Receding Horizon Recursive State Estimation

K. V. Ling and K. W. Lim

***Abstract*—This paper describes a receding horizon discrete-time state observer using the deterministic least squares framework. The state estimation horizon, which determines the number of past measurement samples used to reconstruct the state vector, is introduced as a tuning parameter for the proposed state observer. A stability result concerning the choice of the state estimation horizon is established. It is also shown that the fixed memory receding horizon state observer can be related to the standard dynamic observer by using an appropriate end-point state weighting on the estimator cost function.**

***Index Terms*—Kalman filter, least squares techniques, receding horizon estimation, state observers.**

I. Introduction

Moving window or receding horizon state estimation can be considered as a form of limited memory filtering first discussed in [4] to overcome the problem of divergence in the state estimates due to modeling errors. There has been renewed interest in receding horizon state estimation recently [9], [15]. The idea in receding horizon state estimation is to estimate the state vector based on a finite number of past measurement samples, so that the oldest measurement sample is discarded as a new sample becomes available. The memory of the state estimator is thus fixed. A detailed study of fixed memory filters for discrete-time state space models in the stochastic framework can be found in [5].

Receding horizon state estimation based on batch least squares techniques has been proposed elsewhere; however, it is not straightforward to select appropriate values for the weighting matrices in the state observer of [11] and [12]. In [7], the state estimation horizon, which determines the number of past measurement samples used to reconstruct the state vector, is chosen as an alternative tuning parameter for the batch least squares state observer. This use of estimation horizon is in a fashion akin to the role of the prediction horizon in the receding horizon predictive control strategies represented by, for example, [1] and [2]. While the use of prediction horizon in predictive control is now well-established, the use of the state estimation horizon in state observer design is relatively unexplored. In the context of an observer-based state feedback system studied in [8], it is shown that the state estimation horizon can be employed as a convenient and useful parameter for adjusting online, in both single rate and multirate situations, the robustness of a closed-loop system against high-frequency modeling error.

In this paper, the property of the receding-horizon state observer is further explored. A stability result concerning the choice of the state estimation horizon is established. It is also shown that the fixed memory receding horizon state observer can be related to the standard dynamic observer in terms of an appropriate end-point state weighting on the estimator cost function.

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II. The Deterministic Least Squares State Observer

Consider the discrete-time system[1] represented by

$$x_{k+1} = Ax_k \quad \text{and} \quad y_k = cx_k. \tag{1}$$

The problem considered here is how to obtain an estimate $\hat{x}$ of the state vector x by using a finite number of samples of the system output y. Throughout the paper, it is assumed that the constant matrices A and c are known, $[c, A]$ is observable, and A nonsingular.

A. Batch Least Squares

Suppose we set up $N + 1$ equations

$$Y_{k,N} = M_N x_{k-N}$$

where

$$Y_{k,N} = [y_k \; y_{k-1} \; \cdots \; y_{k-N}]^T \tag{2}$$

$$M_N = [(cA^N)^T \; (cA^{N-1})^T \; \cdots \; c^T]^T \tag{3}$$

and N is the tuning parameter to be called the state estimation horizon.

Since there are in general more equations than unknowns, a least squares solution is sought, resulting in

$$\hat{x}_k = A^N \hat{x}_{k-N} = A^N P_N M_N^T Y_{k,N} \tag{4}$$

where

$$\begin{aligned} P_N &= (M_N^T M_N)^{-1} \\ &= (P_{N-1}^{-1} + (cA^N)^T cA^N)^{-1} \\ &= P_{N-1} - P_{N-1}(cA^N)^T (I + cA^N P_{N-1}(cA^N)^T)^{-1} \\ &\quad \cdot cA^N P_{N-1}. \end{aligned} \tag{5}$$

As the intent of the proposed state observer design is to use N as a tuning parameter, at the next sampling instance $k + 1$, the state estimate is computed as

$$\hat{x}_{k+1} = A^N P_N M_N^T Y_{k+1,N} \tag{6}$$

where $Y_{k+1,N} = [y_{k+1} \;\; y_k \;\; \cdots \;\; y_{k+1-N}]^T$. That is, the oldest observation is discarded as a new observation becomes available. This is a receding horizon or moving window approach and the resulting state observer is termed the moving window or receding horizon state observer.

Note that the formulation above can easily be generalized to include a weighting matrix. In the interest of simplicity, we omit this weighting matrix in the rest of the paper.

B. Recursive Solution 1

Equation (4) is clearly an inefficient way of actually computing an estimate. It is also not suitable if the estimation horizon N is to be used as an online tuning parameter for the performance of the proposed state observer. In many engineering applications, the observations are obtained sequentially. It is desirable to compute, as new observations are obtained, the current state estimate based on the previous state estimate. This can be done by deriving a recursive solution for the moving window state observer as follows.

[1] Here, the external input term is ignored for simplicity since we assume that its influence on x is known exactly.

From (2) and (3), the following recursion can be established:

$$Y_{k,N} = \begin{bmatrix} Y_{k,N-1} \\ y_{k-N} \end{bmatrix} = \begin{bmatrix} y_k \\ Y_{k-1,N} \end{bmatrix}$$
$$M_N = \begin{bmatrix} M_{N-1}A \\ c \end{bmatrix} = \begin{bmatrix} cA^N \\ M_{N-1} \end{bmatrix}.$$

Thus, from (4) and (6), we have

$$\begin{aligned}\hat{x}_{k+1} - \hat{x}_k &= A^N P_N((cA^N)^T y_{k+1} + M_{N-1}^T Y_{k,N-1}) \\ &\quad - A^N P_N((M_{N-1}A)^T Y_{k,N-1} + c^T y_{k-N}) \\ &= A^N P_N (cA^N)^T y_{k+1} + A^N P_N (I - A^T) M_{N-1}^T \\ &\quad \cdot Y_{k,N-1} - A^N P_N c^T y_{k-N}.\end{aligned}$$

Since from (4), $M_{N-1}^T Y_{k,N-1} = A^{-T}(A^N P_N)^{-1}\hat{x}_k - A^{-T}c^T y_{k-N}$, the moving window state observer can be written as

$$\begin{aligned}\hat{x}_{k+1} &= A^N P_N A^{-T}(A^N P_N)^{-1}\hat{x}_k + A^N P_N (cA^N)^T y_{k+1} \\ &\quad - A^N P_N A^{-T} c^T y_{k-N}. \end{aligned} \tag{7}$$

Remark 1: In contrast to (6), (7) gives an efficient algorithm for implementing the moving window state observer as new observations are obtained. The dimension of the matrix P_N is independent of the window length N. It should be noted, however, that the memory requirements are not reduced, since at any one time all the data within the current window must be accessible.

C. Recursive Solution 2

Equation (7) is derived for the moving window strategy. Another technique commonly employed in the system identification literature to derive a recursive solution for (6) is the so-called growing memory strategy. It will be shown in the following that this gives the conventional dynamic state observer equation.

Rewrite (6) as

$$\hat{x}_{k+1} = A^N P_N M_{N-1}^T Y_{k,N-1} + A^N P_N (cA^N)^T y_{k+1}$$

and let

$$\begin{aligned}L_N &= A^N P_N (cA^N)^T \\ &= A^N (P_{N-1} - P_{N-1}(cA^N)^T (I + cA^N P_{N-1}(cA^N)^T)^{-1} \\ &\quad \cdot cA^N P_{N-1})(cA^N)^T \\ &= A^N P_{N-1}(cA^N)^T (I + cA^N P_{N-1}(cA^N)^T)^{-1}.\end{aligned}$$

Then

$$\begin{aligned}&A^N P_N M_{N-1}^T Y_{k,N-1} \\ &= A^N (P_{N-1} - P_{N-1}(cA^N)^T (I + cA^N P_{N-1}(cA^N)^T)^{-1} \\ &\quad \cdot cA^N P_{N-1}) M_{N-1}^T Y_{k,N-1} \\ &= (A - A^N P_{N-1}(cA^N)^T (I + cA^N P_{N-1}(cA^N)^T)^{-1} cA) \\ &\quad \cdot A^{N-1} P_{N-1} M_{N-1}^T Y_{k,N-1} \\ &= (A - L_N cA)\hat{x}_k\end{aligned}$$

where the substitution $\hat{x}_k = A^{N-1} P_{N-1} M_{N-1}^T Y_{k,N-1}$ has been made.

Thus, an alternative recursive formula for (6) is

$$\hat{x}_{k+1} = (A - L_N cA)\hat{x}_k + L_N y_{k+1} \tag{8}$$

which corresponds to the conventional dynamic state observer.

Remark 2: In the literature on recursive estimation, N is usually associated with the time index k so that as k increases, N the window size increases. A well-known problem with this strategy is that the state observer becomes insensitive to new observations as more data are collected. In this paper, this link is decoupled and the estimation horizon, N, is being used as a tuning parameter analogous to the prediction horizon in predictive controllers. An application example highlighting the role of N in improving the robustness of the closed loop is given in [8].

III. A Unification

The observers described by (6)–(8) can be unified and generalized.

Proposition 1: The following state observers are equivalent:

- $$\begin{aligned}\hat{x}_{k+1} &= (A - L_N cA)(I + \tilde{L}_{N-1} cA^{-N})\hat{x}_k + L_N y_{k+1} \\ &\quad - \tilde{L}_N y_{k-N}\end{aligned} \tag{9}$$
- $$\hat{x}_{k+1} = (A - L_N cA)\hat{x}_k + L_N y_{k+1} - \tilde{L}_N (y_{k-N} - cA^{-N}\hat{x}_k) \tag{10}$$
- $$\begin{aligned}\hat{x}_{k+1} &= A^N P_N A^{-T}(P_N^{-1} - (1-\alpha)c^T c)A^{-N}\hat{x}_k + L_N y_{k+1} \\ &\quad - \tilde{L}_N y_{k-N}\end{aligned} \tag{11}$$

where α is a continuous scalar variable,

$$\begin{aligned}L_N &= A^N P_N (cA^N)^T = A^N P_{N-1}(cA^N)^T (I + cA^N P_{N-1} \\ &\quad \cdot (cA^N)^T)^{-1}\end{aligned} \tag{12}$$

and

$$\tilde{L}_N = \alpha A^N P_N (cA^{-1})^T. \tag{13}$$

Proof: See the Appendix.

Proposition 1 shows the connection between the conventional dynamic state observer ($\alpha = 0$) and the moving window state observer ($\alpha = 1$). α also plays a role in ensuring the stability of the proposed state observer (see Proposition 3). In addition, (10) suggests that the moving window state observer can be interpreted as a conventional dynamic state observer with an appropriate weighting on the end-point observation error, $y_{k-N} - cA^{-N}\hat{x}_k$. (See also Section IV-B.)

Remark 3: Numerically reliable algorithms such as the QR method [6] should be used to compute the state estimate given in Proposition 1.

IV. Properties of the Proposed State Observer

A. Stability

Since the state estimation horizon N has been chosen as a tuning parameter for the proposed state observer design, the stability of the resulting observer with respect to N is of particular interest. The stability of the observer when $\alpha = 0$ is ensured by the following proposition.

Proposition 2: If $[c, A]$ is observable, A nonsingular, then the matrix, $A - L_N cA$, has all its eigenvalues strictly within the unit circle for all finite $N \geq n - 1$ where n is the dimension of the state vector.

Proof: See the Appendix.

The next proposition highlights the role of α in the proposed design.

Proposition 3: For $N \geq n - 1$, there exists $\alpha < \alpha_{\max}$, $\alpha \neq 1$ such that the proposed state observer is stable.

Proof: See the Appendix.

Remark 4: The structure of L_N obtained in (12) or (17) (in the proof of Proposition 2) suggests the existence of a Kalman filter interpretation for the proposed state observer obtained through a deterministic least squares framework. This is not unexpected since the Kalman filter can be regarded as a kind of stochastic least squares state observer [13], [14].

B. Relation with End-Point Weighting

In this section, it is shown that the state observer presented in Proposition 1 can also be interpreted as a particular recursive solution of the following deterministic least squares (DLS) problem.

Consider the cost function

$$J_{\gamma,\beta,N} = \|Y_{k,N} - M_N\hat{x}_{k-N}\|^2 + \gamma\|Y_o - \beta\hat{x}_{k-N}\|^2$$

where Y_o is a known vector, γ is a scalar, and β is a user selectable weighting matrix of appropriate dimensions.

The state estimate $\hat{x}_{k-N}$ that minimizes $J_{\gamma,\beta,N}$ is

$$\hat{x}_{k-N} = (M_N^T M_N + \gamma\beta^T\beta)^{-1}(M_N^T Y_{k,N} + \gamma\beta^T Y_o). \tag{14}$$

To see the relation with the proposed state observer of (10), write (14) as

$$M_N^T M_N\hat{x}_{k-N} = M_N^T Y_{k,N} + \gamma\beta^T Y_o - \gamma\beta^T\beta\hat{x}_{k-N} \tag{15}$$

or

$$\begin{aligned}\hat{x}_k &= A^N\hat{x}_{k-N}\\ &= A^N P_N M_N^T Y_{k,N} + \gamma A^N P_N\beta^T Y_o - \gamma A^N P_N\beta^T\beta\hat{x}_{k-N}.\end{aligned}$$

If $\gamma = -\alpha$, $Y_o = y_{k-N-1}$, $\beta = cA^{-1}$, and $A_N P_N M_N^T Y_{k,N}$ is replaced by its "conventional dynamic state observer" recursive solution (8), then (10) is recovered. It can thus be said that the conventional and the moving window state observers correspond to a particular recursive solution of the DLS problem with appropriate end-point weighting.

There are therefore at least two ways to view a moving window (recursive) state observer. The first, as shown in Section II-B, is to adopt a moving horizon estimation horizon with no end-point weighting. The second, as in this section, is to augment a conventional "growing" memory observer with end-point state weighting.

V. Conclusions

In this paper, the design of a state observer is treated as a problem of determining a solution to a system of linear equations. The state observer is readily obtained by using the techniques of least squares. The state estimation horizon N is introduced as a tuning parameter for the proposed state observer, and a stability result concerning the choice of N is established.

Due to the conceptual simplicity of this deterministic formulation which involves only solving a system of linear equations, multivariable, or even multirate systems can be handled in a similar way without much difficulty. In fact, the view of an observer as a solution to a system of linear equations can also be extended to nonlinear systems to derive observers for nonlinear systems [10]. It would be interesting to study the effect of the state estimation horizon in this nonlinear setting.

Appendix

A. Proof of Proposition 1

- To show the equivalence of (9) and (10), we need to show that $(A - L_N cA)\tilde{L}_{N-1}cA^{-N} = \tilde{L}_N cA^{-N}$. This is done through the following calculation:

$$\begin{aligned}&(A - L_N cA)\tilde{L}_{N-1}cA^{-N}\\ &\quad= \alpha(A - A^N P_{N-1}(cA^N)^T(I + cA^N P_{N-1}(cA^N)^T)^{-1}cA)\\ &\qquad\cdot A^{N-1}P_{N-1}(cA^{-1})^T cA^{-N}\\ &\quad= \alpha A^N(P_{N-1} - P_{N-1}(cA^N)^T(I + cA^N P_{N-1}(cA^N)^T)^{-1}\\ &\qquad\cdot cA^N P_{N-1})(cA^{-1})^T cA^{-N}\\ &\quad= \alpha A^N P_N(cA^{-1})^T cA^{-N}\\ &\quad= \tilde{L}_N cA^{-N}.\end{aligned}$$

- The equivalence of (9) and (10) is established through

$$\begin{aligned}&(A - L_N cA)(I + \tilde{L}_{N-1}cA^{-N})\\ &\quad= A - L_N cA + \tilde{L}_N cA^{-N}\\ &\quad= A - A^N P_N(cA^N)^T cA + \alpha A^N P_N(cA^{-1})^T cA^{-N}\\ &\quad= A^N P_N A^{-T}(A^T P_N^{-1}A - (A^{N+1})^T c^T cA^{N+1} + \alpha c^T c)A^{-N}\\ &\quad= A^N P_N A^{-T}(P_N^{-1} - (1-\alpha)c^T c)A^{-N},\end{aligned}$$

where we recall $P_N^{-1} = (A^N)^T c^T cA^N + M_{N-1}^T M_{N-1} = A^T M_{N-1}^T M_{N-1}A + c^T c$.

B. Proof of Proposition 2

Let $\overline{P}_N = A^{N+1}P_N(A^{N+1})^T$, then (5) can be written as an algebraic Riccati equation

$$\begin{aligned}\overline{P}_{N-1} = A\overline{P}_{N-1}A^T - A\overline{P}_{N-1}c^T(I + c\overline{P}_{N-1}c^T)^{-1}\\ \cdot c\overline{P}_{N-1}A^T + Q_{N-1}\end{aligned} \tag{16}$$

where $Q_{N-1} = \overline{P}_{N-1} - \overline{P}_N$.

Also, the estimator gain L_N (12) can be written as

$$L_N = \overline{P}_{N-1}c^T(I + c\overline{P}_{N-1}c^T)^{-1}. \tag{17}$$

Applying [3, Th. 3.2], we have

$$\tilde{A}_o = A - A\overline{P}_{N-1}c^T(c\overline{P}_{N-1}c^T + I)^{-1}c = A - AL_N c$$

which has all its eigenvalues strictly within the unit circle if

1) $[c, A]$ is detectable;
2) $[A, Q_{N-1}^{1/2}]$ is stabilizable[2];
3) $Q_{N-1} \geq 0$.

Since

$$\begin{aligned}Q_{N-1} &= \overline{P}_{N-1} - \overline{P}_N\\ &= A^N(P_{N-1} - AP_N A^T)(A^N)^T\\ &= A^N(P_{N-1} - A(A^T M_{N-1}^T M_{N-1}A\\ &\qquad + c^T c)^{-1}A^T)(A^N)^T\\ &= A^N(P_{N-1} - (P_{N-1}^{-1} + A^{-T}c^T cA^{-1})^{-1})(A^N)^T\\ &= A^N(P_{N-1} - P_{N-1} + P_{N-1}A^{-T}c^T(I + cA^{-1}P_{N-1}\\ &\qquad \cdot A^{-T}c^T)^{-1}cA^{-1}P_{N-1})(A^N)^T\\ &= \lambda A^N\overline{Q}_{N-1}(A^N)^T\\ &\geq 0 \qquad \text{if } P_{N-1} \text{ and } A^{-1} \text{ exists}\end{aligned}$$

where $\overline{Q}_{N-1} = P_{N-1}A^{-T}c^T cA^{-1}P_{N-1}$ and $\lambda = (I + cA^{-1}P_{N-1}A^{-T}c^T)^{-1}$ is a nonnegative scalar.

Recall that

$$P_{N-1} = \left(\begin{bmatrix} cA^{N-1}\\ cA^{N-2}\\ \vdots\\ c\end{bmatrix}^T \begin{bmatrix} cA^{N-1}\\ cA^{N-2}\\ \vdots\\ c\end{bmatrix}\right)^{-1}.$$

[2] Or $[A, Q_{N-1}^{1/2}]$ has no unstabilizable mode on the unit circle.

Thus, P_{N-1} exists if $[c, A]$ is observable and $N \geq n$ where n is the dimension of the state vector.

As for condition (2), for simplicity but without loss of generality, consider

$$\overline{Q}_{N-1}^{1/2} = P_{N-1} A^{-T} c^T.$$

If $[c, A]$ is observable, then it can be shown that $[A, \overline{Q}_{N-1}^{1/2}]$ is controllable if P_{N-1} and A^{-1} exist. This implies that $[A, Q_{N-1}^{1/2}]$ is also controllable. In addition, stability of $\tilde{A}_o$ implies stability of $A - L_N cA$.

Finally, since the formula for the state observer gain (12) specializes to the Ackermann formula for deadbeat state observer when $N = n - 1$, we conclude that $N \geq n - 1$ is sufficient for stability.

C. Proof of Proposition 3

Standard error analysis gives

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1} &= Ax_k - (A - L_N cA)\hat{x}_k - L_N cAx_k \\ &\quad + \tilde{L}_N cA^{-N} x_k - \tilde{L}_N cA^{-N} \hat{x}_k \\ &= (A - L_N cA + \tilde{L}_N cA^{-N})(x_k - \hat{x}_k). \end{aligned}$$

When $\alpha \neq 0$, a sufficient condition for stability is that

$$\begin{aligned} &\overline{\sigma}(A - L_N cA + \tilde{L}_N cA^{-N}) \\ &\quad \leq \overline{\sigma}(A - L_N cA) + \overline{\sigma}(\tilde{L}_N cA^{-N}) < 1 \end{aligned}$$

where $\overline{\sigma}$ denotes the maximum singular value.

This implies that

$$\begin{aligned} &\overline{\sigma}(\tilde{L}_N cA^{-N}) < 1 - \overline{\sigma}(A - L_N cA) \\ \Rightarrow\ &\alpha \overline{\sigma}(A^N P_N (cA^{-1})^T cA^{-N}) < 1 - \overline{\sigma}(A - L_N cA) \\ \Rightarrow\ &\alpha < \frac{1 - \overline{\sigma}(A - L_N cA)}{\overline{\sigma}(A^N P_N (cA^{-1})^T cA^{-N})} = \alpha_{\max}. \end{aligned}$$

Note that the existence of P_N requires that $N \geq n - 1$.

Note also that if $\alpha = 1$

$$A - L_N cA + \tilde{L}_N cA^{-N} = A^N P_N A^{-T} P_N^{-1} A^{-N}$$

which has eigenvalues equal to that of A^{-T}, and the observer will be unstable if A has eigenvalues within the unit circle!

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Finite-Dimensional Nonlinear Output Feedback Dynamic Games and Bounds for Sector Nonlinearities

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***Abstract*—In general, nonlinear output feedback dynamic games are infinite-dimensional. This paper treats a class of minimax games when the nonlinearities enter the dynamics of the unobservable states. An information state approach is introduced to recast these games as one of full information in infinite dimensions. Explicit solutions of the first-order partial differential information state equation are derived in terms of a finite-number of sufficient statistics. When the nonlinearities are sector bounded, suboptimal finite-dimensional strategies are derived.**

***Index Terms*—Information states, minimax games, output feedback.**

I. INTRODUCTION

As is well known, the theory of minimax dynamic games is an alternative approach in solving the sensitivity minimization problem posed by Zames [1]. In this game (see [2]), there are two opposing players: the disturbance input and the control input. Extensions to nonlinear problems are also possible, by interpreting the $\mathcal{H}^\infty$ operator norm in terms of the L^2-gain of the system, thus making the theory of dissipative systems applicable (see [3] and [5]).

However, in the case of output feedback minimax dynamic games, several issues associated with the controller design are not so well developed and understood. This is due, in part, to the fundamental difficulty that for general nonlinear output feedback minimax games one has to introduce an observer state (summarizing the observation history), on which the control action should be based, and in part

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