Contractive Model Predictive Control for Constrained Nonlinear Systems

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***Abstract*—This paper addresses the development of stabilizing state and output feedback model predictive control (MPC) algorithms for constrained continuous-time nonlinear systems with discrete observations. Moreover, we propose a nonlinear observer structure for this class of systems and derive sufficient conditions under which this observer provides asymptotically convergent estimates.**

The MPC scheme proposed here consists of a basic finite horizon nonlinear MPC technique with the introduction of an additional state constraint, which has been called a *contractive constraint*. The resulting MPC scheme has been denoted *contractive MPC* (CNTMPC). This is a Lyapunov-based approach in which a Lyapunov function chosen a priori is decreased, not continuously, but discretely; it is allowed to increase at other times (between prediction horizons). We will show in this work that the implementation of this additional constraint into the on-line optimization makes it possible to prove strong nominal stability properties of the closed-loop system. In the absence of disturbances, it can be shown that the presence of the contractive constraint renders the closed-loop system exponentially stable in the state feedback case and uniformly asymptotically stable in the output feedback case.

***Index Terms*—Constrained control, model predictive control, nonlinear control.**

I. Introduction

MOST practical control problems are dominated by process constraints and nonlinearities. Nonlinearities can be quantified (as, e.g., in [4] and [5]), and it may be that while a linear controller design is satisfactory for a "weakly nonlinear" system, it will probably fail for a system with stronger nonlinearities. Besides the intrinsic nonlinear dynamics of the process, constraints on the state and/or manipulated variables may also introduce nonlinearities into the closed-loop system. Constraints on the manipulated variables are present in the vast majority of processes and they generally result from physical limitations of the actuators. Constraints on the states may be imposed for reasons of safety or productivity.

There are very few design techniques that can be proven to stabilize processes in the presence of nonlinearities and constraints. Model predictive control (MPC)—an optimal control based method—is one of these techniques. Other important features of MPC are its variable structure in the event of faults and its ability to handle multi-input multi-output (MIMO) systems with minor changes in the formulation compared to the single-input single-output (SISO) case.

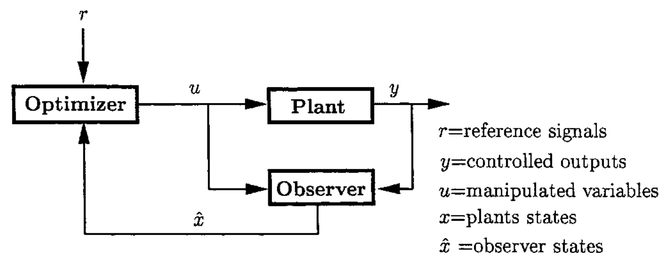

Fig. 1. Inherent structure in all MPC schemes.

The various implementations of MPC are identical in their global structure but differ in the details. The general structure of MPC schemes is shown in Fig. 1.

In Fig. 1, the selected observer uses the input and output information (u and y, respectively) and computes the state estimate $\hat{x}$. With this estimate, one can use an optimization scheme to predict the trajectory of the controlled variables y over some prediction (or output) horizon P with the manipulated variables u changed over some control (or input) horizon $M(M \leq P)$. This prediction step is depicted in Fig. 2.

In Fig. 2, the optimizer is used to compute the present and future manipulated variable moves at time step k, $u(k|k), \ldots, u(k+M-1|k)$, such that the predicted outputs follow the selected reference trajectories in a satisfactory manner. The optimizer takes into account the input and output constraints which may exist by incorporating them directly into the optimization. Only $u(k|k)$, the first control move of the sequence, is implemented on the real plant from time step k to $k+1$. At time step $k+1$ the measurement $y(k+1)$ is used together with $u(k|k)$ by the observer to compute the new estimate $\hat{x}(k+1)$, the horizons M and P are shifted ahead by one step and a new optimization problem is solved at time step $k+1$ with the new initial condition $\hat{x}(k+1)$. This procedure results in a so-called *moving* or *receding horizon* type of strategy.

Regarding the formulation of the optimal control problem, MPC algorithms can be divided into the following main categories:

1) finite prediction horizon $[P \in (0, \infty)]$ for:
   a) linear plants [17], [20], [29];
   b) nonlinear plants [11], [12], [26], [28];

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2) infinite prediction horizon $[P \to \infty]$ for:
   a) linear plants [36], [63], [72];
   b) nonlinear plants [1], [3], [51];
3) finite prediction horizon with end constraints[1] (also known as stability constraints) for:
   a) linear plants [8], [15], [19], [30], [36], [57], [69], [70], [72];
   b) nonlinear plants [1], [2], [18], [33], [34], [44]–[48], [50], [52], [53], [71].

In the first category, a simple finite horizon objective function is employed that does not, per se, guarantee stability. In [13], the authors underline the poor stability properties of finite prediction horizon schemes.

In the second category, Lyapunov arguments are used to show asymptotic stability of the infinite prediction horizon scheme. Although these are simple and powerful results, one of the great restrictions of setting $P \to \infty$ for nonlinear systems is naturally computational.

In the third category, asymptotic stability is added to finite prediction horizon schemes via the introduction of end constraints. The main difference amongst the various works in this area is the type of end constraint used. Besides, some works address discrete-time and others continuous-time linear/nonlinear systems. The end constraint most commonly used is $x(k+P_k|k) = 0$, i.e., the states of the prediction model are set exactly to zero at the end of the prediction horizon (which can be an extra decision variable and is, therefore, a function of k). The main problem with this widely used constraint is that, because it is an equality end constraint, it cannot be satisfied in a finite number of algorithm iterations. In other words, it is not implementable. In [45], a relaxed version of this constraint, $x(k+P_k|k) \in W$ (where W is a neighborhood of the origin), is proposed. In this case, the MPC strategy loses its stabilizing properties inside W. Thus, a linear locally stabilizing controller designed for the linearized system is used inside W. The resulting “hybrid” controller is shown to be globally stabilizing. The main difficulty in actually implementing this scheme is the computation of the region W.

Since only the nonlinear MPC schemes in the third category effectively provide stability guarantees but most of them are difficult to implement, some authors have proposed alternative finite horizon MPC algorithms aimed at removing terminal constraints while preserving stability (see, e.g., [21] and [58]).

A rather comprehensive review of most of these methods can be found in [37].

In this paper, we will address issues such as stability and implementability of MPC by proposing a different inequality end constraint called a *contractive constraint*. Because this is an inequality end constraint, it can be satisfied in a finite number of algorithm iterations. Moreover, contrary to the inequality end constraint in [45], no region of attraction W needs to be computed. The contractive constraint per se is a Lyapunov function for the closed-loop system and stability can be proven in a straightforward manner. Here we will address only the nominal case. If all the states are measurable (state feedback), then the closed-loop system can be proven exponentially stable. In the output feedback case, if the nonlinear estimation scheme generates asymptotically convergent estimates, then the closed loop can be shown uniformly asymptotically stable (UAS).

[1]By end constraint we mean any state constraint imposed at the end of the prediction horizon.

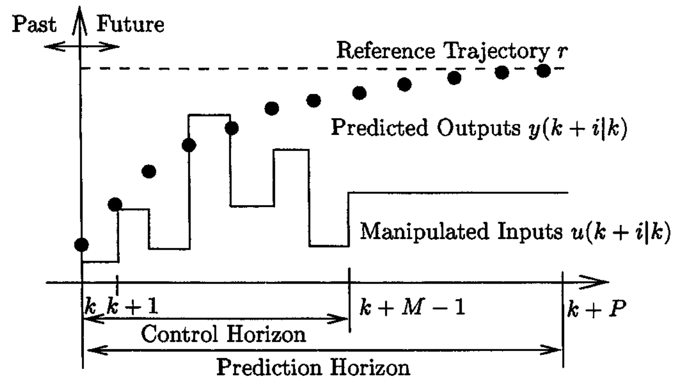


Fig. 2. Optimization problem at time k.

The idea of controllers which have user parameters that are intimately connected to the decay rate of the closed-loop system is also present in [14], [42], and [43]. These papers propose state feedback controllers with a contraction property for linear discrete-time unconstrained systems. The extension of this idea of imposing a contractive property on the closed-loop behavior to a nonlinear MPC framework has also been reported in [71]. In that work, the authors present a continuous-time formulation of the optimization problem which is of little practical value. Here we provide a discrete-time formulation of the optimization for continuous-time systems. Furthermore, those authors fail to recognize the fact that the contractive constraint leads to exponential stability and not only asymptotic stability of the closed loop. The output feedback case is also addressed there but the assumptions are many and difficult to check and the stability proof is very intricate. Here we examine closely the influence of the contractive constraint on the closed-loop stability and performance characteristics. We also propose the formulation of a nonlinear observer for continuous-time systems with discrete observations and show under which conditions this observer provides asymptotically stable estimates and generates a UAS closed-loop when associated with contractive MPC (CNTMPC).

The idea of explicitly imposing a decay rate on the closed-loop system trajectories is also present in a successful industrial MPC algorithm, namely, *Robust Model Predictive Control Technology* (RMPCT), a Honeywell product (this is reported in [62]). The RMPCT algorithm applies the so-called MPC *based on a funnel*. This algorithm attempts to keep each controlled variable (CV) within a user-defined zone. When the CV goes outside the zone, the RMPCT algorithm defines a CV funnel to bring the CV back within its range. The slope of the funnel is determined by a user-defined performance ratio (much similar to the parameter of contraction in CNTMPC if the CVs of RMPCT are the system states). In practical applications, it has been noticed that this type of constraint adds robustness to the closed-loop response.

The remainder of this paper is divided into three main parts (Section s II–IV). Section II deals with the state feedback control problem in the absence of disturbances, Section III examines the state feedback problem in the presence of asymptotically decaying disturbances, and Section IV looks into the output feedback problem and the design of an asymptotically convergent observer for nonlinear systems. In Section V, we detail the CNTMPC algorithm implementation. In Sections VI and VII, the CNTMPC controller is applied to a well-known chemical process example and to a nonholonomic system (model of a car with no trailers), respectively. In Section VIII, we present some concluding remarks. The proofs of all lemmas and theorems in Sections II and III can be found in Appendixes A and B, respectively.

II. State Feedback and no Disturbances

A. Description of the System

In this section, we assume that the plant is nonlinear time-invariant and described by the following differential equations:

$$\dot{x}(t) = f(x(t),\, u(t)) \tag{1}$$

where $f : \Re^n \times \Re^m \to \Re^n \in C^1$.

Throughout this paper, we assume that the manipulated variables $u(t)$ are subject to the following hard constraints:

$$u(t) \in \mathcal{U} = \{u \in \Re^m\colon u_{\min} \le u \le u_{\max}\}, \quad \forall\, t \in [0,\, \infty); \\ u_{\min},\, u_{\max} \in \Re^m \text{ known.} \tag{2}$$

Other very commonly used constraints which are considered here are linear bounds on the rate of change of the manipulated variables.

The solution of (1) at time t, corresponding to the initial state x_0 at time t_0 and the input $u(\tau)$ for $\tau \in [t_0,\, t]$, is denoted by $x(t - t_0,\, x_0,\, u)$ or, in a simplified notation, $x_0(t)$.

B. Notation

The following notation will be adopted throughout the paper and is not restricted to Section II.

The symbol := represents that the left-hand side of an equation is defined as the right-hand side. The converse applies to =:.

Regarding norms, $|.|$ represents the scalar norm, $\|\,.\,\|$ denotes the Euclidean norm of a vector and $\|\,x\,\|_{\tilde{P}} := \sqrt{x'\tilde{P}x}$, with $\tilde{P} \in \Re^{n\times n}$ positive definite ($\tilde{P} > 0$), is the weighted Euclidean norm of $x \in \Re^n$. $\|\,.\,\|$ and $\|\,.\,\|_{\tilde{P}}$ also denote the Euclidean and weighted Euclidean norms of a matrix in $\Re^{n\times n}$, respectively. $|||\,.\,|||$ is the induced norm on tensors corresponding to the vector norm $\|\,.\,\|$ on $\Re^n$. The induced norm of a tensor $\tilde{T} \in \Re^{n\times n\times n}$ is defined as

$$\begin{aligned} \left|\left|\left|\tilde{T}\right|\right|\right| &:= \sup_{x,\, y\neq 0,\, x,\, y\in\Re^n} \frac{\left\|y'\tilde{T}x\right\|}{\|y\|\,\|x\|} \\ &= \sup_{\|x\|=\|y\|=1} \left\|y'\tilde{T}x\right\| \\ &= \sup_{\|x\|\le 1,\, \|y\|\le 1} \left\|y'\tilde{T}x\right\|. \end{aligned} \tag{3}$$

$\mathcal{Z}_+$ and $\mathcal{Z}_+^* := \mathcal{Z}_+ - \{0\}$ are the sets of nonnegative and positive integers, respectively.

T denotes the sampling time, t_0 the initial time for computations, and P, M the prediction and control horizons, respectively.

Given a matrix $\tilde{P} > 0$, $\lambda_{\min}(\tilde{P})$ and $\lambda_{\max}(\tilde{P})$ denote the smallest and largest eigenvalues of $\tilde{P}$, respectively.

Here we adopt the following notation for the sampling times: $t_k^j := t_0 + (j + kP)T$, with j varying in the interval $j = 0, \ldots, P$, while k is kept constant at $k = 0, 1, 2 \ldots$. Moreover, we adopt $t_k := t_k^0$, and, $t_{k+1} := t_k^P, \quad \forall\, k \in \mathcal{Z}_+$. Thus, we have the following sequence of sampling times: $\{t_0, t_0^1, \ldots, t_0^{P-1}, t_1 = t_0^P, t_1^1, \ldots, t_1^{P-1}, t_2 = t_1^P, \ldots\}$. Although we realize that this is not standard MPC notation, we have found it most convenient because we are interested in emphasizing that our CNTMPC problems are solved as sets of P problems where the contractive constraint remains constant. Thus, it is important that the time variable has two indexes, one that remains constant for each set of P problems (k) and another that varies to represent each sampling time within the set corresponding to a given $k(j)$.

The following notation is used for the state and input trajectories: $x_k := x_k^0 := x(t_k^0 - t_0,\, x_0,\, u), x_k^j := x(t_k^j - t_k,\, x_k,\, u)$, $x_k^j(t) := x(t - t_k^j,\, x_k^j,\, u)$, and $u_k^j(t)$ is the computed control law for $t \in [t_k^j,\, t_k^{j+P}]$. In order to conform to MPC's usual implementation scheme, let us consider a control law of the kind $u_k^j(t) = u(t_k^{j+i}|t_k^j)$ for $t \in [t_k^{j+i},\, t_k^{j+i+1}]$, $i = 0, \ldots, P - 1$, i.e., $u_k^j(t)$ is constant during one sampling time and equal to $u(t_k^{j+i}|t_k^j)$, which symbolizes the predicted value of the control law at time t_k^{j+i} computed with information up to time t_k^j. Moreover, $u(t_k^{j+i}|t_k^j) = u(t_k^{j+M-1}|t_k^j)$, $\forall\, i = M, \ldots, P - 1$, i.e., the computed control variable remains constant after M steps. The rate of change of the manipulated variables is defined as $\Delta u(t_k^j|t_k^j) := u(t_k^j|t_k^j) - u(t_k^{j-1}|t_k^{j-1})$ and $\Delta u(t_k^{j+i}|t_k^j) := u(t_k^{j+i}|t_k^j) - u(t_k^{j+i-1}|t_k^j)$ with i varying in the interval $i = 1, \ldots, P - 1$, while j is kept constant at each of the values in the interval $j = 0, \ldots, P - 1$, and $k \in \mathcal{Z}_+$.

$\mathcal{P}(t_k^j,\, x_k^j)$ denotes the optimal control problem to be solved at time t_k^j with initial condition x_k^j.

C. Optimization Step of CNTMPC

The optimization problem at time t_k, $\mathcal{P}(t_k,\, x_k)$, $\forall\, k \in \mathcal{Z}_+$, is represented by

$$\min_{u(t_k|t_k),\, \ldots,\, u(t_k^{M-1}|t_k)} V(t_k,\, x_k) \tag{4}$$

with the objective function $V(t_k,\, x_k)$ defined as follows:

$$\begin{aligned} V(t_k,\, x_k) := \int_{t_k}^{t_{k+1}} \|x_k(t)\|_Q^2\, dt + \sum_{i=0}^{P-1} \|u(t_k^i|t_k)\|_R^2 \\ + \sum_{i=0}^{P-1} \|\Delta u(t_k^i|t_k)\|_S^2 \end{aligned} \tag{5}$$

subject to

$$\begin{cases} \dot{x}_k(t) = f(x_k(t), u_k(t)), & \text{with } u_k(t) = u(t_k^i|t_k) \\ & \text{for } t \in [t_k^i, t_k^{i+1}] \\ & \forall i = 0, \ldots, P-1 \\ u_{\min} \le u(t_k^i|t_k) \le u_{\max}, & i = 0, \ldots, M-1 \\ \Delta u(t_k^i|t_k) = 0, & i = M, \ldots, P-1 \\ |\Delta u_l(t_k^i|t_k)| \le \Delta u_{\max,l}, & i = 0, \ldots, M-1 \\ & \text{with } l = 1, \ldots, m; \\ & \Delta u_{\max} > 0 \\ \|x_{k+1}\|_{\hat{P}} = \|x_k(t_{k+1})\|_{\hat{P}} & \\ \qquad \le \alpha \|x_k\|_{\hat{P}} & \alpha \in [0, 1), \hat{P} > 0. \end{cases} \tag{6}$$

Remark 1: The last inequality constraint in the list of constraints (6) is the so-called *contractive constraint*. This denomination comes from the fact that the constraint imposed in the optimization at time t_k requires that the system states at the end of the prediction horizon, x_{k+1}, are "contracted" in norm with respect to the states at the beginning of the prediction, x_k. The two additional controller parameters (considering MPC's usual set of parameters) which determine how much contraction is required are the so-called *contractive parameter*, $\alpha \in [0, 1)$, and the positive definite matrix $\hat{P}$. Thus, "contraction" occurs at every P steps. As it will be demonstrated later, the imposition of such constraint guarantees global exponential stability of the closed-loop system. Therefore, the contractive constraint is a stabilizing constraint in the proposed algorithm.

D. CNTMPC Implementation

In Section II, the CNTMPC controller is implemented according to the following scheme.

Control Algorithm 1

Data: Initial Condition: x_0 at t_0; Controller Parameters: horizons $P, M(1 \le M \le P < \infty)$, weights $Q, R, S, \hat{P} > 0$, contractive parameter $\alpha \in [0, 1)$, sampling time $T \in (0, \infty)$ and control constraints $u_{\min}, u_{\max}, \Delta u_{\max} \in \Re^m$.

Step 0: Set $k = 0$.

Step 1: Solve $\mathcal{P}(t_k, x_k)$ specified by the sets of equations (4)–(6). The result of this step is a sequence of control moves $\{u(t_k|t_k), \ldots, u(t_k^{M-1}|t_k)\}$.

Step 2: Apply the whole sequence of computed control moves to plant (1) for $t \in [t_k, t_{k+1}]$ and measure the states at t_{k+1}, x_{k+1}.

Step 3: Set $k = k+1$ and go back to *Step 1*.

Remark 2: In *Step 1* we assume that the control problem $\mathcal{P}(t_k, x_k)$ is feasible for all $k \in \mathcal{Z}_+$. This means that there always exists a solution which satisfies all the constraints and renders the objective function finite. It is important to emphasize that a global optimal solution is not required. Feasible or local optimal solutions (i.e., solutions that minimize the objective function only locally in the presence of the constraints imposed) are acceptable solutions of the optimization problem at t_k. The fact that we are not restricting acceptance to global optimal solutions only means that performance will most probably be compromised but the stability properties of the algorithm are not weakened. The reasoning behind this argument is that a simple feasible solution already satisfies the stabilizing contractive constraint in the presence of the remaining constraints. Thus, optimality is not a requirement for stability although it is important for achievement of optimal performance.

Remark 3: Notice that *Control Algorithm 1* is not implemented in the usual moving (or receding) horizon fashion of most MPC algorithms because in Section II we are dealing with the case in which there are no disturbances and all the states are measurable. In this case, feedback at every sampling time is not necessary since the prediction is exact. In Sections III and IV, however, when disturbances and state estimation errors are considered, new control algorithms implemented in the usual moving horizon style are proposed.

E. Basic Assumptions and Definitions

Without loss of generality, let us consider the regulation problem where the desired operating point is the origin. Then, the following assumptions are needed to ensure local stability:

Assumption 1: $(x, u) = (0, 0)$ is an equilibrium point of (1), i.e., $f(0, 0) = 0$.

Assumption 2: The linearization of the model dynamics around the origin is stabilizable, i.e., $\{(\partial f/\partial x)(0, 0), (\partial f/\partial u)(0, 0)\}$ is a stabilizable pair.

Assumption 3: There exists a constant $\rho \in (0, \infty)$ such that for all $x_k \in B_\rho := \{x \in \Re^n \,|\, \|x\|_{\hat{P}} \le \rho\}$, the optimization problem at t_k, $\mathcal{P}(t_k, x_k)$, is feasible for all $k \in \mathcal{Z}_+$. In other words, for all $x_k \in B_\rho$, we can find a contractive parameter $\alpha \in [0, 1)$ so that with the chosen finite horizon P all the constraints on the inputs and states can be satisfied and the objective function is finite.

Remark 4: *Assumption 3* is not very restrictive since all that it establishes is that there exists a nonempty convex and compact set of initial conditions for which the optimization problem at every P time steps is feasible.

If $[(\partial f/\partial x)(\partial f/\partial u)] \in \overline{\Omega}, \forall x, u$ (where $\overline{\Omega} \subseteq \Re^{n\times(n+m)}$), then the nonlinear system can be replaced by a time-varying linear system (an idea that is implicit in the early works of Lur'e and Postnikov [40], [41] and Popov [61]), and this approach is known as *global linearization*. Once the nonlinear system is represented in linear differential inclusion form, sufficient conditions for satisfaction of *Assumption 3* can be stated (see [10], where sufficient conditions for exponential stability and an induced L_2-norm performance objective are given) by using a single quadratic Lyapunov function approach.

Remark 5: Since the contractive constraint implies that $\|x_{k+1}\|_{\hat{P}} \le \alpha\|x_k\|_{\hat{P}}$, if $x_0 \in B_\rho$, then $x_k \in B_{\alpha^k\rho} \subset B_\rho$. Thus, using *Assumption 3*, if $\mathcal{P}(t_0, x_0)$ is feasible then the sequence of control problems $\mathcal{P}(t_k, x_k), \quad \forall k \in \mathcal{Z}_+^*$, is feasible as well.

Assumption 4: There exists a constant $\beta \in (0, \infty)$ so that the transient states of the model remain inside the set $B_{\beta\|x_k\|_{\hat{P}}} :=$

$\{x \in \Re^n \mid \|x\|_{\hat{P}} \leq \beta \|x_k\|_{\hat{P}}\}$, i.e., $\|x_k(t)\|_{\hat{P}} \leq \beta \|x_k\|_{\hat{P}}$, $\forall\, t \in [t_k, t_{k+1}]$, $k \in \mathcal{Z}_+$.

Remark 6: Since u is constrained, *Assumption 4* is always satisfied except for systems with finite escape time. So, nonlinear systems with finite escape time are ruled out from our investigation.

Definition 1: Under *Assumption 3*, the reachable set $\mathcal{X}$ is defined by

$$\begin{aligned}\mathcal{X} := \{x(t) \in \Re^n \mid x(t) = x(t - t_0, x_0, u(t))\\ t \in [t_0, \infty);\, x_0 \in B_\rho,\, u(t) \in \mathcal{U}\}.\end{aligned} \tag{7}$$

Thus, the reachable set is the subspace of $\Re^n$, which contains all the trajectories of the closed-loop system.

Remark 7: Under the conditions in Section II (i.e., state feedback and no disturbances), *Assumptions 3* and *4*, the reachable set $\mathcal{X}$ is a finite dimensional subspace of $\Re^n$, which is contained in $B_{\beta\rho}$.

F. Basic Idea Behind the Controller Design

Fig. 3 illustrates the behavior of the closed-loop system generated by CNTMPC under the conditions in Section II. This figure displays the exponential decay of the state trajectories with state contraction occurring at every P steps.

G. Stability Analysis of CNTMPC in Section II

Theorem 1 (Exponential Stability): Suppose that *Assumptions 1–4* are satisfied. Then, given $\alpha \in [0, 1)$ and $\rho, \beta \in (0, \infty)$ satisfying *Assumptions 3* and *4*, respectively, *Control Algorithm 1* renders the closed-loop system exponentially stable on the set B_ρ, i.e., for any $x_0 \in B_\rho$, the resulting trajectory $x(t) := x(t - t_0, x_0, u)$ satisfies the following inequality:

$$\|x(t)\| \leq a\|x_0\| e^{-(1-\alpha)((t-t_0)/PT)}, \qquad \text{with } a \geq \beta e^{(1-\alpha)}. \tag{8}$$

Proof: The proof can be found in Appendix A. □

Now that exponential stability has been proven, we will show that, under certain assumptions on the control law originated by CNTMPC, the objective function (5) is an exponentially decaying Lyapunov function for the closed-loop system.

Before we start showing the conditions under which this is true, let us point out that the objective function being a Lyapunov function is not a necessary condition for stability of the closed-loop under the CNTMPC controller, as it is for the majority of MPC schemes. The closed-loop is stabilized by the CNTMPC controller because the quadratic function that defines the contractive constraint is itself a Lyapunov function that decreases discretely, not continuously, at intervals of prediction horizons.

We will see in the next theorem that in order for the objective function (5) to be an exponentially decaying Lyapunov function, stronger assumptions are needed on the computed control law and on the dynamics of the nonlinear system. Here we want to establish which and how strong these assumptions are because they are *necessary* for proving *exponential* (not asymptotic) stability of the MPC scheme with the equality end constraint $x(k + P|k) = 0$ (see [65]). In other words, we want to emphasize that we are able to prove exponential stability under much less restrictive assumptions when the contractive constraint, rather than the equality end constraint, is used.

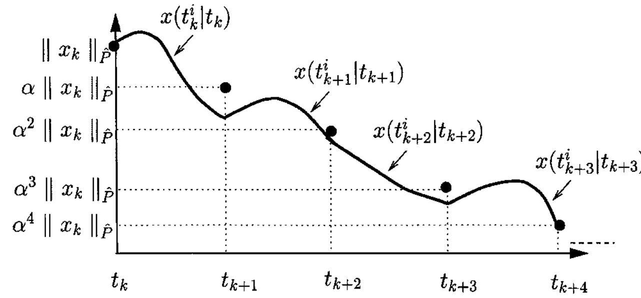


Fig. 3. Exponential decay of the state trajectory.

The additional assumptions needed to prove *Theorem 2* are enunciated below.

Let $u_k(t) =: \eta(x(t - t_k, x_k, u)) = \eta(x_k(t))$ be the feedback law computed at t_k for $t \in [t_k, t_{k+1}]$ and for all $x_k \in B_\rho$, $\forall\, k \in \mathcal{Z}_+$. Thus, we have $u_k(t) = u(t^i_k|t_k) = \eta(x_k(t))$ for $t \in [t^i_k, t^{i+1}_k]$, $\forall\, i = 0, \ldots, M - 1$.

Let us also define the function $F(x_k(t)) := f(x_k(t), u_k(t)) = f(x_k(t), \eta(x_k(t)))$..

Assumption 5: Let us assume that the feedback law computed at t_k is such that $\eta(0) = 0$ and that $\eta(.)$ is Lipschitz continuous inside the set $B_{\beta\rho}$, i.e., there exists $L_\eta > 0$ such that the following inequality is satisfied:

$$\|\eta(x) - \eta(y)\| \leq L_\eta \|x - y\|, \quad \forall\, x, y \in B_{\beta\rho}. \tag{9}$$

Assumption 6: Suppose that f is C^q for some integer $q \geq 1$ and that, for some finite constant $\xi \geq 0$, we have

$$\left\| \frac{df(x_k(t), u_k(t))}{dx} \right\| = \left\| \frac{dF(x_k(t))}{dx} \right\| \leq \xi, \quad \forall\, x_k(t) \in B_{\beta\rho}. \tag{10}$$

Assumption 7: Let the matrices $Q > 0$ and $\hat{P} > 0$ be chosen such that the following inequality is satisfied:

$$\overline{\alpha} := \alpha \sqrt{\frac{\lambda_{\max}(\hat{P})\lambda_{\max}(Q)}{\lambda_{\min}(\hat{P})\lambda_{\min}(Q)}} < 1. \tag{11}$$

Remark 8: Assumption 7 ensures that if the contractive constraint is satisfied, then the following inequality holds:

$$\|x_{k+1}\|_Q \leq \overline{\alpha} \|x_k\|_Q, \qquad \forall\, x_k \in B_\rho,\, k \in \mathcal{Z}_+ \tag{12}$$

with $\overline{\alpha} \in [0, 1)$.

Theorem 2 (Objective Function as a Lyapunov Function): Let *Assumptions 1–7* be satisfied. Then, the objective function in (5), which, due to the fact that $u_k(t) = u(t^i_k|t_k)$ for $t \in [t^i_k, t^{i+1}_k]$, $\quad \forall\, i = 0, \ldots, P - 1$, can be rewritten as

$$V(t_k, x_k) = \int_{t_k}^{t_{k+1}} [\|x_k(t)\|^2_Q + \|u_k(t)\|^2_{R^*}]\, dt \tag{13}$$

where $R^* := R/T$ and we considered $S = 0$ for simplicity, is a Lyapunov function for the closed-loop system. This means that there exist constants $c, d, e, l > 0$ such that

1. $c\|x_k\|_{\hat{P}}^2 \le V(t_k, x_k) \le d\|x_k\|_{\hat{P}}^2$
2. $\dot{V}(t_k, x_k) := \dfrac{dV}{dt_k}(t_k, x_k) = \dfrac{\partial V}{\partial t_k}(t_k, x_k) \le -e\|x_k\|_{\hat{P}}^2$
3. $\left\| \dfrac{\partial V}{\partial x_k}(t_k, x_k) \right\|_{\hat{P}} \le l\|x_k\|_{\hat{P}}, \quad \forall x_k \in B_\rho, \qquad \forall k \in \mathcal{Z}_+$.

Proof: The proof can be found in Appendix A. □

III. State Feedback and Asymptotically Decaying Disturbances

A. Description of the System

In this section, we assume that the plant is described by the following differential equations [instead of (1)]:

$$\dot{x}^p(t) = f(x^p(t), u(t)) + d(t) \tag{14}$$

where the unknown additive disturbance $d(t)$ is bounded and asymptotically decaying. Thus, $x^p(t)$ represents the trajectory of the states of the plant with $x_k^p(t) := x^p(t - t_k, x_k^p, u, d)$ and $x_k^p := x^p(t_k - t_0, x_0^p, u, d), \quad \forall k \in \mathcal{Z}_+$.

The model used in the control computations is then given by

$$\dot{x}(t) = f(x(t), u(t)). \tag{15}$$

Thus, model (15) neglects the influence of the disturbances $d(t)$, which cannot be measured.

An important reason for studying the influence of additive asymptotically decaying disturbances on closed-loop stability is the fact that such disturbances can be produced by introduction of an asymptotically convergent state estimator into the closed-loop. In this section, we will see how the results obtained in Section II will help us in the stability analysis of the output feedback case.

B. Optimization Step of CNTMPC

The optimization problem at time t_k^j, $\mathcal{P}(t_k^j, x_k^{p,j})$, $\forall j = 0, \ldots, P-1$, $k \in \mathcal{Z}_+$, is represented by

$$\min_{u(t_k^j|t_k^j), \ldots, u(t_k^{j+M-1}|t_k^j)} \int_{t_k^j}^{t_k^{j+P}} \|x_k^j(t)\|_Q^2 \, dt + \sum_{i=0}^{P-1} \|u(t_k^{j+i}|t_k^j)\|_R^2 + \sum_{i=0}^{P-1} \|\Delta u(t_k^{j+i}|t_k^j)\|_S^2 \tag{16}$$

subject to

$$\begin{cases} \dot{x}_k^j(t) = f(x_k^j(t), u_k^j(t)), & \text{with } x_k^j = x_k^{p,j} \\ & \text{and } u_k^j(t) = u(t_k^{j+i}|t_k^j) \\ & \text{for } t \in [t_k^{j+i}, t_k^{j+i+1}], \\ & \forall i = 0, \ldots, P-1 \\ u_{\min} \le u(t_k^{j+i}|t_k^j) \le u_{\max}, & i = 0, \ldots, M-1 \\ \Delta u(t_k^{j+i}|t_k^j) = 0, & i = M, \ldots, P-1 \\ |\Delta u_l(t_k^{j+i}|t_k^j)| \le \Delta u_{\max, l}, & i = 0, \ldots, M-1 \\ & \text{with } l = 1, \ldots, m; \\ & \Delta u_{\max} > 0 \\ \|\overline{x}_k^j(t_{k+1})\|_{\hat{P}} \le \alpha \|x_k^p\|_{\hat{P}}, & \alpha \in [0, 1), \hat{P} > 0 \end{cases} \tag{17}$$

where

$$\dot{\overline{x}}_k^j(t) = f(\overline{x}_k^j(t), u_k^j(t))$$

with

$$\overline{x}_k^0 := x_k^p$$

and

$$\overline{x}_k^j = \overline{x}_k^{j-1}(t_k^j), \qquad \text{for } j \ge 1 \tag{18}$$

is the trajectory of the model used in the computation of the contractive constraint. The trajectory $\overline{x}_k^j(t)$ is obtained via numerical integration of the model in (18) with initial condition $\overline{x}_k^j$ from time t_k^j up to t_{k+1} (notice that it is not necessary to integrate up to t_k^{j+P} because this model is used only for computation of $\overline{x}_k^j(t_{k+1})$, not in the prediction). The recursive definition in (18) shows that at t_k the initial condition for integration are the measured states x_k^p. However, at t_k^j, for $j = 1, \ldots, P-1$, the initial condition is given by the trajectory of the same model computed at the previous time step, $\overline{x}_k^{j-1}(t)$, evaluated at t_k^j. Thus, the model in (18) only takes feedback into account at every P steps (as opposed to the prediction model in the set of equations (6), which is updated with the plant states at every sampling time).

C. CNTMPC Implementation

The state feedback CNTMPC controller is implemented according to the following scheme.

Control Algorithm 2

Data: Initial Condition: x_0^p at t_0; Controller Parameters: $P, M, Q, R, S, \hat{P}, \alpha, T, u_{\min}, u_{\max}, \Delta u_{\max}$.

Step 0: Set $k = 0, j = 0$.

Step 1: Assuming that the optimal control problem $\mathcal{P}(t_k^j, x_k^{p,j})$ is feasible for the chosen set of parameters, then at $t = t_k^j$ solve $\mathcal{P}(t_k^j, x_k^{p,j})$ specified by the sets of equations (16)–(18). Local optimal solutions or even feasible solutions are accepted. The result of this step is an optimal (or feasible) sequence of control moves $\{u(t_k^j|t_k^j), \ldots, u(t_k^{j+M-1}|t_k^j)\}$.

Step 2: Apply the first control move, $u(t_k^j|t_k^j)$, to plant (14) for $t \in [t_k^j, t_k^{j+1}]$ and measure the states at t_k^{j+1}, $x_k^{p,j+1}$. Set $x_k^{j+1} = x_k^{p,j+1}$ and $\overline{x}_k^{j+1} = \overline{x}_k^j(t_k^{j+1})$.

Step 3: If $j < P-1$, set $j = j+1$ and go back to *Step 1*. If $j = P-1$ set $x_{k+1}^{p,0} =: x_{k+1}^p = x_k^{p,P}$, $k = k+1$, $j = 0$, and go back to *Step 1*.

Remark 9: Notice that both the contractive constraint and its location (at time $t = t_{k+1}$ and with respect to x_k^p) do not change for a fixed k as j varies in the interval $j = 0, \ldots, P-1$. This means that if at time t_k it is possible to find a control sequence which makes the objective function finite and satisfies all the constraints (i.e., $\mathcal{P}(t_k, x_k^p)$ is feasible) and if the constraints remain unaltered for a fixed k while j varies from 0 to $P-1$, then the subsequent $P-1$ control problems (corresponding to the different values of j) will be feasible as well. So, all we need to be concerned about is the feasibility of $\mathcal{P}(t_k, x_k^p), \quad \forall k \in \mathcal{Z}_+$.

Remark 10: Notice that, opposite to *Control Algorithm 1*, *Control Algorithm 2* is implemented in the usual moving horizon fashion. In Section III, since disturbances are considered, the moving horizon implementation will most likely result in improved performance while retaining certain stability properties (which will be examined in the stability analysis that follows).

Remark 11: In order to avoid feasibility problems in this section, we should consider α and P to be time-varying tuning parameters (α_k, P_k) that could be adjusted in the case of a large asymptotically decaying disturbance entering the system between t_k and t_{k+1} and leading to infeasibility of the control problem at t_{k+1}, $\mathcal{P}(t_{k+1}, x_{k+1}^p)$. However, we did not write these parameters as time-varying for the sake of simplicity of the notation and of the resulting controller. Instead, in the stability analysis that follows we derive an upper bound on the allowable disturbance norm (only a sufficient condition) so that $\mathcal{P}(t_k, x_k^p)$ is feasible for every $k \in \mathcal{Z}_+$.

D. Basic Assumptions

Besides *Assumptions 1–4*, the following additional assumptions will be made in the derivation of the results in Section III.

Assumption 8: The disturbance satisfies the following boundedness condition:

$$d_k(t) \in B_{\overline{\rho}_k} := \{d \in \Re^n : \|d\|_{\hat{P}} \le \overline{\rho}_k\} \qquad \text{for } \overline{\rho}_k \in [0, \infty), \quad t \in [t_k, t_{k+1}], \quad \forall k \in \mathcal{Z}_+ \tag{19}$$

where $d_k(t) := d(t)$ for $t \in [t_k, t_{k+1}], \forall k \in \mathcal{Z}_+$.

Furthermore, the asymptotic properties of $d(t)$ are described as follows: for any $\epsilon > 0$, $\exists$ a finite $:= \overline{k}(\epsilon) \in \mathcal{Z}_+$ so that $\overline{\rho}_k \le \epsilon, \quad \forall k \in [\overline{k}, \infty)$, and $\overline{k}(\epsilon) \to \infty$ if $\epsilon \to 0$.

Assumption 9: The function $f : \Re^n \times \Re^m \to \Re^n$ is Lipschitz continuous in a compact set $\Omega \subset \Re^n$, i.e., there exists $L > 0$ such that

$$\|f(x^p, u) - f(\overline{x}, u)\|_{\hat{P}} \le L\|x^p - \overline{x}\|_{\hat{P}}, \quad x^p, \overline{x} \in \Omega \text{ and } u \in \mathcal{U}. \tag{20}$$

Remark 12: Let the reachable set for the closed-loop system resulting from implementation of *Control Algorithm 2* to plant (14), using model (15) for prediction, be defined by

$$\begin{aligned} \mathcal{X} := \{ & x^p(t), x(t) \text{ and } \overline{x}(t) \in \Re^n \mid \\ & x^p(t) = x^p(t - t_0, x_0^p, u(t), d(t)) \\ & x(t) = x(t - t_0, x_0^p, u(t), 0) \\ & \overline{x}(t) = \overline{x}(t - t_0, x_0^p, u(t), 0), \, t \in [t_0, \infty); \\ & x_0^p \in B_\rho, \, u(t) \in \mathcal{U}, \, d(t) \in B_{\overline{\rho}}\} \end{aligned} \tag{21}$$

where $\overline{\rho} := \max_{k \in \mathcal{Z}_+} \overline{\rho}_k$. $\mathcal{X}$ is then the union of the reachable sets of three different state equations, namely, (14), (15) and (18).

Then, (20) only needs to be satisfied for $x^p, \overline{x} \in \mathcal{X}$. The reason why we do not state *Assumption 9* in this way is because the set $\mathcal{X}$ is not known *a priori*. Thus, in this form, condition (20) cannot be checked and we need to impose it on a set Ω large enough so that $\mathcal{X} \subset \Omega$.

E. Stability Analysis of CNTMPC in Section III

Theorem 3 (Stabilizing Properties of Control Algorithm 2 in the Presence of Asymptotically Decaying Disturbances): Let *Assumptions 1–4, 8,* and *9* be satisfied and let $x_k^p, \overline{x}_k \in B_\rho, \quad \forall k \in \mathcal{Z}_+$. Then, the closed-loop system resulting from application of *Control Algorithm 2* to system (14) is UAS.

Proof: The proof can be found in Appendix B. □

We have proven *Theorem 3* under the assumption of feasibility. We shall now proceed to derive a sufficient condition on the disturbance $d(t)$ under which $x_k^p, \overline{x}_k$ stay inside B_ρ for all $k \in \mathcal{Z}_+$.

Theorem 3 (Feasibility Condition): Under *Assumptions 8* and *9*, if $\overline{\rho} < (\rho(1-\alpha)/PTe^{\mathrm{LPT}})$ [with $\overline{\rho}$ defined right below equation (21)], then there exists $\rho_0 \in (0, \rho]$ such that for all $x_0^p \in B_{\rho_0}$, the sequences $\{x_k^p\}_{k=0}^\infty$ and $\{\overline{x}_k\}_{k=0}^\infty$ resulting from implementation of *Control Algorithm 2* stay inside the set B_ρ.

Proof: The proof can be found in Appendix B. □

Thus, we have proven that *Control Algorithm 2* applied to plant (14) is well defined and produces a UAS closed-loop. Next we propose a recursive dynamic nonlinear observer and study the conditions under which this observer is asymptotically stable. Then, using the results obtained in Section III we show that, under these conditions, the combination of our exponentially stabilizing CNTMPC with the proposed observer generates a UAS closed-loop.

IV. OUTPUT FEEDBACK

A. Dynamic Observers for Nonlinear Systems

1) Observer Design: The formulation of stable nonlinear observers is an active area of research. An observer based on minimization of a moving horizon cost function is presented in [55], [64]. A variation of this technique for continuous-time systems is presented in [54].

In our design we follow guidelines similar to the ones used to derive the extended Kalman filter (EKF), a well-known standard linearization method for approximate nonlinear filtering. The available literature is vast and we refer the reader to [32], [38], and the references therein. In the context of parameter estimation for linear stochastic systems, a fairly systematic and comprehensive convergence analysis of the EKF is presented in [39].

Let us consider a continuous-time system with discrete observations and a nonlinear output map

$$\begin{aligned} \dot{x}_k(t) &= F(x_k(t)), \quad x_k(t_k) =: x_k, \quad t \in [t_k, t_{k+1}] \\ y_k &= H(x_k), \quad \forall\, k \in \mathcal{Z}_+ \end{aligned} \tag{22}$$

with $x \in \Re^n$, $y \in \Re^{n_y}$ and $n_y \leq n$.

Remark 13: We will envision system (22) as resulting from implementation of *Control Algorithm 2* to the following original system:

$$\begin{aligned} \dot{\psi}_k^x(t) &= f(\psi_k^x(t), u_k(t)) \\ \psi_k^y(t_k) &= h(\psi_k^x(t_k), u_k(t_k)) \end{aligned} \tag{23}$$

where we have made a distinction between the states $x(t)$ and $\psi^x(t)$ to allow for dynamic feedback (instead of restricting ourselves to static feedback).

We assume that $F : \Re^n \to \Re^n$ and $H : \Re^n \to \Re^{n_y}$ are at least twice differentiable, and therefore we define

$$A(x) := DF(x)$$

and

$$C(x) := DH(x)$$

with

$$D(.) := \frac{d}{dx}(.) \tag{24}$$

Associate the following "noisy" system with (22):

$$\begin{aligned} \dot{z}_k(t) &= F(z_k(t)) + R_w w_k(t), \quad z_k(t_k) =: z_k, \\ & \quad t \in [t_k, t_{k+1}], \\ \zeta_k &= H(z_k) + R_v v_k, \qquad \forall\, k \in \mathcal{Z}_+. \end{aligned} \tag{25}$$

As usual, we assume that z_0, $w(t)$ and $v(t)$ are jointly Gaussian and mutually independent. Furthermore, $z_0 \sim \mathcal{N}(z_0, \overline{P}_0^{-1})$, $w(t) \sim \mathcal{N}(0, I_n)$, and $v(t) \sim \mathcal{N}(0, I_p)$, where the notation $s \sim \mathcal{N}(s_0, S)$ means that s is an stochastic variable with mean value s_0 and covariance S. We also assume that the design variables R_w, R_v and $\overline{P}_0$ are such that R_w has rank n and R_v and $\overline{P}_0$ are positive definite.

Then let us propose the following structure for the nonlinear observer for the associated "noisy" system (25):

$$\begin{aligned} &\dot{\hat{x}}_k(t) = F(\hat{x}_k(t)) + \overline{P}_k(t)^{-1} C_k' (R_v R_v')^{-1} [\zeta_k - H(\hat{x}_k)], \\ &\hat{x}_0(t_0) =: \hat{x}_0 \text{ (chosen)} \end{aligned} \tag{26}$$

and $\overline{P}_k(t)$ satisfies the following differential Riccati equation:

$$\begin{aligned} \dot{\overline{P}}_k(t) =& -\overline{P}_k(t) A(\hat{x}_k(t)) - A(\hat{x}_k(t))' \overline{P}_k(t) \\ & - \overline{P}_k(t) R_w R_w' \overline{P}_k(t) + Q_k' Q_k \\ \overline{P}_0(t_0) =&: \overline{P}_0 \end{aligned} \tag{27}$$

where $C_k := C(\hat{x}_k)$ and $Q_k := R_v^{-1} C_k$.

2) Basic Assumptions:

Assumption 10: The linearized system determined by $(A(\hat{x}_k(t)), C(\hat{x}_k))$ along the estimated trajectory of the observer (26), (27), is uniformly observable, that is, $(A(\hat{x}_k(t)), C_k)$ satisfies the uniform observability condition presented in [9], [24] for linear systems and in [67] for linear and nonlinear systems.

Assumption 11: The following induced operator norms are bounded:

$$\|A\| := \sup_{z \in \Re^n} \|A(z)\| \quad \text{and } \|C\| := \sup_{z \in \Re^n} \|C(z)\| \tag{28}$$

and

$$|||D^2 F||| := \sup_{z \in \Re^n} |||D^2 F(z)|||$$

and

$$|||D^2 H||| := \sup_{z \in \Re^n} |||D^2 H(z)|||. \tag{29}$$

Remark 14: Under *Assumptions 10* and *11* it is possible to show that the error covariance $\overline{P}(t)^{-1}$ and its inverse $\overline{P}(t)$ are uniformly bounded (see [9] for derivation of these bounds), i.e.,

$$\|\overline{P}(t)^{-1}\| \leq \overline{q} \quad \text{and } \|\overline{P}(t)\| \leq \overline{p}, \quad \forall\, t \geq t_0 \tag{30}$$

where $\overline{p}, \overline{q} \in [0, \infty)$ depend on $\overline{P}_0$, R_v and R_w and on the functions F and H.

Assumption 12: There exists a constant $G \in [0, \infty)$ such that the function $G(x, y)$ defined as $G(x, y) := H(x) - H(y) - C(y)(x - y)$ satisfies the following bound: $\|G(x, y)\| \leq G\, |||D^2 H|||\, \|x - y\|^2$ for all $x, y \in \Re^n$.

Remark 15: Assumption 12 can be interpreted in the following way: the function $G(x, y)$ is the difference between $H(x)$ and the expansion in Taylor series of $H(x)$ around y truncated after the first-order term. In other words, $G(x, y)$ expresses the error between $H(x)$ and its first-order approximation. We know that $\lim_{y \to x} G(x, y) = 0$. Then, for arbitrary $x, y \in \Re^n$ we assume that there exists $G \in [0, \infty)$ such that the norm of the error is bounded by the norm of the second-order terms ignored in the Taylor series expansion of $H(x)$ scaled by a large enough constant G.

3) Asymptotic Stability of the Observer:

Remark 16: Similar proofs of asymptotic convergence of nonlinear observers applied to both continuous-time systems with continuous observations and linear output maps and discrete-time systems with discrete observations can be found in the literature [9], [66], [67], but here we are addressing the observer design problem for nonlinear continuous-time systems with discrete observations and nonlinear output maps (a more general situation).

Theorem 5 (Stability Region for the Nonlinear Observer): Let us assume that

$$\|x_0 - \hat{x}_0\| \, \varphi(D^2F, \, D^2H, \, \overline{P}_0, \, R_v, \, R_w) < \frac{r}{\overline{p}^{1/2} \, \overline{q}^2 \|\overline{P}_0^{1/2}\|} \tag{31}$$

for some $r > 0$, $\overline{p}$, $\overline{q}$ defined as in *Remark 14* and with the function φ defined as:

1) linear output map ($y_k = Cz_k = H(z_k)$ and, consequently, $D^2H = 0$)

$$\varphi(D^2F, \, D^2H, \, \overline{P}_0, \, R_v, \, R_w) := |||D^2F||| \tag{32}$$

2) nonlinear output map

$$\begin{aligned}\varphi(D^2F, \, &D^2H, \, \overline{P}_0, \, R_v, \, R_w) \\ &:= |||D^2F||| + 2\left(\frac{\overline{q}}{\overline{p}}\right)^{1/2} \|C\|\kappa_v \|R_v R_v^{'}\|^{-1} G |||D^2H|||\end{aligned} \tag{33}$$

where κ_v is the condition number of the nonsingular symmetric matrix $R_v R_v^{'}$ and R_w is such that $R_w R_w^{'} \geq rI_n$.

Then the dynamical system (26), (27) is an asymptotically stable observer for the nonlinear system (22) provided that *Assumptions 10–12* hold. Let $e(t) := x(t) - \hat{x}(t)$ be the state estimation error at time $t \geq t_0$. Then, there exists a constant $\delta > 0$ such that

$$\begin{aligned}\overline{V}(t_k + T) &- \overline{V}(t_k) \\ &:= \|\overline{P}^{1/2}(t_k + T) e(t_k + T)\|^2 \\ &\quad - \|\overline{P}^{1/2}(t_k) e(t_k)\|^2 \leq -\delta, \qquad \forall \, k \in \mathcal{Z}_+\end{aligned} \tag{34}$$

i.e., $\overline{V}(t) := \|\overline{P}(t)^{1/2} e(t)\|^2$ is a Lyapunov function for the closed-loop system that decreases discretely, at sampling times, for all initial estimates satisfying (31).

Proof: Theorem 5 will not be proven here because similar proofs can be found in the literature (see, e.g., [9] and [67]). For a detailed proof, the reader is referred to [22]. □

B. MPC Algorithm with State Estimation

In order to include state estimation, *Control Algorithm 2* is modified in the following way.

Control Algorithm 3

Data: Initial Condition: $\hat{x}_0$ at t_0; Controller Parameters: $P, \, M, \, Q, \, R, \, S, \, \hat{P}, \, \alpha, \, T, \, u_{\min}, \, u_{\max}, \, \Delta u_{\max}$; Observer Parameters: $\overline{P}_0, \, R_v, \, R_w$; Output measurement at t_0: ζ_0.

Step 0: Set $k = 0, j = 0$.

Step 1: Solve the optimal control problem $\mathcal{P}(t_k^j, \, \hat{x}_k^j)$ specified by

$$\begin{aligned}\min_{u(t_k^j|t_k^j), \cdots, u(t_k^{j+M-1}|t_k^j)} &\int_{t_k^j}^{t_k^{j+P}} \|x_k^j(t)\|_Q^2 \, dt \\ &+ \sum_{i=0}^{P-1} \|u(t_k^{j+i}|t_k^j)\|_R^2 + \sum_{i=0}^{P-1} \|\Delta u(t_k^{j+i}|t_k^j)\|_S^2\end{aligned} \tag{35}$$

subject to

$$\begin{cases} \dot{x}_k^j(t) = f(x_k^j(t), \, u_k^j(t)), & \text{with } x_k^j = \hat{x}_k^j \\ & \text{and } u_k^j(t) = u(t_k^{j+i}|t_k^j) \\ & \text{for } t \in [t_k^{j+i}, \, t_k^{j+i+1}], \\ & \forall \, i = 0, \, \ldots, \, P-1 \\ u_{\min} \leq u(t_k^{j+i}|t_k^j) \leq u_{\max}, & i = 0, \, \ldots, \, M-1 \\ \Delta u(t_k^{j+i}|t_k^j) = 0, & i = M, \, \ldots, \, P-1 \\ |\Delta u_l(t_k^{j+i}|t_k^j)| \leq \Delta u_{\max, \, l}, & i = 0, \, \ldots, \, M-1 \\ & \text{with } l = 1, \, \ldots, \, m; \\ & \Delta u_{\max} > 0 \\ \|\overline{x}_k^j(t_{k+1})\|_{\hat{P}} \leq \alpha \|\hat{x}_k\|_{\hat{P}} & \alpha \in [0, \, 1), \, \hat{P} > 0 \end{cases} \tag{36}$$

where

$$\dot{\overline{x}}_k^j(t) = f(\overline{x}_k^j(t), \, u_k^j(t))$$

with

$$\overline{x}_k^0 := \hat{x}_k$$

and

$$\overline{x}_k^j = \overline{x}_k^{j-1}(t_k^j), \qquad \text{for } j \geq 1 \tag{37}$$

is the trajectory of the model, which is only updated with the states of the estimator at $t = t_k$.

Step 2: Apply the first control move, $u(t_k^j|t_k^j)$, to the plant for $t \in [t_k^j, \, t_k^{j+1}]$, measure the output at t_k^{j+1}, ζ_k^{j+1}, and estimate the states of the system at t_k^{j+1} (i.e., obtain $\hat{x}_k^{j+1}$) using the following equations with initial conditions $\hat{x}_k^j$ and $\overline{P}_k^j$:

$$\begin{aligned}\dot{\hat{x}}_k^j(t) &= f(\hat{x}_k^j(t), \, u(t_k^j|t_k^j)) + \overline{P}_k^j(t)^{-1} (C_k^j)' \\ &\quad \cdot (R_v R_v^{'})^{-1} [\zeta_k^j - H(\hat{x}_k^j, \, u(t_k^j|t_k^j))]\end{aligned} \tag{38}$$

with

$$\begin{aligned}\dot{\overline{P}}_k^j(t) &= -\overline{P}_k^j(t) A(\hat{x}_k^j(t)) - A(\hat{x}_k^j(t))' \overline{P}_k^j(t) \\ &\quad - \overline{P}_k^j(t) R_w R_w^{'} \overline{P}_k^j(t) + (Q_k^j)' Q_k^j, \\ &\quad t \in [t_k^j, \, t_k^{j+1}].\end{aligned} \tag{39}$$

Result of the estimation: $\hat{x}_k^{j+1} := \hat{x}_k^j(t_k^{j+1})$ and $\overline{P}_k^{j+1} := \overline{P}_k^j(t_k^{j+1})$.

`Step 3:` If $j < P-1$, set $j = j+1$ and go back to `Step 1`. If $j = P-1$ set $\hat{x}_{k+1}^0 =: \hat{x}_{k+1} = \hat{x}_k^P$, $k = k+1$, $j = 0$, and go back to `Step 1`

where we have used the notation $\zeta_k := \zeta_k^0$ and $\overline{P}_0 := \overline{P}_0^0$.

C. Stability Analysis of CNTMPC in Section IV

In the state feedback case, the state evolution of the model used in the prediction step of the CNTMPC algorithm at time step k is given by

$$x_k(t) = x_k + \int_{t_k;\, x_k}^{t} f(x_k(\tau),\, u_k(\tau))\, d\tau, \quad \forall\, t \in [t_k,\, t_{k+1}] \tag{40}$$

where the subscript of the integral means that the integration of the function $f(x_k(\tau), u_k(\tau))$ starts from t_k with initial states x_k for evaluation of the state trajectory.

In the output feedback case, the trajectory of the model is given by

$$x_k(t) = \hat{x}_k + \int_{t_k;\, \hat{x}_k}^{t} f(x_k(\tau),\, u_k(\tau))\, d\tau, \quad \forall\, t \in [t_k,\, t_{k+1}]. \tag{41}$$

The difference between the two model dynamics can be represented by an additive disturbance, i.e., the output feedback case is equivalent to the state feedback case modified to

$$\dot{x}_k(t) = f(x_k(t),\, u_k(t)) + d_k(t) \quad \text{with } x_k(t_k) = x_k. \tag{42}$$

If $d_k(t) = d_k =$ constant for $t \in [t_k,\, t_k + T]$, integration of (42) results in

$$x_k(t) = x_k + \int_{t_k;\, x_k}^{t} f(x_k(\tau),\, u_k(\tau))\, d\tau + (t - t_k)\, d_k. \tag{43}$$

Thus, we want to compute d_k so that it represents the difference in the dynamic behavior of the model caused by the estimation, i.e., the states in (43) have to be equal to the states in (41) at $t = t_k + T$. Thus, by subtracting (41) from (43) and evaluating at $t = t_k + T$, we have

$$d_k T = \hat{x}_k - x_k + \int_{t_k;\, \hat{x}_k}^{t_k+T} f(x_k(\tau),\, u_k(\tau))\, d\tau - \int_{t_k;\, x_k}^{t_k+T} f(x_k(\tau),\, u_k(\tau))\, d\tau. \tag{44}$$

Therefore, since $e_k = x_k - \hat{x}_k$ we obtain

$$d_k = \frac{-e_k + \overline{F}(\hat{x}_k + e_k, u_k) - \overline{F}(\hat{x}_k, u_k)}{T} \tag{45}$$

where the function $\overline{F} : \Re^n \times \Re^m \to \Re^n$ is such that

$$\int_{t_1}^{t_2} f(x(\tau),\, u(\tau))\, d\tau = \overline{F}(x(t_2),\, u(t_2)) - \overline{F}(x(t_1),\, u(t_1)). \tag{46}$$

If $\overline{F}$ is Lipschitz continuous, i.e., if there exists $\tilde{L}_F \in [0, \infty)$ such that

$$\|\overline{F}(x_1,\, u) - \overline{F}(x_2,\, u)\| \le \tilde{L}_F \|x_1 - x_2\|, \quad \forall\, x_1,\, x_2 \in B_\rho, \text{ and } u \in \mathcal{U} \tag{47}$$

then, from (45), we have the following bound:

$$\|d_k\| \le \hat{L}_F \|e_k\| \quad \text{with } \hat{L}_F := \frac{\tilde{L}_F + 1}{T}. \tag{48}$$

From the results of *Theorem 5*, we can then find $L_F \in [0, \infty)$ such that

$$\|d_k\| \le L_F \frac{\|\overline{P}_k^{1/2} e_k\|}{\overline{p}^{1/2}} \le L_F \frac{\|\overline{P}_0^{1/2}\|}{\overline{p}^{1/2}} \|e_0\| =: \overline{\rho}, \quad \forall\, k \in \mathcal{Z}_+. \tag{49}$$

So, onto this paper's main result.

Theorem 6 (Closed-loop Stability with Output Feedback): Let $\overline{\rho}, \rho \in (0, \infty)$ be as defined in (49) and *Assumption 3*, respectively. Let $x_0 \in B_{\rho_0}$, with $\rho_0 := \rho - (\overline{\rho} P T e^{\mathrm{LPT}}/(1-\alpha))$, and let the initial estimate of the states $\hat{x}_0$ be such that e_0 and the functions F and H [i.e., the system dynamics after implementation of the control law—see equation (22)] satisfy condition (31) for the chosen observer parameters $\overline{P}_0, R_v, R_w$. Then, if $\overline{\rho} < (\rho(1-\alpha)/PTe^{\mathrm{LPT}})$, the control problem is well-posed (i.e., $x_k, \hat{x}_k, \overline{x}_k \in B_\rho, \ \forall\, k \in \mathcal{Z}_+$), the observer produces asymptotically convergent estimates and the resulting closed-loop system is uniformly asymptotically stable.

Proof: The proof follows straightforwardly from *Theorems 3–5* (and the assumptions made in their derivations) and (49). □

V. Algorithm Implementation

The CNTMPC algorithm was implemented using a preliminary version of the "mpc" package in MATLAB written as the result of a semester thesis developed in the Institut für Automatik at the Eidgenössische Technische Hochschule-Zürich (ETH-Zürich). This package is a combination of the well-known codes DASSL ([59]) and NPSOL ([31]) in MATLAB. DASSL is used for integration of the sets of algebraic and ordinary differential equations which describe the nonlinear dynamics of the model and the plant and NPSOL is used for solving the nonlinear optimization problem.

VI. Example 1

Example 1 has been widely used in the literature (see, e.g., [25] and [27]) for testing control schemes due to its underlying dynamic characteristics, which give rise to well-known control difficulties.

The objective is to exemplify the implementation of CNTMPC and to show the stabilizing effect of the contractive constraint and how it influences the closed-loop performance. For the purpose of comparison, we will also show simulations with a standard nonlinear MPC controller—STNMPC (which is equivalent to CNTMPC without the contractive constraint or with $\alpha \to \infty$).

A. Reactor Dynamics

The process consists of an ideal continuous stirred tank reactor (CSTR) where the reversible exothermic reaction $A \rightleftharpoons R$ is carried out. The reactor is modeled by the following differential equations:

$$\frac{dC_A}{dt} = \frac{C_{Ai} - C_A}{\tau} - k_1 C_A + k_{-1} C_R \tag{50}$$

$$\frac{dC_R}{dt} = \frac{C_{Ri} - C_R}{\tau} + k_1 C_A - k_{-1} C_R \tag{51}$$

$$\frac{dT_r}{dt} = \frac{-\Delta H}{\rho C_p}[k_1 C_A - k_{-1} C_R] + \frac{T_i - T_r}{\tau}. \tag{52}$$

The states C_A, C_R, and T_r represent the concentrations of A and R and the temperature in the reactor, respectively. The manipulated variable is the feed temperature T_i and the controlled variable is C_R. The reaction rate expressions are $k_1 := K_1 e^{-(Q_1/\mathcal{R}T_R)}$ and $k_{-1} := K_{-1} e^{-(Q_{-1}/\mathcal{R}T_R)}$ and the specific parameters and operating conditions used in our simulations are shown in Table I.

The reactor equilibrium conversion as a function of temperature has a well-defined maximum (see Fig. 4) at which the steady-state gain is zero (for a detailed explanation of this condition see [27]). Thus, because of the conversion maximum, the system gain changes sign from one side of the maximum to the other.

TABLE I
PARAMETERS AND OPERATING CONDITIONS FOR THE REACTOR

$\tau = 60$ s (time constant for the reactor)
$\mathcal{R} = 1.987$ cal mol^{-1} K^{-1} (Arrehnius constant)
$K_1 = 5 \times 10^3$ s^{-1} (kinetic constant of the forward reaction)
$K_{-1} = 1 \times 10^6$ s^{-1} (kinetic constant of the backward reaction)
$Q_1 = 1 \times 10^4$ cal mol^{-1} (kinetic factor of the forward reaction)
$Q_{-1} = 1.5 \times 10^4$ cal mol^{-1} (kinetic factor of the backward reaction)
$\Delta H = -5 \times 10^3$ cal mol^{-1} (enthalpy of reaction)
$\rho = 1$ kg/l (density)
$C_p = 1 \times 10^3$ cal kg^{-1} K^{-1} (specific heat capacity)
$C_{Ai} = 1$ mol/l (feed concentration of species A)
$C_{Ri} = 0$ mol/l (feed concentration of species R)

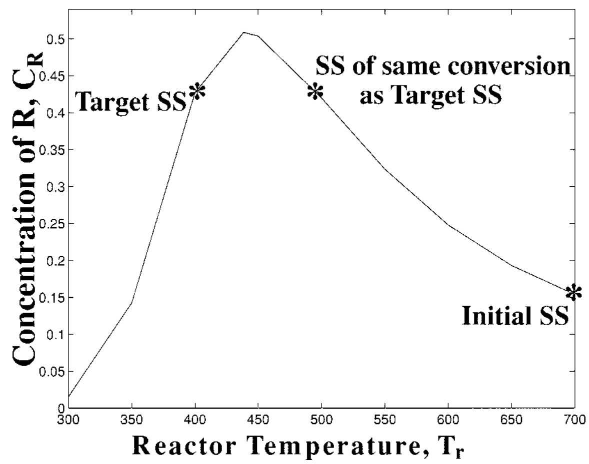


Fig. 4. Equilibrium diagram for the reactor (SS = steady state).

TABLE II
COORDINATES OF THE INITIAL AND TARGET STEADY STATES

State and input coordinates of the initial steady state			
$C_A = 0.8461$ mol/l	$C_R = 0.1539$ mol/l	$T_r = 700$ K	$T_i = 699.2305$ K
State and input coordinates of the target steady state			
$C_A = 0.5729$ mol/l	$C_R = 0.4271$ mol/l	$T_r = 400$ K	$T_i = 397.8644$ K

B. Simulation Results

The simulations shown here represent a steady-state change from an equilibrium point of low conversion and high temperature (located to the right side of the point of maximum conversion on the equilibrium curve) to a target equilibrium point of lower temperature and significantly higher conversion (located to the left of the point of maximum conversion), a much more desirable operating condition. The state and input coordinates of these two equilibria are shown in Table II, and they are also marked on the equilibrium curve in Fig. 4.

The simulation results for different values of the contractive parameter α are shown in Fig. 5.

The controller parameters used in all the simulations are given in Table III.

For $\alpha = 0$ the resulting controller becomes a "dead-beat" unstable controller where a very large control action is generated and the algorithm stops at the first time step. For $\alpha \in (0, 1)$ the control problem is feasible at all time steps and the closed-loop system is stable. As shown in Fig. 5, the performance is improved for smaller values of α. For this example, if $\alpha = 0.9$, the input variable T_i quickly reaches its desired steady-state value and remains there. The system is basically left in open loop for values of α in the range $\alpha \in [0.9, 1)$.

Again from Fig. 5, we notice that simulations with STNMPC (equivalent to CNTMPC with $\alpha \to \infty$) show a very aggressive and large control action with the output C_R reaching its setpoint value but the input temperature T_i and the reactor temperature T_r displaying large offsets. This occurrence can be easily understood since from the equilibrium curve in Fig. 4, we can see that there are always two steady states corresponding to the same conversion C_R (except for the maximum conversion), one of low and another of higher temperature. Because neither the input T_i nor the reactor temperature T_r are weighted in the objective function (as we can see from Table III), and the contractive constraint is not present to stabilize the reactor temperature deviation with respect to the desired steady state value to zero, the controller simply drives the system to a steady state of same target conversion C_R but of much higher temperature (about 100 K) than the desired one. This point is also marked on the equilibrium curve in Fig. 4.

The results with $\alpha \geq 1$ are equivalent to the ones obtained with STNMPC, i.e., the contractive constraint is no longer active for $\alpha \geq 1$. Thus, we have demonstrated that for the given choice of control parameters (output weight only and $P = M$, which

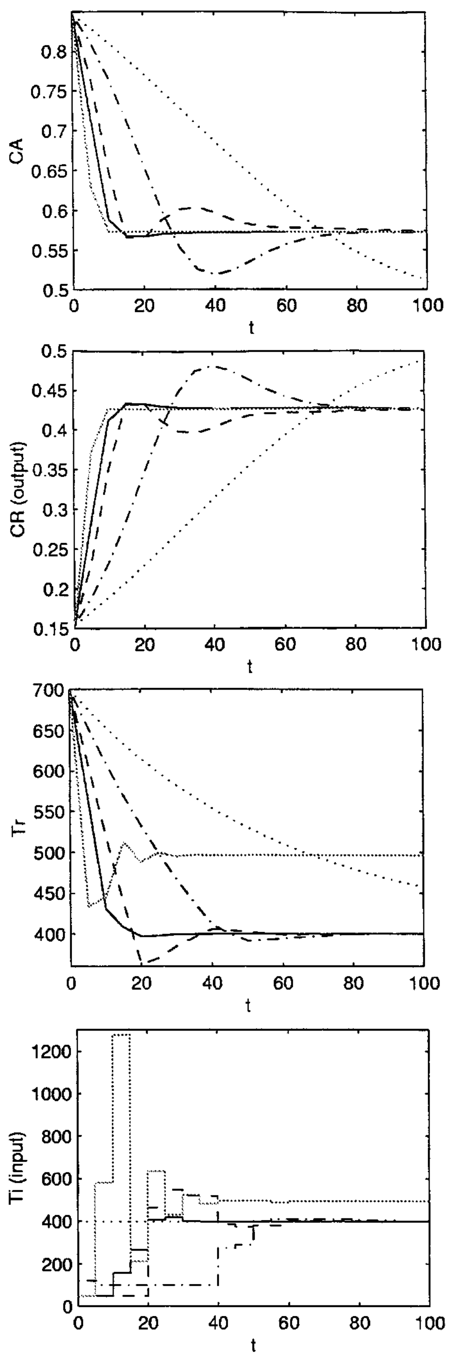


Fig. 5. State and control responses for CNTMPC with different values of α: $\alpha = 0.1$ (solid), $\alpha = 0.4$ (dashed), $\alpha = 0.7$ (dashdot), $\alpha = 0.9$ (dotted), $\alpha \geq 1$ (solid, gray).

TABLE III
CONTROLLER PARAMETERS FOR SIMULATIONS IN Fig. 5

$Q = \mathrm{diag}([0\ 1\ 0])$	$R = 0$	$S = 0$	$T = 5$
$P = 2$	$M = 2$	$u_{min} = 50$	$u_{max} = \infty$

means perfect inversion of the system dynamics for $\alpha \to \infty$), STNMPC drives the system to a steady state of unnecessarily large temperatures while the implementation of the contractive constraint introduces stability guarantees to the target steady state.

We do not mean to say here that STNMPC is unable to provide satisfactory performance under a different choice of controller parameters. In fact, if $P = 2$ and $M = 1$, using the same weights in the objective function, offset in T_r is no longer observed. Thus, what we are trying to show with these results is that while an inadequate choice of controller parameters leads to an unacceptable response by STNMPC, the contractive constraint compensates for this and guarantees that the states not present in the objective function still reach their desired steady-state values without degradation of the output response. Thus, the parameter choice was deliberate to show a case in which the contractive constraint has decisive stabilizing and performance improving effects on the closed loop.

VII. EXAMPLE 2

Example 2 is a nonholonomic system, which is the model of a car with no trailers. This model violates Brockett's necessary condition for smooth or even continuous stabilization [16], and that is what makes the control design problem for this system (and nonholonomic systems, in general) a real challenge. Since MPC can automatically generate a discontinuous control law, we expect this controller to be suited for the class of nonholonomic systems. Moreover, this system is not controllable on the manifold of its equilibrium points, which also represents a challenge from the control point of view. We will see shortly what difficulties are encountered by CNTMPC due to this fact.

In practice, the obstacle for on-line implementation of computationally demanding controllers like CNTMPC to systems with "fast" dynamics such as the car is naturally computational. So, the reader should examine the simulation results which follow keeping this into perspective. In order to sidestep this practical problem in the implementation of nonlinear MPC schemes (especially in the case of "fast" dynamical systems), some approximate methods have been proposed. Some of the ideas are to use a local linear approximation of the nonlinear system in the prediction step of the MPC algorithm [22], to employ a neural net to simulate the closed-loop behavior and implement the nonlinear MPC control law off-line [58], or to use a combination of the input/output feedback linearization and MPC techniques [23], [56].

A. Car Dynamics

This system can be represented by the following set of equations:

$$\begin{aligned}\dot{x} &= \cos\theta \cdot v\\ \dot{y} &= \sin\theta \cdot v\\ \dot{\theta} &= w\end{aligned} \tag{53}$$

where

- (x, y) represents the Cartesian position of the center of mass of the car;
- θ is the inclination of the car with respect to the horizontal axis;
- v and w are its forward and angular velocities, respectively.

The coordinate system for the car is illustrated in Fig. 6.

The inputs determined by the control law are v and w and the outputs are the state variables x, y and θ. The objective is to

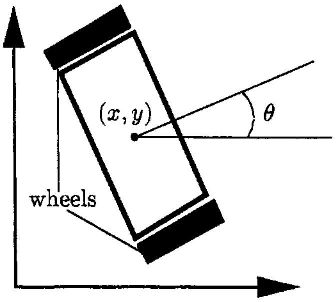


Fig. 6. Coordinate system for the car.

drive the system from any given initial condition to the origin with a satisfactory level of performance.

B. Simulation Results

1) Comparison Between CNTMPC and Astolfi's Discontinuous Controller (Unconstrained Case): Here we will compare CNTMPC with Astolfi's discontinuous controller [6], [7], an analytic control design devoted especially to stabilization of nonholonomic systems.

Fig. 7 shows the resulting paths in the xy-plane of the controlled car using CNTMPC and Astolfi's discontinuous controller (we have reproduced the plot for Astolfi's controller from page 36 in [6] with the same control gains).

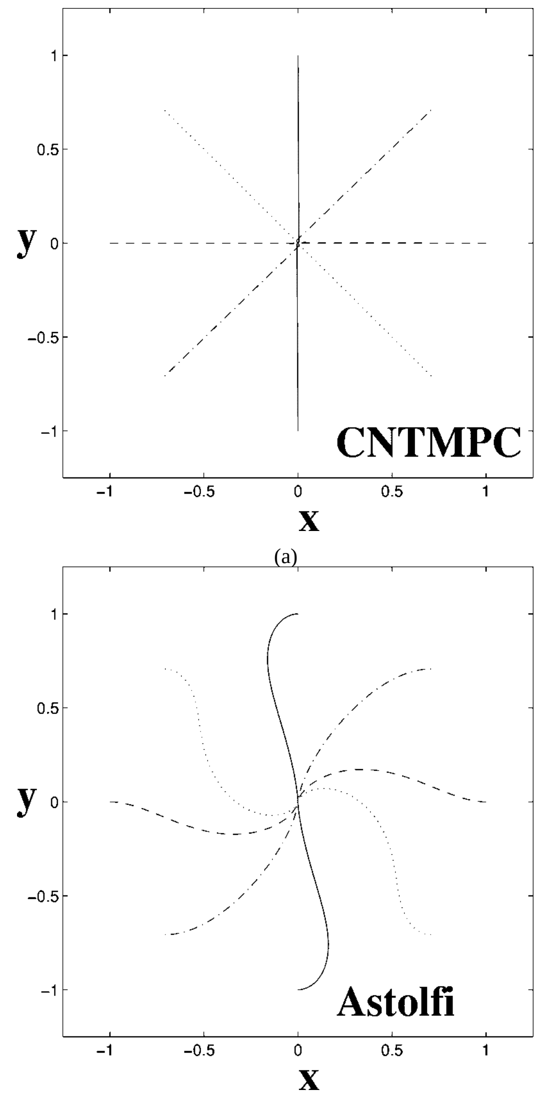
The CNTMPC controller parameters used in these simulations are given in Table IV.

We should emphasize that the time taken by Astolfi's discontinuous controller to compute these trajectories is less than a second, while CNTMPC took between 9–12 min on average (using the preliminary version of the "mpc" package in MATLAB discussed in Section V).

Moreover, because of the lack of controllability of this system at the origin, the CNTMPC algorithm is only able to drive the system to a very close neighborhood of the origin and then it stops (this effect cannot be really noticed in Fig. 7 due to scales). What happens is that, once the car is driven very near to the origin, the control action generated in the optimization step is very large—due to the lack of controllability—and the controller is stopped.

We can see from Fig. 7 that for both controllers the car performs its maneuver toward the origin of the coordinate system in a very natural way and without ever inverting its motion. Hence, the floor trajectories do not contain any cusps. This response can be anticipated for the analytic discontinuous controller because the control signal v is constructed to always have a constant sign. In the case of CNTMPC, the controller just automatically generates such a response.

We also observe that CNTMPC generates trajectories which approach the origin in an almost straight path. It is clear that the analytic discontinuous controller cannot match this performance. This is not surprising since the construction of the analytic controller does not take into account performance but only stabilization. CNTMPC, on the other hand, minimizes a performance criterion at every time step therefore improving the closed-loop response.



(a)

(b)

Fig. 7. Resulting paths in the xy-plane using CNTMPC and Astolfi's discontinuous controller when the car is initially on the unit circle and parallel to the x-axis.

TABLE IV
CONTROLLER PARAMETERS FOR SIMULATIONS IN Fig. 7

$Q = \mathrm{diag}([1\ 1\ 0])$	$R = 0$	$S = 0$	$T = 0.1$
$P = 5$	$M = 3$	$\alpha = 0.9$	

2) Comparison Between CNTMPC and Some Classic Controllers (Constrained Case): Here we want to compare the constrained closed-loop response generated by CNTMPC with some classic analytic control design techniques for nonholonomic systems. These techniques do not take into account process constraints but, since the control response for the given initial condition remains between the bounds used in the simulations with CNTMPC, the comparison is fair.

The simulation results are shown in Fig. 8.

The controller parameters used in the simulations with CNTMPC are shown in Table V.

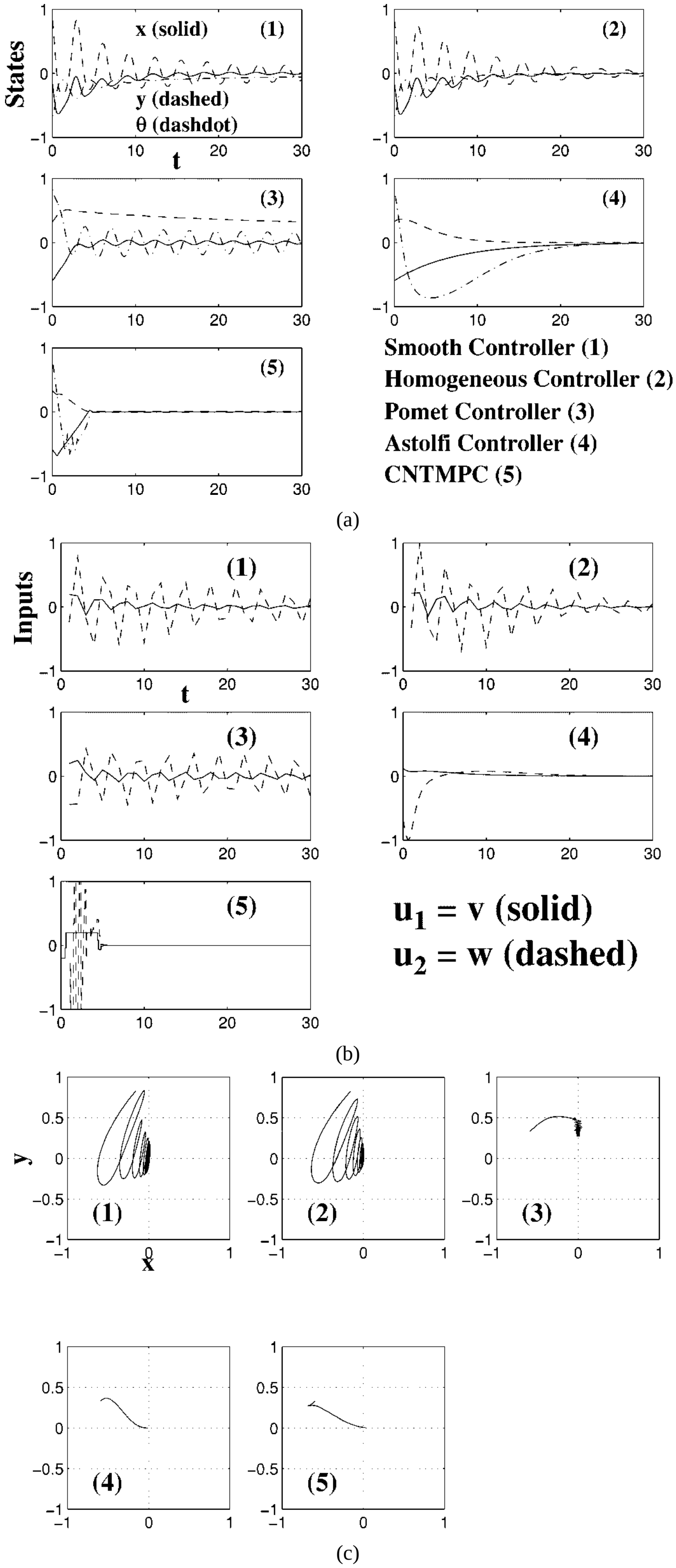


Fig. 8. Comparison of CNTMPC with classic controllers for nonholonomic systems.

The initial condition chosen was used in [49] by M'Closkey in his experiments at the Department of Mechanical Engineering at Caltech in 1993 and is given in Table VI.

From Fig. 8, we can see that the smooth [68] and Pomet's controllers [60] are not able to stabilize the car. The homogeneous controller [49] performs better than the two previous controllers but once again the states oscillate indefinitely. Astolfi's discontinuous controller [6], [7] is undoubtedly the best amongst these four analytic controllers, and it can actually stabilize the system to the origin without oscillations. However, the comparison with CNTMPC shows that the response time is five times longer within approximately the same control bounds.

VIII. Conclusions

In this paper, we have proposed a finite horizon MPC scheme with an end constraint, which has been denoted *contractive MPC*. We have shown that the addition of the contractive constraint to the optimal control problem introduces strong stability guarantees in both the state and output feedback nominal cases. Furthermore, we have shown that contractive MPC is easily implementable by applying it to a challenging chemical process example and to a nonholonomic system.

Appendix A
Proofs for Section II

Proof of Theorem 1: From *Assumption 3*, we have that if $x_0 \in B_\rho$, the optimal control problems $\mathcal{P}(t_k, x_k), \forall k \in \mathcal{Z}_+$, are feasible. Thus, it follows that

$$\|x_k\|_{\hat{P}} \leq \alpha^k \|x_0\|_{\hat{P}}, \quad \forall k \in \mathcal{Z}_+ \tag{54}$$

and, therefore, from *Assumption 4*, $x_k(t)$ satisfies the following inequality:

$$\|x_k(t)\|_{\hat{P}} \leq \beta\alpha^k \|x_0\|_{\hat{P}}, \, t \in [t_k, t_{k+1}], \quad \forall k \in \mathcal{Z}_+. \tag{55}$$

Since $e^{(\alpha-1)} - \alpha \geq 0 \Longleftrightarrow \alpha^k \leq e^{-(1-\alpha)k}, \forall \alpha \in [0, 1)$ and $\forall k \in \mathcal{Z}_+$, it results that

$$\|x_k\|_{\hat{P}} \leq \|x_0\|_{\hat{P}} e^{-(1-\alpha)k}$$

and

$$\|x_k(t)\|_{\hat{P}} \leq \beta \|x_0\|_{\hat{P}} e^{-(1-\alpha)k}. \tag{56}$$

Although (56) establishes an exponentially discretely decaying bound on the states for all times $t \geq t_0$, our proof of exponential stability for system (1) is not yet concluded. In order to satisfy the condition for exponential stability for continuous-time systems, we must find an exponentially continuously decaying function that bounds $\beta e^{-(1-\alpha)k}$ for all $t \in [t_k, t_{k+1}]$, $k \in \mathcal{Z}_+$.

The discrete bounds on the states (56) and the continuous upper bound are graphically represented in Fig. 9.

So, as shown in Fig. 9, we want to find the least conservative continuously exponentially decreasing bound that matches the discrete bound exactly at the end of horizons.

Since $k = ((t_k - t_0)/PT)$ and $((t - t_0)/PT) < ((t_k - t_0)/PT), \forall t \in [t_0, t_k)$, we can easily see that

$$\beta e^{-(1-\alpha)k} \leq a e^{-(1-\alpha)((t-t_0)/PT)}, \quad \forall a \geq \beta e^{(1-\alpha)}. \tag{57}$$

Thus, using (56) and (57), we finally have

$$\|x(t)\| \leq a\|x_0\| e^{-(1-\alpha)((t-t_0)/PT)}, \qquad \text{with } a \geq \beta e^{(1-\alpha)} \tag{58}$$

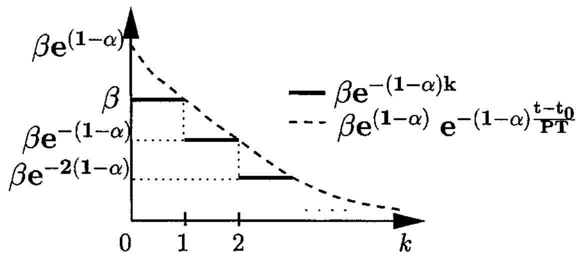


Fig. 9. Discrete- and continuous-time exponentially decaying upper bounds for the state trajectory.

TABLE V
CNTMPC CONTROLLER PARAMETERS FOR SIMULATIONS IN Fig. 8

$Q = \text{diag}([1\ \ 8\ \ 0.1])$	$R = 0.01\, I_m$	$S = 0$	$T = 0.1$
$P = 20$	$M = 6$	$\alpha = 0.9$	
$u_{min} = [-0.2\quad -1.0]$		$u_{max} = [0.2\quad 1.0]$	

TABLE VI
INITIAL CONDITION FOR SIMULATIONS IN Fig. 8

$x_0 = -0.5945$	$y_0 = 0.3299$	$\theta_0 = 0.8262$

which is what we wanted to prove. This means that if $x_0 \in B_\rho$, then the sequence of optimal control problems $\{\mathcal{P}(t_k, x_k)\}_{k\in\mathcal{Z}_+}$ is feasible and the origin is an exponentially stable equilibrium point inside the reachable set $\mathcal{X} \subset B_{\beta\rho}$. $\square$

Proof of Theorem 2: The proof is constructive, i.e., we will compute the constants $c, d, e, l > 0$ such that the statement of the theorem holds.

- *Upper Bound on* $V(t_k, x_k)$: This means computing a value for $d > 0$. Using *Assumption 4*, it results that

$$V(t_k, x_k) \le \frac{\lambda_{\max}(Q)}{\lambda_{\min}(\hat{P})}\beta^2 PT\|x_k\|_{\hat{P}}^2 + \int_{t_k}^{t_{k+1}} \|u_k(t)\|_{R^*}^2\, dt. \tag{59}$$

Since $\|u_k(t)\| = \|\eta(x_k(t))\| \le L_\eta\|x_k(t)\|$, $\forall t \in [t_k, t_{k+1}]$ (this inequality follows from *Assumption 5*), and using *Assumption 4*, we have

$$V(t_k, x_k) \le \frac{\beta^2 PT}{\lambda_{\min}(\hat{P})}[\lambda_{\max}(Q) + L_\eta^2\lambda_{\max}(R^*)] =: d\|x_k\|_{\hat{P}}^2. \tag{60}$$

- *Upper Bound on the Gradient of* $V(t_k, x_k)$, $\dot{V}(t_k, x_k)$: Taking the derivative from (13) we have

$$\begin{aligned}\dot{V}(t_k, x_k) = &\|x_k(t_k + PT)\|_Q^2 - \|x_k\|_Q^2\\ &+ \|u_k(t_k + PT)\|_{R^*}^2 - \|u_k\|_{R^*}^2.\end{aligned} \tag{61}$$

Using our index notation for the control law, it follows that

$$\begin{aligned}\dot{V}(t_k, x_k) = &\|x_{k+1}\|_Q^2 - \|x_k\|_Q^2 + \|u(t_k^{P-1}|t_k)\|_{R^*}^2\\ &- \|u(t_{k-1}^{P-1}|t_{k-1})\|_{R^*}^2.\end{aligned} \tag{62}$$

Due to *Assumption 7*, we get

$$\begin{aligned}\dot{V}(t_k, x_k) \le &-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\hat{P})}(1 - \overline{\alpha}^2)\|x_k\|_{\hat{P}}^2 + \|u(t_k^{P-1}|t_k)\|_{R^*}^2\\ &- \|u(t_{k-1}^{P-1}|t_{k-1})\|_{R^*}^2.\end{aligned} \tag{63}$$

Now, if $u(t_k^i|t_k) = 0, \forall i = M, \ldots, P-1$, and $M < P$, then it immediately follows from (63) that

$$\begin{aligned}&\dot{V}(t_k, x_k)\\ &\quad\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\hat{P})}(1 - \overline{\alpha}^2)\|x_k\|_{\hat{P}}^2 - \|u(t_{k-1}^{P-1}|t_{k-1})\|_{R^*}^2\\ &\quad\le -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(\hat{P})}(1 - \overline{\alpha}^2)\|x_k\|_{\hat{P}}^2\\ &\quad=: -e\|x_k\|_{\hat{P}}^2.\end{aligned} \tag{64}$$

- *Upper Bound on* $(\partial V/\partial x_k)(t_k, x_k)$: From (13), we have

$$\begin{aligned}&\frac{\partial V}{\partial x_k}(t_k, x_k)\\ &= \int_{t_k}^{t_{k+1}} \left[\frac{d\|x_k(t)\|_Q^2}{dx}\frac{\partial x_k(t)}{\partial x_k} + \frac{d\|u_k(t)\|_{R^*}^2}{du}\frac{\partial u_k(t)}{\partial x_k}\right] dt.\end{aligned} \tag{65}$$

Thus

$$\begin{aligned}&\left\|\frac{\partial V}{\partial x_k}(t_k, x_k)\right\|_{\hat{P}}\\ &\le \lambda_{\max}(\hat{P})\int_{t_k}^{t_{k+1}}\left[\frac{d\|x_k(t)\|_Q^2}{dx}\left\|\frac{\partial x_k(t)}{\partial x_k}\right\|\right.\\ &\qquad\left. + \frac{d\|u_k(t)\|_{R^*}^2}{du}\left\|\frac{\partial u_k(t)}{\partial x_k}\right\|\right] dt.\end{aligned} \tag{66}$$

Then, after some algebraic manipulations, it results that

$$\begin{aligned}&\left\|\frac{\partial V}{\partial x_k}(t_k, x_k)\right\|_{\hat{P}}\\ &\le 2\lambda_{\max}(\hat{P})\int_{t_k}^{t_{k+1}}\left[\|x_k(t)\|_Q e^{\xi(t-t_k)}\right.\\ &\qquad\left. + \|u_k(t)\|_{R^*}\frac{\lambda_{\max}^{1/2}(\hat{P})}{\lambda_{\min}^{1/2}(\hat{P})}L_\eta\beta\right] dt\\ &\le 2\beta\frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}^{1/2}(\hat{P})}\left[\sqrt{\frac{\lambda_{\max}(R^*)\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})}}L_\eta^2\beta PT\right.\\ &\qquad\left. + \lambda_{\max}^{1/2}(Q)\int_{t_k}^{t_{k+1}} e^{\xi(t-t_k)}\, dt\right]\|x_k\|_{\hat{P}}\\ &\le 2\beta\frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}^{1/2}(\hat{P})}\left[\sqrt{\frac{\lambda_{\max}(R^*)\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})}}L_\eta^2\beta PT\right.\\ &\qquad\left. + \lambda_{\max}^{1/2}(Q)\frac{(e^{\xi PT} - 1)}{\xi}\right]\|x_k\|_{\hat{P}}\\ &=: l\|x_k\|_{\hat{P}}.\end{aligned} \tag{67}$$

This follows from

1)

$$x_k(t) = x_k + \int_{t_k}^{t} f(x_k(\tau), u_k(\tau))\, d\tau = x_k + \int_{t_k}^{t} F(x_k(\tau))\, d\tau \tag{68}$$

which means that

$$\frac{\partial x_k(t)}{\partial x_k} = 1 + \int_{t_k}^{t} \frac{dF(x_k(\tau))}{dx} \frac{\partial x_k(\tau)}{\partial x_k}\, d\tau. \tag{69}$$

Thus, using *Assumption 6*, we have

$$\left\| \frac{\partial x_k(t)}{\partial x_k} \right\| \le 1 + \xi \int_{t_k}^{t} \left\| \frac{\partial x_k(\tau)}{\partial x_k} \right\| d\tau. \tag{70}$$

Using the Bellman–Grownwall (BG) inequality, it results that

$$\left\| \frac{\partial x_k(t)}{\partial x_k} \right\| \le e^{\xi(t-t_k)}. \tag{71}$$

2) Since $u_k(t) = \eta(x_k(t)), \forall\, t \in [t_k, t_{k+1}]$, and $\eta(.)$ satisfies *Assumption 5*, we have

$$\begin{aligned} \|u_k(t)\| &\le L_\eta \|x_k(t)\| \\ &\le \frac{L_\eta \beta}{\lambda_{\min}^{1/2}(\hat{P})} \|x_k\|_{\hat{P}} \\ &\le \frac{\lambda_{\max}^{1/2}(\hat{P})}{\lambda_{\min}^{1/2}(\hat{P})} L_\eta \beta \|x_k\| \Longrightarrow \left\| \frac{\partial u_k(t)}{\partial x_k} \right\| \\ &\le \frac{\lambda_{\max}^{1/2}(\hat{P})}{\lambda_{\min}^{1/2}(\hat{P})} L_\eta \beta. \end{aligned} \tag{72}$$

- *Lower Bound on* $V(t_k, x_k)$: As a result of *Assumption 6*, we have

$$\begin{aligned} \|x_k(t)\|_{\hat{P}} &\ge \|x_k\|_{\hat{P}} - \xi \int_{t_k}^{t} \|x_k(\tau)\|_{\hat{P}}\, d\tau \\ &\ge [1 - \xi\beta(t-t_k)] \|x_k\|_{\hat{P}}. \end{aligned} \tag{73}$$

It then follows, for example, that

$$\|x_k(t)\|_{\hat{P}} \ge \frac{\|x_k\|_{\hat{P}}}{2} \qquad \text{for } t \in \left[t_k,\, t_k + \frac{1}{2\xi\beta}\right]. \tag{74}$$

Thus, we have two cases to consider.

1)

$$t_{k+1} \le t_k + \frac{1}{2\xi\beta} \Longleftrightarrow PT \le \frac{1}{2\xi\beta.}$$

In this case

$$\begin{aligned} V(t_k, x_k) &\ge \int_{t_k}^{t_{k+1}} \left(\frac{\lambda_{\min}(Q)}{4\lambda_{\max}(\hat{P})} \|x_k\|_{\hat{P}}^2 + \|u_k(t)\|_{R^*}^2 \right) dt \\ &\ge \frac{\lambda_{\min}(Q) PT}{4\lambda_{\max}(\hat{P})} \|x_k\|_{\hat{P}}^2. \end{aligned} \tag{75}$$

2)

$$t_{k+1} \ge t_k + \frac{1}{2\xi\beta} \Longleftrightarrow PT \ge \frac{1}{2\xi\beta.}$$

In this case

$$\begin{aligned} V(t_k, x_k) &\ge \int_{t_k}^{t_k + (1/2\xi\beta)} \left(\frac{\lambda_{\min}(Q)}{4\lambda_{\max}(\hat{P})} \|x_k\|_{\hat{P}}^2 + \|u_k(t)\|_{R^*}^2 \right) dt \\ &\ge \frac{\lambda_{\min}(Q)}{8\xi\beta\lambda_{\max}(\hat{P})} \|x_k\|_{\hat{P}}^2. \end{aligned} \tag{76}$$

Thus, it follows that

$$\begin{aligned} V(t_k, x_k) &\ge \min\left\{ \frac{\lambda_{\min}(Q) PT}{4\lambda_{\max}(\hat{P})},\, \frac{\lambda_{\min}(Q)}{8\xi\beta\lambda_{\max}(\hat{P})} \right\} \|x_k\|_{\hat{P}}^2 \\ &=: c\|x_k\|_{\hat{P}}^2. \end{aligned} \tag{77}$$

So, we have shown that, under the assumptions in the statement of the theorem, the quadratic performance criterion (5) subject to the constraints in (6), with $u(t_k^i | t_k) = 0$ for $i = M, \ldots, P-1$ and $M < P$, is an exponentially decaying Lyapunov function for the closed-loop system.

□

APPENDIX B
PROOFS FOR SECTION III

The following lemma will be very useful for our stability analyses in Sections III and IV.

Lemma 1: Consider the discrete-time linear system

$$z_{k+1} \le a_k z_k + b_k, \quad k \in \mathcal{Z}_+. \tag{78}$$

If $a_k \in [0, 1)$ and $b_k \ge 0, \forall\, k \in \mathcal{Z}_+$, then system (78) is stable in the practical sense, i.e.,

1. $z_k < z_0 + \dfrac{b_{\max}}{1 - a_{\max}} \quad \forall\, k \in \mathcal{Z}_+$
2. $\lim_{k\to\infty} z_k \le \dfrac{b_{\max}}{1 - a_{\max}}$

where $a_{\max} := \max_{k \in \mathcal{Z}_+} a_k$ and $b_{\max} := \max_{k \in \mathcal{Z}_+} b_k$.

Proof: The proof is straightforward and can be found in [22]. □

Proof of Theorem 3: The difference between the dynamics of the plant and of the model used in the computation of the contractive constraint for $t \in [t_k, t_{k+1}]$ is given by

$$\begin{aligned} \dot{x}_k^{p,j}(t) - \dot{\overline{x}}_k^j(t) &= f(x_k^{p,j}(t), u_k^j(t)) \\ &\quad - f(\overline{x}_k^j(t), u_k^j(t)) + d_k^j(t), \\ &\quad \forall\, j = 0, \ldots, P-1. \end{aligned} \tag{79}$$

Then, since the states of this model are updated with the states of the plant at every $t_k, \forall\, k \in \mathcal{Z}_+$, we can integrate (79) for any $j = 0, \ldots, P-1$ (thus we will omit the superscript j) and obtain

$$\begin{aligned} x_k^p(t) - \overline{x}_k(t) &= \int_{t_k}^{t} [f(x_k^p(\tau), u_k(\tau)) - f(\overline{x}_k(\tau), u_k(\tau))]\, d\tau \\ &\quad + \int_{t_k}^{t} d_k(\tau)\, d\tau. \end{aligned} \tag{80}$$

Therefore, using (19) and (20), we have

$$\begin{aligned}&\|x_k^p(t)-\overline{x}_k(t)\|_{\hat{P}}\\&\quad\le\int_{t_k}^{t}\|f(x_k^p(\tau),u_k(\tau))-f(\overline{x}_k(\tau),u_k(\tau))\|_{\hat{P}}\,d\tau\\&\qquad+\int_{t_k}^{t}\|d_k(\tau)\|_{\hat{P}}\,d\tau\\&\quad\le L\int_{t_k}^{t}\|x_k^p(\tau)-\overline{x}_k(\tau)\|_{\hat{P}}\,d\tau+\overline{\rho}_k(t-t_k).\end{aligned}\tag{81}$$

Using the BG inequality, it results that

$$\|x_k^p(t)-\overline{x}_k(t)\|_{\hat{P}}\le\overline{\rho}_kPTe^{\mathrm{LPT}},\quad\forall\,t\in[t_k,t_{k+1}].\tag{82}$$

The contractive constraint imposes that $\|\overline{x}_{k+1}\|_{\hat{P}}\le\alpha\|x_k^p\|_{\hat{P}}$, $\forall\,k\in\mathcal{Z}_+$. Thus, by applying the triangle inequality, we have

$$\|x_{k+1}^p\|_{\hat{P}}\le\alpha\|x_k^p\|_{\hat{P}}+\overline{\rho}_kPTe^{\mathrm{LPT}}.\tag{83}$$

Using *Assumption 8* and (83), it follows that

$$\|x_{k+1}^p\|_{\hat{P}}\le\alpha\|x_k^p\|_{\hat{P}}+\epsilon PTe^{\mathrm{LPT}},\quad\forall\,k=\overline{k},\,\overline{k}+1,\ldots.\tag{84}$$

Then, since $\alpha\in[0,1)$, we can use the results of *Lemma 1* to obtain

$$\begin{aligned}\left\|x_{\overline{k}(\epsilon)+l}^p\right\|_{\hat{P}}&\le\alpha^l\left\|x_{\overline{k}(\epsilon)}^p\right\|_{\hat{P}}+\left(\sum_{i=0}^{l-1}\alpha^i\right)\epsilon PTe^{\mathrm{LPT}}\\&<\alpha^l\left\|x_{\overline{k}(\epsilon)}^p\right\|_{\hat{P}}+\frac{\epsilon PTe^{\mathrm{LPT}}}{1-\alpha},\quad\forall\,l\in\mathcal{Z}_+^*.\end{aligned}\tag{85}$$

Thus, by taking the limit as $\epsilon\to0$, we have

$$\lim_{\epsilon\to0}\left\|x_{\overline{k}(\epsilon)+l}^p\right\|_{\hat{P}}<\alpha^l\lim_{\epsilon\to0}\left\|x_{\overline{k}(\epsilon)}^p\right\|_{\hat{P}}\tag{86}$$

and if we now take the limit as $l\to\infty$ knowing that $\overline{k}(\epsilon)\to\infty$ for $\epsilon\to0$ and that $\alpha^l\to0$ exponentially fast as $l\to\infty$, we finally obtain

$$\begin{aligned}\lim_{l\to\infty}\left[\lim_{\epsilon\to0}\left\|x_{\overline{k}(\epsilon)+l}^p\right\|_{\hat{P}}\right]&<\left(\lim_{l\to\infty}\alpha^l\right)\left[\lim_{\epsilon\to0}\left\|x_{\overline{k}(\epsilon)}^p\right\|_{\hat{P}}\right]\\&=0\end{aligned}$$

or

$$\lim_{k\to\infty}\|x_k^p\|_{\hat{P}}=0\tag{87}$$

which means asymptotic convergence to the origin.

So, we have shown that as long as $x_k^p,\,x_k\in B_\rho$, it follows that $x_k^p\to0$ as $k\to\infty$, which means that the origin is an attractive equilibrium point.

Let us now proceed to show that $(x,u)=(0,0)$ is actually uniformly attractive. From (82), (84), and the triangle inequality, we get

$$\|x_k^p(t)\|_{\hat{P}}\le\|\overline{x}_k(t)\|_{\hat{P}}+\epsilon PTe^{\mathrm{LPT}},\quad\forall\,t\ge t_{\overline{k}}:=t_{\overline{k}(\epsilon)}.\tag{88}$$

Using *Assumption 4*, for each $x_0^p\in B_\rho$, $t_0>0$, we have

$$\begin{aligned}&\|x^p(t-t_0,x_0^p,u,d)\|_{\hat{P}}\\&\quad\le\beta\rho+\epsilon PTe^{\mathrm{LPT}}=:\epsilon^*,\\&\qquad\forall\,t\ge t_{\overline{k}(\epsilon)}=:\overline{T}(\epsilon^*)+t_0.\end{aligned}\tag{89}$$

This means that the equilibrium $(x,u)=(0,0)$ is *uniformly attractive*.

Let us now proceed to show that $(x,u)=(0,0)$ is also uniformly stable. Using (82) and the triangle inequality, we get

$$\|x_k^p(t)\|_{\hat{P}}\le\|\overline{x}_k(t)\|_{\hat{P}}+\overline{\rho}_kPTe^{\mathrm{LPT}}.\tag{90}$$

From *Assumption 4* and (90) it results that

$$\|x_k^p(t)\|_{\hat{P}}\le\beta\rho+\overline{\rho}PTe^{\mathrm{LPT}}=:\overline{\epsilon}\tag{91}$$

where $\overline{\rho}=\max_{k\in\mathcal{Z}_+}\overline{\rho}_k$.

Therefore, from (91), it follows that $\rho=((\overline{\epsilon}-\overline{\rho}PTe^{\mathrm{LPT}})/\beta)=:\delta(\overline{\epsilon})$.

With these definitions, we finally arrive at the following result. For

$$\begin{aligned}&\|x_k^p\|_{\hat{P}}<\rho=:\delta(\overline{\epsilon})t_k\ge0\\&\qquad\Longrightarrow\|x^p(t-t_k,x_k,u,d)\|_{\hat{P}}<\overline{\epsilon},\\&\qquad\forall\,t\in[t_k,t_{k+1}],\quad\forall\,k\in\mathcal{Z}_+\end{aligned}\tag{92}$$

it follows that $(x,u)=(0,0)$ is a *uniformly stable* equilibrium point of the closed-loop system.

Since we have shown that $(x,u)=(0,0)$ is both uniformly attractive and uniformly stable, we finally conclude that (0, 0) is a *uniformly asymptotically stable* equilibrium point. □

Proof of Theorem 4: In the proof of *Theorem 3*, we have derived that

$$\|x_{k+1}^p\|_{\hat{P}}\le\alpha\,\|x_k^p\|_{\hat{P}}+\overline{\rho}_kPTe^{\mathrm{LPT}}\le\alpha\,\|x_k^p\|_{\hat{P}}+\overline{\rho}PTe^{\mathrm{LPT}}.\tag{93}$$

By defining $z_k:=\|x_k^p\|_{\hat{P}}$, $a:=\alpha$, and $b:=\overline{\rho}PTe^{\mathrm{LPT}}$ and using *Lemma 1*, we get

$$\|x_k^p\|_{\hat{P}}<\|x_0^p\|_{\hat{P}}+\frac{\overline{\rho}PTe^{\mathrm{LPT}}}{1-\alpha}.\tag{94}$$

Now, if $x_0^p\in B_{\rho_0}$, the application of *Control Algorithm 2* to system (14) assures that the states at the end of prediction horizons are bounded by

$$\|x_k^p\|_{\hat{P}}<\rho_0+\frac{\overline{\rho}PTe^{\mathrm{LPT}}}{1-\alpha}.\tag{95}$$

Therefore, a sufficient condition for $\{x_k^p\}_{k\in\mathcal{Z}_+}\in B_\rho$, is given by

$$\rho>\rho_0+\frac{\overline{\rho}PTe^{\mathrm{LPT}}}{1-\alpha}\Longleftrightarrow\rho_0<\rho-\frac{\overline{\rho}PTe^{\mathrm{LPT}}}{1-\alpha}.\tag{96}$$

Thus, since $\overline{\rho}<(\rho(1-\alpha)/PTe^{\mathrm{LPT}})$, it follows that $\rho_0>0$ (or, equivalently, B_{ρ_0} is a nonempty set).

So, we conclude that there exists $\rho_0<\rho-(\overline{\rho}PTe^{\mathrm{LPT}}/(1-\alpha))$ (thus, $\rho_0<\rho$) so that if $x_0^p\in B_{\rho_0}\subset B_\rho$, then $x_k^p\in B_\rho$, with $\rho>(\overline{\rho}PTe^{\mathrm{LPT}}/(1-\alpha))$, and $\mathcal{P}(t_k,x_k^p)$ is well posed for all $k\in\mathcal{Z}_+$.

Since from the contractive constraint we have $\|\overline{x}_{k+1}\|_{\hat{P}}\le\alpha\|x_k^p\|_{\hat{P}}$, then $\overline{x}_1\in B_{\alpha\rho_0}\subset B_\rho$ and $\overline{x}_k\in B_{\alpha\rho}\subset B_\rho$, $\forall\,k\in\mathcal{Z}_+^*-\{1\}$.

Thus, we have proven that under the conditions of the theorem, the sequences $\{x_k^p\}_{k\in\mathcal{Z}_+}$ and $\{\overline{x}_k\}_{k\in\mathcal{Z}_+}$ remain inside the set of feasible initial conditions, B_ρ. □

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