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A hybrid strategy for the feedback stabilization of nonholonomic mobile robots*

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Abstract

A hybrid strategy for the control of nonholonomic mobile robots is proposed and assessed, using both theoretical arguments and simulation experiments. The control law is a discontinuous state feedback that combines the advantadges of time invariant smooth feedback control far from the target and of time-varying smooth feedback control close to the target, while avoiding their respective drawbacks.

1 Introduction

Our concern, in this paper, is to discuss the feedback control of the class of mobile robots (see fig.1) described by the state space model :

$$\begin{aligned} \dot{x} &= -\sin\theta u_1 \\ \dot{y} &= \cos\theta u_1 \\ \dot{\theta} &= u_2 \end{aligned} \tag{1}$$

where x, y denote the cartesian coordinates of the robot in the plane and θ denotes its angular orientation. The control inputs u_1 and u_2 are, respectively, the longitudinal and angular velocities.

In the sequel, we shall often use the following terminology : - the pair (x, y) of cartesian coordinates is called the *cartesian position* of the robot in the plane,
- the angle θ is called the *attitude* of the robot,
- the triplet (x, y, θ) is called the *configuration.*

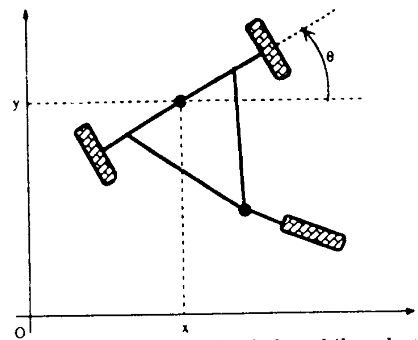

Figure 1 : the wheeled mobile robot..

The purpose of the feedback control is to automatically drive the robot from any arbitrary initial configuration (x_0, y_0, θ_0) to a given final configuration (x_f, y_f, θ_f), called the target configuration, which we assume to be the origin of the generalised coordinates, i.e. $x_f = 0$, $y_f = 0$ and $\theta_f = 0$. We are only interested here in *feedback* control strategies, in contrast with "path planning", which consists in finding a trajectory depending on the initial conditions and then tracking it with local feedback. More technically, the problem is thus to find bounded feedback control functions $u_1(x, y, \theta, t)$ and $u_2(x, y, \theta, t)$ such that, starting from (x_0, y_0, θ_0), the state $(x(t), y(t), \theta(t))$ is bounded and asymptotically converges to zero :

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \theta(t) = 0 \tag{2}$$

It is well known that this control problem *cannot* be solved by using smooth time invariant static state feedback(see e.g. [2],[4], [8]). In fact, in the case of the mobile robot (1), the best that can be done with a smooth time-invariant feedback is to make attractive a one-dimentional submanifold of the configuration space (see e.g.[4]).

The complete solution of the problem requires to use discontinuous and/or time varying feedback con-

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trols (see e.g. [1]). A nice example of a discontinuous feedback controller for mobile robots of the form (1) is given in [3]. On the other hand, examples of time varying feedback controllers for various classes of mobile robots can be found in [8]; this approach has been extended to a more general class systems in [7]. A critical drawback of this solution is however to produce trajectories that can be very unsatisfactory (fairly erratic, poor convergence), as we shall illustrate later.

In this paper, our main contribution is to show that *satisfactory trajectories* can be obtained by using a hybrid strategy that combines the advantages of time invariant control far from the target configuration and of time varying control close to the target, while avoiding the drawbacks of both solutions.

A *satisfactory trajectory* is defined as a trajectory that satisfies the following four specifications :

S1- The control objective as stated before is achieved, that is, on all trajectories,
- boundedness of configuration $[x(t), y(t), \theta(t)]$,
- boundedness of the control inputs,
- asymptotic convergence : see (2).

S2- The distance $\sqrt{x^2+y^2}$ between the cartesian position of the robot and target position is decreasing along the trajectory.

S3- The cartesian trajectory is as close as possible to the straight line from (x_0, y_0) to the origin.

S4- The attitude of the robot is bounded as follows along the trajectory :

$$\theta_0 - 2\pi \le \theta(t) \le \theta_0 + 2\pi \quad \forall t .$$

The paper is organised as follows : section 2 gives a description of the closed-loop behaviour resulting from partially linearizing feedback laws; section 3 presents a method for Lyapunov design of time-varying feedback laws for system (1), and investigates in details, on the basis of various simulation experiments, the influence of some design parameters; finally, section 4 is devoted to a "hybrid" strategy combining the advantages of both designs.

2 Partial Feedback Linearization

The largest feedback linearizable subsystem may be shown to have dimension 2. In other words, it is possible to assign a linear behaviour to two independent functions of the state (x, y, θ). It is unfortunately impossible (see [4] for details) to take these two independant functions to be x and y, but it turns out that the functions X and Y given by

$$\begin{array}{rcl} X &=& x + L\sin\theta \\ Y &=& y - L\cos\theta \end{array} \tag{3}$$

may be linearized and decoupled if $L \neq 0$. The nonsingular feedback transformation

$$\begin{array}{rcrcl} u_1 &=& -\; v_1\sin\theta &+& v_2\cos\theta \\ u_2 &=& \frac{1}{L}v_1\cos\theta &+& \frac{1}{L}v_2\sin\theta \end{array} \tag{4}$$

yields $\dot{X} = v_1$, $\dot{Y} = v_2$. We might assign, for instance, $\dot{X} = -X$ and $\dot{Y} = -Y$ by taking $v_1 = -X$ and $v_2 = -Y$, i.e. $u_1 = X\sin\theta - Y\cos\theta$ and $u_2 = -\frac{1}{L}(X\cos\theta + Y\sin\theta)$. Since this results in having a control which goes to infinity when (x, y) goes to infinity, we will prefer to divide these controls by a function which goes to infinity at infinity :

CONTROL LAW CL1 :

$$\begin{array}{rcl} u_1 &=& \alpha(x,y,\theta)\,(\,x\sin\theta \;-\; y\cos\theta \;+\; L\,) \\ u_2 &=& -\frac{1}{L}\,\alpha(x,y,\theta)\,(\,x\cos\theta \;+\; y\sin\theta\,) \end{array} \tag{5}$$

with

$$\begin{array}{rcl} \alpha(x,y,\theta) &=& \dfrac{k_1}{k_2 + \sqrt{X^2+Y^2}} \\ &=& k_1\,\left[k_2 + (x^2 + y^2 \right. \\ && \left. + L^2 + 2L(x\sin\theta - y\cos\theta))^{\frac{1}{2}}\right]^{-1} \end{array} \tag{6}$$

Note that :
- Instead of $\frac{k_1}{k_2+\sqrt{X^2+Y^2}}$, one may use for α any positive function, nonzero at zero, and going to zero at infinity in such a way that the controls are bounded.
- The control law given by (5) is not smooth (actually not differentiable) at zero. It is however locally lipschitzian, so that solutions of the closed-loop system are defined and are unique. It will be used,in section 4, only outside a neighborhood the origin, where it is smooth.

The closed-loop is then (α is given by (6))

$$\begin{array}{c} \dot{X} \;=\; -\alpha X \;\;,\;\; \dot{Y} \;=\; -\alpha Y \;, \\ \dot{\theta} \;=\; -\dfrac{\alpha}{L}(X\cos\theta \;+\; Y\sin\theta)\;\;. \end{array} \tag{7}$$

This obviously forces (X, Y) to go straight to $(0, 0)$ and yields (this was not at all predictible from linearization theory) a bounded θ. This was noticed in [5], and it may actually be checked that θ converges to a limit value θ_∞ which does not differ from the initial conditions by more than 2π. If $X_o + iY_o = \rho_o e^{i\varphi_o}$,

$$\theta_\infty = \theta_o - 2\,\mathrm{Arctan}[\left(1 - e^{-\frac{\rho_o}{L}}\right)\tan(\frac{\theta_o - \varphi_o}{2} + \frac{\pi}{4})] \quad (8)$$

The simulation experiment reported in fig. 2 is done with L equal to the length of the cart, i.e. (X, Y) is the extreme point on the cart axis. The dashed curve represents the trajectory of (X,Y), a straight line as predicted above, and the solid curve is the trajectory of the cartesian position (x, y). Drawings of the cart itself have been added; they give an idea of the evolution of the attitude θ. Specifications S2, S3 and S4 are met since the cartesian position is confined within an a bandwidth of size $2L$ and the attitude θ remains in an interval $[\pm 2\pi]$. The drawback is that specification S1 is not satisfied since the cartesian position (x, y) converges to a point on the circle with radius L and center zero, while the attitude θ goes to a value we cannot control.

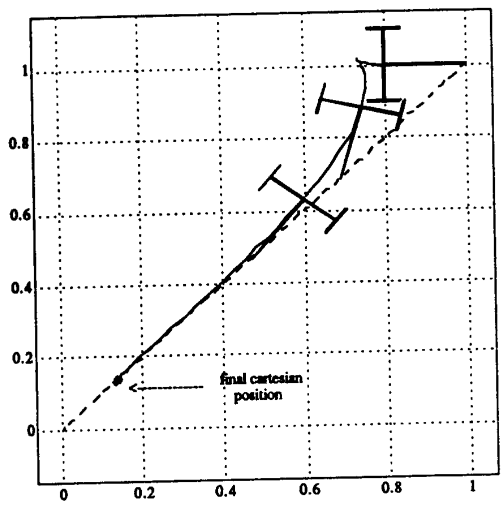


Figure 2 : EXP1 (feedback linearization).

3 Time-varying stabilizing laws

3.1 Design method

We follow a procedure similar to the one proposed in [7]. We allow however a slightly different choice of the Lyapunov function candidate :

$$V(t, x) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}(\frac{\psi(\theta)}{\lambda} + x\cos t)^2 \quad (9)$$

with λ a positive real number and ψ a real smooth function with $\psi' > 0$ and $\psi(\theta) = 0 \Leftrightarrow \theta = 0$. The derivative of V along (1) may, if we set

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\lambda}{\psi'} x \sin t \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} , \quad (10)$$

be written as :

$$\begin{aligned} \dot{V} &= (-x\sin\theta + y\cos\theta)\, w_1 \\ &\quad + (\frac{\psi(\theta)}{\lambda} + x\cos t) \\ &\qquad [\frac{\psi'(\theta)}{\lambda} w_2 - \cos t \sin\theta\, w_1] \quad (11) \\ &= [-x\sin\theta + y\cos\theta \\ &\qquad -(\frac{\psi(\theta)}{\lambda} + x\cos t)\cos t\sin\theta\,]\, w_1 \\ &\quad + (\frac{\psi(\theta)}{\lambda} + x\cos t)\frac{\psi'(\theta)}{\lambda} w_2 \ . \quad (12) \end{aligned}$$

Following faithfully the design proposed in [7], one may make both terms in (12) negative with :

CONTROL LAW CL2 :

$$\begin{aligned} u_1 &= x\sin\theta - y\cos\theta \\ &\quad -(\tfrac{\psi(\theta)}{\lambda} + x\cos t)\cos t \sin\theta \quad (13) \\ \tfrac{\psi'(\theta)}{\lambda} u_2 &= x\sin t - (\tfrac{\psi(\theta)}{\lambda} + x\cos t) \ , \end{aligned}$$

and the following expression of $\dot{V}$ then results :

$$\begin{aligned} \dot{V} &= -[x\sin\theta + y\cos\theta \\ &\qquad +(\frac{\psi(\theta)}{\lambda} + x\cos t)\cos t\sin\theta\,]^2 \\ &\quad - (\frac{\psi(\theta)}{\lambda} + x\cos t)^2 \ . \quad (14) \end{aligned}$$

This control may be proved to yield convergence of the configuration (x, y, θ) to zero, using LaSalle invariance principle, because the only invariant set contained in $\{(t, x, y, \theta) \in \mathbb{R}^4, \dot{V} = 0\}$ is $\{(t, 0, 0, 0), t \in \mathbb{R}\}$. See [7] for further details. Hence specification S1 is satisfied, but we would like to achieve also the other specifications, in particular S2. The above analysis does not give much information on this : if x_0 and y_0 are small and θ_0 is large, (14) does not prevent x and y to grow very large before going to zero. This phenomenon will be observed in fig. 3 hereafter.

It is possible to avoid such a behaviour by modifying the control laws, designing them to force both V and $x^2 + y^2$ to decrease : if $\dot{V}$ is expressed according to (11) instead of (12), the first term is exactly the derivative of $x^2 + y^2$, and it is possible to make both terms negative with the following controls :

CONTROL LAW CL3 :	
$$\begin{aligned} u_1 &= x\sin\theta - y\cos\theta \\ \frac{\psi'(\theta)}{\lambda}u_2 &= x\sin t + \cos t\sin\theta\, u_1 \\ &\quad -\left(\frac{\psi(\theta)}{\lambda} + x\cos t\right). \end{aligned}$$	(15)

We then have :

$$\dot{V} = -(-x\sin\theta + y\cos\theta)^2 - \left(\frac{\psi(\theta)}{\lambda} + x\cos t\right)^2 , \tag{16}$$

$$\frac{d}{dt}\left[\tfrac{1}{2}(x^2+y^2)\right] = -(-x\sin\theta + y\cos\theta)^2 . \tag{17}$$

Specification S1 is again satisfied since the closed-loop convergence can be proved exactly in the same way as before.Eq. (17) clearly implies that specification S2 is also satisfied. Furthermore, property (17) will prove to be crucial to give a theoretical background to the hybrid stategy proposed in section 4.

The simulation experiment on fig. 3 illustrates the superiority of **CL3**. The same initial conditions and the same design parameters $\lambda = 1$ and $\psi(\theta) = \theta$ are used in both cases. The dashed curve (EXP2) represents the trajectory obtained with the control **CL2**, and does violate specification S2 since the cart goes far from the origin before coming back; the solid curve (EXP3) is obtained with **CL3**, and clearly satisfies S2. The improvement is dramatic when using **CL3**, and this may be observed when comparing (13) and (15) with other choices of ψ and λ. **We will use from now on the control laws CL3 given by (15).**

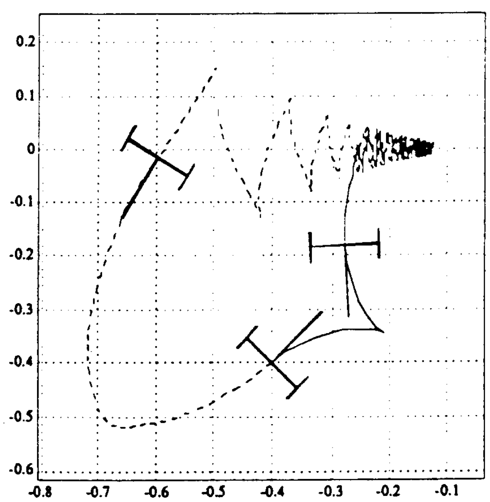


Figure 3 : Solid curve : EXP 2 (**CL2**), *dashed curve :* EXP 3 (**CL3**)

3.2 Influence of the design parameters

Here, we will present various simulation results in order to demonstrate the influence of the various parameters that are at the user's disposal in the design of the control law presented above, namely the function ψ and the positive number λ.

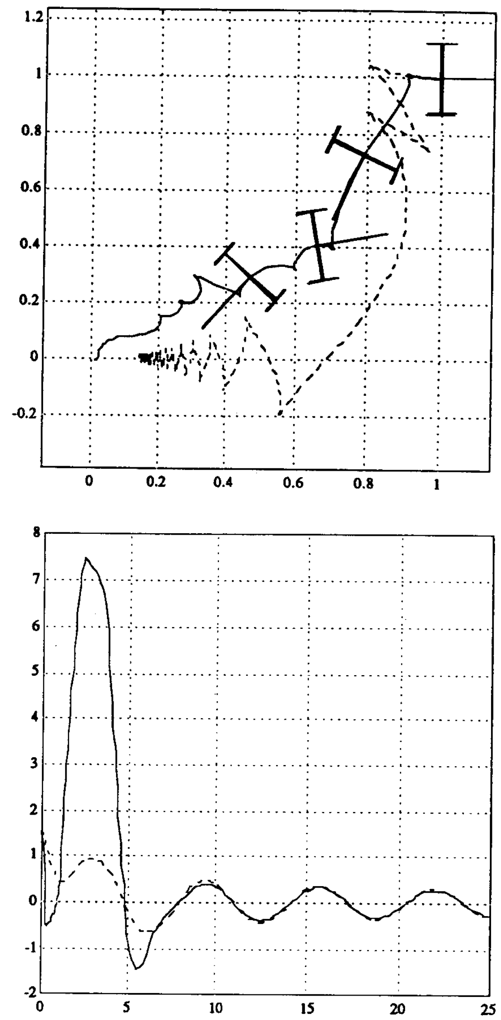


Figure 4 : Dashed curve : EXP 4 ($\lambda = 1$), *solid curve :* EXP 5 ($\lambda = 20$).
Top: cartesian trajectories, bottom: θ versus time.

We shall first study the influence of λ with the simplest choice of ψ : $\psi(\theta) = \theta$:

CONTROL LAW CL3, with $\psi(\theta) = \theta$:

$$\begin{aligned} u_1 &= x\sin\theta - y\cos\theta \\ u_2 &= \lambda\,[x\sin t + (x\sin\theta - y\cos\theta)\cos t\sin\theta] \\ &\quad -(\theta + \lambda x\cos t) \end{aligned} \tag{18}$$

In fig.4, two experiments appear; both use control law **CL3**, $\psi(\theta) = \theta$, but λ is set to one for EXP4 (dashed curve) and to 20 in EXP5 (solid curve). It appears that a larger λ increases the convergence speed of the position (x, y) to zero, and that with a small gain $\lambda = 1$, we satisfy S4 ($\theta \in [\pm\frac{\pi}{2}]$) but not S3 (the motion is not close to a straight line) while with $\lambda = 20$, we satisfy, more or less, S3, but not S4 : the motion is very erratic and characterized by a large number of cusps and an increase of θ up to more than seven radians, as seen on bottom of fig. 4.

One way to prevent θ from growing high is to take a Lyapunov function which is infinite for θ outside some bounds, e.g. by taking $\psi(\theta) = \tan\frac{\theta}{a}$ with $a > 2$:

CONTROL LAW CL3, with $\psi(\theta) = \tan\frac{\theta}{a}$:

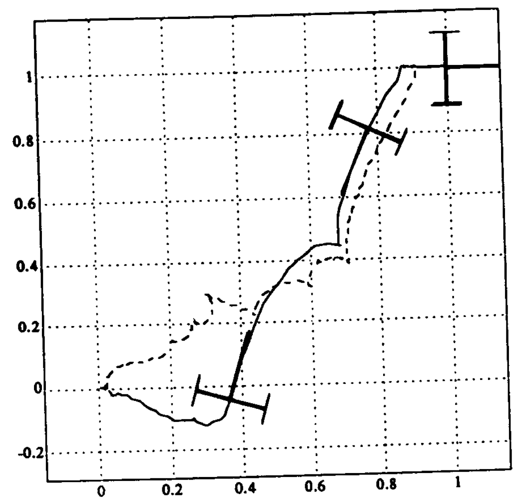
$$\begin{aligned} u_1 &= x\sin\theta - y\cos\theta \\ u_2 &= -\,a\,\cos\tfrac{\theta}{a}\sin\tfrac{\theta}{a} \\ &\quad + a\lambda\cos^2\tfrac{\theta}{a}[\,x(\sin t - \cos t) \\ &\qquad +(x\sin\theta - y\cos\theta)\cos t\sin\theta\,] \end{aligned} \tag{19}$$

This control law is well defined everywhere. However, the Lyapunov function is infinite when $\tan\frac{\theta}{a}$ is infinite and the stability analysis is therefore only valid inside θ in $]-\frac{a}{2}\pi, \frac{a}{2}\pi[$. Since the fact that V decrease still holds, and V is infinite when $\theta = \pm\frac{a}{2}\pi$, one can show that θ stays in $]-\frac{a}{2}\pi, \frac{a}{2}\pi[$ if it starts therein.

In figure 5, where we only consider initial conditions with θ in $]\pm\frac{a}{2}\pi[$, The experiment EXP5 is reproduced from fig. 4, in order to compare it with EXP6. The only difference is the choice of ψ : $\psi(\theta) = \theta$ for EXP5, $\psi(\theta) = \tan\frac{\theta}{2}$ for EXP6. The latter allows us to satisfy S4 (since $\theta \in [\pm 2\pi]$) but at the price of a worst achievement of S3.

4 A hybrid strategy

We have described in the pevious section several time-varying control laws which do bring x, y, and θ to zero –and this type of control is, to our knowledge, the only type of continuous feedback able to do so– but they do not yield a very satisfactory behaviour of the cartesian position (x, y), in particular far from zero : instead of going straight to zero, the cart keeps turning back and forth, because of the time-dependance of the Lyapunov function. On the other hand, partial linearization yields a satisfactory behaviour of (x, y), i.e. almost a straight line (at most one cusp), leading to $(-L\sin\theta_\infty, L\cos\theta_\infty)$ (instead of $(0,0)$) : this type of control at least gives a nice way to get (x, y) closer to $(0,0)$: since it makes (x, y) converge to a point located on a circle with radius L centered at the origin, it brings (x, y) in finite time into, say, a disk with radius $\sqrt{\frac{11}{10}L}$) with a "smooth" behaviour of θ (see (8) and above).



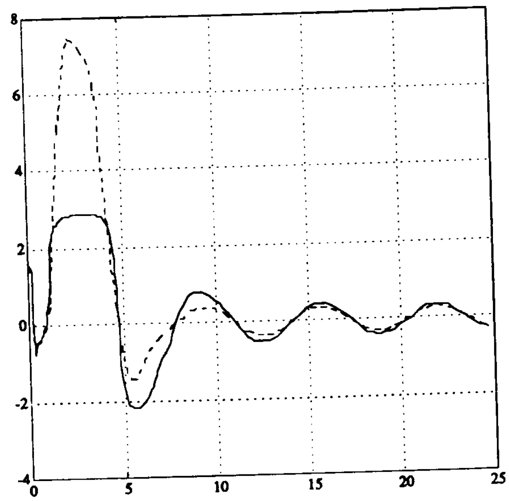


Figure 5 : Dashed curve : EXP 5, same as fig. 4 ($\psi(\theta) = \theta$), solid curve : EXP 6 ($\psi(\theta) = \tan\frac{\theta}{2}$). Top: cartesian trajectories, bottom: θ versus time.

A natural idea is therefore to use partial linearizing control when (x, y) is outside the disk with radius $\sqrt{\frac{11}{10}}L$, and time varying control inside this disk. This is still a feedback stategy, i.e. the control is a function of the state (and of time for (x, y) inside the disk) but with a discontinuity on the surface $\{x^2 + y^2 = \frac{11}{10}L\}$:

CONTROL LAW CL4, with $\psi(\theta) = \tan\frac{\theta}{a}$:
If $x^2 + y^2 > 1.1\,L^2$,

$$\begin{aligned} u_1 &= \alpha(x,y,\theta)\,(\,x\sin\theta\ -\ y\cos\theta\ +\ L\,) \\ u_2 &= -\tfrac{1}{L}\,\alpha(x,y,\theta)\,(\,x\cos\theta\ +\ y\sin\theta\,) \end{aligned} \tag{20}$$

If $x^2 + y^2 \leq 1.1\,L^2$,

$$\begin{aligned} u_1 &= x\sin\theta - y\cos\theta \\ u_2 &= -\,a\,\cos\tfrac{\theta}{a}\sin\tfrac{\theta}{a} \\ &\quad +\,a\lambda\cos^2\tfrac{\theta}{a}[\,x(\sin t - x\cos t) \\ &\quad +(x\sin\theta - y\cos\theta)\cos t\sin\theta\,] \end{aligned} \tag{21}$$

It must be pointed out that this strategy does not create an infinite number os switching or limit cycles, since control law **CL 3**, used inside the disk, has been proved in (17) to force the norm of (x, y) to decrease. Since the control law used outside (i.e. in the region where $x^2 + y^2 > 1.1L^2$) brings any solution to the border of the inside region in finite time, it is easy to conclude that a solution starting from a certain initial (x_0, y_0, θ_0) at time 0 is such that there is a time $T \geq 0$ ($T = 0$ if $x_0^2 + y_0^2 \leq 1.1L^2$) such that $x(t)^2 + y(t)^2 > 1.1L^2$ for $0 \leq t < T$ and $x(t)^2 + y(t)^2 \leq 1.1L^2$ for $t \geq T$.

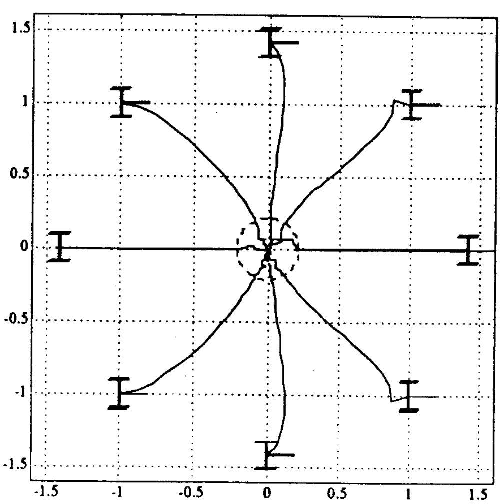


Figure 6 : "hybrid" control strategy.

Fig. 6 shows trajectories obtained from simulation experiments with this control law and various initial conditions. L is 0.2 and the gain λ is set to 40. Initial conditions are : (x, y) at eight equidistant positions on the circle with radius $\sqrt{2}$, $\theta = \frac{\pi}{2}$. The dashed circle is the circle with radius $\sqrt{1.1}$ where the switching between (20) and (21) occurs in **CL4**.

Note that the discontinuity here is quite different from the discontinuity in the (time invariant) feedback proposed in [1] or in [3] since it does not cross the origin, and is not at all important as far as local stabilization is concerned. We only introduce this discontimuity in order to profit by the smooth behaviour of the partially linearizing control laws **CL2** far from zero.

References

[1] A.M. Bloch, N.H. McClamroch, M. Reyhanoglu : Controllability and Stabilizability properties of a Nonholonomic Control System,*Proc. CDC, Honolulu, 1991*,pp. 1312-1314.

[2] R. P. Brockett : Asymptotic stability and feedback stabilization, in:*Differential Geometric Control Theory*, Birkäuser, 1983.

[3] O.J. Sordalen, C. Canudas : Exponential Control law for a mobile robot : extension to path following, *Proc IEEE Conf. on Robotics and Automation*, Nice 1992.

[4] G. Campion, B. d'Andréa-Novel and G. Bastin : Controllability and State-Feedback Stabilization of Non Holonomic Mechanical Systems, *Lect. Notes in Cont. & Inf. Sc., Vol 162. Advanced Robot Control*, pp.106-124, Springer, 1991.

[5] B. d'Andréa-Novel, G. Campion and G. Bastin : Modelling and Control of Nonholonomic wheeled Mobile Robots, *IEEE Conf. on Robotics and Automation*, Sacramento, pp.1130-1135, 1991.

[6] J.-P. LaSalle : Stability theory for ordinary differential equations. *J. of diff. eq.* Vol. 4 (1968) 57-65.

[7] J.-B. Pomet : Explicit design of time-varying stabilizing control laws for a class of time-varying controllable systems without drift. To appear in *Systems & Control Letters*, 1991.

[8] C. Samson : Velocity and torque feedback control a nonholonomic cart. *Lect. Notes in Cont. & Inf. Sc., Vol 162. Advanced Robot Control*, pp.125-151, Springer, 1991.