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Exponential Stability of Constrained Receding Horizon Control with Terminal Ellipsoid Constraints

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***Abstract*—In this correspondence, state- and output-feedback receding horizon controllers are proposed for linear discrete time systems with input and state constraints. The proposed receding horizon controllers are obtained from the finite horizon optimization problem with the finite terminal weighting matrix and the artificial invariant ellipsoid constraint, which is less restrictive than the conventional terminal equality constraint. Both hard constraints and mixed constraints are considered in the state-feedback case, and mixed constraints are considered in the output-feedback case. It is shown that all proposed state- and output-feedback receding horizon controllers guarantee the exponential stability of closed-loop systems for all feasible initial sets using the Lyapunov approach.**

***Index Terms*—Discrete linear system with input and state constraints, exponential stability, output-feedback control, receding horizon control.**

I. Introduction

The receding horizon control has emerged as a powerful strategy for constrained systems with limitations on inputs, states, and outputs [4]–[6], [8]–[11]. Especially, the stability issue of the receding horizon control for constrained systems has been focused on in recent literatures [4], [6], [8], [11].

For the state-feedback case, the terminal equality constraint has been utilized to guarantee the closed-loop stability of the receding horizon controller for unconstrained systems [2], [3] and for constrained systems [8], [11]. This artificial constraint is satisfied by driving the state (or unstable mode) to the origin at the finite terminal time. This terminal equality constraint, however, is rather restrictive, because it is generally more difficult to drive a state to a specified point than into a specified set such as an ellipsoid or a ball. Moreover, this approach may make the optimization problem infeasible under the hard state constraint. Hence, the horizon size may have to be made longer so as to make the problem feasible. Even though the mixed constraint has been introduced to relax the hard state constraint [11], issues still need to be covered regarding feasibility and stability, because a somewhat restrictive terminal equality constraint should still be satisfied under the input constraint, and only "attractivity" rather than "asymptotic" or "exponential" stability has been shown in existing results [8], [11]. Recently, efforts have been made to overcome restriction of the terminal equality constraint [4], [7]. In [4], the invariant ellipsoid constraint has been introduced. This artificial constraint is satisfied by putting the state (or unstable mode) into an invariant ellipsoid. In this result, however, exponential stability is shown only for initial states inside the invariant ellipsoid defined by the terminal weighting matrix. In [7], it is shown that the receding horizon control with a sufficiently long horizon size can guarantee attractivity without any artificial constraint. The long horizon size is, however, needed to make the state sufficiently small at the final time, which may increase computational burden and may be regarded as a kind of artificial constraint. Moreover, in this result, only attractivity is shown.

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For the output-feedback case, state-feedback constrained receding horizon controllers have been combined with asymptotic observers [5], [6], [11]. It is noted that only the terminal equality constraint has been utilized for output-feedback receding horizon controllers in conventional results [5], [6], [11]. In the output-feedback case, the mismatch between the estimated state and the predicted state can be regarded as a disturbance that may cause infeasibility of the optimization problem. Hence, the relaxation of the terminal artificial constraint is more crucial than the state-feedback case. Two approaches show stability properties of the output-feedback constrained receding horizon controllers. One of them is the perturbed system approach using Lipschitz continuity [5], [6]. In this approach, asymptotic stability rather than exponential stability is established by showing stability and attractivity. In order to utilize this approach, the optimal cost value and the state-feedback receding horizon control law with the terminal equality constraint should be Lipschitz continuous. This Lipschitz continuity property, however, is satisfied only when gradients of binding constraints of the quadratic program are linearly independent. We can easily find cases where this condition is not satisfied for mixed constraints. Moreover, it is not still shown that this property is satisfied in case of the optimization problem with the invariant ellipsoid constraint, and in these results, the region of attraction is not considered. The other approach is the Lyapunov approach [11]. In this approach, stability can be shown by choosing a proper Lyapunov candidate. In [11], however, attractivity is shown only for stable systems.

In this correspondence, state-feedback and output-feedback constrained receding horizon controllers are proposed that are obtained from finite horizon optimization problems with finite terminal weighting matrices. In order to relax the conventional terminal equality constraint, the *invariant ellipsoid constraint* is utilized. For the state-feedback case, both hard constraints and mixed constraints are considered. For the output-feedback case, mixed constraints are considered, and the proposed state-feedback controller and an asymptotic observer are combined. It is noted that it is not so easy to satisfy hard constraints in the output-feedback case, because the mismatch between the predicted state and the estimated state may cause infeasibility. Hence, only mixed constraints are considered in the output-feedback case. Using the Lyapunov approach, it is shown that the proposed state-feedback and output-feedback receding horizon controllers guarantee the exponential stability of closed-loop systems for all feasible initial states rather than initial states inside the invariant ellipsoid. Because stability is shown by the Lyapunov approach, the proposed receding horizon controller and the corresponding optimal cost value need not satisfy the Lipschitz continuity. It is shown that regions of attraction for both cases are characterized by the corresponding feasible initial sets.

This correspondence is organized as follows. In Section II, some preliminaries are introduced. In Section III, it is shown that the state-feedback receding horizon controller with the invariant ellipsoid constraint guarantees exponential stability for both hard constrained systems and mixed constrained systems. In Section IV, an output-feedback receding horizon controller is proposed for mixed constrained systems and its exponential stability is shown. Finally, Section V concludes this correspondence.

II. PRELIMINARIES

Throughout this correspondence, the following time-invariant discrete linear system with input and state constraints will be considered:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \tag{1}$$

subject to

$$\begin{cases} u^- \le u_k \le u^+, & k = 0, 1, \cdots, \infty \\ g^- \le Gx_k \le g^+, & k = 0, 1, \cdots, \infty \end{cases} \tag{2}$$

where $u^-, u^+ \in R^m$, $G \in R^{n_g \times n}$, and $g^-, g^+ \in R^{n_g}$. It is assumed that $u_k = 0$ and $Gx_k = 0$ satisfy the constraint (2). The output constraint $y^- \le y_k \le y^+$ can be expressed as the state constraint in (2) with $G = C$. We denote $\mathcal{U}$ and $\mathcal{X}$ as the feasible sets for the above input and state constraint, respectively.

Denoting that $x_{k+i|k}$ and $u_{k+i|k}$ are predicted variables at the time k with $x_{k|k} = x_k$, we define the following finite horizon cost function, which should be optimized at every current time k:

$$J(x_k, k) = \sum_{i=0}^{N-1} (x'_{k+i|k} Q x_{k+i|k} + u'_{k+i|k} R u_{k+i|k}) + x'_{k+N|k} \Psi x_{k+N|k}$$

where $Q > 0$, $R > 0$, $\Psi > 0$, and N is a finite positive integer. We assume that the terminal weighting matrix Ψ satisfies the following inequality condition:

$$\Psi \ge (A + BH)' \Psi (A + BH) + Q + H'RH \tag{3}$$

where H is a free parameter and will be specified later.

First, we introduce a state-feedback receding horizon controller for unconstrained systems, which was proposed in [4] and is defined as the first solution of the following optimization problem:

$$\underset{u_{k|k}, \cdots, u_{k+N-1|k}}{\text{Minimize}} \quad J(x_k, k). \tag{4}$$

Then, the closed-loop stability of this state-feedback receding horizon controller is guaranteed by the following theorem.

Theorem 1: [4] Suppose that the inequality condition (3) is satisfied. Then the receding horizon control $u_k = u^*_{k|k}$ where $u^*_{k+i|k}$, $i = 0, \cdots, N-1$ is the optimal solution of the optimization problem (4), exponentially stabilizes the system (1).

In order to handle constraints, we introduce the so-called *invariant ellipsoid* property that can be interpreted in terms of quadratic stability [1]. This property will be used to construct constrained receding horizon controllers in the following sections. Suppose that a $K \in R^{m \times n}$ and a positive-definite matrix $P \in R^{n \times n}$ exist such that

$$(A + BK)' P (A + BK) - P < 0 \tag{5}$$

and define an ellipsoid $\mathcal{E}_P$ centered at the origin

$$\mathcal{E}_P = \{x_0 \in R^n | x'_0 P x_0 \le 1\}.$$

Then, for every initial state $x_0 \in \mathcal{E}_P$, the state trajectory $x_k (\forall\, k > 0)$ with the state-feedback control $u_k = Kx_k$ remains in the ellipsoid $\mathcal{E}_P$. Based on this property, we introduce the following lemma for the stability of the system (1) subject to the constraint (2) with a state-feedback controller.

Lemma 1: Suppose that $P > 0$ and K satisfying (5) also satisfy the following linear matrix inequalities(LMI's) for some Z and V with $X = P^{-1}$ and $Y = KX$

$$\begin{bmatrix} Z & Y \\ Y' & X \end{bmatrix} \ge 0, \; Z_{jj} \le \overline{u}_j^2, \qquad j = 1, 2, \cdots, m \tag{6}$$

$$GXG' \le V, \; V_{jj} \le \overline{g}_j^2, \qquad j = 1, 2, \cdots, n_g \tag{7}$$

where $\overline{g}_j$ and $\overline{u}_j$ are defined by $\overline{g}_j = \min(-g_j^-, g_j^+)$ and $\overline{u}_j = \min(-u_j^-, u_j^+)$, and g_j^-, g_j^+, u_j^-, and u_j^+ are the jth elements of

g^-, g^+, u^-, and u^+, respectively. Then, the state-feedback controller $u_k = Kx_k$ exponentially stabilizes the system for all $x_0 \in \mathcal{E}_P$, while satisfying the constraint (2), and the resultant state trajectory x_k always remains in the region $\mathcal{E}_P$.

Proof: The proof is a simple extension of the result in [1], which is based on continuous time systems. ■

III. STATE-FEEDBACK RECEDING HORIZON CONTROL

In this section, we investigate the exponential stability property of the state-feedback receding horizon controller with the invariant ellipsoid constraint for both cases of hard constraints and mixed constraints. Consider the following optimization problem subject to the additional constraint of invariant ellipsoid:

$$\underset{u_{k|k},\,\cdots,\,u_{k+N-1|k}}{\text{Minimize}} \quad J(x_k,\,k), \tag{8}$$

$$\text{subject to} \begin{cases} u^- \le u_{k+i|k} \le u^+, & i = 0,\,1,\,\cdots,\,N-1 \\ g^- \le Gx_{k+i|k} \le g^+, & i = 0,\,1,\,\cdots,\,N \\ x'_{k+N|k}\Psi x_{k+N|k} \le 1 \end{cases} \tag{9}$$

where we assume that the terminal weighting matrix Ψ satisfies the following assumption.

Assumption 1: The finite terminal weighting matrix Ψ satisfies the following LMI's with $X = \Psi^{-1}$:

$$\begin{bmatrix} X & (AX+BY)' & (Q^{1/2}X)' & (R^{1/2}Y)' \\ (AX+BY) & X & 0 & 0 \\ Q^{1/2}X & 0 & I & 0 \\ R^{1/2}Y & 0 & 0 & I \end{bmatrix} > 0$$
$$X > 0,$$

while satisfying (6) and (7), simultaneously, where X, Y, Z, and V are variables that should be found.

It can be easily verified that Ψ satisfying Assumption 1 also satisfies (3) with $H = YX^{-1}$, and hence, $A + BH$ is stable. It is also noted that X and Y in Assumption 1 satisfy (5), (6), and (7) with $P = X^{-1}$. Then, the state-feedback receding horizon controller with the invariant ellipsoid constraint is defined as the first optimal solution of the constrained optimization problem (8) with the terminal weighting matrix satisfying Assumption 1. We also define the feasible initial states set of the constrained optimization problem (8):

$$\mathcal{F}(\Psi,\,N) = \{x_0 \in R^n \mid \text{ there exists } u_i \in \mathcal{U}, \quad i = 0,\,\cdots,\,N-1, \\ \text{such that } x_{i+1} \in \mathcal{X} \text{ and } x_N \in \mathcal{E}_\Psi\}.$$

It is noted that $\mathcal{F}(\Psi,\,N)$ contains $\mathcal{E}_\Psi$ and, hence, contains a closed ball. In order to show the exponential stability of the state-feedback receding horizon controller stemming from the optimization problem (8), we need the following lemma.

Lemma 2: Suppose that $x_k \in \mathcal{F}(\Psi,\,N)$. Then, $\kappa > 0$ and $u_{k+i|k} \in \mathcal{U}, i = 0,\,\cdots,\,N-1$ exist such that $|u_{k+i|k}|^2 \le \kappa|x_k|^2$, $x_{k+i|k} \in \mathcal{X} \forall i = 0,\,\cdots,\,N-1$ and $x_{k+N|k} \in \mathcal{E}_\Psi$.

Proof: We consider the case that $x_k \ne 0$ because $x_k = 0$ gives the trivial solution $u_{k+i|k} = 0$. Let $B(\gamma)$ be a closed ball with a radius $\gamma > 0$ such that $B(\gamma) \subset \mathcal{F}$. If $x_k \in B(\gamma)$, define $\alpha(x_k) \in [1,\,\infty)$ such that $\alpha(x_k)x_k \in \partial B(\gamma)$, where $\partial B(\gamma)$ denotes the boundary of $B(\gamma)$. Otherwise, define $\alpha(x_k) \in [1,\,\infty)$ such that $\alpha(x_k)x_k \in \mathcal{F} - B(\gamma)$ Then, a control sequence $\hat{u}_{k+i|k} \in \mathcal{U}$ exists that drives $\alpha(x_k)x_k$ into the ellipsoid $\mathcal{E}_\Psi$ in N steps while satisfying the state constraint. Because the system is linear, $1/\alpha(x_k)\hat{u}_{k+i|k} \in \mathcal{U}$ drives x_k into $\mathcal{E}_\Psi$ while satisfying the state constraint. Denoting $\overline{u} = \max\{|u^-|_\infty,\,|u^+|_\infty\}$, we obtain $|\hat{u}_{k+i|k}|^2 \le m\overline{u}^2$. Hence, it holds that $|(1/\alpha(x_k))\hat{u}_{k+i|k}|^2 \le (1/\alpha(x_k)^2)m\overline{u}^2 \le (m\overline{u}^2/\gamma^2)|x_k|^2$, which completes the proof. ■

Now, we are ready to state the result on the exponential stability of the state-feedback receding horizon controller with the invariant ellipsoid constraint.

Theorem 2: Suppose that Assumption 1 holds. Then, the optimization problem minimizing $J(x_k,\,k)$ subject to the constraint (9) is always feasible for all $k \ge 0$ and for all initial states $x_0 \in \mathcal{F}(\Psi,\,N)$. Also, $x_k = 0$ is the exponential stable equilibrium of the closed-loop system with the state-feedback receding horizon controller stemming from this optimization problem, for all initial states $x_0 \in \mathcal{F}(\Psi,\,N)$.

Proof: Suppose that the optimal solution $u^*_{k+i|k}$ at the current time k exists, and let $H = YX^{-1}$. Then, at the next time step $k+1$, consider the following control sequence:

$$\left.\begin{array}{ll} u_{k+i|k+1} = u^*_{k+i|k}, & i = 1,\,2,\,\cdots,\,N-1 \\ u_{k+N|k+1} = Hx_{k+N|k+1} & \end{array}\right\}. \tag{10}$$

Then, the above control sequence gives a feasible solution for the optimization problem (8) subject to (9) at the next time step $k+1$, because $x^*_{k+N|k}(=x_{k+N|k+1})$ remains in $\mathcal{E}_\Psi$ and the control input $u_{k+N|k+1} = Hx_{k+N|k+1}$ satisfies the invariant ellipsoid property. Hence, by induction, we observe that the optimization problem is always feasible for all k and every initial state $x_0 \in \mathcal{F}(\Psi,\,N)$. In order to show the exponential stability of the closed-loop system, we will show that $a,\,b,\,c > 0$ exist such that $a|x_k|^2 \le J^*(x_k,\,k) \le b|x_k|^2$, $\Delta J^* < -c|x_k|^2$, and hence, $J^*(x_k,\,k)$ serves as a Lyapunov functional for exponential stability. We can easily show that the following inequality holds:

$$J^*(x_k,\,k) \ge x'_kQx_k \ge \lambda_{\min}(Q)|x_k|^2. \tag{11}$$

From Lemma 2, a feasible $\tilde{u}_{k+i|k}$, $i = 0,\,\cdots N-1$ and $\kappa > 0$ exist such that $|\tilde{u}_{k+i|k}|^2 \le \kappa|x_k|^2$. Denoting $\tilde{x}_{k+i|k}$ as the resultant state trajectory with this control sequence, we obtain

$$\begin{aligned} J^*(x_k,\,k) \le \sum_{i=0}^{N-1} &(\tilde{x}'_{k+i|k}Q\tilde{x}_{k+i|k} + \tilde{u}'_{k+i|k}R\tilde{u}_{k+i|k}) \\ &+ \tilde{x}'_{k+N|k}\Psi\tilde{x}_{k+N|k} \le [(N+1)\mathcal{A}^2(1+N\|B\|\sqrt{\kappa})^2 \\ &\cdot \max\{\lambda_{\max}(Q),\,\lambda_{\max}(\Psi)\} + \kappa\lambda_{\max}(R)]|x_k|^2 \end{aligned} \tag{12}$$

where $\mathcal{A} := \max_{i=1,\,\cdots,\,N} \|A^i\|$.

Next, let $\overline{J}$ be the cost value when the control sequence (10) is implemented at the next time step $k+1$. Then, we can easily obtain that $J^*(x_k,\,k) \ge x'_kQx_k + u'_kRu_k + \overline{J} \ge x'_kQx_k + u'_kRu_k + J^*(x^*_{k+1},\,k+1)$, which can be represented as

$$\Delta J^*(x_k) \le -x'_kQx_k - u'_kRu_k \le -\lambda_{\min}(Q)|x_k|^2. \tag{13}$$

Hence, the inequalities (11)–(13) show that $J^*(x_k,\,k)$ serves as a Lyapunov functional for exponential stability. ■

Generally, input constraints are imposed by physical limitations of actuators, valves, pumps, and etc., whereas state constraints are often desirable. Often, cases exist in which state constraints cannot be satisfied all of the time, and hence, some violations of state constraints are allowable. Especially, even if state constraints are satisfied in nominal operations, unexpected disturbances may put the states aside from the feasible region where state constraints are satisfied. In this case, it may happen that some violations of state constraints are unavoidable, whereas input constraints can be still satisfied. Hence, if some violations of state constraints are allowed, we can guess larger feasible initial sets. Moreover, if the terminal ellipsoid constraints are relaxed, we can guess much larger feasible initial sets. This strategy makes more sense in the output-feedback case, because it is difficult to satisfy the hard state constraint. More details for the output-feedback case will be followed in the next section. Hence, we introduce the mixed constraints,[1] which are given by

$$\begin{cases} u^- \le u_k \le u^+, & k = 0, 1, \cdots, \infty \\ Gx_k \le g + \epsilon_k, & k = 0, 1, \cdots, \infty \end{cases}$$

where $\epsilon_k \ge 0$ denotes tolerance for violation of state constraints.

In order to enlarge feasibility, we partition the system matrix A into stable and unstable modes as follows:

$$A = VJV^{-1} = [V_u \quad V_s] \begin{bmatrix} J_u & 0 \\ 0 & J_s \end{bmatrix} \begin{bmatrix} \tilde{V}_u \\ \tilde{V}_s \end{bmatrix}$$

where J_u's eigenvalues are unstable and J_s's eigenvalues are stable. Then, the unstable mode $z^u = \tilde{V}_u x$ satisfies $z^u_{k+1} = J_u z^u_k + \tilde{V}_u B u_k$. Now, the idea is to drive only the unstable mode into the ellipsoid instead of full states.

We assume that $X_u > 0$ and Y_u exist satisfy the following couple of LMI's and denote that $\Psi_u = X_u^{-1}$ and $H_u = Y_u \Psi_u$.

Assumption 2: Ψ_u exist that satisfies the LMI's (see the equation at the bottom of the page) where $X_u = \Psi_u^{-1}$, $B_u = V^{-1}B$, and $Q_u = V_u' Q V_u$.

Because J_s is stable, $\Psi_s > 0$ exists satisfying $\Psi_s \ge J_s' \Psi_s J_s + V_s' Q V_s$. Now, denoting

$$\Psi = V \begin{bmatrix} \Psi_u & 0 \\ 0 & \Psi_s \end{bmatrix} V^{-1} \quad \text{and} \quad H = [H_u \; 0] V^{-1} \tag{14}$$

it can be easily verified that Ψ and H satisfy the inequality condition (3). Using Ψ in (14) as the terminal weighting matrix and introducing a cost function for violations of state constraints, we modify the cost function and the corresponding optimization problem as

$$J_\epsilon(x_k, k) = J(x_k, k) + \epsilon(k)' S \epsilon(k)$$
$$\underset{u_{k|k}, \cdots, u_{k+N-1|k}, \epsilon(k)}{\text{Minimize}} \; J_\epsilon(x_k, k) \tag{15}$$

subject to

$$\begin{cases} u^- \le u_{k+i|k} \le u^+, & i = 0, 1, \cdots, N-1 \\ Gx_{k+i|k} \le g + \epsilon(k), & i = 1, \cdots, N \\ G(A+BH)^j x_{k+N|k} \le g + \epsilon(k), & j = 1, \cdots, \infty \\ x'_{k+N|k} \tilde{\Psi} x_{k+N|k} \le 1 \end{cases} \tag{16}$$

where $\tilde{\Psi} = \tilde{V}_u' \Psi_u \tilde{V}_u$, $\epsilon(k) \ge 0$. It is noted that the third constraint in (16) can be checked within a finite step because $A + BH$ is stable [8]. We can observe that the terminal ellipsoid constraint in this optimization is relaxed from that in the optimization in the previous section, because we only have to put unstable modes instead of full states into the ellipsoid. We also define the feasible initial states set of the optimization problem (15)

$$\mathcal{F}_u(\tilde{\Psi}, N) = \{x_0 \in R^n \,|\, \text{there exists } u_i \in \mathcal{U}, \; i = 0, \cdots, N-1, \text{ such that } x_N \in \mathcal{E}_{\tilde{\Psi}}\}.$$

It is noted that $\mathcal{F}_u(\tilde{\Psi}, N) \supseteq \mathcal{F}(\Psi, N)$, because only input constraint exists and the terminal ellipsoid constraint is much relaxed.

Theorem 3: Suppose that Assumption 2 is satisfied and the terminal weighting matrix Ψ is defined by (14). Then, the optimization problem minimizing $J_\epsilon(x_k, k)$ subject to the constraint (16) is always feasible for all $k \ge 0$ and for all initial states $x_0 \in \mathcal{F}_u(\tilde{\Psi}, N)$. Also, $x_k = 0$ is the exponentially stable equilibrium of the closed-loop system with the receding horizon controller stemming from this optimization problem, for all initial states $x_0 \in \mathcal{F}_u(\tilde{\Psi}, N)$.

Proof: Let $u^*_{k+i|k}$ and $\epsilon^*(k)$ be the optimal solution at the current time step k. Then, the following sequence gives a feasible solution at the next time step $k+1$:

$$\left.\begin{array}{ll} u_{k+i|k+1} = u^*_{k+i|k}, & i = 1, \cdots, N-1 \\ u_{k+N|k+1} = Hx_{k+N|k+1}, & \\ \epsilon(k+1) = \epsilon^*(k) & \end{array}\right\}$$

where H is defined by (14). Now, we will show the exponential stability using the optimal cost value $J^*_\epsilon(x_k, k)$ as a Lyapunov candidate. Denoting J^*_ϵ as the optimal cost of J_ϵ, we can easily obtain the inequality $J^*_\epsilon(x_k, k) \ge x_k' Q x_k + u_k' R u_k + J^*_\epsilon(x_{k+1}, k+1)$, which shows that $c > 0$ exists such that $\Delta J^*_\epsilon(x_k, k) \le -c|x_k|^2$, and we can easily show that $a > 0$ exists such that $J^*_\epsilon(x_k, k) \ge a|x_k|^2$. As in Lemma 2, we can easily show that a feasible control sequence $\tilde{u}_{k+i|k}$ exists for $x_k \in \mathcal{F}_u(\tilde{\Psi}, N)$ such that $|\tilde{u}_{k+i|k}|^2 \le \kappa |x_k|^2$, and observe that $\tilde{\epsilon}(k) = \max\{1, \overline{\mathcal{A}}\} \mathcal{A}(1 + N\|B\|\sqrt{\kappa})|x_k|[1 \cdots 1]'$, where $\overline{\mathcal{A}} := \max_{i=1,\cdots,\infty} \|(A+BH)^i\|$ is a feasible solution. It is noted that $\overline{\mathcal{A}}$ is finite because $A + BH$ is stable. Hence, using the solution $\tilde{u}_{k+i|k}$ and $\tilde{\epsilon}_k$, we can show that $b > 0$ exists such that $J^*_\epsilon(x_k, k) \le b|x_k|^2$. ■

IV. Output-Feedback Receding Horizon Control

In this section, we propose a new output-feedback receding horizon controller that is applicable not only to stable systems, but also to unstable systems. We assume that the state is estimated by the following asymptotic observer:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L\{C[A\hat{x}_k + Bu_k] - y_{k+1}\}$$

where $\hat{x}_k$ denotes the estimated state at time k and the observer gain L is chosen such that $(I + LC)A$ is stable. It is noted that we can always find such an L if the pair $[A \; C]$ is detectable. Then, the dynamic equation of the error $e_k = \hat{x}_k - x_k$ is represented as

$$e_{k+1} = (I + LC)Ae_k.$$

In the output-feedback case, mixed constraints are recommended, because it is difficult to satisfy the hard state constraint exactly. Moreover, the mismatch between the predicted state and the estimated state

[1]We will represent the hard state constraint $g^- \le Gx_k \le g^+$ as $Gx_k \le g$, which can be easily derived by modifying G.

$$\begin{bmatrix} X_u & (J_u X_u + B_u Y_u)' & (Q_u^{1/2} X)' & (R^{1/2} Y_u)' \\ (J_u X_u + B_u Y_u) & X_u & 0 & 0 \\ Q_u^{1/2} X_u & 0 & I & 0 \\ R^{1/2} Y_u & 0 & 0 & I \end{bmatrix} > 0 \quad X_u > 0$$
$$\begin{bmatrix} Z & Y_u \\ Y_u' & X_u \end{bmatrix} \ge 0, \; Z_{jj} \le \overline{u}_j^2, \quad j = 1, 2, \cdots, m$$

may cause infeasibility. Hence, we consider the following cost function and the corresponding optimization problem based on the estimated state $\hat{x}_k$:

$$J_\epsilon(\hat{x}_k) = \sum_{i=0}^{N-1} (\hat{x}'_{k+i|k} Q \hat{x}_{k+i|k} + u'_{k+i|k} R u_{k+i|k}) + \hat{x}'_{k+N|k} \Psi \hat{x}_{k+N|k} + \epsilon'(k) S \epsilon(k)$$

$$\underset{u_{k|k}, \cdots, u_{k+N-1|k}, \epsilon(k)}{\text{Minimize}} \; J_\epsilon(\hat{x}_k, k)$$

$$\text{subject to} \begin{cases} \hat{x}_{k+i+1|k} = A\hat{x}_{k+i|k} + Bu_{k+i|k}, & i = 0, 1, \cdots, N-1 \\ u^- \le u_{k+i|k} \le u^+, & i = 0, 1, \cdots, N-1 \\ G\hat{x}_{k+i|k} \le g + \epsilon(k), & i = 0, 1, \cdots, N_\epsilon \\ \hat{x}'_{k+N|k} \tilde{\Psi} \hat{x}_{k+N|k} \le 1 \end{cases} \tag{17}$$

where

$\hat{x}_{k|k}$ $= \hat{x}_k$;

$\tilde{\Psi}$ satisfies Assumption 2;

Ψ defined by (14).

We also denote u^*, ϵ^* as the optimal solution and $\hat{x}^*$ the resultant state trajectory as in Section III.

Then, a new output-feedback receding horizon controller for constrained systems is defined as the first solution of the optimization problem (17). In order to obtain the dynamic equation of the mismatch between the predicted state and the estimated state, we define a variable ξ as

$$\xi_{k+i|k+1} = \hat{x}_{k+i|k+1} - \hat{x}^*_{k+i|k}, \quad i = 1, \cdots, N-1.$$

Because it holds that $\hat{x}_{k+1|k+1} = A\hat{x}_k + Bu_k + L\{C[A\hat{x}_k + Bu_k] - y_{k+1}\}$ and $\hat{x}_{k+i+1|k+1} = A\hat{x}_{k+i|k+1} + Bu_{k+i|k+1}$, where $\hat{x}_k = \hat{x}^*_{k|k}$ and $u_k = u^*_{k|k}$, we obtain

$$\begin{aligned} &\xi_{k+i+1|k+1} = A\xi_{k+i|k+1} + Bv_{k+i|k+1}, \quad i = 1, \cdots, N-1 \\ &\xi_{k+1|k+1} = LCAe_k, \end{aligned}$$

where $v_{k+i|k+1} = u_{k+i|k+1} - u^*_{k+i|k}$. In order to compensate this mismatch and make the optimization problem (17) feasible even at the next time step $k+1$, a sequence $v_{k+i|k+1}$ should exist such that $x^*_{k+N|k} + \xi_{k+N|k+1} \in \mathcal{E}_{\tilde{\Psi}}$. We define the feasible error set for which such a sequence exists:

$$\begin{aligned} \mathcal{F}_e(\tilde{\Psi}, N, k) = \{ e_k \in R^n \mid & \text{ there exists } v_{k+i|k+1} \\ & \text{such that } x^*_{k+N|k} + \xi_{k+N|k+1} \in \mathcal{E}_{\tilde{\Psi}} \\ & \text{and } u^- - u^*_{k+i|k} \le v_{k+i|k+1} \le u^+ \\ & - u^*_{k+i|k}, \; i = 1, \cdots, N-1 \}. \end{aligned}$$

Then, it holds that $\hat{x}_{k+1} \in \mathcal{F}_u$, if and only if $\hat{x}_k \in \mathcal{F}_u$ and $e_k \in \mathcal{F}_e(\tilde{\Psi}, N, k)$. We also define the feasible initial set for which the optimization problem (17) is feasible

$$\begin{aligned} \Omega = \{ (\hat{x}_0, e_0) \in R^n \times R^n | & \hat{x}_k \in \mathcal{F}_u \\ & \text{and } e_k \in \mathcal{F}_e(\tilde{\Psi}, N, k) \forall k \}. \end{aligned}$$

In order to show the exponential stability of the output-feedback receding horizon controller, we present several lemma.

Lemma 3: Suppose that $e_k \in \mathcal{F}_e(\tilde{\Psi}, N, k)$. Then, a sequence $v_{k+i|k+1}$ and $\kappa_v > 0$ exist, such that

$$\left. \begin{aligned} &x^*_{k+N|k} + \xi_{k+N|k+1} \in \mathcal{E}_{\tilde{\Psi}} \\ &u^- - u^*_{k+i|k} \le v_{k+i|k+1} \le u^+ - u^*_{k+i|k} \\ &|v_{k+i|k+1}|^2 \le \kappa_v |e_k|^2 \end{aligned} \right\}. \tag{18}$$

Proof: The proof is similar to that of Lemma 2. ■

Lemma 4: Suppose that $e_k \in \mathcal{F}_e(\tilde{\Psi}, N, k)$ and $v_{k+i|k+1}$ is chosen to satisfy (18), and denote

$$\eta_k := \max\{ \max_{i=1,\cdots,N} |G\xi_{k+i|k+1}|_\infty, \; \|G\| \overline{\mathcal{A}} |\xi_{k+N|k+1}|_\infty \} [1 \cdots 1]' \tag{19}$$

where $\overline{\mathcal{A}}$ is defined in Section III. Then, $\gamma_1, \gamma_2 > 0$ exist such that $|\xi_{k+i|k+1}|^2 \le \gamma_1 |e_k|^2$ and $|\eta_k|^2 \le \gamma_2 |e_k|^2$.

Proof: It can be easily verified that the following inequality holds:

$$|\xi_{k+i|k+1}| \le \mathcal{A}(\|LCA\| + \|B\|\kappa_v)|e_k|$$

where $\mathcal{A}$ is defined in Section III. This inequality shows the existence of $\gamma_1 > 0$ and $\gamma_2 > 0$. ■

Lemma 5: Suppose that P is a symmetric matrix and a and b are vectors with appropriate dimensions. Then, the following inequality holds for any scalar $\delta > 0$:

$$(a+b)'P(a+b) \le (1+\delta)a'Pa + \left(1 + \frac{1}{\delta}\right) b'Pb.$$

$$\begin{aligned} J^*_\epsilon(\hat{x}_{k+1}) &\le J_\epsilon(\hat{x}_{k+1}) \\ &= \sum_{i=1}^{N-1} \left((\hat{x}^*_{k+i|k} + \xi_{k+i|k+1})' Q (\hat{x}^*_{k+i|k} + \xi_{k+i|k+1}) + (u^*_{k+i|k} + v_{k+i|k+1})' R (u^*_{k+i|k} + v_{k+i|k+1}) \right) \\ &\quad + \hat{x}'_{k+N|k} M \hat{x}_{k+N|k} + (\epsilon^*(k) + \eta_k)' S (\epsilon^*(k) + \eta_k) \\ &\le (1+\delta) \left(\sum_{i=1}^{N-1} (\hat{x}^{*'}_{k+i|k} Q \hat{x}^*_{k+i|k} + u^{*'}_{k+i|k} R u^*_{k+i|k}) + \hat{x}^{*'}_{k+N|k} M \hat{x}^*_{k+N|k} + \epsilon^{*'}(k) S \epsilon^*(k) \right) \\ &\quad + \left(1 + \frac{1}{\delta}\right) \left(\sum_{i=1}^{N-1} (\xi'_{k+i|k+1} Q \xi_{k+i|k+1} + v'_{k+i|k+1} R v_{k+i|k+1}) + \xi'_{k+N|k+1} M \xi_{k+N|k+1} + \eta'_k S \eta_k \right) \\ &= (1+\delta)(J^*_\epsilon(\hat{x}_k) - \hat{x}^{*'}_{k+N|k} (\Psi - M) \hat{x}^*_{k+N|k} - \hat{x}'_k Q \hat{x}_k - u'_k R u_k) \\ &\quad + \left(1 + \frac{1}{\delta}\right) \left(\sum_{i=1}^{N-1} (\xi'_{k+i|k+1} Q \xi_{k+i|k+1} + v'_{k+i|k+1} R v_{k+i|k+1}) + \xi'_{k+N|k+1} M \xi_{k+N|k+1} + \eta'_k S \eta_k \right) \end{aligned} \tag{19a}$$

Proof: It is easily derived from the fact that

$$0 \le \left(a - \frac{b}{\delta}\right)' P \left(a - \frac{b}{\delta}\right) = a'Pa + \frac{1}{\delta^2} b'Pb - \frac{2}{\delta} a'Pb.$$

■

Now, we are ready to present the main result on the exponential stability of the proposed output-feedback receding horizon controller.

Theorem 4: The origin is the exponentially stable equilibrium of the closed-loop system with the output-feedback receding horizon controller stemming from the optimization problem (17) for all $(\hat{x}_0, e_0) \in \Omega$.

Proof: To prove the exponential stability, we consider the following Lyapunov candidate:

$$\begin{aligned} V(\hat{x}_k, e_k) &= J_\epsilon^*(\hat{x}_k) + \sum_{i=0}^{\infty} e'_{k+i|k} O e_{k+i|k} \\ &= J_\epsilon^*(\hat{x}_k) + e'_k \overline{O} e_k \end{aligned}$$

where

$J_\epsilon^*(\cdot)$ optimal value of $J_\epsilon(\cdot)$,

O positive-definite matrix,

$\overline{O}$ unique positive-definite solution satisfying $\overline{O} = A'(I + LC)'\overline{O}(I + LC)A + O$.

It is noted that such an $\overline{O}$ always exists for given $O > 0$, because $(I + LC)A$ is stable.

Similar to the state-feedback case, we can show that $a, b > 0$ exist such that $a(|\hat{x}_k|^2 + |e_k|^2) \le V(\hat{x}_k, e_k) \le b(|\hat{x}_k|^2 + |e_k|^2)$. Now, we will show that $c > 0$ exists such that $\Delta V(\hat{x}_k, e_k) \le -c(|\hat{x}_k|^2 + |e_k|^2)$. Consider the following sequence:

$$\left.\begin{aligned} u_{k+i|k+1} &= u^*_{k+i|k} + v_{k+i|k+1}, \quad 1 = 1, 2, \cdots, N-1 \\ u_{k+N|k+1} &= H\hat{x}_{k+N|k+1} \\ \epsilon(K+1) &= \epsilon^*(k) + \eta_k \end{aligned}\right\}$$

where $v_{k+i|k+1}$ is chosen to satisfy (18) and η_k is defined by (19). This sequence gives a feasible solution at the next time step $k + 1$. Using Lemma 5, we obtain following inequalities as shown in (19a) at the bottom of the previous page, where $M = Q + H'RH + (A + BH)'\Psi(A+BH)$. It is noted that $\Psi - M \ge 0$,because Ψ and H satisfy the inequality (3). From Lemma 4 and the fact that $b_1 > 0$ exists such that $J_\epsilon^*(\hat{x}_k) \le b_1|\hat{x}_k|^2$, we obtain the following inequalities:

$$\begin{aligned} \Delta J_\epsilon^*(\hat{x}_k) &= J_\epsilon^*(\hat{x}_{k+1}) - J_\epsilon^*(\hat{x}_k) \\ &\le -(\lambda_{\min}(Q) - \delta(b_1 - \lambda_{\min}(Q)))|\hat{x}_k|^2 + \left(1 + \frac{1}{\delta}\right) \\ &\quad \cdot \left(\sum_{i=1}^{N} (\lambda_{\max}(Q)|\xi_{k+i|k+1}|^2 + \lambda_{\max}(R)|v_{k+i|k+1}|^2)\right. \\ &\quad \left. + \lambda_{\max}(M)|\xi_{k+N|k+1}|^2 + \lambda_{\max}(S)|\eta_k|^2\right) \\ &\le -\alpha|\hat{x}_k|^2 + \beta|e_k|^2 \end{aligned}$$

where $\alpha = \lambda_{\min}(Q) - \delta(b_1 - \lambda_{\min}(Q))$ and $\beta = (1 + (1/\delta))(N(\gamma_1\lambda_{\max}(Q) + \kappa_v\lambda_{\max}(R)) + \gamma_1\lambda_{\max}(M) + \gamma_2\lambda_{\max}(S))$. Thus, it can be shown that $c > 0$ exists such that

$$\begin{aligned} \Delta V(\hat{x}, e_k) &= -\alpha|\hat{x}_k|^2 + \beta|e_k|^2 - \lambda_{\min}(O)|e_k|^2 \\ &\le -c(|\hat{x}_k|^2 + |e_k|^2) \end{aligned}$$

if δ and O are chosen such that $\delta < (\lambda_{\min}(Q)/b_1)$ and $\beta < \lambda_{\min}(O)$. Hence, we obtain the exponential stability. ■

V. Conclusions

In this correspondence, proposed are new state- and output-feedback receding horizon controllers for discrete linear systems with input and state constraints. The proposed controllers are obtained from finite horizon optimization problems with the terminal ellipsoid constraint. For the state-feedback case, both of hard and mixed constrained systems are considered. For the output-feedback case, mixed constraints are considered. We showed that the proposed controllers guarantee exponential stability in the Lyapunov sense, for all feasible initial states.

Regarding conventional results on the state-feedback case, only attractivity has been shown [7], [8], [11] or exponential stability has been shown only for the initial sets inside an invariant ellipsoid [4]. Regarding conventional results on the output-feedback case, no result has existed that shows exponential stability for unstable systems in the Lyapunov sense and only asymptotic stability property has been shown by the perturbed system approach [5]. It is also noted that no result has existed on the output-feedback receding horizon control with the terminal ellipsoid constraint. Because the proposed controllers utilize the invariant ellipsoid constraint that is less restrictive than the conventional terminal equality constraint, we can guess larger feasible initial sets or less computational burden than the conventional receding horizon controllers with the terminal equality constraint.

One of our future studies will be to find the minimum horizon size such that feasibility is guaranteed, which will be useful when disturbances appear.

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