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Steering nonholonomic systems in chained form*

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Abstract

In this paper we introduce a nilpotent form, called chained form, for nonholonomic control systems. For the case of a nonholonomic system with two inputs, we give constructive conditions for the existence of a feedback transformation which puts the system into chained form, and show how to steer the system between arbitrary states. Examples are presented for steering a car and a car with a trailer attached; other examples can be found in the areas of space robotics and multi-fingered robot hands. The results of this paper also have applications in the area of nilpotentization of distributions of vector fields on $\mathbb{R}^n$.

1 Introduction

Consider the problem of steering a system with configuration $x \in \mathbb{R}^n$ subject to a set of independent kinematic constraints having the form

$$\omega_j(x)\dot{x} = 0 \quad j = 1, \cdots, k. \tag{1}$$

We assume the ω_j's are smooth and linearly independent over the ring of smooth functions. Formally, these constraints are exterior differential one-forms on $\mathbb{R}^n$. Such constraints can arise when two surfaces roll against each other, such as the rolling between a wheel and the road, or in space-based systems where the total angular momentum of the system is conserved. Although strictly speaking this latter case is not a "constraint", it can be treated with the same set of tools.

To study such a system, we convert the path planning problem into a control problem. Let Δ be a distribution of dimension $m = n - k$ which is annihilated by the constraints. We represent this distribution with respect to a basis of vector fields:

$$\Delta = \text{span}\{g_1, g_2, \cdots, g_m\} \qquad g_i(x) \in \mathbb{R}^n \tag{2}$$

In coordinates, the constraint one-forms can be written as an $k \times n$ matrix and the g_i's are a basis for the right null space of this matrix. The path planning problem can then be restated as finding an input function, $u(t) \in \mathbb{R}^m$ such that the control system

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \tag{3}$$

is driven from x_0 to x_1. As a consequence of our assumptions on the ω_j's, the g_i's are also smooth and linearly independent.

The condition for the existence of a path between two configurations is given by Chow's theorem. We let $[f, g]$ be the Lie bracket between two vector fields,

$$[f, g] = \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g,$$

and define the involutive closure of a distribution Δ as the closure of Δ under Lie bracketing. Briefly, Chow's theorem states that if the involutive closure of the distribution associated with equation (3) spans $\mathbb{R}^n$ at each configuration, the system can be steered between any two configurations. It is not apparent how the path can be explicitly constructed; in this paper we propose techniques for generating such paths.

We say that a system is *holonomic* if the kinematic constraints in (1) restrict the motion of the system to a manifold of dimension $n - k$. In this case, the constraints on the system can be rewritten as an algebraic constraint on the configuration variables x. A system is *nonholonomic* if it is not constrained to lie on a manifold of the same dimension as the input space. In particular, we are most interested in systems which are *maximally nonholonomic*: any point in the configuration space can be reached. This is equivalent to saying that the corresponding control system is controllable. If the system is not maximally nonholonomic, it can still be treated by restricting the initial and final configurations to lie on the same leaf of the foliation generated by the distribution.

It is possible to classify nonholonomic systems based on the way in which controllability is achieved. Define $G_1 = \Delta$ and

$$G_i = G_{i-1} + [G_1, G_{i-1}]$$

where

$$[G_1, G_{i-1}] = \text{span}\{[g, h] : g \in G_1, h \in G_{i-1}\}$$

A system is *regular* in a neighborhood U of x_0 if

$$\text{rank } G_i(x) = \text{rank } G_i(x_0) \qquad \forall x \in U.$$

If a system is regular, there exists an integer $p < n$ such that $G_i = G_{p+1}$ for all $i \geq p + 1$. We refer to p as the *degree of nonholonomy* of the distribution. The *growth vector* for a regular system is defined as $r \in \mathbb{Z}^{p+1}$, where $r_i = \text{rank } G_i$. We define the *relative growth vector* $\sigma \in \mathbb{Z}^{p+1}$ as $\sigma_i = r_i - r_{i-1}$ and $r_0 := 0$. The growth vector for a (regular) system is a convenient way to represent information about the associated control Lie algebra. See [5, 21] for more details.

Nonholonomic control systems cannot be studied using the usual linear control techniques. The linearization of a nonholonomic control system is degenerate; linearizing the system about a point gives:

$$\dot{x} = Ax + Bu \qquad A = 0,\ B = [g_1(0) \cdots g_m(0)]$$

which is clearly not controllable. Furthermore, the feedback linearization conditions fail for such systems due to the lack of a term which is independent of u. In particular, if the system is controllable, the distribution

$$\Delta_0 = \text{span}\{g_1, g_2\}$$

is not involutive, violating the necessary conditions for full state feedback linearization [8].

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Related work

There has been a large interest in the control of nonholonomic systems in the control and robotics literature. We mention here only a few of the papers which have influenced our work. A more complete set of citations can be found in [19, 22].

Much of the early work in nonholonomic motion planning was devoted to path planning for mobile robots. Laumond used a set of canonical paths to steer a cart, and later a cart with a trailer, to an arbitrary location in the presence of obstacles [13, 14, 15]. More recent work has been used for a mobile robot with bounded input constraints in addition to the nonholonomic constraint [9]. The approach presented in this latter paper may be applicable to more general systems. Another general algorithm, developed by Barraquand and Latombe, can reportedly handle any set of nonholonomic constraints [1]. Sample paths are presented for a front-wheel drive car and a car pulling a trailer. The paths generated are locally time-optimal, but can be computed only in sufficiently low-dimensional spaces. Yet another approach has been to train a neural net to park a car pulling a trailer [23].

Control theoretic approaches to to nonholonomic motion planning problems have also been explored, beginning with the work of Brockett [4]. Brockett showed that for a class of systems with degree of nonholonomy 1, the optimal controls consist of sinusoids at integrally related frequencies. In the robotics literature, Li and Canny studied the motion of a fingertip rolling on an object without slipping [16]. This problem has also been investigated using some of the methods presented here [20]. Later work by Li and others studied a hopping robot flipping in mid-air by using conservation of angular momentum to construct paths on a reduced space [17]. Similar techniques have also been used for studying the motion of coupled rigid bodies and space manipulators [25, 18]. Sussmann *et al.* have recently used Lie algebraic techniques for generating nonholonomic motions [12] and for approximating arbitrary trajectories with feasible ones [26].

Overview

The paper is organized as follows. Section 2 reviews our previous results in steering nonholonomic systems and constructs a special class of nonholonomic systems, called chained systems. Using the concept of chained systems, we give an algorithm which can be used to steer a system in chained form to an arbitrary location. Section 3 considers the problem of converting a nonholonomic system into chained form. We consider only the 2 input case, since most of the examples which have motivated our research are of this form. Section 3 also contains examples of how to convert systems into chained form.

2 Chained form

In our previous work [21], we used sinusoidal inputs to steer a class of nonholonomic systems. These systems had a special triangular form which allowed Fourier series techniques to be used to analyze the motion resulting from inputs which consisted of sines and cosines at integrally related frequencies. As a simple example, consider the system

$$
\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
\dot{x}_4 &= x_3 u_1 \\
\dot{x}_5 &= x_2 u_1
\end{aligned}
$$

For this system, the distribution

$$\{g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_2, [g_1, g_2]]\}$$

spans $\mathbb{R}^5$ and hence the system is controllable. The system has degree of nonholonomy 2 since it takes two levels of brackets to span the tangent space to the configuration manifold.

This system can be steered using sinusoidal inputs. The algorithm proceeds as follows. Using constant inputs, steer x_1 and x_2 to their desired values. During this motion, x_3, x_4 and x_5 will drift. Next, use $u_1 = a_3 \sin t$ and $u_2 = b_3 \cos t$ to move x_3 to its desired value. These inputs describe a closed loop in x_1 and x_2 and hence at the end of this motion the first 3 variables have obtained their desired values. Finally, using $u_1 = a_4 \sin t$, $u_2 = b_4 \cos 2t$ and $u_1 = a_5 \cos 2t$, $u_2 = b_5 \sin t$, it is possible to steer x_4 and x_5 while leaving all other states unchanged. Hence we have succeeded in steering the system to its desired location. Full details are contained in [21].

There are many advantages to using sinusoidal inputs for steering nonholonomic systems. The paths generated are piecewise smooth and consist of a relatively small number of segments (4 for the example above). Furthermore, the use of sinusoids allows some nonlinear systems to be treated with minor modifications to the algorithm given above. This leads to exact steering algorithms for certain nonlinear systems, in contrast to the approximate (iterative) method proposed by Lafferriere and Sussmann (for non-nilpotent systems).

It can be shown that simple sinusoids cannot be effectively used for steering arbitrary nonholonomic systems. In particular, it is possible to build systems using techniques formulated by Grayson and Grossman [6] which cannot be steered using sinusoidal inputs at single frequencies [19, 22]. The construction of these systems relies on the use of a P. Hall basis for a Lie algebra of with a fixed number of generators and a given degree of nilpotency.

Rather than explore the use of more complicated inputs for steering nonholonomic systems, we consider instead a simpler class of systems. The justification for changing the class of systems is simple—most of the systems encountered as examples do not have the complicated structure of the general case. Thus there may be a simpler class of systems which is both steerable using simple sinusoids and representative of systems in which we are interested.

Consider a two input system of the following form:

$$
\begin{aligned}
\dot{x}_0 &= u_1 & \dot{y}_0 &= u_2 \\
\dot{x}_1 &= y_0 u_1 & (y_1 &:= -x_1) \\
\dot{x}_2 &= x_1 u_1 & \dot{y}_2 &= y_1 u_2 \\
\dot{x}_3 &= x_2 u_1 & \dot{y}_3 &= y_2 u_2 \\
&\vdots & &\vdots \\
\dot{x}_{n_x} &= x_{n_x - 1} u_1 & \dot{y}_{n_y} &= y_{n_y - 1} u_2
\end{aligned}
\tag{4}
$$

or more compactly

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = X u_1 + Y u_2 \qquad
\begin{aligned}
X &= \frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} x_{i-1} \frac{\partial}{\partial x_i} \\
Y &= \frac{\partial}{\partial y_0} + \sum_{j=2}^{n} y_{j-1} \frac{\partial}{\partial y_j}
\end{aligned}
$$

where $y_1 := -x_1$ to account for skew-symmetry of the Lie bracket. We refer to this system as a *two-chain system*. The first item is to check the controllability of these systems. To this end, denote iterated Lie products as $\mathrm{ad}_X^k Y$:

$$\mathrm{ad}_X Y = [X, Y] \qquad \mathrm{ad}_X^k Y = [X, \mathrm{ad}_X^{k-1} Y] = [X, [X, \cdots, [X, Y] \cdots]]$$

Lemma 1 (Lie bracket calculations)
For the vector fields in equation (4)

$$\begin{aligned} \mathrm{ad}_X^k Y &= (-1)^k \frac{\partial}{\partial x_k} \\ \mathrm{ad}_Y^k X &= (-1)^k \frac{\partial}{\partial y_k} \end{aligned} \qquad k > 1$$

Proof. By induction. Since the first level of brackets is irregular, we begin by expanding $[X, Y]$ and $[X, [X, Y]]$.

$$\begin{aligned} [X,Y] =& \left(\frac{\partial}{\partial x_0} + y_0\frac{\partial}{\partial x_1} + \sum x_{i-1}\frac{\partial}{\partial x_i}\right)\left(\frac{\partial}{\partial y_0} + \sum y_{j-1}\frac{\partial}{\partial y_j}\right) - \\ & \left(\frac{\partial}{\partial y_0} + \sum y_{j-1}\frac{\partial}{\partial y_j}\right)\left(\frac{\partial}{\partial x_0} + y_0\frac{\partial}{\partial x_1} + \sum x_{i-1}\frac{\partial}{\partial x_i}\right) \\ =& 0 - \frac{\partial}{\partial x_1} \\ [X,[X,Y]] =& X(-\frac{\partial}{\partial x_1}) + \frac{\partial}{\partial x_1}(X) = 0 + \frac{\partial}{\partial x_2} \end{aligned}$$

Now assume that $\mathrm{ad}_X^k Y = (-1)^k \frac{\partial}{\partial x_k}$. Then

$$\begin{aligned} \mathrm{ad}_X^{k+1} Y &= [X, \mathrm{ad}_X^k Y] \\ &= (-1)^k \left(X(\frac{\partial}{\partial x_k}) - \frac{\partial}{\partial x_k}(X)\right) = (-1)^{k+1}\frac{\partial}{\partial x_{k+1}} \end{aligned}$$

The proof for $\mathrm{ad}_Y^k X$ is identical using the facts $[Y, X] = -[X, Y]$ and $y_1 := -x_1$. □

Proposition 1 (Controllability of the two-chain system)
The two-chain system (4) *is maximally nonholonomic (controllable).*

Proof. There are $2n-1$ coordinates in (4) and the $2n-1$ Lie products

$$\{X, Y, \mathrm{ad}_X^i Y, \mathrm{ad}_Y^j X\} \qquad i \geq 1, \quad j \geq 2$$

are independent using Lemma 1. We require $j \geq 2$ since $\mathrm{ad}_Y X = -\mathrm{ad}_X Y$ and hence those Lie products can never be independent. □

To steer this system, we use sinusoids at integrally related frequencies. Roughly speaking, if we use $u_1 = \sin t$ and $u_2 = \cos kt$ then $\dot{x}_1$ will have components at frequency $k-1$, $\dot{x}_2$ at frequency $k-2$, etc. $\dot{x}_k$ will have a component at frequency zero and when integrated we get motion in x_k while all previous variables return to their starting values. In the y variables, all frequency components will be of the form $m \cdot k \pm 1$ and hence we get no motion for $k > 1$. (For $k = 1$, y_1 and x_1 are the same variable). We make this precise with the following algorithm.

Algorithm 1

1. Steer x_0 and y_0 to their desired values.
2. For each x_k, $k \geq 1$, steer x_k to its final value using $u_1 = a \sin t$, $u_2 = b \cos kt$, where a and b satisfy
$$x_k(2\pi) - x_k(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi$$
3. For each y_k, $k \geq 2$, steer y_k to its final value using $u_1 = b \cos kt$, $u_2 = a \sin t$, where a and b satisfy
$$y_k(2\pi) - y_k(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi$$

Proposition 2 *Algorithm 1 can steer* (4) *to an arbitrary configuration.*

Proof. The proof is constructive. It suffices to consider only step 2 since step 3 can be proved by switching x and y in what follows. We must show 2 things:

1. moving x_k does not affect x_j, $j < k$
2. moving x_k does not affect y_j, $j = 1, \cdots, n_y$

To verify that using $u_1 = a \sin t$, $u_2 = b \cos kt$ produces motion only in x_k, we integrate the x states. If x_{k-1} has terms at frequency ω_i, then x_k has corresponding terms at $\omega_i \pm 1$ (by expanding products of sinusoids as sums of sinusoids). Since the only way to have $x_i(2\pi) \neq x_i(0)$ is to have x_i have a component at frequency zero, it suffices to keep track only of the lowest frequency component in each variable; higher components will integrate to zero. Direct computation starting from the origin yields

$$\begin{aligned} x_0 &= a(1 - \cos t) \\ x_1 &= \int \frac{ab}{k} \sin kt \sin t \\ &= \frac{1}{2}\frac{ab}{k(k-1)}\sin(k-1)t + \frac{1}{2}\frac{ab}{k(k+1)}\sin(k+1)t \\ x_2 &= \frac{1}{2^k}\frac{a^2 b}{k(k-1)(k-2)}\sin(k-2)t + \cdots \\ &\vdots \\ x_k &= \int \left(\frac{a^k b}{2^{k-1}k!}\sin^2 t + \cdots\right) dt = \frac{a^k b}{2^{k-1}k!}\frac{t}{2} + \cdots \end{aligned}$$

$x_k(2\pi) = x_k(0) + \frac{(a/2)^k b}{k!}\pi$ and all earlier x_i's are periodic and hence $x_i(2\pi) = x_i(0)$, $i < k$. If the system does not start at the origin, the initial conditions generate extra terms of the form $x_{i-1}(0)u_2$ in the i^{th} derivative and this integrates to zero, giving no net contribution.

To show that we get no motion in the y variables, we show that all frequency components in the y's have the form $mk \pm 1$ where m is some integer. This is true for $y_1 := -x_1$ from the calculation above. Assume it is true for y_i:

$$\begin{aligned} \dot{y}_{i+1} &= y_i u_2 \\ &= \sum_m \alpha(m)\sin(mk \pm 1)t \cdot \cos kt \\ &= \sum_m \frac{\alpha(m)}{2}\big(\sin((m+1)k \pm 1)t + \sin((m-1)k \pm 1)t\big) \end{aligned}$$

Hence y_{i+1} only has components at non-zero frequencies $m'k \pm 1$ and therefore $y_i(2\pi) = y_1(0)$. □

To include systems with more than two inputs, we replicate the structure of (4) for each additional input. Let h_{ij}^k represent the motion corresponding to the Lie product $\mathrm{ad}_{X_i}^k X_j$. In the two input case, $x_k = h_{21}^k$ and $y_k = h_{12}^k$. The following system on $\mathbb{R}^n$ is the *m-chain system*:

$$\begin{aligned} \dot{h}_j^0 &= u_j & j &= 1, \cdots, m \\ \dot{h}_{ij}^1 &= h_i^0 u_j & i &> j \text{ and } h_{ji}^1 := -h_{ij}^1 \\ \dot{h}_{ij}^k &= h_{ij}^{k-1} u_j \end{aligned} \tag{5}$$

Proposition 3 (Multi-chain system controllability)
The multi-chain system of (5) *is maximally nonholonomic and can be steered using sinusoids.*

Proof. The system (5) can be rewritten

$$\dot{h} = X_1 u_1 + \cdots + X_m u_m$$

with

$$X_j = \frac{\partial}{\partial h_j^0} + \sum_{\substack{i=1\\i<j}}^{m} h_i^0 \frac{\partial}{\partial h_{ij}^1} + \sum_k \sum_i h_{ij}^{k-1} \frac{\partial}{\partial h_{ij}^k}$$

Given any two X_i, X_j, their Lie product expansions only involve terms of the form h_{ij}^k for some k. But this is precisely the vector fields from Lemma 1 and hence

$$\mathrm{ad}_{X_i}^k X_j = (-1)^k \frac{\partial}{\partial h_{ij}^k}$$

Taking these terms for all possible i, j, k we get a set of independent Lie products just as in the proof of Theorem 1.

To show that the system can be steered using sinusoids, pick any $i, j \in \{1, \cdots, m\}$, $i < j$. Fix $u_l = 0$ for all $l \neq i, j$. The resulting system is identical to (4) can be steered using algorithm 1. By choosing all possible combinations of i and j, we can move to any position. □

3 Converting Systems to Chained Form

In this section we introduce a set of sufficient conditions for determining if a system can be converted to chained form. This set of conditions gives a constructive method for building a feedback transformation which accomplishes the conversion. We concentrate on the two input case with a single chain.

Proposition 4 (Converting systems to two-chained form) *Consider a controllable system*

$$\dot{x} = g_1(x) u_1 + g_2(x) u_2,$$

with g_1, g_2 linearly independent and smooth and having the special form

$$g_1(x) = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} g_1^i(x) \frac{\partial}{\partial x_i}$$
$$g_2(x) = \sum_{i=2}^{n} g_2^i(x) \frac{\partial}{\partial x_i}$$

(by appropriate change of basis, if necessary). Define

$$\Delta_0 := \mathrm{span}\{g_1, g_2, \mathrm{ad}_{g_1} g_2, \cdots, \mathrm{ad}_{g_1}^{n-2} g_2\}$$
$$\Delta_1 := \mathrm{span}\{g_2, \mathrm{ad}_{g_1} g_2, \cdots, \mathrm{ad}_{g_1}^{n-3} g_2\}$$

If for some open set U, $\Delta_0(x) = \mathbb{R}^n$ for all $x \in U \subset \mathbb{R}^n$ and Δ_1 is involutive on U, then there exists a local feedback transformation

$$\xi = \phi(x) \quad u = \beta(x) v$$

such that the transformed system is in chained form:

$$\begin{aligned}
\dot{\xi}_1 &= v_1\\
\dot{\xi}_2 &= v_2\\
\dot{\xi}_3 &= \xi_2 v_1\\
&\vdots\\
\dot{\xi}_n &= \xi_{n-1} v_1
\end{aligned}$$

Proof. Since Δ_1 is an involutive distribution of dimension $n-2$, there exists a function h such that $dh \cdot \Delta_2 = 0$ and $dh \cdot \mathrm{ad}_{g_1}^{n-2} g_2 \neq 0$. Define the map $\phi : x \mapsto \xi$ as

$$\begin{aligned}
\xi_1 &= x_1\\
\xi_2 &= L_{g_1}^{n-2} h\\
&\vdots\\
\xi_{n-1} &= L_{g_1} h\\
\xi_n &= h
\end{aligned}$$

To verify that ϕ is a valid change of coordinates, we use the fact that

$$L_{[f,g]} h = L_f L_g h - L_g L_f h$$

so that

$$\begin{aligned}
L_{\mathrm{ad}_{g_1}^{n-2} g_2} h &= L_{g_1} L_{\mathrm{ad}_{g_1}^{n-3} g_2} h - L_{\mathrm{ad}_{g_1}^{n-3} g_2} L_{g_1} h\\
&= (-1)^{n-2} L_{g_2} L_{g_1}^{n-2} h \neq 0
\end{aligned}$$

and $L_{\mathrm{ad}_{g_1}^k g_2} h = 0$ for $k < n-2$ by the same reasoning. Using this calculation,

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} dh \\ dL_{g_1}^{n-2} h \\ \vdots \\ dL_{g_1} h \end{bmatrix} \qquad \frac{\partial \phi}{\partial x} \Delta_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & \pm a(x) & * & \cdots & * \\ * & 0 & \pm a(x) & & \vdots \\ \vdots & \vdots & & \ddots & * \\ * & 0 & \cdots & 0 & \pm a(x) \end{bmatrix}$$

where $a(x) = L_{g_2} L_{g_1}^{n-2} h \neq 0$.

Evaluating the derivatives of the coordinate transformation, we define

$$\begin{aligned}
v_1 &:= u_1\\
v_2 &:= (L_{g_1}^{n-1} h) u_1 + (L_{g_2} L_{g_1}^{n-2} h) u_2
\end{aligned}$$

Since $L_{g_2} L_{g_1}^{n-2} h \neq 0$, this change of inputs is invertible and the resulting system is in chained form. □

This proposition gives a set of sufficient conditions for converting a system with relative growth vector $\sigma = (2, 1, \cdots, 1)$ into chained form (locally). In order to apply the results, however, we must modify the original inputs to the system such that one of the states is controlled directly by the input. Such a change of input is always possible due to the assumption that the input vector fields are linearly independent. This change of input is not unique.

One corollary to Proposition 4 is that all systems with relative growth vector $\sigma = (2, 1)$ can be converted to chained form. This is a direct consequence of the fact that all 1 dimensional distributions are involutive.

Example 1 (Kinematic car) Consider as our first example, the kinematic model of an automobile. The equations governing the motion of the system are [22]:

$$\begin{aligned}
\dot{x} &= \cos\theta\, u_1\\
\dot{y} &= \sin\theta\, u_1\\
\dot{\phi} &= u_2\\
\dot{\theta} &= \frac{1}{l} \tan\phi\, u_1
\end{aligned} \tag{6}$$

To convert the system to chained form, we first scale the inputs so that u_1 enters $\dot{x}$ directly. Reusing the symbol u_1, the kinematics become:

$$\begin{aligned}
\dot{x} &= u_1\\
\dot{y} &= \tan\theta\, u_1\\
\dot{\phi} &= u_2\\
\dot{\theta} &= \frac{1}{l} \sec\theta \tan\phi\, u_1
\end{aligned}$$

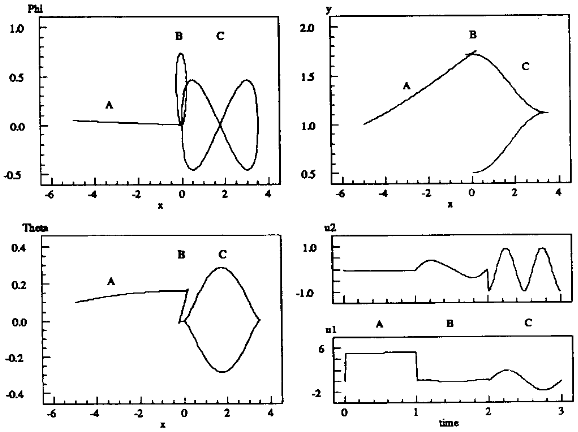

Figure 1: Sample trajectories for steering a $(2,1,1)$ system. The trajectory shown is a three stage path which moves the system from the initial configuration to the origin.

Choose the y position of the car as the function h; it is easy to verify that this function satisfies the conditions of Proposition 4. The resulting change of coordinates is

$$\begin{aligned}
\xi_1 &= x & u_1 &= v_1 \\
\xi_2 &= \frac{1}{l}\sec^3\theta\tan\phi & u_2 &= -\frac{3}{l}\sin^2\phi\sin\theta v_1 + \frac{1}{l}\cos^2\theta\cos^3\phi v_2 \\
\xi_3 &= \tan\theta \\
\xi_4 &= y
\end{aligned}$$

And the transformed system has the form:

$$\begin{aligned}
\dot\xi_1 &= v_1 \\
\dot\xi_2 &= v_2 \\
\dot\xi_3 &= \xi_2 v_1 \\
\dot\xi_4 &= \xi_3 v_1
\end{aligned}$$

This system can now be steered using the sinusoidal algorithm of the previous section or another method, such as Lafferriere and Sussmann's algorithm for generating motions for nilpotent systems. The motion is implemented as a feedback pre-compensator which converts the v inputs into the actual system inputs, u. This feedback transformation agrees with that used in Lafferriere and Sussmann to nilpotentize the kinematic car example. Their formulation of the feedback transformation was not presented, although it seems clear that a similar approach must have been used.

Figure 1 shows the results of using chained form to steer an automobile. These trajectories are qualitatively similar to those in [21], but do not require the calculation of Fourier coefficients for determining open loop trajectories.

Example 2 (Car with N trailers) Consider first the case of a car pulling a single trailer. The equations of motion are identical to those of the car, with an additional equation specifying the motion of the attached trailer [22]:

$$\dot\theta_1 = \sin(\theta_0 - \theta_1)u_1$$

By solving the partial differential equations in the statement of the proposition above, it can be shown that the function

$$h(y,\theta_1) = y - \log\left(\frac{1+\sin\theta_1}{\cos\theta_1}\right)$$

generates a chained set of coordinates. Again we can locally steer the trailer using sinusoidal inputs or other methods.

When additional trailers are added, the distribution Δ_1 is no longer involutive and hence the procedure outlined above does not apply. Since the conditions in the proposition are only sufficient conditions, this does not mean that a car with N trailers cannot be steered using sinusoids. But a more complicated change of basis would be required in order to convert the vector fields to the necessary form. This example points out the weaknesses of the theorem and provides directions for future research.

4 Discussion and extensions

The conditions given in Proposition 2 give a constructive set of sufficient conditions for converting a nonholonomic control system into chained form. Using chained form, it is possible to efficiently find paths for steering a system between arbitrary configurations. We have presented one such method based on sinusoidal inputs, although other techniques can be used. We have presented only the simplest case (2 inputs, 1 non-zero length chain) here, but it is possible to extend this result in several directions. The method proposed here is also useful in the more general area of local nilpotentization of distributions of vector fields [7]. In particular, if the conditions of Proposition 2 hold for a set of two vector files g_1 and g_2 on $\mathbb{R}^n$, then there is a nilpotent basis for the corresponding distribution.

Converting a system to chained form is very closely related to the exact linearizability conditions for a general nonlinear system. As we noted in the introduction, linear control techniques cannot be applied to nonholonomic control systems due to the lack of a drift term. However, many of the underlying geometric tools on which exact linearization techniques are based can be applied effectively to nonholonomic systems. We see this in the application of Proposition 2. In particular, we note that if the distribution

$$\{g_2, \mathrm{ad}_{g_1} g_2, \cdots, \mathrm{ad}_{g_1}^{n_x-1} g_2\}$$

is involutive, then we are guaranteed of the existence of a function h which annihilates the distribution. Finding a specific h which satisfies this requires solving a set of first order partial differential equations.

There are many open questions which are currently being studied by ourselves and others. These include the introduction of a drift vector field into the control system and feedback control of nonholonomic systems. Some initial results in these areas can be found in the work of Bloch and McClamroch [2, 3] and Samson [24]. We also note that approximate versions of Proposition 4 can be formulated using tools similar to those developed by Krener [10, 11].

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