6. Representing Rotation

Mechanics of Manipulation

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Outline.

- Generalities
- Axis-angle
- Rodrigues's formula
- Rotation matrices
- Euler angles

Why representing rotations is hard.

- Rotations do not commute.
- The topology of spatial rotations does not permit a smooth embedding in Euclidean three space.

Choices

- More than three numbers
  - Rotation matrices
  - Unit quaternions. (aka Euler parameters)
- Many-to-one
  - Axis times angle (matrix exponential)
- Unsmooth and many-to-one
  - Euler angles
- Unsmooth and many-to-one and more than three numbers
  - Axis-angle

Axis-angle

Recall Euler's theorem: every spatial rotation leaves a line of fixed points: the rotation axis.

Let O, $\hat{\mathbf{n}}$, θ, be ...

Let $\mathrm{rot}(\hat{\mathbf{n}}, \theta)$ be the corresponding rotation.

Many to one:

$\mathrm{rot}(-\hat{\mathbf{n}}, -\theta) = \mathrm{rot}(\hat{\mathbf{n}}, \theta)$

$\mathrm{rot}(\hat{\mathbf{n}}, \theta + 2k\pi) = \mathrm{rot}(\hat{\mathbf{n}}, \theta)$, for any integer k.

When $\theta = 0$, the rotation axis is indeterminate, giving an infinity-to-one mapping.

Representation

What do we want from a representation? For a start:

- Rotate points;

  Rodrigues's formula

- Compose rotations;

  Using axis-angle? Ugh.

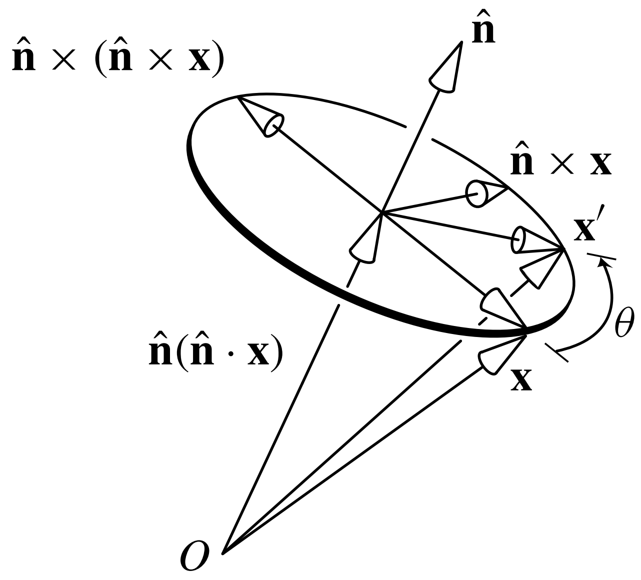
- (Convert to other representations.)

Rodrigues's formula

Others derive Rodrigues's formula using rotation matrices, missing the geometrical aspects.

Given point $\mathbf{x}$, decompose into components parallel and perpendicular to the rotation axis

$$\mathbf{x} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) - \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$



Only $\mathbf{x}_\perp$ is affected by the rotation, yielding *Rodrigues's formula*:

$$\mathbf{x}' = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \sin\theta\ (\hat{\mathbf{n}} \times \mathbf{x}) - \cos\theta\ \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

A common variation:

$$\mathbf{x}' = \mathbf{x} + (\sin\theta)\ \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos\theta)\ \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Rotation matrices

Choose O on rotation axis. Choose frame $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$.

Let $(\hat{\mathbf{u}}_1', \hat{\mathbf{u}}_2', \hat{\mathbf{u}}_3')$ be the image of that frame.

Write the $\hat{\mathbf{u}}_i'$ vectors in $\hat{\mathbf{u}}_i$ coordinates, and collect them in a matrix:

$$\hat{\mathbf{u}}_1' = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_1' \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1' \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}_1' \end{pmatrix}$$

$$\hat{\mathbf{u}}_2' = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2' \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_2' \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}_2' \end{pmatrix}$$

$$\hat{\mathbf{u}}_3' = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_3' \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_3' \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}_3' \end{pmatrix}$$

So many numbers

A rotation matrix has nine numbers,

but spatial rotations have only three degrees of freedom,

leaving six excess numbers ...

There are six constraints that hold among the nine numbers.

$$|\hat{\mathbf{u}}'_1| = |\hat{\mathbf{u}}'_2| = |\hat{\mathbf{u}}'_3| = 1$$

$$\hat{\mathbf{u}}'_3 = \hat{\mathbf{u}}'_1 \times \hat{\mathbf{u}}'_2$$

i.e. the $\hat{\mathbf{u}}'_i$ are unit vectors forming a right-handed coordinate system.

Such matrices are called *orthonormal* or *rotation* matrices.

Rotating a point

Let (x_1, x_2, x_3) be coordinates of $\mathbf{x}$ in frame $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$.

Then $\mathbf{x}'$ is given by the same coordinates taken in the $(\hat{\mathbf{u}}'_1, \hat{\mathbf{u}}'_2, \hat{\mathbf{u}}'_3)$ frame:

$$\begin{aligned}
\mathbf{x}' &= x_1 \hat{\mathbf{u}}'_1 + x_2 \hat{\mathbf{u}}'_2 + x_3 \hat{\mathbf{u}}'_3 \\
&= x_1 A \hat{\mathbf{u}}_1 + x_2 A \hat{\mathbf{u}}_2 + x_3 A \hat{\mathbf{u}}_3 \\
&= A(x_1 \hat{\mathbf{u}}_1 + x_2 \hat{\mathbf{u}}_2 + x_3 \hat{\mathbf{u}}_3) \\
&= A\mathbf{x}
\end{aligned}$$

So rotating a point is implemented by ordinary matrix multiplication.

Rotating a point

Let A and B be coordinate frames. Notation:

x	a point
$\mathbf{x}$	a geometrical vector, directed from an origin O to the point x or, a vector of three numbers, representing x in an unspecifie frame
${}^{A}\mathbf{x}$	a vector of three numbers, representing x in the A frame

Let ${}^{B}_{A}R$ be the rotation matrix that rotates frame B to frame A.

Then (see previous slide) ${}^{B}_{A}R$ represents the rotation of the point x:

$$ {}^{B}\mathbf{x}' = {}^{B}_{A}R \; {}^{B}\mathbf{x} $$

Note presuperscripts all match. Both points, and xform, must be written in same coordinate frame.

Coordinate transform

There is another use for ${}^B_A R$:

${}^A\mathbf{x}$ and ${}^B\mathbf{x}$ represent the same point, in frames A and B resp.

To transform from A to B:

$${}^B\mathbf{x} = {}^B_A R \, {}^A\mathbf{x}$$

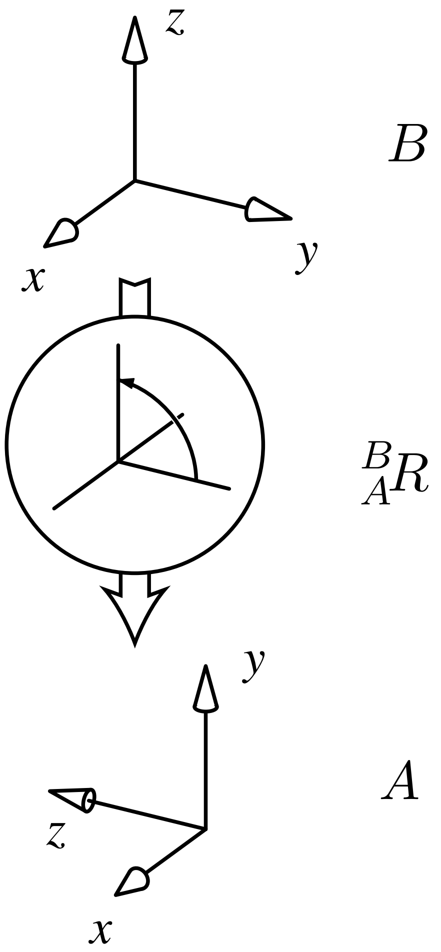
For coord xform, matrix subscript and vector superscript "cancel".

Rotation from B to A is the same as coordinate transform from A to B.

Example rotation matrix

$$
{}^B_AR = \left(\ {}^B\mathbf{x}_A \ \middle| \ {}^B\mathbf{y}_A \ \middle| \ {}^B\mathbf{z}_A \ \right)
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}
$$



How to remember what ${}^B_AR$ does? Pick a coordinate axis and see. The x axis isn't very interesting, so try y:

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Nice things about rotation matrices

- Composition of rotations: $\{R_1; R_2\} = R_2 R_1$.
  ($\{x; y\}$ means do x then do y.)
- Inverse of rotation matrix is its transpose ${}^B_A R^{-1} = {}^A_B R = {}^B_A R^T$.
- Coordinate xform of a rotation matrix:

$$^{B}R = {}^B_A R \, {}^A R \, {}^A_B R$$

Converting $\mathrm{rot}(\hat{\mathbf{n}}, \theta)$ to R

Ugly way: define frame with $\hat{\mathbf{z}}$ aligned with $\hat{\mathbf{n}}$, use coordinate xform of previous slide.

Keen way: Rodrigues's formula!

$$\mathbf{x}' = \mathbf{x} + (\sin\theta)\, \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos\theta)\, \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Define "cross product matrix" N:

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

so that

$$N\mathbf{x} = \hat{\mathbf{n}} \times \mathbf{x}$$

. . . using Rodrigues's formula . . .

Substituting the cross product matrix N into Rodrigues's formula:

$$\mathbf{x}' = \mathbf{x} + (\sin\theta)N\mathbf{x} + (1-\cos\theta)N^2\mathbf{x}$$

Factoring out $\mathbf{x}$

$$R = I + (\sin\theta)N + (1-\cos\theta)N^2$$

That's it! Rodrigues's formula in matrix form. If you want to you could expand it:

$$\begin{pmatrix} n_1^2 + (1-n_1^2)c\theta & n_1 n_2(1-c\theta) - n_3 s\theta & n_1 n_3(1-c\theta) + n_2 s\theta \\ n_1 n_2(1-c\theta) + n_3 s\theta & n_2^2 + (1-n_2^2)c\theta & n_2 n_3(1-c\theta) - n_1 s\theta \\ n_1 n_3(1-c\theta) - n_2 s\theta & n_2 n_3(1-c\theta) + n_1 s\theta & n_3^2 + (1-n_3^2)c\theta \end{pmatrix}$$

where $c\theta = \cos\theta$ and $s\theta = \sin\theta$. Ugly.

Rodrigues's formula for differential rotations

Consider Rodrigues's formula for a differential rotation $\mathrm{rot}(\hat{\mathbf{n}}, d\theta)$.

$$\begin{aligned}\mathbf{x}' &= (I + \sin d\theta N + (1 - \cos d\theta) N^2)\mathbf{x} \\ &= (I + d\theta N)\mathbf{x}\end{aligned}$$

so

$$\begin{aligned}d\mathbf{x} &= N\mathbf{x}\, d\theta \\ &= \hat{\mathbf{n}} \times \mathbf{x}\, d\theta\end{aligned}$$

It follows easily that differential rotations are vectors: you can scale them and add them up. We adopt the convention of representing angular velocity by the unit vector $\hat{\mathbf{n}}$ times the angular velocity.

Converting from R to $\mathrm{rot}(\hat{\mathbf{n}}, \theta) \ldots$

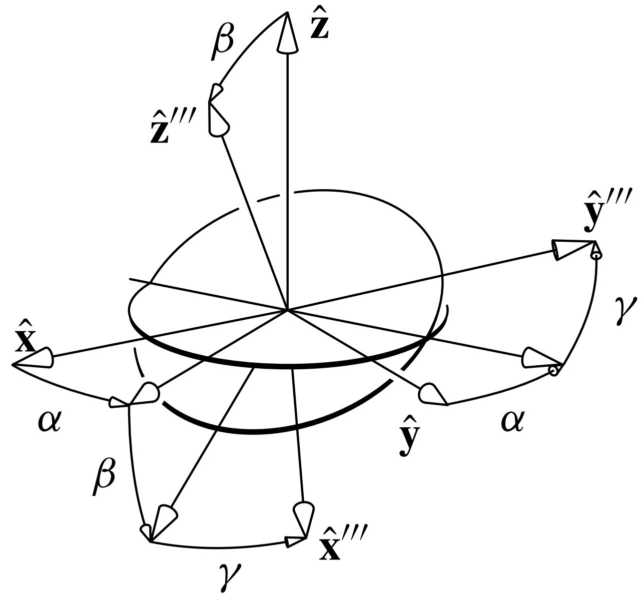
Problem: $\hat{\mathbf{n}}$ isn't defined for $\theta = 0$.

We will do it indirectly. Convert R to a unit quaternion (next lecture), then to axis-angle.

Euler angles

Three numbers to describe spatial rotations. ZYZ convention:

$$(\alpha, \beta, \gamma) \mapsto \mathrm{rot}(\gamma, \hat{\mathbf{z}}'')\,\mathrm{rot}(\beta, \hat{\mathbf{y}}')\,\mathrm{rot}(\alpha, \hat{\mathbf{z}})$$



Can we represent an arbitrary rotation?

Rotate α about $\hat{\mathbf{z}}$ until
    $\hat{\mathbf{y}}' \perp \hat{\mathbf{z}}'''$;
Rotate β about $\hat{\mathbf{y}}'$ until
    $\hat{\mathbf{z}}'' \parallel \hat{\mathbf{z}}'''$;
Rotate γ about $\hat{\mathbf{z}}''$ until
    $\hat{\mathbf{y}}'' = \hat{\mathbf{y}}'''$.

Note two choices for $\hat{\mathbf{y}}'$ ...

...except sometimes infinite choices.

From (α, β, γ) to R

Expand $\mathrm{rot}(\alpha, \hat{\mathbf{z}})\,\mathrm{rot}(\beta, \hat{\mathbf{y}})\,\mathrm{rot}(\gamma, \hat{\mathbf{z}})$

(Why is that the right order?)

$$\begin{pmatrix} \mathsf{c}\alpha & -\mathsf{s}\alpha & 0 \\ \mathsf{s}\alpha & \mathsf{c}\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{c}\beta & 0 & \mathsf{s}\beta \\ 0 & 1 & 0 \\ -\mathsf{s}\beta & 0 & \mathsf{c}\beta \end{pmatrix} \begin{pmatrix} \mathsf{c}\gamma & -\mathsf{s}\gamma & 0 \\ \mathsf{s}\gamma & \mathsf{c}\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathsf{c}\alpha\,\mathsf{c}\beta\,\mathsf{c}\gamma - \mathsf{s}\alpha\,\mathsf{s}\gamma & -\mathsf{c}\alpha\,\mathsf{c}\beta\,\mathsf{s}\gamma - \mathsf{s}\alpha\,\mathsf{c}\gamma & \mathsf{c}\alpha\,\mathsf{s}\beta \\ \mathsf{s}\alpha\,\mathsf{c}\beta\,\mathsf{c}\gamma + \mathsf{c}\alpha\,\mathsf{s}\gamma & -\mathsf{s}\alpha\,\mathsf{c}\beta\,\mathsf{s}\gamma + \mathsf{c}\alpha\,\mathsf{c}\gamma & \mathsf{s}\alpha\,\mathsf{s}\beta \\ -\mathsf{s}\beta\,\mathsf{c}\gamma & \mathsf{s}\beta\,\mathsf{s}\gamma & \mathsf{c}\beta \end{pmatrix} \tag{1}$$

From R to (α, β, γ) the ugly way

Case 1: $r_{33} = 1$, $\beta = \pi$. $\alpha - \gamma$ is indeterminate.

$$R = \begin{pmatrix} \cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & 0 \\ \sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Case 2: $r_{33} = -1$, $\beta = -\pi$. $\alpha + \gamma$ is indeterminate.

$$R = \begin{pmatrix} -\cos(\alpha-\gamma) & -\sin(\alpha-\gamma) & 0 \\ -\sin(\alpha-\gamma) & \cos(\alpha-\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For generic case: solve 3rd column for β. (Sign is free choice.)
Solve third column for α and third row for γ.

. . . but there are numerical issues . . .

From R to (α, β, γ) the clean way

Let

$$\sigma = \alpha + \gamma$$
$$\delta = \alpha - \gamma$$

Then

$$r_{22} + r_{11} = \cos\sigma(1 + \cos\beta)$$
$$r_{22} - r_{11} = \cos\delta(1 - \cos\beta)$$
$$r_{21} + r_{12} = \sin\delta(1 - \cos\beta)$$
$$r_{21} - r_{12} = \sin\sigma(1 + \cos\beta)$$

(No special cases for $\cos\beta = \pm 1$?)

Solve for σ and δ, then for α and γ, then finally

$$\beta = \tan^{-1}(r_{13}\cos\alpha + r_{23}\sin\alpha, r_{33})$$