A Sufficient Condition for the Stability of Nonlinear Model Predictive Controllers

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Abstract

We propose a Model Predictive Control (MPC) framework to generate feedback controls for time-varying nonlinear systems with input constraints. We provide for this framework a set of conditions on the design parameters that are sufficient to guarantee the stability of the closed-loop response. This sufficient condition for stability allows MPC practitioners to verify *a priori* that a given set of design parameters will lead to stability. The generality of the framework gives us an increased flexibility to choose a set of stabilizing design parameters. This flexibility can be explored to reduce the terminal constraints that are traditionally imposed on the optimal control problems (OCPs). This results in OCPs that can solved more efficiently by current optimization algorithms.

1 Introduction

This work concerns the construction of stabilizing feedbacks for time-varying nonlinear systems by the Model Predictive Control (MPC) method, which is also known as Receding-Horizon or Moving-Horizon Control. This method obtains the feedback control by solving a sequence of open-loop optimal control problems, each of them using the measured state of the plant as its initial state.

The study of MPC stabilizing schemes has been the subject of intense research in recent years. The first stability results for nonlinear systems required the optimal control problems to constrain the terminal state to be at the origin [8, 10]. These works were succeeded by other important contributions like the dual-mode approach [12], the contractive constraints [15], and more recently approaches based on the use of an appropriate terminal cost in the open-loop optimal control problems [3, 4, 7]. The importance of the terminal cost to guarantee stability was first noticed in [14] in a context of linear systems. The use of a terminal cost has been having its importance recognized in the last years and it is also of key importance in our approach. Further details and references can be found in surveys on nonlinear model predictive control schemes focusing on stability: [9], [11] and [2].

Traditionally, MPC schemes with guaranteed stability for nonlinear systems impose conditions on the open loop optimal control problem that either lead to some demanding hypotheses on the system or make the on-line computation of the open loop optimal control very hard. In previous works, these conditions take the form of a terminal state constrained to the origin, or an infinite horizon, or else impose some rather conservative controllability conditions on the system near the origin. On the other hand, most practitioners of MPC methods know that for some systems, by an appropriate choice of some parameters of the objective function and horizon (obtained by trial-and-error and some empirical rules), it is possible to obtain stabilizing trajectories without imposing demanding artificial constraints. However, their achievements cannot often be supported by any theoretical result to date, and “playing” with the design parameters is an option frequently criticized by researchers (see e.g. [1]). Here we intend to reduce this gap between theory and practice.

We propose a very general framework of MPC for systems satisfying very mild hypotheses. The *design parameters* of the MPC strategy are chosen in order to satisfy a certain (sufficient) *stability condition*, and hence the resulting closed-loop system will have the desirable stability properties guaranteed.

From a practical point of view, we give a stability condition that can be verified *a priori* (i.e. one not requiring trial-and-error, simulations) to guarantee that a particular set of design parameters will lead to stability. The generality of the framework gives us an increased flexibility to choose a set of stabilizing design parameters. This flexibility can be explored to reduce the terminal constraints that are traditional imposed on the optimal control problems (OCPs). This results in OCPs that are solved much more efficiently by current optimization algorithms.

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2 The Model Predictive Control Framework

We consider a nonlinear plant with input constraints, where the evolution of the state after time t is predicted by the following model.

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \geq t \tag{1a}$$
$$x(t) = x_t \tag{1b}$$
$$u(s) \in U(s). \tag{1c}$$

The data of this model comprise a set $X_0 \subset \mathbb{R}^n$ containing all possible initial states, a vector $x_t \in X_0$ that is the state of the plant measured at time t, a given function $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a multifunction $U : \mathbb{R}^+ \rightrightarrows \mathbb{R}^m$ of possible sets of control values. These data combined with a particular control function $u : [t, +\infty) \to \mathbb{R}^m$ define a trajectory $x : [t, +\infty) \to \mathbb{R}^n$.

We assume this system to be asymptotically controllable on X_0.

Our objective is to obtain a feedback law that (asymptotically) drives the state of our plant to the origin. This task is accomplished by using a MPC strategy. Consider a sequence of sampling instants $\{t_i\}_{i \geq 0}$ with a constant intersampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T)$ at each sampling instant t_i, every time using the current measure of the state of the plant x_{t_i}.

$\mathcal{P}(t, x_t, T)$: Minimize

$$\int_t^{t+T} L(s, x(s), u(s))ds + W(t+T, x(t+T)) \tag{2}$$

subject to:

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \in [t, t+T] \tag{3}$$
$$x(t) = x_t$$
$$u(s) \in U(s) \quad \text{a.e. } s \in [t, t+T]$$
$$x(t+T) \in S. \tag{4}$$

We clarify the notation adopted here. The variable t represents real time while we reserve s to denote the time variable used in the prediction model. The vector x_t denotes the actual state of the plant measured at time t. The process (x, u) is a pair trajectory/control obtained from the model of the system. The trajectory is sometimes denoted as $s \mapsto x(s; t, x_t, u)$ when we want to make explicit the dependence on the initial time, initial state, and control function. The pair $(\bar{x}, \bar{u})$ denotes our optimal solution to an open-loop optimal control problem (OCP). The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the time horizon T, the running and terminal costs functions L and W, and the terminal constraint set $S \subset \mathbb{R}^n$.

The MPC conceptual algorithm consists of performing the following steps at a certain instant t_i.

1. Measure the current state of the plant x_{t_i}.
2. Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x_{t_i}, T)$.
3. The control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta)$ is applied to the plant, (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).
4. The procedure is repeated from (1.) for the next sampling instant t_{i+1} (the index i is incremented by one unit).

The resultant control law is a feedback control since during each sampling interval, the control u^* is dependent on the state x_{t_i}.

It is a well-known fact that (for fixed finite horizon) the closed-loop trajectory of the system (x^*) does not necessarily coincide with the open-loop trajectory ($\bar{x}$) solution to the OCP. Hence, the fact that MPC will lead to a stabilizing closed-loop system is not guaranteed *a priori*, and is highly dependent on the *design parameters* of the MPC strategy.

We show that we can guarantee stability of the resultant closed loop system, by choosing the design parameters to satisfy a certain *stability condition*.

In the next section, we provide stability results for systems complying with the following hypotheses.

H1 For all $t \in \mathbb{R}^n$ the set $U(t)$ contains the origin, and $f(t, 0, 0) = 0$.

H2 The function f is continuous, and $x \mapsto f(t, x, u)$ is locally Lipschitz continuous for every pair (t, u).

H3 The set $U(t)$ is compact for all t, and for every pair (t, x) the set $f(t, x, U(t))$ is convex.

H4 The function f is compact on compact sets of x, more precisely given any compact set $X \subset \mathbb{R}^n$, the set $\{\, \|f(t, x, u)\| : t \in \mathbb{R}_+, x \in X, u \in U(t) \,\}$ is compact.

3 Main Results

The stability result provided in this section asserts that the feedback controller resulting from the application of the MPC strategy is a stabilizing controller, as long as the design parameters satisfy the stability condition below. For a proof a this result we refer to [6, 5].

Consider the following stability condition SC:

SC For system (1) the design parameters: time horizon T, objective functions L and W, and terminal constraint set S, satisfy:

SC1 The set S is closed and contains the origin.

SC2 The function L is continuous, $L(\cdot,0,0)=0$, and there is a continuous positive definite and radially unbounded function $M:\mathbb{R}^n\to\mathbb{R}_+$ such that $L(t,x,u)\geq M(x)$ for all $(t,u)\in\mathbb{R}_+\times\mathbb{R}^m$. Moreover, the "extended velocity set" $\{(v,\ell)\in\mathbb{R}^n\times\mathbb{R}_+ : v=f(t,x,u),\ \ell\geq L(t,x,u),\ u\in U(t)\}$ is convex for all (t,x).

SC3 The function W is positive semi-definite and continuously differentiable.

SC4 The time horizon T is such that, the set S is reachable in time T from any initial state and from any point in the generated trajectories: that is, there exists a set X containing X_0 such that for each pair $(t_0,x_0)\in\mathbb{R}_+\times X$ there exists a control $u:[t_0,t_0+T]\to\mathbb{R}^m$, with $u(s)\in U(s)$ for all $s\in[t_0,t_0+T]$, satisfying

$$x(t_0+T;t_0,x_0,u)\in S.$$

Also, for all control functions u in the conditions above

$$x(t;t_0,x_0,u)\in X \quad \text{for all } t\in[t_0,t_0+T].$$

SC5 There exists a scalar $\epsilon>0$ such that for each time $t\in[T,\infty)$ and each $x_t\in S$, we can choose a control function $\tilde{u}:[t,t+\epsilon]\to\mathbb{R}^m$ satisfying

$$\begin{aligned}&W_t(t,x_t)+W_x(t,x_t)\cdot f(t,x_t,\tilde{u}(t))\\ &\quad\leq -L(t,x_t,\tilde{u}(t)),\end{aligned} \qquad (SC5a)$$

and

$$x(t+r;t,x_t,\tilde{u})\in S \qquad (SC5b)$$

for all $r\in[0,\epsilon]$.

Theorem 3.1 *Assume the system satisfies hypotheses H1–H4. Choose the design parameters to satisfy SC . Then for a sufficiently small inter-sample time δ the closed-loop system resulting from the application of the MPC strategy is asymptotically stable.*

The set of conditions in SC can be seen as divided into two types: the first type consists of the conditions guaranteeing the existence of solutions to the OCPs; the other type comprises the conditions ensuring that the closed-loop trajectory is actually driven towards the origin. This second type of condition naturally has similarities with the conditions requiring W to be a control Lyapunov function. At first, it might appear that finding W satisfying (SC5a) together with all the other conditions is a task as hard as finding a control Lyapunov function as in [7, 13]. However, a distinguishing feature of our approach is that (SC5a) is only required to be satisfied on a subset S, which we have considerable freedom to choose. An appropriate choice of S, as is shown in the next section, makes it easier to choose the remaining design parameters to satisfy SC.

The task of choosing design parameters to satisfy all the conditions of SC might seem formidable at first. But one should not be discouraged by the generality of SC. In fact, the stability condition greatly simplifies for some standard choices of part of the design parameters. Typically, we might choose the objective function to be quadratic (making SC2 and SC3 trivially satisfied); choose the set S to be the whole space $\mathbb{R}^n$ (makes SC4 trivially satisfied); or make SC5 trivially satisfied by choosing S to be the set of points that satisfy SC5. It can be shown, with the help of examples, how stabilizing design parameters can be easily chosen for some nonlinear systems. (see [6, 5])

References

[1] R. R. Bitmead, M. Gevers, and V. Wertz. *Adaptive Optimal Control: The Thinking Man's GPC*. Prentice Hall, New York, 1990.

[2] H. Chen and F. Allgöwer. Nonlinear model predictive control schemes with guaranteed stability. In R. Berber, editor, *Nonlinear Model Based Process Control*. Kluwer, 1998.

[3] H. Chen and F. Allgöwer. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34(10), 1998.

[4] G. De Nicolao, L. Magni, and R. Scattolini. Stabilizing receding-horizon control of nonlinear time-varying systems. *IEEE Transactions on Automatic Control*, 43(7):1030–1036, 1998.

[5] F. A. C. C. Fontes. A general framework to design satabilizing nonlinear model predictive controllers. *Submmitted for publication*, 1999.

[6] F. A. C. C. Fontes. *Optimisation-Based Control of Constrained Nonlinear Systems*. PhD thesis, Imperial College of Science Technology and Medicine, University of London, London SW7 2BY, U.K., 1999.

[7] A. Jadbabaie, J. Yu, and J. Hauser. Stabilizing receding horizon control of nonlinear systems. In *American Control Conference*, San Diego, USA, June 1999.

[8] S. Keerthi and E. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon aproximations. *Journal of Optimization Theory and Applications*, 57:265–293, 1988.

[9] D. Q. Mayne. Nonlinear model predictive control: An assessment. In *5th International Conference on Chemical Process Control*, volume 93 of *AIChe Symposium Series*, pages 217–231, 1997.

[10] D. Q. Mayne and H. Michalska. Receding horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, 35:814–824, 1990.

[11] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Model predictive control: A review. *Automatica*, to appear, 1998.

[12] H. Michalska and D. Q. Mayne. Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 38:1623–1633, 1993.

[13] J. A. Primbs, V. Nevistić, and J. C. Doyle. On control Lyapunov functions and receding horizon control: Part II. Submitted to IEEE Transactions on Automatic Control, 1998.

[14] J. B. Rawlings and K. R. Muske. The stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, 38(10):1512–1516, 1993.

[15] T. H. Yang and E. Polak. Moving horizon control of nonlinear systems with input saturation, disturbances and plant uncertainty. *International Journal of Control*, 58(4):875–903, 1993.