V. Conclusion Remarks

For a class of tridiagonal matrices, the textured iterative algorithm for linear equations is proved strictly faster than the corresponding classical iterative algorithms. The spectral radiuses of both algorithms and some bounds are found. The results of this paper can be extended to multiple $(m > 2)$ splitting and multiple block approximation schemes.

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Solution Approximation in Infinite Horizon Linear Quadratic Control

Irwin E. Schochetman and Robert L. Smith

Abstract— **We consider the problem of choosing a discounted-cost minimizing infinite-stage control sequence under nonstationary positive semidefinite quadratic costs and linear constraints. Specific cases include the nonstationary LQ tracker and regulator problems. We show that the optimal costs for finite-stage approximating problems converge to the optimal infinite-stage cost as the number of stages grows to infinity. Under a state reachability condition, we show that the set unions of all controls optimal to all feasible states for the finite-stage approximating problems converge to the set of infinite-stage optimal controls. A tie-breaking rule is provided that selects finite-stage optimal controls so as to force convergence to an infinite horizon optimal control.**

I. Introduction

Consider the following infinite stage linear quadratic control problem:

$$\min \sum_{k=0}^{\infty}[(z_k - r_k)^t Q_k (z_k - r_k) + u_k^t R_k u_k]$$

subject to $\qquad (\mathcal{C})$

$$x_{k+1} = A_k x_k + B_k u_k + d_k,$$
$$z_k = C_k x_k, \qquad k = 0, 1, 2, \cdots,$$

and

$$u_k \in U_k \subseteq \mathbb{R}^m,$$
$$x_k \in X_k \subseteq \mathbb{R}^n,$$
$$z_k \in \mathbb{R}^p, \qquad k = 0, 1, 2, \cdots,$$

where x_0 in $\mathbb{R}^n$ is given. We make the following assumptions on the problem data:

A) The data are deterministic.
B) Each matrix Q_k and R_k is symmetric and positive *semidefinite.*
C) Each X_k is a closed convex subset of $\mathbb{R}^n$.
D) Each U_k is a *compact* convex subset of $\mathbb{R}^m$.
E) The resulting feasible region is nonempty.
F) The objective function converges absolutely and uniformly over the feasible region.

As usual, x_k is the state, u_k is the control and z_k is the output. The quantity d_k is a known exogenous parameter (e.g., demand in production-inventory models).

If we set d_k to zero, all k, then we get the nonstationary *LQ tracker problem* [13] as a special case. If we also set C_k to the identity matrix and r_k to zero, all k, then we get the nonstationary *LQ regulator problem* [13].

We will show that under a suitable controllability assumption, a sequence of selected solutions to the finite-stage approximation problems converges to an optimal solution of $(\mathcal{C})$. In [15], for a general mathematical programming problem, we showed how to obtain such solution convergence under a state reachability assumption and certain regularity conditions. Our plan here is to establish the regularity conditions and obtain this state reachability property as a consequence of the assumed controllability condition. The familiar planning horizon approach to such infinite horizon problems requires *finite* control sets and the existence of a *unique* solution. Our *approximation* approach is more realistic in that it does not require such assumptions. Moreover, it is more consistent with common practice.

The general infinite horizon LQ problem $(\mathcal{C})$ is challenging in several respects. First, the presence of nonstationary data, as well as constraints on the admissible controls, presents a problem that is acknowledged to be difficult to solve [2], [10]. It is nonetheless almost certain that in practice there will be bounds on feasible controls and states. Also, generalizing previous work [3], [10], we do not require the control space in each period to be finite. Second, Assumption B) allows the quadratic control costs to be positive *semidefinite,* so that in general there will be *multiple* infinite horizon optima. This complicates the task of approximating an infinite horizon optimal solution through finite horizon truncations (as in [3], [10], [11]), since, in general, finite horizon optimal controls will not converge. In the presence of a state reachability property similar to that of [11], we show how to select finite horizon optimal controls so as to force convergence to an infinite horizon optimal control strategy. In related work, in a more abstract framework than ours, [6] establishes existence of an optimal infinite horizon solution and [8] proves convergence of optimal values for a "moving horizon" sequence of approximating solutions (see also [7] for related stability results).

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In Section II, we introduce the finite horizon (or finite stage) approximations $(\mathcal{C}(N))$ to $(\mathcal{C})$ consisting of the first N controls and N constraints of $(\mathcal{C})$. We then establish that the optimal values of $(\mathcal{C}(N))$ converge to the optimal value of $(\mathcal{C})$ (i.e., optimal value convergence). We also show that for positive *definite* control costs, the unique optimal controls for the $(\mathcal{C}(N))$ converge to the unique optimal control strategy of $(\mathcal{C})$, (i.e., solution convergence). In Section III, in the presence of a state reachability property, we show that the best approximations in the sets of optimal controls for the $(\mathcal{C}(N))$ converge to the best approximation in the set of infinite horizon optimal controls. This allows for an arbitrarily close approximation to an infinite horizon optimal solution by solving a sufficiently long finite horizon version of $(\mathcal{C})$. Finally, in Section IV, we illustrate the preceding development with an application to multiproduct production planning.

II. Value and Solution Convergence

Our first objective is to characterize the set of states V_k at stage k that are reachable by admissible controls from the initial state x_0. That is, let

$$V_1 = \{A_0 x_0 + B_0 u + d_0 : u \in U_0\}$$

and

$$V_{k+1} = \{A_k x + B_k u + d_k : x \in V_k, u \in U_k\}, \qquad k = 1, 2, \cdots.$$

Then it is easy to see that each reachable set V_k is a compact, convex, nonempty subset of $\mathbb{R}^n$.

Next define the set S_k of all *feasibly* reachable states at stage k from the initial state x_0. Let

$$T_1 = \{A_0 x_0 + B_0 u + d_0 : u \in U_0\},$$
$$S_1 = T_1 \cap X_1,$$

and

$$T_{k+1} = \{A_k x + B_k u + d_k : x \in S_k, u \in U_k\}$$
$$S_{k+1} = T_{k+1} \cap X_{k+1}, \qquad k = 0, 1, 2, \cdots.$$

Clearly, each T_k and S_k is a compact, convex subset of $\mathbb{R}^n$. Since there exists a feasible solution to $(\mathcal{C})$ by Assumption E), both T_k and S_k are nonempty, $k = 1, 2, \cdots$. If $X_j = \mathbb{R}^n$, all j, then of course $S_k = T_k = V_k$, all k. For an explicit construction of the feasibly reachable states when controls and states are linearly constrained, see [12].

Lemma 2.1: For each $k = 0, 1, 2, \cdots$, the following are equivalent:

i) $x \in S_{k+1}$.

ii) $x \in X_{k+1}$ and there exist $u_j \in U_j$, $0 \le j \le k$ and $x_j \in S_j$, $1 \le j \le k$, such that $x_{j+1} = A_j x_j + B_j u_j + d_j$, for $j = 0, 1, \cdots, k$, where $x_{k+1} = x$.

Finally, define the set of reachable outputs Z_k by

$$Z_k = \{C_k x : x \in V_k\}, \qquad k = 1, 2, \cdots.$$

Then each Z_k is a compact, convex, nonempty subset of $\mathbb{R}^p$.

For convenience, we let $y_k = (u_{k-1}, x_k, z_k)$, $Y_k = U_{k-1} \times V_k \times Z_k$, $k = 1, 2, \cdots$, and $Y = \prod_{k=1}^{\infty} Y_k$. Then each Y_k is compact, convex and nonempty.

As in [15], we embed Y in a Hilbert space formed by the weighted Hilbert sum of its component spaces. Letting $q = m + n + p$, we have that $Y_k \subseteq \mathbb{R}^q$, $k = 1, 2, \cdots$. Since the Y_k are compact, for each k, there exists $r_k > 0$ such that $\|y_k\| \le r_k$, where $\|\cdot\|$ is the usual norm on $\mathbb{R}^q$. Fix $0 < \beta_k < 1$, such that $\sum_{k=1}^{\infty} \beta_k^2 r_k < \infty$; for example, set $\beta_k = 1/k r_k$. Let

$$H = \left\{ (y_k) : y_k \in \mathbb{R}^q, k = 1, 2, \cdots, \text{ and } \sum_{k=1}^{\infty} \beta_k^2 \|y_k\|^2 < \infty \right\}.$$

Then H becomes a Hilbert space contained in $\prod_{k=1}^{\infty} \mathbb{R}^q$ with inner product $\langle x, y \rangle$ given by

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \beta_k^2 (x_k, y_k),$$

where $(\cdot, \cdot)$ is the usual inner product on $\mathbb{R}^q$. From our choice of the β_k, it follows that $Y \subseteq H$, so that Y inherits the Hilbert metric ρ of H, where $\rho(x, y) = (\langle x - y, x - y \rangle)^{1/2}$, $x, y \in H$. It was noted in [15, lemma 2.1] that the ρ-metric topology on Y is the same as the product topology, so that Y is a compact metric space relative to ρ by the Tychonoff theorem. Moreover, a sequence $\{y^n\}$ in Y converges to y relative to ρ if and only if for each k, $\{y_k^n\}$ converges to y_k in $\mathbb{R}^q$ relative to the usual Euclidean metric.

Define $\mathcal{K}(Y)$ to be the space of all compact, nonempty subsets of Y and let D denote the Hausdorff metric on $\mathcal{K}(Y)$ derived from ρ [5]. In this way, $\mathcal{K}(Y)$ becomes a compact metric space, so that convergence in $\mathcal{K}(Y)$ is relative to D; equivalently [5], [9], convergence in $\mathcal{K}(Y)$ is *Kuratowski convergence.*

The feasible region F for $(\mathcal{C})$ is clearly a closed, convex subset of Y. Thus, F is also compact and nonempty by Assumption E), so that $F \in \mathcal{K}(Y)$. From this it follows that the set S_k of feasibly reachable states at stage k is nonempty, for each k. If $x = (x_k) \in F$, then by Lemma 2.1, $x_k \in S_k$, all k. If we also write $y = (y_k)$ and abbreviate the objective function for $(\mathcal{C})$ by

$$C(y) = \sum_{k=0}^{\infty} [(z_k - r_k)^t Q_k (z_k - r_k) + u_k^t R_k u_k],$$

then $(\mathcal{C})$ becomes

$$\min_{y \in F} C(y).$$

In order to establish continuity for the objective function, we let

$$c_1(y_1) = c_1(u_0, x_1, z_1) = (z_0 - r_0)^t Q_0 (z_0 - r_0) + u_0^t R_0 u_0 + (z_1 - r_1)^t Q_1 (z_1 - r_1)$$

and

$$\begin{aligned} c_k(y_k) &= c_k(u_{k-1}, x_k, z_k) \\ &= (z_k - r_k)^t Q_k (z_k - r_k) \\ &\quad + u_{k-1}^t R_{k-1} u_{k-1}, \qquad k = 2, 3, \cdots, \end{aligned}$$

so that each c_k is a continuous, convex function on $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$. Then $C(y) = \sum_{k=1}^{\infty} c_k(y_k)$, $y \in Y$. Since the U_k and Z_k are compact, we may define

$$\mu_k = \max_{u_k \in U_k} \|u_k\|, \qquad k = 0, 1, 2, \cdots,$$

and

$$\zeta_k = \max_{z_k \in Z_k} \|z_k - r_k\|, \qquad k = 1, 2, \cdots.$$

Throughout this note we assume $\sum_{k=1}^{\infty} \mu_k^2 \|R_k\|_2 < \infty$ and $\sum_{k=1}^{\infty} \zeta_k^2 \|Q_k\|_2 < \infty$, where $\|A\|_2$ denotes the usual spectral norm of matrix A. For example, if the μ_k, ζ_k are bounded and the Q_k, R_k are of the form

$$Q_k = \alpha^k Q_k', \qquad R_k = \alpha^k R_k',$$

where $0 < \alpha < 1$ is a discount factor and the Q_k', R_k' are bounded, then these series are dominated by convergent geometric series and therefore the condition holds.

If $\|c_k\|_\infty = \sup_{y_k \in Y_k} c_k(y_k)$ denotes the supremum norm of c_k as a function on Y_k, then

$$\sum_{k=1}^{\infty} \|c_k\|_\infty \le \sum_{k=0}^{\infty} \mu_k^2 \|R_k\|_2 + \sum_{k=1}^{\infty} \zeta_k^2 \|Q_k\|_2 < \infty,$$

i.e., the series is absolutely convergent. Consequently, the correspondence $y \to C(y)$ defines a continuous, real-valued function C on Y which is the uniform limit of the sequence $\{\sum_{k=1}^{N} c_k\}_{N=1}^{\infty}$ of continuous partial sums.

Since C is continuous on Y and F is a compact, nonempty subset of Y, the objective function C attains its minimum C^* on F. If we let

$$F^* = \{y \in F: C(y) = C^*\},$$

then F^* is a compact, nonempty subset of Y, i.e., $F^* \in \mathcal{K}(Y)$. Thus, under our assumptions, $(\mathcal{C})$ is a convex programming problem with continuous objective function C, compact, convex, nonempty feasible region F, optimal objective value C^* and compact, nonempty optimal solution set F^*. Note that since C is not required to be strictly convex, F^* will not be a singleton in general, so that there may be multiple optimal solutions y^*.

Our primary objective in this note is to approximate the optimal value C^* and an optimal solution y^* of $(\mathcal{C})$ by corresponding quantities obtained from finite-stage subproblems of $(\mathcal{C})$. To this end, let N be a positive integer and define the problem $(\mathcal{C}(N))$ as follows:

$$\min \sum_{k=0}^{N-1} [(z_k - r_k)^t Q_k (z_k - r_k) + u_k^t R_k u_k] + (z_N - r_N)^t Q_N (z_N - r_N)$$

subject to $\qquad (\mathcal{C}(N))$

$$\begin{aligned} &x_{k+1} = A_k x_k + B_k u_k + d_k, \qquad k = 0, 1, \cdots, N-1,\\ &z_k = C_k x_k, \qquad k = 1, \cdots, N,\\ &u_k \in U_k, \qquad x_k \in X_k, \qquad z_k \in Z_k, \qquad k = 1, 2, \cdots, \end{aligned}$$

where x_0 (hence z_0) is given.

In order that feasible solutions for $(\mathcal{C}(N))$ be comparable to those of $(\mathcal{C})$, we have defined $(\mathcal{C}(N))$ to be an infinite stage problem with constraints and objective function depending only on the first N stages. The feasible solutions are just *arbitrary* extensions of the feasible solutions to the N-stage truncation of $(\mathcal{C})$. For each $N = 1, 2, \cdots$, the feasible region $F(N)$ for $(\mathcal{C}(N))$ is contained in Y. Moreover, $F(N+1) \subseteq F(N)$ and $F \subseteq F(N)$, so that $F(N)$ is nonempty, $N = 1, 2, \cdots$. Also, each $F(N)$ is compact and convex and $F = \lim_{N\to\infty} F(N)$. If $x \in F(N)$, then automatically $x_k \in S_k$, $k = 1, \cdots, N$.

For convenience, denote the objective function for $(\mathcal{C}(N))$ by $C(\cdot; N)$. Then $(\mathcal{C}(N))$ becomes

$$\min_{y \in F(N)} C(y; N).$$

By our assumptions, the continuous functions $C(\cdot, N)$ converge uniformly to C on the compact space Y. Moreover, each $C(\cdot; N)$ is convex, so that $(\mathcal{C}(N))$ is again a convex programming problem. We set

$$C^*(N) = \min_{y \in F(N)} C(y; N)$$

and

$$F^*(N) = \{y \in F(N): C(y; N) = C^*(N)\}, \qquad N = 1, 2, \cdots,$$

so that $C^*(N)$ is the optimal objective function value and $F^*(N)$ is the set of optimal solutions for $(\mathcal{C}(N))$. Clearly, each $F^*(N)$ is a compact, nonempty subset of $F(N)$.

Theorem 2.2: i) (Optimal Value Convergence). The optimal values of the finite-stage problems $(\mathcal{C}(N))$ converge to the optimal value of the infinite-stage problem $(\mathcal{C})$, i.e.,

$$C^*(N) \to C^*, \quad \text{as} \quad N \to \infty.$$

ii) (Optimal Solution Convergence). If Q_k and R_k are positive definite, for all $k = 1, 2, \cdots$, then C is strictly convex. In this event, $(\mathcal{C})$ has a unique optimal solution y^*, i.e., $F^* = \{y^*\}$. If $y^*(N)$ is the optimal solution to $(\mathcal{C}(N))$, then the sequence $\{y^*(N)\}$ converges to y^* in the product topology of Y. In particular, optimal controls $\{u_k^*(N)\}_{k=1}^{\infty}$ of the finite-stage problems $(\mathcal{C}(N))$ converge to the optimal control $\{u_k^*\}_{k=1}^{\infty}$ of the infinite-stage problem $(\mathcal{C})$, i.e.,

$$u_k^*(N) \to u_k^*, \quad \text{as} \quad N \to \infty,$$

for $k = 1, 2, \cdots$.

Proof: Part i) follows from the Maximum Theorem [1, p. 116] applied to a canonically defined set-valued mapping. (See [15, theorem 4.1]). For part ii), observe that $\limsup F^*(N) = F^* = \{y^*\}$ by the Maximum Theorem and compactness. Now apply [16, corollary 2.2], noting that $F^*(N) = \{y^*(N)\}$, for $N = 1, 2, \cdots$. (See [15, theorem 4.5]).

Remark: More generally, the last part of this theorem is valid whenever F^* is a singleton [15].

In view of this result, it remains to study the question of optimal solution convergence when F^* is *not* a singleton, i.e., when there exist multiple optimal solutions to $(\mathcal{C})$. The difficulty is that in this case of multiple infinite-stage optima, optimal controls to the finite-stage subproblems may not converge. In order to allow for convergence, we need to enlarge the set of finite-stage controls from which we select. In particular, we enlarge $F^*(N)$ to include optimal controls to *all* states. In this regard, note that if y is feasible for $(\mathcal{C}(N))$, then the only connection between the constraints of $(\mathcal{C}(N))$ and the remaining constraints of $(\mathcal{C})$ is the value $A_N x_N$, which is the dynamic programming state associated with y at stage N. This is the motivation for what follows.

If s is any element of $\mathbf{R}^n$, define $F(N, s)$ to be the set of $(\mathcal{C}(N))$-feasible solutions having dynamic programming state s at stage N, i.e.,

$$F(N, s) = \{y \in F(N): A_N x_N = s\}, \qquad N = 1, 2, \cdots.$$

Thus, $F(N, s)$ is a (possibly empty) compact subset of $F(N)$. Define

$$S(N) = A_N S_N \equiv \{s \in R^n: A_N x = s, \text{ some } x \in S_N\}, \qquad N = 1, 2, \cdots.$$

Then $S(N)$ is the set of all $(\mathcal{C}(N))$-feasible dynamic programming states at stage N.

Lemma 2.3: For each $N = 1, 2, \cdots$,

$$\begin{aligned} S(N) &= \{s \in \mathbf{R}^n: F(N, s) \neq \phi\}\\ &= \{s \in \mathbf{R}^n: A_N x_N = s, \text{ for some } y \in F(N)\}. \end{aligned}$$

Proof: Follows from Lemma 2.1. ∎

For each $N = 1, 2, \cdots$ and $s \in S(N)$, consider the mathematical program $(\mathcal{C}(N), s)$ given by

$$\min_{y \in F(N, s)} C(y; N). \qquad \text{(C(N), s)}$$

Since the minimum value $C^*(N, s)$ is attained, we may define

$$F^*(N, s) = \{y \in F(N, s): C(y; N) = C^*(N, s)\},$$

which represents the set of all solutions optimal to dynamic programming state $s \in S(N)$ for problem $(\mathcal{C}(N))$. Each such $F^*(N, s)$ is a compact, nonempty subset of $F(N)$, i.e., an element of $\mathcal{K}(Y)$.

In order to construct a sequence of $(\mathcal{C}(N))$-feasible solution sets which converges to F^* in $\mathcal{K}(Y)$, define

$$\mathcal{F}^*(N) = \bigcup_{s\in S(N)} F^*(N, s), \qquad N = 1, 2, \cdots.$$

For technical reasons, we also define $\overline{\mathcal{F}}^*(N)$ to be the closure of $\mathcal{F}^*(N)$ in Y, so that $\overline{\mathcal{F}}^*(N) \in \mathcal{K}(Y)$, all N.

III. Controllability, Reachability and Solution Convergence Via Best Approximation

We turn next to the task of establishing conditions under which the sets $\mathcal{F}^*(N)$ converge to F^* in $K(Y)$. When this happens, the sequence of best-approximations from the $\mathcal{F}^*(N)$ (relative to any point in Y) is guaranteed to converge to the best-approximation in F^* [15]. (A *best-approximation* from a set $K \in \mathcal{K}(Y)$ with respect to a point $p \in Y$ is a point in K closest in the Hilbert metric to p.)

As we saw in [15], a sufficient condition for this convergence was a state reachability property. As we shall see, this property in turn can be derived from a suitable controllability property for the constraint system.

In order to define controllability for this problem, we must first express x_{k+1} in terms of $x_j, u_j, \cdots, u_k$, for $0 \le j \le k$. As is customary, for $0 \le j \le k+1$, define the matrix

$$\Gamma(k, j) = \begin{cases} A_k \cdots A_j, & j \le k, \\ I, & j = k+1. \end{cases}$$

Also, for $0 \le j \le k$ define the matrices

$$\Phi(k, j) = \begin{cases} [B_k,\, A_k B_{k-1},\, A_k A_{k-1} B_{k-2}, \cdots, A_k \cdots A_{j+1} B_j], \\ \quad j < k, \\ B_k, \quad j = k, \end{cases}$$

and

$$\Psi(k, j) = \begin{cases} [I,\, A_k,\, A_k A_{k-1}, \cdots, A_k \cdots A_{j+1}], & j < k. \\ I, \quad j = k. \end{cases}$$

Lemma 3.1: We have the following properties for Γ, Φ, and Ψ:

i) $\Gamma(k, j) = A_k \Gamma(k-1, j)$, $0 \le j \le k$.

ii) $\Phi(k, j) = [B_k, A_k * \Phi(k-1, j)]$, $0 \le j < k$, where $A_k * \Phi(k-1, j)$ is the partitioned matrix $\Phi(k-1, j)$ with each matrix in the partition premultiplied by A_k.

iii) $\Psi(k, j) = [I, A_k * \Psi(k-1, j)]$, $0 \le j < k$, where $A_k * \Psi(k-1, j)$ is defined as in ii).

Proof: These follow immediately from the definitions. ■

Lemma 3.2: For each $0 \le j \le k$, suppose

$$x_{l+1} = A_l x_l + B_l u_l + d_l, \qquad l = j, \cdots, k,$$

where $x_l \in R^n$, for all l. Then

$$x_{k+1} = \Gamma(k, j) x_j + \Phi(k, j)[u_k, \cdots, u_j]^t + \Psi(k, j)[d_k, \cdots, d_j]^t.$$

Proof: By induction on j and k with $k \ge j$. ■

We are now ready to define controllability. Let $0 \le j \le k$. The constraint system for $(\mathcal{C})$ is (j, k)*-controllable* if, for each $x_j \in S_j$ and $x_{k+1} \in S_{k+1}$, there exists $u_i \in U_i$, $j \le i \le k$, such that:

i) $x_{k+1} = \Gamma(k, j) x_j + \Phi(k, j)[u_k, \cdots, u_j]^t + \Psi(k, j)[d_k, \cdots, d_j]^t$ and

ii) $\Gamma(i, j) x_j + \Phi(i, j)[u_i, \cdots, u_j]^t + \Psi(i, j)[d_i, \cdots, d_j]^t \in S_{i+1}, j \le i \le k-1$.

Remarks: Part i) says that, for any pair of feasible states x_j and x_{k+1} at states j and k+1 respectively, there exists a sequence $u_j, \cdots, u_k$ of controls which transforms x_j into x_{k+1}. Alternately, the $U_j \times \cdots \times U_k$—span of the columns of $\Phi(k, j)$ contains the subset of $\mathbb{R}^n$ given by $S_{k+1} - \Gamma(k, j) S_j - \Psi(k, j)[d_k, \cdots, d_j]^t$. Part ii) says that the states s_i obtained in the intermediate stages satisfy $s_i \in S_i$, $j < i \le k$, i.e., they are feasible. In the special case where each X_i is $\mathbb{R}^n$, this is automatically the case. Then, the constraint system of $(\mathcal{C})$ is (j, k)—controllable if i) holds.

We say that the constraint system for $(\mathcal{C})$ is *controllable* if, for each $j = 0, 1, 2, \cdots$, there exists $k_j > j$ such that the constraint system for $(\mathcal{C})$ is (j, k)—controllable for each $k \ge k_j$. Hence, the constraint system is controllable if from any feasible state, we can eventually reach all subsequent feasible states. This is a difficult property to verify in general, in most cases requiring exploitation of a specific problem's structure and data.

The key assumption of [15] which guaranteed that $\lim \mathcal{F}(N) = F^*$ was the notion of reachability in the dynamic programming sense. Our next objective is to define reachability in a control-theoretic sense (which will imply the dynamic programming reachability property of [15]) and compare it with the property of controllability.

Let k be a nonnegative integer and $s_k \in S_k$. Then the sequence of all feasible states $\{S_N\}$ is *reachable from the state* s_k *at stage* k if, given any sequence of feasible states $\{t_N\}$ with $t_N \in S_N$, $N = 1, 2, \cdots$, there exists $N_k > k$ sufficiently large such that for each $N \ge N_k$, there exists $y^N \in F(N)$ satisfying $x_N^N = t_N$ and $x_k^N = s_k$. (Note that $F(N) \subseteq F(k)$, for all N.) The feasible sequence $\{S_N\}$ is *reachable from all finite-stage feasible states* if it is reachable from all $s_k \in S_k$, all $k = 0, 1, 2, \cdots$.

Remark: It is not difficult to verify that this notion of reachability implies that of [15]. Also, reachability implies that

$$F(N, s_N) \to F, \quad \text{as} \quad N \to \infty,$$

for all feasible state sequences $\{s_N\}$.

Lemma 3.3: If the constraint system of problem $(\mathcal{C})$ is controllable, then $\{S_N\}$ is reachable from all finite-stage feasible control states.

We are now ready to state our main result.

Theorem 3.4: Suppose the constraint system of $(\mathcal{C})$ is controllable. Then:

i) $\lim_{N\to\infty} \overline{\mathcal{F}}^*(N) = F^*$ in $\mathcal{K}(Y)$, i.e., $\lim_{N\to\infty} \mathcal{F}^*(N) = F^*$, in the sense of Kuratowski.

ii) For each point p in Y, the sequence $\{y_p^*(N)\}$ converges to y_p^*, where $y_p^*(N)$ is any best-approximation in $\overline{\mathcal{F}}^*(N)$ to p and y_p^* is the unique best-approximation in F^* to p. In particular, if $y_p^*(N) = ((u_p^*)_{k-1}(N), (x_p^*)_k(N), (z_p^*)_k(N))_{k=1}^\infty$ and $y_p^* = ((u_p^*)_{k-1}, (x_p^*)_k, (z_p^*)_k)_{k=1}^\infty$, then $(u_p^*)_{k-1}(N) \to (u_p^*)_{k-1}$, as $N \to \infty$, for $k = 1, 2, \cdots$.

Proof: By our hypothesis and Lemma 3.3, it follows that $\{S_N\}$ is reachable from all finite horizon feasible states. It is then easy to see that this implies the reachability condition of [15, theorem 5.4] (for the sets $S(N) = A_N S_N$). For part i), this reachability condition implies that all accumulation points of the sequences of solutions from the $\mathcal{F}^*(n)$, i.e., the points of $\limsup \overline{\mathcal{F}}^*(n)$, are in F^*. The other inclusion is true in general. Part ii) follows (as in the proof of Theorem 2.2 above) from [16, corollary 2.2]. (See [15, corollary 5.5]). ■

Theorem 3.4 says that we can arbitrarily well approximate an infinite-stage optimal control by solving a finite stage subproblem of sufficiently long horizon. From the standpoint of implementation, if we approximate the continuous control spaces U_k by uniformly bounded discrete control sets, then two simplifications follow [14]. First, $\mathcal{F}^*(N) = \mathcal{F}^*(N)$, thus avoiding the necessity to form the closure of $\mathcal{F}^*(N)$. Therefore, a standard forward dynamic programming algorithm will, as N increases, automatically generate the set $\mathcal{F}^*(N)$ of optimal controls to all feasible states. Second, for (β_k) sufficiently small, the best-approximation $y_p^*(N)$ (with p the origin) is the lexicomin, i.e., the lexicographically smallest element of the finite set $\mathcal{F}^*(N)$. By Theorem 3.4, the lexicomin first-stage control from

$\mathcal{F}^*(N)$ will eventually lock-in and agree with the infinite horizon first-stage optimal control for that and all subsequent horizons. In this manner, the forward dynamic programming algorithm will recursively recover the lexicomin infinite horizon optimal control. A stopping rule is provided in [15] that determines how large the horizon N must be to guarantee agreement with the infinite horizon first-stage optimal control.

IV. Multiproduct Production Planning

In this section, we illustrate the previous development with an application to production planning. Consider the problem of scheduling production of one or more products to meet a nonstationary, deterministic demand over an infinite horizon. The objective is to optimally balance the economies of scale of production against the cost of carrying inventory. If we assume convex quadratic costs for production and inventory holding, then the problem may be formulated by the following mathematical program $(\mathcal{P})$ [4], where vector inequalities are interpreted componentwise:

$$\min \sum_{k=0}^{\infty} \alpha^k [x_{k+1}^t Q_{k+1} x_{k+1} + u_k^t R_k u_k]$$

subject to $(\mathcal{P})$

$$\begin{aligned} &x_{k+1} = x_k + u_k - d_k, \\ &-b \le x_k \le a, \\ &0 \le u_k \le q, \qquad k = 0, 1, 2, \cdots, \end{aligned}$$

where (for convenience) the initial inventory $x_0 = 0$, $x_k \in \mathbf{R}^n$ is the multiproduct inventory ending period k, $u_k \in \mathbf{R}^n$ is the multiproduct production in period k, and $d_k \in \mathbf{R}^n$ is the multiproduct demand for production in period k. The quantity α is the discount factor reflecting the time-value of money, where $0 < \alpha < 1$. We require that $q > 0$, $a > 0$, $b \ge 0$ and $d_k \ge 0$, $k = 0, 1, 2, \cdots$. If a component of b is positive, then backlogging is allowed for the corresponding product. We impose the following assumptions on $(\mathcal{P})$.

Assumptions

I) For each k, Q_k, and R_k are positive *semidefinite*. Moreover,

$$\sum_{k=0}^{\infty} \alpha^k (\|Q_{k+1}\|_2 + \|R_k\|_2) < \infty.$$

II) Program $(\mathcal{P})$ is feasible.

Assumption I) is similar to the cost assumptions of $(\mathcal{C})$. A sufficient condition for the series to converge is that the sequences $\{\|Q_k\|_2\}$ and $\{\|R_k\|_2\}$ be uniformly bounded. Then the series is essentially the geometric series with $0 < \alpha < 1$.

Assumption II) asserts that the feasible region F for $(\mathcal{P})$ is nonempty. A necessary condition for feasibility is that $Nq \ge \sum_{k=0}^{N-1} d_k - b$, $N = 1, 2, 3, \cdots$, i.e., up to each period, the maximum production is at least equal to the total demand less the amount backordered. However, in general this is not a sufficient condition for feasibility since the upper bound a on allowable inventory can be violated. As in [15], it is easy to see that $d_k \le q$, all k, is a sufficient condition.

Clearly, $(\mathcal{P})$ is a special case of $(\mathcal{C})$ with $r_k = 0$, $A_k = B_k = C_k = I$, $U_k = [0, q]$, $X_k = [-b, a]$ and $z_k = x_k$, $k = 0, 1, 2, \cdots$. Thus, $(\mathcal{P})$ satisfies all the hypotheses of Section I. Consequently, C^* denotes the optimal objective value of $(\mathcal{P})$ and F^* the nonempty set of optimal solutions.

For each $N = 1, 2, \cdots$, the finite-stage approximation $(\mathcal{P}(N))$ to $(\mathcal{P})$ is given by

$$\min \sum_{k=0}^{N} \alpha^k [x_{k+1}^t Q_{k+1} x_{k+1} + u_k^t R_k u_k]$$

subject to $(\mathcal{P}(N))$

$$\begin{aligned} x_{k+1} &= x_k + u_k - d_k, && k = 0, 1, \cdots, N-1, \\ -b &\le x_k \le a, && k = 1, 2, \cdots, \\ 0 &\le u_k \le q, && k = 0, 1, 2, \cdots. \end{aligned}$$

Since the inventory levels $x_1, \cdots, x_N$ are defined by the production schedule $u_0, \cdots, u_N$ via

$$x_k = \sum_{j=0}^{k} u_j - \sum_{j=1}^{k} d_j, \qquad k = 1, \cdots, N,$$

we will say that $(u_0, \cdots, u_N, \cdots)$ is *feasible* for $(\mathcal{P}(N))$ if $0 \le u_k \le q$, all k, and

$$-b + \sum_{j=1}^{k} d_j \le \sum_{j=0}^{k} u_j \le a + \sum_{j=1}^{k} d_j, \qquad k = 1, \cdots, N.$$

As in Section II, we let $F(N)$ denote the nonempty feasible region of $(\mathcal{P}(N))$, $F^*(N)$ the nonempty set of optimal solutions to $(\mathcal{P}(N))$ and $C^*(N)$ the optimal objective value for $(\mathcal{P}(N))$. As in Section III, let S_N denote the set of feasible control states at stage N, $\mathcal{F}^*(N)$ the set of $(P(N))$—feasible solutions which are optimal to some state $s \in S_N$ and $\overline{\mathcal{F}}^*(N)$ the closure of $\mathcal{F}^*(N)$ in $\mathcal{K}(Y)$. Note that S_N is identical to both (1) the set of feasible inventories ending period N and (2) the set of feasible dynamic programming states $S(N)$ at stage N since $A_N = I$.

We next turn to the problem of establishing controllability for the constraint system of $(\mathcal{P})$. For the given data, it is clear that $\Gamma(k, j) = I$, $k \ge j - 1$, and

$$\Phi(k, j) = \overbrace{[I, \cdots, I]}^{k-j+1} = \Psi(k, j),$$

for $0 \le j \le k$. Thus, for $0 \le j \le k$ and $x_j \in \mathbf{R}^n$, the equation in Lemma 3.2 becomes

$$x_{k+1} = x_j + \sum_{i=j}^{k} u_i - \sum_{i=j}^{k} d_i,$$

as expected. Hence, for $0 \le j \le k$, the constraint system of $(\mathcal{P})$ is (j, k)—controllable if, for each $s_j \in S_j$ and $s_{k+1} \in S_{k+1}$, there exist $0 \le u_i \le q$, $i = j, \cdots, k$, such that

i) p
ii) $s_{k+1} = s_j + \sum_{i=j}^{k} u_i - \sum_{i=j}^{k} d_i$ and
iii) $s_{i+1} \in S_{i+1}$, where $s_{i+1} = s_j + \sum_{l=j}^{i} u_l - \sum_{l=j}^{i} d_l$, $i = j, \cdots, k-1$.

We are now ready to state the main result of this section.

Theorem 4.1: Suppose the constraint system of $(\mathcal{P})$ is controllable. Then the convergence results for $(\mathcal{C})$ in Theorem 3.4 are valid for $(\mathcal{P})$.

Remarks: 1) As in [15], the following conditions are sufficient for controllability of $(\mathcal{P})$: $\liminf d_k < q$, $\limsup d_k > 0$ and $0 \le d_k \le q$, $k = 0, 1, 2, \cdots$.

2) If there is only one product, then the objective function is simply

$$\sum_{k=0}^{\infty} \alpha^k [Q_{k+1} x_{k+1}^2 + R_k u_k^2],$$

where Q_{k+1} and R_k are nonnegative real *numbers*. If they are both positive for *all* k, then the objective function is strictly convex, thus making the need for selection unnecessary by Theorem 2.3. However, if Q_{k+1} or R_k is zero, for *some* k, then the objective function need *not* be strictly convex, thus requiring best-approximation selections to obtain convergence.

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Loop Transfer Recovery Design Using Biased and Unbiased Controllers

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***Abstract*—Loop transfer recovery is considered to be a special form of loop sensitivity shaping in this note. This viewpoint suggests a design strategy which relaxes the requirement that the estimator or controller be unbiased. This strategy is illustrated using a stable, SISO example with a nonminimum phase zero. The approach still faces the design tradeoffs and limitations inherent in all feedback systems including those which apply to nonminimum phase plants. The formulation used here, however, suggests a different approach for dealing with these issues.**

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I. Introduction

Linear quadratic optimal control with state variable feedback is known to have remarkable sensitivity and robustness properties [1]. However, these sensitivity and robustness properties are lost when full state feedback is replaced by state estimate feedback [2]. To overcome this difficulty, a procedure known as loop transfer recovery (LTR) was developed [2]–[4]. It can be shown that, *if the system transfer function is minimum phase and left invertible*, the sensitivity of the loop to input disturbances with state estimate feedback can be made to approach the sensitivity achievable with full state feedback. Results are available for this design strategy in both continuous [2]–[4] and discrete time [5], [14]. The intuitive appeal of these results is very persuasive, and hence there has been a great deal of interest in this method [6]–[14].

Beginning from a different perspective, there has been a great deal of work on the topics of design tradeoffs and fundamental limits on sensitivity shaping in feedback systems [3], [4], [15], [16]. These fundamental limits are inescapable and therefore apply, mutatis mutandis, to the loop transfer recovery problem. For example, nonminimum phase zeros and time delays limit the achievable bandwidth and hence influence the ability to shape transfer functions [15].

Here we regard loop transfer recovery as a particular form of sensitivity shaping as in [4]. From this perspective, we critically examine the implicit constraints employed in the usual application of the LTR method, most notably the constraint that the estimator should be unbiased. We argue that this constraint can be relaxed, and this leads to alternative procedures for sensitivity shaping based on the use of a biased estimator. Motivated by the characteristics of LTR designs for minimum phase, left invertible plants, we replace the unbiasedness constraint by a requirement that the state estimate be decoupled from the desired input. We note that the optimization of both the biased and unbiased estimator problems can be formulated using a range of criteria including $\mathcal{H}_2$ and $\mathcal{H}_\infty$. The design problem is set up to synthesize the dynamic elements of the controller directly without passing through the intermediate step of designing an estimator. Finally, we compare a number of design strategies to traditional LTR using a stable, SISO example with a nonminimum phase zero.

II. Problem Formulation

Consider the following Laplace transformed state-space model with zero initial conditions:

$$sx = Ax + Bu' + Bd; \qquad y = Cx \tag{2.1}$$

with $x \in \mathbb{R}^n$, $u' \in \mathbb{R}^r$, $d \in \mathbb{R}^r$, $y \in \mathbb{R}^m$. The variable u' is a control input, d is a disturbance, and s is the Laplace variable. Since we intend to use input disturbance sensitivity as a vehicle to explore several design options, it is not necessary to include output noise in the model. We assume that a full state feedback gain K has been found such that the control law $u' = -Kx$ results in a closed-loop system (Fig. 1) where the transfer function from d to u (i.e., the sensitivity function) has satisfactory disturbance attenuation. This transfer function is given by $S_o = [I + K\Phi B]^{-1}$, where $\Phi = (sI - A)^{-1}$.

Next consider a design using a linear, stable, time-invariant observer which produces an estimate $\hat{x}$ of the state in the form $\hat{x} = T_1(s)u' + T_2(s)y$. In this expression, T_1 and T_2 are proper, stable, rational transfer functions, y is the measurement, and u' is the *desired*