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Control of Mechanical Systems with Classical Nonholonomic Constraints

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Abstract

A theoretical framework for the control of mechanical systems with m ≥1 classical nonholonomic constraints is established. In particular, we emphasize certain control properties for mechanical systems with nonholonomic constraints that have no counterpart in systems with holonomic constraints. Conditions for smooth stabilization of an m-dimensional equilibrium manifold are presented; we also demonstrate that smooth stabilization of a single equilibrium solution is not possible. The development is illustrated using two physical examples: the control of a knife edge moving on a plane surface and the control of a wheel rolling without slipping on a plane surface; the physical significance of the obtained theoretical results is described for these examples.

1. Introduction

Numerous papers have been published in recent years on the control of mechanical systems with holonomic constraints. Work of the authors includes Bloch and McClamroch in [1], McClamroch and Bloch in [2], and McClamroch and Wang in [3]. Earlier work which deals with control of systems which are nonholonomic in nature is described by Brockett in [4]. Several models for the control of robotic fingers have recently been proposed by Cole, et. al. in [5] and by Hui and Goldenberg in [6]; these models include classical nonholonomic constraints. A recent paper by Bloch [7] has examined several control theoretic issues which pertain to both holonomic and nonholonomic constraints in a very general form. In this paper, we present several interesting results that relate specifically to the control of mechanical systems with classical nonholonomic constraints.

In this paper, we extend our methods for holonomic systems to the nonholonomic case. However, our previous consideration of control problems for mechanical systems with holonomic constraints can only provide general guidance for the class of problems considered here. In particular, the methods used in our previous work must be modified in several important ways for the class of problems studied in this paper. More interestingly, there are new control phenomena that can occur for mechanical systems with nonholonomic constraints that have no counterpart in systems with holonomic constraints. The purpose of this paper is to describe some of these interesting phenomena, how control theoretic problems for this class of problems should be formulated and how some of these control problems can be solved.

2. Models of Mechanical Systems with Classical Nonholonomic Constraints

We consider the class of mechanical systems with classical nonholonomic constraints which are described by the equations

$$M(q)\ddot{q} + F(q,\dot{q}) = J'(q)\lambda + B(q)u \tag{2.1}$$

$$J(q)\dot{q} = 0 \tag{2.2}$$

where q is an n-vector of configuration variables, $\dot{q}$ is an n-vector of velocity variables, and $\ddot{q}$ is an n-vector of acceleration variables; in addition, u is an r-vector of control input variables and λ is an m-vector of constraint multipliers. The nxn matrix function M(q) is assumed to be symmetric and positive definite, $F(q,\dot{q})$ is an n-vector function, J(q) denotes an mxn matrix function which is assumed to have full rank, and B(q) is an nxr matrix function. All of these functions are assumed to be smooth analytic functions defined on appropriate open subsets of the $(q,\dot{q})$ phase space. The formulation could be given in terms of specified smooth manifolds; we have not made such a generalization since it is direct. Various assumptions about the control input variables are indicated subsequently.

Differential-algebraic equations of the above form are known to arise from (controlled) mechanical systems with classical nonholonomic constraints, for example by use of appropriate variational principles; see Neimark and Fufaev [8] for many examples. The m algebraic constraint equations are said to be classical nonholonomic constraints since they are linear in the velocity variables and since the constraints are assumed not to be integrable. The assumption that the constraints are not integrable precludes the case that the constraints are linear; consequently, each system in this class is intrinsically nonlinear.

We now introduce the constraint manifold in the $(q,\dot{q})$ phase space defined by

$$\mathbf{M} = \{(q,\dot{q}) \in R^n x R^n \mid J(q)\dot{q} = 0\} \tag{2.3}$$

Note that since the constraints are nonintegrable, there is in fact no restriction on the value of the configuration variables. This manifold **M** plays a critical role in the formulation and solution of control problems associated with (2.1)-(2.2).

Since we assume that the mxn matrix J(q) has full rank, there is no loss of generality in assuming that the configuration variables are ordered so that

$$J(q) = [\, J_1(q)\ J_2(q) \,] \tag{2.4}$$

where $J_1(q)$ is an (n-m)xm matrix function and $J_2(q)$ is an mxm invertible matrix function.

Our development is given in terms of equations (2.1)-(2.2) which are associated with a Lagrangian formulation for mechanical systems with classical nonholonomic constraints. We remark that the development could also be expressed in terms of a Hamiltonian formulation for mechanical systems with classical nonholonomic constraints.

We are interested in feedback control of the form $u = U(q,\dot{q})$ where $U: \mathbf{M} \to R^r$; the corresponding closed loop is described by

$$M(q)\ddot{q} + F(q,\dot{q}) = J'(q)\lambda + B(q)U(q,\dot{q}) \tag{2.5}$$

$$J(q)\dot{q} = 0 \tag{2.6}$$

We point out the obvious fact that the closed loop is still defined in terms of the nonholonomic constraint equation.

3. Solution Properties

We begin by making clear that equations (2.1)-(2.2) and equations (2.5)-(2.6) do represent well posed models in the sense that an associated initial value problem has a unique solution, at least

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locally. A pair of vector functions $(q(t),\lambda(t))$ defined on an interval $[0,T)$ is a solution of the initial value problem defined by equations (2.1)-(2.2) [or equations (2.5)-(2.6)] and the initial condition data $(q_0,\dot{q}_0)$ if $q(t)$ is at least twice differentiable, $\lambda(t)$ is integrable, the vector functions $(q(t),\lambda(t))$ satisfy the differential-algebraic equations (2.1)-(2.2) [or equations (2.5)-(2.6)] almost everywhere on their domain of definition and the initial conditions $(q(0),\dot{q}(0)) = (q_0,\dot{q}_0)$.

The following existence and uniqueness result can be stated.

Theorem 3.1. (a) Assume that the control input function $u: [0,T) \rightarrow R^r$ is a given bounded and measurable function for some $T > 0$. If the initial condition data satisfy $(q_0,\dot{q}_0) \in \mathbf{M}$, then there exists a unique solution (at least locally defined) of the initial value problem corresponding to equations (2.1)-(2.2) which satisfies $(q(t),\dot{q}(t)) \in \mathbf{M}$ for each t for which the solution is defined.

(b) Assume that the feedback function $u = U(q,\dot{q})$ where $U: \mathbf{M} \rightarrow R^r$ is a smooth function. If the initial condition data satisfy $(q_0,\dot{q}_0) \in \mathbf{M}$, then there exists a unique solution (at least locally defined) of the initial value problem corresponding to equations (2.5)-(2.6) which satisfies $(q(t),\dot{q}(t)) \in \mathbf{M}$ for each t for which the solution is defined.

Proof: Following the development by Rheinboldt in [9], it is easily demonstrated that the differential-algebraic equations define a smooth vector field on the differentiable manifold $\mathbf{M}$; the results follow according to [10].

Since the differential-algebraic equations (2.1)-(2.2) [or equations (2.5)-(2.6)] define a smooth vector field on the manifold $\mathbf{M}$, a number of other results could be stated, including conditions for continuous dependence of the solution on initial conditions and parameters, conditions for non-existence of finite escape times, etc. Such results are important, but they are not given here since they are easily obtained. We use the notation $(Q(t,q_0,\dot{q}_0),\Lambda(t,q_0,\dot{q}_0))$ to denote the solution of equations (2.1)-(2.2) [or equations (2.5)-(2.6)] at time $t \geq 0$ corresponding to the initial conditions $(q_0,\dot{q}_0)$. Thus for each initial condition $(q_0,\dot{q}_0) \in \mathbf{M}$ and each bounded, measurable input function $u: [0,\infty) \rightarrow R^r$ [or each smooth feedback function $u = U(q,\dot{q})$], $(Q(t,q_0,\dot{q}_0),\dot{Q}(t,q_0,\dot{q}_0)) \in \mathbf{M}$ holds for all $t \geq 0$ where the solution is defined.

A particularly important class of solutions are the equilibrium solutions of (2.1)-(2.2) [or equations (2.5)-(2.6)]. A solution is an equilibrium solution if it is a constant solution. The following result should be clear.

Theorem 3.2. (a) Suppose that $u(t) = 0$, $t \geq 0$. Equations (2.1)-(2.2) have a smooth manifold of equilibrium solutions which is defined by the set $\{(q,\lambda) \in R^n x R^m \mid F(q,0) - J'(q)\lambda = 0\}$. The manifold of equilibrium solutions is a smooth submanifold of the configuration space, given by

$$\{q \in R^n \mid F(q,0) - J'(q)\lambda = 0 \text{ for some } \lambda \in R^m\}.$$

(b) Suppose that $u = U(q,\dot{q})$ where $U: \mathbf{M} \rightarrow R^r$ is a smooth function. Equations (2.5)-(2.6) have a smooth manifold of equilibrium solutions which is defined by the set

$$\{(q,\lambda) \in R^n x R^m \mid F(q,0) - J'(q)\lambda = B(q)U(q,0)\}.$$

The manifold of equilibrium solutions is a smooth submanifold of the configuration space, given by

$$\{q \in R^n \mid F(q,0) - J'(q)\lambda = B(q)U(q,0) \text{ for some } \lambda \in R^m\}.$$

(c) In either case the equilibrium submanifold of the configuration space has dimension that is at least m.

The remainder of this paper is concerned with the development of conditions for smooth stabilization of an m-dimensional equilibrium manifold and with consideration of (non-smooth) stabilization of a single equilibrium solution. The development is illustrated using two physical examples: the control of a knife edge moving on a plane surface and the control of a vertical wheel rolling without slipping on a plane surface; the physical significance of the obtained theoretical results is described for these examples.

4. Formulation of Stabilization Problem

Our goal is to formulate a stabilization problem for mechanical systems with nonholonomic constraints as described by equation (2.1)-(2.2). A suitable stability definition is first introduced.

Definition 4.1. Assume that $u = U(q,\dot{q})$. Let $\mathbf{M_s}$ be a smooth manifold in $\mathbf{M}$. $\mathbf{M_s}$ is locally stable if for any neighborhood $\mathbf{U} \supset \mathbf{M_s}$ there is a neighborhood $\mathbf{V}$ of $\mathbf{M_s}$ with $\mathbf{U} \supset \mathbf{V} \supset \mathbf{M_s}$ such that if $(q_0,\dot{q}_0) \in \mathbf{V} \cap \mathbf{M}$ then the solution of equations (2.5)-(2.6) satisfies $(Q(t,q_0,\dot{q}_0),\dot{Q}(t,q_0,\dot{q}_0)) \in \mathbf{U} \cap \mathbf{M}$ for all $t \geq 0$. If, in addition, $(Q(t,q_0,\dot{q}_0),\dot{Q}(t,q_0,\dot{q}_0)) \rightarrow (q_s,\dot{q}_s)$ as $t \rightarrow \infty$ for some $(q_s,\dot{q}_s) \in \mathbf{M_s}$ then we say that $\mathbf{M_s}$ is locally asymptotically stable.

Note that if $(Q(t,q_0,\dot{q}_0),\dot{Q}(t,q_0,\dot{q}_0)) \rightarrow (q_s,\dot{q}_s)$ as $t \rightarrow \infty$ for some $(q_s,\dot{q}_s) \in \mathbf{M_s}$, it follows that there is $\lambda_s \in R^n$ such that $\Lambda(t,q_0,\dot{q}_0) \rightarrow \lambda_s$ as $t \rightarrow \infty$.

The usual definition of local stability correspond to the case that $\mathbf{M_s}$ is a single equilibrium solution; the more general case is required in the present paper.

The existence of a feedback function for which a certain manifold is stabilized is of particular interest; hence we introduce the following.

Definition 4.2. The system defined by equation (2.1)-(2.2) is said to be locally stabilizable to a smooth manifold $\mathbf{M_s}$ in $\mathbf{M}$ if there exists a feedback function $U: \mathbf{M} \rightarrow R^r$ such that, for the associated closed loop equations (2.5)-(2.6), $\mathbf{M_s}$ is locally asymptotically stable.

If there exists such a feedback function which is smooth on $\mathbf{M}$ then we say that equations (2.1)-(2.2) are smoothly stabilizable to $\mathbf{M_s}$; of course it is possible that equations (2.1)-(2.2) might be stabilizable to $\mathbf{M_s}$ but not smoothly stabilizable to $\mathbf{M_s}$.

The following important consequences follow from Theorem 3.2, part (c). Smooth feedback cannot be used to make the dimension of the equilibrium manifold for equations (2.5)-(2.6) less than m. In particular, if $m > 0$ there is no smooth feedback for which equations (2.5)-(2.6) have an isolated equilibrium solution. Consequently, smooth feedback cannot be used to stabilize an isolated equilibrium solution of equations (2.5)-(2.6).

5. Decomposition Transformation

The fundamental approach, described in previous work by McClamroch and Wang in [3] on control of mechanical systems with holonomic constraints, is to introduce a certain coordinate transformation so that the constraints have a trivial specification; once the constraints are enforced explicitly a set of decomposed differential-algebraic equations are obtained: the reduced differential equations characterize the (control dependent) motion on the constraint manifold independently of the constraint force and the algebraic equations characterize the (control dependent) constraint force in terms of the motion on the constraint manifold. In this work where nonholonomic constraints are studied, the same approach is successful. However, unlike the case for holonomic constraints, the resulting differential equations for the constrained motion do not have the same basic Lagrangian type structure of the unconstrained equations. In addition, the particular structure which these reduced equations do have implies certain results regarding control and stabilization which are not observed in the holonomic situation.

It should be emphasized that the subsequent development is assumed to be carried out locally; this should be understood even if it is not always explicitly stated. Consider the diffeomorphism

$$x_1 = q_1$$

$$x_2 = \dot{q}_1$$

$$x_3 = q_2$$

$$x_4 = \dot{q}_2 + J_2^{-1}(q)\, J_1(q)\, \dot{q}_1$$

with local inverse given by

$$q_1 = x_1$$

$$\dot{q}_1 = x_2$$

$$q_2 = x_3$$

$$\dot{q}_2 = x_4 - J_2^{-1}(x)\, J_1(x) x_2$$

where we have used the partition $q' = (q_1,q_2)'$ which is consistent with equation (2.4), that is $q_1 \varepsilon R^{n-m}$ and $q_2 \varepsilon R^m$. In the above note that $J_1(x)$ and $J_2(x)$ actually depend only on x_1 and x_3. In the transformed variables the nonholonomic constraint is given by the trivial equation $x_4 = 0$; upon enforcing this constraint we obtain the 2n-m reduced order differential equations

$$\dot{x}_1 = x_2 \qquad (5.1)$$

$$\dot{x}_2 = \overline{F}(x) + \overline{B}(x)u \qquad (5.2)$$

$$\dot{x}_3 = - J_2^{-1}(x)\, J_1(x)\, x_2 \qquad (5.3)$$

In the above equations it should be noted that there is no dependence on λ; this variable drops out due to the explicit form of the transformation. In addition, the notation $x' = (x_1,x_2,x_3)'$ is used where $x_1 \varepsilon R^{n-m}$, $x_2 \varepsilon R^{n-m}$ and $x_3 \varepsilon R^m$ in the above equations. The n-m vector function $\overline{F}(x)$ and the (n-m)xr matrix function $\overline{B}(x)$ can easily be determined from the indicated transformation.

Equations (5.1)-(5.3) have a very special structure which we examine in the sequel. Note that these equations are a special case of the normal form equations studied by Byrnes and Isidori in [11]. In particular, note that the zero dynamics equation of (5.1)-(5.3) is given by

$$\dot{x}_3 = 0, \qquad (5.4)$$

and it is not locally asymptotically stable. The fact that the zero dynamics is described by equation (5.4), which is a linear system with all zero eigenvalues, means that the equations (5.1)-(5.3) are critically minimum phase at (0,0,0); this has some important implications in terms of local stabilizabilty of (2.1)-(2.2).

6. Stabilization to an Equilibrium Manifold using Smooth Feedback

In this section we introduce the problem of stabilization of equations (2.1)-(2.2) to the smooth equilibrium submanifold $\{(q_1,q_2) \mid q_1 = 0\}$ of the configuration space which corresponds to the submanifold of **M** defined by

$$\mathbf{N_e} = \{(q_1,q_2,\dot{q}_1,\dot{q}_2) \mid q_1 = 0,\ \dot{q}_1 = 0,\ \dot{q}_2 = 0\} \qquad (6.1)$$

where the partitioning of the variables is as indicated in Section 5. This is an m dimensional smooth submanifold of **M**. We show that, with appropriate assumptions, there exists a smooth feedback such that the closed loop is locally asymptotically stable to $\mathbf{N_e}$.

The smooth stabilization problem is to give conditions so that there exists a smooth feedback function U: $\mathbf{M} \rightarrow R^r$ such that $\mathbf{N_e}$ is locally asymptotically stable according to Definition 4.1. Of course, our interest is not only to demonstrate that such a smooth feedback exists but also to indicate how such a stabilizing smooth feedback can be constructed.

Our basic approach is to make use of the decomposition introduced in the previous section, namely the differential equations (5.1)-(5.3). In particular, it is clear that the stabilization problem for equations (2.1)-(2.2) is equivalent to a stabilization problem associated with equations (5.1)-(5.3). Thus our approach is to indicate that the latter problem has a solution which can be used to construct a solution for the former.

Theorem 6.1. Equations (2.1)-(2.2) are locally stabilizable to $\mathbf{N_e}$ using smooth feedback if the matrix $\overline{B}(x)$ defined according to equation (5.2) has a smooth right inverse.

Proof: Suppose that $\overline{B}(x)$ has a smooth right inverse denoted by $B^r(x)$. Let the controller be given by

$$u = B^r(x)\, [V(x_1,x_2) - \overline{F}(x)]\,.$$

Then, the closed loop system, expressed in terms of the local coordinates introduced in Section 5, is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = V(x_1,x_2)$$

$$\dot{x}_3 = - J_2^{-1}(x)\, J_1(x)\, x_2.$$

The function $V(x_1,x_2)$ can be chosen to be smooth and to guarantee that $x_1(t) \rightarrow 0$ and $x_2(t) \rightarrow 0$ as $t \rightarrow \infty$, exponentially, for initial conditions sufficiently close to (0,0). Consequently, it follows that $\dot{x}_3(t) \rightarrow 0$ and $x_3(t)$ tends to some (unspecified) constant value; thus $(q(t),\dot{q}(t)) \rightarrow \mathbf{N_e}$ as $t \rightarrow \infty$ locally.

Our development in this section has indicated that, under appropriate assumptions, equations (2.1)-(2.2) can be smoothly stabilized to the m dimensional equilibrium manifold specified by (6.1). However, the definition of this equilbrium manifold depends on the specific partitioning of the variables as indicated in Section 5. More generally, if the function V, which may be chosen as a function of x_1, x_2, and x_3, is chosen as in the proof of Theorem 6.1, then the system may be stabilized to any submanifold of the configuration space satisfying $V(x_1,0,x_3) = 0$.

7. Stabilization to an Equilibrium Solution using Nonsmooth Feedback

The results in the previous section demonstrate that smooth feedback can be used to stabilize the smooth manifold $\mathbf{N_e}$ in **M**, where the dimension of $\mathbf{N_e}$ exceeds the number m of independent constraints. Consequently, those results do not guarantee smooth stabilization to a single equilibrium solution if $m \geq 1$.

In fact, there is no smooth feedback which can stabilize the closed loop to a single equilibrium solution. For suppose that there is a smooth feedback which stabilizes, for example, the origin. Then it follows from the algebraic equations for equilibrium that there is a smooth equilibrium manifold of dimension m containing the origin; that is, the origin is not isolated which contracts the assumptions that it is asymptotically stable. We state this formally.

Theorem 7.1. Let $m \geq 1$ and let $\mathbf{N_e}$ consist of a single equilbrium solution in **M**. Equations (2.1)-(2.2) are not smoothly stabilizable to $\mathbf{N_e}$.

This follows directly from the fact that equations (5.1)-(5.3) do not satisfy necessary conditions for smooth stabilization given by

Brockett in [12]. The violation of these necessary conditions can be traced directly to the special structure of equations (5.1)-(5.3) that arises from the nonholonomic constraints.

Of course, it may be that a single equilibrium solution simply cannot be stabilized or it may be that any stabilizing feedback is necessarily nonsmooth. The paper [13] by Sontag provides a survey of a number of theoretical developments that relate to nonsmooth stabilization.

8. Two Example Stabilization Problems

In this section we briefly describe mathematical models for the control of the motion of two physical examples, each of whose motion must satisfy nonholonomic constraints. These examples, as well as many other examples, are studied by Neimark and Fufaev in [8], although not in terms of control issues. For simplicity in both examples, all numerical constants are set to unity.

We first consider the control of a knife edge moving in point contact on a plane surface. Let x and y denote the coordinates of the point of contact of the knife edge on the plane and let ϕ denote the heading angle of the knife edge, measured from the x-axis. Then the equations of motion are given by

$$\ddot{x} = \lambda \sin\phi + u_1 \cos\phi \tag{8.1}$$

$$\ddot{y} = - \lambda \cos\phi + u_1 \sin\phi \tag{8.2}$$

$$\ddot{\phi} = u_2 \tag{8.3}$$

where u_1 denotes the control force in the direction defined by the heading angle, u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the scalar (m=1) nonholonomic constraint

$$\dot{x} \sin\phi - \dot{y} \cos\phi = 0 \tag{8.4}$$

It is clear that the constraint manifold is a five-dimensional manifold and is defined by

$$\mathbf{M} = \{(\phi,x,y,\dot{\phi},\dot{x},\dot{y}) \mid \dot{x} \sin\phi - \dot{y} \cos\phi = 0\}$$

and any configuration variable is an equilibrium if the controls are zero. We now introduce the transformation that results in the decomposition described in Section 5. Define the variables

$$x_1 = \phi,\ x_2 = x,\ x_3 = \dot{\phi},\ x_4 = \dot{x},\ x_5 = y,\ x_6 = \dot{y} - \dot{x} \tan\phi$$

so that the reduced differential equations are given by

$$\dot{x}_1 = x_3,$$

$$\dot{x}_2 = x_4,$$

$$\dot{x}_3 = u_1,$$

$$\dot{x}_4 = - x_3 x_4 \tan x_1 + u_2 \cos x_1 ,$$

$$\dot{x}_5 = x_4 \tan x_1.$$

Denote the above by

$$\dot{x} = f(x) + u_1\, g_1(x) + u_2\, g_2(x)$$

where f is the drift vector field and g_1 and g_2 are the two control vector fields. The following conclusions are based on analysis of the above differential equations.

Proposition 1. The knife edge dynamics described by equations (8.1)-(8.4) have the following properties:

1. There is a smooth feedback which stabilizes the closed loop to any smooth one dimensional equilibrium manifold in **M**. Construction of a smooth feedback function U: $\mathbf{M} \to R^2$ which stabilizes a given one dimensional smooth manifold is quite easy in this case.
2. There is no smooth feedback which stabilizes any single equilibrium solution, for example (0,0,0,0,0,0).
3. The above differential equations are strongly accessible from (0,0,0,0,0) since the space spanned by the vectors g_1, g_2, $[f,g_1]$, $[f,g_2]$ and $[[f,[f,g_2]],g_1]$ has dimension 5 at (0,0,0,0,0).

As a second example, we consider the control of a vertical wheel rolling without slipping on a plane surface. Let x and y denote the coordinates of the point of contact of the wheel on the plane, let ϕ denote the heading angle of the wheel, measured from the x-axis and let θ denote the rotation angle of the wheel due to rolling, measured from a fixed reference. Then the equations of motion are given by

$$\ddot{x} = \lambda_1 \tag{8.5}$$

$$\ddot{y} = \lambda_2 \tag{8.6}$$

$$\ddot{\theta} = -\lambda_1 \cos\phi - \lambda_2 \sin\phi + u_1 \tag{8.7}$$

$$\ddot{\phi} = u_2 \tag{8.8}$$

where u_1 denotes the control torque about the rolling axis of the wheel and u_2 denotes the control torque about the vertical axis through the point of contact; the components of the forces of constraint arise from the (m=2) nonholonomic constraints

$$\dot{x} = \cos\phi\, \dot{\theta} \tag{8.9}$$

$$\dot{y} = \sin\phi\, \dot{\theta} \tag{8.10}$$

The constraint manifold is six-dimensional and is given by

$$\mathbf{M} = \{(\theta,\phi,x,y,\dot{\theta},\dot{\phi},\dot{x},\dot{y}) \mid \dot{x} = \cos\phi\, \dot{\theta},\ \dot{y} = \sin\phi\, \dot{\theta}\}$$

and any configuration variable is an equilibrium if the controls are zero. We now introduce the transformation that results in the decomposition described in Section 5. Define the variables

$$x_1 = \theta,\ x_2 = \phi,\ x_3 = \dot{\theta},\ x_4 = \dot{\phi},$$
$$x_5 = x,\ x_6 = y,\ x_7 = \dot{x} - \dot{\theta} \cos\phi,\ x_8 = \dot{y} - \dot{\theta} \sin\phi$$

so that the reduced differential equations are given by

$$\dot{x}_1 = x_3,$$

$$\dot{x}_2 = x_4,$$

$$\dot{x}_3 = \frac{1}{2} u_1,$$

$$\dot{x}_4 = u_2,$$

$$\dot{x}_5 = x_3 \cos x_2,$$

$$\dot{x}_6 = x_3 \sin x_2.$$

Denote the above by

$$\dot{x} = f(x) + u_1\, g_1(x) + u_2\, g_2(x)$$

where f is the drift vector field and g_1 and g_2 are the two control vector fields. The following conclusions are based on analysis of the above differential equations.

Proposition 2. The rolling wheel dynamics described by equations (8.5)-(8.10) have the following properties:

1. There is a smooth feedback which stabilizes the closed loop to any smooth two dimensional equilibrium manifold in **M**. Construction of a smooth feedback function U: $\mathbf{M} \rightarrow R^2$ which stabilizes a given two dimensional smooth manifold is easy in this case.
2. There is no smooth feedback which stabilizes any smooth manifold of dimension less than two, for example (0,0,0,0,0,0,0,0).
3. The above differential equations are strongly accessible from (0,0,0,0,0,0) since the space spanned by the vectors g_1, g_2, $[f,g_1]$, $[f,g_2]$, $[[f,[f,g_1]],g_2]$ and $[[[f,[f,[f,g_2]]],g_2],g_1]$ has dimension 6 at (0,0,0,0,0,0).

In the above two examples, it is interesting that a single equilbrium solution, e.g. the origin, cannot be stabilized using any smooth feedback; in particular, a single equilibrium solution cannot be stabilized using any linear feedback. The strong accessibility results indicated above, see for example [14] by Sussman and Jurdjevic, are mentioned since they are suggestive, but accessibility does not imply controllability or stabilizability for nonlinear systems as discussed by Sontag in [15], for example. However, a nonsmooth feedback can be constructed algorithmically which does stabilize the above systems to the origin. In order to do this, however, controls are required which direct the system away from the neighborhood of the origin, i.e. the control strategy is nonlocal in nature. This suggests that the systems may not be small time locally controllable. These issues will be investigated further in forthcoming work. As noted earlier, the fundamentally nonlinear characteristics of these problems arise directly from the nature of the nonholonomic constraints.

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