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Robust Receding Horizon Control - Analysis & Synthesis

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Abstract— **In this paper, a combined *Receding Horizon Control* (RHC) analysis and synthesis technique for constrained linear time-invariant systems subject to polytopic or additive uncertainty is introduced. The proposed method consists of two components. First, the maximum robust control invariant set $\tilde{\mathcal{C}}_\infty$ is computed along with the associated piecewise affine (PWA) state feedback law. This computation is based on an iterative solution of *multi-parametric Quadratic Programs* (mp-QP). Second, the obtained PWA feedback law is analyzed for robust stability. This analysis is based on linear matrix inequalities (LMI). The combination of these two components guarantees robust stability and constraint satisfaction for all time. Extensive numerical examples show a decrease in controller complexity by a factor of up to 250 versus established controller design methods [15], [2], [7] at the cost of a degradation in performance of less than 0.5% on average.**

Index Terms— **Robust Stability, Feasibility, Optimal Control, Receding Horizon Control, LMI**

I. Introduction

The focus of this paper is on obtaining a robust piecewise affine (PWA) state feedback control law that can be computed off-line by solving an optimization problem as a *multi-parametric program* [2]. For any given state, the input sequence obtained with the feedback law is identical to the solution of the associated finite time constrained optimal control problem. In *Receding Horizon Control* (RHC), this input sequence is computed at each time step but only the first input of the sequence is applied to the plant. Stability of the nominal closed-loop system can be guaranteed by solving an infinite-horizon optimization problem [16] or by imposing terminal set constraints [15]. Recently methods to obtain robust feedback controllers based on min-max optimization with terminal set constraints have been published [1], [11]. However, large horizons are needed to cover the maximum control invariant set which inherently results in significant computational complexity. Furthermore, the approaches in [1], [11] are restricted to linear optimization objectives. The primary focus will be on achieving robust stability and feasibility. Performance of the nominal system is the secondary objective, which may be set by adjusting the prediction horizon N in the RHC formulation. The prediction horizon N may be seen as a tuning parameter which allows a tradeoff between controller complexity and closed-loop performance. Furthermore, we extend the stability analysis methods first presented in [10], [6] to deal with systems subject to polytopic and additive uncertainty. The methods proposed here consist of two components. First, the maximum robust control invariant set is computed along with the associated feedback law. This procedure is based on an iterative solution of multi-parametric programs. Second, the obtained controller is analyzed for robust stability.

II. Problem Formulation

In this section we will give a brief overview of the optimal control problem considered in this paper. We will postpone treatment of uncertainty to the next section and focus here on the nominal linear, time-invariant, discrete-time system

$$x(t+1) \quad = \quad Ax(t)+Bu(t), \qquad (1)$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and (A, B) controllable. Let $x(t)$ denote the measured state at time t and $x_{t+k|t}$ denote the predicted state at time $t+k$ given the state at time t. For brevity we will denote $x_{k|0}$ as x_k. Assume now that the states and the inputs of system (1) are subject to the constraints

$$x \in \mathbb{X} \subset \mathbb{R}^n, \qquad u \in \mathbb{U} \subset \mathbb{R}^m, \qquad (2)$$

where $\mathbb{X}$ and $\mathbb{U}$ are polyhedral sets containing the origin in their interior. Consider the finite-time constrained LQR problem

$$J_N^*(x(0)) = \min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} \left(u_k'\mathcal{R}u_k + x_k'\mathcal{Q}x_k\right) + x_N'\mathcal{Q}_f x_N, \qquad (3a)$$

$$\begin{aligned}
\text{subj. to} \quad & x_k \in \mathbb{X}, \qquad \forall k \in \{1,\dots,N\}, && (3b)\\
& u_{k-1} \in \mathbb{U}, \quad \forall k \in \{1,\dots,N\}, && (3c)\\
& x_{k+1} = Ax_k + Bu_k, \quad x_0 = x(0), && (3d)\\
& \mathcal{Q} = \mathcal{Q}' \succeq 0, \quad \mathcal{Q}_f = \mathcal{Q}_f' \succeq 0, \quad \mathcal{R} = \mathcal{R}' \succ 0. && (3e)
\end{aligned}$$

Before going further, we will introduce the following definitions and theorems.

Definition 1. *We define the N-step feasible set $\mathcal{X}_f^N \subseteq \mathbb{R}^n$ as the set of initial states $x(0)$ for which the optimal control problem (3) is feasible, i.e. for which problem (3) has a solution satisfying the constraints.*

$$\begin{aligned}\mathcal{X}_f^N = \{x(0) \in \mathbb{R}^n | \exists U_N^* = [u_0', \dots, u_{N-1}']' \in \mathbb{R}^{Nm},\\ x_k \in \mathbb{X},\ u_{k-1} \in \mathbb{U},\ \forall k \in \{1,\dots,N\}\}.\end{aligned}$$

Because problem (3) depends on $x(0)$, it can be solved either by solving the quadratic program (3) for a given $x(0)$, or as shown in [2], by solving problem (3) for all $x(0)$ within a polyhedral set of values, i.e., by considering (3) as a *multi-parametric Quadratic Program* (mp-QP).

Theorem 1. *[2] Consider the finite time constrained LQR problem (3). Then, the set of feasible parameters $\mathcal{X}_f^N$ is convex, the optimizer $U_N^* : \mathcal{X}_f^N \mapsto \mathbb{R}^{Nm}$ is continuous and piecewise affine (PWA), i.e.*

$$\begin{aligned}
U_N^*(x(0)) &= \hat{F}_r x(0) + \hat{G}_r \quad \text{if} \quad x(0) \in \mathcal{P}_r,\\
\mathcal{P}_r &= \{x \in \mathbb{R}^n | H_r x \le K_r\},\ r = 1,\dots,R,
\end{aligned}$$

and the optimal solution $J_N^* : \mathcal{X}_f^N \mapsto \mathbb{R}$ is continuous, convex and piecewise quadratic (PWQ).

Henceforth, we will denote the feedback law which provides the first input as $u_0 = F_r x + G_r$. Analysis of the mp-QP solution [2] yields that $\{\mathcal{P}_r\}_{r=1}^R$ is a polyhedral partition of $\mathcal{X}_f^N$. Henceforth we will denote the polyhedron $\mathcal{P}_r$ as region r. Since the upper bound on the number of regions is exponential in the prediction horizon [8], large horizons are, in general, computationally prohibitive. Although the procedure in [2] has recently been extended to cope with uncertainties for optimization problems with linear objectives [11], [1], the robust approaches published to date suffer from computational requirements which are even more severe than for the nominal case. In order to create a basis for comparison of the various algorithms, the following definition is introduced here:

Definition 2. *[3] The set $\tilde{\mathcal{C}}_\infty(\mathbb{R}^n)$ is the maximal robust control invariant set contained in $\mathbb{R}^n$ for the system $x_{k+1} = f(x_k, u_k, w_k)$ with the constraints in (2) if and only if $\tilde{\mathcal{C}}_\infty(\mathbb{R}^n)$ is control invariant and contains all the robust invariant sets contained in $\mathbb{R}^n$ which uphold the constraints in (2).*

Henceforth we will assume that the set $\tilde{\mathcal{C}}_\infty(\mathbb{R}^n)$ is bounded because of the constraints in (2) and therefore simply refer to $\tilde{\mathcal{C}}_\infty$.

III. Polytopic Uncertainty

In this section we will generalize the system in (1) to incorporate polytopic uncertainty which results in the system

$$x_{k+1} = A(\lambda)x_k + B(\lambda)u_k, \tag{4}$$

with $A(\lambda) \in \mathbb{R}^{n\times n}$, $B(\lambda) \in \mathbb{R}^{n\times m}$ and

$$[A(\lambda)|B(\lambda)] \in \Omega, \tag{5a}$$

$$\Omega \triangleq Conv\{[A^{(1)}|B^{(1)}], \dots, [A^{(L)}|B^{(L)}]\}, \tag{5b}$$

i.e., there exist L nonnegative coefficients λ_l $(l = 1, \dots, L)$ such that

$$\sum_{l=1}^{L} \lambda_l = 1 , \qquad [A(\lambda)|B(\lambda)] = \sum_{l=1}^{L} \lambda_l [A^{(l)}|B^{(l)}]. \tag{6}$$

This type of uncertainty has been studied for many years [5] and is a flexible tool to describe uncertain systems. In Section III-A we will present methods to compute the PWA feedback law that produces the maximum robust invariant set. In section III-B, an analysis procedure capable of testing robust stability for RHC is introduced.

A. Feasibility for Polytopic Uncertainty

In order to ascertain robust constraint satisfaction, it is necessary to augment the constraints in (2) and (3) to take all the possible dynamics in (5) into account, i.e.

$$J_N^*(x(0)) = \min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1} \left(u_k'\mathcal{R}u_k + x_k'\mathcal{Q}x_k\right) + x_N'\mathcal{Q}_f x_N, \tag{7a}$$

$$\begin{aligned}
\text{subj. to } & x_1^{(l)} \in \mathbb{X}, \quad x_1^{(l)} \in T_{\text{set}}, && \text{(7b)}\\
& x_k \in \mathbb{X}, \qquad \forall k \in \{1, \dots, N\}, && \text{(7c)}\\
& u_{k-1} \in \mathbb{U}, \quad \forall k \in \{1, \dots, N\}, && \text{(7d)}\\
& x_1^{(l)} = A^{(l)}x_0 + B^{(l)}u_0, \quad \forall l \in \{1, \dots, L\}, && \text{(7e)}\\
& x_{k+1} = Ax_k + Bu_k, \quad x_0 = x(0), && \text{(7f)}\\
& \mathcal{Q} = \mathcal{Q}' \succeq 0, \quad \mathcal{Q}_f = \mathcal{Q}_f' \succeq 0, \quad \mathcal{R} = \mathcal{R}' \succ 0, && \text{(7g)}
\end{aligned}$$

where $[A \mid B] \in \Omega$ denote the nominal dynamics, $[A^{(l)}|B^{(l)}]$ denote the vertices of the uncertain dynamics Ω and T_{set} in (7b) denotes an additional constraint on the first state, as will be explained below. The constraints are imposed for all vertices of the uncertainty set for the first time step (7b). Because of linearity, this implies constraint satisfaction for all dynamics $[A(\lambda)|B(\lambda)] \in \Omega$ in the first time step. For all subsequent steps the nominal dynamics are assumed (7c). In order to compute the maximum robust invariant set $\tilde{\mathcal{C}}_\infty$ as well as the associated feedback law, the following iterative algorithm is applied:

Algorithm 1.

1) *Initialize the set constraint $T_{set} = \mathbb{R}^n$.*
2) *Solve (7) with the set constraint T_{set} as an mp-QP with prediction horizon N to obtain the region partition $\mathcal{P}_r = \{x \in \mathbb{R}^n | H_r x \leq K_r\}$, $\forall r \in \{1, \dots, R\}$ with the convex union [2] $\mathcal{X}_f^N \equiv \bigcup_1^R \mathcal{P}_r$ and the associated feedback laws $u = F_r x + G_r$.*
3) *If $T_{set} \neq \mathcal{X}_f^N$, set $T_{set} = \mathcal{X}_f^N$ and goto 2, else return.*

The system uncertainty is only considered in the first step in order to avoid infeasibility of (7), which may result when adding robust constraints to an open-loop optimization problem. However, the iterative approach in Algorithm 1 ensures that the resulting PWA controller keeps the state within $\tilde{\mathcal{C}}_\infty$ for all time.

Theorem 2. *The polyhedral-invariant-set Algorithm 1 converges to the maximum invariant set $\tilde{\mathcal{C}}_\infty$ for system (4).*

Proof. At each iteration i, the set constraint $T_{\text{set}}(i)$, is a subset of the set constraint at the previous iteration. If the feasible set at iteration i is denoted by $\mathcal{X}_f^N(i)$, it follows that $\mathcal{X}_f^N(i) \supseteq \mathcal{X}_f^N(j)$, if and only if $i \leq j$. If $\mathcal{X}_f^N(i) \equiv \mathcal{X}_f^N(j)$ with $i < j$, it follows from (7) that $\mathcal{X}_f^N(i) \equiv T_{\text{set}}$ and the algorithm terminates. The set is invariant by design of (7) and due to the fact that $\mathcal{X}_f^N \equiv T_{\text{set}}$, the set $\mathcal{X}_f^N \equiv \tilde{\mathcal{C}}_\infty$ at the point of termination [12]. ■

By shrinking the set constraint T_{set} with a scalar parameter $0 < \gamma < 1$, finite time termination is always attained [3]. This is because in Algorithm 1, the volume decrease of the feasible set $\mathcal{X}_f^N$ from iteration to iteration is not arbitrarily small, such that the final volume is obtained in finite time. The advantage of the presented algorithm over other methods [15], [11], [1], lies in the fact that the resulting controller covers the maximum robust control invariant set regardless of the optimization horizon N. The prediction horizon N serves merely as a tuning parameter which trades off complexity for performance. Furthermore the imposed set constraint $\mathcal{T}_{\text{set}}$ is the weakest constraint guaranteeing robust feasibility and will be satisfied by any robust controller. Therefore, $\mathcal{T}_{\text{set}}$ has the minimal possible impact on performance. Note that although the control invariant set guarantees feasibility for all time, no statement on stability can be made.

B. Stability for Polytopic Uncertainty

As shown in [14], it is sufficient to find a Lyapunov function which guarantees stability for all the vertices $[A^{(l)}|B^{(l)}] \in \Omega$ in (6) to prove stability for all $[A(\lambda)|B(\lambda)] \in \Omega$. An LMI formulation for computing a piecewise quadratic (PWQ) Lyapunov function

$$P_{\text{PWQ}}(x) = x'Q_r x + x'L_r + C_r, \qquad \text{for} \quad H_r x \le K_r, \quad (8)$$

guaranteeing exponential stability for uncertain systems of type (4) will be presented in this section. As shown in [6], this function does not need to be continuous to imply stability for discrete time systems. If a state moves from region s to region t in one time step for the system dynamics $[A^{(l)}|B^{(l)}] \in \Omega$, the decay rate of the Lyapunov function $\Delta V_{st}^{(l)}(x_0)$ is defined as:

$$\begin{aligned}\Delta V_{st}^{(l)}(x_0) &= P_{\text{PWQ}}(x_1^{(l)}) - P_{\text{PWQ}}(x_0),\\ &= x_0' \, \Delta Q_{st}^{(l)} \, x_0 + 2x_0' \, \Delta L_{st}^{(l)} + \Delta C_{st}^{(l)},\end{aligned}$$

where the definitions of $\Delta Q_{st}^{(l)}$, $\Delta L_{st}^{(l)}$ and $\Delta C_{st}^{(l)}$ follow from (4), (8) and Theorem 1. We will assume that all transitions from the subset $\mathcal{T}_{st}^{(l)} = \{x \in \mathbb{R}^n | H_{st}^{(l)} \, x \le K_{st}^{(l)}\}$ of region s, enter region t in one time step for the dynamics $[A^{(l)} \mid B^{(l)}] \in \Omega$. The computation of $H_{st}^{(l)}$ and $K_{st}^{(l)}$ corresponds to a reach-set computation which can be implemented efficiently with the methods in [9]. The set of all transitions st is stored in the set denoted as $\Pi^{(l)} = \{s, t \in \mathbb{N}^+ \mid \mathcal{T}_{st}^{(l)} \ne \varnothing\}$. We aim to find the function $P_{\text{PWQ}}(x)$ which satisfies the following conditions:

$$P_{\text{PWQ}}(x) \ge \epsilon ||x||^2, \quad \epsilon > 0, \quad (9a)$$

$$\Delta V_{st}^{(l)}(x) \le -\rho ||x||^2, \quad \rho > 0, \quad (9b)$$

$$\forall st \in \Pi^{(l)}, \ \forall x \in \mathcal{X}_f^N \text{ and } \forall l \in \{1, \dots, L\}. \quad (9c)$$

The conditions in (9) can be reformulated along the lines of [5], [6], [10] by considering only the vertices of the uncertainty Ω to obtain a formulation suitable for LMI solvers:

$$\text{find } P_{\text{PWQ}}, \ N_{st}^{(l)}, \ N_r, \ \rho, \ \epsilon, \ \text{s.t.} \quad (10a)$$

$$\forall r \in \{1, \dots, R\}, \ \forall st \in \Pi^{(l)} \ \text{ and } \forall l \in \{1, \dots, L\},$$

$$\begin{bmatrix} -\Delta Q_{st}^{(l)} - \rho I & -\Delta L_{st}^{(l)} \\ -(\Delta L_{st}^{(l)})' & -\Delta C_{st}^{(l)} \end{bmatrix} \succeq \begin{bmatrix} -(H_{st}^{(l)})' \\ (K_{st}^{(l)})' \end{bmatrix} N_{st}^{(l)} \, [-H_{st}^{(l)} \ \ K_{st}^{(l)}], \quad (10b)$$

$$\begin{bmatrix} Q_r - \epsilon I & 0.5 L_r \\ 0.5 L_r' & C_r \end{bmatrix} \succeq \begin{bmatrix} -H_r' \\ K_r' \end{bmatrix} N_r \, [-H_r \ \ K_r], \quad (10c)$$

$$N_{st}^{(l)} \ge 0, \quad N_r \ge 0, \quad \rho > 0, \quad \epsilon > 0, \quad (10d)$$

$$N_r = N_r', \quad N_r \in \mathbb{R}^{d_r \times d_r}, \ C_1 = 0, \quad (10e)$$

$$N_{st}^{(l)} = (N_{st}^{(l)})', \quad N_{st}^{(l)} \in \mathbb{R}^{d_{st} \times d_{st}}, \ L_1 = 0 \in \mathbb{R}^n. \quad (10f)$$

Here (10b) induces $\Delta V_{st}^{(l)}(x) \le -\rho ||x||^2$, (10c) ascertains that the PWQ Lyapunov function is positive and (10d) ensures that ρ, ϵ and all elements of the matrices N_r and $N_{st}^{(l)}$ are nonnegative, with d_r and d_{st} denoting the number of rows of H_r and $H_{st}^{(l)}$ respectively. The scalar parameter ϵ is greater than zero in order to enforce a positive Lyapunov function, whereas $\rho > 0$ is needed to ascertain exponential stability. Equation (10f) ensures that a quadratic upper bound for the Lyapunov function exists, by enforcing a quadratic Lyapunov function around the origin, which is covered by region $r = 1$.

Theorem 3. *If a PWQ function $P_{\text{PWQ}}(x)$ is found by solving (10), the system in (4) is exponentially stable for all dynamics $[A(\lambda)|B(\lambda)] \in \Omega$.*

Proof. Follows from [6] and [14]. ∎

It should be noted that due to the conservative nature of the LMI analysis, the proposed procedure may not lead to a solution in all cases. If the LMI computation fails to identify a Lyapunov function, the weight matrices $\mathcal{Q}$ and $\mathcal{R}$ and the prediction horizon N in (3) can serve as tuning parameters which may increase the likelihood of successful Lyapunov function identification. However, as shown in Section V, the proposed LMI formulation is far less conservative than may be expected.

IV. Additive Uncertainty

In this section we will augment the system in (1) to incorporate bounded additive uncertainty

$$x_{k+1} = Ax_k + Bu_k + w_k \quad (11)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $w_k \in \mathbb{W} \subset \mathbb{R}^n$ where $\mathbb{W}$ is polyhedral. In the following, we will demonstrate how an infinite-time feasible controller can be found for the system in (11) and how stability can be shown by solving an LMI.

A. Feasibility for Additive Uncertainty

Analogously to Section III-A it is necessary to 'tighten' the constraints on the state in (2) by applying the following changes:

$$x_1 \in \tilde{T}_{\text{set}} \triangleq \{x \in T_{\text{set}} |\ x + w \in T_{\text{set}},\ \forall w \in \mathbb{W}\}, \quad (12a)$$

$$x_k \in \tilde{\mathbb{X}} \triangleq \{x \in \mathbb{X} |\ x + w \in \mathbb{X},\ \forall w \in \mathbb{W}\}, \quad (12b)$$

$$u_k \in \mathbb{U} \subseteq \mathbb{R}^m. \quad (12c)$$

Here, (12a) denotes a set constraint on the first step. The sets $\tilde{T}_{\text{set}}$ in (12a) and $\tilde{\mathbb{X}}$ in (12b) correspond to the states where the additive uncertainty cannot lead to infeasibility, and are known as Pontryagin difference sets [12]. The polytopes $\tilde{T}_{\text{set}}$ and $\tilde{\mathbb{X}}$ can easily be computed by solving d LPs, where d denotes the number of rows of T_{set} or $\mathbb{X}$, respectively. The following algorithm will yield a PWA state feedback controller which covers $\tilde{\mathcal{C}}_\infty$:

Algorithm 2.

1) *Initialize the set constraint* $\tilde{T}_{set} = \mathbb{R}^n$.
2) *Solve the optimization problem in (3) with the constraints in (12) as an mp-QP with prediction horizon* N *to obtain the region partition* $\mathcal{P}_r = \{x \in \mathbb{R}^n | H_r x \leq K_r\}$, $\forall r \in \{1, \dots, R\}$ *with the convex union [2]* $\mathcal{X}_f^N \equiv \bigcup_1^R \mathcal{P}_r$ *and the associated feedback laws* $u = F_r x + G_r$.
3) *If* $T_{set} \neq \mathcal{X}_f^N$, *set* $T_{set} = \mathcal{X}_f^N$, *compute* $\tilde{T}_{set}$ *and goto 2, else return.*

We refer to section III-A for the proof of convergence. Note that the algorithm can be speeded up by only imposing robust constraints on x_1 in step 3, i.e., $x_1 \in \tilde{\mathbb{X}}$ and $x_k \in \mathbb{X}$, $\forall k \in \{2, \dots, N\}$.

B. Stability for Additive Uncertainty

If the system is subjected to additive uncertainty of the type $w_k \in \mathbb{W}$, where $\mathbb{W} = \{x \in \mathbb{R}^n \mid H_w x \leq K_w\}$, sufficient conditions for stability will be presented in this section. Specifically, we are looking for the following quadratic Lyapunov function,

$$\text{find } P_L \succ 0, \text{ s.t. } \forall x_0 \in \mathcal{X}_f^N \setminus \tilde{\mathcal{X}}_I,\ \forall w \in \mathbb{W}, \quad (13a)$$

$$x_1' P_L x_1 - x_0' P_L x_0 \leq -\rho ||x||^2,\ \rho > 0, \quad (13b)$$

$$x_1 = (A + BF_r)x_0 + BG_r + w \text{ if } H_r x_0 \leq K_r. \quad (13c)$$

The Lyapunov decay rate $\Delta V_r^w(x_0)$ for region r over one step is:

$$\Delta V_r^w(x_0) = x_1' P_L x_1 - x_0' P_L x_0 = v' \Delta Q_r^w v,$$

where $v = [x_0\ w\ 1]$ and

$$\Delta Q_r^w = \begin{bmatrix} (A+BF_r)' P_L (A+BF_r) - P_L & * & * \\ \frac{1}{2} P(A+BF_r) & P_L & * \\ G_r' B' P (A+BF_r) & \frac{1}{2} G_r' B' P' & G_r' B' P B G_r \end{bmatrix}. \quad (14)$$

Here $*$ is a placeholder for the elements of the symmetric matrix ΔQ_r^w. Obviously, the region $\mathcal{P}_1$ containing the origin will never satisfy the Lyapunov stability conditions, since $\Delta V_r^w(x_0) > 0$ if $x_0 = 0$ and $\mathbb{W} \neq 0$. Therefore, it is necessary to extract a robust control invariant set $\tilde{\mathcal{X}}_I \subseteq \mathcal{P}_1$ around the origin for which stability cannot be proven [13], [4]. Note that it would be possible to show input-to-state (ISS) stability for $\tilde{\mathcal{X}}_I$. If $\mathcal{P}_1$ is not completely covered by $\tilde{\mathcal{X}}_I$, i.e $\mathcal{P}_1 \supset \tilde{\mathcal{X}}_I$, it is necessary to divide the uncovered regions $\mathcal{P}_1 \setminus \tilde{\mathcal{X}}_I$ into convex sets for which the conditions in (13) have to hold. Division of the remaining set can be easily achieved by extending the facets of the polyhedral robust control invariant set $\tilde{\mathcal{X}}_I$ as is depicted in Figure 1. The LMI (15) is subsequently solved for all regions $r > 1$.

Problem (13) can be reformulated as an LMI by applying the S-procedure [5]:

$$\text{find } P_L,\ N_r,\ \rho \text{ s.t. } \forall r \in \{2, \dots, R\}, \quad (15a)$$

$$-\Delta Q_r^w - \rho \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq \begin{bmatrix} -\tilde{H}_r' \\ \tilde{K}_r' \end{bmatrix} N_r [-\tilde{H}_r\ \ \tilde{K}_r], \quad (15b)$$

$$N_r \geq 0,\ N_r = N_r' \in \mathbb{R}^{d_r \times d_r},\ P_L \succ 0,\ \rho > 0, \quad (15c)$$

with

$$\tilde{H}_r = \begin{bmatrix} H_r & 0 \\ 0 & H_w \end{bmatrix}, \qquad \tilde{K}_r = \begin{bmatrix} K_r \\ K_w \end{bmatrix}.$$

Equation (15c) signifies that P_L is positive definite and all elements of the matrix N_r are nonnegative with d_r denoting the number of rows in $\tilde{H}_r$. The scalar parameter $\rho > 0$ is added to enforce exponential stability. Note that setting the diagonal elements of N_r to zero reduces the computational complexity without influencing the result.

Theorem 4. *If a quadratic matrix* P_L *satisfying (13) (and therefore also (15)) is found, the system in (11) is exponentially stable over the entire region partition* $\mathcal{X}_f^N \setminus \tilde{\mathcal{X}}_I$ *for all types of disturbances* $w \in \mathbb{W}$. *Furthermore, the state will enter the robust invariant set* $\tilde{\mathcal{X}}_I$ *containing the origin in finite time.*

Proof. The proof for exponential stability follows from [6] and will be omitted for brevity. It follows from exponential stability that the state will enter the set $\tilde{\mathcal{X}}_I$ in finite time, since the origin is contained in the interior of $\tilde{\mathcal{X}}_I$. This is because the Lyapunov decay rate will be negative but not be arbitrarily large for all initial states $x_0 \in \tilde{\mathcal{C}}_\infty \setminus \tilde{\mathcal{X}}_I$. ■

V. Numerical Examples

In this section we will demonstrate how Algorithms 1 and 2 can be used to obtain controllers of low complexity.

Example 1. *Consider the second order system in [11]*

$$x(t+1) = \begin{pmatrix} 1 & 0.8 \\ 0 & 0.7 \end{pmatrix} x_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

The task is to regulate the system to the origin while fulfilling the constraints

$$\begin{array}{rcll} -3 \leq & u(t) & \leq 3, & \forall t \geq 0 \\ -10 \leq & x_{1,2}(t) & \leq 10, & \forall t \geq 1 \end{array}$$

The weight on the state is set to $\mathcal{Q} = I$ *and the input-weight is* $\mathcal{R} = 1$.

TABLE I

THE TABLE DISPLAYS THE AMOUNT OF UNCERTAINTY FOR WHICH STABILITY AND INFINITE-TIME FEASIBILITY CAN BE SHOWN WITH THE CONTROLLER OBTAINED WITH ALGORITHM 1. THE TABLE SHOWS WHETHER THE LYAPUNOV ANALYSIS WAS SUCCESSFUL (PWQ: YES/NO) AND THE NUMBER OF REGIONS R IN THE RESULTING CONTROL LAW.

Prediction Horizon $N=1$	Example 1 (PWQ / R)	Example 2 (PWQ / R)
$\delta = 10\%$	Yes / 3	Yes / 3
$\delta = 20\%$	Yes / 3	Yes / 3
$\delta = 30\%$	Yes / 3	Yes / 3
$\delta = 40\%$	Yes / 3	No / 3
$\delta = 50\%$	Yes / 3	No / 3
$\delta = 60\%$	No / 3	No / 3

Example 2. *Consider the system*

$$x(t+1) = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} \\ 1 & 0 \end{pmatrix} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$

The task is to regulate the system to the origin while fulfilling the input constraint

$$\begin{aligned} -5 \leq \quad & u(t) \quad \leq 5, & \forall t \geq 0 \\ -20 \leq \quad & x_{1,2}(t) \quad \leq 20, & \forall t \geq 1 \end{aligned}$$

The weight on the state is set to $\mathcal{Q} = 10I$ *and the input-weight is* $\mathcal{R} = 0.1$.

A. Polytopic Uncertainty

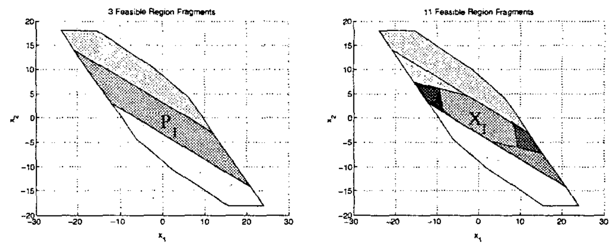
We will assume a simple type of polytopic uncertainty given by $L = 2$ and:

$$[A^{(1)}|B^{(1)}] = [(1-\delta)A \mid B], \quad [A^{(2)}|B^{(2)}] = [(1+\delta)A \mid B]$$

For various values of δ and a prediction horizon of $N = 1$, Algorithm 1 is used to compute a PWA control law. The results are given in Table I. With the algorithm in [1], example 1 was solved for $\delta = 0.4$ in 35 seconds and 154 regions covering $\tilde{\mathcal{C}}_\infty$ were obtained. Note that [1] enforces robust performance whereas the algorithm presented here only enforces nominal performance. With the algorithm in this paper, the computation took under 3 seconds and 3 regions were obtained. Note that both controllers cover $\tilde{\mathcal{C}}_\infty$ and guarantee stability and feasibility for all time.

B. Additive Uncertainty

We will assume additive uncertainty according to (11) as $\mathbb{W} = \{ w \in \mathbb{R}^n | \ ||w||_\infty \leq \gamma\}$. With the LMIs in (15), stability for the maximum robust invariant set $\tilde{\mathcal{C}}_\infty \setminus \tilde{\mathcal{X}}_I$ can be shown for $|\gamma| \leq 1.6$ in Example 1. In [11], Example 1 was solved for $\gamma = 0.1$ and horizon $N = 2$. The authors in [11] obtained 71 regions which took 23 seconds of computation time[1]. The resulting PWA feedback controller enforced robust performance as well as stability and feasibility. The maximum robust invariant set $\tilde{\mathcal{C}}_\infty$ was not covered by the 71 regions. With our new approach, computation of 3 regions covering the entire set $\tilde{\mathcal{C}}_\infty$ with stability and feasibility



(a) Original partition obtained by solving an mp-QP for $N = 1$

(b) The region containing the origin is divided into the robust invariant set and a finite number of other convex sets.

Fig. 1. Procedure for extracting the robust invariant set from the partition. The partition was obtained from Example 1 for additive noise ($w \in \mathbb{R}^2$, $||w||_\infty \leq 1.6$) in Section V.

guarantees took about 2 seconds[2]. In Example 2, stability and infinite-time feasibility of $\tilde{\mathcal{C}}_\infty$ can be shown $\forall w$, s.t. $||w||_\infty \leq 1.8$.

C. Complexity and Performance Analysis

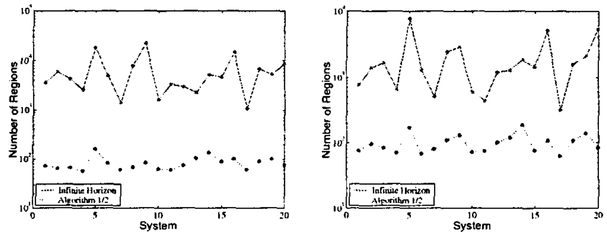
We will now examine the complexity decrease and degradation in performance incurred by Algorithm 1 based on 20 random stable systems with $n = 4$ states and $m = 2$ inputs. The inputs for all systems were constrained to $-1 \leq u_{1,2} \leq 1$ and the states were limited to $-10 \leq ||x||_\infty \leq 10$. Two different performance objectives in (3) were considered: small and large weights on the input, i.e., $\mathcal{R}_1 = 0.1I$ and $\mathcal{R}_2 = 10I$. $\mathcal{Q} = I$ was used throughout. The LMI analysis succeeded in identifying a PWQ Lyapunov function for all problems considered. Note that the success rate for Algorithm 2 is expected to be lower than for Algorithm 1, though the impact on complexity and performance will remain the same. This is because Algorithm 2 aims at identifying a quadratic instead of a PWQ Lyapunov function. As can be gathered from Figure 2, the decrease in controller complexity is substantial (e.g. Figure 2(a), system 9: $R = 22529$ for [7] vs. $R = 85$ for Alg. 1). On average, Algorithm 1 decreases complexity relative to [7] by a factor of 50. At the same time, the average decrease in closed-loop performance is around 0.23% (see Figure 3). Performance was measured by gridding the state space and computing the closed-loop trajectory cost to the origin. A more extensive complexity analysis of a similar method as Algorithm 1 is given in [8].

VI. CONCLUSION

We have shown how to compute low complexity feedback controllers for linear discrete-time systems subject to polytopic and additive uncertainty which satisfy constraints on states and input. The procedure requires the off-line computation of the feedback solution of the finite horizon constrained optimal control problem. However, when implementing a controller, the obtained results may be used

[1] AMD Athlon and CDD LP solver

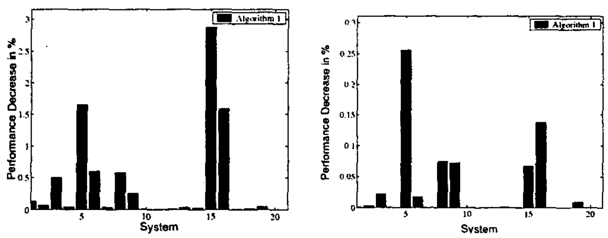
[2] Pentium IV, NAG Library LP solver



(a) $\mathcal{Q} = I$ and $\mathcal{R} = 0.1I$ in (3).

(b) $\mathcal{Q} = I$ and $\mathcal{R} = 10I$ in (3).

Fig. 2. Complexity comparison of Algorithms 1 with the infinite horizon solution [7] for 20 random fourth order systems with two inputs ($n = 4, m = 2$). On average the number of regions is decreased by a factor of about 50, with a peak decrease factor of 265.



(a) $\mathcal{Q} = I$ and $\mathcal{R} = 0.1I$ in (3).

(b) $\mathcal{Q} = I$ and $\mathcal{R} = 10I$ in (3).

Fig. 3. Performance comparison of Algorithms 1 with the infinite horizon solution [7] for 20 random fourth order systems with two inputs ($n = 4, m = 2$). On average the performance is decreased by about 0.23%, with a peak decrease of 2.87%.

for on-line optimization as well. The key contribution of this paper is a computation scheme for optimal controllers which does not require terminal set constraints and a framework for applying stability analysis to systems subject to additive and polytopic uncertainty. The numerical results in Section V have shown that the presented methods can be used to obtain feedback controllers of very low complexity with robustness and feasibility guarantees. Furthermore, extensive simulations have shown that controller complexity can be decreased by a factor of up to 250 for fourth order systems compared to established controller computation techniques [15], [2], [7]. Recent results indicate that this factor grows with increasing system size. At the same time, closed-loop performance decreases by less than 0.5% on average.

The presented algorithms can be downloaded from: `http://control.ee.ethz.ch/~grieder/pg_downloads.msql`

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