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Tracking Control of Wheeled Mobile Robots with Unknown Dynamics

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Abstract

This paper considers the tracking problem of nonholonomic wheeled mobile robots with unknown dynamics. A new adaptive robust global dynamic controller is presented based on the canonical form of the wheeled mobile robots. This novel controller has low dimension and no singular points. Simulations show the effectiveness of the control scheme.

1 Introduction

In recent years, there has been a growing interest in design of feedback control laws for wheeled mobile robots (*WMR*) subjected to nonholonomic constraints. Due to Brockett's theorem, it is well-known that the nonholonomic *WMR* cannot be asymptotically stabilized to a rest configuration by differentiable pure-state feedback laws [1]. However, several approaches have been proposed for stabilizing such systems. For details, see the survey paper [14].

Another control problem of *WMR* is trajectory tracking problem. For the tracking problem of the kinematic model (i.e. control inputs are generalized velocities), several authors have presented some results. In [3][16][13][11], the tracking problem of individual *WMR* is solved by designing a controller which makes the robot follow a virtual reference robot. However, these results do not fit all nonholonomic *WMR*. Since the kinematic model of *WMR* can be put into the chained system locally or globally [15], some authors have discussed the tracking problem of the chained system. A solution based on a linear approximation of the kinematic nonholonomic system around sufficiently exciting reference trajectories was described and analyzed in [18]. Using results on "differentially flat" nonlinear systems [9], an n-dimensional single-generator, one-chain system can be dynamically linearized by adding $(n-2)$ integrators. Therefore, the tracking problem of *WMR* can be solved by an $(n-2)$-dimensional dynamic controller [7]. Unfortunately, the control law has singular points, and the dimension of the controller is high if n is large. With change of the time scale, the singular points may be avoided [10], but the dimension of the controller can not be reduced. With aid of backstepping technique, a static controller was also presented for the tracking problem of the chained system in [12]. Since there is a unknown large enough number λ in the controller, it may be difficult to implement the control in practice.

In many cases, tracking control of the dynamic model of *WMR* with uncertainty are important for practical applications. Few results on the subject are presented now. [17][5][4] studied the tracking problem of dynamic nonholonomic systems with uncertainty. However, in these papers configuration tracking for the given trajectory still has not been solved, since only partial state (i.e. generalized velocity) tracking was achieved.

Our purpose in this paper is to investigate how to construct a controller which can make all states of the closed-loop dynamic model of *WMR* with unknown dynamics globally track a given trajectory. To this end, generic structures of *WMR* model equations are reviewed, and the canonical form is introduced based on them. With aid of the canonical form and the well-known backstepping technique, a new adaptive robust tracking controller is presented. Our result is novel in the following aspects: 1) The tracking problem of the nonholonomic *WMR* with unknown dynamics is solved, which has not been addressed in the existing literature to our knowledge. 2) Our global controller can make all states of the closed loop system asymptotically track the given desired trajectory, while in paper [17][5][4] only partial states (i.e. generalized velocities) can track the desired trajectory. 3) Our controller is simple in structure and easy to be implemented, since no information and calculation on the system dynamics are involved.

2 Model Equations and Problem Statement

Consider the nonholonomic *WMR* discussed in [2][7], let $\xi = [x, y, \theta]^T$, where x, y are the coordinates of a reference point P on the frame in a fixed orthonormal inertial basis $\{O, \vec{I}_1, \vec{I}_2\}$, and θ is the orientation of an arbitrary basis $\{\vec{x}_1, \vec{x}_2\}$ attached to the frame with respected to the inertial basis $\{\vec{I}_1, \vec{I}_2\}$, β is the angle of the orientation wheel. It is shown in [2] that the mobility of any *WMR* can be characterized by two integers δ_m and δ_s. The interesting nonholonomic *WMR* identified by (δ_m, δ_s) are type $(2,0), (2,1), (1,1), (1,2)$ robots. Generally, the dynamic model of the nonholonomic *WMR* can be expressed in the following form [2]:

$$
\begin{aligned}
J(X)\dot{X} &= 0 && (1)\\
M(X)\ddot{X} + C(X, \dot{X})\dot{X} &= A(X)\tau + J^T(X)\lambda && (2)
\end{aligned}
$$

where the generalized coordinate $X = \xi$ if $\delta_s = 0$ or $X = (\xi, \beta)$ if $\delta_s \neq 0$, $M(X)$ is a bounded positive definite symmetric inertia matrix, $C(X, \dot{X})\dot{X}$ presents the vectors of centripetal and Coriolis torques, $A(X)$ is input transformation matrix, $J(X)$ is a full rank matrix, λ is Lagrange multiplier, τ is $(\delta_m + \delta_s)$-dimensional control input and the superscript T denotes the transpose. The constraint (1) is assumed to be completely nonholonomic, and (2) satisfies the following two properties [6]:

Property 1: $\dot{M} - 2C$ is skew-symmetric for a suitable definition of C.

Property 2: There exist positive constants $c_i > 0 (1 \leq i \leq 3)$ such that $\forall X$ and $\forall \dot{X}$, $\|M(X)\| \leq c_1$, $\|C(X, \dot{X})\| \leq c_2 + c_3\|\dot{X}\|$.

Given a desired differentiable trajectory $X^*(t)$ which satisfies the nonholonomic constraint

$$J(X^*)\dot{X}^* = 0 \tag{3}$$

the dynamic tracking problem is defined as follows: For the system (1)-(2) with $M(X)$ and $C(X, \dot{X})$ unknown and a given desired trajectory $X^*(t)$ satisfying (3), find a feedback law τ such that $\lim_{t\to\infty}(X - X^*) = 0$ and $\lim_{t\to\infty}(\dot{X} - \dot{X}^*) = 0$. To make this problem resolvable, the following assumptions about the desired trajectory X^* are made:

Assumption 1: $x^*(t)$ and $y^*(t)$ are bounded, X^* does not contain singular points listed in Table 2.

Assumption 2: For type (2,0), (1,1) and (1,2) robots, $\dot{\theta}^*(t)$ and $\ddot{\theta}^*(t)$ are bounded. For type (2,1) robot, $\dot{\theta}^*(t) + \dot{\beta}^*(t)$ and $\ddot{\theta}^*(t) + \ddot{\beta}^*(t)$ are bounded.

Assumption 3: There exists a time diverging sequence $\{t_i\}_{i\in N} (N = 1, 2, \cdots)$, and $|t_i - t_{i-1}| \leq T_0 < \infty$, such that for type (2,0), (1,1) and (1,2) robots $\lim_{i\to\infty} \inf |\dot{\theta}^*(t_i)| = \epsilon > 0$, for type (2,1) robot $\lim_{i\to\infty} \inf |\dot{\theta}^*(t_i) + \dot{\beta}^*(t_i)| = \epsilon > 0$,

Table 1. The kinematic models of nonholonomic WMR

Type	Model equation: $\dot{X} = B(X)u$
(2,0)	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}-\sin\theta & 0\\ \cos\theta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$
(2,1)	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-\sin(\theta+\beta) & 0 & 0\\ \cos(\theta+\beta) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \zeta_1\end{bmatrix}$
(1,1)	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-L\sin\theta\sin\beta & 0\\ L\cos\theta\sin\beta & 0\\ \cos\beta & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \zeta_1\end{bmatrix}$
(1,2)	$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\beta}_1\\ \dot{\beta}_2\end{bmatrix} = \begin{bmatrix}-L[\sin\beta_1\sin(\theta+\beta_2)+\sin\beta_2\sin(\theta+\beta_1)] & 0 & 0\\ L[\sin\beta_1\cos(\theta+\beta_2)+\sin\beta_2\cos(\theta+\beta_1)] & 0 & 0\\ \sin(\beta_2-\beta_1) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1\\ \zeta_1\\ \zeta_2\end{bmatrix}$

Following [1], it is shown in [2] that, after eliminating Lagrange multiplier, (1)-(2) can be written as

$$
\begin{aligned}
\dot{X} &= B(X)u && (4)\\
M_1(X)\dot{u} + C_1(X, \dot{X})u &= A_1(X)\tau && (5)
\end{aligned}
$$

where (4) is listed in Table 1 for each type of WMR, $M_1(X) = B^T(X)M(X)B(X)$, $C_1(X, \dot{X}) = B^T(X)M(X)\dot{B}(X) + B^T(X)C(X, \dot{X})B(X)$, $A_1(X) = B^T(X)A(X)$. Similarly, (3) can be written as

$$\dot{X}^*(t) = B(X^*(t))u^*(t) \tag{6}$$

where $u^*(t)$ is a known virtual input, (6) is called the virtual reference system. Since (4)-(5) describes the motion of the system (1)-(2), the dynamic tracking problem can be discussed based on (4)-(6).

3 Controller Design

To solve the dynamic tracking problem, the system (4)-(5) are converted into the extended chained form:

$$
\begin{cases}
\dot{q}_1 = v_1, \quad \dot{q}_{i,j} = v_1 q_{i+1,j} \ (2 \leq i \leq n_j - 1)\\
\dot{q}_{n_j,j} = v_{1+j} \ (1 \leq j \leq m)
\end{cases} \tag{7}
$$

$$M_2(q)\dot{v} + C_2(q, \dot{q})v = A_2(q)\tau \tag{8}$$

By the diffeomorphic state transformation:

$$
\begin{cases}
q = [q_1, q_{2,1}, \ldots, q_{n_1,1}, \ldots, q_{2,m}, \ldots, q_{n_m,m}]^T\\
\quad = T_1(X)\\
v = [v_1, \ldots, v_{m+1}]^T = T_2^{-1}(X)u
\end{cases} \tag{9}
$$

where $M_2(q) = g^T(X)M(X)g(X)|_{X=T_1^{-1}(q)}$, $C_2(q, \dot{q}) = [g^T(X)M(X)\frac{d}{dt}g(X) + g^T(X)C(X, \dot{X}) \cdot$

Table 2. Canonical Forms of Kinematic Models of restricted mobility robots and Corresponding Transformations

Type	State transformation $q = T_1(X)$	Input transformation $v = T_2^{-1}(X)u$	Canonical form	Singular points
(2,0)	$q_1 = \theta$ $q_{2,1} = x\cos\theta + y\sin\theta$ $q_{3,1} = -x\sin\theta + y\cos\theta$	$v_1 = u_2$ $v_2 = u_1 - (x\cos\theta + y\sin\theta)u_2$	$\dot{q}_1 = v_1$ $\dot{q}_{2,1} = q_{3,1}v_1$ $\dot{q}_{3,1} = v_2$	
(2,1)	$q_1 = \theta + \beta$ $q_{2,1} = -x\cos(\theta+\beta) - y\sin(\theta+\beta)$ $q_{3,1} = x\sin(\theta+\beta) - y\cos(\theta+\beta)$ $q_{2,2} = \beta$	$v_1 = u_2 + u_3$ $v_2 = -u_1 + (u_2 + u_3)(x\cos(\theta+\beta) + y\sin(\theta+\beta))$ $v_3 = u_3$	$\dot{q}_1 = v_1$ $\dot{q}_{2,1} = q_{3,1}v_1$ $\dot{q}_{3,1} = v_2$ $\dot{q}_{2,2} = v_3$	$\theta + \beta = 0 (\mathrm{mod}\,\pi)$
(1,1)	$q_1 = \theta$ $q_{2,1} = x\cos\theta + y\sin\theta$ $q_{3,1} = -x\sin\theta + y\cos\theta$ $q_{4,1} = L\tan\beta - q_{2,1}$	$v_1 = u_1\cos\beta$ $v_2 = u_1\cos\beta(x\sin\theta - y\cos\theta) + \frac{Lu_2}{\cos^2\beta}$	$\dot{q}_1 = v_1$ $\dot{q}_{2,1} = q_{3,1}v_1$ $\dot{q}_{3,1} = q_{4,1}v_1$ $\dot{q}_{4,1} = v_2$	$\beta = \frac{\pi}{2} (\mathrm{mod}\,\pi)$
(1,2)	$q_1 = \theta$ $q_{2,1} = x\cos\theta + y\sin\theta$ $q_{3,1} = -x\sin\theta + y\cos\theta - 2L\frac{\sin\beta_1\sin\beta_2}{\sin(\beta_2-\beta_1)}$ $q_{2,2} = x\sin\theta - y\cos\theta$ $q_{3,2} = x\cos\theta + y\sin\theta - L\frac{\sin(\beta_1+\beta_2)}{\sin(\beta_2-\beta_1)}$	$v_1 = u_1\sin(\beta_2-\beta_1)$ $v_2 = -q_{3,2}u_1\sin(\beta_2 - \beta_1) - \frac{2Lu_2\sin^2\beta_2}{\sin^2(\beta_2-\beta_1)} + \frac{2Lu_3\sin^2\beta_1}{\sin^2(\beta_2-\beta_1)}$ $v_3 = q_{3,1}u_1\sin(\beta_2 - \beta_1) - \frac{Lu_2\sin(2\beta_2)}{\sin^2(\beta_2-\beta_1)} + \frac{Lu_3\sin(2\beta_1)}{\sin^2(\beta_2-\beta_1)}$	$\dot{q}_1 = v_1$ $\dot{q}_{2,1} = q_{3,1}v_1$ $\dot{q}_{3,1} = v_2$ $\dot{q}_{2,2} = q_{3,2}v_1$ $\dot{q}_{3,2} = v_3$	$\beta_1 = \beta_2 (\mathrm{mod}\,\pi)$ $\beta_1 = 0 (\mathrm{mod}\,\pi)$ $\beta_2 = 0 (\mathrm{mod}\,\pi)$

$g(X)]|_{X=T_1^{-1}(q)}$, $A_2(X) = g^T(X)A(X)|_{X=T_1^{-1}(q)}$, and $g(X) := B(X)T_2(X)$, $T_1(X)$ and $T_2(X)$ are given in [15] and listed in Table 2 for each type of *WMR*.

Remark 1: Canonical form of (2, 1) robot is a special case of (7). From Table 2, there are singular points in some transformations. If a robot rests on a singular point at the initial time, a disturbance should be exerted on it such that it leaves the point before applying the transformation.

By the same transformation as (9), i.e.

$$\left\{ \begin{array}{rcl} q^* & = & [q_1^*, q_{2,1}^*, \ldots, q_{n_1,1}^*, \ldots, q_{2,m}^*, \ldots, \\ & & q_{n_m,m}^*]^T = T_1(X^*) \\ v^* & = & [v_1^*, \ldots, v_{m+1}^*]^T = T_2^{-1}(X^*)u^* \end{array} \right. \tag{10}$$

the given reference system (6) can be put into

$$\left\{ \begin{array}{l} \dot{q}_1^* = v_1^*, \quad \dot{q}_{i,j}^* = v_1^* q_{i+1,j}^* \ (2 \le i \le n_j - 1) \\ \dot{q}_{n_j,j}^* = v_{1+j}^* \ (1 \le j \le m) \end{array} \right. \tag{11}$$

where v^* is a known vector. Additionally, Assumption 1~3 about $X^*(t)$ can be rephrased as follows:

Assumption 4: Q_1^* is bounded, Q_1^* denotes the remainder vector of q^* after element q_1^* is removed.

Assumption 5: v_1^* and $\dot{v}_1^*$ are bounded.

Assumption 6: There exists a time diverging sequence $\{t_i\}_{i \in N}$, and $|t_i - t_{i-1}| \le T_0 < \infty$, such that $\lim_{i\to\infty} \inf |v_1^*(t_i)| = \epsilon > 0$.

With the transformations (9) and (10), it is easy to verify that the dynamic tracking problem is equivalent to find a feedback law τ such that $\lim_{t\to\infty}(q(t) - q^*(t)) = 0$ and $\lim_{t\to\infty}(\dot{q}(t) - \dot{q}^*(t)) = 0$. Let $e = \Psi(q - q^*) := [e_1, e_{2,1}, \ldots, e_{n_1,1}, \ldots, e_{2,m}, \ldots, e_{n_m,m}]^T$, where $\Psi = \mathrm{diag}[\Psi^1, \Psi_1^2, \ldots, \Psi_1^m]$, $\Psi_1^l (2 \le l \le m)$ is the resulting matrix eliminating the first row and the first column of the matrix $\Psi^l = \{\psi_{i,j}^l\} \in R^{n_l \times n_l}$, $\psi_{i,j}^l$ is defined as follows.

$$\begin{array}{l} \psi_{i,i}^l = 1 (1 \le i \le n_l), \quad \psi_{i,j}^l = 0 (i < j; 1 \le i, j \le n_l) \\ \psi_{i,1}^l = 0 (2 \le i \le n_l), \quad \psi_{i,j}^l = 0 (i \ne j (\mathrm{mod} 2)) \\ \psi_{i,j}^l = k_{i-3,l}\psi_{i-2,j} + \psi_{i-1,j-1} (4 \le i \le n_l; 2 \le j \le n_l) \end{array}$$

and constants $k_{i,l} > 0 (1 \le i \le n_l - 3)$, the following lemma can be proved.

Lemma: Consider the system (7) and a given desired trajectory q^* in (11), under Assumption 4~6, the control law

$$\begin{array}{rcl} \dot{p} & = & -\mu_2 p - \mu_1 e_1 - \displaystyle\sum_{l=1}^{m}\sum_{j=2}^{n_l-1}\sum_{i=2}^{j} \frac{e_{j,l}\psi_{j,i}^l q_{i+1,l}}{k_{0,l}k_{1,l}\cdots k_{j-2,l}} \\ & & -\displaystyle\sum_{l=1}^{m}\sum_{j=2}^{n_l-1} \frac{e_{n_l,l}\psi_{n_l,j}^l q_{j+1,l}}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}} \end{array} \tag{12}$$

$$\begin{array}{rcl} v & = & \left[\begin{array}{l} v_1^* + p \\ v_2^* - \mu_{3,1}e_{n_1,1} - k_{n_1-2,1}v_1^* e_{n_1-1,1} \\ \qquad -v_1^* \displaystyle\sum_{i=2}^{n_1-1} \psi_{n_1,i}^1 (q_{i+1,1} - q_{i+1,1}^*) \\ \qquad \vdots \\ v_{m+1}^* - \mu_{3,m}e_{n_m,m} - k_{n_m-2,m}v_1^* e_{n_m-1,m} \\ \qquad -v_1^* \displaystyle\sum_{i=2}^{n_m-1} \psi_{n_m,i}^m (q_{i+1,m} - q_{i+1,m}^*) \end{array} \right] \\ & =: & \sigma \end{array} \tag{13}$$

makes $q(t)$ and $\dot{q}(t)$ asymptotically converge to $q^*(t)$ and $\dot{q}^*(t)$ respectively, where $\mu_1 > 0$, $\mu_2 > 0$, $\mu_{3,l} > 0$, $k_{0,l} = 1$, $k_{n_l-2,l} > 0$ $(1 \le l \le m)$.

Proof: The closed-loop system of (7), (12) and (13) can be written as

$$\begin{array}{l} \dot{e}_1 = p, \quad \dot{e}_{2,l} = v_1^* e_{3,l} + p q_{3,l} \\ \dot{e}_{j+3,l} = v_1^*(-k_{j+1,l}e_{j+2,l} + e_{j+4,l}) + p\displaystyle\sum_{i=2}^{j+3}\psi_{j+3,i}^l q_{i+1,l} \\ \qquad (0 \le j \le n_l - 4; 1 \le l \le m) \\ \dot{e}_{n_l,l} = -\mu_{3,l}e_{n_l,l} - k_{n_l-2,l}v_1^* e_{n_l-1,l} + p\displaystyle\sum_{i=2}^{n_l-1}\psi_{n_l,i}^l q_{i+1,l} \end{array}$$

$$\dot{p} = -\mu_2 p - \mu_1 e_1 - \sum_{l=1}^{m}\sum_{j=2}^{n_l-1}\left[\sum_{i=2}^{j}\frac{e_{j,l}\psi_{j,i}^l q_{i+1,l}}{k_{0,l}k_{1,l}\cdots k_{j-2,l}} + \frac{e_{n_l,l}\psi_{n_l,j}^l q_{j+1,l}}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}}\right]$$

Let $V = 0.5[p^2 + \mu_1 e_1^2 + \sum_{l=1}^{m}\sum_{j=2}^{n_l} e_{j,l}^2/(k_{0,l}k_{1,l}\cdots k_{j-2,l})]$, differentiating V along the closed loop system yields

$$\dot{V} = -\mu_2 p^2 - \sum_{l=1}^{m}\frac{\mu_{3,l}e_{n_l,l}^2}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}} \leq 0 \quad (14)$$

thus V is non-increasing and has limit $V_{lim} \geq 0$. Noting the expression of V, so p and e are bounded. By Assumption 4~5, $q_{i,l}(3 \leq i \leq n_l; 1 \leq l \leq m)$, $\dot{e}$ and $\dot{p}$ are bounded. Since $\frac{d}{dt}\dot{V} = -2\mu_2 p\dot{p} - \sum_{l=1}^{m}(2\mu_{3,l}e_{n_l,l}\dot{e}_{n_l,l})/(k_{1,l}k_{2,l}\cdots k_{n_l-2,l})$ is bounded, $\dot{V}$ is uniformly continuous. By Barbalat's lemma, $\dot{V} \to 0$, hence p and $e_{n_l,l}(1 \leq l \leq m)$ tend to zero, respectively.

Since v_1^* is bounded, $v_1^{*2}e_{n_l,l}(1 \leq l \leq m)$ tend to zero. Differentiating $v_1^{*2}e_{n_l,l}$ yields $\frac{d}{dt}(v_1^{*2}e_{n_l,l}) = -k_{n_l-2,l}v_1^{*3}e_{n_l-1,l} + [2v_1^*e_{n_l,l}\dot{v}_1^* - \mu_{3,l}v_1^{*2}e_{n_l,l} + v_1^{*2}p\sum_{i=2}^{n_l-1}\psi_{n_l,i}^l q_{i+1,l}]$, where the first term is uniformly continuous, since its derivative $\frac{d}{dt}(-k_{n_l-2,l}v_1^{*3}e_{n_l-1,l}) = -3k_{n_l-2,l}\dot{v}_1^* v_1^{*2}e_{n_l-1,l} - k_{n_l-2,l}v_1^{*3}\dot{e}_{n_l-1,l}$ is bounded. The other terms tend to zero (since v_1^*, $\dot{v}_1^*$, and $q_{i,l}(3 \leq i \leq n_l; 1 \leq l \leq m)$ are bounded, $v_1^* e_{n_l,l}$ and p tend to zero). By Barbalat's lemma, $\frac{d}{dt}(v_1^{*2}e_{n_l,l}) \to 0$, thus $v_1^{*3}e_{n_l-1,l} \to 0$. Furthermore, $v_1^{*2}e_{n_l-1,l}$ and $v_1^*e_{n_l-1,l}$ converge to zero.

Differentiating $v_1^{*2}e_{j,l}(j = n_l-1,\ldots,2)$ and repeating the above procedure, it can be proved $v_1^{*2}e_{j,l}$ and $v_1^*e_{j,l}$ $(j = n_l-1,\ldots,2)$ converge to zero, respectively.

Again v_1^* is bounded and p tends to zero, so $v_1^{*2}p$ converges to zero. Differentiating $v_1^{*2}p$, yields

$$\frac{d}{dt}(v_1^{*2}p) = -\mu_1 v_1^{*2}e_1 + 2v_1^*\dot{v}_1^* p - \mu_2 v_1^{*2}p - \sum_{l=1}^{m}\sum_{j=2}^{n_l-1}\left[\sum_{i=2}^{j}\frac{v_1^{*2}e_{j,l}\psi_{j,i}^l q_{i+1,l}}{k_{1,l}k_{2,l}\cdots k_{j-2,l}} + \frac{v_1^{*2}e_{n_l,l}\psi_{n_l,j}^l q_{j+1,l}}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}}\right]$$

where the first term is uniformly continuous (since its time derivative is bounded), the other terms tend to zero (since v_1^* and $q_{j,l}^*(3 \leq j \leq n_l; \leq l \leq m)$ are bounded, p and $v_1^*e_{j,l}(2 \leq j \leq n_l; 1 \leq l \leq m)$ tend to zero). By Barbalat's lemma, $\frac{d}{dt}(v_1^{*2}p)$ tends to zero, so $v_1^{*2}e_1$ tends to zero. Furthermore $v_1^*e_1$ tends to zero.

Considering $v_1^*e_1$, $v_1^*e_{j,l}(2 \leq j \leq n_l, 1 \leq l \leq m)$ and p tend to zero, so $v_1^{*2}V$ tends to zero. Since V has limit $V_{lim} \geq 0$ and v_1^* does not tend to zero (by Assumption 6), V_{lim} is necessarily equal to zero. Therefore e_1, $e_{j,l}(2 \leq j \leq n_l; 1 \leq l \leq m)$ and p tend to zero. Since Ψ is a nonsingular constant matrix, q and $\dot{q}$ asymptotically converge to q^* and $\dot{q}^*$, respectively. ◇

Remark 2: By the proof, in the Lemma Assumption 6 can be relaxed as: $v_1^* \not\to 0$ as $t \to \infty$. By the inverse state transformation, Assumption 3 can be replaced by: For type (2,0), (1,1) and (1,2) robots, $\dot{\theta}^*(t) \not\to 0$ as $t \to \infty$. For type (2,1) robot, $\dot{\theta}^*(t) + \dot{\beta}^*(t) \not\to 0$ as $t \to \infty$.

With aid of the Lemma and the well-known backstepping technique, the following theorem can be proved.

Theorem: Consider the system (7)-(8) with unknown dynamics and the virtual reference system (11), under Assumption 4~6, the control law (12),

$$\tau = A_2^{\#}\left[-K_p(v-\sigma) - \frac{\widehat{a}\gamma_2^2\chi^2(\sigma,\dot{\sigma})(v-\sigma)}{\gamma_2\chi(\sigma,\dot{\sigma})\|v-\sigma\| + \gamma(t)} - \Lambda\right] \quad (15)$$

and the adaptive law

$$\dot{\widehat{a}} = \frac{\gamma_1\gamma_2^2\chi^2(\sigma,\dot{\sigma})\|v-\sigma\|^2}{\gamma_2\chi(\sigma,\dot{\sigma})\|v-\sigma\| + \gamma(t)} \quad (16)$$

make $q(t)$ and $\dot{q}(t)$ asymptotically converge to $q^*(t)$ and $\dot{q}^*(t)$ respectively, and $\widehat{a}$ is bounded, where $^{\#}$ is any left inverse, K_p is a positive matrix, constants $\gamma_2 \geq 1$ and $\gamma_1 > 0$, $\gamma(t) > 0$ and such that

$$\int_0^\infty \gamma(t)dt = d_1 < \infty \quad (17)$$

σ is defined in Lemma and $\Lambda = [\Lambda_1,\ldots,\Lambda_{m+1}]^T$

$$\Lambda = \begin{bmatrix} \mu_1 e_1 + \sum_{l=1}^{m}\sum_{j=2}^{n_l-1}\left[\sum_{i=2}^{j}\frac{e_{j,l}\psi_{j,i}^l q_{i+1,l}}{k_{0,l}k_{1,l}\cdots k_{j-2,l}} + \frac{e_{n_l,l}\psi_{n_l,j}^l q_{j+1,l}}{k_{1,l}k_{2,l}\cdots k_{n_l-2,l}}\right] \\ \frac{e_{n_1,1}}{k_{1,1}k_{2,1}\cdots k_{n_1-2,1}} \\ \vdots \\ \frac{e_{n_m,m}}{k_{1,m}k_{2,m}\cdots k_{n_m-2,m}} \end{bmatrix}$$

$$\chi(\sigma,\dot{\sigma}) := \|g(T_1^{-1}(q))\| \cdot [\|\tfrac{d}{dt}(g(T_1^{-1}(q))\sigma)\| + (1 + \|\tfrac{d}{dt}T_1^{-1}(q)\|) \cdot \|g(T_1^{-1}(q))\sigma\|]$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\mu_{3,j} > 0$, $k_{0,l} = 1$, $k_{n_l-2,l} > 0$ $(1 \leq l \leq m)$.

Proof: Let $w = [w_1, w_2]^T = v - \sigma$, $a = \max\{c_1, c_2, c_3\}/\gamma_2$, $\tilde{a} = \widehat{a} - a$, the closed-loop system of (7), (8), (12), (15) and (16) can be written as

$$\begin{cases} \dot{e}_1 = p + w_1, \quad \dot{e}_{2,l} = v_1^* e_{3,l} + (p + w_1) q_{3,l} \\ \dot{e}_{j+3,l} = v_1^*(-k_{j+1,l} e_{j+2,l} + e_{j+4,l}) + (p \\ \quad + w_1) \sum_{i=2}^{j+3} \psi_{j+3,i}^l q_{i+1,l}, (0 \le j \le n_l - 4; 1 \le l \le m) \\ \dot{e}_{n_l,l} = -\mu_{3,l} e_{n_l,l} - k_{n_l-2,l} v_1^* e_{n_l-1,l} + w_2 + (p \\ \qquad + w_1) \sum_{i=2}^{n_l-1} \psi_{n_l,i}^l q_{i+1,l} \\ M_2(q)\dot{w} = -C_2(q,\dot{q})w - K_p w - \Lambda - \Phi(\sigma,\dot{\sigma}) \\ \qquad - \dfrac{\hat{a}\gamma_2^2 \chi^2(\sigma,\dot{\sigma}) w}{\gamma_2 \chi(\sigma,\dot{\sigma}) \|w\| + \gamma} \\ \dot{\hat{a}} = \dfrac{\gamma_1 \gamma_2^2 \chi^2(\sigma,\dot{\sigma}) \|w\|^2}{\gamma_2 \chi(\sigma,\dot{\sigma}) \|w\| + \gamma}, \quad \dot{p} = -\mu_2 p - \Lambda_1 \end{cases} \tag{18}$$

where $\Phi(\sigma,\dot{\sigma}) = M_2(q)\dot{\sigma} + C_2(q,\dot{q})\sigma$. Let

$$V = \frac{1}{2}[p^2 + \mu_1 e_1^2 + \sum_{l=1}^{m} \sum_{j=2}^{n_l} \frac{e_{j,l}^2}{k_{0,l} k_{1,l} \cdots k_{j-2,l}} + w^T M_2 w + \tilde{a}^2/\gamma_1]$$

Differentiating V along (18) yields $\dot{V} = -\mu_2 p^2 - \sum_{l=1}^m (\mu_{3,l} e_{n_l,l}^2)/(k_{1,l} k_{2,l} \cdots k_{n_l-2,l}) - w^T K_p w + R$, where

$$\begin{aligned} R &= -w^T \Phi - \frac{\hat{a}\gamma_2^2 \chi^2 \|w\|^2}{\gamma_2 \chi \|w\| + \gamma} + \frac{\tilde{a}\gamma_2^2 \chi^2 \|w\|^2}{\gamma_2 \chi \|w\| + \gamma} \\ &\le a\gamma_2 \chi \|w\| - \frac{a\gamma_2^2 \chi^2 \|w\|^2}{\gamma_2 \chi \|w\| + \gamma} = \frac{a\gamma_2 \chi \gamma \|\tilde{v}\|}{\gamma_2 \chi \|\tilde{v}\| + \gamma} \le a\gamma \end{aligned}$$

therefore

$$\dot{V} \le -\mu_2 p^2 - \sum_{l=1}^{m} \frac{\mu_{3,l} e_{n_l,l}^2}{k_{1,l} k_{2,l} \cdots k_{n_l-2,l}} - w^T K_p w + a\gamma$$

Since γ satisfies (17), integrating both sides of the above inequation gives

$$\begin{aligned} V(t) - V(0) &= \int_0^t \Big[-\mu_2 p^2(s) - w^T(s) K_p w(s) \\ &\quad + a\gamma(s) - \sum_{l=1}^{m} \frac{\mu_{3,l} e_{n_l,l}^2(s)}{k_{1,l} k_{2,l} \cdots k_{n_l-2,l}}\Big] ds \le a d_1 \end{aligned} \tag{19}$$

thus V is bounded, which implies that $p \in L_\infty$, $e \in L_\infty$, $w \in L_\infty$, and $\tilde{a} \in L_\infty$. By Assumption 4-5, $q_{i,l} (3 \le i \le n_l; 1 \le l \le m)$, $\dot{e}$, and $\dot{p}$ are bounded. So M_2, C_2, and Λ are bounded, furthermore $\dot{w}$ is bounded. From (19),

$$\int_0^t \Big[-\mu_2 p^2(s) - \sum_{l=1}^{m} \frac{\mu_{3,l} e_{n_l,l}^2(s)}{k_{1,l} k_{2,l} \cdots k_{n_l-2,l}} - w^T(s) K_p w(s)\Big] ds \le V(0) - V(t) + a d_1$$

so $p \in L_2$, $e_{n_l,l} \in L_2 (1 \le l \le m)$, and $w \in L_2$. Therefore, $p \to 0$, $e_{n_l,l} \to 0 (1 \le l \le m)$, and $w \to 0$ as $t \to \infty$, respectively.

Mimicking Proof of the Lemma, by differentiating $v_1^{*2} e_{i,l} (i = n_l, n_l - 1, \ldots, 2; 1 \le l \le m)$ and $v_1^{*2} p$, we can prove $v_1^* e_{i,l} (i = n_l - 1, n_l - 2, \ldots, 2; 1 \le l \le m)$ and $v_1^* e_1$ tend to zero step by step. With respect to (18), $\dot{e}_1$ and $\dot{e}_{i,l} (2 \le i \le n_l; 1 \le l \le m)$ tend to zero, therefore $\dot{q}$ asymptotically converges to $\dot{q}^*$. Noting Assumption 6, the sequence $\{e_{i,l}(t_j)\}_{j \in N}$ and $\{e_1(t_j)\}_{j \in N}$ tend to zero respectively. Using Taylor expansion, $e_{i,l}(t) = e_{i,l}(t_s) + (t - t_s)\dot{e}_{i,l}(t')$, where $|t - t_s| \le T_0$, t' is some time between t and t'. When $t \to \infty$, then $t_s \to \infty$, $e_{i,l}(t) \to 0$, and $\dot{e}_{i,l}(t') \to 0$. Therefore, $e_{i,l}(t) \to 0$ as $t \to \infty$. Similarly, we can prove e_1 tends to zero. Thus q asymptotically converges to q^*. Additionally, $\hat{a}$ being bounded is guaranteed by boundedness of $\tilde{a}$. ◇

Remark 3: In the control law, $\gamma(t)$ may be $1/(1+t)^{d_2} (d_2 \ge 2)$, $e^{-d_3 t} (d_3 > 0)$, or anything else which satisfies (17). By inverse transformation, the controller for the original system can be easily obtained, so it is omitted here.

4 Simulation

Consider the tracking problem of type (2,0) robot moving on a horizontal plane whose dynamic model equation is described in [8], given a desired trajectory $X^* = [X_1^*, X_2^*, X_3^*]^T$, where $X_1^* = \cos t$, $X_2^* = \sin t$, X_3^* is determined by the nonholonomic constraint $\dot{X}_1^* \cos X_3^* + \dot{X}_2^* \sin X_3^* = 0$. Following Section 2, the control law can be easily derived step by step, due to space limit, it is omitted here.

In order to simulate, suppose in (2) $M(X) = \text{diag}[5,5]$ and $C(X,\dot{X}) = 0$. In the simulation, let $[x(0), y(0), \theta(0)] = [1.2, -0.3, 0.2]$, $[\dot{x}(0), \dot{y}(0), \dot{\theta}(0)] = [-0.002, 0.01, 0.1]$, $p(0) = 0.5$, $\hat{a}(0) = 0$. Select $\mu_1 = 12.25$, $\mu_2 = 7$, $\mu_{3,1} = 6$, $k_1 = 5$, $K_p = \text{diag}(50, 50)$ and $\gamma = 1/(1+t)^2$, $\gamma_1 = 1$, $\gamma_2 = 1$ in the feedback law and the adaptive law. Simulation results are depicted in Figure 1-2 respectively.

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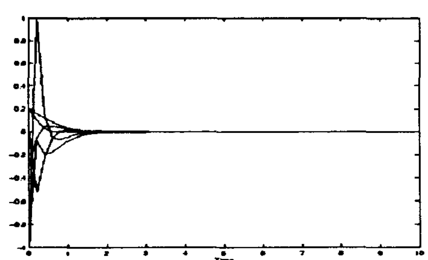

Figure 1. Response of $X - X^*$ and $\dot{X} - \dot{X}^*$.

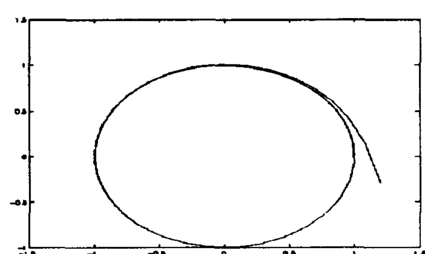

Figure 2. Geometric trajectory of x^*-y^* and x-y