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Steady-State Target Optimization in Linear Model Predictive Control

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Abstract
For practical model predictive control applications, a receding horizon regulator must be able to handle non-zero controlled variable targets and non-square systems in a consistent manner. This work presents a steady-state target optimization that determines the steady-state state and input targets for a constrained, linear, state-space regulator.

1. Introduction

The linear, time-invariant, discrete system is

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \qquad k = 0,1,2,\ldots \\ y_k &= Cx_k \end{aligned} \tag{1}$$

in which $y \in \Re^p$ is the output or measured variable, $u \in \Re^m$ is the input or manipulated variable, and $x \in \Re^n$ is the state of the system. A model predictive controller based on the minimization of the following infinite horizon open-loop quadratic objective function is described by Muske and Rawlings [1].

$$\min_{v_j} \sum_{j=0}^{\infty} \left((z_j - x_s)^T C^T Q C (z_j - x_s) + (v_j - u_s)^T R (v_j - u_s) + \Delta v_j^T S \Delta v_j \right) \tag{2}$$

Subject to:

$$\begin{aligned} z_0 &= \hat{x}_{k|k} \\ z_{j+1} &= Az_j + Bv_j, \quad j = 0,1,\ldots, \\ v_{-1} &= u_{k-1} - u_s \\ \Delta v_j &= v_j - v_{j-1} \\ y_{\min} &\leq Cz_j \leq y_{\max}, \quad j = j_1, j_1+1, \ldots \\ u_{\min} &\leq v_j \leq u_{\max}, \quad j = 0,1,\ldots,N-1 \\ \Delta_{\min} &\leq \Delta v_j \leq \Delta_{\max} \quad j = 0,1,\ldots,N \\ v_j &= u_s, \quad j \geq N \end{aligned} \tag{3}$$

In this objective function, x_s and u_s are the steady-state state and input corresponding to the desired controlled variable target. The determination of x_s and u_s for a given controlled variable target is discussed in this work.

2. Controlled Variables

The controlled variables considered in the sequel are determined linearly from the state of the system. These variables can consist of some or all of the measured outputs. They may also be other linear combinations of the state vector. These controlled variables are defined as

$$y_k^c = \acute{C} x_k \tag{4}$$

in which $\acute{C}$ is not necessarily equal to or the same row dimension as C. If the regulator is to bring the controlled variables to a nonzero target y_t^c, steady-state state and input vectors, x_s and u_s, are required such that the system reaches y_t^c at steady state.

$$\begin{aligned} x_s &= Ax_s + Bu_s \\ y_t^c &= \acute{C} x_s \end{aligned} \tag{5}$$

In order to have a well-posed target tracking regulator, x_s and u_s must be uniquely determined from the linear system matrices $(A, B, \acute{C})$, and the controlled variable target y_t^c. If the intersection of the null spaces of $(I - A)$ and $\acute{C}$ is a vector space containing more than the zero vector, x_s cannot be determined uniquely as shown from rearranging Eq. 5.

$$\acute{\mathcal{O}} x_s = \begin{bmatrix} Bu_s \\ y_t^c \end{bmatrix}, \qquad \acute{\mathcal{O}} = \begin{bmatrix} (I - A) \\ \acute{C} \end{bmatrix} \tag{6}$$

Therefore, a necessary restriction on $\acute{C}$ is that the matrix $\acute{\mathcal{O}}$ is full rank. This matrix is full rank under the conditions stated in Theorem 1.

Theorem 1 *The matrix $\acute{\mathcal{O}}$ in Eq. 6 is full rank if and only if the integrating modes of A are in the observable subspace of $(\acute{C}, A)$.*

Proof: See Appendix A.

3. Target Tracking

The determination of the steady-state state and input vectors x_s and u_s is discussed in this section. Non-square systems with more controlled variables than inputs are handled by minimizing the steady-state deviation from the controlled variable target in a least squares sense. In order to guarantee a unique steady state for non-square systems with more inputs than controlled variables, input targets are specified to remove the additional degrees of freedom.

3.1. Perfect Target Tracking

Steady-state state and input vectors that track the controlled variable target exactly can be determined from the solution of the following quadratic program.

$$\min_{[x_s, u_s]^T} \quad (u_s - u_t)^T R_s (u_s - u_t) \tag{7}$$

Subject to:

$$\begin{bmatrix} I-A & -B \\ \acute{C} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_t^c \end{bmatrix} \tag{8}$$

In this quadratic program, u_t is the desired value of the input at steady state and R_s is a positive definite penalty matrix for the deviation of the steady-state input from the input target. The equality constraints in Eq. 8 guarantee a steady-state solution, offset free tracking of the controlled variable target such that $\acute{C}x_s = y_t^c$, and provide a unique x_s and u_s.

Theorem 2 *The feasible quadratic program in Eq. 7 with the equality constraint in Eq. 8 has a unique solution for $\acute{O}$ in Eq. 6 full rank and R_s positive definite.*

Proof: See Appendix B.

3.2. Least Squares Target Tracking

It may not always be possible to achieve a given y_t^c as required by the equality constraint in Eq. 8. If the system is non-square with more controlled variables than inputs, there normally are not enough degrees of freedom in the system to bring the controlled variables exactly to their target. The quadratic program in Eqs. 7 and 8 will then be infeasible. In this case, the controlled variable target is tracked in a least squares sense.

The solution of the following quadratic program can be used to determine x_s and u_s in which Q_s is a positive definite penalty matrix on the controlled variable tracking error that specifies the relative importance of each controlled variable.

$$\min_{[x_s, u_s]^T} \quad (y_t^c - \acute{C}x_s)^T Q_s (y_t^c - \acute{C}x_s) + (u_s - u_t)^T R_s (u_s - u_t) \tag{9}$$

$$\text{Subject to:} \quad \begin{bmatrix} I-A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0 \tag{10}$$

A steady-state solution is guaranteed by the equality constraint in Eq. 10. The actual steady-state value of the controlled variables is $y_s^c = \acute{C}x_s$ in which x_s and u_s are determined uniquely.

Theorem 3 *The quadratic program in Eq. 9 with the equality constraint in Eq. 10 has a unique solution for $\acute{O}$ in Eq. 6 full rank and R_s, Q_s positive definite.*

Proof: See Appendix C.

The preceding quadratic program penalizes the deviations of both the controlled variables and the inputs from their steady-state targets. The result may not be the perfect tracking solution when perfect controlled variable target tracking is possible. In order to bring the controlled variables as close as possible to their target, in a least squares sense, the penalty on the inputs must not effect the controlled variable tracking error. The following input penalty penalizes only the input space that does not influence the controlled variable tracking error in which $\hat{R}_s$ is the desired input penalty restricted to be a symmetric matrix.

$$\begin{aligned} R_s &= \hat{R}_s \mathcal{N}_u \alpha \alpha^T \mathcal{N}_u^T \hat{R}_s & (11) \\ \alpha &= \mathrm{Null}\,(\mathcal{N}_x^T \acute{C}^T \acute{C} \mathcal{N}_x) \\ \mathcal{N} &= \mathrm{Null}\,([(I-A) \;\; -B]) = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix} & (12) \end{aligned}$$

The input penalty in Eq. 11, which is a positive semi-definite matrix in general, is determined from the null space of the reduced Hessian for the quadratic program in Eqs. 9 and 10 with $Q_s = I$ and $R_s = 0$. The solution of the quadratic program in Eqs. 9 and 10 using this input penalty results in a unique x_s and u_s that minimizes the deviation from the controlled variable target in a least squares sense. If possible, this formulation reaches the controlled variable target exactly.

Theorem 4 *The quadratic program in Eq. 9 with the equality constraint in Eq. 10 has a unique solution for $\acute{O}$ in Eq. 6 full rank, Q_s positive definite, and R_s computed by Eq. 11.*

Proof: See Appendix D.

3.3. Minimal Transfer Function Realizations

An important class of state-space process models that can be considered are those determined from a minimal realization of a transfer function matrix in which the controlled variables are taken as the measured outputs. In this case, $\acute{C} = C$ and the state-space model is both controllable and observable. The target tracking quadratic program in Section 3.1 can then be used for systems in which the number of outputs is less than or equal to the number of inputs, $p \leq m$. For systems in which the number of outputs is greater than or equal to the number of inputs, $p \geq m$, the quadratic program in Section 3.2 with $R_s = 0$ can be used. This is the method presented in [1] and provides a unique solution for both cases without requiring the solution of Eq. 11 to determine R_s.

Theorem 5 *The perfect target tracking quadratic program in Eqs. 7 and 8 with R_s positive definite has a* *unique solution for all state-space models determined from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \leq m$.*

Proof: See Appendix E.

Theorem 6 *The quadratic program in Eqs. 9 and 10 for least squares target tracking with Q_s positive definite and $R_s = 0$ has a unique solution for all state-space models determined from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \geq m$.*

Proof: See Appendix F.

4. Constraints

The target tracking receding horizon regulator quadratic program is subject to the constraints on the input and controlled variables in Eq. 3. In this discussion, these constraints are specified as maximum and minimum position limits on the input and controlled variables.

$$u_{\min} \leq \quad v_j \quad \leq u_{\max} \tag{13}$$

$$-\Delta u_{\min} \leq \quad v_j - v_{j-1} \quad \leq \Delta u_{\max} \tag{14}$$

$$y^c_{\min} \leq \quad \acute{C} z_j \quad \leq y^c_{\max} \tag{15}$$

A consistent constraint set is specified when the following restrictions are imposed on the position and rate of change limits.

$$\begin{bmatrix} u_{\max} \\ -u_{\min} \\ y^c_{\max} \\ -y^c_{\min} \\ \Delta u_{\max} \\ \Delta u_{\min} \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

These restrictions guarantee a convex feasible region containing the origin. The constraints can always be constructed in this form by shifting the linear system such that the nominal operating point of the process model is the origin.

The constraints on the input and controlled variable must also be respected in the target tracking quadratic program in order for the receding horizon regulator quadratic program to have a feasible solution. The following constraints are included in the quadratic program for the determination of x_s and u_s in Eqs. 9 and 10.

$$u_{\min} < \quad u_s \quad < u_{\max} \tag{17}$$

$$y^c_{\min} < \quad \acute{C} x_s \quad < y^c_{\max} \tag{18}$$

Since these constraints contain a neighborhood of the origin, the origin is a feasible solution of the target tracking quadratic program with objective Eq. 9 and constraints Eqs. 10, 17, and 18. Therefore, a solution to the target tracking quadratic program exists with the additional constraints. However, perfect tracking may not be attainable when these constraints are present.

In order to guarantee that the open-loop input trajectory can reach any feasible steady-state input, the following restriction is imposed on the input constraints and control horizon length N.

$$(N+1) \begin{bmatrix} \min(\Delta u^1_{\min}, \Delta u^1_{\max}) \\ \vdots \\ \min(\Delta u^m_{\min}, \Delta u^m_{\max}) \end{bmatrix} \geq u_{\max} - u_{\min} \tag{19}$$

If the input at time $k-1$ does not satisfy the position constraints in Eq. 13, an additional constraint is required on the steady-state input to ensure feasibility of the receding horizon regulator quadratic program.

$$-(N+1)\Delta u_{\min} < u_s - u_{k-1} < (N+1)\Delta u_{\max} \tag{20}$$

In the sequel, it will be assumed that the previous input satisfies the input position constraints.

5. Examples

Consider the following discrete linear system with two inputs and one controlled variable.

$$A = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \acute{C} = [1 \;\; 1]$$

The controlled variable target is $y^c_t = 1$ and the input target is $u_t = [0 \;\; 0]^T$. The desired input penalty is $\hat{R}_s = I$. Using Eq. 11 to determine R_s and solving the quadratic program in Eqs. 9 and 10 results in the following steady-state input and controlled variable.

$$R_s = \begin{bmatrix} 0.53 & -0.41 \\ -0.41 & 0.31 \end{bmatrix}, \quad u_s = \begin{bmatrix} 0.074 \\ 0.097 \end{bmatrix}, \quad y^c_s = 1.0$$

If it is desired to move the second input closer to its target, the penalty on that input can be increased.

$$\hat{R}_s = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \qquad u_s = \begin{bmatrix} 0.171 \\ 0.022 \end{bmatrix}, \qquad y^c_s = 1.0$$

The same results are obtained by solving the quadratic program in Eqs. 7 and 8 with $R_s = \hat{R}_s$.

We now consider this linear system with two controlled variables and the following controlled variable target.

$$\acute{C} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad y^c_t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad Q_s = I$$

In this case, $R_s = 0$ from Eq. 11 and the solution to the quadratic program in Eqs. 9 and 10 is the following.

$$u_s = \begin{bmatrix} -1.1 \\ 1.0 \end{bmatrix}, \qquad y^c_s = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

The same result is obtained as the only solution to the equality constraint in Eq. 8.

If there is a constraint on the input, the perfect target tracking solution may not be achieved. Consider the previous example with a minimum input position constraint of -1. The solution of the quadratic program in Eqs. 9, 10, and 17 is then the following.

$$u_{\min} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad u_s = \begin{bmatrix} -1.0 \\ 0.927 \end{bmatrix}, \quad y_s^c = \begin{bmatrix} 1.025 \\ 0.927 \end{bmatrix}$$

In this case, there is no solution to the equality constraint in Eq. 8 when the input constraint in Eq. 17 is also considered.

If we now consider the same linear system with three controlled variables and the following controlled variable target,

$$\acute{C} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad y_t^c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad Q_s = I$$

perfect target tracking is not possible even with an unconstrained input. The quadratic program in Eqs. 9 and 10 results in the following steady-state input and controlled variable with $R_s = 0$ from Eq. 11.

$$u_s = \begin{bmatrix} -0.600 \\ 0.667 \end{bmatrix}, \qquad y_s^c = \begin{bmatrix} 1.333 \\ 0.667 \\ 0.333 \end{bmatrix}$$

There is no solution to the equality constraint in Eq. 8 for this example.

A. Proof of Theorem 1

The sufficient condition is proved first. If the integrating modes of A are in the observable subspace of $(\acute{C}, A)$, $\mathcal{O}v_i \neq 0$ in which $\mathcal{O}$ is the observability matrix of $(\acute{C}, A)$ and v_i are the integrating modes of A. Since v_i are integrating modes, $Av_i = v_i$. Therefore, $\mathcal{O}v_i \neq 0$ implies $\acute{C}v_i \neq 0$. Since the only vectors in the null space of $(I - A)$ are v_i and $\acute{C}v_i \neq 0$, $\acute{\mathcal{O}}$ does not have a nonzero null space. Therefore, the matrix is full column rank. The necessary condition is proved as follows. If $\acute{\mathcal{O}}$ is full rank, there is no vector in the null space of both $(I-A)$ and $\acute{C}$. Therefore the integrating modes of A, which are in the null space of $(I - A)$, are not in the null space of $\acute{C}$. This implies that the integrating modes are contained in the observable subspace of $(\acute{C}, A)$.

B. Proof of Theorem 2

It is sufficient to show that the quadratic program is a strictly convex programming problem [3]. Feasibility of the equality constraint in Eq. 8 implies that this equation has a solution. If the matrix

$$\mathcal{A} = \begin{bmatrix} (I - A) & -B \\ \acute{C} & 0 \end{bmatrix}$$

is full column rank, the solution is unique which proves the theorem. If this matrix is not full column rank, the set of solutions to the equality constraint can be expressed as

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \mathcal{R}z_r + \mathcal{N}z_n, \qquad \mathcal{R}z_r = \begin{bmatrix} 0 \\ y_t \end{bmatrix}$$

in which $\mathcal{R}$ consists of the basis vectors of the range of $\mathcal{A}$ and $\mathcal{N}$ consists of the basis vectors of the null space of $\mathcal{A}$. Substituting this expression into the quadratic objective in Eq. 7 results in the following reduced quadratic function.

$$z_n^T \mathcal{N}^T H \mathcal{N} z_n + 2(H\mathcal{R}z_r - u_t)^T \mathcal{N} z_n + (H\mathcal{R}z_r - u_t)^T \mathcal{R}z_r$$

If the reduced Hessian, $\mathcal{N}^T H \mathcal{N}$, is positive definite, a unique solution exists [3]. The reduced Hessian is expressed as follows.

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix}, \qquad \mathcal{N}^T H \mathcal{N} = \mathcal{N}_u^T R_s \mathcal{N}_u$$

Since R_s is positive definite, the reduced Hessian is positive definite if $\mathcal{N}_u$ is full column rank [4]. Since $\mathcal{N}z_n$ is in the null space of $\mathcal{A}$, the following relationship holds for all z_n.

$$\begin{aligned} (I - A)\mathcal{N}_x z_n - B\mathcal{N}_u z_n &= 0 \\ \acute{C}\mathcal{N}_x z_n &= 0 \end{aligned}$$

Since $\acute{\mathcal{O}}$ is full rank and $\mathcal{N}_x z_n$ is in the null space of $\acute{C}$, $\mathcal{N}_x z_n$ cannot be in the null space of $(I - A)$. Therefore, $B\mathcal{N}_u z_n = 0$ only when $\mathcal{N}_x z_n = 0$. Since $\mathcal{N}$ is full column rank, $\mathcal{N}z_n \neq 0$ for all $z_n \neq 0$. If $\mathcal{N}_x z_n = 0$, then $\mathcal{N}_u z_n \neq 0$ from full column rank of $\mathcal{N}$. Therefore, $\mathcal{N}_u z_n \neq 0$ for all $z_n \neq 0$ which implies full column rank of $\mathcal{N}_u$ and proves the theorem.

C. Proof of Theorem 3

As in the proof of Theorem 2, the quadratic program is shown to be a strictly convex programming problem. Since the origin is a feasible solution to the equality constraint in Eq. 10, the quadratic program is feasible. The set of solutions to the equality constraint can be expressed as

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix} z_n$$

in which $\mathcal{N}$ consists of the basis vectors of the null space of $[(I - A) \ \ -B]$. If the reduced Hessian, $\mathcal{N}^T H \mathcal{N}$, is positive definite, a unique solution exists [3]. The reduced Hessian is expressed as follows.

$$\mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x + \mathcal{N}_u^T R_s \mathcal{N}_u$$

Since $\mathcal{N}z_n$ is in the null space of $[(I - A) \ \ -B]$, the following relationship holds for all z_n.

$$(I - A)\mathcal{N}_x z_n - B\mathcal{N}_u z_n = 0$$

For all z_n such that $\mathcal{N}_x z_n$ is not in the null space of $(I-A)$, the equality implies $\mathcal{N}_u z_n \neq 0$. Therefore, the reduced Hessian is positive and nonzero. For all z_n such that $\mathcal{N}_x z_n$ is in the null space of $(I-A)$, full rank of $\acute{\mathcal{O}}$ ensures that $\acute{C}\mathcal{N}_x z_n \neq 0$. Therefore, the reduced Hessian is also positive and nonzero. Since the reduced Hessian is positive and nonzero for all $z_n \neq 0$, it is a positive definite matrix which proves the theorem.

D. Proof of Theorem 4

The quadratic program is shown to be a strictly convex programming problem. Since the origin is a feasible solution to the equality constraint in Eq. 10, the quadratic program is feasible. The set of solutions to the equality constraint can be expressed as

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathcal{N}_x \\ \mathcal{N}_u \end{bmatrix} z_n$$

in which $\mathcal{N}$ consists of the basis vectors of the null space of $[(I-A) \quad -B]$. If the reduced Hessian, $\mathcal{N}^T H \mathcal{N}$, is positive definite, a unique solution exists [3]. The reduced Hessian is expressed as follows

$$\mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x + \mathcal{N}_u^T \hat{R}_s \mathcal{N}_u \alpha \alpha^T \mathcal{N}_u^T \hat{R}_s \mathcal{N}_u$$

in which α is a basis for the null space of $\mathcal{N}_x^T \acute{C} Q_s \acute{C} \mathcal{N}_x$. Consider the following expansion for z_n in which $\alpha^\perp$ is a basis for the subspace orthogonal to α.

$$z_n = \alpha v_1 + \alpha^\perp v_2$$

Since $\mathcal{N} z_n$ is in the null space of $[(I-A) \quad -B]$, the following relationship holds for all v_1, v_2.

$$(I-A)\mathcal{N}_x(\alpha v_1 + \alpha^\perp v_2) - B\mathcal{N}_u(\alpha v_1 + \alpha^\perp v_2) = 0$$

Since $\acute{C}\mathcal{N}_x \alpha = 0$, full rank of $\acute{\mathcal{O}}$ ensures that the null space of $(I-A)$ is not contained in $\mathcal{N}_x \alpha$. Therefore, $\mathcal{N}_u \alpha v_1 \neq 0$ for all $v_1 \neq 0$ which implies $\mathcal{N}_u \alpha$ is full column rank. Therefore the matrix $\alpha^T \mathcal{N}_u^T \hat{R}_s \mathcal{N}_u \alpha$ is positive definite [4]. Since $\alpha^\perp$ is orthogonal to α, $\acute{C}\mathcal{N}_x \alpha^\perp v_2 \neq 0$ for all $v_2 \neq 0$ which implies $\acute{C}\mathcal{N}_x \alpha^\perp$ is full column rank. Therefore the matrix $\alpha^{\perp T} \mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x \alpha^\perp$ is also positive definite. The reduced Hessian contains

$$v_1^T (\alpha^T \mathcal{N}_u^T \hat{R}_s \mathcal{N}_u \alpha)^2 v_1 + v_2^T (\alpha^{\perp T} \mathcal{N}_x^T \acute{C}^T Q_s \acute{C} \mathcal{N}_x \alpha^\perp) v_2$$

which is a nonzero positive number for all $v_1 \neq 0$ and $v_2 \neq 0$. Since the reduced Hessian is positive and nonzero for all v_1 and v_2, it is a positive definite matrix which proves the theorem.

E. Proof of Theorem 5

Since the state-space system came from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \leq m$, $C(I-A)^{-1}B$ is full row rank. Choose α_1 and α_2 such that

$$\alpha_1^T(I-A) + \alpha_2^T C = 0 \implies \alpha_1^T = -\alpha_2^T C(I-A)^{-1}$$

which implies $-\alpha_1^T B = \alpha_2^T C(I-A)^{-1}B \neq 0$ for all α_1 from full row rank of $C(I-A)^{-1}B$. Therefore, the following matrix is full row rank.

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \tag{21}$$

The equality constraint in Eq. 8 is then feasible for all y_t and uniqueness of the solution follows from Theorem 2.

F. Proof of Theorem 6

Since the state-space system came from a minimal realization of a discrete transfer function matrix with a full rank steady-state gain matrix, no integrating modes, and $p \geq m$, $C(I-A)^{-1}B$ is full column rank. Choose α_1 and α_2 such that

$$(I-A)\alpha_1 - B\alpha_2 = 0 \implies \alpha_1 = (I-A)^{-1}B\alpha_2$$

which implies $C\alpha_1 = C(I-A)^{-1}B\alpha_2 \neq 0$ for all α_1 from full column rank of $C(I-A)^{-1}B$. Therefore, the matrix in Eq. 21 is full column rank and, from $\mathcal{N}$ full column rank, there exists no $v \neq 0$ that satisfies the following equation.

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \mathcal{N} v = 0$$

Since $\mathcal{N}$ is a basis for the null space of $[(I-A) \quad -B]$ defined in Eq. 12

$$[(I-A) \quad -B]\mathcal{N} v = 0 \quad \forall \quad v$$

Therefore, $C\mathcal{N}_x v \neq 0$ for all $v \neq 0$ from full column rank of the matrix in Eq. 21. This implies $C\mathcal{N}_x$ is full column rank, the matrix $\mathcal{N}_x^T C Q_s C \mathcal{N}_x$ is positive definite and, consequently, $R_s = 0$. Uniqueness of the solution follows from Theorem 4.

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