Optimisation-Based Control of Constrained Nonlinear Systems

by

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Abstract

This thesis addresses the control of nonlinear systems whose inputs or trajectory are subject to constraints. These constraints represent a major obstacle in the application of traditional methods to design adequate controllers. Among the methodologies well-suited to deal with this problem we distinguish the optimisation-based ones, in particular model predictive control and the open-loop optimal control methods. Developments concerning these two methodologies are the subject of this research.

We provide a method to constructively generate stabilising feedbacks for a universal class of time-varying nonlinear systems with input constraints. The method is based on Model Predictive Control (MPC) which generates a feedback law by solving a sequence of open-loop optimal control problems. We propose a new MPC framework. Within this framework, we develop a set of conditions on the design parameters that are sufficient for the stability of the closed loop-trajectory. The novel sufficient conditions for stability can be used to analyse *a priori* the stability properties of most MPC schemes. Some important classes of nonlinear systems, like nonholonomic systems, can now be stabilised using MPC. In addition, we can exploit the increased flexibility in the choice of the MPC design parameters to avoid constraints that make the optimal control problems involved difficult to solve. This facilitates, if not ensures, feasibility and improves the performance of the optimisation algorithms. Further developments of the MPC framework for a certain class of nonlinear systems enable us to guarantee specified rates of exponential stability.

A well-established tool for solving open-loop optimal control problems is necessary conditions in the form of a Maximum Principle. For certain state constrained optimal control problems of interest, the Maximum Principle is degenerate in the sense that the necessary conditions are satisfied by a set of multipliers that conveys no useful information about the optimal solution. Among these problems are the ones in which the fixed initial state lies on the boundary of the state constraint set, which have particular relevance to MPC. New necessary conditions of optimality, which overcome this degeneracy phenomenon under a suitable constraint qualification, are developed. Subsequent refinements give normality conditions, which can even provide additional information about minimisers.

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Notation

HJE	Hamilton-Jacobi equation, ... 21
MPC	Model Predictive Control, ... 11, 34
NCO	Necessary Conditions of Optimality, ... 18, 26
OCP	open-loop optimal control problem, ... 22
$(\bar{x}, \bar{u})$	process solving an open-loop optimal control problem, ... 18, 22
$\dot{x}(t)$	total derivative with respect to time, ... 17
$a \cdot b$	inner product of a and b, ... 18
$f_x(x, u)$	partial derivative with respect to x, ... 18
$\|x\|$	Euclidean norm of x, ... 25
$\|x\|_X$	norm of x in the space X, ... 125
$\emptyset$	empty-set, ... 123
$\mathbb{B}$	closed unit ball, ... 25
$\{x\} + \delta\mathbb{B}$	ball of radius δ centred at x, ... 25
$\operatorname{dom} f$	domain of the function f, ... 26
$\operatorname{epi} f$	epigraph of the function f, ... 26, 127
$\operatorname{supp}\{\mu\}$	support of the measure μ, ... 27, 128
$\operatorname{co} A$	convex hull of the set A, ... 26, 131
$\operatorname{bdy} A$	boundary of the set A, ... 106

Chapter 1

Introduction

Every restriction corresponds to a law of nature, a regularization of the universe. The more restrictions there are on what matter and energy can do, the more knowledge human beings can attain."

Carl Sagan, *Broca's Brain*

1.1 Scope and Motivation

The aim of this thesis is the study of methods to control nonlinear dynamical systems whose inputs or trajectory are subject to constraints.

Control problems with state constraints along the trajectory arise naturally and frequently in practice every time certain regions of the state space should not be entered for safety reasons, reliability of operation, or physical restrictions that are not expressed in the dynamical equation nor in the control constraints. Typical examples of these state constraints are minimum altitude or velocity in a plane, maximum temperature or pressure in a chemical reactor, obstacles to be avoided by a vehicle or robot, amongst many others.

The input constraints are present in most dynamical systems modelling physical phenomena. Virtually every actuator has a restriction on maximum displacement, force, or power. These restrictions need to be accounted for in the model.

Considering the model to be nonlinear might be beneficial, or even necessary in some applications. Common hard nonlinearities of a discontinuous nature do not allow linear approximation. Even for systems that are adequately described by a linear model in some range of operation, the use of a nonlinear model can broaden the valid range of operation as well as improve the accuracy of the model. These benefits often result in enhanced performances.

The methodologies to synthesise the control for nonlinear systems can be divided in two groups: the methods that use optimisation as a main procedure in finding the desired control; and those methods in which optimisation is not used or does not have a prominent role.

The latter methods typically explore particular structures of the system: known examples are passivity, sliding modes, and methods for affine or linear systems. Even general analysis methods like Lyapunov stability theory are not easily transformable into general synthesis methods and involve trial-and-error procedures. If we consider that the nonlinear system also has input or state constraints, the scarcity of adequate methods is even greater.

The optimisation-based methods typically are less restricted to particular structures of the model and naturally incorporate constraints. Moreover, the designer just has to translate the set of specifications and performance criteria into a set of constraints and an objective function; the remaining part of the process can be automated. Finally, the method not only finds a control that drives the system to meet the specifications, but also finds a good, if not the best, control according to criteria of our own choice.

This thesis concentrates on the optimisation-based methods to control nonlinear systems. Amongst the optimisation-based methods we distinguish the following:

- Open-loop optimal control methods: These include necessary conditions of optimality in the form of a Maximum Principle and Nonlinear Programming of the discretized model.

- Closed-loop optimal control methods: Dynamic Programming and the Hamilton-Jacobi equation are known tools to obtain the optimal closed-loop control.

- Model Predictive Control (MPC): It generates a “feedback” control by solving on-line a sequence of open-loop optimal control problems.

For open-loop optimal control problems there is a well-developed body of theory addressing a variety of features like nonsmooth data and state or control constraints. There are also several algorithms capable of solving even large dimension problems in an efficient way. The handicap is precisely that we obtain an open-loop and not a closed-loop control. An exception is the linear quadratic regulator.

The closed-loop methods have as one of their main advantages precisely the fact that they provide a feedback control. The superiority of the feedback control is a key idea, if not the key idea in control engineering: as the model is never an exact representation of the real dynamical system, it is preferable to have a feedback control that acts based on the information of "how the system is" rather than having an open-loop control that acts based on the information of "how the model predicts the system to be". However, the price we pay to obtain the "Holy Grail"—the optimal feedback control—is computationally very high. The Hamilton-Jacobi partial differential equation as well as the Dynamic Programming Bellman recursion, are computationally very hard both in time and memory requirements. These requirements grow exponentially with the dimension of the problem and soon become impossible to satisfy with current technology. In addition, the Hamilton-Jacobi equation provides just sufficient conditions of optimality: in the cases when these sufficient conditions are never satisfied, they will be of no help in finding the optimum.

Model predictive control combines the best of both worlds. It generates a feedback control by using open-loop optimisation methods that are computationally efficient. The drawback is that these open-loop optimal control problems have to be solved on-line: hence MPC can only be applied when the time to solve these optimisation problems is low in comparison with the time constants of the system. Nevertheless, the number of applications of MPC, specially in the chemical process industries, is vast[1] and is rapidly increasing with the increase of the available computational power.

In the first part of this thesis we concentrate on the study of Model Predictive Control. In the second part, the focus is on developments of necessary conditions of optimality for open-loop control problems, which, in particular, are relevant to MPC.

When designing a controller the most basic, yet most important requirement, is stability. An unstable system is typically useless and might even be dangerous. Being able to guarantee stability is therefore, in many applications, a major concern. Despite that, the MPC strategy when naively used does not guarantee stability. The stability of MPC for linear systems is a well studied subject, but the nonlinear case is a different story altogether. The existent results on the stability of the MPC for nonlinear systems require conditions (typically terminal state constrained to the origin) that not only limit the class of systems that can be controlled in this way, but also increase the complexity of the

[1]Industrial surveys [QB97, QB98] done with five MPC software vendors identified more than 2200 applications, 86 nonlinear, and the largest being a process with 603 outputs and 283 inputs.

optimisation algorithms solved on-line. There is clearly a need to find conditions under which we can guarantee stability without compromising the efficiency of the algorithms and, more importantly, without restricting the class of systems that can be addressed. This need is explored in this work. Only recently it was proved that the class of systems that can be stabilised by feedback coincides with the class of asymptotically controllable systems [CLSS97]. However, the proof of existence of a feedback was not constructive. Here, presented for the first time, is a constructive way to obtain a stabilising feedback for the class of uniformly asymptotically controllable nonlinear systems. This constitutes one of the main contributions of this research.

The advantages and potential of MPC in solving engineering control problems renew and further motivate the interest in studying methods to solve open-loop optimal control problems. Prominent among such methods are the necessary conditions of optimality in the form of a Maximum Principle. These methods, most of the time, identify a small set of candidates to minimisers; so that some further elimination, if necessary, can easily identify the desired optimum. Yet, for certain state constrained optimal control problems of interest, the maximum principle can be satisfied with a set of multipliers, called degenerate, which conveys no useful information about the minimisers. This degeneracy phenomenon occurs, for example, when the fixed initial state lies on the boundary of the state constraint set. When addressing state constrained problems using MPC, we have to account for the possibility of hitting the boundary of the state constraint set. This possibility is even likely to happen since in many engineering applications where optimisation is involved, the resources or the safety limits (modelled by state constraints) are, in the optimum, used at their maximum allowed value. In state constrained problems where MPC is used and the boundary of the state constraint is hit, we have to solve an open-loop optimal control problem with the fixed initial state on the boundary of the state constraint set. But this is precisely a case when the traditional versions of the Maximum Principle degenerate, and give us no useful information.

Nondegenerate versions of the Maximum Principle for state constrained optimal control problems are the subject of the second part of this thesis.

The study of these nondegenerate conditions, apart from its relevance to MPC, has a motivation of its own. Necessary conditions of optimality for control problems have been a subject of intense research since the 1960s. Researchers have generalised in several ways the original Pontryagin Maximum Principle to include new features like dynamics modelled by partial differential equations, nonsmooth data, or discontinuous trajectories. Another line of research has concentrated on strengthening the necessary conditions by studying second order conditions, singular problems, and in recent years nondegenerate conditions.

We emphasise the importance attached to nondegenerate conditions by reference to their history in Mathematical Programming ([Aba67, Man69]). Some less known necessary conditions of optimality for problems with inequality constraints are the Fritz John conditions in which there is a nonnegative multiplier associated with the objective function. A weakness in these conditions is that if this multiplier is chosen to be zero, no information is given to help us in selecting the optimum. In 1951 Kuhn and Tucker developed a nondegenerate version of these conditions. Their result states that under a suitable constraint qualification the multiplier associated with the objective function can be chosen to be positive, or simply equal to 1. These nondegenerate conditions are the famous Kuhn-Tucker conditions [KT51], and are the best known optimality conditions for Mathematical Programming problems having inequality constraints. This illustrates the significance of nondegenerate versions of necessary conditions of optimality.

1.2 Overview

This thesis is organised as follows.

In *Chapter 2*, we define the optimal control problem and review some of its most important solution techniques. These techniques will be used extensively in this research.

Part I of the thesis is on Model Predictive Control and comprises *Chapters 3* and *4*. The former gives an algorithm to generate stabilising feedback control for a universal class of nonlinear time-varying systems: the class of the open-loop uniformly asymptotically controllable systems. The algorithm is based on the MPC technique, and the stability results for this technique are extended in several ways. The MPC framework proposed also provides important practical advantages relating to the choice of design parameters and efficiency of the optimisation algorithms.

In *Chapter 4* we develop the previous MPC framework for a more restricted class of nonlinear systems while still considering a large class of systems of practical interest. Here, the aim is to analyse, and to some extent choose, how fast the output decays. We show that the controller can be designed to achieve a prescribed degree of exponential stability. A major feature of this framework is that the fixed-finite horizon open-loop optimal control problem (OCP) to be solved does not have any terminal state constraints imposed. This feature brings about some important advantages: it significantly improves the efficiency of the optimisation algorithms used to implement the strategy, and more importantly the usual hypothesis on the existence of solutions to the OCP (in general difficult to verify in the presence of terminal state constraints) is automatically satisfied.

In *Part II*, comprising *Chapters 5* to *8*, we address the degeneracy phenomenon occurring when traditional versions of the Maximum Principle are applied to control problems with state constraints along the trajectory. *Chapter 5* explains this degeneracy phenomenon, reviews some of the most significant literature, and provides the context and motivation for our contributions.

In *Chapter 6*, we derive nondegenerate necessary conditions of optimality for control problems in which the fixed initial state lies on the boundary of the state constraint set. The study of this case is particularly relevant to MPC.

In *Chapter 7*, we introduce the definition of the particular type of degeneracy associated with pathwise state constraints, the q-degeneracy. We generalise the results of the previous chapter to problems with arbitrarily (closed) initial set, and give results on how to avoid additional cases of q-degeneracy. By combining the conditions (constraint qualifications) under which we can avoid each type of q-degeneracy, we deduce a strengthened form of the necessary conditions of optimality that does not allow any set of q-degenerate multipliers.

Using the nondegeneracy results obtained in the previous chapters we develop in *Chapter 8* necessary conditions of optimality in normal form. These conditions are refinements of our earlier ones and can provide even more information about minimisers in some cases.

In *Chapter 9*, we conclude this thesis by providing a summary of contributions and posing some related open questions to motivate further research.

Finally, we offer in the Appendix a brief review of relevant background material in functional analysis and nonsmooth analysis.

Chapter 2

Optimal Control

> *This problem, is it good to know that it is not, as it may seem, purely speculative and without practical use. Rather it even appears, and this may be hard to believe, that it is very useful also for other branches of science than mechanics*
>
> Johann Bernoulli, 1696, Posing the Brachystochrone Problem

Here we introduce the optimal control problem and the main methods that enable us to identify its solution. These methods form the main set of tools used to derive the results obtained in this thesis. We start by presenting the origins and the classical results in an informal setting. Some recent developments, that will actually be of use in this thesis, are covered in later sections.

2.1 Introduction

Origins: Optimal control theory can be regarded as a generalisation of the Calculus of Variations which was initiated when Bernoulli challenged in 1696 "the sharpest minds of the globe" with the Brachystochrone Problem[1]. The Calculus of Variations has, since then, been built up by numerous famous mathematicians. The most influential contributions, like the ones of Euler, Lagrange, Legendre, Hamilton, Jacobi, and Weierstrass can still be found (in different formats) in modern optimality conditions for control problems, which are identified by the names of their creators. (see e.g. [Bry96, SW97])

In the 1950s Bellman developed the concept of Dynamic Programming which can be

[1] The problem of finding the shape of a curve in the vertical plane such that an object sliding along the curve under the influence of its own weight transverses the distance between two given endpoints in minimum time.

used to solve discrete optimal control problems [Bel57]. Later Kalman [Kal60] solved a problem with linear dynamics and integral quadratic cost function, showing that the optimal control is a linear feedback. This important instance of the optimal control problem was later named the Linear Quadratic Regulator (LQR). Two years later, in 1962, Pontryagin and his collaborators published the monograph [PBGM62] with the first necessary conditions of optimality for nonlinear control problems, known as the Maximum Principle. These conditions were subsequently generalised in many ways, in particular to allow for the possibility of addressing problems with nondifferentiable data. Some of the most influential generalisations were the ones based on Nonsmooth Analysis initiated in the late 70s by Francis Clarke. The main results are compiled in the book [Cla83], which is one of the main references for the work developed here.

Initially, in the 50s and 60s, most of the applications that drove research in Optimal Control came from the aerospace industry. Today the applications covered include a wide range of advanced industrial design problems from process systems to robotics.

The Optimal Control Problem: In the optimal control problem addressed here we are given a dynamical system modelled on some time interval $[a, b] \subset \mathbb{R}_+$ by a set of first order ordinary differential equations together with an initial condition

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(a) = x_0.$$

The evolution of the state $x(t) \in \mathbb{R}^n$ is determined (uniquely, if f satisfies some hypotheses) by the given function f, the given initial state x_0, and a control function $u \in \mathcal{U}([a, b])$ where $\mathcal{U}([a, b])$ is the set of functions on $[a, b]$ whose values are chosen from a given subset $\Omega(t)$ of $\mathbb{R}^m$. The Optimal Control Problem consists of choosing a control function $u \in \mathcal{U}([a, b])$ (thereby defining the trajectory x) in such a way that the pair (x, u) minimises a given performance criterion represented by a functional of the type

$$g(x(b)) + \int_a^b L(t, x(t), u(t))dt.$$

This functional is denoted the *objective function* and it can represent very general costs : running costs like energy/fuel consumption or time spent, and terminal costs like distance from target, among others. The control function u is merely required to be measurable and the set Ω can be defined in very general terms. Our wide freedom to specify the set of possible controls combined with the possibility of dealing with general objective functions gives this formulation the ability to cover a wide range of control engineering problems.

Additional features can be added to this basic formulation to further expand the range of problems that can be addressed. Namely, we can have the initial state x_0 to be chosen from a given set C_0 instead of being fixed *a priori*. We can also consider the free-time problem, where the problem is defined on an interval $[a, a+T]$ and T is a decision variable. Additional constraints can be added to the problem, for instance requiring the final state $x(b)$ to be within a given set C_1, or state constraints along the path like the ones represented by a functional inequality $h(t, x(t)) \leq 0$ which are the object of study in a major part of this thesis. For the moment we will refrain from these complications and proceed to present in an expository manner the optimality conditions for this problem in a smooth setting. First we introduce a necessary condition of optimality in the form of the celebrated Maximum Principle. After that, we give conditions based on dynamic programming. Dynamic programming arguments will lead us to the Hamilton-Jacobi equation, whose existence of a smooth solution provides a sufficient condition for optimality.

In later sections of this chapter we provide a formal specification of the optimal control problem and modern optimality conditions involving much less restrictive hypotheses on the data.

Necessary Conditions of Optimality (NCO): The literature on Necessary conditions for optimal control problems, in the form of a Maximum Principle, is vast. (See e.g. [PBGM62], [War72],[FR75], [IT79], and [Cla83]) Early versions of the Maximum Principle for the basic optimal control problem described above assert (under smoothness hypotheses) that if $(\bar{x}, \bar{u})$ is a minimiser then there exists an absolutely continuous function p satisfying the Euler-Lagrange equation

$$-\dot{p}(t) = p(t) \cdot f_x\left(t, \bar{x}(t), \bar{u}(t)\right) - L_x(t, \bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t \in [a, b],$$

the transversality condition

$$p(b) = -g_x(\bar{x}(b)),$$

and the Maximisation of the Hamiltonian or Weierstrass condition: $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto p(t) \cdot f\left(t, \bar{x}(t), u\right) - L(t, \bar{x}(t), u)$$

for almost every $t \in [a, b]$.

If we now define the (unmaximised) Hamiltonian as

$$H(t, x, p, u) = p \cdot f(t, x, u) - L(t, x, u),$$

the problem of finding an optimal solution can be restated as solving the Hamiltonian system of equations

$$\dot{p}(t) = -H_x(t, \bar{x}(t), p(t), \bar{u}(t)) \tag{2.1}$$

$$\dot{x}(t) = H_p(t, \bar{x}(t), p(t), \bar{u}(t)), \tag{2.2}$$

with boundary conditions

$$x(a) = x_0 \tag{2.3}$$

$$p(b) = -g_x(\bar{x}(b)), \tag{2.4}$$

where $\bar{u}(t)$ maximises over $\Omega(t)$ the function

$$u \mapsto H(t, \bar{x}(t), p(t), u) \tag{2.5}$$

for almost every $t \in [a, b]$.

The problem of finding an actual pair of functions $(\bar{x}, \bar{u})$ satisfying the equations (2.1) – (2.5) above is known as the *two point boundary value problem* and its numerical solution is a well studied subject (see for example [Atk89]).

The proof of the Maximum Principle in its general form is long, but there are several interesting informal interpretations of this result. There are some authors that see it as an extension of the Euler-Lagrange condition in the calculus of variations (see e.g [Loe93]). Some explore the links with the more intuitive Dynamic Programming; the interpretation of [Dre65] in this sense is reproduced below. In a discrete time context some authors apply the Kuhn-Tucker conditions to a reformulation of the OCP as a mathematical programming problem [Var72]. Finally, some explore the interesting geometric interpretation that the adjoint vector p is an outward normal to a hyperplane moving along the optimal trajectory. This hyperplane is the support hyperplane of a convex cone constructed on the basis of the effects of perturbations to the optimal control [AF66]. In linear problems this hyperplane supports the set of reachable states [Var72].

Necessary Conditions of Optimality valid under much weaker hypothesis are provided in a later section.

Sufficient Conditions of Optimality: Having identified a set of candidates for minimisers for our problem (using the NCO or any other, perhaps *ad hoc*, method) we might be interested in verifying if a particular candidate is in fact a minimiser. This is when the sufficient conditions come into play.

Here, we concentrate on global sufficient conditions of optimality. We introduce the concepts involved starting from dynamic programming.

The concept of dynamic programming (see [Bel57]) is based on Bellman's Principle of Optimality which states:

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

The simplicity and power of this principle is striking. Applying the Principle of Optimality to continuous time optimal control problems, we obtain the Hamilton-Jacobi equation, and under some (somewhat strong) assumptions a very simple derivation of the Maximum Principle is possible.

Define the *Value Function* as the infimum cost from a initial pair time/state $(t_0, x(t_0))$

$$V(t_0, x(t_0)) = \inf_{u \in \mathcal{U}([t_0,b])} \left\{ g(x(b)) + \int_{t_0}^{b} L(s, x(s), u(s)) ds \right\}.$$

From the Principle of Optimality, we deduce that for any time subinterval $[t, t+\delta] \subset [a, b]$ $(\delta > 0)$ we have that for a trajectory x corresponding to u

$$V(t, x(t)) = \inf_{u \in \mathcal{U}([t,t+\delta])} \left\{ \int_{t}^{t+\delta} L(s, x(s), u(s)) ds + V(t+\delta, x(t+\delta)) \right\}, \tag{2.6}$$

with the boundary condition

$$V(b, x(b)) = g(x(b)).$$

Assuming the existence of a process $(\bar{x}, \bar{u})$ defined on $[t, t+\delta]$ which is actually a minimiser for the equation (2.6), we can write

$$-V(t, \bar{x}(t)) + \int_{t}^{t+\delta} L(s, \bar{x}(s), \bar{u}(s)) ds + V(t+\delta, \bar{x}(t+\delta)) = 0,$$

and for all pairs (x, u)

$$-V(t, x(t)) + \int_{t}^{t+\delta} L(s, x(s), u(s)) ds + V(t+\delta, x(t+\delta)) \geq 0.$$

Assume that $\bar{u}$ and u are continuous from the right. Suppose also that V is continuously differentiable and L is continuous. Now, add and subtract $V(t+\delta, x(t))$ to the equations above. Dividing by δ, and taking the limit as $\delta \downarrow 0$, we obtain the Hamilton-Jacobi equation (HJE)

$$\begin{aligned} &V_t(t, x(t)) + \min_{u \in \Omega(t)} \{V_x(t, x(t)) \cdot f(t, x(t), u(t)) + L(t, x(t), u(t))\} = 0, \\ &V(b, x(b)) = g(x(b)). \end{aligned} \tag{2.7}$$

This is usually written as

$$V_t(t,x) - \max_{u\in\Omega(t)} H(t,x,-V_x(t,x),u) = 0, \quad V(b,x) = g(x),$$

where, as before,

$$H(t,x,p,u) = p\cdot f(t,x,u) - L(t,x,u)$$

is the (unmaximised) Hamiltonian.

The above analysis relates the Hamilton-Jacobi equation and the Value function. The Hamilton-Jacobi equation also features in the following sufficient condition for a pair $(\bar{x},\bar{u})$ to be a minimiser.

If we can find a continuously differentiable function V such that the Hamilton-Jacobi equation is satisfied and

$$V_t(t,\bar{x}(t)) - H(t,\bar{x}(t),-V_x(t,\bar{x}(t)),\bar{u}(t)) = V_t(t,\bar{x}(t)) - \max_{u\in\Omega(t)} H(t,\bar{x}(t),-V_x(t,\bar{x}(t)),u),$$

then $(\bar{x},\bar{u})$ is a local minimiser.

As we have seen, if the value function is continuously differentiable it is a natural candidate for the function V in the above sufficient condition. The main limitation of this result is that a continuously differentiable function V may not exist. To remedy this, some developments of the Hamilton-Jacobi theory using Nonsmooth Analysis techniques were carried out (see e.g. [AF90, CLSW98, Vin99]).

Even if we do not have just a small set of candidates for minimisers, we can still use the HJE to help us find the solution. In this case we should express the control as a function of the time and state defined by the relationship $u(t,x) = \arg\max_u H(t,x,-V_x,u)$, and solve the partial differential equation (2.7) (see [RV91a]). This has the advantage that we obtain the optimal control already in feedback form. However the analytical solution of the HJE is in general not possible (an exception is the Linear Quadratic Regulator), and a numerical solution is computationally very hard. Thus, in general, only very low dimensional problems can be solved in reasonable time. This constitutes the main practical limitation of the Hamilton-Jacobi / Dynamic-Programming approach.

Assuming that V is C^2, derivation of the conditions of the Maximum Principle is very simple [Dre65]. Let $(\bar{x},\bar{u})$ be a minimiser satisfying the Hamilton-Jacobi equation. Define

$$p(t) = -V_x(t,\bar{x}(t)).$$

Since for free terminal state OCP's the boundary condition can be written as $V(b,x) = g(x)$ for all x in the domain of g, we have that $V_x(b,\bar{x}(b)) = g_x(\bar{x}(b))$. It follows immediately that

$$p(b) = -g_x(\bar{x}(b)).$$

From the HJE, we obtain

$$p(t) \cdot f(t, \bar{x}(t), \bar{u}(t)) - L(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in \Omega(t)} \{ p(t) \cdot f(t, \bar{x}(t), u) - L(t, \bar{x}(t), u) \}.$$

We have deduced the transversality condition and the maximisation of the Hamiltonian condition. It remains to obtain the Euler-Lagrange equation. Since the HJE equals zero for any x, differentiating with respect to x we obtain

$$V_{tx}(t, \bar{x}(t)) + V_{xx}(t, \bar{x}(t)) \cdot f(t, \bar{x}(t), \bar{u}(t)) + V_x(t, \bar{x}(t)) \cdot f_x(t, \bar{x}(t), \bar{u}(t)) + L_x(t, \bar{x}(t), \bar{u}(t)) = 0. \tag{2.8}$$

It follows using (2.8) that the derivative of p is

$$\begin{aligned} \dot{p}(t) &= -\frac{d}{dt} V_x(t, \bar{x}(t)) \\ &= -[V_{tx}(t, \bar{x}(t)) + V_{xx}(t, \bar{x}(t)) \cdot f(t, \bar{x}(t), \bar{u}(t))] \\ &= -[V_x(t, \bar{x}(t)) \cdot f_x(t, \bar{x}(t), \bar{u}(t)) + L_x(t, \bar{x}(t))], \end{aligned}$$

or written equivalently we obtain the Euler-Lagrange equation

$$-\dot{p}(t) = p(t) \cdot f_x(t, \bar{x}(t), \bar{u}(t)) + L_x(t, \bar{x}(t), \bar{u}(t)).$$

2.2 The Optimal Control Problem

We address the fixed time open-loop optimal control problem (OCP), adopting the following formulation:

$$(P) \quad \text{Minimise} \quad g(x(0), x(1)) \tag{2.9}$$

subject to

$$\dot{x}(t) = f\,(t, x(t), u(t)) \quad \text{a.e. } t \in [0,1] \tag{2.10}$$

$$x(0) \in C_0$$

$$x(1) \in C_1$$

$$u(t) \in \Omega(t) \quad \text{a.e. } t \in [0,1]. \tag{2.11}$$

The data of this problem comprise functions $g : \mathbb{R}^n \to \mathbb{R}$, $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the sets $C_0, C_1 \in \mathbb{R}^n$, and a multifunction $\Omega : [0,1] \rightrightarrows \mathbb{R}^m$. The set of *control functions* for (P) is

$$\mathcal{U} := \{u : [0,1] \to \mathbb{R}^m : u \text{ is a measurable function, } u(t) \in \Omega(t) \text{ a.e. } t \in [0,1]\}.$$

A *state trajectory* is an absolutely continuous function which satisfies (2.10) for some control function u. The domain of the above optimisation problem is the set of *admissible processes*, namely pairs (x, u) comprising a control function u and a corresponding state trajectory x which satisfy the constraints of (P). We say that an admissible process $(\bar{x}, \bar{u})$ is a *strong local minimiser* if there exists $\delta > 0$ such that

$$g(\bar{x}(0), \bar{x}(1)) \leq g(x(0), x(1))$$

for all admissible processes (x, u) satisfying

$$\|x(t) - \bar{x}(t)\|_{L^\infty} \leq \delta.$$

In considering the objective function without the integral term we are not sacrificing generality since the Bolza/Lagrange formulation (with integral term) can be transformed into an equivalent Mayer formulation (without integral term) simply by considering an additional state x_{n+1} which is added to the objective function with dynamics $\dot{x}_{n+1}(t) = L(t, x(t), u(t))$. Also, in the case of a fixed time interval we can, without loss of generality, consider the interval $[0, 1]$ as "time" domain for our problem. The free-time problem (for example minimum time problems) can be fitted to this formulation by means of state-augmentation. This transformation is not possible if $t \mapsto f(t, x, u)$ is not Lipschitz (because even using the weakest hypotheses for the NCO given below the function $x \mapsto f(t, x, u)$ is required to be Lipschitz). In this case, we can use NCO with a slightly different form, having an extra condition on the value of the Hamiltonian. (see e.g. [RV91b] for details)

2.3 Existence of Optimal Controls

In the search for an optimal solution, especially when the NCO are the tool of choice, to guarantee the existence of a solution is of prime importance. The logical chain of reasoning in what is called the *deductive method* in optimisation proceeds as follows [Cla89]:

1. A solution to the problem exists.

2. The necessary conditions are applicable, and they identify certain candidates – extrema.

3. Further elimination, if necessary, identifies a solution.

Obviously, if the first step is ignored we might end up selecting an element from the set of candidates given by the NCO when this set does not contain the minimiser we seek.

Among several results on existence of solutions that can be found in the literature, we provide the following well-known result that will be especially useful later in the framework of Model Predictive Control. For a proof we refer to [FR75, Vin87].

Theorem 2.3.1 *Assume that the data of problem (P) satisfy:*

1. *The function g is continuous.*
2. *The set C_0 is compact and C_1 is closed.*
3. *There exists at least one admissible process.*
4. *The function $t \mapsto f(t,x,u)$ is measurable for all (x,u), and $(x,u) \mapsto f(t,x,u)$ is continuous for all $t \in [0,1]$.*
5. *The function $x \mapsto f(t,x,u)$ is globally Lipschitz for all $t \in [0,1]$ and all $u \in \Omega(t)$. (With a Lipschitz constant k not depending on t nor u.)*
6. *The "velocity set" $f(t,x,\Omega(t)) := \{v \in \mathbb{R}^n : v = f(t,x,u), u \in \Omega(t)\}$ is convex for all $(t,x) \in [0,1] \times \mathbb{R}^n$.*

Then there exists an optimal process.

The continuity, compactness, and non-emptiness requirements in the first four conditions are natural in any existence result based on the Weierstrass theorem. The last condition on the convexity of the velocity set has no parallel in finite dimensional optimisation problems, but ensures that the minimum is not achieved by limits of rapid switching controls that do not exist in a conventional sense. The measurability and Lipschitz properties of the velocity function f guarantee that the trajectory is uniquely defined by the control.

For problems with Bolza-type objective function

$$\text{Minimise } g(x(1)) + \int_0^1 L(t,x(t),u(t))\,dt,$$

subject to the same constraints as (P), existence of minimisers is guaranteed if we modify the last three conditions to (see [FR75, Thm. III.4.1])

4'. The functions $t \mapsto f(t,x,u)$ and $t \mapsto L(t,x,u)$ are measurable for all (x,u); and the functions $(x,u) \mapsto f(t,x,u)$ and $(x,u) \mapsto L(t,x,u)$ are continuous for all $t \in [0,1]$.

5'. The functions $x \mapsto f(t,x,u)$ and $x \mapsto L(t,x,u)$ are globally Lipschitz for all $t \in [0,1]$ and all $u \in \Omega(t)$. (With a Lipschitz constant k not depending on t nor u.)

6'. The "extended velocity set" $\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(t, x, u), \ell \geq L(t, x, u), u \in \Omega(t)\}$ is convex for all $(t, x) \in [0, 1] \times \mathbb{R}^n$.

We note that the last condition on the convexity of the "extended velocity set" can be easily established in the cases when L and Ω are convex and f is affine in the control (i.e. $f(t, x, u) = \alpha(t, x) + \beta(t, x) \cdot u$), and also in the case when $f(t, x, \Omega)$ is convex and L does not depend on u.

2.4 The Maximum Principle

We provide here a version of the maximum principle under much weaker hypotheses. In fact the hypotheses under which this problem is treated are the minimum hypothesis in which it makes sense to talk about a control problem [Cla76]. They are denoted here and throughout as the Basic Hypotheses.

Basic Hypotheses For the strong local minimizer $(\bar{x}, \bar{u})$ of interest, the following hypotheses will be invoked. There exists a positive scalar δ' such that:

H1 The function $(t, u) \mapsto f(t, x, u)$ is $\mathcal{L} \times \mathcal{B}$ measurable for each x. ($\mathcal{L} \times \mathcal{B}$ denotes the product σ-algebra generated by the Lebesgue subsets $\mathcal{L}$ of $[0, 1]$ and the Borel subsets of $\mathbb{R}^m$.)

H2 There exists a $\mathcal{L} \times \mathcal{B}$ measurable function $k(t, u)$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|$$

for $x, x' \in \{\bar{x}(t)\} + \delta'\mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0, 1]$.

H3 The function g is Lipschitz continuous on $\{(\bar{x}(0), \bar{x}(1))\} + \delta'\mathbb{B}$.

H4 The end-point constraint sets C_0 and C_1 are closed.

H5 The graph of Ω is $\mathcal{L} \times \mathcal{B}$ measurable.

In the above, $\mathbb{B}$ denotes the closed unit ball, $\mathbb{B} := \{\xi \in \mathbb{R}^n : \|\xi\| \leq 1\}$.

Proposition 2.4.1 (Necessary Conditions of Optimality) *Assume hypotheses H1 – H5. If $(\bar{x}, \bar{u})$ is an optimal process then there exist an absolutely continuous function $p : [0, 1] \to \mathbb{R}^n$, and a scalar $\lambda \geq 0$ such that*

$$||p||_{L_\infty} + \lambda > 0, \tag{2.12}$$

$$-\dot{p}(t) \in \operatorname{co} \partial_x \left(p(t) \cdot f\left(t, \bar{x}(t), \bar{u}(t)\right)\right) \quad a.e.\ t \in [0,1], \tag{2.13}$$

$$(p(0), -p(1)) \in N_{C_0}(\bar{x}(0)) \times N_{C_1}(\bar{x}(1)) + \lambda \partial g(\bar{x}(0), \bar{x}(1)), \tag{2.14}$$

and for almost every $t \in [0,1]$, $\bar{u}(t)$ *maximises over* $\Omega(t)$

$$u \mapsto p(t) \cdot f\left(t, \bar{x}(t), u\right). \tag{2.15}$$

Here, $N_C(x)$ denotes the limiting normal cone to the closed set $C \subset \mathbb{R}^n$ at $x \in C$ defined as

$$\begin{aligned} N_C(x) := \{\lim y_i : \text{there exist } x_i \xrightarrow{C} x,\ \{M_i\} \subset \mathbb{R}_+ \text{ s.t.} \\ y_i \cdot (z - x_i) \le M_i \|z - x_i\|^2\ \forall z \in C,\ \forall i\}, \end{aligned}$$

and $\partial f(x)$ is the limiting subdifferential of lower semicontinuous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ defined as

$$\partial f(x) := \{y : (y, -1) \in N_{\operatorname{epi} f}(x, f(x))\}.$$

Further details of the nonsmooth analysis involved are provided in the appendix. The proposition above is a particular instance of the result for state constrained problems that is given next.

2.5 State Constrained Problems

Consider now the following fixed-time optimal control problem with state constraints.

$$\begin{aligned}
(P) \quad \text{Minimise} \quad & g(x(0), x(1)) && \qquad (2.16)\\
\text{subject to} \quad & && \\
& \dot{x}(t) = f\left(t, x(t), u(t)\right) \quad && \text{a.e. } t \in [0,1] \qquad (2.17)\\
& x(0) \in C_0 && \\
& x(1) \in C_1 && \\
& u(t) \in \Omega(t) && \text{a.e. } t \in [0,1] \\
& h\left(t, x(t)\right) \le 0 && \text{for all } t \in [0,1]. \qquad (2.18)
\end{aligned}$$

The novelty in this problem is the (pathwise) state constraint

$$h\left(t, x(t)\right) \le 0, \quad \text{for all } t \in [0,1],$$

where the function $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following hypothesis.

H6 The function h is upper semicontinuous and there exists a scalar $K_h > 0$ such that the function $x \mapsto h(t,x)$ is Lipschitz of rank K_h for all $t \in [0,1]$.

Proposition 2.5.1 (NCO for State Constrained Problems) *Assume H1 – H6. If $(\bar{x},\bar{u})$ is an optimal process then there exist an arc $p:[0,1]\to \mathbb{R}^n$, a measurable function γ, a nonnegative Radon measure $\mu \in C^*([0,1],\mathbb{R})$, and, a scalar $\lambda \geq 0$ such that*

$$\mu\{[0,1]\} + ||p||_{L_\infty} + \lambda > 0, \tag{2.19}$$

$$-\dot{p}(t) \in \operatorname{co}\partial_x \left(\left(p(t) + \int_{[0,t)} \gamma(s)\mu(ds)\right)\cdot f\left(t,\bar{x}(t),\bar{u}(t)\right)\right) \quad a.e.\ , \tag{2.20}$$

$$\left(p(0), -p(1) - \int_{[0,1]} \gamma(s)\mu(ds)\right) \in N_{C_0}(\bar{x}(0)) \times N_{C_1}(\bar{x}(1)) + \lambda\partial g(\bar{x}(0),\bar{x}(1)), \tag{2.21}$$

$$\gamma(t) \in \partial_x^{>} h(t,\bar{x}(t)) \quad \mu\text{-}a.e.\ , \tag{2.22}$$

$$\operatorname{supp}\{\mu\} \subset \left\{t \in [0,1] : h\left(t,\bar{x}(t)\right) = 0\right\}, \tag{2.23}$$

and for almost every $t \in [0,1]$, $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto \left(p(t) + \int_{[0,t)} \gamma\left(s\right)\mu(ds)\right)\cdot f\left(t,\bar{x}(t),u\right). \tag{2.24}$$

Here, $\partial_x^{>} h(t,x)$ denotes the hybrid partial subdifferential of h in the x-variable defined as

$$\begin{aligned}\partial_x^{>} h(t,x) := \operatorname{co}\{\xi : &\text{ there exists } (t_i,x_i) \to (t,x) \text{ s.t.}\\ &h(t_i,x_i) > 0\,\forall i,\ h(t_i,x_i) \to h(t,x),\ \text{and } \nabla_x h(t_i,x_i) \to \xi\}.\end{aligned}$$

Necessary conditions of optimality for nonsmooth and state constrained control problems were introduced in [VP82]. The particular result given here, and the proof, can be found in [VZ98, Vin99].

Part I

Model Predictive Control

Chapter 3

A Universal Constructor of Stabilising Feedbacks

In this work we present an algorithm to generate stabilising feedback controls for every open-loop (uniformly asymptotically) controllable nonlinear system. The algorithm is based on the Model Predictive Control (MPC) technique which naturally allows the controls to be constrained. The stability results for this technique are extended in several ways. The MPC framework proposed, besides covering important new classes of nonlinear systems and being able to analyse stability of most previous MPC schemes, also brings some important practical advantages. The sufficient stability condition presented makes possible a more flexible choice of the design parameters for the sequence of open-loop optimal control problems involved. This flexibility permits reduction of the terminal constraints traditionally imposed, resulting in optimal control problems that can be solved more efficiently by current optimisation algorithms.

3.1 Introduction

A longstanding open question in nonlinear control theory was whether every asymptotically (open-loop) controllable system could be stabilised by means of some feedback law. This question was recently answered positively by Clarke *et al.* [CLSS97] in the context of time-invariant systems. Some other important questions remained open.

(a) Is it possible to extend their result to time-varying systems?

(b) Is the result still valid if we consider the controllability and stability concepts without the Lyapunov stability condition? That is, does the open-loop attractiveness property on its own imply the closed-loop attractiveness?

And more importantly:

(c) A stabilising feedback exists, but can we find it? Can we devise an algorithm to construct the stabilising feedback for every asymptotically controllable system?

In this work, we address the questions above. We present an algorithm to generate stabilising feedback controls for every uniformly asymptotically controllable nonlinear time-varying system. The concepts of controllability and stability used do not include the usual condition of Lyapunov stability. Rather, they are based only on the asymptotic condition known as attractiveness. We argue that this alternative concept of stability is better suited to the large class of nonlinear systems considered.

There is a vast literature concerning methods to construct feedback controls for nonlinear systems. Most of the literature considers a particular class of nonlinear system and does not allow constrained controls (see e.g. [Rya91, Son89]). The few universal methods available are based on the assumption that we can solve the Hamilton-Jacobi equation (see e.g. [RV91a]). However, except for problems with very small dimension or particular structure we cannot expect to solve the Hamilton-Jacobi partial differential equation in a reasonable time or using a reasonable amount of computer memory.

The method used here to construct stabilising feedbacks for general nonlinear systems is Model Predictive Control (MPC), also known as Receding-Horizon or Moving-Horizon Control. This method obtains the feedback control by solving a sequence of open-loop optimal control problems, each of them using the measured state of the plant as its initial state.

In order to be able to address a general class of nonlinear systems it was important to realize the following fact. A main feature in Clarke *et al.*'s work is the definition of feedback solutions under discontinuous controllers. This is done using "sampling-feedback", a technique there defined. The MPC framework, with a small positive inter-sampling time, used here naturally produces "sampling-feedback" solutions of the dynamic equation, overcoming in this way the inherent difficulty of defining solutions of differential equations with discontinuous feedback.

Other contributions of this work are to MPC itself, both on the theoretical side by providing a general framework to analyse stability, and on the practical side in that the flexibility of the framework allows the use of open-loop optimal control problems that can be solved more efficiently.

The study of MPC stabilising schemes has been the subject of intense research in recent years. For continuous-time nonlinear systems the first stability results, using terminal equality constraints, were developed by Mayne and Michalska in 1990 [MM90]. These were succeeded by other important contributions like [MM93] with the dual-mode approach, [YP93] and [dOM97] using contractive constraints, and more recently [CA98b] with the quasi-infinite horizon. This last work uses the terminal cost as an important ingredient to prove stability. The importance of the terminal cost to guarantee stability was first noticed in [RM93] in a context of linear systems. The use of a terminal cost in the open loop optimal control problem is also of key importance in our approach. Recent surveys on nonlinear model predictive control schemes focusing on stability are [May97], [MRRS98] and [CA98a]. The success of MPC in dealing with some of the difficult control engineering problems can be confirmed in the industrial surveys of [QB97] and [QB98].

Traditionally, MPC schemes with guaranteed stability for nonlinear systems impose conditions on the open loop optimal control problem that either lead to some demanding hypotheses on the system or make the on-line computation of the open loop optimal control very hard. In previous works, these conditions take the form of a terminal state constrained to the origin, or an infinite horizon, or else impose some rather conservative controllability conditions on the system near the origin. These approaches considerably restrict the applicability of the MPC method, not only by narrowing the classes of systems to which it can be applied, but also by making very difficult to verify whether some hypotheses are satisfied for a particular nonlinear system.

Most practitioners of MPC methods know that, by appropriate choice of some parameters of the objective function (obtained by trial-and-error and some empirical rules), it is possible to obtain stabilising trajectories without imposing demanding artificial constraints. However, their achievements can not often be supported by any theoretical result to date, and “playing” with the design parameters is an option criticised by researchers (see e.g. [BGW90]). Here we intend to reduce this gap.

We propose a very general framework of MPC for systems satisfying very mild hypotheses. The *design parameters* of the MPC strategy are chosen in order to satisfy a certain (sufficient) *stability condition*, and hence the resulting closed-loop system will have the desirable stability properties guaranteed.

From a theoretical point of view, we provide a unifying framework for stable MPC schemes. Most MPC schemes can be constructed from our framework and their stability properties deducted from our stability results. But perhaps more important is that with the insight obtained by using the general framework we are able to construct novel MPC schemes capable of dealing with new classes of nonlinear systems.

From a practical point of view, we give a stability condition that can be verified *a priori* (i.e. one not requiring trial-and-error, simulations) and guarantees that a particular set of design parameters will lead to stability. In particular, the design parameters can always be chosen in such a way that the open-loop optimal control problem is a free terminal state one, which leads to more efficient optimisation algorithms to solve this problem. Alternatively, the design parameters can be very easily found, if we impose a mild constraint on the terminal state. Several examples are examined to show how to choose the design parameters in order to satisfy the stability condition. These include some instances of nonholonomic systems for which none of the cited MPC schemes is able to guarantee stability.

3.2 Preliminaries on Discontinuous Feedback and Stability

A common assumption in most previous MPC approaches was the continuity of the controls resulting from the open-loop optimal control problems. This assumption, in addition to being very difficult to verify, was a major obstacle in enabling MPC to address a broader class of nonlinear systems. This is because some nonlinear systems cannot be stabilised by a continuous feedback as was first noticed in [SS80] and [Bro83].

However, if we allow discontinuous feedbacks, it would not be clear what should be the solution (in a classical sense) of the dynamic differential equation. Consider a time-varying feedback control $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$. The classical definition of a trajectory of the system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), k(t, x(t))), \quad t \in \mathbb{R} \\ x(t_0) &= x_0, \end{aligned}$$

depends on certain properties of the function f, as well as on the requirement that the feedback $x \mapsto k(t, x)$ is continuous.

This motivated the development of new concepts of solution to differential equations under a discontinuous feedback. Previous attempts to deal with discontinuous controls in a MPC context are [MV94] using Filippov solutions and approaches avoiding the continuity problem by considering a discrete-time framework (e.g. [MHER95]). Recently, Ryan [Rya94] and Coron and Rosier [CR94] have shown that Filippov solutions cannot lead to stability results for general nonlinear systems. A successful approach that deals with discontinuous feedbacks to stabilise general nonlinear systems is to use the "sampling-feedbacks" of Clarke *et al.*[CLSS97]. In their definition of trajectory, the feedback is not a function of the state on *every* instant of time, rather it is a function of the state at the last sampling instant. But this coincides with the trajectories defined by our MPC framework. Consider a sequence of sampling instants $\pi := \{t_i\}_{i\geq 0}$ with a constant inter-sampling time $\delta > 0$ such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. Let the function $t \mapsto \lfloor t \rfloor_\pi$ give the greatest sampling instant less than or equal to t, that is

$$\lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\}.$$

The π-trajectories of the system under the feedback k are obtained by

$$\dot{x}(t) = f(t, x(t), k(t, x(\lfloor t \rfloor_\pi))), \quad t \in \mathbb{R} \tag{3.1a}$$

$$x(t_0) = x_0. \tag{3.1b}$$

The π-trajectories are, under mild conditions, defined even for discontinuous feedback. We proceed to define stability in this framework.

Definition 3.2.1 The sampling-feedback $k(t, x(t_i)) \in U(t)$ is said to *asymptotically stabilise* the system (3.1) on X_0 if there exists a sufficiently small inter-sample time δ such that the following condition is satisfied. For any $\gamma > 0$ we can find a scalar $M > 0$ such that for any pair $(t_0, x_0) \in \mathbb{R} \times X_0$ we have $\|x(s + t_0; t_0, x_0, k)\| < \gamma$ for $s \geq M$.

Note that this definition does not imply existence of a feedback which makes the system stable in the usual Lyapunov sense. The concept of stabilisability defined above is better suited to many nonlinear systems for which a controller cannot simultaneously satisfy our objective of driving the state to the origin together with the Lyapunov notion of keeping the state arbitrarily close to the origin. A well known example is a car-like vehicle (investigated in section 3.6.2 below). It can be easily seen that even if we are arbitrarily close to our objective, we may have to manoeuvre the vehicle to a certain minimal distance away from our target in order to drive to it. This is what we usually do to park a car sideways.

In order to prove that a particular system can be stabilised we naturally have to assume that there exists at least an open-loop control driving the system to the origin. More precisely, we require *uniformly asymptotically controllability* defined as follows.

Definition 3.2.2 A system is said to be *uniformly asymptotically controllable on* some set $X_0 \subset \mathbb{R}^n$ if for any $\gamma > 0$, there exists a scalar $M > 0$ such that for all pairs $(t_0, x_0) \in \mathbb{R} \times X_0$, we can find a piecewise-continuous control function $u : [t_0, +\infty) \to \mathbb{R}^m$ such that $u(s + t_0) \in U(s + t_0)$ for all $s \geq 0$ and $\|x(s + t_0; t_0, x_0, u)\| \leq \gamma$ for all $s \geq M$.

Note that this is just an attractiveness requirement ($\|x(s+t_0; t_0, x_0, u)\| \to 0$ as $s \to \infty$ uniformly on t_0 and x_0) and, as before, there is no requirement on Lyapunov stability.

3.3 The Model Predictive Control Framework

We shall consider a nonlinear plant with input constraints, where the evolution of the state after time t is predicted by the following model.

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \geq t \tag{3.2a}$$

$$x(t) = x_t \tag{3.2b}$$

$$u(s) \in U(s). \tag{3.2c}$$

The data of this model comprise a set $X_0 \subset \mathbb{R}^n$ containing all possible initial states, a vector $x_t \in X_0$ that is the state of the plant measured at time t, a given function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a multifunction $U : \mathbb{R} \rightrightarrows \mathbb{R}^m$ of possible sets of control values. These data combined with a particular measurable control function $u : [t, +\infty) \to \mathbb{R}^m$ define an absolutely continuous trajectory $x : [t, +\infty) \to \mathbb{R}^n$.

We assume this system to be *uniformly asymptotically controllable on* X_0.

Our objective is to obtain a feedback law that (asymptotically) drives the state of our plant to the origin. This task is accomplished by using a MPC strategy. Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online open-loop optimal control problems $\mathcal{P}(t_i, x_{t_i}, T)$ at each sampling instant t_i, every time using the current measure of the state of the plant x_{t_i}.

$$\mathcal{P}(t, x_t, T) \quad \text{Minimise} \quad \int_t^{t+T} L(s, x(s), u(s))ds + W(t + T, x(t + T)) \tag{3.3}$$

subject to

$$\dot{x}(s) = f(s, x(s), u(s)) \quad \text{a.e. } s \in [t, t + T] \tag{3.4}$$

$$x(t) = x_t$$

$$
\begin{aligned}
& u(s) \in U(s) && \text{a.e. } s \in [t, t+T] \\
& x(t+T) \in S. && (3.5)
\end{aligned}
$$

The domain of this optimisation problem is the set of admissible processes, namely pairs (x, u) comprising a measurable control function u and the corresponding absolutely continuous state trajectory x which satisfy the constraints of $\mathcal{P}(t, x_t, T)$. A process $(\bar{x}, \bar{u})$ is said to solve $\mathcal{P}(t, x_t, T)$ if it globally minimises (3.3) among all admissible processes.

We pause to clarify the notation adopted here. The variable t will represent real time while we reserve s to denote the time variable used in the prediction model. The vector x_t will denote the actual state of the plant measured at time t. The process (x, u) is a pair trajectory/control obtained from the model of the system. The trajectory will sometimes be denoted as $s \mapsto x(s; t, x_t, u)$ when we want to make explicit the dependence on the initial time, initial state, and control function. The pair $(\bar{x}, \bar{u})$ denotes an optimal solution to an open-loop optimal control problem (OCP). The process (x^*, u^*) is the closed-loop trajectory and control resulting from the MPC strategy. We call *design parameters* the variables present in the open-loop optimal control problem that are not from the system model (i.e. variables we are able to choose); these comprise the time horizon T, the running and terminal costs functions L and W, and the terminal constraint set $S \subset \mathbb{R}^n$.

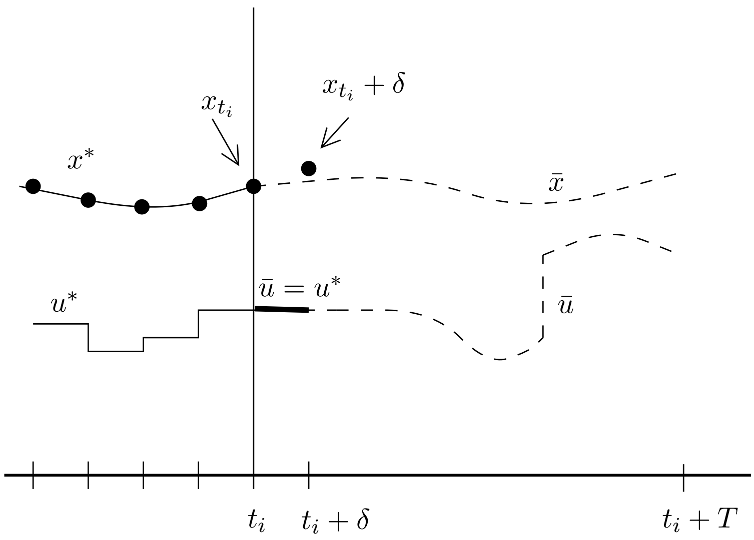

Figure 3.1: The MPC strategy.

The MPC conceptual algorithm consists of performing the following steps at a certain instant t_i (see Fig. 3.1).

1. Measure the current state of the plant x_{t_i}.

2. Compute the open-loop optimal control $\bar{u} : [t_i, t_i + T] \to \mathbb{R}^n$ solution to problem $\mathcal{P}(t_i, x_{t_i}, T)$.

3. The control $u^*(t) := \bar{u}(t)$ in the interval $[t_i, t_i + \delta)$ is applied to the plant, (the remaining control $\bar{u}(t), t \geq t_i + \delta$ is discarded).

4. The procedure is repeated from (1.) for the next sampling instant t_{i+1} (the index i is incremented by one unit).

The resultant control law, called the MP controller, is a feedback control since during each sampling interval, the control u^* is dependent on the state x_{t_i}.

It is a well-known fact that (for fixed finite horizon) the closed-loop trajectory of the system (x^*) does not necessarily coincide with the open-loop trajectory ($\bar{x}$) solution to the OCP. Hence, the fact that MPC will lead to a stabilising closed-loop system is not guaranteed *a priori*, and is highly dependent on the *design parameters* of the MPC strategy.

We will show that we can guarantee stability of the resultant closed loop system, by choosing the design parameters to satisfy a certain *stability condition*. We anticipate here some of the key steps in our result. The crucial element of the stability condition is the requirement that the design parameters are chosen in such away that for all states x belonging to the set of possible terminal states of the OCP (which is, of course, a subset of S), there exists a control value $\tilde{u}$ such that

$$W_x(x) \cdot f(t, x, \tilde{u}) \;\leq\; -L(t, x, \tilde{u}). \tag{3.6}$$

This condition is important in establishing that a certain function V^δ constructed from value functions of the OCP's involved is "decreasing". Then, using Lyapunov-type arguments, we are able to prove the stability of the closed-loop system.

It is interesting to note that Mayne *et al* [MRRS98], in an independent development, identified a condition similar to (3.6) for discrete time systems as a "common ingredient" to most stabilising MPC schemes. Here, we also show how stability of other MPC schemes can be verified with the help of (3.6). But a perhaps more important consequence is that this generalisation enables us to construct new MPC schemes guaranteeing stability for new important classes of nonlinear systems, like for example nonholonomic systems.

In the next section, we show that we can guarantee stability of the resultant closed-loop trajectory for all systems complying with the following hypotheses.

H1 For all $t \in \mathbb{R}^n$ the set $U(t)$ contains the origin, and $f(t, 0, 0) = 0$.

H2 The function f is continuous, and $x \mapsto f(t, x, u)$ is locally Lipschitz continuous for every pair (t, u).

H3 The set $U(t)$ is compact for all t, and for every pair (t, x) the set $f(t, x, U(t))$ is convex.

H4 The function f is compact on compact sets of x, more precisely given any compact set $X \subset \mathbb{R}^n$, the set $\{\, \|f(t, x, u)\| : t \in \mathbb{R}, x \in X, u \in U(t)\,\}$ is compact.

H5 Let $X_0 \subset \mathbb{R}^n$ be a compact set containing all possible initial states. The system (3.2) is uniformly asymptotically controllable on X_0.

Remark 3.3.1 *A controllability hypothesis, such as H5, is inevitable to obtain stability of the closed-loop system. It should be noted however that in other respects, all the hypotheses are expressed directly in terms of the data of the nonlinear model. This has not been the case in most previous MPC literature where assumptions on the feasibility of the OCP or on properties of the value function are standard.*

Hypothesis H1 should not be seen as a restrictive one, since most systems can be made to satisfy it after an appropriate change of coordinates similar to the one discussed in Section 3.5.1.

The assumption H3 (together with H2) is necessary to guarantee the existence of solution to the OCP. Some variations are possible. For example, if we allow relaxed controls (see [War72, You69, Art83]), the convexity of the velocity set $f(t, x, U(t))$ is no longer required.

The condition in H4 is automatically satisfied for time-invariant systems since the image of compact sets under a continuous function is compact.

3.4 Stability Results

In this section, we state two of our results. The main result asserts that the feedback controller resulting from the application of the MPC strategy is a stabilising controller, as long as the design parameters satisfy the stability condition below. The other result states that we can always find design parameters satisfying the stability condition for systems complying with assumptions H1–H5.

The important consequence of these two results, expressed in the corollary below, is that we are able to design a stabilising feedback controller for any nonlinear system belonging to the large class of systems that satisfies the assumptions H1–H5, using the MPC strategy.

Consider the following stability condition SC:

SC For system (3.2) the design parameters: time horizon T, objective functions L and W, and terminal constraint set S, satisfy:

SC1 The set S is closed and contains the origin.

SC2 The function L is continuous, $L(\cdot,0,0)=0$, and there is a continuous positive definite and radially unbounded function $M:\mathbb{R}^n \to \mathbb{R}_+$ such that $L(t,x,u) \geq M(x)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^m$. Moreover, the "extended velocity set" $\{(v,\ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(t,x,u),\ \ell \geq L(t,x,u),\ u \in U(t)\}$ is convex for all (t,x).

SC3 The function W is positive semi-definite and continuously differentiable.

SC4 The time horizon T is such that, the set S is reachable in time T from any initial state and from any point in the generated trajectories: that is, there exists a set X containing X_0 such that for each pair $(t_0,x_0) \in \mathbb{R} \times X$ there exists a control $u:[t_0,t_0+T] \to \mathbb{R}^m$ satisfying

$$x(t_0+T;t_0,x_0,u) \in S.$$

Also, for all control functions u in the conditions above

$$x(t;t_0,x_0,u) \in X \quad \text{for all } t \in [t_0,t_0+T].$$

SC5 There exists a scalar $\epsilon > 0$ such that for every time $t \in [T,\infty)$ and each $x_t \in S$, we can choose a control function $\tilde{u}$ continuous from the right at t satisfying

$$W_t(t,x_t) + W_x(t,x_t) \cdot f(t,x_t,\tilde{u}(t)) \leq -L(t,x_t,\tilde{u}(t)), \tag{SC5a}$$

and

$$x(t+r;t,x_t,\tilde{u}) \in S \quad \text{for all } r \in [0,\epsilon]. \tag{SC5b}$$

The main result on stability is the following.

Theorem 3.4.1 *Assume the system satisfies hypotheses H1–H5. If we choose the design parameters to satisfy SC then the closed loop system resulting from the application of the MPC strategy is asymptotically stable.*

Next, we state a result, that says that it is always possible to choose the design parameters satisfying SC. Furthermore, the result has the appealing consequence that it is always possible to design a stabilising MPC strategy using free terminal state optimal control problems.

Theorem 3.4.2 *Assume H1–H5. Then it is always possible to find design parameters S, T, L, and W to satisfy SC. In particular this choice can be made with $S = \mathbb{R}^n$,(i.e. free end-point problem).*

The proof of these results is supplied in a later section. The following result is an immediate consequence of the two theorems above.

Corollary 3.4.3 *Using MPC we are able to construct an asymptotic stabilising feedback for any nonlinear system complying with hypotheses H1-H5.*

Remark 3.4.4 *The requirement in SC1 that the set S is closed is necessary to guarantee the existence of a solution to the open-loop optimal control problem.*

The first part of condition SC2 and condition SC3 are trivially satisfied for the usual quadratic objective function $L(x, u) = x^T Qx + u^T Ru$, with $Q > 0$ and $R \geq 0$ and $W(x) = x^T Px$, with $P \geq 0$. The second part of SC2 on the convexity of the "extended velocity set" is a well known requirement for existence of solution in OCP with integral cost term. Given H3, it is automatically satisfied if L is convex and f depends linearly on u (i.e. affine systems, see [FR75]) or if L does not depend on u. The latter is a consequence of both $f(t, x, U(t))$ and $\{\ell : \ell \geq L(t, x)\}$ being convex sets.

Condition SC4 is obviously necessary for the existence of solution to the sequence of OCP's.

Condition SC5a is a key requirement for establishing the existence of a decreasing "Lyapunov" function, and thus asymptotic stability. It can be interpreted as the existence of a control $\tilde{u}$ that drives the state towards inner level sets of W at rate L. In the case of quadratic functions W the level sets are ellipsoids centred at the origin and the velocity vector should point inwards. However, W need not to be restricted to quadratic functions, and this freedom can be used in our advantage as we shall see below.

The condition SC5b, which states that the trajectory does not leave the set S immediately, can be written as the easier to verify condition

$$\zeta \cdot f(t, x_t, \tilde{u}(t)) \leq 0 \quad \text{for all } \zeta \in N_S^P(x(t)),$$

where $N_S^P(x)$ denotes the proximal normal cone to the set S at $x \in S$ defined as

$$N_S^P(x) = \{\zeta \in \mathbb{R}^n : \textit{there exists } \sigma \geq 0 \textit{ s.t. } \zeta \cdot (y - x) \leq \sigma \|y - x\|^2,\, y \in S\}.$$

See [CLSW98] for details.

The task of choosing design parameters to satisfy all the conditions of SC might seem formidable at first. But one should not be discouraged by the generality of SC. In fact, the stability condition greatly simplifies for some standard choices of part of the design parameters. Typically, we might choose the objective function to be quadratic (making SC2 and SC3 trivially satisfied); choose the set S to be the whole space $\mathbb{R}^n$ (makes SC4 trivially satisfied); or make SC5 trivially satisfied by choosing S to be the set of points that satisfy SC5. This issue will be explored in a later section where we show, with the help of some examples, how stabilizing design parameters can be easily chosen.

3.5 Tracking and Economic Objectives

Up until now we have considered our objective to be regulating the state to a particular point we called the origin. This approach, adopted for reasons of simplicity might hide the full potential of the MPC framework given here. In fact, using straighforward changes of coordinates, the problem of tracking (i.e. following a specific trajectory) can be dealt with. As we will see, the time-varying capabilities of the framework play an important role, since the tracking problem of even time-invariant systems is converted into a time-varying regulating (i.e. drive to the origin) problem.

A criticism that is often directed at MPC frameworks such as ours, in which the design parameters are chosen to satisfy stability criteria, is that they do not give the designer the freedom to incorporate in the objective function the economic/performance objectives that he ultimately wants to achieve. We argue that this criticism is unjustified in MPC frameworks that are capable of dealing with the tracking problem. The reason is that the economic objectives can be expressed in an objective function of an OCP that is solved initially off-line. The solution to this initial OCP will be a process (y^{ref}, v^{ref}) that maximises our economic objectives. The MPC strategy will then enter at a later stage. The design parameters (including the objective function of the OCP's to be solved on-line) are then chosen to obey stability criteria to ensure that the system is actually tracking the desired reference process (y^{ref}, v^{ref}). This procedure guarantees that the process that maximises our economic criteria is closely followed.

3.5.1 Tracking

Suppose our objective is to make the system

$$\dot{y}(t) = \phi(y(t), v(t)) \quad \text{a.e.} \tag{3.7a}$$

$$y(0) = y_0 \tag{3.7b}$$

$$v(t) \in \Omega(t) \quad \text{a.e.} \tag{3.7c}$$

track a given (feasible) reference trajectory $y^{ref} : [0, +\infty) \to \mathbb{R}^n$. This problem is easily convertible into one of driving to the origin the trajectory x defined as

$$x(t) := y(t) - y^{ref}(t) \quad \text{for all } t.$$

Associated with the reference trajectory y^{ref} we select a reference control function v^{ref} (such that $\dot{y}^{ref}(t) = \phi(y^{ref}(t), v^{ref}(t))$ a.e.) and define a new control function $u : [0, +\infty) \to \mathbb{R}^m$ as

$$u(t) := v(t) - v^{ref}(t).$$

The dynamics of the process are given by

$$\begin{aligned} \dot{x}(t) = f(t, x(t), u(t)) \quad &:= \quad \dot{y}(t) - \dot{y}^{ref}(t) \\ &= \quad \phi(x(t) + y^{ref}(t), u(t) + v^{ref}(t)) - \phi(y^{ref}(t), v^{ref}(t)). \end{aligned}$$

Define also

$$x_0 = y_0 - y^{ref}(0),$$

and

$$U(t) := \{u \in \mathbb{R}^m : u + v^{ref}(t) \in \Omega(t)\}.$$

We have constructed the system (3.2) in such a way that $f(t, 0, 0) = 0$, the origin is in $U(t)$ for all t (hypothesis H1 is satisfied), and our objective is now driving the trajectory x to the origin.

3.5.2 Economic Objectives

Suppose that our plant is still represented by the model (3.7), and that our ultimate objective is to maximise a given economic or performance criteria represented by a functional J of the trajectory and control on some suitable time interval I.

The first step is to solve off-line the following initial optimal control problem (iOCP).

$$
\begin{array}{lll}
(iOCP) & \text{Maximise} & J(y,v) \\
& \text{subject to} & \\
& & \dot{y}(t) = \phi(y(t), v(t)) \qquad \text{a.e. } t \in I \\
& & y(0) = y_0 \\
& & v(t) \in \Omega(t) \qquad \text{a.e. } t \in I.
\end{array}
$$

Next, define the process (y^{ref}, v^{ref}) as a solution to this problem. Finally, apply the MPC strategy to track this reference trajectory as seen in the previous subsection. The stability results of the MPC strategy guarantee that the process (y^{ref}, v^{ref}) that maximises the criterion J is closely followed.

3.6 Choosing Stabilising Design Parameters

We analyse three different strategies for choosing a set of design parameters satisfying the stability condition SC. We start by the natural and easiest choice; setting the objective function to be quadratic and the terminal set as the whole space. This simplified framework works for general linear systems and for some nonlinear systems as is shown in the examples below. The quadratic objective function and, most significantly, the free terminal state of the open-loop optimal control problem make possible a more efficient computation of the optimal controls. Therefore, it might be worth checking, for the particular system we have at hand, whether we can satisfy SC just using this simplified framework. A more thorough discussion of this choice is provided in the next chapter. There, we identify a class of nonlinear systems that can be stabilised with this simplified framework, and explore additional properties that can be established, such as a prescribed degree of exponential stability.

Although Thm. 3.4.2 guarantees the existence of design parameters satisfying SC for $S = \mathbb{R}^n$, it might be difficult to find these parameters for certain nonlinear systems with such a large set S. This task, in particular choosing W to satisfy SC5, can be simplified if we restrict the set S to be just a subset of $\mathbb{R}^n$ containing the origin, for example a linear subspace or a closed ball centred at the origin.

We propose two strategies to define a convenient set S. The first of these strategies is based on physical knowledge of the system. This insight combined with a geometric interpretation of SC5 enable us to immediately identify a set of states where SC5 is satisfied and that can be reached in a known finite time. Some examples, including nonholonomic systems, are presented.

The last method we propose for choosing a set of stabilising design parameters is to define S simply to be the set in which SC5 is satisfied. This procedure, as we will see later, is much less restrictive than the contractive constraint methods. If this choice is made, then some care must be taken to ensure that the choice of T satisfies SC4.

The examples provided below are of academic nature and deliberately simple; there are undoubtedly methods, some ad-hoc, to design adequate stabilising controllers for each of the examples. The purpose of these examples is to illustrate firstly how to choose a convenient set of design parameters. Secondly, the examples illustrate how our MPC framework succeeds in guaranteeing stability, where previous ones fail. Finally, we also illustrate how a single framework can address all the examples in a unified way.

3.6.1 Method A: Quadratic Objective Function and Free Terminal State

Here we set S to be the whole space $S = \mathbb{R}^n$, and the objective function to be a quadratic positive definite function — where $L(x, u) = x^T Qx + u^T Ru$ (with $Q > 0$ and $R \geq 0$), and $W(x) = kx^T Px$ (with k a positive scalar and P a positive definite matrix). If f is affine in u or R is chosen to be zero then conditions SC2 and SC3 are satisfied. If additionally, we set S to be $\mathbb{R}^n$ then SC4 and SC5b are also trivially satisfied and SC reduces to the following simplified version of SC5a.

SC' The positive scalar k and the positive definite symmetric matrix P are such that for each $x \in \mathbb{R}^n$ belonging to the set of possible terminal states of the OCP, we can choose a control value $u \in U$ satisfying

$$2kx^T Pf(x, u) \leq -(x^T Qx + u^T Ru).$$

In this simplified framework, the stability result, Thm. 3.4.1 holds with SC replaced by SC'.

Next we present some examples intended to show how we can easily choose the design parameters in order to satisfy SC', with the appealing property of free terminal state open-loop optimal control problems (i.e. $S = \mathbb{R}^n$). The first one, a linear system, has the purpose of giving the first insight into how SC' can be achieved, and confirms the results of [RM93].

In Example 2, a deliberately simple nonlinear system, we can start to see the power of this frame work. For this system, none of the previously cited MPC methods is able to guarantee stability, since the origin cannot be reached in finite time (equality end-point constraint methods [MM90] fail) and the linearization of the system is uncontrollable at the origin ([MM93], [YP93], [dOM97] and [CA98b] require stabilizability of the linearization of the system at the origin, thus cannot be used).

Example 1: A Linear System Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{a.e. .}$$

We start by choosing $S = \mathbb{R}^n$. The horizon T can then be chosen as an arbitrary positive number. Choose $L(x, u) = x^T Q x + u^T R u$ with Q and R any symmetric positive definite matrices. Select also $W(x) = k x^T P x$ in which the scalar k and the matrix P will be defined below.

From assumption H5 and by the definition of stabilizable linear system there exist a matrix F and a linear feedback control $u = Fx$ such that the closed-loop system $\dot{x}(t) = \mathcal{A}x(t)$ (with $\mathcal{A} = A + BF$) is stable. Since $\dot{x}(t) = \mathcal{A}x(t)$ is stable, the matrix P can be chosen as a positive definite solution to the Lyapunov equation

$$P\mathcal{A} + \mathcal{A}^T P = -I.$$

This last expression can be written as

$$2x^T P\mathcal{A}x = -\|x\|^2.$$

Now if we choose k satisfying

$$k \geq \lambda_{max}(Q + F^T R F),$$

we have that

$$2kx^T P\mathcal{A}x = -k\|x\|^2 \leq -x^T(Q + F^T R F)x,$$

or written equivalently as

$$2kx^T P\mathcal{A}x \leq -x^T Q x + u^T R u,$$

we obtain SC', guaranteeing stability of the closed-loop system for this choice of parameters.

An alternative way, without having to find the matrix F *a priori*, would be to set

$$F = -R^{-1} B^T P.$$

The condition SC' would then be

$$2kx^T P(A + BF)x \leq -x^T(Q + F^T RF)x$$

or equivalently

$$2kx^T(PA - PBR^{-1}B^T P)x \leq -x^T(Q + PBR^{-1}B^T P)x.$$

Selecting $k = 1$ we get

$$x^T(2PA - PBR^{-1}B^T P + Q)x \leq 0. \tag{3.8}$$

We can easily see that for instance choosing $P > 0$ satisfying the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0,$$

we satisfy (3.8) as equality confirming previous stability results on the infinite horizon MPC for linear systems and the results in [RM93]. However, here we have a much wider choice for the design parameters, being able to guarantee stability under weaker conditions. This freedom, in particular the possibility of choosing $k > 1$ will be explored in the next chapter.

Example 2: A simple nonlinear system Consider the nonlinear system with control constraints:

$$\dot{x}(t) = x(t) \cdot u(t), \qquad u(t) \in [-1, 1].$$

We can easily see that it is impossible to drive the state to the origin in a finite time, and that the linearization of the system is uncontrollable at the origin. Hence, trying to satisfy a terminal-state constraint such as is required in the classical stability results for MPC would fail. Also, the dual-mode, contractive constraint, or the quasi-infinite approach cannot be used since they require stabilizability of the linearised system at the origin.

Despite that, we can easily find design parameters such that SC' is satisfied for this system. For example, if we set the design parameters

$$Q = I_n, \quad R = 0, \quad P = I_n, \text{ and } k = \frac{1}{2},$$

then, there exist a control $u = -1$ such that the stability condition SC' is satisfied:

$$\begin{aligned} 2kx^T Pf(x, u) &= -x^T Px = -\|x\|^2 \\ &= -x^T Qx. \end{aligned}$$

Thus, this simple choice of design parameters guarantees closed-loop stability.

3.6.2 Method B: Set S Chosen Using Physical Knowledge of the System

Recall the geometric interpretation of SC5a given in Remark 3.4.4. Of course, for some systems whose linearization around the origin gives a stabilizable system we might be able to choose S to be a neighbourhood of the origin, and W to be a Lyapunov function for the linearised system, using some convenient stabilising controller. This is the approach followed in [MM93] and [CA98b]. One of the powerful features of our framework is the ability to choose different types of terminal-set, and thus be able to tackle a large class of systems whose linearization is not stabilizable, including some interesting instances of nonholonomic systems.

Example 3: A nonlinear system Consider the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + 2x_1(t)u(t), \end{cases}$$

with the control constraint

$$u(t) \in [0,1] \quad \text{a.e. } t.$$

This system cannot be driven to the origin in finite time, hence MPC schemes having terminal state constrained to the origin do not apply. Moreover, linearising around the origin we obtain the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x,$$

having poles over the imaginary axis at $\{-j, j\}$. Hence it is not stabilizable, and all the other cited MPC schemes fail to guarantee closed loop stability.

However, our framework enables us to almost trivially find the feedback control for this system. Simply notice that in the subspace

$$S := \{(x_1, x_2) : x_1 = -x_2\}$$

the control $u = 1$ drives the system towards the origin through S. Furthermore, this subspace can be reached with a well-determined finite horizon (see Fig. 3.2)

$$T = 2\pi,$$

because choosing $u = 0$, the trajectories are

$$\begin{cases} x_1(t) = x_1(0)\sin(t) \\ x_2(t) = x_2(0)\cos(t). \end{cases}$$

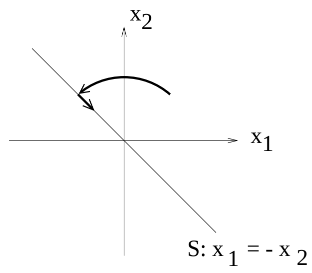

Figure 3.2: Example 3: Reaching set S.

Hence the trajectories will certainly meet at S within time 2π.

As to the objective function, the simple choice

$$L(x) = \|x\|^2 \text{ and } W(x) = \|x\|^2$$

is able to satisfy SC5 since

$$\dot{W} = 2x^T \cdot f(x, 1) = -4x_1^2 \;\leq\; -L(x) = -2x_1^2,$$

and

$$x(t + r, t, x_t, 1) \in S \quad \text{for all } r \geq 0,$$

if $x_t \in S$.

It follows from our main stability result that this choice of design parameters guarantees the stability of the closed-loop trajectory.

From this example we may conclude that using the terminal set to be a neighbourhood of the origin (as done in most MPC schemes) is clearly not the best choice for some systems, namely the important class of nonholonomic systems. This same conclusion can also be drawn from the next example.

Example 4: A car-like vehicle Consider the car-like vehicle of Fig. 3.3, steered by two front directional wheels, represented by the following model.

$$\begin{cases} \dot{x} = v \cdot \cos\theta \\ \dot{y} = v \cdot \sin\theta \\ \dot{\theta} = v \cdot c \end{cases}$$

where the control inputs v and c satisfy

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

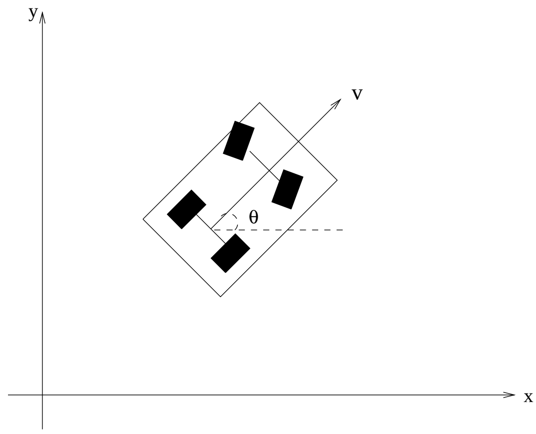

Figure 3.3: A car-like vehicle.

Here (x, y) represents the location in the plane of a point in the car (the mid-point of the axle between the two rear wheels), and θ the angle of the car body with the x axis. The control v represents the linear velocity and c the curvature which is the inverse of the turning radius. It should be noted that the vehicle has a minimum turning radius ($R_{min} = c_{max}^{-1}$).

Our objective is to find a feedback controller to drive the vehicle to the origin ($x = y = 0$ and also $\theta = 0$). This objective cannot be achieved by any of the MPC methods cited: firstly because the linearization of the system around the origin is not stabilizable; secondly because the system cannot be stabilised by a continuous feedback, since it is a nonholonomic system. (This last issue makes the stabilisation of this system challenging; see [Ast96] for a discussion of this point and [Ast95] for a non MPC controller for a car-like vehicle.)

To define S we should look for possible trajectories approaching the origin. One possibility for such a set is represented in Fig. 3.4.

We define S to be the union of all semi-circles with radius greater than or equal to R_{min}, with centre lying on the y axis, passing through the origin, and lying in the left half-plane. In order to make this set reachable in finite time from any point in the space, we add the set of trajectories that are horizontal lines of distance more than $2R_{min}$ from the x axis, and lie in the right half-plane. More precisely

$$S := S_1 \cup S_2 \cup S_3$$

where

$$S_1 \quad := \quad \{(x, 0, 0) : x \leq 0\}$$

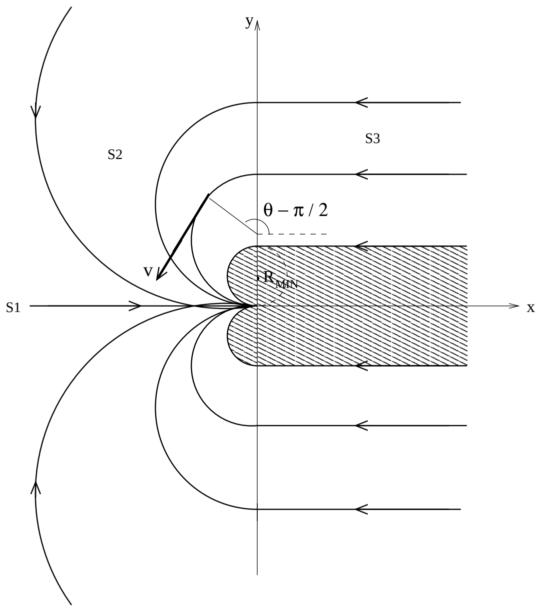

Figure 3.4: Set S of trajectories approaching the origin.

$$
\begin{aligned}
S_2 \; &:= \; \Big\{(x,y,\theta) \in \mathbb{R}^2 \times [-\pi,\pi] : x \leq 0,\; x^2 + (y-r)^2 = r^2,\; r \geq R_{min}, \\
&\quad \tan\left(\theta - \frac{\pi}{2}\right) = \frac{y-r}{x}\Big\} \\
S_3 \; &:= \; \{(x,y,\theta) \in \mathbb{R}^2 \times [-\pi,\pi] : x > 0,\; |y| \geq 2R_{min},\; \theta = \pi\}
\end{aligned}
$$

The set S as defined above can be reached in a well-determined finite time-horizon. This horizon is the time to complete a circle of minimum radius at maximum velocity, that is

$$T = 2\pi R_{min}/v_{max}.$$

Conditions SC1 and SC4 are satisfied.

Choose L as

$$L(x,y,\theta) := x^2 + y^2 + \theta^2,$$

and W as

$$W(x_0,y_0,\theta_0) := \int_0^{\bar{t}} L(x(t),y(t),\theta(t))dt$$

where $\bar{t}$ is the time to reach the origin with the controls chosen to be the maximum velocity and the curvatures depicted in Fig. 3.4. Of course, if we are already at the origin the value chosen for the velocity is zero. That is

$$v_{x,y,\theta} = \begin{cases} v_{max} & \text{if } (x,y,\theta) \neq 0 \\ 0 & \text{if } (x,y,\theta) = 0, \end{cases}$$

and

$$c_{x,y,\theta} = \begin{cases} 0 & \text{if } (x,y,\theta) \in S_1 \cup S_3 \\ -\text{sign}(\theta)/r & \text{if } (x,y,\theta) \in S_2 \cup \bar{S}, \end{cases}$$

where

$$r = \frac{x^2+y^2}{2|y|}.$$

An explicit formula for W is, as is derived below

$$W(x,y,\theta) = \begin{cases} \dfrac{-x^3}{3v_{max}} & \text{if } (x,y,\theta) \in S_1 \\ \dfrac{r}{3v_{max}}[6r^2\theta + \theta^3 - 6rx + 3\theta x^2 + 3\theta y^2 & \\ \qquad +6r\,(x-\theta y)\cos(\theta) + 6r\,(-r+\theta x + y)\sin(\theta)] & \text{if } (x,y,\theta) \in S_2 \\ \dfrac{r}{3v_{max}}\left[x^2 + 3\pi^2 x + r\pi^3 + 30\pi r^2\right] & \text{if } (x,y,\theta) \in S_3. \end{cases}$$

We can easily see that SC2 and SC3 are satisfied and as we show below SC5 also is fulfilled.

It follows from our main stability result that this choice of design parameters guarantees the stability of the closed-loop trajectory.

We proceed to a detailed verification of SC5. We verify this condition separately for each of the subsets S_1, S_2, and S_3. Starting by S_1, we choose the controls

$$\begin{cases} v = v_{max} \\ c = 0 \end{cases}$$

from which follows immediately that

$$f(x,y,\theta) = \begin{cases} \dot{x} = v_{max} \\ \dot{y} = 0 \\ \dot{\theta} = 0, \end{cases}$$

the trajectories are

$$\begin{cases} x(t) = x_0 + v_{max}t \\ y(t) = 0 \\ \theta(t) = 0. \end{cases}$$

and the time to reach the origin is

$$\bar{t} = -x_0/v_{max},$$

satisfying the second part of SC5, since starting in S_1 with these controls we remain inside this set.

Expanding W we obtain

$$W(x_0, y_0, \theta_0) = \int_0^{-x_0/v_{max}} x^2(t)\, dt = \frac{-x_0^3}{3v_{max}},$$

and

$$\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 \leq -L(x, y, \theta),$$

satisfying SC5a.

At S_2 we choose the controls

$$\begin{cases} v = v_{max} \\ c = -\text{sign}(\theta)/r, \end{cases}$$

We analyse the case in which θ is positive, the remaining case can be analysed in a similar way. It follows that

$$f(x, y, \theta) = \begin{cases} \dot{\theta}(t) = -v_{max}/r \\ \dot{x} = v_{max} \cos \theta(t) \\ \dot{y} = v_{max} \sin \theta(t), \end{cases}$$

the trajectories are

$$\begin{cases} \theta(t) = \theta_0 - v_{max} t/r \\ x(t) = x_0 + r \sin \theta_0 - r \sin(\theta_0 - v_{max} t/r) \\ y(t) = x_0 - r \cos \theta_0 + r \cos(\theta_0 - v_{max} t/r), \end{cases}$$

and the time to reach the origin is

$$\bar{t} = \theta_0 r/v_{max},$$

satisfying the second part of SC5, since starting in S_2 with these controls, we remain inside S_2.

Expanding W we obtain

$$\begin{aligned} W(x, y, \theta) &= \int_0^{\theta_0 r/v_{max}} [x^2(t) + y^2(t) + \theta^2(t)]\, dt & (3.9) \\ &= \frac{r}{3v_{max}} \left[6r^2\theta + \theta^3 - 6rx + 3\theta x^2 + 3\theta y^2 \right. \\ &\qquad \left. + 6r\,(x - \theta y)\cos(\theta) + 6r\,(-r + \theta x + y)\sin(\theta)\right] & (3.10) \end{aligned}$$

and

$$\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 - y^2(t) - \theta^2(t) \leq -L(x, y, \theta),$$

satisfying SC5a.

Finally, if we are in S_3, we choose the controls

$$\begin{cases} v = v_{max} \\ c = 0 \end{cases}$$

from which follows immediately that

$$f(x, y, \theta) = \begin{cases} \dot{x} = -v_{max} \\ \dot{y} = 0 \\ \dot{\theta} = 0, \end{cases}$$

the trajectories are

$$\begin{cases} x(t) = x_0 - v_{max}t \\ y(t) = y_0 \\ \theta(t) = \pi, \end{cases}$$

and the time to reach the y axis is

$$\tilde{t} = x_0/v_{max},$$

satisfying the second part of SC5, since starting in S_3 with these controls we remain inside it for some interval of time.

Expanding W we obtain

$$W(x_0, y_0, \theta_0) = \int_0^{\tilde{t}} x^2(t) + \pi^2\, dt + W(0, 2r, \pi) = \frac{r}{3v_{max}} \left[x_0^2 + 3\pi^2 x_0 + r\pi^3 + 30\pi r^2\right],$$

where $W(0, 2r, \pi)$ is given by the expression of W in S_2 (3.10), when the state is on the y axis. Finally, we confirm SC5a since

$$\nabla W(x, y, \theta) \cdot f(x, y, \theta) = -x^2 \leq -L(x, y, \theta).$$

3.6.3 Method C: Set S Defined as Satisfying SC5

It might happen that we are faced with a nonlinear system whose complexity does not allows us to easily use the methodologies described above to choose the design parameters. Alternatively we may want to build a MPC package that works for several nonlinear systems with minimal intervention by the user, who might not necessarily have the knowledge to perform an analysis similar to the one above. In this case we define the terminal set to be the set of states that satisfy SC5, eliminating one major decision: to select the design parameter S.

One possible approach is to choose L and W as some functions satisfying SC2 and SC3 (for example quadratic). Define the function h as

$$h(t,x,u) = W_x(x) \cdot f(t,x,u) + L(t,x,u),$$

and solve the following optimal control problem for some small scalar $r > 0$

$$\begin{array}{lll} \mathcal{P}_r(s,x_s) & \text{Minimise} & \displaystyle\int_s^{s+T} L(t,x(t),u(t))dt + W(x(s+T)) \\ & \text{subject to} & \\ & & \dot{x}(t) = f(t,x(t),u(t)) \qquad \text{a.e. } t \in [s, s+T+r] \\ & & x(s) = x_s \\ & & u(t) \in U(t) \qquad \text{a.e. } t \in [s, s+T+r] \\ & & h(t,x(t),u(t)) \leq 0 \qquad \text{for all } t \in [s+T, s+T+r]. \end{array} \quad (3.11)$$

Note that the satisfaction of constraint (3.11) implies that SC5 holds. The first part of SC5 is guaranteed by $h(t,x(t),u(t)) \leq 0$ for $t = s+T$, and the second part by $h(t,x(t),u(t)) \leq 0$ for $t > s+T$.

Only condition SC4 is not yet guaranteed. But if SC4 does not hold, this is immediately evident when we try to solve $\mathcal{P}_r(s,x_s)$ because it would not have a feasible solution.

However, we still have one parameter to play with — the horizon T. Of course, we can try to analyse the feasibility of problem $\mathcal{P}_r(s,x_s)$ to try to find a suitable horizon; but if we want an analysis-free framework/package, with minimal user intervention, the parameter T can always be chosen by trial-and-error until $\mathcal{P}_r(s,x_s)$ is feasible for a significant set of possible initial states.

We should note that in this last situation, stability is only guaranteed if $\mathcal{P}_r(s,x_s)$ is feasible for *all* possible initial states. However, all other MPC methods with fixed-horizon and guaranteed stability also impose a terminal constraint that in some cases can make the required feasibility of the OCP difficult to verify. The advantage of our method is that the terminal constraint used, (SC5) or (3.11), is less restrictive than in other MPC approaches, increasing the possibilities of feasibility. This point is discussed in detail in the next section.

The described method, method C, can be used for all systems that can be addressed by methods A and B, provided we choose T, L and W in the same way.

3.7 Comparison with alternative MPC approaches

In this section we show how most of the previous approaches can be seen as particular cases of the general framework proposed, and the prominent role of the stability conditions, mainly SC5, for the stabilising properties of the MPC strategy.

Terminal state constrained to the origin This is also known as the classical approach. The stabilising properties for continuous-time nonlinear systems were first proved in [MM90]. To cover this approach by our framework we would choose

$$
\begin{aligned}
&L(x,u) = x^T Q x + u^T R u,\\
&W = 0,\\
&S = \{0\},
\end{aligned}
$$

The stabilising properties of this approach can be confirmed by using our stability conditions. Condition SC5 is satisfied since $L(0,\cdot) = 0 = \dot{W}$, and so stability is guaranteed provided that S is reachable in time T. The main drawback of this approach is precisely the terminal state constraint, because the assumption on the existence of an admissible solution to the open loop optimal control problem is not always easy to verify, and also because of the difficulty of computing an exact solution to the constrained optimal control problem on-line.

Infinite horizon This approach was mainly discussed in the context of linear systems (see e.g. [BGW90]). We have

$$
\begin{aligned}
&L(x,u) = x^T Q x + u^T R u,\\
&W = 0,\\
&S = \mathbb{R}^n,\\
&T \to \infty.
\end{aligned}
$$

Here, the assumption on the existence of an admissible solution implies a finite cost solution, therefore we must have $\lim_{T\to\infty} x(T) = 0$. We would then obtain an equivalent open loop optimal control problem if we set $S = \{0\}$, and SC would also be satisfied as in the previous case, confirming the stabilising properties of the approach. The disadvantage of this approach is that (apart from the problem concerning existence of solutions to the OCP) computing the solution to an infinite horizon nonlinear (or constrained linear) optimal control problem is very hard, which limits the applicability of this method.

Dual mode approach This approach was first described in [MM93]. In this case, outside a neighbourhood S of the origin, we have to solve the open loop optimal control problem with

$$\begin{aligned}
&L(x,u) = x^T Q x + u^T R u, \\
&W = 0, \\
&S = \epsilon \mathrm{B} \\
&T \text{ free}
\end{aligned}$$

After the set S is reached we switch to a linear stabilising feedback controller for the linearised system.

Before we reach S, we have a free time problem. It follows therefore from the Bellman principle of optimality, that the trajectory resulting from the MPC strategy coincides with the open loop trajectory of the solution to the optimal control problem. This has two important consequences: firstly it is certain that the closed loop trajectory reaches the set S and secondly, in terms of performance, the MPC trajectory is actually the one that minimises the objective function of the optimal control problem.

Naturally, we would have to define S in such a way that the linear feedback controller stabilises the actual system, which may not be easy if we have strong nonlinearities, or even impossible if the linearization of the system is not stabilizable. The other disadvantage is that having to solve free-time optimal control problems leads to less efficient computational procedures since apart from having to deal with an extra decision variable, the solution may lead to a very long horizon.

Despite the above disadvantages, this approach was the most promising until recently. The quasi-infinite approach discussed next is an evolution of this concept that does not require switching between controllers. But we can easily see that our general framework subsumes this particular approach since satisfying SC5 within a small ball S centred at the origin is a much weaker requirement than being able to find a stabilising linear feedback controller (see Example 1.)

Quasi-infinite horizon The quasi-infinite horizon approach that was recently developed by Chen and Allgower [CA98b] also uses a terminal cost in the objective function as a key element to achieve stability. Their approach has a very intuitive explanation that might also help to give some insight in our analytical results. The central idea is to choose the terminal cost such that it exceeds the running cost till infinity,

$$W(x(t+T)) \geq \int_{t+T}^{\infty} L(x(s), u(s))\, ds. \tag{3.12}$$

This would imply that the value function for this problem is greater than the value function for the infinite horizon problem. Thus – using the value function as a Lyapunov function – stability can be easily proved. As determining W satisfying the above inequality might be difficult in general, W is computed in some neighbourhood of the origin (which we choose to be the terminal constraint set S), on which the controller is a stabilising linear feedback for the linearization of the system around the origin.

Condition (3.12) is closely related to our condition in SC5a. Let $(\tilde{x}, \tilde{u})$ be a stabilising process defined in $[t_f, \infty)$, and

$$W(x(t_f)) = \int_{t_f}^{\infty} L(\tilde{x}(s), \tilde{u}(s))\, ds.$$

Differentiating with respect to time, we obtain

$$\dot{W}(x(t_f)) = -L(\tilde{x}(t_f), \tilde{u}(t_f)),$$

which yields our condition SC5a.

There are two main disadvantages in this quasi-infinite approach. Firstly, it might be difficult to find a finite horizon T such that the chosen neighbourhood of the origin S is reachable from any possible initial condition. Secondly, if the linearization of the system around the origin is not stabilizable then this approach cannot be used.

Contraction constraints This is the approach followed in [YP93] and [dOM97]. In this approach a constraint is introduced into the OCP which requires that

$$w(x(t+T)) \leq \alpha w(x(t)).$$

Here $\alpha \in (0, 1)$ and $w(x) = x^T P x$ for some positive definite matrix P. Moreover, there is an assumption that there exist a scalar $\beta \in [1, \infty)$ such that

$$\sup_{s \in [t, t+T]} \|\bar{x}(s)\| \leq \beta \|\bar{x}(t)\| \quad \text{for all } t.$$

With these conditions it might be very difficult to guarantee *a priori* feasibility of the OCP for a fixed horizon T (see [May97]). We argue that our condition SC5a is weaker since, as was explained in Remark 3.4.4, ours is a local and pointwise condition merely requiring the state to be driven towards inner level sets of W. The contractive constraint requires the terminal state to be in an inner ellipsoid to the one that passes through the initial state. That is, some components of the state have to decrease and this requirement is global, hence to be met it certainly had to be satisfied locally in some interval of time of non-zero measure. More precisely, the contractive constraint can be written as

$$w(x(t+T)) - w(x(t)) \leq -(1-\alpha)w(x(t))$$

or equivalently

$$\int_t^{t+T} \dot{w}(x(s))ds \leq -(1-\alpha)w(x(t)).$$

The interpretation is that $\dot{w}$ has to be lower than a negative quadratic term in an interval of time with non-zero measure. This will of course be recognised as a stronger requirement than our condition SC5a: when we choose W and L to be a positive quadratic terms our condition merely requires that $\dot{W}$ is less than a negative quadratic term at a specific point, namely the final state.

3.8 Proof of the results

3.8.1 Existence of Design Parameters satisfying SC (Thm.3.4.2)

Set $S = \mathbb{R}^n$ and T an arbitrary positive number. The conditions SC1, SC4 (with $X = \mathbb{R}^n$) and SC5b are trivially satisfied.

For any pair $(t_0, x_0) \in \mathbb{R} \times X_0$ define v_{t_0,x_0} to be a right-continuous control function that drives $x(t; t_0, x_0, v_{t_0,x_0})$ asymptotically to the origin and satisfies

$$v_{t_1,x_1}(t) = v_{t_0,x_0}(t) \quad \text{for all } t \in [t_1, \infty)$$

if $x_1 = x(t_1; t_0, x_0, v_{t_0,x_0})$. (i.e. the choices of control functions are consistent with previous ones). From H5 these control functions exist. The right-continuity of the controls does not cause any difficulty since the existence of a piecewise-continuous stabilising control u implies the existence of a right-continuous one v, that is also stabilising. To see this, consider the sequence $\{t_i\}$ of all points of discontinuity of u. Let $v(t_i) = \lim_{t \downarrow t_i} u(t)$ for all i, and $v(t) = u(t)$ on all remaining points. As the points of discontinuity form a set of null Lebesgue measure, $v(t) = u(t)$ a.e. , and therefore the trajectories corresponding to each of these controls coincide.

The construction of L and W to satisfy the remaining conditions of SC has similarities with the proof of converse Lyapunov theorems. We will borrow a definition and couple of lemmas from this area. See e.g. [Vid93, Chap. 5].

A function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of *class L* if it is continuous, strictly decreasing, bounded and $\sigma(r) \to 0$ as $r \to \infty$.

Lemma 3.8.1 *If $\|x(s+t_0; t_0, x_0, v_{t_0,x_0})\| \to 0$ as $s \to \infty$ uniformly on t_0 and x_0, then there exists a function σ of class L such that*

$$\|x(s+t_0; t_0, x_0, v_{t_0,x_0})\| \leq \sigma(s) \quad \textit{for all } s \geq 0 \textit{ and all } x_0 \in X_0.$$

Lemma 3.8.2 (Massera) *Let σ be a given function of class L. There exist a C^1, positive definite, strictly increasing and radially unbounded function γ such that*

$$\int_0^\infty \gamma(\sigma(r))\, dr < \infty.$$

An attentive reader might note that the radially unbounded property is not part of the usual Massera's lemma. However, the modification is trivial. Pick a point $r' > 0$ and define the function $\gamma'(y)$ to be equal to $\gamma(y)$ for $y \leq \sigma(r')$ and some radially unbounded C^1 extension of γ for $y > \sigma(r')$ (say, $\gamma'(y) = \gamma(y) + (y - \sigma(r'))^2$ for $y > \sigma(r')$). Obviously, if γ satisfies Massera's Lemma then γ' will satisfy the lemma above since

$$\int_0^\infty \gamma'(\sigma(r))\, dr = \int_0^{r'} \gamma'(\sigma(r))\, dr + \int_{r'}^\infty \gamma(\sigma(r))\, dr < \infty.$$

Let $L(\cdot, x, \cdot) := \gamma(\|x\|)$. By the previous lemmas SC2 is satisfied. Define

$$W(t_0, x_0) := \int_{t_0}^\infty \gamma(\|x(s; t_0, x_0, v_{t_0,x_0})\|) ds.$$

Since γ is an increasing function we have that

$$W(t_0, x_0) \leq \int_{t_0}^\infty \gamma(\sigma(s - t_0)) ds < \infty.$$

The time-derivative of W is given by

$$\begin{aligned}
\frac{d}{dt} W(t, x(t; t_0, x_0, v_{t_0,x_0})) &= \frac{d}{dt} \int_t^\infty \gamma(\|x(s; t, x(t; t_0, x_0, v_{t_0,x_0}), v_{t,x(t)})\|) ds \\
&= \frac{d}{dt} \int_t^\infty \gamma(\|x(s; t_0, x_0, v_{t_0,x_0})\|) ds \\
&= -\gamma(\|x(t; t_0, x_0, v_{t_0,x_0})\|) \\
&= -L(\cdot, x(t), \cdot).
\end{aligned}$$

This implies condition SC5a, since the controls were chosen to be right-continuous. As L is continuous, $\frac{d}{dt} W$ is also continuous. As W is clearly positive semidefinite, SC3 is satisfied as well. All conditions of SC are satisfied.

3.8.2 Existence of Solutions to the OCP's

Proposition 3.8.3 (Existence of Solution) *Assume hypotheses H1–H5. Assume also that the design parameters satisfy SC. Then for any* $(t_0, x_0) \in \mathbb{R} \times X_0$ *a solution to the open loop optimal control problem* $\mathcal{P}(t_0, x_0, T)$ *exists.*

Consider the sequence of pairs $\{t_i, x_i\}$ *such that* x_i *is the value at instant* t_i *of a trajectory solving* $\mathcal{P}(t_{i-1}, x_{i-1}, T)$*, then a solution to* $\mathcal{P}(t_i, x_i, T)$ *for all* $i \geq 1$ *also exists.*

Moreover, the trajectory x^* *constructed by MPC has no finite escape times.*

Proof.

Consider first $\mathcal{P}(t_0, x_0, T)$. Noticing that f and the objective function are continuous, $x \mapsto f(t, x, u)$ is Lipschitz, U is compact and non-empty, the "extended velocity set" is convex, the terminal set S is closed and nonempty, and SC4 guarantees the existence of an admissible process, we are in conditions to apply a well-known existence result on solution to OCP's (see e.g. [FR75]). The first assertion follows.

Assume that the solution to $\mathcal{P}(t_{i-1}, x_{i-1}, T)$ exists and that $x_{i-1} \in X$ (X defined as in SC4), pick a pair (t_i, x_i) from the trajectory solving this latter problem. Then from SC4, we have that $x_i \in X$, and we also satisfy all the conditions for $\mathcal{P}(t_i, x_i, T)$ to have existence of solution guaranteed. The second assertion follows by induction.

It remains to prove the third assertion. Notice that implicit in the existence of solution, we have that if $\bar{x}$ is a trajectory from a solution to $\mathcal{P}(t_i, x_i, T)$ then

$$\|\bar{x}\|_{L^\infty[t_i, t_i+T]} < \infty \quad \text{for all } i \geq 0.$$

As the MPC trajectory x^* is constructed with the concatenation of solutions to a sequence of problems $\mathcal{P}(t_i, x_i, T)$ in the conditions of the second assertion, we deduce that for all $M \geq t_0$ there exists $M_2 \in \mathbb{R}$ such that

$$\|x^*\|_{L^\infty[t_0, M]} < M_2,$$

as required. □

Next, we shall prove that the closed loop system resulting from the MPC strategy is asymptotically stable.

3.8.3 Main Stability Result (Thm. 3.4.1)

We show that the "MPC value function" $V^\delta(t, x)$ constructed with value functions of OCP's, satisfies a decrescence condition implying that the closed loop system is asymptotically stable as required.

Consider the sampling interval $[t_i, t_i+\delta)$. Choose $(\bar{x}, \bar{u})$ to be a solution to $\mathcal{P}(t_i, x_{t_i}, T)$. By definition of the MPC strategy we have that

$$u^*(t) = \bar{u}(t) \quad \text{for all } t \in [t_i, t_i + \delta).$$

Assuming the plant behaves as predicted by the model in this interval (we are considering just nominal stability, not robust stability) we have also that

$$x^*(t) = \bar{x}(t) \quad \text{for all } t \in [t_i, t_i + \delta). \tag{3.13}$$

For $t \in [t_i, t_i + \delta)$ define

$$V_{t_i}(t, x_t)$$

to be the value function for problem $\mathcal{P}(t, x_t, T - (t - t_i))$ (the usual OCP but where we shrink the horizon by $t - t_i$). We have that $V_{t_i}(t, x^*(t)) = V_{t_i}(t, \bar{x}(t))$ and by Bellman's Principle of Optimality the solution to $\mathcal{P}(t, \bar{x}(t), T-(t-t_i))$ coincides with the remaining trajectory of $(\bar{x}, \bar{u})$ (because for all $t \in [t_i, t_i+\delta)$ all these problems terminate at the same instant $t_i + T$), therefore

$$\begin{aligned} V_{t_i}(t, x^*(t)) &= \int_t^{t_i+T} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T, \bar{x}(t_i + T)) \\ &= V_{t_i}(t_i, \bar{x}(t_i)) - \int_{t_i}^{t} L(s, \bar{x}(s), \bar{u}(s))ds. \end{aligned} \tag{3.14}$$

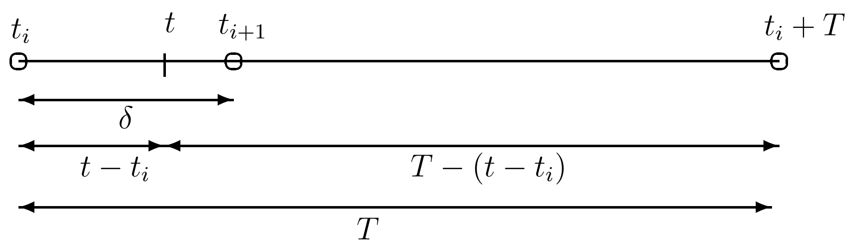

Figure 3.5: Time intervals involved in problems $\mathcal{P}(t, x_t, T - (t - t_i))$.

Finally define the "MPC Value function" to be

$$V^\delta(t, x_t) := V_\tau(t, x_t)$$

where τ is the sampling instant immediately before t, that is $\tau = \max_i\{t_i : t_i \leq t\}$.

We show that for the closed-loop trajectory x^* the function $t \mapsto V^\delta(t, x^*(t))$ converges to zero as $t \to \infty$ and thus x^* converges to zero as well. From (3.14) we know that this function is decreasing on each interval $(t_i, t_i + \delta)$ for any i. The next lemma establishes that V^δ is smaller at t_{i+1} than at t_i (see Fig. 3.6).

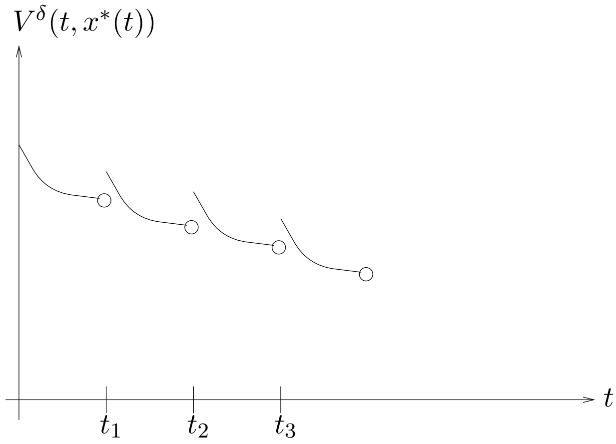

Figure 3.6: "Decreasing" behaviour of the MPC value function $V^\delta(t, x^*(t))$.

Lemma 3.8.4 *There exists an inter-sample time $\delta > 0$ small enough such that*

$$V_{t_{i+1}}(t_{i+1}, x^*(t_{i+1})) - V_{t_i}(t_i, x^*(t_i)) \leq -\int_{t_i}^{t_{i+1}} M(x^*(s))ds \quad \textit{for all } i \in \mathbb{N}.$$

Proof. First notice that due to the assumption on the accuracy of the model (3.13)

$$V_{t_{i+1}}(t_{i+1}, x^*(t_{i+1})) - V_{t_i}(t_i, x^*(t_i)) = V_{t_{i+1}}(t_{i+1}, \bar{x}(t_{i+1})) - V_{t_i}(t_i, \bar{x}(t_i)).$$

The value function for $\mathcal{P}(t_i, x_{t_i}, T)$ is

$$V_{t_i}(t_i, \bar{x}(t_i)) = \int_{t_i}^{t_i+T} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T, \bar{x}(t_i + T)).$$

Choose δ smaller than ϵ of SC5. Extend the process $(\bar{x}, \bar{u})$ to $[t_i, t_i + T + \delta])$ in such a way that $\bar{u} : [t_i + T, t_i + T + \delta] \to \mathbb{R}^m$ satisfies SC5. To this control will correspond the trajectory $\bar{x} : [t_i + T, t_i + T + \delta] \to \mathbb{R}^n$. Since this process is not necessarily optimal for $\mathcal{P}(t_i + \delta, x_{t_i+\delta}, T)$ we have

$$V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) \leq \int_{t_i+\delta}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds + W(t_i + T + \delta, \bar{x}(t_i + T + \delta)),$$

whence

$$\begin{aligned} V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) - V_{t_i}(t_i, \bar{x}(t_i)) \quad \leq \quad & -\int_{t_i}^{t_i+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ & + \int_{t_i+T}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ & + W(t_i + T + \delta, \bar{x}(t_i + T + \delta)) \\ & - W(t_i + T, \bar{x}(t_i + T)). \end{aligned}$$

Our choice of δ and SC5b imply

$$\bar{x}(t_i + T + r) \in S \quad \text{for all } r \in [0, \delta].$$

Integrating SC5a we obtain

$$W(t_i + T + \delta, \bar{x}(t_i + T + \delta)) - W(t_i + T, \bar{x}(t_i + T)) + \int_{t_i+T}^{t_i+T+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \leq 0.$$

Finally, recalling the condition on M in SC2 we obtain

$$\begin{aligned} V_{t_i+\delta}(t_i + \delta, \bar{x}(t_i + \delta)) - V_{t_i}(t_i, \bar{x}(t_i)) &\leq -\int_{t_i}^{t_i+\delta} L(s, \bar{x}(s), \bar{u}(s))ds \\ &\leq -\int_{t_i}^{t_i+\delta} M(x^*(s))ds. \end{aligned}$$

The lemma is proved. □

Lemma 3.8.5

$$V^\delta(t, x^*(t)) + \int_0^t M(x^*(s))ds \leq V^\delta(0, x^*(0)), \quad \textit{for all } t \geq 0. \tag{3.15}$$

Proof.

Let $t_i = i \cdot \delta$. From Lemma 3.8.4 we easily obtain

$$V^\delta(t_i, x^*(t_i)) - V^\delta(0, x^*(0)) \leq -\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} M(x^*(s))ds$$

or

$$V^\delta(t_i, x^*(t_i)) \leq V^\delta(0, x^*(0)) - \int_0^{t_i} M(x^*(s))ds.$$

Using equality (3.14)

$$\begin{aligned} V^\delta(t, x^*(t)) &= V_{t_i}(t, x^*(t)) \\ &\leq V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^{t} M(x^*(s))ds \\ &\leq V_{t_i}(t_i, x^*(t_i)) - \int_{t_i}^{t} M(x^*(s)) \\ &\leq V^\delta(0, x^*(0)) - \int_0^t M(x^*(s))ds \end{aligned}$$

□

Now, from the last lemma, since M is positive definite, the function $t \mapsto V^\delta(t, x^*(t))$ is bounded for all $t \in [0, \infty)$. We may also deduce from (3.15) that $\int_0^t M(x(s))ds$ is bounded as well. We have that x^* is bounded and from the properties of f that $\dot{x}^*$ is also bounded. These facts combine with the following well known lemma (the proof of which can be found in e.g. [MV94]) to prove asymptotic convergence.

Lemma 3.8.6 *Let M be a continuous, positive definite function and x be an absolutely continuous function on $\mathbb{R}_+$. If*

$$\|x(\cdot)\|_{L^\infty(0,\infty)} < \infty, \qquad \|\dot{x}(\cdot)\|_{L^\infty(0,\infty)} < \infty, \qquad \textit{and} \lim_{T\to\infty} \int_0^T M(x(t))\,dt < \infty,$$

then $x(t) \to 0 \quad$ as $\quad t \to \infty$.

The result of Thm. 3.4.1 follows immediately.

Chapter 4

Specified Rate of Exponential Stability

We propose a Model Predictive Control framework under which we can prescribe guaranteed rates of exponential stability for a class of nonlinear systems. A major feature of this framework is that the fixed-finite horizon open-loop optimal control problem (OCP) to be solved does not have any terminal state constraints imposed. This feature provides some important advantages: it significantly improves the efficiency of the optimisation algorithm and more importantly the usual hypothesis on the existence of solution to the OCP (in general difficult to verify in the presence of terminal state constraints) is automatically satisfied.

4.1 Introduction

In the previous chapter we provided a general framework for MPC of nonlinear systems. There it was shown that we can find a set of design parameters that satisfy conditions ensuring that the control obtained is a stabilising feedback, even when we use free terminal state problems. Although we have shown some examples on how a convenient set of design parameters without terminal state constraints could be obtained, the task of choosing such set of design parameters may prove itself difficult for some complex nonlinear systems. This should come as no surprise, since we were dealing with a very large class of nonlinear time varying systems, for which there is no alternative general approach to find a stabilising feedback control.

Here, restricting the set of nonlinear systems considered, by adding some additional hypotheses (while still considering a large class of systems with interest in practice), we develop a direct and easy method to find a set of stabilising design parameters to be used in a free terminal state OCP. Moreover, we prove that the controlled system can be designed to achieve a prescribed degree of exponential stability.

Several advantages arise from using a free terminal state OCP. The main advantage is perhaps the fact that typically the optimisation algorithms used perform considerably better when the OCP has no terminal state constraints [BH75, CA98b]. Also feasibility, the existence of an admissible trajectory, is trivially guaranteed. In addition, the regularity of the value function for these OCP's facilitates the stability analysis.

De Oliveira and Morari [dOM97], using a Contractive MPC framework, are also able to guarantee exponential stability for a class of nonlinear systems. However, their degree of exponential stability cannot be controlled, and can even be difficult to find since it involves an implicitly defined parameter β relating the norm of the final state with the supremum of the norm of any point in the trajectory and assumed to have value less than one.

De Nicolao *et al.* [DNMS98] using a quasi-infinite horizon MPC approach once again establish exponentially stability for discrete-time nonlinear systems. But, in their work, the rate of exponential decay cannot be controlled nor related with the design parameters. Besides, their terminal state constraint (in the form of a terminal cost set to infinity if the final state is outside a certain set) makes the feasibility of the OCP's difficult to guarantee.

Our approach differs from the above ones in the sense that, in addition to having the advantages of a free terminal state, we can choose — to a certain extent — the rate of exponential decay. Another novel feature is the way we avoid assuming continuity of the optimal control by an appropriate definition of the control that is applied to the plant.

This chapter is organised as follows. In the next section we provide the MPC framework and the set of hypotheses that the addressed nonlinear system has to satisfy. Then, in section 3, we give the main result on exponential stability. Section 4 provides some intermediate results, building up to the proof of the stability result given in section 5.

We adopt here the following notation. The variable t represents real time while we reserve s to denote the time variable used in the prediction model. The vector x_t denotes the actual state of the plant measured at time t. The process (x, u) is a pair trajectory/control obtained from the model of the system. The pair $(\bar{x}, \bar{u})$ denotes the process solving the open-loop optimal control problem. The process (x^*, u^*) is the trajectory and control resulting from the MPC strategy. The set $\mathbb{B} := \{\xi \in \mathbb{R}^n : \|\xi\| \leq 1\}$ is the closed unit ball. The quadratic form $x^T Mx$ if often represented as $\|x\|_M^2$.

4.2 The Model Predictive Control Framework

We shall consider a nonlinear plant with input constraint whose state evolution after time t is predicted by the following model

$$\dot{x}(s) = f(x(s), u(s)) \quad \text{a.e. } s \geq t \tag{4.1a}$$

$$x(t) = x_t \tag{4.1b}$$

$$u(s) \in \Omega \quad \text{a.e. } s \geq t, \tag{4.1c}$$

where the data comprise a vector $x_t \in \mathbb{R}^n$ that is the state of the plant measured at time t, a given function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and a set Ω of possible control values. We consider also a ball $M\mathbb{B}$ where all possible initial states lie. These data together with a measurable control function $u : [t, +\infty) \to \mathbb{R}^m$ define an absolutely continuous trajectory $x : [t, +\infty) \to \mathbb{R}^n$ which is sometimes written as $x(s; x_t, u)$ when we want to make explicit the dependence on the initial state and control function.

Our objective is to obtain a feedback law that drives the state of this system to the origin. This task is accomplished by using a MPC strategy. Consider a sequence of sampling instants $\pi := \{t_i\}_{i \geq 0}$ with a constant inter-sampling time $\delta > 0$ (smaller than the horizon T) such that $t_{i+1} = t_i + \delta$ for all $i \geq 0$. The feedback control is obtained by repeatedly solving online, at each sampling instant t_i, open-loop optimal control problems $\mathcal{P}(x_{t_i}, T)$ with fixed horizon T and free terminal-state, every time using the current measure of the state of the plant x_{t_i} as the initial state.

$$\mathcal{P}(x_t, T) \quad \text{Minimise} \quad \int_t^{t+T} (\|x(s)\|_Q^2 + \|u(s)\|_R^2) ds + k\|x(t+T)\|_P^2 \tag{4.2}$$

subject to

$$\dot{x}(s) = f(x(s), u(s)) \qquad \text{a.e. } s \in [t, t+T] \tag{4.3}$$

$$x(t) = x_t$$

$$u(s) \in \Omega \qquad \text{a.e. } s \in [t, t+T].$$

Here the matrices Q and P are positive definite, the matrix R is positive semi-definite, and k is a positive scalar. The domain of this optimisation problem is the set of admissible processes, namely pairs (x, u) comprising a measurable control function u and the corresponding absolutely continuous state trajectory x which satisfy the constraints of

$\mathcal{P}(x_t, T)$. A process $(\bar{x}, \bar{u})$ is said to solve $\mathcal{P}(x_t, T)$ if it globally minimises (4.2) among all admissible processes.

The control $u^*(s) := \bar{u}(s)$ is applied to the plant on the interval $[t, t+\delta]$, (the remaining control $\bar{u}(s), s > t + \delta$ is discarded) and the procedure is repeated for the next sampling instant $t_{i+1} = t_i + \delta$, every time using the newly measured state of the plant. As the sampling interval δ is typically much smaller than the horizon T, we consider in our stability analysis applying the MPC strategy with $\delta \downarrow 0$. The resultant control law u^*, called the MP control, is a feedback control since at every sampling time, the control is dependent on the state x_{t_i}. It is a well-known fact that (for fixed finite horizon) the closed-loop trajectory of the system (x^*) does not necessarily coincide with the open-loop trajectory $(\bar{x})$ solution to the OCP. Hence, the fact that MPC leads to a stabilising closed-loop system is not guaranteed *a priori*, and is highly dependent on the *design parameters* of the MPC strategy.

We show that we can guarantee stability of the resulting closed loop trajectory for all systems complying with the following hypotheses.

H1 The set Ω contains the origin in its interior, and $f(0,0) = 0$.

H2 The function $f(x, u)$ is continuous. In addition, $x \mapsto f(x, u)$ is globally Lipschitz for all $u \in \Omega$, and is continuously differentiable at $(0, 0)$.

H3 The set Ω is compact and convex, and for every x the velocity set $f(x, \Omega) := \{v \in \mathbb{R}^n : v = f(x, u),\ u \in \Omega\}$ is convex.

H4 The set of initial states $M\mathbb{B}$ is bounded.

H5 There exists a trajectory that can be bounded by a trajectory of some stable linear system. More precisely, there exist a positive semi-define matrix R, and positive scalars M_1, M_2, and α such that for any initial state $x_0 \in M\mathbb{B}$ we can find a control $\tilde{u} : [s, \infty) \to \mathbb{R}^m$ satisfying

$$\|\tilde{x}(t; x_0, \tilde{u})\|^2 \leq M_1 \|x_0\|^2 \mathrm{e}^{-\alpha t}$$

and

$$\|\tilde{u}(t)\|_R^2 \leq M_2 \|x_0\|^2 \mathrm{e}^{-\alpha t}.$$

H6 The linearization of the system around the origin is stabilizable, i.e. the partial derivatives of f at the origin, the matrices $A = f_x(0,0)$ and $B = f_u(0,0)$, exist and the unstable modes of the pair (A, B) are controllable.

It should be noted that all the hypotheses only involve data of the the nonlinear system and not of the optimal control problem. Assumptions concerning existence of the solution to the OCP, or properties of the value function are standard in previous MPC literature. Here, instead of assuming those difficult to test hypotheses, we show that they easily follow from the fact that the OCP has a free terminal state.

Another common assumption, present in virtually all continuous-time MPC approaches is the continuity of the optimal control $\bar{u}$ at its initial time. This regularity was necessary because the optimal control obtained by solving problem $\mathcal{P}(x,T)$ is in fact a class of functions that may differ on a finite number of points. Therefore, if we take a value of $\bar{u}$ in a particular instant, this value can in fact be anything (even not in Ω), being of little use if applied to the system. When we consider the sampling interval $\delta \to 0$, then some care is required to define the control that is actually applied to the plant. For the value of the control $u^*(t)$, selected at instant t, to be meaningful we define it as the one satisfying

$$f(\bar{x}(t), u^*(t)) = \lim_{\delta_i \to 0} \frac{1}{\delta_i} \int_t^{t+\delta_i} f(\bar{x}(s), \bar{u}(s))\, ds$$

for some sequence $\{\delta_i\}$ converging to zero. By the convexity hypothesis (H3) such control value exists and $u^*(t) \in \Omega$.

4.3 Main Stability Result

We now show that design parameters can chosen to give exponential stability, and provide an estimate of the degree. The following theorem involves the value $\bar{S} := \max\{\, \|f(x,u)\| : x \in M\mathbb{B},\, u \in \Omega\,\}$. (Such a maximum exists since the sets involved are compact and f is continuous.)

Theorem 4.3.1 *For any scalar $\mu > 0$ for which the pair $(A+\mu I, B)$ is stabilizable, we can find design parameters (T, Q, R, P, k) with any $k > 1$ of our choice such that the trajectory resulting from applying the MPC satisfies the exponential convergence rate*

$$\|x(t+t_0)\| \leq \sqrt{\frac{4k\lambda_{\max}(P)}{\rho\lambda_{\min}(Q)}}\|x(t_0)\|\mathrm{e}^{-\frac{\mu}{k}t} \quad \textit{for all } t_0 \geq T.$$

where $\rho = \min\left\{T, \dfrac{\|x(t_0)\|}{2\bar{S}}\right\}$.

The proof of this theorem can be found in a later section. Furthermore, strengthening the stabilizability assumption to controllability we could attain any prescribed degree of stability.

How to choose the design parameters to attain this degree of stability is described in the next section.

4.4 Choice of Design Parameters

In order to choose design parameters guaranteeing a prescribed rate of exponential stability we can follow the following steps.

1. Choose the desired rate μ conditioned to $(A + \mu I, B)$ being stabilizable.

2. Choose the scalar $k > 1$.

3. Choose some positive definite matrices $\hat{R}$ and $\hat{Q}$.

4. Find P to solve the algebraic Riccati equation (ARE)

$$(A + \mu I)^T P + P(A + \mu I) - PB\hat{R}^{-1}B^T P + \hat{Q} = 0.$$

5. Let $Q = \hat{Q} + 2\mu P$.

6. Choose $R \geq 0$ such that it satisfies H5, $\hat{R} - R \geq 0$ and the the "extended velocity set" $\{(v, \ell) \in \mathbb{R}^n \times \mathbb{R}_+ : v = f(x, u),\, \ell \geq x^T Qx + u^T Ru,\, u \in \Omega\}$ is convex. (Always true if $R = 0$.)

It remains to choose the horizon T. This task is not as trivial, and we are only able here to give a very conservative bound. However, for most systems the results should also hold for much smaller values of T than the one obtained as indicated below.

1. Choose a size $w > 0$ for a ball centred at the origin as in Prop.4.5.3
   (a) Select ϵ according to (4.7).
   (b) Select w satisfying (4.6) and (4.8).

2. Choose T to satisfy Prop.4.5.2.
   (a) Choose T to satisfy (4.4) with $\beta = w$.

Obviously the larger we choose μ the faster the rate of convergence. In choosing k we have a compromise. The nearer it is of 1 the faster the convergence rate μ/k. But since the difference $(k - 1)$ will accommodate the differences between the nonlinear and the linearised model, the smaller k is, the smaller will w be, leading to longer horizons T.

4.5 Intermediate Results

Proposition 4.5.1 (Existence of Solution) *Assume H1–H4. The solution to the open loop optimal control problem $\mathcal{P}(x_t, T)$ exists.*

Proof. Since $\mathcal{P}(x_t, T)$ is a free terminal state problem, the existence of a solution follows from a standard theorem under the stated hypotheses. Note that f and the objective function are continuous, Ω is compact and non-empty, and the "extended velocity set" is convex. We are in conditions to apply a well-known existence result (see e.g. [FR75, Thm. III.4.1]). □

Proposition 4.5.2 *For any scalar $w > 0$, we can choose a horizon $T > 0$ such that for any initial state x_{t_0} satisfying H4, the terminal state of a trajectory that solves the open-loop optimal control problem satisfies*

$$\bar{x}(t_0 + T) \in w\mathbb{B}.$$

Proof.

Step 1. Assume for simplicity and without loss of generality that $t_0 = 0$. Fix any $\beta > 0$, then for a sufficient large horizon T there exists an instant $\bar{t} \in [0, T]$ such that the trajectory $\bar{x}$ solution to the open loop optimal control problem satisfies

$$\|\bar{x}(\bar{t})\| \leq \beta.$$

Suppose in contradiction that this latter assertion was not true, then

$$\begin{aligned} J(\bar{x}, \bar{u}) &= \int_0^T (\|\bar{x}(t)\|_Q^2 + \|\bar{u}(t)\|_R^2) dt + k\|\bar{x}(T)\|_P^2 \\ &\geq T\lambda_{\min}(Q)\beta^2 + k\lambda_{\min}(P)\beta^2. \end{aligned}$$

But, by H5 we know that there is a process (x', u') with $x'(0) = \bar{x}(0) = x_0$ such that

$$\begin{aligned} J(x', u') &= \int_0^T (\|x'(t)\|_Q^2 + \|u'(t)\|_R^2) dt + k\|x'(T)\|_P^2 \\ &\leq \int_0^T \left(\lambda_{\max}(Q) M_1 \|x_0\|^2 \mathrm{e}^{-\alpha t} + M_2 \|x_0\|^2 \mathrm{e}^{-\alpha t}\right) dt \\ &\quad + k\lambda_{\max}(P) M_1 \|x_0\|^2 \mathrm{e}^{-\alpha T} \\ &\leq \frac{1}{\alpha}\|x_0\|^2 (\lambda_{\max}(Q) M_1 + M_2) + k\lambda_{\max}(P) M_1 \|x_0\|^2 \mathrm{e}^{-\alpha T}. \end{aligned}$$

By optimality we should have

$$J(\bar{x}, \bar{u}) \leq J(x', u'),$$

implying that

$$\begin{aligned} T\lambda_{\min}(Q)\beta^2 + k\lambda_{\min}(P)\beta^2 \quad \leq \quad & \frac{1}{\alpha}\|x_0\|^2(\lambda_{\max}(Q)M_1 + M_2) \\ & + k\lambda_{\max}(P)M_1\|x_0\|^2 \mathrm{e}^{-\alpha T}. \end{aligned}$$

But this leads to a contradiction if we choose a horizon T to satisfy

$$\begin{aligned} T \quad > \quad \max & \left\{ \frac{1}{\alpha\lambda_{\min}(Q)\beta^2\|x_0\|^2(\lambda_{\max}(Q)M_1 + M_2)}, \right. \\ & \left. \frac{1}{\alpha} \ln\left(\frac{\lambda_{\max}(P)M_1\|x_0\|^2}{\lambda_{\min}(P)\beta^2} \right) \right\} \end{aligned} \tag{4.4}$$

Step 2. Suppose in contradiction to what we want to prove that there exist a scalar $\gamma > \beta > 0$ such that

$$\|\bar{x}(T)\| \geq \gamma.$$

As the image of compact sets by a continuous function is compact we can find a bound on f as $\bar{S} := \sup\{\|f(x,u)\| : x \in M\mathbb{B}, u \in \Omega\}$. Since $\|\bar{x}(\bar{t})\| \leq \beta$ and $\|f(\cdot,\cdot)\| \leq \bar{S}$, we have that (see figure below)

$$T - \bar{t} \geq \frac{\gamma - \beta}{\bar{S}},$$

and

$$\int_{\bar{t}}^{T} \|\bar{x}(t)\|_Q^2 dt \geq \int_0^{\frac{\gamma-\beta}{\bar{S}}} \lambda_{\min}(Q)(\beta + \bar{S}t)^2 dt = \frac{\lambda_{\min}(Q)}{3\bar{S}}(\gamma^3 - \beta^3).$$

Hence

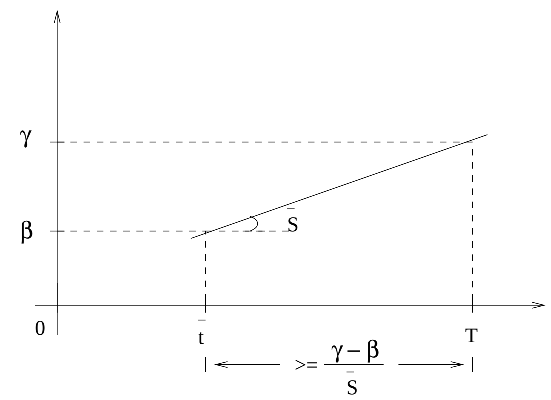

Figure 4.1: Minimum time to grow

$$J(\bar{x},\bar{u}) \geq \int_0^{\bar{t}} (\|\bar{x}(t)\|_Q^2 + \|\bar{u}(t)\|_R^2)dt + \frac{\lambda_{\min}(Q)}{3\bar{S}}(\gamma^3 - \beta^3) + k\lambda_{\min}(P)\gamma^2.$$

On the other hand at instant $\bar{t}$ we can choose the remaining part of the trajectory (x', u') to satisfy H5, obtaining the cost

$$\begin{aligned} J(x', u') &\leq \int_0^{\bar{t}} (\|\bar{x}(t)\|_Q^2 + \|\bar{u}(t)\|_R^2)dt \\ &\quad + \int_{\bar{t}}^T \beta^2 (M_1 \lambda_{\max}(Q) + M_2) \mathrm{e}^{-\alpha(t-\bar{t})} dt + k\lambda_{\max}(P)\beta^2 \\ &\leq \int_0^{\bar{t}} (\|\bar{x}(t)\|_Q^2 + \|\bar{u}(t)\|_R^2)dt \\ &\quad + \frac{\beta^2}{\alpha}(M_1 \lambda_{\max}(Q) + M_2) + k\lambda_{\max}(P)\beta^2. \end{aligned}$$

By optimality we have

$$J(\bar{x}, \bar{u}) \leq J(x', u'),$$

implying that

$$\frac{\lambda_{\min}(Q)}{3\bar{S}}(\gamma^3 - \beta^3) + k\lambda_{\min}(P)\gamma^2 \leq \frac{\beta^2}{\alpha}(M_1 \lambda_{\max}(Q) + M_2) + k\lambda_{\max}(P)\beta^2$$

But if β is chosen small enough to satisfy

$$\beta^2 < \frac{\alpha k \lambda_{\min}(P)\gamma^2}{(M_1 \lambda_{\max}(Q) + M_2) + k\lambda_{\max}(P)}$$

we get a contradiction. □

Proposition 4.5.3 *Consider the matrices $A = f_x(0,0)$ and $B = f_u(0,0)$. Fix the design parameters with the scalar $k > 1$ and the matrix P solving the Algebraic Riccati Equation*

$$A^T P + PA - PB\hat{R}^{-1}B^T P + Q = 0,$$

then we can find a scalar $w > 0$ and a control value $u \in \Omega$ such that

$$\begin{aligned} 2kx^T P f(x, u) &\leq -(x^T Q x + u^T \hat{R} u) \\ &\leq -(x^T Q x + u^T R u) \quad \textit{for all } x \in w\mathbb{B}. \end{aligned} \tag{4.5}$$

Proof. Consider the first order expansion of the dynamics

$$f(x, u) = f_x(0,0)x + f_u(0,0)u + f_1(x, u).$$

The higher order terms $f_1(x, u)$ are such that

$$\lim_{\|(x,u)\| \to 0} \frac{\|f_1(x, u)\|}{\|(x, u)\|} = 0.$$

If the control u satisfies $\|u\| \leq F\|x\|$ for some fixed scalar F, it follows that

$$\lim_{\|x\|\to 0} \sup_{\|u\|\leq F\|x\|} \frac{\|f_1(x,u)\|}{\|x\|} = 0.$$

An equivalent expression for the limit above is: For all $\epsilon > 0$ there exists a $\delta_1 > 0$ such that if $\|x\| < \delta_1$ and $\|u\| \leq F\|x\|$, then $\|f_1(x,u)\| < \epsilon\|x\|$.

In particular, we consider the feedback control

$$u = -\hat{R}^{-1}B^T P x.$$

By choosing w small enough we can guarantee that

$$u = -\hat{R}^{-1}B^T P x \in \Omega, \text{ for all } x \in w\mathbb{B}. \tag{4.6}$$

Using the defined feedback control, we have

$$\begin{aligned}
2kx^T Pf(x,u) &= 2kx^T P(Ax+Bu) + 2kx^T Pf_1(x,u) \\
&= 2kx^T(PA - PB\hat{R}^{-1}B^T P)x + 2kx^T Pf_1(x,u) \\
&= kx^T(A^T P + PA - 2PB\hat{R}^{-1}B^T P)x + 2kx^T Pf_1(x,u).
\end{aligned}$$

Let the matrix P solve the Algebraic Riccati Equation, then

$$\begin{aligned}
2kx^T Pf(x,u) &= kx^T(-Q - PB\hat{R}^{-1}B^T P)x + 2kx^T Pf_1(x,u) \\
&= -k(x^T Qx + u^T\hat{R}u) + 2kx^T Pf_1(x,u).
\end{aligned}$$

Recall that we want to prove that

$$2kx^T Pf(x,u) \leq -(x^T Qx + u^T\hat{R}u),$$

which can be written equivalently as

$$(1-k)(x^T Qx + u^T\hat{R}u) + 2kx^T Pf_1(x,u) \leq 0.$$

Now, as $k > 1$, choose a scalar ϵ to satisfy

$$\epsilon \leq \frac{(k-1)\lambda_{\min}(Q + PB\hat{R}^{-1}B^T P)}{2k\lambda_{\max}(P)}, \tag{4.7}$$

and find a $\delta_1 > 0$ such that

$$\|f_1(x,u)\| \leq \epsilon\|x\| \quad \text{for all } x \in \delta_1\mathbb{B}. \tag{4.8}$$

It follows that if $w \leq \delta_1$

$$
\begin{aligned}
(1-k)(x^T Q x + u^T \hat{R} u) + 2k x^T P f_1(x,u) &\leq (1-k)\lambda_{\min}(Q + PB\hat{R}^{-1}B^T P)\|x\|^2 \\
&\quad +2k\lambda_{\max}(P)\|x\|^2\epsilon \\
&\leq 0
\end{aligned}
$$

for all $x \in w\mathbb{B}$, as we wanted to show. □

Next, we quote a well-known result, that is useful later, on regularity of the value function for OCP with free terminal state (for proof see e.g. [FR75, Thm. 4.2]).

Lemma 4.5.4 *The value function $V : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous.*

The following proposition is of key importance in establishing the stability results.

Proposition 4.5.5 *The value function satisfies*

$$\frac{d}{dt}V(x^*(t)) < -x^*(t)^T Q x^*(t) \quad a.e.\ t \in \mathbb{R}_+.$$

Proof. As V is Lipschitz and x is absolutely continuous, the composition $V \circ x$ is also absolutely continuous and thus differentiable almost everywhere. Take a point t where $V(x(t))$ is differentiable. Recall that the pair $(\bar{x}, \bar{u})$ defined in the interval $[t, t+T]$ denotes a solution of the optimal control problem $\mathcal{P}(x_t, T)$, and the pair (x^*, u^*) defined in $\mathbb{R}_+$ denotes the trajectory and control resulting from the application of the MPC strategy. By definition of MPC strategy $x^*(t) = \bar{x}(t) = x_t$ and $u^*(t)$ satisfies $f(\bar{x}(t), u^*(t)) = lim_{\delta_i \to 0} \frac{1}{\delta_i} \int_t^{t+\delta_i} f(\bar{x}(s), \bar{u}(s))\, ds$ for some sequence $\{\delta_i\}$ converging to zero.

We start by showing (step 1) that

$$\frac{d}{dt}V(x^*(t)) = \lim_{h \downarrow 0} \frac{1}{h}[V(\bar{x}(t+h)) - V(\bar{x}(t))].$$

Next we show (step 2) that

$$\lim_{h \downarrow 0} \frac{1}{h}[V(\bar{x}(t+h)) - V(\bar{x}(t))] \leq -\bar{x}(t)^T Q \bar{x}(t).$$

To simplify notation define

$$m(x,u) := \|x\|_Q^2 + \|u\|_R^2$$

and

$$W(x) := k\|x\|_P^2.$$

Step 1. Since $x^*(t) = \bar{x}(t)$ we have

$$\frac{d}{dt}V(x^*(t)) = \lim_{h\downarrow 0} \frac{1}{h}[V(x^*(t+h)) - V(\bar{x}(t))].$$

Adding and subtracting $V(\bar{x}(t+h))$,

$$\frac{d}{dt}V(x^*(t)) = \lim_{h\downarrow 0} \frac{1}{h}[V(\bar{x}(t+h)) - V(\bar{x}(t))] + \lim_{h\downarrow 0} \frac{1}{h}[V(x^*(t+h)) - V(\bar{x}(t+h))].$$

It remains to show that the second term vanishes

$$\begin{aligned}\lim_{h\downarrow 0} \frac{1}{h}[\,V(x^*(t+h)) - V(\bar{x}(t+h))\,] = \\ \lim_{h\downarrow 0} \frac{1}{h}\left[V\left(\bar{x}(t) + \int_t^{t+h} f(x^*(s), u^*(s))ds\right)\right. \\ \left. -V\left(\bar{x}(t) + \int_t^{t+h} f(\bar{x}(s), \bar{u}(s))ds\right)\right].\end{aligned}$$

Using the fact that V is Lipschitz (of rank K_V), and the definition of u^* we obtain

$$\begin{aligned}\lim_{h\downarrow 0} \frac{1}{h}|V(x^*(t+h)) - V(\bar{x}(t+h))| &\leq K_V \lim_{h\downarrow 0} \frac{1}{h}\left\| \int_t^{t+h} f(x^*(s), u^*(s))ds \right. \\ &\qquad \left. - \int_t^{t+h} f(\bar{x}(s), \bar{u}(s))ds \right\| \\ &= K_V \| f(x^*(t), u^*(t)) - f(\bar{x}(t), u^*(t)) \| \\ &= 0.\end{aligned}$$

Step 2. The value function for $\mathcal{P}(x_t, T)$ is

$$V(\bar{x}(t)) = \int_t^{t+T} m(\bar{x}(s), \bar{u}(s))ds + W(\bar{x}(t+T)).$$

Consider the pair $(\bar{x}, \bar{u})$ solution to $\mathcal{P}(x_t, T)$ in the interval $[t, t+T]$. Extend it to $t+T+h$ in such a way that it satisfies (4.5) and $\bar{u}$ is right-continuous at $t+T$. As this extended process is an admissible but not necessarily optimal process for $\mathcal{P}(s+h, \bar{x}(s+h), T+h)$ we have

$$V(\bar{x}(t+h)) \leq \int_{t+h}^{t+T+h} m(\bar{x}(s), \bar{u}(s))ds + W(\bar{x}(t+T+h)).$$

The difference between the two value functions is

$$\begin{aligned}V(\bar{x}(t+h)) - V(\bar{x}(t)) &\leq -\int_t^{t+h} m(\bar{x}(s), \bar{u}(s))ds \\ &\quad + \int_{t+T}^{t+T+h} m(\bar{x}(s), \bar{u}(s))ds \\ &\quad + W(\bar{x}(t+T+h)) - W(\bar{x}(t+T)).\end{aligned}$$

As $V \circ x$ is absolutely continuous the limit as $h \downarrow 0$ exists almost everywhere in $\mathbb{R}_+$. Moreover, since m, W, and $\bar{x}$ are continuous, and $\bar{u}$ was chosen to be right-continuous at $t+T$, it follows

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}[V(\bar{x}(t+h)) - V(\bar{x}(t))] \;\; \leq \;\; & -\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} m(\bar{x}(s), \bar{u}(s))ds \\
& + m(\bar{x}(s+T), \bar{u}(s+T)) \\
& \dot{W}(\bar{x}(s+T)) \quad \text{a.e. } s \in \mathbb{R}_,
\end{aligned}$$

By equation (4.5) the sum of last two terms is non-positive and we obtain,

$$\begin{aligned}
\lim_{h\downarrow 0} \frac{1}{h}[V(\bar{x}(t+h)) - V(\bar{x}(t))] \;\; & \leq \;\; -\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} [\bar{x}^T(s)Q\bar{x}(s) + \bar{u}^T(s)R\bar{u}(s)]ds \\
& \leq \;\; -\bar{x}^T(t)Q\bar{x}(t),
\end{aligned}$$

as we intend to show. □

4.6 Proof of the Main Stability Result

Proof of Theorem 4.3.1 We start by showing that (step 1) the value function satisfies

$$\rho\lambda_{\min}(Q)\frac{\|x_0\|^2}{4} \leq V(x_0) \leq k\lambda_{\max}(P)\|x_0\|^2,$$

and

$$\dot{V}(x^*(t)) \leq \frac{-2\mu}{k} V(x^*(t)).$$

Finally, we show (step 2) that

$$\|x^*(t+t_0)\| \leq \sqrt{\frac{4k\lambda_{\max}(P)}{\rho\lambda_{\min}(Q)}} \|x^*(t_0)\| \mathrm{e}^{-\frac{\mu}{k}t}.$$

Step 1. Choose some positive definite matrices $\hat{R}$ and $\hat{Q}$, and find the matrix P that solves the ARE

$$(A+\mu I)^T P + P(A+\mu I) - PB\hat{R}^{-1}B^T P + \hat{Q} = 0. \tag{4.9}$$

Choose the matrix Q to be

$$Q = \hat{Q} + 2\mu P,$$

then, the ARE above can be equivalently written as

$$A^T P + PA - PB\hat{R}^{-1}B^T P + Q = 0.$$

Choose $R \geq 0$ such that $\hat{R} - R \geq 0$. As in the proof of Prop. 4.5.3, we can show using similar arguments that if $u = \hat{R}^{-1}B^T P x$ for all $x \in w\mathbb{B}$ then

$$2kx^T P f(x,u) \leq -(x^T Q x + u^T \hat{R} u) \leq -(x^T Q x + u^T R u).$$

With this inequality we are able to establish a quadratic bound on the cost of using this control

$$\begin{aligned}
J(x,u) &= \int_0^T [x(t)^T Q x(t) + u^T(t) R u(t)]dt + kx(T)^T P x(T) \\
&\leq \int_0^T -2kx(t)^T P f(x(t),u(t))dt + kx(T)^T P x(T) \\
&= -\int_0^T \frac{d}{dt}[kx(t)^T P x(t)]dt + kx(T)^T P x(T) \\
&= kx_0^T P x_0.
\end{aligned}$$

Hence the value function

$$\begin{aligned}
V(x_0) &= \min_u \left\{ \int_0^T [x(t)^T Q x(t) + u^T(t) R u(t)]dt + kx(T)^T P x(T) \right\} \\
&\leq J(x,u) \\
&\leq kx_0^T P x_0.
\end{aligned}$$

On the other hand, we can also establish a quadratic lower bound on the value function

$$V(x_0) \geq \lambda_{\min}(Q) \min_{\|\dot{x}(t)\| \leq \bar{S}} \int_0^T \|x(t)\|^2 dt,$$

where $\bar{S}$ is a bound on the function f when it takes values on a bounded set containing $\{(x,u) : \|x\| \leq M_1, \|u\| \leq M_2\}$. Note that (see Fig. 4.2)

$$\|x(t)\| \geq \frac{\|x_0\|}{2} \quad \text{for all } t \in \left[0, \frac{\|x_0\|}{2\bar{S}}\right].$$

Defining $\rho = \min\left\{T, \frac{\|x_0\|}{2\bar{S}}\right\}$, we obtain

$$\begin{aligned}
V(x_0) &\geq \lambda_{\min}(Q) \min_{\|\dot{x}(t)\| \leq \bar{S}} \int_0^T \|x(t)\|^2 dt \\
&\geq \lambda_{\min}(Q) \int_0^\rho \frac{\|x_0\|^2}{4} dt \\
&\geq \rho \lambda_{\min}(Q) \frac{\|x_0\|^2}{4}.
\end{aligned}$$

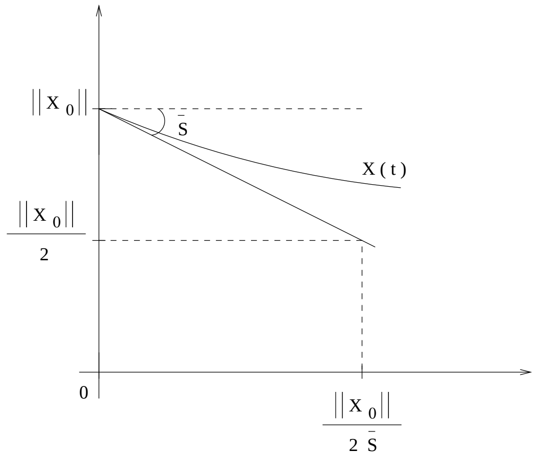

Figure 4.2: Minimum area under the trajectory

Recalling that in Prop. 4.5.5 we have shown that if x^* is the trajectory obtained by using the MPC strategy then

$$\dot{V}(x^*(t)) \leq -[x(t)^T Q x(t) + u^T(t) R u(t)] \quad \text{a.e. } t.$$

It follows that

$$\begin{aligned}
\dot{V}(x^*(t)) &\leq -(x(t)^T Q x(t) + u^T(t) R u(t)) \\
&= -2\mu x(t)^T P x(t) - x(t)^T \hat{Q} x(t) - u^T(t) R u(t) \\
&\leq -2\mu x(t)^T P x(t) \\
&\leq \frac{-2\mu}{k} V(x^*(t)),
\end{aligned}$$

for a.e. t such that $x^*(t) \in w\mathbb{B}$.

Step 2. Define

$$Z(t) := \dot{V}(x^*(t)) + \frac{2\mu}{k} V(x^*(t)) \quad \text{a.e. } t.$$

We can write the solution of this ODE as

$$V(x^*(t+t_0)) = V(x^*(t_0)) \mathrm{e}^{-\frac{2\mu}{k}t} + \int_{t_0}^{t_0+t} \mathrm{e}^{-\frac{2\mu}{k}(t-\tau)} Z(\tau) d\tau.$$

From the last inequality we have that $Z(t) \leq 0$ a.e. t, hence

$$V(x^*(t+t_0)) \leq V(x^*(t_0)) \mathrm{e}^{-\frac{2\mu}{k}t}$$

for all $x^*(t_0) \in w\mathbb{B}$. Finally

$$\begin{aligned}\|x^*(t+t_0)\|^2 &\leq \frac{4}{\rho\lambda_{\min}(Q)} V(x^*(t+t_0)) \\ &\leq \frac{4}{\rho\lambda_{\min}(Q)} V(x^*(t_0)) \mathrm{e}^{-\frac{2\mu}{k}t}\end{aligned}$$

for all $x^*(t_0) \in w\mathbb{B}$. Equivalently we can write

$$\|x^*(t+t_0)\| \leq \sqrt{\frac{4k\lambda_{\max}(P)}{\rho\lambda_{\min}(Q)}} \|x^*(t_0)\| \mathrm{e}^{-\frac{\mu}{k}t}$$

for all $t_0 > T$. The proof is complete.

Part II

Nondegenerate Necessary Conditions for State Constrained Optimal Control Problems

Chapter 5

The Degeneracy Phenomenon

Once again, to derive the more important necessary optimality conditions, constraint qualifications are needed.

Olvi L. Mangasarian, 1969, *Nonlinear programming.*

In this chapter we introduce the Degeneracy Phenomenon of the necessary conditions of optimality. We start by giving the main concepts in a framework of Mathematical Programming. Later we review the main literature on nondegenerate necessary conditions for optimal control problems and provide the context and motivation for the results in the forthcoming chapters.

5.1 Introduction

The purpose of all Necessary Conditions of Optimality (NCO) is to enable us to identify a *small* set of candidates to local minimisers among the overall set of admissible solutions. It is thus natural the interest to construct *stronger NCO* that further reduce the set of candidates identified to an even smaller set while still identifying all the local minimisers. The simplest and best known example of strengthening a necessary condition of optimality is, in the case of minimising a function of a real variable, to combine the renowned Fermat's rule

$$\frac{d}{dx}f(x) = 0,$$

with the second order condition

$$\frac{d^2}{dx^2}f(x) \geq 0,$$

to obtain a more reduced set of candidates to local minimisers.

In constrained optimisation problems it may happen that, even when there are admissible solutions leading to different costs, the set of candidates to minimisers that satisfy certain NCO coincides with the set of all admissible solutions. When this is the case, the NCO give us no information and thus all its purpose is lost. Such phenomenon reveals itself in Mathematical Programming when the multiplier λ associated with the objective function takes the value zero. To see this, consider the following Mathematical Programming problem.

$$\begin{array}{lll} (MP) & \text{Minimise} & g(x) \\ & \text{subject to} & h(x) \leq 0. \end{array}$$

If $\bar{x}$ is a solution to this problem then the following NCO (known as the Fritz John conditions, [Joh48] in [Aba67, Man69]) guarantee the existence of multipliers $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$\begin{aligned} &(\lambda, \mu) \geq 0 \\ &\lambda \nabla g(\bar{x}) + \mu \cdot \nabla h(\bar{x}) = 0 \\ &\mu \cdot h(\bar{x}) = 0 \\ &h(\bar{x}) \leq 0. \end{aligned}$$

Choose the multiplier $\lambda = 0$. If we also choose $\mu = 0$, then the first three conditions are automatically satisfied. Therefore, in this case, the set of candidates satisfying these conditions is the same we had before applying the NCO, that is $\{x : h(x) \leq 0\}$.

In the situation when the NCO give us no information although not all admissible solutions lead to the same cost, the NCO are said to be *degenerate*. If we want the NCO to be of any use in such cases, we need to strengthen them with additional conditions. The importance of finding strengthened forms of the NCO is then even more apparent in this degenerate case. The strengthened forms of the NCO capable of avoiding some type of degeneracy are called *nondegenerate forms*, and these forms for control problems will be the object of study in this part of the thesis.

The first thing we might think of to remedy the problem is to add a condition, like $\lambda \neq 0$, that does not allow the multiplier associated with the objective function to vanish, forcing the objective function to be involved in the Lagrangian. In fact choosing $\lambda = 1$ we obtain the famous Kuhn-Tucker conditions [KT51]. But, we have to guarantee that the NCO are still satisfied at local minima, that is, we have to ensure that for every local minimiser we can find a set of multipliers (λ and μ) satisfying $\lambda \neq 0$. However, the existence of nondegenerate multipliers cannot be guaranteed in every problem. Consider for example the following problem.

Example 5.1.1 (adapted from [Lue69]).

$$\begin{array}{lll} (MP1) & \text{Maximise} & x_1 \\ & \text{subject to} & x_2 + (x_1 - 1)^3 \leq 0 \\ & & -x_2 \leq 0. \end{array}$$

It can easily be seen that $\bar{x} = (1, 0)$ is the unique solution to this problem. Nevertheless, the NCO cannot be written for this problem with the multiplier $\lambda > 0$. This is due to

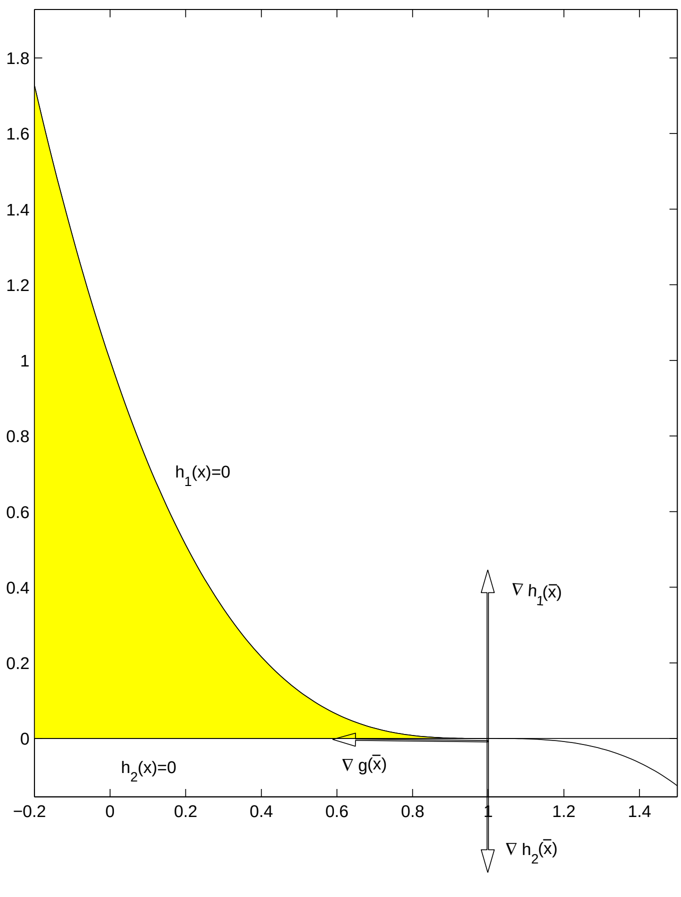

Figure 5.1: Example of degeneracy in Mathematical Programming

the fact that we cannot express $\nabla g(\bar{x})$ as a linear combination of $\nabla h_1(\bar{x})$ and $\nabla h_2(\bar{x})$ (see Fig. 5.1). (We will present below an example of an optimal control problem in which the degenerate multipliers are also the only possible choice.)

Fortunately, the type of problems that only admit degenerate multipliers are a special case and, in practice, most problems admit NCO with $\lambda > 0$. This explains the interest in finding a condition that identifies this type of problems and then use the NCO with $\lambda > 0$. The conditions that identify the problems in which we can guarantee the existence of nondegenerate multipliers are called *constraint qualifications*. Only under these constraint qualifications we can safely strengthen the NCO to avoid degeneracy.

The literature on the study of nondegenerate forms of NCO and constraint qualifications is abundant, and has remote origins in the case of Mathematical Programming (see e.g.[KT51], [Man69] or [Cla83] for details). Some of the best known examples of a constraint qualification are: the condition that the gradients of the active constraints are linearly independent; and the condition that for every local minimiser $\bar{x}$ there exists a vector $v \in \mathbb{R}^m$ such that

$$\nabla h_i(\bar{x}) \cdot v < 0 \quad \text{if } h_i(\bar{x}) = 0, \quad i = 1, 2, ..., m.$$

It is immediate to see that any of these conditions eliminates problems like the one above. The famous NCO of the paper [KT51] are precisely a nondegenerate version of the Fritz John conditions valid under a suitable constraint qualification [Aba67, Man69]. It states that we can choose $\lambda = 1$ for all problems complying with the constraint qualification.

5.2 Degeneracy in Optimal Control

Recently there has been a growing interest and literature on nondegenerate NCO for control problems, coming under the keywords of normality, calmness (see [Cla83]) and nondegeneracy (see [AA97, AA95, FV94] and [FFV99]). The term normality is used when the NCO for control problems (the Maximum Principle Prop. 2.5.1) can be written with the multiplier $\lambda \neq 0$. The term "degeneracy phenomenon" has been used in the optimal control literature to describe a particular type of degeneracy occurring due to the presence of pathwise state constraints in the problem (see [FV94, AA97]). Here the value of the multiplier associated with the pathwise state constraint has a crucial role in making the set of multipliers degenerate. Having this in mind we use hereafter, within the framework of state constrained optimal control problems, the terms degeneracy/nondegeneracy in this spirit and differentiate them from the terms normality/abnormality to clarify the discussion.

Note that the term nondegenerate can mislead us into thinking that it is stronger than it really is. Traditionally, the term nondegenerate NCO has been used in conditions that eliminate *a particular* kind of degeneracy occurring with the traditional NCO, but are not necessarily informative since other kinds of degeneracy, like abnormality, might be allowed.

Consider a fixed left-endpoint optimal control problem

$$
\begin{aligned}
(P) \quad & \text{Minimise} \quad g(x(1)) && && (5.1)\\
& \text{subject to} && && \\
& && \dot{x}(t) = f\left(t, x(t), u(t)\right) && \text{a.e. } t \in [0,1] \qquad (5.2)\\
& && x(0) = x_0 && \\
& && x(1) \in C && \\
& && u(t) \in \Omega(t) && \text{a.e. } t \in [0,1] \\
& && h\left(t, x(t)\right) \leq 0 && \text{for all } t \in [0,1], \qquad (5.3)
\end{aligned}
$$

in which the pathwise state constraint is active in the initial instant of time

$$h(0, x_0) = 0. \tag{5.4}$$

If $h_x(0, x_0) \neq 0$, this is the case when x_0 lies in the boundary of the state region. (We are assuming for the moment that h is smooth to simplify the exposition.) It can be verified that the necessary conditions of Prop. 2.5.1 are satisfied at *any* feasible process $(\bar{x}, \bar{u})$ (local minimiser or not) for the choice of multipliers

$$\lambda = 0, \quad \mu = \delta_{\{0\}}, \quad p = -h_x(0, x_0), \tag{5.5}$$

or scalar multiples of them. This can be easily seen by noting that the quantity $p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds)$ vanishes almost everywhere and all the conditions of the maximum principle are satisfied independently of the value of $\bar{x}$ or $\bar{u}$. In this case no useful information is supplied about minimisers.

The case (5.4) is encountered in certain applications of interest, namely in Model Predictive Control. A further discussion of this point can be seen in the next chapter.

To avoid this type of degeneracy we might think of strengthening the nontriviality condition of the maximum principle to

$$\mu\{(0,1]\} + \lambda + \operatorname*{ess\,sup}_t \left\| p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds) \right\| > 0.$$

This strengthening has the advantage that (apart from the trivial multipliers) it eliminates only the degenerate multipliers (5.5). But, as is the case with mathematical programming, the answer to the problem is not as simple as the above might lead us to believe. We obviously have to guarantee that the NCO strengthened in this way are still satisfied for all local minimisers. The importance of this point is apparent in the next example where the degenerate multipliers are the only possible choice and then can not be eliminated, otherwise the NCO would not be valid.

Example 5.2.1 (Dubovitskii, in [AA97])

$$\begin{array}{lll} \text{Minimise} & x_2(1) & \\ \text{subject to} & & \\ & (\dot{x}_1(t), \dot{x}_2(t)) = (tu(t), u(t)) & \text{a.e. } t \in [0,1] \\ & (x_1(0), x_2(0)) = (0,0) & \\ & u(t) \in [-1,1] & \text{a.e. } t \in [0,1] \\ & x_1(t) \geq 0 & \text{for all } t \in [0,1], \end{array}$$

The pair $(\bar{x}, \bar{u}) = (0,0)$ is the unique solution to this problem. Applying the NCO (Prop. 2.5.1), it is not difficult to show that they are satisfied only with the degenerate multipliers (5.5) or scalar multiples of them. This example shows the need for a condition, a constraint qualification, that, by ruling out problems like the one above, identifies the problems under which we can eliminate the degenerate multipliers by strengthening the NCO.

There is a growing literature on refinements of necessary conditions which assert the existence of multiplier sets in addition to the degenerate ones, under a suitable constraint qualification. (See [AA97, AA95] and [FV94].) Among these, the important work of Arutyunov and Aseev [AA97] provides references to the vast earlier Russian literature on the subject. In fact, the results therein generalise most of the work that had been done with the exception of the paper of Ferreira and Vinter [FV94] which is capable of treating problems in which the data is merely measurable in time.

In [AA97] the NCO are strengthen to avoid the degenerate multipliers by adding the condition

$$H\left(t, \bar{x}(t), p(t) + \int_{[0,t)} h_x(s, \bar{x}(s))\mu(ds)\right) = H\left(t, \bar{x}(t), p(t) + \int_{[0,t)\cup\{t\}} h_x(s, \bar{x}(s))\mu(ds)\right) \tag{5.6}$$

where $H(t,x,q) := \max_u q^T \cdot f(t,x,u)$ is the true (maximised) Hamiltonian. This strengthening is valid for problems complying with the constraint qualification

$$\max_{u \in \Omega(t)} \{-h_x(0, x_0) \cdot f(0, x_0, u)\} > 0.$$

Loosely speaking, this is the requirement that there exist control functions pushing the state away from the state constraint boundary at the initial time (see Fig. 5.2). Condition

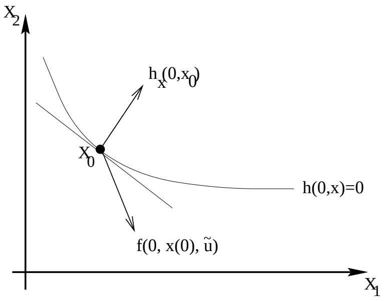

Figure 5.2: Arutyunov-Aseev constraint qualification

(5.6), combined with the constraint qualification, eliminates the degenerate multipliers, since using the multipliers (5.5) in equation (5.6), the left hand side of this equation gives us

$$H(0, \bar{x}(0), p(0)) = \max_{u \in \Omega} \{-h_x(0, x_0) \cdot f(0, x_0, u)\} > 0.$$

On the other hand the right hand side becomes

$$H\left(0, \bar{x}(0), p(0) + \int_{\{0\}} h_x(s, \bar{x}(s))\mu(ds)\right) = 0,$$

which is a contradiction.

However, for the [AA97] results to be valid the NCO have to be weakened by substituting the cone normal to the terminal set $N_C(x)$ by a coarser one $N_{C \cap G}(x)$, where $G := \{x : h(1, x) \leq 0\}$. Furthermore, in the derivation of these results it is required that the velocity set $f(t, x, \Omega(t))$ is convex and the data is Lipschitz continuous with respect to the time variable.

The weaknesses of the results above are overcome by our approach, presented in the next chapters, that can handle nonconvex problems and data measurable in time. Such generalisations can be achieved at the expense of constraint qualifications more difficult to verify, requiring the existence of a control function pulling the state away from the boundary of the state constraint set faster than the optimal control (see Fig. 5.3). These conditions first appeared in [FV94] and are of the kind

$$\inf_{u \in \Omega(t)} h_x(t, x_0) \cdot (f(t, x_0, u) - f(t, x_0, \bar{u}(t))) < 0, \tag{5.7}$$

for t near 0. The difficulty in verifying this type of constraint qualification is precisely due

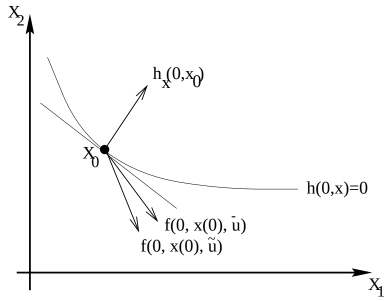

Figure 5.3: Ferreira-Vinter constraint qualification

to its dependence on the optimal control $\bar{u}$. In the remaining chapters of the thesis we shall develop some nonsmooth uniform versions of condition (5.7).

The next chapter deals with the type of degeneracy presented in the beginning of the current section, that occurs in the fixed left-endpoint problem. Its main results are reproduced in [FFV99]. Building upon this result we are able, in Chapter 7, to address more general problems (with the initial point belonging to a set) and additional types of degeneracy that are covered in a concept we call *q-degeneracy*. Finally, in Chapter 8, we deduce conditions under which we can strengthen even further the NCO and guarantee normality.

Chapter 6

Nondegeneracy in the Fixed Left-endpoint Problem

Traditional versions of the Maximum Principle for state constrained optimal control problems are satisfied with a set of degenerate multipliers in the case when the left-endpoint is fixed and lies on the boundary of the admissible region of the state constraint set. This degeneracy of the multipliers implies that no information is given about the optimal solution by the necessary conditions, rendering this important tool useless in many applications of interest. In this chapter we will develop results that, in the fixed left-endpoint case, identify problems in which we can guarantee the existence of a set of multipliers in addition to the degenerate one. We provide a strengthened, nondegenerate form of the NCO for such a class of problems.

6.1 The Fixed Left-endpoint Degeneracy

Consider a fixed left-endpoint OCP in which the trajectory starts on the boundary of the state constraint set, that is

$$h(0, x_0) = 0.$$

It can be easily verified that the choice of degenerate multipliers (or scalar multiples of them)

$$\lambda = 0, \quad \mu \equiv \delta_{\{t=0\}}, \quad p \equiv -\zeta \quad \text{with } \zeta \in \partial_x^{>} h(0, x_0), \tag{6.1}$$

satisfy the necessary conditions (Prop. 2.5.1) but bring us no information whatsoever about the optimal solution (note that the expression $p(t) + \int_{[0,t)} \gamma(s)\, \mu(ds)$ with $\gamma(0) = \zeta$ vanishes and all the equations of the Maximum Principle are automatically satisfied for any pair (x, u)). In this case the necessary conditions of optimality do not eliminate any candidate to solution we may test and hence they are useless.

Apart from the obvious motivation to strengthen the NCO to avoid the degeneracy phenomenon, strengthened NCO are particularly pertinent in the framework of Model Predictive Control of state constrained systems. In this framework we solve a collection of optimal control problems with a fixed left end point constraint $x(t_0) = \xi$, where ξ will take values along the resulting trajectory. If the trajectory hits the boundary of the state space, as expected, we are in the conditions for the degeneracy phenomenon to occur, and all the construction will crumble. For a discussing of other applications motivating the development of nondegenerate NCO see [FV94].

As seen in a previous chapter there is a growing literature on refinements of earlier necessary conditions which assert existence of multiplier sets in addition to the degenerate one, under a suitable constraint qualification. A variety of these nondegenerate necessary conditions have been derived, covering problems with nonsmooth as well as smooth data, problems in which the dynamic constraint involves a differential inclusion, or a differential equation, and in which the state constraint is formulated as a set inclusion as well as a functional inequality. A feature of earlier work, treating nonsmooth data, is the need to impose hypotheses requiring

(a) the velocity set $f(t, x, \Omega(t))$ is convex,

(b) the data are Lipschitz continuous with respect to the time variable.

In [AA97], for example, those hypotheses have an important role in ensuring the closure of certain sets of functions and that certain perturbation terms introduced in the analysis can be suitably estimated.

New methods are introduced here for proving nondegenerate necessary conditions, based on applying standard necessary conditions to the optimal control problem (P), after an appropriate modification of the data "near" to the left endpoint has been made. Their main advantage is that they are valid even when hypotheses (a) and (b) above are violated. The price we pay for reducing the hypotheses in this way is that the constraint qualification, similarly to the one which was imposed in [FV94], is dependent on the optimal control (in its initial time interval) and so is not, in general, directly verifiable. However, in certain cases *a priori* regularity properties of optimal controls permit verification of this hypothesis (see [FV94]).

6.2 Nondegeneracy Results

Consider the fixed left-endpoint optimal control problem

$$(P) \quad \text{Minimise} \quad g(x(1)) \tag{6.2}$$

subject to

$$\dot{x}(t) = f\left(t, x(t), u(t)\right) \quad \text{a.e. } t \in [0,1] \tag{6.3}$$

$$x(0) = x_0$$

$$x(1) \in C$$

$$u(t) \in \Omega(t) \quad \text{a.e. } t \in [0,1]$$

$$h\left(t, x(t)\right) \leq 0 \quad \text{for all } t \in [0,1]. \tag{6.4}$$

In addition to the basic hypothesis H1–H6 (see section 2.4) we further impose a strengthened form of H2 so that f is, in an initial time interval, Lipschitz of rank K_f not depending on t or u. More precisely we assume in addition to H1–H6 the following hypothesis.

H2′ there exist scalars $K_f > 0$ and $\epsilon' > 0$ such that

$$\|f(t,x,u) - f(t,x',u)\| \leq K_f \|x - x'\|$$

for $x, x' \in \{\bar{x}(0)\} + \delta' \mathbb{B}$, $u \in \Omega(t)$ a.e. $t \in [0, \epsilon']$.

In deriving a strengthened form of the NCO we will also make reference to the following constraint qualification.

CQ (*constraint qualification*) Let $(\bar{x}, \bar{u})$ be an optimal process. If $h(0, x_0) = 0$ then there exist positive constants K_u, ϵ, ϵ_1, δ, and a control $\tilde{u} \in \mathcal{U}$ such that for a.e. $t \in [0, \epsilon)$

$$\|f(t, x_0, \bar{u}(t))\| \leq K_u, \quad \|f(t, x_0, \tilde{u}(t))\| \leq K_u,$$

and

$$\zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta$$

for all $\zeta \in \partial_x^{>} h(s,x),\ s \in [0, \epsilon),\ x \in \{x_0\} + \epsilon_1 \mathbb{B}$.

This constraint qualification identifies the class of problems in which we can strengthen the NCO to avoid the left-end point degeneracy. The main result here is that if H1 – H6, H2′ and CQ are satisfied, then the necessary conditions of optimality (Prop. 2.5.1) hold with the non-triviality condition strengthened to

$$\mu\{[0,1] \backslash \{0\}\} + ||q||_{L_\infty} + \lambda > 0, \tag{6.5}$$

where $q(t) = p(t) + \int_{[0,t)} \gamma(s)\mu(ds)$.

Note that the set of degenerate multipliers

$$\lambda = 0, \quad \mu \equiv \beta\delta_{\{t=0\}}, \quad \text{and } p \equiv -\beta\zeta \quad \text{with } \zeta \in \partial_x^{>} h(0, x_0) \text{ for some } \beta > 0. \tag{6.6}$$

satisfies the traditional nontriviality condition

$$\mu\{[0,1]\} + ||p||_{L_\infty} + \lambda > 0, \tag{6.7}$$

but not (6.5). Thus, we have guaranteed the existence of a set of multipliers distinct from the degenerate set. This result is expressed in the following nondegenerate necessary conditions of optimality.

Theorem 6.2.1 *Let $(\bar{x}, \bar{u})$ be a strong local minimiser for (P). Assume that hypotheses H1–H6 and H2′ are satisfied. Assume also that the constraint qualification CQ is satisfied. Then there exist an absolutely continuous function $p : [0,1] \mapsto \mathbb{R}^n$, a measurable function γ, a nonnegative Radon measure $\mu \in C^*([0,1], \mathbb{R})$, and a scalar $\lambda \geq 0$ such that*

$$-\dot{p}(t) \in \bar{\partial}_x(q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))) \quad a.e.\ t \in [0,1], \tag{6.8}$$

$$-q(1) \in N_C(\bar{x}(1)) + \lambda\partial g(\bar{x}(1)), \tag{6.9}$$

$$\gamma(t) \in \partial_x^{>} h(t, \bar{x}(t)) \quad \mu\text{-}a.e.\ , \tag{6.10}$$

$$\text{supp}\{\mu\} \subset \{t \in [0,1] : h(t, \bar{x}(t)) = 0\}, \tag{6.11}$$

for almost every $t \in [0,1]$, $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto q(t) \cdot f(t, \bar{x}(t), u) \tag{6.12}$$

and,

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda > 0, \tag{6.13}$$

where

$$q(t) = \begin{cases} p(t) + \displaystyle\int_{[0,t)} \gamma(s)\mu(ds) & t \in [0,1) \\ p(1) + \displaystyle\int_{[0,1]} \gamma(s)\mu(ds) & t = 1. \end{cases}$$

Remark 6.2.2 *In the case where h is continuously differentiable, ζ can be replaced by $h_x(0, x_0)$, simplifying the constraint qualification.*

Remark 6.2.3 *The case of multiple state constraints*

$$h_i(t, x) \leq 0 \quad i = 1, 2, \cdots, N$$

can be handled defining

$$h(t, x) := \max\{h_i(t, x) : i = 1, 2, \cdots, N\}.$$

Since if each h_i satisfies H6, it can be easily verified that h will also satisfy it.

Remark 6.2.4 *Consider a variation on problem (P) by replacing the state constraint by the implicit constraint*

$$x(t) \in X(t)$$

in which $X : [0, 1] \rightrightarrows \mathbb{R}^n$ is a given multifunction with closed graph.

Taking the function h to be the distance function to the set X

$$h(t, x) = d_{X(t)}(x) := \inf\{\|x - y\| : y \in X(t)\}$$

we return to a problem like (P).

One natural concern in this approach is that if $0 \in \partial_x^> d_{X(t)}(x)$ then the inequality of the constraint qualification CQ can never be satisfied. We note that $0 \in \partial_x d_{X(t)}(x)$ for every $x \in \operatorname{cl} X(t)$, but the same is not true when we consider $\partial_x^> d_{X(t)}(x)$.

Under some additional assumption below, A1, we can even ensure that $0 \notin \partial_x^> d_{X(t)}(x)$ for every $x \in (\{x_0\} + \epsilon_1 \mathbb{B})$ (see [LR91, Prop. 2.3]).

A1 *Let $\hat{N}_{X(t)}(x) := \limsup \big\{N_{X(s)}(y) : y \in X(s),\ (s, y) \to (t, x)\big\}$. For all $t \in [0, \epsilon]$, $x \in (\{x_0\} + \epsilon_1 \mathbb{B}) \cap \operatorname{bdy} X(t)$*

$$\operatorname{co} \hat{N}_{X(t)}(x) \text{ is pointed.} \tag{6.14}$$

We recall that a convex cone N is said to be pointed if for any nonzero elements $x_1, x_2 \in N \setminus \{0\}$, we have $x_1 + x_2 \neq 0$.

6.3 Proof of Theorem 6.2.1

In what follows we shall assume that $h(0, x_0) = 0$, since, otherwise, the conditions of Theorem 6.2.1 cannot be satisfied by the trivial multipliers (6.6), and in this case $\mu\{\{0\}\} = 0$

and the theorem would be no different from the standard necessary conditions of optimality.

The key idea of the proof is to replace the original control problem by one in which the state constraint is inactive on $[0, \alpha]$ for arbitrary small α. The multipliers for this new problem are nondegenerate. We then obtain a set of multipliers for the original problem by passing to the limit $\alpha \downarrow 0$. Our construction is of such a nature that the limiting multipliers are still nondegenerate.

For some small scalar α in $(0, 1]$ consider measurable functions v, and absolutely continuous functions x satisfying

$$(S) \begin{cases} \dot{x}(t) = f(t, x(t), \bar{u}(t)) + v(t) \cdot \Delta f(t, x(t)) & \text{a.e. } t \in [0, \alpha] \\ x(0) = x_0 & \\ x(t) \in \{\bar{x}(t)\} + \delta' \mathbb{B} & \text{all } t \in [0, \alpha] \\ v(t) \in \{0\} \cup \{1\} & \text{a.e. } t \in [0, \alpha] \end{cases} \tag{6.15}$$

where we define

$$\Delta f(t, x) := f(t, x, \tilde{u}(t)) - f(t, x, \bar{u}(t)). \tag{6.16}$$

This system describes a trajectory based on the optimal process perturbed, in the beginning of the time interval, towards the inside of the admissible region, by the control $\tilde{u}$ chosen in the constraint qualification CQ. In Lemma 6.3.2 we prove that, in fact, the trajectory is inside the admissible region for some interval of time and for all possible choices of v. This enable us to write in Lemma 6.3.3, a sequence of perturbed equivalent problems (in the sense that they have the same solution). When we apply the necessary conditions of optimality to these new problems, a sequence of multipliers is obtained converging to a nondegenerate set of multipliers for the original problem.

We start by proving, in Lemma 6.3.1, some simple results on the growth of the trajectory that are needed later.

Lemma 6.3.1 *Consider a pair of functions (x, v) solving the system of equations (S), and $\bar{x}$ solving (P). There exist positive constants A and B such that for some α small enough*

$$\begin{aligned} \|x(t) - x_0\| &\leq At, \\ \|x(t) - \bar{x}(t)\| &\leq B \int_0^t v(s) ds \end{aligned}$$

for all $t \in [0, \alpha]$.

Proof. Choose an α smaller than the ϵ of CQ and smaller than the ϵ' of (H2). Integrating x we have that

$$\begin{aligned}
\|x(t) - x_0\| &\leq \int_0^t \|f(s, x(s), \bar{u}(s)) + v(s) \cdot \Delta f(s, x(s))\| \, ds \\
&\leq \int_0^t (1 - v(s)) \, \|f(s, x(s), \bar{u}(s)) - f(s, x_0, \bar{u}(s))\| \\
&\quad + v(s) \|f(s, x(s), \tilde{u}(s)) - f(s, x_0, \tilde{u}(s))\| \\
&\quad + (1 - v(s)) \|f(s, x_0, \bar{u}(s))\| \\
&\quad + v(s) \|f(s, x_0, \tilde{u}(s))\| \, ds \\
&\leq \int_0^t K_f \|x(s) - x_0\| \, ds + K_u t.
\end{aligned}$$

Applying the Gronwall-Bellman inequality (see e.g. [War72]) yields

$$\begin{aligned}
\|x(t) - x_0\| &\leq K_u t + \mathrm{e}^{K_f t} \int_0^t K_f K_u s \, ds \\
&\leq K_u t + \frac{1}{2} K_f K_u \mathrm{e}^{K_f t} t^2,
\end{aligned}$$

and as $0 \leq t \leq \alpha \leq 1$, we obtain

$$\|x(t) - x_0\| \leq K_u t + \frac{1}{2} K_f K_u \mathrm{e}^{K_f} t = At, \tag{6.17}$$

where $A := K_u + \frac{1}{2} K_f K_u \mathrm{e}^{K_f}$. The first assertion is proved.

Similarly

$$\begin{aligned}
\|x(t) - \bar{x}(t)\| &\leq \int_0^t \|f(s, x(s), \bar{u}(s)) + v(s) \cdot \Delta f(s, x(s)) \\
&\quad - f(s, \bar{x}(s), \bar{u}(s))\| \, ds \\
&= \int_0^t \|f(s, x(s), \bar{u}(t)) - f(s, \bar{x}(s), \bar{u}(s)) \\
&\quad + v(s) \cdot [f(s, x(s), \tilde{u}(s)) - f(s, x_0, \tilde{u}(s))] \\
&\quad - v(s) \cdot [f(s, x(s), \bar{u}(s)) - f(s, x_0, \bar{u}(s))] \\
&\quad + v(s) \cdot [f(s, x_0, \tilde{u}(s)) - f(s, x_0, \bar{u}(s)]\| \; ds \\
&\leq \int_0^t [K_f \|x(s) - \bar{x}(s)\| + 2v(s) K_f \|x(s) - x_0\|] \; ds \\
&\quad + 2K_u \int_0^t v(s) ds \\
&\leq \int_0^t K_f \|x(s) - \bar{x}(s)\| \, ds + 2AK_f \int_0^t v(s) s \, ds \\
&\quad + 2K_u \int_0^t v(s) ds \\
&\leq \int_0^t K_f \|x(s) - \bar{x}(s)\| \, ds + (2K_f A + 2K_u) \int_0^t v(s) ds.
\end{aligned}$$

Applying once again the Gronwell-Bellman inequality and defining $A_1 = 2K_f A + 2K_u$ we obtain

$$\begin{aligned} \|x(t) - \bar{x}(t)\| &\leq A_1 \int_0^t v(s)ds + \mathrm{e}^{K_f t} \int_0^t K_f A_1 \int_0^\tau v(s)ds\, d\tau \\ &\leq (A_1 + \mathrm{e}^{K_f} K_f A_1) \int_0^t v(s)ds, \end{aligned}$$

or equivalently

$$\|x(t) - \bar{x}(t)\| \leq B \int_0^t v(s)ds,$$

where $B := A_1 + \mathrm{e}^{K_f} K_f A_1$, proving the second assertion. □

The following lemma is of key importance. It establishes that every trajectory x associated with the system of equations (S) satisfies the state constraint on some initial interval of time.

Lemma 6.3.2 *By reducing the size of α if necessary we can ensure that*

$$h(t, x(t)) \leq 0 \quad \textit{for all } t \in [0, \alpha], \tag{6.18}$$

for all trajectories x solving system (S).

Proof.

Choose an α satisfying

$$\alpha < \min\left\{ \frac{\delta}{8K_h K_f (A+B)}, \frac{\epsilon_1}{A}, \epsilon \right\}. \tag{6.19}$$

Suppose, in contradiction, that for some fixed $t \in [0, \alpha]$

$$h(t, x(t)) > 0. \tag{6.20}$$

Define for $\beta \in [0, 1]$

$$r(\beta) := h(t, \bar{x}(t) + \beta(x(t) - \bar{x}(t))).$$

In view of the properties of h as a function of x, r is continuous. We have also that

$$\begin{aligned} r(0) &= h(t, \bar{x}(t)) \leq 0, \\ r(1) &= h(t, x(t)) > 0. \end{aligned}$$

It follows that the set

$$D := \{\beta \in [0, 1] : r(\beta) = 0\}$$

is non-empty, closed, and bounded. We can therefore define

$$\beta_m := \max_{\beta \in D} \beta.$$

Since $r(1) > 0$, we have $\beta_m < 1$.

Take any $\beta \in (\beta_m, 1]$.

Applying the Lebourg Mean-Value Theorem ([Cla83]), we obtain

$$\begin{aligned} h(t, x(t)) - r(\beta) &= \zeta_t \cdot [x(t) - \bar{x}(t) - \beta(x(t) - \bar{x}(t))] \\ &= (1-\beta)\zeta_t \cdot [x(t) - \bar{x}(t)] \end{aligned}$$

for some $\zeta_t \in \bar{\partial}_x h(t, \hat{x})$ and $\hat{x}$ in the segment $(x(t), \bar{x}(t) + \beta[x(t) - \bar{x}(t)])$.

As $r(\beta) > 0$ for all $\beta \in (\beta_m, 1]$, we have that $h(t, \hat{x}) > 0$, which implies that $\bar{\partial}_x h(t, \hat{x}) \subset \partial_x^{>} h(t, \hat{x})$. It follows that $\zeta_t \in \partial_x^{>} h(t, \hat{x})$.

Expanding the expression above yields

$$\begin{aligned} & h(t, x(t)) - r(\beta) \\ & \quad = (1-\beta)\,\zeta_t \cdot \int_0^t [f(s, x(s), \bar{u}(s)) + v(s)\Delta f(s, x(s)) \\ & \qquad\qquad -f(s, \bar{x}(s), \bar{u}(s))]\, ds \\ & \quad \leq (1-\beta)\left(\zeta_t \cdot \int_0^t v(s)\Delta f(s, x(s))\, ds\ + \|\zeta_t\| K_f \int_0^t \|x(s) - \bar{x}(s)\|\, ds\right) \\ & \quad \leq (1-\beta)\left(\int_0^t v(s)\zeta_t \cdot \Delta f(s, x_0)\, ds + 2K_f \|\zeta_t\| \int_0^t v(s)\|x(s) - x_0\|\, ds \right. \\ & \qquad\qquad \left. + K_h K_f \int_0^t \|x(s) - \bar{x}(s)\|\, ds\right) \\ & \quad \leq (1-\beta)\left(-\delta \int_0^t v(s)\, ds + 2K_f K_h A t \int_0^t v(s)\, ds \right. \\ & \qquad\qquad \left. + K_h K_f B \int_0^t \int_0^s v(\tau) d\tau\, ds\right) \\ & \quad \leq (1-\beta)\,(-\delta + K_h K_f (2A + B)t) \int_0^t v(s) ds \\ & \quad \leq 0 \quad \text{for all } \beta \in (\beta_m, 1]. \end{aligned}$$

Here we have used the fact that the norm of every element of the subdifferential is bounded by the Lipschitz rank of the function. In the last two inequalities we have used CQ and (6.19).

Since r is continuous and $r(\beta_m) = 0$ it follows that If we take the limit as $\beta \to \beta_m^+$ of both terms of the inequality

$$h(t, x(t)) - r(\beta) \leq 0,$$

we obtain

$$h(t, x(t)) \leq 0.$$

This contradicts (6.20). The proof is complete. □

Take a decreasing sequence $\{\alpha_i\}$ on $(0, \alpha)$, converging to zero. Associate with each α_i the following problem (P_i), in which satisfaction of the state constraint is enforced only on the subinterval $[\alpha_i, 1]$.

$$(P_i) \quad \text{Minimise} \quad g(x(1)) \tag{6.21}$$

subject to

$$\dot{x}(t) = f(t, x(t), \bar{u}(t)) + v(t) \cdot \Delta f(t, x(t)) \quad \text{a.e. } t \in [0, \alpha_i)$$

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [\alpha_i, 1] \tag{6.22}$$

$$x(0) = x_0$$

$$x(1) \in C$$

$$u(t) \in \Omega(t) \quad \text{a.e. } t \in [\alpha_i, 1]$$

$$v(t) \in \{0\} \cup \{1\} \quad \text{a.e. } t \in [0, \alpha_i)$$

$$h(t, x(t)) \leq 0 \quad \text{for all } t \in [\alpha_i, 1]. \tag{6.23}$$

Note that we can write the first dynamic equation as

$$\dot{x}(t) = f(t, x(t), \hat{u}(t)) \quad \text{a.e. } t \in [0, \alpha_i)$$

where

$$\hat{u}(t) = \begin{cases} \bar{u}(t) & \text{if } v(t) = 0 \\ \tilde{u}(t) & \text{if } v(t) = 1. \end{cases}$$

The function $\hat{u}$ is a measurable function and $\hat{u} \in \Omega(t)$. These facts combine with the previous lemma to ensure that all admissible state trajectories x for (P_i) such that $\|x(t) - \bar{x}(t)\|_{L^\infty} < \delta'$ are contained in the set of admissible trajectories of (P). Moreover, the process $(x, (u, v)) \equiv (\bar{x}, (\bar{u}, 0))$ for (P_i) has cost identical to that of (P). We have proved the following lemma.

Lemma 6.3.3 *For each i, the process $(\bar{x}, (\bar{u}, 0))$ is a strong local minimizer for (P_i).*

The necessary conditions for problem (P_i) assert the existence of an arc $p_i : [0, 1] \mapsto \mathbb{R}^n$, a nonnegative Radon measure $\mu_i \in C^*([\alpha_i, 1], \mathbb{R})$, a measurable function γ_i, and a scalar $\lambda_i \geq 0$ such that

$$\mu_i\{[\alpha_i, 1]\} + ||p_i|| + \lambda_i > 0, \tag{6.24}$$

$$-\dot{p}_i(t) \in \begin{cases} \bar{\partial}_x \left(p_i(t) \cdot f(t, \bar{x}(t), \bar{u}(t))\right) & \text{a.e. } t \in [0, \alpha_i) \\ \bar{\partial}_x \left(\left(p_i(t) + \int_{[\alpha_i, t)} \gamma_i(s)\mu_i(ds) \right) \right. & \\ \qquad \left. \cdot f(t, \bar{x}(t), \bar{u}(t)) \right) & \text{a.e. } t \in [\alpha_i, 1] \end{cases} \tag{6.25}$$

$$-\left(p_i(1) + \int_{[\alpha_i, 1]} \gamma_i(s)\mu_i(ds) + \lambda_i \xi_i \right) \in N_C(\bar{x}(1)) \tag{6.26}$$

where $\xi_i \in \partial_x g(\bar{x}(1))$,

$$\gamma_i(t) \in \partial_x^> h(t, \bar{x}(t)) \quad \mu_i\text{-a.e. }, \tag{6.27}$$

$$\text{supp}\{\mu_i\} \subset \{t \in [\alpha_i, 1] : h\left(t, \bar{x}(t)\right) = 0\}, \tag{6.28}$$

for almost every $t \in [0, \alpha_i)$, $v(t) = 0$ maximises over $\{0\} \cup \{1\}$

$$\nu \mapsto \nu\, p_i(t) \cdot [f(t, \bar{x}(t), \tilde{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))], \tag{6.29}$$

and for almost every $t \in [\alpha_i, 1]$, $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto \left(p_i(t) + \int_{[\alpha_i, t)} \gamma_i(s)\mu_i(ds) \right) \cdot f\left(t, \bar{x}(t), u\right). \tag{6.30}$$

It remains to pass to the limit as $i \to \infty$ and thereby to obtain a set of nondegenerate multipliers for the original problem.

Without changing the notation, we extend μ_i as a regular Borel measure on $[0, 1]$

$$\mu_i(B) = \mu_i(B \cap [\alpha_i, 1]) \quad \text{for all } \text{ Borel set } B \subset [0, 1].$$

Extend also γ_i, originally defined on $[\alpha_i, 1]$, arbitrarily to the interval $[0, 1]$ as a Borel measurable function. With these extensions, noting that $\mu([0, \alpha_i)) = 0$, we can write

$$-\dot{p}_i(t) \in \bar{\partial}_x \left(\left(p_i(t) + \int_{[0, t)} \gamma_i(s)\mu_i(ds) \right) \cdot f(t, \bar{x}(t), \bar{u}(t)) \right) \quad \text{a.e. } t \in [0, 1]$$

By scaling the multipliers we can then ensure that

$$\|p_i\| + \|\mu_i\| + \lambda_i = 1.$$

The multifunction $\partial_x^> h$ is uniformly bounded, compact, convex, and has a closed graph. As $\{p_i\}$ is uniformly bounded and $\{\dot{p}_i\}$ is uniformly integrally bounded, we can arrange by means of subsequence extraction [Cla83, Thm. 3.1.7, Prop. 3.1.8] that

$$p_i \to p \text{ uniformly}, \quad \gamma_i d\mu_i \to \gamma d\mu \text{ weak}^*, \quad \lambda_i \to \lambda, \quad \xi_i \to \xi$$

where μ is the weak* limit of μ_i, γ is a measurable selection of $\partial_x^> h(t, \bar{x}(t))$ μ a.e., and $\xi \in \partial g(\bar{x}(1))$. To obtain ξ we have used the fact that $\partial g(\bar{x}(1))$ is a compact set.

It follows that the conditions (6.11), (6.13), and (6.8) for problem (P) are satisfied. Also as $N_C(\bar{x}(1))$ is closed, (6.9) also holds.

Consider the set $S_i = [\alpha_i, 1] \backslash \Omega_i$ where Ω_i is a null Lebesgue measure set in $[\alpha_i, 1]$ containing all times where the maximisation of (6.30) is not achieved at $\bar{u}$. We can then write

$$\left(p_i(t) + \int_{[\alpha_i,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f\left(t, \bar{x}(t), u\right) \leq$$
$$\left(p_i(t) + \int_{[\alpha_i,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f\left(t, \bar{x}(t), \bar{u}(t)\right),$$

for all $t \in S_i$ and for all $u \in \Omega(t)$. Now consider the full measure set $S = (0, 1] \backslash \bigcup_i \Omega_i$. Fix some t in S. Then for all $i > N$, where N is such that $\alpha_N \leq t$ we have

$$\left(p_i(t) + \int_{[0,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f\left(t, \bar{x}(t), u\right) \leq$$
$$\left(p_i(t) + \int_{[0,t)} \gamma_i(s)\mu_i(ds)\right) \cdot f\left(t, \bar{x}(t), \bar{u}(t)\right).$$

for all $u \in \Omega(t)$. Applying limits to both sides of this inequality we obtain (6.12).

At this point we have established that the set of multipliers (p, μ, λ), obtained as a limit of the subsequence (p_i, μ_i, λ_i) satisfies the necessary conditions of optimality for the original problem (P).

It remains to verify

$$\mu\{(0, 1]\} + ||q||_{L^\infty} + \lambda > 0. \tag{6.31}$$

This is accomplished with the help of the following lemma, that will also be useful later.

Lemma 6.3.4 *For any scalar $\epsilon' > 0$, we can find $\alpha' > 0$ such that*

$$p(t) \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] \leq \epsilon' \quad \textit{for a.e.} \quad t \in [0, \alpha').$$

Proof. The maximisation condition on v (6.29) implies

$$p_i(t) \cdot [f(t, \bar{x}(t), \tilde{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] \leq 0 \quad \text{a.e. } t \in [0, \alpha_i). \tag{6.32}$$

It follows that

$$\begin{aligned}
&p(t) \cdot \Delta f(t, x_0) \\
&\qquad = p_i(t) \cdot \Delta f(t, \bar{x}(t)) + [p(t) - p_i(t)] \cdot \Delta f(t, x_0) \\
&\qquad\quad + p_i(t) \cdot [\Delta f(t, x_0) - \Delta f(t, \bar{x}(t))] \\
&\qquad \leq 2K_u \|p_i(t) - p(t)\| + 2K_f \|\bar{x}(t) - x_0\| \|p_i(t)\| \\
&\qquad \leq 2K_u \|p_i(t) - p(t)\| + 2K_f At \|p_i(t)\|
\end{aligned}$$

By the uniform convergence of p_i, we can make $\|p_i - p\| < \dfrac{1}{2K_u}\dfrac{\epsilon'}{2}$ for any $\epsilon' > 0$ of our choice provided we choose a sufficient large i (say $i > N$, some $N > 0$). Moreover as $\|p_i\| \leq 1$ limiting t to $t \leq \dfrac{1}{2K_f A}\dfrac{\epsilon'}{2}$ we obtain

$$p(t) \cdot \Delta f(t, x_0) \leq \epsilon' \quad \text{for a.e. } \; t \in [0, \alpha')$$

if $\alpha' \leq \min \left\{ \alpha_N, \dfrac{\epsilon'}{4K_f A} \right\}$ □

In view of the constraint qualification, there exists a constant $\delta > 0$ such that, for all i and for a.e. $t \in [0, \alpha_i)$,

$$\zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta,$$

for all $\zeta \in \partial_x^> h(s, x)$, $\; s \in [0, \epsilon)$, $\; x \in \{x_0\} + \epsilon_1 \mathbb{B}$.

Suppose, in contradiction to (6.31), that

$$\mu\{(0, 1]\} + ||q||_{L^\infty} + \lambda = 0. \tag{6.33}$$

Since $(\lambda, \mu, p) \neq 0$, we must have

$$\begin{aligned}
&\lambda = 0, \\
&\mu = \beta \delta_{\{0\}}, \\
&p(t) = -\beta\zeta \quad \text{for some } \beta > 0 \text{ and } \zeta \in \partial_x^> h(0, x_0).
\end{aligned} \tag{6.34}$$

The constraint qualification CQ implies

$$p(t) \cdot \Delta f(t, x_0) = -\beta\zeta \cdot \Delta f(t, x_0) > \delta\beta \quad \text{a.e. } \; t \in [0, \alpha_i).$$

But this last expression contradicts Lemma 6.3.4 if we choose $\epsilon' < \delta\beta$. The proof is complete.

Chapter 7

q-Degeneracy: A Phenomenon of the State Constrained Optimal Control Problems

In this chapter we introduce the definition of a particular type of degeneracy associated with the pathwise state constraints which we call *q-degeneracy*. We generalise the results of the previous chapter to problems with arbitrary (closed) initial set and give results for additional cases of degeneracy covered by our notion of q-degeneracy. By combining the conditions (constraint qualifications) under which we can avoid each type of q-degeneracy we deduce a strengthened form of the NCO that does not allow any set of q-degenerate multipliers.

7.1 q-Degeneracy

Consider the following optimal control problem with both end-point constraints in the form of a set inclusion.

$$
\begin{aligned}
(P)\quad & \text{Minimise} \quad && g(x(0),x(1)) && && (7.1)\\
& \text{subject to} && && && \\
& && \dot{x}(t) = f\,(t,x(t),u(t)) && \text{a.e. } t \in [0,1] && (7.2)\\
& && x(0) \in C_0 && && \\
& && x(1) \in C_1 && && \\
& && u(t) \in \Omega(t) && \text{a.e. } t \in [0,1] &&
\end{aligned}
$$

$$h\left(t, x(t)\right) \leq 0 \qquad \text{for all } t \in [0, 1]. \tag{7.3}$$

For this problem we provide strengthened forms of the NCO of Prop. 2.5.1 , that we recall here for convenience.

Proposition 7.1.1 *Assume H1 – H6. If $(\bar{x}, \bar{u})$ is an optimal process then there exist an arc $p : [0, 1] \mapsto \mathbb{R}^n$, a measurable function γ, a nonnegative Radon measure $\mu \in C^*([0, 1], \mathbb{R})$, and, a scalar $\lambda \geq 0$ such that*

$$\mu\{[0, 1]\} + ||p||_{L^\infty} + \lambda > 0, \tag{7.4}$$

$$-\dot{p}(t) \in \bar{\partial}_x \left(q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))\right) \quad a.e.\ t \in [0, 1], \tag{7.5}$$

$$(p(0), -q(1)) \in N_{C_0}(\bar{x}(0)) \times N_{C_1}(\bar{x}(1)) + \lambda \partial g(\bar{x}(0), \bar{x}(1)), \tag{7.6}$$

$$\gamma(t) \in \partial_x^> h(t, \bar{x}(t)) \quad \mu\, a.e.\ , \tag{7.7}$$

$$\text{supp}\{\mu\} \subset \left\{t \in [0, 1] : h\left(t, \bar{x}(t)\right) = 0\right\}, \tag{7.8}$$

and for almost every $t \in [0, 1]$, $\bar{u}(t)$ maximises over $\Omega(t)$

$$u \mapsto q(t) \cdot f\left(t, \bar{x}(t), u\right), \tag{7.9}$$

where

$$q(t) = \begin{cases} p(t) + \displaystyle\int_{[0,t)} \gamma(s)\mu(ds) & t \in [0, 1) \\ p(1) + \displaystyle\int_{[0,1]} \gamma(s)\mu(ds) & t = 1. \end{cases}$$

We have seen in a previous chapter that literature on strengthened forms of the Maximum Principle avoiding some form of degeneracy is substantial. The terms normality, calmness, nonsingularity and degeneracy have been used in the literature to describe works on how to avoid this phenomenon. However, in particular the term degeneracy has been reserved in the literature of optimal control problems with state constraints to describe one specific kind of "degeneracy"phenomenon where the multiplier associated with the state constraint plays a crucial role in making the set of multipliers degenerate. See [AA95, AA97, FV94, FFV99].

The motivation in this chapter is to identify the common features of this degeneracy specific to state constrained OCP's, define them mathematically, and explore how the NCO should be strengthened to avoid this phenomenon. Since the most distinguished feature of this specific degeneracy is that the function of bounded variation $t \mapsto p(t) + \int_{[0,t)} \gamma(s)\mu(ds)$ — which is usually denoted by q — vanishes for almost every t, we denote this concept by *q-degeneracy* and define it as follows.

Definition 7.1.1 A set of multipliers (λ, p, μ) associated with Necessary Conditions of Optimality for state constrained control problems is said to be *q-degenerate* if it satisfies

$$\lambda = 0, \tag{7.10}$$

and

$$p(t) + \int_{[0,t)} \gamma(s)\mu(ds) = 0, \quad \text{for a.e. } t \in [0,1] \tag{7.11}$$

for some $\gamma(s) \in \partial_x^> h(s, \bar{x}(s))$ satisfying the NCO of Prop. 7.1.1. The Necessary Conditions are said to be q-degenerate when they are satisfied with q-degenerate multipliers and are q-nondegenerate when they do not allow any q-degenerate multipliers.

Furthermore, we classify the q-degeneracy phenomenon in three distinct cases:

Left-endpoint q-degeneracy when the multipliers are q-degenerate and

$$\mu\{(0,1]\} = 0.$$

Right-endpoint q-degeneracy when the multipliers are q-degenerate and

$$\mu\{[0,1)\} = 0.$$

Intermediate q-degeneracy when the multipliers are q-degenerate and

$$\mu\{\{0\} \cup \{1\}\} = 0.$$

In the last chapter, we have seen that the left-endpoint q-degeneracy occurs when the initial state starts on the boundary of the admissible region (that is, $h(0, \bar{x}(0)) = 0$). In problems "symmetric" to this one (i.e. $h(1, \bar{x}(1)) = 0$), the right-endpoint q-degeneracy might occur. Consider the following example which is capable of illustrating two different types of q-degeneracy.

Example 7.1.2

$$\text{Minimise} \quad \int_0^1 x(t)dt$$

subject to

$$\begin{aligned}
&\dot{x}(t) = u(t) && \text{a.e. } t \in [0,1]\\
&x(0) = 1 &&\\
&x(1) = 0 &&\\
&u(t) \in [-2,2] && \text{a.e. } t \in [0,1]\\
&-x(x-\frac{1}{2})^2 \leq 0 && \text{for all } t \in [0,1].
\end{aligned}$$

We can easily see that the unique solution to this problem is (see Fig. 7.1)

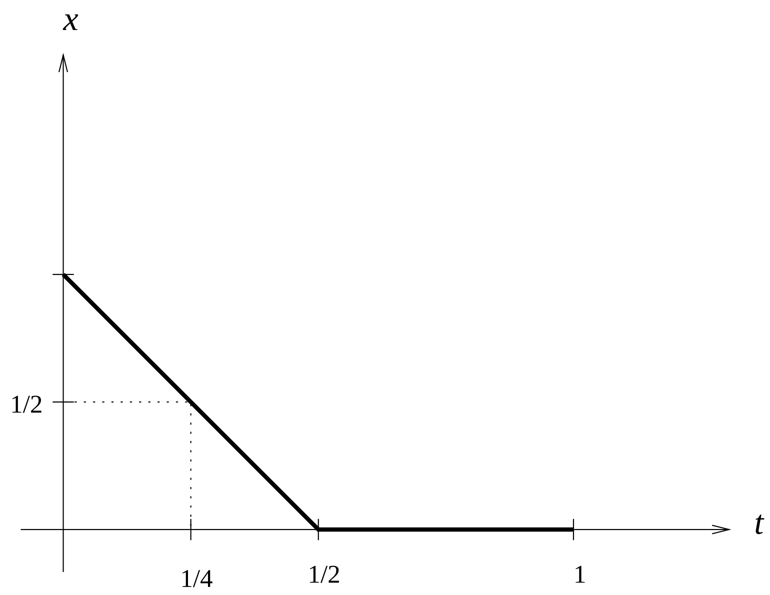

Figure 7.1: An example where q-degeneracy occurs

$$u(t) = -2, \quad x(t) = 1-2t, \quad \text{for } t \in [0, \tfrac{1}{2}),$$

and

$$u(t) = 0, \quad x(t) = 0, \quad \text{for } t \in [\tfrac{1}{2}, 1].$$

However, the right-endpoint q-degenerate set of multipliers

$$\lambda = 0, \quad \mu = \delta_{\{1\}}, \quad p \equiv 0,$$

satisfies the NCO for any admissible pair (x, u).

The intermediate q-degeneracy occurs when both the state constraint and its gradient vanish at some instant in the interior of the time interval. To illustrate this, consider once again the problem above. Since $x(0) = 1$ and $x(1) = 0$, by continuity any admissible trajectory satisfies $x(\tau) = \frac{1}{2}$ for some $\tau \in (0, 1)$, (in the optimal case $\tau = \frac{1}{4}$). Notice that $h(\tau, \frac{1}{2}) = 0$ and $h_x(\tau, \frac{1}{2}) = 0$. It follows that for any admissible pair (x, u) the NCO are satisfied with the intermediate q-degenerate set of multipliers

$$\lambda = 0, \quad \mu = \delta_{\{\tau\}}, \quad p \equiv 0.$$

In the next sections we provide conditions (constraint qualifications) that identify the class of problems where each type of q-degeneracy can be avoided and we derive the corresponding strengthened forms of the NCO. We start by generalising the results on the left-endpoint q-degeneracy of the previous chapter by allowing an arbitrary initial set. Next, we address the remaining cases of q-degeneracy. Finally, by combining the results found we will be able to write a q-nondegenerate necessary condition. Conditions for normality will not be analysed until the next chapter.

7.2 q-Nondegenerate Conditions

In addition to the basic hypothesis H1–H6 of Section 2.4 we further impose a strengthened form of H2 so that f is in an initial and final time interval Lipschitz of rank K_f not depending on t or u. More precisely we assume in addition to H1–H6 the following hypothesis.

H2″ There exist scalars $K_f > 0$ and $\epsilon' > 0$ such that

$$\|f(t, x, u) - f(t, x', u)\| \leq K_f \|x - x'\|$$

for $x, x' \in (\{\bar{x}(0)\} + \delta'\mathbb{B}) \cup (\{\bar{x}(1)\} + \delta'\mathbb{B})$, $u \in \Omega(t)$ a.e. $t \in [0, \epsilon'] \cup [1 - \epsilon', 1]$.

In deriving the q-nondegenerate NCO we will make reference to the following constraint qualifications. We start by a condition on the left end-point.

CQ$_L$ (Constraint Qualification which excludes left endpoint q-degeneracy)

Consider the set $H_0 := \{x \in \mathbb{R}^n : h(0, x) \leq 0\}$ and $x_0 := \bar{x}(0)$.

At least one of the following conditions is satisfied:

CQ$_L$1 $x_0 \in \operatorname{int} H_0$.

CQ$_L$2 $x_0 \in \operatorname{int} C_0$ and

$$0 \notin \partial_x^> h(0, x_0).$$

CQ$_L$3 x_0 is in the set $(\operatorname{bdy} H_0) \cap (\operatorname{bdy} C_0)$, the set $(\operatorname{bdy} H_0) \cap (\operatorname{bdy} C_0) \cap (\{x_0\} + \beta\mathbb{B})$ is a singleton for some $\beta > 0$, and there exist positive constants K_u, ϵ, ϵ_1, δ, and a control $\tilde{u} \in \mathcal{U}$ such that for a.e. $t \in [0, \epsilon)$

$$\|f(t, x_0, \bar{u}(t))\| \leq K_u, \quad \|f(t, x_0, \tilde{u}(t))\| \leq K_u,$$

and

$$\zeta \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta$$

for all $\zeta \in \partial_x^> h(s, x),\ s \in [0, \epsilon),\ x \in \{x_0\} + \epsilon_1\mathbb{B}$.

CQ$_L$4 x_0 is in the set $(\operatorname{bdy} H_0) \cap (\operatorname{bdy} C_0)$, the set $(\operatorname{bdy} H_0) \cap (\operatorname{bdy} C_0) \cap (\{x_0\} + \beta\mathbb{B})$ is not a singleton for any $\beta > 0$, and

$$(-N_{C_0}(x_0)) \cap \partial_x^> h(0, x_0) = \emptyset.$$

CQ$_L$5 There exists $\epsilon > 0$ such that

$$h(t, \bar{x}(t)) < 0 \quad \text{for all } t \in (0, \epsilon).$$

Remark 7.2.1 *Since x_0 belongs to both sets C_0 and H_0, we distinguish the cases when $x_0 \in \operatorname{int} H_0$, $x_0 \in \operatorname{int} C_0$, and when $x_0 \in (\operatorname{bdy} H_0) \cap (\operatorname{bdy} C_0)$. (Note that x_0 is certainly in one of these cases.) In the first two cases we should test conditions CQ_L1 or CQ_L2 respectively. In the latter case, when x_0 is on the boundary of both sets we further divide this case in two alternatives: the case when the intersection of the boundaries of both sets together with some neighbourhood of x_0 is a singleton; and the case when this intersection is not a singleton. In these cases we should test conditions CQ_L3 or CQ_L4 respectively.*

Condition CQ_L5 is a different type of constraint qualification whose satisfaction can be tested independently. It was pointed out in [FV94], but it can be easily extended to the more general problem and nonsmooth context explored here.

Similarly, we have a constraint qualification in the right end-point.

CQ$_R$ (Constraint Qualification which excludes right end-point q-degeneracy)

Consider the set $H_1 := \{x \in \mathbb{R}^n : h(1, x) \leq 0\}$ and $x_1 = \bar{x}(1)$.

At least one of the following conditions is satisfied:

CQ$_R$1 $x_1 \in \operatorname{int} H_1$.

CQ$_R$2 $x_1 \in \operatorname{int} C_1$ and

$$0 \notin \partial_x^{>} h(1, x_1).$$

CQ$_R$3 x_1 is in the set $(\operatorname{bdy} H_1) \cap (\operatorname{bdy} C_1)$, the set $(\operatorname{bdy} H_1) \cap (\operatorname{bdy} C_1) \cap (\{x_1\} + \beta\mathbb{B})$ is a singleton for some $\beta > 0$, and there exist positive constants K_u, ϵ, ϵ_1, δ, and a control $\tilde{u} \in \mathcal{U}$ such that for a.e. $t \in (1-\epsilon, 1]$

$$\|f(t, x_1, \bar{u}(t))\| \leq K_u, \quad \|f(t, x_1, \tilde{u}(t))\| \leq K_u,$$

and

$$\zeta \cdot [f(t, x_1, \tilde{u}(t)) - f(t, x_1, \bar{u}(t))] > \delta$$

for all $\zeta \in \partial_x^{>} h(s, x),\ s \in (1-\epsilon, 1],\ x \in \{x_1\} + \epsilon_1\mathbb{B}$.

CQ$_R$4 x_1 is in the set $(\operatorname{bdy} H_1) \cap (\operatorname{bdy} C_1)$, the set $(\operatorname{bdy} H_1) \cap (\operatorname{bdy} C_1) \cap (\{x_1\}) + \beta\mathbb{B})$ is not a singleton for any $\beta > 0$, and

$$N_{C_1}(x_1) \cap \partial_x^{>} h(1, x_1) = \emptyset.$$

CQ$_R$5 There exists $\epsilon > 0$ such that

$$h(t, \bar{x}(t)) < 0 \quad \text{for all } t \in (1-\epsilon, 1).$$

Finally the constraint qualification on intermediate points.

CQ$_I$ (Constraint Qualification which excludes intermediate q-degeneracy)

$$0 \notin \partial_x^{>} h(t, \bar{x}(t)), \quad \text{for all } t \in \{t \in (0,1) : h(t, \bar{x}(t)) = 0\}.$$

Theorem 7.2.2 (left-endpoint q-nondegeneracy) *Assume H1–H6 and H2″ are satisfied. Assume additionally that the constraint qualification CQ$_L$ holds. Then the NCO (Prop. 7.1.1) can be strengthened by replacing (7.4) by*

$$\lambda + \|q\|_{L^\infty} + \mu\{(0, 1]\} > 0.$$

A similar result avoiding right-endpoint q-degeneracy is obtained by assuming CQ$_R$.

Theorem 7.2.3 (right-endpoint q-nondegeneracy) *Assume H1–H6 and H2″ are satisfied. Assume additionally that the constraint qualification CQ$_R$ holds. Then the NCO (Prop. 7.1.1) can be strengthened by replacing (7.4) by*

$$\lambda + \|q\|_{L^\infty} + \mu\{[0, 1)\} > 0.$$

Theorem 7.2.4 (intermediate q-nondegeneracy) *Assume H1–H6 and H2$''$ are satisfied. Assume additionally that the constraint qualification CQ_I holds. Then the NCO (Prop. 7.1.1) can be strengthened by replacing (7.4) by*

$$\lambda + \|q\|_{L^\infty} + \mu\{\{0\} \cup \{1\}\} > 0.$$

Finally, we give the q-nondegenerate form of the NCO under the conditions CQ_L, CQ_R, and CQ_I.

Theorem 7.2.5 (q-nondegenerate Necessary Conditions) *Suppose H1–H6 and H2$''$ are satisfied. Assume additionally that the constraint qualifications CQ_L, CQ_R, and CQ_I hold. Then the NCO (Prop. 7.1.1) can be strengthened by replacing (7.4) by*

$$\lambda + \|q\|_{L^\infty} > 0.$$

7.3 Proof of the q-Nondegeneracy Results

7.3.1 Proof of Theorem 7.2.2 (left-endpoint q-nondegeneracy)

Suppose in contradiction that

$$\lambda + \|q\|_{L^\infty} + \mu\{(0,1]\} = 0.$$

By the nontriviality condition (7.4), we must have the set of q-degenerate multipliers

$$\lambda = 0, \quad \mu \equiv \beta\delta_{\{t=0\}}, \quad p \equiv -\beta\zeta \quad \text{with } \zeta \in \partial_x^{>} h(0, x_0) \text{ and some } \beta > 0. \tag{7.12}$$

Now, suppose the condition CQ_L1 is satisfied, then $\partial_x^{>} h(0, x_0)$ is an empty set. Therefore, for condition (7.7) to be meaningful, we must have

$$\mu\{\{0\}\} = 0.$$

Hence the set of left-endpoint q-degenerate multipliers (7.12) is not allowed.

Suppose condition CQ_L4 is satisfied, then condition (7.6)

$$p(0) \in N_{C_0}(x_0)$$

is not satisfied for $p = -\beta\zeta$. The degenerate multipliers do not satisfy the NCO.

If condition CQ_L2 is satisfied then

$$N_{C_0}(x_0) = \{0\}$$

and hence CQ_L4 is also satisfied.

If condition CQ_L5 is satisfied then arguments similar to the ones used in [FV94, pp 462–464] can be used to show that we can find a set of multipliers with $\mu\{\{0\}\} = 0$.

It remains to verify condition CQ_L3. If x_0 is in the set $(\text{bdy}\, H_0) \cap (\text{bdy}\, C_0)$, and the set $(\text{bdy}\, H_0) \cap (\text{bdy}\, C_0) \cap (\{x_0\} + \beta\mathbb{B})$ is a singleton for some $\beta > 0$ then we can write the problem equivalently as a fixed left-endpoint one. We are then in the condition to apply Theorem 6.2.1. The result follows. The proof is complete.

7.3.2 Proof of Theorem 7.2.3 (right-endpoint q-nondegeneracy)

First note the symmetry of problem (P) with respect to each endpoint: both endpoints constraints are in the form of a set inclusion and the objective function depends on the value of the state at each endpoint.

The right-end point q-degeneracy can be transformed in the left-end point q-degeneracy by the change of time variable $t = 1-s$. Rewriting the problem and CQ_L in the "new" time variable s, rewriting the constraint qualification in the original time variable we obtain the condition CQ_R that guarantees the existence of a set of multipliers different from the one causing right-end point degeneracy.

7.3.3 Proof of Theorem 7.2.4 (intermediate q-nondegeneracy)

Suppose in contradiction that CQ_I holds and the Maximum Principle degenerates with some $\gamma(s) \in \partial_x^{>}(s, \bar{x}(s))$ and multipliers satisfying

$$\lambda = 0, \tag{7.13a}$$

$$p(t) + \int_{[0,t)} \gamma(s)\mu(ds) = 0, \quad \text{for a.e. } t \in [0,1], \tag{7.13b}$$

$$\mu\{\{0\} \cup \{1\}\} = 0. \tag{7.13c}$$

By (7.13b) and the adjoint differential inclusion (7.5), we can write

$$dp(t) = -\gamma(t)\mu(dt),$$

and

$$\dot{p}(t) = 0 \quad \text{a.e. } t \in [0,1].$$

As p is absolutely continuous we deduce

$$p \text{ is constant,} \tag{7.14}$$

$$\gamma(a)\mu(\{a\}) = 0 \quad \text{for every scalar } a \in (0,1), \tag{7.15}$$

$$\int_S \gamma(s)\mu(ds) = 0 \quad \text{for every set } S \subset (0,1). \tag{7.16}$$

Moreover, since $\mu\{\{0\}\} = 0$ we can also deduce that $p \equiv 0$. Now combining (7.13) with the nontriviality condition (7.4) we must have

$$\mu\{(0,1)\} > 0$$

Hence, there exists a set $A \subset (0,1)$ of nonzero Lebesgue measure such that

$$h(t, \bar{x}(t)) = 0 \quad \text{for all } t \in A,$$

$$\mu(A) \neq 0,$$

and by (7.15) and (7.16)

$$\gamma(t) = 0 \quad \text{for a.e. } t \in A \tag{7.17}$$

But this last condition contradicts CQ_I. The proof is complete

7.3.4 Proof of Theorem 7.2.5 (q-nondegenerate necessary conditions)

We have seen that CQ_L, CQ_R and CQ_I imply respectively

$$\mu\{(0,1]\} + ||q||_{L^\infty} + \lambda > 0,$$

$$\mu\{[0,1)\} + ||q||_{L^\infty} + \lambda > 0,$$

$$\mu\{\{0\} \cup \{1\}\} + ||q||_{L^\infty} + \lambda > 0.$$

These conditions can be written equivalently as a disjunction: either

$$||q||_{L^\infty} + \lambda > 0 \tag{7.18}$$

or

$$\begin{cases} ||q||_{L^\infty} + \lambda = 0 \\ \mu\{(0,1]\} > 0 \\ \mu\{[0,1)\} > 0 \\ \mu\{\{0\} \cup \{1\}\} > 0. \end{cases} \tag{7.19}$$

Recall that we want to prove the existence of multipliers satisfying (7.18). We show that we can always find a set of multipliers that do not satisfy (7.19) and hence must satisfy (7.18).

Now, conditions (7.19) can be written again as a disjunction: either

$$\begin{cases} ||q||_{L^\infty} + \lambda = 0 \\ \mu\{\{0\}\} > 0 \\ \mu\{(0,1]\} > 0 \end{cases} \tag{7.20}$$

or

$$\begin{cases} ||q||_{L^\infty} + \lambda = 0 \\ \mu\{\{1\}\} > 0 \\ \mu\{[0,1)\} > 0. \end{cases} \tag{7.21}$$

Assume the first of these cases (7.20). Then $\mu\{\{0\}\} = \beta\delta_{\{0\}}$ for some $\beta > 0$ and as $q \equiv 0$ we have that $p(0) = -\beta\zeta$ for some $\zeta \in \partial_x^> h(0, x_0)$.

Using the arguments in the proof of Thm. 7.2.2, we conclude that CQ_L1, CQ_L4 and CQ_L2 lead to a contradiction with (7.20).

If CQ_L3 is satisfied, we meet the requirements of Thm. 6.2.1 and Lemma 6.3.4 establishes that for any $\epsilon' > 0$, there exists α' such that

$$p(t) \cdot \Delta f(t, x_0) \leq \epsilon' \quad \text{a.e. } t \in [0, \alpha'). \tag{7.22}$$

Furthermore CQ_L3 can be written, after α' has been reduced in size if necessary, as

$$\zeta \cdot \Delta f(t, x_0) < -\delta \quad \text{a.e. } t \in [0, \alpha').$$

It follows that

$$\begin{aligned} p(t) \cdot \Delta f(t, x_0) &= p(0) \cdot \Delta f(t, x_0) + [p(t) - p(0)] \cdot \Delta f(t, x_0) \\ &\geq -\beta\zeta - 2K_u \|p(t) - p(0)\|. \end{aligned}$$

As p is continuous, we have that for α' small enough we can make $\|p(t) - p(0)\| < \dfrac{1}{2K_u}\dfrac{\beta\delta}{4K_f A}$ for all $t \leq \alpha'$. Hence

$$p(t) \cdot \Delta f(t, x_0) > \frac{\beta\delta}{2} \quad \text{a.e. } t \in [0, \alpha').$$

But this contradicts (7.22) if we choose $\epsilon' < \dfrac{\beta\delta}{2}$.

Similar arguments using CQ_R can now be used to show that assuming (7.21) also leads to a contradiction. Hence (7.19) is not satisfied and we must have

$$||q||_{L^\infty} + \lambda > 0, \tag{7.23}$$

as was intended to show.

Chapter 8

Normality

Based on the nondegeneracy results obtained in the previous chapters we develop here even stronger necessary conditions of optimality. These conditions are guaranteed to supply non-trivial information about the minimisers, in the sense that they guarantee that the objective function is taken into account to select candidates to optimal processes.

8.1 Normality Condition

Consider the Maximum Principle for optimal control problems with either terminal state constraints or pathwise state constraints. In its conditions we cannot assume, in general, that the scalar multiplier associated with the objective function, λ, is nonzero. But, if $\lambda = 0$, then the necessary conditions merely state a relationship between the constraints and make no reference to the objective function. In these circumstances, we cannot expect the necessary conditions to generate useful information about minimisers. Therefore, it is not surprising that there is a longstanding interest in deriving conditions under which the problem is normal, that is, the NCO can be written with λ positive. (See e.g. [Hes66, Cla83].) However, available results typically explore the influence of the terminal state constraint on normality. The only exception, to the best of our knowledge, that also explores the influence of the pathwise state constraint on normality is the work of Ferreira and Vinter [FV94]. In this chapter our concern is also to explore ways of avoiding abnormality arising from pathwise state constraints. Our results generalise those in [FV94] in that they have a weaker constraint qualification and allow problems with nonsmooth data. The influence of the terminal state constraint is not analysed here (we assume $\bar{x}(1) \in \operatorname{int} C$), but we recall that a terminal state constraint can easily be transformed into a pathwise constraint.

The main result in this chapter is that for all problems satisfying the constraint qualification CQn below, we can write the NCO with the assurance that the multiplier λ (the scalar associated with the objective function) can always be chosen to be positive.

CQn (*Constraint Qualification for Normality*) We have that

$$\bar{x}(1) \in \text{int } C. \tag{8.1}$$

Furthermore, for any abnormal set of multipliers $(\lambda = 0, \mu, p)$ satisfying the conditions of Thm. 6.2.1 and such that $\tau > 0$, where

$$\tau = \inf \left\{ t \in [0,1] : \int_{[t,1]} \mu(ds) = 0 \right\},$$

there exist a positive constant ϵ and a control $\hat{u} \in \mathcal{U}$ such that for a.e. $t \in (\tau - \epsilon, \tau]$

$$\|f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))\| \leq K_u,$$

and

$$\zeta \cdot [f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))] < -\delta, \tag{8.2}$$

for all $\zeta \in \partial_x^{>} h(s, \bar{x}(s))$, when $s \in (\tau - \epsilon, \tau]$ μ-a.e. .

The interiority condition (8.1) is automatically satisfied for a free right-end-point problem. The last condition (8.2) in CQn says that there is a control that can pull the trajectory away from the boundary (faster than the optimal control) near to the last instant that the measure is active.

Theorem 8.1.1 *Assume hypotheses H1–H6 and H2". Assume also that the constraint qualifications CQ and CQn hold. Then the Necessary Conditions of Optimality 6.2.1 can be written with* $\lambda = 1$.

A somewhat stronger but easier to verify condition would be the following

CQn' We have that

$$\bar{x}(1) \in \text{int } C.$$

Furthermore, there exist a positive constant ϵ and a control $\hat{u} \in \mathcal{U}$ such that for a.e. $t \in [0,1]$

$$\|f(t, \bar{x}(t), \bar{u}(t)) - f(t, \bar{x}(t), \hat{u}(t))\| \leq K_u,$$

and

$$\zeta \cdot [f(t,\bar{x}(t),\hat{u}(t)) - f(t,\bar{x}(t),\bar{u}(t))] < -\delta,$$

for all $\zeta \in \partial_x^{>} h(s,\bar{x}(s))$, when $s \in \{t \in [0,1] : h(t,\bar{x}(t)) = 0\}$.

Note that as we do not need to determine τ the condition becomes independent of the multipliers, making this condition much easier to verify *a priori*. It is easy to see that if Equation (8.2) is satisfied for a.e. $t \in [0,1]$ and for $s \in \{t \in [0,1] : h(t,\bar{x}(t)) = 0\}$, then it certainly is satisfied on the interval $(\tau - \epsilon, \tau]$. This is because the measure μ is supported on the set of points where the constraint is active. We have proved the following corollary.

Corollary 8.1.2 *Assume hypotheses H1–H6 and H2". Assume also that the constraint qualifications CQ and CQn' hold. Then the Necessary Conditions of Optimality 6.2.1 can be written with $\lambda = 1$.*

8.2 Proof of Theorem 8.1.1

In the presence of CQ consider the Maximum Principle of Thm. 6.2.1. Define a bounded variation function q relating with the usual adjoint vector p as

$$q(t) = \begin{cases} p(t) + \displaystyle\int_{[0,t)} \gamma(s)\mu(ds) & t \in [0,1) \\ p(1) + \displaystyle\int_{[0,1]} \gamma(s)\mu(ds) & t = 1. \end{cases} \tag{8.3}$$

The adjoint inclusion can be written equivalently as

$$-\dot{p}(t) \in \bar{\partial}_x \left(q(t) \cdot f(t,\bar{x}(t),\bar{u}(t)) \right) \quad \text{a.e. } t \in [0,1].$$

Expanding the internal product and applying a well-known nonsmooth calculus rule (see [Cla83, Prop. 2.3.3]) we obtain

$$\begin{aligned} -\dot{p}(t) &\in \bar{\partial}_x \left(\sum_{i=1}^{n} q_i(t) f_i(t,\bar{x}(t),\bar{u}(t)) \right) \\ &\subset \sum_{i=1}^{n} q_i(t) \bar{\partial}_x f_i(t,\bar{x}(t),\bar{u}(t)) \quad \text{a.e. } t \in [0,1]. \end{aligned}$$

Define the matrix $\xi(t) = \begin{bmatrix} \xi_1(t) \\ \cdots \\ \xi_n(t) \end{bmatrix}$ for some $\xi_i(t) \in \bar{\partial}_x f_i(s,\bar{x}(s),\bar{u}(s))$ conveniently selected such that

$$-\dot{p}(t) = q(t) \cdot \xi(t) \quad \text{a.e. } t \in [0,1].$$

It follows that

$$p(t) = p(1) + \int_t^1 q(s)\xi(s)ds$$

or equivalently

$$q(t) = q(1) + \int_t^1 q(s)\xi(s)ds - \int_{[t,1]} \gamma(s)\mu(ds).$$

Noting also that the interiority condition in CQn implies $N_C(\bar{x}(1)) = \{0\}$ we can establish the following necessary conditions of optimality:

If $(\bar{x}, \bar{u})$ is an optimal process then there exist: a function q of bounded variation and continuous from left, a scalar $\lambda \geq 0$, and a nonnegative Radon measure $\mu \in C^*([0,1], \mathbb{R})$ such that:

$$\mu\{[0,1]\backslash\{0\}\} + ||q||_\infty + \lambda > 0, \tag{8.4}$$

$$q(t) = q(1) + \int_t^1 q(s)\xi(s)ds - \int_{[t,1]} \gamma(s)\mu(ds)$$

where $\gamma(t) \in \partial_x^> h(t, \bar{x}(t)) \quad \mu\text{-a.e.}$,

$$-q(1) \in \lambda \partial_x g(\bar{x}(1)),$$

$$\text{supp}\{\mu\} \subset \{t \in [0,1] : h(t, \bar{x}(t)) = 0\},$$

and

$$q(t) \cdot [f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))] \leq 0 \quad \text{for all } u \in \Omega(t), \quad \text{a.e. } t \in [0,1]. \tag{8.5}$$

Now suppose in contradiction that $\lambda = 0$. In this case we can write $q(1) = 0$ and

$$q(t) = \int_t^1 q(s)\xi(s)ds - \int_{[t,1]} \gamma(s)\mu(ds).$$

Let $\tau = \inf\{t \in [0,1] : \int_{[t,1]} \mu(ds) = 0\}$. If $\tau = 0$ then $\int_{(0,1]} \mu(ds) = 0$. This implies that $q(t) = 0$ for a.e. $t \in [0,1]$. Hence $\mu\{[0,1]\backslash\{0\}\} + ||q||_\infty + \lambda = 0$ and we arrive at a contradiction with the nontriviality condition (8.4).

It remains to consider the case when $\tau > 0$. We show that when $\lambda = 0$ and CQn is verified, the maximisation condition (8.5) can not be satisfied.

Defining $\Phi(t,s)$ as the transition matrix for the linear system $\dot{z}(t) = \xi(t)z(t)$, the function q can be written as

$$q(t) = -\int_{[t,1]} \gamma(s)\Phi(s,t)\mu(ds).$$

Let $\Delta f(t, \bar{x}(t)) = f(t, \bar{x}(t), \hat{u}(t)) - f(t, \bar{x}(t), \bar{u}(t))$, where $\hat{u}$ is the control function chosen in CQn and consider $t \in (\tau - \epsilon, \tau]$. We have

$$\begin{aligned}
q(t) \cdot \Delta f(t, \bar{x}(t) &= -\int_{[t,1]} \gamma(s)\Phi(s,t)\Delta f(t, \bar{x}(t))\mu(ds) \\
&= -\int_{[t,\tau]} \gamma(s)\Phi(s,t)\Delta f(t, \bar{x}(t))\mu(ds) \\
&= -\int_{[t,\tau]} \gamma(s)\Delta f(t, \bar{x}(t))\mu(ds) \\
&\quad -\int_{[t,\tau]} \gamma(s)[\Phi(s,t) - \Phi(\tau,\tau)]\Delta f(t, \bar{x}(t))\mu(ds) \\
&> \delta\mu\{[t,\tau]\} - \int_{[t,\tau]} \gamma(s)[\Phi(s,t) - \Phi(\tau,\tau)]\Delta f(t, \bar{x}(t))\mu(ds).
\end{aligned}$$

As Φ is continuous we can assure the existence of a positive scalar δ_1 such that $\|\Phi(s,t) - \Phi(\tau,\tau)\| < \dfrac{\delta}{2k_u k_h}$ for all (s,t) satisfying $\|(s,t) - (\tau,\tau)\| < \delta_1$. Hence, for a.e. $t \in (\tau - \epsilon, \tau] \cap (\tau - \delta_1, \tau]$ we have

$$q(t) \cdot \Delta f(t, \bar{x}(t) > \delta\mu\{[t,\tau]\} - \frac{\delta}{2}\mu\{[t,\tau]\} > 0$$

contradicting the maximisation condition (8.5).

Chapter 9

Conclusion

This chapter summarises the main contributions of this thesis. It also outlines some of the future developments this research suggests.

9.1 Contributions

The contributions of this thesis to the problem of controlling a nonlinear system subject to constraints are twofold. On the one hand we have introduced new MPC schemes and studied the stability of their closed-loop responses. On the other, we have contributed to techniques which describe and identify the solutions of the open-loop optimal control problems involved.

Part I: Model Predictive Control. The stability properties of Model Predictive Control (MPC) schemes have been extended in several ways, with contributions relevant to both theory and practice of nonlinear control.

The literature on Nonlinear Model Predictive Control schemes, with focus on stability properties, has been reviewed and compared with our developed frameworks. We highlight the following contributions:

- We have proposed a new framework for MPC of input-constrained and time-varying nonlinear systems:
  - This framework covers most previous MPC frameworks that use finite fixed horizon optimal control problems.
  - It allows a rigorous stability analysis in continuous-time while permitting finite sampling rates.

  - It also allows discontinuous feedbacks, which are necessary to stabilise important classes of nonlinear systems, such as the nonholonomic systems.

- For this new framework we have developed a set of conditions on the design parameters that are sufficient for stability:

  - These conditions allow us to analyse the stability of most previous MPC schemes.
  - They allow the practitioners to verify *a priori* that a given set of design parameters will lead to stability without trial-and-error procedures using simulations.
  - They permit more flexibility in the choice of stabilising design parameters. This flexibility makes possible the reduction of terminal constraints (or their elimination altogether), resulting in optimal control problems that can be solved more efficiently by current optimisation algorithms.
  - Different strategies to choose stabilising design parameters were given, for different levels of the compromise generality versus ease-of-use. These strategies ranged from showing that it is always possible to stabilise the system without using a terminal state constraint, to an easy-to-use method requiring minimal intervention from the user in choosing stabilising design parameters.

- Furthermore, it has been shown that our MPC framework is in fact a universal constructor of stabilising feedbacks. To the best of our knowledge, there is no alternative constructive method to generate a stabilising feedback for the large class of nonlinear systems we have considered.

- Another line of research taken was, instead of broadening the range of application of a particular property, to consider a particular class of systems and obtain for that class stronger properties. We have considered a more restricted class of nonlinear systems and developed a MPC scheme featuring:

  - The ability to analyse, and to a certain extent specify, the decay of the transient response of the closed loop system. More precisely, we are able to guarantee a specified rate of exponential stability.
  - The fixed-horizon open loop OCP to be solved does not impose any terminal state constraints. This feature provides some important advantages: it significantly improves the efficiency of the optimisation algorithm to be solved on-line; and it trivially guarantees feasibility of the optimal control problems involved.

– A simple way to choose a set of stabilising design parameters: a set of rules to direct this choice was provided.

– A novel way to avoid imposing regularity of the optimal controls.

Part II: Nondegenerate Necessary Conditions for State Constrained Optimal Control Problems. We have studied the problem of degeneracy occurring when trying to solve certain optimal control problems with constraints along the trajectory. New nondegenerate necessary conditions of optimality were developed and the results were compared with existent literature on the degeneracy phenomenon. The results here developed improve on the existent literature in the sense that not only we address new types of degeneracy, but also more general problems are covered. Namely, we address problems with nonsmoothness, with nonconvex velocity sets, and with data merely measurable in time. More specifically:

- Necessary conditions that avoid the fixed left-end point degeneracy were developed. These conditions are of particular relevance to MPC of systems with state constraints. This work was done in collaboration and the results are reproduced in the publication [FFV99].

- A new concept of degeneracy — q-degeneracy — was introduced and rigorously defined. It covers the different types of degeneracy that can occur due to pathwise state constraints.

- The results on the left-end point degeneracy were generalised to problems with both endpoint constraints in the form of set inclusion. For this more general problem, constraint qualifications covering all types of q-degeneracy were derived. The corresponding necessary conditions were also developed.

- Our nondegenerate conditions were further strengthened to guarantee normality. This stronger property guarantees that the necessary conditions of optimality take the objective function into account to select a minimiser.

9.2 Future Work

The research described here naturally leads on to several open questions and suggests some future developments. We start by enumerating some points more related to MPC.

- The potential of MPC, as an effective tool in control engineering, has been increased. This encourages even more developments on optimal control theory and algorithms.
- We have concentrated our efforts in studying stability. This might provide the basis for studying the also important properties of robustness and performance of the MPC framework.
- Implicit throughout the work was the assumption that the full state could be measured. When the output available for measurement has, at an instant, less information than the state, observers are needed. The theory of nonlinear observers and its satisfactory integration with feedback controllers provides an opening for research in this area. Can arguments similar to the ones used to prove stability of the MPC be adapted to show the convergence of receding horizon observers?
- We have developed a set of conditions on the design parameters that are sufficient for stability. Can we derive a set of necessary conditions for stability that would help us find a set of stabilising design parameters in a systematic way for a general nonlinear system?
- Another clear avenue for further research is in the domain of applications. Since our MPC framework addresses new classes of nonlinear systems and might enable more efficient implementations for some of the already addressed, there is a large quantity of applications that might benefit from this research.
- Since our MPC framework combines continuous-time data with discrete-time sampling and control updates, it would be interesting to explore links with the fast developing area of hybrid dynamical systems, and thereby further broaden the MPC application domain.
- A natural extension this whole thesis suggests is to explore stability of the MPC for systems with state constraints along the trajectory.

As to the future developments more related to the nondegenerate conditions we distinguish the following.

- The majority of our constraint qualifications involve information about the optimal control. This dependence on the optimal control makes, in general, more difficult to test the constraint qualification. Other works in the literature managed not to use such information for a more restricted class of problems. Is there a class of problems among the ones we have addressed such that it is inevitable to involve the optimal control in the constraint qualifications? If yes, it would be interesting to identify such class or at least to provide an example. If not, our constraint qualifications could have been written in a simpler, and less restrictive way, but the development would require completely different arguments.

- The constraint qualifications for the right and left-endpoint q-degeneracies were written as an assertion of alternatives. Is there a cleaner unified way to write those?

- The nondegenerate necessary conditions are needed when the standard first order necessary conditions do not provide enough information. Provided our problem has adequate regularity, an alternative way to complement the lack of information of the first order conditions is to consider higher order conditions. These provide another whole field of research.

- Another interesting development would be to consider different types of state constraints: higher index or involving the control. The derivation of nondegenerate necessary conditions for such state constraints provides another promising future research direction.

Appendix A

Background

The material provided here is intended as a concentrated summary of definitions and results that are used throughout this thesis. It also aims to define the notation used.

This material is by no means new and can be found spread in several textbooks on mathematics and optimisation like [KT66, Roy88, Lue69, HL69, Bil68, Bil86] where we can find a more detailed description of the concepts of functional analysis below. The books [AF90, Cla83, Cla89, Loe93, Vin99] were the main references used for nonsmooth analysis. Finally, we provide a summary of stability theory extracted mainly from [Hah67, Vid93].

A.1 Functional Analysis

An import tool in optimisation, namely in establishing existence of solution in function spaces, is the study of convergence, continuity and compacticity in topologies weaker than the usual *strong topology* induced by the Euclidean norm in $\mathbb{R}^n$, it is then natural to start this chain of concepts by this basic definition of topology.

A.1.1 Topologies

A **topology** $\mathcal{T}$ for a set X is a family of subsets of X satisfying:

1. The empty set $\emptyset$, and X belong to $\mathcal{T}$;
2. The intersection of a finite collection of sets of $\mathcal{T}$ is again in $\mathcal{T}$;
3. The union of a countable collection of sets of $\mathcal{T}$ is again in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called **open sets** and $(X, \mathcal{T})$ is called a **topological space**.

Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ for a set X, $\mathcal{T}_1$ is said to be a **weaker topology** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$. In this case $\mathcal{T}_2$ is said to be a **stronger topology** than $\mathcal{T}_1$.

This notion of open set enables us to define closed sets, neighbourhoods, continuity, convergence, and compactness.

A set F is said to be a **closed set** if its complement $X \backslash F$ is an open set. We will denote by cl$A$ the **closure** of a set A, that is the smallest closed set containing A.

An open set containing a point x is called a **neighbourhood** of x.

Consider the topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$. A function $f : X \mapsto Y$ is **continuous at a point** x in X if to each neighbourhood V of $f(x)$ there is a neighbourhood U of x such that $f(U) \subset V$. The function is called **continuous** if it is continuous at each $x \in X$, or equivalently, if the inverse image of every open set is open, that is $f^{-1}(O) \in \mathcal{T}$ for every $O \in \mathcal{S}$.

A sequence $\{x_n\}$ in X (i.e. $\{x_n \in X : n = 1, 2, \cdots\}$) is said to **converge** to a point x in X (denoted $x_n \to x$, or $\lim_{n\to\infty} x_n = x$) if for any neighbourhood V of x there is a positive integer such that $x_n \in V$ for all $n > m$.

Let S be a subset of X. A family $\mathcal{C}$ of open sets of X is said to be a **open covering** of S if the union of the elements of $\mathcal{C}$ contains the set S. The set S is **compact** if every open covering of S includes a finite subfamily which covers S. The set S is called **sequentially compact** if every sequence $\{x_n\}$ in S contains a subsequence $\{x_{n_k}\}$ convergent to an element $x \in S$.

It turns out that for metric spaces (defined below) sequentially compactness is equivalent to compactness providing an alternative and perhaps more useful definition. In finite dimensional spaces this definition simplifies even further – a compact set is one that is simply closed and bounded.

A.1.2 Metric Spaces

In most applications the topology on a space X is determined by a distance function or a **metric**, which is a real-valued function d defined on pairs of elements of X satisfying:

1. $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$;

2. $d(x, y) = 0$ if and only if $x = y$;

3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The metric topology is defined by **open balls** $\mathbb{B}_\epsilon(x) := \{y : d(x,y) < \epsilon\}$ [1] which are neighbourhoods of x. A set O is open (i.e. is in the metric topology) if for every $x \in X$ there is some $\epsilon > 0$ such that $B_\epsilon(x) \subset O$. The space (X, d) defined this way is called a **metric space**.

A sequence $\{x_n\}$ in a metric space is called a **Cauchy sequence** if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

If a sequence is convergent then it is Cauchy, by the triangle inequality. A metric space (X, d) is said to be **complete** if the converse also holds, namely if every Cauchy sequence converges to an element of X.

A.1.3 Linear Spaces

A set X is a **linear space** if operations of addition and scalar multiplication are defined and if X is closed under these operations, that is for any pair of elements x, y in X, and for any pair of scalars α, β, the element $\alpha x + \beta y$ is again in X.

The set X is a **normed linear space** if there is a function, called the **norm**, that assigns a nonnegative real number $\|x\|$ to each element x of X and satisfies

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

We also denote the norm by $\|x\|_X$ when we want to make explicit the normed space we are using.

The norm defines a metric $d(x, y) = \|x - y\|$ and hence a metric topology. If the normed linear space X is complete in this metric, then it is a **Banach space**.

Examples of Banach spaces include the Euclidean spaces $\mathbb{R}^n$ with the usual Euclidean norm $\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, and some function spaces such as L^p for $p = 1, 2, \ldots$, that are further described below.

A set D is said to be **dense** in a normed space X if there are point of D arbitrarily close to each point of X, or equivalently if the closure of D is X.

Examples of dense sets are the set of rationals in the real line, and the space of polynomials that is dense in the space of continuous functions. This latter result is known as the Weierstrass approximation theorem.

[1] In the context of set-valued analysis we will use the notation $\{x\} + \epsilon\mathbb{B}$ where $\mathbb{B}$ denotes the open unit ball centred at the origin.

A normed space is **separable** if it contains a countable dense set.

Most of the spaces to be used are separable but L^∞ is an example of one that is not.

Dimension of a linear space is the minimum number of elements of the space such that their linear combination can generate any element of the space.

Consider the normed linear space X. A **functional** on X is a scalar valued linear function defined on X. A **linear functional** f is a functional satisfying $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

The set of all linear continuous functionals on X is also a linear space. This space is called the **normed dual** (or simply **dual**) of X and is denoted X^*. The norm of an element $f \in X^*$ is

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|.$$

Now, we define the **inner product**. Given two vector vectors $x, y \in X$, the inner product $x \cdot y$ is a real number, and satisfies the following:

1. $x \cdot y = y \cdot x$;
2. $(x + y) \cdot z = x \cdot y + x \cdot z$;
3. $(\lambda x) \cdot y = \lambda(x \cdot y)$;
4. $x \cdot x \geq 0$, and $x \cdot x = 0$ if and only if $x = 0$.

A Banach space X together with an inner product defined on $X \times X$ is called a **Hilbert space**.

In a Hilbert space X, the functional $f(x) = x \cdot y$ for a fixed $y \in X$ defines a bounded linear functional on x. Thus, in Hilbert Spaces the elements of the dual can be generated by elements of the space itself. Some well known spaces that illustrate this result are $\mathbb{R}^n$ and L_2 which are dual of themselves.

A.1.4 Continuity and Differentiability

Consider a function $f : X \mapsto Y$. If both X and Y are normed linear spaces, then continuity of f at a point $x_0 \in X$ is equivalent to the following popular assertion: for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - x_0\|_X < \delta$ implies $\|f(x) - f(x_0)\|_Y < \epsilon$.

We say that f is **uniformly continuous** on $X' \subset X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X'$ with $\|x - y\|_X < \delta$ we have $\|f(x) - f(y)\|_Y < \epsilon$.

Consider a sequence of functions $\{f_n\}$ and a function f, each one a map from X to Y. The sequence $\{f_n\}$ is said to **converge pointwise** to f on X if for every $x \in X$ we have

$f(x) = \lim f_n(x)$; that is, if, for any $x \in X$ and any $\epsilon > 0$, there is an N such that for all $n \geq N$ we have $\|f(x) - f_n(x)\|_Y < \epsilon$. If we can choose N as above independently of x we say that the sequence $\{f_n\}$ **converges uniformly** to f on X; that is, if, for any $\epsilon > 0$, there is an N such that for all $x \in X$ and all $n \geq N$ we have $\|f(x) - f_n(x)\|_Y < \epsilon$.

Consider now f to be a real valued function. We define

$$\limsup_{x \to y} f(x) = \inf_{\delta > 0} \left\{ \sup_{0 < |x-y| < \delta} f(x) \right\}, \text{ and}$$

$$\liminf_{x \to y} f(x) = \sup_{\delta > 0} \left\{ \inf_{0 < |x-y| < \delta} f(x) \right\}.$$

We have that $\liminf_{x \to y} f(x) \leq \limsup_{x \to y} f(x)$ with equality if and only if the standard limit $\lim_{x \to y} f(x)$ exists.

The function f is called **lower semi-continuous** at a point y of its domain if $f(y) \leq \liminf_{x \to y} f(x)$. It is called upper-semicontinuous if $(-f)$ is lower semi-continuous.

Let the **epigraph** of f (denoted $\operatorname{epi} f$) be the set of points on and above the graph of f. More precisely $\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$. The $\operatorname{epi} f$ is a closed set if and only if f is lower semi-continuous. This last result is of importance in the context of nonsmooth analysis of semi-continuous functions.

A real-valued function f defined on a real interval $[a, b]$ is said to be **absolutely continuous** if given any $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \text{ implies } \sum_{i=1}^{n} |f(b_i) - (a_i)| < \epsilon$$

for every finite collection of disjoint intervals (a_i, b_i) contained in $[a, b]$. This notion extends to vector-valued functions in the obvious way, by imposing that each of its component functions is absolutely continuous. This concept is of prime importance in control theory since the solutions to ordinary differential equations are absolutely continuous, and from this some important properties can be deduced. Namely if f is absolutely continuous, it is continuous, it is a function of bounded variation, it is differentiable almost everywhere, and is equal to the indefinite integral of its derivative. An absolutely continuous function $f : [a, b] \mapsto \mathbb{R}^n$ is also called an **arc**.

Another very import notion is the slightly stronger condition of Lipschitz continuity which is usually easier to verify. A function $f : X \mapsto \mathbb{R}$ is **Lipschitz continuous** on $A \subset \mathbb{R}$ if there is some nonnegative scalar K satisfying

$$|f(x) - f(y)| \leq K\|x - y\| \quad \text{for all } x, y \in A.$$

The function f is **globally Lipschitz** if it is Lipschitz on X, it is **locally Lipschitz** if it is Lipschitz on any compact subset A of X, and it is called **Lipschitz near** $x \in X$ if it is Lipschitz for some neighbourhood of x. The locally Lipschitz property is equivalent to being Lipschitz near every x in X.

A locally Lipschitz continuous function is also absolutely continuous, therefore it is also differentiable almost everywhere. This last result is known as the Rademacher's theorem and formed the basis for the nonsmooth analysis in its earlier developments.

A.1.5 Measure Theory

Consider a subset A of the real numbers. Consider also countable collections of open intervals $\{I_n\}$ that cover A. The length of each of this intervals is a real number: the length of an interval (a, b) is simply $b - a$. For each collection consider the sum of the lengths of all the intervals in the collection. We define the **Lebesgue** (outer) measure of A to be the infimum amongst all collections of all such sums. That is, the Lebesgue measure of A is given by $\inf\{\sum \text{length}(I_n) : A \subset \bigcup I_n\}$.

We say that an equation or property is satisfied **almost everywhere**, or a.e., on a set $A \subset \mathbb{R}$, if it is satisfied on every point of A except on a set a Lebesgue measure zero. We use the notation μ-a.e. , if a measure μ is used instead of the Lebesgue measure.

We say that a real-valued function f defined on $[0, 1]$ is **Lebesgue measurable**, or simply measurable, if there exists a sequence of continuous functions $\{f_i\}$ such that

$$\lim f_i(t) = f(t) \quad \text{a.e. } t \in \mathbb{R}.$$

A set S is said to be the **support of a measure** μ if and only if S is smallest closed set whose complement has μ measure 0.

To define measurability in a more abstract sense, we need the following definitions. A collection $\mathcal{A}$ of sets in X is called an algebra in X, if (i) X is in $\mathcal{A}$, (ii) any finite number of unions in $\mathcal{A}$ is again in $\mathcal{A}$, and (iii) the complement of any set in $\mathcal{A}$ is again in $\mathcal{A}$. If, in addition, any countable union in $\mathcal{A}$ is again in $\mathcal{A}$, it is called a **σ-algebra**.

If $\mathcal{A}$ is a **σ-algebra** in X, then $(X, \mathcal{A})$, or simply X, is called a measurable space, and the members of $\mathcal{A}$ are called the **measurable sets** in X.

If $(X, \mathcal{A})$ is a measurable space, Y a topological space, and f is a mapping of X into Y, then f is said to be measurable on $(X, \mathcal{A})$ provided that $f^{-1}(V)$ is a measurable set in $(X, \mathcal{A})$ for every open set V in Y.

Note that the intersection of any collection of closed sets, or the union of a finite collection of closed sets is closed. However, a countable union of closed sets might not be closed. Thus if we are interested in σ-algebras of closed sets, we must consider more general types of closed sets: the Borel sets. A collection $\mathcal{B}$ of **Borel sets** is the smallest σ-algebra which contains all the open sets.

A function $f : [0,1] \times \mathrm{I\!R}^m \mapsto \mathrm{I\!R}$ is said to be $\mathcal{L} \times \mathcal{B}$ **measurable** if it is measurable on the space $([0,1] \times \mathrm{I\!R}^m, \mathcal{L} \times \mathcal{B})$, where $\mathcal{L} \times \mathcal{B}$ denotes the σ-algebra of subsets of $[0,1] \times \mathrm{I\!R}^m$ generated by product sets of the Lebesgue measurable subsets of $[0,1]$ and the Borel subsets of $\mathrm{I\!R}^m$.

A.1.6 Function Spaces

We define $L_p([0,1]; \mathrm{I\!R}^n)$, also denoted simply L_p as the set of (Lebesgue) measurable functions $x : [0,1] \mapsto \mathrm{I\!R}^n$ for which $\int_0^1 \|x(t)\|^p dt < \infty$. The norm in this spaces is given by $\|f\|_{L_p} := \left(\int_0^1 \|x(t)\|^p dt\right)^{1/p}$. The space L_1 is simply the set of Lebesgue integrable functions on $[0,1]$. We denote by L_∞ the space of all bounded measurable functions on $[0,1]$. The norm in this space is defined as $\|f\|_{L_\infty} := \text{ess sup}\, \|f(t)\|$. The L_p spaces are Banach spaces.

Let p and q be two positive extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p$ and L_q, then

$$\int_0^1 |f(t) \cdot g(t)| dt \leq \|f\|_{L_p} \|g\|_{L_q}.$$

This result is known as the **Hölder inequality**.

The dual of L_p is L_q since fixing an element $y \in L_q$ we can define for all $x \in L_q$ the bounded linear functional

$$f(x) = \int_0^1 x(t) \cdot y(t) dt,$$

having norm $\|f\| = \|y\|_{L_q}$.

We denote by $C^k([0,1], \mathrm{I\!R}^n)$, or C^k, $k = 0, 1, \cdots, \infty$ the space of all k-times continuously differentiable functions from $[0,1]$ to $\mathrm{I\!R}^n$. The space C^0 is simply denoted as C. These spaces are also Banach Spaces.

Another space of particular interest is the dual space of $C([0,1], \mathrm{I\!R})$. A Radon measure is defined by linear functionals on $C([0,1], \mathrm{I\!R})$ (i.e. defined in $C^*([0,1], \mathrm{I\!R})$) through the **Riesz Representation Theorem**: *Let ϕ be a positive linear functional on $C([0,1], \mathrm{I\!R})$. Then there exists a σ-algebra $\mathcal{A}$ in $[0,1]$ which contains all Borel sets in $[0,1]$, and there exists a unique positive measure μ on $\mathcal{A}$ which represents ϕ in the sense that for every*

$f \in C$

$$\phi(f) = \int_{[0,1]} f d\mu.$$

The measure μ defined as above is a regular Borel measure or **Radon measure**.

A.1.7 Weak and Weak* Convergence

Let X be a normed linear space. Consider the following **Weierstrass Existence Theorem**: *A lower-semicontinuous function defined on a compact set $K \in X$ attains its minimum on K.*

Since compactness in some functional spaces is difficult to establish in the norm topology (the closed unit ball in an infinite-dimensional normed linear space is not compact in the norm topology), it might be useful to consider weaker topologies. The weaker the topology we choose the less number of open sets it will have, therefore there are more convergent sequences and the compactness will be easier to establish. On the other hand continuity will be harder to establish in weaker topologies.

Consider the **weak topology** generated by X^*, which is the weakest topology for X under which the elements of X^* are still continuous. In this topology the ϵ-neighbourhoods of a point $x_0 \in X$ are the sets

$$\{x \in X : x' \cdot (x - x_0) < \epsilon, x' \in X^*\}.$$

Also of interest is the **weak* topology**. In this topology the closed unit ball of X^* is compact. In the weak* topology the ϵ-neighbourhoods of a point $x'_0 \in X^*$ are the sets

$$\{x' \in X^* : (x' - x'_0) \cdot x < \epsilon, x \in X\}.$$

In this way a sequence $\{x_i\}$ of elements of X is **weak convergent** to $x \in X$ if for all $x' \in X^*$, the sequence of real numbers $x' \cdot x_i$ converges to $x' \cdot x$.

Similarly a sequence $\{x'_i\}$ of elements of X^* is **weak* convergent** to $x' \in X^*$ if for all $x \in X$, the sequence of real numbers $x'_i \cdot x$ converges to $x' \cdot x$.

A set S in X^* is **weak* compact** if all sequences of elements in S contain a subsequence that is weak* convergent. A useful result, known as the **Alaoglu's Theorem**, is the following: *the closed unit ball in X^* is a weak* compact set.*

A.1.8 Cones and Convexity

A set K in X is said to be a **convex set** if given two points $x, y \in K$ the "line segment" $\alpha x + (1 - \alpha)y$, for $\alpha \in [0, 1]$ also belongs to K.

The **convex hull** of a set K, denoted $\operatorname{co} K$, is the smallest convex set containing K. In other words, $\operatorname{co} K$ is the intersection of all convex sets containing K.

A function f is **convex** if the "line segment" joining two points of its graph is on or above the graph. That is, if the epigraph of f is a convex set, or alternatively if for all $x, y \in \operatorname{dom} f$ and for all $\alpha \in [0,1]$, we have that $\alpha f(x) + (1-\alpha) f(y) \geq f(\alpha x + (1-\alpha) y)$.

A set K in X is said to be a **cone** with vertex at the origin if $x \in K$ implies that $\alpha x \in K$ for all $\alpha \geq 0$.

A convex cone N is said to be **pointed** if for any nonzero elements $x_1, x_2 \in N \setminus \{0\}$, we have $x_1 + x_2 \neq 0$.

A.1.9 Differential Equations

Consider the system of first order ordinary differential equations

$$\dot{x}(t) = f(t, x(t)) \quad \text{a.e. } [t_0, T], \tag{A.1}$$

with initial condition

$$x(t_0) = x_0, \tag{A.2}$$

where $x : [t_0, T] \mapsto \mathbb{R}^n$, $f : [t_0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$, and $\dot{x}(t)$ denotes $\frac{d}{dt} x(t)$.

Assume that f is measurable in t and continuous in x. Assume additionally the existence of a real-valued function $\phi \in L_1$ such that $\sup_x \|f(t,x)\| \leq \phi(t)$. This suffices to apply a **local existence theorem**, which establishes the existence of arc satisfying (A.1) and (A.2) on an interval $[t_0, t_0 + \delta)$ for some $\delta > 0$.

However, to guarantee **uniqueness of solution** we have to assume in addition that f is locally Lipschitz as a function of x.

In order to guarantee **existence of a global solution**, i.e. solution on $[t_0, +\infty)$, we have to assume linear growth, that is that there exist nonnegative constants a and b such that $\|f(t,x)\| \leq a\|x\| + b$ for all (t, x).

The last is a consequence of the Gronwall-Bellman lemma. Since this lemma is also of independent interest it is provided below.

Lemma A.1.1 (Gronwall-Bellman) *Consider the functions $\Phi : T \mapsto \mathbb{R}$ nonnegative and integrable, h and u continuous on T, where h is also nonnegative and $T = [t_0, t_1]$. Let*

$$u(t) \leq \int_{t_0}^{t} \Phi(s) u(s)\, ds + h(t) \quad \textit{for all } t \in T,$$

then

$$u(t) \leq h(t) + C \int_{t_0}^{t_1} \Phi(s)h(s)\, ds \quad \textit{for all } t \in T,$$

with $C = \exp\left(\int_{t_0}^{t_1} \Phi(s)\, ds\right)$.

A.2 Nonsmooth Analysis

We start by providing some basic concepts of set-valued analysis. Consider two sets A and B and a scalar λ. The scalar multiplication and (Minkowski) set addition are understood in the following way:

$$\lambda A := \{\lambda a : a \in A\},$$

and

$$A + B := \{a + b : a \in A,\, b \in B\}.$$

A **multifunction** F from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a mapping such that for each $x \in \mathbb{R}^n$, $F(x)$ is a subset of $\mathbb{R}^m$.

A.2.1 Normal Cones

Consider a closed set S and a point y not in S. If x is the point in S that is the nearest point to y, then the direction $y - x$ is called a **proximal normal direction** to S at x. Any nonnegative multiple $\lambda(y - x)$, $\lambda \geq 0$, of such direction is also a proximal normal to S at x. It turns out that if the boundary of S is not smooth at a certain point then the set might admit more than one (or none) proximal normal directions at that point. The set of all proximal normals of S at x forms a cone and is called the **proximal normal cone**, denoted as $N_S^P(x)$.

$$N_S^P(x) = \{\, \zeta \in \mathbb{R}^n : \zeta = \lambda(y - x),\, \lambda > 0,\, \exists\, y \in \mathbb{R}^n \text{ s.t. } \|y - x\| = \min\{\|y - x'\| : x' \in S\}\,\}.$$

An alternative definition valid in a Hilbert Space X is the following

$$N_S^P(x) = \{\zeta \in X : \text{there exists } \sigma \geq 0 \text{ s.t. } \zeta \cdot (y - x) \leq \sigma \|y - x\|^2,\ \forall y \in S, \forall i\}.$$

Another normal cone obtained from taking limits of a sequence of proximal normal cones is the **limiting normal cone**. We define the limiting normal cone $N_S(\bar{x})$ to the closed set $S \subset \mathbb{R}^n$ at $\bar{x} \in S$ as:

$$\begin{aligned} N_S(x) := \{\lim \zeta_i : \text{there exist } x_i \xrightarrow{S} x,\, \{\sigma_i\} \subset \mathbb{R}_+ \text{ s.t.} \\ \zeta_i \cdot (y - x_i) \leq \sigma_i \|y - x_i\|^2 \text{ for all } y \in S\}. \end{aligned}$$

Finally we define the **Clarke's normal cone**, which can be obtained from the limiting by taking convex hull and closure. The Clarke's normal cone $\bar{N}_S(\bar{x})$ to the closed set $S \subset \mathbb{R}^n$ at $\bar{x} \in S$ is defined as

$$\bar{N}_S(x) := \operatorname{cl}\operatorname{co} N_S(x).$$

We obviously have that $N_S^P(x) \subset N_S(x) \subset \bar{N}_S(x)$.

A.2.2 Subdifferentials

Let f be a continuously differentiable real-valued function. It is a well known calculus fact that the gradient of a function is related to a normal to its graph: the normal vectors to the epigraph of f at the point $(x, f(x))$ are given by $\lambda(\nabla f(x), -1)$ for $\lambda > 0$.

Similarly we can define generalized gradients via normal directions for all functions for which the epigraph is a closed set, i.e. for all lower semicontinuous functions.

Based on the normal cones above we define the corresponding proximal, limiting and Clarke's **subdifferentials** (denoted $\partial^P f(x)$, $\partial f(x)$, and $\bar{\partial} f(x)$ respectively) of lower semi-continuous function $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} f$ to be the following multifunctions.

$$\partial^P f(x) := \left\{\zeta : (\zeta, -1) \in N^P_{\operatorname{epi} f}(x, f(x))\right\},$$

$$\partial f(x) := \{\zeta : (\zeta, -1) \in N_{\operatorname{epi} f}(x, f(x))\},$$

$$\bar{\partial} f(x) := \left\{\zeta : (\zeta, -1) \in \bar{N}_{\operatorname{epi} f}(x, f(x))\right\}.$$

We call subgradients to the elements of a subdifferential.

If f is Lipschitz the Clarke's subdifferential can also be given by the following limit of gradients:

$$\bar{\partial} f(x) := \operatorname{co}\left\{\lim_{i\to\infty} \nabla f(x_i) : x_i \to x, x \notin A, x \notin B\right\}$$

where A is the set points in the neighbourhood of x where f fails to be differentiable (since f is Lipschitz this set has measure zero) and B any other set of measure zero.

In the context of optimal control problems with inequality state constraints we consider also $\partial_x^{>} h(t, x)$ which denotes the "hybrid" partial subdifferential of h in the x-variable and is defined as

$$\begin{aligned}\partial_x^{>} h(t, x) := \operatorname{co}\{\xi : \text{there exists } (t_i, x_i) \to (t, x) \text{ s.t.}\\ h(t_i, x_i) > 0\,\forall i,\ h(t_i, x_i) \to h(t, x),\ \text{and } \nabla_x h(t_i, x_i) \to \xi\}.\end{aligned}$$

A.2.3 Nonsmooth Calculus

Consider two Lipschitz functions f and g and a scalar λ. Some important calculus rules using the Clarke's subdifferential are the following

- **Scalar multiples:**
$$\bar{\partial}(\lambda f(x)) = \lambda \bar{\partial} f(x).$$

- **Sum rule:**
$$\bar{\partial}(f + g)(x) \subset \bar{\partial} f(x) + \bar{\partial} g(x).$$

- **Lebourg Mean Value Theorem**: There exist a point u in the line segment (x, y) such that
$$f(y) - f(x) \in \bar{\partial} f(u) \cdot (y - x).$$

- If $\bar{x}$ is a **local minimum** for f, then $0 \in \bar{\partial} f(\bar{x})$.

- **Constrained local minimum:** Let $\bar{x}$ be a local minimum for f when $\bar{x}$ is constrained to be in C. Then $0 \in \bar{\partial} f(\bar{x}) + \bar{N}_C(\bar{x})$.

One of the advantages of the limiting subdifferential over the Clarke's one is that the former distinguishes better between maximum and minimum and between f and $-f$. For example $\partial(|x|) = \bar{\partial}(|x|) = \bar{\partial}(-|x|) = [-1, 1]$, but $\partial(-|x|) = \{-1, 1\}$. However the Clarke's subdifferential has a much cleaner calculus rules mainly due to the fact of being convex.

A.3 Stability

Consider once again the dynamical system (A.1) and (A.2). Let 0 be an equilibrium point of this system, that is $f(t, 0) = 0$ for all t. The equilibrium point 0 is **Lyapunov Stable** at t_0 if for any $R > 0$, there exists a positive scalar $r = r(R, t_0)$ such that

$$\|x(t_0)\| < r \text{ implies } \|x(t)\| < R \quad \text{for all } t \geq t_0.$$

The equilibrium 0 is **attractive** if $x(t + t_0; t_0, x_0)$ converges to 0 as $t \to \infty$. It is **uniformly attractive** if $x(t + t_0; t_0, x_0)$ converges to 0 uniformly in t_0 and x_0.

Traditional literature defines a system to be Asymptotic Stable at zero, if zero is both attractive and Lyapunov stable [Hah67, Vid93]. Attractiveness it typically established using a continuous differentiable Lyapunov functions and this regularity gives also Lyapunov Stability as a bonus. Here we consider stability just with attractiveness as in [CLSW98, pag. 208], which, we believe, is better suited to the large class of nonlinear systems we consider.

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