# 4. Spatial kinematics. Constraint.

## *Mechanics of Manipulation*

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# Outline.

- Spherical kinematics
  - Euler’s theorem
  - Cones
- Spatial kinematics
  - Chasles’ theorem
  - Screws and twists
  - Axodes
- Kinematic constraint
  - Overview
  - Taxonomy and terminology
  - Reuleaux’ method for unilateral constraints

# About spherical kinematics

- Why study motions of the sphere? Because it corresponds to rotations about a given point of $\mathbb{E}^3$.

# About spherical kinematics

- Why study motions of the sphere? Because it corresponds to rotations about a given point of $\mathbb{E}^3$.
- There is a close connection to planar kinematics. Let the radius of the sphere approach infinity . . .

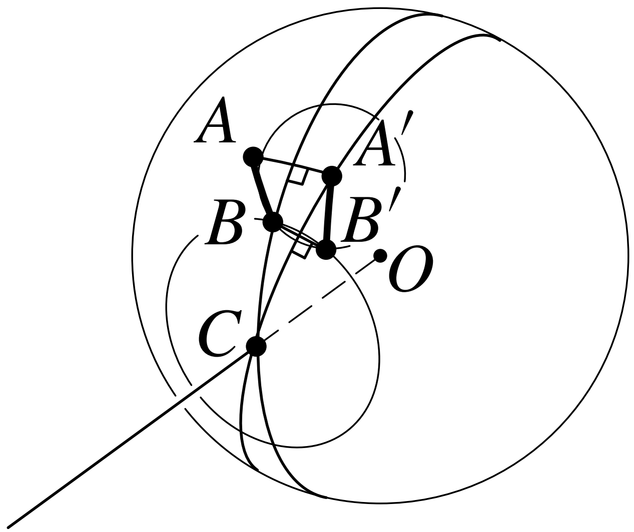
# Two not-antipodal points enough

Theorem 2.5: A displacement of the sphere is completely determined by the motion of any two points that are not antipodal.

Proof: Construct a coordinate frame ...

# Euler's theorem

Theorem 2.6: For every spatial rotation, there is a line of fixed points. In other words, every rotation about a point is a rotation about a line, called the *rotation axis*.



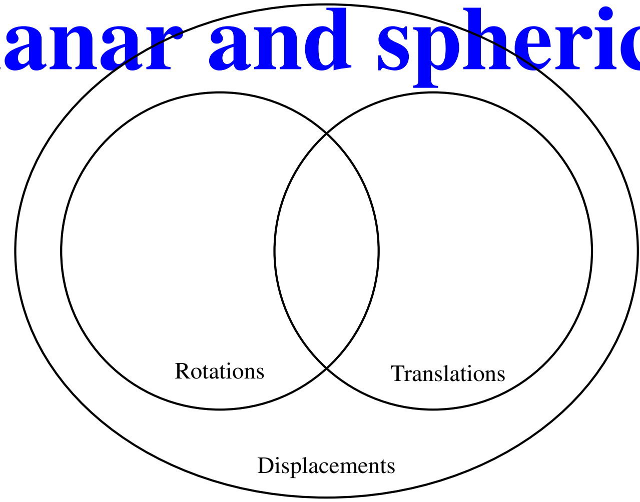

Proof:

> Prove that every displacement of the sphere has a fixed point.
>
> Define $A$, $\perp AA'$, $B$, $B'$, $\perp BB'$.
>
> Define $C$ to be either intersection of $\perp AA'$ with $\perp BB'$.
>
> Let $R$ be the rotation mapping $A$ to $A'$ and $C$ to itself.
>
> Show $R$ maps $B$ to $B'$, so $R$ is the given displacement.

# Review of displacements: planar and spherical

For the Euclidean plane, are there . . .

. . . rotations that are not translations?

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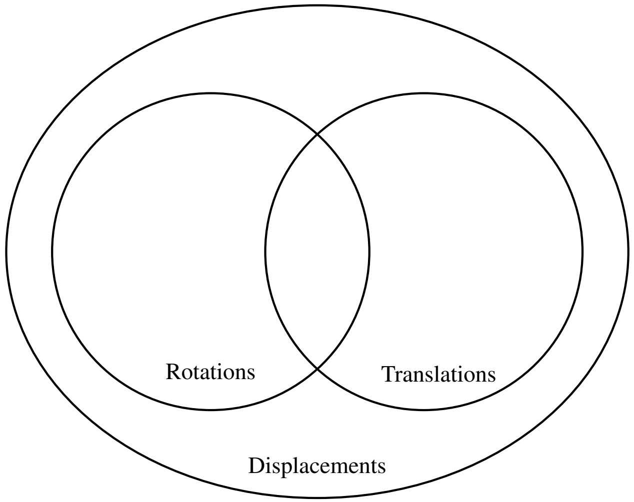


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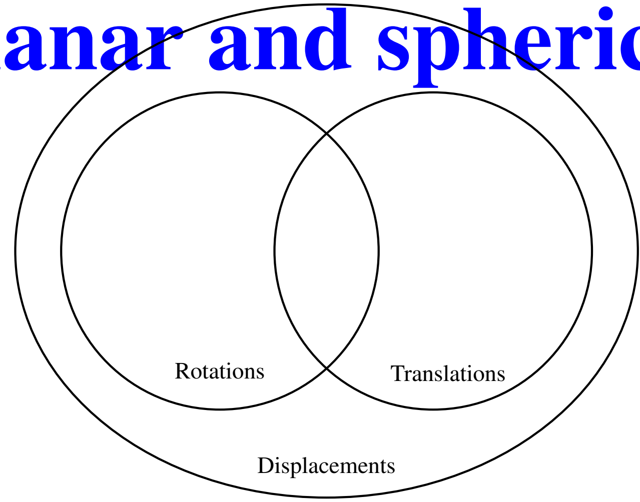


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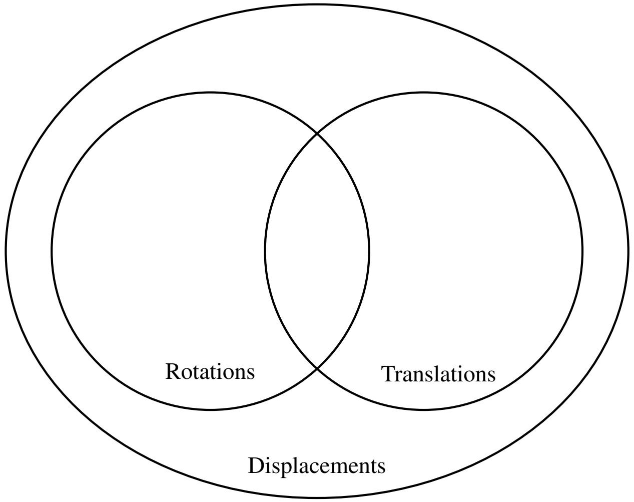


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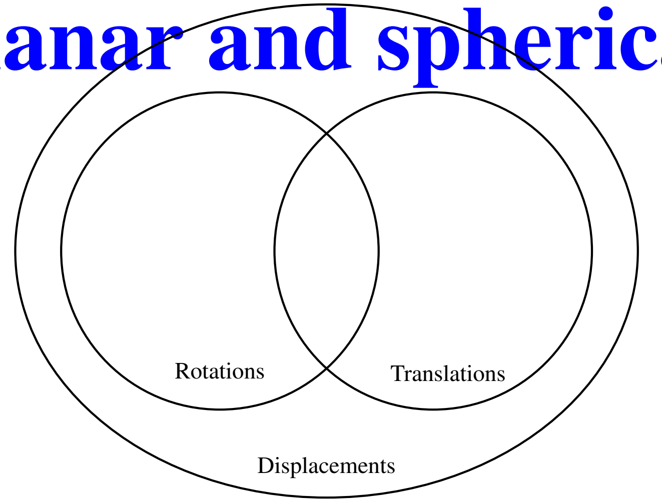


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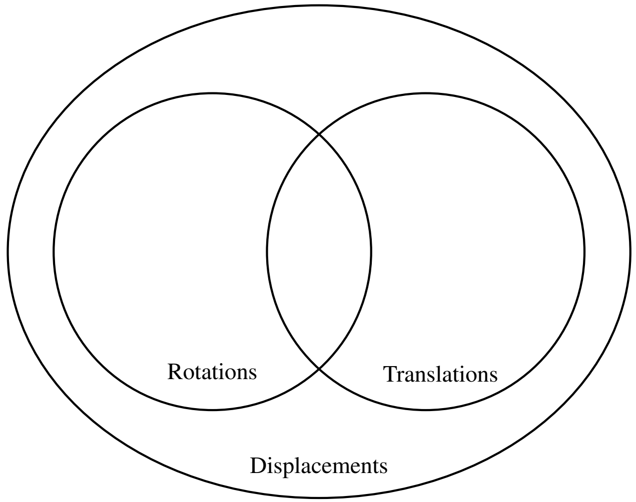


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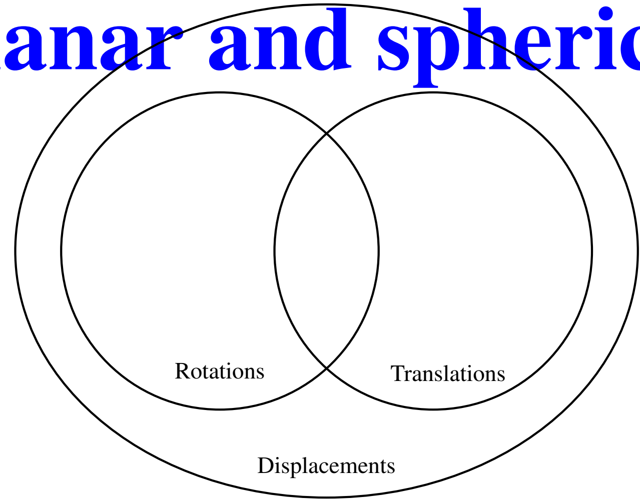


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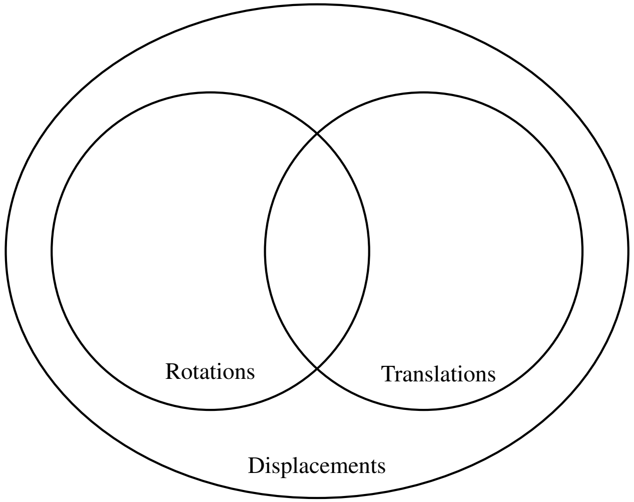


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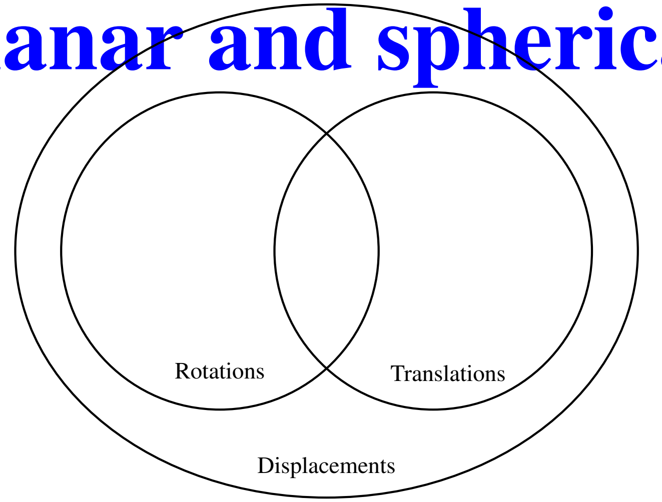


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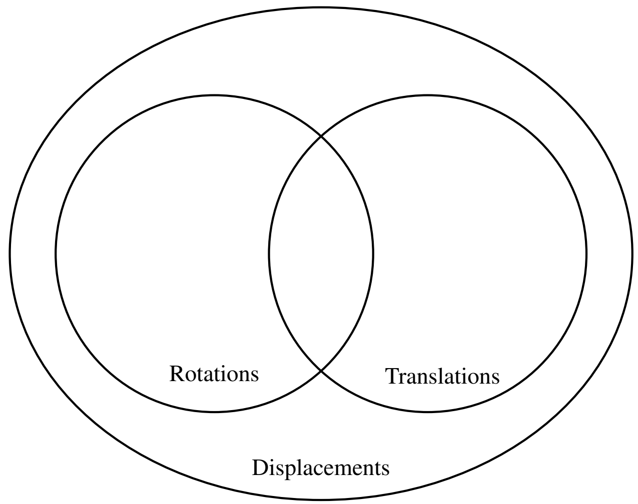


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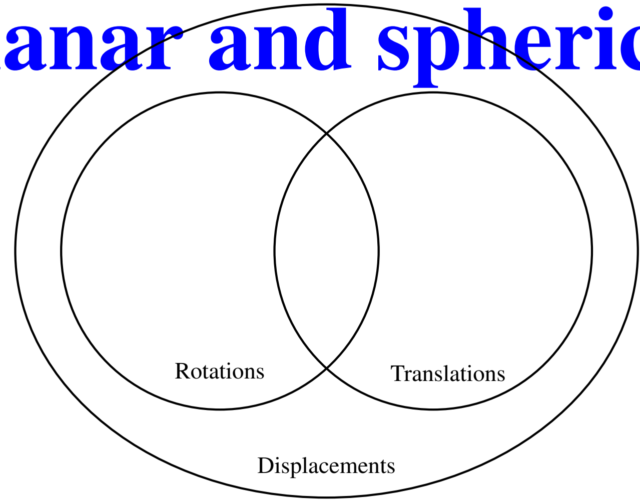


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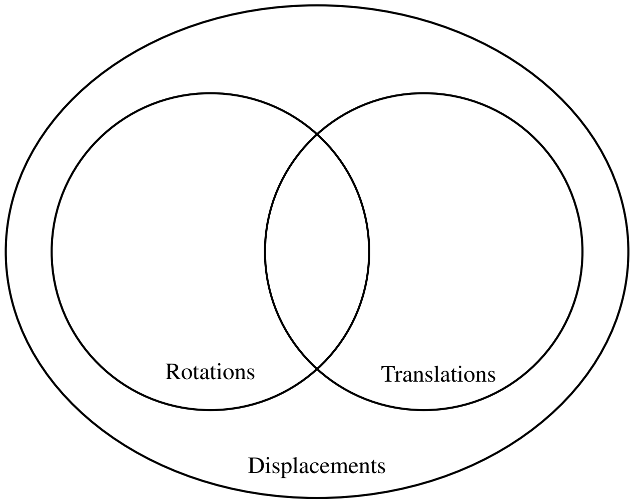


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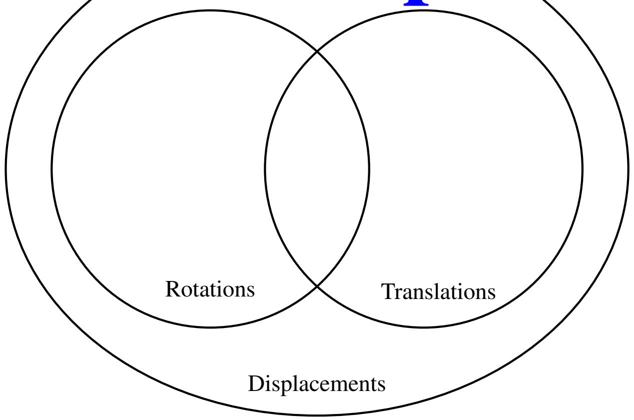


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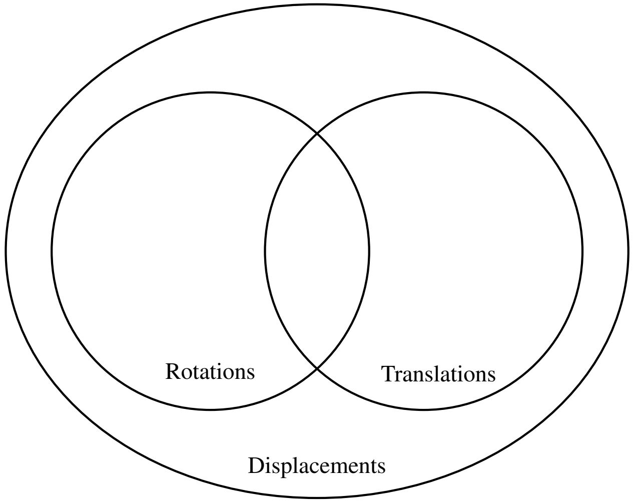


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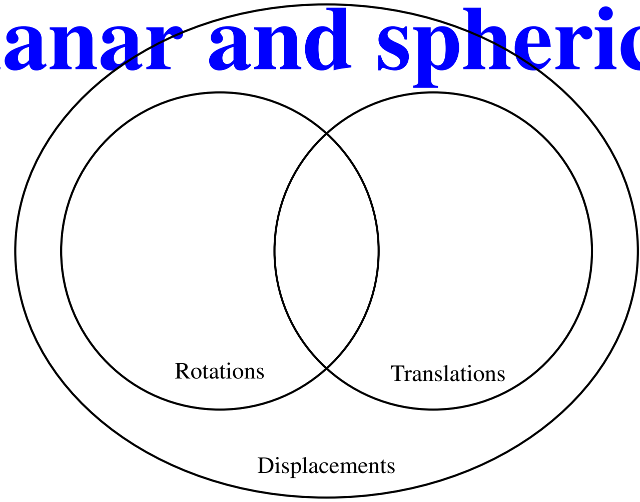


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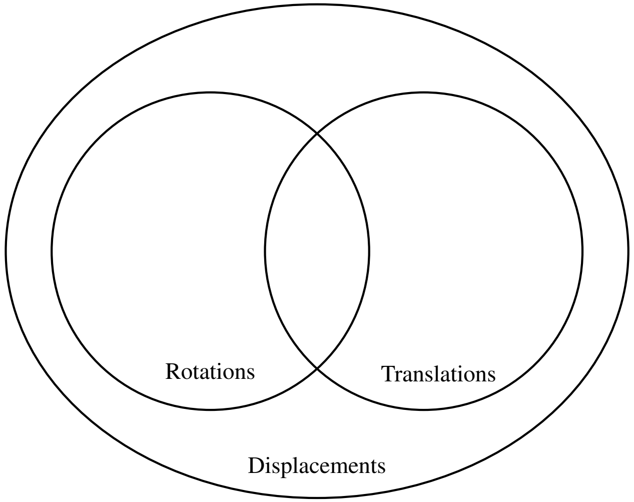


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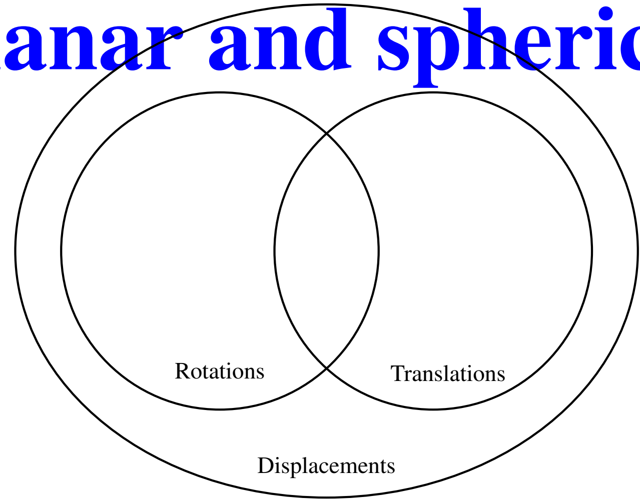


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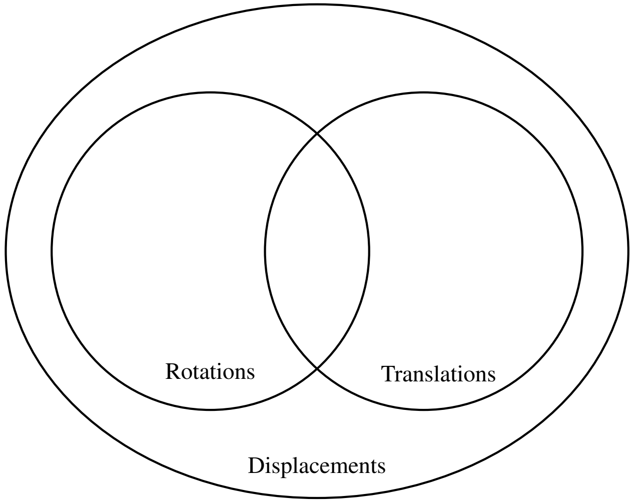


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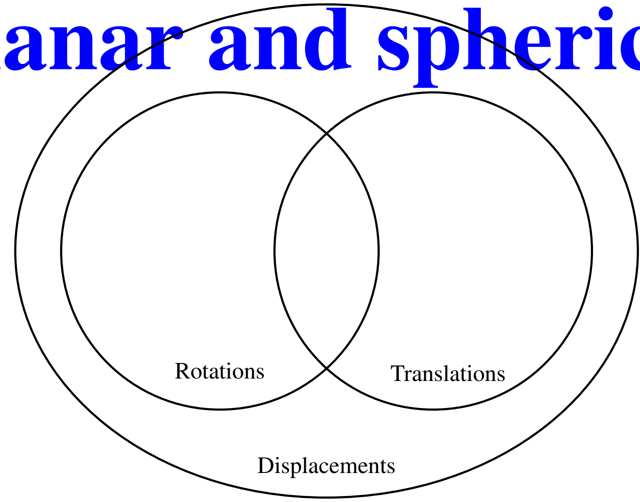


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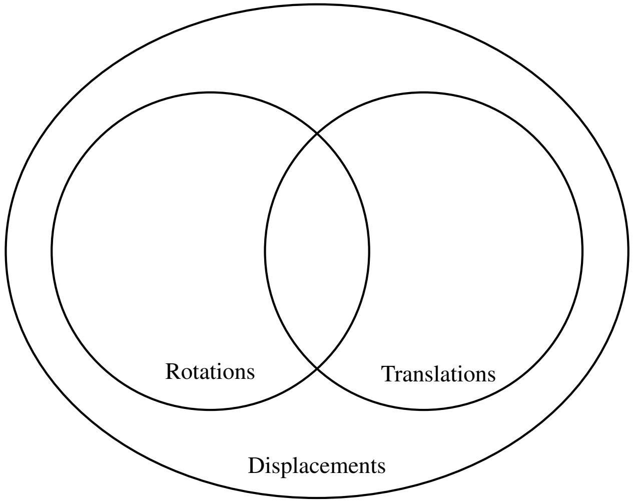


Lots!

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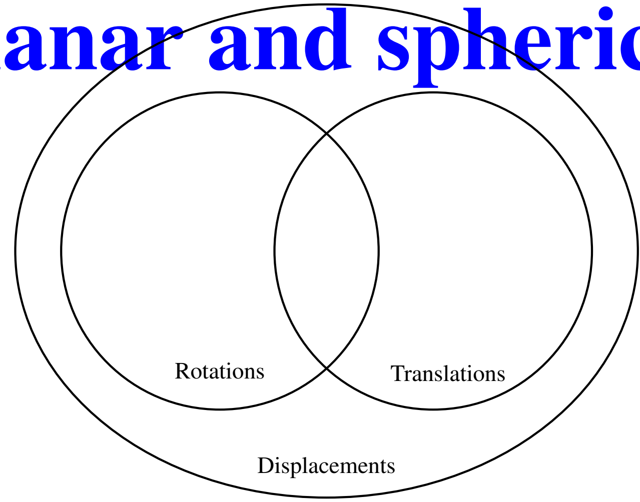


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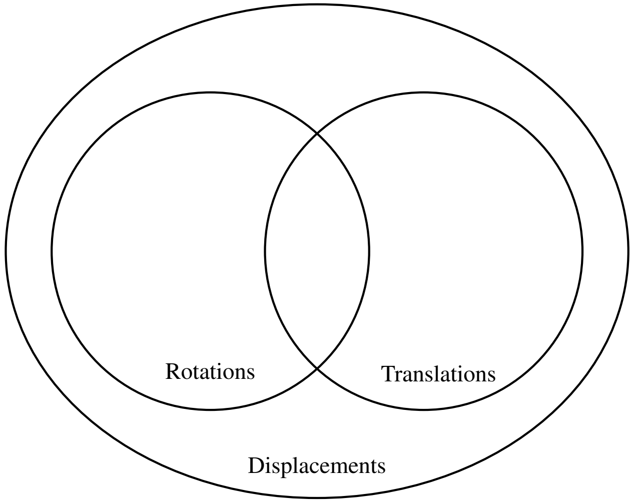


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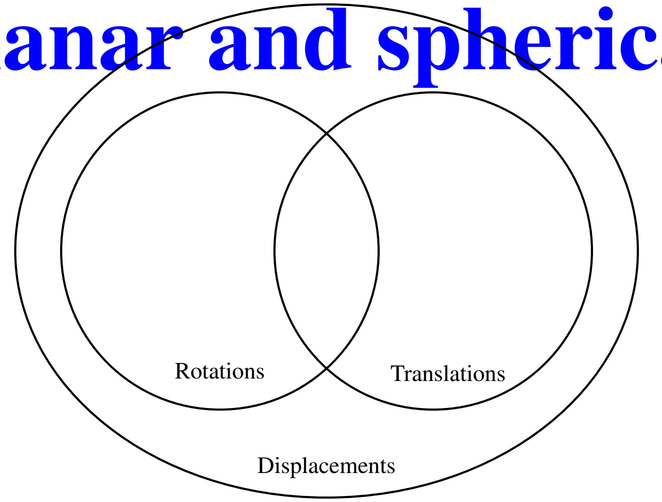


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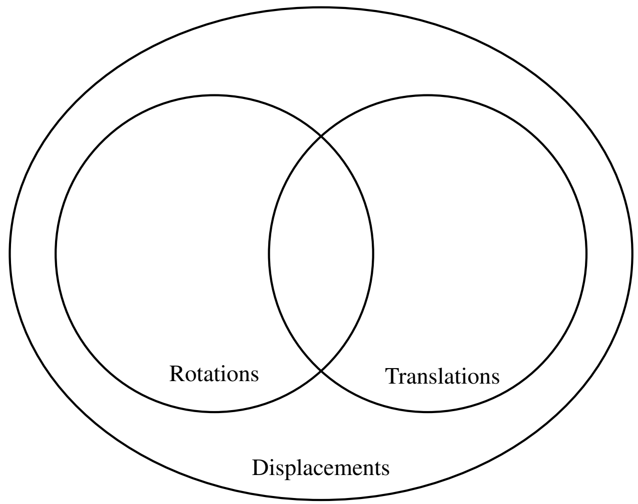


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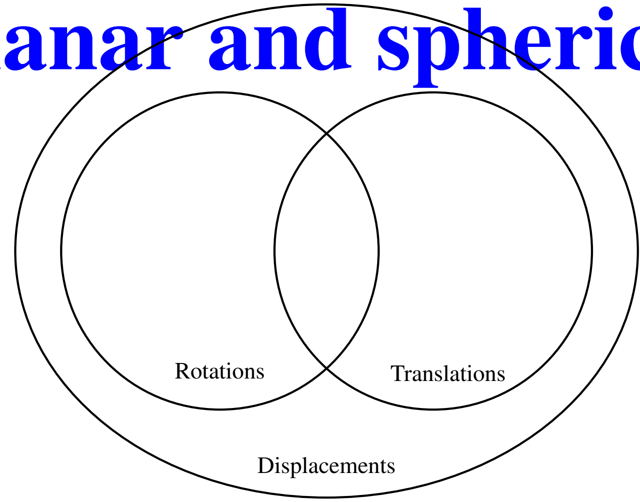


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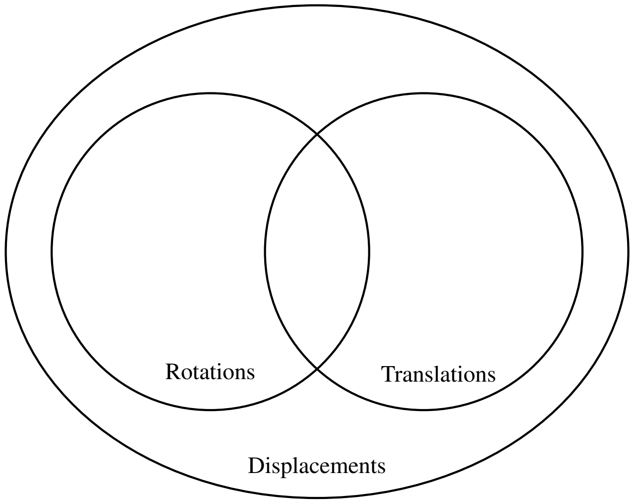


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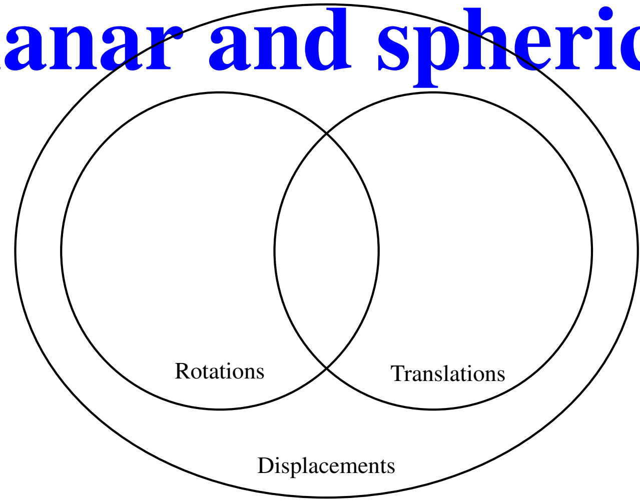


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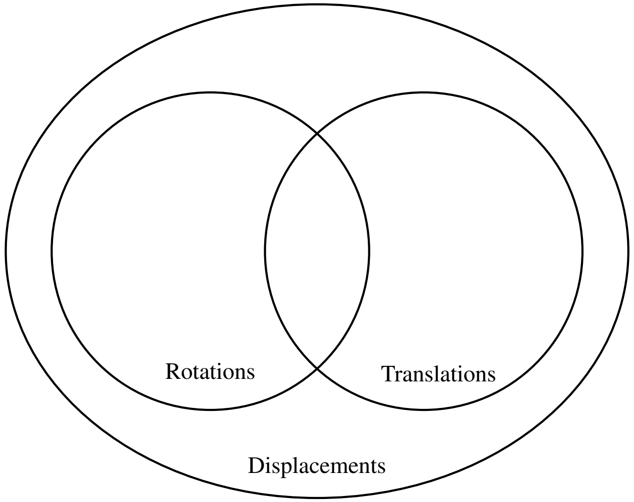


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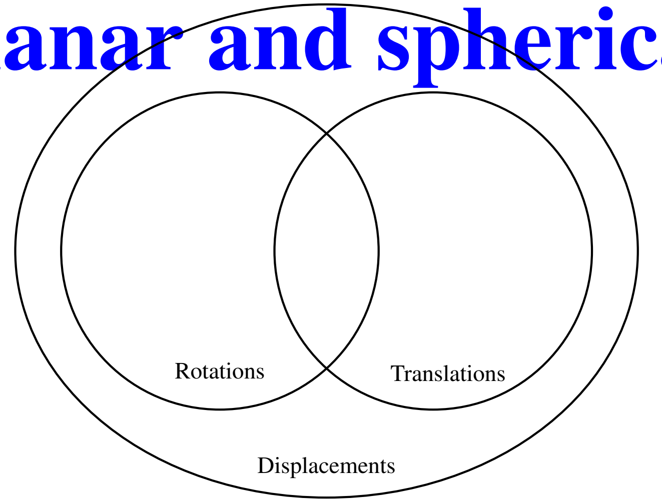


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Lots!

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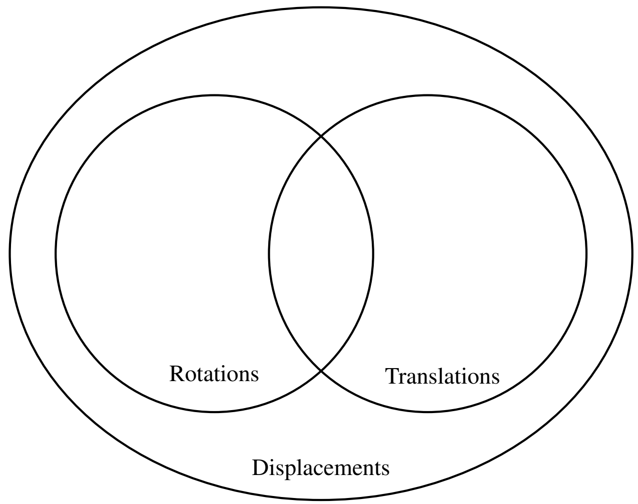


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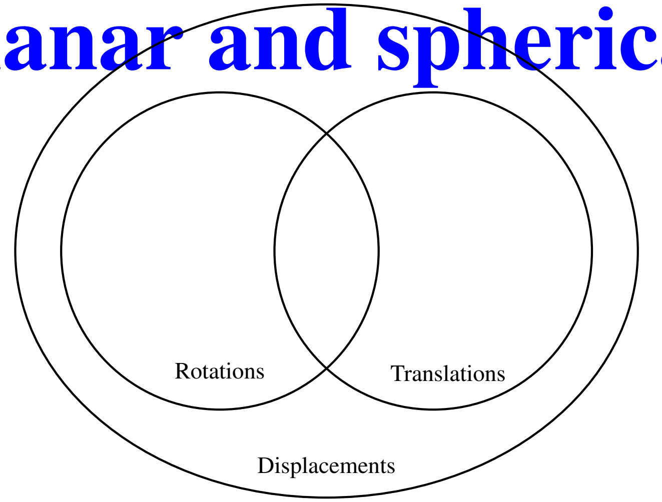


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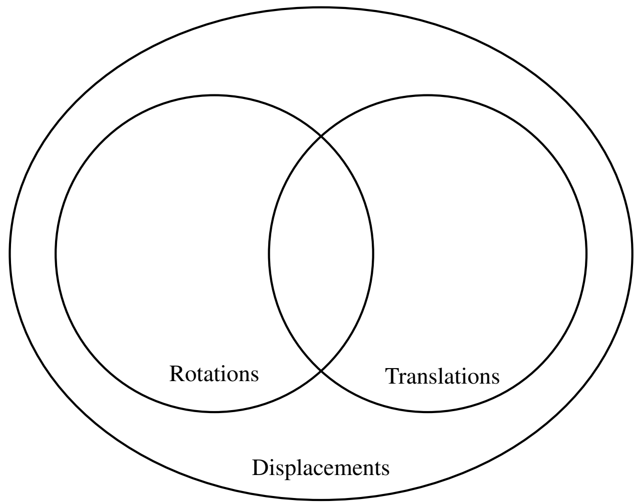


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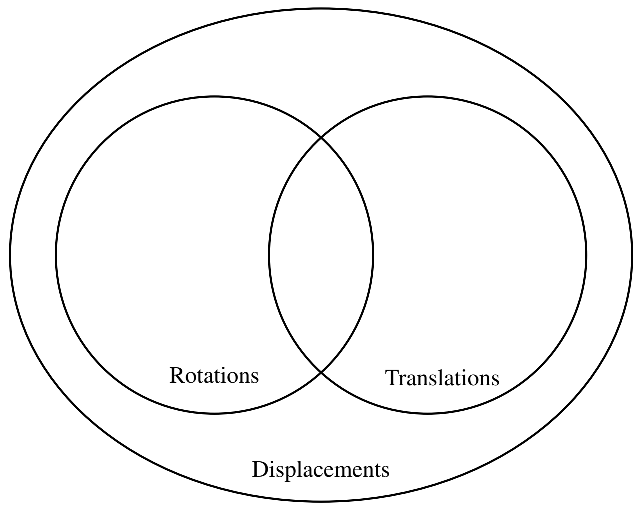


Euclidean three space

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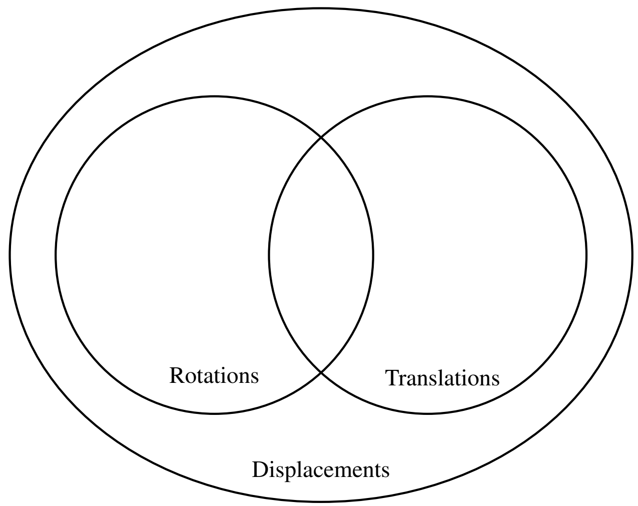


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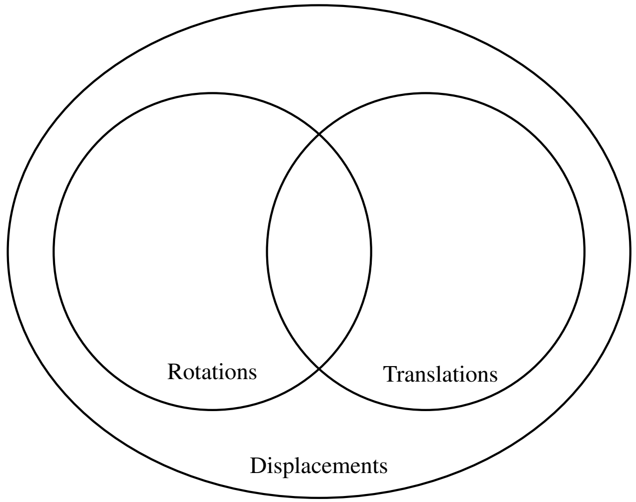


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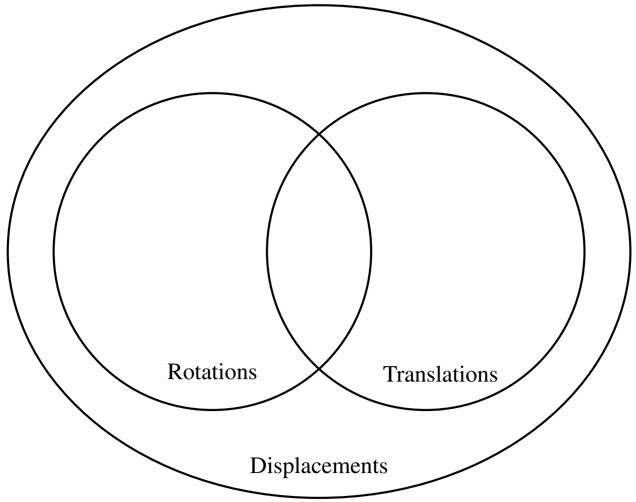


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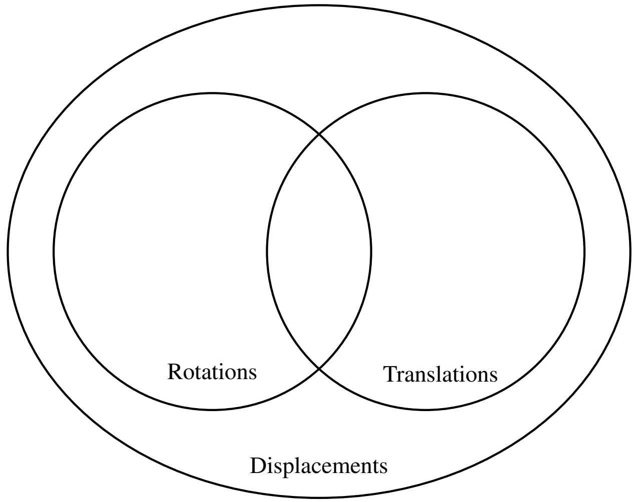


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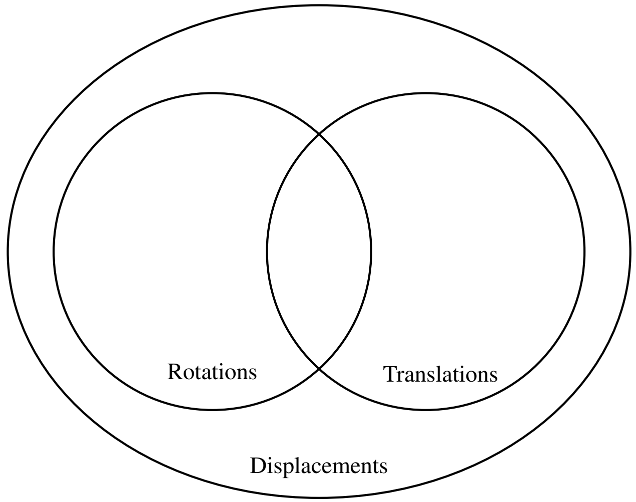


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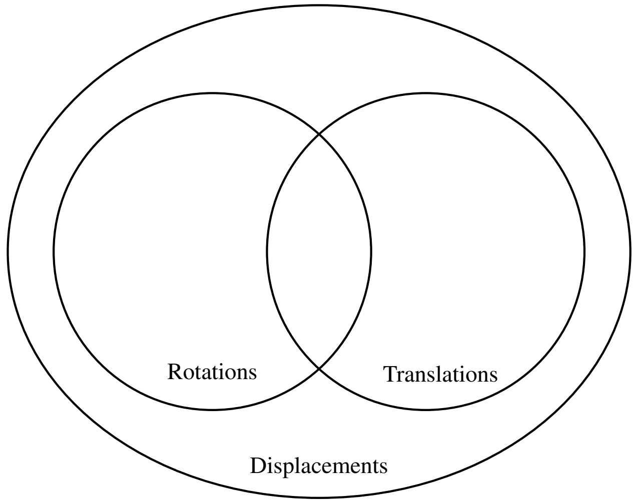


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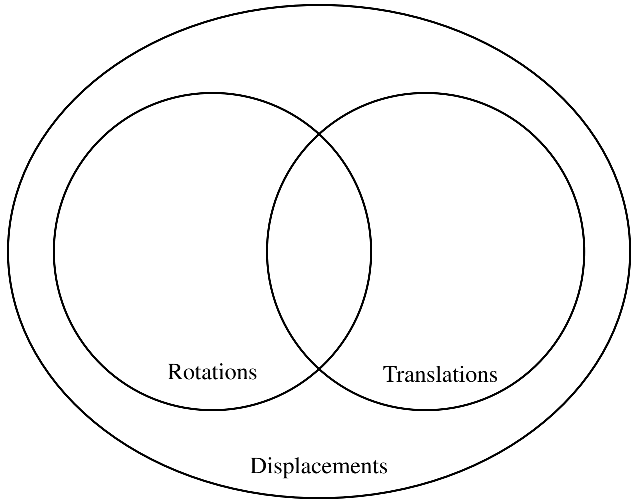


Lots! Screws!

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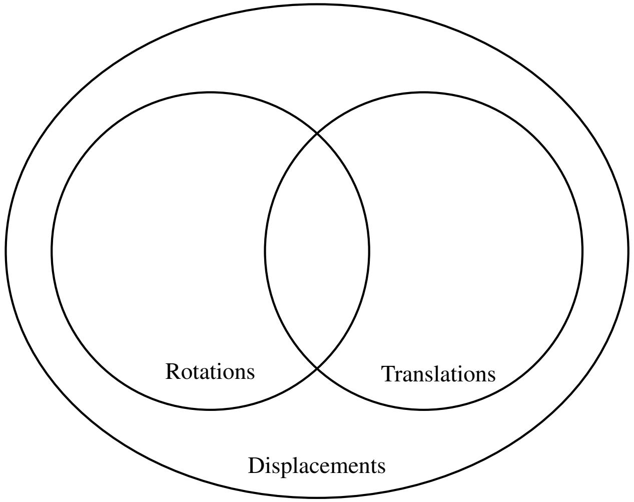


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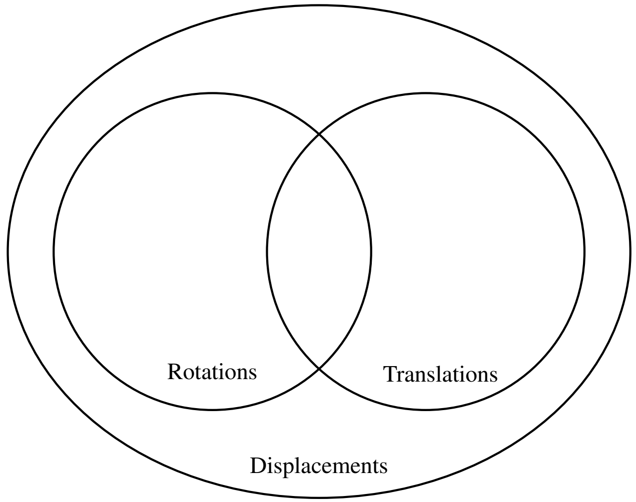
No. Chasles' theorem




Screws! Lots! Lots!

Euclidean three space

# Chasles's theorem

Theorem 2.7: Every spatial displacement is the composition of a rotation about some axis, and a translation along the same axis.

Proof:

Assume arbitrary displacement $D$ is given.

Use theorem 2.2 to decompose $D = R \circ T$.

Decompose $T$ into components parallel to and perpendicular to axis of $R$: $D = R \circ T_{\perp} \circ T_{\|}$.

Note that $R \circ T_{\perp}$ is planar! Every plane perpendicular to rotation axis is mapped rigidly to itself.

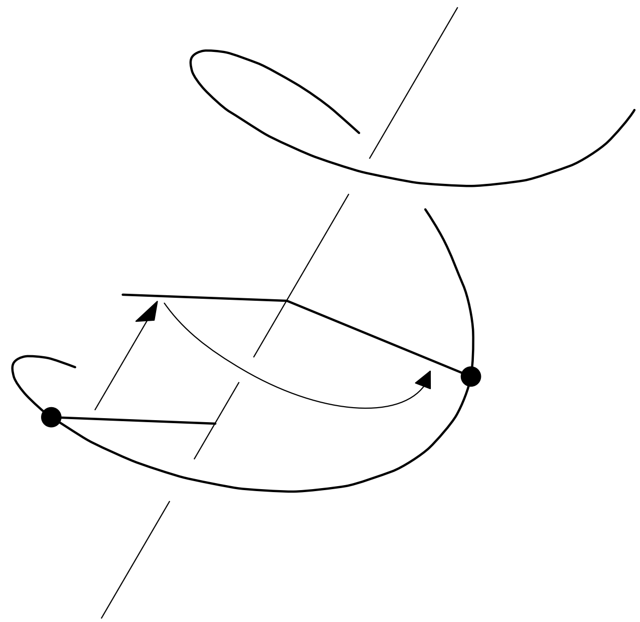
If $R \circ T_{\perp}$ is a translation the theorem follows immediately. Otherwise $R \circ T_{\perp}$ is a rotation about some axis parallel to the rotation axis of $R$.

So $D = (R \circ T_{\perp}) \circ T_{\|}$ is the desired decomposition.

# Screws.

A *screw* is a line in space with an associated pitch, which is a ratio of linear to angular quantities.

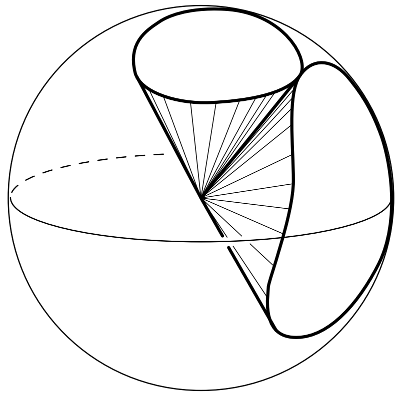
A *twist* is a screw plus a scalar magnitude, giving a rotation about the screw axis plus a translation along the screw axis. The rotation angle is the twist magnitude, and the translation distance is the magnitude times the pitch. Thus the pitch is the ratio of translation to rotation.



# Analogous to centrodes . . .

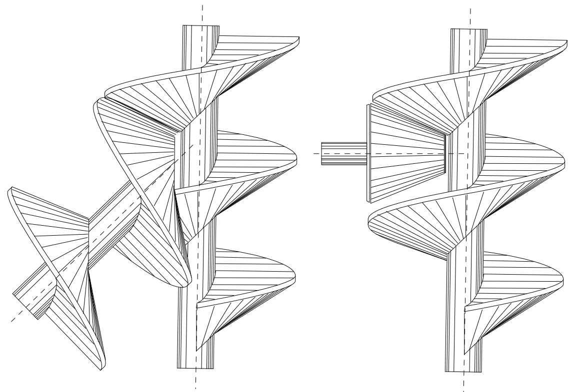
On the sphere . . .

Plotting the instantaneous rotation axis in the fixed and moving frames gives *fixed and moving cones*.



In three space . . .

Plotting the instantaneous screw axis in the fixed and moving frames gives *fixed and moving axodes*.



# Kinematic constraint

One of the best manipulation tricks!

In simple cases, freedoms and constraints are just a matter of counting unknowns and equations.

$$\begin{aligned} &\text{nominal DOFs} \\ -&\text{independent constraints} \\ =&\text{ DOFs} \end{aligned}$$

Things to worry about:

If an equation reduces DOFs by 1, does an inequation reduce DOFs by 1/2?

Identifying dependencies and singular cases.

Constraints on velocity versus on configuration.

# Constraint in general

Consider constraints of the form

$$f(q, \dot{q}, t) = 0$$

or

$$f(q, \dot{q}, t) \geq 0$$

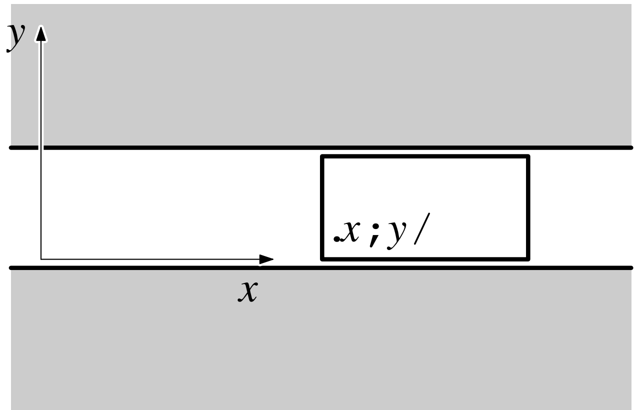
where

$$\begin{array}{lcll} q & \in & Q & \text{configuration space, e.g. } (x, y, \theta) \\ \dot{q} & \in & TQ & \text{tangent space, e.g. } (\dot{x}, \dot{y}, \dot{\theta}) \\ t & = & \text{time} & \end{array}$$

# Constraint: taxonomy and examples

**bilateral**

Expressed as an equation. Two sided.

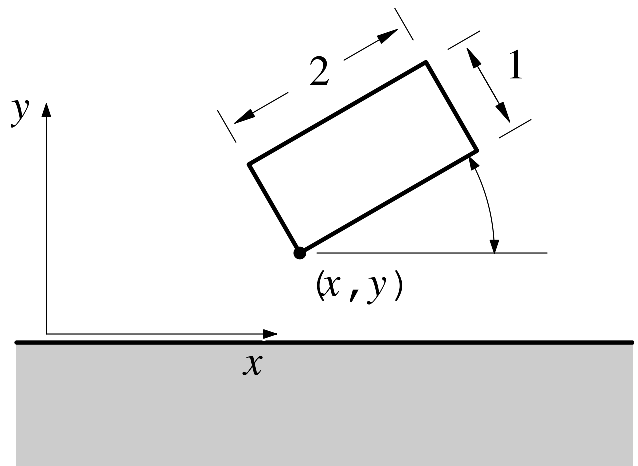



$$y = 0$$

$$\theta = 0$$

**unilateral**

Expressed as an inequation. One sided.




$$y \geq 0$$

$$y + 2\sin\theta \geq 0$$

$$y + 2\sin\theta + \cos\theta \geq 0$$

$$y + \cos\theta \geq 0$$

# Constraint: taxonomy and examples

**scleronomic**

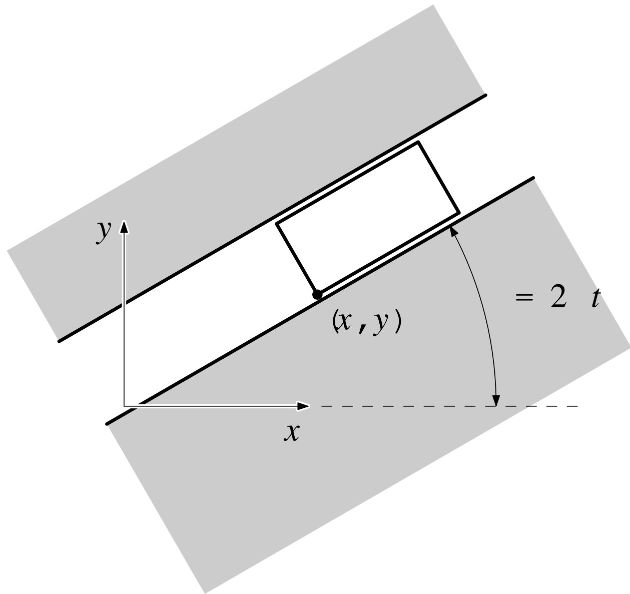
Independent of $t$. Stationary.

**rheonomic**

Depends on $t$.

$$x\sin(2\pi t) - y\cos(2\pi t) = 0$$

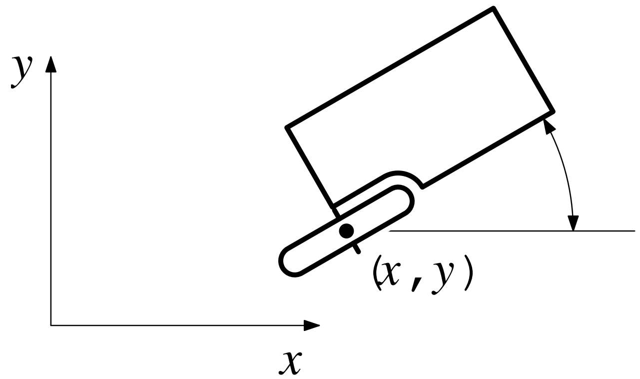
$$\theta = 2\pi t$$




**holonomic**

Independent of $\dot{q}$ and bilateral.

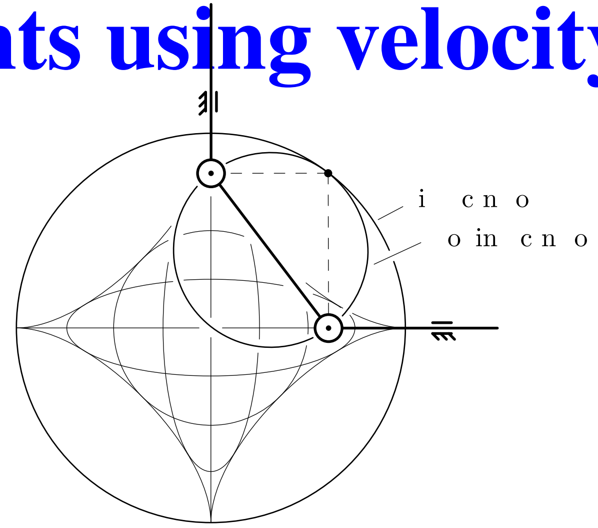
$$f(q,t) = 0$$



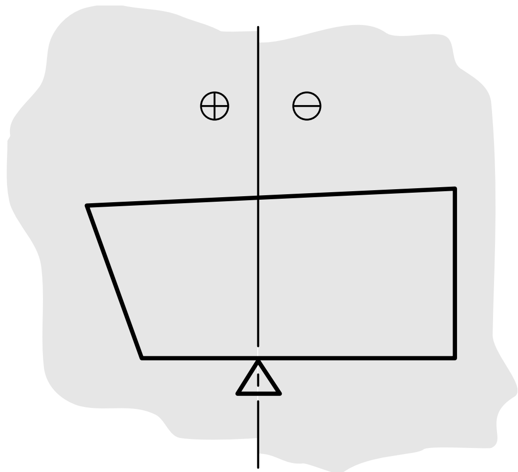

**nonholonomic**

# Analysis of planar constraints using velocity cent

Bilateral: recall technique from previous lecture. Construct perpendicular to allowed velocity at point. IC must be at intersection of perpendiculars.

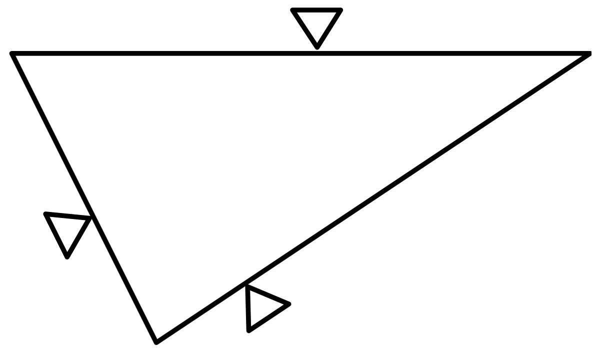


Extension to unilateral. Perpendicular to constraint divides plane into positive IC's, negative IC's, and IC's of either sign.



# Multiple unilateral constraints (Reuleaux's meth

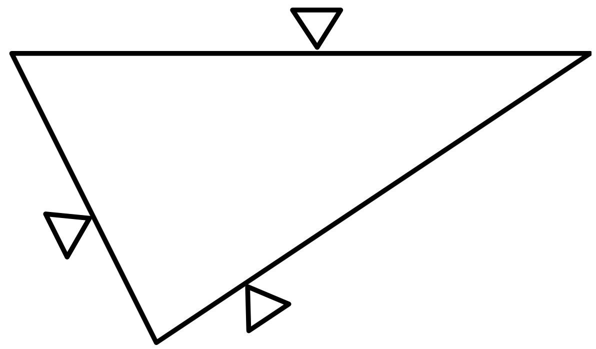
Can this triangle move?



# Multiple unilateral constraints (Reuleaux's meth

Can this triangle move?

Construct positive and negative half-planes for each contact.



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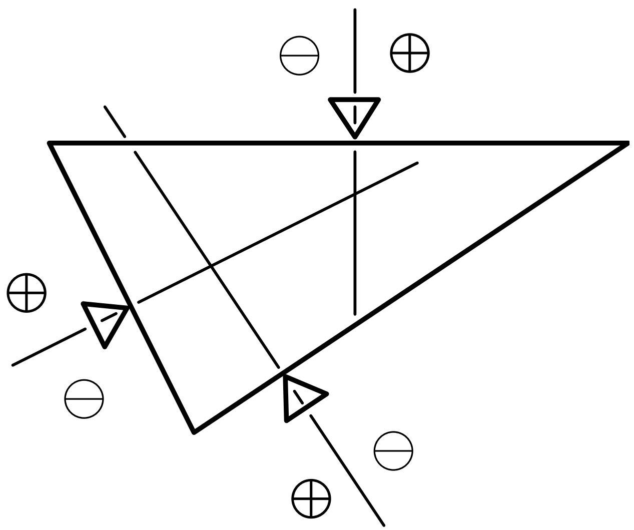
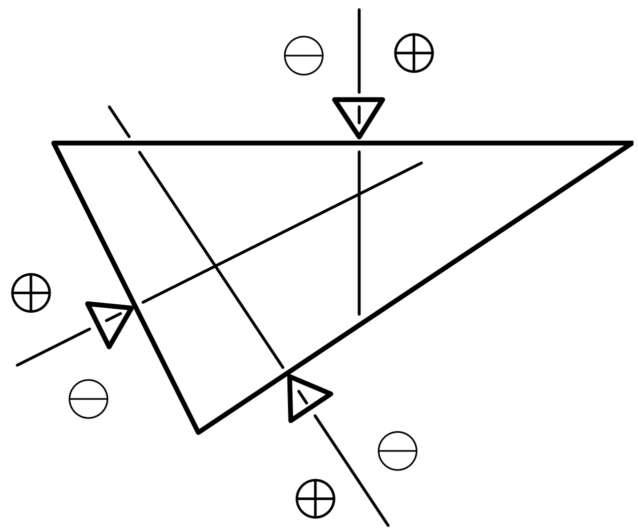


# Multiple unilateral constraints (Reuleaux's meth

Can this triangle move?

Construct positive and negative half-planes for each contact.

Keep consistently labelled points.

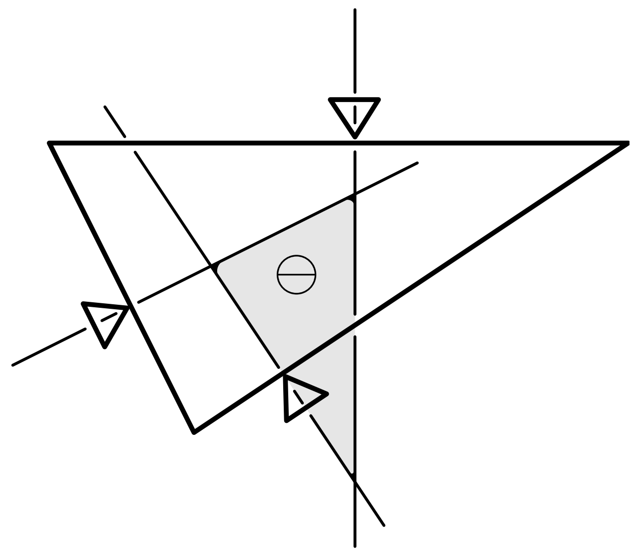


# Multiple unilateral constraints (Reuleaux's meth

Can this triangle move?

Construct positive and negative half-planes for each contact.

Keep consistently labelled points.



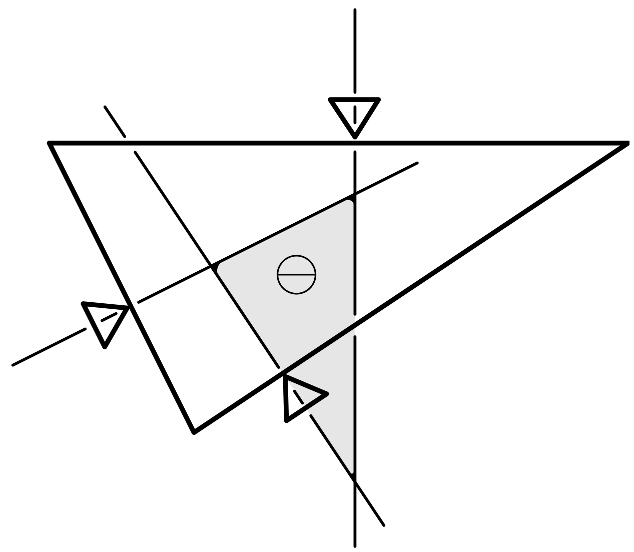
# Multiple unilateral constraints (Reuleaux's meth

Can this triangle move?

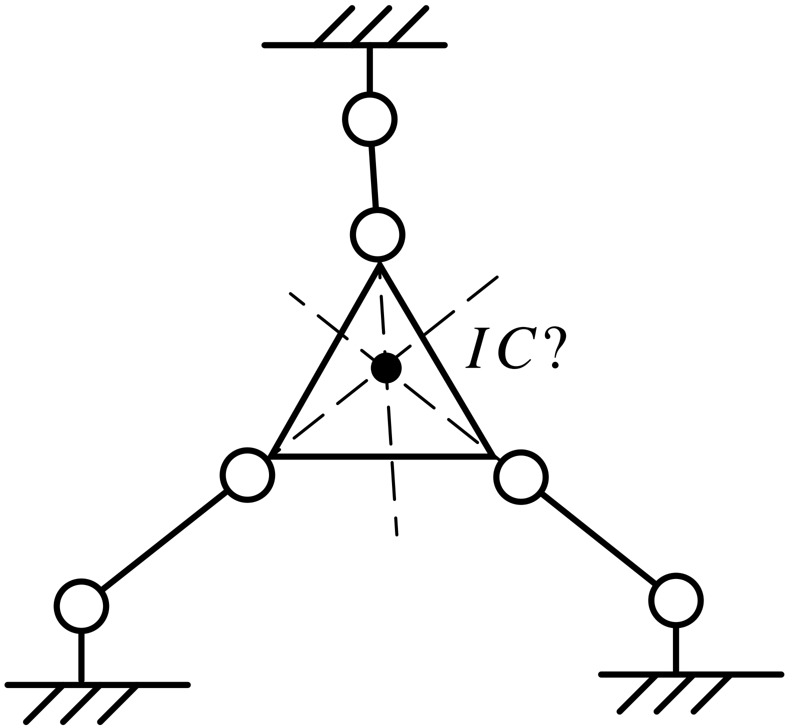
Construct positive and negative half-planes for each contact.

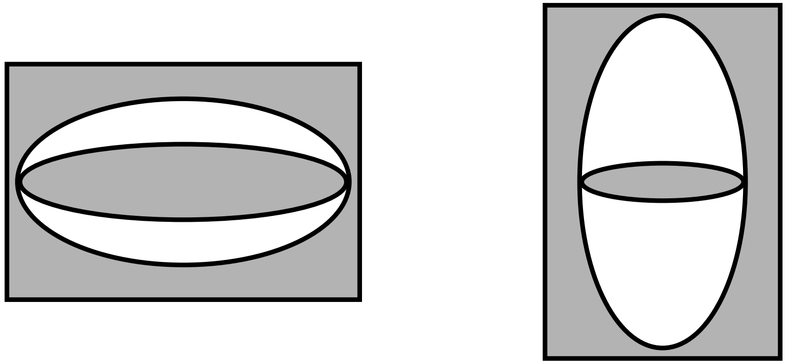
Keep consistently labelled points.

Triangle can rotate CW about any $-$ point.



# But watch for false positives



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