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Efficient nonlinear model predictive control algorithms

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Abstract

This paper provides a review of computationally efficient approaches to nonlinear model predictive control. The methods considered cover the following areas: tailoring of nonlinear programming algorithms to the structure of the online optimization, use of the optimal control formulation of the receding horizon problem, constraint and cost approximations based on state space partitioning, and reparameterization of the degrees of freedom in predictions.

Keywords: Receding horizon control; Nonlinear programming; Computational efficiency

1. Introduction

Model Predictive Control (MPC) requires the online solution of a receding horizon optimization problem, and in order to be implementable in practice, the optimization must be performed within the time constraints dictated by the sampling period of an application. Computational efficiency is a critical consideration in the design of nonlinear model predictive control (NMPC) strategies since the incorporation of nonlinear (as opposed to linear) dynamic models into the MPC formulation results in a nonconvex nonlinear programming problem of significantly increased complexity. The development of NMPC formulations and computationally efficient algorithms for their implementation have therefore proceeded simultaneously, and computational efficiency has now become a major driving force behind NMPC research.

This paper aims to review the wide range of currently available computationally efficient NMPC strategies. Dividing this range into approaches based on the development of optimization algorithms (which generally derive from the optimization literature) and methods based on modifying the NMPC formulation (generally appearing in the control literature), we give below an overview of the following areas.

- Direct optimization methods, which adapt nonlinear programming algorithms to the NMPC receding horizon problem. These are variants of sequential quadratic programming (SQP) algorithms applied either sequentially (in which predicted trajectories of the model are computed at each iteration, including methods based on successive linearization of the prediction model), or simultaneously (in which the system state is treated as an optimization variable, and the model dynamics are incorporated as constraints).
- Euler–Lagrange and Hamilton–Jacobi–Bellman (HJB) approaches, which avoid the direct optimization of predicted control trajectories used in conventional MPC strategies by searching online for numerical solutions to optimal control formulations of the NMPC receding horizon problem.
- Methods based on modifying or approximating the cost and constraints of the receding horizon optimization. These include the use of feasible sets (computed offline) for optimization variables in conjunction with cost approximations, and feasible sets used in combination with cost bounds corresponding to partitions of the state space determined offline.
- NMPC strategies in which the the degrees of freedom in predictions are re-parameterized in order to reduce the

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size of the receding horizon optimization, including strategies employing interpolation between feedback laws computed offline.

1.1. Problem formulation

A discrete-time system model is assumed, with dynamics and constraints on state variables ($x \in \mathbb{R}^n$) and inputs ($u \in \mathbb{R}^m$) of the form:

$$x(k+1) = f(x(k), u(k)) \tag{1a}$$

$$x(k) \in \mathcal{X}, \quad u(k) \in \mathcal{U} \tag{1b}$$

where f is twice continuously differentiable, and $\mathcal{U}$, $\mathcal{X}$ are polytopic sets defined by linear inequalities. The objective in the NMPC cost is taken to be quadratic, and (unless otherwise stated) has the form:

$$J(\boldsymbol{x}(k), \boldsymbol{u}(k)) = \sum_{i=0}^{N-1} (\|x(k+i|k)\|_Q^2 + \|u(k+i|k)\|_R^2) + \|x(k+N|k)\|_{\bar{Q}}^2 \tag{2}$$

where $x(k+i|k)$, $u(k+i|k)$ are predicted values at time k of $x(k+i)$, $u(k+i)$, and $\boldsymbol{x}, \boldsymbol{u}$ are vectors of predicted state and input sequences:

$$\begin{aligned} \boldsymbol{x}(k) &= [x(k+1|k)^T \cdots x^T(k+N|k)]^T \\ \boldsymbol{u}(k) &= [u(k|k)^T \cdots u^T(k+N-1|k)]^T. \end{aligned}$$

For simplicity Q, R are taken to be positive definite matrices, $\bar{Q}$ defines a suitable terminal weight, and $\|x\|_Q^2 = x^T Q x$. The general form of NMPC law corresponding to (1) and (2) is then defined by the solution at each sampling instant of the following problem

$$\min_{\boldsymbol{u}(k), \boldsymbol{x}(k)} \quad J(\boldsymbol{x}(k), \boldsymbol{u}(k)) \tag{3a}$$

$$\text{s.t.} \quad x(k+i|k) \in \mathcal{X}, \quad u(k+i|k) \in \mathcal{U} \tag{3b}$$

$$x(k+i+1|k) = f(x(k+i|k), u(k+i|k)), \quad i = 0, \ldots, N-1 \tag{3c}$$

$$x(k+N|k) \in \Omega_T \tag{3d}$$

where $x(k+N|k) \in \Omega_T$ is a terminal stability constraint, for some suitable ellipsoidal or polytopic set Ω_T.

2. Direct methods

A variety of techniques have been developed for the direct solution of (3) based on modifying SQP algorithms for general nonlinear programming to the particular form of the receding horizon NMPC problem. These approaches attempt to compute an optimal control sequence by splitting (3) into a sequence of convex Quadratic Programming (QP) problems, which are solved to generate a sequence of iterates converging to an optimal solution.

Consider (3) rewritten in the form

$$\min_z \quad J(z) \tag{4a}$$

$$\text{s.t.} \quad c_{\mathrm{E}}(z) = 0, \quad c_{\mathrm{I}}(z) \le 0 \tag{4b}$$

where $z = [\boldsymbol{x}^T \boldsymbol{u}^T]^T$ with $\boldsymbol{x}$ and $\boldsymbol{u}$ defined as a vectors of state and input predictions at current time, and where $c_{\mathrm{E}} = 0$, $c_{\mathrm{I}} \le 0$ invoke the equality and inequality constraints implied by (3c) and (3b,d) respectively. An optimal solution for (4) satisfies the first order necessary (Kuhn–Tucker) conditions:

$$\nabla \mathcal{L}(z^*, \lambda^*, s^*) = 0, \quad s^* \ge 0, \quad \lambda_{\mathrm{I}}^* \ge 0, \quad \lambda_{\mathrm{I}}^{*T} s^* = 0 \tag{5}$$

where s is a vector of slack variables, $\lambda = [\lambda_{\mathrm{E}}^T \lambda_{\mathrm{I}}^T]^T$, and

$$\nabla = \begin{bmatrix} \nabla_z \\ \nabla_\lambda \end{bmatrix}, \qquad \mathcal{L}(z, \lambda, s) = J(z) + \lambda^T \begin{bmatrix} c_{\mathrm{E}}(z) \\ c_{\mathrm{I}}(z) + s \end{bmatrix}. \tag{6}$$

Here $\mathcal{L}$ is the Lagrange function for (4) and $\lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}$ are Lagrange multipliers associated with the equality and inequality constraints respectively in (4). SQP approaches can be derived from the application of Newton's method to the problem of determining a stationary point of $\mathcal{L}$ satisfying (5) (see, e.g. Fletcher, 1987). In its basic form this results in a sequence of iterates $\{(z^{(k)}, \lambda^{(k)}, s^{(k)})\}$, in which $(z^{(k+1)}, \lambda^{(k+1)}, s^{(k+1)})$ are determined by solving at the kth iteration for $(d^{(k)}, \delta^{(k)}, s^{(k+1)})$ satisfying:

$$\nabla^2 \mathcal{L}(z^{(k)}, \lambda^{(k)}, s^{(k+1)}) \begin{bmatrix} d^{(k)} \\ \delta^{(k)} \end{bmatrix} = -\nabla \mathcal{L}(z^{(k)}, \lambda^{(k)}, s^{(k+1)}),$$

$$s^{(k+1)} \ge 0, \quad \lambda_{\mathrm{I}}^{(k)} + \delta_{\mathrm{I}}^{(k)} \ge 0, \quad (\lambda_{\mathrm{I}}^{(k)} + \delta_{\mathrm{I}}^{(k)})^T s^{(k+1)} = 0 \tag{7a}$$

and defining

$$\begin{aligned} z^{(k+1)} &= z^{(k)} + d^{(k)} \\ \lambda^{(k+1)} &= \lambda^{(k)} + \delta^{(k)}. \end{aligned} \tag{7b}$$

To limit the step size (and hence the error introduced by the quadratic approximation to $\mathcal{L}$ implicit in (7)), the iteration (7a) is usually formulated as an equivalent QP subproblem:

$$\min_{d^{(k)}} \quad \nabla_z J(z^{(k)})^T d^{(k)} + \frac{1}{2} d^{(k)^T} B(z^{(k)}, \lambda^{(k)}) d^{(k)} \tag{8a}$$

$$\text{s.t.} \quad \nabla_z c_{\mathrm{E}}(z^{(k)})^T d^{(k)} + c_{\mathrm{E}}(z^{(k)}) = 0 \tag{8b}$$

$$\nabla_z c_{\mathrm{I}}(z^{(k)})^T d^{(k)} + c_{\mathrm{I}}(z^{(k)}) \le 0 \tag{8c}$$

The solution $d^{(k)}$ determines a search direction which can be used to determine the next iterate $z^{(k+1)}$ via a line search. In (8) B is either taken to be the Hessian w.r.t. z of the Lagrange function $\mathcal{L}$, i.e.

$$B(z, \lambda) = \nabla_z^2 \mathcal{L} = \nabla_z^2 J(z) + \sum_i \lambda_i \nabla_z^2 c_i(z, \lambda),$$

or B is defined as a suitable approximation of $\nabla_z^2 \mathcal{L}$. Use of an approximate Hessian for B can enable the calculation of second order derivatives of the cost and constraints to be avoided, thus allowing a reduction in computational complexity, though at the expense of reduced rate of convergence near the solution (see the discussion in Biegler (2000)). More importantly, $\nabla_z^2 \mathcal{L}$ is not positive definite in general, and use of a quasi-Newton approximation for B results in a convex and easily solvable SQP subproblem (8).

The NMPC optimization (3) is a nonconvex problem due to the inclusion of the nonlinear prediction dynamics (3c), and this causes significant difficulties for the online implementation of direct methods based on SQP. In particular a control sequence must be computed online within strict time constraints, whereas the number of iterations required to reach a given solution accuracy is unknown. Secondly, convergence to a globally optimal solution cannot generally be guaranteed, and this has implications for the closed-loop stability of the NMPC strategy since stability guarantees are dependent on a sufficient rate of decrease in the value of the receding horizon objective (Mayne, Rawlings, Rao, & Scokaert, 2000; Scokaert, Mayne, & Rawlings, 1999). Clearly such requirements are unlikely to be met if the NMPC optimization may converge to any one of several locally optimal solutions with different associated prediction costs. To overcome these difficulties a number of techniques have been developed to customize the SQP approach to the NMPC problem (3), a summary of which is as follows.

- *Warm starting*. In this approach the solver is initialized at an approximate solution, usually determined from the solution of the receding horizon optimization computed at a previous sampling instant. As well as providing a feasible (or near feasible) initial point and reducing the number of iterations needed to obtain a given solution tolerance, this can prevent the solver converging to a suboptimal local solution and therefore allows the conventional MPC stability guarantees based on the recurrence of feasibility to hold.
- *Early termination*. If the solver retains a feasible solution at each SQP iteration, then it is possible to enforce compliance with constraints on the computation time for the online NMPC optimization by terminating the optimization procedure before convergence to an optimal solution is reached. When combined with warm starting based on a feasible suboptimal solution, and implemented using an approach that accounts for approximation errors, this approach can enable the closed-loop stability and convergence of the MPC scheme to be guaranteed even if time constraints in fast sampling applications restrict the solver to a single iteration on each sampling interval (Cannon & Kouvaritakis, 2002; Lee, Kouvaritakis, & Cannon, 2003).
- *Exploitation of SQP subproblem structure*. In this approach the computational burden of each SQP iteration is reduced by exploiting the structure of the Hessian $\nabla^2 \mathcal{L}$ appearing in the Newton iteration (7). In order to solve the SQP subproblem (8) it is necessary to factorize this matrix (or a reduced form if the equality constraints (8b) are eliminated), and the use of computationally efficient sparse factorization routines can therefore provide significant reductions in the computational burden of the SQP subproblem.

The solution methods used in direct optimization in NMPC can be classified according to how they handle the nonlinear equality constraints (3c) associated with the evolution of state predictions. In sequential approaches the equality constraints (3c) are eliminated from the SQP subproblem by simulating the model dynamics at each iteration to generate a sequence of state predictions, together with additional information on the derivatives of the cost and inequality constraints. Simultaneous approaches employ the alternative strategy of retaining state predictions as optimization variables, and searching simultaneously for a solution to both the model equations and the optimal control sequence. The two approaches have distinct advantages and disadvantages, which are summarized in the following section.

2.1. Sequential methods

The sequential approach, which is also known as single shooting, ensures that a feasible (or near feasible) solution to the model equations is available at each iteration. By restricting the step size in the SQP iteration (e.g. through an appropriate line search or trust region method (Biegler, 2000)), it is therefore possible to retain at each iteration a control sequence that is feasible with respect to input/state constraints (3b) and stability constraints (3d), provided that a feasible control sequence is available for initializing the solver. This makes it possible to use suboptimal early termination strategies with sequential SQP solvers.

If the constraint curvature terms in $\nabla_z^2 \mathcal{L}$ are omitted from B, so that B has the simpler (Gauss-Newton) form in (8):

$$B(z, \lambda) = B(z) = \nabla_z^2 J(z),$$

then the sequential SQP method becomes equivalent to a successive linearization strategy. This is an iterative approach in which the nonlinear model dynamics are linearized at each iteration about the predicted trajectories for the state and control variables obtained from the previous iteration, and the resulting linear time varying system is used in place of (3c) in the receding horizon optimization (3). If the terminal constraint set T is either polytopic (Lee et al., 2003) or a target state (Brooms & Kouvaritakis, 2000; Kouvaritakis, Cannon, & Rossiter, 1999), then with this modification (3) becomes a convex QP, which is solved at each iteration.

In order to ensure closed-loop asymptotic stability, the cost of (3a) computed on the basis of the nonlinear model

dynamics rather than an approximate linearized system must decrease sufficiently rapidly along closed-loop trajectories under the NMPC law. Without limits on the step size $\|d^{(k)}\|$, successive linearization methods may therefore require many iterations of the SQP subproblem before the linearization errors in the approximate cost become sufficiently small, requiring the solution of a large number of QP problems on each sampling interval. This difficulty is overcome in Lee et al. (2003), which requires the solution of only a single SQP-type subproblem at each sample, by adding variable polytopic bounds on predicted states and inputs in the SQP subproblem. Using local Lipschitz bounds on the nonlinear model, the additional state and input bounds are used to formulate an upper bound on the cost (3a) for the nonlinear model, which then is minimized in place of (3a) in the SQP subproblem, thus allowing a guarantee of closed-loop stability to be obtained on the basis of the SQP subproblem cost regardless of the magnitude of linearization errors. The additional input and state bounds introduced in this approach are treated as optimization variables, and the solution is therefore nonconservative in the limit of convergence to an optimum point for (3).

A disadvantage of the sequential approach is that, by eliminating the state variables from the SQP subproblem, it generally results in non-sparse gradient and Hessian matrices, which prevents the use of efficient factorization methods in the solution of (8). The computational complexity of each SQP iteration therefore increases approximately as N_u^3(Fletcher, 1987), where N_u is the number of degrees of freedom in predicted control sequences, making the sequential approach unattractive for large scale problems involving long horizons and/or many manipulated inputs. Furthermore, unstable model dynamics can result in numerical instability when initialized at poor starting points (although this problem can be avoided if a stabilizing reparameterization of the control variables of the form $u(k) = \kappa(x(k), v(k))$ is employed). If the system model is defined in continuous time, then a further disadvantage of sequential SQP is that the numerical solution for the predicted state trajectory at each iteration can be computationally costly (Martinsen, Biegler, & Foss, 2002).

2.2. Simultaneous methods

The SQP subproblem for the simultaneous approach involves a greater number of optimization variables than are needed in the sequential approach since the equality constraints and variables associated with the evolution of the model state are retained in the subproblem (8). However, this results in an almost block diagonal structure of the Hessian $\nabla^2\mathcal{L}$ which allows for the use of efficient sparse factorization routines (Biegler, 2000; Bock, Diehl, Leineweber, & Schloder, 2000), or Riccati-like recursion algorithms (Rao et al., 1998) in the solution of the SQP subproblem. Consequently the computational complexity of the SQP subproblem varies only linearly in the horizon length N, making the approach attractive for large scale problems. Further computational advantages may be obtained by using interior point solvers for the SQP subproblem (Bartlett, Wachter, & Biegler, 2000).

Unstable prediction dynamics are also handled effectively since the simultaneous approach uses a boundary value problem formulation. Simultaneous methods are likewise advantageous in the case of continuous time prediction models since numerical discretization methods (e.g. multiple shooting or orthogonal collocation) can be efficiently incorporated in the SQP algorithm (Bock et al., 2000; Martinsen et al., 2002; Muske et al., 2000). However a major disadvantage of the simultaneous approach is that early termination strategies cannot be used since the system model is only satisfied at the solution of the optimization problem, and feasible solution estimates are therefore not available prior to convergence.

3. Euler–Lagrange and HJB approaches

The conventional MPC strategy of directly optimizing predicted performance over a set of admissible predicted control trajectories is attractive due to its wide applicability, transparency, and constraint handling capabilities. However, the direct optimization approaches discussed in the previous section do not make use of the structure implicit in the optimal control formulation of the NMPC receding horizon optimization. As a result, naive parameterizations of predicted input trajectories generally require an infinite number of degrees of freedom in order to recover the continuous time optimal control law, even in the absence of input and state constraints. Another consequence is that a large number of iterations may be needed in a direct optimization approach (or the optimization routine may not even converge to a minimum point) before near optimal predicted performance is reached.

This section reviews several approaches to the online computation of numerical solutions to the optimal control formulation of NMPC. For simplicity we consider here a continuous time system subject to input constraints

$$\dot{x}(t) = f(x(t), u(t)) \tag{9a}$$

$$u(t) \in \mathcal{U}. \tag{9b}$$

The predictions $x(\tau, t)$, $u(\tau, t)$ of $x(t+\tau)$, $u(t+\tau)$ at time t are to be optimized with respect to a continuous time cost:

$$J(t) = \int_0^T l(x(\tau, t), u(\tau, t))d\tau + \phi(x(T, t)). \tag{10}$$

3.1. Euler–Lagrange approach

Define the Hamiltonian function H as

$$H(x, u, \lambda, \mu) = l(x, u) + \lambda^T f(x, u) + \mu^T c(u) \tag{11}$$

where $\lambda = \lambda(\tau, t)$ denotes the co-state, $\mu = \mu(\tau, t)$ is a multiplier for the control inequality constraint (9b), and $c(u(\tau, t)) \leq 0$ is equivalent to the constraint $u(\tau, t) \in \mathcal{U}$. Then the first order necessary conditions for the minimization of (10) subject to (9) are obtained as a two-point boundary value problem as (Bryson & Ho, 1975):

$$\frac{\partial}{\partial \tau} x(\tau, t) = \nabla_\lambda \mathcal{H}, \quad x(0, t) = x(t) \tag{12a}$$

$$\frac{\partial}{\partial \tau} \lambda(\tau, t) = -\nabla_x \mathcal{H}, \quad \lambda(T, t) = \nabla_x \phi(x(T, \tau)) \tag{12b}$$

$$\nabla_u \mathcal{H} = 0, \quad \mu(\tau, t) \begin{cases} = 0 & \text{if} c(u(\tau, t)) \leq 0 \\ \geq 0 & \text{if} c(u(\tau, t)) = 0 \end{cases} \tag{12c}$$

If the initial value of the co-state, $\lambda(0, t)$, is known, then the optimal predicted state and co-state trajectories can be obtained by simulating (12a) and (12b) forward over the horizon $\tau \in [0, T]$, with the optimal input trajectory $u(\tau, t)$ determined at each prediction time τ from (12c). However, $\lambda(0, t)$ is determined implicitly by the terminal condition in (12b), and $\lambda(0, t)$ cannot therefore be determined analytically in general.

To overcome this problem, in Ohtsuka and Fujii (1997), the terminal condition is considered as a nonlinear equation in $\lambda(0, t)$:

$$F(\lambda(0, t), x(t), T) = \lambda(T, t) - \nabla_x \phi(x(T, \tau)), \tag{13}$$

which is solved online for $\lambda(0, t)$ using a continuation method starting from the trivial solution

$$\lambda(0, 0) = \nabla_x \phi(x(0)) \tag{14}$$

corresponding to $T = 0$ at $t = 0$. The prediction horizon $T(t)$ is specified as a smooth function of time such that $T(0) = 0$ and $T(t) \to T_f$ as $t \to \infty$, where T_f is a desired horizon length for the receding horizon optimization. The continuation method is based on the implicit function theorem, and determines $\lambda(0, t)$ as a solution of the differential equation $\mathrm{d}F/\mathrm{d}t = 0$. To correct for errors introduced by numerical integration, a stabilization term: $-\zeta F$, for $\zeta > 0$, is added to the RHS, and the resulting differential equation for $\lambda(0, t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \lambda(0, t) = -(\nabla_\lambda F)^{-1} \left[\zeta F + \nabla_x F \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \frac{\partial F}{\partial T} \frac{\mathrm{d}}{\mathrm{d}t} T(t) \right] \tag{15}$$

A receding horizon control law can be determined online at each sampling instant through the following procedure:

Algorithm 1.

(i) Use the estimate of $\lambda(0, t)$ determined at the previous sample to solve (12) over the horizon T and compute the partial derivatives of $(x(T, t), \lambda(T, t))$ w.r.t. $(x(0, t), \lambda(0, t))$;
(ii) update the estimate of $\lambda(0, t)$ using (15) on the basis of this information;
(iii) compute $u(t) = u(0, t)$ from (12c).

Although the approach is described here in continuous-time, it can be implemented (Ohtsuka, 2004) using a discrete-time formulation based on finite-difference approximations to (12a–c). The development in Ohtsuka and Fujii (1997) and Ohtsuka (2004) does not consider inequality constraints explicitly, although extending the treatment to the input constraints considered here appears to present no difficulties. However, the approach suffers from the significant drawback that terminal state constraints are not included in the problem formulation. Moreover, the method of initially setting the horizon T to zero appears to be incompatible with the function of the terminal stability constraints employed in MPC, since it requires that the initial state $x(0)$ lies within the terminal region. As a result the receding horizon control law has no stability guarantee.

3.2. HJB approach

Let $J^*(x)$ denote the optimal value of the cost (10) subject to the model dynamics and constraints (9) with initial state x, and assume that J^* is continuously differentiable w.r.t. x. Also define $\hat{J}^*(x(\tau, t), \tau)$ as the optimal value of the cost to go at prediction time $t + \tau$ along trajectories predicted at time t:

$$\hat{J}(x(\tau, t), \tau) = \int_\tau^T l(x(s, t), u(s, t)) \mathrm{d}s + \phi(x(T, t)). \tag{16}$$

Then $J^*(x) = \hat{J}^*(x, 0)$, where, from the principle of optimality, $\hat{J}^*(x, \tau)$ necessarily satisfies the Hamilton–Jacobi–Bellman (HJB) nonlinear partial differential equation:

$$\min_{u \in \mathcal{U}} \left\{ \nabla_x \hat{J}^*(x, \tau)^T f(x, u) + l(x, u) \right\} = -\frac{\partial}{\partial \tau} \hat{J}^*(x, \tau), \tag{17}$$

and the minimizing argument $u^*(x, \tau)$ defines the associated optimal control law. Inclusion of a terminal constraint $\psi(x(T, t)) = 0$ in the problem leads to boundary conditions of the form $\hat{J}(x, T) = \phi(x)$ for all x such that $\psi(x) = 0$. Knowledge of the solution $\hat{J}^*(x, \tau)$ of the HJB equation would therefore allow the optimal receding horizon control law $u^*(x) = u^*(x, 0)$ to be computed online as a state feedback law simply by solving the minimization on the LHS of (17).

Closed-form solutions to (17) are not available in general, but numerical solutions for $\hat{J}^*(x, \tau)$ can be computed using a variety of techniques including the method of characteristics, finite element methods (Gonzalez & Rofman, 1985), or finite difference methods (Capuzzo Dolcetta, 1983). However, approaches of this kind, which attempt to find solutions to the HJB equation through a discretization of the state space, are extremely computationally demanding and require storage of very large quantities of data which must be recovered online to determine the control law $u^*(x)$.

An alternative approach that successively improves the performance of a stabilizing control law can be derived from the generalized Hamilton–Jacobi–Bellman (GHJB) equation:

$$\nabla_x J(x)^T f(x,u) + l(x,u) = 0. \tag{18}$$

The method is based on the following result. Given $J(x) = J^{(k)}(x)$ satisfying (18) for a control law $u = u^{(k)}(x)$ which is stabilizing on some compact set Ω, the function $u^{(k+1)}(x)$ that minimizes the LHS of (18) defines a control law with the property that the solution $J(x) = J^{(k+1)}(x)$ of (18) with $u = u^{(k+1)}(x)$ satisfies (Saridis & Lee, 1979)

$$J^{(k+1)}(x) \le J^{(k)}(x) \quad \forall x \in \Omega. \tag{19}$$

Furthermore the solution $J^{(k)}(x)$ of the GHJB equation (18) with $u = u^{(k)}(x)$ is the value of the cost (10) evaluated over an infinite horizon ($T \to \infty$) under the control law $u^{(k)}(x)$, and (19) therefore implies that $u^{(k+1)}(x)$ achieves a cost no worse than that associated with $u(x)$.

This suggests an iterative procedure of successively solving (18) for $J^{(k)}(x)$ and recomputing $u^{(k)}(x)$:

Algorithm 2 (*iterative GHJB solution*).

(i) Solve (18) for $J(x) = J^{(k)}(x)$ given $u = u^{(k)}(x)$;
(ii) Define $u^{(k+1)}(x)$ via

$$u^{(k+1)}(x) = \arg\min_{u \in \mathcal{U}} \left\{ \nabla_x J^{(k)}(x,\tau)^T f(x,u) + l(x,u) \right\};$$

(iii) Set $k := k+1$ and return to (i).

Using (19), it can be shown that this iteration results in a sequence of control laws $u^{(k)}(x)$ converging (uniformly on Ω) to the optimal control law $u^*(x)$ for the infinite horizon control problem.

The iterative GHJB approach is attractive since (18) defines a linear PDE in $J(x)$ and is therefore easier to solve than the nonlinear HJB equation (17). The control law $u(x)$ computed at each iteration also asymptotically stabilizes (9) at $x = 0$ with region of attraction Ω if the stage cost $l(x,u)$ is positive definite in x (or if the state of (9) is observable from $l(x,u)$), and the procedure can therefore be terminated prior to convergence to $u^*(x)$. However, a major disadvantage of the approach is that it requires the solution of (18) at each iteration. In Beard, Saridis, and Wen (2000) the solution of the GHJB is approximated offline over all of Ω via Galerkin's method of weighted residuals. A more efficient online solution method is proposed in Curtis and Beard (2001), which uses orthogonal collocation to derive a quadratic approximation to $J^*(x)$ in a small local region containing the current state. The approach is modified to allow for input constraints (9b) in (Abu-Khalaf & Lewis, 2002) through the use of a specific form of stage cost.

Compared with conventional NMPC (which does not require the optimal value of predicted cost to be determined as a function of x), the computational burden of currently available methods for the GHJB successive approximation approach remains prohibitive.

4. Cost and constraint approximation

In order to construct NMPC strategies suitable for fast sampling applications that require the receding horizon optimization to be performed in times of the order of milliseconds, the online computational burden must be reduced to a level less than or comparable to that of a small quadratic or linear programming problem. This can be achieved at the cost of a degree of suboptimality by determining offline approximations of the the objective and feasible regions for optimization variables. Here we consider several methods that use Affine Difference Inclusion (ADI) representations of the model (1) to compute invariant or reachable sets. These enable convex representations of the cost and constraints in the online NMPC optimization to be determined.

Using ADI, the model (1) is represented locally on compact sets $\Pi_x \subset \mathcal{X}$, $\Pi_u \subset \mathcal{U}$ of the state and input space (here Π_x, Π_u are taken to be polytopic) as an uncertain affine time-varying system the inclusion

$$f(x,u) \in \mathrm{Co}\{f_0 + A_i x + B_i u, i = 1,\dots,p\}$$
$$\forall\, (x,u) \in \Pi_x \times \Pi_u \tag{20}$$

where Co denotes the convex hull. This representation considerably simplifies the construction of invariant or reachable sets since the conditions for inclusion of $x(k+1)$ within a polytopic or ellipsoidal set then reduce to linear or quadratic constraints on $x(k)$, $u(k)$. Procedures for computing suitable sets of linear models exist, e.g. if f is continuously differentiable w.r.t. x, u (Bacic, Cannon, & Kouvaritakis, 2003; Boyd, Diehl, Leineweber, & Schloder, 1994).

Invariant sets are necessarily defined with respect to the dynamics of a closed-loop model, and in order to approximate input and state constraints over the MPC prediction horizon using ellipsoidal invariant sets Kothare, Balakrishnan, and Morari (1996) parameterizes predictions in terms of a variable linear state feedback law. This results in an online semi-definite programming (SDP) problem, the computational burden of which is prohibitive for fast-sampling applications. An alternative which allows for significant reductions in online computational load (Cannon, Kouvaritakis, Lee, & Brooms, 2001; Kouvaritakis, Cannon, & Rossiter, 1999) is to introduce the degrees of freedom in predictions (denoted $c(i|k)$, $i = 0,\dots,N-1$) as perturbations on a fixed-term state feedback law $u = k(x)$:

$$u(k+i|k) = \begin{cases} \kappa(x(k+i|k)) + c(i|k) & i = 0,\dots,N-1 \\ \kappa(x(k+i|k)) & i \ge N \end{cases} \tag{21}$$

and then determine an invariant set for the model state augmented by the vector of degrees of freedom in

predictions:

$$z(k) = \begin{bmatrix} x(k) \\ \boldsymbol{c}(k) \end{bmatrix}, \qquad \boldsymbol{c}(k) = \begin{bmatrix} c(0|k) \\ \vdots \\ c(N-1|k) \end{bmatrix},$$

$$z(k+1) = \begin{bmatrix} f(x(k), \kappa(x(k)) + c(0|k)) \\ M\boldsymbol{c}(k) \end{bmatrix} \tag{22}$$

where M can be any matrix with eigenvalues in the closed unit disc. Use of the ADI representation (20) (with $f_0 = 0$) enables an ellipsoidal invariant set $\mathcal{E}$ for the augmented state z to be computed offline through a SDP which maximizes the volume of the projection $\mathcal{E}_x$ of $\mathcal{E}$ onto the x-subspace:

$$\mathcal{E}_x = \{x : \exists \boldsymbol{c}, [x^T \boldsymbol{c}^T]^T \in \mathcal{E}\}$$

subject to invariance, feasibility w.r.t. input and state constraints (3b,d), and $\mathcal{E}_x \subset \Pi_x$. If $u = \kappa(x)$ is optimal for the unconstrained system, then the NMPC online optimization can be formulated as the solution of:

$$\boldsymbol{c}(k) = \arg\min_{\boldsymbol{c}} \quad \|\boldsymbol{c}\|_W^2 \tag{23a}$$

$$\text{s.t.} \quad z = \begin{bmatrix} x(k) \\ c \end{bmatrix} \in \mathcal{E} \tag{23b}$$

for suitably chosen weight W. It can be shown (Cannon, Kouvaritakis, Grimble, & Bulut, 2003a) that the corresponding control law (21) asymptotically stabilizes $x = 0$ and forces $u(k)$ to converge in finite time to $u = \kappa(x)$. Although conservative due to the use of a constraint approximation based on a global ADI and a global quadratic cost approximation, the approach has the advantage that the online optimization is performed at a fraction of the computational cost of a single QP. The online optimization consists of a univariate Newton–Raphson iteration with complexity $O(N)$ and guaranteed quadratic convergence rate from a known initial point (Kouvaritakis et al., 2002).

Polytopic invariant sets allow less conservative constraint approximations than ellipsoidal sets since the use of a sufficiently large number of vertices allows for an arbitrarily close approximation of the feasible sets for the model state and degrees of freedom in predictions. Furthermore the conservativeness of ADI representations can be reduced by matching the polytopic region of validity of the ADI with the boundary of the polytopic invariant or reachable set. An algorithm for determining low complexity polytopic invariant sets of the form

$$\Pi = \{x : \|Vx\|_\infty \leq 1\}, \quad V \in \mathbb{R}^{n \times n}$$

and linear associated feedback laws concurrently with a global ADI is proposed in Cannon, Deshmukh, and Kouvaritakis (2003b). Extensions to the case of invariant polytopes with arbitrary pre-specified numbers of vertices:

$$\Pi = \{x : Vx \leq 1\}, \quad V \in \mathbb{R}^{r \times n},$$

and corresponding nonlinear control laws determined using local ADI are given in Cannon, Kouvaritakis, and Deshmukh (2004). In this work the maximization of the volume of the polytopic set subject to bounds on closed-loop cost is performed by solving a sequence of linear programming (LP) problems, and the procedure is used iteratively to determine a sequence of polytopic reachable sets corresponding to a sequence of cost bounds. A nonlinear feedback law is constructed online by interpolating between controls associated with vertices through the solution an online LP, and the resulting feedback law is optimal with respect to bounds on the closed-loop cost. Alternatively the sequence of partially invariant sets and interpolated feedback laws are incorporated into an MPC law based on a piecewise linear terminal cost and polytopic terminal state constraint.

Similar approaches are used to construct polytopic sets that are invariant under feedback linearizing control laws in Bacic, Cannon, and Kouvaritakis (2003a) for bilinear systems, and in Bacic, Cannon, and Kouvaritakis (2003c) for more general input affine systems of the form

$$x(k+1) = f(x(k)) + g(x(k))u(k).$$

This approach is motivated by the fact that, in the absence of input and state constraints, feedback linearization provides a computationally convenient means of determining the optimal control law with respect to a performance cost penalizing the plant output $y(k) = h(x(k))$ alone:

$$J(k) = \sum_{i=0}^{\infty} \|y(k+i|k)\|_Q^2. \tag{24}$$

The absence of a penalty term on the input in this cost implies that the feedback linearizing control law may not stabilize the plant state for all initial conditions in the desired operating region. This is addressed in Bacic et al. (2003a) by switching to a stabilizing feedback law whenever a state convergence condition is violated.

An offline dynamic programming algorithm is used in Bacic et al. (2003b) to reduce the online receding horizon optimization to the minimization of a single-stage cost. This involves partitioning the operating region of state space into simplexes through a sequential optimization procedure that balances the requirements of:

(i) maximal simplex volume;
(ii) minimal stage cost under a control law that maps the simplex subject to constraints (3b) into an invariant set constructed from previously computed simplexes.

Using local ADI model representations, this procedure can be formulated as a sequence of QP problems. For each simplex, an upper bound on stage cost of the state transition to every other simplex is then computed via a sequence of QPs. These upper bounds are used to construct a directed graph giving the optimal path from each simplex and an upper bound on the associated stage cost. Online, the current stage cost (computed using the actual nonlinear model) is

minimized subject to constraints (3b) and the constraint that $x(k+1)$ lie in the optimal one-step-ahead simplex. For input affine systems the online optimization therefore reduces to a QP problem. This approach provides a method of solving approximately the HJB equation using a discretization of the state space. However, the construction of state space partitions as reachable sets enables performance to be traded off against the size of each partition without affecting closed-loop stability under the control law computed online, thus allowing for the possibility of using small numbers of partitions.

5. Degrees of freedom in predictions

The parameterization of predictions in terms of degrees of freedom in the receding horizon optimization directly affects the size of the online optimization problem, and therefore also the computational burden of the NMPC strategy. Conventional MPC algorithms, which employ the values of future control variables as the degrees of freedom in the online optimization, are subject to a trade-off of performance and size of stabilizable set against computational load: short horizons are computationally less expensive but result in poor performance and small stabilizable initial condition sets. However, this compromise can be avoided to a large extent by parameterizing input and state predictions in terms of predicted trajectories under given feedback laws.

An extremely computationally efficient univariate online optimization can be derived by specifying predicted inputs as an interpolation between control laws (Bloemen, Cannon, & Kouvaritakis, 2002; Cannon & Kouvaritakis, 2001; Kouvaritakis, Rossiter, & Cannon, 1998; Kouvaritakis, Rossiter, & Schuurmans, 2000):

$$u(k+i|k) = (1-c(k))\kappa^o(x(k+i|k)) + c(k)\kappa^f(x(k+i|k)) \quad (25)$$

where $\kappa^o(x)$, $\kappa^f(x)$ are fixed feedback laws computed offline. If $\kappa^o(x)$ is optimal in the absence of constraints (but may violate constraints (3b)), and $\kappa^f(x)$ has a large associated stabilizable set (but may give poor closed-loop performance), then the interpolation MPC law defined by the optimization of predicted performance over $c(k)$ inherits both the desirable optimality and feasibility properties of $\kappa^o(x)$, $\kappa^f(x)$. Closed-loop stability can be ensured by incorporating a suitable terminal cost in the receding horizon objective or by including an explicit convergence constraint on the closed-loop system in the receding horizon optimization. By considering open-loop and inverse optimality properties, these approaches are compared in Cannon and Kouvaritakis (2001) for the case of input affine systems, with $\kappa^o(x)$ computed using feedback linearization. An interpolation NMPC law in which $\kappa^o(x)$ is computed using the successive linearization method is described in Cannon et al. (2003a).

Additional degrees of freedom are introduced in the receding horizon optimization proposed in Magni, De Nicolao, Magnani, and Scattolini (2001). This approach considers the predicted inputs over a horizon of N_c samples as degrees of freedom, then switches to an auxiliary control law $\kappa^f(x)$ for the remainder of a prediction horizon of length $N_p > N_c$. The receding horizon optimization has the conventional form of (3) with $N = N_p$. Use of a long horizon N_p is computationally cheap provided N_c remains small. Invoking the terminal constraint (3d) N_p (rather than N_c) steps ahead therefore reduces the dependence of the stabilizable set for the NMPC law on the size of the target set Ω_T at little additional computational cost.

6. Conclusions

Advances in efficient SQP optimization techniques have widened the applicability of NMPC to encompass large scale problems involving high order systems with many manipulated inputs. The use of constraint and cost approximation approaches likewise makes suboptimal NMPC strategies viable for applications with sampling intervals of the order of milliseconds. Further improvements in computational efficiency are likely to be obtained through techniques that transfer more of the online computational burden to offline procedures. Methods based on the underlying optimal control problem present a promising means of achieving this without compromising the robustness of the NMPC law.

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