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Model Predictive Control for Uncertain Systems

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ABSTRACT

A new model predictive control (MPC) algorithm for uncertain system is presented. This approach is less conservative than "robust MPC" and is equivalent to standard model algorithm control (MAC) in the case of the certain system. A useful simplified algorithm being more acceptable to industrial process is also given. The advantage of the new algorithm is illustrated by two examples.

1. INTRODUCTION

Mathematical models of real processes, especially control models, can only describe the dynamics of the process in an approximate way. The objective of robust control is to design controllers which preserve stability and performance in spite of the model's inaccuracies or uncertainties.

In the last 10 years model predictive control (MPC) has achieved a significant level of industrial success in practical process control applications. A fairly complete discussion of several design techniques based on MPC can be found in the review article [4]. Perhaps the principal shortcoming of existing MPC-based control techniques is their inability to explicitly incorporate plant model uncertainty. Thus nearly all known formulations of MPC minimize, on-line, a nominal objective function, using a single linear time-invariant (LTI) model to predict the future plant behavior. Feedback, in the form of plant measurement at the next sampling time, is expected to account for plant model uncertainty. Needless to say, such control systems that provide 'optimal' performance for a particular model may perform very poorly when implemented on a physical system that is not exactly described by the model.

Currently, there is an increasingly visible interest in the research community in revisiting the predictive control design techniques with the intention of including robustness features that guarantee stability or adequate performance when the plant model is uncertain [1, 2, 5, 7, 9, 11]. A new technique for the synthesis of a robust MPC law, using linear matrix inequalities (LMIs) is proposed by Kothare et al. [6]. The technique allows incorporation of a large class of plant uncertainty descriptions, and is shown to be robustly stabilizing.

However, we don't know the conservatasion of these robust MPC design methods. It is improper to do optimization only in the worst case when we know more information about the uncertainty. In this paper, we present a new model predictive control algorithm for uncertain systems, which is called uncertain system model predictive control (USMPC). An analytic solution can be achieved for unconstrained linear systems by using 2-norm objective function which includes uncertain parameters. This algorithm is very simple. This approach has some robustness, but it doesn't lead to conservative design as min-max method since the distribution information of the uncertain parameters is considered. In the case of no uncertainty, our control law is equivalent to that of standard model algorithm control (MAC), i.e. the latter is a special case of the former. A simplified algorithm of USMPC is also given in purpose of practical application.

2. FORMULATION OF USMPC

Consider the finite impulse response (FIR) model, for which $g_i = 0$, $\forall i>N$, where g_i is the coefficients of the impulse response. We denote by Π the family of possible models. Parametrizing Π in terms of a vector of unknown parameters, θ_j, with $|\theta_j| \le \delta_j$, $j=1,\cdots q$, we can write

$$\Pi = \left\{ g_i(\Theta) \middle| \Theta = \left[\theta_1,\cdots,\theta_q\right], \forall i = 1,\cdots,N \right\}$$

we define the nominal model as $g_i(\mathbf{0})$, then the predictor can be written as follows [1]

$$\hat{\mathbf{y}}(k) = \mathbf{G}_1\mathbf{u}_1(k) + \mathbf{G}_2\mathbf{u}_2(k) + \tilde{\mathbf{e}}(k) \qquad (1)$$

where

$$\hat{\mathbf{y}}(k)=\left[\hat{y}(k+1|k),\cdots,\hat{y}(k+P|k)\right]^T$$
$$\mathbf{u}_1(k)=\left[u(k),\cdots,u(k+M-1)\right]^T$$
$$\mathbf{u}_2(k)=\left[u(k-1),\cdots,u(k-N+1)\right]^T$$
$$\tilde{\mathbf{e}}(k)=\left[e(k|k),\cdots,e(k|k)\right]^T$$
$$e(k|k)=y(k)-\sum_{i=1}^{N}g_i(0)u(k-i)$$
$$\mathbf{G}_1(\Theta)=\begin{bmatrix} g_1(\Theta) & 0 & \cdots & & \mathbf{0} \\ g_2(\Theta) & g_1(\Theta) & \ddots & & \\ & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & g_1(\Theta) & 0 \\ \vdots & \vdots & \cdots & & g_1(\Theta) \\ & & & \vdots & \vdots \\ g_P(\Theta) & g_{P-1}(\Theta) & \cdots & & g_1(\Theta)+\cdots+g_{P-M+1}(\Theta) \end{bmatrix}$$
$$\mathbf{G}_2(\Theta)=\begin{bmatrix} g_2(\Theta) & \cdots & \cdots & & g_N(\Theta) \\ g_3(\Theta) & & \cdots & g_N(\Theta) & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \vdots \\ g_{P-1}(\Theta) & \cdots & g_N(\Theta) & 0\ \cdots & 0 \end{bmatrix}$$

$\hat{y}(k+l|k)$ = predicted value of the output at time k+l based on information available at time k

$e(k|k)$ = predicted value of additive disturbances at the output at time k based on information available at time k

$u(k+l)$ = input at time $k+l$

P = tracking error horizon length

M = the control horizon

The objective function of USMPC is now described as

$$J(k)=\int f(\Theta)\left(\left\|\hat{\mathbf{y}}(k)-\mathbf{w}(k)\right\|_{\mathbf{Q}}^2\right)d\theta_1\cdots d\theta_q+\left\|\mathbf{u}_1(k)\right\|_{\mathbf{R}}^2 \qquad (2)$$

where $\mathbf{w}$ is a given reference signal and $f(\Theta)$ is the density function of Θ, $\mathbf{Q}$ and $\mathbf{R}$ are the weighted matrices. The method to choose $f(\Theta)$ will be discussed later. Let us first derive the control law.

Theorem 1 For the objective function (2) of uncertain system (1), the minimum of $J(k)$, assuming there are no constraints on the control signals, can be achieved by

$$\mathbf{u}_1=-\mathbf{P}_{11}^{-1}\mathbf{P}_1 \qquad (3)$$

where

$$\mathbf{P}_{11}=\int f(\Theta)\left[\mathbf{F}_{11}+\mathbf{R}\right]d\theta_1\cdots d\theta_q \qquad (4)$$

$$\mathbf{P}_1=\int f(\Theta)\left[\mathbf{F}_{12}\mathbf{u}_2+\mathbf{F}_1\mathbf{s}\right]d\theta_1\cdots d\theta_q \qquad (5)$$

with $\mathbf{F}_{i,j}=\mathbf{G}_i^T(\Theta)\mathbf{Q}\mathbf{G}_j(\Theta)$, $\mathbf{F}_i=\mathbf{G}_i^T(\Theta)\mathbf{Q}$, $\mathbf{s}=\tilde{\mathbf{e}}(k)-\mathbf{w}(k)$.

Proof It is easy to show that

$$\left\|\hat{\mathbf{y}}(k)-\mathbf{w}(k)\right\|_Q^2=\mathbf{u}_1^T\mathbf{F}_{11}\mathbf{u}_1+2\mathbf{u}_1^T\left(\mathbf{F}_{12}\mathbf{u}_2+\mathbf{F}_1\mathbf{s}\right)+\left(\mathbf{u}_2^T\mathbf{F}_{22}\mathbf{u}_2+2\mathbf{u}_2^T\mathbf{F}_2\mathbf{s}+\mathbf{s}^T\mathbf{Q}\mathbf{s}\right)$$

Only $\mathbf{G}_1(\Theta)$ and $\mathbf{G}_2(\Theta)$ in $\mathbf{F}_{ij}$, and $\mathbf{F}_i(i, j=1,2)$ depend on Θ, we have

$$\begin{aligned} J(k) &= \int f(\Theta)\mathbf{u}_1^T\left[\mathbf{F}_{11}+\mathbf{R}\right]\mathbf{u}_1 d\theta_1\cdots d\theta_q \\ &\quad+\int f(\Theta)\mathbf{u}_1^T\left[\mathbf{F}_{12}\mathbf{u}_2+\mathbf{F}_1\mathbf{s}\right]d\theta_1\cdots d\theta_q+\mathbf{P}_2 \\ &= \mathbf{u}_1^T\mathbf{P}_{11}\mathbf{u}_1+2\mathbf{u}_1^T\mathbf{P}_1+\mathbf{P}_2 \end{aligned} \qquad (6)$$

where

$$\mathbf{P}_2=\int f(\Theta)\left[\mathbf{u}_2^T\mathbf{F}_{22}\mathbf{u}_2+2\mathbf{u}_2^T\mathbf{F}_2\mathbf{s}+\mathbf{s}^T\mathbf{Q}\mathbf{s}\right]d\theta_1\cdots d\theta_q$$

Thus the minimum of J can be found by making the gradient of J equal to zero, which leads to

$$\mathbf{u}_1=-\mathbf{P}_{11}^{-1}\mathbf{P}_1. \qquad \square$$

Based on this, an USMPC algorithm is presented as follows

Algorithm 1

1. Obtain the nominal model from some identification methods [8], then compute $\mathbf{P}_{11}$, $\mathbf{P}_1$ by equation (4-5) for a given $f(\Theta)$.
2. At each sampling time k, get the error between the output signal and the reference signal and then compute $\mathbf{u}_1(k)$ by equation (3), and only the first element of $\mathbf{u}_1(k)$ is sent to the process.
3. Compute the predicted output signal $\hat{\mathbf{y}}(k)$ by equation (1).
4. Return step 2 at the next sampling time k+1.

Remark 1 With no loss of generality, Θ is assumed to have only one parameter θ. In general, it is impossible to know the density function $f(\theta)$ of θ, usually it can be chosen as follows

(1) Normal distribution: If θ is normal distribution, $f(\theta)$ is $f(\theta)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\eta)^2}{2\sigma^2}}$, where η and σ are the expectation and variance of θ respectively.

(2) Uniform distribution: In the worst case, we only know that θ varies in one range of $[-\delta, \delta]$ stochastically, $f(\theta)$ can be chosen as $f(\theta)=1/2\delta, \theta\in[-\delta,\delta]; f(\theta)=0, \theta\notin[-\delta,\delta]$,

(3) Discrete distribution : the distribution can also be chosen as discrete distribution, et. $f(\theta)$ is a constant in

the range of θ, for example

$$f(\theta)=1/3, \theta\in[-\delta,\ 0];\ f(\theta)=2/3, \theta\in[0,\ \delta]$$

In summary, $f(\theta)$ can be arbitrarily chosen so long as it represents the distribution information of θ.

Remark 2 Certain model can be considered as the parametric uncertain model in which the density function of each parameter satisfy

$$f(\theta)=0 \text{ if } \theta\neq 0 \text{ and } \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} f(\theta)=1.$$

Compute $\mathbf{P}_{11}$ and $\mathbf{P}_1$ in theorem 1 for such $f(\theta)$, we have

$$\mathbf{P}_{11}=\mathbf{G}_1^T(0)\mathbf{Q}\mathbf{G}_1(0)+\mathbf{R} \tag{7}$$

$$\mathbf{P}_1=\mathbf{G}_1^T(0)\mathbf{Q}\mathbf{G}_2(0)\mathbf{u}_2+\mathbf{G}_1^T(0)\mathbf{Q}\mathbf{s} \tag{8}$$

Thus the controller $\mathbf{u}_1=-\mathbf{P}_{11}^{-1}\mathbf{P}_1$ is the same as standard MAC [4].

3. A SIMPLIFIED ALGORITHM OF USMPC

There are few effective methods to obtain the density function of uncertain parameters, therefore we have to choose $f(\Theta)$ by practical experience and this is a drawback for a wider dissemination of USMPC. However a simplified approach can be used if we have obtained the expectation and variance of the parameters by some new identification tools. The simplified algorithm is based on the following lemma [10].

Lemma 1 Let x be a random variable whose expectation and variance are η, σ^2 respectively. If the second derivative of $f(x)$ exists, then

$$E\{f(x)\}\approx f(\eta)+f''(\eta)\frac{\sigma^2}{2} \tag{9}$$

where E{ • } is expectation function.

For simplicity, Θ is also assumed to have only one parameter θ, Let

$$H(\theta)=\|\hat{\mathbf{y}}(k)-\mathbf{w}(k)\|_Q^2+\|\mathbf{u}_1(k)\|_R^2.$$

The second derivative of H(θ) exits obviously, since the expectation and variance of θ are 0 and σ^2 respectively, by lemma 1 we have

$$E\{H(\theta)\}\approx H(0)+H''(0)\frac{\sigma^2}{2} \tag{10}$$

By some matrix manipulations, it is easy to show that

$$H''(0)=\mathbf{u}_1^T\overline{\mathbf{F}}_{11}\mathbf{u}_1+2\mathbf{u}_1^T\overline{\mathbf{F}}_{12}\mathbf{u}_2+\mathbf{u}_2^T\overline{\mathbf{F}}_{22}\mathbf{u}_2 \tag{11}$$

where $\overline{\mathbf{F}}_{ij}=\mathbf{G}_i^T(0)\mathbf{Q}\mathbf{G}_j(0)$, i,j=1,2. Then the objective function can be rewritten as

$$J(k)=H(0)+H''(0)\frac{\sigma^2}{2}. \tag{12}$$

Theorem 3 ( ) For an uncertain system (1), assume $g_1(\theta)=g_1+\theta$, $g_i(\theta)=g_i$, i=2,…N, where θ is a random value whose expectation and variance are 0 and σ^2 respectively, and g_i is the impulse response coefficient of the nominal model, then the minimum of $J(k)$, assuming there are no constraints on the control signals, can be achieved by

$$\mathbf{u}_1=-\overline{\mathbf{P}}_{11}^{-1}\overline{\mathbf{P}}_1 \tag{13}$$

where

$$\overline{\mathbf{P}}_{11}=\overline{\mathbf{F}}_{11}(1+\sigma^2)+\mathbf{R} \tag{14}$$

$$\overline{\mathbf{P}}_1=\overline{\mathbf{F}}_{12}(1+\sigma^2)\mathbf{u}_2+\overline{\mathbf{F}}_1\mathbf{s} \tag{15}$$

with $\overline{\mathbf{F}}_{ij}=\mathbf{G}_i^T(0)\mathbf{Q}\mathbf{G}_j(0)$, i,j=1,2 and $\overline{\mathbf{F}}_i=\mathbf{G}_i^T(0)\mathbf{Q}$.

Proof From (11) and (12) we get

$$\min_{\mathbf{u}_1} J(k)=\min_{\mathbf{u}_1}\Big\{\mathbf{u}_1^T\big[\overline{\mathbf{F}}_{11}(1+\sigma^2)+\mathbf{R}\big]\mathbf{u}_1 + 2\mathbf{u}_1^T\big[\overline{\mathbf{F}}_{12}(1+\sigma^2)\mathbf{u}_2+\overline{\mathbf{F}}_1\mathbf{s}\big]+[\cdots]\Big\} \tag{16}$$

The terms not including u_1 are omitted in equation (16), because they don't effect the optimal control law. Therefore, just as the same as the proof of theorem 1, we have $\mathbf{u}_1=-\overline{\mathbf{P}}_{11}^{-1}\overline{\mathbf{P}}_1$. □

Lemma 1 and theorem 3 have obvious extensions to the multi-parameter case when $g_i(\Theta)=g_i+\theta_i$, $i=1,\cdots,N$. This treatment will not be repeated here; instead, a statement of the results is listed below.

Lemma 2 Assume $\theta_i\,(i=1,\cdots,N)$ are independent random variable (the expectation and variance of θ_i are 0 and σ^2 respectively), then the following approximation exists

$$J(k)=E\{H(\Theta)\} \approx H(\mathbf{0})+\frac{1}{2}\left(\sum_{i=1}^{n}\sigma_i^2\frac{\partial^2 H(\Theta)}{\partial\theta_i^2}\right)\Bigg|_{\mathbf{0}} \tag{16}$$

where $H(\theta)=\|\hat{\mathbf{y}}(k)-\mathbf{w}(k)\|_Q^2+\|\mathbf{u}_1(k)\|_R^2$.

Theorem 3 For uncertain system (1), assume $g_i(\theta)=g_i+\theta_i$, $i=1,\cdots,N$, (the expectation and variance of θ_i are 0 and σ_i^2 respectively), then the minimum of $J(k)$ can be achieved by

$$\mathbf{u}_1=-\overline{\mathbf{P}}_{11}^{-1}\overline{\mathbf{P}}_1 \tag{17}$$

where

$$\overline{\mathbf{P}}_{11}=\overline{\mathbf{F}}_{11}\left(1+\sum_{i=1}^{n}\sigma_i^2\right)+\mathbf{R} \tag{18}$$

$$\overline{\mathbf{P}}_1=\overline{\mathbf{F}}_{12}\left(1+\sum_{i=1}^{n}\sigma_i^2\right)\mathbf{u}_2+\overline{\mathbf{F}}_1\mathbf{s} \tag{19}$$

with $\overline{\mathbf{F}}_{ij} = \mathbf{G}_i^T(0)\mathbf{Q}\mathbf{G}_j(0)$, i,j=1,2 and $\overline{\mathbf{F}}_i = \mathbf{G}_i^T(0)\mathbf{Q}$.

Algorithm 2

1. Obtain the nominal model and the variance of the parameter, compute $\overline{\mathbf{P}}_{11}$, $\overline{\mathbf{P}}_1$ by equation (18) and (19).
2. At each sampling time k, get the error between the output signal and the reference signal and then compute $\mathbf{u}_1(k)$ by equation (17), and only the first element of $\mathbf{u}_1(k)$ is sent to the process.
3. Compute the predicted output signal $\hat{\mathbf{y}}(k)$ by equation (1).
4. Return step 2 at the next sampling time k+1.

4. SIMULATION RESULTS

An example will be given in this section to demonstrate the characteristics of the algorithms presented in this paper compared to standard MAC. The nominal plant is given by G(z)=1/(z-0.5).The impulse response description of the real plant is

$$g_i = g_{i0} + \theta_i ,$$

where g_{i0} is the coefficient of the nominal plant and θ_i is the uncertainty. Let q=N=20, P=8, M=4, $\mathbf{Q} = \mathbf{I}$, $\mathbf{R} = \mathbf{0}$, the set-point signal w is taken as a square wave.

Example 1 Let $\theta_i = (1/2)^i d_i$ ($i=1, \cdots n$), where d_i are independent uniformly distributed random variables over the interval (-0.5,0.5). Fig.1 shows the impulse response coefficients of the nominal plant and the real plant. The results of the closed-loop simulation test for the MAC and the simplified USMPC are depicted in Fig. 2 and 3 respectively. The output trajectory shown in Fig. 2 is unstable, but the USMPC can work well.

Example 2 Let $\theta_i = (1/2)^i d_i$, where d_i are independent unit normal random variables. The same results as example 1 are shown in Fig. 4, 5 and 6. In this case, the simplified USMPC provides a better response than the MAC since there is a static error via MAC [4].

5. CONCLUSION

We have developed a new model predictive control formulation for uncertain systems. An analytic solution can be obtained in the unconstrained case. A simplified case of the algorithm is also given in purpose of more practical use. Finally, two examples illustrate the efficiency of the algorithms developed in this paper. However the USMPC can not guarantee the stability. It would be a good project to study the stability of USMPC for future research.

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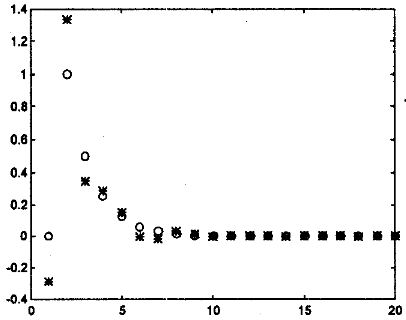

Fig. 1 Impulse response coefficients
(o: the nominal plant, ✳: the real plant)

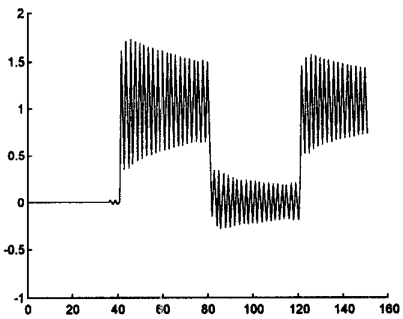

Fig. 2 Behavior of the MAC

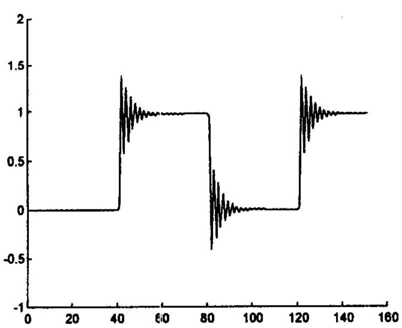

Fig. 3 Behavior of the SUSMPC

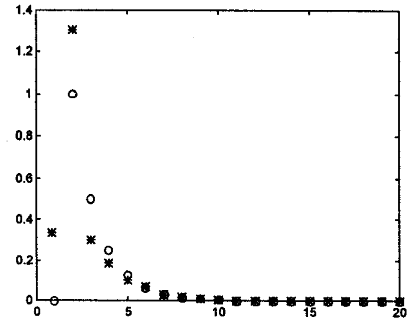

Fig. 4 Impulse response coefficients
(o: the nominal plant, ✳: the real plant)

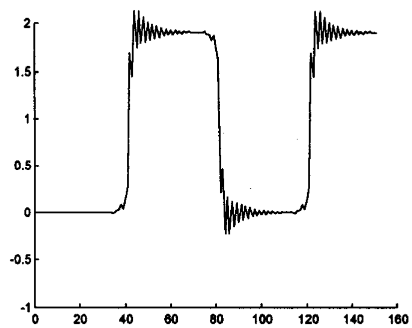

Fig. 5 Behavior of the MAC

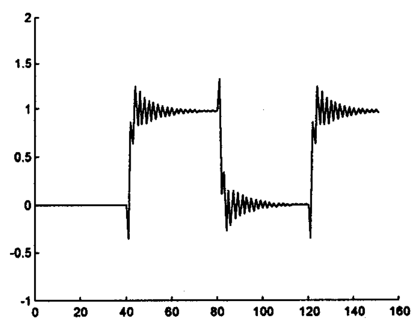

Fig. 6 Behavior of the SUSMPC