Discontinuous Control for Exponential Stabilization of Wheeled Mobile Robots*

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Abstract

This paper presents an exponentially stabilizing controller for a unicycle-like mobile robot, by means of a new coordinates transformation and Lyapunov Method. The proposed control law allows to drive the vehicle in forward or backward motion from any initial configuration to any final one without inverting the direction of motion, provided that the robot configuration is, initially, into a large domain excluding the free-y axis. The enclosed simulation results show the effectiveness of the proposed controller.

1 Introduction

The problem of stabilizing nonholonomic wheeled mobile robots has received a great attention of many researchers in the last few years. The main difficulty for dealing with this class of systems may be attributed to the restricted mobility, in the direction of the wheels axes, which prevents the vehicle to move sideways. In fact, as it has been proved by Brockett [5], this class of systems, in Cartesian space, cannot be stabilized via any smooth time-invariant state feedback. To overcome this difficulty, discontinuous or (and) time-varying state feedback have been proposed in the literature. The main drawback of the elegant time-varying methods is that the vehicle reaches the goal in several steps by inverting the direction of motion. Hence, the resulting paths are nonsmooth and oscillating, which is not suitable in practice (see for example, [9-13]). Recently, discontinuous time-invariant control laws, yielding exponential stabilization, have been proposed by means of adequate coordinates transformations. In fact, these controllers generate nonoscillating paths which seem more realistic than those obtained via time-varying methods, (see [1-3] and [6-7] and references therein). The authors in [3] have introduced, for the first time, the polar representation which allows to overcome the problem of stabilizability pointed out by Brockett and proposed a locally stabilizing, fuzzy-tuned, state feedback. The authors in [7] have proposed a globally stabilizing state feedback using the same polar representation. However, this controller allows the vehicle to reach the goal in forward motion only and generates, for certain initial configurations, discontinuous paths due to the inversion of the direction of motion. In [1], the author has proposed a linear state feedback , using also the same representation, where local and global stability issues have been discussed. Though this controller generates continuous paths, it suffer from the necessity of using two different model for the forward and the backward motion.

In this paper, we propose a discontinuous, semi-global, state feedback control law yielding exponential convergence to the final configuration, without inverting the direction of motion, i.e. in one step. The target configuration can be reached in forward or backward motion along a smooth trajectory, provided that the initial conditions are into an open and large domain Ω. This controller is based on a particular polar representation, called '*signed polar representation*', defined over the domain Ω and leading to a unified kinematics model for both negative and positive linear velocities.

The paper is organized as follows: In section 2, we derive the kinematics model of the unicycle in the new coordinates. In section 3, we present a discontinuous state feedback control law and we discuss the choice of the control design parameters. In section 4, simulation results are given for illustration while section 5 concludes the paper.

2 Kinematics Model

The mobile robot under consideration is a unicycle-like vehicle. The motion control of this vehicle can be achieved by dealing with the linear and rotational velocities (v, $\dot{\theta}$) as shown in Fig. 1. It is assumed that the vehicle moves without slipping on a horizontal ground.

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2.1. Coordinates Transformation

Rather than using the Cartesian coordinates x, y and θ to describe the kinematics behavior of the vehicle, we will use the *signed distance* d, and the angles γ and ψ, which are defined as follows

$$d = sign(y)\sqrt{x^2 + y^2} \quad (1)$$

$$\psi = \begin{cases} arcsin\left(\frac{y}{d}\right) mod(\frac{\pi}{2}) & if \quad x.y \geq 0 \text{ and } y \neq 0 \\ \pi - \left(arcsin\left(\frac{y}{d}\right) mod(\frac{\pi}{2})\right) & if \quad x.y < 0 \end{cases} \quad (2)$$

$$\gamma = \psi - \theta \quad (3)$$

where $x \in IR$, $y \in IR/\{0\}, \theta \in [-\pi, \pi]$.

The new coordinates are defined over the domain $\Omega = \Omega_1 \cup \Omega_2$,
with

$$\Omega_1 = \{(d,\gamma,\psi) \mid (d,\gamma,\psi) \in]0,+\infty[\times [-\pi,\pi] \times]0,\pi[\}$$

$$\Omega_2 = \{(d,\gamma,\psi) \mid (d,\gamma,\psi) \in]-\infty,0[\times [-\pi,\pi] \times]0,\pi[\}$$

The variables x and y are the coordinates of the point M, located at mid-distance of the rear-wheels, in the global frame $(O, \vec{i}_0, \vec{j}_0)$, and θ is the orientation of the vehicle, taken counterclockwise from the global x-axis (see Fig. 1).

2.2. Kinematics Representation in the New Coordinates

Now, we will derive a suitable kinematics model, in the new coordinates, for solving the point-stabilization problem.

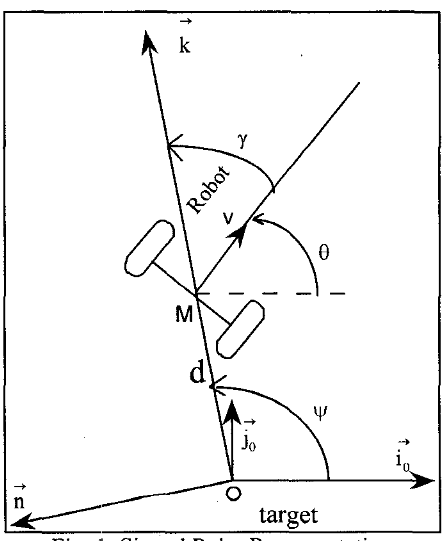

Fig. 1. Signed Polar Representation

The position error vector $\vec{OM}$ can be written as follows

$$\vec{OM} = d\,\vec{k} \quad (4)$$

Differentiating (4) with respect to time leads to

$$\frac{d\vec{OM}}{dt} = \dot{d}\,\vec{k} + d\frac{d\vec{k}}{dt} = \dot{d}\,\vec{k} + d\dot{\psi}\,\vec{n} \quad (5)$$

Furthermore, one has

$$\frac{d\vec{OM}}{dt} = v\cos\gamma\;\vec{k} - v\sin\gamma\;\vec{n} \quad (6)$$

Identifying (5) and (6) gives

$$\begin{cases} \dot{d} = v\cos\gamma \\ \dot{\psi} = -\frac{v}{d}\sin\gamma \end{cases}$$

Using the expression of θ in (3) yields

$$\dot{\gamma} = \dot{\psi} - \dot{\theta} = -\frac{v}{d}\sin\gamma - \dot{\theta}$$

Finally we obtain the following state space representation

$$\begin{cases} \dot{d} = v\cos\gamma \\ \dot{\gamma} = -\frac{v}{d}\sin\gamma - \dot{\theta} \\ \dot{\psi} = -\frac{v}{d}\sin\gamma \end{cases} \quad (7)$$

The linear and rotational velocities of the vehicle, namely v and $\dot{\theta}$ in (7), are the control variables.
If we consider u_1 and u_2 as the new control variables

$$\begin{cases} u_1 = \frac{v}{d} \\ u_2 = \dot{\theta} \end{cases}$$

then, system (7) becomes

$$\begin{cases} \dot{d} = d\,u_1\cos\gamma \\ \dot{\gamma} = -u_1\sin\gamma - u_2 \\ \dot{\psi} = -u_1\sin\gamma \end{cases} \quad (8)$$

Remark 1. The difference between this particular polar representation and the classical one is that the second half of the plane usually described by $\psi \in \left]\pi, 2\pi\right[$ and $d > 0$ is substituted by $\psi \in \left]0, \pi\right[$ and $d < 0$. This representation allows the vehicle to move forward if $(d < 0)$ and backward if $(d > 0)$, as it will be shown later.

Remark 2. The Brockett necessary conditions for the stabilizability via smooth stationary state feedback are fulfilled if and only if $d \neq 0$ (see [5])

3 Exponentially Stabilizing Controller

3.1. Controller Synthesis

In this section, we focus our attention to the design of a discontinuous control law for the point stabilization problem. One has the following result

Proposition. Assume that $d(0) \neq 0$. Then, the equilibrium point $(d = 0,\ \gamma = 0,\ \psi = 0)$ is asymptotically stable under the following control law

$$\begin{cases} u_1 = -k_1 \\ u_2 = k_2\gamma - (k_3\ \psi + \gamma)\dfrac{\sin\gamma}{\gamma} u_1 \end{cases} \tag{9}$$

where, k_1, k_2 and k_3 are strictly positive parameters. □

Proof. Let us consider the following Lyapunov function

$$V(\gamma, \psi) = \frac{1}{2}(k_3\psi^2 + \gamma^2) \tag{10}$$

Differentiating this expression with respect to time gives

$$\dot{V}(\gamma, \psi) = -u_2\gamma - (k_3\psi + \gamma)u_1 \sin\gamma \tag{11}$$

Substituting u_1 and u_2 into (11) we obtain

$$\dot{V}(\gamma, \psi) = -k_2\gamma^2 \tag{12}$$

It is clear that $\dot{V}(\gamma, \psi)$ is always negative semi-definite since k_2 is positive, therefore $V(\gamma, \psi)$ decreases with respect to time and tends to a finite positive value $V_{lim} \geq 0$, when t tends to infinity. Thus, $\dot{V}(\gamma, \psi)$ tends to zero, and the convergence of γ to zero immediately follows and so do $\dot{\gamma}$ when t tends to infinity. Therefore, from (8), u_2 tends to zero. Moreover, from (9), u_2 tends to $k_1 k_3\ \psi$. Hense, the convergence of ψ to zero readily follows. Finally, one can say that the largest invariant set defined by $\dot{V}(\gamma, \psi) = 0$ is restricted to $\{\ \psi = 0, \gamma = 0\ \}$.

Now, consider the first equation of (8) under the control law (9)

$$\dot{d} = -k_1 d \cos\gamma\ .$$

From this equation the exponential convergence of d to zero occurs when $\cos\gamma > 0$ (i.e. when γ reaches $\left]-\pi/2, \pi/2\right[$). □

Remark 3. The signed distance $d(t)$ approaches exponentially zero without never crossing the plane $d = 0$, whenever $d(0) \neq 0$

Remark 4. The assumption $d(0) \neq 0$ is not very restrictive as it is always possible to apply an open loop control, for an arbitrary small period of time, to drive the vehicle away from $d = 0$ and then switch to the feedback (9).

3.2. Choice of Control Parameters

In this section we will give some guidelines for the choice of the free control design parameters, namely k_1, k_2 and k_3, such that the trajectory curvature remains bounded and goes to zero when t tends to infinity.
The curvature of the generated trajectory under the proposed control law is given by

$$C = \frac{\dot{\theta}}{v} = -\frac{k_2\ \gamma}{d\ k_1} - \frac{(k_3\psi + \gamma)\sin\gamma}{d\ \gamma} \tag{13}$$

which can be bounded as follows

$$|C| \leq \left(\frac{k_2}{k_1} + 1\right)\frac{|\gamma|}{|d|} + k_3\frac{|\psi|}{|d|} \tag{14}$$

Since the proposed controller ensures the boundedness of the whole state, it is sufficient to guarantee that both $\frac{|\gamma|}{|d|}$ and $\frac{|\psi|}{|d|}$ tend to zero when t tends to infinity. In other words, we must ensure that the angles γ and ψ converge to zero faster than the distance d in the neighborhood of the equilibrium point $(0,0,0)$. To this end, let us consider the resulting closed-loop system (8)-(9):

$$\begin{cases} \dot{d} = -k_1\ d\cos\gamma \\ \dot{\gamma} = -k_2\ \gamma - k_1 k_3\ \psi\dfrac{\sin\gamma}{\gamma} \\ \dot{\psi} = k_1\ \sin\gamma \end{cases} \tag{15}$$

which can be approximated, in the neighborhood of the equilibrium point, by the following partially decoupled linear system

$$\begin{cases} \dot{d} = -k_1 d \\ \dot{\gamma} = -k_2 \gamma - k_1 k_3 \psi \\ \dot{\psi} = k_1 \gamma \end{cases} \tag{16}$$

The latter bring into evidence that the signed distance d converges to zero as $(e^{-k_1 t})$, while γ and ψ converge to zero faster than $(e^{-\lambda t})$, where $(-\lambda)$ is the real part of the dominant pole of the following subsystem

$$\begin{pmatrix} \dot{\gamma} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} -k_2 & -k_1 k_3 \\ k_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \psi \end{pmatrix} \tag{17}$$

Now, it is possible to conclude that the vehicle reaches the goal asymptotically to the direction of $\vec{i}_0$ (i.e. γ and ψ converge to zero faster than d) if $\lambda > k_1$. To satisfy this condition it is sufficient to choose $k_2 = 2k_1\sqrt{k_3}$ and $k_3 > 1$. In this case, the curvature becomes bounded as follows

$$|C| \le \left(2\sqrt{k_3} + 1\right)\frac{|\gamma|}{|d|} + k_3 \frac{|\psi|}{|d|} \tag{18}$$

Hence, k_3 can be tuned to control the bounds of the curvature, while k_1 can be tuned to control the linear velocity of the vehicle.

4 Simulation Results

In this section, we present some simulation results, for different initial configurations, to show the vehicle motion under the proposed control law, with $k_1 = 0.5$, $k_2 = \sqrt{3}$ and $k_3 = 3$.

The configuration space is divided into two subspaces Ω_1(i.e. $y > 0$) and Ω_2 (i.e. $y < 0$). As shown in Fig. 2, the vehicle reaches the final configuration $(0,0,0)$ in forward motion whenever its initial configuration is into the subspace Ω_2, and in backward motion if its initial configuration is into the subspace Ω_1.

Figure. 3 and Fig. 6 show the time evolution of the state variables and the generated paths for the following particular initial configurations

$$d = 2,\ \gamma = \frac{\pi}{2},\ \psi = \frac{\pi}{2} \quad \text{and} \quad d = -2,\ \gamma = \pi,\ \psi = \frac{\pi}{4}$$

Figure. 4 and Fig. 7 show the time evolution of the control variables v and $\dot{\theta}$, while Fig. 5 and Fig. 8 show the vehicle motion for the previous initial configurations.

The presented simulations clearly show the boundedness of the control variables and the smoothness of the generated paths (i.e. no inversion of motion occurs).

Remark 5. It is worth noticing that the coordinates transformation proposed in *section 2.1* is not defined for $y = 0$. Furthermore, exponential convergence of the signed distance d does not imply systematically exponential convergence of the variable y. These facts, can instigate harmful effects whenever y is brought to cross the axis $y = 0$. However, if the vehicle is, initially, in the neighborhood of the free-y axis, one must apply an open loop control to drive the vehicle, 'sufficiently', within one of the subspaces Ω_1 and Ω_2 and then switch to the feedback (9).

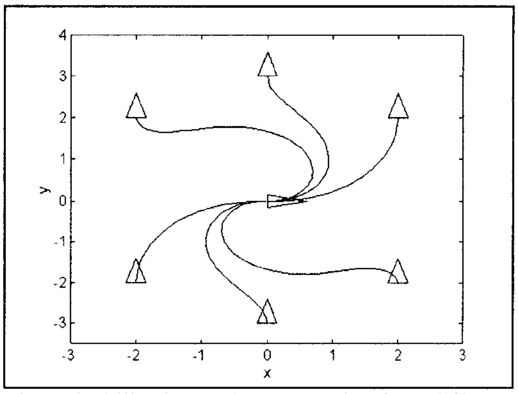

Fig. 2. Stabilization at $(0,0,0)$ starting from different configurations.

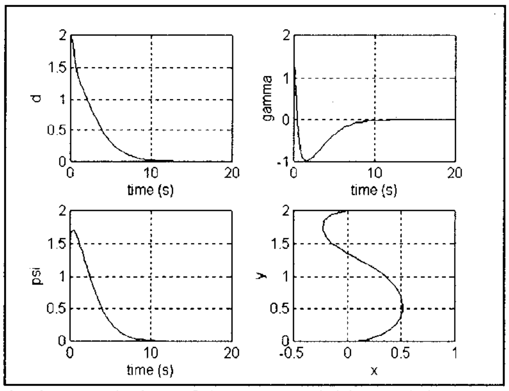

Fig. 3 Time evolution of the state variables and the generated trajectory starting from the configuration $(2, \frac{\pi}{2}, \frac{\pi}{2})$.

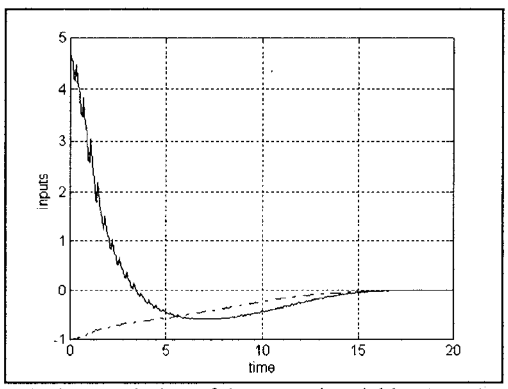

Fig. 4 Time evolution of the control variables (v $\cdots$) and ($\dot{\theta}$ —) for the initial conditions $(2, \frac{\pi}{2}, \frac{\pi}{2})$.

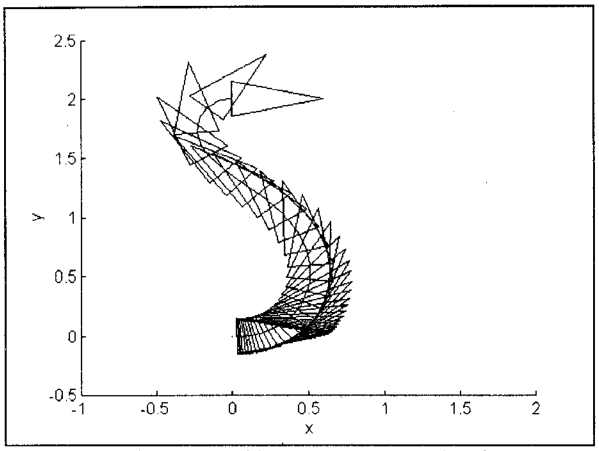

Fig. 5. Vehicle in parking maneuver, starting from the configuration $(2, \frac{\pi}{2}, \frac{\pi}{2})$.

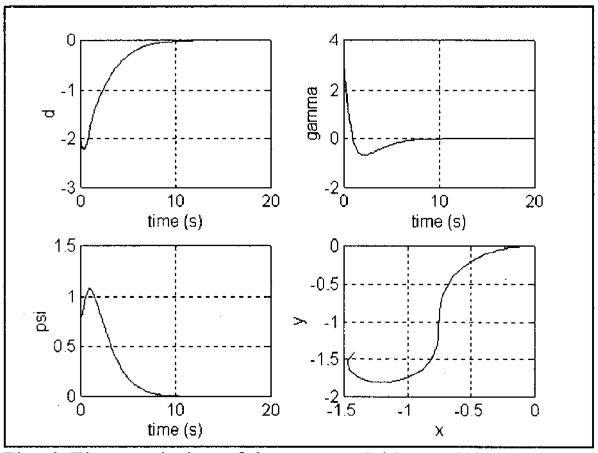

Fig. 6. Time evolution of the state variables and the generated trajectory starting from the configuration $(-2, \pi, \frac{\pi}{4})$.

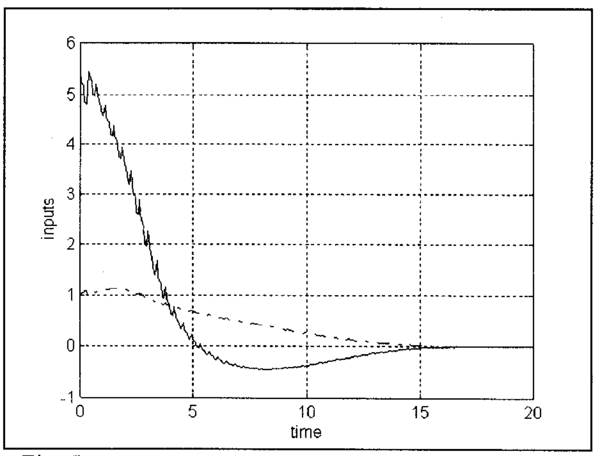

Fig. 7 Time evolution of the control variables ($v \cdots$) and ($\dot{\theta}$ —) for the initial conditions $(-2, \pi, \frac{\pi}{4})$

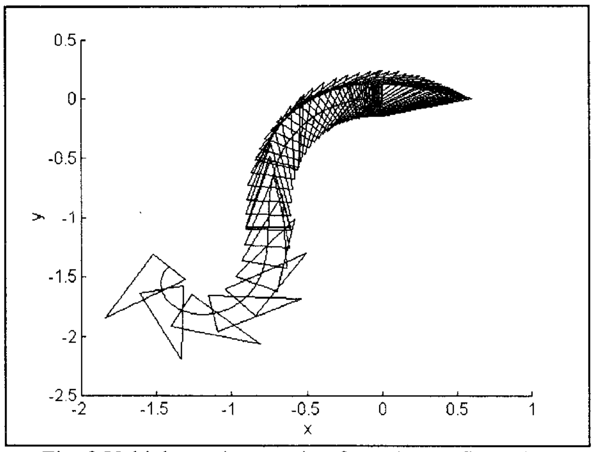

Fig. 8 Vehicle motion starting from the configuration $(-2, \pi, \frac{\pi}{4})$.

5 Conclusion

In this paper, an exponentially stablizing controller based on a new polar representation and Lyapunov method has been proposed for the stabilization of a unicycle-like vehicle. This controller allows to drive the vehicle, in forward or backward motion, from any initial configuration to any final one without inverting the direction of motion, provided that the initial conditions are into the domain Ω.

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