Pergamon

Chemical Engineering Science, Vol. 53, No. 2, pp. 273-292, 1998

Printed in Great Britain
0009-2509/98 $19.00 + 0.00

PII: S0009-2509(97)00284-4

Shortest-prediction-horizon non-linear model-predictive control

Sairam Valluri,* Masoud Soroush*,† and Masoud Nikravesh‡

*Department of Chemical Engineering, Drexel University, Philadelphia, PA 19104, U.S.A.;
‡Earth Science Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, U.S.A.

(Received 19 July 1996; accepted in revised form 26 May 1997)

Abstract—This article concerns non-linear control of single-input–single-output processes with input constraints and deadtimes. The problem of input–output linearization in continuous time is formulated as a model-predictive control problem, for processes with full-state measurements and for processes with incomplete state measurements and deadtimes. This model-predictive control formulation allows one (i) to establish the connections between model-predictive and input–output linearizing control methods; and (ii) to solve *directly* the problems of constraint handling and windup in input–output linearizing control. The derived model-predictive control laws have the shortest possible prediction horizon and explicit analytical form, and thus their implementation does not require on-line optimization. Necessary conditions for stability of the closed-loop system under the constrained dynamic control laws are given. The connections between (a) the developed control laws and (b) the model state feedback control and the modified internal model control are established. The application and performance of the derived controllers are demonstrated by numerical simulations of chemical and biochemical reactor examples. © 1997 Elsevier Science Ltd

Keywords: Non-linear control; model-predictive control; constrained control; input–output linearization; model-based control; windup compensation.

1. INTRODUCTION

Since the early 1980s, nonlinear model-based control has made significant advances. These advances have been mainly within the two frameworks of model-predictive control (MPC) and differential geometric control.

In model-predictive control, constraints are explicitly accounted for, and the controller action is the solution to a constrained optimization problem. Thus, at each time instant, an MPC action is the 'best' feasible control action over a desirable horizon into the future. The recent powerful theoretical results on the closed-loop stability of constrained processes under MPC (e.g. Mayne and Michalska, 1993; Oliveira and Biegler, 1994; Rawlings and Muske, 1993; Vuthandam *et al.*, 1995; Zafiriou, 1990; Zheng and Morari, 1995), together with the other attractive features of MPC such as its inherent constraint handling capability, have made MPC the most appealing available model-based control methodology.

On the other hand, differential geometric control is a feedback linearization-based control methodology which leads to a control law with an analytical form. Because feedback linearization is only possible in the absence of constraints, much of the pioneering work on differential geometric control has been focused on unconstrained processes. As a matter of fact, in terms of constraint handling and windup, the differential geometric controllers suffer, more or less, from the same limitations that any analytical (non-MPC) feedback controller does.

Windup is a controller-performance-degradation phenomenon that is associated with actuator saturation. Although this phenomenon has been studied extensively, only a few attempts have been made to define it precisely. Furthermore, while closed-loop-response quality indices such as response time and overshoot have been used to indicate the presence of windup, at the present time there is no specific measure to quantify windup. To characterize the classes of controllers that do not exhibit windup, we adopt the following definitions:

Definition 1 (*Campo and Morari*, 1990). A dynamic controller does not exhibit windup, if and only if the states of the controller are not driven by the error when the actuator is in saturation.

Definition 2 (*Kapoor and Daoutidis*, 1996a). A dynamic controller does not exhibit windup, if and only

†Corresponding author. Tel: +1 215 895 1710. Fax: +1 215 895 5837. E-mail: masoud.soroush@coe.drexel.edu.

if when the actuator is in saturation, the closed-loop behavior under the controller is identical to that under a static state feedback. This definition is based on the realization that windup is not associated with static feedback controllers.

Definition 3. A dynamic controller does not exhibit windup, if and only if the controller action is solution to a moving-horizon constrained optimization problem.

According to Definition 1 any controller whose states are not driven by the error when the actuator is in saturation, does not exhibit windup. However, according to Definition 3 only model-predictive controllers have a windup-free performance. Definition 2 is neither as broad as Definition 1 nor as strict as Definition 3. In other words, if $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, respectively, represent the sets of the controllers that do not exhibit windup according to Definitions 1, 2 and 3, then $\mathscr{C} \subset \mathscr{B} \subset \mathscr{A}$.

In linear analytical control, the issues of windup and constraint handling as well as closed-loop stability in the presence of input constraints have been studied extensively (e.g. Åström and Rundqwist, 1989; Hanus *et al.*, 1987; Kapoor and Daoutidis, 1995; Kothare *et al.*, 1994; Suarez *et al.*, 1991; Walgama and Sternby, 1990). In particular, in linear analytical model-based control, powerful results are available in the frameworks of internal model control (Zheng *et al.*, 1994) and model state feedback control (Coulibaly *et al.*, 1992, 1995).

In nonlinear analytical model-based control, the issues of input constraint handling and windup have received considerable attention in recent years. More specifically, there have been several approaches to the problem of integral windup in input–output linearizing control methods. These include:

- Use of conditional integration (i.e. turning off integration when a constraint is active). This approach was employed in real-time non-linear control of pilot-scale polymerization reactors (e.g. Soroush and Kravaris, 1992a).
- Use of an MPC formulation of input–output linearization for a limited class of non-linear processes (Soroush and Kravaris, 1992b).
- Use of the input constraint mapping proposed by Calvet and Arkun (1988), to map the constraints on manipulated input to 'state-dependent' constraints on the reference input to the input–output linearizing state feedback. This mapping, together with the input–output linearizing state feedback, converts the non-linear system with input constraints to a 'linear' system with the state-dependent input constraints. To regulate the 'linearized' system with the state-dependent input constraints, Oliveira *et al.* (1995) and Kurtz and Henson (1996) have used linear MPC, and Doyle (1995) has employed the modified linear internal model control (Zheng *et al.*, 1994).
- Development of an observer-based anti-windup approach with a non-linear gain for non-linear processes (Kapoor and Daoutidis, 1996a).

In this article, the problem of input–output linearization is *formulated* as a model-predictive control problem. Model-predictive controllers that in the absence of constraints can force the controlled output to follow perfectly a linear reference trajectory are derived. Because under such model-predictive controllers the relation between the set-point and controlled output is governed by the linear ordinary differential equation that describes the reference trajectory, these model-predictive controllers are input–output linearizing. In particular, shortest possible prediction horizon is used to establish the connections between MPC and input–output linearizing control methods; this formulation is a continuous-time non-linear analog of that of the linear model algorithmic control (Mehra and Rouhani, 1980). The problems of constraint handling and windup in input–output linearizing control methods are solved via the model-predictive control formulation of input–output linearization. The derived dynamic control laws do not exhibit windup in terms of Definitions 1–3. The formulation also simplifies the analysis of the closed-loop stability. The connections between (a) the developed control laws and (b) the model state feedback control (Coulibaly *et al.*, 1992, 1995) and the modified internal model control (Zheng *et al.*, 1994) are established. Some of the results included in this article were presented in Soroush and Nikravesh (1996).

The scope of this work is first described, followed by some mathematical preliminaries. In Section 3, input–output linearization is formulated as a model-predictive control problem first for constrained non-linear processes with full-state measurements and then for constrained non-linear processes with incomplete-state measurements and deadtimes. In Section 4, dynamic input–output linearizing control laws that can handle constant disturbances and model errors are derived. In Section 5, the derived control laws are first applied to linear systems to establish the connections between (a) the resulting linear control laws and (b) the modified internal model control (IMC) and the model state feedback control (MSFC). They are then parameterized to obtain non-linear modified IMC and MSFC laws. In Section 6, the application and performance of the non-linear control laws are demonstrated by chemical and biochemical reactor examples with input constraints.

2. SCOPE AND MATHEMATICAL PRELIMINARIES

The focus of this article is on single-input–single-output, continuous-time, non-linear processes described by a state-space model of the form

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = 0 \\
y(t) &= h(x(t - \theta))
\end{aligned}
\tag{1}
$$

where θ is the deadtime, $x = [x_1 \cdots x_n]^T$ is the vector of state variables, u is the manipulated input, and y is the controlled output. Without loss of generality, it is assumed that all the variables are in the form of deviations from nominal steady-state values, and thus the origin is the equilibrium point. It is also assumed that (a) $x \in X \subset \mathbb{R}^n$, where X is an open connected set which contains the equilibrium point; (b) $u \in U = \{u \mid u_{\min} \leqslant u \leqslant u_{\max}\} \subset \mathbb{R}$, where $u_{\min}$ and $u_{\max}$ are scalar constants which satisfy $u_{\min} < 0 < u_{\max}$; (c) $f(x)$ and $g(x)$ are analytic vector functions on X; and (d) $h(x)$ is an analytic scalar function on X.

In the presence of an active input constraint, the actual input to a process, $u(t)$, is different from the controller output, say $w(t)$. In mathematical terms, u is related to w according to

$$u(t) = \operatorname{sat}\{w(t)\} \triangleq \begin{cases} u_{\min} & \text{if } w(t) < u_{\min} \\ w(t) & \text{if } u_{\min} \leqslant w(t) \leqslant u_{\max} \\ u_{\max} & \text{if } w(t) > u_{\max} \end{cases} \quad (2)$$

where sat{.} is the saturation function.

In practical situations, deadtime is often present in both input and output:

$$\dot{\eta}(\zeta) = f(\eta(\zeta)) + g(\eta(\zeta))u(\zeta - \theta_u)$$
$$y(\zeta) = h(\eta(\zeta - \theta_y)).$$

This process model can be written in the general form of eq. (1) with $\theta = \theta_u + \theta_y$, $t = \zeta - \theta_u$, $x(t) = \eta(t + \theta_u)$. Therefore, throughout this article, our focus will be on the process models of the form of eq. (1). The system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) \\ y^*(t) &= h(x(t)) \end{aligned} \quad (3)$$

will be referred to as the delay-free part of the system described by eq. (1). The delay-free system is assumed to be minimum phase (to have asymptotically stable zero dynamics). The actual process is represented by

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + g(\bar{x}(t))u(t), \quad \bar{x}(0) = 0$$
$$\bar{y}(t) = h(\bar{x}(t - \theta)) + d(t)$$

where $\bar{x}$ and $\bar{y}$ represent the vector of the measured state variables and the measured controlled output, respectively, and d is the unmeasurable constant disturbance of the process.

For a process with a model of the form of eq. (1), given the present values of the state variables and the manipulated input, $x(t)$ and $u(t)$, approximate future values of the controlled output over a sufficiently short time horizon beyond the deadtime, $[\tau - (t + \theta)]$, can be obtained by using a truncated Volterra series (Isidori, 1989):

$$y(\tau) \approx h(x(t)) + \sum_{\ell=1}^{r} L_f^{\ell} h(x(t)) \frac{[\tau - (t + \theta)]^{\ell}}{\ell!} + L_g L_f^{r-1} h(x(t)) u(t) \frac{[\tau - (t + \theta)]^r}{r!} \quad (4)$$

where r represents the relative order of the delay-free output y^* (i.e. the smallest integer r for which $L_g L_f^{r-1} h(x) \not\equiv 0$), and L_f and L_g are Lie derivative (in the direction of the vectors f and g, respectively) operators. Equation (4) is also a truncated Taylor series expansion of $y(\tau)$ with respect to $(t + \theta)$; when the horizon $[\tau - t - \theta]$ is very small, an rth-order truncated Volterra series of the process model is equivalent to an rth-order truncated Taylor series expansion of $y(\tau)$ with respect to $(t + \theta)$, where $\mathrm{d}^{\ell} y(t + \theta)/\mathrm{d}t^{\ell} = L_f^{\ell} h(x(t))$, $\ell = 0, \ldots, r - 1$, and $\mathrm{d}^r y(t + \theta)/\mathrm{d}t^r = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t))\, u(t)$.

For a process with a model of the form of eq. (1), the nth-order non-linear system

$$\dot{\Theta}(t) = f(\Theta(t)) + g(\Theta(t)) \frac{v(t) - h(\Theta(t)) - \sum_{\ell=1}^{r} \gamma_\ell L_f^{\ell} h(\Theta(t))}{\gamma_r L_g L_f^{r-1} h(\Theta(t))},$$
$$\Theta(0) = 0$$
$$u(t) = \frac{v(t) - h(\Theta(t)) - \sum_{\ell=1}^{r} \gamma_\ell L_f^{\ell} h(\Theta(t))}{\gamma_r L_g L_f^{r-1} h(\Theta(t))} \quad (5)$$

represents a minimal-order state-space realization of the inverse of the delay-free part of the process together with an rth-order filter of the from $1/(\gamma_r s^r + \cdots + \gamma_1 s + 1)$ with $\gamma_r \neq 0$; when the system of eq. (5) is placed in series with the process model of eq. (1), the relationship between v and y is given by

$$y(t) + \gamma_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \cdots + \gamma_r \frac{\mathrm{d}^r y(t)}{\mathrm{d}t^r} = v(t - \theta).$$

The MPC laws are obtained by solving one-dimensional quadratic optimization problems of the form

$$\min_{u(t)} \{\|y_d(\tau) - \hat{y}(\tau)\|_p^2 + \rho' |u(t)|^2\} \quad (6)$$

subject to the constraints

$$u_{\min} \leqslant u(t) \leqslant u_{\max}$$

where (a) t represents the present time; (b) ρ' ($\rho' \geqslant 0$) is an input penalty; (c) $\|\omega(\tau)\|_p$ denotes the p-function norm of the scalar function $\omega(\tau)$ over the finite time interval $[t + \theta, t + \theta + T_h]$:

$$\|\omega(\tau)\|_p \triangleq \left[\int_{t+\theta}^{t+\theta+T_h} |\omega(\tau)|^p \,\mathrm{d}\tau \right]^{(1/p)}, \quad p \geqslant 1$$

(d) $y_d(\tau)$ is a reference trajectory; and (e) $\hat{y}(\tau)$ is the predicted value of the output. As we will see, instantaneous solutions to the preceding optimization problem lead to continuous-time analytical control laws.

3. MPC FORMULATION OF INPUT–OUTPUT LINEARIZATION

In this section, the problem of input–output linearization is formulated as a model-predictive control problem. An approach similar to the one used in (Soroush and Kravaris, 1992b), is employed to derive an input–output linearizing static-state feedback.

First constrained non-linear processes with full-state measurements are considered, followed by constrained non-linear processes with incomplete-state measurements and deadtimes. A fraction of the results presented in this section are from Soroush and Kravaris (1992b); they will serve as a basis for the new theoretical results that will be presented in the subsequent section.

3.1. *Processes with full-state measurements*

Consider deadtime-free processes with completely measurable state variables and with a model of the form of eq. (1) ($\theta = 0$):

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = 0$$
$$y(t) = h(x(t)). \tag{7}$$

For such a process, using the truncated Volterra series of eq. (4), over a sufficiently small prediction horizon of $(\tau - t)$, the predicted value of the controlled output is calculated from the prediction equation

$$\hat{y}(\tau) = \bar{y}(t) + \sum_{\ell=1}^{r} L_f^{\ell} h(\bar{x}(t)) \frac{(\tau - t)^{\ell}}{\ell!} + L_g L_f^{r-1} h(\bar{x}(t)) u(t) \frac{(\tau - t)^r}{r!} + \text{higher-order terms (h.o.t.)} \tag{8}$$

where $\hat{y}(\tau)$ denotes the predicted value of the output at time τ.

3.1.1. *Reference trajectory.* A reference trajectory is a desired trajectory, which the controller (in the absence of penalty and constraints on the manipulated input) will try to force the controlled output to follow. For a given process, a reference trajectory must be 'trackable' in the absence of constraints and input penalty, i.e. there must exist a feasible (finite magnitude) smooth controller action which can force the output to follow the trajectory exactly.

The reference trajectory, $y_d(\tau), \tau \geqslant t$, is defined as the solution of

$$y_d(\tau) + \gamma_1 \frac{\mathrm{d}y_d(\tau)}{\mathrm{d}\tau} + \cdots + \gamma_r \frac{\mathrm{d}^r y_d(\tau)}{\mathrm{d}\tau^r} = y_{sp}(t) \tag{9}$$

subject to the initial conditions: $y_d(t) = \bar{y}(t)$ and $\mathrm{d}^{\ell} y_d(t)/\mathrm{d}t^{\ell} = L_f^{\ell} h(\bar{x}(t)), \ell = 1, \ldots, r-1$. Here $\gamma_1, \ldots, \gamma_r$ are scalar constants which are chosen such that all the roots of the equation $\gamma_r s^r + \cdots + \gamma_1 s + 1 = 0$ lie in the left-half of the complex plane, and y_{sp} is the output set-point. The set-point is assumed to be *feasible* in the sense that there exits a $u_o \in \mathrm{int}(U)$, which satisfies $f(\zeta) + g(\zeta)\, u_0 = 0$, where $\zeta \in X$ and $h(\zeta) = y_{sp}$.

This reference trajectory is an exponential-type trajectory, for which there exists a feasible (finite magnitude) smooth controller action that can force the output to follow the trajectory: $y_d(\tau)$ is a trackable exponential trajectory, because the process along this trajectory is *output functional controllable* (Hirschorn, 1979). The ordinary differential equation (ODE) of the reference trajectory is, indeed, the lowest-order ODE whose solution can be tracked.

3.1.2. *Input–output linearizing state feedback*

Theorem 1. *For a delay-free process with a model of the form of eq.* (7), *the exact solution to the constrained minimization problem of eq.* (6) *with a sufficiently small time horizon of T_h is given by*

$$u(t) = \mathrm{sat}\{\Psi_\rho[\bar{x}(t), e(t) + h(\bar{x}(t))]\} \tag{10}$$

where $e(t) \triangleq y_{sp}(t) - \bar{y}(t)$,

$$\Psi_\rho[\bar{x}(t), \zeta(t)] \triangleq \frac{L_g L_f^{r-1} h(\bar{x}(t))}{(L_g L_f^{r-1} h(\bar{x}(t)))^2 + \rho} \times \frac{\zeta(t) - h(\bar{x}(t)) - \sum_{\ell=1}^{r} \gamma_\ell L_f^{\ell} h(\bar{x}(t))}{\gamma_r}.$$

The proof is given in Appendix A.

The static-state feedback of eq. (10) in its unconstrained form was derived by Rangel *et al.* (1990), in the context of keeping u bounded at the points of singularity x_s where $L_g L_f^{r-1} h(x_s) = 0$.

Remark 1. In the case that the input is not constrained and that there is no input penalty ($\rho = 0$), the state feedback of eq. (10) is an input–output linearizing state feedback:

$$u = \Psi_0[\bar{x}, e + h(\bar{x})] \triangleq \frac{[e + h(\bar{x})] - h(\bar{x}) - \sum_{\ell=1}^{r} \gamma_\ell L_f^{\ell} h(\bar{x})}{\gamma_r L_g L_f^{r-1} h(\bar{x})} \tag{11}$$

which induces the linear input–output closed-loop response

$$\bar{y} + \gamma_1 \frac{\mathrm{d}\bar{y}}{\mathrm{d}t} + \cdots + \gamma_r \frac{\mathrm{d}^r \bar{y}}{\mathrm{d}t^r} = y_{sp}. \tag{12}$$

3.2. *Processes with incomplete state measurements*

In this subsection, analog of the results presented in the previous subsection are derived for the non-linear processes described by a state-space model of the form of eq. (1), whose state variables are not completely measurable.

Consider a process of the from of eq. (1). Given the present measurement of the controlled output, $\bar{y}(t)$, and the estimated history of the state variables, $x(\kappa)$, $\kappa \leqslant t$, then for $t \leqslant \tau \leqslant t + \theta$, the predicted value of the output is obtained from

$$\hat{y}(\tau) \triangleq \bar{y}(t) + \underbrace{[y(\tau) - y(t)]}_{\text{predicted change in } y \text{ by model}} = \bar{y}(t) + h(x(\tau - \theta)) - h(x(t - \theta)).$$

This yields

$$\hat{y}(t + \theta) = \bar{y}(t) + h(x(t)) - h(x(t - \theta)). \tag{13}$$

For $\tau \geqslant t + \theta$, the predicted value of the output is calculated from

$$\hat{y}(\tau) \triangleq \hat{y}(t+\theta) + \underbrace{[y(\tau) - y(t+\theta)]}_{\text{predicted change in } y \text{ by model}}$$

$$= \bar{y}(t) - h(x(t-\theta)) + \sum_{\ell=0}^{r} L_f^\ell h(x(t)) \frac{[\tau - (t+\theta)]^\ell}{\ell!}$$

$$+ L_g L_f^{r-1} h(x(t)) u(t) \frac{[\tau - (t+\theta)]^r}{r!} + \text{h.o.t.} \quad (14)$$

where the state estimates are obtained from on-line simulation of process model $[\dot{x} = f(x) + g(x)u]$.

For these processes, the reference trajectory is defined as in the time-delay-free case, except that it will refer to the desired output response beyond the process deadtime θ. It is an exponential trajectory which at the time instant $\tau = t + \theta$, satisfies the conditions $\mathrm{d}^\ell y_d(\tau)/\mathrm{d}\tau^\ell = \mathrm{d}^\ell \hat{y}(\tau)/\mathrm{d}\tau^\ell$, $\ell = 0, \ldots, r-1$. Thus, the reference trajectory $y_d(\tau)$, $\tau \geqslant t + \theta$, is the solution of

$$y_d(\tau) + \gamma_1 \frac{\mathrm{d}y_d(\tau)}{\mathrm{d}\tau} + \cdots + \gamma_r \frac{\mathrm{d}^r y_d(\tau)}{\mathrm{d}\tau^r} = y_{sp}(t) \quad (15)$$

subject to the initial conditions: $y_d(t+\theta) = \hat{y}(t+\theta)$ and $\mathrm{d}^\ell y_d(t+\theta)/\mathrm{d}t^\ell = L_f^\ell h(x(t))$, $\ell = 1, \ldots, r-1$. In this case, the controller will try to match the predicted output with the reference trajectory beyond the deadtime.

3.2.1. *Input–output linearizing dynamic error-feedback*

Theorem 2. *For a process with a model of the form of eq.* (1), *the exact solution to the constrained minimization problem of eq.* (6) *with a sufficiently small time horizon of* T_h *is given by*

$$\dot{x}(t) = f(x(t)) + g(x(t))\mathrm{sat}\{\Psi_\rho[x(t), e(t) + h(x(t-\theta))]\},$$

$$x(0) = 0$$

$$u(t) = \mathrm{sat}\{\Psi_\rho[x(t), e(t) + h(x(t-\theta))]\}. \quad (16)$$

The proof is given in Appendix A.

Remark 2. In the case that the input is not constrained and that there is no input penalty ($\rho = 0$), the dynamic feedback of eq. (16) is an input–output linearizing controller:

$$\dot{x}(t) = f(x(t)) + g(x(t))\Psi_0[x(t), e(t) + h(x(t-\theta))],$$

$$x(0) = 0$$

$$u(t) = \Psi_0[x(t), e(t) + h(x(t-\theta))] \quad (17)$$

which induces the linear input–output closed-loop response

$$\bar{y}(t) + \gamma_1 \frac{\mathrm{d}\bar{y}(t)}{\mathrm{d}t} + \cdots + \gamma_r \frac{\mathrm{d}^r \bar{y}(t)}{\mathrm{d}t^r} = y_{sp}(t-\theta). \quad (18)$$

4. NON-LINEAR CONTROLLER SYNTHESIS FOR PROCESSES WITH CONSTANT DISTURBANCES AND MODEL ERRORS

In the previous section, two model predictive control problems were solved, leading to a static-state feedback and to a dynamic error-feedback control law. In this section, these results with $\rho = 0$ are used to derive input–output linearizing control laws that can eliminate asymptotically the effect of constant unmeasurable disturbances and model errors and that can effectively handle input constraints. Again, we follow the same sequence, i.e. non-linear processes with full-state measurements are first considered, followed by non-linear processes with incomplete-state measurements and deadtimes.

4.1. *Processes with full-state measurements*

The control law of Remark 1 is a pure static state-feedback, and therefore in the presence of a model error or an unmeasurable disturbance, the closed-loop response exhibits offset. To ensure an offsetless response, we have to add integral action to the state feedback.

In MPC, the customary approach to introduce integral action, as described in Muske and Rawlings (1993), has been: (a) addition of a constant, unknown, fictitious disturbance to the model, (b) use of an estimator/observer to estimate the value of the unknown disturbance, and (c) synthesis of an MPC law on the basis of the model augmented with the observer. Here integral action is added to the state feedback of eq. (11) by using a different approach.

As in IMC and MSFC, an estimate of the disturbance-free controlled output is calculated by simulating the process model and then the difference between the measured controlled output and the estimated disturbance-free controlled output is fed back to the controller as an estimate of the process disturbance. However, here instead of simulating the process model, the nominal linear closed-loop response of eq. (12), which is of order r, is simulated. The following system is used to represent a minimal-order state-space realization of the nominal linear closed-loop response of eq. (12):

$$\dot{\eta} = A_c \eta + b_c y_{sp}, \quad \eta(0) = 0$$
$$y = c_c \eta \quad (19)$$

where $\eta \in \mathbb{R}^r$, and A_c, b_c, and c_c are constant matrices. A representative matrix triplet (A_c, b_c, c_c) is given in Appendix B.

Theorem 3. *For a process with a model of the form of eq.* (7), *the dynamic mixed error- and state-feedback control law:*

$$\dot{\eta} = A_c \eta + b_c \Phi(\bar{x}, u), \quad \eta(0) = 0$$
$$u = \mathrm{sat}\{\Psi_0(\bar{x}, c_c \eta + e)\} \quad (20)$$

where

$$\Phi(\bar{x}, u) = h(\bar{x}) + \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(\bar{x}) + \gamma_r L_g L_f^{r-1} h(\bar{x}) u$$

(a) *in the presence or absence of the constraints, is the solution to the minimization problem of eq.* (6) *with a zero* ρ;

(b) *in the absence of the constraints, induces the linear input–output closed-loop response of eq.* (12);

(c) *has integral action: in the presence of constant disturbances and model errors, induces an offsetless closed-loop response.*

The proof is given in Appendix A.

The controller of eq. (20) (whose block diagram is shown in Fig. 1) belongs to the class $\mathscr{C}$ of controllers; it does not exhibit windup in the sense of Definitions 1–3 (given in the Introduction), because:

- its states, $\eta_1, \dots, \eta_r$, are not driven by the error when the actuator is in saturation; its states are updated by the *actual input* to the process not by the error;
- the closed-loop behavior under the controller is identical to that under the static state feedback of eq. (11), since $c_c\eta(t) + e(t) \equiv e(t) + h(\bar{x}(t))$;
- its action is solution to the moving-horizon constrained optimization problem of eq. (6) with a zero ρ, as Theorem 3 states.

We will refer to this controller as the *modified mixed error- and state-feedback controller.* This controller is of order r (consists of a system of order r). In the absence of model errors and disturbances, η in the control law of eq. (20) is simply the vector of the r observable modes of the process model under an input–output linearizing controller. Thus, the quantity $c_c\eta = \eta_1$ is simply the model-predicted value of the controlled output $[y = h(x)]$, and $e + c_c\eta = e + y = y_{sp} - d$.

Remark 3. The controller of Theorem 3 can also be derived using the anti-windup observer-based approach of Åström and Rundqwist (1989). Consider the following dynamic input–output linearizing controller with integral action (Daoutidis and Kravaris, 1992):

$$\begin{aligned} \dot{\eta} &= \tilde{A}\eta + \tilde{b}e, \quad \eta(0) = 0 \\ u &= \mathrm{sat}\{\Psi_0(\bar{x}, \tilde{c}\eta + e)\} \end{aligned} \tag{21}$$

where the linear system $(\tilde{A}, \tilde{b}, \tilde{c}, 1)$ is a minimal-order state-space realization of the transfer function

$$\frac{\gamma_r s^r + \cdots + \gamma_1 s + 1}{\gamma_r s + \cdots + \gamma_1 s}$$

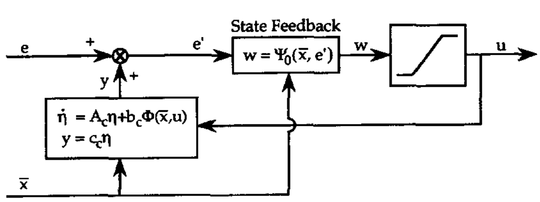


Fig. 1. Modified mixed error- and state-feedback control structure.

i.e. $\tilde{A} = A_c + b_c c_c$, $\tilde{b} = b_c$, and $\tilde{c} = c_c$. In the absence of input constraints, the controller of eq. (21) induces the same linear input–output closed-loop response of eq. (12). In the presence of an active input constraint, however, the controller of eq. (21) exhibits integral windup in the sense of Definition 1. Application of the anti-windup observer-based approach of Åström and Rundqwist to the preceding controller leads to

$$\begin{aligned} \dot{\eta} &= \tilde{A}\eta + \tilde{b}e + K(u - w), \quad \eta(0) = 0 \\ u &= \mathrm{sat}\{\Psi_0(\bar{x}, \tilde{c}\eta + e)\} \end{aligned}$$

where $w = \Psi_0(\bar{x}, \tilde{c}\eta + e)$. If the observer gain is chosen to be $K = \tilde{b}\gamma_r L_g L_f^{r-1} h(\bar{x})$, then the preceding controller takes the form (using the definition of Ψ_0):

$$\begin{aligned} \dot{\eta} = [\tilde{A} - \tilde{b}\tilde{c}]\eta + \tilde{b}\Bigg[& h(\bar{x}) + \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(\bar{x}) \\ & + \gamma_r L_g L_f^{r-1} h(\bar{x}) u \Bigg], \quad \eta(0) = 0 \\ u = \mathrm{sat}\{\Psi_0(\bar{x}, & \tilde{c}\eta + e)\} \end{aligned} \tag{22}$$

which is *exactly* the modified mixed error- and state-feedback control law of eq. (20). Kapoor and Daoutidis (1996a) have derived an anti-windup observer-based approach for non-linear systems of the form of eq. (7) recast in a normal form, leading to a windup compensation method with the same non-linear compensator gain K.

Remark 4. In the case of processes with the relative order $r = 1$, the control law of eq. (20) becomes

$$\begin{aligned} \dot{\eta}_1 &= -\frac{1}{\gamma_1}\eta_1 + \frac{1}{\gamma_1}\Phi(\bar{x}, u), \quad \eta_1(0) = 0 \\ u &= \mathrm{sat}\{\Psi_0(\bar{x}, \eta_1 + e)\}. \end{aligned}$$

This non-linear control law implicitly includes a PI controller with an integral windup compensator. The PI controller has a unity gain and provides the required integral action. From a practical point of view, the preceding non-linear control law is sufficient to ensure an offsetless response even in general processes with a relative order r greater than one.

4.2. *Processes with incomplete state measurements*

Equation (17) describes a dynamic error-feedback controller with integral action, and therefore in the presence of constant disturbances and model errors, the corresponding closed-loop response does not exhibit offset. To establish the connections between another class of differential geometric controllers and MPC, for a moment, let us assume that the dynamic error-feedback of eq. (17) lacks integral action. In this case, to add integral action to the feedback, we pursue an approach similar to the one used for the deadtime-free case.

Theorem 4. *For a process with a model of the form of eq.* (1), *the dynamic error-feedback control law*

$$\dot{\eta} = A_c\eta + b_c\Phi(x, u), \qquad \eta(0) = 0$$
$$\dot{x} = f(x) + g(x)\ \mathrm{sat}\{\Psi_0(x, c_c\eta + e^*)\}, \qquad x(0) = 0 \quad (23)$$
$$u = \mathrm{sat}\{\Psi_0(x, c_c\eta + e^*)\}$$

where

$$e^*(t) \triangleq e(t) + h(x(t-\theta)) - h(x(t)),$$

(a) *in the presence or absence of the constraints, is the solution to the minimization problem of eq.* (6) *with a zero* ρ;

(b) *in the absence of the constraints, induces the linear input–output closed-loop response of eq.* (18);

(c) *has integral action: in the presence of constant disturbances and model errors, induces an offsetless closed-loop response.*

The proof is given in Appendix A.

The control law of eq. (23) [whose block diagram is shown in Fig. 2(a)] also belongs to the class $\mathscr{C}$ of controllers (see Definition 3 given in the Introduction), because it does not exhibit windup in the sense of Definitions 1–3:

- Its states, $\eta_1, \ldots, \eta_r, x_1, \ldots, x_n$, are not driven by the error when the actuator is in saturation; its states are updated by the *actual input* to the process not by the error.
- The closed-loop behavior under the controller is identical to that under the static state feedback of eq. (11), since $c_c\eta(t) + e^*(t) \equiv e(t) + h(\bar{x}(t))$.
- Its action is solution to the moving-horizon constrained optimization problem of eq. (6) with a zero ρ, as stated by Theorem 4.

We will refer to this control law as the *modified error-feedback control law.*

In the case of non-linear processes operating at an unstable steady state and having unmeasurable state variables, one can use the 'closed-loop' reduced-order observer design method described in Soroush (1996), to reconstruct the state variables. Using this closed-loop reduced-order observer instead of the full-order open-loop observer $[\dot{x} = f(x) + g(x)u]$, in the control law of eq. (23) leads to a non-linear dynamic output–feedback control law that can be used for the aforementioned class of non-linear processes.

Under nominal conditions (no model errors and no disturbances), the control law of eq. (23) has r redundant modes [e.g. $\eta_1(t) \equiv h(x(t))$]. In this case, $e^*(t) + c_c\eta(t) \equiv e(t) + h(x(t-\theta)) \equiv e(t) + y(t)$, and thus the controller action can be calculated without a need for the value of η. Elimination of the unnecessary/redundant modes, $\eta_1, \ldots, \eta_r$, leads to the error-feedback control law of eq. (17):

$$\dot{x} = f(x) + g(x)\ \mathrm{sat}\{\Psi_0(x, e + y)\}, \quad x(0) = 0$$
$$u = \mathrm{sat}\{\Psi_0(x, e + y)\}. \qquad (24)$$

The block diagram of the corresponding control structure is depicted in Fig. 2(b), which has been called the reduced-order error-feedback globally linearizing control structure (Soroush and Kravaris, 1992c). The preceding control law has implicit integral action which is a consequence of using the full-order open-loop observer. Thus, if the state variables are reconstructed by using such a full-order observer, the reduced-order non-linear control law of eq. (24) is sufficient to ensure an offsetless closed-loop response (full-order open-loop observer provides the required integral action).

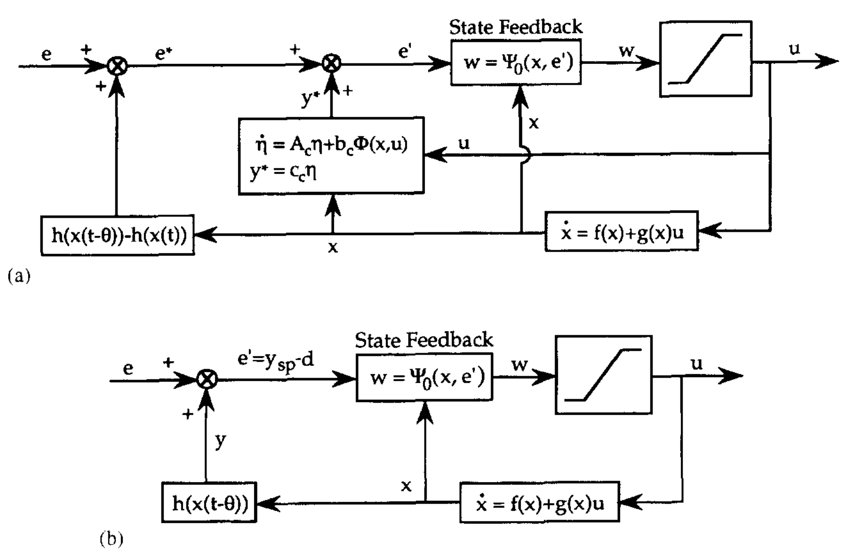


Fig. 2(a). Modified error-feedback control structure. (b). Reduced-order error-feedback control structure.

Remark 5. In the case of processes with the relative order $r = 1$, the control law of eq. (23) becomes

$$\dot{\eta}_1 = -\frac{1}{\gamma_1}\eta_1 + \frac{1}{\gamma_1}\Phi(x, u), \quad \eta(0) = 0$$

$$\dot{x} = f(x) + g(x)u, \quad x(0) = 0$$

$$u = \text{sat}\{\Psi_0[x, \eta_1 + e^*]\}.$$

This non-linear control law implicitly includes a PI controller with an integral windup compensator. From a practical point of view, the preceding non-linear control law is sufficient to ensure an offsetless response even in general processes with a relative order r greater than one.

4.3. *Closed-loop stability*

The nominal asymptotic stability of the closed-loop system under the non-linear control laws of eqs (20), (23) and (24) are as follows. By nominal, we mean when there are no model errors and no disturbances.

Consider the following conditions: (i) the parameters $\gamma_1, \ldots, \gamma_r$ are chosen such that all the roots of the characteristic equation $\gamma_r s^r + \cdots + \gamma_1 s + 1 = 0$ lie in the left-half of the complex plane; (ii) the delay-free part of the system of eq. (1) is minimum-phase (has asymptotically stable zero dynamics); and (iii) the system of eq. (1) is asymptotically (open-loop) stable. In the absence of input constraints, the closed-loop system will be asymptotically stable: under the control law of eq. (20), if the conditions (i) and (ii) hold; under the control laws of eqs (23) and (24), if the conditions (i)–(iii) hold.

In the presence of input constraints, however, the aforementioned conditions are certainly necessary, but may not be sufficient to guarantee closed-loop stability. The problem of closed-loop stability under a constrained analytical non-linear static state feedback has not been solved completely yet. For a special class of non-linear processes, one can use the dynamical systems approach developed by Kapoor and Daoutidis (1996b) to determine a subset of the region of closed-loop stability.

5. CONNECTIONS WITH MODIFIED IMC AND MSFC

To show that the non-linear control law of eq. (24) is indeed a non-linear model state feedback control law and a reduced-order non-linear modified internal model control law, we first apply the control law of eq. (24) to the class of general time-invariant linear processes, to show that the resulting linear controller is exactly a model state feedback controller (Coulibaly *et al.*, 1995) and a reduced-order modified internal model controller (Zheng *et al.*, 1994). The control law of eq. (20) is also applied to the linear systems and the resulting linear controller is shown to be a reduced-order model state feedback controller. The derived non-linear control laws of eqs (20) and (24) are then parameterized according to the MSFC structure and the modified IMC structure. These parameterizations allow us to interpret the developed non-linear control laws in these two linear controller synthesis frameworks.

5.1. *Application to linear systems—modified IMC and MSFC*

Consider time-invariant, linear systems described by a state-space model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \quad x(0) = 0 \\ y(t) &= cx(t-\theta) \end{aligned} \tag{25}$$

where A, b and c are $n \times n$, $n \times 1$ and $1 \times n$ constant matrices, respectively. This class of systems is a special case of eq. (1) for $f(x(t)) = Ax(t)$, $g(x(t)) = b$, $h(x(t-\theta)) = cx(t-\theta)$. The relative order r of the delay-free part of this system is the smallest integer for which $cA^{r-1}b \neq 0$. It is assumed that the delay-free part of the system of eq. (25) has a finite relative order r and is minimum phase (all of its zeros lie in the left-half of the complex plane).

Application of the control law of eq. (24) to the linear processes of the form of eq. (25) leads to the linear controller:

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$u(t) = \text{sat}\left\{-\frac{1}{\gamma_r cA^{r-1}b}[c + \gamma_1 cA + \cdots + \gamma_r cA^r]x(t) + \frac{1}{\gamma_r cA^{r-1}b}e'(t)\right\} \tag{26}$$

where $e' = e + y = y_{sp} - d$. The controller of eq. (26) in the absence of the constraints can be written as

$$\begin{aligned} \dot{x}(t) &= \left\{A - b\frac{1}{\gamma_r cA^{r-1}b}[c + \gamma_1 cA + \cdots + \gamma_r cA^r]\right\} \\ &\quad \times x(t) + b\frac{1}{\gamma_r cA^{r-1}b}e'(t) \\ u(t) &= -\frac{1}{\gamma_r cA^{r-1}b}[c + \gamma_1 cA + \cdots + \gamma_r cA^r]x(t) \\ &\quad + \frac{1}{\gamma_r cA^{r-1}b}e'(t) \end{aligned} \tag{27}$$

whose parameterization according to the IMC structure [shown in Fig. 3(a)] leads to

$$P: \begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = cx(t-\theta) \end{cases} \tag{28}$$

$$Q: \begin{cases} \dot{x}(t) = \left\{A - b\dfrac{1}{\gamma_r cA^{r-1}b}[c + \gamma_1 cA + \cdots + \gamma_r cA^r]\right\}x(t) + b\dfrac{1}{\gamma_r cA^{r-1}b}e'(t) \\ u(t) = -\dfrac{1}{\gamma_r cA^{r-1}b}[c + \gamma_1 cA + \cdots + \gamma_r cA^r]x(t) + \dfrac{1}{\gamma_r cA^{r-1}b}e'(t). \end{cases} \tag{29}$$

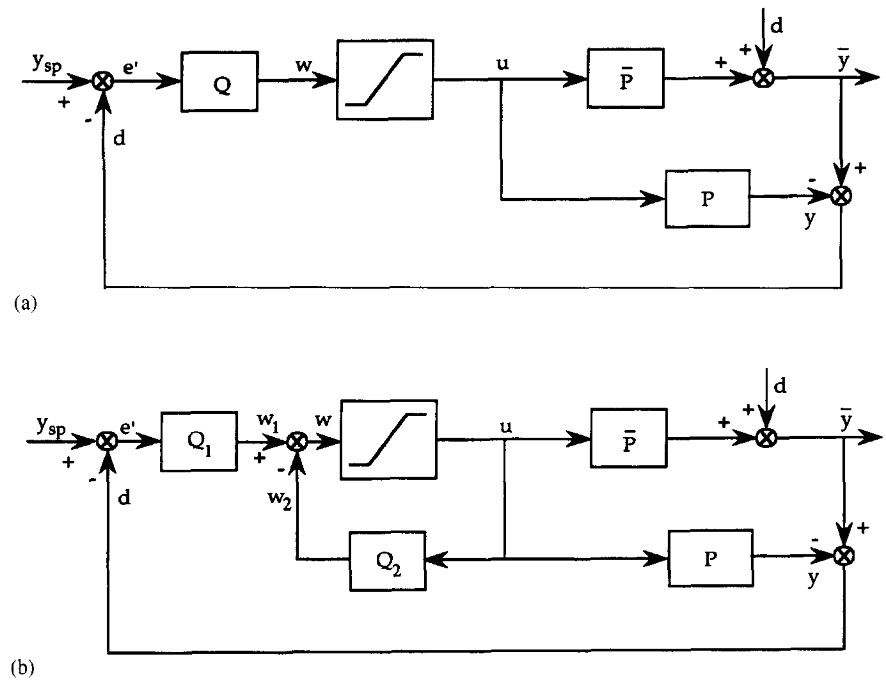


Fig. 3(a). Internal model control structure. (b). Modified internal model control structure.

The systems of equations (28) and (29) are, respectively, the process model and a minimal-order state-space realization of the inverse of the delay-free part of the process together with an rth-order filter of the form $1/(\gamma_r s^r + \cdots + \gamma_1 s + 1)$. The system of equation (29) is a minimal-order state-space realization of what is called controller, $Q(s) = 1/[c(sI - A)^{-1}b(\gamma_r s^r + \cdots + \gamma_1 s + 1)]$, in the IMC structure. As pointed out in Section 2, when the system of equation (29) is placed in series with the process model of eq. (25), the relationship between e' and y is given by

$$y(t) + \gamma_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \cdots + \gamma_r \frac{\mathrm{d}^r y(t)}{\mathrm{d}t^r} = e'(t - \theta)$$

i.e. the relationship is governed by the IMC filter.

5.1.1. *Modified IMC parameterization.* As depicted in Fig. 3(b), in modified IMC structure, the IMC controller $Q(s)$ is factored into two components $Q_1(s)$ and $Q_2(s)$ so that in the absence of input constraints $Q(s) = Q_1(s)\,[1 + Q_2(s)]^{-1}$. The modified IMC structure has been proposed by Zheng *et al.* (1994), to address the problem of integral windup in the original IMC.

The linear controller of eq. (26) can be easily parameterized according to the modified IMC structure:

$$u = \mathrm{sat}\{w_1 - w_2\} = \mathrm{sat}\{Q_1 e' - w_2\}$$

where $e' = y_{sp} - (\bar{y} - y)$, P is given by eq. (28),

$$Q_1\!: \left\{ w_1(t) = \frac{1}{\gamma_r c A^{r-1} b} e'(t) \right.$$

Q_2:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ w_2(t) = -\dfrac{1}{\gamma_r c A^{r-1} b}\,[c + \gamma_1 cA + \cdots + \gamma_r c A^r]\,x(t). \end{cases}$$

In the s-domain, the three controller components (linear systems) take the forms

$$P(s) = c(sI - A)^{-1} b e^{-\theta s}, \quad Q_1(s) = \frac{1}{\gamma_r c A^{r-1} b},$$

$$Q_2(s) = \frac{1}{\gamma_r c A^{r-1} b}\left[c + \sum_{\ell=1}^{r} \gamma_\ell c A^\ell\right](sI - A)^{-1} b.$$

Thus, when the control law of eq. (24) is applied to linear systems, the resulting linear controller [eq. (26)] is a *minimal-order* state-space realization of the modified IMC with a constant $Q_1 = 1/[\lim_{s\to\infty} \gamma_r s^r c(sI - A)^{-1} b] = Q(\infty)$, where $Q(s)$ is the transfer function of the unconstrained IMC controller.

5.1.2. *MSFC parameterization.* As depicted in Fig. 4(a), the MSFC structure is similar to that of the modified IMC, but the resulting controller in MSFC does not have n redundant modes; once the state variables are estimated by simulating the process model P, they are fed to Q_2'. Thus, MSFC can provide the same controller performance that the modified IMC can induce, but its order is lower by n.

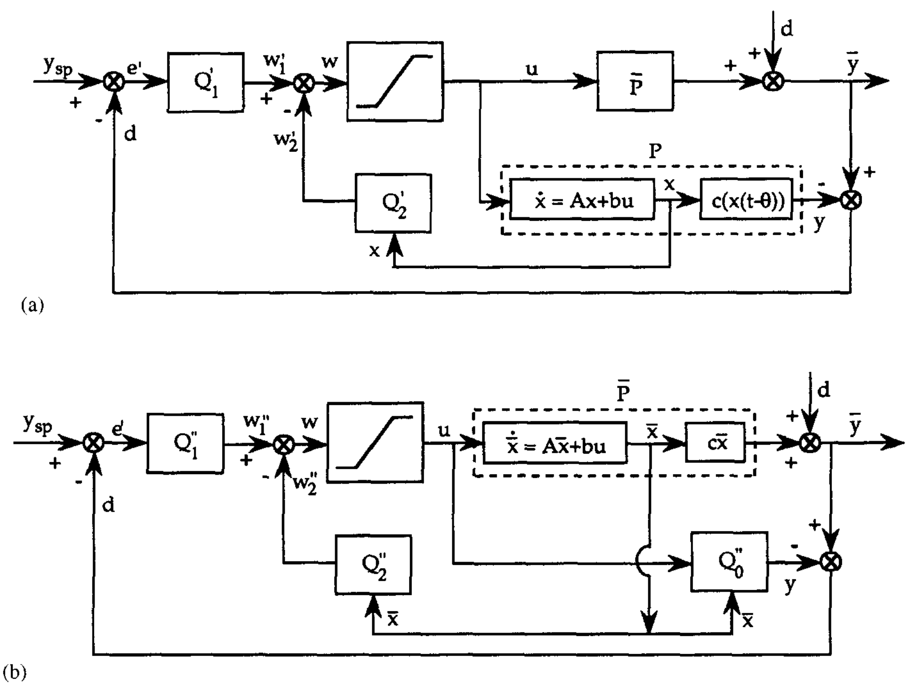


Fig. 4(a). Model state-feedback control structure. (b). Reduced-order model state-feedback control structure.

Parameterization of the linear controller of eq. (26) according to the MSFC structure leads to

$$u = \text{sat}\{w'_1 - w'_2\} = \text{sat}\{Q'_1 e' - Q'_2 x\}$$

where $e' = y_{sp} - (\bar{y} - y)$, P is given by eq. (28),

$$Q'_1: \left\{ w_1(t) = \frac{1}{\gamma_r c A^{r-1} b} [y_{sp} - \bar{y}(t) + y(t)] \right.$$

Q'_2:

$$\left\{ w_2(t) = -\frac{1}{\gamma_r c A^{r-1} b} [c + \gamma_1 c A + \cdots + \gamma_r c A^r] x(t). \right.$$

In the s-domain, the three controller components (linear systems) take the forms

$$P(s) = c(sI - A)^{-1} b e^{-\theta s}, \qquad Q'_1(s) = \frac{1}{\gamma_r c A^{r-1} b},$$

$$Q'_2(s) = \frac{1}{\gamma_r c A^{r-1} b}\left[c + \sum_{\ell=1}^{r} \gamma_\ell c A^\ell \right].$$

Thus, when the control law of eq. (24) is applied to linear systems, the resulting linear control law [eq. (26)] is exactly that of MSFC.

5.1.3. *Reduced-order MSFC parameterization.* Application of the control law of eq. (20) to the linear processes of the from of eq. (25) with complete state measurements and no deadtime leads to the linear controller

$$\begin{aligned} \dot{\eta}(t) &= A_c \eta(t) + b_c[(c + \gamma_1 c A + \cdots \\ &\quad + \gamma_r c A^r)\bar{x}(t) + \gamma_r c A^{r-1} b u(t)] \\ u(t) &= \text{sat}\left\{ -\frac{1}{\gamma_r c A^{r-1} b} [c + \gamma_1 c A + \cdots + \gamma_r c A^r] \bar{x}(t) \right. \\ &\quad \left. + \frac{1}{\gamma_r c A^{r-1} b} [c_c \eta(t) + e(t)] \right\} \end{aligned} \tag{30}$$

which is of order r, while the control law of eq. (26) is of order n. Parameterization of the linear controller of eq. (30) according to the MSFC structure leads to

$$u = \text{sat}\{w''_1 - w''_2\} = \text{sat}\{Q''_1 e' - Q''_2 \bar{x}\}$$

where $e' = y_{sp} - (\bar{y} - y)$,

$$Q''_0: \left\{ \begin{aligned} \dot{\eta}(t) &= A_c \eta(t) + b_c[(c + \gamma_1 c A + \cdots \\ &\quad + \gamma_r c A^r)\bar{x}(t) + \gamma_r c A^{r-1} b u(t)] \\ y(t) &= c\eta(t) \end{aligned} \right.$$

$$Q''_1 = Q_1, \qquad Q''_2 = Q'_2.$$

In the s-domain, the three controller components (linear systems) take the forms

$Q''_0(s)$

$$= c_c(sI - A_c)^{-1} b_c \left[\left(c + \sum_{\ell=1}^{r} \gamma_\ell c A^\ell \right) \bar{x} + \gamma_r c A^{r-1} b u \right],$$

$$Q''_1(s) = Q_1(s), \quad Q''_2(s) = Q'_2(s).$$

Thus, when the control law of eq. (20) is applied to linear systems, the resulting linear control law [eq. (30)] is a reduced-order MSFC; it is the lowest-order model-based controller with integral action but without windup (in the sense of Definition 2), that can induce the closed-loop response of eq. (12).

Example 1. Consider the same first-order linear example in Zheng *et al.*, (1994): $y(s) = [2/(100s + 1)]u(s)$, and let us request the same unconstrained desired closed-loop response as in Zheng *et al.* (1994):

$$\frac{y(s)}{y_{sp}(s)} = \frac{1}{20s + 1}. \tag{31}$$

A minimal-order state-space realization of the example is $\dot{x} = -0.01x + 0.02u$, $y = x$. For this example, $r = 1$, $cA^{r-1}b = 0.02$, $\gamma_r = 20$, and the modified IMC and MSFC parameterizations of the linear controller of eq. (26) and reduced-order MSFC parameterization of the linear controller of eq. (30) lead to the controller components given in Table 1. Note that the transfer functions $Q_1(s)$ and $Q_2(s)$ are the same as those reported in Zheng *et al.* (1994) and that in the absence of constraints, the preceding two controllers irrespective of their parameterizations induce the same closed-loop response of eq. (31).

Table 1. Modified IMC, MSFC and reduced-order MSFC parameterizations corresponding to Example 1

P	Q_1	Q_2
$y = \frac{2}{100s + 1} u$	$w_1 = 2.5e'$	$w_2 = \frac{4}{100s + 1} u$
P	Q'_1	Q'_2
$y = \frac{2}{100s + 1} u$	$w'_1 = 2.5e'$	$w'_2 = 2x$
Q''_0	Q''_1	Q''_2
$y = \frac{1}{100s + 5}[4\bar{x} + 2u]$	$w''_1 = 2.5e'$	$w''_2 = 2\bar{x}$

5.2. *Nonlinear modified IMC and non-linear MSFC*

As in the linear case, the non-linear control laws derived in the previous section are parameterized according to the modified IMC and MSFC structures, to interpret the developed non-linear control laws in these two linear controller synthesis frameworks.

5.2.1. *Nonlinear modified IMC parameterization.* Parameterization of the error-feedback control law of eq. (24) according to the modified IMC structure leads to the controller components given in Table 2, where $e' = y_{sp} - d$ and $u = \text{sat}\{w_1 - w_2\}$. This parameterization is the same as the modified IMC parameterization depicted in Fig. 3(b), but in this non-linear case, Q_1 is a dynamic system (depends on the state x and thus on u). For this reason, this control structure will be referred to as the non-linear modified IMC structure, whose block diagram is shown in Fig. 5. As we have seen in the previous subsection, when process model is linear, Q_1 will be simply a constant equal to $Q(\infty)$, where $Q(s)$ is the transfer function of the unconstrained IMC controller, and Q_2 will be a minimal-order state-space realization of the transfer function $Q_2(s)$ in the modified IMC structure. The above parameterization indicates that the implementation of the control law of eq. (24) according to the modified IMC structure will increase the order of the non-linear controller by $2n$. Thus, the control law of eq. (24) is a minimal-order state-space realization of a reduced-order non-linear modified IMC controller.

Table 2. Non-linear modified IMC, non-linear MSFC, and non-linear reduced-order MSFC parameterizations

P	Q_1	Q_2
$\dot{x} = f(x) + g(x)u$ $y(t) = h(x(t-\theta))$	$\dot{x} = f(x) + g(x)u$ $w_1 = \frac{e'}{\gamma_r L_g L_f^{r-1} h(x)}$	$\dot{x} = f(x) + g(x)u$ $w_2 = \frac{h(x) + \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(x)}{\gamma_r L_g L_f^{r-1} h(x)}$
P	Q'_1	Q'_2
$\dot{x} = f(x) + g(x)u$ $y(t) = h(x(t-\theta))$	$w'_1 = \frac{e'}{\gamma_r L_g L_f^{r-1} h(x)}$	$w'_2 = \frac{h(x) + \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(x)}{\gamma_r L_g L_f^{r-1} h(x)}$
Q''_0	Q''_1	Q''_2
$\dot{\eta} = A_c\eta + b_c\Phi(\bar{x}, u)$ $y = c_c\eta$	$w''_1 = \frac{e'}{\gamma_r L_g L_f^{r-1} h(\bar{x})}$	$w''_2 = \frac{h(\bar{x}) + \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(\bar{x})}{\gamma_r L_g L_f^{r-1} h(\bar{x})}$

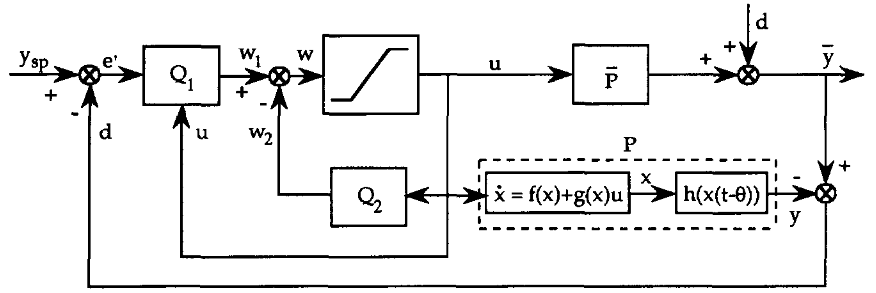


Fig. 5. Non-linear modified internal model control structure.

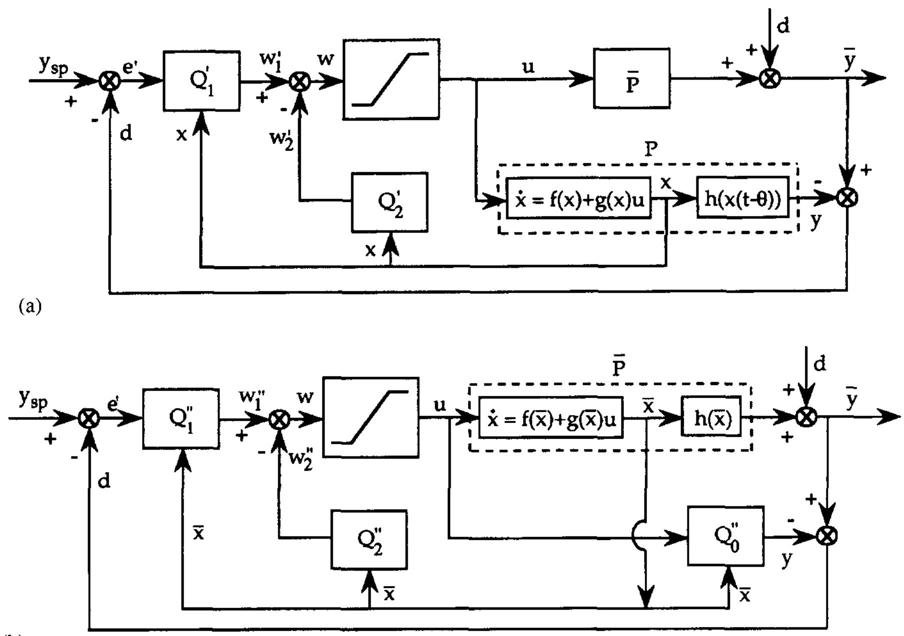


Fig. 6(a). Non-linear model state-feedback control structure. (b). Reduced-order non-linear model state-feedback control structure.

5.2.2. *Nonlinear MSFC parameterization.* Parameterization of the error-feedback control law of eq. (24) according to the MSFC structure leads to the controller components given in Table 2, where $u = \text{sat}\{w_1' - w_2'\}$. This parameterization is the same as the MSFC parameterization, but in this non-linear case, Q_1' is a non-linear function of x (is not a constant). For this reason, this control structure will be referred to as the non-linear MSFC structure whose block diagram is shown in Fig. 6(a). As we have seen in the previous subsection, when process model is linear, Q_1' will be simply a constant with the same value as in MSFC, and Q_2' will be the linear static state feedback of the MSFC (Coulibaly *et al.*, 1992). Thus, eq. (24) describes a non-linear MSFC law.

5.2.3. *Reduced-order non-linear MSFC parameterization.* The modified mixed error- and state-feedback control law of eq. (20) can be parameterized according to MSFC structure, leading to the control structure shown in Fig. 6(b) which will be called the reduced-order MSFC structure. The controller components are given in Table 2, where $u = \text{sat}\{w_1'' - w_2''\}$. In the previous subsection, it was shown that when process model is linear, Q_1'' will be simply a constant with the same value as in MSFC, Q_2'' will be the linear static state feedback of MSFC (Coulibaly *et al.*, 1992), but instead of the process model P, Q_0'' (which is of order r) is simulated. For this reason the control law of eq. (20) describes a reduced-order non-linear MSFC law.

6. APPLICATION TO NON-LINEAR REACTORS

6.1. *Example* 1: *a bioreactor*

Consider the same continuous stirred-tank bioreactor described in Dibiasio *et al.* (1981), in which the substrate methanol is utilized for the growth of *Methylomonas* strain. The bioreactor model is of the form

$$\frac{dS}{dt} = -\sigma(S)X + D(S_f - S)$$
$$\frac{dX}{dt} = \mu(S)\ X - DX \tag{32}$$

where the specific growth rate, $\mu(S)$, and substrate consumption rate, $\sigma(S)$, are given by

$$\mu(S) = \frac{1.4 \times 10^{-4} S(1 - 0.204S)}{(0.000849 + S + 0.406S^2)}$$

$$\sigma(S) = \frac{2.78 \times 10^{-4} S(1.32 + 3.86S - 0.661S^2)}{(0.000849 + S + 0.406S^2)}.$$

The operating conditions of the reactor and the corresponding multiple steady states are given in Tables 3 and 4, respectively. The control problem is to perform reactor start-up and operate the reactor at the unstable steady state $(S_{ss}, X_{ss}) = (0.4, 0.24)$, by controlling the substrate concentration in the reactor, S, and by manipulating the dilution rate, D. It is assumed that the dilution rate is bounded $(0 \leqslant D \leqslant 1.389 \times 10^{-4}\ \mathrm{s}^{-1})$ and that both state variables can be measured on-line.

The process model of eq. (32) can be recast in the form of eq. (1) with $y = S$, $x = [S\ X]^{\mathrm{T}}$, $u = D$. For this problem, $r = 1$ $(L_g h = (S_f - S) \not\equiv 0)$, and there is no deadtime.

- *Mixed error- and state-feedback controller.* For this example, the control law of eq. (21) takes the form

$$\frac{d\eta_1}{dt} = \frac{1}{\gamma_1} e, \quad \eta_1(0) = S(0)$$
$$u = \mathrm{sat}\left\{\frac{\eta_1 + e - S + \gamma_1 \sigma(S) X}{\gamma_1 (S_f - S)}\right\} \tag{33}$$

where $u_{\min} = 0$, and $u_{\max} = 1.389 \times 10^{-4}\ \mathrm{s}^{-1}$.

- *Mixed error- and state-feedback controller with conditional integration.* This controller has the same non-linear state feedback of the controller of eq. (33) but $\dot{\eta}_1 = Ie/\gamma_1$, where $I = 1$ when $u_{\min} < u < u_{\max}$, otherwise $I = 0$.
- *Modified mixed error- and state-feedback controller.* Application of the control law of eq. (20) results in

$$\frac{d\eta_1}{dt} = -\frac{1}{\gamma_1}\eta_1 + \frac{1}{\gamma_1}[S - \gamma_1 \sigma(S) X + \gamma_1 (S_f - S) u],$$
$$\eta_1(0) = S(0) \tag{34}$$
$$u = \mathrm{sat}\left\{\frac{\eta_1 + e - S + \gamma_1 \sigma(S) X}{\gamma_1 (S_f - S)}\right\}$$

where $u_{\min} = 0$, and $u_{\max} = 1.389 \times 10^{-4}\ \mathrm{s}^{-1}$.

6.1.1. *Simulation results.* Using the preceding three controllers and the operating conditions given in Table 3, the closed-loop process was simulated. In all the simulation cases, the controller tuning parameter $\gamma_1 = 630$ s.

Table 3. Nominal operating conditions of the bioreactor example

$S_f = 1.80 \times 10^{0}$	W/V%
$S(0) = 0.00 \times 10^{0}$	W/V%
$X(0) = 1.00 \times 10^{-2}$	W/V%
$S_{ss} = 4.00 \times 10^{-1}$	W/V%
$X_{ss} = 2.40 \times 10^{-1}$	W/V%
$D_{ss} = 1.11 \times 10^{-4}$	s^{-1}

Table 4. Multiple steady states of the bioreactor example for $D_{ss} = 1.11 \times 10^{-4}\,\mathrm{s}^{-1}$

Steady state (S_{ss}, X_{ss})	Stability
$(1.8 \times 10^{0}, 0.0 \times 10^{0})$	Stable
$(4.0 \times 10^{-1}, 2.4 \times 10^{-1})$	Unstable
$(3.3 \times 10^{-3}, 6.8 \times 10^{-1})$	Stable

Start-up performance. Figure 7(a) depicts the start-up profiles of the controlled output and manipulated input. A list of simulation cases is given in Table 5. The closed-loop responses (solid line) under the three controllers were identical for the unconstrained process. The closed-loop performance (dashed line) under mixed error- and state-feedback controller deteriorated in the presence of input constraints and it exhibited significant overshoot, undershoot, longer settling time. However, the modified mixed error- and state-feedback controller eliminated these undesired effects, and the closed-loop performance (dotted line) was close to that of unconstrained controller response (solid line). Also the modified mixed error- and state-feedback controller performed better compared to the closed-loop response (dot-dashed line) obtained using the controller with conditional integration. In terms of manipulated input profile, modified mixed error- and state-feedback controller kept the value of dilution rate at the upper constraint for 'optimal' amount of time. The controller with conditional integration left the upper constraint sooner, whereas the mixed state- and error-feedback controller overstayed, leading to degradation in the closed-loop performance.

Regulatory performance. In order to ascertain the regulatory performance of the modified mixed error- and state-feedback controller, when the bioreactor was operating at steady state, a step change of -20% was introduced in the the substrate feed concentration, S_f. This disturbance was considered to be unmeasurable. Figure 7(b) depicts the profiles of the controlled output and manipulated input. A list of simulation cases is given in Table 5. The relative regulatory performance of the three controllers was very much similar to their relative start-up performance: the modified mixed error- and state-feedback controller exhibited the best regulatory performance.

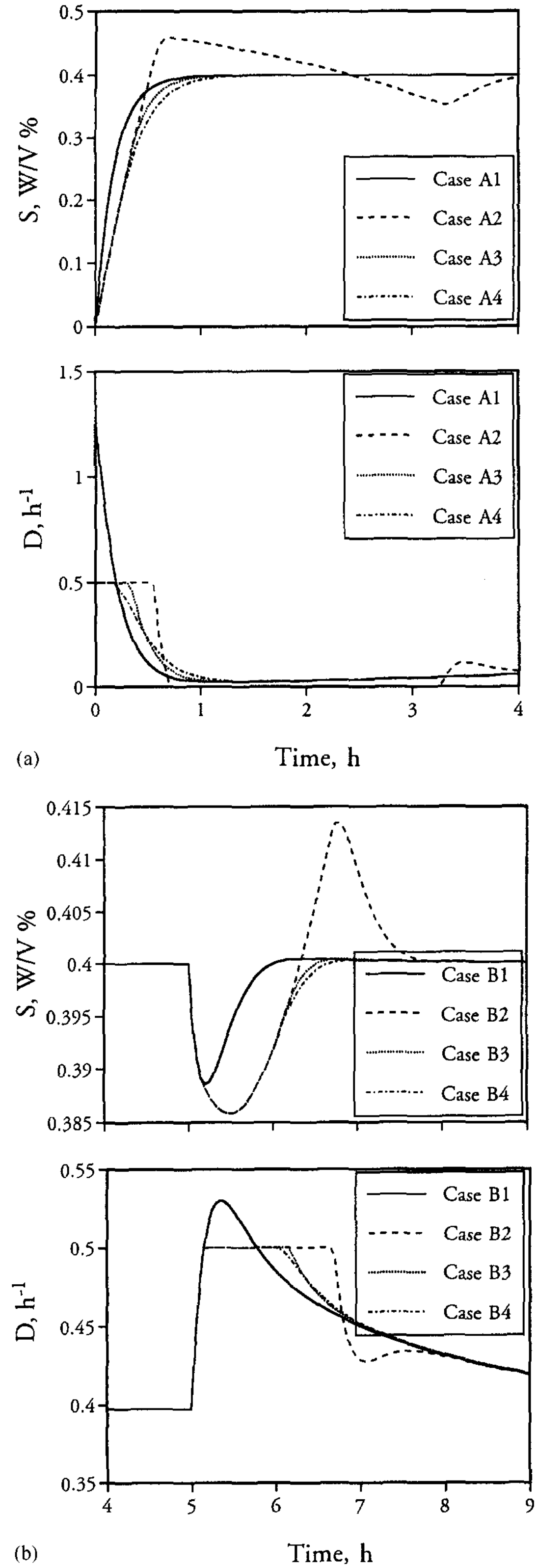


Fig. 7(a). Startup profiles of controlled and manipulated variables of the bioreactor example (list of the simulation cases is given in Table 5). (b). Profiles of controlled and manipulated variables of the bioreactor example in the presence of the unmeasurable disturbance (list of the simulation cases is given in Table 5).

Table 5. List of simulation cases for the bioreactor example

Cases	Controller	Input constraints
A1, B1	Mixed error and state feedback	No
	Modified mixed error and state feedback	No
A2, B2	Mixed error and state feedback	Yes
A3, B3	Modified mixed error and state feedback	Yes
A4, B4	Mixed error and state feedback with conditional integration	Yes

6.2. *Example 2: application to a chemical reactor*

Consider the same continuous stirred-tank reactor example described in Soroush and Kravaris (1992c). The reactor has a model of the form

$$\frac{dC_A}{dt} = R_A(C_A, T) + \frac{C_{A_i} - C_A}{\tau}$$
$$\frac{dT}{dt} = \frac{R_H(C_A, T)}{\bar{\rho}c} + \frac{T_i - T}{\tau} + \frac{Q}{\bar{\rho}cV}. \tag{35}$$

This exothermic reactor has three multiple steady states. The control problem is to perform reactor start-up and operate the reactor at the locally stable high-temperature high-conversion steady state $(C_{A_{ss}}, T_{ss}) = (1.3, 400)$, by controlling the reactant concentration, C_A, and by manipulating the heat input to the jacket, Q. It is assumed that (a) the heat input is bounded ($|Q| \leqslant 25$ kJ s^{-1}), and (b) C_A is measured with a time delay of 100 s.

The process model of eq. (35) can be recast in form of eq. (1): $y(t) = C_A(t-100)$, $x = [C_A\ T]^T$, $u = Q$. For this process, $r = 2$ ($L_g h \equiv 0$ and $L_g L_f h = (1/\bar{\rho}cV)\partial R_A(C_A, T)/\partial T \not\equiv 0$).

- *Error-feedback controller.* For this example, the reduced-order error-feedback controller of eq. (24) takes the form

$$\frac{dC_A}{dt} = R_A(C_A, T) + \frac{C_{A_i} - C_A}{\tau}, \quad C_A(0) = C_{A_0}$$

$$\frac{dT}{dt} = \frac{R_H(C_A, T)}{\bar{\rho}c} + \frac{T_i - T}{\tau} + \frac{1}{\bar{\rho}cV}u, \quad T(0) = T_0$$

$$u = \mathrm{sat}\left\{\frac{e' - C_A - \gamma_1 f_1(C_A, T) - \gamma_2 L_f f_1(C_A, T)}{\gamma_2 \dfrac{1}{\bar{\rho}cV}\dfrac{\partial R_A(C_A, T)}{\partial T}}\right\}. \tag{36}$$

where $u_{min} = -25$ kJ s^{-1}, $u_{max} = +25$ kJ s^{-1}, $e'(t) = e(t) + C_A(t-100)$,

$$f_1(C_A, T) = R_A(C_A, T) + \frac{C_{A_i} - C_A}{\tau},$$

and the initial conditions C_{A_0} and T_0 are the same as those given in Soroush and Kravaris (1992).

• *'Nonlinear IMC' controller.* The following controller is a non-linear IMC controller parameterized according to the original IMC structure for processes with input constraints:

$$\frac{dC_A}{dt} = R_A(C_A, T) + \frac{C_{A_i} - C_A}{\tau}, \quad C_A(0) = C_{A_0}$$

$$\frac{dT}{dt} = \frac{R_H(C_A, T)}{\bar{\rho}c} + \frac{T_i - T}{\tau} + \frac{1}{\bar{\rho}cV}\,\mathrm{sat}\{w\}, \quad T(0) = T_0$$

— — — — — — — — — — — — — — — —

$$\frac{d\xi_1}{dt} = R_A(\xi_1, \xi_2) + \frac{C_{A_i} - \xi_1}{\tau}, \quad \xi_1(0) = C_{A_0}$$

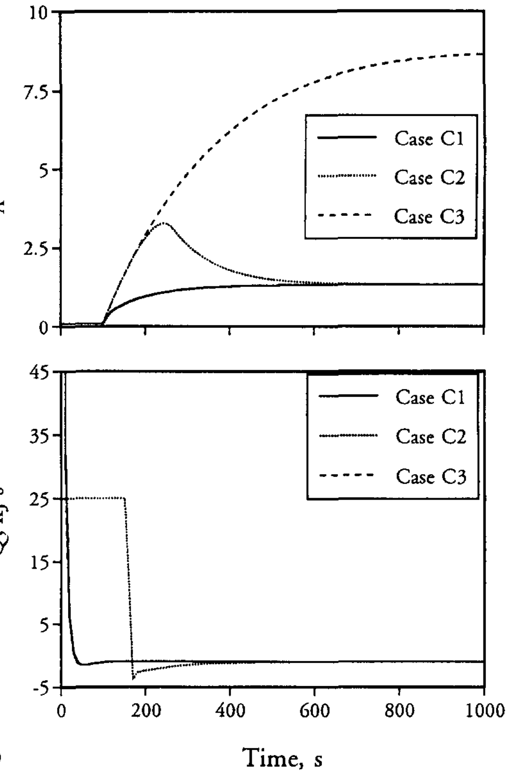
$$\frac{d\xi_2}{dt} = \frac{R_H(\xi_1, \xi_2)}{\bar{\rho}c} + \frac{T_i - \xi_2}{\tau} + \frac{1}{\bar{\rho}cV}w, \quad \xi_2(0) = T_0$$

$$w = \frac{e' - \xi_1 - \gamma_1 f_1(\xi_1, \xi_2) - \gamma_2 L_f f_1(\xi_1, \xi_2)}{\gamma_2 \dfrac{1}{\bar{\rho}cV}\dfrac{\partial R_A(\xi_1, \xi_2)}{\partial T}}. \tag{37}$$

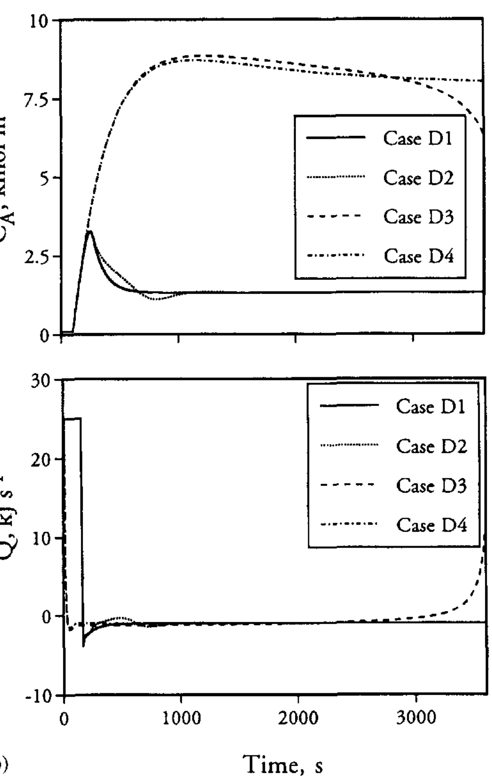
where $u_{\min} = -25\ \mathrm{kJ\,s^{-1}}$, $u_{\max} = +25\ \mathrm{kJ\,s^{-1}}$, and $e'(t) = e(t) + C_A(t - 100)$. The preceding controller consists of two subsystems: (i) the upper subsystem [above the dashed line and with the state variables C_A and T driven by $\mathrm{sat}(w)$] represents the process model (P) in the IMC structure, and (ii) the lower subsystem (below the dashed line and with the state variables ξ_1 and ξ_2 driven by w) represents the process model inverse plus the IMC filter $[1/(\gamma_2 s^2 + \gamma_1 s + 1)]$.

6.2.1. *Simulation results.* Using the preceding two controllers and the same operating conditions given in Soroush and Kravaris (1992a), the closed-loop process was simulated. In all the simulation cases, the controller tuning parameters $\gamma_1 = 1.1 \times 10^2$ s, and $\gamma_2 = 1.0 \times 10^3\ \mathrm{s}^2$.

Start-up under nominal conditions. Figure 8(a) depicts the start-up profiles of the controlled output and manipulated input. A list of simulation cases is given in Table 6. In the absence of input constraints, the closed-loop performance (solid line) under the two controllers was precisely the requested second-order response, but this was achieved by an extremely high rate of heat input at the early stage of start-up. Under the assumed limits on the heat input rate, u, the closed-loop response obtained using the error-feedback controller (dotted line) was close to that of the unconstrained case (solid line). The overshoot in the concentration was due to the lower reactor temperature during the early stage of the start-up—the upper constraint limited the necessary rate of heat input—leading to the accumulation of the reactant in the reactor. Subsequently, the reactant concentration, C_A, started decreasing after the temperature approached a higher value of about 400 K.



(a)



(b)

Fig. 8(a). Startup profiles of controlled and manipulated variables of the chemical reactor example (list of the simulation cases is given in Table 6). (b). Startup profiles of controlled and manipulated variables of the chemical reactor example in the presence of model errors (list of the simulation cases is given in Table 7).

Table 6. List of simulation cases for the start-up performance of the chemical reactor example

Case	Controller	Input constraints
C1	Nonlinear IMC	No
	Error feedback	No
C2	Error feedback	Yes
C3	Non-linear IMC	Yes

Table 7. List of simulation cases for the start-up performance of the chemical reactor example in the presence of model errors

Case	Controller	Model errors	Input constraints
D1	Error feedback	No	Yes
D2	Error feedback	Yes	Yes
D3	Non-linear IMC	Yes	Yes
D4	Non-linear IMC	No	Yes

In the presence of the input constraints, the non-linear IMC controller (profiles indicated by dashed line), however, operated the process asymptotically at an undesired, low-temperature low-conversion steady state. This poor performance was due to that the process model inverse plus filter [lower subsystem of the controller of eq. (37)] was unaware of the saturation when a constraint was active, leading to significant discrepancies between C_A and ξ_1 and between T and ξ_2. As pointed out by Zheng *et al.* (1994), the original IMC simply 'clips' the unconstrained controller action [see Fig. 8(a)]. This led to an insufficient supply of heat to the reactor during the start-up, and thus the non-linear IMC controller operated the reactor asymptotically at an undesired low-temperature low-conversion steady state corresponding to $Q_{ss} = -1.03\,\mathrm{kJ\,s^{-1}}$, not at the desired high-temperature high-conversion steady state.

Start-up in the presence of model errors. The performance of the error-feedback controller was tested in the presence of considerable modeling errors, i.e. 10% increase in all the frequency factors used in the controllers. Figure 6(b) depicts the start-up profiles of the controlled output and manipulated input under the two controllers. A list of simulation cases is given in Table 7. The closed-loop response under error-feedback controller (dotted line) was satisfactory when compared to the nominal closed-loop response (solid line). This figure also shows the closed-loop nominal and non-nominal responses under the non-linear IMC controller. The non-nominal response was similar to that of nominal response (dot-dashed line) over a long period of time, but unlike the nominal response, it did not settle down at the low-conversion low-temperature steady-state. In the non-nominal case, after 3000 s, the IMC controller calculated higher values for heat input that exceeded the upper constraint, and thus the IMC controller broke down (controller state variables became unbounded).

7. CONCLUSIONS

The model-predictive control formulation of input–output linearization allowed to establish the connections between MPC and input–output linearizing control: an input–output linearizing state feedback is simply a shortest prediction-horizon MPC law. The MPC formulation was used to address the problems of constraint handling and windup in input–output linearizing control for non-linear processes with full-state measurements and for non-linear processes with incomplete-state measurements and deadtime. In the absence of constraints and penalties on input, the derived control laws are input–output linearizing.

The connections between the developed non-linear control laws and the modified internal model control and the model state feedback control were established. When the derived reduced-order non-linear error-feedback control law [given by eq. (24)] is applied to a linear process, the resulting linear controller will be exactly a minimal-order state-space realization of MSFC and a minimal-order state-space realization of a reduced-order modified IMC.

The controller synthesis results presented in this article have already been extended to multi-input–multi-output (MIMO) processes (Valluri, 1997). The derived MIMO control laws consist of two distinct components: (i) an input–output linearizing controller that inherently includes an integral windup compensator and (ii) an optimal directionality compensator. The optimal directionality compensator is a quadratic program that is easily solvable on-line. In the case that the characteristic matrix (decoupling) matrix of process is diagonal or that process is single-input–single-output, the optimal directionality compensator is identical to 'clipping'; in these two cases, a closed analytical solution to the optimization problem exists. For general processes, however, neither clipping nor direction preservation can optimally compensate for process directionality.

Acknowledgment

Acknowledgment is made to the donors of The Petroleum Research Fund, administered by the ACS, for support of this research.

NOTATION

A	reactant
c	heat capacity of reacting mixture, $\mathrm{kJ\,kg^{-1}\,K^{-1}}$
C_A	concentration of reactant A, $\mathrm{kmol\,m^{-3}}$
C_{A_i}	inlet concentration of reactant A, $\mathrm{kmol\,m^{-3}}$
$C_{A_{ss}}$	steady-state concentration of A, $\mathrm{kmol\,m^{-3}}$
d	unmeasured disturbance
D	dilution rate, $\mathrm{s^{-1}}$
D_{ss}	steady-state dilution rate, $\mathrm{s^{-1}}$

r	relative order of controlled output
R_A	rate of production of A, kmol m^{-3} s^{-1}
R_H	overall rate of heat production by reactions, kJ m^{-3} s^{-1}
Q	rate of heat input to the chemical reactor, kJ s^{-1}
S	substrate concentration, W/V%
S_{ss}	steady-state substrate concentration, W/V%
S_f	substrate concentration in the feed, W/V%
t	time, s
T	reactor temperature, K
T_h	prediction time horizon, s
T_i	temperature of inlet stream, K
T_{ss}	steady-state reactor temperature, K
u	manipulated input
V	volume of the reacting mixture, m^3
x	vector of model-calculated values of state variables
$\bar{x}$	vector of measured state variables
X	cell concentration, W/V%
X_{ss}	steady-state cell concentration, W/V%
y	model calculated value of the controlled output
$\bar{y}$	measured controlled variable
$\hat{y}$	predicted future value of the controlled variable
y_d	reference trajectory
y_{sp}	output set-point

Greek letters

γ_j	jth tunable parameter of controller, s^j
$\mu(S)$	specific growth rate, s^{-1}
ρ	penalty on the manipulated input
$\bar{\rho}$	density of reacting mixture, kg m^{-3}
$\sigma(S)$	substrate consumption rate, s^{-1}
θ	deadtime, s
τ	CSTR residence time; time, s

Math symbols

int	interior
$L_f h(x)$	Lie derivative of the scalar field $h(x)$ with respect to the vector field $f(x)$ $\left[L_f^1 h(x) \triangleq \sum_{\ell=1}^{n} \frac{\partial h(x)}{\partial x_\ell} f_\ell(x)\right]$
$L_f^{\ell+1} h(x)$	Lie derivative of scalar field $L_f^\ell h(x)$ with respect to vector field $f(x)$
$L_g L_f^\ell h(x)$	Lie derivative of scalar field $L_f^\ell h(x)$ with respect to vector field $g(x)$
$\lVert\omega(t)\rVert_p$	the p-function norm of $\omega(t)$ over a finite time interval $[a, b]$, where $p \geqslant 1$ $(\int_a^b \lvert\omega(t)\rvert^p \, dt)^{1/p}$

REFERENCES

Åström, K. J. and Rundqwist, L. (1989) Integrator windup and how to avoid it. *Proceedings of the ACC*, Pittsburgh, PA, p. 1693.

Calvet, J. P. and Arkun, Y. (1988) Feedforward and feedback linearization of nonlinear systems and its implementation using internal model control. *Ind. Engng Chem. Res.* **27**, 1822.

Campo, P. J. and Morari, M. (1990) Robust control of processes subject to saturation nonlinearities. *Comput. Chem. Engng* **14**, 343.

Coulibaly, E., Maiti, S. and Brosilow, C. (1995) Internal model predictive control (IMPC). *Automatica* **31**, 1471.

Coulibaly, E., Maiti, S. and Brosilow, C. (1992) Internal model predictive control (IMPC). Preprints of AIChE Annual Meeting, Paper 123j.

Daoutidis, P. and Kravaris, C. (1992) Dynamic output feedback control of minimum-phase nonlinear processes. *Chem. Engng Sci.* **47**, 837.

Doyle, F. J. (1995) An anti-windup input–output linearization scheme for SISO systems, *Automatica* (submitted).

Dibiasio, D., Lim, H. C. and Weigand, W. A. (1981) An experimental investigation of stability and multiplicity of steady states in a biological reactor. *A.I.Ch.E. J.* **27**, 284.

Hanus, R., Kinnaert, M. and Henrotte, J. L. (1987) Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica* **23**, 729.

Hirschorn, R. M. (1979) Invertibility of nonlinear control systems. *SIAM J. Control Optim.* **17**, 289.

Isidori, A. (1989) *Nonlinear Control Systems: An Introduction*, 2nd Edn. Springer-Verlag, Berlin.

Kapoor, N. and Daoutidis, P. (1996a) Windup compensation for nonlinear processes with input constraints. *Int. J. of Contr.* (submitted).

Kapoor, N. and Daoutidis, P. (1996b) Control of nonlinear processes with input constraints: a dynamical systems approach. *Chem. Engng Sci.* (submitted).

Kapoor, N. and Daoutidis, P. (1997) Stabilization of systems with input constraints. *Int. J. of Control* **66**(5), 653.

Kothare, M. V., Campo, P. J., Morari, M. and Nett, C. (1994) A unified framework for the study of anti-windup designs. *Automatica* **30**, 1869.

Kurtz, J. M. and Henson, M. A. (1996) Linear model predictive control of input–output linearized processes with constraints. Preprints of CPC V.

Mayne, D. Q. and Michalska, H. (1993) Robust receding horizon control of constrained nonlinear systems. *IEEE Trans. Automat Control* **AC-38**, 1623.

Mehra, R. K. and Rouhani, R. (1980) Theoretical considerations on model algorithmic control for nonminimum phase systems. *Proceedings of ACC*, p. TA8-B.

Muske, K. R. and Rawlings, J. B. (1993) Model predictive control with linear models. *A.I.Ch.E J.* **39**, 1923.

Oliveira, N. M. C. and Biegler, L. T. (1994) Constraint handling and stability properties of model-predictive control. *A.I.Ch.E J.* **40**, 1138.

Oliveira, S. L., Nevistic, V. and Morari, M. (1995) Control of nonlinear systems subject to input constraints. *Preprints of Nonlinear Control Systems Design Symposium*, p. 15.

Rangel, D., Lien, C. Y. and Wang, T. W. (1990) Application of feedback linearization to the control of nonlinear chemical reactors. *Preprints of AIChE Annual Meeting.*

Rawlings, J. B. and Muske, K. R. (1993) Stability of constrained receding horizon control. *IEEE Trans. Automat Control* **38**, 1512.

Soroush. M. (1997) Nonlinear state-observer design with application to reactors. *Chem. Engng Sci.* **52**(3), 387.

Soroush, M. and Nikravesh, M. (1996) Shortest-prediction-horizon nonlinear model predictive control. *Proceedings of the IFAC World Congress*, San Francisco, CA.

Soroush, M. and Kravaris, C. (1992a) Nonlinear control of a batch polymerization reactor: an experimental study. *A.I.Ch.E J.* **38**, 1429.

Soroush, M. and Kravaris, C. (1992b) A continuous-time formulation of nonlinear model predictive control. *Proceedings of the ACC*, Chicago, p. 1561.

Soroush, M. and Kravaris, C. (1992c) Discrete-time nonlinear controller synthesis by input/output linearization. *A.I.Ch.E J.* **38**, 1923.

Suarez, R., Alvarez, J. and Alvarez, J. (1991) Linear systems with single saturated input: stability analysis. *Proceedings of the 30th Conference on Decision and Control*, Brighton, U.K., p. 223.

Valluri, S. (1997) Nonlinear control of processes with actuator saturations. Ph.D. thesis, Drexel University, Philadelphia.

Vuthandam, P., Genceli, H. and Nikolaou, M. (1995) Performance bounds for robust quadratic dynamic control with end condition. *A.I.Ch.E J.* **41**, 2083.

Walgama, K. S. and Sternby, J. (1990) The inherent observer property in a class of anti-windup compensators. *Int. J. Control* **52**, 705.

Zafiriou, E. (1990) Robust model predictive control of processes with hard constraints. *Comput. Chem. Engng* **14**, 359.

Zheng, A., Kothare, M. V. and Morari, M. (1994) Anti-windup design for internal model control. *Int. J. Control* **60**, 1015.

Zheng, A. and Morari, M. (1995) Control of linear unstable systems with constraints. *Proceedings of ACC*, p. 3704.

APPENDIX A: PROOFS

Proof of Theorem 1. For a sufficiently short time horizon $(\tau - t)$, the solution of eq. (9) can be approximated by the following $O[(\tau - t)^r]$ truncated power series:

$$y_d(\tau) = \bar{y}(t) + \sum_{\ell=1}^{r-1} L_f^\ell h(\bar{x}(t)) \frac{(\tau - t)^\ell}{\ell!} + \frac{y_{sp}(t) - \bar{y}(t) - \sum_{\ell=1}^{r-1} \gamma_\ell L_f^\ell h(\bar{x}(t))}{\gamma_r} \frac{(\tau - t)^r}{r!} + \text{h.o.t.} \tag{A1}$$

Substituting for $y_d(\tau)$ and $\hat{y}(\tau)$ [from eqs (A1) and (8), respectively] in eq. (6), the optimization problem (for a sufficiently small time horizon T_h) becomes

$$\min_{u(t)} \left\{ \left\| \left(\frac{y_{sp}(t) - \bar{y}(t) - \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(\bar{x}(t))}{\gamma_r} - L_g L_f^{r-1} h(\bar{x}(t)) u(t) \right) \frac{(\tau - t)]^r}{r!} \right\|_p^2 + \rho' |u(t)|^2 \right\}$$

subject to $u_{\min} \leqslant u(t) \leqslant u_{\max}$, which is equivalent to

$$\min_{u(t)} \left\{ \left(\frac{e(t) - \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(\bar{x}(t))}{\gamma_r} - L_g L_f^{r-1} h(\bar{x}(t)) u(t) \right)^2 + \rho |u(t)|^2 \right\}$$

subject to $u_{\min} \leqslant u(t) \leqslant u_{\max}$, where

$$\rho \triangleq \frac{\rho'(r!)^2}{\|(\tau - t)^r\|_p^2}.$$

This is a one-dimensional quadratic minimization problem with the analytical solution

$$u(t) = \text{sat}\left\{ \frac{L_g L_f^{r-1} h(\bar{x}(t))}{(L_g L_f^{r-1} h(\bar{x}(t)))^2 + \rho} \times \frac{[e(t) + h(\bar{x}(t))] - h(\bar{x}(t)) - \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(\bar{x}(t))}{\gamma_r} \right\}. \tag{A2}$$

Proof of Theorem 2. For a sufficiently short time horizon $[\tau - (t + \theta)]$, the solution of eq. 15 can be approximated by the following $O[(\tau - (t + \theta))^r]$ truncated power series:

$$y_d(\tau) = \bar{y}(t) - h(x(t - \theta)) + \sum_{\ell=0}^{r-1} L_f^\ell h(x(t)) \frac{[\tau - (t + \theta)]^\ell}{\ell!} + \frac{y_{sp}(t) - [\bar{y}(t) - h(x(t - \theta)) + h(x(t)) + \sum_{\ell=1}^{r-1} \gamma_\ell L_f^\ell h(x(t))]}{\gamma_r} \times \frac{[\tau - (t + \theta)]^r}{r!} + \text{h.o.t.} \tag{A3}$$

Substituting for $y_d(\tau)$ and $\hat{y}(\tau)$ [from eqs (A3) and (14), respectively] in eq. (6), the optimization problem (for a sufficiently small time horizon T_h) becomes

$$\min_{u(t)} \left\{ \left\| \left(\frac{e(t) + h(x(t - \theta) - h(x(t)) - \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(x(t))}{\gamma_r} - L_g L_f^{r-1} h(x(t)) u(t) \right) \frac{[\tau - (t + \theta)]^r}{r!} \right\|_p^2 + \rho'' |u(t)|^2 \right\}$$

subject to $u_{\min} \leqslant u(t) \leqslant u_{\max}$. The preceding optimization problem is equivalent to

$$\min_{u(t)} \left\{ \left(\frac{e(t) + h(x(t - \theta)) - h(x(t)) - \sum_{\ell=1}^{r} \gamma_\ell L_f^\ell h(x(t))}{\gamma_r} - L_g L_f^{r-1} h(x(t)) u(t) \right)^2 + \rho |u(t)|^2 \right\}$$

subject to $u_{\min} \leqslant u(t) \leqslant u_{\max}$, where

$$\rho \triangleq \frac{\rho''(r!)^2}{\|[\tau - (t+\theta)]^r\|_p^2}.$$

This is again a one-dimensional quadratic minimization problem with the analytical solution

$$u(t) = \operatorname{sat}\left\{\frac{L_gL_f^{r-1}h(x(t))}{(L_gL_f^{r-1}h(x(t)))^2 + \rho}\right.$$

$$\left.\times \frac{e(t) + h(x(t-\theta)) - h(x(t)) - \sum_{\ell=1}^{r}\gamma_\ell L_f^\ell h(x(t))}{\gamma_r}\right\}. \tag{A4}$$

Proof of Theorem 3. (a) Upon substitution for matrices A_c, b_c, and c_c (see the Appendix B), the control law of Theorem 3 takes the form

$$\begin{aligned}
\dot\eta_1 &= \eta_2 \\
&\vdots \\
\dot\eta_{r-1} &= \eta_r
\end{aligned} \tag{A5}$$

$$\dot\eta_r = -\frac{1}{\gamma_r}\eta_1 - \frac{\gamma_1}{\gamma_r}\eta_2 - \cdots - \frac{\gamma_{r-1}}{\gamma_r}\eta_r + \frac{1}{\gamma_r}\Phi(\bar{x}, u)$$

$$u = \operatorname{sat}\{\Psi_0(\bar{x}, \eta_1 + e)\}$$

where

$$\begin{aligned}
\Phi(\bar{x}, u) &= h(\bar{x}) + \sum_{\ell=1}^{r}\gamma_\ell L_f^\ell h(\bar{x}) + \gamma_r L_g L_f^{r-1}h(\bar{x})u \\
&= y + \gamma_1\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots + \gamma_r\frac{\mathrm{d}^r y}{\mathrm{d}t^r}.
\end{aligned}$$

Thus,

$$\gamma_r\dot\eta_r = -\eta_1 - \gamma_1\eta_2 - \cdots - \gamma_{r-1}\eta_r + \left[y + \gamma_1\frac{\mathrm{d}y}{\mathrm{d}t} + \cdots + \gamma_r\frac{\mathrm{d}^r y}{\mathrm{d}t^r}\right]$$

$$(y - \eta_1) + \gamma_1\frac{\mathrm{d}(y-\eta_1)}{\mathrm{d}t} + \cdots + \gamma_r\frac{\mathrm{d}^r(y-\eta_1)}{\mathrm{d}t^r} = 0.$$

Since it was assumed that $\eta_{\ell+1}(0) = L_f^\ell h(\bar{x}(0) = 0$, $\ell = 0, \ldots, r-1$, $\eta_1(t) = y(t) = h(\bar{x}(t))$, $\forall t \geqslant 0$, and thus

$$c_c\eta(t) + e(t) = \eta_1(t) + e(t) = e(t) + h(\bar{x}(t)), \forall t \geqslant 0.$$

Thus, the controller action calculated by eq. (20) is exactly equal to that calculated by eq. (11) which is a solution to the constrained optimization problem of eq. (6) with a zero ρ.

(b) In the absence of input constraints and by using part (a) of this theorem, we see that the controller of Theorem 3 is identical to the input–output linearizing feedback of eq. (11) which induces the linear response of eq. (12).

(c) When the closed-loop control system is asymptotically stable and the process is subjected to 'rejectable' [in the sense that always $u_{ss} \in \operatorname{int}(U)$] constant disturbances and model errors, the closed-loop system reaches a steady state, and therefore

$$\begin{aligned}
\eta_{1_{ss}} &= \Phi(\bar{x}_{ss}, u_{ss}) = h(\bar{x}_{ss}) + \sum_{\ell=1}^{r}\gamma_\ell L_f^\ell h(\bar{x}_{ss}) \\
&\quad + \gamma_r L_g L_f^{r-1}h(\bar{x}_{ss})u_{ss}, \\
\eta_{\ell_{ss}} &= 0, \ell = 2, \ldots, r
\end{aligned}$$

and

$$u_{ss} = \Psi_0[\bar{x}_{ss}, \Phi(\bar{x}_{ss}, u_{ss}) + e_{ss}].$$

Substituting for Ψ_0 and Φ, by using their definitions, in the preceding equation, we see that

$$u_{ss} = \frac{[h(\bar{x}_{ss}) + \sum_{\ell=1}^{r}\gamma_\ell L_f^\ell h(\bar{x}_{ss}) + \gamma_r L_g L_f^{r-1}h(\bar{x}_{ss})u_{ss}] + e_{ss} - h(\bar{x}_{ss}) - \sum_{\ell=1}^{r}\gamma_\ell L_f^\ell h(\bar{x}_{ss})}{\gamma_r L_g L_f^{r-1}h(\bar{x}_{ss})}$$

which implies that $e_{ss} = 0$.

Proof of Theorem 4. (a) Since $x(0) = \bar{x}(0)$, $x(t) = \bar{x}(t)$, $\forall t \geqslant 0$. Similar to the proof of part (a) of Theorem 3, one can prove that $\eta_1(t) = h(x(t))$, $\forall t \geqslant 0$. Thus,

$$\begin{aligned}
c_c\eta(t) + e^*(t) &= \eta_1(t) + e(t) + h(x(t-\theta)) - h(x(t)) \\
&= e(t) + h(x(t-\theta)), \quad \forall t \geqslant 0.
\end{aligned}$$

Thus, the controller action calculated by eq. 23 is exactly equal to that calculated by eq. 17 which is a solution to the constrained optimization problem of eq. (6) with a zero ρ.

(b) In the absence of input constraints and by using part (a) of this theorem, we see that the controller of Theorem 4 is identical to the input–output linearizing controller of eq. (17) which induces the linear response of eq. (18).

(c) When the closed-loop control system is asymptotically stable and the process is subjected to rejectable constant disturbances and model errors, the closed-loop system reaches a steady state, and therefore $e^*_{ss} = e_{ss}$,

$$\eta_{1_{ss}} = \Phi(x_{ss}, u_{ss}), \quad \eta_{\ell_{ss}} = 0, \ell = 2, \ldots, r,$$

and

$$u_{ss} = \Psi_0[x_{ss}, \Phi(x_{ss}, u_{ss}) + e_{ss}].$$

Using the preceding equation and the definitions of Ψ_0 and Φ, as in the proof of Theorem 3, it is straightforward to show that $e_{ss} = 0$.

APPENDIX B: REPRESENTATIVE MATRIX TRIPLET

(A_c, b_c, c_c)

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{1}{\gamma_r} & -\dfrac{\gamma_1}{\gamma_r} & -\dfrac{\gamma_2}{\gamma_r} & \cdots & -\dfrac{\gamma_{r-1}}{\gamma_r} \end{bmatrix}, \qquad b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dfrac{1}{\gamma_r} \end{bmatrix}, \qquad c_c = [1\ 0\ \ldots\ 0]$$