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Transformation of the kinematic models of restricted mobility wheeled mobile robots with a single platform into chain forms

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Abstract

We are concerned in this paper with the transformation of the kinematic models of restricted mobility wheeled mobile robots with one platform into chain forms. For this purpose we take benefit of the existing transformation results concerning nonholonomic systems with two and three inputs. However, the existing sufficient conditions failed in the case of a two-steering wheeled mobile robot and have motivated the establishment of a new set of sufficient conditions for a particular class of 3-inputs chain form systems. We also give a stabilizing control with an exponential rate of convergence for the two-steering wheeled mobile robot.

Keywords: wheeled mobile robot, chain form, stabilization.

1 Introduction

Restricted mobility Wheeled Mobile Robots (futher mentioned as WMR) made of a single platform can be classified into four categories in terms of their degree of mobility δ_m and their number δ_s of steering wheels which can be oriented independently [3]. The corresponding kinematic equations of motion are given below:

- $(\delta_m, \delta_s) = (2,0)$ - the unicycle WMR:

$$\begin{pmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

- $(\delta_m, \delta_s) = (2,1)$:

$$\begin{pmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -\sin(\theta+\beta_c) & 0 \\ \cos(\theta+\beta_c) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\dot{\beta}_c = u_3$$

- $(\delta_m, \delta_s) = (1,1)$ - the car-like WMR:

$$\begin{pmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -L\sin\theta\sin\beta_c \\ L\cos\theta\sin\beta_c \\ \cos\beta_c \end{pmatrix} u_1$$
$$\dot{\beta}_c = u_2$$

- $(\delta_m, \delta_s) = (1,2)$ - the two-steering WMR:

$$\begin{pmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -2L\cos\theta s_1 s_2 - L\sin\theta s_{12} \\ -2L\sin\theta s_1 s_2 + L\cos\theta s_{12} \\ \sin(\beta_{c2}-\beta_{c1}) \end{pmatrix} u_1$$
$$\dot{\beta}_{c1} = u_2$$
$$\dot{\beta}_{c2} = u_3$$

where s_i, $i = 1,2$ and s_{12} denote respectively $sin\beta_{ci}$ and $\sin(\beta_{c1}+\beta_{c2})$ and where $x = (\xi^T, \beta_c^T)^T$ is the state vector and u the input vector. More precisely, the posture ξ is composed of the cartesian coordinates $(x_p, y_p)^T$ of a point P of the WMR platform and of the orientation θ of the platform with respect to the horizontal axis. β_c is the vector of the orientation angles of the independent steering wheels.

It has been shown that all these WMR are completely linearizable by means of dynamic state feedback and therefore that they can exponentially follow reference trajectories with non zero velocity [1]. Another problem consists in stabilizing these systems about a rest configuration, and we know, according to a theorem due to Brockett, that it is not feasable by a continuous pure-state feedback. This stabilization problem has essentially been solved for chain form nonholonomic systems with two inputs and has yielded two types of control laws leading to stabilization with an exponential rate of convergence: time-varying piecewise continuous control [10] and time-varying continuous control [9, 8]. It appears that the chain form structure, introduced by Murray et al. [11] to realize motion planning using sinusoids [7], is a powerful tool to systematically compute such stabilizing controls. Our concern in what proceeds, is therefore to show that the four types of WMR can be transformed into chain form. For this purpose, we use results given in [7, 2] and complete them by our main result theorem 4.1.

This article is structured as follows: we first recall in section 2 the chained structure and give some of its properties as well as two sets of sufficient conditions to transform a nonholonomic system into chain form.

Section 3 concerns the application of these transformation tools to the four types of WMR mentioned above. It concludes to the possible transformation of the three first type WMR. However, the set of sufficient conditions for 3-inputs chain form systems given in [2] does not allow to conclude about the WMR $(1,2)$. For this purpose, we establish in section 4 theorem 4.1, a set of weaker sufficient conditions for a particular class of 3-inputs chain form systems. In section 5, the WMR $(1,2)$ is shown by application of theorem 4.1 to be also transformable into chain form. Finally, we propose a stabilizing time-varying piecewise continuous feedback control law for the WMR $(1,2)$. Corresponding Scilab simulation results are displayed.

2 Chained structure systems and their properties

From the general definition given in [7], we only need for our purpose to consider the following two chained strutures:

- 2 inputs, 1 chain, 1 generator

$$\begin{array}{ll} \dot{z}_1^0 = v_1 & \dot{z}_2^0 = v_2 \\ & \dot{z}_{21}^1 = z_2^0 v_1 \\ & \vdots \\ & \dot{z}_{21}^n = z_{21}^{n-1} v_1 \end{array} \tag{2.1}$$

- 3 inputs, 2 chains, 1 generator

$$\begin{array}{lll} \dot{z}_1^0 = v_1 & \dot{z}_2^0 = v_2 & \dot{z}_3^0 = v_3 \\ & \dot{z}_{21}^1 = z_2^0 v_1 & \dot{z}_{31}^1 = z_3^0 v_1 \\ & \vdots & \vdots \\ & \dot{z}_{21}^{n_2} = z_{21}^{n_2-1} v_1 & \dot{z}_{31}^{n_3} = z_{31}^{n_3-1} v_1 \end{array} \tag{2.2}$$

where $z = (z_1^0, z_2^0, z_{21}^1, \ldots, z_{21}^n)^T$ (respectively $z = (z_1^0, z_2^0, z_{21}^1, \ldots, z_{21}^{n_2}, z_3^0, z_{31}^1, \ldots, z_{31}^{n_3})^T$) is the state vector with dimension $n+2$ (respectively $3 + n_2 + n_3$) and where $v = (v_1, v_2)^T$ (respectively $v = (v_1, v_2, v_3)^T$) is the input vector.

Systems (2.1), (2.2) can be rewritten as:

$$\dot{z} = \sum_{i=1}^{i=m} g_i(z) v_i$$

where $m = dim(v)$ and g_i, $i = 1, \cdots, m$ are suitably defined vector fields. We now define the relative growth vector σ associated to the Lie algebra built from the distribution $\Delta = span\{g_i,\ i = 1, \cdots, m\}$. For this purpose, we first introduce the filtration associated to the distribution Δ as follows:

$$\begin{array}{l} G_0 = \Delta, \\ G_i = G_{i-1} + [G_0, G_{i-1}], \end{array}$$

where $[G_0, G_{i-1}] = span\{[g,h]/\ g \in G_0,\ h \in G_{i-1}\}$. Let r be the vector composed of the integer r_i defined as $r_i = rankG_i$. Then, we have:

Definition 2.1 *The relative growth vector σ associated to the distribution Δ is given as:*

$$\begin{array}{l} \sigma_0 = r_0, \\ \sigma_i = r_i - r_{i-1},\ i > 0. \end{array}$$

Remark 2.1 *There exists a bijection relation between the relative growth vector and the structure of the chained system. Indeed, σ_0 gives the dimension of the input vector v, σ_1 gives the number of the so called "chains" and the other σ_i $(i > 1)$ give information about the chain lengths.*

We are now interested in the transformation of nonholonomic driftless systems into chain forms. Results exist in the cases of two inputs [7, prop. 7] and three inputs [2, prop. 1] giving sufficient conditions for these tranformations. Both results are based on the same concept; thus we only detail the one concerning the three inputs case.

Proposition 2.1 *[2] Consider the control system:*

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3$$

with x the n-dimensional state space vector, $u = (u_1, u_2, u_3)^T$ the 3-dimensional input vector and g_1, g_2, g_3 three independent smooth vector fields. Let us define the distributions:

$$\begin{array}{l} \Delta_0 = span\,\{g_1, ad_{g_1}^j g_2, ad_{g_1}^k g_3, \\ \qquad 0 \le j \le n_2,\ 0 \le k \le n_3\} \\ \Delta_1 = span\,\{ad_{g_1}^j g_2, ad_{g_1}^k g_3, \\ \qquad 0 \le j \le n_2,\ 0 \le k \le n_3\} \\ \Delta_2 = span\,\{ad_{g_1}^j g_2, ad_{g_1}^k g_3, \\ \qquad 0 \le j \le n_2 - 1,\ 0 \le k \le n_3\} \\ \Delta_3 = span\,\{ad_{g_1}^j g_2, ad_{g_1}^k g_3, \\ \qquad 0 \le j \le n_2 - 1,\ 0 \le k \le n_3 - 1\} \end{array}$$

where $n = 3 + n_2 + n_3$. If for some open set $U \subset R^n$, $\Delta_0(x) = R^n$, $\forall x \in U$, and Δ_1, Δ_2 and Δ_3 are involutive on U, then there exists a local feedback transformation on U : $z = \Phi(x)$, $v = \beta(x)u$, such that the transformed system is a two-chains, single-generator chain form system (2.2).

From a practical point of view, solving the problem of the conversion of a driftless nonholonomic system into its equivalent chain form consists first in checking the involutivity of the distributions Δ_i, $i = 0, \ldots, 3$ and then in finding functions $h_1(x)$, $h_2(x)$ and $h_3(x)$ such that:

$$\begin{array}{ll} dh_1 \cdot \Delta_1 \equiv 0, & dh_1 \cdot g_1 \ne 0, \\ dh_2 \cdot \Delta_2 \equiv 0, & dh_2 \cdot ad_{g_1}^{n_2} g_2 \ne 0, \\ dh_3 \cdot \Delta_3 \equiv 0, & dh_3 \cdot ad_{g_1}^{n_3} g_3 \ne 0. \end{array}$$

Then, the chained system is obtained by diffeomorphism $\Phi(x)$ and state feedback $u(x)$:

$$\begin{array}{l} \Phi(x) = (h_1, L_{g_1}^{n_2} h_2, L_{g_1}^{n_2-1} h_2, \ldots, h_2, \\ \qquad\qquad L_{g_1}^{n_3} h_3, L_{g_1}^{n_3-1} h_3, \ldots, h_3)^T, \end{array}$$

$$v = \begin{pmatrix} 1 & 0 & 0 \\ L_{g_1}^{n_2+1} h_2 & L_{g_2} L_{g_1}^{n_2} h_2 & L_{g_3} L_{g_1}^{n_2} h_2 \\ L_{g_1}^{n_3+1} h_3 & L_{g_2} L_{g_1}^{n_3} h_3 & L_{g_3} L_{g_1}^{n_3} h_3 \end{pmatrix} u.$$

Remark 2.2 *Notice that this procedure may be simplified by taking the vector fields g_1, g_2, g_3 of the following form:*

$$\begin{array}{l} g_1(x) = \frac{\partial}{\partial x_1} + \sum_{i=2}^n g_1^i(x) \frac{\partial}{\partial x_i} \\ g_2(x) = \sum_{i=2}^n g_2^i(x) \frac{\partial}{\partial x_i} \\ g_3(x) = \sum_{i=2}^n g_3^i(x) \frac{\partial}{\partial x_i} \end{array} \tag{2.3}$$

which is always possible due to the independence of the three vector fields. This way, choosing $h_1(x) = x_1$ and noticing that Δ_1 is always involutive, the problem reduces to check the involutivity of Δ_2, Δ_3 and to solve the corresponding differential equations. This vector field manipulations may however introduce a subset of R^n where the simplified system would be singular.

3 Transformation of WMR $(2,0)$, $(1,1)$ and $(2,1)$

In this section, we apply the results of section 2 to the four types WMR kinematic models. We first consider WMR with two inputs: $(2,0)$ and $(1,1)$.

3.1 WMR with two inputs

Applying remark 2.2 and [7, prop. 7] with $h_1(x) = \theta$ and $h_2(x) = x_p \cos\theta + y_p \sin\theta$ for both WMR $(2,0)$ and WMR $(1,1)$ leads to the diffeomorphisms, feedback laws and chain forms given in the following tables.

WMR(2,0)
$z_1^0 = \theta$ $z_2^0 = -x_p \sin\theta + y_p \cos\theta$ $z_{21}^1 = x_p \cos\theta + y_p \sin\theta$
$v = \begin{pmatrix} 0 & 1 \\ 1 & -x_p\cos\theta - y_p \sin\theta \end{pmatrix} u$
$\dot z_1^0 = v_1 \quad \dot z_2^0 = v_2$ $\dot z_{21}^1 = z_2^0 v_1$

WMR(1,1)
$z_1^0 = \theta$ $z_2^0 = -x_p \cos\theta - y_p \sin\theta + L\tan\beta_c$ $z_{21}^1 = -x_p \sin\theta + y_p \cos\theta$ $z_{21}^2 = x_p \cos\theta + y_p \sin\theta$
$v = \begin{pmatrix} \cos\beta_c & 0 \\ \cos\beta_c\,(x_p \sin\theta - y_p\cos\theta) & -L\frac{1}{\cos^2\beta_c} \end{pmatrix} u$
$\dot z_1^0 = v_1 \quad \dot z_2^0 = v_2$ $\dot z_{21}^1 = z_2^0 v_1$ $\dot z_{21}^2 = z_{21}^1 v_1$

We may however notice that the transformation for WMR $(1,1)$ is not defined over the subset $\{x \in R^4 \ / \ \beta_c = \frac{\pi}{2} mod \pi\}$ of R^4. The reason comes from the input transformation $\tilde u_1 = \cos(\beta_c) u_1$ used to put the new system into the form (2.3) (see remark 2.2).

3.2 WMR with three inputs

First, we consider the WMR $(2,1)$ for which proposition 2.1 applies since its relative growth vector is $\sigma = (3,1)$. Unfortunately (see [6]) the resulting transformation is not defined on the subset $\{x \in R^4 / \ \theta + \beta_c = 0 mod \pi\}$.
Actually, a non singular transformation is obtained simply by observing the similar role played by θ and β_c in the original kinematic equations (see section 1). The resulting chain form and the corresponding diffeomorphism and feedback law are reported in the table below.

WMR(2,1)
$z_1^0 = \theta + \beta_c$ $z_2^0 = x_p \sin(\theta+\beta_c) - y_p\cos(\theta+\beta_c)$ $z_{21}^1 = -x_p\cos(\theta+\beta_c) - y_p \sin(\theta+\beta_c)$ $z_3^0 = \beta_c$
$v = \begin{pmatrix} 0 & 1 & 1 \\ -1 & b(x) & b(x) \\ 0 & 0 & 1 \end{pmatrix} u$
$\dot z_1^0 = v_1 \quad \dot z_2^0 = v_2 \quad \dot z_3^0 = v_3$ $\dot z_{21}^1 = z_2^0 v_1$

where $b(x) = x_p\cos(\theta+\beta_c) + y_p \sin(\theta+\beta_c)$.

Let us now consider the application of proposition 2.1 to the WMR $(1,2)$.

Proposition 3.1 *The application of proposition 2.1 does not allow to conclude about the transformability into chain form of the kinematic equations of the WMR $(1,2)$ given in section 1.*

Proof: First, we rewrite the kinematic equations considering the state vector $\kappa = (\theta, x_p, y_p, \beta_{c1}, \beta_{c2})^T$ and the following input transformation μ:

$$\dot\kappa = g_1(\kappa)\mu_1 + g_2(\kappa)\mu_2 + g_3(\kappa)\mu_3, \qquad (3.4)$$

where $\mu = \begin{pmatrix} \sin(\beta_{c2}-\beta_{c1}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} u$ and

$$g_1(\kappa) = \begin{pmatrix} 1 \\ g_{1x} \\ g_{1y} \\ 0 \\ 0 \end{pmatrix}, \ g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ g_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with $g_{1x} = (-2L\cos\theta s_1 s_2 - L\sin\theta s_{12})/\sin(\beta_{c2}-\beta_{c1})$, $g_{1y} = (-2L\sin\theta s_1 s_2 + L\cos\theta s_{12})/\sin(\beta_{c2}-\beta_{c1})$.
We also point out the following two remarks.

1. The system dimension is five therefore, according to proposition 2.1, n_2 and n_3 can only take values: $n_2 = n_3 = 1$ or $n_2 = 2$, $n_3 = 0$. This means that only two iterative constructions of the Lie algebra are considered: $\sigma = (3,2)$ and $\sigma = (3,1,1)$.
2. Secondly, vector fields g_2 and g_3 play symmetric roles when considering Lie bracket manipulations.

Besides, we give below the computations of a set of Lie brackets that will be useful in the sequel.

$$[g_1, g_2] = \begin{pmatrix} 0 \\ g_{12x} \\ g_{12y} \\ 0 \\ 0 \end{pmatrix} \quad [g_1, g_3] = \begin{pmatrix} 0 \\ g_{13x} \\ g_{13y} \\ 0 \\ 0 \end{pmatrix} \qquad (3.5)$$

$g_{12x} = (2L\cos\theta s_2^2 + L\sin\theta\sin(2\beta_{c2}))/\sin^2(\beta_{c2}-\beta_{c1})$,
$g_{12y} = (2L\sin\theta s_2^2 - L\cos\theta\sin(2\beta_{c2}))/\sin^2(\beta_{c2}-\beta_{c1})$,
$g_{13x} = (-2L\cos\theta s_1^2 - L\sin\theta\sin(2\beta_{c1}))/\sin^2(\beta_{c2}-\beta_{c1})$,
$g_{13y} = (-2L\sin\theta s_1^2 + L\cos\theta\sin(2\beta_{c1}))/\sin^2(\beta_{c2}-\beta_{c1})$,

and $[g_2, [g_1, g_3]] = \frac{-2}{\sin^2(\beta_{c2}-\beta_{c1})}(0, g_{1x}, g_{1y}, 0, 0)^T$,
$[g_3, [g_1, g_3]] = 2\frac{\cos(\beta_{c2}-\beta_{c1})}{\sin(\beta_{c2}-\beta_{c1})}[g_1, g_3]$.

Then, according to these calculations and to the second point mentioned above, lemma 3.1 follows.

Lemma 3.1 *[6] The vector fields $[g_1, g_2]$, $[g_1, g_3]$ are independent to each other as well as $[g_1, g_2]$ with $[g_3, [g_1, g_2]]$ and $[g_1, g_3]$ with $[g_2, [g_1, g_3]]$.*

From the expression of $[g_3, [g_1, g_3]]$ and since $[g_2, g_3] = 0$, we can conclude that the vector field g_3 cannot, and by symmetry nor can g_2, be the generator in proposition 2.1 (where it is denoted g_1). Therefore, we are left with the evaluation of the sufficient conditions of proposition 2.1 with g_1 as generator in the two possible cases:

- $n_2 = n_3 = 1$

$$\begin{array}{ll} \Delta_0 = \{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3]\}, & \\ \Delta_1 = \{g_2, g_3, [g_1, g_2], [g_1, g_3]\}, & \\ \Delta_2 = \{g_2, g_3, [g_1, g_3]\}, & \rightarrow \text{non involutive because} \\ \Delta_3 = \{g_2, g_3\} & \quad \text{of } [g_2, [g_1, g_3]] \text{ (lemma 3.1)} \end{array}$$

- $n_2 = 2$, $n_3 = 0$

$$\begin{array}{ll} \Delta_0 = \{g_1, g_2, g_3, [g_1, g_2], ad^2_{g_1} g_2\}, & \\ \Delta_1 = \{g_2, g_3, [g_1, g_2], ad^2_{g_1} g_2\}, & \\ \Delta_2 = \{g_2, g_3, [g_1, g_2]\}, & \rightarrow \text{non involutive because} \\ \Delta_3 = \{g_2, [g_1, g_2]\} & \quad \text{of } [g_3, [g_1, g_2]] \text{ (lemma 3.1)} \end{array}$$

Thus, the sufficient conditions of proposition 2.1 are not satisfied. $\diamondsuit$

To overcome the non conclusive result of proposition 3.1, we state in the following section, a theorem similar to proposition 2.1 but with weaker sufficient conditions. We shall however restrict the problem to systems with relative growth vectors of the form $\sigma = (3, 2, \cdots, 2)$.

4 Main result

Theorem 4.1 *Consider the control system:*

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3$$

with x the n-dimensional state space vector, $u = (u_1, u_2, u_3)^T$ the 3-dimensional input vector and g_1, g_2, g_3 three independent continuous vector fields. Define the distributions:

$$\begin{array}{l} \Delta_0 = span\{g_1, ad^j_{g_1} g_2, ad^k_{g_1} g_3,\ 0 \le j, k \le n_2\} \\ \Delta_1 = span\{ad^j_{g_1} g_2, ad^k_{g_1} g_3,\ 0 \le j, k \le n_2\} \\ \Delta_2 = span\{ad^j_{g_1} g_2, ad^k_{g_1} g_3,\ 0 \le j, k \le n_2 - 1\} \end{array}$$

where $n = 3 + 2n_2$. If for some open set $U \subset R^n$, $\Delta_0(x) = R^n$, $\forall x \in U$ and Δ_1, Δ_2 are involutive on U, then there exists a local feedback transformation on U: $z = \Phi(x)$, $v = \beta(x)u$, such that the transformed system is in two-chains, single-generator chain form (2.2) (with $n_2 = n_3$).

Proof. According to [5], since on an open set U of R^n $\Delta_0 = R^n$ and Δ_1 is a $n-1$ dimensional involutive distribution ($\Delta_1 \subset \Delta_0$), there exists a function $h_1(x)$ such that:

$$L_{\Delta_1} h_1 \equiv 0 \ \text{ and } L_{g_1} h_1 = a(x) \neq 0. \tag{4.6}$$

Tacking into account the linear independence of the vector fields g_1, g_2, g_3, we may as well consider these latters to have the form (2.3) and have $a(x) = 1$. In the same manner, since Δ_2 is involutive and $\Delta_2 \subset \Delta_1$ ($dim\Delta_1 - dim\Delta_2 = 2$), there exist two independent functions $h_2(x)$ and $h_3(x)$ such that:

$$L_{\Delta_2} h_2 = L_{\Delta_2} h_3 \equiv 0. \tag{4.7}$$

Their independence feature implies that the matrix $\begin{pmatrix} dh_2 \\ dh_3 \end{pmatrix} \begin{pmatrix} ad^{n_2}_{g_1} g_2 & ad^{n_2}_{g_1} g_3 \end{pmatrix}$ is nonsingular. In other words:

$$L_{ad^{n_2}_{g_1} g_2} h_2 L_{ad^{n_2}_{g_1} g_3} h_3 - L_{ad^{n_2}_{g_1} g_2} h_3 L_{ad^{n_2}_{g_1} g_3} h_2 \neq 0 \tag{4.8}$$

Let us consider the change of coordinates $z = \Phi(x)$:

$$\begin{array}{lll} z^0_1 = h_1 & z^0_2 = L^{n_2}_{g_1} h_2 & z^0_3 = L^{n_2}_{g_1} h_3 \\ & z^1_{21} = L^{n_2-1}_{g_1} h_2 & z^1_{31} = L^{n_2-1}_{g_1} h_3 \\ & \vdots & \vdots \\ & z^{n_2}_{21} = h_2 & z^{n_2}_{31} = h_3 \end{array}$$

To check that Φ is a diffeomorphism in a neighborhood of x, we compute $\frac{\partial \Phi}{\partial x}\Delta_0$ where Φ and Δ_0 are rewritten respectively as: $\Phi(x) = \left(z^0_1, z^0_2, z^0_3, z^1_{21}, z^1_{31}, \ldots, z^{n_2}_{21}, z^{n_2}_{31}\right)^T$ and $\Delta_0 = span\{g_1, g_2, g_3, [g_1, g_2], [g_1, g_3], \ldots, ad^{n_2}_{g_1} g_2, ad^{n_2}_{g_1} g_3\}$. According to (4.6) and (4.7), we have, omitting the argument x (see [6] for details):

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ * & b_{22} & b_{23} & \cdots & * & * \\ * & b_{32} & b_{33} & \cdots & * & * \\ \vdots & & & \ddots & & \vdots \\ * & 0 & 0 & & (-1)^{n_2} b_{22} & (-1)^{n_2} b_{23} \\ * & 0 & 0 & \cdots & (-1)^{n_2} b_{32} & (-1)^{n_2} b_{33} \end{bmatrix}$$

where $b_{22}(x) = L_{g_2} L^{n_2}_{g_1} h_2$, $b_{23}(x) = L_{g_3} L^{n_2}_{g_1} h_2$, $b_{32}(x) = L_{g_2} L^{n_2}_{g_1} h_3$, $b_{33}(x) = L_{g_3} L^{n_2}_{g_1} h_3$. Its determinant is equal to the left term of (4.8) to the n_2^{th} power. Thus it is nonsingular and so is the Jacobian matrix $\frac{\partial \Phi}{\partial x}$ since Δ_0 has full rank n. Therefore Φ is a local diffeomorphism.

In order to check that the system has the desired structure in the transformed coordinates z, we simply derive the components of z. Using the zero entries of the above matrix $\frac{\partial \Phi}{\partial x}\Delta_0$, we get:

$$\begin{array}{l} \dot{z}^0_1 = (L_{g_1} h_1) u_1 \\ \dot{z}^0_2 = (L^{n_2+1}_{g_1} h_2) u_1 + (L_{g_2} L^{n_2}_{g_1} h_2) u_2 + (L_{g_3} L^{n_2}_{g_1} h_2) u_3 \\ \dot{z}^1_{21} = (L^{n_2}_{g_1} h_2) u_1 \\ \vdots \\ \dot{z}^{n_2}_{21} = (L_{g_1} h_2) u_1 \\ \dot{z}^0_3 = (L^{n_2+1}_{g_1} h_3) u_1 + (L_{g_2} L^{n_2}_{g_1} h_3) u_2 + (L_{g_3} L^{n_2}_{g_1} h_3) u_3 \\ \dot{z}^1_{31} = (L^{n_2}_{g_1} h_3) u_1 \\ \vdots \\ \dot{z}^{n_2}_{31} = (L_{g_1} h_3) u_1 \end{array}$$

and taking the feedback law:

$$v = \begin{pmatrix} 1 & 0 & 0 \\ L_{g_1}^{n_2+1}h_2 & L_{g_2}L_{g_1}^{n_2}h_2 & L_{g_3}L_{g_1}^{n_2}h_2 \\ L_{g_1}^{n_2+1}h_3 & L_{g_2}L_{g_1}^{n_2}h_3 & L_{g_3}L_{g_1}^{n_2}h_3 \end{pmatrix} u$$

we obtain the desired chained structure (2.2) with $n_2 = n_3$. ◊

Remark 4.1 *The sufficient conditions of theorem 4.1 are weaker to those of proposition 2.1 because they need to check the involutivity condition of only distributions Δ_0, Δ_1, Δ_3 defined in proposition 2.1.*

5 Application to the WMR $(1,2)$ and simulation results

Let us apply theorem 4.1 to the kinematic model of the WMR $(1,2)$ (see section 1). Since the system's dimension is five, we take $n_2 = 1$. If we consider the transformed system (3.4) with the Lie products (3.5), we easily check that the distributions

$$\begin{aligned} \Delta_0 &= span\{g_1, g_2, g_3, [g_1,g_2], [g_1,g_3]\}, \\ \Delta_1 &= span\{g_2, g_3, [g_1,g_2], [g_1,g_3]\}, \\ \Delta_2 &= span\{g_2, g_3\} \end{aligned}$$

are all involutive (with $dim\Delta_0 = 5$). Therefore, the WMR $(1,2)$ is also transformable into chain form. Following the same steps as the constructive proof of theorem 4.1 yields the diffeomorphism $z = \Phi(x)$:

$$\begin{aligned} z_1^0 &= \theta \\ z_2^0 &= -x_p \sin\theta + y_p \cos\theta - 2L\tfrac{\sin\beta_{c1}\sin\beta_{c2}}{\sin(\beta_{c2}-\beta_{c1})} \\ z_{21}^1 &= x_p\cos\theta + y_p \sin\theta \\ z_3^0 &= x_p\cos\theta + y_p\sin\theta - L\tfrac{\sin(\beta_{c1}+\beta_{c2})}{\sin(\beta_{c2}-\beta_{c1})} \\ z_{31}^1 &= x_p \sin\theta - y_p\cos\theta \end{aligned}$$

and the control feedback law $v = \beta(x)u$ or:

$$v = \begin{pmatrix} 1 & 0 & 0 \\ -z_3^0 & \frac{-2L\sin^2\beta_{c2}}{\sin^2(\beta_{c2}-\beta_{c1})} & \frac{2L\sin^2\beta_{c1}}{\sin^2(\beta_{c2}-\beta_{c1})} \\ z_2^0 & \frac{-L\sin(2\beta_{c2})}{\sin^2(\beta_{c2}-\beta_{c1})} & \frac{L\sin(2\beta_{c1})}{\sin^2(\beta_{c2}-\beta_{c1})} \end{pmatrix} \mu$$

where μ is the vector input defined in (3.4).

The resulting "3 inputs, 2 chains, 1 generator" chain form is therefore:

$$\begin{matrix} \dot z_1^0 = v_1 & \dot z_2^0 = v_2 & \dot z_3^0 = v_3 \\ & \dot z_{21}^1 = z_2^0 v_1 & \dot z_{31}^1 = z_3^0 v_1 \end{matrix} \tag{5.9}$$

Notice that the above transformation is not defined over the subset $\{x \in R^5 / \ \beta_{c1} = \beta_{c2} mod\pi\}$ and is singular over the subset $\{x \in R^5 / \ \beta_{c1} = 0 mod\pi$ or $\beta_{c2} = 0 mod \pi\}$ since $det\beta(x) = -4L^2\frac{\sin\beta_{c1}\sin\beta_{c2}}{\sin^3(\beta_{c2}-\beta_{c1})}$.

As stated in the introduction, two types of stabilizing control laws with exponential rates of convergence have been developed for chained systems with 2 inputs and 1 chain. Taking as reference the time-varying piecewise continuous control applied to the particular case of the 3-dimensional system with 2-inputs WMR $(2,0)$ in [4] (which in turn is inspired by [10]), we wish to extend this stabilizing control to the case of the 3-inputs, 2-chains kinematic chained model (5.9) of WMR $(1,2)$. We may notice that more recent results have also been obtained for stabilization of WMR $(1,2)$ for example in [12].

The action of the stabilizing control established in [10] in the case of 2-inputs, 1-chain systems of the form (2.1) can be roughly summerized by the following points:

- The control v_1 is computed such that the subsystem made of the chain be linear and time-varying in time intervals of the form $[k\delta, (k+1)\delta[$, $k = 0, 1, \cdots$, where $j\delta$, $j = 0, 1, \cdots$ are discrete instants of time. The purpose of v_1 is also to make $z_1(t)$ exponentially converge to zero as $||Z_2||$ converges to zero where $Z_2 = (z_2^0, z_{21}^1, \cdots)^T$ is a vector composed by the state variables of the chain.
- The input v_2 is chosen such that $||Z_2||$ exponentially converges to zero.

Therefore, the extension of the time-varying piecewise continuous control to the case of 3-inputs, 2-chains systems may simply be performed by taking into account the influence of both Z_2 and Z_3 (Z_3 is defined analogously to Z_2) in the computation of v_1 and choose v_3 in the same manner as v_2 i.e., such that $||Z_3||$ exponentially converges to zero.

Before treating the case of the chained system (5.9), we define Z, Z_2, Z_3 as follows:

$$\begin{aligned} Z &= (z_2^0, z_{21}^1, z_3^0, z_{31}^1)^T, \\ Z_2 &= (z_2^0, z_{21}^1)^T, \\ Z_3 &= (z_3^0, z_{31}^1)^T. \end{aligned}$$

Then, we propose the following time-varying piecewise continuous control (see [6] for details):

$$\begin{aligned} v_1 &= k_1(z(k\delta))f(t), \\ v_2 &= -(\lambda_2 + \lambda_3 f^3)z_2^0 - \lambda_3(\lambda_2 f^2 + 2f\dot f)\tfrac{1}{k_1(z(k\delta))}z_{21}^1, \\ v_3 &= -(\lambda_4 + \lambda_5 f^3)z_3^0 - \lambda_5(\lambda_4 f^2 + 2f\dot f)\tfrac{1}{k_1(z(k\delta))}z_{31}^1, \end{aligned} \tag{5.10}$$

with

$$\begin{aligned} & f(t) = \tfrac{1-\cos\omega t}{2}, \ \omega = \tfrac{2\pi}{\delta}, \\ & k_1(z(k\delta)) = -\left[z_1^0(k\delta) + sgn(z_1^0(k\delta))\gamma\left(||Z(k\delta)||\right)\right]\beta, \\ & \beta^{-1} = \textstyle\int_{k\delta}^{(k+1)\delta} f(\tau)d\tau, \\ & \gamma\left(||Z(k\delta)||\right) = \\ & \qquad \alpha\left(z_2^0(k\delta)^2 + z_{21}^1(k\delta)^2 + z_3^0(k\delta)^2 + z_{31}^1(k\delta)^2\right)^{\frac{1}{4}} \end{aligned}$$

and λ_2, λ_3, λ_4, λ_5, δ, α are positive constants.

Simulation results are displayed in figure 5.1 where the graphics are successively time plots of the state in the original coordinates x and in the transformed coordinates z and the phase plot of y_p versus x_p. One can check that in the z-coordinates, the convergence is ensured with an exponential rate (see [6]).

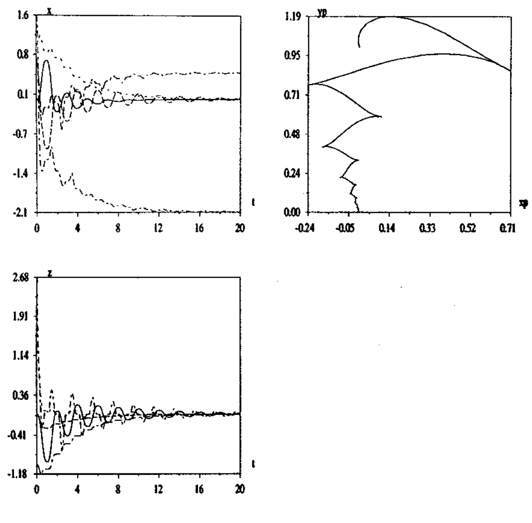

Figure 5.1: Resulting trajectories when control (5.1) is applied to WMR $(1, 2)$ over 20 seconds, with initial conditions $x_0 = (0, 1, 0, \pi/2, \pi/4)^T$, with desired configurations $x_d = (0, 0, 0, 1[\pi], .5[\pi])^T$ and $L = 1$, $\lambda_2 = \lambda_4 = 4$, $\lambda_3 = \lambda_5 = 1$, $\delta = 1$ second, $\alpha = 1$.

6 Conclusion

We have shown that the kinematic models of all restricted mobility WMR made of a single platform can be transformed generically by diffeomorphism and state feedback into chain form. Actually, the term "generically" can be avoided in the case of the WMR $(2, 0)$ (unicycle-type) and $(2, 1)$ because their transformation is always defined and non singular. Therefore any result such as those concerning open-loop strategies [7, 2, 13] can be applied to any WMR considered in section 1.

Those concerning stabilizing control laws with exponential rates of convergence in the case of 2-inputs, 1-chain chained systems [10, 8] can directly be applied to WMR with two inputs (WMR $(2, 0)$ and $(1, 1)$).
Notice that the equivalent kinematic chain form of the WMR $(2, 1)$ (see section 3.2) can be viewed as two independent subsystems: a 3-dimensional, 2-inputs, 1-chain chained system and a 1-dimensional linear system. Thus, its stabilization can be treated with existing exponentially stabilizing control laws.
Moreover, we have proposed a straightforward extension of a stabilizing control law for the WMR $(1, 2)$ modeled in chain form. Therefore, by means of transformation into chain form, we have given a way to stabilize all restricted mobility WMR with a single platform with an exponential rate of convergence.

Finally, it would be interesting to study the use of the transformations into chain form technics together with the cascade structure properties to deduce stabilizing control laws for the dynamical models of restricted mobility WMR.

References

[1] B. d'Andréa-Novel, G. Campion, and G. Bastin. Control of nonholonomic wheeled mobile robots by state feedback linearization. To appear in *International Journal of Robotics Research*, 1995.

[2] L. G. Bushnell, D. M. Tilbury, and S. S. Sastry. Steering three-input chained form nonholonomic systems using sinusoids: the fire truck example. In *ECC*, pages 1432–1437, Grenoble, 1993.

[3] G. Campion, G. Bastin, and B. d'Andréa-Novel. Structural properties and classification of kinematic and dynamic models of wheeled mobile robots. To appear in the *Transactions on Robotics and Automation*, 1995.

[4] C. Canudas de Wit, H. Khennouf, C. Samson, and O. J. Sørdalen. *Nonlinear control design for mobile robots*. World scientific, 1993. Recent Trends in Mobile Robots.

[5] A. Isidori. *Nonlinear control systems*. Springer-Verlag, second edition, 1989.

[6] W. Leroquais. Transformation of the kinematic models of restricted mobility wheeled mobile robots with a single platform into chain form. Research report, ENSMP/Paris, 1995.

[7] R. M. Murray and S. S. Sastry. Nonholonomic motion planning: steering using sinusoids. *IEEE TAC*, 38(5):700–716, May 1993.

[8] J. B. Pomet and C. Samson. Time-varying exponential stabilization of nonholonomic systems in power form. Rapport de recherche 2126, INRIA/Sophia-Antipolis, December 1993.

[9] C. Samson. Mobile robot control, part 2: control of chained systems and application to path following and time-varying point-stabilization of wheeled vehicles. Rapport de recherche 1994, INRIA/Sophia-Antipolis, July 1993.

[10] O. Sørdalen and O. Egeland. Exponential stabilization of chained nonholonomic systems. In *ECC*, pages 1438–1443, Grenoble, 1993.

[11] A. Teel, R. M. Murray, and G. Walsh. Nonholonomic control systems: from steering to stabilization with sinusoids. In *IEEE CDC*, pages 1603–1609, Tucson, December 1992.

[12] B. Thuilot, B. d'Andréa-Novel, and A. Micaelli. Modeling and feedback control of mobile robots equipped with several steering wheels. Submitted to the *IEEE Transactions on Robotics and Automation*. First review, 1995.

[13] D. Tilbury and A. Chelouah. Steering a three-input nonholonomic system using multi-rate controls. In *ECC*, pages 1428–1431, Grenoble, 1993.