Exponential Stabilization of Nonholonomic Chained Systems

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***Abstract*—This paper presents a feedback control scheme for the stabilization of two-input, driftless, chained nonholonomic systems, also called chained form. These systems are controllable but not asymptotically stabilizable by a smooth static-state feedback control law. In addition, exponential stability cannot be obtained with a smooth, time-varying feedback control law. Here, global, asymptotical stability with exponential convergence is achieved about any desired configuration by using a nonsmooth, time-varying feedback control law. The control law depends, in addition to the state and time, on a function which is constant except at predefined instants of time where the function is recomputed as a nonsmooth function of the state. The inputs are differentiable with respect to time and tend exponentially toward zero. For use in the analysis, a lemma on the exponential convergence of a stable time-varying nonlinear system perturbed by an exponentially decaying signal is presented. Simulation results are also shown.**

I. Introduction

Nonholonomic chained systems can be used to represent a large class of mechanical systems. Important and well-known examples are unicycles, four-wheeled cars and n-trailer systems. Accordingly, control schemes for nonholonomic chained systems has a potential of being applied to a large number of mechanical systems. The problem of designing stabilizing feedback controllers for nonholonomic chained systems is a challenging one since the system is not stabilizable by a smooth static-state feedback law [4]. Moreover, the problem of exponential stabilization, which is the topic of this paper, cannot be solved by any smooth feedback law [10].

The nonholonomic chained system considered in this paper is the chained form [26]

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_i &= x_{i-1}u_1, \quad i \in \{3, \cdots, n\}. \end{aligned}$$

A constructive procedure to transform a nonholonomic system with two inputs into this chained form was presented by [25] under certain conditions on the input vectors. Necessary and sufficient conditions for converting a nonholonomic system into chained form were derived in [23] based on the theory of exterior differential systems using the Goursat normal form theorem. One of the results from this work is that all two-input, regular nonholonomic systems in three and four dimensions are locally feedback equivalent to a nonholonomic chained form. In [33], the kinematic model of a car with n trailers was converted into chained form. This conversion was global in the position of the system and local in the orientations. Another conversion into chained form was proposed in the framework of exterior differential systems in [35] based on [33], [23], and [30]. This conversion of the kinematics of the n-trailer system allowed any orientation of the last trailer. Control strategies for nonholonomic chained systems can, therefore, be used for the control of a broad class of nonholonomic, mechanical systems.

The control of chained form and of more general nonholonomic systems is a very active field of research. The problem of nonholonomic motion planning was introduced by [17] who proved that a car-like robot with one nonholonomic constraint is controllable. Open-loop planners for low-dimensional mobile robots have been proposed in [18], [1], [19]. Because of the invertible transformation in [33], these planners can also be used to plan a path for low-dimensional chained systems. Other open-loop strategies have explored control theoretic approaches using differential geometry tools to control nonholonomic systems. Sinusoids were proposed by [24] to steer in open-loop nonholonomic systems on a special canonical form including chained form. A generalization of the use of sinusoidal inputs to generate motion at a given level of Lie brackets of the input vectors was given in [16], [15], [12], [22] for nilpotent and nilpotentizable systems using an extended system with additional input vectors corresponding to higher order Lie brackets of the original system. Since the nonholonomic chained system is nilpotent, these open-loop strategies can also be used to steer such a system in open loop.

To make the control more robust with respect to disturbances and errors in the initial condition, some stabilizing closed-loop approaches have also been proposed. Characteristic for the nonholonomic systems encountered in robotics is that they cannot be stabilized by a smooth static-state feedback control law which has been shown with Brockett's theorem [4]. This has been further discussed by [2] and [8] for nonholonomic mechanical systems. Therefore, a discontinuous feedback control law was proposed by [5] to make the kinematic model of a three-dimensional robot globally, exponentially converge to a given configuration. This system is equivalent to a three-dimensional chained form. Another

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discontinuous feedback approach was proposed by [3] for the same system with acceleration inputs instead of velocity inputs. This approach makes the system reach the origin in finite time in the case of no disturbances.

The use of time-varying feedback control laws is another approach to stabilize nonholonomic systems about a constant configuration. This approach was first studied by [30] for the stabilization of a cart. This approach was further developed in [29] for a car-like mobile robot with a steering wheel. This system is locally equivalent to a four-dimensional chained form. Constructive approaches were presented in [31], [28]. The existence of stabilizing time-varying feedback control laws for more general nonholonomic systems was studied in [6]. The design methods in [28] were extended in [7] to the more general situation given in [6]. An algorithm for computing time-periodic feedback solutions for nonholonomic motion planning was presented in [10]. This approach considered the extended system using Lie bracket completion vectors such as in the open-loop strategy of [16]. This feedback algorithm was based on multi-scaling averaging techniques and highly oscillatory inputs. These time-dependent approaches also work for chained form. Another time-varying smooth feedback control law to stabilize chained form was derived in [34] for power forms based on the work on the use of sinusoids.

These smooth time-varying feedback control laws yield asymptotic stability but not exponential since time-periodic smooth feedback cannot be exponentially stabilizing [9], [10]. To improve the rate of convergence, a feedback control law was proposed by [27] to obtain local exponential convergence to a neighborhood of the origin for a three-dimensional chained form. The asymptotic behavior was obtained by letting the control law be time varying. The exponential convergence to the neighborhood was obtained by letting the control law be nonsmooth at the origin. No feedback control law, however, has been presented in previous work that exponentially stabilizes a nonholonomic chained form of an arbitrary dimension about any constant configuration.

In this paper, the problem of exponential stabilization is addressed and a new feedback approach is proposed for chained form. The stabilization is achieved by letting the state feedback control law depend on time and on a function which is nonsmooth with respect to the state at discrete instants of time. The proposed feedback control law globally stabilizes the system about the origin with exponential convergence. Asymptotical stability about any desired configuration is obtained by using a coordinate transformation. The resulting closed-loop system is not exponentially stable as defined by [14, p. 168], but it is shown to have a property which will be termed $\mathcal{K}$-exponential stability.

The paper is organized as follows: The concept of $\mathcal{K}$-exponential stability is proposed in Section II. A lemma on the exponential convergence of a nonlinear time-varying system which is perturbed by an exponentially decaying signal is given in Section III. The nonholonomic chained system is presented in Section IV. The control law is presented in Section V. The convergence of a part of the system is analyzed in Section VI. The stability of the total system is analyzed in Section VII. In Section VIII, a coordinate transformation is presented such that a control law to control the chained form to the zero configuration can be used for the control to any desired configuration. The stabilizing feedback strategy is illustrated by a simulation example in Section IX. The conclusions are given in Section X.

II. $\mathcal{K}$-Exponential Stability

In this section, the concept of $\mathcal{K}$-exponential stability is introduced. It will be shown in later sections that the controller proposed in this paper makes the chained system $\mathcal{K}$-exponential stable. First, we need the following notion [11, Definition 2.5].

Definition 1: A continuous function $\alpha: R^+ \to R^+$ is said to be of class $\mathcal{K}$ (or belong to class $\mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$.

One property of class $\mathcal{K}$ functions is that, [14] and [11]

$$\alpha_1, \alpha_2 \in \text{ class } \mathcal{K} \Rightarrow \alpha_1 \circ \alpha_2 \in \text{ class } \mathcal{K}. \tag{1}$$

We then define the following.

Definition 2: Consider the nonlinear, time-varying system

$$\dot{x} = f(x,t) \quad x \in D \subset R^n, \quad t \geq t_0. \tag{2}$$

System (2) is $\mathcal{K}$-exponentially stable about x_p iff there exist a neighborhood $\Omega_p \subset D$ about x_p, a positive constant λ, and a function $h(\cdot)$ of class $\mathcal{K}$ such that all solutions $x(t)$ of (2) satisfy

$$\forall x(t_0) \in \Omega_p \forall t \geq t_0, \|x(t) - x_p\| \leq h(\|x(t_0) - x_p\|)e^{-\lambda(t-t_0)} \tag{3}$$

where the constant λ and the neighborhood Ω_p are independent of t_0, and $\|\cdot\|$ denotes a norm in R^n.

If (3) is satisfied for $\Omega_p = D$, then system (2) is globally, $\mathcal{K}$-exponentially stable about x_p. According to this definition, if system (2) is $\mathcal{K}$-exponentially stable at x_p, then it is uniformly asymptotically stable as defined by e.g., [14, 4.3], and in addition it has an exponential rate of convergence.

The concept of $\mathcal{K}$-exponential stability is called exponential stability in [21]. The term exponential stability is however often used for the special case where the function $h(\cdot)$ is linear, [11] and [14], i.e.,

$$h(\|x(t_0) - x_p\|) = r\|x(t_0) - x_p\|.$$

Here, r is a positive constant independent of t_0 and $x(t_0)$. This means that $\mathcal{K}$-exponential stability corresponds to a weaker form of stability than the usual concept of exponential stability. $\mathcal{K}$-exponential stability and exponential stability are equal, however, with respect to the rate of convergence. Therefore, the notion "exponential stabilization" is used in the title of this paper; although, only $\mathcal{K}$-exponential stability will be proved.

Locally, a similar definition of exponential stability was introduced in [13] in terms of homogeneous norms. This concept has been used in [20] to investigate the convergence rates for controllers for low dimensional nonholonomic systems in so-called power form which is equivalent to chained form, [34].

III. A Lemma on Exponential Convergence

The following lemma is useful for establishing exponential convergence for a class of time-varying systems. It will be used in the convergence analysis of the control law.

Lemma 1: Consider the nonlinear, one-dimensional, time-varying system

$$\dot{x} = -a(x,t)x + d(x,t), \quad t \geq t_0, \quad x(t_0) \in R \tag{4}$$

under the following assumptions:

- There exists a solution $x(t)$ of (4) for any $x(t_0)$ and any $t \geq t_0$; see Remark 2 below.
- $a(x,t)$ has the property that for all $x(t)$

$$\left| \int_{t_0}^{t} (a(x(\tau),\tau) - \lambda) d\tau \right| \leq P, \quad \forall t \geq t_0 \tag{5}$$

  where λ and P are positive constants.
- The signal $d(x,t)$ is bounded for any $t \geq t_0$ and any $x(t)$ by

$$|d(x(t),t)| \leq De^{-\gamma(t-t_0)} \tag{6}$$

  for some positive constants D and γ.

Then

$$\forall \varepsilon > 0, \quad |x(t)| \leq c(|x(t_0)| + D)e^{-(\alpha-\varepsilon)(t-t_0)}$$

where

$$\alpha = \min\{\lambda, \gamma\} > 0, \quad c = \max\left\{e^P, \frac{e^{2P}}{\varepsilon}\right\}.$$

Proof: We denote

$$F(t) = \int_{t_0}^{t} a(x(\tau),\tau) d\tau$$

where $x(t)$ is a solution of (4). Multiplying (4) with $e^F(t)$ gives

$$\frac{d}{dt}(x(t)e^{F(t)}) = d(x(t),t)e^{F(t)}.$$

This implies

$$x(t)e^{F(t)} = x(t_0) + \int_{t_0}^{t} e^{F(\tau)} d(x(\tau),\tau) d\tau.$$

Dividing by $e^F(t)$ then gives the following (implicit) expression for $x(t)$

$$x(t) = e^{-F(t)} \left(x(t_0) + \int_{t_0}^{t} e^{F(\tau)} d(x(\tau),\tau) d\tau \right). \tag{7}$$

Property (5) implies that

$$|F(t) - \lambda(t - t_0)| \leq P \tag{8}$$

which is equivalent to

$$-P + \lambda(t - t_0) \leq F(t) \leq P + \lambda(t - t_0). \tag{9}$$

By using (7), (9), and (6) we get

$$|x(t)| \leq e^P e^{-\lambda(t-t_0)} \left(|x(t_0)| + D \int_{t_0}^{t} e^P e^{(\lambda-\gamma)(\tau-t_0)} d\tau \right).$$

In the case that $\lambda = \gamma$ we find

$$|x(t)| \leq e^P |x(t_0)| e^{-\lambda(t-t_0)} + De^{2P}(t - t_0)e^{-\lambda(t-t_0)}.$$

In the case that $\lambda \neq \gamma$ we find

$$|x(t)| \leq e^P |x(t_0)| e^{-\lambda(t-t_0)} + \frac{De^{2P}}{\lambda - \gamma}(e^{-\gamma(t-t_0)} - e^{-\lambda(t-t_0)}).$$

Since

$$1 - e^{-b} \leq b, \quad \forall b \geq 0$$

then

$$|x(t)| \leq e^P |x(t_0)| e^{-\lambda(t-t_0)} + De^{2P}(t - t_0)e^{-\alpha(t-t_0)} \tag{10}$$

for all $\lambda > 0$ and $\gamma > 0$ where $\alpha = \min\{\lambda, \gamma\}$. From (10) we have that

$$\begin{aligned} |x(t)| &\leq (\rho(t - t_0) + \sigma)e^{-\alpha(t-t_0)} \\ &= (\rho(t - t_0) + \sigma)e^{-\varepsilon(t-t_0)} e^{-(\alpha-\varepsilon)(t-t_0)} \end{aligned} \tag{11}$$

where

$$\rho = De^{2P}, \quad \sigma = e^P |x(t_0)| \tag{12}$$

since $e^{-(\lambda-\alpha)(t-t_0)} \leq 1$ for $t \geq t_0$. By comparison we find that

$$(\rho t + \sigma)e^{-\varepsilon t} \leq \xi \quad \forall t \geq 0 \tag{13}$$

if

$$\xi = \begin{cases} \frac{\rho}{\varepsilon e} e^{\frac{\varepsilon\sigma}{\rho}}, & \rho > \varepsilon\sigma \\ \sigma, & \rho \leq \varepsilon\sigma. \end{cases}$$

Since

$$e^{\frac{\varepsilon\sigma}{\rho}} < e$$

when $\rho > \varepsilon\sigma$, (13) will also be satisfied with the following choice of ξ

$$\xi = \begin{cases} \frac{\rho}{\varepsilon}, & \rho > \varepsilon\sigma \\ \sigma, & \rho \leq \varepsilon\sigma \end{cases}. \tag{14}$$

Then, from (11), (12), (13) and (14) we get that

$$\begin{aligned} |x(t)| &\leq \xi e^{-(\alpha-\varepsilon)(t-t_0)} \\ &\leq \left(\frac{\rho}{\varepsilon} + \sigma \right) e^{-(\alpha-\varepsilon)(t-t_0)} \\ &\leq c(|x(t_0)| + D)e^{-(\alpha-\varepsilon)(t-t_0)} \end{aligned}$$

where $\alpha = \min\{\lambda, \gamma\}$ and $c = \max\{e^P, \frac{e^{2P}}{\varepsilon}\}$. □

This lemma implies that a solution $x(t)$ of (4) converges exponentially to zero if $a(x,t)$ and $d(x,t)$ have the properties (5) and (6).

Remark 1: By choosing $\varepsilon = \alpha$ we see that

$$\max_{t \geq t_0} |x(t)| \leq (|x(t_0)| + D) \max\left\{e^P, \frac{e^{2P}}{\alpha}\right\}$$

where $\alpha = \min\{\lambda, \gamma\} > 0$.

Remark 2: If $a(x,t)$ and $d(x,t)$ are continuous in x and t, then there exists at least one solution of (4), [21, Theorem 2.3].

IV. The System

The nonholonomic chained system considered in this paper is the so-called chained form introduced in [26]

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_2 u_1\\ &\vdots\\ \dot{x}_n &= x_{n-1}u_1.\end{aligned} \tag{15}$$

The input vector fields are

$$\begin{aligned}g_1(x) &= [1, 0, x_2, x_3, \cdots, x_{n-1}]^T,\\ g_2(x) &= [0, 1, 0, 0, \cdots, 0]^T\end{aligned}$$

where $x = [x_1, x_2, \cdots, x_n]^T$. Repeated Lie brackets can be defined recursively by

$$\mathrm{ad}^0_{g1}g_2 = g_2, \quad \mathrm{ad}^i_{g1}g_2 = [g_1, \mathrm{ad}^{i-1}_{g_1}g_2].$$

It can easily be shown that for system (15)

$$\mathrm{ad}^{j-2}_{g_1}g_2 = (-1)^j e_j, \quad j \in \{2, 3, \cdots, n\}$$

where $e_j \triangleq [\delta_{1j}, \delta_{2j}, \cdots, \delta_{nj}]^T$ where δ_{ij} is the Kronecker delta. Since

$$\forall x \in R^n, \quad \mathrm{span}\{g_1, \mathrm{ad}^0_{g1}g_2, \cdots, \mathrm{ad}^{n-2}_{g1}g_2\}(x) = R^n$$

the chained form (15) is completely controllable and the degree of nonholonomy is $n-1$. In spite of the controllability, (15) cannot be stabilized by a smooth static-state feedback control law as stated by Brockett's Theorem [4]. From [10] we know that to exponentially stabilize (15), the controller cannot be smooth, even if it is time dependent.

V. The Control Law

A. Outline of the Control Law and Analysis

In this section, we will propose a control law to globally, $\mathcal{K}$-exponentially stabilize the nonholonomic chained system (15) about the origin. Since there is no smooth static-state feedback control law which can stabilize (15), we let the feedback control law be time dependent, as first proposed by [32] for a nonholonomic cart. The main idea of the control scheme to be presented is based on the observation that if u_1 is a function of time only, then the state variables

$$z \triangleq [x_2, \cdots, x_n]^T \tag{16}$$

satisfy a linear time-varying state space model with input u_2. This linear time-varying system with input u_2 and state z can be made $\mathcal{K}$-exponentially stable using feedback from z, which will be shown in the following. The problem which remains to be solved is how to achieve $\mathcal{K}$-exponentially stability when x_1 is included into the state vector, that is, when the complete system with state vector x and inputs u_1 and u_2 is considered. The solution proposed in this paper is as follows: Define a sequence $(t_0, t_1, t_2, \cdots)$ where the equidistant terms are defined by

$$t_i \triangleq iT \tag{17}$$

where T is a strictly positive constant. Then, the control u_1 is given by $u_1 = u_1(t, x(t_i)), t \in [t_i, t_{i+1})$, that is, u_1 is a function of time and not of the state whenever $t_i < t < t_{i+1}$. By appropriately selecting $u_1(t, x(t_i))$, it is then possible to achieve exponential convergence in $x_1(t)$ and, in addition, by patching together the solutions for $z(t)$, $\mathcal{K}$-exponentially stability for the complete system can be established. To obtain exponential convergence, the time-varying feedback control law is nonsmooth in the origin with respect to the state $x(t_i)$.

In this section the control laws are presented without further explanation. Then, in the following sections the stability of the closed-loop system is analyzed, and it is shown that the proposed control laws have the desired properties. The analysis is done in two steps: First the stability of the linear system with state vector z and input u_2 is analyzed. To this end a vector $z^d(x, t)$: $R^n + R_+ \rightarrow R^{n-1}$ is defined, and it is shown that for the proposed control law the vector $\tilde{z}(x, t) \triangleq z - z^d(x, t)$ converges exponentially to zero. Then, this result is used to prove exponential convergence of $z(t)$. At this point the state x_1 is included, and $\mathcal{K}$-exponentially stability of the complete system with state vector x is established.

Concerning the notation, throughout the paper the one-norm will be used, i.e., the norm of an n dimensional vector $x = [x_1, \cdots, x_n]^T$ is given by

$$\|x\| \triangleq \sum_{j=1}^{n} |x_j|. \tag{18}$$

Furthermore, the index i will refer to terms in the time sequence $(t_0, t_1, \cdots)$.

B. The Control Law for u_1

Let $k(\cdot)$ be a bounded function

$$k\colon R^n \rightarrow R \tag{19}$$

which satisfies for a strictly positive constant K

$$\forall x \in R^n |k(x)| \leq K, \quad \forall x \in R^n/\{0\} k(x) \neq 0, \quad k(0) = 0. \tag{20}$$

The idea is now to let u_1 be given by a time-varying function multiplied with the function $k(x(t_i))$ such that the part of the system which is represented by the state variables z with input u_2 is linear and time-varying in the time interval (t_i, t_{i+1}). The input u_2 is then used to make $[x_2(t), \cdots, x_n(t)]^T$ exponentially converge to zero. The sign and magnitude of $k(x(t_i))$ will be chosen such that $x_1(t)$ also converges exponentially to zero, and the inputs remain bounded.

We introduce a time-varying function f: $R_+ \rightarrow R$ which has the following properties:

P1) $f(t) \in C^\infty[t_0, +\infty)$

P2) $0 \leq f(t) \leq 1, \forall t \geq t_0$

P3) $f(t_i) = 0,\ t_i \in \{t_0, t_1, t_2, \cdots\}$

P4) for any $j \in \{3, \cdots, n\}$, there are positive constants η_j and P_j such that

$$\forall t_p \in \{t_0, t_1, \cdots\},\ \forall t \geq t_p \quad \left| \int_{t_p}^{t} [f^{2j-3}(\tau) - \eta_j] d\tau \right| \leq P_j.$$

A function satisfying conditions P1)–P4) is

$$f(t) = (1 - \cos \omega t)/2, \quad \omega = \frac{2\pi}{T} \tag{21}$$

where $T = t_{i+1} - t_i$ is a constant, (17). Incidentally, the function $f(t)$ is not restricted to be time-periodic to satisfy P1)–P4).

The control law for u_1 is defined by

$$u_1 = k(x(t_i))f(t), \quad t \in [t_i, t_{i+1}) \tag{22}$$

where

$$k(x) = \text{sat}(-[x_1 + \text{sgn}(x_1)G(\|z\|)]\beta, K). \tag{23}$$

Here

$$\text{sat}(q, k) \triangleq \begin{Bmatrix} q, & |q| < K \\ K\ \text{sgn}(q), & |q| \geq K \end{Bmatrix}$$

and

$$G(\|z\|) = \kappa \|z\|^{\frac{1}{2n-4}} \tag{24}$$

$$\beta = \frac{1}{\int_{t_i}^{t_i+1} f(\tau) d\tau} \tag{25}$$

where κ is a positive constant and $\text{sgn}(x_1)$ is defined as

$$\text{sgn}(x_1) = \begin{cases} 1, & x_1 \geq 0 \\ -1, & x_1 < 0 \end{cases}.$$

All the norms are one-norms as defined in (18). The saturation function is used to guarantee global stability. The idea of using saturation functions was also used in [34].

Note that due to P1)–P3) and the bound on $k(\cdot)$, the control u_1 is bounded and continuous with respect to time.

C. The Control Law for u_2

With the input u_1 given by (22), we get form (15) and (16) that $\dot{z}$ is given by

$$\begin{aligned} \dot{x}_2 &= u_2 \\ \dot{x}_3 &= k(x(t_i))f(t)x_2 \\ &\ \vdots \\ \dot{x}_n &= k(x(t_i))f(t)x_{n-1}. \end{aligned} \tag{26}$$

In this subsection, we derive a feedback control law for u_2 to make $z(t) = [x_2(t), \cdots, x_n(t)]^T$ globally, exponentially converge to zero.

Define $g_{jm}\colon R_+ \to R$ for $j, m \in \{2, \cdots, n\}$ by

$$g_{n-1,n} = -\lambda_n \tag{27}$$

$$\begin{aligned} g_{j-1,m}(t) = &-g_{jm}[\lambda_j f^{2j-2}(t) + 2(j-1)\dot{f}(t)] \\ &+ f(t)[\dot{g}_{jm}(t) + g_{j,m+1}(t)f(t)] \end{aligned} \tag{28}$$

$$g_{j-1,j}(t) = -\lambda_j + f^2(t)g_{j,j+1}(t) \tag{29}$$

$$g_{jp} = 0 \quad \text{if } p \leq j \text{ or } p = n+1 \tag{30}$$

where λ_j, $j \in \{2, \cdots, n\}$, are any positive constants. Incidentally, we see from this definition of $g_{jm}(t)$ and P1) that $g_{jm}(t)$ is smooth.

The definition of $g_{jm}(t)$ from (27)–(30) can be illustrated by the following diagram where $a \to b$ means that b depends on a

$$\begin{array}{ccccccccc} g_{2,3} & \cdots & & g_{2,n-2} & & g_{2,n-1} & & g_{2,n} \\ & \nwarrow \cdots & \nwarrow & \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\ & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ & & \nwarrow & \uparrow & \nwarrow & \uparrow & \nwarrow & \uparrow \\ & & & g_{n-3,n-2} & & g_{n-3,n-1} & & g_{n-3,n} \\ & & & & \nwarrow & \uparrow & \nwarrow & \uparrow \\ & & & & & g_{n-2,n-1} & & g_{n-2,n} \\ & & & & & & \nwarrow & \uparrow \\ & & & & & & & g_{n-1,n}. \end{array}$$

The control law for u_2 is defined for $t \in [t_i, t_{i+1})$ by

$$u_2 = \begin{cases} \Gamma(k(x(t_i)), t)^T z, & x(t_i) \neq 0 \\ 0, & x(t_i) = 0 \end{cases} \tag{31}$$

where

$$z = [x_2, \cdots, x_n]^T \in R^{n-1},$$
$$\Gamma(k, t) = [\Gamma_2(k, t), \cdots, \Gamma_n(k, t)]^T \in R^{n-1}$$

and for $j \in \{3, \cdots, n\}$

$$\Gamma_2(k, t) = -\lambda_2 + f^3 g_{2,3}$$
$$\Gamma_j(k, t) = f(\lambda_2 f g_{2j} + 2\dot{f} g_{2j} + f\dot{g}_{2j} + f^2 g_{2,j+1})\frac{1}{k^{j-2}}.$$

The smooth functions $g_{2j}(t)$ are given by (27)–(30). The function $k(x(t_i))$ is given by (23).

The feedback control law (22) and (31) depends on $k(\cdot)$ which is a nonsmooth function of the state $x(t_i)$. One might, therefore, raise the question about the existence and uniqueness of the solution $x(t)$ of (15). Since in each time interval $[t_i, t_{i+1})$ the closed-loop system (15) with the control law (22) and (31) becomes a linear, time-varying system, global existence and uniqueness of $x(t)$ can be shown in any time interval $[t_i, t_{i+1})$ for any $x(t_i)$ by applying e.g., Theorem 2.3 in [14]. The time intervals can then be patched together to show existence and uniqueness of $x(t)$ for all $t \in R_+$ and for all initial conditions $x(t_0)$.

VI. Convergence Analysis of $z(t)$

In this section, it is shown that the control law (22) and (31) makes $z(t)$ (16) globally exponentially converge to zero.

To this end, define the functions $z^d = [x_2^d, \cdots, x_n^d]^T\colon R^n \times R_+ \to R^{n-1}$ as follows for $t \in [t_i, t_{i+1})$

$$x_n^d \triangleq 0 \tag{32}$$

$$x_j^d(x, t) \triangleq \begin{cases} f^{2j-2}(t) \sum_{m=j+1}^{n} g_{jm}(t) \frac{1}{k^{m-j}(x(t_i))} x_m, & x(t_i) \neq 0 \\ 0, & x(t_i) = 0 \end{cases}. \tag{33}$$

Note from this definition of x_2^d that the control law for u_2 (31) satisfies

$$u_2 = -\lambda_2(x_2 - x_2^d) + \dot{x}_2^d \tag{34}$$

whenever $x(t_i) \neq 0$. This relation will be used in the convergence analysis.

By assuming that the control laws yield a continuous solution $x(t)$ of (15), we see from the definition (33) of $x_j^d(x,t)$, P1), and P3) that $x_j^d(x(t),t)$ is continuous for all $t > 0$ and $x_j^d(x(t_i),t_i) = 0$. In addition, z^d has the properties shown in the following lemma.

Lemma 2: Let $z^d = [x_2^d, \cdots, x_d^n]^T$ be defined by (32) and (33). Then, for $j \in \{3,\cdots,n\}$, $t \in [t_i, t_{i+1})$

$$\dot{x}_n^d = 0 \tag{35}$$

$$x_{j-1}^d k(x(t_i)) f(t) = -\lambda_j f^{2j-3}(t)[x_j - x_j^d(x,t)] + \dot{x}_j^d(x,t). \tag{36}$$

Proof: Equation (35) is trivially satisfied from (32). Equation (36) will be proved by calculating $\dot{x}_j^d(x,t)$. The arguments of the functions $f(t)$, $k(x(t_i))$, and $x_j^d(x,t)$ will be omitted in the proof for simplicity. From the definition of x_j^d, (33), we find for $t \in (t_i, t_{i+1})$ in the case $x(t_i) \neq 0$ by using $\dot{x}_m = x_{m-1} k f$ from (15)

$$\dot{x}_j^d = (2j-2) f^{2j-3} \dot{f} \sum_{m=j+1}^{n} g_{jm} \frac{1}{k^{m-j}} x_m + f^{2j-2} \sum_{m=j+1}^{n} \left[\dot{g}_{jm} \frac{1}{k^{m-j}} x_m + g_{jm} \frac{f}{k^{m-j-1}} x_{m-1} \right]. \tag{37}$$

We see that by assuming a continuous solution $x(t)$ of (15), we have due to P1) and P3)

$$\lim_{t \to t_i^+} \dot{x}_j^d(x(t),t) = \lim_{t \to t_i^-} \dot{x}_j^d(x(t),t) = 0.$$

Therefore, (37) is valid for all $t > 0$. Now, we will express x_{j-1}^d by using $\dot{x}_j^d$ from (37). From (33), (28)–(30) we find

$$\begin{aligned} x_{j-1}^d &= f^{2j-4} \sum_{m=j}^{n} g_{j-1,m} \frac{1}{k^{m-(j-1)}} x_m \qquad (38) \\ &= f^{2j-4} \Big\{ x_j \Big(-\frac{\lambda_j}{k} + f^2 g_{j,j+1} \frac{1}{k} \Big) \\ &\quad + \sum_{m=j+1}^{n} \frac{x_m}{k^{m-j+1}} [(\lambda_j f^{2j-2} + 2(j-1)\dot{f}) g_{jm} \\ &\quad + f(\dot{g}_{jm} + g_{j,m+1} f)] \Big\} \\ &= -\frac{\lambda_j}{k} f^{2j-4} \Big(x_j - f^{2j-2} \sum_{m=j+1}^{n} g_{jm} \frac{1}{k^{m-j}} x_m \Big) \\ &\quad + f^{2j-4} \Big\{ (2j-2)\dot{f} \sum_{m=j+1}^{n} g_{jm} \frac{1}{k^{m-j+1}} x_m \\ &\quad + f \sum_{m=j+1}^{n} \Big[\dot{g}_{jm} \frac{1}{k^{m-j+1}} x_m + g_{jm} \frac{f}{k^{m-j}} x_{m-1} \Big] \Big\} \\ &= -\frac{\lambda_j}{k} f^{2j-4} (x_j - x_j^d) + \frac{1}{kf} \dot{x}_j^d. \end{aligned}$$

Equation (36) follows readily. □

Define the functions $\tilde{z} = [\tilde{x}_2, \cdots, \tilde{x}_n]^T$: $R^n \times R_+ \to R^{n-1}$ by

$$\tilde{x}_j(x,t) \triangleq x_j - x_j^d(x,t), \quad j \in \{2,\cdots,n\}. \tag{39}$$

From the model (15), Lemma 2, and the control law (22) and (31) with the expression (34) for the input u_2, we find that $\tilde{z}$ satisfies

$$\begin{aligned} \dot{\tilde{x}}_2 &= -\lambda_2 \tilde{x}_2 \qquad (40) \\ \dot{\tilde{x}}_3 &= \lambda_3 f^3(t) \tilde{x}_3 + k f(t) \tilde{x}_2 \\ &\vdots \\ \dot{\tilde{x}}_n &= -\lambda_n f^{2(n-2)+1}(t) \tilde{x}_n + k f(t) \tilde{x}_{n-1}. \qquad (41) \end{aligned}$$

The following lemma shows that $\tilde{x}_j(x(t),t)$ tends exponentially toward zero.

We denote

$$\underline{z}_j = [x_2, x_3, \cdots, x_j]^T, \quad j \in \{2,\cdots,n\} \tag{42}$$

and use the one-norm $\|\cdot\|$ as defined by (18).

Lemma 3: Consider system (15) with the control law (22) and (31). Let $\tilde{z}(x(t),t)$ be defined by (39) where $x(t)$ is the solution of (15). Then, for any $j \in \{2,\cdots,n\}$, there is a $\delta > 0$ such that

$$\begin{gathered} \forall \varepsilon_2, \cdots, \varepsilon_j \in (0,\delta), \\ \exists c_j > 0, \exists \gamma_j > 0, \\ \forall t_p \in \{t_0, t_1, \cdots\}, \forall \underline{z}_j(t_p) \in R^{j-1} \\ |\tilde{x}(x(t),t)| \le c_j \|\underline{z}_j(t_p)\| e^{-\gamma_j (t-t_p)}, \quad \forall t \ge t_p \end{gathered} \tag{43}$$

where the constants γ_j can be defined as

$$\gamma_j = \alpha_j - \varepsilon_j > 0 \tag{44}$$

$$\alpha_q = \min\{\lambda_q \eta_q, \alpha_{q-1} - \varepsilon_{q-1}\}, \quad q \in \{3,\cdots,j\} \tag{45}$$

$$\alpha_2 = \lambda_2. \tag{46}$$

The constants η_q are found from Property P4) of $f(t)$.

Proof: The proof will be given by induction. Assume that (43) is satisfied for $j = m-1 \in \{2,\cdots,n-1\}$ for a $t_p \in \{t_0, t_1, \cdots\}$. Then, from (40)–(41) we have that

$$\begin{aligned} \dot{\tilde{x}} &= -\lambda_m f^{2m-3}(t) \tilde{x}_m + k(x(t_i)) f(t) \tilde{x}_{m-1} \\ &= -a(t) \tilde{x}_m + d(t) \end{aligned}$$

where

$$\begin{aligned} a(t) &= \lambda_m f^{2m-3}(t) \\ d(t) &= k(x(t_i)) f(t) \tilde{x}_{m-1}(x(t),t). \end{aligned}$$

From Property P4) of $f(t)$ we have that for all $t_p \in \{t_0, t_1, \cdots\}$ and for all $t \ge t_p$

$$\left| \int_{t_p}^{t} (a(\tau) - \lambda_m \eta_m) d\tau \right| \le \lambda_m P_m.$$

Since (43) is assumed to be satisfied for $j = m-1$ and since $|f(t)| \le 1$ from P1), and $|k(x)| \le K$ from (20) and (23) we have

$$|d_m(t)| \le D_m e^{-\gamma_{m-1}(t-t_p)}$$

where $D_m \triangleq Kc_{m-1}\|\underline{z}_{m-1}(t_p)\|$. From the definition of $\tilde{z}(x,t)$ (39) and since $x_j^d(x(t_p),t_p) = 0$ for all $t_p \in \{t_0, t_1, \cdots\}$, Section V-C, we have $\tilde{x}_m(x(t_p),t_p) = x_m(t_p) = x_m(t_p)$. From Lemma 1 we can then conclude that

$$\begin{aligned}
&\forall \varepsilon_m > 0\, \exists_{c_m} \exists_{\gamma_m} \forall \underline{z}_{m-1} \mid \\
&|\tilde{x}_m(t)| \le c_m(\|\underline{z}_{m-1}(t_p)\| + |\tilde{x}_m(t_p)|)e^{-\gamma_m(t-t_p)} \\
&\quad = c_m\|\underline{z}_{m-1}(t_p)\|e^{-\gamma_m(t-t_p)}
\end{aligned}$$

where the constants γ_m and c_m can be defined as

$$\begin{aligned}
\gamma_m &= \alpha_m - \varepsilon_m \\
\alpha_m &= \min\{\lambda_m \eta_m, \gamma_{m-1}\} \\
c_m &= \max\left\{e^{\lambda_m P_m}, e^{2\lambda_m P_m}\frac{Kc_{m-1}}{\varepsilon_m}\right\}.
\end{aligned}$$

The constant γ_m is strictly positive if

$$0 < \varepsilon_j < \alpha_j, \quad j \in \{2, \cdots, m\}.$$

Consequently, there exists a constant δ such that $\varepsilon_2, \cdots, \varepsilon_n \in (0, \delta)$ implies that $\gamma_m > 0$.

We have, therefore, proved that if (43) is satisfied for $j = m-1$ for a $t_p \in \{t_0, t_1, \cdots\}$, then (43) is satisfied for $j = m$, for all $m \in \{3, \cdots, n\}$. The induction is completed by showing that (43) is satisfied for $j = 2$. From (40) we find that for all $x(t_p), t_p \in \{t_0, t_1, \cdots\}$

$$\begin{aligned}
|\tilde{x}_2(x(t),t)| &= |\tilde{x}_2(x(t_p),t_p)|e^{-\lambda_2(t-t_p)} \\
&= \|\underline{z}_2(t_p)\|e^{-\lambda_2(t-t_p)}
\end{aligned}$$

since $x_j^d(x(t_p),t_p) = 0$ for $t_p \in \{t_0, t_1, \cdots\}$ and hence $\tilde{x}_j(x(t_p),t_p) = x_j(t_p)$. Equation (43) is then satisfied for $j = 2$ by choosing $c_2 = 1$. (In this case, (43) is also satisfied for $\varepsilon_2 = 0$.) Since (43) is satisfied for all $t_p \in \{t_0, t_1, \cdots\}$ when $j = 2$ then (43) is proved by induction to be satisfied for all $t_p \in \{t_0, t_1, \cdots\}$. □

Remark: Note from (45) that arbitrarily large values for $\alpha_2, \cdots, \alpha_n$ can be obtained by choosing $\lambda_2, \cdots, \lambda_n$ appropriately, which means that the exponential rate of convergence $\gamma_j = \alpha_j - \varepsilon_j$ can be chosen arbitrarily large since ε_j can be chosen arbitrarily close to zero.

The definitions of x^d (32)–(33) and $\tilde{x}$ (39) can be used to show that the state x_j can be expressed as a weighted sum of the functions $\tilde{x}_r, r \in \{j, \cdots, n\}$ as stated in the following lemma.

Lemma 4: Let $x_j^d(x,t)$ be defined by (32)–(33) and $\tilde{x}_j(x,t)$ by (39) for $j \in \{2, \cdots, n\}$ and $t \in [t_i, t_{i+1})$. If $x(t_i) \neq 0$ then x_j satisfies for all $j \in \{2, \cdots, n-1\}$

$$x_j = \tilde{x}_j(x,t) + f^{2j-2}(t)\sum_{r=j+1}^{n} \tilde{g}_{jr}(t)\frac{1}{k^{r-j}(x(t_i))}\tilde{x}_r(x,t). \tag{47}$$

The functions $\tilde{g}_{jr}$: $R_+ \to R$ are bounded and smooth.

Proof: See Appendix A. □

From Lemmas 3 and 4 we can now show global, exponential convergence of $z(t)$ to zero.

Theorem 1: Consider system (15) with the control law (22) and (31) and z defined in (16). There are a function h_z: $R_+ \to R_+$ of class $\mathcal{K}$ and a constant $\gamma_z > 0$ such that for any initial condition $z(t_0) \in R^{n-1}$

$$\|z(t)\| \le h_z(\|z(t_0)\|)e^{-\gamma_z(t-t_0)}, \quad \forall t \ge t_0. \tag{48}$$

The constant γ_z can be given by

$$\gamma_z = \frac{\gamma_n}{2}$$

where γ_n is given from (44)–(46).

Proof: See Appendix A. □

VII. Stability Analysis of $x(t)$

In this section, we prove global $\mathcal{K}$-exponential stability of the state $x(t)$ of the system (15) about zero.

By showing that $u_1 = k(x(t_i))f(t)$, (22), makes $x_1(t)$ converge to zero as $z(t)$ converge to zero, Theorem 1 can be used to show the following lemma.

Lemma 5: Let the control law be given by (22) and (31). Then, system (15) is $\mathcal{K}$-exponentially stable about the origin, i.e., in a neighborhood Ω about $x = 0$ there exist a function $h(\cdot;T)$ of class $\mathcal{K}$ and a constant $\gamma > 0$ such that

$$\forall x(t_0) \in \Omega\, \|x(t)\| \le h(\|x(t_0);T\|)e^{-\gamma(t-t_0)}, \quad \forall t \ge t_0 \tag{49}$$

where the function $h(\cdot;T)$ depends on time parameter $T = t_{i+1} - t_i$, (17).

Proof: See Appendix A. □

The neighborhood Ω is given by

$$\Omega = \left\{x \mid |x_1| < \frac{K}{2\beta'}, \quad G(h_2(\|z\|)) < \frac{K}{2\beta}\right\} \tag{50}$$

where $h_z(\cdot)$ is a function of class $\mathcal{K}$ from Theorem 1 and $z = [x_2, \cdots, x_n]^T$ (16). The function $G(\cdot)$ is defined in (24), and β is defined in (25). The constant γ can be defined as

$$\begin{aligned}
\gamma &= \frac{\gamma_z}{2n-4} > 0 \\
\gamma_z &= \frac{\gamma_n}{2}
\end{aligned}$$

where γ_n is given from Lemma 3.

Remark 1: The exponential convergence rate γ in (49) can be selected arbitrarily large by choosing $\lambda_2, \cdots, \lambda_n$ appropriately, that is large enough. See also the remark to Lemma 3 in Section VI. For all $\lambda_2, \cdots, \lambda_n > 0$ the system is $\mathcal{K}$-exponentially stable for all positive T where $T = t_{i+1} - t_i$. The class $\mathcal{K}$ function $h(\cdot;T)$ increases exponentially, however, with the parameter T. This is due to the fact that it takes at least the time T for the input u_1 to drive x_1 to an arbitrarily small neighborhood about zero.

Remark 2: The neighborhood Ω can be chosen arbitrarily large by choosing the parameters K, $\mathcal{K}$ and β (or $T = t_{i+1} - t_i$) appropriately. Therefore, the stability is semi-global.

Remark 3: The bounds on $\|x(t)\|$ given by the function $h(\cdot;T)$, (88), and γ_z can be very conservative and do not in general provide quantitative information.

Lemma 5 does not prove global, $\mathcal{K}$-exponential stability because of the saturation function in the definition of $k(x(t_i))$, (23). A function $h_x(\cdot)$ of class $\mathcal{K}$, however, can be constructed to yield also global, $\mathcal{K}$-exponential stability as shown in the following theorem.

Theorem 2: Let the control law be given by (22) and (31). Then system (15) is globally, $\mathcal{K}$-exponentially stable about the origin, i.e., there exist a function $h_x(\cdot;T)$ of class $\mathcal{K}$ and a constant $\gamma > 0$ such that $\forall t \geq t_0$

$$\forall x(t_0) \in R^n \|x(t)\| \leq h_x(\|x(t_0)\|; Te^{-\gamma(t-t_0)}. \quad (51)$$

Proof: See Appendix A. □

As for Lemma 5, the constant γ can be defined as follows

$$\gamma = \frac{\gamma_z}{2n-4} > 0$$
$$\gamma_z = \frac{\gamma_n}{2}$$

where γ_n is given from Lemma 3.

The remarks after Lemma 5 on the rate of convergence γ and the dependence of T for the class $\mathcal{K}$ function $h_x(\cdot;T)$ also apply here.

VIII. Stabilization About Abritrary Configuration

We have shown that the controller (22) and (31), and (23) makes system (15) globally, $\mathcal{K}$-exponentially stable about the origin, i.e., about $[x_1,\cdots,x_n]^T = 0$. In this section we show that the same control law can be used to $\mathcal{K}$-exponentially stabilize the chained from (15) about any configuration. Indeed, here we show that any strategy to control the chained form to the origin can be used to control it to any desired configuration.

Let the desired constant configuration be given by

$$x^p = [x_1^p, x_2^p, \cdots, x_n^p]^T \in R^n, \quad \dot{x}^p \equiv 0.$$

Now we introduce the following variables

$$x_m^r \triangleq x_m^p + \sum_{j=2}^{m-1} x_j^p \frac{1}{(m-j)!}(x_1 - x_1^p)^{m-j}, \quad m \in \{1,\cdots,n\}. \quad (52)$$

Here, x_i is a state variable of the chained form (15) satisfying $\dot{x}_1 = u_1$. The vector $x^r = [x_1^r,\cdots,x_n^r]^T$ is thus given from (52) as a smooth function $x^r = \phi(x_1;x^p)$.

Lemma 6: Let $x^r = [x_1^r,\cdots,x_n^r]^T$ be given by (52). Then

$$\dot{x}_1^r = 0 \quad (53)$$
$$\dot{x}_2^r = 0 \quad (54)$$
$$\dot{x}_m^r = x_{m-1}^r u_1, \quad m \in \{3,\cdots,n\}. \quad (55)$$

Proof: From (52) we have that $x_1^r = x_1^p$ and $x_2^r = x_2^p$. Since x_1^p and x_2^p are constants, (53)–(54) follow readily. Equation (55) can be proved by induction. Assume that there is an index $m \in \{3,\cdots,n-1\}$ such that

$$\dot{x}_m^r = x_{m-1}^r u_1.$$

Since $\dot{x}_m^p = 0$, $m \in \{1,\cdots,n\}$, differentiating x_{m+1}^r from (52) gives

$$\begin{aligned}\dot{x}_{m+1}^r &= \sum_{j=2}^{m} x_j^p \frac{1}{(m+1-j)!}(m+1-j) \\ &\quad \times (x_1 - x_1^p)^{m-j}(\dot{x}_1 - \dot{x}_1^p) \\ &= \sum_{j=2}^{m} x_j^p \frac{1}{(m-j)!}(x_1 - x_1^p)^{m-j} u_1 \\ &= \left[x_m^p + \sum_{j=2}^{m-1} x_j^p \frac{1}{(m-j)!}(x_1 - x_1^p)^{m-j} \right] u_1 \\ &= x_m^r u_1.\end{aligned}$$

Consequently, if (55) is satisfied for index m then (55) is satisfied for $m+1$, too. The proof is then completed by showing that $\dot{x}_3^r = x_2^r u_1$. From (52) we get

$$\dot{x}_3^r = \frac{d}{dt}[x_2^p(x_1 - x_1^p) + x_3^p] = x_2^p u_1 = x_2^r u_1.$$

□

We now define

$$\overline{x}_m \triangleq x_m - x_m^r, \quad m \in \{1,\cdots,n\}. \quad (56)$$

From system (15) and Lemma 6 it follows that

$$\dot{\overline{x}}_1^r = u_1 \quad (57)$$
$$\dot{\overline{x}}_2^r = u_2 \quad (58)$$
$$\dot{\overline{x}}_m^r = \overline{x}_{m-1}^r u_1, \quad m \in \{3,\cdots,n\}. \quad (59)$$

This system has the same structure as the chained form (15). A control law for (15) controlling $x = [x_1,\cdots,x_n]^T$ to zero can be used to control $\overline{x} = [\overline{x}_1,\cdots,\overline{x}_n]^T$ to zero. The coordinate transformation between x and $\overline{x}$ is then given from (52) and (56) by

$$\begin{aligned}\overline{x} &= x - x^r = x - \phi(x_1;x^p) \triangleq \tau(x;x^p) \\ x &= \overline{x} + x^r = \overline{x} + \phi(x_1;x^p) \\ &= \overline{x} + \phi(x_1 + x_1^p; x^p) \triangleq \overline{\tau}(\overline{x};x^p)\end{aligned}$$

where $\tau(\cdot;x^p)$ and $\overline{\tau}(\cdot;x^p)$ are smooth functions.

We can then conclude with the following theorem.

Theorem 3: Given the system (57)–(59) where $\overline{x}_m$ and x_m^r are given by (56) and (52). Then, a control law for (57)–(59) making $\overline{x} = [\overline{x}_1,\cdots,\overline{x}_n]^T$ converge to zero makes x converge to the desired configuration x^p. The convergence of x to x^p is exponential if $\overline{x}$ converges exponentially to zero.

Proof: From (56) and (52) we have that

$$\begin{aligned}x_m = x_m^r + \overline{x}_m = x_m^p + \sum_{j=2}^{m-1} x_j^p \frac{1}{(m-j)!}\overline{x}_1^{m-j} \\ + \overline{x}_m, \ m \in \{1,\cdots,n\}.\end{aligned}$$

Therefore, the control of $\overline{x}$ to zero implies the control of x to x^p. We also see that exponential convergence of $\overline{x}$ to zero implies exponential convergence of x to x^p. □

IX. Simulations

A simulation with $n = 4$ was done in MATLAB at a SPARC Station 1. The control law for u_1 was chosen as follows

$$u_1 = k(x(t_i))f(t), \quad f(t) = (1 - \cos t)/2$$

where

$$k = \operatorname{sat}(-[x_1(t_i) + \operatorname{sgn}(x_1(t_i))G(\|z(t_i)\|)]\beta, K), \quad K = 2$$

as defined in (23). By studying the time integral of $f^3(t)$ and $f^5(t)$, we see that Property P4) of $f(t)$ is satisfied by choosing

$$\eta_3 = \frac{5}{16}, \; \eta_4 = \frac{63}{256}, \; P_3 = \frac{1}{2}, \; P_4 = \frac{1}{2}.$$

The controller parameter κ in $G(\cdot)$, (24), was taken to $\kappa = 3$. The constant β is given by (25)

$$\beta = 1/\int_0^T f(\tau)d\tau = \frac{1}{\pi}$$

where T is the time-period of the function $f(t)$, i.e., $T = 2\pi$. The instants of time t_i where $k(x(t_i))$ may switch are given by the set $\{0, 2\pi, 4\pi, 6\pi, \cdots\}$.

We find the control law for u_2 from (31) and (27)–(30)

$$\begin{aligned} u_2 = &- (\lambda(1 + f^3 + f^5))x_2 \\ &- (f\lambda(f\lambda + f^3\lambda + f^6\lambda + 2\dot{f} + 8f^2\dot{f})/k)x_3 \\ &- (f\lambda(f^5\lambda^2 + 4f\lambda\dot{f} + 6f^4\lambda\dot{f} + 8\dot{f}^2 + 4f\ddot{f})/k^2)x_4. \end{aligned}$$

Here, we have chosen

$$\lambda = \lambda_2 = \lambda_3 = \lambda_4.$$

In the simulation, λ was taken to $\lambda = 1$.

The initial state was chosen as

$$(x_1(0), x_2(0), x_3(0), x_4(0)) = (0, -0.1, 0.1, 1).$$

Euler's method was applied for the numerical integration where the time-step was taken to 0.05. In Fig. 1, $z(t) = [x_2(t), x_3(t), x_4(t)]^T$ is plotted versus time showing the convergence to zero. The state variable $x_1(t)$ is plotted in Fig. 2. We see that $x_1(t)$ converges to zero, too. Note, however, from the time-axes that the rate of convergence of $x_1(t)$ is slower than the one of $z(t)$. This coincides with the relation between γ and γ_z in Lemma 5 where $\gamma = \gamma_z/4$. To show that the convergence is exponential, $\log \|x(t)\|$ is plotted in Fig. 3 where the one-norm is chosen. We see from this figure that $\log \|x(t)\|$ decreases toward $-\infty$ implying that $\|x(t)\|$ exponentially converges to zero.

By using the coordinate transformation from [33], we can interpret the variables x_1 and x_4 as the x- and y-position of the midpoint of the rear axle of a four-wheeled car. The path $(x(t), y(t))$ is presented in Fig. 4. We see that the motion seems natural when interpreted as a parking maneuver.

In Figs. 5 and 6 the inputs $u_1(t)$ and $u_2(t)$ are shown as functions of time. We see that they are continuous and converge exponentially to zero.

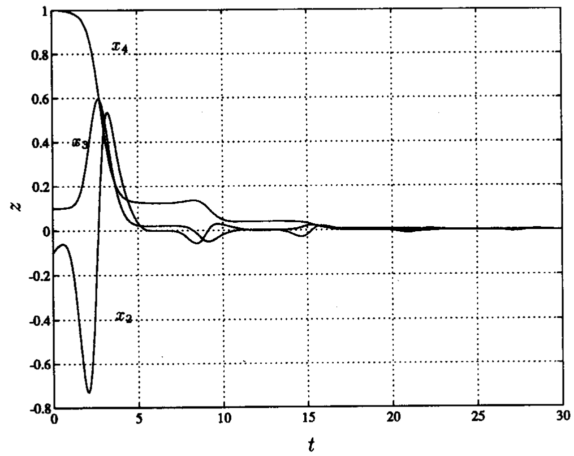

Fig. 1. Exponential convergence of $z(t) = [x_2, x_3, x_4]^T$ to zero.

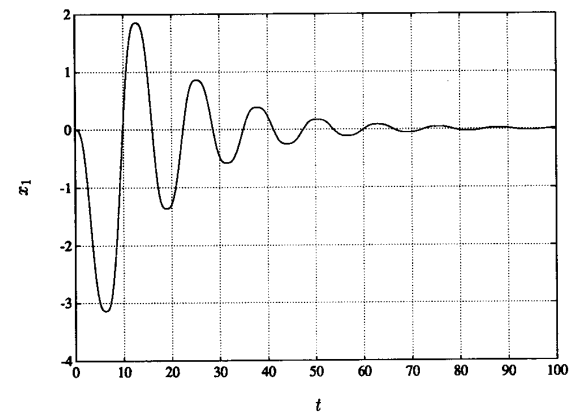

Fig. 2. Exponential convergence of $x_1(t)$ to zero.

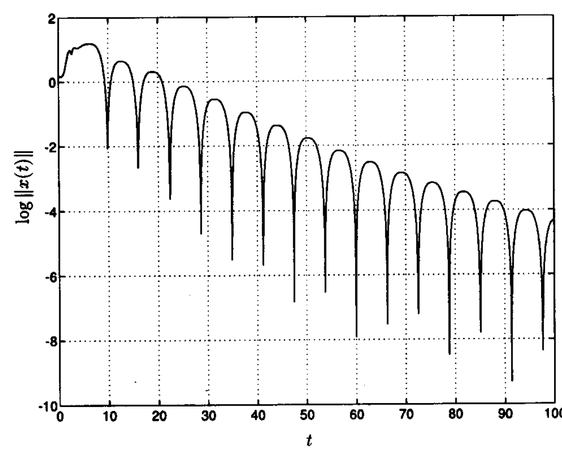

Fig. 3. Convergence of $\log \|x(t)\|$ to $-\infty$.

X. Summary and Conclusion

A feedback control law has been proposed to globally stabilize a chained nonholonomic system of any dimension.

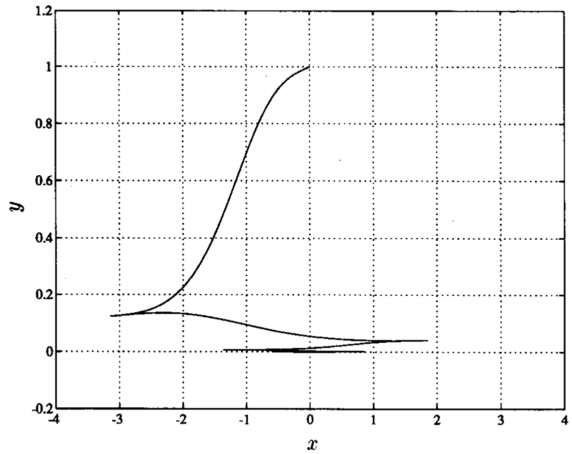

Fig. 4. The resulting path in the xy—plane when applying the exponential convergent control law. The variables $x = x_1$ and $y = x_4$ are interpreted as the planar position of a four-wheeled car.

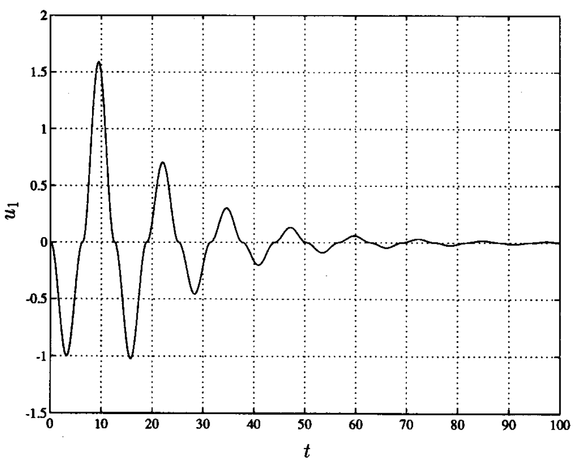

Fig. 5. The input $u_i(t)$ from the exponentially convergent control law.

The resulting rate of convergence is exponential and global, $\mathcal{K}$-exponential stability was achieved. $\mathcal{K}$-exponential stability is a weaker form of stability than exponential stability in the usual sense, but it possesses the same rate of convergence.

The feedback control law depends on a time-varying function, not necessarily periodic, and on a function which is nonsmooth with respect to the state at predefined discrete instants of time. The exponential rate of convergence can be arbitrarily fast by choosing the controller parameters appropriately. The controller cannot, however, drive the system to an arbitrary small neighborhood of the desired configuration in shorter time than T i.e., the length of the time intervals used in the control law. The class $\mathcal{K}$ function which defines a initial-state-dependent upper bound of the time-evolution of the system, has T as a parameter. Simulation of a four dimensional chained form indicated the exponential convergence. By using a coordinate transformation similar to the transformation from chained form to power form, a strategy to control the state to zero can be used to control the state of any desired configuration.

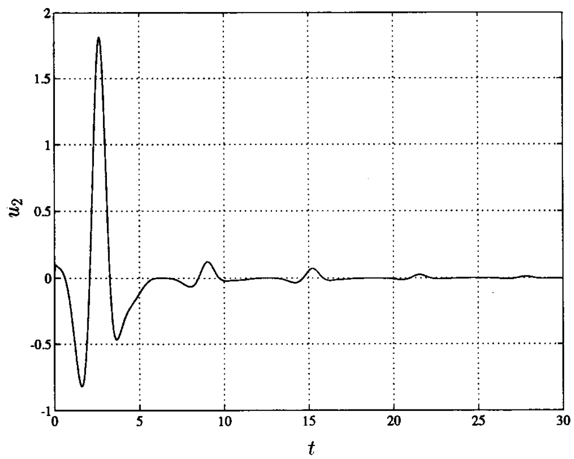

Fig. 6. The input $u_2(t)$ from the exponentially convergent control law.

The idea of introducing a function which varies with the state at discrete instants of time seems, therefore, to be a useful approach to obtain good stability properties for nonholonomic systems. The control law has a simple structure, though the stability analysis is quite involved because of few existing mathematical tools for such systems.

A drawback of this approach is the possibility for numerical problems at digital computers as the parameter $k(x(t_i))$ approaches zero, since there are divisions by $k(x(t_i))$ in the control law for u_2. (These divisions do not cause unbounded quantities, since $k(x(t_i))$ always dominates the numerator.) Such numerical problems were, however, not observed during the simulations. Since the parameter $k(x(t_i))$ only changes at $t_i \in \{t_0, t_1, \cdots\}$, the control of x_1 may be more sensitive to disturbances and modeling imperfections than if $k = k(x(t))$ i.e., function of the state at all $t \geq t_0$. Depending on the physical application, feedback from the state only at $t_i \in \{t_0, t_1, \cdots\}$ may be sufficient to make x_1 exponentially converge to zero. Note that the state variables $z = [x_2, \cdots, x_n]^T$ are controlled by the control law $u_2 = u_2(z(t), k)$ which depends on $z(t)$ for all $t \geq t_0$.

In further work, an extension from $k = k(x(t_i))$ to $k = k(x(t))$ by redefining z^d, (32)–(33), can be studied. The structure of $k(x(t_i))$ in the present control law indicates that the use of $k = k(x(t))$ results in a nonsmooth, time-varying feedback control law. Also, more mathematical tools are needed to analyze such systems. The need for the definition of $\mathcal{K}$-exponential stability for the stabilization with exponential convergence of chained systems and other nonholonomic systems should also be analyzed.

APPENDIX
PROOFS

Proof of Lemma 4: The arguments of $k(x(t_i)), \tilde{x}(x,t)$, $x^d(x,t), f(t)$, and $g_{jr}(t)$ will mostly be omitted for simplicity. From the definition (39) of $\tilde{x}_{n-1}$ we have that

$$x_{n-1} = \tilde{x}_{n-1} + x_{n-1}^d. \tag{60}$$

From (32) we have that $x_n^d = 0$ which means that $\tilde{x}_n = x_n$. It is trivially seen from Lemma 2 assuming that $x(t_i) \neq 0$ that

$$x_{n-1} = \tilde{x}_{n-1} + x_{n-1}^d = \tilde{x}_{n-1} + f^{2n-4}(-\lambda_n)\frac{1}{k}\tilde{x}_n \quad (61)$$

which implies that

$$\tilde{g}_{n-1,n} = g_{n-1,n} = -\lambda_n. \quad (62)$$

We can then calculate from the definition (33) of x_{n-2}^d and (60)–(62)

$$\begin{aligned} x_{n-2} &= \tilde{x}_{n-2} + x_{n-2}^d \\ &= \tilde{x}_{n-2} + f^{2n-6}\Big(g_{n-2,n-1}\frac{x_{n-1}}{k} + g_{n-2,n}\frac{x_n}{k^2}\Big) \\ &= \tilde{x}_{n-2} + f^{2n-6}\bigg[g_{n-2,n-1}\frac{1}{k}\tilde{x}_{n-1} \\ &\quad + (g_{n-2,n} + g_{n-2,n-1}\tilde{g}_{n-2,n}f^{2n-4})\frac{x_n}{k^2}\bigg] \\ &= \tilde{x}_{n-2} + f^{2n-6}\left[\tilde{g}_{n-2,n-1}\frac{1}{k}\tilde{x}_{n-1} + \tilde{g}_{n-2,n}\frac{1}{k^2}x_n\right] \end{aligned} \quad (63)$$

where

$$\tilde{g}_{n-2,n-1} = g_{n-2,n-1} \quad (64)$$
$$\tilde{g}_{n-2,n} = g_{n-2,n} + g_{n-2,n-1}\tilde{g}_{n-1,n}f^{2n-4}. \quad (65)$$

The rest of the proof will be given by induction. Assume that (47) is satisfied for $j+1, j+2, \cdots, n-1$, i.e.,

$$x_{m+1} = \tilde{x}_{m+1} + f^{2m}\sum_{r=m+2}^{n}\tilde{g}_{m+1,r}\frac{1}{k^{(r-m-1)}}\tilde{x}_r \quad (66)$$

where $m \in \{j, j+1, \cdots, n-2\}$. We then find from the definition (33) of x^d assuming that $x(t_i) \neq 0$

$$\begin{aligned} x_j &= \tilde{x}_j + x_j^d \\ &= \tilde{x}_j + f^{2j-2}(t)\sum_{r=j+1}^{n} g_{jr}\frac{1}{k^{r-j}}x_r \\ &= \tilde{x}_j + f^{2j-2}(t)\sum_{r=j+1}^{n} g_{jr}\frac{1}{k^{r-j}} \\ &\quad \times \left[\tilde{x}_r + f^{2r-2}\sum_{m=r+1}^{n}\tilde{g}_{rm}\frac{1}{k^{m-r}}\tilde{x}_m\right] \\ &= \tilde{x}_j + f^{2j-2}(t)\sum_{r=j+1}^{n} g_{jr} \\ &\quad \times \left[1 + f^{2r-2}\sum_{m=j+1}^{r-1}\tilde{g}_{mr}\right]\frac{1}{k^{r-j}}\tilde{x}_r \\ &= \tilde{x}_j + f^{2j-2}(t)\sum_{r=j+1}^{n}\tilde{g}_{jr}\frac{1}{k^{r-j}}\tilde{x}_r \end{aligned} \quad (67)$$

where

$$\tilde{g}_{jr} = g_{jr}\left[1 + f^{2r-2}\sum_{m=j+1}^{r-1}\tilde{g}_{mr}\right]. \quad (68)$$

Equation (67) shows that if (47) is satisfied for $j+1, j+2, \cdots, n$, then (47) is satisfied for j as well. Therefore, (61)–(62) and (67) imply that (47) is satisfied for all $j \in \{2, \cdots, n-1\}$. We see that $\tilde{g}_{jr}$ is given by $\tilde{g}_{mr}$ and g_{jr} which are known. Because of the smoothness and boundedness of $f(t), g_{jr}(t)$ and $\tilde{g}_{mr}(t)$ for $m \in \{j+1, \cdots, r-1\}, \tilde{g}_{jr}(t)$ will be smooth and bounded as well.

Proof of Theorem 1: The point of time t_i denotes the largest term in the sequence $(t_0, t_2, \cdots)$ such that $t \geq t_i$. If $x(t_i) = 0$ then $[u_1(t), u_2(t)]^T = 0$ from the control law (22) and (31) for $t \in [t_i, t_{i+1})$. From the model (15) we then see that $x(t) = 0$, for $t \in [t_i, t_{i+1}]$, implying that $[u_1(t), u_2(t)]^T = 0$ for $t \in [t_{i+1}, t_{i+2})$, etc. and $x(t) = 0$ for all $t \geq t_i$. In that case, relation (48) is satisfied for all $t \geq t_i$ for any function $h_z(\cdot)$ of class $\mathcal{K}$. In the following, we analyze the case where $x(t_i) \neq 0$. The norm $\|\cdot\|$ denotes the one-norm (18).

From Lemma 4 we have that for $j \in \{2, \cdots, n-1\}$

$$\begin{aligned} x_j(t) &= \tilde{x}_j(x(t), t) + f^{2j-2}(t) \\ &\quad \times \sum_{r=j+1}^{n}\tilde{g}_{jr}(t)\frac{1}{k^{r-j}(x(t_i))}\tilde{x}_r(x(t), t) \end{aligned}$$

where the functions $\tilde{g}_{jr}(t)$ are bounded, say by the constant G, such that $|\tilde{g}_{jr}(t)| \leq G$ for all $t \geq t_0$. Since $|f(t)| \leq 1$, Property P2) of $f(t)$, we get

$$|x_j(t)| \leq |\tilde{x}_j(x(t), t)| + \sum_{r=j+1}^{n} G\frac{1}{|k(x(t_i))|^{r-j}}|\tilde{x}_r(x(t), t)|, \quad j \in \{2, \cdots, n-1\}. \quad (69)$$

Now, we will show that there are constants κ_j such that

$$|k(x(t_i))| = K \text{ or } |k(x(t_i))| \geq \kappa_r|\tilde{x}_r(x(t), t)|^{\frac{1}{2n-4}} \quad (70)$$

for all $j \in \{3, \cdots, n\}$. From (23) we find that if $|k(x(t_i))| < K$ then

$$|k(x(t_i))| = [|x_1(t_i))| + G(\|z(t_i)\|)]\beta.$$

Combining this with the definition of $G(\cdot)$, (24), implies

$$\begin{aligned} |k(x(t_i))| \geq G(\|z(t_i)\|)\beta &= \beta\kappa\|z(t_i)\|^{\frac{1}{2n-4}} \\ &\geq \beta\kappa\|\underline{z}_j(t_i)\|^{\frac{1}{2n-4}} \\ &\quad j \in \{2, \cdots, n\}. \end{aligned} \quad (71)$$

Lemma 3 implies

$$\begin{aligned} |k(x(t_i))| &\geq \beta\kappa\left(\frac{1}{c_j}|\tilde{x}_j(x(t), t)|\right)^{\frac{1}{2n-4}} \\ &= \beta\kappa c_j^{\frac{-1}{2n-4}}|\tilde{x}_j(x(t), t)|^{\frac{1}{2n-4}}, \quad j \in \{3, \cdots, n\}. \end{aligned} \quad (72)$$

Therefore, (70) is satisfied by defining $\kappa_j = \beta\kappa c_j^{\frac{-1}{2n-4}}$.

From (70) and (69) we then get

$$\begin{aligned} |x_j(t)| \leq |\tilde{x}_j(x(t), t)| + G\sum_{r=j+1}^{n}&\bigg[\frac{1}{K^{r-j}}|\tilde{x}_r(x(t), t)| \\ &+ \frac{1}{\kappa_r^{r-j}}|\tilde{x}_r(x(t), t)|^{1-\frac{r-j}{2n-4}}\bigg] \end{aligned} \quad (73)$$

for all $j \in \{2, \cdots, n-1\}$. From Lemma 3 we get, with $t_p = t_0$

$$\begin{aligned} |x_j(t)| &\le c_j \|\underline{z}_j(t_0)\| e^{-\gamma_j (t-t_0)} \\ &+ G \sum_{r=j+1}^{n} \left[\frac{1}{K^{r-j}} c_r \|\underline{z}_r(t_0)\| e^{-\gamma_r(t-t_0)} \right. \\ &+ \left. \left(\frac{c_r}{\kappa_r^{r-j}} \|\underline{z}_r(t_0)\| e^{-\gamma_r(t-t_0)} \right)^{1-\frac{r-j}{2n-4}} \right] \\ &\le h_j(\|z(t_0)\|) e^{-\zeta_j(t-t_0)} \end{aligned}$$

for all $j \in \{2, \cdots, n-1\}$, since $\|z(t_0)\| \ge \|\underline{z}_r(t_0)\|$, where

$$h_j(q) = c_j q + G \sum_{r=j+1}^{n} \left[\frac{1}{K^{r-j}} c_r q + \left(\frac{c_r}{\kappa_r^{r-j}} q \right)^{1-\frac{r-j}{2n-4}} \right]$$

$$\zeta_j = \min_r \left\{ \gamma_r \left(1 - \frac{r-j}{2n-4} \right) \right\}, \quad r \in \{j, \cdots, n\}. \quad (74)$$

From (44)–(46) we see that

$$0 < \gamma_n < \gamma_{n-1} < \cdots < \gamma_2 < \lambda_2.$$

Equation (74) then implies that

$$\zeta_j = \gamma_n \left(1 - \frac{n-j}{2n-4} \right).$$

Note that the function $h_j(q)$ is of class $\mathcal{K}$. Since $x_n^d \triangleq 0$ we have from Lemma 3

$$|x_n(t)| = |\tilde{x}_n(x(t), t)| \le c_n \|z(t_0)\| e^{-\gamma_n(t-t_0)}.$$

We define $h_n(\|z(t_0)\|) = c_n \|z(t_0)\|$ and $\zeta_n = \gamma_n$ such that from Lemma 3

$$|x_n(t)| = |\tilde{x}_n(t)| \le h_n(\|z(t_0)\|) e^{-\gamma_n(t-t_0)}.$$

The proof is completed by noting that

$$\begin{aligned} \|z(t)\| &= |x_2(t)| + \cdots |x_n(t)| \\ &\le \sum_{j=2}^{n} h_j(\|z(t_0)\|) e^{-\zeta_j(t-t_0)} = h_z(\|z(t_0)\|) e^{-\gamma_z(t-t_0)} \end{aligned}$$

where

$$h_z(q) = \sum_{j=2}^{n} h_j(q) \quad (75)$$

$$\gamma_z = \min\{\zeta_2, \cdots, \zeta_n\} = \zeta_2 = \frac{\gamma_n}{2}. \quad (76)$$

□

Proof of Lemma 5: The outline of the proof is to show that we can find a small enough neighborhood Ω of the origin such that the controller u_1 does not get saturated due to the convergence of $z(t)$ from Theorem 1. It will then be shown that the control law $u_1 = k(x(t_i)) f(t)$ is chosen such that $x_1(t)$ converges to zero as $z(t)$ converges. A function $h(\cdot; T)$ will be constructed. The norm $\|\cdot\|$ denotes the one-norm (18).

Let the neighborhood be given by

$$\Omega = \left\{ x \mid |x_1| < \frac{K}{2\beta}, G(h_z(\|z\|)) < \frac{K}{2\beta} \right\}$$

where $h_z(\cdot)$ is a function of class $\mathcal{K}$ from Theorem 1 and $z = [x_2, \cdots, x_n]^T$ (16). The function $G(\cdot)$ is defined in (24), and β is defined in (25). By induction, we will show that if $x(t_0) \in \Omega$, then

$$|k(x(t_i))| < K, \quad \forall t_i \in \{t_0, t_1, \cdots\} \quad (77)$$

which implies from the definition of $k(\cdot)$, (23)

$$\begin{aligned} k(x(t_i)) &= -[x_1(t_i) + \mathrm{sgn}(x_1(t_i)) G(\|z(t_i)\|)]\beta, \\ &\forall t_i \in \{t_0, t_1, \cdots\}. \end{aligned} \quad (78)$$

Note from Theorem 1 that

$$\|z(t)\| \le h_z(\|z(t_0)\|), \quad \forall t \ge t_0.$$

Since $G(\cdot)$ is of class $\mathcal{K}$, (24), this implies

$$G(\|z(t)\|) \le G(h_z(\|z(t_0)\|)), \quad \forall t \ge t_0.$$

Assume for a $t_m \in \{t_0, t_1, \cdots\}$ that

$$k(x(t_m)) = -[x_1(t_m) + \mathrm{sgn}(x_1(t_m)) G(\|z(t_m)\|)]\beta.$$

Integrating $\dot{x}_1 = u_1 = k(x(t_m)) f(t)$ from t_m to t_{m+1} then gives

$$x_1(t_{m+1}) = -\mathrm{sgn}(x_1(t_m)) G(\|z(t_m))\|) \quad (79)$$

which implies that

$$|x_1(t_{m+1})| = G(\|z(t_m)\|) \le G(h_z(\|z(t_0)\|)).$$

By assumption, $x(t_0) \in \Omega$ which implies that $G(h_z(\|z(t_0)\|)) < \frac{K}{2\beta}$ and therefore

$$|x_1(t_{m+1})| < \frac{K}{2\beta}.$$

Since

$$[|x_1(t_{m+1})| + G(\|z(t_{m+1})\|)]\beta < \left(\frac{K}{2\beta} + \frac{K}{2\beta} \right)\beta = K$$

then from the definition of $k(\cdot)$, (23)

$$k(x(t_{m+1})) = -[x_1(t_{m+1}) + \mathrm{sgn}(x_1(t_{m+1})) G(\|z(t_{m+1})\|)]\beta.$$

Equations (77) and (78) are proved by noting that since $x(t_0) \in \Omega$, then

$$[|x_1(t_0)| + G(\|z(t_i)\|)]\beta < \left(\frac{K}{2\beta} + \frac{K}{2\beta} \right)\beta = K$$

which implies that

$$k(x(t_0)) = -[x_1(t_0) + \mathrm{sgn}(x_1(t_0)) G(\|z(t_0)\|)]\beta.$$

Integrating $\dot{x}_1 = u_1$ in (15) from t_i to $t < t_{i+1}$ with $u_1 = k(x(t_i)) f(t)$ and k constant gives

$$\begin{aligned} |x_1(t)| &\le |x_1(t_i)| + |k| \int_{t_i}^{t} f(\tau) d\tau \\ &\le |x_1(t_i)| + |k| \int_{t_i}^{t_{i+1}} f(\tau) d\tau \\ &= |x_1(t_i)| + |x_1(t_{i+1}) - x_1(t_i)| \\ &\le 2|x_1(t_i)| + |x_1(t_{i+1})|. \end{aligned} \quad (80)$$

Since $x(t_0) \in \Omega$ then $k(x(t_i))$ is given by (78) and we get from (79) and (80)

$$|x_1(t)| \leq 2G(||z(t_{i-1})||) + G(||z(t_i)||), \quad i \in \{1, 2, \cdots\}. \tag{81}$$

From Theorem 1 we have that

$$||z(t_k)|| \leq h_z(||z(t_0)||)e^{-\gamma_z(t_k - t_0)}. \tag{82}$$

From the definition of $G(\cdot)$, (24), we have

$$G(ae^{-bt}) = \kappa(ae^{-bt})^{\frac{1}{2n-4}} = e^{\frac{-bt}{2n-4}}G(a) \tag{83}$$

where a and b are positive constants. We denote for simplicity

$$q = h_z(||z(t_0)||).$$

Equation (83) combined with (81) and (82) implies

$$\begin{aligned} |x_1(t)| &\leq 2G(qe^{-\gamma_z(t_{i-1}-t_0)}) + G(qe^{-\gamma_z(t_i - t_0)}) \\ &= 2G(q)e^{\frac{-\gamma_z}{2n-4}(t_{i-1}-t_0)} + G(q)e^{\frac{-\gamma_z}{2n-4}(t_i - t_0)} \\ &\leq 3G(q)e^{\frac{-\gamma_z}{2n-4}(t_{i-1}-t_0)}. \end{aligned}$$

By convention we have for all $i \in \{1, 2, \cdots\}$

$$t_{i-1} = t_{i+1} - 2T \geq t - 2T$$

since $t \in [t_i, t_{i+1})$. This implies

$$|x_1(t)| \leq 3G(q)e^{\frac{-\gamma_z 2T}{2n-4}}e^{\frac{-\gamma_z(t-t_0)}{2n-4}}, \quad t \geq t_1. \tag{84}$$

If $t \in [t_0, t_1)$ we find the following bound on $|x_1(t)|$ by integrating $\dot{x}_1 = k(x(t_0))f(t)$ where k is given by (78) and $f(t) \geq 0$

$$\begin{aligned} |x_1(t)| &\leq |x_1(t_0)| + [|x_1(t_0)| + G(||z(t_0)||)]\frac{\int_{t_0}^{t} f(\tau)d\tau}{\int_{t_0}^{t_1} f(\tau)d\tau} \\ &\leq 2|x_1(t_0)| + G(||z(t_0)||). \end{aligned}$$

Since $t \in [t_0, t_1)$ this implies

$$|x_1(t)| \leq [2|x_1(t_0)| + G(||z(t_0)||]e^{-\frac{\gamma_z(t-t_1)}{2n-4}}, \quad t \in [t_0, t_1). \tag{85}$$

Combining (84) and (85) and using $t_1 - t_0 = T$ implies

$$\begin{aligned} |x_1(t)| \leq \{&3G(h_z(||z(t_0)||))e^{\frac{\gamma_z 2T}{2n-4}} \\ &+ [2|x_1(t_0)| + G(||z(t_0)||)]e^{\frac{\gamma_z T}{2n-4}}\}e^{\frac{-\gamma_z(t-t_0)}{2n-4}} \\ \leq\ &h_1(||x(t_0)||; T)e^{\frac{-\gamma_z(t-t_0)}{2n-4}}, \quad t \geq t_0 \end{aligned} \tag{86}$$

where

$$\begin{aligned} h(||x(t_0)||; T) = &\ 3G(h_z(||x(t_0)||))e^{\frac{\gamma_z 2T}{2n-4}} \\ &+ [2||x(t_0)|| + G(||x(t_0)||)]e^{\frac{\gamma_z T}{2n-4}}. \end{aligned}$$

Since $G(\cdot)$ and $h_z(\cdot)$ are functions of class $\mathcal{K}$, $h_1(\cdot; T)$ is also of class $\mathcal{K}$, (1). The proof is completed by noting from Theorem 1 and (86) that

$$\begin{aligned} ||x(t)|| &= |x_1(t)| + ||z(t)|| \\ &\leq h_1(||x(t_0)||; T)e^{\frac{-\gamma_z(t-t_0)}{2n-4}} + h_z(||z(t_0)||)e^{-\gamma_z(t-t_0)} \\ &\leq [h_1(||x(t_0)||; T) + h_z(||z(t_0)||)]e^{\frac{-\gamma_z(t-t_0)}{2n-4}} \\ &= h(||x(t_0)||; T)e^{-\gamma(t-t_0)}, \quad t \geq t_0 \end{aligned} \tag{87}$$

where

$$h(||x(t_0)||; T) = h_1(||x(t_0)|| + h_z(||x(t_0)||) \tag{88}$$
$$\gamma = \frac{\gamma_z}{2n-4}$$

where γ_z is given from Theorem 1. □

Proof of Theorem 2: The theorem will be proved by showing that $z(t) = [x_2(t), \cdots, x_n(t)]^T$ and $x_1(t)$ are bounded and reach the neighborhood Ω defined in Lemma 5 in finite time. After this finite time, exponential convergence of $x(t)$ is ensured by Lemma 5. A function $h_x(||x(t_0)||; T)$ is constructed by using expressions for the finite time and the bounds on $||x(t)||$ and $h(||x(t_0)||; T)$ in Lemma 5. The norm $||\cdot||$ denotes the one-norm (18).

We will seek a time $\tau(||x(t_0)||)$ such that for all $t \geq \tau, x(t) \in \Omega$. First, define the time function $T_z : R_+ \to R_+$ as

$$T_z(q) \triangleq \begin{cases} \dfrac{1}{\gamma_z}\ln G(h_z(h_z(q)))\dfrac{2\beta}{K}, & G(h_z(h_z(q))) \geq \dfrac{K}{2\beta} \\ 0, & G(h_z(h_z(q))) < \dfrac{K}{2\beta} \end{cases}. \tag{89}$$

Consider $G(h_z(||z(t)||))$. By inserting (48) and accounting for $||z|| \leq ||x||$ and the fact that $G(\cdot)$ and $h_z(\cdot)$ are of class $\mathcal{K}$, it is seen that

$$\forall t > T_z(||x(t_0)||) + t_0, \quad G(h_z(||z(t)||)) < \frac{K}{2\beta}. \tag{90}$$

Then, integrating $\dot{x}_1 = u_1 = k(x(t_i))f(t)$ from t_i to t_{i+1} gives

$$x_1(t_{i+1}) = \begin{cases} -\text{sgn}(x_1(t_i))G(||z(t_i)||, & |k(x(t_i))| < K \\ x_1(t_i) - \text{sgn}(x_1(t_i))\dfrac{K}{\beta}, & |k(x(t_i))| = K \end{cases}. \tag{91}$$

Define the time function τ: $R_+ \to R_+$ as

$$\tau(q) = \max\left\{T_z(q), \frac{\beta}{K}q\right\}. \tag{92}$$

One can show from (90) and (91) that for all $t_m \in \{t_0, t_1, \cdots\}$ which satisfies $t_m > t_0 + \tau(||x(t_0)||)$

$$|x_1(t_m)| < \frac{K}{2\beta}. \tag{93}$$

By integrating $\dot{x}_1 = k(x(t_m))f(t)$ one can show that $x_1(t)$ is a linear interpolation between $x_1(t_m)$ and $x_1(t_{m+1})$ for all $t \in [t_m, t_{m+1}]$. From the definition of Ω, (50), and from (90) and (93) we find

$$\forall t > t_0 + \tau(||x(t_0)||), \quad x(t) \in \Omega.$$

Denote by t_p the smallest element in the sequence $(t_0, t_1, \cdots)$ such that $t_p > t_0 + \tau(||x(t_0)||)$. From Lemma 5 it then follows that

$$\forall t \geq t_p, \quad ||x(t)|| \leq h(||x(t_p)||; T)e^{-\gamma(t-t_p)}. \tag{94}$$

To include $t \in [t_0, t_p)$, an expression for the maximum of $||x(t)||$ must be found.

We can show from (91) by using the convergence property of $z(t)$, Theorem 1, that

$$\begin{aligned}\max_{t\geq t_0}|x_1(t)| &\leq \max\{|x_1(t_0)|, G(h_z(||z(t_0)||))\}\\ &\leq \max\{||x(t_0)||, G(h_z(||x(t_0)||))\}\\ &\triangleq x_{1m}(||x(t_0)||)\end{aligned} \tag{95}$$

since $||x(t_0)|| \geq ||z(t_0)||$, and $h_z(\cdot)$ are of class $\mathcal{K}$. Here, we have defined the function x_{1m}: $R_+ \rightarrow R_+$ which is of class $\mathcal{K}$.

From (95) and Lemma 5 we have

$$\begin{aligned}||x(t)|| &\leq x_{1m}(||x(t_0)|| + h_z(||x(t_0)||\\ &\triangleq x_{mx}(||x(t_0)||), \quad \forall t \geq t_0.\end{aligned}$$

Here, we have defined the function x_{mx}: $R_+ \rightarrow R_+$ which is of class $\mathcal{K}$. The bound in (94) can then be extended to an exponential bound for all $t \geq t_0$ and all $x(t_0) \in R^n$

$$||x(t)|| \leq [h(x_{mx}(q);T) + x_{mx}(q)]e^{\gamma\tau(q)}e^{-\gamma(t-t_0)}$$

where q denotes $||x(t_0)||$. From the definition of $\tau(\cdot)$, (92), we then obtain for all $t \geq t_0$

$$\forall x(t_0) \in R^n, \quad ||x(t)|| \leq h_x(||x(t_0)||;T)e^{-\gamma(t-t_0)}$$

where the class $\mathcal{K}$ function h_x: $R_+ \rightarrow R_+$ is defined as

$$h_x(q;T) \triangleq [h(x_{mx}(q);T) + x_{mx}(q)]e^{\gamma\tau(q)}$$

and q denotes $||x(t_0)||$. □

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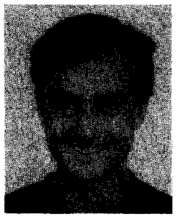

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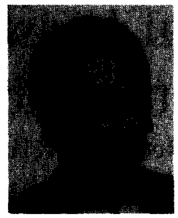

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