Constrained multvariable cautious stable predictive control

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Abstract: The characterisation of the class of stable predictions has been used to derive single-input single-output model based predictive control laws with guaranteed stability. Here it is shown that this work can be extended to the multivariable case.

1 Introduction

Model-based predictive control (MBPC) has gained in popularity due to: first, its simple strategy of minimising a cost that penalises predicted tracking errors and control activity; and secondly, its ability to handle system constraints (see reviews in [1, 2]). Early work [3] lacked a general stability theory, but recent algorithms [4–7] have provided the missing guarantees of stability even in the presence of system constraints [8]. This was achieved by introducing terminal constraints which require the predicted tracking errors to be zero beyond a given output horizon and the predicted control moves to be zero beyond a given input horizon.

These terminal constraints are restrictive: they are based on conditions which are sufficient but not necessary for stability. In this paper we remedy this by utilising the entire class of stable predictions and hence maximising the control authority available for improving performance while respecting input constraints. The treatment of these concepts as they apply to single-input single-output (SISO) plants can be found in [9–11]; here this work is extended to the multivariable case. As with earlier stable algorithms [12, 13], the extension is straightforward, but the following issues must be addressed: first, derivation of necessary and sufficient conditions for the stability of multivariable predictions; and secondly, proof that constraint satisfaction over an infinite horizon for the multivariable case can be treated in the same way as for SISO systems.

A convenient way to guarantee the stability of MBPC is to force the predicted error and control increment trajectories to be finite length sequences (FLS). Recent work relaxed this by allowing the predicted error trajectories to be stable infinite length sequences (ILS) [14] while further work allowed the predicted control increment trajectories also to be stable ILS [11]. In fact the development of necessary and sufficient conditions for stable predictions involves ILS, rather than FLS predictions for both errors and control increments. This development is a straightforward process for SISO plant, but is more involved in the multivariable case. Here, to obtain the necessary and sufficient conditions, we use the determinant and adjoint matrix description of the inverse of the transfer function matrix numerator and denominator polynomial matrices and then define conditions based only on the 'unstable' portion of the determinant.

In the presence of physical constraints, ILS trajectories lead to a difficulty: the constraints must be invoked over an infinite horizon. This problem can be overcome through the use of suitable input bounds. Such bounds have been given in a state space setting for state constraints [15]; an alternative which handles input constraints is presented in [10]. Here, we use multivariable predictions which are ILS for both inputs and outputs, and are concerned with input constraints only: therefore, we require bounding results on ILS inputs rather than outputs. For the derived class of stable input predictions, we show that the scalar bounding techniques [10] can be introduced into the multivariable case to provide an efficient means of invoking the constraints over a infinite horizon by enforcing them over a finite horizon. The multivariable necessary and sufficient conditions developed are introduced into a new algorithm, constrained multivariable cautious stable control (CMCaSC), whose efficacy is illustrated by a numerical example.

2 System model, objective and notation

Let the left coprime factorisation of a matrix transfer function model with inputs $u^{(i)}$ and outputs $y^{(i)}$ for $i = 1, 2, ..., m$ be

$$\mathbf{y}_t = z^{-1}[A(z)]^{-1}B0z\mathbf{u}_t;$$
$$\mathbf{y}_t = \left[y_t^{(1)} \quad \cdots \quad y_t^{(m)}\right]^T,$$
$$\mathbf{u}_t = \left[u_t^{(1)} \quad \cdots \quad u_y^{(m)}\right]^T \tag{1}$$

$A(z)$, $B(z)$ are nth, $(n-1)$th order matrix polynomials in the delay operator, z^{-1}, such that for a general $M(z)$

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$$M(z) = M_0 + M_1 z^{-1} + \cdots + M_{n_M} z^{-n_M} \quad (2)$$

with $n_M = \delta\{M(z)\}$ being the degree of $M(z)$, i.e. the highest power of z^{-1} with nonzero coefficient. Denote the determinant and adjoint of $M(z)$ by $m(z)$ and $M_d(z)$, such that $M^{-1}(z) = M_d(z)/m(z)$. We assume that factors common to all the elements of $M_d(z)$ and to $m(z)$ have been removed. Let $m(z) = m^+(z)m^-(z)$, where the roots of $m^+(z)$ [$m^-(z)$] are on or outside (strictly inside) the unit circle.

MBPC minimises a cost J_t, subject to input/state constraints, implements the current optimal control and repeats at the next time instant. The cost J_t is to be minimised over the future control increments, $\Delta\mathbf{u}_{t+i} = \mathbf{u}_{t+i} - \mathbf{u}_{t+i-1}$, $i = t, t+1, \ldots$ and is

$$J_t = \sum_{i=1}^{\infty} \|\mathbf{e}_{t+i}\|_2^2 + \lambda \sum_{i=0}^{\infty} \|\Delta\mathbf{u}_{t+i}\|_2^2 \quad (3)$$

$\mathbf{e}_{t+i} = \mathbf{r}_{t+i} - \mathbf{y}_{t+i}$ denotes the predicted output error vectors ($\mathbf{r}_{t+i}$ being future setpoint vectors), λ is the control weighting. Without loss of generality, we let $\mathbf{r}_{t+i} = \mathbf{r}_0$ where $\mathbf{r}_0$ is a constant vector. To guarantee stability J_t must be finite and thus behave like a Lyapunov function; this is possible iff both the sequences $\mathbf{e}_{t+i}$ and $\Delta\mathbf{u}_{t+i}$, converge to $\mathbf{0}$.

This can be achieved within finite output/input horizons, n_y and n_u, through the use of a stabilising (deadbeat) inner loop [12]. Here we relax the finite horizon requirement, demanding only asymptotic convergence, and adopt an approach which uses information on past values of inputs/outputs and the future setpoint to define the class of predicted control increments/errors which converge to $\mathbf{0}$. As a by-product we derive an explicit characterisation of the degrees of freedom which are available for the minimisation of the cost J_t.

Following common practice we replace the model of eqn. 1 with the incremental model

$$\begin{aligned} &D(z)\mathbf{y}_t = z^{-1}B(z)\Delta\mathbf{u}_t; \\ &D(z) = A(z)\Delta(z); \\ &\Delta(z) = 1 - z^{-1} \end{aligned} \quad (4)$$

Next, define the Toeplitz–Hankel matrices, C_M, H_M, for the polynomial matrix, $M(z)$, as

$$C_M = \begin{bmatrix} M_0 & 0 & \cdots & \cdots & 0 \\ M_1 & M_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{n_M} & \vdots & \vdots & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & M_0 \end{bmatrix};$$

$$H_M = \begin{bmatrix} M_1 & M_2 & \cdots & M_{n_M} \\ M_2 & \cdots & M_{n_M} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ M_{n_M} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (5)$$

where the dimensions of C_M, H_M are as implied by the context. Simulating the incremental model of eqn. 4 forward in time, we obtain the input/output prediction equation

$$C_D\mathbf{Y} = C_B\Delta\mathbf{U} + \mathbf{P}; \quad \mathbf{P} = H_B\Delta\mathbf{U}_{past} - H_D\mathbf{Y}_{past} \quad (6)$$

where the output/input prediction and past vectors are defined as

$$\begin{aligned} \mathbf{Y} &= [\mathbf{y}_{t+1} \quad \mathbf{y}_{t+2} \quad \cdots]^T \\ \Delta\mathbf{U} &= [\Delta\mathbf{u}_t \quad \Delta\mathbf{u}_{t+1} \quad \cdots]^T \\ \mathbf{Y}_{past} &= [\mathbf{y}_t \quad \cdots \quad \mathbf{y}_{t-n}]^T \\ \Delta\mathbf{U}_{past} &= [\Delta\mathbf{u}_{t-1} \quad \cdots \quad \Delta\mathbf{u}_{t-n+1}]^T \end{aligned} \quad (7)$$

3 Stable predictions

Eqn. 6 can be rewritten in z-transform form as

$$D(z)\mathbf{y}(z) = B(z)\Delta\mathbf{u}(z) + \mathbf{p}(z),$$
$$\mathbf{p}(z) = [I, Iz^{-1}, \ldots, Iz^{-n+1}]\mathbf{P}; \quad (8a)$$

$$\begin{cases} \mathbf{y}(z) = \mathbf{y}_1 + \mathbf{y}_2 z^{-1} + \cdots \\ \Delta\mathbf{u}(z) = \Delta\mathbf{u}_0 + \Delta\mathbf{u}_1 z^{-1} + \cdots \end{cases} \quad (8b)$$

where $\mathbf{y}(z)$ $\Delta\mathbf{u}(z)$ involve future output/input values only, whereas $\mathbf{p}(z)$ involves past values only. Noting that $\mathbf{r}(z) = \mathbf{r}_0/\Delta(z)$ and subtracting $D(z)\mathbf{r}(z)$ from both sides of eqn. 8a gives

$$-D(z)\mathbf{e}(z) = B(z)\Delta\mathbf{u}(z) + \mathbf{q}(z);$$
$$\mathbf{q}(z) = \mathbf{p}(z) - A(z)\mathbf{r}_0 \quad (9)$$

and use has been made of the fact that $D(z)/\Delta(z) = A(z)$. Eqn. 9 enables derivation of necessary and sufficient conditions for the stability of the error/input predictions.

Theorem: The entire class of stable error/input increment predictions, $\mathbf{e}(z)$, $\Delta\mathbf{u}(z)$, is given as

$$\mathbf{e}(z) = -\frac{B_1(z)}{d^-(z)}\mathbf{c}(z) - \phi'(z); \quad (10a)$$

$$\Delta\mathbf{u}(z) = \frac{D_1(z)}{b^-(z)}\mathbf{c}(z) + \psi'(z) \quad (10b)$$

where $B_1(z)$ and $D_1(z)$ are right coprime matrices which satisfy

$$b^-(z)D(z)B_1(z) = d^-(z)B(z)D_1(z) \quad (11)$$

$\psi'(z)$, $\phi'(z)$ are the minimal order vector solutions to the diophantine equation

$$D(z)\phi'(z) - B(z)\psi'(z) = \mathbf{q}(z) \quad (12)$$

and $\mathbf{c}(z)$ is an arbitrary 'stable' polynomial vector (i.e. with coefficients that form a sequence of vectors which converge to $\mathbf{0}$) and contains all the available degrees of freedom.

Proof: Premultiplication of both sides of eqn. 9 by $-[D(z)]^{-1}$, gives

$$\begin{aligned} \mathbf{e}(z) &= -[D(z)]^{-1}[B(z)\Delta\mathbf{u}(z) + \mathbf{q}(z)] \\ &= -\frac{D_d(z)[B(z)\Delta\mathbf{u}(z) + \mathbf{q}(z)]}{d(z)} \end{aligned} \quad (13)$$

and thus, it is obvious that $\mathbf{e}(z)$ will be stable iff

$$D_d(z)[B(z)\Delta\mathbf{u}(z) + \mathbf{q}(z)] = d^+(z)\phi(z) \quad (14a)$$
$$\Rightarrow \quad \mathbf{e}(z) = -\phi(z)/d^-(z) \quad (14b)$$

where $\phi(z)$ is an arbitrary stable polynomial vector. Premultiplying both sides of the constraint (eqn. 14a) with $D(z)$ and then dividing by $d(z)$ we derive the following equivalent constraint:

$$B(z)\Delta\mathbf{u}(z) + \mathbf{q}(z) = \frac{D(z)\phi(z)}{d^-(z)}$$

or

$$\Delta\mathbf{u}(z) = \frac{B_d(z)}{b(z)}\left[\frac{D(z)\phi(z)}{d^-(z)} - \mathbf{q}(z)\right] \quad (15)$$

where we have used the identity $D(z)D_d(z) = d(z)I_m$. By similar reasoning, $\Delta\mathbf{u}(z)$ will be stable if

$$B_d(z)\left[\frac{D(z)\phi(z)}{d^-(z)} - \mathbf{q}(z)\right] = b^+(z)\psi(z) \quad (16a)$$

$$\Rightarrow \quad \Delta\mathbf{u}(z) = \frac{\psi(z)}{b^-(z)} \quad (16b)$$

where $\psi(z)$ is an arbitrary stable polynomial vector. Premultiplying eqn. 16a by $d^-(z)B(z)/b^+(z)$ gives the following linear polynomial matrix diophantine equation:

$$b^-(z)D(z)\phi(z) - d^-(z)B(z)\psi(z) = d^-(z)b^-(z)\mathbf{q}(z) \quad (17)$$

where use has been made of the identity: $B(z)B_d(z) = b(z)I_m$.

The solution to eqn. 17 can be obtained from a particular solution, $\boldsymbol{\phi}_p(z)$, $\boldsymbol{\psi}_p(z)$, and a solution, in terms of the available degrees of freedom, defined by eqn. 17 when $d^-(z)b^-(z)\mathbf{q}(z)$ is replaced by $\mathbf{0}$. Of these two components, the latter is given by $\boldsymbol{\phi}_k(z) = B_1(z)\mathbf{c}(z)$, $\boldsymbol{\psi}_k(z) = D_1(z)\mathbf{c}(z)$, where $B_1(z)$, $D_1(z)$ and $\mathbf{c}(z)$ are as defined in the statement of the theorem. Furthermore, $B_1(z)$, $D_1(z)$ can be computed using Kucera's algorithm 7.10.4 [16]. A useful particular solution, on the other hand, is given by $\boldsymbol{\phi}_p(z) = d^-(z)\boldsymbol{\phi}'(z)$ and $\boldsymbol{\psi}_p(z) = b^-(z)\boldsymbol{\psi}'(z)$, where $\boldsymbol{\phi}'(z)$, $\boldsymbol{\psi}'(z)$ are as defined in the statement of the theorem. Given that $A(z)$ and $B(z)$ (and hence $D(z)$ and $B(z)$) are left coprime, the solutions $\boldsymbol{\phi}'(z)$, $\boldsymbol{\psi}'(z)$ exist and are easily obtained. Writing down the convolutional sum involving the coefficients of the polynomials in eqn. 12 one can derive a matrix vector equation which can be solved for the vector comprising the coefficients of $\boldsymbol{\phi}'(z)$, $\boldsymbol{\psi}'(z)$ thereby defining explicitly the dependence of $\boldsymbol{\phi}'(z)$, $\boldsymbol{\psi}'(z)$ on $\mathbf{q}(z)$.

Combining the components $[\boldsymbol{\phi}_k(z), \boldsymbol{\psi}_k(z)]$, $[\boldsymbol{\phi}_p(z), \boldsymbol{\psi}_p(z)]$ we obtain the general solution to eqn. 17

$$\phi(z) = B_1(z)\mathbf{c}(z) + d^-(z)\phi'(z);$$
$$\psi(z) = D_1(z)\mathbf{c}(z) + b^-(z)\psi'(z) \quad (18)$$

Substitution of eqn. 18 into eqns. 14b and 16b yields the result.

To obtain the entire class of stable predictions, $\mathbf{c}(z)$ must be of infinite order but for any practical constrained optimisation of J_t, the number of degrees of freedom must be finite, hence we let $\delta\{\mathbf{c}(z)\} = n_c - 1$. Expanding the class of vector polynomials (eqn. 10) into vectors of coefficients we obtain

$$\mathbf{E} = -\Gamma_{I_m/d^-}\Gamma_{B_1}\mathbf{C} - \mathbf{\Phi}'; \quad \Delta\mathbf{U} = \Gamma_{I_m/b^-}\Gamma_{D_1}\mathbf{C} + \mathbf{\Psi}' \quad (19)$$

$\mathbf{E}$ and $\mathbf{C}$ are the vectors of future errors and degrees of freedom, respectively, and are defined in a manner analogous to $\mathbf{Y}$; Γ_M has $n_c + n_m$ block-rows and is the matrix formed out of the first n_c block-columns of C_M; Γ_{I_m/m^-} is the matrix formed out of the first $n_c + n_m$ block-columns of the infinite dimensional matrix, C_{I_m/m^-}, whose columns are defined from the Taylor series expansion of $I_m/m^-(z)$; Φ', Ψ' are defined from the coefficient vectors of $\boldsymbol{\phi}'(z)$, $\boldsymbol{\psi}'(z)$ in a manner analogous to eqn. 7 and are padded on the bottom with an infinite number of zeros. Then using eqn. 19, the cost J_t of eqn. 3 can be written in the quadratic form below, and can be minimised over $\mathbf{C}$ subject to input constraints

$$\begin{aligned} J_t &= \|\mathbf{E}\|^2 + \lambda\|\Delta\mathbf{U}\|^2 \\ &= \mathbf{C}^T(\Gamma_{B_1}^T\Gamma_{I_m/d^-}^T\Gamma_{I_m/d^-}\Gamma_{B_1} \\ &\quad + \lambda\Gamma_{D_1}^T\Gamma_{I_m/b^-}^T\Gamma_{I_m/b^-}\Gamma_{D_1})\mathbf{C} \\ &\quad + \mathbf{C}^T(\Gamma_{B_1}^T\Gamma_{I_m/d^-}^T\Phi' - \lambda\Gamma_{D_1}^T\Gamma_{I_m/b^-}^T\Psi') \\ &\quad + \|\Phi'\|^2 + \lambda\|\Psi'\|^2 \end{aligned} \quad (20)$$

Remark: Γ_{I_m/m^-} has an infinite number of rows, but always appears in J_t either, as part of a finite dimensional quadratic product, or transposed and postmultiplied with a vector which has only a finite number of nonzero elements. Computationally the latter poses no problem, and as $I_m/m^-(z)$ is diagonal, the former can be computed in precisely the same way as in the SISO case [11], i.e. through the use of a Lyapunov equation, or through the solution of matrix/vector equations that arise out of Laurent expansions of Hermitian forms of transfer functions [11].

4 Input constraints and the multivariable algorithm

A typical set of input constraints are given as

$$\Delta\underline{u}^{(j)} \le \Delta u_i^{(j)} \le \Delta\overline{u}^{(j)}; \quad \underline{u}^{(j)} \le u_i^{(j)} \le \overline{u}^{(j)}$$
$$\forall i \ge 0, j = 1, \ldots, m \quad (21)$$

where we make the usual assumption that $\mathbf{0}$ is strictly in the interior of the constraint region for $\Delta\mathbf{u}$, and that the steady state value of $\mathbf{u}$, $\mathbf{u}_{ss} = [B(1)]^{-1}A(1)\mathbf{r}_0$, which is required if $\mathbf{y}$ is to reach $\mathbf{r}_0$, lies strictly in the interior of the constraint region for $\mathbf{u}$. The sequence of inputs is of infinite length, and thus constraints (eqn. 21) must be invoked over an infinite horizon. However, the sequence is stable and therefore there exists a finite horizon, say n_{con}, such that satisfaction of the constraints over this finite horizon implies satisfaction over the infinite horizon. As $I_m/b^-(z)$ is diagonal, the system inputs can be handled by considering the inputs of m independent scalar systems.

The problem of finding n_{con}, for the scalar case [10] can be extended to the multivariable case as follows. Rewriting eqn. 10b and writing the related equation for $\mathbf{u}(z)$ as

$$\Delta\mathbf{u}(z) = \frac{D_1(z)\mathbf{c}(z)}{b^-(z)} + \psi'(z);$$
$$\mathbf{u}(z) = \frac{D_1(z)\mathbf{c}(z)}{\Delta(z)b^-(z)} + \frac{\psi'(z) + \mathbf{u}_{t-1}}{\Delta(z)} \quad (22)$$

Since $\boldsymbol{\psi}'(z)$ is a known polynomial and of finite degree, say n_ψ, it makes: first, a zero contribution to all future values $\Delta\mathbf{u}_i$ for $i > n_\psi$; and secondly, a constant known contribution to all future values $\mathbf{u}_i$ for $i > n_\psi$. Thus, providing that n_{con} is chosen larger than n_ψ, $\boldsymbol{\psi}'(z)$ poses no particular difficulty. Then, to choose n_{con}, we need to

(i) Calculate, for a given i, bounds on the size of the jth, $\forall j \ge i$, elements of the impulse response of $h(z) = 1/b^-(z)$ (for $\Delta\mathbf{u}(z)$) and $h(z) = 1/\Delta(z)b^-(z)$ (for $\mathbf{u}(z)$)

$$\underline{H}_i = \min_j h_j(j \ge i); \quad \overline{H}_i = \max_j h_j(j \ge i) \quad (23)$$

(these bounds are time invariant, can be calculated offline, and saved in a look-up table).

(ii) Use the first $n_{D_1} + n_c$ constraints on $\Delta\mathbf{u}(z)$ and $\mathbf{u}(z)$ to stipulate necessary bounds on the elements of the $n_{D_1} + n_c$ vector coefficients of $\mathbf{f}(z) = D_1(z)\mathbf{c}(z)$, say $\underline{f}_k \le f_k \le \overline{f}_k$, $k = 0 \ldots n_{D_1} + n_c - 1$.

(iii) Calculate necessary bounds on $\Delta\mathbf{u}_j$ (for all $j \ge i$) based on (i)–(ii), i.e. if $\mathbf{g}(z) = h(z)\mathbf{f}(z)$, then

$$\underline{G}_i^{(l)} \le g_j^{(l)} \le \overline{G}_i^{(l)} \quad (j \ge i)$$

$$\begin{cases} \underline{G}_i^{(l)} = \displaystyle\sum_{k=0}^{n_f} \min[\underline{H}_{i-k}\underline{f}_k^{(l)}, \overline{H}_{i-k}\underline{f}_k^{(l)}, \\ \qquad\qquad \underline{H}_{i-k}\overline{f}_k^{(l)}, \overline{H}_{i-k}\overline{f}_k^{(l)}] \\ \overline{G}_i^{(l)} = \displaystyle\sum_{k=0}^{n_f} \max[\underline{H}_{i-k}\underline{f}_k^{(l)}, \overline{H}_{i-k}\underline{f}_k^{(l)}, \\ \qquad\qquad \underline{H}_{i-k}\overline{f}_k^{(l)}, \overline{H}_{i-k}\overline{f}_k^{(l)}] \end{cases} \tag{24}$$

and, repeat for $\mathbf{u}_j$, using: first, $h(z) = 1/\Delta(z)b^{-}(z)$; and secondly, the steady-state value of the second term on the RHS of eqn. 22*b*, $\boldsymbol{\psi}'(1) + \mathbf{u}_{t-1}$.

(iv) Increment i until the bounds of (iii) (for $l = 1 \ldots m$) are inside constraints (eqn. 21); then set $n_{con} = i$.

An alternative (possibly less) conservative approach, is to bound each of the $2m^2$ transfer functions, $h^{(l,p)}(z)$, in $D_1(z)/b^{-}(z)$ and $D_1(z)/\Delta(z)b^{-}(z)$ as per (i) and the first n_c vector coefficients of $\mathbf{f}(z) = \mathbf{c}(z)$ as per (ii) and then calculate necessary bounds on $\Delta\mathbf{u}_j$ and $\mathbf{u}_j$ as per (iii), using superposition:

$$\underline{G}_i^{(l)} \le g_j^{(l)} \le \overline{G}_i^{(l)} \quad (j \ge i)$$

$$\begin{cases} \underline{G}_i^{(l)} = \displaystyle\sum_{k=0}^{n_f}\sum_{p=1}^{m} \min[\underline{H}_{i-k}^{(l,p)}\underline{f}_k^{(p)}, \overline{H}_{i-k}^{(l,p)}\underline{f}_k^{(p)}, \\ \qquad\qquad \underline{H}_{i-k}^{(l,p)}\overline{f}_k^{(p)}, \overline{H}_{i-k}^{(l,p)}\overline{f}_k^{(p)}] \\ \overline{G}_i^{(l)} = \displaystyle\sum_{k=0}^{n_f}\sum_{p=1}^{m} \max[\underline{H}_{i-k}^{(l,p)}\underline{f}_k^{(p)}, \overline{H}_{i-k}^{(l,p)}\underline{f}_k^{(p)}, \\ \qquad\qquad \underline{H}_{i-k}^{(l,p)}\overline{f}_k^{(p)}, \overline{H}_{i-k}^{(l,p)}\overline{f}_k^{(p)}] \end{cases} \tag{25}$$

The receding horizon CMCaSC algorithm is then defined as follows.

Algorithm (CMCaSC): Minimise performance index (eqn. 20) with respect to $\mathbf{C}$ and subject to input constraints (eqn. 21) for $0 \le i \le n_{con}$. Of the optimal input increments, $\Delta\mathbf{U}$, implement $\Delta\mathbf{u}_0$ and then recompute the optimisation at the next time instant with new plant data.

Remark: As with the scalar case it can be shown that the cost of the above algorithm forms a monotonically decreasing function of time, and this ensures closed loop stability. However, implicit in these arguments is the assumption that the CMCaSC optimisation is feasible, namely that there exists at least one member of the prediction class of eqn. 19 which meets the constraints (eqn. 21). It is easy to show that (in the absence of uncertainty) feasibility at the start time implies feasibility at all future times. Furthermore, since the prediction class of the algorithm is derived from conditions which are necessary and sufficient for stability, it is known that so long as the setpoint can be achieved within the physical input constraints, there will exist a finite (but possibly large) n_c for which feasibility can be guaranteed. Significantly, the use of predicted input trajectories which are not deadbeat (unlike those of earlier algorithms) implies that constraints are easier to meet (i.e. they can be met with a smaller number of degrees of freedom). This particular point is illustrated in the numerical example of Section 5.

5 Numerical example

Let the system of eqn. 1 be defined by

$$\begin{aligned} A(z) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2.4 & 0 \\ 0 & 1.9 \end{bmatrix} z^{-1} \\ &\quad + \begin{bmatrix} 0.68 & 0 \\ 0 & 0.28 \end{bmatrix} z^{-2} \\ &\quad + \begin{bmatrix} 0.24 & 0 \\ 0 & 0.48 \end{bmatrix} z^{-3} \\ B(z) &= \begin{bmatrix} 0.219 & 0.6789 \\ 0.047 & 0.6793 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0.0766 & -0.05702 \\ 0.00165 & -0.5706 \end{bmatrix} z^{-1} \\ &\quad - \begin{bmatrix} 0.982 & 0.6430 \\ 0.211 & 0.6434 \end{bmatrix} z^{-2} \end{aligned} \tag{26}$$

with input constraints (eqn. 21) defined by $\Delta\bar{\mathbf{u}} = -\Delta\underline{\mathbf{u}} = [0.5, 0.3]^T$, $\bar{\mathbf{u}} = -\underline{\mathbf{u}} = [1, 0.5]^T$, and control parameters, $n_c = 3$, $\lambda = 1$, and $\mathbf{r}_0 = [1\ ,0]^T$. For this example it was found that the bounds of step (iii) in the procedure of Section 4 lie inside constraints for $i = 6$; this was chosen to be the value of n_{con}. Figs. 1–3 show, respectively, the output, input-rate and input responses for CMSGPC [12] and Figs. 4–6 give the corresponding responses for CMCaSC. Both are stable, but CMSGPC has more active controls which are pushing against the limits. For a setpoint $\mathbf{r}_0 = [1, -1]^T$, CMSGPC is unstable (not shown); this is because the relevant constrained optimisation is infeasible, and as the input constraints are hard, its stable (FLS) input/output predictions are invalid. CMCaSC still performs admirably (Figs. 7–9).

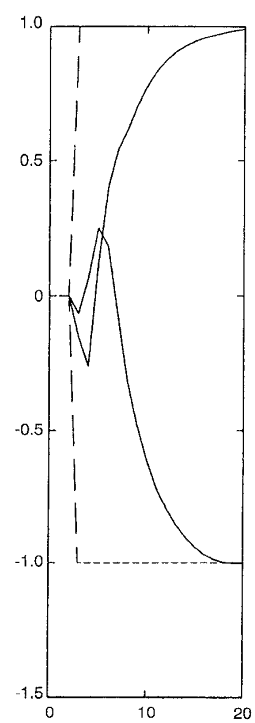

Fig. 1 *CMSGPC response to $r_0 = [1, 0]^T$*
– – – – setpoint trajectories
——— output responses

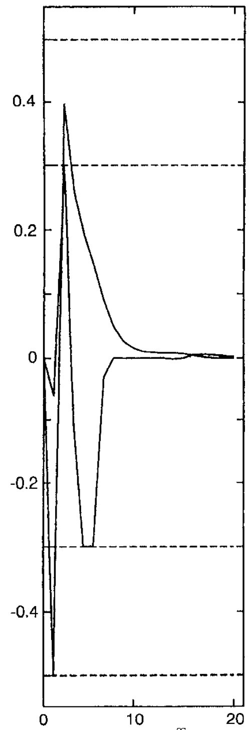

Fig. 2 *CMSGPC response to $r_0 = [1, 0]^T$*
——— control increments
– – – – corresponding rate limits

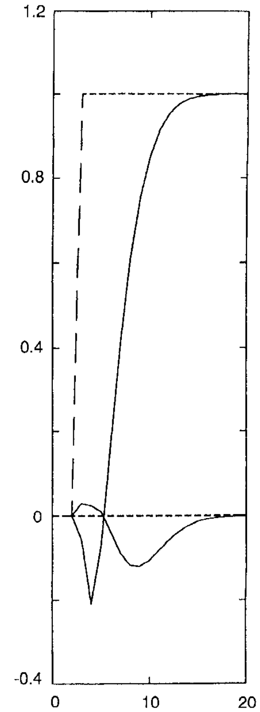

Fig. 4 *CMCaSC response to $r_0 = [1, 0]^T$*
– – – – setpoint trajectories
——— output responses

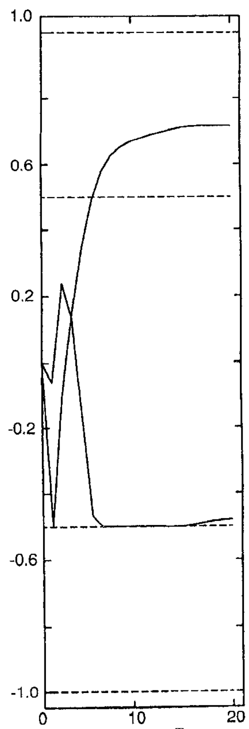

Fig. 3 *CMSGPC response to $r_0 = [1, 0]^T$*
——— input trajectories
– – – – corresponding limits

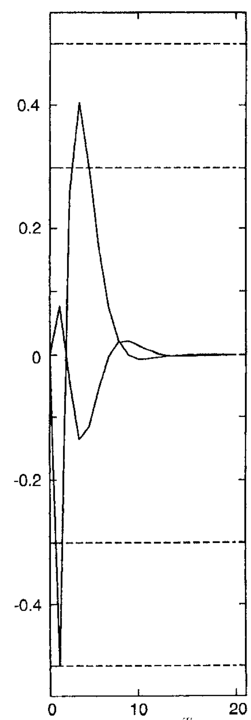

Fig. 5 *CMCaSC response to $r_0 = [1, 0]^T$*
——— control increments
– – – – corresponding rate limits

6 Conclusions

Necessary and sufficient conditions for the stability of error and control increment prediction equations have been given for the scalar case. This paper extended these to the multivariable case and proposed an algorithm for the deployment of the relevant degrees of freedom in the optimisation of a cost subject to hard input constraints. This was done in Sections 3 and 4, and the efficacy of the derived algorithm was illustrated by means of a numerical example in Section 5.

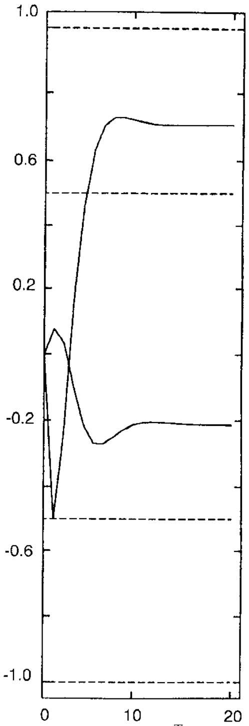

Fig. 6 *CMCaSC response to* $r_0 = [1, 0]^T$
——— input trajectories
– – – – corresponding limits

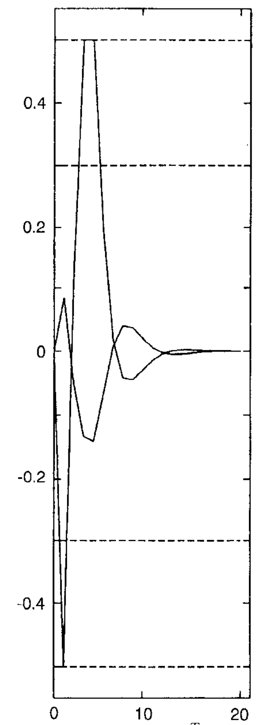

Fig. 8 *CMCaSC response to* $r_0 = [1, -1]^T$
——— control increments
– – – – corresponding rate limits

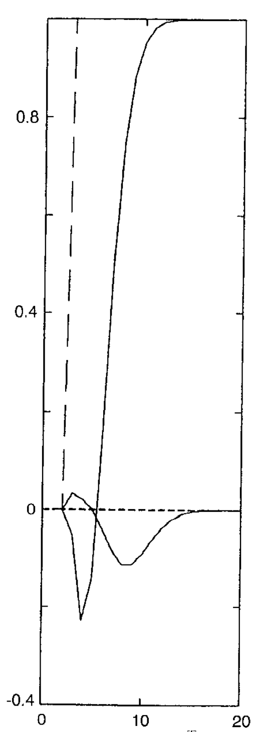

Fig. 7 *CMCaSC response to* $r_0 = [1, -1]^T$
– – – – setpoint trajectories
——— output responses

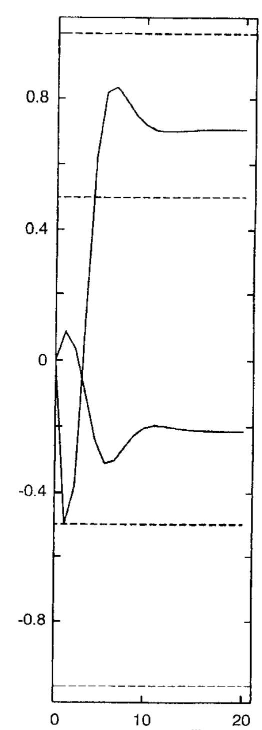

Fig. 9 *CMCaSC response to* $r_0 = [1, -1]^T$
——— input trajectories
– – – – corresponding limits

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