y(0.05) \times Y) \subset O_\infty(A, C, Y)$ . Note that for small  $\epsilon$ ,  $t^*$  appears to increase logarithmically with respect to  $\epsilon^{-1}$ . This result can be confirmed by a detailed analysis of the example.

#### VI. CONTINUOUS-TIME SYSTEMS

The characterization of  $O_\infty^c$  is difficult, even for simple systems of low order. As can be seen from (1.8), there is a continuum of constraints and generally they are active over all  $t \in \mathcal{R}^+$ . This means that the powerful finite determination results of the preceding sections do not apply. However, there are some connections between the discrete-time and continuous-time systems which can be exploited. The most obvious of these is the easily verified inclusion

$$O_\infty^c(A, C, Y) \subset O_\infty(e^{AT}, C, Y), \quad T \in \mathcal{R}^+. \quad (6.1)$$

Actually, it turns out that when  $T > 0$  is sufficiently small the inclusion becomes an approximation. Thus, our ability to evaluate  $O_\infty^c(e^{AT}, C, Y)$  becomes a tool in characterizing  $O_\infty^c(A, C, Y)$ . Before making these comments precise, it is

TABLE II  
EXAMPLE 4.1:  $t^*$  VERSUS  $\epsilon$

$\epsilon$	1.0	0.5	0.25	0.10	0.05	$1 \times 10^{-2}$	$1 \times 10^{-3}$	$1 \times 10^{-4}$	$1 \times 10^{-5}$	$1 \times 10^{-6}$
$t^*$	2	3	3	4	5	7	10	13	16	19

worth noting that some of our prior results extend, with little modification, to continuous-time systems.

*Remark 6.1:* Theorem 2.1 applies to  $O_\infty^c(A, C, Y)$  with no change except for the obvious modification in the definition of Lyapunov stability ( $\text{Re } \lambda_i(A) \leq 0$ ,  $i = 1, \dots, n$  and  $\text{Re } \lambda_i(A) = 0$  implies  $\lambda_i(A)$  is simple). The line of reasoning is exactly the same as before except for part ii) where the following steps are used: there exists a  $T > 0$  such that  $C, e^{AT}$  is observable [16], by Theorem 2.1  $O_\infty(e^{AT}, C, Y)$  is bounded, (6.1) implies  $O_\infty^c(A, C, Y)$  is bounded. Similarly, Theorem 2.3 and its proof go through with minor changes. Specifically,  $O_\infty^c$  replaced  $O_\infty$  in (2.5) and  $g_i$  is defined by

$$g_i(x) = \sup \{f_i(Ce^{At}x) : t \in \mathcal{R}^+\}. \quad (6.2)$$

Finally, (2.10) extends to  $O_\infty^c$ . Therefore, (2.11)–(2.14), and Remark 2.5 are also valid for  $O_\infty^c$ .

*Theorem 6.1:* Assume  $C, A$  is observable,  $A$  is Lyapunov stable and assumptions i)–iii) of Theorem 5.1 hold. Then for each  $\epsilon \in (0, \epsilon_0]$  there exists a  $T(\epsilon) > 0$  such that

$$O_\infty(e^{AT(\epsilon)}, C, Y(\epsilon)) \subset O_\infty^c(A, C, Y) \subset O_\infty(e^{AT(\epsilon)}, C, Y). \quad (6.3)$$

*Proof:* The right inclusion follows from (6.1). We begin the proof of the left inclusion by establishing several technical facts. By the Lyapunov stability, there exists a  $\gamma > 0$  such that  $\|Ce^{At}\| \leq \gamma$ ,  $t \in \mathcal{R}^+$ . By the observability of  $C, A$  there exists a  $T_0 > 0$  such that for all  $T \in (0, T_0]$ ,  $C, e^{AT}$  is observable [16]. Hence, by Theorem 2.1 there exists an  $r > 0$  such that  $O_\infty(e^{AT_0}, C, Y(\epsilon)) \subset S(r)$ . Let  $T_j = j^{-1}T_0$  where  $j \in \mathcal{J}^+$  and  $j > 0$ . It is then clear that for all such  $j$ ,  $O_\infty(e^{AT_j}, C, Y(\epsilon)) \subset O_\infty(e^{AT_0}, C, Y(\epsilon)) \subset S(r)$ . Let  $\delta_0 > 0$ . By the continuity of the  $f_i$ ,  $Y$  is compact and therefore  $Y + S(\delta_0)$  is compact. Because of the compactness the  $f_i$  are uniformly continuous on  $Y + S(\delta_0)$  and, in fact, there is a single modulus of continuity  $\delta(\epsilon)$  which holds for  $i = 1, \dots, s$ . Now let  $x \in O_\infty(e^{AT}, C, Y(\epsilon))$ . We need to show that there is a  $T > 0$ , independent of  $x$  and  $i$ , such that for all  $t \in \mathcal{R}^+$ ,  $f_i(Ce^{At}x) \leq 0$ . Stated differently, we must find  $T$  such that for all  $k \in \mathcal{J}^+$  and  $\sigma \in [0, T]$ ,  $f_i(Ce^{AkT}x + \Delta y(\sigma)) \leq 0$ , where  $\Delta y(\sigma) = Ce^{AkT}(e^{A\sigma} - I_n)x$ . Noting that  $f_i(Ce^{AkT}x) \leq -\epsilon$ , and applying our above series of facts, leads to the desired result. Choose  $T = T_j = T(\epsilon)$  so that

$$\gamma \|e^{A\sigma} - I_n\| r \leq \max \{\delta_0, \delta(\epsilon)\} \quad \forall \sigma \in [0, T]. \quad (6.4)$$

The analyticity of  $e^{A\sigma}$  shows that (6.3) is satisfied if  $j$  is sufficiently large.  $\square$

*Remark 6.2:* Of course,  $O_\infty(e^{AT(\epsilon)}, C, Y(\epsilon))$  is an output admissible set for the continuous-time system and is  $e^{AT(\epsilon)}$ -invariant. In general, it is not  $A$ -invariant or positively invariant for (1.6).

While the theorem shows that a sensible discrete-time

approximation of  $O_\infty^c$  exists, it does not provide a practical scheme for obtaining  $T(\epsilon)$ . Verifying the left inclusion of (6.3) by a series of guesses for  $T(\epsilon)$  is a workable idea, but it presents serious computational difficulties. In particular, the testing of the left inclusion of (6.3) for each  $T$  requires the solution of the following optimization problems: maximize  $F_i(x)$  subject to  $x \in O_\infty(e^{AT}, C, Y(\epsilon))$  where

$$F_i(x) = \sup \{f_i(Ce^{At}x) : t \in \mathcal{R}^+\}, \quad (6.5)$$

The inclusion holds if and only if for  $i = 1, \dots, s$  the resulting maxima satisfy  $F_i^* \leq 0$ . While the constraint  $x \in O_\infty(e^{AT}, C, Y(\epsilon))$  can be represented easily by using an obvious modification of (2.5)–(2.7), the evaluation of  $F_i(x)$  is computationally troublesome because it involves a supremum over  $\mathcal{R}^+$ . Certain simplifications occur when  $Y$  satisfies the Minkowski assumptions (see the next section). At the very least, the proof of the theorem supports an obvious intuitive criterion for the choice of  $T$ . In particular, (6.4) is satisfied if  $|\lambda_i(A)| \ll 1$  for  $i = 1, \dots, n$ .

## VII. THE MINKOWSKI ASSUMPTIONS

Many of the prior results are strengthened when  $Y$  satisfies the following *Minkowski* assumptions: a)  $0 \in \text{int } Y$ , b)  $Y$  is compact, c)  $Y$  is convex. Under these assumptions there is a corresponding Minkowski distance function, defined by

$$\mu_Y(y) = \inf \{\alpha : \alpha > 0, y \in \alpha Y\}. \quad (7.1)$$

The key properties of the distance function are summarized in the following theorem. (See, e.g., [19].)

*Theorem 7.1:* Suppose  $Y \in \mathcal{R}^p$  satisfies the Minkowski assumptions. Then i)  $\mu_Y : \mathcal{R}^p \rightarrow \mathcal{R}$  is defined and  $\mu_Y(0) = 0$ ; ii)  $\mu_Y$  is convex; iii) for all  $y \in \mathcal{R}^p$  and  $\alpha \in \mathcal{R}^+$ ,  $\mu_Y(\alpha y) = \alpha \mu_Y(y)$ ; iv) for all  $y_1, y_2 \in \mathcal{R}^p$ ,  $\mu_Y(y_1 + y_2) \leq \mu_Y(y_1) + \mu_Y(y_2)$ ; v) there exist  $r_1, r_2 > 0$  such that for all  $y \in \mathcal{R}^p$ ,  $r_1 \|y\| \leq \mu_Y(y) \leq r_2 \|y\|$ ; vi)  $Y = \{y : \mu_Y(y) \leq 1\}$ ; vii) for all  $\alpha > 0$ ,  $\mu_{\alpha Y} = \alpha^{-1} \mu_Y$  and  $\alpha Y = \{y : \mu_Y(y) \leq \alpha\}$ .  $\square$

Properties i)–v) show that the distance function is very much like a norm. In fact, if  $Y$  is symmetric,  $\mu_Y$  is a norm. The constraint set considered in [6] satisfies the Minkowski assumptions because there  $\mu_Y$  is the infinity norm in  $\mathcal{R}^p$ . Most practical physical constraints lead to constraint sets  $Y$  which satisfy the Minkowski assumptions and whose distance functions are readily determined. Key consequences of the Minkowski assumptions are contained in the following theorem.

*Theorem 7.2:* Assume  $C, A$  is observable,  $A$  is Lyapunov stable, and  $Y$  satisfies the Minkowski assumptions. Then  $O_\infty(A, C, Y)$  and  $O_\infty^c(A, C, Y)$  satisfy the Minkowski assumptions and their distance functions are given by

$$\mu_{O_\infty}(x) = \sup \{\mu_Y(CA^t x) : t \in \mathcal{J}^+\}, \quad (7.2)$$

$$\mu_{O_\infty^c}(x) = \sup \{\mu_Y(Ce^{At} x) : t \in \mathcal{R}^+\}, \quad (7.3)$$



*Proof:* It is an immediate consequence of Theorem 2.1 and Remark 6.1 that  $O_\infty(A, C, Y)$  and  $O_\infty^c(A, C, Y)$  satisfy the Minkowski assumptions. By the definitions of the Minkowski distance function and  $O_\infty$  and parts vi) and iii) of Theorem 7.1,  $\mu_{O_\infty}(x) = \inf\{\alpha > 0: \alpha^{-1}x \in O_\infty\} = \inf\{\alpha > 0: CA^t\alpha^{-1}x \in Y, \forall t \in \mathcal{J}^+\} = \inf\{\alpha > 0: \mu_Y(CA^t\alpha^{-1}x) \leq 1, \forall t \in \mathcal{J}^+\} = \inf\{\alpha > 0: \mu_T(CA^t x) \leq \alpha, \forall t \in \mathcal{J}^+\}$ . From this, (7.2) is evident. The proof of (7.3) is essentially the same.  $\square$

*Remark 7.1:* Result (7.2) may be interpreted as a special case of Theorem 2.3 where  $s = 1$ ,  $f_1 = \mu_Y - 1$ , and  $g_1 = \mu_{O_\infty} - 1$ .

*Remark 7.2:* The characterization of  $Y$  by  $f_1(y) = \mu_Y(y) - 1 \leq 0$ , may be used to define  $Y(\epsilon)$  in Theorem 5.1 and 6.1. Then, by part vii) of Theorem 7.1

$$Y(\epsilon) = \{y: \mu_Y(y) \leq 1 - \epsilon\} = (1 - \epsilon)Y. \quad (7.4)$$

Thus, by part iv) of Theorem 2.1

$$O_\infty(A, C, Y(\epsilon)) = (1 - \epsilon)O_\infty(A, C, Y). \quad (7.5)$$

When this identity is substituted into (5.3) and (6.3) Theorems 5.1 and 6.1 are strengthened. Specifically, the accuracy of the indicated approximations is measured directly in terms of  $O_\infty$  rather than indirectly in terms of  $Y(\epsilon)$ .

*Remark 7.3:* Suppose for a given  $T > 0$  we wish to find the smallest  $\epsilon > 0$  such that  $O_\infty(e^{AT}, C, Y(\epsilon)) \subset O_\infty^c(A, C, Y)$ . Using the representation (7.5), this corresponds to finding the smallest  $\epsilon$  such that  $O_\infty(e^{AT}, C, Y) \subset (1 - \epsilon)^{-1}O_\infty^c(A, C, Y)$ . The actual determination of the smallest  $\epsilon$  can be carried out by solving an optimization problem. Maximize  $\mu_{O_\infty^c}(x)$  subject to  $x \in O_\infty(e^{AT}, C, Y)$  and let the maximum value be denoted by  $\mu^*$ . Then the smallest  $\epsilon$  is given by  $(1 - \epsilon)\mu^* = 1$ . The result has an obvious application in Theorem 6.1. By solving a single optimization problem it is possible to determine, for a specified  $T$ , the tightest inclusion of the form

$$(1 - \epsilon)O_\infty(e^{AT}, C, Y) \subset O_\infty^c(A, C, Y) \\ \subset O_\infty(e^{AT}, C, Y).$$

*Remark 7.4:* The optimization problem in the preceding remark is, in general, not easy to solve numerically. As noted in (7.2), the evaluation  $\mu_{O_\infty^c}(x)$  requires a supremum over  $\mathcal{R}^+$ . Moreover, the problem is not convex because it involves the *maximization* of the convex function  $\mu_{O_\infty^c}$ . It is not clear at the present time what computational procedure should be used and how the special structure of the optimization problem may be exploited. We sketch one approach which has worked on simple problems. Suppose,  $O_\infty(e^{AT}, C, Y)$  is a polytope, determined by Algorithm 3.2. This is certainly the situation if  $C, A$  is observable,  $A$  is asymptotically stable, and  $Y$  is a polytope. Suppose further, that all the vertices,  $v_1, \dots, v_n$ , of  $O_\infty(e^{AT}, C, Y)$  are known; algorithms are available for computing them [20]–[22] when  $O_\infty(e^{AT}, C, Y)$  is defined by a set of linear inequalities. It is not hard to show that the optimization problem has its solution on at least one of the vertices. Thus for  $i = 1, \dots, N$ , solve the following one-dimensional optimization

problems: maximize  $\mu_Y(Ce^{At}v_i)$  over  $t \in \mathcal{R}^+$ . This can be done efficiently by mapping  $\mathcal{R}^+$  onto  $[0, 1]$  and applying one of the many available algorithms for maximizing a function on an interval. Let the maximum values be denoted by  $\mu_i^*$ . Then,  $\mu^* = \max\{\mu_i^*, i = 1, \dots, N\}$ . In general, the number of vertices  $N$  grows very rapidly with  $n$ . Thus, the approach is by no means a panacea.

*Remark 7.5:* It is easy to see from the  $A$ -invariance of  $O_\infty = \{x: \mu_{O_\infty}(x) \leq 1\}$  and the fact that  $\mu_{O_\infty}$  is a Minkowski distance that the sequence  $\mu_{O_\infty}(x(t)), t \in \mathcal{J}^+$ , is nonincreasing on any solution of  $x(t+1) = Ax(t)$ . This and property v) of Theorem 7.1 make  $\mu_{O_\infty}$  useful as a Lyapunov function. For example, if  $A$  is asymptotically stable,  $O_\infty$  is a domain of attraction such that the output constraint is not violated.

## VIII. AN APPLICATION OF $O_\infty$ TO THE DESIGN OF A NONLINEAR CONTROLLER

Maximal output admissible sets have important applications in system analysis and controller design. A simple example of an application to analysis is the regulator control system described in the first paragraph of Section I. Suppose, for  $i = 1, \dots, p$ , the actuator for the  $i$ th component of  $u$  saturates when  $|u^i| > 1$  and it is desired to estimate the domain of attraction for the closed-loop system in the presence of this nonlinearity. Let (1.2) take the form  $u(t) \in Y = \{y: \|y\|_\infty \leq 1\}$ . Then for any  $x(0) \in O_\infty(A + BK, K, Y)$  saturation is avoided and the motion is described by the linear equations (1.1). Thus, if  $A + BK$  is asymptotically stable,  $x(t) \rightarrow 0$  and  $O_\infty$  is a subset of the domain of attraction.

Kapasouris, Athans, and Stein [6]–[8] have more interesting applications. They allow dynamic compensators and exploit the properties of  $O_\infty$  to obtain nonlinear controllers which avoid actuator saturation and give a much improved overall response to large inputs. Here, we present a discrete-time modification of their continuous-time error governor scheme [6].

Our control system configuration is shown in Fig. 3. To avoid any confusion,  $t$  represents continuous time and  $k$  represents discrete time. The error governor generates a scalar gain  $\kappa$ , multiplying the sampled error signal  $e(k) = r(kT) - c(kT)$ . We assume, as is in [6], that the plant is linear and asymptotically stable and that a linear Lyapunov stable compensator has been designed by some methodology so that with  $\kappa = 1$  good linear-system closed-loop performance is obtained. If  $\kappa$  stays fixed at 1 and  $\|u(k)\|_\infty > 1$ , the resulting actuator saturation may cause a serious degradation in the closed-loop response. Reset windup [6] is but one example of this sort of difficulty; see [1] for additional discussion. The idea of the error governor is to adjust  $\kappa$  downward if  $\kappa = 1$  has the potential to create immediate or subsequent saturation. It is argued in [6] that such gain reductions will be relatively infrequent and that when they do occur that their effect on the response is much less damaging than actuator saturation.

In our implementation of the error governor,  $\kappa(k) \in [0, 1]$  is adjusted according to the state of the discrete-time compensator so that for all  $k \in \mathcal{J}^+$ ,  $\|u(k)\|_\infty \leq 1$ . Assume that the

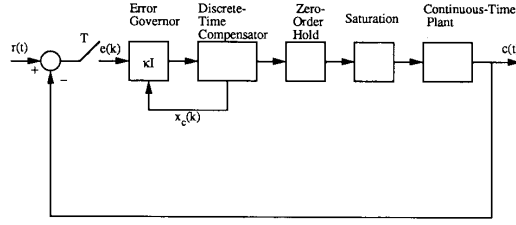


Fig. 3. Implementation of the modified error governor.

compensator and error governor are described by

$$\begin{aligned} x_c(k+1) &= A_c x_c(k) + B_c \kappa(k) e(k), \\ u(k) &= C_c x_c(k). \end{aligned} \quad (8.1)$$

Set  $Y = \{y: \|y\|_\infty \leq 1\}$  and use Algorithm 3.2 to obtain a characterization of  $O_\infty(A_c, C_c, Y)$  of the form (2.5)–(2.7). If  $A_c$  has characteristic roots at  $\lambda = 1$  (assume there are no others on the unit circle), it may be necessary to replace

$$\begin{aligned} A_c &= \text{diag} \left[ \begin{bmatrix} 0.2642 & 0.3223 \\ -0.3223 & 0.2642 \end{bmatrix}, \begin{bmatrix} 0.1417 & 0.4709 \\ -0.4709 & 0.1417 \end{bmatrix}, 0.2167, 0.9993, 1, 1 \right], \\ B_c^T &= \begin{bmatrix} -2.6526 & -1.0503 & 1.5410 & -4.3306 & 0.5860 & -2.3701 & 0.0318 & 0.4376 \\ -0.0792 & -2.3646 & -6.5588 & 0.5859 & -0.2589 & 0.9062 & -0.0319 & -0.5221 \end{bmatrix}, \\ C_c &= \begin{bmatrix} 1.4802 & -0.1946 & -0.1451 & -1.3393 & 0.2037 & 0 & 1 & 0 \\ -0.0188 & 1.2752 & -0.0352 & -0.2546 & -0.3914 & -0.0002 & 0 & 1 \end{bmatrix}. \end{aligned}$$

$O_\infty(A_c, C_c, Y)$  by an approximation  $O_\infty(A_c, \hat{C}_c, Y(\epsilon) \times Y)$  as described in Section V. Set

$$\kappa(k) = \max \{ \kappa \in [0, 1] : A_c x_c(k) + B_c \kappa e(k) \in O_\infty \}. \quad (8.2)$$

Then it is obvious that  $x_c(k) \in O_\infty$  implies  $x_c(k+1) \in O_\infty$ . Thus, if  $x_c(0) \in O_\infty$ , which is a reasonable assumption for a compensator,  $\|u(k)\|_\infty \leq 1$ ,  $k \in \mathcal{J}^+$ . Moreover, it is certain that the maximization problem has a solution because  $O_\infty$  is closed and  $A_c x_c(k) + B_c \kappa e(k) \in O_\infty$  for  $\kappa = 0$ . It should be noted that the on-line implementation of (8.2) is straightforward. Since  $O_\infty$  is defined by a system of linear inequalities, the upper limit imposed on  $\kappa$  by each inequality is given by a simple formula. The least of these limits is  $\kappa(k)$ . The real-time implementation of the overall control strategy requires that the computation of  $\kappa(k)$  and  $x_c(k+1)$  in (8.1) be done in less than  $T$  seconds.

The intuitive basis for the operation of the error governor is clear. If  $x_c(k) \in \text{int } O_\infty$  and  $e(k)$  is sufficiently small,  $\kappa(k) = 1$ . Consequently, when both  $r(t)$  and  $e(k)$  are reasonably small, the closed-loop system satisfies the linear equations of motion. When  $e(k)$  is large, or when  $A_c x_c(k)$  is near the boundary of  $O_\infty$  and  $B_c \kappa e(k)$  points toward the boundary,  $\kappa(k) < 1$  and the compensator action is reduced to avoid saturation. As noted in [6], the basis for the error governor is entirely intuitive. There is no theory which actually proves that the error governor has better response characteristics. Because of the asymptotic stability of the plant it is easy to show that the closed-loop system is

bounded-input bounded-output stable. The argument is essentially the same as the one used in [6].

We have tested the modified error governor controller on the aircraft control problem described in [6]. There, state models are given for both the longitudinal motion of the aircraft (4th-order) and the continuous-time compensator (8th-order). There are two inputs to the aircraft: an elevator and a flaperon. The physical limits on both of the inputs are  $\pm 25$  degrees. Thus, saturation occurs when  $\|u\|_\infty > 25$ . The compensator in [6] was obtained by adding integral control and using the LQG/LTR methodology. Let  $A_a, B_a, C_a$  denote the system matrices for this continuous-time compensator. Our discrete-time compensator was derived from it by the usual zero-order hold approach:  $A_c = e^{A_a T}$ ,  $B_c = \int_0^T e^{A_a t} dt B_a$ ,  $C_c = C_a$ . The choice  $T = 0.05$  provided a reasonable approximation to the continuous-time controller. A minimal realization of the resulting compensator, using modal coordinates for the state, is given by the matrices

The characteristic roots of  $A_c$  are evident from its real block diagonal form.

The compensator is Lyapunov stable with two characteristic roots of  $A_c$  on the unit circle at  $\lambda = 1$ . Thus, an approximation of  $O_\infty(A_c, C_c, Y)$  was considered. The system equations already have the form (4.1), so Algorithm 3.2 was applied to the determination of  $O_\infty = O_\infty(A_c, \hat{C}_c, Y(\epsilon) \times Y)$ . The set  $Y$  was defined by (2.3) with  $f_i(y) = 0.04 |y_i| - 1$  and  $s = 2$ . Note that the Minkowski assumptions are satisfied and  $\mu_Y(y) = 0.04 \|y\|_\infty$ . The required optimization problems were solved by linear programming for several values of  $\epsilon$ . Table III summarizes the results. It appears that  $O_\infty(A_c, \hat{C}_c, Y \times Y)$  is not finitely determined. However,  $\epsilon = 0.004$  gives a very good approximation; by Remark 7.4, it follows that the error of the approximation is at most 0.4%. The computational time required by Algorithm 3.2 is not large. For  $\epsilon = 0.004$  and an Apollo DN 3500, it is 6.1 s; for  $\epsilon = 0.00004$ , it is 11.7 s.

The actual characterization of  $O_\infty(A_c, \hat{C}_c, Y(\epsilon) \times Y)$  is quite simple. For  $\epsilon = 0.004$ , it is given by the representation (2.7)

$$\begin{aligned} O_\infty &= \{x_c: |x_c^7| \leq 24.9, |x_c^8| \leq 24.9, \\ &\quad |C_{c1} A_c^j x_c| \leq 25, |C_{c2} A_c^k x_c| \leq 25, \\ &\quad j = 0, \dots, 7, k = 0, \dots, 9\} \end{aligned} \quad (8.3)$$

where  $C_{c1}$  and  $C_{c2}$  are the rows of  $C_c$ . Suppose that the  $C_{c1} A_c^j$  and  $C_{c2} A_c^k$  are precomputed. Then the testing of  $x \in O_\infty$  requires 18 inner product evaluations and 20 compar-

TABLE III  
AIRCRAFT CONTROL SYSTEM:  $t_1^*$  AND  $t_2^*$  VERSUS  $\epsilon$

$\epsilon$	$4 \times 10^{-2}$	$4 \times 10^{-3}$	$4 \times 10^{-4}$	$4 \times 10^{-5}$	$4 \times 10^{-6}$	$4 \times 10^{-7}$	$4 \times 10^{-8}$	$4 \times 10^{-9}$
$t_1^*$	5	7	9	11	13	14	16	17
$t_2^*$	8	9	11	12	14	15	17	19

isons. The computational time is dominated by the  $18 \times 8 = 144$  multiplies which are required. With appropriate precomputed data, the evaluation of (8.2) requires only a few more comparisons and multiplies.

The closed-loop system was simulated with the step input  $r_1(t) = r_2(t) = 10$ . Fig. 4 shows the output response  $c_1(t)$  and  $c_2(t)$  for three different situations: i)  $\kappa = 1$  and no saturation, ii)  $\kappa = 1$  and saturation, iii)  $\kappa(k)$  determined by the error governor ( $\epsilon = 0.004$ ). As expected in i), the resulting linear sampled-data feedback system performs well and has essentially the same response as the corresponding continuous-time system in [6]. The response ii) shows the effects of adding saturation to the same system. Both overshoot and settling time are badly degraded. With the error governor in place saturation does not occur and the response iii) is much improved. Fig. 5 shows  $\kappa(k)$ . For  $k \geq 15$  ( $t \geq 0.75$ ),  $\kappa(k) = 1$  and the system behaves as a linear system.

The results shown in Figs. 4 and 5 are very close to those in [6]. Thus, our discrete-time modification appears to maintain the nice properties of the continuous-time mechanization in [6]. However, there is a substantial saving in the computational time required for the controller. On a Macintosh 512K the simulation of our implementation takes 13.1 s; the corresponding time reported in [8] is approximately 29,000 s. The reason for the large time is not clear from the discussion in [8], but it is certainly related to the complexity of testing whether or not a point  $x$  belongs to  $O_\infty^c(A_a, C_a, Y)$ . On an Apollo DN 3500 the simulation takes 1.24 s, which is about four times faster than real time. Hence, for systems of significant complexity, it appears that the implementation of practical on-line controllers is feasible.

It is perhaps worthwhile to compare briefly the error governor approach to some others which have been proposed for treating the control of linear plants with state and control constraints. The papers [3], [9]–[13] are concerned with state regulator problems similar to the one described by (1.1), (1.2) where  $Y$  is a polyhedron. While they do give attention to desirable choices for the state feedback matrix  $K$ , the positively invariant set of allowed initial conditions must be specified a priori and it is, in general, not maximal. Other researchers formulate the state regulator problem, with its state and control constraints, as an optimal control problem. See, e.g., [23]–[25] and the many papers cited in these references. The optimal control generated by such an approach allows the capture of the largest possible set of initial conditions. A serious disadvantage is the difficulty in implementing the on-line computation of the resulting nonlinear feedback control. It appears that feasible implementations are only possible for systems of very low order ( $n \leq 3$ ). The error governor has a number of important advantages: it can be applied to systems with time-varying inputs, it begins with

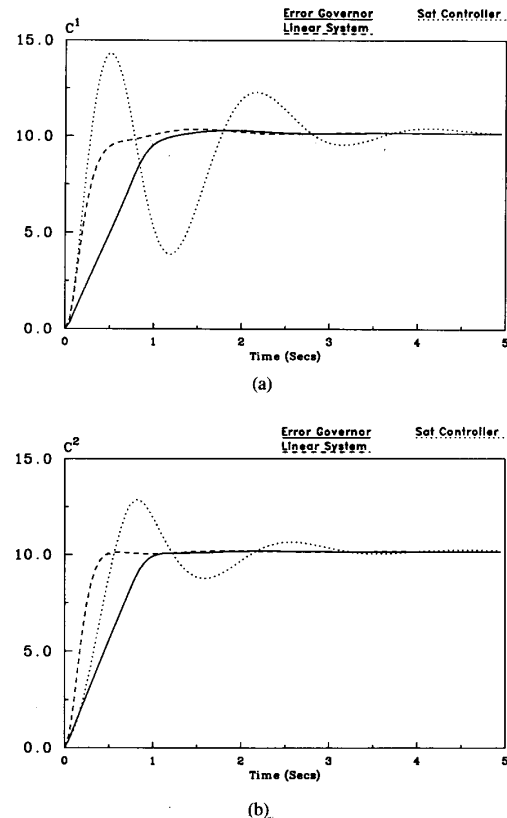


Fig. 4. Output responses for aircraft control system with step input  $r_1 = r_2 = 10$ .

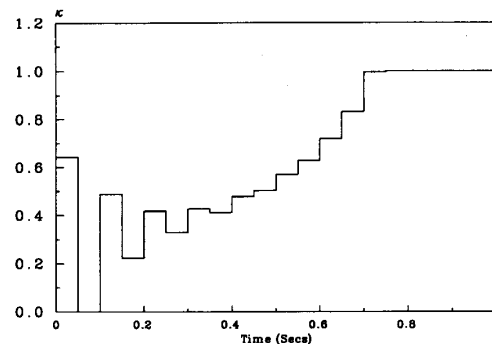


Fig. 5. Gain produced by error governor for aircraft control system with step input  $r_1 = r_2 = 10$ .

a specified linear design which can be carried out using dynamic compensation and the most advanced design methods, the nonlinear control strategy is implemented by a structurally simple modification of the linear control system, it appears feasible to implement the nonlinear control for

systems of fairly high order. Unfortunately, relatively little can be said theoretically about the dynamic response characteristics of error governor system. Moreover, when there are constraints on the state of the plant, or the plant is open-loop unstable, the error governor will not work. These limitations can be largely overcome by using different control strategies, such as modifications of the reference governor of [7]. Results concerning these issues will be reported in the future.

#### IX. CONCLUSION

In this paper, we have developed a general theory which pertains to the maximal output admissible sets  $O_\infty$  and  $O_\infty^c$ . Our most important contributions concern the algorithmic characterization of  $O_\infty$ . If  $Y$  is bounded,  $0 \in \text{int } Y$  and  $A$  has no characteristic roots on the unit circle,  $O_\infty$  may be determined in a finite number of steps. In most cases, the steps can be implemented by solving several easily formulated mathematical programming problems. When  $A$  has characteristic roots on the unit circle, it may turn out that the computations are not finite. If this happens and the characteristic roots on the unit circle are all at  $\lambda = 1$ , there is a practical way out: by introducing slightly stronger constraints a close approximation of  $O_\infty$  may be obtained finitely. While many of the properties of  $O_\infty$  carry over to  $O_\infty^c$ , the characterization of  $O_\infty^c$  is by no means as simple. A practical solution of this difficulty exists too. By taking  $T > 0$  sufficiently small, it is possible to approximate as closely as desired  $O_\infty^c(A, C, Y)$  by  $O_\infty(e^{AT}, C, Y)$ . Many of the preceding results, including the measure of approximation accuracy, are strengthened if  $Y$  satisfies the Minkowski assumptions of Section VII.

When  $Y$  is polyhedron the determination of  $O_\infty$  by Algorithm 3.2 is especially straightforward; it involves the solution of a sequence of linear programming problems. With the easy availability of good software and powerful computers it should, therefore, be possible to treat a variety of interesting systems of high order and considerable complexity. When  $Y$  is not polyhedral, computational difficulties may arise. Here, there is need for further research. Perhaps, polyhedral approximations of  $Y$  will prove useful.

Using representations of the form (2.7)–(2.9) it is possible to test numerically whether or not  $x$  belongs to  $O_\infty$ . The computational effort is reasonable, even for systems of fairly high order, and the structure of computations is suitable for parallel processing. This situation makes it feasible to implement on-line feedback control laws for linear plants with a variety of state and control constraints. The example in Section VII effectively demonstrates this potential. The error governor control methodology of [6] has been modified to the discrete-time case and applied to the 12th-order aircraft control problem considered in [6]. The resulting nonlinear discrete-time controller performs in the same effective way as the controller in [6] and it reduces computational load on the controller by several orders of magnitude. Other nonlinear control strategies, which are also based on  $O_\infty$ , are under investigation. They appear to have certain advantages and the details will be reported in the future.

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**Elmer G. Gilbert** (S'51-A'52-M'57-SM'78-F'79) received the Ph.D. degree in instrumentation engineering from the University of Michigan, Ann Arbor.

Since 1957, he has been a faculty member in the Department of Aerospace Engineering at the University of Michigan, where he is currently a Professor. He is also a member of the Department of Electrical Engineering and Computer Science and a participant in the Center for Research on Integrated Manufacturing at the University. His current re-

search interests include optimization, nonlinear control, and robotics.

Dr. Gilbert's special awards include the O.H. Schuck Award for best paper at the 1978 Joint Automatic Control Conference and a Control Systems Society best paper award in 1979. He is a member of the Johns

Hopkins University Society of Scholars. He has published numerous papers and holds eight patents.



**Kok Tin Tan** (S'88) was born in Singapore, on October 3, 1958. He received the B.Eng(Honors) and M.Eng in mechanical engineering from the National University of Singapore, in 1983 and 1986 and received the M.Sc in aerospace engineering from University of Michigan, Ann Arbor, in 1989.

He is currently a candidate for the Ph.D degree in the Aerospace Engineering Department, the University of Michigan. His research interests include nonlinear control, optimal control, and flight control.