Structural properties and classification of kinematic and dynamic models of wheeled mobile robots

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Abstract

The structure of the kinematic and dynamic models of wheeled mobile robots is analysed. It is shown that, for a large class of possible configurations, they can be classified into 5 types, characterized by generic structures of the model equations. For each class several problem are adressed : irreducibility of the model, configuration of the motorization, feedback equivalence.

1 Introduction

Mobile robots equipped with undeformable wheels (referred to as "Wheeled Mobile Robots", WMR) constitute a particular class of mechanical systems : they are characterized by kinematic constraints that are not integrable and cannot therefore be eliminated from the model equations. The consequence is that the standard control algorithms developped for robotic manipulators without constraints are no more applicable.

Several examples of derivation of kinematic and/or dynamic models for WMR are available in the literature, for particular prototypes of mobile robots (see, for instance, [1], [2], [3]), as well as for general robots equipped with wheels of several types. A systematic procedure for model derivation can be found in ([5], [6]). In this paper we also consider a general WMR, with several wheels of various types, and various motorizations. Our purpose is to emphasize the structural properties of the kinematic and dynamic models, taking into account the restriction to the robot mobility induced by the constraints. We show that, notwithstanding the variety of possible robot constructions and wheel configurations, the set of WMR can be partitioned in 5 classes. The reducibility of these models is discussed, within the framework of nonholonomic systems. The problem of the configuration of the motorization is adressed and we provide a criterion to check if the motorization is sufficient to exploit fully the kinematic mobility. Finally we show that the dynamic model is feedback equivalent to a model whose structure depends only on the kinematic constraints. This simple form is particularly useful for the discussion of the control structural properties, namely the controllability and the stabilizability (e.g. [4],[7])

2 Kinematics of Wheeled Mobile Robots

2.1 Robot position

Without real loss of generality and to keep the mathematical technicalities as simple as possible, we shall assume that the mobile robots under study in this paper are made up of a rigid frame equipped with non deformable wheels and that they are moving on a horizontal plane. The position of the robot on the plane is described as follows (see Fig.1). An arbitrary orthonormal inertial basis $\{0, \vec{I_1}, \vec{I_2}\}$ is fixed in the plane of the motion. An arbitrary reference point P on the frame and an arbitrary basis $\{\vec{x_1}, \vec{x_2}\}$ attached to the frame are defined. The position of the robot is then completely specified by the 3 variables x, y, θ:

- x, y are the coordinates of the reference point P in the inertial basis $\{0, \vec{I_1}, \vec{I_2}\}$.
- θ is the orientation of the basis $\{\vec{x_1}, \vec{x_2}\}$ with respect to the inertial basis $\{\vec{I_1}, \vec{I_2}\}$.

We define the 3-vector ξ describing the robot posture

$$\xi \triangleq (x \;\; y \;\; \theta)^T \tag{1}$$

We also define the following orthogonal rotation matrix

$$R(\theta) \triangleq \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

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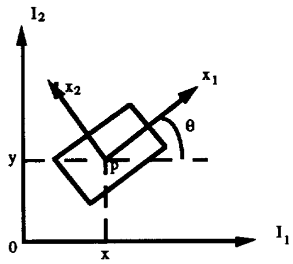


Fig.1 Coordinates

2.2 Wheels description

We distinguish between two basic classes of idealized wheels : the *conventional* wheels and the *swedish* wheels. In each case, it is assumed that the contact between the wheel and the ground is reduced to a single point of the plane.

For a *conventional wheel*, the contact between the wheel and the ground is supposed to satisfy the pure rolling without slipping condition. This means that the velocity of the contact point is equal to zero. For a *swedish* wheel, only *one* component of the velocity of the contact point of the wheel with the ground is supposed to be equal to zero along the motion. The direction of this zero component of the velocity is a priori arbitrary but is fixed with respect to the orientation of the wheel. We now derive explicitly the expressions of the constraints .

2.2.1 Conventional wheels

We consider 3 configurations for conventional wheels : fixed wheels, centered orientable wheels (Fig. 2), and off-centered orientable wheels (Fig. 3), according to the following description. The position of A, a fixed point of the frame, is characterized using polar coordinates by the distance $PA = l$ and the angle α. The plane of the wheel is vertical and attached to the frame either directly at point A (for fixed and centered orientable wheels), or via a rigid rod AB of constant length d (for off-centered wheels). The difference between centered and off-centered wheels is that, for an off-centered wheel, the rotation of the wheel plane is around an axle which does not pass through the center of the wheel (B). The orientation of the wheel with respect to the frame is characterized by the angle β, constant for fixed wheels, and time varying for centered and off-centered orientable wheels. The rotation of the wheel around its horizontal axis is described by the time-varying angle φ.

The pure rolling and non slipping constraint are can then be expressed under the following form :

$$[-\sin(\alpha+\beta)\ \cos(\alpha+\beta)\ l\cos\beta]R(\theta)\dot{\xi} + r\dot{\varphi} = 0 \quad (3)$$

$$[\cos(\alpha+\beta)\ \sin(\alpha+\beta)\ d + l\sin\beta]R(\theta)\dot{\xi} + d\dot{\beta} = 0 \quad (4)$$

where r is the radius of the wheel and, $d = 0$ for fixed and centered orientable wheels, and of course $\dot{\beta} = 0$ for fixed wheels.

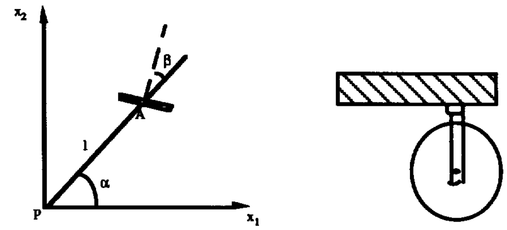


Fig.2 Fixed and conventional centered orientable wheels

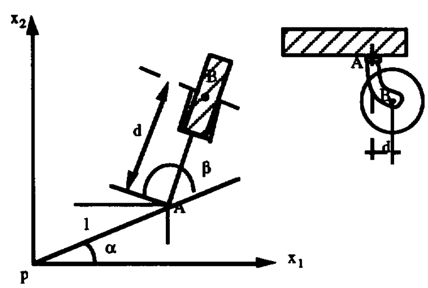


Fig.3 Conventional Off-Centered Orientable Wheels.

2.2.2 Swedish wheels

The position of the wheel with respect to the frame is described, as for the conventional fixed wheel, by the 3 constant parameters : α, β, l. An additional parameter is required to characterize the direction, with respect to the wheel plane, of the zero component of the velocity of the contact point represented by the angle γ . The motion constraint is expressed as :

$$[-\sin(\alpha+\beta+\gamma)\ \ \cos(\alpha+\beta+\gamma)\ \ l\cos(\beta+\gamma)] \\ R(\theta)\dot{\xi} + r\cos\gamma\dot{\varphi} = 0 \quad (5)$$

2.3 Restrictions to the robot mobility

We now consider a general mobile robot, equipped with N wheels of the 4 above described categories. We use the 4 following subscripts to identify quantities relative to these 4 classes : f for conventional fixed wheels, c for conventional centered orientable wheels, oc for conventional off-centered orientable wheels and sw for swedish wheels. The numbers of wheels of each type are denoted N_f, N_c, N_{oc}, N_{sw} with $N_f + N_c + N_{oc} + N_{sw} = N$.

The configuration of the robot is fully described by the following vectors of coordinates :

- Posture coordinates $\xi(t)$.
- Angular coordinates :$\beta_c(t)$for the orientation angles of the centered orientable wheels and $\beta_{oc}(t)$ for the orientation angles of the off-centered orientable wheels
- Rotation coordinates

  $$\varphi(t) \triangleq (\varphi_f(t), \varphi_c(t), \varphi_{oc}(t), \varphi_{sw}(t))^T$$

  for the rotation angles of the wheels around their horizontal axle of rotation.

The whole set of coordinates ξ, β_c, β_{oc} and φ is called the set of *configuration coordinates* in the sequel. The total number of the configuration coordinates is clearly: $N_f + 2N_c + 2N_{oc} + N_{sw} + 3$.

With these notations the constraints can be written under the general matrix form :

$$J_1(\beta_c, \beta_{oc})R(\theta)\dot{\xi} + J_2\dot{\varphi} = 0 \quad (6)$$

$$C_1(\beta_c, \beta_{oc})R(\theta)\dot{\xi} + C_2\dot{\beta}_{oc} = 0 \quad (7)$$

where J_1, J_2, C_1, C_2 are matrices whose rows derive from the constraints (3), (4). Particularly we have

$$C_1(\beta_c, \beta_{oc}) \triangleq \begin{pmatrix} C_1f \\ C_{1c}(\beta_c) \\ C_{1oc}(\beta_{oc} \end{pmatrix} \; C_2 \triangleq \begin{pmatrix} 0 \\ 0 \\ C_{2oc} \end{pmatrix}$$

where C_{1f}, C_{1c}, C_{1oc} are 3 matrices respectively of dimension $(N_f \times 3), (N_c \times 3), (N_{oc} \times 3)$. C_{1f} is constant while C_{1c} and C_{1oc} are time varying. $C_{2oc} = \mathrm{diag}(d)$ for the N_{oc} off-centered orientable wheels.

Consider now the $(N_f + N_c)$ first constraints from (7) and written explicitly as :

$$C_{1f}R(\theta)\dot{\xi} \quad = 0 \quad (8)$$

$$C_{1c}(\beta_c)R(\theta)\dot{\xi} \; = 0 \quad (9)$$

These constraints imply that the vector $R(\theta)\dot{\xi}$ belong to the null space of the following matrix $C_1^*(\beta_c)$:

$$C_1^*(\beta_c) = C_{1f}, C_{1c}(\beta_c)^T \quad (10)$$

$$R(\theta)\dot{\xi} \in \mathcal{N}[C_1^*(\beta_c)] \quad (11)$$

Obviously rank $[C_1^*(\beta_c)] \leq 3$. If rank $[C_1^*(\beta_c)] = 3$, then $R(\theta)\dot{\xi} = 0$ and any motion in the plane is impossible! More generally, the limitations of the mobility of the robot are related to the rank of C_1^*. This point will be discussed in detail hereafter.

Obviously the rank of the matrix $C_1^*(\beta_c)$ depends on the design of the mobile robot. We define the *degree of mobility* δ_m of a mobile robot as

$$\delta_m = \dim \mathcal{N}[C_1^*(\beta_c)] = 3 - \; \mathrm{rank}[C_1^*(\beta_c)]$$

Let us now examine the case $\mathrm{rank}[C_{1f}] = 2$ which implies that the robot has at least 2 fixed wheels and, if there are more than 2, that their axles are concurrent to the center of rotation whose position with respect to the frame is *fixed*. In such a case, it is clear that the only possible motion is a rotation of the robot around this fixed center of rotation. Obviously this limitation is not acceptable in practice and we thus assume that $\mathrm{rank}[C_{1f}] \leq 1$. We assume moreover that the structure robot is **non degenerate** in the sense that :

A1 $\mathrm{rank}[C_1^*(\beta_c)] = \mathrm{rank}[C_{1f}] + \mathrm{rank}[C_{1c}(\beta_c)] \leq 2$

It follows that only *five* types of nonsingular structure are of practical interest and such that :

$$\mathrm{Rank}[C_{1f}] = 0 \text{ or } 1$$
$$\mathrm{Rank}[C_{1c}(\beta_c)] = 0 \text{ or } 1 \text{ or } 2.$$

In the sequel, we shall designate these types of structure by using a denomination of the form : "mobile robot of **Type** (i,j)" where $i \triangleq \mathrm{rank}[C_{1f}], j \triangleq \mathrm{Rank}[C_{1c}(\beta_c)]$.

The main design characteristics of each type of mobile robot are now briefly presented.

Type (0,0).$\delta_m = \dim \mathcal{N}(C_1^*(\beta_c)) = 3$. These robots have *no* conventional fixed wheels ($N_f = 0$) and *no* conventional centered orientable wheels ($N_c = 0$). Such robots are called *omnidirectional* because they have a full mobility in the plane which means that they can move at each instant in any direction without any reorientation. In contrast, the other four types of robot have a restricted mobility (degree of mobility less than 3).

Type (1,0).$\delta_m = \dim \mathcal{N}(C_1^*(\beta_c)) = \dim \mathcal{N}(C_{1f}) = 2$. These robots have *no* conventional centered orientable wheels ($N_c = 0$). They have either one conventional fixed wheel or several conventional fixed wheels but with a single common axle (otherwise $\mathrm{Rank}[C_{1f}]$ would be greater than 1). The mobility of the robot is restricted in the sense that, for any admissible trajectory $\xi(t)$, the velocity $\dot{\xi}(t)$ is constrained to belong to the 2-dimensional distribution spanned by the vector fields $R^T(\theta)s_1$ and $R^T(\theta)s_2$, where s_1 and s_2 are two *constant* vectors spanning $\mathcal{N}(C_{1f})$.

Type (0,1). $\delta_m = \dim \; (C_1^*(\beta_c)) = \dim \mathcal{N}(C_{1c}(\beta_c)) = 2$. These robots have *no* conventional fixed wheel

(N_f = 0) and *one* conventional centered orientable wheel (N_c = 1). The velocity $\dot{\xi}(t)$ is constrained to belong to the 2-dimensional distribution spanned by the vector fields $R^T(\theta)s_1(\beta_c)$ and $R^T(\theta)s_2(\beta_c)$ where $s_1(\beta_c)$ and $s_2(\beta_c)$ are two vectors spanning $\mathcal{N}(C_{1c}(\beta_c))$ and parametrized by the angle β_c of the conventional centered orientable wheel.

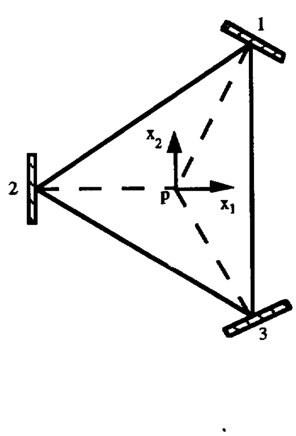
Type (1,1).$\delta_m = \dim\mathcal{N}(C_1^*(\beta_c)) = 1$. These robots have one or several conventional fixed wheels with a single common axle. They have also one or several conventional centered orientable wheels, with the condition that the center of one of them is *not* located on the axle of the conventional fixed wheels (otherwise the structure would be degenerate, see Assumption **A1**). The velocity $\dot{\xi}(t)$ is constrained to belong to a 1-dimensional distribution parametrized by the orientation angle of one arbitrarily chosen conventional centered orientable wheel.

Type (0,2). $\delta_m = \dim\mathcal{N}(C_1^*(\beta_c))=\dim\mathcal{N}(C_{1c}(\beta_c)) = 1$. These robots have *no* conventional fixed wheels (N_f = 0). They have at least *two* conventional centered orientable wheels ($N_c \geq 2$). The velocity $\dot{\xi}(t)$ is constrained to belong to a 1-dimensional distribution parametrized by the orientation angles of two arbitrarily chosen conventional centered orientable wheels of the robot.

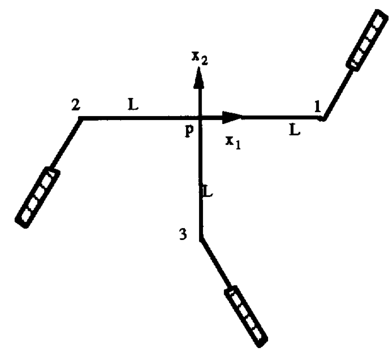
3 Examples

We present in this section six concrete examples of mobile robots to illustrate the five types of nondegenerate structures that have been presented above. We restrict our attention to robots with *three* wheels. Except for one example, we only give here the figures. The complete description can be found in [7].

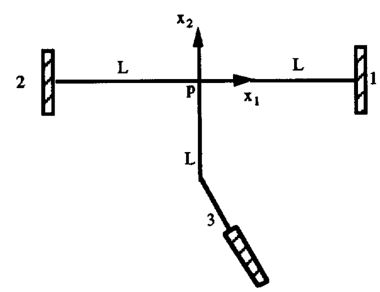
Example 1 : Omnidirectional robots with swedish wheels (Type (0,0)). The considered robot has three swedish wheels located at the vertices of the frame that has the form of an equilateral triangle.



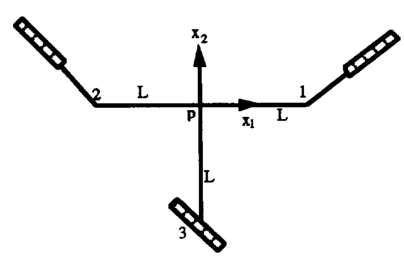
Example 2 : Omnidirectional robots with off-centered orientable wheels (Type (0,0)). The robot has three conventional off-centered orientable wheels .



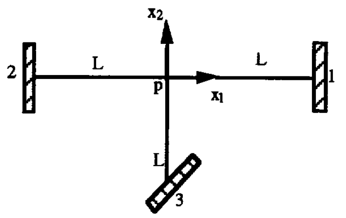
Example 3 : Type(1,0). Robot with two conventional fixed wheels on the same axle and one conventional off-centered orientable wheel .



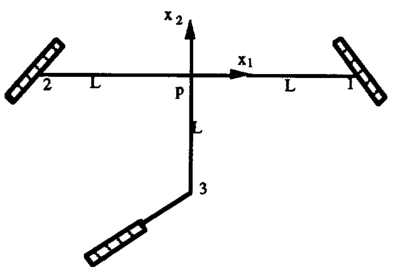
Example 4 : Type (0,1). Robot with one conventional centered orientable wheel and two conventional off-centered orientable wheels.



Example 5 : Type (1,1). Robot with two conventional fixed wheels on the same axle and one conventional centered orientable wheel (like the tricycles of the kids)



Example 6 : Type $(0,2)$. Robot with two conventional centered orientable wheels and one conventional off-centered orientable wheel. We give here the explicit expressions of the constraint matrices.



The constraints have the form (6), (7) where :

$$\begin{aligned} J_1 &= \begin{pmatrix} -\sin\beta_{c1} & \cos\beta_{c1} & L\cos\beta_{c1} \\ \sin\beta_{c2} & -\cos\beta_{c2} & L\cos\beta_{c2} \\ \cos\beta_{oc3} & \sin\beta_{oc3} & -L\cos\beta_{oc3} \end{pmatrix}, \\ J_2 &= \text{diag}\ (r) \qquad (12) \\ C_1 &= \begin{pmatrix} \cos\beta_{c1} & \sin\beta_{c1} & L\sin\beta_{c1} \\ -\cos\beta_{c2} & -\sin\beta_{c2} & L\sin\beta_{c2} \\ \cos\beta_{oc3} & \sin\beta_{oc3} & d+L\cos\beta_{oc3} \end{pmatrix}, \\ C_2 &= (0\ \ 0\ \ d)^T \qquad (13) \end{aligned}$$

4 The kinematic state space model

In this section, the analysis of the mobility, as discussed in Section 2.3, is reformulated into a state space form which will be useful for our subsequent developments.

We have shown that, whatever the type of mobile robot, the velocity $\dot{\xi}(t)$ is restricted to belong to a distribution Δ_c defined as :

$$\dot{\xi}(t) \in \Delta_c \triangleq \text{span}\ \{\text{col}\ R^T(\theta)\Sigma(\beta_c)\} \qquad \forall t$$

where the columns of the matrix $\Sigma(\beta_c)$ form a basis of $\mathcal{N}[C_1^*(\beta_c)]$:

$$\mathcal{N}[C_1^*(\beta_c)] = \text{span}\{\text{col}\Sigma(\beta_c)\}$$

This is trivially equivalent to the following statement : *for all t, there exists a time varying vector $\eta(t)$ such that*

$$\dot{\xi} = R^T(\theta)\Sigma(\beta_c)\eta \qquad (14)$$

The dimension of the distribution Δ_c and hence, of the vector $\eta(t)$ is the degree of mobility δ_m of the robot. Obviously, in the case where the robot has no conventional centered orientable wheels ($N_c = 0$), the matrix Σ is constant and the expression (14) reduces to :

$$\dot{\xi} = R^T(\theta)\Sigma\eta \qquad (15)$$

In the opposite case ($N_c \geq 1$), the matrix Σ explicitely depends on the angular coordinates β_c and the expression (14) can be augmented as follows :

$$\begin{aligned} \dot{\xi} &= R^T(\theta)\Sigma(\beta_c)\eta \qquad (16) \\ \dot{\beta}_c &= \zeta \qquad (17) \end{aligned}$$

This representation (15) or (16)-(17) can be regarded as a state space representation of the system (called the ***kinematic posture state space model***), with the posture coordinates ξ and (possibly) the angular coordinates β_c as state variables while η and ζ (that are homogeneous to velocities) can be interpreted as control inputs entering the model linearly. This interpretation must however be taken with some care since the true physical control inputs of a mobile robot are the torques provided by the embarked motors : the kinematic state space model is in fact only a subsystem of the general dynamical model that will be presented in Section 5.

The kinematic state space model is now illustrated for example 6.

- **Robots of Type** $(0,2)$ - Example 6

  The matrix $\Sigma(\beta_c)$ can be selected as :

$$\Sigma(\beta_c) = \begin{pmatrix} -2L\sin\beta_{c1}\sin\beta_{c2} \\ L\sin(\beta_{c1}+\beta_{c2}) \\ 2\sin\beta_{c2}\cos\beta_{c1} - \sin(\beta_{c1}+\beta_{c2}) \end{pmatrix} \qquad (18)$$

  The kinematic state space model (16)-(17) specializes as :

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} &= R^T(\theta)\Sigma(\beta_c)\eta_1 \\ \dot{\beta}_{c1} &= \zeta_1 \\ \dot{\beta}_{c2} &= \zeta_2 \end{aligned}$$

5 Dynamics of Wheeled Mobile Robots

The aim of this section is the derivation of a general dynamical state space model describing the dynamical relations between the configuration coordinates $\xi, \beta_{oc}, \varphi, \beta_c$ and the torques developped by the embarked motors. The general state space model of mobile robots is made up of six kinds of state equations : one for each of the coordinates $\xi, \beta_{oc}, \varphi, \beta_c$ and one for each of the internal coordinates η and ζ that were introduced in Section 4. The state equations for ξ and β_c have been derived in Section 4 under the form of the so-called kinematic state space model. In the next section, we derive the state equations for β_{oc} and φ. Afterwards, the state equation for η and ζ will be established in Section 5.2 using the Lagrange formalism. Finally, the configuration of the motorisation will be discussed in Section 5.3.

5.1 State equations for β_{oc} and φ

We notice that the angular and rotation velocities $\dot{\beta}_{oc}$ and $\dot{\varphi}$ can be deduced from the kinematic constraints (6) and (7) as follows :

$$\begin{aligned} \dot{\beta}_{oc} &= -C_{2oc}^{-1} C_{1oc}(\beta_{oc}) R(\theta) \dot{\xi} \\ \dot{\varphi} &= -J_2^{-1} J_1(\beta_c, \beta_{oc}) R(\theta) \dot{\xi} \end{aligned}$$

Combining these equations with the kinematic state space model (16)-(17), the state equations for β_{oc} and φ are written as :

$$\begin{aligned} \dot{\beta}_{oc} &= D(\beta_{oc}) \Sigma(\beta_c) \eta \quad (19) \\ \dot{\varphi} &= E(\beta_{oc}, \beta_c) \Sigma(\beta_c) \eta \quad (20) \end{aligned}$$

with the following definitions of $D(\beta_{oc})$ and $E(\beta_{oc}, \beta_c)$

$$\begin{aligned} D(\beta_{oc}) &\triangleq -C_{2oc}^{-1} C_{1oc}(\beta_{oc}) \\ E(\beta_{oc}, \beta_c) &\triangleq -J_2^{-1} J_1(\beta_c, \beta_{oc}) \end{aligned}$$

5.2 The general dynamical model

We assume that the robot is equipped with motors that can force either the orientation of the orientable wheels (angular coordinates β_c and β_{oc}) or the rotation of the wheels (rotation coordinates φ). The torques provided by the motors are denoted : τ_φ for the rotation of the wheels, τ_{oc} for the orientation of the off-centered wheels, τ_c for the orientation of the centered wheels.

Using the Lagrange formalism, the dynamics of wheeled mobile robots are described by the following $(3 + N_{oc} + N + N_c)$ Lagrange equations :

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) - \frac{\partial T}{\partial \xi} &= R^T(\theta) J_1^T(\beta_c, \beta_{oc}) \lambda \\ &\quad + R^T(\theta) C_1^T(\beta_c, \beta_{oc}) \mu \quad (21) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}_{oc}}\right) - \frac{\partial T}{\partial \beta_{oc}} &= C_2^T \mu + \tau_{oc} \quad (22) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} &= J_2^T \lambda + \tau_\varphi \quad (23) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}_c}\right) - \frac{\partial T}{\partial \beta_c} &= \tau_c \quad (24) \end{aligned}$$

where T represents the kinetic energy and λ, μ are the Lagrange coefficients associated with the constraints (6) and (7) respectively.

After elimination of the Lagrange multipliers and other calculations whose details are given in [7], these equations can be rewritten as follows :

$$H(\beta_c, \beta_{oc}) \begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} + f(\beta_c, \beta_{oc}, \eta, \xi) = \begin{pmatrix} \Sigma^T (D^T \tau_{oc} + E^T \tau_\varphi) \\ \tau_c \end{pmatrix} \quad (25)$$

with the following properties

a) The matrix $H(\beta_c, \beta_{oc})$ is positive definite for all β_c, β_{oc}.

b) The vector field $f(\beta_c, \beta_{oc}, \eta, \zeta)$ can be written in the form :

$$C(\beta_c, \beta_{oc}, \eta, \zeta) \begin{pmatrix} \eta \\ \zeta \end{pmatrix}$$

where $C(\beta_c, \beta_{oc}, \eta, \zeta)$ is a $(\delta_m + N_c)$-dimensional square matrix such that:

$$\frac{d}{dt}[H(\beta_c, \beta_{oc})] - 2C(\beta_c, \beta_{oc}, \eta, \zeta)$$

is skew symmetric along the trajectories of the system.

5.3 Configuration of the motorization

In the general dynamical model (25), the vectors τ_φ, τ_{oc} and τ_c represent all the torques that can *potentially* be applied for the rotation and the orientation of the wheels of the robot. In practice however only a limited number of motors will be used, which means that many components of τ_φ, τ_{oc} and τ_c are identically zero. Our concern in this Section is to explicit the configurations of the motorization that allow a full mobility of the robots while requiring a number of motors as limited as possible.

First, it is clear that all the centered orientable wheels must be provided with a motor for their orientation (otherwise, these wheels would just play the role of fixed wheels).

Moreover, to ensure a full robot mobility, N_m additional motors (with $N_m \geq \delta_m$) must be implemented for either the rotation of some wheels or the orientation of some off-centered orientable wheels. The vector of the torques developed by these motors is denoted τ_m and we have :

$$\begin{pmatrix} \tau_{oc} \\ \tau_\varphi \end{pmatrix} = P\tau_m \qquad (26)$$

where P is a $(N_{oc}+N)\times N_m$ elementary matrix which selects the components of (τ_{oc},τ_φ) that are effectively used as control inputs.

Using (26) we see that equation (25) of the general dynamical model is rewritten as :

$$H\begin{pmatrix} \dot\eta \\ \dot\xi \end{pmatrix} + f = \begin{pmatrix} B(\beta_c,\beta_{oc})P & 0 \\ 0 & I \end{pmatrix}\tau_m$$

with $B(\beta_c,\beta_{oc}) \triangleq \Sigma^T(\beta_c)[D^T(\beta_{oc}) \;\; E^T(\beta_c,\beta_{oc})]$.

We introduce the following assumption.

A2 The configuration of the motorization is such that the matrix :

$$B(\beta_c,\beta_{oc})P$$

has full rank for all $(\beta_c,\beta_{oc}) \in \mathbb{R}^{N_c+N_{oc}}$ ■

We now present the minimal admissible motorizations for the example 6. The discussion of the other cases can be found in [7]

Two motors are required for the orientation of the two centered wheels. The matrix $B(\beta_c,\beta_{oc})$ is written as :

$$\begin{aligned} B(\beta_c,\beta_{oc}) &= \Sigma^T(\beta_c)[D^T(\beta_{oc}) \;\; E^T(\beta_c,\beta_{oc})] \\ \text{with } D^T &= \begin{pmatrix} -d^{-1}\sin\beta_{oc3} \\ d^{-1}\cos\beta_{oc3} \\ -1-Ld^{-1}\sin\beta_{oc3} \end{pmatrix} \\ E^T &= -\frac{1}{r}\begin{pmatrix} -\sin\beta_{c1} & \sin\beta_{c2} & \cos\beta_{oc3} \\ \cos\beta_{c1} & -\cos\beta_{c2} & \sin\beta_{oc3} \\ L\cos\beta_{c1} & L\cos\beta_{c2} & L\cos\beta_{oc3} \end{pmatrix} \end{aligned}$$

and $\Sigma(\beta_c)$ defined in (18).

Since $\delta_m = 1$, it would be sufficient to have one column of $B(\beta_c,\beta_{oc})$ being nonzero for all the possible configurations. However, there is no such column. It is therefore necessary to use 2 additional motors, for instance for the rotation of wheels 1 and 2.

6 Structural Properties of the dynamical model

6.1 Feedback equivalence

In the previous chapter, the general dynamical state space model of wheeled mobile robots has been established and can be rewritten in the following more compact form :

$$\begin{pmatrix} \dot\xi \\ \dot\beta_{oc} \\ \dot\varphi \end{pmatrix} = \begin{pmatrix} R^T(\theta) \\ D(\beta_{oc}) \\ E(\beta_c,\beta_{oc}) \end{pmatrix}\Sigma(\beta_c)\eta \qquad (27)$$

$$\dot\beta_c = \zeta \qquad (28)$$

$$H(\beta_c,\beta_{oc})\begin{pmatrix} \dot\eta \\ \dot\zeta \end{pmatrix} + f(\beta_c,\beta_{oc},\eta,\zeta) = F(\beta_c,\beta_{oc})\begin{pmatrix} \tau_m \\ \tau_c \end{pmatrix} \qquad (29)$$

with

$$F(\beta_c,\beta_{oc}) \triangleq \begin{pmatrix} B(\beta_c,\beta_{oc})P & 0 \\ 0 & I \end{pmatrix}$$

where F is full rank under Assumption **A2**. We have the following property.

Property 1. The general state space model of wheeled mobile robots (27)-(29) is feedback equivalent (by a smooth static time-invariant state feedback) to the following system :

$$\begin{pmatrix} \dot\xi \\ \dot\beta_{oc} \\ \dot\varphi \end{pmatrix} = \begin{pmatrix} R^T(\theta) \\ D(\beta_{oc}) \\ E(\beta_c,\beta_{oc}) \end{pmatrix}\Sigma(\beta_c)\eta$$

$$\dot\beta_c = \zeta \qquad (30)$$

$$\begin{cases} \dot\eta = u \\ \dot\zeta = v \end{cases} \qquad (31)$$

where u and v represent auxiliary control inputs of appropriate dimensions.

Proof. Since F is full rank the following smooth static time invariant state feedback is well defined everywhere in the state space :

$$\begin{pmatrix} \tau_m \\ \tau_c \end{pmatrix} = F^\dagger(\beta_c,\beta_{oc})[H\begin{pmatrix} u \\ v \end{pmatrix} - f(\beta_c,\beta_{oc},\eta,\zeta)] \qquad (32)$$

where $F^\dagger$ denotes an arbitrary left inverse of $F(\beta_c,\beta_{oc})$. ■

6.2 Reduction of the general dynamical model

We now address the question of the reducibility of the general equivalent model (30) (31). A state space model is reducible if there exists a change of coordinates such that some of the new coordinates are identically zero along the motion of the system. Due to the structure of subsystem (31) , it is equivalent to address the problem of the reducibility of the kinematic configuration model (30). The reducibility of (30) is related to the dimension of the involutive closure of the following distribution Δ, expressed in local coordinates as :

$$\Delta(\xi,\beta_{oc},\varphi,\beta_c) \triangleq \text{ span } \left\{ \begin{pmatrix} R^T(\theta)\Sigma(\beta_c) & 0 \\ D(\beta_{oc})\Sigma(\beta_c) & 0 \\ E(\beta_c,\beta_{oc})\Sigma(\beta_c) & 0 \\ 0 & I \end{pmatrix} \right\} \tag{33}$$

We have the following property.

Property 2.
The dimension of the involutive closure of Δ satisfies the following inequality :

$$\delta_m + N_c < \dim(\text{inv}\Delta) < 3 + N_c + N + N_{oc}$$

The quantity $M = \dim(\text{inv}\Delta) - (\delta_m + N_c)$ is the degree of nonholonomy of the robot (related to the non-integrability of the pure rolling and nonslipping constraints).
The submodel (30) is reducible and the number of coordinates that can be eliminated is $3 + N_c + N + N_{oc} - \dim(\text{inv}\Delta)$. ■

In addition to the model reduction described in the above Property, a further simplification is of interest from an operational viewpoint. In a context of control design, it is clear that the user will be essentially concerned by controlling the posture of the robot $\xi(t)$ by using the control input u and v. We observe that this implies that we can ignore deliberately the coordinates β_{oc} and φ and restrict our attention to the following dynamical submodel :

$$\begin{cases} \dot{\xi} = R^T(\theta)\Sigma(\beta_c)\eta \\ \dot{\beta}_c = \zeta \\ \dot{\eta} = u \\ \dot{\zeta} = v \end{cases} \tag{34}$$

which is ineducible and fully describes the system dynamics between the control inputs u and v and the posture ξ. The coordinates β_{oc} and φ have, apparently disappeared but it is important to notice that they are in fact hidden in the feedback (32) which allows us to write the model in the above simple form (34).

7 Conclusion

It has been shown that, according to the restriction to the mobility induced by the kinematic constraints, all WMR can be classified into 5 categories, with particular structures of the corresponding kinematic and dynamic models.

It is however possible, with a static state feedback, to reduce the model to a simple form, with a minimum number of state variables, allowing to describe the dynamics of the plane motion of the robot. This form is particularly useful to derive the control structural properties and to design control laws (see for instance [4]). [1]

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