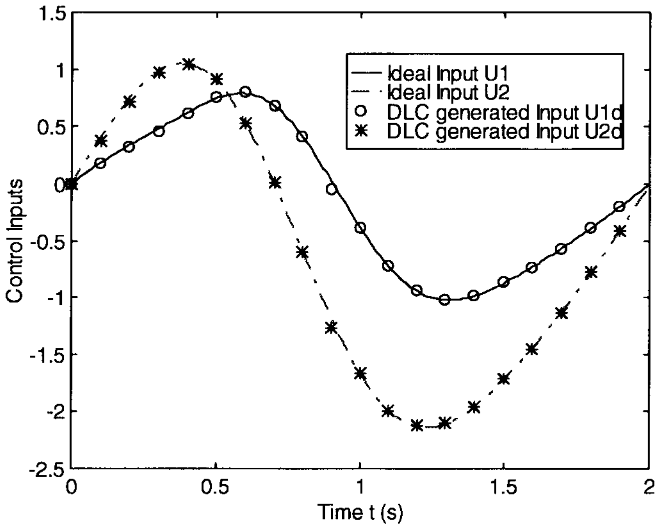

Fig. 1. The desired and directly learned control input profiles.

In the simulation, it is assumed that control knowledge concerning the following three trajectories has been achieved *a priori*.

1)
$$\mathbf{x}_1(t_1) = \begin{bmatrix} 1-\cos(1.5\omega t_1) \\ 1-\cos(1.5\omega t_1) \end{bmatrix}; \quad t_1 = \rho_1(t_d) = \tfrac{2}{3}t_d; \qquad t_1 \in \left[0, \tfrac{4}{3}\right].$$

2)
$$\mathbf{x}_2(t_2) = \begin{bmatrix} 1-\cos\left(\omega \frac{t_2^2}{2}\right) \\ 1-\cos\left(\omega \frac{t_2^2}{2}\right) \end{bmatrix}; \quad t_2 = \rho_2(t_d) = \sqrt{2t_d}; \qquad t_2 \in [0, 2].$$

3)
$$\mathbf{x}_3(t_3) = \begin{bmatrix} 1-\cos\left(\omega \sqrt{2t_3}\right) \\ 1-\cos\left(\omega \sqrt{2t_3}\right) \end{bmatrix}; \quad t_3 = \rho_3(t_d) = \tfrac{t_d^2}{2}; \qquad t_3 \in [0, 2].$$

It is easy to verify that there is one singularity between $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_2)$ at $t_d = 1.125$ s, one singularity between $\mathbf{x}_1(t_1)$ and $\mathbf{x}_3(t_3)$ at $t_d = 2/3$ s, and one singularity between $\mathbf{x}_2(t_2)$ and $\mathbf{x}_3(t_3)$ at about $t_d = 0.8$ s. Hence, an augmented regressor matrix A_3 is constructed to utilize all three given trajectories.

A sampling interval of 0.01 s is used in the simulation. Since the sampling interval is rather small, it is not even necessary to carry out any interpolation. In the simulation, $t_i = \rho_i(t_d)$ is simply rounded off to two decimal places, and the corresponding prestored control input signal is used.

Fig. 1 shows the desired input signals and the directly learned input signals for the control inputs $u_1(t_d)$ and $u_2(t_d)$. The effectiveness of the proposed direct learning scheme is immediately obvious, as the direct learning control (DLC)-generated input profiles are consistent with the desired control input profiles.

V. Conclusion

Direct learning control schemes for a class of nonlinear systems are developed. The proposed control schemes are able to learn from stored control profiles with different time scales and generate the desired control profile directly without any repeated learning process. The singularity problem in direct learning is addressed by means of using more stored control knowledge. The introduction of the new concept *direct learning* and the associated direct learning control methods open a new direction toward solving the nonrepeatable learning control problems. Simulation results confirm the validity of the proposed direct learning control scheme.

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Stabilizing Receding-Horizon Control of Nonlinear Time-Varying Systems

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***Abstract*—A new receding-horizon control scheme for nonlinear time-varying systems is proposed which is based on a finite-horizon optimization problem with a terminal state penalty. The penalty is equal to the cost that would be incurred over an infinite horizon by applying a (locally stabilizing) linear control law to the nonlinear system. Assuming only stabilizability of the linearized system around the desired equilibrium, the new scheme ensures exponential stability of the equilibrium. As the length of the optimization horizon goes from zero to infinity, the domain of attraction moves from the basin of attraction of the linear controller toward the basin of attraction of the infinite-horizon nonlinear controller. Stability robustness in the face of system perturbations is also established.**

***Index Terms*—Nonlinear control, optimal control, predictive control, receding horizon control, time-varying systems.**

I. Introduction

Since the seminal papers [1] and [2], the receding horizon (RH) approach has been shown to be a very effective way to control nonlinear systems. Basically, RH control is developed according to a very simple idea. At time t, a finite horizon (FH) optimal control problem is solved over $[t, t+N]$ and the corresponding

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optimal control $u^o(j),\ t \leq j < t+N$ is computed. Then, the current control is set equal to $u^o(t)$. At the next time instant the whole procedure is repeated. In [1] and [2], closed-loop stability is enforced by complementing the FH optimization problem with the constraint $x(t+N) = 0$ on the final state. Since this nonlinear optimization problem with equality constraint is computationally demanding, in [3] closed-loop stability is ensured by resorting to a terminal inequality constraint requiring that $x(t+N)$ enters a suitable neighborhood of the origin. When such a neighborhood is reached, the nonlinear RH controller switches to a locally stabilizing linear control law designed on the basis of the linearized system. The potential disadvantages of this method are the complex machinery of the algorithm involving variable horizons and the somewhat artificial switching from a nonlinear to a linear controller. A different approach has been proposed in [4], where the RH controller is obtained by solving an FH problem with the quadratic terminal state penalty $ax(t+N)'Px(t+N)$, P being a positive definite symmetric matrix. Although under suitable assumptions there exist values of N, a, and P that ensure closed-loop stability, simple constructive guidelines for tuning these parameters are not available.

In the present paper, a new nonlinear RH control scheme is proposed that circumvents most of the drawbacks of previous methods without giving up guaranteed stability. The first step is the design of a locally stabilizing linear controller. Then, the RH control law is computed through the solution of an FH problem with a (generally nonquadratic) terminal state penalty equal to the cost that would be incurred by applying the linear control law thereafter. Note that, since the RH rationale is adopted, the linear control law is never actually applied, but it is just used to compute the terminal state penalty; this represents a main difference between this method and that presented in [3]. Assuming only stabilizability of the linearized system around the desired equilibrium, the new RH scheme guarantees exponential stability of the equilibrium for any value of the horizon N. Moreover, as N varies from zero to ∞, the basin of attraction of the equilibrium moves from the basin of attraction of the linear controller toward the basin of attraction of the infinite horizon (IH) nonlinear controller. Hence, N regulates the tradeoff between computational complexity, on one side, and extent of the stability region and performance, on the other. Notably, if the linear controller is designed as an IH optimal controller minimizing the same cost functional (of l_2 type, for instance) used in the nonlinear FH problem, then the derivative of the nonlinear RH control law in the equilibrium point coincides with the gain of the linear control law. Therefore, the new scheme provides a method to obtain a consistent nonlinear extension of a locally optimal linear control law.

Since the new algorithm completely avoids the use of equality constraints, its computational feasibility is substantially better than in [1] and [2] and can be compared to that of [3] and [4]. The fact that the stability results here reported are derived in the general time-varying case allows one to address a more general class of stabilization problems, namely those concerning stabilization around a given state movement. Finally, with reference to time-invariant affine systems subject to additive and multiplicative perturbations, it is easy to see that the new algorithm enjoys the same robustness properties established in [5] for a nonlinear RH algorithm with terminal equality constraint. In addition, a new robustness result is derived for a very general class of model perturbations. To enhance readability, all the proofs of lemmas and theorems are reported in the Appendix.

II. Notation and Basic Assumptions

The symbol $\|\cdot\|$ denotes the Euclidean vector norm in R^n (where the dimension n follows from the context); B_r denotes the closed ball of radius r, i.e., $B_r = \{x \in R^n : \|x\| \leq r\}$.

The paper deals with receding-horizon control of the time-varying nonlinear discrete-time dynamic system

$$x(k+1) = f(x(k), u(k), k), \qquad x(t) = \bar{x}, \qquad k \geq t \tag{1}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^m$ is the input, $f(\cdot,\cdot,\cdot)$ is a uniformly C^1 function, and $f(0,0,k) = 0$, $\forall k \geq t$. The state and input vectors are subject to the constraints

$$x(k) \in X(k), \qquad u(k) \in U(k), \qquad k \geq t \tag{2}$$

where $X(k)$ and $U(k)$ are closed subsets of R^n and R^m, respectively, both containing the origin as an interior point. In the following it is assumed that there exist positive constants r_1, r_2 such that $X(k) \supseteq B_{r_1}$, $U(k) \supseteq B_{r_2}$ $\forall k \geq t$.

Assumption A1: There exist positive constants r_3, r_4 such that $B_{r_3} \supseteq X(k)$, $B_{r_4} \supseteq U(k)$, $\forall k \geq t$.

In order to investigate the properties of the linearized system, define

$$A(k) = \left.\frac{\delta f(x,u,k)}{\delta x}\right|_{x=0,u=0} \qquad B(k) = \left.\frac{\delta f(x,u,k)}{\delta u}\right|_{x=0,u=0}$$

$\tilde{f}(x,u,k) = f(x,u,k) - [A(k)x(k) + B(k)u(k)]$.

Assumption A2:

$$\lim_{\|(x,u)\| \to 0} \sup_{k \geq 0} \frac{\|\tilde{f}(x,u,k)\|}{\|(x,u)\|} = 0. \qquad \square$$

According to [6], when A2 holds, the system

$$x_L(k+1) = A(k)x_L(k) + B(k)u(k) \tag{3}$$

is called the linearized system of (1) around the equilibrium point $(x,u) = (0,0)$.

Assumption A3: The time-varying matrices $A(k)$, $B(k)$ are uniformly bounded, $\forall k \geq t$: $\|A(k)\| < M_A$, $\|B(k)\| < M_B$, where M_A and M_B are finite positive constants. Moreover, the pair $(A(\cdot), B(\cdot))$ is uniformly stabilizable [7]. $\square$

With reference to (1) the equilibrium (0, 0) is said exponentially stable if there exist constants $r, a, b > 0$ such that, $\forall \bar{x} \in B_r$

$$\|x(k)\| \leq a\|\bar{x}\| \exp(-b(k-t)), \qquad \forall k \geq t. \tag{4}$$

Given a and b, a closed and bounded set $X_e(t)$ such that (4) holds $\forall \bar{x} \in X_e(t)$ will be said to be an *(a,b)-exponential stability region* of (1). In order to design an exponentially stabilizing control law for the linearized system (3), uniform stabilizability of $(A(\cdot), B(\cdot))$ must be assumed.

Proposition 1 [7]: The pair $(A(\cdot), B(\cdot))$ is uniformly stabilizable if and only if there exists a bounded matrix function $K(\cdot)$, $\|K(\cdot)\| \leq M_K$, $\forall k > t$, where M_K is a finite positive constant, such that $A(\cdot) + B(\cdot)K(\cdot)$ is exponentially stable. $\square$

When a linear feedback law $u(k) = K(k)x(k)$ is applied to (1), the closed-loop dynamics is given by

$$x(k+1) = f_c(x(k), k), \qquad x(t) = \bar{x}, \qquad k \geq t \tag{5}$$

where $f_c(x,k) = f(x, K(k)x, k)$. Define also $F(k) = A(k) + B(k)K(k)$ and $\tilde{f}_c(x,k) = f_c(x,k) - F(k)x$. Note that since $A(\cdot)$, $B(\cdot)$, and $K(\cdot)$ are bounded, $F(\cdot)$ is bounded too, i.e., there exists a finite positive constant M_F such that $\|F(k)\| \leq M_F$, $\forall k \geq t$. In order to use linearization to analyze closed-loop stability, it is essential to demonstrate also that $\tilde{f}_c$ goes to zero uniformly faster than $\|(x,u)\|$.

Lemma 1: Under Assumptions A1–A3

$$\lim_{\|x\| \to 0} \sup_{k \geq 0} \frac{\|\tilde{f}_c(x,k)\|}{\|x\|} = 0. \qquad \square$$

Provided that Assumptions A2 and A3 are satisfied, the stabilizing gain for the linearized system (3) locally stabilizes (1) around the origin as well.

Proposition 2 [6]: Consider the closed-loop nonlinear system (5), where $K(\cdot)$ is a bounded gain such that $F(\cdot) = [A(\cdot) + B(\cdot)K(\cdot)]$ is exponentially stable. Then under A2 and A3, the origin is an exponentially stable equilibrium point. □

In the following, $\varphi(k, t, \bar{x}, \tilde{u}(\cdot))$ will denote the solution $x(k)$ of (1) at time $k \geq t$ with initial condition $x(t) = \bar{x}$ and forcing input $u(k) = \tilde{u}(k)$, $k \geq t$. Analogously, $\varphi_c(k, t, \bar{x})$ is the solution $x(k)$ of (5) at time $k \geq t$ with $x(t) = \bar{x}$.

Given a bounded time-varying gain $K(\cdot)$, which exponentially stabilizes $(A(\cdot), B(\cdot))$ and two positive constants a, b, the symbol $X(K(\cdot), t)$ represents an (a, b)-exponential stability region for the closed-loop system (5) such that $\bar{x} \in X(K(\cdot), t)$ implies the fulfillment of the constraints (2), i.e., $\varphi_c(k, t, \bar{x}) \in X(k)$, $K(k)\varphi_c(k, t, \bar{x}) \in U(k)$, $\forall k \geq t$. In other words, $\bar{x} \in X(K(\cdot), t)$ if the application of the linear control law $u(k) = K(k)x(k)$ to the nonlinear system (1) generates an input sequence guaranteeing the satisfaction of the constraints (2) and driving the state to the origin at a rate specified by (4). In practice, one can empirically verify whether $\bar{x} \in X(K(\cdot), t)$, by integrating (1) with $u(k) = K(k)x(k)$, $k \geq t$ for a sufficiently large number of steps checking that (2) and (4) hold $\forall k \geq t$. Note that, if $A(\cdot) + B(\cdot)K(\cdot)$ is exponentially stable and the assumptions of Proposition 2 hold, then, for a suitable choice of a and b, the set $X(K(\cdot)) := \bigcap_{t \geq 0} X(K(\cdot), t)$ is nonempty and contains the origin as an interior point. Lyapunov stability results that will prove useful in the subsequent stability analysis conclude the section.

Lemma 2 [1]: Let $\tilde{X}(k)$ be a set in R^n such that, for some $r > 0$, $B_r \subseteq \tilde{X}(k)$, $\forall k \geq t$. Consider (5), and let $Y = \{(k, x) : k \geq t, x \in \tilde{X}(k)\}$. Suppose that $\exists V : Y \to R$ and constants α_1, β_1, γ_1, $q \geq 1$, which satisfy the following conditions.

1) $V(k, x) \leq \beta_1 \|x\|^q$, $\forall (k, x) \in Y$.
2) $V(k, x) \geq \alpha_1 \|x\|^q$, $V(k, x) - V(k+1, f_c(x, k)) \geq \gamma_1 \|x\|^q$, $\forall (k, x) \in Y$.

Then, the origin is an exponentially stable equilibrium and $\tilde{X}(k)$ is an exponential stability region.

III. Nonlinear RH Control

The usual way to stabilize the origin of (1) is to design a stabilizing gain for the linearized system (5). Although exponential stability is then guaranteed in a neighborhood of the origin, this neighborhood could be too small and/or performance unsatisfactory when the initial state is not sufficiently close to the origin. In order to enlarge the stability region of the closed-loop system and improve dynamic performance, one might use the nonlinear IH optimal control law $u = \gamma^{\rm IH}(\bar{x}, t)$ obtained through the minimization of

$$J_{\rm IH}(\bar{x}, u(\cdot), t) = \sum_{k=0}^{\infty} h(x(t+k), u(t+k), t+k) \tag{6}$$

subject to (1) and (2). Concerning $h(\cdot, \cdot, \cdot)$, the following assumption is made.

Assumption A4: There exist positive numbers p_1, p_2, and $q \geq 1$ such that $p_1 \|(x, u)\|^q \leq h(x, u, k) = \|(x, u)\|^q_{\bar{Q}}(k) \leq p_2 \|(x, u)\|^q$, $\forall x \in X(k)$, $\forall u \in U(k)$, $\forall k \geq t$. □

The stabilizing properties of the IH controller have been analyzed in [1]. In particular, denoting by $X^{\rm IH}(t)$ the set of initial conditions $\bar{x}$ such that the IH problem is solvable, if $x(t) = \bar{x} \in X^{\rm IH}(t)$, the closed-loop dynamics $x(k+1) = f(x(k), \gamma^{\rm IH}(x(k)), k)$, $k \geq t$, asymptotically drives $x(k)$ to the origin. Moreover, under A4, it is easy to see that $\bar{x} \in X(K(\cdot), t)$ implies $J_{\rm IH}(\bar{x}, K(\cdot)\varphi_c(\cdot, t, \bar{x}), t) < \infty$, so that $X^{\rm IH}(t) \supseteq X(K(t), t)$.

Unfortunately, for a generic nonlinear system, analytic solutions of the IH control problem do not exist, and even the attempt to approximate the IH cost functional by means of a finite-horizon one (by truncating the series in $J_{\rm IH}$) leads to a hard optimization problem in a large dimensional space. This motivated the development of RH control strategies that, without giving up the stability properties of the IH controller, are however computationally feasible [3], [4]. For the same reasons, hereafter, a novel RH strategy, first proposed in [8] for time-invariant systems, is presented and extended to the time-varying case. It is based on the following finite-horizon problem.

Finite-Horizon Optimal Control Problem (FHOCP): Minimize with respect to $u_{t,t+N-1} := [u(t) \;\; u(t+1) \;\; \cdots \;\; u(t+N-1)]$, the cost function

$$\begin{aligned} J(\bar{x}, u_{t,t+N-1}, N, K(\cdot), t) &= \sum_{i=0}^{N-1} h(x(t+i), u(t+i), t+i) \\ &\quad + V_f(x(t+N), K(\cdot), t+N) \end{aligned} \tag{7a}$$

subject to (1) and (2), where the terminal state penalty V_f is defined as

$$V_f(\bar{x}, K(\cdot), t) = \begin{cases} \sum_{k=t}^{\infty} h(\varphi_c(k, t, \bar{x}), K(k)\varphi_c(k, t, \bar{x}), k), & \text{if } \bar{x} \in X(K(\cdot), t) \\ \infty, & \text{otherwise.} \end{cases} \tag{7b}$$

In other words, $V_f(\bar{x}, K(\cdot), t)$ is the cost that is incurred over $[t, \infty)$ by applying the linear control law $u(k) = K(k)x(k)$ to (1), with the convention of assigning an infinite cost when the admissibility constraints (2) are violated. Note that given $K(\cdot)$ such that $A(\cdot) + B(\cdot)K(\cdot)$ is exponentially stable, if $\bar{x}$ belongs to the exponential stability region $X(K(\cdot), t)$, then $V_f(\bar{x}, K(\cdot), t)$ is always finite in view of A4.

In the sequel, given an initial state $\bar{x}$ and an exponentially stabilizing gain $K(\cdot)$, a sequence $u_{t,t+N-1}$ is said *admissible* if, when applied to (1), the following constraints are simultaneously satisfied.

C1) $x(k) \in X(k)$, $u(k) \in U(k)$, $t \leq k < t+N$.
C2) $x(t+N) \in X(K(\cdot), t+N)$.

Nonlinear Receding Horizon (NRH) Control Law: Given $K(\cdot)$ such that $A(\cdot) + B(\cdot)K(\cdot)$ is exponentially stable, at every time instant t define $\bar{x} = x(t)$ and find an admissible control sequence $u^o_{t,t+N-1}$ minimizing the FHOCP. Then, apply the control $u(t) = u^o(\bar{x}, t)$, where $u^o(\bar{x}, t)$ is the first column of $u^o_{t,t+N-1}$. □

The RH control law is defined by the function $u = \gamma^{\rm RH}(x, t)$, where $\gamma^{\rm RH}(x, t) := u^o(x, t)$. Accordingly, $\varphi^{\rm RH}(k, t, \bar{x})$ will denote the solution $x^{\rm RH}(k)$ at time $k \geq t$ of the closed-loop system

$$x^{\rm RH}(k+1) = f(x^{\rm RH}(k), \gamma^{\rm RH}(x^{\rm RH}(k), k), k) \tag{8}$$

with initial condition $x^{\rm RH}(t) = \bar{x}$.

Hereafter, $X_0(N, K(\cdot), t)$ will denote the set of states $\bar{x}$ such that the RH control $u^o(\bar{x}, t)$ is computable (i.e., the set of admissible sequences is nonempty).

Proposition 3: The set $X_0(N, K(\cdot), t)$ enjoys the following properties.

1) $X^{\rm IH}(t) \supseteq X_0(N, K(\cdot), t) \supseteq X(K(\cdot), t)$.
2) $X_0(M+1, K(\cdot), t) \supseteq X_0(M, K(\cdot), t)$, $\forall M > 0$.
3) $\lim_{M \to \infty} X^0(M, K(\cdot), t) = X^{\rm IH}(t)$. □

In view of 1), the boundary of $X_0(N, K(\cdot), t)$ lies somewhere in between the boundaries of $X(K(\cdot), t)$ and $X^{\rm IH}(t)$ (see Fig. 1). Observing that the sets $X_0(N, K(\cdot), t)$ approach $X^{\rm IH}(t)$ as $N \to \infty$, the NRH controller is a kind of compromise between the IH optimal controller and the linear controller, the degree of compromise being tunable through the parameter N. This is even more true in view of the fact that $X_0(N, K(\cdot), t)$ will soon be shown to be an exponential stability region for the closed-loop system (8) (see Theorem 1).

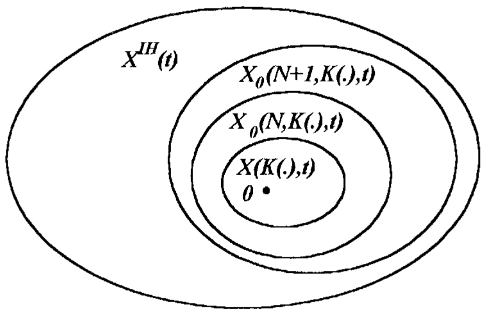

Fig. 1. $X(K(\cdot),t)$: exponential stability region of the linear controller $u(t)=K(t)x(t)$; $X_0(N,K(\cdot),t)$: solvability region of the NRH controller with horizon length N; and $X^{\mathrm{IH}}(t)$: solvability region of the nonlinear IH controller.

Now let $J^o(\bar{x},N,K(\cdot),t)$ be the optimal value of the cost function of the FHOCP. Before stating the main stability result, a technical lemma is needed. In particular, point 2) of the following lemma states that $X_0(N,K(\cdot),t)$ is an invariant set for the closed-loop system (8).

Lemma 3: Assume that $\bar{x}\in X_0(N,K(\cdot),t)$. Then:

1) $J^o(\bar{x},N+1,K(\cdot),t)\le J^o(\bar{x},N,K(\cdot),t)$;
2) $x^{\mathrm{RH}}(t+1)=\varphi^{\mathrm{RH}}(t+1,t,\bar{x})\in X_0(N,K(\cdot),t+1)$;
3) $J^o(x^{\mathrm{RH}}(t+1),N,K(\cdot),t+1)\le J^o(\bar{x},N,K(\cdot),t)-h(\bar{x},\gamma^{\mathrm{RH}}(\bar{x},t),t)$.

Theorem 1: Assume that A1–A4 hold and let $K(\cdot)$ be such that $A(\cdot)+B(\cdot)K(\cdot)$ is exponentially stable. Then, if the RH control law $u=\gamma^{\mathrm{RH}}(x,t)$ is applied to the nonlinear (1), the origin is an exponentially stable equilibrium point of the resulting closed-loop system having $X^0(N,K(\cdot),t)$ as exponential stability region. □

For the sake of simplicity, only the time-invariant case is considered in the remarks below.

Remark 1: A very practical procedure for stabilizing the nonlinear system (1) is to first design a linear control law $u=Kx$ by minimizing J_{IH} subject to the linearized state dynamics (3). In this respect, if one considers the cost function $h(x,u)=x'Qx+u'Ru$, $Q>0$, $R>0$, well-established tools are available for tuning the weighting matrices Q and R so as to achieve the desired specifications for the linearized closed-loop system $x(k+1)=(A+BK)x(k)$. Then, the same $h(x,u)$ is used to implement the nonlinear RH controller. Under regularity assumptions, as $\|x\|\to 0$, it turns out that $\gamma^{\mathrm{RH}}(x)\to\gamma^{\mathrm{IH}}(x)\to Kx$. In particular, $\partial\gamma^{\mathrm{RH}}(x)/\partial x|_{x=0}=K$ so that the NRH control law can be regarded as a consistent nonlinear extension of the linear control law $u=Kx$. In this procedure, once Q and R have been selected, the only free parameter is the optimization horizon N, which can be tuned to trade computational complexity (which grows with N) for performance ($\gamma^{\mathrm{RH}}(x)\to\gamma^{\mathrm{IH}}(x)$ as $N\to\infty$) and extent of the stability region ($X_0(N,K)\to X^{\mathrm{IH}}$ as $N\to\infty$).

Remark 2: As for the computational issue, in the novel NRH controller the critical point is finding an admissible control sequence at the beginning. At each subsequent step an admissible control can always be constructed from the control obtained at the previous iteration [at time $t+1$ just take the last $N-1$ values of the previous optimal control sequence $u^o_{t,t+N-1}$ followed by a single step of control computed through the linear control law $u(t+N)=Kx(t+N)$]. Differently from [3], this "feasibility property" has been obtained without resorting to variable horizons. Finally, in view of the rationale of the proof of Theorem 1, it is not actually necessary to minimize J, as finding at each step a control sequence improving on the available feasible sequence suffices to ensure stability (this is the so-called "improvement property" as defined in [3]).

Remark 3: Although the evaluation of $V_f(\bar{x},K(\cdot))$ can be easily done by integrating (5) for a sufficiently large number of steps, some saving of computation can be achieved as follows. Assume that $h(x,u)=x'Qx+u'Ru$, $Q>0$, $R>0$, and K is given by the LQ_∞ gain $K=-[R+B'PB]^{-1}B'PA$, where P is the unique positive definite solution of the algebraic Riccati equation $P=A'PA+Q-A'BP[R+B'PB]^{-1}B'PA$. Then, the terminal penalty V_f in (7a) can be approximated as

$$\begin{aligned}V_f&(x(t+N),K)\\&\cong\sum_{k=N}^{M-1}x'(t+k)Qx(t+k)+x'(t+k)K'RKx(t+k)\\&\quad+x'(t+M)Px(t+M)\end{aligned}$$

where $M>N$ is "large" compared to the system dynamics. This approximation hinges on the assumption that at time $t+M$ the system has been driven in a sufficiently small neighborhood of the origin by the application of the sequence $u_{t,t+N-1}$ followed by the linear control law $u(t+N+j)=Kx(t+N+j)$, $j=0,\cdots,M-N-1$. In this neighborhood, the behavior of the system is approximately linear, so that, by standard results of LQ control theory

$$\begin{aligned}\sum_{k=t+M}^{\infty}&x'(k)Qx(k)+x'(k)K'RKx(k)\\&\cong x'(t+M)Px(t+M).\end{aligned}$$

Remark 4: The overall NRH algorithm is numerically efficient, but its online implementation may still be not feasible for "fast systems" (e.g., mechanical systems). In this case, one can adopt the procedure proposed in [4] and solve the FHOCP offline for several different values of the initial state so as to collect examples for training a neural network capable of implementing the nonlinear map $u=\gamma^{\mathrm{RH}}(x)$.

Remark 5: The method proposed in [9] and [10] for the solution of the LQ_∞ problem with control and state constraints can be viewed as the specialization of the algorithm here proposed to linear time-invariant systems and quadratic performance indexes. In [9] and [10] the controller gain K is the solution of the unconstrained LQ_∞ problem.

IV. Robustness Analysis

If (1) is invariant and affine in u (i.e., $f(x,u)=f_1(x)+f_2(x)u$) and the cost function is of the type $h(x,u)=l(x)+m(u)$, the analysis reported in [5] for an NRH controller with zero-state terminal constraint can be replicated for the NRH controller of this paper just using $V(x)=J^o(x,N,K)$ as a Lyapunov function. Then, all the sufficient conditions for robustness in the face of gain and additive perturbations derived in [5] are straightforwardly extended.

To analyze robustness in a more general context, assume now that the real system is

$$x(k+1)=f_r(x(k),u(k)),\qquad x(t)=\bar{x},\qquad k\ge t$$

with $f_r(0,0)=0$ and $f_r(\cdot,\cdot)\neq f(\cdot,\cdot)$. In other words, (1) used to compute $\gamma^{\mathrm{RH}}(x)$ through the solution of the FHOCP is only an approximate representation of the real system. The next theorem gives conditions under which the stability property of Theorem 1 extends also to the real closed-loop system

$$x(k+1)=f_r(x(k),\gamma^{\mathrm{RH}}(x(k))),\quad x(t)=\bar{x},\qquad k\ge t.\qquad(9)$$

Theorem 2: Under the Assumptions of Theorem 1, given a positive real number v, let $\bar{X}(K,N) \subseteq X(K,N)$ be a set containing the origin as interior point such that

$$V(\bar{x}) = J^0(\bar{x},K,N) \le v, \qquad \forall \bar{x} \in \bar{X}(K,N).$$

Then, for a given $d > 0$:

1) there exists a positive real number α such that
$$|V(x_1) - V(x_2)| \le \alpha \|x_1 - x_2\|^d, \quad \forall x_1, x_2 \in \bar{X}(K,N);$$

2) for all $f_r(\cdot,\cdot)$ such that
$$\|f_r(x,u) - f(x,u)\|^d \le \frac{\rho p_1}{\alpha}\|(x,u)\|^q, \qquad \forall x \in \bar{X}(K,N), \quad \forall u \in U,$$
where ρ is an arbitrary real number such that $0 < \rho < 1$ and p_1 and q are those of Assumption A4, the origin is an exponentially stable equilibrium of (9) with exponential stability region $\bar{X}(K,N)$.

V. Numerical Example

Consider the following fourth-order continuous-time nonlinear system describing the dynamics of an inverted pendulum mounted on a cart, as in (10), shown at the bottom of the page, where x_1 and x_2 are the cart position and velocity, x_3 and x_4 are the angle and the angle velocity, and the control variable u is the armature voltage of the direct current motor moving the cart. In the (unstable) equilibrium $x = [0\ 0\ 0\ 0]'$ the pendulum stands in an upright position with the cart located at the origin of the longitudinal coordinate. The system is assumed to be controlled by means of a digital controller with sampling period equal to 50 ms.

The locally stabilizing control law K has been designed by solving an LQ_∞ problem with $h(x,u) = x'Qx + u'Ru$, where $Q = \mathrm{diag}\{0.25, 0.0001, 4, 0.0001\}$, $R = 0.0003$. The resulting gain matrix is $K = [-10.62\ \ -15.31\ \ -58.04\ \ -6.68]$.

The NRH controller has been designed using the same $h(x,u)$ with horizon length $N = 5$. Due to the nonlinearity of the system the discretized model is not explicitly available and the discrete dynamics is simulated by performing integration of (10).

In Figs. 2–5 simulations starting from $x(0) = [0\ \ 0\ \ x_3(0)\ \ 0]'$ are illustrated. In Figs. 2 and 3, for $x_3(0) = 50°$ it is seen that the NRH controller yields some improvement with respect to the linear controller $u = Kx$. In particular, the overshoot of both x_1 and x_3 is reduced as well as their settling time. In Figs. 4 and 5 for $x_3(0) = 60°$, it is seen that the NRH controller is still stabilizing, whereas the linear controller fails to achieve stability.

VI. Concluding Remarks

The novel RH controller here presented guarantees closed-loop stability at a reasonable computational cost as it hinges on finite-horizon optimization without terminal equality constraints. These results have been achieved with the introduction of a suitable non-quadratic terminal state penalty in the finite-horizon control problem under the mild assumption of stabilizability of the linearized system. The controller is always stabilizing in a neighborhood of the origin whose extent monotonically depends on length of the optimization horizon N.

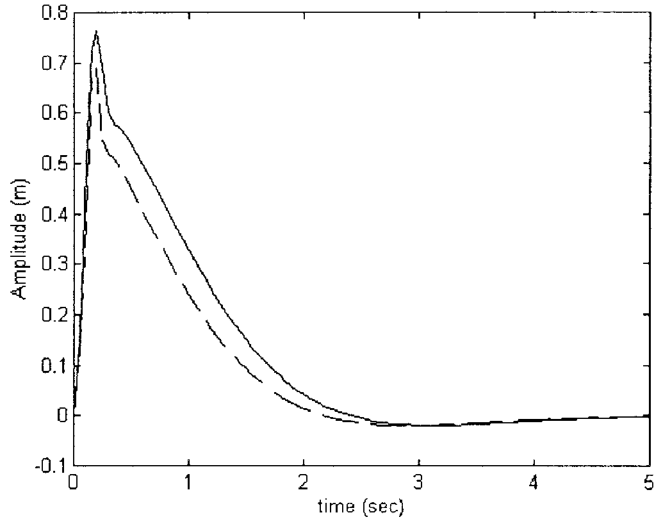

Fig. 2. Closed-loop response of x_1 starting from the initial state $x(0) = [0\ \ 0\ \ 50°\ \ 0]'$ (continuous: linear control law, dashed: NRH).

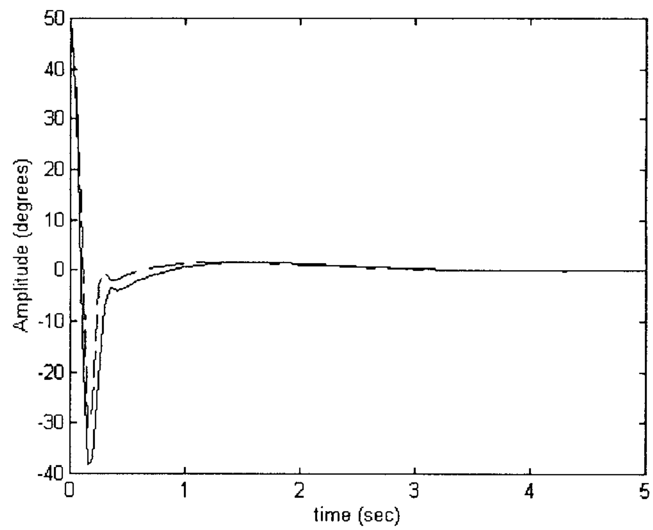

Fig. 3. Closed-loop response of x_3 starting from the initial state $x(0) = [0\ \ 0\ \ 50°\ \ 0]'$ (continuous: linear control law, dashed: NRH).

Appendix

Proof of Lemma 1: The proof is carried out by showing that, $\forall \varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $\|x\| < \delta(\varepsilon)$ implies $\|\tilde{f}_c(x,k)\| \le \varepsilon\|x\|$, $\forall k \ge 0$. First, note that in view of A2,

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= \frac{-2.06\cos(x_3(t))\sin(x_3(t)) + 0.06x_4(t)^2\sin(x_3(t)) - 7.56x_2(t) + 0.01u(t)}{0.66 - 0.21\cos(x_3(t))^2}\\
\dot{x}_3(t) &= x_4(t)\\
\dot{x}_4(t) &= 32.16\sin(x_3(t)) - 3.28\cos(x_3(t)) \times \frac{-2.06\cos(x_3(t))\sin(x_3(t)) + 0.06x_4(t)^2\sin(x_3(t)) - 7.56x_2(t) + 0.01u(t)}{0.66 - 0.21\cos(x_3(t))^2}
\end{aligned} \tag{10}$$

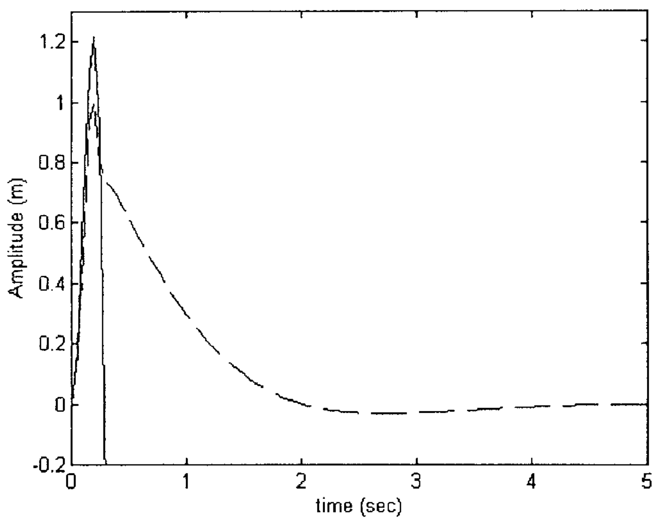

Fig. 4. Closed-loop response of x_1 starting from the initial state $x(0) = [0\ \ 0\ \ 60^\circ\ \ 0]'$ (continuous: linear control law, dashed: NRH).

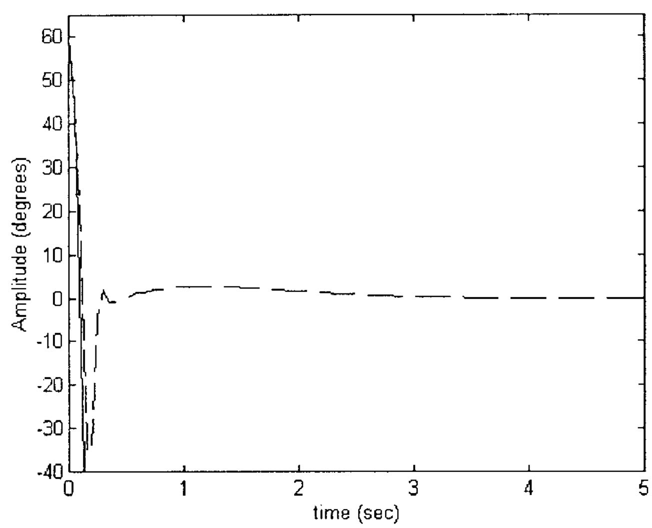

Fig. 5. Closed-loop response of x_3 starting from the initial state $x(0) = [0\ \ 0\ \ 60^\circ\ \ 0]'$ (continuous: linear control law, dashed: NRH).

$\forall \tilde{\varepsilon} > 0$ there exists $\tilde{\delta} = \tilde{\delta}(\tilde{\varepsilon})$ such that $\|(x,u)\| < \tilde{\delta}(\tilde{\varepsilon})$ implies $\|\tilde{f}(x,u,k)\| \le \tilde{\varepsilon}\|(x,u)\|$, $\forall k \ge 0$. Now $\forall \varepsilon > 0$, let

$$\tilde{\varepsilon} = \frac{\varepsilon}{\sqrt{1+M_K^2}}, \qquad \delta(\varepsilon) = \frac{\tilde{\delta}(\tilde{\varepsilon})}{\sqrt{1+M_K^2}}.$$

Then, if $\|x\| < \delta(\varepsilon)$

$$\begin{aligned}\|(x, K(k)x)\| &= \sqrt{\|x\|^2 + \|K(k)x\|^2} \le \sqrt{1+M_K^2}\|x\| \\ &\le \sqrt{1+M_K^2}\,\delta(\varepsilon) = \tilde{\delta}(\tilde{\varepsilon})\end{aligned}$$

and therefore

$$\begin{aligned}\|\tilde{f}_c(x,k)\| &= \|\tilde{f}(x, K(k)x, k)\| \le \tilde{\varepsilon}\|(x, K(k)x)\| \\ &= \frac{\varepsilon}{\sqrt{1+M_K^2}}\|(x, K(k)x)\| \le \varepsilon\|x\|.\end{aligned}$$ □

Proof of Proposition 3:

1) The inclusion $X^{\mathrm{IH}}(t) \supseteq X^0(N, K(\cdot), t)$ follows from the fact that, in view of the very definition of J^{IH} and J, the optimal value $J^{\mathrm{IH}^o}(\bar{x}, t)$ of the IH cost function (6) is such that $J^{\mathrm{IH}^o}(\bar{x}, t) \le J(\bar{x}, u_{t,t+N-1}, N, K(\cdot), t)$. Then, the existence of an admissible sequence for the $\mathrm{FHOCP}(\bar{x}, N, K(\cdot), t)$ implies that the IH optimization problem (6) is solvable as well. As for the other inclusion observe that if $\bar{x} \in X(K(\cdot), t)$ then the sequence $u(k) = K(k)\varphi_c(k, t, \bar{x})$, $t \le k \le t+N-1$, is admissible for the $\mathrm{FHOCP}(\bar{x}, N, K(\cdot), t)$.

2) If $\bar{x} \in X_0(M, K(\cdot), t)$, there exists an admissible sequence $\hat{u}_{t,t+M-1} = [\hat{u}(t)\ \ \hat{u}(t+1)\ \ \cdots\ \ \hat{u}(t+M-1)]$ for the $\mathrm{FHOCP}(\bar{x}, M, K(\cdot), t)$. Then, it is immediately seen that $\tilde{u}_{t,t+M} = [\hat{u}_{t,t+M-1}\ \ K(t+M)\varphi(t+M, t, \bar{x}, \hat{u}(\cdot)]$ is admissible for the $\mathrm{FHOCP}(\bar{x}, M+1, K(\cdot), t)$ (in fact, $J(\bar{x}, \tilde{u}_{t,t+M}, M+1, K(\cdot), t) = J(\bar{x}, \hat{u}_{t,t+M-1}, M, K(\cdot), t))$ so that $\bar{x} \in X_0(M+1, K(\cdot), t)$ as well.

3) Let $\varphi^{\mathrm{IH}}(k, t, \bar{x})$ denote the solution $x(k)$ at time $k \ge t$ of (1) subject to the IH control law $u(k) = \gamma^{\mathrm{IH}}(x(k), k)$. Due to the stabilizing properties of the IH controller [1], $\lim_{k\to\infty}\|\varphi^{\mathrm{IH}}(k, t, \bar{x})\| = 0$. In particular for a given $\bar{x} \in X^{\mathrm{IH}}(t)$ it will happen that there exists a finite $\bar{k} \ge t$ such that $\varphi^{\mathrm{IH}}(\bar{k}, t, \bar{x}) \in X(K(\cdot))$ so that letting $M = \bar{k} - t$ the sequence $u(k) = \gamma^{\mathrm{IH}}(\varphi^{\mathrm{IH}}(k, t, \bar{x}))$, $t \le k \le t + M$ is admissible for the $\mathrm{FHOCP}(\bar{x}, M, K(\cdot), t)$, i.e., $\bar{x} \in X_0(M, K(\cdot), t)$. □

Proof of Lemma 3:

1) Let $u^o_{t,t+N-1} = [u^o(t)\ \ u^o(t+1)\ \ \cdots\ \ u^o(t+N-1)]$ be the optimal sequence for the $\mathrm{FHOCP}(\bar{x}, N, K(\cdot), t)$. Then $\tilde{u}_{t,t+N} = [u^o_{t,t+N-1}\ \ K(t+N)\varphi(t+N, t, \bar{x}, u^o(\cdot))]$ is admissible for the $\mathrm{FHOCP}(\bar{x}, N+1, K(\cdot), t)$ and yields $J(\bar{x}, \tilde{u}_{t,t+N}, N+1, K(\cdot), t) = J^o(\bar{x}, N, K(\cdot), t)$.

2), 3) It suffices to observe that $\hat{u}_{t+1,t+N} = [u^o_{t+1}, u^o_{t+2}, \cdots, u^o_{t+N-1}\ \ K(t+N)\varphi(t+N, t, \bar{x}, u^o(\cdot))]$ is such that $J(x^{\mathrm{RH}}(t+1), \hat{u}_{t+1,t+N}, N, K(\cdot), t+1) = J^o(\bar{x}, N, K(\cdot), t) - h(\bar{x}, \gamma^{\mathrm{RH}}(\bar{x}, t), t)$. □

Proof of Theorem 1: First, observe that $\gamma^{\mathrm{RH}}(0, t) = 0$, $\forall t$. Hence, $x = 0$ is an equilibrium point of the closed-loop system when the RH regulator is applied. Next, it is shown that

$$V(t, x) = J^o(x, N, K(\cdot), t), \qquad x \in X_0(N, K(\cdot), t)$$

is a Lyapunov function for (1) driven by the RH regulator. Clearly, $V(t, 0) = J^o(0, N, K(\cdot), t) = 0$. In view of A1–A4 and the very definition of $X^0(N, K(\cdot), t)$, it follows that $V(t, x)$ satisfies assumption 1) of Lemma 2 for some β_1 [11].

By A4, $V(t, x)$ is a positive-definite function of x such that $V(t, x) \ge p_1\|x\|^q$. Letting $\Delta V(t, x) = V(t, x) - V(t+1, \varphi^{\mathrm{RH}}(t+1, t, x))$, Lemma 3-3) and A4 entail that

$$\begin{aligned}\Delta V(t, x) &\ge h(x, \gamma^{\mathrm{RH}}(x, t), t) \\ &\ge p_1\|(x, \gamma^{\mathrm{RH}}(x, t))\|^q \ge p_1\|x\|^q > 0 \\ &\qquad \forall t, \quad \forall x \in X_0(N, K(\cdot), t), \quad x \ne 0.\end{aligned}$$

Observing that $\Delta V(t, 0) = 0$, this implies that assumption 2) of Lemma 2 holds with $\alpha_1 = \gamma_1 = p_1$ and $\tilde{X}(k) = X_0(N, K(\cdot), k)$. Then, the thesis directly follows from Lemma 2. □

Proof of Theorem 2:

1) This immediately follows from the fact that $0 \le V(x) \le v$, $x \in \bar{X}(K, N)$.

2) In view of the stated assumptions

$$\begin{aligned}&V(f_r(x,u)) - V(f(x,u)) \\ &\le |V(f_r(x,u)) - V(f(x,u))| \\ &\le \alpha\|f_r(x,u) - f(x,u)\|^d \\ &\le \rho p_1\|(x,u)\|^q, \quad \forall x \in \bar{X}(K, N).\end{aligned}$$

Hence, recalling Lemma 3-3)

$$\begin{aligned} V(f_r(x,\gamma^{\mathrm{RH}}(x))) &\le V(f(x,\gamma^{\mathrm{RH}}(x))) + \rho p_1 \|(x,\gamma^{\mathrm{RH}}(x))\|^q \\ &\le V(x) - h(x,\gamma^{\mathrm{RH}}(x)) + \rho p_1 \|(x,\gamma^{\mathrm{RH}}(x))\|^q \\ &\le V(x) - (1-\rho) p_1 \|(x,\gamma^{\mathrm{RH}}(x))\|^q, \quad \forall x \in \bar{X}(K,N) \end{aligned}$$

so that the same arguments of Theorem 1 can be applied to deduce exponential stability of the origin of (9), with exponential stability region $\bar{X}(K,N)$. □

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