Improved MPC Design based on Saturating Control Laws

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Abstract[1]

This paper is concerned with the design of stabilizing MPC controllers for constrained linear systems. This is achieved by obtaining a suitable terminal cost and terminal constraint using a saturating control law as local controller. The system controlled by the saturating control law is modelled by a linear differential inclusion. Based on this, how to determine a Lyapunov function and a polyhedral invariant set which can be used as terminal cost and constraint is shown. The obtained invariant set is potentially larger than the maximal invariant set for the unsaturated linear controller, $\mathcal{O}_\infty$.

Furthermore, considering these elements, a simple dual MPC strategy is proposed. This dual-mode controller guarantees the enlargement of the domain of attraction or, equivalently, the reduction of the prediction horizon for a given initial state. If the local control law is the saturating LQR controller, then the proposed dual-mode MPC controller retains the local infinite horizon optimality. Finally, an illustrative example is given.

Keywords: model predictive control, constrained control, control saturation, invariant sets, polytopic systems, domain of attraction.

1 Introduction

MPC has become a popular control technique both in academy and industry. The main reason for this success is that MPC can be considered as an optimal control technique able to deal with constraints on the states and the manipulated variables in an explicit manner. Furthermore, a theoretical framework to analyze topics such as stability, robustness, optimality, etc. has recently been developed. See [15] for a survey, or [3] for process industry application issues.

It has been proved [15] that closed-loop stability of the MPC controller is guaranteed by adding a terminal cost and a terminal constraint in the optimization problem. The considered terminal cost is a Lyapunov function associated to the system controlled by a local control law and an associated invariant set is chosen as the terminal set. The domain of attraction of the MPC controller is the set of states that can be steered to the terminal region in N steps, where N is the prediction horizon. It has been recently proved that the constrained MPC without terminal constraint asymptotically stabilizes the system in a neighborhood of

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the origin [10, 14].

The most common way of designing a stabilizing MPC controller for a constrained linear system consists in considering the following three ingredients: (i) an LQR local stabilizing controller; (ii) the optimal cost of the unconstrained system as terminal cost; (iii) the maximal admissible invariant set, $\mathcal{O}_\infty$, which is the region where the local controller stabilizes the constrained system [7], as terminal set.

If a saturating control law is used instead, then the region where the local controller is stabilizing is increased. Therefore, the domain of attraction of the MPC controller can be enlarged or, equivalently, the required prediction horizon for a given initial state can be reduced. This idea has previously been presented in [5] for single-input linear systems subject to constraints in the input (but not in the states). That paper presents a region where the saturated LQR is optimal, and the optimal cost is explicitly computed for the system in the maximal invariant set contained in that region. The authors design an MPC controller based on these ingredients. The main disadvantage of using these results is that the derived optimization problem is non-convex, since the maximal invariant set for the system controlled by the saturated controller is probably non-convex. Furthermore, these results can be applied only to single input plants without constraints on the state.

In this paper we use a saturating control law from a different point of view: using a linear differential inclusion representation, how to compute a suitable Lyapunov quadratic function and an invariant polytopic set for the constrained system controlled by the saturating control law is shown. These ingredients can be used to design a stabilizing MPC controller. Since the terminal set is potentially larger than $\mathcal{O}_\infty$, the domain of attraction can be enlarged. If the saturated LQR is chosen as local control law, the designed MPC is not the infinite-horizon optimal controller in a neighborhood of the origin, because the terminal cost is greater than the unconstrained optimal cost; this property, denoted in what follows as local infinite-horizon optimality, is desirable since it provides (locally) the best closed-loop performance. In order to guarantee the enlargement of the domain of attraction as well as the local infinite-horizon optimality, a simple dual MPC technique is proposed: if the LQR-based MPC is feasible, then it is applied; if not then the saturated LQR-based MPC is used.

Consequently, we present a novel procedure to design stabilizing model predictive controllers based on saturating control laws. This method achieves the same properties than the one proposed in [5] (enlargement of the domain of attraction and local infinite-horizon optimality) with the following advantages: it can be applied to multi-input systems subject to constraints on the states; a standard QP must be solved at each sampling time and hence well-known results on multiparametric programming can be applied to compute the explicit control law.

The paper is organized as follows: first, some notations and the problem statement is presented. In section 3, some existing results on the MPC and the stabilizing design are shown. In section 4, we present results on the design of a stabilizing MPC using saturating local control laws based on an LDI representation; moreover, an enhanced MPC strategy is proposed. In the next section an illustrative example is presented. The paper finishes with some conclusions and an appendix with some proofs.

Notations. For any vector $x \in \mathbb{R}^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are non-negative and $x \succ 0$ means that they are strictly positive. $y = |x|$, for $x \in \mathbb{R}^n$, denotes the component-wise absolute value, that is, $y_{(i)} = |x_{(i)}|$. For two vectors x, y of $\mathbb{R}^n$, the notation $x \succeq y$ means that $x - y \succeq 0$. For a symmetric matrix A, $A > 0$ means that it is definite positive. Consequently, for two symmetric matrices, A and B, $A > B$ means that $A - B > 0$. For a definite positive matrix $P > 0$, $\varepsilon(P, \alpha)$ denotes the ellipsoid $\varepsilon(P, \alpha) = \{x \in \mathbb{R}^n : x^T P x \leq \alpha\}$. $A_{(i)}$ denotes the ith row of matrix A, and A^T denotes the transpose of A. I_n denotes the n-order identity matrix. For any vector $x \in \mathbb{R}^n$, $diag(x)$ denotes the diagonal matrix obtained from x. **Co**$\{\cdot\}$ denotes a convex hull. For a given time-invariant dynamic system, $x^{\mathbf{u}}(j, x)$ denotes the predicted state at time j when the initial state is x at time 0 and a sequence of control inputs $\mathbf{u}$ is applied to the system.

2 Problem statement

Let a discrete-time linear system be described by:

$$x^+ = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^m$ is the current input and x^+ is the successor state. The state of the system and the control input applied at sampling time k is denoted as x_k and u_k respectively. The system is subject to hard constraints on state and control:

$$x_k \in X, \quad u_k \in U$$

for all $k \geq 0$. The set X is a polyhedron containing the origin in its interior. Furthermore the set U is given by

$$U = \{u \in \mathbb{R}^m : |u| \preceq \rho\}$$

where vector $\rho \in \mathbb{R}^m$ is such that $\rho \succ 0$.

In this paper we are interested in the study of model predictive control schemes that stabilizes the multi-input system (1) in the presence of the state and control constraints defined respectively by the sets U and X. In particular, we are interested in the enlargement of the region of attraction associated to the MPC controller, without increasing the prediction horizon.

Before presenting the main results of the paper, in the next section we recall some basic concepts concerning stabilizing MPC schemes.

3 Stabilizing MPC

MPC is a well-known control technique able to asymptotically stabilize constrained systems. The control law is obtained by solving at each sampling time an associated optimization problem, where a functional (measure of the closed-loop performance) is minimized. The control law is obtained by means of a receding horizon strategy.

The control action for a given state x is obtained by solving an optimization problem $\mathcal{P}_N(x)$ defined by

$$\begin{aligned} V_N^0(x) \quad &= \quad \min_{\mathbf{u}} V_N(x, \mathbf{u}) \\ s.t. \quad & u(j) \in U,\ j = 0, \cdots, N-1 \\ & x(j) \in X,\ j = 0, \cdots, N-1 \\ & x(N) \in \mathcal{X}_f \end{aligned}$$

where $\mathbf{u} = \{u(0), u(1), \cdots, u(N-1)\}$ is a sequence of N control actions and $V_N(x, \mathbf{u})$ is given by

$$V_N(x, \mathbf{u}) = \sum_{j=0}^{N-1} \ell(x(j), u(j)) + F(x(N))$$

with an stage cost function $\ell(x, u) = \|x\|_Q^2 + \|u\|_R^2$, being Q and R positive definite matrices, and $x(j) = x^{\mathbf{u}}(j, x)$, that is, the state at time j if the initial state is x at time 0 and the control sequence $\mathbf{u}$ is applied to the system. The terms $F(\cdot)$, $\mathcal{X}_f$ are denoted as terminal cost function and terminal set respectively.

Problem $\mathcal{P}_N(x)$ is solved at event (k, x), yielding the minimizer $\mathbf{u}^0$ and the optimal cost $V_N^0(x)$. The MPC control law is implicitly given by

$$u = \kappa_N(x) = u^0(0)$$

There exists several methods to design a stabilizing MPC controller, but the most general is given in [15]. In that paper, sufficient conditions for closed-loop asymptotic stability are stated in the following theorem:

Theorem 1 *[15] If the terminal set $\mathcal{X}_f$ is a positively invariant set for the system controlled by a local control law $u = \kappa_f(x)$ such that $\kappa_f(x) \in U$ for all $x \in \mathcal{X}_f$ and the terminal cost $F(x)$ is an associated Lyapunov function such that*

$$F(Ax + B\kappa_f(x)) - F(x) \leq -\ell(x, \kappa_f(x)), \ \ \forall x \in \mathcal{X}_f$$

then $u = \kappa_N(x)$ asymptotically stabilizes the system for all feasible initial states.

The domain of attraction of the MPC controller, $X_N(\mathcal{X}_f)$, is the set of states which can be steered to the terminal set $\mathcal{X}_f$ in N steps or less by an admissible control law and with an admissible evolution. This set depends on the prediction horizon N, on the model (A, B), on the constraint sets X and U, and on the terminal set $\mathcal{X}_f$. In the case of linear systems, given the polyhedral constraint sets X, U and the polyhedral terminal set $\mathcal{X}_f$, the set X_N is a polyhedron and it can be accurately computed [11].

For a given constrained system, the size of the region of attraction of the controller may be enlarged by increasing the prediction horizon N or by enlarging the size of the terminal set $\mathcal{X}_f$. The first procedure leads to increase the number of decision variables, and hence to a greater computational burden of the controller on line. On the other hand, by enlarging the terminal set, the domain of attraction is enlarged without this drawback [13].

4 Design of stabilizing MPC based on saturating control laws

When the system is linear, the classical design of a stabilizing MPC is to consider the LQR as local controller (i.e. $\kappa_f(x) = K_{LQR}x$) and the unconstrained optimal cost $F(x) = x^T P_{LQR} x$ as terminal cost, both of them obtained from the solution of the Riccati equation. Since there are constraints, the local controller is only admissible in a polyhedral region around of the origin known as the maximal admissible invariant set, $\mathcal{O}_\infty$, which is chosen as terminal set $\mathcal{X}_f$. In this case, the associated optimization problem can be posed as a quadratic programming (QP) problem and the obtained MPC control law is locally infinite horizon optimal, that is $V_N^0(x) = V_\infty^0(x)$ in a neighborhood of the origin containing $\mathcal{O}_\infty$. Recent results prove that the control law of this MPC design is piece-wise affine and it can be explicitly computed off-line by means of multi-parametric programming [1].

In the recent paper [5], a stabilizing MPC is designed using as local controller the saturating control law of the LQ regulator. Based on the previous results [6], where it is proved that the saturated LQR is locally optimal, the infinite horizon optimal cost and the region where it is defined are determined. This region is an invariant set for the system controlled by the saturated LQR and it contains the maximal admissible invariant set associated to the (non-saturated) LQR, $\mathcal{O}_\infty$. By using these ones as a terminal cost and a terminal set, a stabilizing MPC is obtained.

The main advantage of this choice is that the domain of attraction is enlarged maintaining the local infinite-horizon optimality of the obtained control law. The main drawbacks are the following: the result can only be applied to single-input systems; the system must be unconstrained on the states; the obtained terminal cost is convex, but not quadratic and consequently the optimization problem to be solved is not a QP; and finally, the obtained terminal set is not a convex region and hence the optimization problem to be solved is not convex. Furthermore, the results on multi-parametric programming can not be used to compute explicitly the control law.

Consequently, the domain of attraction of the MPC is enlarged at the expense of solving a non-quadratic and non-convex optimization problem. In order to circumvent the non-convexity drawback, the authors propose to use a large enough control horizon such that the optimal solution reaches the terminal region. With this choice the optimization problem is convex, but at the expense of a probably large number of decision variables of the non-quadratic optimization problem. When the control horizon is increased, the domain of attraction of the MPC is enlarged, but in this case, the contribution of the terminal set in the enlargement is not as significative as when the control horizon is lower.

The problems above motivate the proposition of an alternative method for designing stabilizing MPC based on saturating control laws. With this aim, the idea is to use a linear differential inclusion (LDI) to represent the behavior of a closed-loop system, obtained by the application of a saturating state feedback.

This LDI modelling will allow us to obtain a suitable quadratic terminal cost and a polyhedral terminal set to design a stabilizing MPC. Differently from the previous approach, these ones can be computed for multi-input systems subject to constraints on the states. Moreover, the associated optimization problem is a QP and the control law can be explicitly computed.

The drawbacks of this technique is that the obtained terminal set potentially includes the $\mathcal{O}_\infty$, but it is not ensured. Moreover, if the saturated LQR is used as terminal controller, the terminal cost is an upper bound of the optimal cost, and hence the local infinite-horizon optimality is lost. In order to circumvent these points, a simple MPC strategy is proposed in the section 4.3.

In the following, the LDI representation to the saturating control law as well as the computation of the terminal cost and constraints are presented.

4.1 LDI representation of a saturating control law

In this section, we present known results about the description of a linear system controlled by a saturating control law by means of an LDI. These results have been previously presented in [9] and they constitutes the base of the design of a stabilizing MPC.

Consider that system (1) is stabilized by a state feedback law $u = Kx$, that is, K is such that the eigenvalues of $(A + BK)$ are placed inside the unit disk. Note that this control law is admissible inside a polyhedral region defined as:

$$R_L = \{x \in \mathbb{R}^n \ : \ |Kx| \preceq \rho\} \tag{2}$$

and hence, the control law $u = Kx$ is only able to stabilize the constrained system in a subset of $R_L \cap X$. This subset is the maximal positively invariant set for $x^+ = (A + BK)x$ contained in $R_L \cap X$ (also called maximal admissible set) and denoted as $\mathcal{O}_\infty$ [7].

The controller stabilizing properties can be extended outside this region, considering that the effective control law to be applied to the system is a saturated state feedback, where each component of the control vector is defined as follows:

$$u_{(i)} = \begin{cases} -\rho_{(i)} & \text{if } K_{(i)}x < -\rho_{(i)} \\ K_{(i)}x & \text{if } -\rho_i \leq K_{(i)}x \leq \rho_{(i)} \\ \rho_{(i)} & \text{if } K_{(i)}x > \rho_{(i)} \end{cases} \tag{3}$$

For all $i = 1, \cdots, m$. This control law will be denoted as $u = sat(Kx)$. In this case, the closed-loop system becomes

$$x^+ = Ax + Bsat(Kx) \tag{4}$$

which is a non-linear system.

Note that each component of the control law defined by (3) can also be written as[9, 8, 16]:

$$u_{(i)} = sat(K_{(i)}x) = \alpha(x)_{(i)} K_{(i)} x \tag{5}$$

where

$$\alpha(x)_{(i)} \triangleq \begin{cases} \frac{-\rho_{(i)}}{K_{(i)}x} & \text{if } K_{(i)}x < -\rho_{(i)} \\ 1 \quad \text{if} & -\rho_{(i)} \leq K_{(i)}x \leq \rho_{(i)} \\ \frac{\rho_{(i)}}{K_{(i)}x} & \text{if } K_{(i)}x > \rho_{(i)} \end{cases} \tag{6}$$

with $0 < \alpha(x)_{(i)} \leq 1$, $i = 1, \ldots, m$.

The coefficient $\alpha(x)_{(i)}$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, the smaller $\alpha(x)_{(i)}$, the farther the state vector from region R_L given by (2). Notice that $\alpha(x)_{(i)}$ is a function of the current state x. For the sake of simplicity, in the sequel we denote $\alpha(x)_{(i)}$ as $\alpha_{(i)}$. Hence, defining both $\alpha \in \mathbb{R}^m$ as a vector for which the ith entry is $\alpha_{(i)}$, $i = 1, \ldots, m$, and a diagonal matrix $D(\alpha) \triangleq diag(\alpha)$, the system (4) can be re-written as

$$x^+ = (A + BD(\alpha)K)x \tag{7}$$

Note that the matrix $(A + BD(\alpha)K)$ depends on the current state x since α does.

Consider now a vector $\underline{\alpha} \in \mathbb{R}^m$ such that $\underline{\alpha}_{(i)} \in (0, 1]$ and define the following polyhedral region

$$R_L(\underline{\alpha}) = \{x \in \mathbb{R}^n : |Kx| \preceq \rho(\underline{\alpha})\} \tag{8}$$

where $\rho(\underline{\alpha})_{(i)} = \frac{\rho_{(i)}}{\underline{\alpha}_{(i)}}$, $i = 1, \ldots, m$. Hence, if $x \in R_L(\underline{\alpha})$, it follows that $\underline{\alpha}_{(i)} \leq \alpha_{(i)} \leq 1$, $\forall i = 1, \ldots, m$. Note that $R_L \subseteq R_L(\underline{\alpha})$.

From convexity arguments, for all x belonging to $R_L(\underline{\alpha})$, it follows that

$$D(\alpha) \in \mathbf{Co}\{D_1(\underline{\alpha}), D_2(\underline{\alpha}), \ldots, D_{2^m}(\underline{\alpha})\} \tag{9}$$

where $D_j(\underline{\alpha})$ are diagonal matrices whose diagonal elements can assume the value 1 or $\underline{\alpha}_{(i)}$. Hence, the system (4) can be locally represented by a polytopic model [16] ,[9] (or a polytopic linear differential inclusion) as stated in the following lemmas.

Lemma 1 *Consider system (4) and a vector $\underline{\alpha} \in \mathbb{R}^m$ whose components $\underline{\alpha}_{(i)}, i = 1, \ldots, m$ belong to the interval $(0, 1]$. If $x \in R_L(\underline{\alpha})$, then successor state x^+ derived from the system (4) can be computed by the following polytopic model:*

$$x^+ = \sum_{j=1}^{2^m} \lambda_j A_j(\underline{\alpha}) x \tag{10}$$

with $\sum_{j=1}^{2^m} \lambda_j = 1$, $\lambda_j \geq 0$ and where

$$A_j(\underline{\alpha}) \triangleq A + BD_j(\underline{\alpha})K$$

Proof:

$x \in R_L(\underline{\alpha})$ it follows that $0 < \underline{\alpha}_{(i)} \leq \alpha_{(i)} \leq 1$, $\forall i = 1, \ldots, m$, and hence there exists $\lambda_j > 0$, $j = 1, \ldots, 2^m$, which may depend on the state of the system x, satisfying $\sum_{j=1}^{2^m} \lambda_j = 1$ and such that $u = sat(Kx) = D(\alpha)Kx = \sum_{j=1}^{2^m} \lambda_j D_j(\underline{\alpha})Kx$. ■

Lemma 2 *If a set $\mathcal{S} \in \mathbb{R}^n$ contained in the region $R_L(\underline{\alpha})$ is positively invariant for the polytopic system (10), then $\mathcal{S}$ is positively invariant for the saturated system (4) .*

Proof:

If $\mathcal{S} \subseteq R_L(\underline{\alpha})$ is positively invariant for (10), $\forall x \in \mathcal{S}$ it follows that $(\sum_{j=1}^{2^m} \lambda_j A_j(\underline{\alpha})x) \in \mathcal{S} \subseteq R_L(\underline{\alpha})$, $\forall \lambda_j > 0$, $j = 1, \ldots, 2^m$ such that $\sum_{j=1}^{2^m} \lambda_j = 1$, and from Lemma 1, the proof is complete. ■

4.2 Terminal cost and region calculation

In this section, we show how to compute a suitable quadratic terminal cost and a positively invariant set for system (4), based on the polytopic representation presented in the previous subsection. It is worth noticing that the results presented in this section can be easily extended to any LDI description of the saturated system (4).

In the following lemma, sufficient conditions to find a quadratic Lyapunov function for the saturated system (4) based on the polytopic differential inclusion representation (10), are given. This Lyapunov function is computed in such a way it satisfies the assumptions of theorem 1.

Lemma 3 *Consider system (4), a vector $\underline{\alpha} \in \mathbb{R}^m$ with $\underline{\alpha}_{(i)} \in (0, 1]$, and positive definite matrices $R \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$. If a positive definite matrix $P \in \mathbb{R}^{n\times n}$ exists satisfying the following LMI*

$$A_j(\underline{\alpha})^T P A_j(\underline{\alpha}) - P + K^T D_j(\underline{\alpha})^T R D_j(\underline{\alpha}) K + Q < 0 \tag{11}$$

for all $j = 1, \dots, 2^m$, then the function $F(x) = x^T P x$ verifies

$$F(x^+) - F(x) \le -x^T Q x - sat(Kx)^T R sat(Kx) \tag{12}$$

for all $x \in R_L(\underline{\alpha})$, where $x^+ = Ax + Bsat(Kx)$.

Proof: By using Schur's complement and convexity arguments it follows that

$$\left[\sum_{j=1}^{2^m} \lambda_j A_j(\underline{\alpha})\right]^T P \left[\sum_{j=1}^{2^m} \lambda_j A_j(\underline{\alpha})\right] + \left[\sum_{j=1}^{2^m} \lambda_j D_j(\underline{\alpha}) K\right]^T R \left[\sum_{j=1}^{2^m} \lambda_j D_j(\underline{\alpha}) K\right] - P + Q < 0 \tag{13}$$

$\forall \lambda_j > 0$, $j = 1, \dots, 2^m$, satisfying $\sum\limits_{j=1}^{2^m} \lambda_j = 1$.

Consider now system (4) and suppose that $x \in R_L(\underline{\alpha})$. From Lemma 1, it follows that λ_j, $j = 1, \dots, 2^m$ exists such that $u = sat(Kx) = D(\alpha)Kx = \sum_{j=1}^{2^m} \lambda_j D_j(\underline{\alpha}) Kx$. Hence, from (13) one can conclude that

$$(x^+)^T P (x^+) - x^T P x \le -x^T Q x - sat(Kx)^T R sat(Kx)$$

■

For the computation of the invariant set, it is necessary to define the so-called one step set, $Q(\Omega)$: the one-step set of Ω is the set of states that reach Ω in one step [2, 11]. The one-step set to a polyhedron $\Omega = \{x \in \mathbb{R}^n : Hx \preceq h\}$ for the polytopic system (10) is another polyhedron given by

$$Q(\Omega) = \bigcap_{j=1}^{2^m} Q_j(\Omega) \tag{14}$$

where $Q_j(\Omega)$ is the one-step set to Ω for the system $x^+ = A_j(\underline{\alpha})x$, that is, $Q_j(\Omega) = \{x \in \mathbb{R}^n : HA_j(\underline{\alpha})x \preceq h\}$.

Consider the following sequence of admissible sets for the system (10) , given by the recursion [2]

$$C_k = Q(C_{k-1}) \cap X_L,\ k \ge 1 \tag{15}$$

where $C_0 = X_L$. Set C_k is the region of initial states from which the system evolution remains in X_L for the next k sampling time. The sequence of admissible sets satisfies $C_{k+1} \subseteq C_k$. Set C_∞ is the set of states that are kept in X_L for the entire time, and hence it is the maximum positively invariant set contained in X_L for the polytopic system. Note that the admissible set C_k is a polyhedron, since it is the intersection of several polyhedra.

In the next theorem, it is stated that the maximum invariant set C_∞ is finitely determined and hence it is a compact polyhedron (that is, a polytope), but first the following lemma is presented:

Lemma 4 *If a vertex matrix of the polytopic model $A_j(\underline{\alpha})$ exists such that the pair $(K, A_j(\underline{\alpha}))$ is observable, then set C_{n-1} is a compact polytope, where n is the order of the system.*

Theorem 2 *Let $V(x) = x^T P x$ a Lyapunov function for system (10) such that $V(x^+) \leq \mu V(x)$ for all $x \in R_L(\underline{\alpha})$, where $\mu \in (0, 1)$. If there exists at least one vertex matrix $A_j(\underline{\alpha})$ such that $(K, A_j(\underline{\alpha}))$ is observable then:*

(i) C_∞ is finitely determined.

(ii) C_∞ is the maximal invariant set for the system (10) contained in X_L.

(iii) For all initial state in C_∞, the system (4) is asymptotically stable and satisfies the constraints.

The proofs of these results are in the appendix of the paper. In the context of polytopic LDIs for modeling uncertain systems, similar results can be found in [4].

Note that the observability condition on $(K, A_j(\underline{\alpha}))$ is not necessary if the set X_L is compact. This can be guaranteed, for instance, if X or $R_L(\underline{\alpha})$ are compact.

The computed set is the maximum invariant set for the polytopic system and, in virtue of lemma 2, it is a polytopic invariant set for the saturated system (4). Note however that the obtained invariant set is not necessarily the maximal invariant set for the saturated system. As mentioned before, it is not possible to ensure that the maximum invariant set for the unsaturated control law $\mathcal{O}_\infty$ will always be contained in C_∞. However, as could be seen in the numerical example, this inclusion often occurs or, at least, C_∞ is potentially larger that $\mathcal{O}_\infty$.

Note that C_∞ is contained in $R_L(\underline{\alpha})$ and in lemma 3, $\underline{\alpha}$ is supposed to be given. In particular, the suitable $\underline{\alpha}$ can be obtained by an iterative LMI feasibility test of (11) on a grid in $\underline{\alpha}$. In order to limit the grid, we can determine a lower bound for $\underline{\alpha}$, by considering the maximum value of $\underline{\alpha}$, denoted $\underline{\alpha}^*$, such that $X_M(\mathcal{O}_\infty) \subset R_L(\underline{\alpha})$ for a given $M \geq 1$. Hence we consider $\underline{\alpha}^*_{(i)} \leq \underline{\alpha}_{(i)} \leq 1$. In the 1-input or 2-input cases, by applying a grid search, one can easily determine $\underline{\alpha}$ with components as smaller as possible. Note that smaller are the components of $\underline{\alpha}$, larger is $R_L(\underline{\alpha})$ and, in consequence, larger tends to be C_∞. Considering the generic multi-input systems, iterative schemes, as proposed in [9, 8], can be used.

For the computation of the admissible sets using the proposed procedure, an algorithm for removing redundant inequalities and another one for subset testing of polyhedra are necessary. Efficient algorithms exist for these tasks [11]. A different algorithm for the computation of C_∞ based on linear programming schemes is given in [16].

4.3 Improved MPC design

In the last section, a suitable terminal cost $F(x) = x^T P x$ where P is the solution of the LMI (11) and terminal set $\mathcal{X}_f = C_\infty$ has been computed. Both of them satisfy the assumption of theorem 1, and thus it can be used for designing a stabilizing MPC. This result can be summarized in the following theorem:

Theorem 3 *Consider a locally stabilizing controller $u = sat(Kx)$ and let P be the matrix solution of the equation (11) for the given weighting matrices Q and R and for a given vector $\underline{\alpha}$. Suppose that C_∞ defined from the sequence (15) is finitely determined. Then the MPC controller obtained by considering as terminal cost $F(x) = x^T P x$ and as terminal set $\mathcal{X}_f = C_\infty$ stabilizes the system asymptotically for all initial state in $X_N(C_\infty)$.*

The proof follows directly from lemmas 2, 3 and theorem 1.

Note that the resultant optimization problem is a convex quadratic programming one. Furthermore, the domain of attraction of the proposed MPC is potentially larger than the one based on the unconstrained local controller.

The enlargement of the domain of attraction is ensured if and only if $\mathcal{O}_\infty \subseteq C_\infty$, which is not guaranteed. Moreover, if a saturated LQR, $u = sat(K_{LQR}x)$, is used as local controller, the designed MPC may not be the locally infinite-horizon optimal in a neighborhood of the origin. This is due to the fact that the considered terminal cost is a conservative approach to the optimal cost of the controller and hence, $P \geq P_{LQR}$. In order to reduce this conservativeness, the matrix P is computed by solving (11) minimizing its trace.

In order to overcome these drawbacks and guarantee both the local infinite-horizon optimality and the enlargement of the domain of attraction, we propose the following simple procedure:

1. Make $F(x) = x^T \cdot P_{LQR} \cdot x$ and $\mathcal{X}_f = \mathcal{O}_\infty$.

   If $\mathcal{P}_N(x)$ is feasible, then make $mode = LQR$; else make $mode = SAT$.

2. If $mode == SAT$, then make $F(x) = x^T \cdot P \cdot x$ and $\mathcal{X}_f = C_\infty$, solve $\mathcal{P}_N(x_k)$ and apply the solution.

3. If $mode == LQR$, make $F(x) = x^T \cdot P_{LQR} \cdot x$ and $\mathcal{X}_f = \mathcal{O}_\infty$, solve $\mathcal{P}_N(x)$ and apply the solution.

In what follows, we will denote the saturated LQR-based MPC (with $F(x) = x^T \cdot P \cdot x$ and $\mathcal{X}_f = C_\infty$) as SLMPC and the (unsaturated) LQR-based MPC (with $F(x) = x^T \cdot P_{LQR} \cdot x$ and $\mathcal{X}_f = \mathcal{O}_\infty$) as ULMPC.

Note that the first step is equivalent to check if $x_k \in X_N(\mathcal{O}_\infty)$. Thus, if $x_k \in X_N(\mathcal{O}_\infty)$, then the ULMPC design is considered ($mode = LQR$). If it is not feasible, then the SLMPC is considered ($mode = SAT$). The control law derived from this procedure stabilize the system in $X_N(\mathcal{O}_\infty) \cup X_N(C_\infty)$ and yields to a dual-mode controller; if the initial state $x_0 \in X_N(C_\infty)$, then the SLMPC is applied until the system reaches $X_N(\mathcal{O}_\infty)$, where the ULMPC is applied all the time and the system is steered to the origin without leaving $X_N(\mathcal{O}_\infty)$. If the initial state $x_0 \in X_N(\mathcal{O}_\infty)$, then the ULMPC is always applied.

Since the resultant dual-mode control law asymptotically stabilizes the system for all initial state in $X_N(C_\infty) \cup X_N(\mathcal{O}_\infty)$, the enlargement of the domain of attraction is ensured. Moreover, the local infinite horizon optimality property of MPC holds since for all $x \in X_N(\mathcal{O}_\infty) \supseteq \mathcal{O}_\infty$ the ULMPC is applied.

The feasibility condition $x \in X_N(\mathcal{O}_\infty)$ can easily be checked: computing off-line the polyhedron $X_N(\mathcal{O}_\infty)$ [11] or posing the feasibility problem as an LP problem to be solved at each sampling time.

If set $\mathcal{O}_\infty \subset C_\infty$, then the domain of attraction of the dual MPC controller and the one of the SLMPC controller are the same, that is, it corresponds to $X_N(C_\infty)$. However, if compared with the SLMPC, (i.e. if mode==SAT is employed all the time), the dual-mode strategy ensures the local infinite-horizon optimality (i.e. it converges to the constrained LQ controller in a finite number of steps); this property implies an improvement of the performance of the closed loop system, considering the infinite-horizon optimal cost $V_\infty^0(x)$ as a measure of the closed-loop performance. If $\mathcal{O}_\infty \not\subset C_\infty$ then the dual-mode controller provides both a larger domain of attraction and the local infinite-horizon optimality (i.e. a better closed-loop performance).

It is worth remarking that, since the dual-mode controller is based on two MPC designs which both optimization problems are QPs, each control law can be explicitly computed off-line using multi-parametric programming, and hence the dual-mode control law can be also obtained.

This technique is simple and allows us to improve the standard MPC design (SLMPC and ULMPC). However, it may require the solution of two optimization problems at each sampling time for a given interval of samples (in the case that the feasibility is checked by solving an LP). This technique can be simplified even more to solve a single QP (for $k > 0$) maintaining the desirable properties of the last method. The procedure is detailed bellow.

First, the maximal ellipsoidal admissible invariant set $\varepsilon(P, \beta) \subset \mathcal{O}_\infty$ is computed. Defining the polyhedral set $X_K = \{x \in X : K_{LQR}{\cdot}x \in U\}$, which is the intersection of a finite number of linear constraints given by $a_i{\cdot}x \leq 1$. Then the maximum β is given by

$$\beta = \min_i \left\{ \frac{1}{a_i^T{\cdot}P^{-1}{\cdot}a_i} \right\}$$

Note that any invariant set for the saturated system contained in X_K is an invariant set for the unsaturated system, and hence is contained in $\mathcal{O}_\infty$. Therefore, $\varepsilon(P, \beta) \subset \mathcal{O}_\infty$.

Consider now the constant

$$d = \lambda_{min}(P^{-1/2}{\cdot}Q{\cdot}P^{-1/2}){\cdot}\beta \tag{16}$$

where $\lambda_{min}(\cdot)$ denotes the minimum real eigenvalue. This constant satisfies the following condition: for all $x \notin \varepsilon(P, \beta)$, the stage cost satisfies $\ell(x, u) = x^T{\cdot}Q{\cdot}x + u^T{\cdot}R{\cdot}u > d$ (the proof can be found in [12]).

Based on these constants, the following lemma can be stated:

Lemma 5 *Assume that P is a feasible solution to (11), the set $\mathcal{C}_\infty$ is computed and the SLMPC is designed with these ingredients. Consider an ellipsoid $\varepsilon(P, \beta) \subset \mathcal{O}_\infty$ and a constant d given by (16). Then for all x such that $V_N^0(x) \leq N{\cdot}d + \beta$ we have that $x \in X_N(\mathcal{O}_\infty)$.*

Proof:

If $V_N^0(x) \leq N{\cdot}d + \beta$, we derive that

$$V_N^0(x) = \sum_{j=0}^{N-1} \|x^0(j)\|_Q^2 + \|u^0(j)\|_R^2 + \|x^0(N)\|_P^2 \leq N{\cdot}d + \beta$$

where $x^0(j) = x^{\mathbf{u}^0}(j;x)$. Assume that $x^0(N)$ is not contained in $\varepsilon(P_{LQR}, \beta)$, then by optimality [14] we derive that $x^0(i)$ is not in $\varepsilon(P, \beta)$ for all $i = 0, \cdots, N-1$ and hence

$$\|x^0(N)\|_P^2 \leq \beta - \sum_{j=0}^{N-1} (\|x^0(j)\|_Q^2 + \|u^0(j)\|_R^2 - d) \leq \beta$$

In consequence, $x^0(N) \in \varepsilon(P, \beta)$, which contradicts the initial assumption. From this fact we derive that $x^0(N) \in \varepsilon(P, \beta) \subseteq \mathcal{O}_\infty$ and then we conclude that $x \in X_N(\varepsilon(P, \beta)) \subseteq X_N(\mathcal{O}_\infty)$. ■

Based on this lemma, and once we have computed off-line the constants β and d, we can therefore propose the following modified procedure:

1. If $k == 0$, make $F(x) = x^T \cdot P_{LQR} \cdot x$ and $\mathcal{X}_f = \mathcal{O}_\infty$.

   If $\mathcal{P}_N(x)$ is feasible, then make $mode = LQR$; else make $mode = SAT$.

2. If $mode == SAT$, then make $F(x) = x^T \cdot P \cdot x$ and $\mathcal{X}_f = C_\infty$ and solve $\mathcal{P}_N(x_k)$.

   If $V_N^0(x_k) \leq (N \cdot d + \beta)$ then make $mode = LQR$.

   Else, apply the solution.

3. If $mode == LQR$ make $F(x) = x^T \cdot P_{LQR} \cdot x$ and $\mathcal{X}_f = \mathcal{O}_\infty$, solve $\mathcal{P}_N(x)$ and apply the solution.

At $k = 0$, we check if the ULMPC is feasible. If so, then the ULMPC is applied for all time ($mode = LQR$). But if it is not feasible, then the SLMPC is applied ($mode = SAT$). This controller is applied until the optimal cost satisfy $V_N^0(x_k) \leq (N \cdot d + \beta)$, which implies that $x_k \in X_N(\mathcal{O}_\infty)$ and then the control law switches to the ULMPC ($mode = LQR$).

Note that both proposed dual-mode MPC strategies guarantee asymptotic stability to the origin maintaining the desired properties of using the saturated LQR. The effect of the obtained switching control law does not affect to the stability and feasibility of the closed-loop system; moreover, since the difference between the SLMPC optimal cost and the ULMPC optimal cost is bounded and decreases along the system evolution, the change in the control action is bounded and probably small.

5 Numerical Example

Consider a system $x^+ = Ax + Bu$ given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}$$

where the inputs are constrained to $\|u\|_\infty \leq 0.3$ and $\|x\|_\infty \leq 2$. For this system, an LQR controller with $Q = I_2$ and $R = I_2$ is computed. The controller $u = Kx$ and the optimal cost $F(x) = x^T P_{LQR} x$ are given by

$$K = \begin{bmatrix} -0.0037 & -0.5850 \\ -0.5919 & -0.8844 \end{bmatrix}$$

$$P_{LQR} = \begin{bmatrix} 2.1801 & 1.1838 \\ 1.1838 & 2.7688 \end{bmatrix}$$

The maximal invariant set for this controller, $\mathcal{O}_\infty$, is shown in figure 1.

Following the technique proposed in the paper, a Lyapunov function and a positively invariant set for the saturating control law derived from the LQR controller is computed. First, a vector $\underline{\alpha}$ has been chosen such that the LMI (11) is feasible. The obtained vector is $\underline{\alpha} = [0.25, 0.2]$ and the calculated quadratic terminal cost $F(x) = x^T P x$ is given by

$$P = \begin{bmatrix} 33.5508 & 28.2391 \\ 28.2391 & 208.3942 \end{bmatrix}$$

As it can be seen, this terminal cost is more conservative than the optimal cost derived from the LQR (that is $P > P_{LQR}$). This fact implies that the standard MPC with $F(x) = x^T P x$ as terminal cost have a higher cost (and consequently a worse performance) than the one based on $F(x) = x^T P_{LQR} x$ in a neighborhood of the origin. This is overcome by using the proposed dual technique, as it will be shown.

Next, the maximal invariant set, C_∞, for the polytopic system contained in $X_L = R_L(\underline{\alpha}) \cap X$ is calculated. Both sets are depicted in figure 1. Note that $\mathcal{O}_\infty \subset C_\infty$.

Considering the terminal cost and terminal set obtained, a stabilizing MPC can be computed. The domain of attraction of the MPC based on the unsaturated LQR controller, $X_N(\mathcal{O}_\infty)$, is contained in the one of the MPC based on the saturating control law, $X_N(C_\infty)$. In figure 2 this enlargement for an MPC with $N = 2$ is shown. Note that the enlargement of the domain of attraction is equivalent to a reduction of the prediction horizon. In figure 3, it is shown that $X_4(\mathcal{O}_\infty) \subset X_2(C_\infty)$. Therefore, every states stabilizable by the LQR based MPC with $N = 4$ is stabilizable by the MPC based on the saturated LQR with $N = 2$. Furthermore, in this case, $X_4(C_\infty)$ is equal to the maximal stabilizable set X_∞. Hence, for $N = 4$, the MPC based on the saturating control law is able to stabilize all stabilizable set, while the MPC based on the LQR is not.

In order to improve the performance of the controller, the proposed dual MPC is used. In figure 4, the evolution of the dual MPC and the standard MPC based on the saturating control law is compared for four initial states. The cost associated to the evolution of the closed loop system for both controllers is computed. The results are given in the table 1. As was demonstrated, the dual MPC controller presents a lower cost, and hence, the performance of the closed loop system is better.

6 Conclusion

In this paper we present a technique to design a stabilizing MPC controller for constrained linear systems, which is based on the saturating control law. Describing the saturated closed loop system by a polytopic differential inclusion, a quadratic terminal cost and a polytopic invariant set can be efficiently computed. This invariant set is potentially larger than the maximal invariant set for the system controlled by the unsaturated controller. These ones can be used to design a stabilizing MPC controller with an associated convex QP optimization problem. In addition, a dual MPC controller is presented. This approach guarantees the enlargement of the domain of attraction as well as the local infinite-horizon optimality property derived from using a saturated LQR as local controller.

Appendix

Proof of lemma 4

Let C^j_{n-1} be the admissible set in $n-1$ steps for system $x^+ = A_j(\underline{\alpha})x$ in set $R_L(\underline{\alpha})$, that is the set of states from which the system evolution remains in $R_L(\underline{\alpha})$ for the next $n-1$ steps.

For any state in C_{n-1}, the system evolution remains in it for the next $n-1$ steps for any convex combination of the polytopic model. Hence, since $A_j(\underline{\alpha})$ is a vertex matrix of the polytopic model (10) and

$X_L \subseteq R_L(\underline{\alpha})$, for any initial state in C_{n-1}, system $x^+ = A_j(\underline{\alpha})x$ evolves remaining in $C_{n-1} \subseteq X_L \subseteq R_L(\underline{\alpha})$, and therefore $C_{n-1} \subseteq C^j_{n-1}$.

It is easy to see that set C^j_{n-1} is given by

$$C^j_{n-1} = \{x \in \mathbb{R}^n : |K\, A_j(\underline{\alpha})^i x| \preceq \rho(\underline{\alpha}),\ \forall i = 0, \cdots, n-1\} \tag{17}$$

Taking into account that the observability matrix of $(K, A_j(\underline{\alpha}))$ is full-rank, C^j_{n-1} is compact and hence C_{n-1}. ■

Proof of Theorem 2

(i) It is derived from lemma 4 that set C_{n-1} is compact, and hence C_k is compact for all $k \geq n-1$, since $C_k \subseteq C_{n-1}$.

Let $\varepsilon(P, \beta)$ denote the ellipsoid $\{x \in \mathbb{R}^n : x^T P x \leq \beta\}$. Since C_{n-1} is compact, there is a finite β such that $C_{n-1} \subset \varepsilon(P, \beta)$.

Let $\varepsilon(P, \gamma)$ be the maximum ellipsoid such that $\varepsilon(P, \gamma) \subset X_L$. Since $x^T P x$ is a Lyapunov function strictly decreasing for all $x \in X_L$, this set is a μ-contractive positively invariant set for the polytopic system and hence $\varepsilon(P, \gamma) \subset C_\infty$.

Note that $\varepsilon(P, \gamma) \subset C_\infty \subseteq C_{n-1} \subset \varepsilon(P, \beta)$ and hence $\beta \geq \gamma$. Let M be a constant such that $\beta\mu^M \leq \gamma$. Therefore, for all $x \in \varepsilon(P, \beta)$ the system reach $\varepsilon(P, \gamma)$ in M steps or less.

Consider $i \geq n-1+M$, then $C_i \subseteq C_{n-1} \subset \varepsilon(P, \beta)$. Consequently, for all $x \in C_i$, the system evolution remains in X_L and reaches $\varepsilon(P, \gamma) \subset C_\infty$ in M steps or less. Therefore, for all $x \in C_i$, the system remains in X_L for all the time and hence $C_i \subseteq C_\infty$. This yields $C_\infty \subseteq C_i \subseteq C_\infty$ which proves that $C_i = C_\infty$ and, therefore, it is finitely determined.

(ii) C_∞ is the set of states from which the polytopic system (10) evolves remaining in X_L for the entire time. Then for all $x \in C_\infty$, the successor state remains in C_∞ and hence it is a positively invariant set. Furthermore it is the maximal invariant set since any invariant set contained in X_L is admissible for all time and hence it is contained in C_∞.

(iii) Lemma 2 guarantees that C_∞ is positively invariant for the system (4). Lemma 3 guarantees that $V(x) = x^T P x$ is a Lyapunov function strictly decreasing for all $x \in C_\infty \subseteq X_L$ and hence the local exponential stability of the system in C_∞ is ensured .

■

Figures

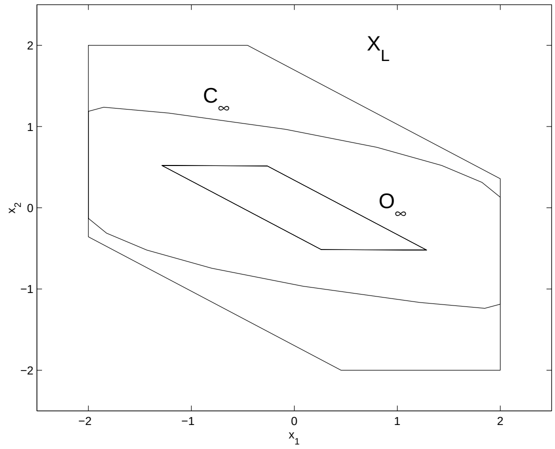

Figure 1: Terminal sets $\mathcal{O}_\infty$ and C_∞.

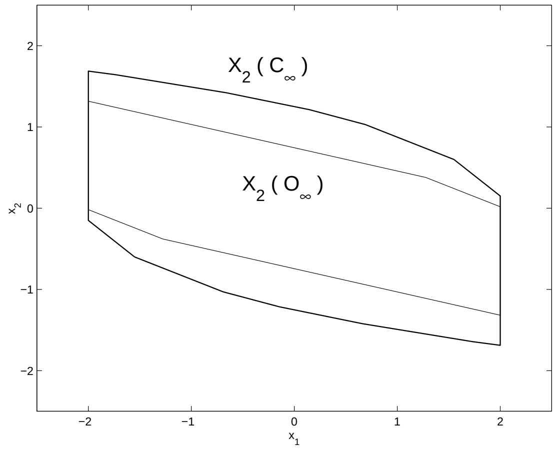

Figure 2: Domain of attraction of the MPC controller with each terminal set.

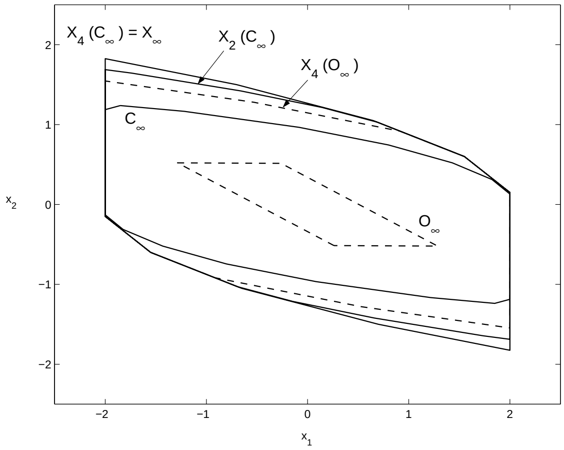

Figure 3: Comparison between domains of attraction of the MPC for several prediction horizons.

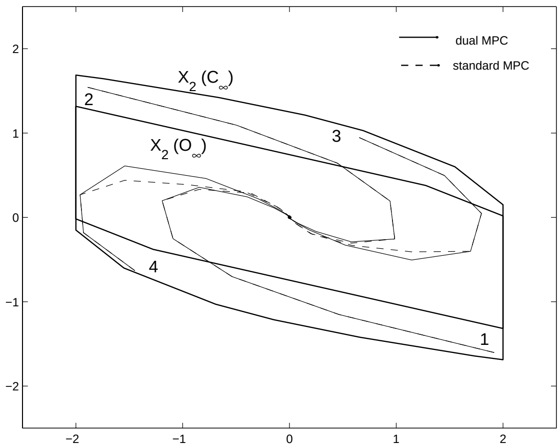

Figure 4: Closed loop state portrait of the dual MPC and the standard MPC.

Tables

x_0	dual	standard
1	13.2857	13.3125
2	11.3655	11.3759
3	12.7831	12.8275
4	14.6420	14.8448

Table 1: Comparison of the cost of proposed and standard MPC

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