Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991

# Exponential Stabilization of Mobile Robots with Nonholonomic Constraints

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## Abstract

*This paper presents an exponentially stable controller for a two degrees of freedom robot with nonholonomic constraints. Although this type of system is open loop controllable, this system has been shown to be nonstabilizable via pure smooth feedback [2], [7]. In this paper, a particular class of piecewise continuous controllers is shown to stabilize the system exponentially. This controller has the feature that it does not require infinitely fast switching, as required, for example, by sliding controllers.*

## 1 Introduction

Path tracking precision is essential in Wheeled Mobile Robots (WMR) performing tasks such as welding, painting, gluing, drawing, etc., where some point in the WMR is in contact with the navigation surface and has to accurately follow a given path. Also, good path following capabilities are required when the navigation is effectuated in cluster obstacle environments. The control problem of mobile robots with nonholonomic constraints has been addressed from two points of view:

(a) Open-loop strategies seek to find a bounded sequence of control inputs steering the cart from any initial position to any other arbitrary configuration. The existence of such sequences has been indicated by [11], as a consequence of the local controllability and reachability of this type of systems. [16] has proposed analytic tools based on Lie algebra and geometrical considerations to find the required control sequence. [3] has worked on optimal sinusoids-type inputs for canonical systems for which controllability is obtained by first order Lie brackets. [12] has extended this work to non-canonical forms requiring a high degree of bracketing to achieve controllability. This proposal results in sub-optimal sinusoids-type inputs. These strategies have been studied in connection with the motion planning of mobile robots.

(b) Closed-loop strategies consist of designing feedback loops that stabilize the cart about an arbitrary point in the state space. Although, the cart model is locally controllable and locally reachable, it has been shown by [7] and [9] based on the work of [2] and [1], that there is no pure smooth state feedback law (i.e. $C^{\infty}$) which can locally stabilize this class of systems. [10] has suggested looking for attractors including the equilibrium point, and has proposed a method to stabilize the closed loop motion about such an attractor. Although the mobile robot model is not considered, [10] deals with attitude stabilization of a rigid spacecraft in failure mode which belongs to the same class of systems as those considered here. Nonlinear controllers for tracking a moving virtual cart (or reference cart) were proposed by [6] and [7], among others. The requirement of non-zero motion excludes the stabilization problem. Extension of the work of [7] including stopping phases, was studied by [8]. They proposed a continuous state feedback law depending on an exogenous time variable. This control scheme yields asymptotic stabilization of the origin. An alternative to time-dependent smooth controllers are the discontinuous or piecewise smooth controllers. [13] has proposed discontinuous controllers for the well known Brockett's example, [2], and for the system considered in [10] concerning a rigid spacecraft in failure mode. [14] and [15] have presented a discontinuous controller for the knife edge example. Their idea consists of first constructing an open loop strategy to steer the system state from any initial condition to the origin. This results in a set of manifolds which are then made invariant through a set of discontinuous feedbacks.

In this paper we propose a "piecewise" smooth controller to make the origin exponentially stable for any initial condition in the state space. The main difference with respect to other approaches can be summarized as follows: The proposed scheme does not seek to render the discontinuous surface invariant, as opposed to the principles of sliding control, but rather to make this surface non-attractive. Infinite switching in the control law and the undesirable "chattering" phenomenon, can thus be avoided. Furthermore, this control law yields exponential stability so that the convergence can be chosen arbitrarily fast.

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## 2 Coordinate transformation

The kinematics of a mobile robot with two degrees of freedom is given as:

$$\begin{aligned} \dot{x} &= \cos\theta(v_1+v_2)/2 = \cos\theta v \\ \dot{y} &= \sin\theta(v_1+v_2)/2 = \sin\theta v \\ \dot{\theta} &= (v_1-v_2)/(2c_r) = \omega \end{aligned} \tag{1}$$

where the state of the system (1) is described by the triplet $q=[x,y,\theta]^T$, indicating the position of the wheel axis center, $(x,y)$, and the cart orientation , $\theta$, with respect to the $x$-axis. The distance between the point $(x,y)$ and each of the wheel locations is $c_r$. The velocities (or inputs) $v_1$ and $v_2$ are the tangent velocities of each wheel at its rotation center, (i.e. motor velocities time wheel radius). The control variables $v$ and $\omega$ are the tangent and angular cart velocities, respectively. They are related to the wheel velocities as:

$$u=\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c_r} & -\frac{1}{2c_r} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2}$$

By stabilization of system (1), we understand the design of a control law, $u(q)$, so that the closed loop system

$$\dot{q} = G(q)u(q) = f(q), \quad G(q) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{3}$$

converges for any initial condition[1],$q(0)$ , to an equilibrium point in $\mathcal{O}$,

$$\mathcal{O} = \{(x,y,\theta)=(0,0,2\pi n); \quad n=0,\pm1,\pm2,\ldots\}$$

Note that all points in $\mathcal{O}$ are equivalent in terms of position and orientation of the cart.

Introduce the circle family $\mathcal{P}$,

$$\mathcal{P} = \{(x,y) \mid x^2+(y-r)^2=r^2\} \tag{4}$$

as the set of circles passing through the origin, centered on the $y$-axis and with $\frac{\partial y}{\partial x}=0$ in the origin. Let $\theta_d$ be the angle of the tangent of $\mathcal{P}$ at $(x,y)$, defined as:

$$\theta_d(x,y) = \begin{cases} 2\arctan(y/x) & ; \quad (x,y)\neq(0,0) \\ 0 & ; \quad (x,y)=(0,0) \end{cases} \tag{5}$$

$$r(x,y) = \frac{x^2+y^2}{2y} \tag{6}$$

where $r$ is the radius of the circles defined by $\mathcal{P}$ and $\theta_d$ is taken by definition to belong to $(-\pi,\pi]$. Hence $\theta_d$ has discontinuities on the $y$-axis with respect to x. The discontinuity surface is defined as:

$$\mathcal{D} = \{(x,y,\theta) \mid x=0,\ y\neq0\} \tag{7}$$

[1] In general, conditions for stabilization are not required to be global, but in this paper we add this requirement.

In view of these definitions, introduce the arclength, $a$, and the orientation error, $\alpha$, as:

$$a(x,y) = r\theta_d = \frac{x^2+y^2}{y}\arctan(y/x) \tag{8}$$

$$\alpha(x,y,\theta) = e - 2\pi n(e), \quad e=\theta-\theta_d \tag{9}$$

where $\alpha \in (-\pi,\pi]$, is a periodic and piecewise continuous function with respect to $e$. $n$ takes values in $\{0,\pm1,\pm2,\ldots\}$ in such a way that $\alpha$ belongs to $(-\pi,\pi]$. $\alpha$ is introduced so that all the elements in $\mathcal{O}$ are mapped into the unique point $(a,\alpha)=(0,0)$. $\mathcal{E}$ is the set of the points in $q$ where $\alpha(q)$ is discontinuous, i.e.,

$$\mathcal{E} = \{(x,y,\theta) \mid \alpha(x,y,\theta)=\pi\} \tag{10}$$

Note that $a(x,y)$ defines the arclength from the origin to $(x,y)$ along a circle which is centered at the $y$-axis and passes through these two points. $a(x,y)$ may be positive or negative according to the sign of $x$. In the degenerate case, where $y=0$, we define $a(x,0)=0$ which makes $a(x,y)$ continuous with respect to $y$. Discontinuities in $a(x,y)$ take only place at the $y$-axis. An illustation of these definitions is shown in figure 1.

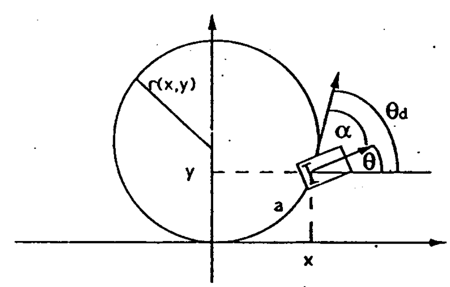


Figure 1: Illustration of the coordinate transformation.

Let us now introduce the function $F(\cdot): R^3 \to R \times (-\pi,\pi]$, mapping the state space coordinates, $q \in R^3$, into the two dimensional space, $z \in R \times (-\pi,\pi]$:

$$z=F(q); \quad F(q)=\begin{bmatrix} a(x,y) \\ \alpha(x,y,\theta) \end{bmatrix} \tag{11}$$

This transformation has several useful properties listed in the following lemma.

**Lemma 1** *The mapping $F(\cdot): R^3 \to R\times(-\pi,\pi]$ has the following properties:*

1. $F(0)=0$
2. $a^2(q), \alpha^2(q), ||F(q)||^2$ *are continuous in* $q$.
3. $||[x,y]^T|| \leq ||z|| \leq |a|+|\alpha|$

*where* $||\cdot||$ *denotes the Euclidian norm.*

**Proof:** The proof is simple and is left to the reader. □

# 3 Control design and stability analysis

This section proposes a piecewise smooth controller and analyses the stability of the closed loop system . The stability analysis is first performed in an open continuous subspace, and then the analysis is extended to the whole state space including discontinuities.

## 3.1 Dynamics in the $\Psi$-space

Let us first consider the case where $q \in \Psi$ where $\Psi$ is defined as the open set $\Psi = R^3 - (\mathcal{D} \cup \mathcal{E})$. $F(\cdot)$ is differentiable in $\Psi$, i.e. $F(q) \in C^\infty, \forall q \in \Psi$.

In $\Psi$ we have,

$$\dot{z} = \frac{\partial F}{\partial q}\dot{q} = J(q)\dot{q} \;\; ; \quad J(q) \in R^{2\times 3} \tag{12}$$

and $J(q)$, with $\beta = y/x$, is given as:

$$J(q) = \begin{bmatrix} \frac{\theta_d}{\beta} - 1 & \frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta} & 0 \\ \frac{2\beta}{(1+\beta^2)x} & -\frac{2}{(1+\beta^2)x} & 1 \end{bmatrix} \tag{13}$$

together with equation (3), we get:

$$\dot{z} = J(q)G(q)u = B(q)u; \;\; B = \begin{bmatrix} b_1 & 0 \\ b_2 & 1 \end{bmatrix} \tag{14}$$

with

$$b_1 = \cos\theta(\frac{\theta_d}{\beta} - 1) + \sin\theta\left(\frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta}\right) \tag{15}$$

$$b_2 = \cos\theta\frac{2\beta}{(1+\beta^2)x} - \sin\theta\frac{2}{(1+\beta^2)x} \tag{16}$$

By noting that $\cos\theta = \cos(\alpha + \theta_d)$, $\sin\theta = \sin(\alpha + \theta_d)$ and $\cos\theta_d = \frac{1-\beta^2}{1+\beta^2}$, $\sin\theta_d = \frac{2\beta}{1+\beta^2}$, we can rewrite $b_1$ as:

$$b_1(\alpha, \beta) = \cos\alpha + \left(-\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \cos\theta_d\left(\frac{\theta_d}{2}(1 - \frac{1}{\beta^2}) + \frac{1}{\beta}\right)\right)\sin\alpha \tag{17}$$

**Lemma 2** *The functions $b_i(q)$ have the following properties for any $x$ and $y$, with $\beta = \frac{y}{x}$:*

1. $b_{min}(\alpha) \leq b_1(\alpha, \beta) \leq b_{max}(\alpha)$
2. *$b_1(\alpha, \beta)$ is continuous in $\alpha$.*
3. $\lim_{\alpha \to 0} b_1(\alpha, \beta) = 1$
4. *$\mid b_2(q)a(q) \mid \leq N$ for some constant $N > 0$.*

*where*

$$b_{min}(\alpha) = \cos\alpha - \frac{\pi}{2}|\sin\alpha|$$

$$b_{max}(\alpha) = \cos\alpha + \frac{\pi}{2}|\sin\alpha|$$

**Proof:** see appendix. □

The properties in lemma 2 will be useful in establishing exponential stability of the closed loop equations.

Taking the following control law, with $\gamma > 0$ and $k > 0$,

$$v = -\gamma b_1 a \tag{18}$$

$$\omega = -b_2 v - k\alpha \tag{19}$$

gives the closed loop equations:

$$\begin{aligned} \dot{a} &= b_1 v &&= -\gamma b_1^2 a \\ \dot{\alpha} &= b_2 v + \omega &&= -k\alpha \end{aligned} \tag{20}$$

which have the following solutions for $a(t)$ and $\alpha(t)$:

$$\begin{aligned} a(t) &= a(0)\exp(-\gamma\kappa(t)) \\ \alpha(t) &= \alpha(0)\exp(-kt) \end{aligned} \tag{21}$$

with,

$$\kappa(t) = \int_0^t b_1^2(q(\tau))d\tau \tag{22}$$

From these equations we have:

$$||z(t)||^2 \leq ||z(0)||^2 \exp(-2\eta(t)) \tag{23}$$

where,

$$\eta(t) = \min(\gamma\kappa(t), kt) \quad \forall t \geq \tag{24}$$

which indicates bounds on the norm of $z(t)$ in the continuous set $\Psi$.

The following subsection extends the boundness of $z(t)$ to the region including the discontinuities by showing that the discontinuous surfaces $\mathcal{D}$ and $\mathcal{E}$ are not invariant and that the norm of $z(t)$ remains constant when traversing the discontinuous surfaces.

## 3.2 Dynamic behaviour on the surfaces $\mathcal{D}$ and $\mathcal{E}$

Motion (or impossibility of motion) on the discontinuous surfaces $\mathcal{D}$ and $\mathcal{E}$ can be investigated by analysing the direction of the vector field $f(q)$ from (3) in the neighbourhood of the discontinuities.

Let us first consider the behaviour on the surface $\mathcal{D}$.

**Lemma 3** *Any trajectory $q(t)$, solution of the closed loop system*

$$\dot{q} = f(q)$$

*cannot stay in $\mathcal{D}$ in a closed time interval $I = [t_1, t_2]$, $t_2 > t_1$.*

**Proof:** To prove that no motion is possible in $\mathcal{D}$, we can first compute $f(q)$ in the neighbourhood of $\mathcal{D}$ as:

$$f^+(q) = \lim_{x \to 0^+} f(q) = \begin{bmatrix} \cos\theta v^+ \\ \sin\theta v^+ \\ \omega^+ \end{bmatrix}$$

$$f^-(q) = \lim_{x \to 0^-} f(q) = \begin{bmatrix} \cos\theta v^- \\ \sin\theta v^- \\ \omega^- \end{bmatrix}$$

where,

$$\begin{aligned} v^{\pm} &= -\gamma(\mp\cos\theta + \frac{\pi}{2}\sin\theta)\frac{\pi}{2}|y| \\ \omega^{\pm} &= -\cos\theta\gamma\pi(\mp\cos\theta + \frac{\pi}{2}\sin\theta)\mathrm{sgn}(y) \\ &\quad -k(\theta - \pi - 2\pi n) \end{aligned}$$

and then show that there exists no convex combination of $f^-(q)$ and $f^+(q)$ which makes $q(t)$ stay in $\mathcal{D}$. In other words, there do not exist a $q \in \mathcal{D}$, $\delta \in [0,1]$ and $\mu \in R$ such that

$$\mu f_i = \delta f^+(q) + (1-\delta)f^-(q) \tag{25}$$

for all $t \in I$, where $f_i$ indicates the directions of possible motions in $\mathcal{D}$. Note that in order to remain in $\mathcal{D}$ during a time interval $I$, the cart should perform either a motion along the $y$-axis, a pure rotation in a fixed point $y$, or stand still. By allowing $\mu$ to be equal to zero, these possibilities are represented by the following directions:

$$f_1 = [0,\ 1,\ 0]^T \quad \text{or} \quad f_2 = [0,\ 0,\ 1]^T$$

The direction indicated by $f_1$, is equivalent to the situation where the cart is oriented in the $y$-direction, i.e. $\theta$ is constant equal to 90° or 270°. In this case it is simple to see that the last line in condition (25) cannot be satisfied. In the $f_2$-direction, the two first lines in condition (25) cannot be verified for all $t \in I$.□

We have shown that trajectories $q(t)$ cannot stay in $\mathcal{D}$. However, it should be noticed that $\mathcal{D}$ can be traversed. This is not the case for the surface $\mathcal{E}$ which is shown to be a repulsive discontinuity by the following lemma.

**Lemma 4** *Any trajectory $q(t)$, solution of the closed loop equations,*

$$\dot{q} = f(q)$$

*starting in $\mathcal{E}$, i.e. $q(0) \in \mathcal{E}$, or in its neighbourhood, will be repelled from $\mathcal{E}$.*

**Proof:** To prove this, we need only to show that for any $q \in \mathcal{E}$, the projection of the vector field $f(q)$ on the normal of $\mathcal{E}$ points out from both sides of the surface. In other words, the inner products of $f(q)$ and the outpointing normal at each side of the discontinuous surface are strictly positive.

Let $s(q) = 0$ denote the discontinuity surface $\mathcal{E}$,

$$s(q) = \theta - \theta_d(x,y) - 2\pi n - \pi$$

Then the normal to $s(q) = 0$ is:

$$n(q) = \frac{\partial s(q)}{\partial q} = \begin{bmatrix} \frac{2y}{x^2+y^2} \\ -\frac{2x}{x^2+y^2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{r} \\ -\frac{x}{yr} \\ 1 \end{bmatrix} \tag{26}$$

We define for $q \in \mathcal{E}$:

$$\begin{aligned} f^+(q) &= \lim_{s\to 0^+} f(q) = \begin{bmatrix} -\cos\theta_d\gamma r\theta_d \\ -\sin\theta_d\gamma r\theta_d \\ -\gamma\theta_d + k\pi \end{bmatrix} \\ f^-(q) &= \lim_{s\to 0^-} f(q) = \begin{bmatrix} -\cos\theta_d\gamma r\theta_d \\ -\sin\theta_d\gamma r\theta_d \\ -\gamma\theta_d - k\pi \end{bmatrix} \end{aligned}$$

Then we have

$$\begin{aligned} < f^+, n > &= k\pi > 0 && (27) \\ < f^-, -n > &= k\pi > 0 && (28) \end{aligned}$$

This means that in the neighbourhood of $\mathcal{E}$, or when $q(0) \in \mathcal{E}$, the field vector, $f(q)$, will always have a component driving the system away from $\mathcal{E}$. This can also be seen by studying the "potential" function, $V(s) = \frac{1}{2}s^2$.

$$\dot{V} = s\dot{s} = s < n, f^{\pm} >= s \cdot \mathrm{sgn}(s)2\pi = 2\pi|s| \geq 0 \tag{29}$$

Therefore $V(t)$ and hence $|s(t)|$ will always increase. □

### 3.3 Dynamics in the complete space

Lemma 3 and lemma 4 allow us to extend the properties of the dynamics of the closed loop system to the space including discontinuities. The following lemma summarizes these results.

**Lemma 5** *For any $q \in R^3$, $z \in R\times(-\pi,\pi]$, and $\forall t \geq 0$, we have:*

$$\begin{aligned} \|z(t)\| &\leq \|z(0)\|e^{-\eta(t)} && (30) \\ \|a(t)\| &\leq \|a(0)\|e^{-\kappa(t)} && (31) \\ \|\alpha(t)\| &\leq \|\alpha(0)\|e^{-kt} && (32) \end{aligned}$$

*where $\eta(t)$ and $\kappa(t)$ are defined by (24) and (22).*

The following theorem establishes our main result.

**Theorem 1** *There exist positive constants, $T$, $\eta_0(T)$, $\sigma_0(T)$ and $\varepsilon(T)$, so that the norm of $z(t)$ satisfies,*

$$\|z(t)\|^2 \leq \sigma_0^2(T)e^{-2\eta_0(T)t}, \quad \forall t \geq 0 \tag{33}$$

*where, $1 > \varepsilon(T) > 0$, and $\sigma_0(T)$, $\eta_0(T)$ are given as:*

$$\begin{aligned} \eta_0(T) &= \min(\gamma(1-\varepsilon(T)), k) && (34) \\ \sigma_0(T) &= \max(|a(0)|e^{\gamma(1-\varepsilon(T))T}, |\alpha(0)|) && (35) \end{aligned}$$

*with arbitrary, positive constants $\gamma$ and $k$.*

**Proof:** Lemma 2 gives upper and lower bounds on $b_1(\alpha,\beta)$ and shows that when $\alpha$ approaches zero, $b_1(\alpha,\beta)$ tends continuously towards one. Lemma 5 shows that $\alpha(t)$ decreases exponentially to zero. Therefore, for all $t \geq T$, there exists a small enough $\varepsilon(T)$ so that,

$$|b_1^2(\alpha(t),\beta(t)) - 1| \leq \varepsilon(T); \quad \forall t \geq T$$

which gives the following bounds on $b_1^2$,

$$1-\varepsilon(T) \le b_1^2(\alpha(t),\beta(t)) \le 1+\varepsilon(T); \quad \forall t \ge T$$

In view of lemma 5, we have for all $t \ge 0$,

$$\begin{aligned} |a(t)| &\le |a(0)|e^{-\kappa(t)} \\ &\le |a(0)|e^{-\gamma\int_T^t(1-\varepsilon(T))d\tau-\gamma\int_0^T b_1^2(\tau)d\tau} \\ &= a_0(T)e^{-\gamma(1-\varepsilon(T))t} \end{aligned}$$

where,

$$a_0(T) = |a(0)|e^{\gamma(1-\varepsilon(T))T}$$

And therefore,

$$\begin{aligned} \|z(t)\|^2 &= a^2(t)+\alpha^2(t) \\ &\le a_0^2(T)e^{-2\gamma(1-\varepsilon(T))t}+\alpha^2(0)e^{-2kt} \\ &\le \max(a_0^2(T),\alpha^2(0))e^{-2\min(\gamma(1-\varepsilon(T)),k)t} \\ &= \sigma_0^2(T)e^{-2\eta_0(T)t} \qquad \square \end{aligned}$$

It can now be established that exponential convergence of $z(t)$ to zero implies exponential convergence of the $q$-trajectories to any of the members of $\mathcal{O}$.

**Theorem 2** *For any initial condition $q(0) \in R^3$, the solutions $q(t)$, $t>0$, of the closed loop equations*

$$\dot{q} = f(q)$$

*converge exponentially to any of the elements in $\mathcal{O} = \{(0,0,2\pi n),\quad n=0,\pm1,\pm2,\ldots\}$.*

**Proof:** The proof will be based on basic properties of the norms of $q$ and $z$. Note first, from lemma 1, property 3, that the distance from $(x,y)$ to the origin is upper bounded by the arclength $|a|$,

$$\|[x(t),y(t)]^T\|^2 \le \|a(t)\|^2; \quad \forall t \ge 0$$

Since $a(t)$ tends to zero exponentially, the norm of $[x,y]^T$ will converge exponentially to zero. It remains to show that the cart orientation, $\theta$, converge to a point in $\mathcal{O}$. For this purpose, we recall that $\theta$ can be written as a function of $\alpha$ as, (9):

$$\theta(t) = \alpha(t)+\theta_d(t)+2\pi n$$

where $n$ increments when the $y$-axis (or $\mathcal{D}$) is crossed from the right to the left and decrements when $\mathcal{D}$ is traversed in the opposite direction.

Since $\alpha(t)$ tends exponentially to zero, the behaviour of $\theta(t)$ will be determined by the behaviour of $\theta_d(t)$. Simple arguments can be used to show this. $\theta_d$ is by definition the tangent angle to the circles defined by $\mathcal{P}$, and $\alpha$ is the error between the actual orientation and this tangent angle. Since $\alpha(t)$ converges exponentially to zero, the motion of the cart will converge exponentially to a motion along one of the circles defined by $\mathcal{P}$. From theorem 1 we have that the distance, $|a|$, from the origin to the position of the cart along this circle, converges exponentially to zero. We have therefore that the position of the cart converges to the origin exponentially along a circle. Therefore, $\theta_d(x(t),y(t))$ converges exponentially to its limit, $\theta_d(0,0)=0$. Since $\theta(t)$ converges exponentially to $\theta_d$, $\theta(t)$ will converge exponentially to zero. $\square$

**Corollary 1** *The control inputs $v(q)$ and $\omega(q)$ remain bounded for any $q \in R^3$.*

**Proof:** Boundness of $v$ and $\omega$ follows from the properties 1 and 4 of $b_1$ and $b_2 a$ listed in lemma 2, and the fact that $a$ and $\alpha$ are bounded quantities. It should also be observed that both the inputs $v$ and $\omega$ tend to zero as time goes to infinity.$\square$

Theorem 1 gives bounds on the convergence rate, $\eta_0$, and on the magnitude of the norm of $z(t)$, $\sigma_0$. Note, however, that when $T$ is high, $\sigma_0$ may describe a too conservative bound on $\|z(t)\|$ since $\sigma_0$ grows exponentially as $T$ increases. Design guide lines for choosing the control gains can be established from lemma 5 and theorem 1.

# 4 Conclusions

Giving up the requirements for pure smooth feedback, a piecewise smooth controller has been proposed. The particularity of these controllers is that infinite high frequency components as well as the well known problem of "chattering" are avoided. The cart converges exponentially to the origin with zero orientation for any initial condition. This is achieved by letting the motion of the cart converge to one of the circles which pass through the origin and are centered on the $y$-axis. The circles were chosen because they yield a new change of coordinates which is geometrical meaningful. However, other types of paths may also be possible.

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## Appendix: proof of lemma 2

1. From equation (17) we have, with $\beta = \frac{y}{x}$ :

$$b_1(\alpha,\beta) = \cos\alpha + B(\beta)\sin\alpha \qquad (36)$$

where

$$\begin{aligned} \theta_d &= \theta_d(\beta) = 2\arctan(\beta) \\ B(\beta) &= -\sin\theta_d(\frac{\theta_d}{\beta} - 1) + \\ &\quad \cos\theta_d(\frac{\theta_d}{2}(1-\frac{1}{\beta^2}) + \frac{1}{\beta}) \\ &= \frac{1}{\beta} - (1+\frac{1}{\beta^2})\arctan\beta \end{aligned}$$

Here, we have used the fact that $\cos\theta_d = \frac{1-\beta^2}{1+\beta^2}$, $\sin\theta_d = \frac{2\beta}{1+\beta^2}$ and $\tan\theta_d = \frac{2\beta}{1-\beta^2}$.

In order to find the maximum and minimum values of $B(\beta)$, we analyze the derivative, $B'(\beta)$:

$$\begin{aligned} B'(\beta) &= -\frac{1}{\beta^2} + \frac{2}{\beta^3}\arctan\beta - (1+\frac{1}{\beta^2})\frac{1}{1+\beta^2} \\ &= \frac{2}{\beta^2}(\frac{\arctan\beta}{\beta} - 1) \; < \; 0 \qquad \forall\beta \end{aligned}$$

We note that $B'(\beta)$ is continuous in $\beta = 0$, if we define $B'(0) = -\frac{2}{3}$.

Since $B'(\beta)$ is negative for all $\beta \in R$, we find that

$$\begin{aligned} \lim_{\beta\to\infty} B(\beta) \le \; & B(\beta) \; \le \lim_{\beta\to-\infty} B(\beta) \\ & \Downarrow \\ -\frac{\pi}{2} \le \; & B(\beta) \; \le \frac{\pi}{2} \end{aligned}$$

Therefore we get:

$$\cos\alpha - \frac{\pi}{2}|\sin\alpha| \le b_1(\alpha,\beta) \le \cos\alpha + \frac{\pi}{2}|\sin\alpha|$$

**2.** From **1** we have that

$$b_1(\alpha,\beta) = \cos\alpha + B(\beta)\sin\alpha$$

where $B(\beta)$ is bounded. Since $\cos\alpha$ and $\sin\alpha$ are continuous in $\alpha$, and $B(\beta)$ is bounded, it is clear that $b_1(\alpha,\beta)$ is also continuous in $\alpha$.

**3.** By the continuity property from **2**, we find

$$\lim_{\alpha\to 0} b_1(\alpha,\beta) = \lim_{\alpha\to 0}\cos\alpha + \lim_{\alpha\to 0} B(\beta)\sin\alpha = 1$$

**4.** From the definition of $a(x,y)$, (8), we have:

$$a = r\theta_d = \frac{x^2+y^2}{y}\arctan\frac{y}{x} = x\frac{1+\beta^2}{\beta}\arctan\beta$$

From the definition of $b_2$, (16), we get:

$$\begin{aligned} |b_2 a| &= |\left(\cos\theta\frac{2\beta}{(1+\beta^2)x} - \sin\theta\frac{2}{(1+\beta^2)x}\right)\cdot \\ &\quad x\frac{1+\beta^2}{\beta}\arctan\beta| \\ &= |2\cos\theta\arctan\beta - 2\sin\theta\frac{\arctan\beta}{\beta}| \\ &\le \pi|\cos\theta| + 2|\sin\theta| \; \le \pi + 2 \; = \; N \end{aligned}$$

$\square$