Necessary Conditions for Global Feedback Control

Efthimios Kappos*

∗ EECS Department, University of California, Berkeley CA 94720,USA
on leave from the University of Sheffield, England.

June 5, 1996

ABSTRACT

In this paper we employ the topological concepts of index of an equilibrium point of a vector field and of the Conley index of an isolated invariant set of a vector field with the purpose of deriving necessary conditions for the design of global control dynamics. Our results are a generalization of those of Brockett, Coron and others, which were derived for the special case of stabilization.

1 Introduction

Recent interest in local stabilization theory (see, for example, the book by Bacciotti [1]) has led to a consideration of some simple topological conditions; an example is the necessary condition for smooth stabilization of Brockett [2] that roughly states that if an equilibrium point is to be stabilizable, all directions must be available through control near this equilibrium. This is generalized in [9] to arbitrary attractors. Coron [3] derived some finer conditions using homology groups; in this paper we derive similar results for more general equilibria. One purpose of the present work is to show that much remains to be done in this direction and that a considerable amount of interesting topological problems arise in connection with control systems. We re-visit the index of a zero of a vector field and use it to derive global necessary conditions; these considerations are then generalized by using the Conley index to arbitrary invariant sets. The propaganda part of this paper is in convincing the reader of the appropriateness and power of these index methods.

*This research was supported by the UK EPSRC Grant GR/K83205.

2 Control dynamics

For our purposes, a smooth control system on the manifold M^n is a pair (X, D), where $X \in \mathcal{X}^\infty(M^n)$ is a smooth vector field and $D \subset TM^n$ is a smooth sub-bundle of the tangent bundle (a distribution.) We assume, unless stated otherwise, that D is of constant rank m and involutive. A smooth feedback control U is a section of the bundle D. The resulting control vector field X_U is equal to $X + U$. We define $\Gamma(D)$ to be the set of all smooth sections of D and we denote by $X + \Gamma(D) \subset \mathcal{X}^\infty(M^n)$ the set of all control vector fields. We also define the subset

$$\Sigma(X, D) = \{x \in M^n \ ; \ X(x) \in D(x)\}$$

of the state space which we call the *singular set* of the control pair (X, D). It is generically a submanifold of M^n of dimension m (see [8].)

2.1 Local obstructions

Now suppose the manifold M^n endowed with a Riemannian metric. The *unit tangent bundle* SM^n is the fibre bundle obtained by taking at each point $x \in M^n$ the set of all tangent vectors of unit length. Thus each fibre has dimension $n-1$. It is in general a non-trivial bundle (not a product bundle), unless, for example, M^n is Euclidean, $\mathbf{R}^n$. If it is trivial, then we have a canonical identification of each sphere fibre with the standard unit sphere S^{n-1}. In the control context, we would like to think of the unit sphere attached to every point as the *set of all directions*, each 'direction' being thought of as a unit outward normal to some hypersurface.

For any vector field $Y \in \mathcal{X}$ and any subset $W \subset M^n$ such that Y is nowhere zero on W, the *Gauss map* is the map

$$G : W \to SM^n \ ; \ x \mapsto \frac{Y(x)}{|Y(x)|},$$

Typically, W is either a closed submanifold of M^n or an open set. One can define the Gauss map as a map from $TM^n \setminus s_0(M^n)$ to SM^n in the obvious way (here $s_0 : M^n \to TM^n$ is the zero section.)

If now X_U is a control vector field and x is any point of M^n such that $X_U(x) \neq 0$, we consider the set

$$C_U(x) = \{v \in S_x M^n \ ; \ X_U(x) \cdot v(x) < 0\}.$$

$C(x)$ is an open hemi-sphere of $S_x M^n$. If $X_U(x) = 0$, we set $C_U(x) = \emptyset$. We take the union of these hemispheres $C(x)$ over all the elements of $X + \Gamma(D)$

$$C(x) = \bigcup_{U \in \Gamma(D)} C_U(x).$$

The complement of this set, $\mathcal{O}(x) = S_x M^n - C(x)$, is called the *obstruction set at* x.

Proposition 1 *If $x \notin \Sigma(X, D)$, then $\mathcal{O}(x)$ is a hemisphere of dimension $(n - m - 1)$. If $x \in \Sigma(X, D)$, then $\mathcal{O}(x)$ is a sphere of dimension $(n - m - 1)$.*

If $m > 0$ and $x \notin \Sigma(X, D)$, then $\mathcal{O}(x)$ has the topological type of Euclidean space; in particular, it is contractible. If $m > 1$ and $x \in \Sigma(X, D)$, then $\pi_0(\mathcal{O}(x)) = 0$, in other words $\mathcal{O}(x)$ is path connected. If $m = 1$ and $x \in \Sigma(X, D)$, then $\mathcal{O}(x)$ has two components, which we call $S_{\pm}^{n-2}(x)$.

The proof follows from a consideration of the corresponding subspaces and then mapping back to the unit sphere.

The above Proposition forms the basis of our topological investigation of global controllability (see also [9].) Note that the above pointwise properties can be extended to local neighbourhoods away from the (closed) singular set; in other words for every $x \notin \Sigma(X, D)$, we can find a neighborhood of x such that the obstruction sets are of the type described in the Proposition for every point in that neighorhood.

3 Topological necessary conditions

The fundamental aim of a topological investigation of control systems is to understand the classes of dynamics contained in the set of all control vector fields X_U.

As an elementary (but not that simple) example, we ask: given an equilibrium point x of the state dynamics X (of arbitrary 'stability' index), is there an element of $X_U \in X + \Gamma(D)$ such that x is an asymptotically stable attractor of X_U? (the local stabilization problem.)

The approach taken in [9] is to first specify the desirable dynamics, using so-called 'Morse specifications,' and then to consider all functions that are possible Lyapunov functions for some element in the class of desirable dynamics (the 'Morse-Lyapunov functions.')

A fundamental problem that has to be addressed is: can one decide whether the chosen dynamics are achievable or not? A set of *necessary conditions* on the control pair (X, D) is helpful in making a preliminary selection. For the case of constant control distribution, some results are given in [9], including some sufficient conditions for stabilization.

We proceed in this paper assuming that a class of dynamics has been specified in the mammer of [9]; for example, if a gradient-like system is desired, we specify the location and index of each equilibrium of the desirable dynamics; in addition, we may fix the location of the connecting orbits and separatrices.

3.1 The topological index of an equilibrium

For an isolated equilibrium x of a vector field, the 'topological' index $\text{ind}(x)$ (to be distinguished from the stability index, which is defined only for hyperbolic equilibria and is equal to the dimension of their unstable manifold) is defined as follows:

Fix a small ball $B_\epsilon(x)$ around x avoiding all other equilibria; let $S_\epsilon(x)$ be its bounding sphere. Since the sphere bundle is locally trivial, the Gauss map restricted to $S_\epsilon(x)$ gives a map of the sphere to itself; this map is determined, up to homotopy, by an integer, which is then the index of x. It is shown to be independent of the ball around x.

If the equilibrium x is nondegenerate, then the index is either $+1$ or -1. This indicates roughly that the vector field is locally described by its linearization and for a nondegenerate equlibrium of a linear vector field Ax, the equation $Ax = v$ has a unique solution, in other words every direction occurs exactly once on the small sphere near x. Furthermore,

Lemma 1 *The index of a hyperbolic equilibrium of stability index k is equal to $(-1)^{n-k}$.*

The index provides a good, but only partially successful, global tool for studying dynamics. The transition to global results is given by the following theorem, a variant of the *Poincaré-Hopf* theorem (see [5], [10].)

Theorem 1 *Suppose $M^n = \mathbf{R}^n$. Let W be a compact subset of $\mathbf{R}^n$ with open interior and whose boundary ∂W is a smooth, connected, oriented submanifold of $\mathbf{R}^n$ of dimension $n-1$. Suppose a vector field X is given in $\mathbf{R}^n$ that is nonzero on ∂W and is such that W contains a finite number of equilibrium points of X. Then the integer $\deg G|_{\partial W}$ is defined, where $G : \partial W \to S^{n-1}$ is the Gauss map restricted to ∂W and furthermore*

$$\deg G|_{\partial W} = \sum_i \text{ind } (x_i)$$

where the sum is over all equilibrium points in W.

Suppose now that W^n is an arbitrary compact manifold (possibly with boundary) with a vector field defined in it such that in addition X points inwards along ∂W; then

$$\sum_i \text{ind } (x_i) = (-1)^n \chi(W)$$

where $\chi(W)$ is the Euler characteristic of W. In particular, the above sum is independent of the vector field X and is a topological invariant.

Notice that the first part of the above theorem gives a way of deriving necessary conditions for control systems defined in $\mathbf{R}^n$. The class of dynamics that is relevant is that of gradient-like systems (they have a finite number of nondegenerate equilibria and there is no other chain-recurrent behavior.) More specifically, the method is as follows:

Select a set W satisfying the conditions of the theorem. This is easy, since we have assumed that the locations of the equilibrium points of our desired dynamics are fixed. On the boundary ∂W of W, compute the degree of the Gauss map using the right hand side of the equation. Thus, for example, if W encloses two attractors and a 1-saddle, then $\deg G = (-1)^n \neq 0$. This means in particular that the Gauss map is **onto**. Hence, as in the simpler case of stabilization, if there is a control direction (i.e. a point in $S_x M^n$, with $x \in \partial W$) that is not in $C(X + D(x))$, then the control pair (X, D) cannot achieve the specified dynamics using smooth feedback. Stronger results using the fact that the index is ± 1 will be given later, in section IV.B.

Notice that if the degree is found to be zero, for example if one attractor and one 1-saddle are enclosed, then we cannot draw any conclusions about whether the dynamics are achievable or not.

4 Local stabilization

We apply some of our methods to the more familiar setting of local asymptotic stabilization.

Having fixed an equilibrium point of X, say 0 for simplicity, the control bundle is locally trivial, so, in a neighborhood V of 0, we write it as a product $V \times \mathbf{R}^m$ and, selecting local coordinates, we may as well consider it as an open subset of $\mathbf{R}^n \times \mathbf{R}^m$, call it $D_V \cong D|_V$.

We say that 0 is *locally asymptotically stabilizable* (LAS) if there exists some neighborhood $N(0)$ of $0 \in \mathbf{R}^n$ and a local *continuous* section of D (a local feedback control) $U : N(0) \to D_V$ such that:

(i) there exist local (in $N(0)$), unique solutions of the vector field X_U,

(ii) 0 is a locally asymptotically stable equilibrium of X_U and

(iii) U(0) = 0.

Given the above U, it will be helpful to consider the set, for $\epsilon > 0$,

$$\Sigma_V = \{(x, v) \in D_V \ ; \ X(x) + v = 0\}$$

and the sequence

$$\begin{aligned} V \setminus \{0\} \xrightarrow{\mathrm{gr}U} V \times \mathbf{R}^m \setminus \Sigma_V \overset{\iota}{\hookrightarrow} T\mathbf{R}^n|_V \setminus \{0\} \\ \xrightarrow{G} S\mathbf{R}^n|_V \xrightarrow{\pi} S^{n-1} \end{aligned}$$

where $\mathrm{gr}U(x) = (x, U(x))$, ι is the inclusion map, G is the Gauss map and π is the obvious projection in the trivial local sphere bundle.

Since 0 is an isolated equilibrium of X_U, $X_U \neq 0$ in $V \setminus \{0\}$ and the above is well-defined.

4.1 A reinterpretation of Coron's condition

With the tools we have at our disposal, it is now easy to give a more geometric interpretation of the necessary condition for local feedback stabilization given in [3]:

We start by noticing that $V \setminus \{0\}$ is homotopically equivalent to S^{n-1}. Thus the composed map defined by the above sequence, call it ϕ,

$$\phi : V \setminus \{0\} \to S^{n-1}$$

has a well-defined degree, since 0 is asymptotically stable for X_U, and this degree is equal to $(-1)^n$. This means that, at the level of, for example, homology (or homotopy), the generator, call it α, of $H_{n-1}(S^{n-1}) \cong \mathbf{Z}$ is in the image of ϕ. In other words, if 0 is LAS, then there is some local section such that the degree of the above map is defined and the image of the corresponding homomorphism at the level of homology is the whole of $H_{n-1}(S^{n-1})$. This is essentially Coron's result. Consider the commutative diagram

$$\begin{tikzcd} V \setminus \{0\} \arrow[r] \arrow[d] & S^{n-1} \\ D_V \setminus \Sigma_V \arrow[ur] & \end{tikzcd}$$

where the vertical map is inclusion and the map from $D_V \setminus \Sigma_V$ to S^{n-1} will be denoted also by X_U and is given by the composition $(x, v) \mapsto X(x) + v \mapsto G(X(x) + v)$. We have that

$$\phi_*(H_{n-1}(V \setminus \{0\})) = H_{n-1}(S^{n-1}).$$

Theorem 2 (Coron, 1990) *If the system (X, D) is locally asymptotically stabilizable, then*

$$(X_U)_*(H_{n-1}(D_V \setminus \Sigma_V)) = H_{n-1}(S^{n-1}).$$

4.2 Generalizations

The simple reasoning that led to Coron's result can be generalized to equilibrium points that are not attractors, but have a well-defined stability index. In [9] we gave a generalization of Brockett's condition that requires the local Gauss map to be onto for *any* asymptotic attractor (limit cycle etc.); we also found that a requirement for turning an equilibrium point into a saddle of any index is again that the Gauss map is onto, just as for a stable equilibrium. What we would like to express here, though, is the fact that if a local feedback control exists yielding an equilibrium of a desired index, then in fact for that system the degree is known and equal to either one or minus one (remember from section 3 that it is equal to $(-1)^{n-k}$) and thus the Gauss map on the graph of this system gives an onto map *in homology!*

We summarize this discussion in the following theorem, whose proof proceeds along similar lines to the discussion above pertaining to the Coron result:

Theorem 3 *Let 0 be an equilibrium of the state dynamics X of the control pair (X, D). If there is a continuous local feedback that yields dynamics X_U with 0 an equilibrium of index k, $0 \leq k \leq n$, then*

$$(X_U)_*(H_{n-1}(D_V \setminus \Sigma_V)) = H_{n-1}(S^{n-1}).$$

5 The Conley index and refinements

The Conley index is a far-reaching generalization of the stability index (sometimes called the Morse index) of a hyperbolic equilibrium point. It can be computed for arbitrary *isolated invariant sets* (IIS) using so-called *index pairs* (consisting of an isolating block and its exit set.) The simplest example of an IIS is the saddle equilibrium of the planar system $\dot{x} = x$ and $\dot{y} = -y$, isolated by the unit square –the exit set consisting of the left and right sides of the square (see [7] for an elementary exposition.)

A crucial result from Conley's theory of the topological index is that *global Lyapunov functions exist for all dynamical systems*, provided we quotient out the chain-recurrent part (these are strict Lyapunov functions in the system terminology: they strictly decrease along trajectories; of course, one has to confine attention to a compact subset or start with a compact manifold.) In particular, if $\mathcal{S}$ is an IIS, then there exist Lyapunov functions defined at least locally near $\mathcal{S}$ (making sure we avoid the other IIS's.)

The crucial fact that led to the necessary conditions of the previous two subsections was that in the neighborhood of an asymptotic attractor, a Lyapunov function exists and the Gauss map is in some sense one-to-one (in addition, maps from the sphere to itself are completely classified up to homotopy by the single integral invariant, the degree.)

Since we have in place a more general setting from section IV above, we present some novel necessary conditions applicable to an arbitrary compact, connected IIS $\mathcal{S}$, isolated by the set $V \subset M^n$. More explicitly, we assume that there is a local feedback $U : V \to D$ such that X_U has an IIS $\mathcal{S}$, whose dynamical structure is known (for example, $\mathcal{S}$ as a set consists of a number of equilibria and limit cycles and their connecting orbits.) Notice that $X_U \neq 0$ in $V \setminus \mathcal{S}$.

Lemma 2 *Endow M^n with a Riemannian metric. There is a function h defined on $V \setminus \mathcal{S}$ such that its gradient vector field ∇h is topologically equivalent to X_U and such that the Gauss maps of ∇h and X_U induce the same homomorphisms on homology, both $G_{\nabla h}$ and G_{X_U} mapping*

$$H_*(V \setminus \mathcal{S}) \to H_*(SM^n|_{V \setminus \mathcal{S}}).$$

Now, as we did for the case of local stabilization, we have the map ϕ defined by the composite map below

$$D_{V \setminus \mathcal{S}} \setminus \Sigma_V \xrightarrow{X+v} TM^n \setminus \Sigma \xrightarrow{G} SM^n$$

which induces the map ϕ_* in homology

$$H_*(D_{V\setminus\mathcal{S}} \setminus \Sigma_V) \to H_*(SM^n|_{V\setminus\mathcal{S}}).$$

We now have the result

Theorem 4 *If the control pair (X, D) can achieve dynamics with IIS $\mathcal{S}$ isolated by the set V, then*

$$G_{\nabla h}(H_*(D_{V\setminus\mathcal{S}} \setminus \Sigma_V)) \subset \phi_*(H_*(D_{V\setminus\mathcal{S}} \setminus \Sigma_V)).$$

We shall give a more extended presentation of this material, together with examples, in a forthcoming paper.

References

[1] A. Bacciotti, Local Stabilizability of Nonlinear Control Systems, World Scientific, 1992.

[2] R. Brockett, "Asymptotic Stability and Feedback Stabilization," in R. Brockett, R. Millmann, H. Sussmann eds: Differential Geometric Control Theory, Birkhäuser, 1983, pp.181–191.

[3] J.M. Coron, "A Necessary Condition for Feedback Stabilization," System and Control Letters **14**, 1990, pp.227–232.

[4] J. Dugundji, Topology, Allyn and Bacon, 1966.

[5] A.T. Fomenko, Variational Problems in Topology, Gordon and Breach, 1990.

[6] S.-T. Hu, Homotopy Theory, Academic Press, 1959.

[7] E. Kappos, "The Conley index and Global Bifurcations," Int.J. Bif. and Chaos, **5**–4, 1995, pp.937–953.

[8] E. Kappos, "A Global ,Geometrical, Input-Output Linearization Theory," IMA J. Math. Con. and Info., **9**–1, 1992, pp.1–21.

[9] E. Kappos, "The Role of Morse-Lyapunov Functions in the Design of Nonlinear Global Feedback Dynamics," in A. Zinober ed.: Variable Structure and Lyapunov Control, Lect. Notes in Control and Infor. Sciences **193**, Springer, 1994, pp.249–267.

[10] J. Milnor, Topology from the Differentiable Viewpoint, The University Press of Virginia, 1965.

[11] M. Vidyasagar, Nonlinear System Theory, Prentice–Hall, 1978.