EXPONENTIAL STABILIZATION OF DRIFTLESS NONLINEAR CONTROL SYSTEMS USING HOMOGENEOUS FEEDBACK

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ABSTRACT. This paper focuses on the problem of exponential stabilization of controllable, driftless systems using time-varying, homogeneous feedback. The analysis is performed with respect to a homogeneous norm in a non-standard dilation that is compatible with the algebraic structure of the control Lie algebra. Using this structure, we show that any continuous, time-varying controller that achieves exponential stabilization relative to the Euclidean norm is necessarily non-Lipschitz. Despite these restrictions, we provide a set of constructive, sufficient conditions for extending smooth, asymptotic stabilizers to homogeneous, exponential stabilizers. The modified feedbacks are everywhere continuous, smooth away from the origin, and can be extended to a large class of systems with torque inputs. The feedback laws are applied to an experimental mobile robot and show significant improvement in convergence rate over smooth stabilizers.

1. INTRODUCTION

In this paper we consider the stabilization problem for driftless control systems of the form

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m \qquad x \in \mathbb{R}^n.$$

We assume that the vector fields X_i are analytic on $\mathbb{R}^n$ and that they are pointwise linearly independent. We further assume that the system is completely controllable: given any two points x_0 and x_1 and a time $T > 0$ there exists a control u defined on the time interval $[0, T]$ which steers the system between x_0 and x_1. Controllability is easily checked using the Lie algebra rank condition for nonlinear control systems (see, for example, Nijmeijer and van der Schaft [38] or Isidori [21]).

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Controllable, driftless control systems arise in the study of mechanical systems with symmetries and nonholonomic constraints, and represent the dynamics consistent with the kinematic constraints placed on the system by the presence of conservation laws or constraints. Typically, the inputs for a driftless control system correspond to the velocities of a mechanical system. Although in practice one almost always controls forces and torques in a mechanical system instead of velocities, in many instances it is possible to extend controllers that prescribe velocities to controllers that command forces and torques. Hence we initially focus our attention on the driftless case and indicate later how to extend controllers to allow more general inputs.

The stabilization problem for driftless systems represents a challenge for nonlinear control theory because the linearization of the system is not controllable. In fact, as shown by Brockett [8], for this class of systems there does not exist a smooth (or even continuous) control law of the form $u_i = \alpha_i(x)$ which asymptotically stablizes the system to an equilibrium point. As such, one is forced to rely on the use of strongly nonlinear techniques to stabilize the system. Results on asymptotic stability typically rely on the use of discontinuous feedback, time-varying feedback, or a combination of the two.

In this paper we concentrate on the problem of *exponential* stabilization of driftless systems. In this case, it is shown that even if time-varying feedback is allowed, it is still not possible to achieve exponential stability using Lipschitz feedback. Indeed, as we show below, the usual definition of exponential stability does not readily apply to this problem and one must use a broader definition of exponential stability.

The approach to exponential stabilization in this paper makes use of the theory of homogeneous systems with non-standard dilations [15, 19]. Using and extending the tools available from that area, we show how to construct and analyze exponential stabilizers for this class of systems. The extra structure which is available through the use of homogeneous systems allows us to circumvent many of the problems normally associated with the lack of Lipschitz feedback and provide a very complete theory for driftless systems as well as provide tools which hold for more general systems.

The main *direct* applications of the work presented here are control of mobile robots and other robotic systems with nonholonomic constraints (see [34] for introductory theory and examples). However, the basic techniques which we develop here are more broadly applicable and have potential application in a number of areas, including power converters [25], underwater vehicles [14] and novel robotic mechanisms [49]. With these and other applications in mind, we have tried to present many of the results in a context in which they can be applied to other strongly nonlinear stabilization problems.

This paper is organized as follows. In Section 2 we give a short review of the literature on stabilization of driftless control systems. This review is intended to orient the reader who is new to the area and also to describe the context for the results presented in this paper. Section 3 reviews results for homogeneous systems. The properties of homogeneous systems form the basis of the analysis in this paper. Much of the material is well established but a few of the results are new. The limitations of Lipschitz feedback are discussed in Section 4. The main result of this section, which is applicable to general C^1 control systems, shows that solutions of a driftless system cannot satisfy an exponential stability bound when the feedback is Lipschitz continuous in the state. Section 5 presents a method of improving the convergence rate of a driftless system when a smooth stabilizing feedback is already known. The convergence rate with the modified feedbacks is a modified notion of exponential stability. This method is applied to an experimental mobile robot in Section 6. Section 7 shows how to extend the exponentially stabilizing feedbacks through a set of integrators.

Finally, we indicate how the results in this paper apply to more general nonlinear control systems and indicate some of the directions for future work.

Preliminary versions of some of the results in this paper have appeared in [28, 29, 30, 32, 31, 35]. Additional technical results related to this work, as well as a more detailed introduction to homogeneous control systems, can be found in [27].

2. Related work

There have been a number of papers published on stabilization of nonholonomic systems over the past four years. A survey of the field can be found in the recent papers by Sørdalen and Egeland [48] and Samson [45]. We concentrate here on work that is most directly related to the results presented in this paper.

The basic limitations in stabilization of driftless systems were given in a 1983 paper by Brockett [8], where it was shown, among other things, that driftless control systems could not be stabilized to a point using continuous, static state feedback (for a particularly nice proof, see the survey paper by Sontag [47]). In 1990, Samson presented a paper in which he showed how to asymptotically stablize a mobile robot to a point using *time-varying*, smooth state feedback [44]. The use of time-varying feedback avoided the difficulties captured by Brockett's necessary condition. Motivated by these results, Coron proved in 1991 that all controllable driftless systems could be stabilized to an equilibrium point using smooth, periodic, time-varying feedback [11] (This result also follows from Sontag's work on universal controls [46].)

Coron's result opened the door to a constructive approaches for stabilizing a general class of driftless control systems. This first such result was presented by Pomet [40], who developed a synthesis technique based on Coron's proof which held for a fairly general class of systems, including as a special cases mobile robots and mobile robots towing trailers. This result was extended to the general case by Coron and Pomet [12]. Additional techniques were given by Teel et al. [50] for a special class of driftless systems in so-called chained form [36].

A second approach to stabilizing nonholonomic systems involved the use of discontinuous feedbacks. One of the early results in this area was given by Bloch, Reyhanoglu, and McClamroch [7] and involves the use of piecewise analytic feedbacks for stabilizing a nonholonomic mechanical system to a point. Unlike much of the other work on nonholonomic systems, the approach proposed by Bloch et al. allowed the use of either velocity or torque inputs rather than just velocity inputs. Another discontinuous stabilization approach was given by Canudas de Wit and Sørdalen [13], who developed piecewise smooth controllers for a set of low dimensional examples. The main application of their results was to mobile robots and one of the features was that they could guarantee that the control was discontinuous at only a finite number of times. Sørdalen and Egeland extended these results to systems in chained form [48].

One of the advantages of the discontinuous stabilization approaches over the smooth, time-periodic feedbacks is that discontinuous stabilizers usually give exponential convergence or convergence in finite time. Extending his previous work, Coron showed it is possible to generate time-periodic feedbacks which gave finite time convergence and were smooth everywhere except the origin [12]. These results imply that exponential stabilizers exist which are time-periodic and smooth away from the equilibrium point. The necessity of non-differentiable feedbacks even in the time-varying case can be found, for example, in [37], and is based on a straightforward linearization argument. One of the contributions of the present paper is to show more precisely how to construct feedback controllers which give exponential rates of convergence and are smooth everywhere except at the origin.

The exponential stability results presented in this paper rely on the properties of homogeneous systems and build off of several previous results on stability of homogeneous systems. The basic tools for dilations and homogeneous functions and vector fields are given in the monograph by Goodman [15] (see also Bacciotti [3]). Hermes has considered the application of homogeneous systems in control theory and has developed approximations which generalize the usual linear approximation theorems [18, 20]. The use of homogeneous structure in stabilization problems has also been considered by Kawski [22], who presented results for low-dimensional control systems with drift and defined the notion of exponential stability which we make use of here. Other work on homogeneous control systems includes the work of Rosier [43] on converse Lyapunov results for autonomous systems and work by Pomet and Samson [41], who have extended their results on smooth stabilization to give exponential stabilization using tools similar to those presented in this paper.

In addition to time-varying feedback and discontinuous feedback, there have been many other approaches proposed for stabilization of driftless systems. Conditions for stabilization to a submanifold where given by Bloch, Reyhanoglu, and McClamroch [7] (see also Montgomery [33]). Maschke and van der Schaft have generated controllers for stabilization to a submanifold using a Hamiltonian framework [26]. Hybrid strategies, involving the use of both discontinuous and time-varying feedbacks have been proposed by Pomet et al. [42], using a combination of Pomet's time-varying controllers near the origin and discontinuous feedback far away from the origin, and also Oelen, Canudas de Wit, Berghuis, and Nijmeijer [39], who presented stabilizers for systems in chained form. Sørdalen and Egeland [48] have given controllers which involving switches at discrete instants in time and smooth feedback between switches. A similar technique has been used by Kolmanovsky and McClamroch [23], who use a discrete event supervisor to generate switchings for controllers which give finite time convergence. Sliding mode controllers have been explored by Bloch and Drakunov [6] and results on adaptive stabilization have been given by Bastin and Campion [4]. The use of nonsmooth changes of coordinates followed by smooth feedbacks in the transformed coordinates is a promising new direction which is being explored by Astolfi [2] and also Casalino, Aicardi, Bicchi, and Balestrino [9, 1].

Our own work in this area started with convergence analysis for the time-periodic, smooth controllers proposed by Pomet and Teel et al. These results showed that the controllers under consideration converged at a rate proportional to $1/\sqrt{t}$ and hence gave very slow convergence. This motivated our work on exponential convergence and, based on the structure present in both chained and power form, we focused on the use of non-standard dilations and homogeneous structure. Initial analysis tools were presented in [29] and preliminary results on controller synthesis were given in [32]. Experimental results on the application of various feedback control laws to a mobile robot system are described in detail in [28] and will be the subject of a forthcoming article. We have also derived a number of extensions to the basic work described in [32]. In [31] we describe how to extend exponentially stabilizing controllers which command the velocity to exponential controllers using torque inputs. In [35] we describe how to convert smooth, time-periodic, asymptotic stabilizers for driftless systems into exponential stabilizers which are smooth everywhere except the origin. The present paper integrates all of these past results as well as presenting new results on control systems using Lipschitz feedback.

3. Homogeneous Systems

We now introduce some background material. To establish notation, functions will be denoted by lower case letters and vector fields by capital letters. We will occasionally abuse notation and define the differential equation $\dot{x} = X(t, x)$ in local coordinates on $\mathbb{R}^n$ associated with the vector field X. The flow of a differential equation is denoted ϕ where $\phi(t, t_0, x_0)$ is the solution, at time t, which passes through the point x_0 at time t_0. For a linear differential equation, $\phi(t, t_0)$ denotes the principal matrix solution. When it is necessary to distinguish between flows of vector fields a superscript will be used; i.e. ϕ_t^X is the flow of X, ϕ_t^Y is the flow of Y, etc.

3.1. Some definitions.

This section reviews dilations and homogeneous vector fields. A *dilation* ${\Delta_\lambda}^r : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is defined with respect to a fixed choice of coordinates $x = (x_1, x_2, \ldots, x_n)$ on $\mathbb{R}^n$ by assigning n positive rationals $r = (r_1 = 1 \leq r_2 \leq \cdots \leq r_n)$ and positive real parameter $\lambda > 0$ such that

$$\Delta_\lambda^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n), \quad \lambda > 0.$$

We usually write Δ_λ in place of Δ_λ^r.

Definition 1. A continuous function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $l \geq 0$ *with respect to* Δ_λ if $f(t, \Delta_\lambda x) = \lambda^l f(t, x)$.

Definition 2. A continuous vector field $X(t, x) = \sum a_i(t, x) \partial/\partial x_i$ on $\mathbb{R} \times \mathbb{R}^n$ is *homogeneous of degree* $m \leq r_n$ *with respect to* Δ_λ if a_i is degree $r_i - m$ for $i = 1, \ldots, m$.

The variable t represents explicit time dependence and is never scaled in our applications.

Definition 3. A continuous map from $\mathbb{R}^n$ to $\mathbb{R}$, $x \mapsto \rho(x)$, is called a *homogeneous norm* with respect to the dilation $\Delta_\lambda$when

1. $\rho(x) \geq 0, \quad \rho(x) = 0 \iff x = 0,$
2. $\rho(\Delta_\lambda x) = \lambda \rho(x) \quad \forall \lambda > 0.$

The homogeneous norm is called *smooth* when it is smooth on $\mathbb{R}^n \setminus \{0\}$.

For example, a smooth homogeneous norm may always be defined as

$$\rho(x) = |x_1^{c/r_1} + x_2^{c/r_2} + \cdots + x_n^{c/r_n}|^{1/c}, \tag{1}$$

where c is some positive integer *evenly* divisible by r_i. We are primarily interested in the convergence of time dependent functions using a homogeneous norm as a

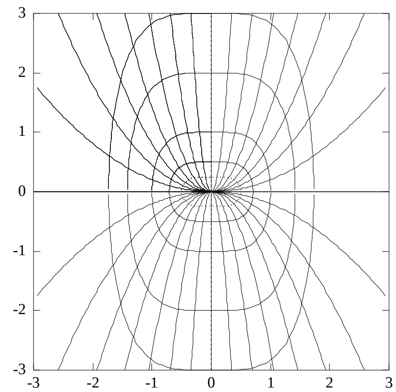

FIGURE 1. Level sets of smooth homogeneous norm and trajectories of the Euler vector field X_E.

measure of their size. When a vector field is homogeneous it is most natural to use a corresponding homogeneous norm as the metric. The usual vector p-norms are homogeneous with respect to the standard dilation ($r_i = 1$).

Definition 4. The Δ-*sphere* is defined as the set

$$S_\Delta = \{x | \rho(x) = 1\},$$

where ρ is a smooth homogeneous norm corresponding to the dilation Δ_λ.

Definition 5. The *Euler vector field corresponding to a dilation* Δ_λ is defined as

$$X_E(x) = \sum r_i x_i \frac{\partial}{\partial x_i}.$$

Thus the images of trajectories of the system $\dot{x} = X_E(x)$ are the rays obtained by scaling the points on the sphere S_Δ with the dilation. Figure 1 show the level sets of the smooth homogeneous norm $\rho = (x_1^4 + x_2^2)^{1/4}$ and the trajectories of the Euler vector field corresponding to the dilation $\Delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2)$.

3.2. **Homogeneous approximations of vector fields.** This section reviews homogeneous approximations of sets of vector fields. The vector fields are the input vector fields of the controllable driftless system

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m. \tag{2}$$

The entire analysis is local so we assume that vector fields are defined on $\mathbb{R}^n$. Furthermore, the vector fields are taken to be analytic. We are interested in obtaining an approximation, in the sense described below, of the set of vector fields $\{X_1, \ldots, X_m\}$. The Lie bracket of vector fields is $[\cdot, \cdot]$.

Let $\mathcal{L}(X_1, \ldots, X_m)$ be the Lie algebra generated by the set $\{X_1, \ldots, X_m\}$. The following definition specifies a special filtration of the Lie algebra of a finite set of generating vector fields.

Definition 6. The *control filtration*, $\mathcal{F}^X$, of $\mathcal{L}(X_1, \dots, X_m)$ is a sequence of subspaces defined as

$$\begin{aligned}
\mathcal{F}_0^X &= \{0\},\\
\mathcal{F}_1^X &= \operatorname{span}\{X_1, \dots, X_m\},\\
\mathcal{F}_2^X &= \operatorname{span}\{X_1, \dots, X_m, [X_1, X_2], \dots, [X_1, X_2], \dots, [X_{m-1}, X_m]\},\\
&\;\vdots\\
\mathcal{F}_k^X &= \operatorname{span}\{\text{all products of i-tuples from } \{X_1, \dots, X_m\}, \text{for } i \le k\},\\
&\;\vdots
\end{aligned} \tag{3}$$

and $\mathcal{F}^X = \{\mathcal{F}_j^X\}_{j \ge 0}$.

The set of vector fields is approximated about a specific point, $x_0 \in \mathbb{R}^n$. This point is the desired equilibrium point in the sequel. Now let $F_i(x_0)$ be the subspace of $\mathbb{R}^n$ (more precisely the tangent space, $T_{x_0}\mathbb{R}^n$, of $\mathbb{R}^n$ at x_0) spanned by $Z(x_0)$ where $Z \in \mathcal{F}_i^X$. This yields an increasing sequence of vector subspaces,

$$\{0\} = F_0(x_0) \subset F_1(x_0) \subset \cdots \subset F_i(x_0) \subset \cdots \subset \mathbb{R}^n.$$

This sequence must be stationary after some integer since it is assumed that the Lie algebra has full rank at x_0. In other words, since the system (2) is controllable $\dim F_k(x_0) = n$ for all k greater that some minimal integer N. Now we count the growth in the dimension of the subspaces and set $n_1 = \dim F_1(x_0), n_2 = \dim F_2(x_0), \dots, n_N = n = \dim F_N(x_0)$. The following dilation is defined:

Definition 7. The *dilation adapted to the filtration* (at the point x_0) is the map

$$\Delta_\lambda^r x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n),$$

where the scalings satisfy $r_i = 1$ for $1 \le i \le n_1$, $r_i = 2$ for $n_1 + 1 \le i \le n_2$, etc.

Henceforth, in order to simplify the notation in the expressions to follow it is assumed that $x_0 = 0$. This is achieved with a translation of the origin of the coordinate system.

Definition 8. The *local coordinates adapted to the filtration* $\mathcal{F}^X$ (denoted by y) are related to the original coordinates (denoted by x) by the local analytic diffeomorphism derived from composing flows of vector fields from the filtration,

$$x = \Phi(y) = \phi_{y_1}^{X_{\pi_1}} \circ \phi_{y_2}^{X_{\pi_2}} \circ \cdots \circ \phi_{y_n}^{X_{\pi_n}}(0), \tag{4}$$

where $\phi_t^X(x_0) = \phi(t, 0, x_0)$ denotes the flow of the vector field X and,

1. $X_{\pi_i} \in \mathcal{F}_j^X$ for $n_{j-1} + 1 \le i \le n_j$,
2. $\dim\{X_{\pi_1}, \dots, X_{\pi_n}\} = n$.

A vector field written in a local coordinate system will explicitly show the dependence, i.e., $X(x)$ is written in x-coordinates while $X(y)$ is the same vector field written in y-coordinates. The importance of the local coordinates adapted to $\mathcal{F}^X$ is explained by the following theorem.

Theorem 3.1 ([20, Theorem 2.1]). *Let $\mathcal{L}$ be a Lie algebra of vector fields on $\mathbb{R}^n$ and $\mathcal{F} = \{\mathcal{F}_j\}_{j \ge 0}$ an increasing filtration of $\mathcal{L}$ at zero with Δ_λ the dilation adapted to $\mathcal{F}$ and y the local coordinates adapted to $\mathcal{F}$. If $X \in \mathcal{F}_l$ then,*

$$X(y) = X^l(y) + X^{l-1}(y) + X^{l-2}(y) + \cdots,$$

where $X^j(y)$ is a vector field homogeneous of degree j with respect to Δ_λ^r.

In other words, if $X(y) \in \mathcal{F}_l$ is expanded in terms of vector fields which are homogeneous with respect to Δ_λ, $X(y) = \sum_{j=r_n}^{-\infty} X^j(y)$, then $X^{r_n}(y) = \cdots = X^{l+1}(y) = 0$ and the "leading order" vector field, $X^l(y)$, is degree l with respect to Δ_λ. This leading order vector field is termed the *$\mathcal{F}$-approximation of $X \in \mathcal{F}_l$ in the $\mathcal{F}$-adapted coordinates.* An useful property of the $\mathcal{F}$-approximation is given by the following proposition.

Proposition 3.2 ([20, Corollary 2.2.1]). *Let $\mathcal{F} = \{\mathcal{F}_j^X\}$ be the control filtration of $\mathcal{L}(X_1, \ldots, X_m)$ and $\{\mathcal{F}_j^Y\}_{j \geq 0}$ be the equivalently defined filtration of $\mathcal{L}(Y_1, \ldots, Y_m)$ where Y_i is the $\mathcal{F}$-approximation of $X_i, i = 1, \ldots, m$. Furthermore, let F_l^X and F_l^Y be the corresponding increasing sequence of vector subspaces of $\mathbb{R}^n$. Then,*

$$F_l^X(0) = F_l^Y(0), \quad l = 0, 1, \ldots.$$

Theorem 3.1 and Proposition 3.2 are very important for analytic driftless control systems: Theorem 3.1 says that a degree one approximation always exists if the original system is controllable and Proposition 3.2 says that the degree one approximation is itself controllable. Thus, for purposes of control, these approximations are the correct ones to take (not the Jacobian linearization). When synthesizing feedbacks for driftless systems we will take advantage of the structure of the approximation.

Remark 3.3. Bellaïche et al. [5] have defined the notion of *local order* which they use to give the approximation a more intrinsic meaning. Their approximation coincides with the $\mathcal{F}$-approximation when the vector fields are written in local coordinates adapted to $\mathcal{F}$.

When implementing a feedback law the equations must be written in some coordinate system. Coordinates adapted to $\mathcal{F}$ are chosen in this paper since the homogeneous structure of the $\mathcal{F}$-approximation is exploited.

3.3. **Stability definitions.** A modified definition of exponential stability is given below. The point $x = 0$ is taken to be an equilibrium point of the differential equation $\dot{x} = X(t, x)$. For vectors $\|\cdot\|$ denotes the Eucldean norm and for matrices it denotes the induced 2-norm.

The concept of exponential stability of a vector field is now defined in the context of a homogeneous norm. This definition was introduced by Kawski [22].

Definition 9. The equilibrium point $x = 0$ is *locally exponentially stable with respect to the homogeneous norm* $\rho(\cdot)$ if there exist two constants $\alpha, \beta > 0$ and a neighborhood of the origin U such that

$$\rho(\phi(t, t_0, x_0)) \leq \beta\rho(x_0)e^{-\alpha(t-t_0)} \quad \forall t \geq t_0, \forall x_0 \in U. \tag{5}$$

This stability type is denoted *ρ-exponential stability* to distinguish it from the usual definition of exponential stability.

This notion of stability is important when considering vector fields which are homogeneous with respect to a dilation. This definition is not equivalent to the usual definition of exponential stability except when the dilation is the standard dilation. This is evident from the following bounds on the Euclidean norm in terms of the smooth homogeneous norm given in equation (1) on the unit cube $\mathcal{C} = \{x : |x_i| < 1, i = 1, \ldots, n\}$ (recall $c \geq 2$ in Definition 1),

$$\rho^{c/2}(x) \leq |x| \leq M\rho(x) \quad \text{for some } M > 0,\ x \in \mathcal{C},$$

for some $M > 0$. Hence, the solutions of a ρ-exponentially stable system which remain in the unit cube also satisfy

$$\|\phi(t, t_0, x_0)\| \leq \beta M \|x_0\|^{\frac{2}{c}} e^{-\alpha(t-t_0)} \tag{6}$$

Thus, each state may be bounded by a decaying exponential envelope except that the size of the envelope does not scale linearly in the initial condition as in the usual definition of exponential stability. Exponential stability with respect to ρ allows for non-Lipschitz dependence on the initial conditions. The expression in equation (5) will in general depend upon the particular coordinate system. However the form of the bound in equation (6) remains the same under smooth diffeomorphism. It is useful to view the relation in (6) as broader notion of exponential stability and ρ-exponential stability as a special case.

3.4. **Properties of homogeneous degree zero vector fields.** Some useful facts concerning degree zero vector fields are reviewed in this section. A homogeneous degree zero vector field $X(t, x)$ is invariant with respect to the dilation since

$$(\Delta_\lambda)_* X(t, x) = X(t, \Delta_\lambda x) \quad \forall \lambda > 0.$$

Thus, solutions scale to solutions with the dilation: $\Delta_\lambda \phi(t, t_0, x_0) = \phi(t, t_0, \Delta_\lambda x_0)$. Some other properties are specified in the lemma below. Let π denote the projection onto the homogeneous sphere $S_\Delta^{(n-1)}$ embedded in $\mathbb{R}^n$, $\pi : \mathbb{R}^n \setminus \{0\} \to S_\Delta^{n-1}$,

$$\pi(x) = \left(\frac{x_1}{\rho^{r_1}(x)}, \ldots, \frac{x_n}{\rho^{r_n}(x)} \right).$$

Lemma 3.4. *Let $X(t, x)$ be a homogeneous degree zero vector field. Then*

1. *X is π-related to a vector field Y defined on $S_\Delta^{(n-1)}$, i.e., $\pi_* X = Y \circ \pi$,*
2. *uniform asymptotic stability is equivalent to global ρ-exponential stability.*

Proof. Suppose the differential equation associated with X is given by the set of equations $\dot{x}_i = a_i(t, x), i = 1, \ldots, n$ where each a_i is a degree r_i function since X is degree zero. Furthermore, the homogeneous norm ρ is taken to be the smooth norm defined in equation (1). The differential equation describing the vector field Y on $S_\Delta^{(n-1)}$ may be explicity constructed by differentiating the coordinate functions of the projection $y_i \doteq x_i/\rho^{r_i}(x)$. Skipping the tedious details it may be shown that the differential equation on the sphere is

$$\dot{y}_i = a_i(t, y) - y_i \sum_{k=1}^{n} \frac{r_i}{r_k} (y_k)^{\frac{c}{r_k} - 1} a_k(t, y), \quad i = 1, \ldots, n.$$

The solutions of the original vector field X are recovered from $x_i(t) = \rho^{r_i}(t) y_i(t)$. Thus the differential equation specifying $\rho(t)$ is required. The equation is

$$\dot{\rho} = \left(\sum_{k=1}^{n} \frac{1}{r_k} y_k^{\frac{c}{r_k} - 1} a_k(t, y) \right) \rho(x), \tag{7}$$

and may be obtained by differentiating $\rho(x(t))$ with respect to time. In the sequel we write $\dot{\rho} = Q(t, y)\rho$ in order to simplify notation.

The second item is proven by noting that the differential equation for ρ is linear in ρ. Hahn [17] observed that uniform asymptotic stability implies that the integral of the coefficient Q in the $\dot{\rho}$ equation has the following bound

$$\int_{t_0}^{t} Q(t, y(t)) \leq K_1 - K_2(t - t_0) \quad K_1 \in \mathbb{R}, K_2 > 0,$$

where K_1 and K_2 are independent to t_0. This bound implies that $\rho \to 0$ exponentially. In other words, $x = 0$ is ρ-exponentially stable. The result is global since any solution of the differential equation has a "local" analog which may be obtained via the dilation. □

3.5. **Uniqueness of solutions.** Uniqueness of solutions of ordinary differential equations is an important property for a mathematical model of any physical process. Uniqueness of solutions gives a precise mathematical interpretation of the physical concept of determinism. The models of the driftless systems considered in this paper are analytic so the only possible way for nonunique solutions to arise occurs when the control designer specifies feedback functions which do not have sufficient regularity to guarantee uniqueness in the closed-loop model. We will see below that in order to obtain exponential convergence the feedbacks must be non-Lipschitz. We establish some sufficient conditions that the feedbacks must satisfy in order to guarantee unique solutions of the closed-loop system.

A homogeneous vector field is completely specified by the values assumed on the set $\{x : \rho(x) = 1\}$ so any smoothness imposed on the vector field here is automatically extended to $\mathbb{R}^n \setminus \{0\}$ via the dilation. We will assume that the vector field is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$, i.e., for every $x \in \mathbb{R}^n \setminus \{0\}$ there exists a neighborhood of x and some $0 < L < \infty$ such that the vector field satisfies $\|X(t,y) - X(t,z)\| \leq L\|y - z\|$ for all y and z in this neighborhood. However, this does not imply that the vector field is Lipschitz in any neighborhood of the origin. Degree zero vector fields are of interest so we will concentrate on this case. The first component of a degree zero vector field is a degree one function (since $r_1 = 1$). Denote this component by $a(t,x)$ and assume that it is a locally Lipschitz function on $\mathbb{R}^n \setminus \{0\}$. For $x \neq 0$ the Dini derivative with respect to the ith variable, $D_i^+ a(t,x)$, exists by virtue of the local Lipschitz bound. A straight forward calculation shows that $D_i^+ a(t, \Delta_\lambda x) = \lambda^{1-r_i} D_i^+ a(t,x)$ for all $\lambda > 0$. If the dilation has some $r_i > 1$ then $\lim_{\lambda \to 0} |D_i^+ a(t, \Delta_\lambda x_0)| \to \infty$ and the vector field is not Lipschitz at the origin. However, it is still possible to conclude uniqueness of solutions in this case. This is proven in the next lemma.

Lemma 3.5. *Suppose $X(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous homogeneous degree zero vector field with respect to the dilation Δ_λ, uniformly bounded with respect to t, and $x = 0$ is an isolated equilibrium point. Furthermore suppose that X is locally Lipschitz everywhere except $x = 0$. Then the flow of X is unique.*

Proof. The point $x = 0$ is the only point where uniqueness may fail since X is not necessarily Lipschitz there. However no solution through a point $p \neq 0$ can reach the origin in finite time because this implies that $\rho(\phi(t, t_0, p) \to 0$ in finite time. This is not possible since the equation describing the evolution of ρ is $\dot{\rho} = Q(t,y)\rho$, where Q is a continuous function of y and uniformly bounded in t. The point y evolves on a compact set so the there always exists a bound

$$m \doteq \sup_{(t,y)} \|Q(t,y)\|.$$

The following inequalities on ρ hold as a result of the bound on Q,

$$c_1 e^{-m(t-t_0)} \leq \rho(x(t-t_0)) \leq c_2 e^{m(t-t_0)},$$

where the c_i's are positive constants. Similarly a solution cannot leave the origin in finite time. If this were possible then the time reversed vector field (which has the same bounds on $\rho(x(t-t_0))$ as its forward time counter part) has a solution which reaches the origin in finite time. This contradicts the above result. Thus solutions cannot leave or reach the origin in finite time and so $x(t) = 0$ for all t is the only solution passing through the origin. □

3.6. **Lyapunov functions for homogeneous degree zero vector fields.** This section reviews converse Lyapunov stability theory for homogeneous systems and gives an extension for degree zero periodic vector fields. These results are important since the feedbacks derived in this paper exponentially stabilize an approximation of the driftless system and the higher order (with respect to a dilation) terms neglected in the approximation process are shown to not locally change the stability of the system. The main theorem by Rosier in [43] states that given an autonomous continuous homogeneous vector field $\dot{x} = f(x)$ with asymptotically stable equilibrium point $x = 0$, there exists a homogeneous Lyapunov function smooth on $\mathbb{R}^n \setminus \{0\}$ and differentiable as many times as desired at the origin. His theorem has been extended to time-periodic degree *zero* systems in [41] and [27]. The extension is simple so the theorem is stated below without proof.

Theorem 3.6. *Suppose the differential equation $\dot{x} = X(t,x)$ satisfies the following properties:*

1. *X is continuous in t and x,*
2. *$X(t,0) = 0$ for all t,*
3. *$X(t+T,x) = X(t,x)$ for all $x \in \mathbb{R}^n$,*
4. *X is homogeneous* degree zero *(in x) with respect to the dilation $\Delta_\lambda = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)$,*
5. *the solution $x(t) = 0$ is asymptotically stable.*

Let k be a positive integer. Then there exists a function $\overline{V} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that,

a) *$\overline{V}(t,x)$ is smooth for $x \in \mathbb{R}^n \setminus \{0\}$,*
b) *$\overline{V}(t,0) = 0$, $\overline{V}(t,x) > 0$ when $x \neq 0$*
c) *$\overline{V}$ is degree k with respect to Δ_λ i.e. $\overline{V}(t,\Delta_\lambda x) = \lambda^k \overline{V}(t,x)$,*
d) *$\overline{V}(t+T,x) = \overline{V}(t,x)$ for all $x \in \mathbb{R}^n$ and smooth with respect to t,*
e) *$\frac{d\overline{V}}{dt}(t,x) = \frac{\partial \overline{V}}{\partial t}(t,x) + \nabla \overline{V}(t,x) \cdot X(t,x) < 0$ for all $x \neq 0$.*

Finally,the following proposition concerning the stability of perturbed degree zero vector fields concludes this section. The proof is elementary and follows the time-invariant case in [43].

Proposition 3.7. *Let $x = 0$ be an asymptotically stable equilibrium point of the T-periodic continuous homogeneous degree zero vector field $\dot{x} = X(t,x)$. Consider the perturbed system*

$$\dot{x} = X(t,x) + R(t,x). \tag{8}$$

Assume each component of $R(t,x)$ may be uniformly bounded by,

$$|R_i(t,x)| \leq m\rho^{r_i+1}(x) \quad i = 1,\dots,n, \quad x \in U,$$

where U is an open neighborhood of the origin and $\rho(\cdot)$ is a homogeneous norm compatible with the dilation that leaves the unperturbed equation invariant. Then $x = 0$ remains a locally exponentially stable equilibrium of the perturbed equation (8).

4. Lipschitz Feedback

The ρ-exponentially stabilizing feedbacks presented in Section 5 are not Lipschitz at the equilibrium point (which is taken to be the origin without loss of generality). However, we show that Lipschitz feedbacks which vanish at the origin cannot exponentially stabilize, in the standard sense, a driftless system. This is a consequence of the main theorem in the section which essentially states that a C^1 system, $\dot{x} = X(x, u)$, which has a linearized uncontrollable mode on the $j\omega$-axis, cannot be exponentially stabilized with Lipschitz feedback that vanishes at the origin. This result is obvious in the case when the feedback is restricted to be continuously differentiable. Our theorem extends this situation to include Lipschitz feedback.

Some results from nonsmooth analysis will be reviewed (see [10]). The *generalized Jacobian* at $x \in \mathbb{R}^n$ of a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$ is defined as the set

$$\partial F(x) \doteq \mathrm{co}\ \{\lim DF(x_i) | x_i \to x, x_i \notin \Omega_F\},$$

where Ω_F is the set of measure zero where the standard Jacobian of F, DF, is not defined. In general, ∂F is a set valued map when F is Lipschitz but not C^1. Some useful properties of ∂F are:

1. ∂F is upper semicontinuous and
2. $\partial F(x)$ is a convex compact subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$.

Additional properties are given in [10]. When $X(t, x)$ is measurable in t and Lipschitz in x denote the flow of the corresponding differential equation $\dot{x} = X(t, x)$ as $\phi(t, \tau, x)$. The *linearization* of X about the trajectory $\phi(t, \tau, x)$ is represented by the *differential inclusion*

$$\dot{y}(s) \in \partial_x X(s, \phi(s, \tau, x))y(s), \quad s \in [\tau, t].$$

Define $\Phi(t, \tau)$ as the set of all linear matrix solutions to this differential inclusion. The *plenary hull* of $\Phi(t, \tau)$, denoted $R(t, \tau)$, is the set

$$R(t, \tau) = \{M | \langle v, Mw\rangle \leq \max[\langle v, Nw\rangle | N \in \Phi(t, \tau)]\ \forall v, w \in \mathbb{R}^n\}. \tag{9}$$

Clarke has established the following relationship between the generalized Jacobian of the flow and the plenary hull of the linearization solutions.

Theorem 4.1 ([10, Theorem 7.4.1])**.** *The map $F(x) \doteq \phi(t, \tau, x)$ is Lipschitz for all t, τ and satisfies $\partial F(x) \subset R(t, \tau)$.*

We will also make use of the following mean value theorem for set valued maps:

Theorem 4.2 ([10, Proposition 2.6.5])**.** *Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map. Then*

$$F(y) - F(x) \in co\ \partial F([x, y])(y - x),$$

where the set co $\{\partial F([x, y])\}$ is the convex hull of all points in $\partial F(z)$ with z on the straight line segment joining x and y.

With these preliminaries established we state the main result of this section. This theorem will be used to demonstrate that solutions of driftless systems with Lipschitz feedback do not satisfy an exponential stability bound. However the theorem is also applicable to general control systems.

Theorem 4.3. *Consider the control system $\dot{x} = X(x,u)$. Assume that X is C^1 in both arguments and $X(0,0) = 0$. Furthermore assume that the linearization around $x = 0$ and $u = 0$ has an uncontrollable mode with real part of the eigenvalue equal to zero. Then solutions of the closed-loop system with Lipschitz feedback $u(t,x)$ and $u(t,0) = 0$ do not satisfy the exponential stability bound $||\phi(t,x_0,t_0)|| \leq \beta ||x_0|| e^{-\alpha(t-t_0)}$ for any $\alpha, \beta > 0$.*

The proof of this theorem is aided by the following proposition.

Proposition 4.4. *Suppose a Lipschitz map $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the bound*

$$||F(x)|| \leq \frac{1}{2}||x||. \tag{10}$$

Then for any $v \in \mathbb{R}^n$ there exists $Z \in \partial F(0)$ such that $||Zv|| \leq 1/2||v||$.

Proof of Proposition 4.4. We first show that given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{co } \partial F([y,x]) \subset \partial F(0) + \epsilon \boldsymbol{B}, \forall ||x|| < \delta, ||y|| < \delta,$$

where $\boldsymbol{B}$ is the unit ball of $n \times n$ matrices measured by the Frobenius norm. From the upper semicontinuity of the generalized Jacobian, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\partial F(x) \subset \partial F(0) + \epsilon \boldsymbol{B}, \forall ||x|| < \delta.$$

Pick x and y with norm less than δ and choose arbitrary elements $X \in \partial F(x), Y \in \partial F(y)$. Combining the following relationships,

$$tX \in t(\partial F(0) + \epsilon \boldsymbol{B})$$
$$(1-t)Y \in (1-t)(\partial F(0) + \epsilon \boldsymbol{B}),$$

yields with $t \in [0,1]$,

$$tX + (1-t)Y \in \text{co } \{\partial F(0) + \epsilon \boldsymbol{B}\}.$$

However, the set $\{\partial F(0) + \epsilon \boldsymbol{B}\}$ is convex since $\partial F(0)$ is convex. Thus the convex combination of any matrices in $\partial F(x)$ and $\partial F(y)$ is also in the set $\partial F(0) + \epsilon \boldsymbol{B}$. Since

$$\text{co } \partial F([y,x]) = \text{co} \left[\bigcup_{z \in [x,y]} \partial F(z) \right],$$

then

$$\text{co } \partial F([x,y]) \subset \partial F(0) + \epsilon \boldsymbol{B}, \quad \forall ||x|| < \delta, ||y|| < \delta.$$

Now choose $v \in \mathbb{R}^n$ and a decreasing sequence $\{\epsilon_i\}$ such that $\lim_{i\to\infty} \epsilon_i = 0$ and $\epsilon_i > 0$. There exists a sequence $\{\delta_i]\}$, $\delta_i > 0$, such that

$$\text{co } \partial F([x,y]) \subset \partial F(0) + \epsilon_i \boldsymbol{B} \quad \forall ||x|| < \delta_i, \forall ||y|| < \delta_i.$$

Define $\lambda_i > 0$ such that $||\lambda_i v|| < \delta_i$ for all i. From Theorem 4.2, $F(\lambda_i v) \in \text{co } \partial F([0, \lambda_i v])\lambda_i v$ for $i = 1, 2, \ldots$. Thus there exists a sequence $\{Z_i\}$ with $Z_i \in \text{co } \partial F([0, \lambda_i v])$ such that $F(\lambda_i v) = Z_i \lambda_i v$ for $i = 1, 2, \ldots$. The bound in equation (10) implies that $||Z_i \lambda_i v|| \leq \frac{1}{2}||\lambda_i v||$ or that $||Z_i v|| \leq \frac{1}{2}||v||$. Define the distance between a point $x \in \mathbb{R}^n$ and set $A \subset \mathbb{R}^n$ as $\text{dist}\,(x,A) = \inf\{||x-y|| \ : \ y \in A\}$. Then, $\text{dist}\,(Z_i, \partial F(0)) \leq \epsilon_i$ since $Z_i \in \partial F(0) + \epsilon_i \boldsymbol{B}$. $\{Z_i\}$ is bounded so a convergent subsequence $\{Z_{\pi_i}\}$, with $\lim_i Z_{\pi_i} = Z$, may be chosen such that $||Z - Z_{\pi_i}|| \leq \epsilon_i$. Now $\text{dist}\,(Z, \partial F(0)) = 0$ since

$$\text{dist}\,(Z, \partial F(0)) \leq \text{dist}\,(Z, Z_{\pi_i}) + \text{dist}\,(Z_{\pi_i}, \partial F(0)) \leq 2\epsilon_{\pi_i},$$

and so $Z \in \partial F(0)$ since $\partial F(0)$ is compact. Lastly, $||Zv|| \leq \frac{1}{2}||v||$ since

$$||Zv|| \leq ||Z_{pi_i}v|| + ||(Z - Z_{\pi_i})v|| \leq (\frac{1}{2} + \epsilon_{\pi_i})||v||.$$

□

Proof of Theorem 4.3. Assume that the linearization of $X(x,u)$ about $(0,0)$ is given by $\dot{\xi} = A\xi + Bv$. Since there is an uncontrollable mode, after a coordinate change, A and B may be partitioned as

$$\begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix} \qquad \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix},$$

where "**0**" represent matrices of zeros of appropriate dimension. Now construct a non-zero positive semidefinite matrix P such that the time derivative of $\xi^T P\xi$ is zero along solutions of the linearization, i.e. $\xi^T(A^T P + PA)\xi + \xi^T PBu + u^T B^T P\xi = 0$. This construction is simple because if A_{22} has a mode with eigenvalue zero and left eigenvector a then choose P as

$$P = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & aa^T \end{bmatrix}.$$

If the mode corresponds to an "oscillator" then there exists a non-zero positive semidefinite $\tilde{P}$ such that $A_{22}^T\tilde{P} + \tilde{P}A_{22} = 0$. This is easily confirmed by placing A_{22} into real Jordan canonical form. In this case P is chosen as

$$P = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{P} \end{bmatrix}.$$

Now consider the original system $\dot{x} = X(x,u)$ with feedback $u(t,x)$. Suppose u is Lipschitz in x and $u(t,0) = 0$ for all t. The linearization of this system about the solution $x(t) = 0$ is the differential inclusion

$$\dot{y}(t) \in \partial_x X(0, u(t,0))y(t).$$

Using the definition of generalized Jacobian and a chain rule for Lipschitz functions, the right hand side of the differential inclusion is

$$\begin{aligned} \partial_x X(0, u(t,0)) &= \{A + B\partial_x u(0,t)\} \\ &= A + \{B\partial_x u(0,t)\}. \end{aligned}$$

The notation is clear: any element of $\partial_x X(0, u(t,0))$ can be written as A plus an element of $B\partial_x u(t,0)$. Thus if $\tilde{\gamma}(t)$ is a measurable selection of $\partial_x X$ then $\tilde{\gamma}(t) = A + \gamma(t)$ where $\gamma(t)$ is a measurable selection of $B\partial_x u(t,0)$. Note that for any element $N \in B\partial_x u(t,0)$, the product $PN = 0$, where P is the matrix constructed above, so $P\gamma(t) = 0$ for all t.

Let $A + \gamma_1(t)$ and $A + \gamma_2(t)$ be two measurable selections. Define $\xi_1(t)$ and $\xi_2(t)$ as the (absolutely continuous) solutions to the corresponding linear differential equation, i.e. $\dot{\xi}_i(t) = (A + \gamma_i(t))\xi_i(t)$ almost everywhere (a.e.). Consider the function $V(t) = \xi_1^T(t)P\xi_2(t)$. The time derivative of V is

$$\begin{aligned} \dot{V} &= \dot{\xi}_1^T P\xi_2 + \xi_1^T P\dot{\xi}_2 \\ &= \xi_1^T(\gamma_1^T P + P\gamma_2 + A^T P + PA)\xi_2 \\ &= 0 \text{ a.e.}. \end{aligned}$$

Thus V is constant since it is absolutely continuous. By choosing arbitrary initial conditions for the two equations we obtain

$$\begin{aligned} V(t) &= \xi_1^T(\tau)\phi_1^T(t,\tau)P\phi_2(t,\tau)\xi_2(\tau) \\ &= \xi_1^T(\tau)\phi_1^T(\tau,\tau)P\phi_2(\tau,\tau)\xi_2(\tau) \\ &= \xi_1^T(\tau)P\xi_2(\tau) \quad \text{for all } t, \tau, \end{aligned}$$

where ϕ_1 and ϕ_2 are principal matrix solutions of the two linear systems ($\phi_1, \phi_2 \in \Phi$). This expression holds for arbitrary ξ_1 and ξ_2 so $N_1^T P N_2 = P$ for all $N_1, N_2 \in \Phi(t,\tau)$.

Next we show that every element M of the plenary hull $R(t,\tau)$ satisfies $M^T P M = P$. Recall that for any $v, w \in \mathbb{R}^n$

$$\langle v, Mw\rangle \leq \max_{N \in \Phi} \langle v, Nw\rangle.$$

Setting $v = PN_0 v_0$ where $N_0 \in \Phi$ and $v_0 \in \mathbb{R}^n$ yields

$$\begin{aligned} v_0^T N_0^T PMw &\leq \max_{N\in\Phi} v_0^T N_0^T PNw \\ &= v_0^T Pw. \end{aligned}$$

Replacing v_0 with $-v_0$ gives us the inequality in the other direction. Thus $v_0^T N_0^T PMw = v_0^T Pw$ for arbitrary v_0 and w, or what is the same

$$N_0^T PM = P.$$

Using this expression, $N_0^T PM = P = N_0^T P N_0$ and since N_0 is invertible

$$PM = PN_0 \implies P^{1/2}M = P^{1/2}N_0 \implies M^T PM = P,$$

where $P^{1/2}$ is the unique square root of P.

If solutions of the original closed-loop systems satisfy the standard exponential stability bound, i.e., $\|\phi(t,t_0,x)\| \leq \beta\|x\|\exp(-\alpha(t-t_0))$, for some $\alpha > 0$ and $\beta > 0$, then the difference $t - t_0$ may be chosen large enough so that the constant $\beta e^{\alpha(t-t_0)} \leq 1/2$. The map $F(x) \doteq \phi(t,t_0,x)$ then satisfies $\|F(x)\| \leq \frac{1}{2}\|x\|$. Choose $w \in \mathbb{R}^n$ to be the eigenvector of P corresponding to the largest eigenvalue. Proposition 4.4 implies that there exists a matrix $Z \in \partial F(0)$ such that $\|Zw\| \leq 1/2\|w\|$. However, from the relations established above $Z^T PZ = P$ since $Z \in R(t,t_0)$. From the choice of w we have $w^T P w = \|P\| \cdot \|w\|^2$. But this value is equal to

$$\begin{aligned} w^T Z^T PZw &= \|P^{1/2}Zw\|^2 \\ &\leq \|P\| \cdot \|Zw\|^2 \\ &\leq \frac{1}{4}\|P\| \cdot \|w\|^2. \end{aligned}$$

This is clearly a contradiction. The contraction property of Z is inconsistent with the property that $Z^T PZ = P$ which was derived from the fact that the linearization of the system has an uncontrollable mode with real part equal to zero. □

The linearization of the driftless system (1) about $x = 0$ and $u = 0$ is

$$\dot{\xi} = X_1(0)v_1 + \cdots + X_m(0)v_m.$$

If the number of input vector fields is less than the state dimension the conditions Theorem 4.3 are always satisfied. Thus a Lipschitz feedback which is zero at the origin cannot exponentially stabilize the origin.

An example of a system with drift vector field is the Euler equations with two "inputs"

$$\begin{aligned} \dot{\omega}_1 &= u_1 \\ \dot{\omega}_2 &= u_2 \\ \dot{\omega}_3 &= \omega_1 \omega_2. \end{aligned} \tag{11}$$

If the origin is an equilibrium point then necessarily $u_i(t, 0) = 0$. The linearization of these equations about $x = 0$, $u = 0$ is the system

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This linearization satisfies the hypothesis of Theorem 4.3 and so cannot be exponentially stabilized with Lipschitz feedback.

A result by Gurvitz and Li [16] states that an exponentially stabilizing (in the standard sense) feedback must be Hölder continuous with Hölder exponent equal to the inverse of number of Lie brackets required to achieve full rank in the control Lie algebra. This result is tighter than Theorem 4.3 when applied to driftless systems but Theorem 4.3 also applies to systems with drift. In addition, the feedbacks derived in the next section ρ-exponentially stabilize the driftless system. It is still an open question as to whether standard exponential stability can be achieved for driftless systems using non-Lipschitz feedback.

5. Synthesis Methods

We now consider how to obtain exponentially stabilizing feedbacks. The use of homogeneous feedback is strongly motivated by the existence of a controllable homogeneous approximating system. If homogeneous degree one control functions $u_i(t, x)$ can be found such that $x = 0$ is a uniformly asymptotically stable equilibrium point of the closed-loop system then $x = 0$ is exponentially stable with respect to the homogeneous norm ρ since the closed-loop vector field is degree zero (Lemma 3.4). Thus, the stability type is not the familiar exponential stability definition but rather ρ-exponential stability. As pointed out in Section 3.3, ρ-exponential stability can be locally recast into the bound $\|\phi(t, t_0, x_0)\|_2 \leq M\|x_0\|_2^{1/\sigma} e^{-\alpha(t-t_0)}$ for some $M > 0, \alpha > 0, \sigma > 1$. Thus each state is bounded by a decaying exponential envelope but the dependence on the initial condition is allowed to be more general than that in the usual definition of exponential stability. The standing assumption in the remainder of the paper is that the system (2) has been transformed to the adapted coordinates and that a degree one homogeneous approximation has been computed. The dilation associated with the input vector field approximations and feedbacks will always have $r_n > 1$ since at least one level of Lie brackets is required to achieve controllability of the system. Thus the degree one feedbacks are not Lipschitz at the origin even though they may be locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$.

Our objective is not to derive methods applicable to general controllable driftless systems but rather to concentrate on cases that model true engineering systems. Several researchers have either given explicit smooth controllers or constructive algorithms that produce smooth time-periodic feedbacks which asymptotically mobile robot and satellite models [50, 40, 51]. A desirable aspect of these methods is the fact that, in many instances, the control laws can be written in terms of algebraic operations between simple functions. The implication of this fact should not be underestimated: implementation of such a control law is very straight forward. However, Theorem 4.3 implies that in some neighborhood of the equilibrium point the convergence of the system in slow. If this neighborhood is sufficiently small for the particular application then no improvement of the convergence rate is required. However, we shall demonstrate in Section 6 that a standard smooth feedback applied to an experimental mobile robot does not perform satisfactorily. The algorithm in Theorem 5.1 gives sufficient conditions under which smooth asymptotically stabilizing feedbacks can be rescaled into homogeneous ρ-exponentially stabilizing feedbacks. The design procedure is systematic in the sense that if the conditions of the theorem are satisfied then the homogeneous feedback may be computed directly from the original feedback. This algorithm is applied to the smooth feedback used for the mobile robot and results in an enormous improvement in convergence rate. Furthermore, implementing the homogeneous feedback requires only slightly more programming effort than the smooth feedback since the rescaling is performed in real-time.

A more direct method of computing ρ-exponential stabilizers, based on an extension of Pomet's original algorithm [40], is briefly noted at the end of this section.

Recall the Euler vector field, $X_E(x)$, corresponding to this dilation is given by the equations $\dot{x}_i = r_i x_i$, $i = 1, \dots, n$. The following theorem specifies the conditions under which an asymptotic stabilizer can be modified into an exponential stabilizer. Most smooth stabilizing controllers are time-periodic so we restrict ourselves to this case.

Theorem 5.1. *Suppose the closed-loop driftless system* $\dot{x} = \sum_{i=1}^{m} X_i(x) u_i(t,x)$ *satisfies the following conditions*

1. *the input vector fields are homogeneous degree one with respect to* Δ_λ,
2. *the feedbacks* $u_i(t,x)$ *are smooth,* T*-periodic and asymptotically stabilize the origin,*
3. *there exists a smooth, positive definite,* T*-periodic function* $V(t,x)$ *such that* $\frac{d}{dt}V(t,x) < 0$ *along non-zero solutions of the closed-loop system,*
4. *for some constant* $C > 0$ *the family of level sets parametrized by* t,
$$G_t^C = \{x | V(t,x) = C\},$$
are transversal to the Euler vector field for all t.

Under these conditions, the original feedbacks may be modified to the following ρ*-exponentially stabilizing feedbacks,*

$$\tilde{u}_i(t,x) = \tilde{\rho}(t,x) u_i(t, \gamma_t(x)) \quad i = 1, \dots, m,$$

where $\tilde{\rho} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ *is a uniquely defined homogeneous degree one function such that*

$$\tilde{\rho}(t,x)|_{x \in G_t^C} = 1,$$

and the map $\gamma_t : \mathbb{R}^n \setminus \{0\} \to G_t^C$ *is defined as*

$$\gamma_t(x) = \delta_\lambda x = \overline{x} \in G_t^C \qquad \textit{for some } \lambda > 0.$$

Remark 5.2. In many cases the stabilizing feedback is derived from Lyapunov analysis and so the closed-loop system has a function which may be tested for the properties given in the theorem.

Remark 5.3. What makes this method attractive from an implementation point of view is the fact that the function $\tilde{\rho}(t,x)$ is easily computed by searching over a single *scalar* parameter λ such that $V(t,\Delta_\lambda x) = C$. In addition $V(t,\Delta_\lambda x)$ is a monotone increasing function of λ in a neighborhood of the λ which satisfies this expression. This search may be performed efficiently in real-time.

Remark 5.4. This theorem also suggests a method for modifying smooth feedbacks for general driftless systems to obtain ρ-exponential stabilizers. The first step is to compute the homogeneous approximation of the input vector fields and write the smooth feedbacks in the new coordinates. If it can be verified with a Lyapunov function that the smooth feedbacks stabilize the approximation, then the Lyapunov function can be tested for the properties in Theorem 5.1. The higher order terms neglected during the approximation process do not affect the local ρ-exponential stability of the original system with the modified feedbacks. This follows from application of Proposition 3.7.

Proof. We first show that $\tilde{\rho}$ and γ_t are well defined quantities. Define the value of the function $g : \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ to be the $\lambda \in \mathbb{R}^+$ which solves

$$F(\lambda, t, x) \doteq V(t, \delta_\lambda x) - C = 0. \tag{12}$$

In other words, $g(t,x) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ returns the dilation scaling factor required to map the point $x \neq 0$ to the point $\overline{x} \in G_t$ on the same homogeneous ray at time t. The point $\overline{x}$ is unique since the transversality condition implies that the projection $\pi|_{G_t^C} : G_T^C \to S_\Delta^{n-1}$ is a local diffeomorphism. Furthermore, since G_t^C is compact and connected [52, Theorem 3.7] there is only one point in the preimage of $(\pi|_{G_t^C})^{-1}(y), y \in S_\Delta^{n-1}$. Hence the projection is a global diffeomorphism between G_t^C and S_Δ^{n-1} for each fixed t. The map from x to $\overline{x}$ is $(\pi|_{G_t^C})^{-1} \circ \pi$ and $g(t,x) = \rho(\overline{x})/\rho(x)$. The smoothness of g is determined with the implicit function theorem as shown below. Suppose that (λ, t, x) satisfies (12), then we compute

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x) &= \left(-\frac{1}{\partial F/\partial \lambda} \frac{\partial F}{\partial t} \right)(t, \Delta_\lambda x) \\ &= \left(-\frac{1}{\partial F/\partial \lambda} \frac{\partial V}{\partial t} \right)(t, \Delta_\lambda x). \end{aligned}$$

The quantity $\partial F/\partial \lambda(t, \Delta_\lambda x)$ is nonzero since

$$\begin{aligned} \frac{\partial F}{\partial \lambda}(t, \Delta_\lambda x) &= \frac{\partial V}{\partial \lambda}(t, \Delta_\lambda x) \\ &= \sum_{i=1}^n \frac{\partial V}{\partial x_i}(t, \Delta_\lambda x) r_i \lambda^{r_i - 1} x_i \\ &= \frac{1}{\lambda} \sum_{i=1}^n \frac{\partial V}{\partial x_i}(t, \Delta_\lambda x) r_i \lambda^{r_i} x_i \\ &= \frac{1}{\lambda} L_{X_E} V(t, \Delta_\lambda x). \end{aligned}$$

The term $L_{X_E} V(t, \Delta_\lambda x)$ is precisely the transversality condition on the set G_t^C so $\partial F/\partial \lambda(t, \Delta_\lambda x) \neq 0$. Similarly, $\partial g/\partial x_i(t,x) \neq 0$ for all $x \in G_t^C$. Thus g is smooth by the implicit function theorem.

We now show that g is *degree* -1. If $g(t,x) = \lambda$, then $g(t, \Delta_\sigma x)$ is the λ_0 that solves $V(t, \Delta_{\lambda_0}\Delta_\sigma x) - C = 0$. Since $\delta_{\lambda_0}\delta_\sigma x = \Delta_{\lambda_0\sigma} x$ then $\lambda = \lambda_0\sigma$ so $g(t, \delta_\sigma x) = \lambda/\sigma = g(t,x)/\sigma$.

The function $\gamma : \mathbb{R} \times \mathbb{R}^n \setminus \{0\} \to G_t$ is,

$$\gamma(t,x) \doteq \Delta_{g(t,x)} x.$$

Note that $\gamma(t, \Delta_\lambda x) = \gamma(t,x)$ for all $\lambda > 0$. $\tilde{\rho} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ is defined as

$$\tilde{\rho}(t,x) \doteq \begin{cases} \frac{1}{g(t,x)} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Furthermore, for any $\overline{x} \in G_t$, $\tilde{\rho}(t, \overline{x}) = 1$ since $\gamma(t, \overline{x}) = \overline{x}$. The definitions may be used to show that $\gamma(t, \cdot)$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and $\tilde{\rho}(t, \cdot)$ is continuous on $\mathbb{R}^n$ and smooth $\mathbb{R}^n \setminus \{0\}$. Furthermore, $\tilde{\rho}$ is homogeneous degree 1. T-periodicity of $\tilde{\rho}$ and γ is evident from the fact that V is T-periodic.

The modified feedbacks are defined as,

$$\tilde{u}_i(t,x) \doteq \tilde{\rho}(t,x) u_i(t, \gamma(t,x)). \tag{13}$$

These functions are degree one since

$$\begin{aligned} \tilde{u}_i(t, \Delta_\lambda x) &= \tilde{\rho}(t, \Delta_\lambda x) u_i(t, \gamma(t, \Delta_\lambda x)) \\ &= \lambda \tilde{\rho}(t,x) u_i(t, \gamma(t,x)) \\ &= \lambda \tilde{u}_i(t,x). \end{aligned}$$

These functions agree with the original feedbacks on G_t^C; i.e., for $\overline{x} \in G_t^C$, $\tilde{u}_i(t, \overline{x}) = u_i(t, \overline{x})$. We now show that the closed-loop system with the newly defined feedbacks is ρ-exponentially stable. The closed-loop system with the new feedbacks is denoted $\dot{x} = \tilde{X}(t,x)$. The closed-loop systems is degree zero since the feedback is degree one and the input vector fields are degree one. Hence, all we need to show is uniform asymptotic stability with the modified feedbacks. This is accomplished by using $\tilde{\rho}(t,x)$ as a Lyapunov function.

First we show that $\tilde{\rho}$ is positive definite and decrescent. The assumptions on $V(t,x)$ imply that there exist two positive definite, strictly increasing functions, ϕ_1 and ϕ_2, such that $\phi_1(\|x\|) \leq V(t,x) \leq \phi_2(\|x\|)$ for all x and t. Any $x \in G_t^C$ must satisfy the bounds, $\phi_2^{-1}(c) \leq \|x\| \leq \phi_1 - 1$. Defining the constants,

$$c_1 = \min_{\|x\| = \phi_2^{-1}(c)} \rho(x) \quad \text{and} \quad c_2 = \max_{\|x\| = \phi_1^{-1}(c)} \rho(x),$$

it is straightforward to verify that $c_1\rho(x) \leq \tilde{\rho}(t,x) \leq c_2\rho(x)$ for all x and t. Thus $\tilde{\rho}$ is positive definite and decrescent. Define the function $\tilde{V}(t,x) = \tilde{\rho}$. The time derivative of $\tilde{V}$ along non-zero solutions of the system with the feedback in equations (13) is

$$\begin{aligned} \frac{d\tilde{V}}{dt}(t,x) &= \left(\frac{d}{dt}\frac{1}{g}\right)(t,x) \\ &= -\frac{1}{g^2(t,x)} \left(\frac{\partial g}{\partial t}(t,x) + D_x g(t,x)(\tilde{X})\right)(t,x) \\ &= -\frac{1}{g^2(t,x)} \left(-\frac{g(t,x)}{L_{X_E}V(t,\overline{x})} \frac{\partial V}{\partial t}(t, \overline{x}) \right. \\ &\quad \left. -\frac{1}{L_{X_E}V(t,\overline{x})} \sum_{i=1}^n g^{r_i+1}(t,x) \frac{\partial V}{\partial x_i}(t,\overline{x}) \tilde{X}_i(t,x)\right) \quad \overline{x} = \delta_{g(t,x)} x \in G_t \end{aligned}$$

$$= \frac{1}{g(t,x) L_{X_E} V(t,\overline{x})} \left(\frac{\partial V}{\partial t}(t,\overline{x}) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,\overline{x}) \tilde{X}_i(t, \delta_{g(t,x)} x) \right)$$
$$= \frac{1}{g(t,x) L_{X_E} V(t,\overline{x})} \left(\frac{\partial V}{\partial t}(t,\overline{x}) + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(t,\overline{x}) X_i(t,\overline{x}) \right)$$
$$= \frac{\tilde{\rho}(t,x)}{L_{X_E} V(t,\overline{x})} \frac{dV}{dt}(t,\overline{x}).$$

The only remaining fact to show is that $L_{X_E} V(t,\overline{x}) > 0$. $L_{X_E} V(t,\overline{x})$ has constant sign from transversity so initially assume that this quantity is negative. For ϵ sufficiently small the points in the sets $G_t^{C+\epsilon}$ and $G_t^{C-\epsilon}$ also satisfy $L_{X_E} V < 0$. As shown above, these sets are diffeomorphic to spheres (for t fixed) and so separate $\mathbb{R}^n$ into an exterior and interior domain. Fix an arbitrary $t_0 \in [0,T)$. The trajectory of X_E pierces each set only once and since $L_{X_E} V < 0$ then we conclude that $G_{t_0}^{C+\epsilon}$ sits inside the interior domain of $G_{t_0}^{C}$ which sits inside the interior domain of $G_{t_0}^{C-\epsilon}$. This holds for all t since t_0 is arbitrary. If we start the system $\dot{x} = X(t,x)$ with an initial condition (τ, x) in the set $G_\tau^{C-\epsilon}$ then at some time later the trajectory enters the ball radius of $\min_{t\in[0,t), x\in G_t^{C+\epsilon}} \|x\|$ by asymptotic stability. Thus at some $\tau' > \tau$ the trajectory crosses $G_{\tau'}^{C+\epsilon}$ but $V(\tau', x(\tau')) = C + \epsilon > V(\tau, x(\tau)) = C - \epsilon$ which contradicts the fact that $\dot{V} < 0$. Hence, $L_{X_E} V(t,\overline{x}) > 0$ and the system with modified feedbacks is uniformly asymptotically stable. ρ-exponential stability follows from the fact that the closed-loop system is degree zero. □

The new feedback is as smooth on $\mathbb{R}^n \setminus \{0\}$ as the original feedback restricted to the level set of the Lyapunov function in the proof of Theorem 5.1. The original feedback is assumed to be smooth and so solutions of the closed-loop system with the modified feedback are unique by Lemma 3.5.

The following example demonstrates the algorithm on the prototype driftless system.

Example 5.5. This example uses the three-dimensional two input driftless system,

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2 u_1. \tag{14}$$

This system is its own homogeneous degree one $\mathcal{F}$-approximation. The dilation is $\Delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$. A smooth asymptotically stabilizing feedback for the system taken from [50] are the functions

$$u_1(t,x) = -x_1 + x_3 \cos t, \tag{15}$$
$$u_2(t,x) = -x_2 + x_3^2 \sin t. \tag{16}$$

Asymptotic stability of the closed-loop system can be shown using the Lyapunov function

$$V(t,x) = \left(x_1 - \frac{x_3}{2}(\cos t + \sin t) \right)^2 + \left(x_2 - \frac{x_3^2}{2}(\sin t - \cos t) \right)^2 + x_3^2.$$

Thus we need to check the transversality condition with a level set of the Lyapunov function. V may be approximated by the quadratic form $\tilde{V} = \langle x, Bx \rangle$ for C sufficiently small, where

$$B = \begin{bmatrix} 1 & 0 & -\frac{1}{2}\alpha \\ 0 & 1 & 0 \\ -\frac{1}{2}\alpha & 0 & 1 + \frac{1}{4}\alpha^2 \end{bmatrix},$$

and $\alpha = \cos t + \sin t \in [-\sqrt{2}, \sqrt{2}]$. The inner product between the level sets of $\tilde{V}$ and the Euler vector field is $L_{X_E}\tilde{V} = \langle x, \mathrm{diag}[r_i]Bx\rangle = \langle x, \tilde{B}x\rangle$, where $\tilde{B}$ is the symmetric matrix

$$\tilde{B} = \begin{bmatrix} 1 & 0 & -\frac{3}{4}\alpha \\ 0 & 1 & 0 \\ -\frac{3}{4}\alpha & 0 & 2+\frac{1}{2}\alpha^2 \end{bmatrix}.$$

Since $\tilde{B}$ is positive definite for all $\alpha \in [-\sqrt{2}, \sqrt{2}]$ the Euler vector field is transverse to any level set of $\tilde{V}$ and hence any level set of V for C sufficiently small. Numerical calculation reveals that value of $C = 1$ works well. The modification of the feedbacks is carried out as specified in the proof. Once the value of λ has been computed which satisfies $V(t, \Delta_\lambda x) = 1$ then we set $\tilde{\rho}(t,x) = 1/\lambda$ and $\overline{x} = \gamma(t,x) = \Delta_\lambda x$. The modified feedbacks are

$$\begin{aligned} \tilde{u}_1(t,x) &= \frac{1}{\lambda}\left(-\overline{x}_1 + \overline{x}_3\cos t\right) \\ &= \frac{1}{\lambda}\left(-\lambda x_1 + \lambda^2 x_3 \cos t\right) \\ &= -x_1 + \lambda x_3 \cos t \\ \tilde{u}_2(t,x) &= \frac{1}{\lambda}\left(-\overline{x}_2 + \overline{x}_3^2 \sin t\right) \\ &= \frac{1}{\lambda}\left(-\lambda x_2 + \lambda^4 x_3^2 \sin t\right) \\ &= -x_2 + \lambda^3 x_3^2 \sin t. \end{aligned} \tag{17}$$

Simulations comparing the performance of these feedbacks with the original smooth feedbacks are shown in Figure 2. The ρ-exponential stabilizer returns the system to a small neighborhood of the origin much faster than the smooth controller from which it was derived. The Euclidean norm of control commands are shown in Figure 3. The maximum effort expended by the ρ-exponentially stabilizing control law is slightly larger than that smooth controller.

The next example applies Theorem 5.1 to a controller derived with Pomet's algorithm [40].

Example 5.6. The control law is derived for the system in equations (14). The reader is referred to [40] for the details on the algorithm. The open-loop periodic generator is chosen as $u_2(t) = \alpha(t,x) = x_3(t)\cos t$. The Lyapunov function defined with this preliminary input is $V(t,x) = 1/2(x_1^2 + (x_2 - x_3\sin t)^2 + x_3^2)$. The asymptotically stabilizing feedbacks are computed to be

$$\begin{aligned} u_1 &= -x_1 - x_2(x_3 - (x_2 - x_3 \sin t)) \\ u_2 &= x_3 \cos t - x_2 + x_3 \sin t. \end{aligned}$$

The closed-loop system is globally asymptotically stable with this feedback. The gradient of V with respect to X_E is a quadratic form

$$L_{X_E}V = [x_1\ x_2\ x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2}\sin t \\ 0 & -\frac{3}{2}\sin t & 2+2\sin^2 t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This quadratic form is positive definite and implies that X_E is transverse to the level sets of V. Note that since V is quadratic the transversality condition holds globally i.e. any level set of V may be chosen as the scaling set. A level set of

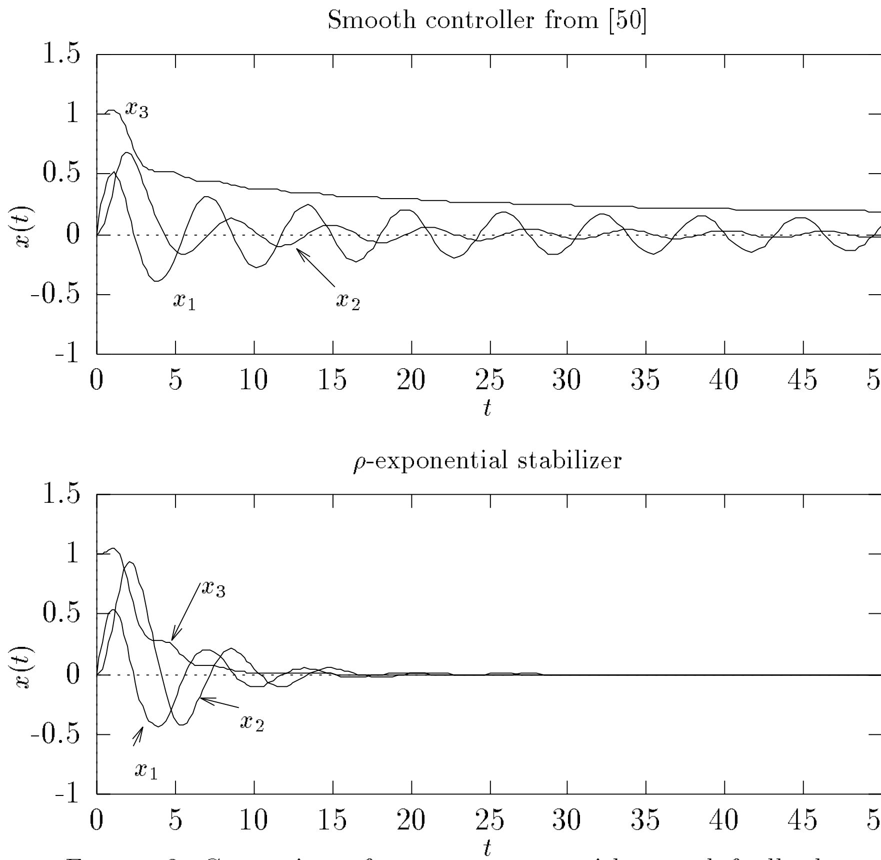

FIGURE 2. Comparison of system response with smooth feedback (top figure) and its modified version (bottom figure).

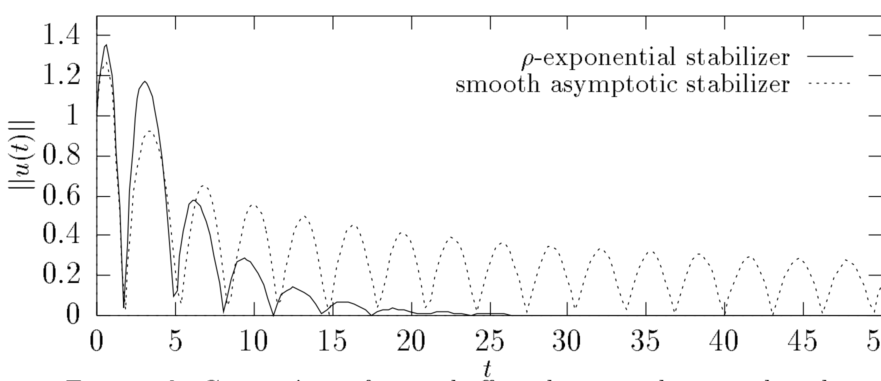

FIGURE 3. Comparison of control efforts between the smooth and modified controllers.

$V(t, x) = 0.5$ is chosen since the initial condition $x(0) = (0, 0, 1)$ is located on this set. The results of the simulations are shown in Figures 4 and 5.

Several comments are in order. The first is about a gain scheduling interpretation of the ρ-exponential stabilizers. In Example 5.5 the smooth controllers are rescaled

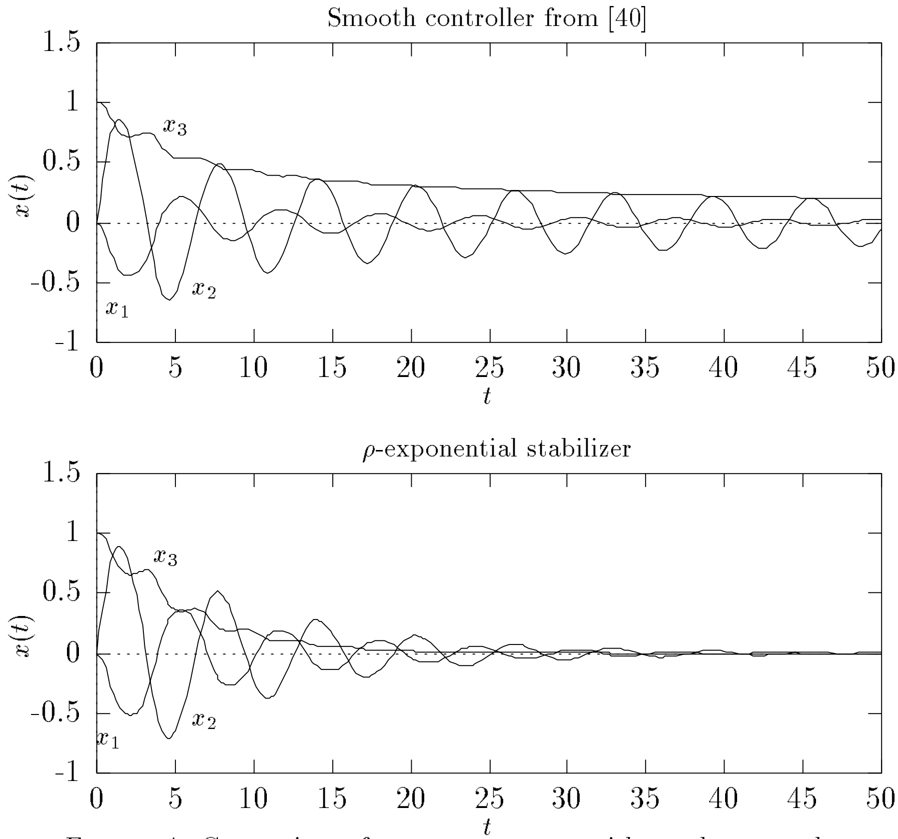

FIGURE 4. Comparison of systems responses with another smooth stabilizer (top figure) and its modified version (bottom figure).

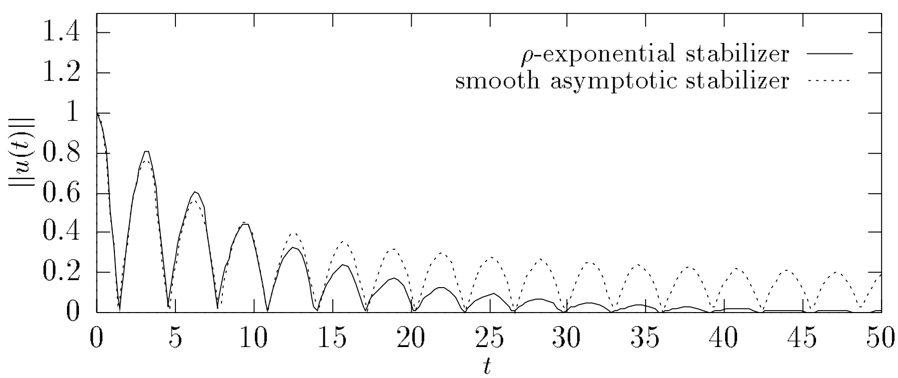

FIGURE 5. Control effort comparison.

into the feedbacks given by equations (17). The parameter λ that scales the x_3 terms is a "gain" that varies in such a manner to ensure exponential convergence of the state. The smooth feedbacks are recovered if λ is set to unity. In the ρ-exponential case, $\lambda \to \infty$ as the state converges to the origin. However, the products "λx_3" in $\tilde{u}_1$ and "$\lambda^3 x_3^2$" in $\tilde{u}_2$ converge to zero as x converges to zero so the control law remains continuous. A similar interpretation may be given to the ρ-exponential stabilizers derived in Example 5.6.

Another important issue is the control effort used in stabilizing the system. Both the maximum control magnitude and the energy in the control signal are useful quantities to consider. The control magnitude will be limited by actuator constraints and the amount of energy available to the controller will be dictated by the physical power source. It is straightforward to verify that

$$\sup_{x\in U_1, t\in\mathbb{R}} |\tilde{u}(t,x)| \leq \sup_{x\in U_2, t\in\mathbb{R}} |u(t,x)|$$

where $U_1 = \cap_t\{x|V(t,x) \leq C\}$ and $U_2 = \cup_t\{x|V(t,x) \leq C\}$. Thus the control effort for the homogeneous feedbacks with initial conditions in U_1 will not exceed the control effort commanded by the original feedbacks with initial conditions in U_2. If U_1 is not much "smaller" than U_2 then the homogeneous feedbacks will ρ-exponentially stabilize the equilibrium point, for approximately the same set of initial conditions as the original controller, with no increase in maximum control magnitude.

Finally, since the homogeneous controllers have a Hölder bound of the form $\|u(t,x)\| \leq \|x\|^{\sigma}$, $\sigma \in (0,1)$, the energy in the control signal is guaranteed to be finite. In the examples above, the rate of convergence of the x_3 variable with the smooth controller is approximately $1/\sqrt{t}$ for large t. Thus the smooth controllers in these examples require an infinite amount of energy to return the system to the origin.

We conclude this section by briefly mentioning another method to synthesize locally ρ-exponentially stabilizing controllers for a class of driftless systems. This method is an extension of Pomets's algorithm [40]. If the homogeneous approximation of the driftless system satisfies the rank condition,

$$\begin{aligned}\operatorname{rank}\Big\{&X^1_{\pi_1}, X^1_{\pi_2}, \ldots, X^1_{\pi_m},\\ &[X^1_{\pi_1}, X^1_{\pi_2}], \ldots, [X^1_{\pi_1}, X^1_{\pi_m}], \ldots,\\ &\operatorname{ad}^j_{X^1_{\pi_1}} X^1_{\pi_2}, \ldots, \operatorname{ad}^j_{X^1_{\pi_1}} X^1_{\pi_m}, \ldots\Big\}(x_0) = n,\end{aligned} \tag{18}$$

for some permutation, π, of the set $\{1,2,\ldots,m\}$, then the steps in [40] may be modified to produce homogeneous feedbacks, smooth on $\mathbb{R}^n \setminus \{0\}$, which ρ-exponentially stabilize the approximating system [32, 27]. This extended method is appealing because it is easy to check the condition in equation (18). The drawback of this approach is that the feedbacks must be stored in look-up tables. This is not an attractive feature for real-time implementation since the number of points which must be computed and stored grows exponentially with the power $n-1$ where n is the state dimension.

Certain driftless control systems may be transformed to exactly a nilpotent homogeneous form. Examples are the "chained form" or "power form" systems [36, 50]. In this case Theorem 5.1 provides a globally δ-exponentially stabilizing feedback since there are no "higher order" perturbing terms.

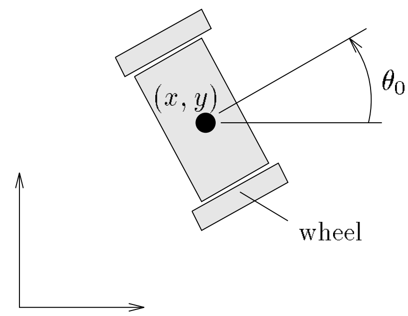

FIGURE 6. Coordinate system for the robot.

6. Experimental Results

Feedbacks derived from the algorithm in Theorem 5.1 are applied to an experimental mobile robot. The objective of the experiments is to verify that the algorithm may be executed in real-time and that the resulting homogeneous feedbacks actually ρ-exponentially stabilize the mobile robot.

The robot is configured so that it models the "kinematic wheel"

$$\begin{aligned}\dot{x} &= \cos\theta\nu\\ \dot{y} &= \sin\theta\nu\\ \dot{\theta} &= \omega\end{aligned} \tag{19}$$

where the coordinates (x, y, θ) are used to describe the position and orientation of the robot (see Figure 6). The control input ν is the forward velocity of the robot and ω its angular velocity. Forward and angular motion of the robot is achieved by changing the relative angular velocities of the wheels. Each wheel is driven by a stepper motor and any desired wheel angular velocity is achieved by commanding the motors to turn the appropriate number of steps per second. Sensing the position and orientation is accomplished with a passive two degree of freedom linkage which is attached to the robot and a fixed base. More details on the experimental apparatus, results with other ρ-exponentially stabilizing controllers and additional trailers, and important controller design issues are related in our paper [30].

A preliminary coordinate change is performed before deriving the feedbacks. Define the new coordinates as

$$\begin{aligned}z_1 &= \theta\\ z_2 &= x\cos\theta + y\sin\theta\\ z_3 &= x\sin\theta - y\cos\theta.\end{aligned} \tag{20}$$

The system equations transform to

$$\begin{aligned}\dot{z}_1 &= \omega\\ \dot{z}_2 &= \nu - z_3\nu\\ \dot{z}_3 &= z_1\nu.\end{aligned} \tag{21}$$

It is simple to verify that the $\mathcal{F}$-approximation of the system in equation (21) is obtained by dropping the $z_3\nu$ term from $\dot{z}_2$. The input vector fields are $X_1(z) = \partial/\partial z_1$ and $X_2(z) = \partial/\partial z_2 + z_1\partial/\partial z_3$ and are homogeneous degree one with respect to the dilation $\Delta_\lambda z = (\lambda z_1, \lambda z_2, \lambda^2 z_3)$. A smooth homogeneous norm is $\rho(z) = (z_1^4 + z_2^4 + z_3^2)^{\frac{1}{4}}$. This example is an instance where an initial transformation places the system into a form very close the to homogeneous approximation. Since the coordinate change in equation (20) is a global diffeomorphism, the feedbacks will perform well over a large domain of the (x, y, θ) state space. Had we taken the homogeneous approximation directly from the original system (19), the resulting coordinate change is generally a local diffeomorphism. This would restrict the region of validity of the control law to the set where the coordinate change is well defined. In practice, it is always desirable to take advantage of these preliminary changes of coordinates if they can be found.

The approximate system is the system in equation (14) (by reordering states and relabeling inputs). A locally stabilizing smooth feedback is given in Equation (16). The response of the mobile robot with this feedback is the top plot in Figure 7. The initial conditions are approximately $(x(0), y(0), \theta(0)) = (0\,\mathrm{m}, 0.3\,\mathrm{m}, 0\,\mathrm{rad})$. The slow convergence rate is evident from this figure. In an effort to improve the convergence rate, the smooth control law is modified to the homogeneous control law as outlined in Example 5.5. The rescaling is performed in real time during the experiment and so the law cannot be written down explicitly. The response of the robot with the rescaled feedback is the second plot in Figure 7. Note that although the transformed variables z satisfy a bound of the form,

$$\rho(z(t)) \le \beta\rho(z(0))e^{-\alpha t} \qquad \text{some } \alpha, \beta > 0,$$

the physical variables x, y, and θ satisfy the bound

$$\|(x, y, \theta)(t)\| \le M\|(x, y, \theta)(0)\|^{\frac{1}{2}}e^{-\alpha t} \qquad \text{some } M > 0$$

The convergence to the origin has shown vast improvement: after fifteen seconds the robot has returned to its desired configuration with the homogeneous feedback whereas the y position of the robot is 8 cm from the desired position with the smooth feedback. Another useful plot is a "top view" of the robot trajectory. This shown in Figure 8 where $y(t)$ is plotted with respect to $x(t)$. The trajectory of the robot with the smooth controller has been reflected about the x-axis in this figure in order to keep the plots uncluttered. The velocities specified by the control laws are shown in Figure 9. Each figure shows the "step rate" commanded to each stepper motor. The maximum control efforts are very close although the homogeneous control law effort exceeds that of the smooth control law by about twenty percent. The motors saturate at about 450 steps per second.

7. Torque inputs and dynamic extension

The mobile robot in the previous section is an example of a mechanical system in which a kinematic model is used for control design. That is, the velocity of the system is assumed to be a direct input which can be manipulated. In physical systems, however, actuators exert forces and not velocities. It is desirable to extend the homogeneous ρ-exponentially stabilizing kinematic controllers to ρ-exponentially controllers that command forces. The models we consider are very simple,

$$\dot{x} = X_1(x)u_1 + \cdots + X_m(x)u_m \qquad x \in \mathbb{R}^n \tag{22}$$

$$\dot{u} = v \qquad u, v \in \mathbb{R}^m. \tag{23}$$

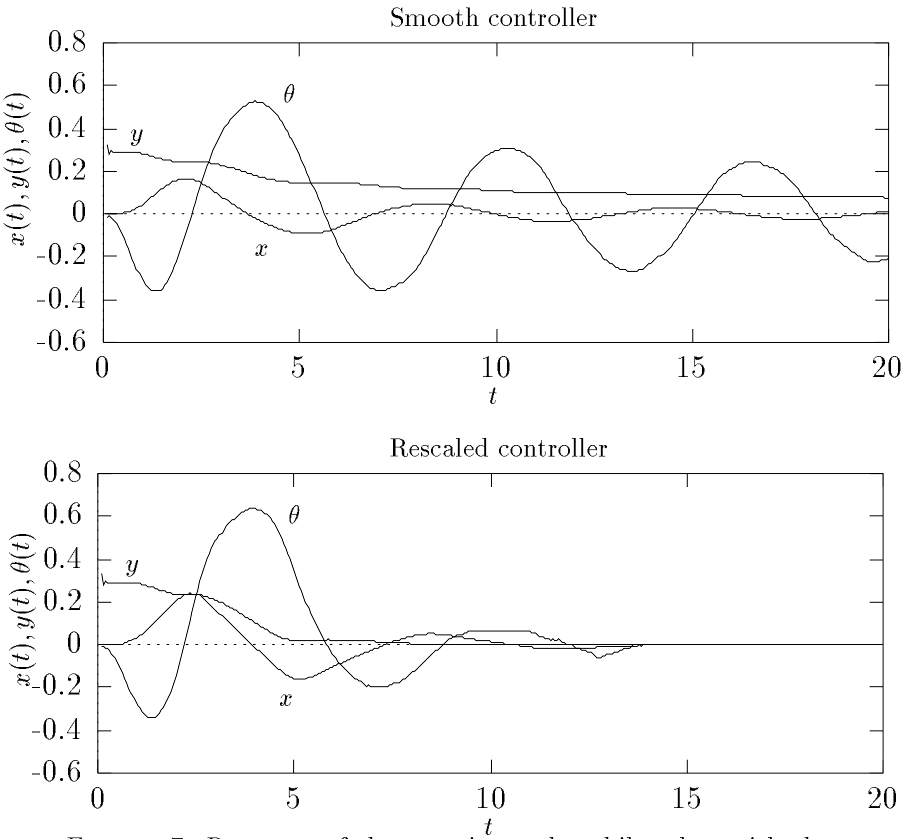

FIGURE 7. Response of the experimental mobile robot with the smooth feedback and the ρ-exponential stabilizer.

Equation (22) is termed the *kinematic* system. Equations (22) and (23) together represent the *dynamic* system. The kinematic system for mobile robots is derived from the Pfaffian constraints which describe the condition that the wheels roll but not slide. We model the dynamic portion of the system via a simple set of integrators. For many systems, more complicated dynamic behavior can be converted to this form using a state feedback control law [7].

The main result of this section gives a set of conditions under which a kinematic controller (i.e., one which assumes the velocities are the inputs) can be converted to a dynamic controller (one which uses the torques as the inputs) and still maintain ρ-exponential stability.

The hypothesis for the systems in this section are:

A1. the vector fields X_i are degree one with respect to a given dilation Δ_λ,

A2. the controls $u_i = \alpha_i(t, x), i = 1, \ldots, m$ are uniformly asymptotically stabilizing feedbacks (for the kinematic system) which are degree one in x with respect to Δ_λ, smooth on $\mathbb{R}^n \setminus \{0\}$ and time-periodic in t,

A3. $\operatorname{rank}[X_1(0) \cdots X_m(0)] = m$.

For smooth controllers, extending kinematic controllers to dynamic controllers has been explored, for example, by Walsh and Bushnell [51]. However, due to the nondifferentiable nature of exponential stabilizers we consider here, the usual

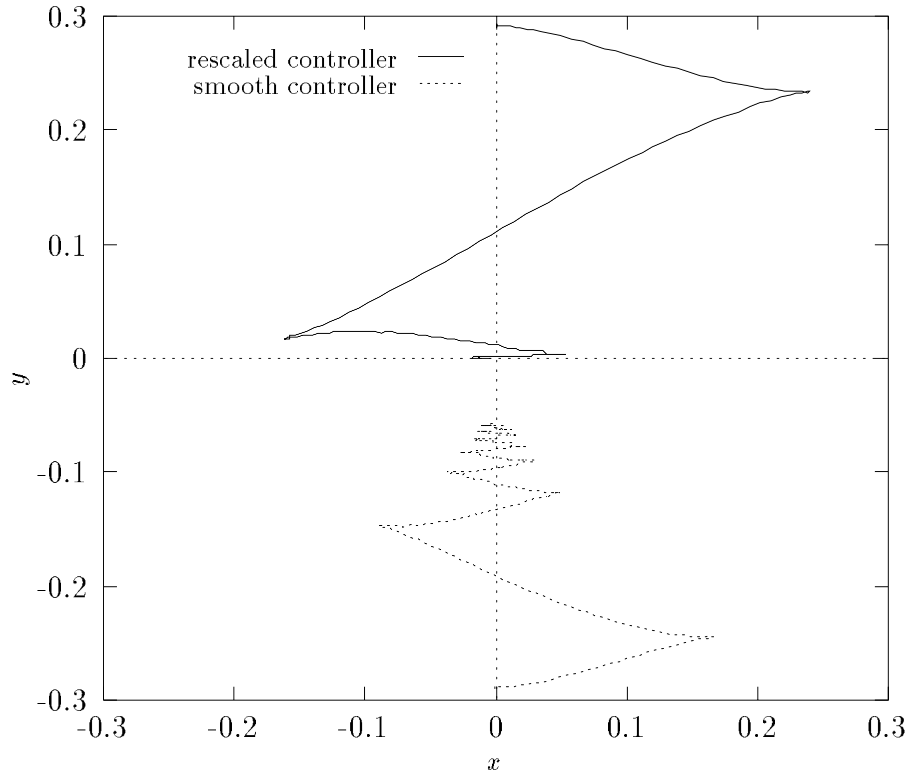

FIGURE 8. Top view of mobile robot trajectories.

control Lyapunov approach does not directly apply and must be modified to verify that the extended controller is well-defined and continuous. The use of continuous functions is important in applications since discontinuous control inputs usually are smoothed by the control electronics and/or the system dynamics and hence cannot be applied in practice, possibly resulting in loss of exponential rate of convergence.

Proposition 7.1. *Let $u = \alpha(t, x)$ be a feedback satisfying the assumptions A1 to A3. Then the feedback*

$$v_i = L_{\alpha X}\alpha_i + \frac{\partial \alpha_i}{\partial t} + k(\alpha_i - u_i), \quad i = 1, \ldots, m \tag{24}$$

globally exponentially stabilizes the dynamic system for $k > 0$ sufficiently large.

The notation αX is used to denote the vector field $\sum_i \alpha_i X_i$. Controller (24) is continuous for all (t, x, u) and smooth for all $x \neq 0$. Furthermore, the control law is homogeneous of degree one with respect to the extended dilation,

$$\tilde{\Delta}_\lambda(x, u) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n, \lambda u_1, \ldots, \lambda u_m). \tag{25}$$

Thus the closed-loop system remains degree zero with this feedback.

Proof. The closed-loop kinematic system is time-periodic, degree zero and asymptotically stable. This implies that there exists a time-periodic homogeneous Lyapunov function $V(t, x)$ such that $V(t, x) > 0$ for all $x \neq 0$ and all t which is strictly decreasing when $u = \alpha(t, x)$. This requires the converse Lyapunov theorem stated

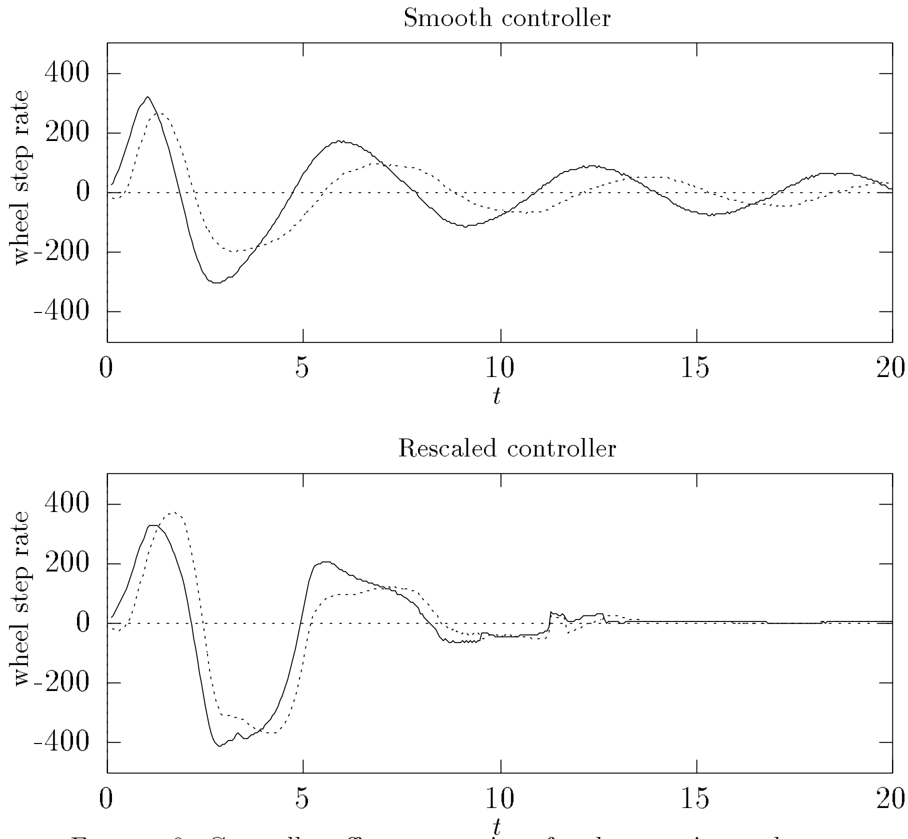

FIGURE 9. Controller effort comparison for the experimental mobile robot.

in Section 3.6. The Lyapunov function is chosen to be degree two with respect to Δ_λ. Thus the following bounds exist:

$$\begin{array}{c} c_1\rho^2(x) \leq V(t,x) \leq c_2\rho^2(x) \\ \frac{dV}{dt}\big|_{\dot{x}=\alpha X}(t,x) \leq -c_3\rho^2(x), \end{array} \tag{26}$$

for some $c_i > 0$ and where ρ is a homogeneous norm with respect to Δ_λ.

For the dynamic system with feedback (24) we use the following function,

$$W(t,x,u) = V(t,x) + \frac{1}{2}\sum_{i=1}^{m}(\alpha_i(t,x) - u_i)^2. \tag{27}$$

This function is positive definite on the extended phase space (x, u) and so is a candidate for a Lyapunov function. W is also degree two with respect to the extended dilation $\tilde{\Delta}$ defined in (25). Continuous partials of W with respect to x exist when $x \neq 0$, so in this case the derivative of (27) along the trajectories of the dynamic system with feedback (24) is

$$\dot{W} = \dot{V} + \sum_{i=1}^{m}\left(\sum_{l=1}^{m}\left(\sum_{j=1}^{n}\frac{\partial\alpha_i}{\partial x_j}X_l^{(j)}\right)u_l + \frac{\partial\alpha_i}{\partial t} - v_i\right)(\alpha_i - u_i) \quad x \neq 0,$$

where $X_l^{(j)}$ represents the j^{th} component of the l^{th} input vector field. Substituting the expression for v_i the derivative maybe expressed as

$$\begin{aligned}\dot{W} =& \dot{V} + \sum_{i=1}^{m}\left(\sum_{l=1}^{m}\left(\sum_{j=1}^{n}\frac{\partial \alpha_i}{\partial x_j}X_l^{(j)}\right)(u_l-\alpha_l) - k(\alpha_i - u_i)\right)(\alpha_i - u_i)\\ =& \dot{V} + (\alpha - u)^T(-kI_m + Q(t,x))(\alpha - u).\end{aligned}$$

I_m denotes the $m \times m$ identity matrix and $Q(t,x)$ is an $m \times m$ matrix with ij^{th} component given by $[Q]_{ij} = -1/2(L_{X_i}\alpha_j + L_{X_j}\alpha_i)$. $L_{X_i}\alpha_j$ is a degree zero function and so is not necessarily defined at $x = 0$.

Assumption A3 guarantees that no non-trivial trajectory of the closed-loop system is contained in the set $Z = \{(x,u) : x = 0, u \neq 0\}$. If a trajectory passes through the set Z at time t^* then $\frac{dW}{dt}(t^*)$ may not be defined, however the continuity of $\frac{dW}{dt}(t^* + \epsilon)$ for $\epsilon > 0$ implies that the upper right Dini derivative of $W(t)$ at t^* is given by $D^+W(t^*) = \lim_{\epsilon\to 0^+} dW/dt(t^* + \epsilon)$. Substituting the original expression for $\dot{W}$ when $x \neq 0$ into the expression for D^+W and noting that $\dot{V}$ is continuous in all arguments yields

$$\begin{aligned}D^+W(t^*) =& \lim_{\epsilon\to 0^+}\left[\frac{dV}{dt} + (\alpha - u)^T(-kI_m + Q)(\alpha - u)\right]_{t=t^*+\epsilon}\\ =& \frac{dV}{dt}(t^*, x(t^*)) - k\|\alpha(t^*, x(t^*)) - u(t^*)\|^2\\ &+ \lim_{\epsilon\to 0^+}\left[(\alpha - u)^T Q(\alpha - u)\right]_{t^*+\epsilon}\\ \leq& \frac{dV}{dt}(t^*, x(t^*)) + (-k+q)|\alpha(t^*, x(t^*)) - u(t^*)|^2\end{aligned} \tag{28}$$

where $\|\cdot\|$ is the Euclidean norm and $q = \sup_{t\in[0,2\pi), x\neq 0}\|Q(t,x)\|$. The bound q is well defined since Q is degree zero and assumes all of its values when restricted to the homogeneous sphere $\{x : \rho(x) = 1\}$. When $x \neq 0$ the expression for the derivative is continuous so the Dini derivative reduces to the actual derivative. Thus the bound in equation (28) is valid for all t, x and u,

$$D^+W(t) \leq dV/dt + (-k+q)\|\alpha - u\|^2 \quad \forall t, x, u.$$

Substituting in $\dot{V}$ yields,

$$D^+W \leq \frac{\partial V}{\partial t} + \sum_{k=1}^{m}\alpha_k L_{X_k}V + \sum_{k=1}^{m}(u_k - \alpha_k)L_{X_k}V + (-k+q)\|\alpha - u\|^2 \qquad \forall t, x, u. \tag{29}$$

The first two terms on the right side of the inequality are the time derivative of V along trajectories of the system when $u = \alpha(t,x)$ and may be bounded by $-c_3\rho^2(x)$ from equation (26). The third term to the right of the inequality may be bounded by $c_4\rho(x)\|u - \alpha\|$ for some $c_4 > 0$. Substituting these bounds into equation (29) yields,

$$\begin{aligned}D^+W \leq& -c_3\rho^2(x) + c_4\rho(x)\|u - \alpha\| + (-k+q)\|\alpha - u\|^2\\ =& \left[\rho(x) \quad \|u - \alpha(t,x)\|\right]\begin{bmatrix} -c_3 & \frac{1}{2}c_4 \\ \frac{1}{2}c_4 & -k+q\end{bmatrix}\begin{bmatrix}\rho(x) \\ \|u - \alpha(t,x)\|\end{bmatrix}.\end{aligned}$$

This bound is negative definite when $k > k^* \doteq q + \frac{1}{4}\frac{c_2^2}{c_1}$. Furthermore the bound is degree two with respect to the dilation $\tilde{\delta}_\lambda$ so $D^+W \leq -\tilde{k}W$ for some $\tilde{k} > 0$ whenever $k > k^*$. The differential inequality from [24, Theorem 1.4.1] implies $W(t) \leq W(0)e^{-\tilde{k}t}$. Hence, the system is asymptotically stable. Exponential stability follows from the fact that the closed-loop system is degree zero with respect to the extended dilation $\tilde{\delta}_\lambda$ defined in equation (25). □

The form of the control law shows that it can be regarded as a combined control law consisting of a feedforward portion, which drives the system along the desired trajectory when $u = \alpha(x,t)$, and a feedback portion, which stabilizes the extended state space equation.

8. Conclusion

Homogeneous feedbacks are an effective means to improve the convergence rate of driftless systems. The feedbacks are non-Lipschitz and the states satisfy a modified definition of exponential stability. An algorithm is presented which gives conditions under which a smooth asymptotically stabilizing controller may be modified into an ρ-exponentially stabilizing feedback. The alogorithm is applied to several examples and an experimental mobile robot.

Theorem 5.1 also has promising extensions to systems with "drift" vector fields as the following example shows. Recall that the Euler equations with two inputs given in equation (11) cannot be exponentially stabilized in the usual sense with Lipschitz feedback. We now show that ρ-exponentially stability is achievable with non-Lipschitz feedback. The system may be written as $\dot{\omega} = X_0 + X_1u_1 + X_2u_2$ where $X_0 = \omega_1\omega_2\partial/\partial\omega_3$, $X_1 = \partial/\partial\omega_1$ and $X_2 = \partial/\partial\omega_2$. Defining the dilation $\Delta_\lambda\omega = (\lambda\omega_1, \lambda\omega_2, \lambda^2\omega_3)$, the drift vector field X_0 is degree zero and the input vector fields, X_1 and X_2, are degree one. For ρ-exponential stability we would like to define u_1 and u_2 to be degree one functions with respect to this dilation since the closed-loop system will be degree zero in this case. A smooth asymptotically stabilizing feedback is

$$\begin{aligned} u_1(\omega) &= -\omega_1 - \omega_3 \\ u_2(\omega) &= -\omega_2 + \omega_3^2. \end{aligned} \tag{30}$$

Asymptotic stability may be verified with the function

$$V = (\omega_1 + \omega_3)^2 + (\omega_2 - \omega_3^2)^2 + \omega_3^2.$$

The Euler vector field

$$X_E = \omega_1\partial/\partial\omega_1 + \omega_2\partial/\partial\omega_2 + 2\omega_3\partial/\partial\omega_3,$$

corresponding to $\Delta_\lambda$is locally transversal to the level sets of V. Thus the smooth feedbacks in equation (30) may be rescaled according to Theorem 5.1 into ρ-exponentially stabilizing degree one feedbacks. These feedbacks are smooth on $\mathbb{R}^n \setminus \{0\}$ but are not Lipschitz at the origin. Figure 10 compares the performance of the smooth feedback in equation (30) versus its rescaled version. The value of the level set used in defining the homogeneous feedback is $V(\omega) = 1$.

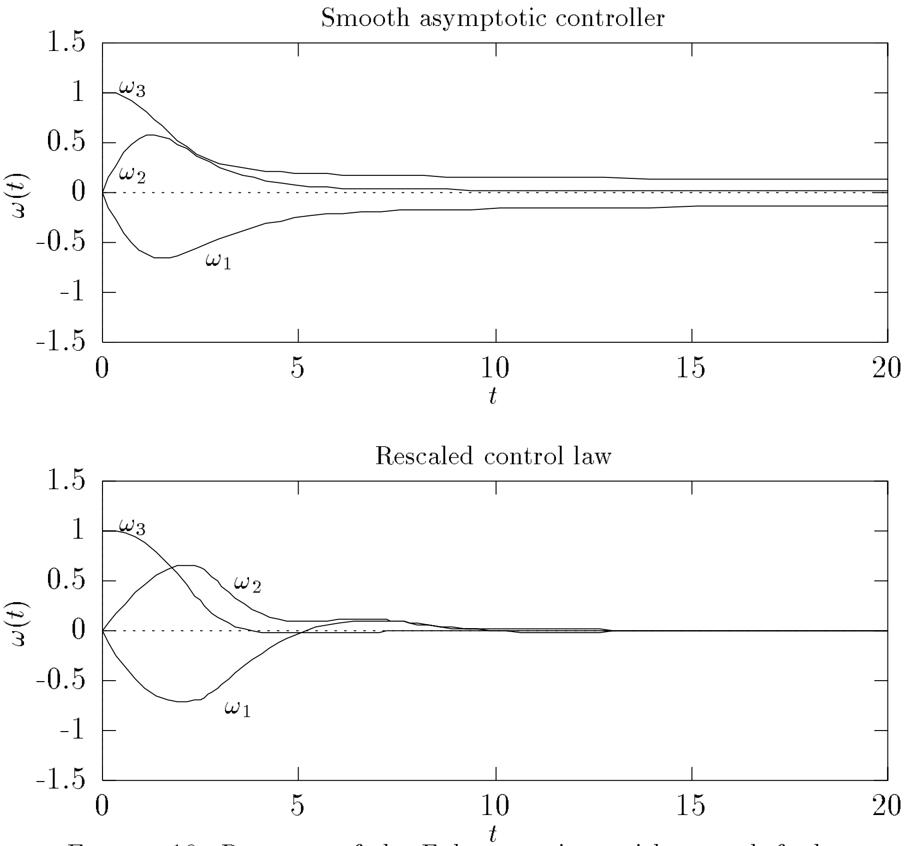

FIGURE 10. Response of the Euler equations with smooth feedback (top) and a ρ-exponential stabilizer (bottom).

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