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A Robust Model Predictive Control Algorithm for Stable Linear Plants

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Abstract.
This paper presents a new robust model predictive control (MPC) algorithm for stable, linear plants that is a direct generalization of the nominally stabilizing regulator presented by Rawlings and Muske (1993). Model uncertainty is parameterized by a list of possible plants. Robust stability is achieved through the addition of constraints that prevent the sequence of optimal controller costs from increasing for the true plant. Asymptotic stability is demonstrated through a Lyapunov argument. Simulation experiments demonstrate the performance of the algorithm for a Continuous Stirred Tank Reactor (CSTR) process.

1.. Introduction

Recent papers by Meadows et al. (1995), Lee (1996) and Mayne (1996) describe several ways that MPC algorithms can be modified to achieve closed loop stability when the plant model is perfect, the so-called *nominal stability* problem. Progress has been much slower for the more difficult problem of closed loop stability in the presence of modeling error, referred to as the *robust stability* problem. A recent survey of industrial MPC technology (Qin and Badgwell, 1996) found that robust stability is a serious concern in industrial MPC applications and is currently addressed through the use of extensive closed loop simulation.

It has been shown that nominally stabilizing MPC algorithms can be made robust by limiting input movement (Vuthandam et al., 1995), adding constraints to restrict future state behavior (Mayne, 1996), or by defining a more conservative objective function (Lee, 1996). This paper presents a new method based on restricting future behavior of the controller cost function. This concept has been applied to the nominally stabilizing regulator of Rawlings and Muske (1993) to develop a new robust MPC algorithm.

2.. Robustly Stabilizing Algorithm

Consider the problem of controlling an uncertain, linear, stable time-invariant system of the form:

$$x_{k+1} = A\, x_k + B\, u_k \tag{1}$$

in which $x \in \Re^n$ and $u \in \Re^m$. The goal of the control system is to bring the state x from an initial value x_0 to the origin when the plant (A, B) is not known exactly. Input and state constraints are not discussed in this paper, although they can be introduced in a straightforward fashion without changing the stability properties of the algorithm (Badgwell, 1996).

Although the plant (A, B) is not known exactly, assume that it is known to lie within a set Ω of p stable plants of the same dimension:

$$(A, B) \in \Omega = \{(A_1, B_1) \cdots (A_p, B_p)\} \tag{2}$$

Let us assume that we have selected one plant from the set Ω as the most likely plant that will be encountered by the controller, and that we will use this as the model for the MPC algorithm. We will refer to this as the *nominal model* $(\tilde{A}, \tilde{B})$. At each time step the controller minimizes a cost function of the form:

$$\Phi = \sum_{j=0}^{N-1} \left(z_j^T\, Q\, z_j + v_j^T\, R\, v_j\right) + \Phi^\infty \tag{3}$$

$$\Phi^\infty = \sum_{j=N}^{\infty} \left(z_j^T\, Q\, z_j\right) \tag{4}$$

Here the z_j are predicted future states and the v_j are future inputs considered by the controller. The weight matrices Q and R are chosen to be positive definite, ensuring that Φ remains nonnegative. The sum Φ^∞ measures the contribution of the freely evolving state to the controller cost function. It can be written as a function of the terminal state z_{k+N} and a terminal weight $\overline{Q}$, computed as the solution to a discrete Lyapunov equation:

$$\Phi^{\infty} = z_N^T \, \overline{Q} \, z_N \qquad \overline{Q} = Q + A^T \, \overline{Q} \, A \quad (5)$$

The cost function 3 depends on a sequence of future inputs that can be conveniently represented by the input vector π_k:

$$\pi_k = \left[v_0^T \cdots v_{N-1}^T\right]^T \qquad (6)$$

Given 6, the controller cost function 3 can be considered to be a function of x_k, π_k, (A, B), Q, R, and N. The controller cost function obtained using the actual plant (A, B) will be referred to as the *plant cost* Φ_k:

$$\Phi_k(\pi_k) = \Phi(x_k, \pi_k, (A, B), Q, R, N) \qquad (7)$$

The controller cost function obtained using the nominal model $(\tilde{A}, \tilde{B})$ will be referred to as the *nominal model cost* $\tilde{\Phi}_k$:

$$\tilde{\Phi}_k(\pi_k) = \Phi(x_k, \pi_k, (\tilde{A}, \tilde{B}), Q, R, N) \qquad (8)$$

The controller cost function obtained using model i from the set Ω will be referred to as the *model i cost* Φ_{ik}:

$$\Phi_{ik}(\pi_k) = \Phi(x_k, \pi_k, (A_i, B_i), Q, R, N) \qquad (9)$$

The controller cost function Φ can be written as a quadratic function of the input vector π_k and the initial state x_k (Badgwell, 1996):

$$\Phi = \pi_k^T \, H \, \pi_k + 2\pi_k^T \, G \, x_k + x_k^T \, L \, x_k \qquad (10)$$

The Hessian matrix H, gradient matrix G, and initial state matrix L can be computed directly from the model (A, B), weight matrices Q and R, and the input horizon length N. Choosing Q and R positive definite leads to a positive definite Hessian matrix H; the function Φ is therefore a strictly convex function of the input vector π_k.

3.. Robustly Stabilizing Regulator

We now define a robustly stabilizing algorithm.

Definition: RLMPC. The *Robust Linear MPC* algorithm (RLMPC) finds the input vector $\overset{*}{\pi}_k$ that minimizes the nominal model cost 8 with Q and R positive definite, and $N \geq 1$:

$$\overset{*}{\pi}_k = \arg\min \left(\tilde{\Phi}_k(\pi_k)\right) \qquad (11)$$

subject to:

$$\Phi_{ik}(\pi_k) \leq \Phi_{ik}(\hat{\pi}_k), \qquad i = 1, p \qquad (12)$$

The first element of optimal input vector $\overset{*}{\pi}_k$ is then injected into the plant:

$$u_k = \overset{*}{v}_0 \qquad (13)$$

□

The input $\hat{\pi}_k$, referred to as the *restriction of the input*, is a shifted version of the previous optimal input $\overset{*}{\pi}_{k-1}$. If the optimal input at time $k-1$ is defined as:

$$\overset{*}{\pi}_{k-1} = \left[\overset{*}{v}_0^T \;\; \overset{*}{v}_1^T \cdots \;\; \overset{*}{v}_{N-1}^T\right]^T \qquad (14)$$

then the restriction of the input at time k is given by:

$$\hat{\pi}_k = \left[\overset{*}{v}_1^T \cdots \;\; \overset{*}{v}_{N-1}^T \;\; 0\right]^T \qquad (15)$$

At the first time step the restriction of the input is initialized to:

$$\hat{\pi}_0 = \left[0^T \cdots 0^T\right]^T \qquad (16)$$

The robustness constraints 12 require the model cost function values for each plant in the family to either remain constant or to decrease at each time step, relative to the cost values computed using the current measured state and the restriction of the input. Because the state and input are required to contract on an *infinite* horizon, this is less conservative than a finite horizon contraction constraint. The robustness constraints 12 are feasible at each time step for the choice $\pi_k = \hat{\pi}_k$.

At each sample time k, the right side of 12 is constant, and the left side is a strictly convex function of the input π_k. The objective function 11 is also strictly convex, which means that the RLMPC algorithm is a special form of nonlinear program called a *strictly convex* program. This is significant because it means that the solution can be found using much simpler numerical methods (Gill et al., 1981), and because existence of the feasible input $\hat{\pi}_k$ ensures the existence of a unique optimal input $\overset{*}{\pi}_k$ at each time step.

The RLMPC algorithm reduces to the Rawlings/Muske regulator (Rawlings and Muske, 1993) for a stable plant with no state or input constraints for the special case in which the uncertainty set Ω contains only the true plant.

The following theorem shows that the RLMPC algorithm brings the state and input of the closed loop system asymptotically to the origin, even though it minimizes the nominal model cost function at each time step.

Theorem: Robust Stability of the RLMPC algorithm. When the input is computed using the RLMPC algorithm, the origin is an asymptotically stable equilibrium point for the closed loop system, with a region of attraction consisting of all $x_0 \in \Re^n$.

Proof: First it is shown that the input and true plant state converge to the origin, and then it is shown that the origin is a stable equilibrium point for the closed loop system. The combination of convergence and stability gives asymptotic stability.

Convergence: Let us assume that we have found the optimal solution at time step k, given by $\overset{*}{\pi}_k$. Although the solution $\overset{*}{\pi}_k$ is found by minimizing the nominal model cost function $\tilde{\Phi}_k$, we can determine the true cost for this solution by computing the plant cost Φ_k, which is based on the true plant (A, B):

$$\overset{*}{\Phi}_k = \Phi(x_k, \overset{*}{\pi}_k, (A,B), Q, R, N) \quad (17)$$
$$= \sum_{j=0}^{\infty} \left(z_j^T \, Q \, z_j + \overset{*}{v}_j^T \, R \, \overset{*}{v}_j \right) \quad (18)$$

Assume that the optimal input is injected into the plant. The state at the next time step is then given by:

$$x_{k+1} = A \, x_k + B \, \overset{*}{v}_0 \quad (19)$$

The plant cost at time $k+1$ using the restriction of the input $\hat{\pi}_{k+1}$ can be written as:

$$\hat{\Phi}_{k+1} = \Phi(x_{k+1}, \hat{\pi}_{k+1}, (A,B), Q, R, N) \quad (20)$$
$$= \sum_{j=1}^{\infty} \left(z_j^T \, Q \, z_j + \overset{*}{v}_j^T \, R \, \overset{*}{v}_j \right) \quad (21)$$

Because they are computed using the true plant (A, B), the state sequences z_j in 18 and 21 are identical. Subtract 18 from 21 to get:

$$\hat{\Phi}_{k+1} - \overset{*}{\Phi}_k = -x_k^T \, Q \, x_k - u_k^T \, R \, u_k \quad (22)$$

We know that the plant lies in the family Ω, so the robustness constraint 12 must be satisfied for the actual plant at time $k+1$:

$$\overset{*}{\Phi}_{k+1} \leq \hat{\Phi}_{k+1} \quad (23)$$

Combine 22 with 23 to get:

$$\overset{*}{\Phi}_{k+1} - \overset{*}{\Phi}_k \leq -x_k^T \, Q \, x_k - u_k^T \, R \, u_k \quad (24)$$

This shows that the sequence of optimal plant cost values $\{\overset{*}{\Phi}_k\}$ is non-increasing. The plant cost is bounded below by zero and thus has a non-negative limit. Therefore as $k \to \infty$, the left hand side of 24 approaches zero. Because Q and R are positive definite, as the left side of 24 approaches zero, the input and state must converge to the origin:

$$x_k \to 0 \text{ and } u_k \to 0 \text{ as } k \to \infty \quad (25)$$

Stability: Stability at the origin can be established using a proof by contradiction, provided that the true plant cost is a continuous function of the state at the origin.

The unique optimal solution $\overset{*}{\pi}_0$ at time $k = 0$ depends on the the initial state x_0. The true plant cost for this solution is therefore a function of x_0:

$$\overset{*}{\Phi}_0(x_0) = \Phi(x_0, \overset{*}{\pi}_k(x_0), (A,B), Q, R, N) \quad (26)$$

The following upper bound for $\overset{*}{\Phi}_0(x_0)$ can be defined:

$$\overset{*}{\Phi}_{\max}(x_0) = \Phi_0(\hat{\pi}_0)$$
$$= 2\hat{\pi}^T \, G \, x_0 + x_0^T \, L \, x_0 \quad (27)$$

The function 27 is an upper bound because the robustness constraint 23 is satisfied at $k = 0$. The following lower bound for $\overset{*}{\Phi}_0(x_0)$ can also be defined:

$$\overset{*}{\Phi}_{\min}(x_0) = x_0^T \, Q \, x_0 \quad (28)$$

Both the upper bound 27 and the lower bound 28 are continuous functions of x_0 and take the value zero at the origin; the function $\overset{*}{\Phi}_0(x_0)$ is therefore continuous in the state at the origin.

Now define a non-decreasing function γ that provides a lower bound on $\overset{*}{\Phi}_k(x_k)$ at each time step k:

$$\gamma(\|x_k\|) = \|Q\| \, \|x_k\|^2 = x_k^T \, Q \, x_k \quad (29)$$

Choose some $\rho > 0$. From continuity of $\overset{*}{\Phi}_0(x_0)$ at the origin, there exists some $\delta < 0$ such that:

$$\|x_0\| < \delta \;\Rightarrow\; \overset{*}{\Phi}_0(x_0) < \gamma(\rho) \quad (30)$$

Because the sequence of optimal plant cost values 24 is non-increasing this implies that:

$$\overset{*}{\Phi}_j(x_j) < \gamma(\rho) \qquad j > 0 \quad (31)$$

Now assume the RLMPC algorithm is not stabilizing. Then for some sufficiently small $r > 0$ we must have $\|x_j\| \geq \rho$ for some value of $j > 0$, even though $\|x_0\| < r$. Since γ is non-decreasing and is a lower bound on $\overset{*}{\Phi}_j(x_j)$, we must have:

$$\gamma(\rho) \leq \gamma(\|x_j\|) \leq \overset{*}{\Phi}_j(x_j) \quad (32)$$

Combining 31 and 32 leads to:

$$\gamma(\rho) \le \gamma(\|x_j\|) \le \overset{*}{\Phi}_j(x_j) < \gamma(\rho) \qquad (33)$$

which is a contradiction. The origin is therefore a stable equilibrium point for the RLMPC algorithm. The combination of convergence and stability implies that the origin is an asymptotically stable equilibrium point for the RLMPC algorithm. **QED**

The convergence proof for the states and inputs given here is a direct generalization of the proof for Theorem 2 in the paper by Rawlings and Muske (1993). The stability proof closely follows that given by Meadows et al. (1995) for nonlinear systems, in which continuity of the optimal controller cost function at the origin is assumed.

We now define a nominally stabilizing regulator to be used in the simulation examples.

Definition: LMPC. The *Linear Model Predictive Control* algorithm (LMPC) is defined as the RLMPC algorithm with the true plant as the only member of the uncertainty set Ω.

4.. Multivariable CSTR Example

Consider an adiabatic, constant level Continuous Stirred Tank Reactor (CSTR) with a single irreversible, first order reaction:

$$\mathrm{A} \xrightarrow{k} \mathrm{B}$$

For this process the inputs are the feed concentration of A and feed temperature; The states are the exit concentration of A and exit temperature. By assuming constant fluid density, heat capacity, reactor volume, heat of reaction and flowrate, we can derive the following linearized, continuous time, dimensionless state-space model:

$$A = \begin{bmatrix} -(1+Da) & -\alpha Da \\ -\beta Da & -(1+\alpha\beta Da) \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The variables used in the model are:

x_1	dimensionless exit concentration of A
x_2	dimensionless exit temperature
u_1	dimensionless feed concentration of A
u_2	dimensionless feed temperature
Da	Damkoehler number
α	dimensionless activation energy
β	dimensionless heat of reaction

Assume that the usual reactor operation results in $\beta = -0.45$ but occasional feed impurities lead to side reactions that can increase the magnitude of β to values as large as -1.00. Other plant parameters are constant at $\alpha = 1.0$ and $Da = 20.0$. Assume also that closed loop simulation with the LMPC algorithm and $x_0 = I$ leads to the following preferred settings for the small β plant with an initial state: $Q = I$, $R = 0.01 * I$, and $N = 2$.

Figure 1 shows the response of the LMPC algorithm for this case. The controller brings the system back to the origin quickly and holds it there. Figure 2 shows the response of the RLMPC algorithm on the same plant. Here the RLMPC algorithm uses the small β plant ($\beta = -0.45$) for its model and both the large β plant ($\beta = -1.00$) and the small *beta* plant for the robustness constraints. Note that there is no difference in performance between the two algorithms for this case. This shows that for this example the robustness constraints have no effect on closed loop response when the model is perfect.

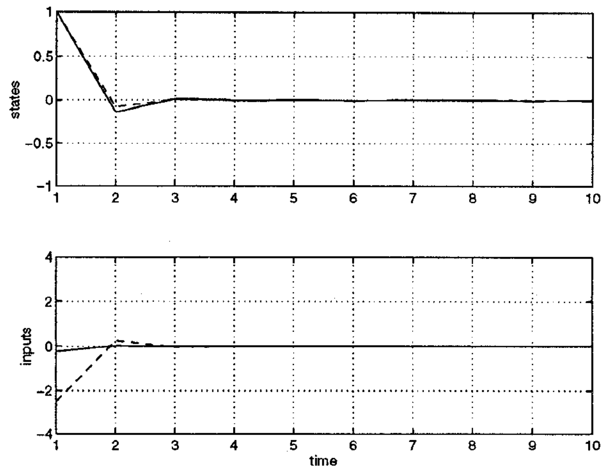

Figure 1: LMPC response for $\beta = -.45$

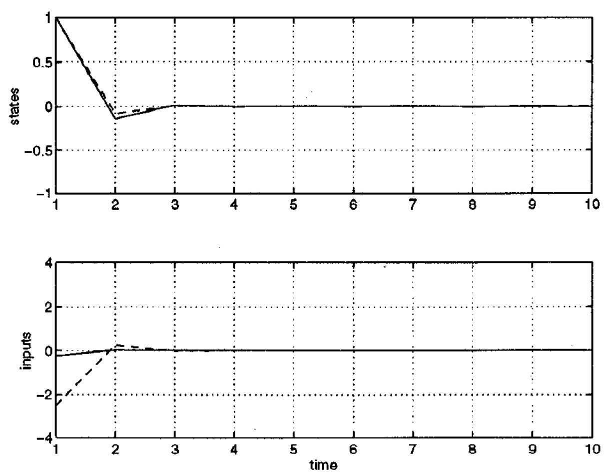

Figure 2: RLMPC response for $\beta = -.45$

Figure 3 shows the response of the LMPC algorithm for the large β plant. The closed loop system goes unstable due to model mismatch. In contrast Figure 4 shows that the RLMPC algorithm stabilizes the closed loop system for this case.

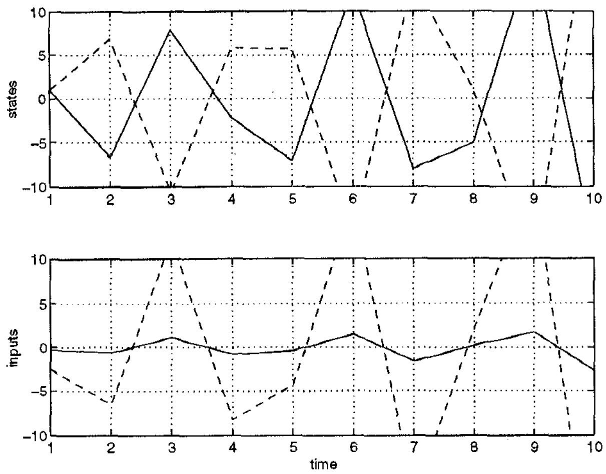

Figure 3: LMPC response for $\beta = -1.00$

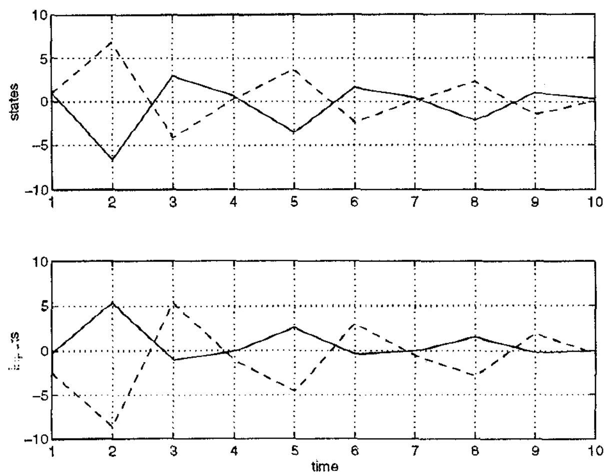

Figure 4: RLMPC response for $\beta = -1.00$

5.. Conclusions

A new robust MPC algorithm (RLMPC) has been presented for stable linear plants. Model uncertainty is parameterized by a list of possible plants. The algorithm is a direct generalization of the nominal stabilizing regulator presented by Rawlings and Muske (1993). Robust stability is achieved by adding constraints that prevent the sequence of optimal controller costs from increasing for the true plant.

Defining the robustness constraints in terms of an infinite horizon controller cost function has several advantages. First the resulting constraints are less restrictive than those based on finite horizon state contraction. For the simulation example shown here the robustness constraints have no effect on the closed loop response when the model is perfect. A second advantage is that no new tuning parameters are required. Finally, robust stability is achieved for all possible values of the tuning parameters.

The RLMPC algorithm can be written in a quasi-QP form consisting of a standard QP with an additional quadratic constraint, resulting in a *strictly convex* nonlinear program. The resulting nonlinear program has been shown to have a unique optimal solution at each time step.

The robust stability results presented here do not require linearity; indeed these results can potentially be extended to stable nonlinear systems.

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