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Brief Paper

Constrained receding horizon predictive control for nonlinear systems☆

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Abstract

The paper concerns the receding horizon predictive control of constrained nonlinear systems and presents an algorithm which relies on the online solution of a simple linear program (LP). Use is made of a finite control horizon in conjunction with a terminal inequality constraint and a predicted cost that includes a terminal penalty term. The optimization procedure is based on predictions made by linearized incremental models at points of a given seed trajectory and the effects of linearization error are taken into account to give a bound on the predicted tracking error. The algorithm is posed in the form of an LP and the proper selection of the terminal penalty term of the predicted cost guarantees the asymptotic stability. The results of the paper are illustrated by means of a simple example.

Keywords: Nonlinear systems; Feasible invariant sets; Linear programming; Terminal weighting; Feasibility

1. Introduction

Receding horizon control (RHC) (e.g. Clarke et al., 1987) is a strategy which, at each instant of time, implements the first element of an input trajectory that is chosen to minimize a performance index. A major advantage of RHC is its ability to account for input/state constraints. Use of infinite horizons provides an automatic guarantee for stability, given that at start time there exists a feasible input trajectory that drives the predicted state to the desired equilibrium. This is because in the worst case RHC can give an open loop implementation of the feasible input trajectory, but in general will do better due to the online optimization. The presence of constraints and/or time-varying and/or nonlinear dynamics precludes the possibility of infinite or even large prediction horizons. This motivated the introduction into RHC of artificial terminal equality constraints, e.g. Kwon and Pearson (1977) for the linear time-varying case, Keerthi and Gilbert (1988) and Mayne and Michalska (1990) for the nonlinear discrete/continuous-time case. The idea was to use a finite horizon but to ensure stability over an infinite predicted horizon by forcing the predicted state to reach its target in a finite number of control moves, and to remain there at all future times. Terminal equality constraints can lead to infeasibility and to avoid this less stringent terminal inequality constraints were introduced e.g. Michalska and Mayne (1993). In this approach (dual-mode control) state predictions over a finite number of control moves are forced to enter a constrained set on which a linear feedback law is stabilizing in the presence of physical constraints. Recently quasi-infinite horizon methods have been developed (De Nicolao, Magni, & Scattolini, 1998; Chen & Allgower, 1998) which modified the dual mode concept by incorporating in the cost a terminal penalty term chosen to be a Lyapunov function for the nonlinear system subject to a local linear control (with the added requirement that the rate of decrease of this Lyapunov function should be sufficiently large). The penalty term is either the cost or an upper bound on the cost associated with the terminal linear control law. Thus, the quasi-infinite horizon cost approximates the infinite horizon cost for the nonlinear system dynamics, but the resulting RHC optimization remains tractable.

Applications of RHC based on nonlinear prediction dynamics (e.g. Michalska & Mayne, 1993; De Nicolao et al., 1998; Chen & Allgower, 1998) are severely restricted by the computational burden of the associated optimization (see e.g. Qin & Badgwell, 2000), which is in general a nonconvex nonlinear programming problem. This motivated

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the development of RHC strategies in which online optimization is approximated as an efficiently solvable convex problem. For example sequential quadratic programming methods (Biegler, 1998; Bock, Diehl, Leineweber, & Schlöder, 2000) search for the solution to the RHC optimization by iteratively approximating local optimization problems as quadratic programs (QP). A disadvantage of this approach is that, if no explicit account is taken of approximation errors, it may require the solution of a large number of quadratic programs at each optimization of the predicted cost. The RHC optimization can alternatively be approximated as a QP by means of linear approximations to the nonlinear prediction dynamics. This approach (e.g. in Kouvaritakis, Cannon, & Rossiter, 1999) linearizes the prediction dynamics about a seed trajectory obtained from the input sequence computed at the previous RHC optimization. The prediction errors resulting from the use of the linearized dynamics in the RHC optimization are accounted for explicitly by perturbing the optimal trajectory computed on the basis of the linearized prediction dynamics in order to obtain a trajectory for the original nonlinear dynamics which satisfies a terminal equality constraint. As a result, the algorithm has guaranteed stability while requiring the solution of a single QP at each online optimization. However it is based on deadbeat predicted responses which are likely to result in suboptimality and infeasibility.

Linearizing the prediction dynamics about a suboptimal seed trajectory is also employed in the current paper. However here we use a terminal inequality constraint in conjunction with a terminal penalty term corresponding to an upper bound on the predicted tracking errors. The inequality constraint ensures that the predicted terminal state belongs to a polytopic invariant set defined with respect to a locally stabilizing state feedback about a reference state. In addition, through the use of Lipschitz conditions, it is possible to derive bounds on the linearization errors, and thus take explicit account of such errors. In order to ensure that the bounds on the linearization error remain valid, despite perturbations on the seed trajectory, a sequence of polytopic constraints are imposed on the linear prediction system state. The parameters of these constraints are deployed as free parameters in the online optimization so that the propagation of the effect of linearization error is not overestimated. The receding horizon optimization problem is thus formulated in terms of linear inequalities, and can be solved efficiently using linear programming (LP). A simple example demonstrates the effectiveness of the resulting RHC strategy.

2. Problem formulation

Consider the discrete-time nonlinear system

$$\mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)), \quad |\mathbf{u}(k)| \leqslant \bar{\mathbf{u}}, \tag{1}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $|\mathbf{u}| = \{|u_i|\}$ and $\bar{\mathbf{u}}$ is the vector of input bounds. We use the notation $\mathbf{x} = \{x_i\}$ for vectors and $A = \{a_{ij}\}$ for matrices and write $A \geqslant 0$ if $a_{ij} \geqslant 0$. For regulation, a convenient RHC strategy is to consider the first N predicted control moves to be free and the remainder to be defined by state feedback $\mathbf{u} = F\mathbf{x}$. The N free moves are deployed in the optimization of a prediction cost

$$J = \sum_{i=0}^{N-1} \|\mathbf{x}(k+i|k)\|_Q^2 + \|\mathbf{u}(k+i|k)\|_R^2 + \|\mathbf{x}(k+N|k)\|_P^2 \tag{2}$$

Q, R and P in the weighted 2-norms above are positive definite; minimization is subject to (1) and the stability inequality constraint that $\mathbf{x}(k+N|k)$ should lie in a terminal set (often taken to be an ellipsoid) defined to be positively invariant for (1) under $\mathbf{u} = F\mathbf{x}$. Provided that P is such that $\mathbf{x}^{\mathrm{T}}P\mathbf{x}$ is a Lyapunov function for (1) under $\mathbf{u} = F\mathbf{x}$, and that rate of decrease of $\mathbf{x}^{\mathrm{T}}P\mathbf{x}$ is large enough (Chen & Allgower, 1998; De Nicolao et al., 1998) this RHC strategy guarantees stability (given feasibility at start time) but due to the explicit use of nonlinear dynamics of (1) the cost J of (2) is nonconvex and its optimization is impracticable. The use of the quadratic penalty term in the cost of (2) can lead to optimal results for linear MPC but for nonlinear systems the use of control Lyapunov functions has distinct advantages (e.g. see Sznaier, Cloutier, Hull, Jacques, & Mraček, 2000 or Primbs, Nevistic, & Doyle, 2000).

Here we consider a tracking problem and adopt an alternative RHC which also uses N free moves to minimize a prediction cost subject to the input constraints of (1) and subject to a stability constraint that the terminal predicted state should lie in a positively invariant target set. However the key to our strategy is that given seed trajectories for the predicted states and inputs, it is possible to consider the N free moves to be perturbations on the input seed trajectory so that the nonlinear dynamics of (1) can be linearized (about the seed trajectories). Constraining the predicted perturbed states and inputs (see Fig. 1) to lie in polytopic sets $X(i|k)$ and $U(i|k)$, provides a mechanism for taking explicit account of the errors in the process of linearization as well as

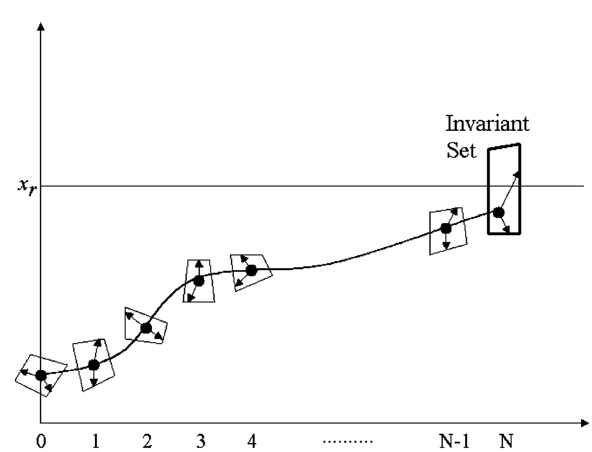

Fig. 1. Schematic of perturbations on a seed trajectory.

defining upper bounds on the error between predicted state and the target state, $\mathbf{x}_r$, which is to be tracked. Summing these bounds over the prediction horizon N provides an alternative cost which conveniently depends linearly on the degrees of freedom. The sets $X(i|k)$ and $U(i|k)$ can also be used to enforce a stability constraint that the terminal state should lie in a polytope X_r that contains $\mathbf{x}_r$ and is positively invariant under a terminal feedback law $\mathbf{u} = \mathbf{u}_r + F_r(\mathbf{x} - \mathbf{x}_r)$ where $\mathbf{u}_r$ is the input that keeps the state vector at its target value $\mathbf{x}_r$. The conditions that constrain the predicted states and inputs to lie in $X(i|k)$, $U(i|k)$ and the terminal state to lie in X_r are linear in the degrees of freedom and hence the on-line optimization reduces to a simple LP. Also, at any time, suitable state and input seed trajectories can be conveniently generated by a forward shift of the state and input trajectories computed at the previous time instant. Thus, provided that feasible seeds can be defined at start time, the proposed RHC is computationally very efficient and guarantees stability and asymptotic tracking. The details of the definitions of the polytopic sets are given in this section whereas the invariance properties and the development of the overall algorithm are given in Sections 3 and 4.

Let a reference state $\mathbf{x}_r$ be an equilibrium state for which the solution $\mathbf{u}_r$ ($|\mathbf{u}_r| < \bar{\mathbf{u}}$) of the equilibrium equation $\mathbf{x}_r = f(\mathbf{x}_r \mathbf{u}_r)$ can be obtained by any algebraic equation solver and define a set of states around $\mathbf{x}_r$ as

$$X_r = \{\mathbf{x} \in R^n \mid |\mathbf{z} - \mathbf{z}_r| \leqslant \bar{\mathbf{z}}_{\delta r}\}, \quad \mathbf{z} = V\mathbf{x},\ \mathbf{z}_r = V\mathbf{x}_r, \tag{3}$$

where V is a nonsingular transformation matrix; possible criteria for the selection of this design parameter are discussed in Remark 3. We assume that at present time k seed trajectories of states $\mathbf{x}^0(i|k)$ $i = 0, 1, \ldots, N$ and inputs $\mathbf{u}^0(i|k)$, $i = 0, 1, \ldots, N-1$ are available, which satisfy

$$\mathbf{x}^0(i+1|k) = f(\mathbf{x}^0(i|k), \mathbf{u}^0(i|k)), \quad \mathbf{x}^0(0|k) = \mathbf{x}(k), \tag{4}$$

$$|\mathbf{u}^0(i|k)| \leqslant \bar{\mathbf{u}}, \quad \mathbf{x}^0(N|k) \in X_r. \tag{5}$$

Then defining $\mathbf{x}_\delta(i|k) = \mathbf{x}(k+i) - \mathbf{x}^0(i|k)$, $\mathbf{u}_\delta(i|k) = \mathbf{u}(k+i) - \mathbf{u}^0(i|k)$ to be state and input deviations from the seed trajectories, and using Taylor's theorem we can linearize (1):

$$\begin{aligned} f(\mathbf{x}(k+i), \mathbf{u}(k+i)) = f(&\mathbf{x}^0(i|k), \mathbf{u}^0(i|k)) \\ &+ A(i|k)\mathbf{x}_\delta(i|k) + B(i|k)\mathbf{u}_\delta(i|k) \\ &+ \mathbf{x}_e(i|k). \end{aligned} \tag{6}$$

$A(i|k) = \partial f / \partial \mathbf{x}(\mathbf{x}^0(i|k), \mathbf{u}^0(i|k))$, $B(i|k) = \partial f / \partial \mathbf{u}(\mathbf{x}^0(i|k), \mathbf{u}^0(i|k))$, and $\mathbf{x}_e(i|k)$ represents the remainder of the Taylor series expansion. Combined use of the state feedback law, $F(i|k)\mathbf{x}_\delta(i|k)$, plus a perturbation $\mathbf{c}(i|k)$:

$$\mathbf{u}_\delta(i|k) = F(i|k)\mathbf{x}_\delta(i|k) + \mathbf{c}(i|k) \tag{7}$$

and application of the transformation $\mathbf{x} = W\mathbf{z}$, $W = V^{-1}$, to (6)–(7) leads to the closed-loop incremental state equation:

$$\begin{aligned} &\mathbf{z}_\delta(i+1|k) = \Phi^W(i|k)\mathbf{z}_\delta(i|k) + B^W(i|k)\mathbf{c}(i|k) + \mathbf{z}_e(i|k), \\ &\mathbf{z}_\delta(i|k) = V\mathbf{x}_\delta(i|k), \quad \mathbf{z}_e(i|k) = V\mathbf{x}_e(i|k); \\ &B^W(i|k) = VB(i|k), \\ &\Phi^W(i|k) = V(A(i|k) + B(i|k)F(i|k))W. \end{aligned} \tag{8}$$

This closed-loop formulation is to be preferred purely on numerical grounds (Rossiter, Kouvaritakis, & Rice, 1998) and thus from a control viewpoint the particular choice of $F(i|k)$ is immaterial; from a numerical viewpoint it would be preferable that $F(i|k)$ make $\Phi^W(i|k)$ stable but for convenience $F(i|k)$ can be chosen as the feedback gain that minimizes the finite horizon cost of (2) for the linearized dynamics described by the pairs $\{A(i|k), B(i|k)\}$ (Bitmead, Gevers, & Wertz, 1990). Note that the magnitude of the remainder term $\mathbf{z}_e(i|k)$ depends on the magnitude of $\mathbf{z}_\delta(i|k)$ and $\mathbf{u}_\delta(i|k)$. In order to obtain guaranteed bounds on this error, we define (for $i = 0, 1, \ldots, N-1$) polyhedral regions of states and inputs as

$$X(i|k) = \{\mathbf{x} \in R^n\colon |\mathbf{z} - \mathbf{z}^0(i|k)| \leqslant \bar{\mathbf{z}}_\delta(i|k)\}, \tag{9}$$

$$U(i|k) = \{\mathbf{u} \in R^m\colon |\mathbf{u} - \mathbf{u}^0(i|k)| \leqslant \bar{\mathbf{u}}_\delta(i|k)\}, \tag{10}$$

where $\mathbf{z}^0(i|k) = V\mathbf{x}^0(i|k)$. Here for the given bounds $\bar{\mathbf{z}}_\delta(i|k)$ and $\bar{\mathbf{u}}_\delta(i|k)$ in (9)–(10), and for all $\mathbf{x} \in X(i|k)$, $\mathbf{u} \in U(i|k)$, $i = 0, 1, \ldots, N-1$ we assume that $\mathbf{z}_e(i|k)$ satisfies

$$|\mathbf{z}_e(i|k)| \leqslant \Gamma_\mathbf{z}(i|k)|\mathbf{z}_\delta(i|k)| + \Gamma_\mathbf{u}(i|k)|\mathbf{u}_\delta(i|k)| \tag{11}$$

for matrices $\Gamma_\mathbf{z}(i|k)$ and $\Gamma_\mathbf{u}(i|k)$ of conformal dimension.

Remark 1. $\Gamma_\mathbf{z}(i|k)$ and $\Gamma_\mathbf{u}(i|k)$ can be determined from the application of Taylor's remainder theorem to $\mathbf{z}_e(i|k)$ when this is viewed as a function of $\mathbf{z}_\delta(i|k)$ and $\mathbf{u}_\delta(i|k)$:

$$\begin{aligned} \mathbf{z}_e(i|k) &= g(\mathbf{z}_\delta(i|k), \mathbf{u}_\delta(i|k)) \\ &= Vf(\mathbf{x}^0(i|k) + \mathbf{x}_\delta(i|k), \mathbf{u}^0(i|k) + \mathbf{u}_\delta(i|k)) \\ &\quad - Vf(\mathbf{x}^0(i|k), \mathbf{u}^0(i|k)) - VA(i|k)W\mathbf{z}_\delta(i|k) \\ &\quad - VB(i|k)\mathbf{u}_\delta(i|k). \end{aligned} \tag{12}$$

Maximizing the moduli of the elements of the Jacobian of f over all $|\mathbf{z}_\delta(i|k)| \leqslant \bar{\mathbf{z}}_\delta(i|k)$ and $|\mathbf{u}_\delta(i|k)| \leqslant \bar{\mathbf{u}}_\delta(i|k)$ yields a choice for $\Gamma_\mathbf{z}(i|k)$ and $\Gamma_\mathbf{u}(i|k)$ (see Section 5). $\Gamma_\mathbf{z}(i|k)$ and $\Gamma_\mathbf{u}(i|k)$ depend on $A(i|k)$ and $B(i|k)$ which may be chosen as in (6). This choice is not unique, and there may exist other choices (a matter that is example dependent) that result in smaller $\Gamma_\mathbf{z}(i|k)$ and $\Gamma_\mathbf{u}(i|k)$.

Remark 2. The computation of $\Gamma_\mathbf{z}(i|k)$, and $\Gamma_\mathbf{u}(i|k)$ of (11) can be performed off-line provided that the bounds $\bar{\mathbf{z}}_\delta(i|k)$ and $\bar{\mathbf{u}}_\delta(i|k)$ are pre-specified as $\bar{\mathbf{z}}_\delta$ and $\bar{\mathbf{u}}_\delta$, respectively. In this case, $\Gamma_\mathbf{z}(i|k)$ and $\Gamma_\mathbf{u}(i|k)$ depend on the values of $\mathbf{x}^0(i|k)$

and $\mathbf{u}^0(i|k)$. Thus, we can consider a uniform gain Γ_X and Γ_U over fixed regions of states, X, and inputs, U, such that

$$\Gamma_X = \max_{\mathbf{x}\in X,\ \mathbf{u}\in U} |\Gamma_{\mathbf{z}}|, \tag{13}$$

$$\Gamma_U = \max_{\mathbf{x}\in X,\ \mathbf{u}\in U} |\Gamma_{\mathbf{u}}|, \tag{14}$$

Γ_X, Γ_U can be computed off-line (as per Remark 1) for different X, U and can be used instead of $\Gamma_{\mathbf{z}}(i|k)$, $\Gamma_{\mathbf{u}}(i|k)$. Also they can be updated online depending on which region X and U the seeds $\mathbf{x}^0(i|k)$ and $\mathbf{u}^0(i|k)$ belong to.

Having defined $X(i|k)$, $U(i|k)$, and X_r we outline the overall MPC algorithm (see Section 4). For an initial state $\mathbf{x}(0)\in X(0|0)$, a control strategy which steers this state to X_r is to move the state through the successive regions $X(i|0)$, $i=1,2,\dots,N-1$ and after the state has entered X_r, use a locally stabilizing state feedback control. Provided the predicted future state and inputs belong to $X(i|0)$ and $U(i|0)$, $i=1,2,\dots,N-1$, we can utilize the linearized model (8) with (11) instead of the nonlinear model (1). Use of (8) with (11) enables us to formulate an optimization problem, which (unlike that based on nonlinear dynamics e.g. (1) (Michalska & Mayne, 1993)) is convex and tractable. To obtain a locally stabilizing control around $\mathbf{x}_r$, the bound $\bar{\mathbf{z}}_{\delta r}$ should be chosen so that: (i) X_r is an invariant set of states with respect to a state feedback gain F_r i.e. $\mathbf{u}=\mathbf{u}_r+F_r(\mathbf{x}-\mathbf{x}_r)$ ensures that $\mathbf{x}\in X_r$; (ii) $\mathbf{u}=\mathbf{u}_r+F_r(\mathbf{x}-\mathbf{x}_r)$ is feasible in X_r. Thus, we wish X_r to be a “positively invariant set” (Gilbert & Tan, 1991). F_r is a design parameter to be chosen according to criteria given in Section 3. At the next time $k=1$, a new seed trajectory of states and inputs can be defined (for $i=0,1,\dots,N-1$) as

$$\mathbf{x}^0(i+1|1) = f(\mathbf{x}^0(i|1),\mathbf{u}^0(i|1)), \tag{15}$$

$$\mathbf{u}^0(i|1) = \mathbf{u}^0(i+1|0)+\mathbf{u}_\delta(i+1|0), \tag{16}$$

$$\mathbf{x}^0(0|1) = f(\mathbf{x}^0(0|0),\mathbf{u}^0(0|0)+\mathbf{u}_\delta(0|0)), \tag{17}$$

$$\mathbf{u}^0(N-1|1) = \mathbf{u}_r + F_r(\mathbf{x}^0(N-1|1)-\mathbf{x}_r) \tag{18}$$

using the control moves $\mathbf{u}_\delta(i|0)$, which were calculated at time 0. Due to the invariance of X_r, $\mathbf{x}^0(N|1)\in X_r$ is guaranteed by (18) provided $\mathbf{u}_\delta(i|0)$, $i=0,1,\dots,N-1$ are selected so that $\mathbf{x}(N|0)\in X_r$. With this new seed trajectory at time 1, we have $\mathbf{x}(1)=\mathbf{x}^0(0|1)$ *i.e.* $\mathbf{z}_\delta(0|1)=0$. These arguments, repeated at all future times, provide an RHC implementation of the overall algorithm to be discussed in Section 4. But first we give the conditions for X_r to be an invariant feasible set with respect to a given state feedback gain F_r.

3. Invariant and feasible set of states X_r

Let $A_r=\partial f/\partial\mathbf{x}(\mathbf{x}_r\mathbf{u}_r)$, $B_r=\partial f/\partial\mathbf{u}(\mathbf{x}_r\mathbf{u}_r)$, $\mathbf{x}_{\delta r}=\mathbf{x}-\mathbf{x}_r$ and $\mathbf{u}_{\delta r}=\mathbf{u}-\mathbf{u}_r=F_r\mathbf{x}_{\delta r}$, then from $\mathbf{x}_r=f(\mathbf{x}_r,\mathbf{u}_r)$, we have

$$\mathbf{x}_{\delta r}(k+1) = (A_r+B_rF_r)\mathbf{x}_{\delta r}(k)+\mathbf{x}_{\mathrm{er}}(k), \tag{19}$$

where $\mathbf{x}_{\mathrm{er}}$ represents the linearization error of the Taylor series expansion about the reference state. Here we assume that $\mathbf{z}_{\mathrm{er}}=V\mathbf{x}_{\mathrm{er}}$ satisfies the following Lipschitz condition:

$$|\mathbf{z}_{\mathrm{er}}| \leqslant \Gamma_{\mathbf{z}r}|\mathbf{z}_{\delta r}|, \quad \forall \mathbf{x}\in X_r. \tag{20}$$

$\Gamma_{\mathbf{z}r}$ can be computed similarly to $\Gamma_{\mathbf{z}}$ (see Remark 1). Then, for any $\mathbf{x}(k)\in X_r$, $\mathbf{u}_{\delta r}(k)=F_r\mathbf{x}_{\delta r}(k)=F_rW\mathbf{z}_{\delta r}(k)$ yields

$$\mathbf{z}_{\delta r}(k+1) = \Phi_r^W\mathbf{z}_{\delta r}(k)+\mathbf{z}_{\mathrm{er}}(k), \tag{21}$$

where $\Phi_r^W=V(A_r+B_rF_r)W$. To guarantee that $\mathbf{x}(k+1)$ remains in X_r for any $\mathbf{x}(k)\in X_r$, we deploy the inequality

$$\begin{aligned}|\mathbf{z}_{\delta r}(k+1)| &\leqslant |\Phi_r^W\mathbf{z}_{\delta r}(k)|+|\mathbf{z}_{\mathrm{er}}(k)|\\ &\leqslant (|\Phi_r^W|+\Gamma_{\mathbf{z}r})\bar{\mathbf{z}}_{\delta r}.\end{aligned} \tag{22}$$

Then, according to (22), the condition

$$(|\Phi_r^W|+\Gamma_{\mathbf{z}r})\bar{\mathbf{z}}_{\delta r} \leqslant \bar{\mathbf{z}}_{\delta r} \tag{23}$$

guarantees the invariance of X_r. Because of the physical limit in (1), we need the following additional feasibility condition:

$$\begin{aligned}|\mathbf{u}_r+\mathbf{u}_{\delta r}(k)| &= |\mathbf{u}_r+F_r\mathbf{x}_{\delta r}(k)|\\ &\leqslant |\mathbf{u}_r|+|F_rW|\bar{\mathbf{z}}_{\delta r} \leqslant \bar{\mathbf{u}}.\end{aligned} \tag{24}$$

Earlier consideration (Dahleh & Diaz-Bobillo, 1995) of polytopic invariant sets required an l^1 optimization which can lead to infinite-dimensional controllers. The approach here is different and is based on the fact that $\bar{\mathbf{z}}_{\delta r}$ satisfying (23) exist iff the Perron–Frobenius norm of $(|\Phi_r^W|+\Gamma_{\mathbf{z}r})$ is less than or equal to 1 (Lee & Kouvaritakis, 2000; Mees, 1981). Note that multiplying $\bar{\mathbf{z}}_{\delta r}$ by a positive scalar does not affect (23). Thus, once $\bar{\mathbf{z}}_{\delta r}$ satisfying (23) is found, the feasibility condition (24) can be satisfied by scaling $\bar{\mathbf{z}}_{\delta r}$.

Remark 3. The above suggests a possible off-line procedure for the selection of F_r, W and $\bar{\mathbf{z}}_{\delta r}$. Thus consider first the special case when $\Gamma_{\mathbf{z}r}$ is diagonal. Then selecting W to be the eigenvector matrix of Φ_r^W will (for simple eigenvalues) make Φ_r^W diagonal so that (23) and (24) show the compromise that F_r has to strike between speed of response and feasibility: (23) requires the closed-loop eigenvalues to be fast, whereas (24) requires that the feedback gain F_r not be too large (for a given $\bar{\mathbf{z}}_{\delta r}$). A convenient way for achieving this compromise is to choose F_r to be LQ optimal for the pair (A_r,B_r) for state and input weights Q and R_0, where $R_0=\lambda R$, and to allow λ to vary; Q and R are as in the cost of (2), except that the cost considered here is over an infinite horizon. A small (large) value for λ would make (23) easier (harder) to satisfy, but would make (24) harder (easier) to satisfy for large $\bar{\mathbf{z}}_{\delta r}$. Now from knowledge of the system dynamics it is possible to decide on a sensible direction $\bar{\mathbf{z}}_{\delta r}^0$ for $\bar{\mathbf{z}}_{\delta r}$ and set $\bar{\mathbf{z}}_{\delta r}=s\bar{\mathbf{z}}_{\delta r}^0$. For any given values of F_r and scaling factor s, one can determine $\Gamma_{\mathbf{z}r}$ and thus check whether (23) and (24) hold true. In the interest of maximizing the control authority available for the optimization of predicted behaviour and/or increasing the applicability of our approach to the largest possible set of initial conditions, it is desirable to make X_r as large as possible. Thus one can look for the largest value of s for which an F_r satisfying (23) and (24) exists. Clearly any F_r satisfying (23) will be stabilizing. A possible overall off-line procedure for the selection of F_r, W and $\bar{\mathbf{z}}_{\delta r}$ could be as follows: select values for λ and s; if (24) is inconsistent increase λ; otherwise find the largest value for s for which both (24) and (23) are satisfied; repeat for a smaller value of λ and stop when s cannot be increased further.

From now on, we assume that $\bar{\mathbf{z}}_{\delta r}$ satisfies (23) and (24) so that X_r is a feasible and invariant set with respect to F_r.

4. Nonlinear receding horizon predictive control

We propose now an RHC algorithm for nonlinear systems, which optimizes predictions in an appropriate sense (see below) over the perturbations $\mathbf{c}(i|k)$ at each k and implements $\mathbf{u}(k|k)=\mathbf{u}^0(0|k)+F(0|k)\mathbf{x}_\delta(0|k)+\mathbf{c}(0|k)$. The feasibility properties of the algorithm are established and conditions for stability are given in terms of a terminal penalty term.

To use the linear dynamics of (8) with (11) instead of (1), it is necessary to ensure that: (R1) the predicted $\hat{\mathbf{x}}(k+i|k)$, $\hat{\mathbf{u}}(k+i|k)$, belong to $X(i|k)$ and $U(i|k)$; and (R2) the terminal predicted state $\hat{\mathbf{x}}(k+N|k)$ belongs to the feasible and invariant set X_r. To invoke (R1) we use (8) to obtain the predicted values $\hat{\mathbf{z}}_\delta(i|k)$. For simplicity it will be assumed that $\mathbf{z}_\delta(0|k)=0$; this would be true after the initial control step and the extension to the nonzero case is trivial. From (8):

$$\hat{\mathbf{z}}_\delta(i|k)=\sum_{j=0}^{i-1}\prod_{l=j+1}^{i-1}\Phi^W(l|k)\{B^W(j|k)\mathbf{c}(j|k)+\mathbf{z}_e(j|k)\}. \tag{25}$$

From (25), (11) we can compute (on-line) bounds on $|\hat{\mathbf{z}}_\delta(i|k)|$ and establish conditions that guarantee membership of $\hat{\mathbf{x}}(k+i|k)$ and $\hat{\mathbf{u}}(k+i|k)$. This is achieved through the use of inequalities which are linear in a set of free parameters: $\mathbf{c}(\cdot|k)$, $\alpha(\cdot|k)$, $\beta(\cdot|k)$, $\mathbf{d}_1(\cdot)$, $\mathbf{d}_2(\cdot)$, $\mathbf{d}_3(\cdot)$, and $\mathbf{d}_4(\cdot)$; linearity allows for an efficient on-line computation of the free parameters.

Theorem 4. *Consider* (1) *and its linearized representation* (8) *with condition* (11). *For given seed trajectories in* (4)–(5) *and sets $X(i|k)$ and $U(i|k)$ in* (9)–(10), *the membership* (*for $i=1,2,\ldots,N-1$*) *$\hat{\mathbf{x}}(k+i|k)\in X(i|k)$, $\hat{\mathbf{u}}(k+i|k)\in U(i|k)$ is guaranteed if $\mathbf{x}(k)\in X(0|k)$ and the linear inequalities*:

$$\left|\sum_{j=0}^{i-1}\prod_{l=j+1}^{i-1}\Phi^W(l|k)\cdot B^W(j|k)\mathbf{c}(j|k)\right|\leqslant \mathbf{d}_1(i), \tag{26}$$

$$\sum_{j=0}^{i-1}\left|\prod_{l=j+1}^{i-1}\Phi^W(l|k)\right|\{\Gamma_{\mathbf{z}}(j|k)\bar{\alpha}(j|k)+\Gamma_{\mathbf{u}}(j|k)\bar{\beta}(j|k)\} \leqslant \mathbf{d}_2(i), \tag{27}$$

$$\mathbf{d}_1(i)+\mathbf{d}_2(i)\leqslant\bar{\alpha}(i|k),\quad \bar{\alpha}(i|k)\leqslant\bar{\mathbf{z}}_\delta(i|k), \tag{28}$$

$$\left|F(i|k)W\sum_{j=0}^{i-1}\left\{\prod_{l=j+1}^{i-1}\Phi^W(l|k)\cdot B^W(j|k)\mathbf{c}(j|k)\right\}+\mathbf{c}(i|k)\right| \leqslant \mathbf{d}_3(i), \tag{29}$$

$$\sum_{j=0}^{i-1}\left|F(i|k)W\prod_{l=j+1}^{i-1}\Phi^W(l|k)\right| \{\Gamma_{\mathbf{z}}(j|k)\bar{\alpha}(j|k)+\Gamma_{\mathbf{u}}(j|k)\bar{\beta}(j|k)\}\leqslant\mathbf{d}_4(i), \tag{30}$$

$$\mathbf{d}_3(i)+\mathbf{d}_4(i)\leqslant\bar{\beta}(i|k),\quad \bar{\beta}(i|k)\leqslant\bar{\mathbf{u}}_\delta(i|k) \tag{31}$$

for $i=0,1,\ldots,N-1$, are satisfied for some positive vectors $\bar{\alpha}(\cdot|k)$ and $\bar{\beta}(\cdot|k)$.

Proof. Appendix A. □

To ensure feasibility beyond the horizon N we invoke requirement (R2) that N feasible moves steer $\hat{\mathbf{x}}(k+N|k)$ into X_r. This is obtained through inequalities which are linear in the parameters of Theorem 4 and a further set, $\mathbf{d}_5(.)$

Theorem 5. *The membership condition of the terminal predicted state $\hat{\mathbf{x}}(k+N|k)=\mathbf{x}^0(N|k)+\hat{\mathbf{x}}_\delta(N|k)\in X_r$ and the feasibility of all predicted input sequences $|\hat{\mathbf{u}}(k+i|k)|\leqslant\bar{\mathbf{u}}$, $i=0,1,\ldots,N-1$ are guaranteed by the following linear inequalities* (*for all $i=0,1,\ldots,N-1$*):

$$\left|\mathbf{z}_r-\mathbf{z}^0(N|k)-\sum_{j=0}^{N-1}\prod_{l=j+1}^{N-1}\Phi^W(l|k)\cdot B^W(j|k)\mathbf{c}(j|k)\right| \leqslant\mathbf{d}_1(N), \tag{32}$$

$$\sum_{j=0}^{N-1}\left|\prod_{l=j+1}^{N-1}\Phi^W(l|k)\right|\{\Gamma_{\mathbf{z}}(j|k)\bar{\alpha}(j|k)+\Gamma_{\mathbf{u}}(j|k)\bar{\beta}(j|k)\} \leqslant\mathbf{d}_2(N), \tag{33}$$

$$\mathbf{d}_1(N)+\mathbf{d}_2(N)\leqslant\bar{\mathbf{z}}_{\delta r}, \tag{34}$$

$$\left|\mathbf{u}^0(i|k)+F(i|k)W\sum_{j=0}^{i-1}\left\{\prod_{l=j+1}^{i-1}\Phi^W(l|k)\cdot B^W(j|k)\mathbf{c}(j|k)\right\} +\mathbf{c}(i|k)\right|\leqslant\mathbf{d}_5(i), \tag{35}$$

$$\mathbf{d}_5(i)+\mathbf{d}_4(i)\leqslant\bar{\mathbf{u}}. \tag{36}$$

Proof. Appendix B. □

Remark 6. (26)–(31) and (32)–(36) have a nonempty feasibility set because they are linear with respect to $\mathbf{c}(\cdot|k)$, $\bar{\alpha}(\cdot|k)$, $\bar{\beta}(\cdot|k)$, $\mathbf{d}_1(\cdot)$, $\mathbf{d}_2(\cdot)$, $\mathbf{d}_3(\cdot)$, $\mathbf{d}_4(\cdot)$, $\mathbf{d}_5(\cdot)$; setting all these to zero provides one obvious feasible solution.

We can now define the cost index to be minimized subject to the constraints developed above. From the earlier definitions and (25), we get the predictions for $\mathbf{z}$ and error in $\hat{\mathbf{z}}$ as

$$\begin{aligned}\hat{\mathbf{e}}_z(i|k) &= \mathbf{z}_r - \hat{\mathbf{z}}(k+i|k),\\ \hat{\mathbf{z}}(k+i|k) &= \mathbf{z}^0(i|k) + \hat{\mathbf{z}}_\delta(i|k),\end{aligned} \tag{37}$$

$$\begin{aligned}|\hat{\mathbf{e}}_z(i|k)| &= |\mathbf{z}_r - \mathbf{z}^0(i|k) - \hat{\mathbf{z}}_\delta(i|k)| \leqslant \Bigg|\mathbf{z}_r - \mathbf{z}^0(i|k)\\ &\quad - \sum_{j=0}^{i-1}\left\{\prod_{l=j+1}^{i-1}\Phi^W(l|k)\cdot B^W(j|k)\mathbf{c}(j|k)\right\}\Bigg|\\ &\quad + \sum_{j=0}^{i-1}\left|\prod_{l=j+1}^{i-1}\Phi^W(l|k)\right|\\ &\quad \times\{\Gamma_{\mathbf{z}}(j|k)\bar{\alpha}(j|k) + \Gamma_{\mathbf{u}}(j|k)\bar{\beta}(j|k)\}\\ &= \mathbf{e}_{\max}(i|k).\end{aligned} \tag{38}$$

Thus, an attainable bound on the 1-norm of the error is

$$J_{\max}(k) = \sum_{i=1}^{N-1}\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\max}(i|k) + \mathbf{p}^{\mathrm{T}}\mathbf{e}_{\max}(N|k), \tag{39}$$

$\mathbf{1} = [1\ 1\ \cdots\ 1]^{\mathrm{T}}$, and $\mathbf{p} = [p_1\ p_2\ \cdots\ p_n]^{\mathrm{T}} > 0$ is a design parameter. The minimization of $J_{\max}$ is an LP:

$$J_{\max}(k) = \sum_{i=1}^{N} t(i), \tag{40}$$

$$\left|\mathbf{z}_r - \mathbf{z}^0(i|k) - \sum_{j=0}^{i-1}\prod_{l=j+1}^{i-1}\Phi^W(l|k)\cdot B^W(j|k)\mathbf{c}(j|k)\right| \leqslant \mathbf{d}_6(i), \tag{41}$$

$$\mathbf{1}^{\mathrm{T}}(\mathbf{d}_6(i) + \mathbf{d}_2(i)) \leqslant t(i), \tag{42}$$

$$\mathbf{p}^{\mathrm{T}}(\mathbf{d}_6(N) + \mathbf{d}_2(N)) \leqslant t(N) \tag{43}$$

for $i = 1, 2, \ldots, N-1$, and subject to the linear inequalities (26)–(28), (29)–(31), (32)–(34),(35)–(36), where $t(\cdot)$ and $\mathbf{d}_6(\cdot)$ are further free parameters. That the minimization of $J_{\max}(k)$ is a LP follows from the linear dependence of the cost and all constraints on the free parameters.

4.1. RHPC for nonlinear system (NLRHPC)

Step 1: For the given seed trajectories of $\mathbf{x}^0(i|k)$ and $\mathbf{u}^0(i|k)$ satisfying (4), define $X(i|k)$ and $U(i|k)$ and obtain the corresponding $\Gamma_{\mathbf{z}}(i|k)$ and $\Gamma_{\mathbf{u}}(i|k)$, which satisfy (11).

Step 2: Optimize future perturbations by solving the LP:

$$\min_{\substack{\mathbf{c}(\cdot|k),\bar{\alpha}(\cdot|k),\bar{\beta}(\cdot|k),\mathbf{d}_1(\cdot),\mathbf{d}_2(\cdot)\\ \mathbf{d}_3(\cdot),\mathbf{d}_4(\cdot),\mathbf{d}_5(\cdot),\mathbf{d}_6(\cdot),t(\cdot)}} \sum_{i=1}^{N} t(i) \tag{44}$$

subject to linear inequalities (26)–(31), (32)–(36), (41)–(43) and use $\mathbf{c}(0|k)$ to compute, from (7), the current control move $\mathbf{u}(k) = \mathbf{u}_\delta(0|k) + \mathbf{u}^0(0|k)$ to be applied to (1).

Step 3: Compute $\hat{\mathbf{x}}(k+i|k)$, $i = 2, 3, \ldots, N-1$ based on the perturbations which were obtained at Step 2 as

$$\hat{\mathbf{x}}(k+i+1|k) = f(\hat{\mathbf{x}}(k+i|k), \mathbf{u}^0(i|k) + \mathbf{u}_\delta(i|k)), \tag{45}$$

$$\hat{\mathbf{x}}(k+1|k) = f(\mathbf{x}(k), \mathbf{u}^0(0|k) + \mathbf{u}_\delta(0|k)), \tag{46}$$

where $\mathbf{u}_\delta(i|k) = F(i|k)(\hat{\mathbf{x}}(k+i|k) - \mathbf{x}^0(i|k)) + \mathbf{c}(i|k)$, and define new seed trajectories $\mathbf{x}^0(i|k+1)$, $i = 0, 1, \ldots, N$ and $\mathbf{u}^0(i|k+1)$, $i = 0, 1, \ldots, N-1$ as

$$\begin{aligned}&\mathbf{x}^0(i|k+1) = \hat{\mathbf{x}}(k+i+1|k), \quad i = 0, 1, \ldots, N-1,\\ &\mathbf{x}^0(N|k+1) = f(\hat{\mathbf{x}}(k+N|k), \mathbf{u}_r + F_r(\hat{\mathbf{x}}(k+N|k) - \mathbf{x}_r)),\\ &\mathbf{u}^0(i-1|k+1) = \mathbf{u}^0(i|k) + \mathbf{u}_\delta(i|k), \quad i = 1, 2, \ldots, N-1,\\ &\mathbf{u}^0(N-1|k+1) = \mathbf{u}_r + F_r(\hat{\mathbf{x}}(k+N|k) - \mathbf{x}_r)\end{aligned}$$

and at the next time repeat the procedure (Steps 1–3).

Thus Step 1 sets up the machinery through which the cost of (40) with (26)–(31), (32)–(36), and (41)–(43) provides an upper bound on the tracking error; (32)–(34) and (43) relate to the error at the end of the prediction horizon, whereas the remainder of the prediction horizon is catered for by the other ten conditions. Then Step 2 minimizes this upper bound subject to the feasibility conditions (35)–(36). The linear nature of the cost and relevant constraints imply that the online optimization is an LP. The recursions of Step 3 provide seed trajectories for the application of Step 1 at the next instant.

Remark 7. From Step 3, it is obvious that the new seed trajectory satisfies (4). Also since X_r is invariant and feasible, the condition that $\mathbf{x}^0(N|k+1) \in X_r$ is guaranteed provided $\hat{\mathbf{x}}(k+N|k) \in X_r$. Thus, the new seed trajectory satisfies both (4) and (5) if the Algorithm is feasible at time k.

Theorem 8. *Consider* (1) *and a seed trajectory* $\mathbf{x}^0(i|0)$, $i = 0, 1, \ldots, N$ *and* $\mathbf{u}^0(i|0)$, $i = 0, 1, \ldots, N-1$ *which satisfy* (4)–(5) *with the Lipschitz condition* (11). *If* (23) *and* (24) *are satisfied, and NLRHPC is feasible for the initial state* $\mathbf{x}(0)$, *then NLRHPC will be feasible at all future instants of time.*

Proof. NLRHPC is feasible at initial time, hence we have $\mathbf{x}(N|0) \in X_r$ and we also have that the seed trajectories $\mathbf{x}^0(\cdot|1)$ and $\mathbf{u}^0(\cdot|1)$ satisfy (4)–(5) as per Remark 7. From (46) $\mathbf{x}^0(0|1)$ will be equal to $\mathbf{x}(1)$ i.e. $\mathbf{z}_\delta(0|1) = 0$, and thus by Remark 6 (26)–(31) and (32)–(36) will be feasible at time $k=1$. This argument applied recursively for $k=1,2,\ldots$ asserts the feasibility of NLRHPC at all future times. □

Remark 9. NLRHPC can be deployed to produce an initial seed trajectory of $\mathbf{x}^0(\cdot|0), \mathbf{u}^0(\cdot|0)$ satisfying (4)–(5). Ignoring the terminal constraint in (5), start with easily computable seed trajectories of state, $\mathbf{x}^0(i|0)$, $i=0,1,\ldots,l$, and input, $\mathbf{u}^0(i|0)$, $i=0,1,\ldots,l-1$, which satisfy $|\mathbf{u}^0(i|k)| \leqslant \bar{\mathbf{u}}$ but not necessarily $\mathbf{x}^0(N|k)\in X_r$. Then Steps 1 and 2 can be applied without reference to the constraint $\hat{\mathbf{x}}(l|0)\in X_r$ to yield optimal perturbations $\mathbf{c}(i|0)$, $i=0,1,\ldots,l-1$. Using these perturbations update the state and input seed trajectory, thereby getting better upper bounds on the tracking error for the same time interval $[0,l]$. This process of refinement can be repeated until the condition $\mathbf{x}^0(l|0)\in X_r$ is met; if this is not possible for the chosen l, increment l by 1 and repeat.

The proper selection (see below) of terminal weights $\mathbf{p}$ help establish the stability properties of NLRHPC.

Theorem 10. *Consider* (1) *and a seed trajectory* $\mathbf{x}^0(i|0)$, $i=0,1,\ldots,N$ *and* $\mathbf{u}^0(i|0)$, $i=0,1,\ldots,N-1$ *which satisfy* (4)–(5) *with the Lipschitz condition* (11). *If the first execution of NLRHPC is feasible with an initial condition* $\mathbf{x}(0)$ *and*

$$\mathbf{p}^{\mathrm{T}} \geqslant \mathbf{p}^{\mathrm{T}}(|\Phi_r^W| + \Gamma_{\mathbf{z}r}) + \mathbf{1}^{\mathrm{T}} \tag{47}$$

then $\mathbf{x}(k)$ *approaches* $\mathbf{x}_r$ *asymptotically as k increases.*

Proof. Appendix C. □

The existence of $\mathbf{p}$ satisfying (47) can be established using the results of Mees (1981) to derive the corollary below.

Corollary 11. *If the Perron–Frobenius norm of* $|\Phi_r^W|+\Gamma_{\mathbf{z}r}$ *is less than* 1, *then* $\mathbf{p}$ *can be chosen so that NLRHPC guarantees asymptotic stability, given feasibility at initial time.*

Proof. There exists $\hat{\mathbf{p}}$ such that $\hat{\mathbf{p}}^{\mathrm{T}} > \hat{\mathbf{p}}^{\mathrm{T}}(|\Phi_r^W| + \Gamma_{\mathbf{z}r})$ provided (Mees, 1981) the Perron–Frobenius norm of $|\Phi_r^W| + \Gamma_{\mathbf{z}r}$ is less than 1. Then scaling $\hat{\mathbf{p}}$, we can find $\mathbf{p}=\gamma\hat{\mathbf{p}}$ ($\gamma>1$) satisfying (47). □

Remark 12. The condition that the Perron–Frobenius norm of $|\Phi^W| + \Gamma_{\mathbf{z}r}$ is less than 1 not only guarantees the existence of $\mathbf{p}$ satisfying (47) but also ensures the existence $\bar{\mathbf{z}}_{\delta r}$ satisfying (23) i.e. the existence of the invariant set X_r.

5. Simulation example

NLRHPC is now used to control a fixed-rotor helicopter with a simplified planar model $\ddot{y}=(u_1+g)\sin\alpha$, $\ddot{z}=(u_1+g)\cos\alpha-g$, $\ddot{\alpha}=u_2$, where y,z,α represent horizontal, vertical and angular displacement, and the inputs u_1,u_2 are proportional to the net thrust and torque acting on the aircraft. The aim is to regulate the height z and angular position α to zero subject to input constraints $|\mathbf{u}| \leqslant \bar{\mathbf{u}}$, $\mathbf{u}=[u_1\ u_2]^{\mathrm{T}}$, $\bar{\mathbf{u}}=[10\ 10]^{\mathrm{T}}$. The inputs are applied via a ZOH with sampling period T seconds, and a discrete-time model with state $\mathbf{x}(k)=[y(kT)\ z(kT)\ \dot{y}(kT)\ \dot{z}(kT)\ \alpha(kT)\ \dot{\alpha}(kT)]^{\mathrm{T}}$ is computed online by numerical integration. The dynamics of perturbations $\mathbf{x}_\delta$ about a seed trajectory $(\mathbf{x}^0,\mathbf{u}^0)$ are also computed via online integration, achieved by setting $u_1=u_1^0+u_{1\delta}$, $u_2=u_2^0+u_{2\delta}$, $\alpha=\alpha^0+\alpha_\delta$, and $\dot{z}=\dot{z}^0+\dot{z}_\delta$ and writing a Taylor series expansion for $\cos(\alpha^0+\alpha_\delta)$ to compute suitable choices for the $\Gamma_{\mathbf{z}},\Gamma_{\mathbf{u}}$ in (11). Repeating this procedure for each element of $\mathbf{x}_\delta$ we obtain the incremental model $\mathbf{x}_\delta(k+1)=A(k)\mathbf{x}_\delta(k)+B(k)\mathbf{u}_\delta(k)+\mathbf{x}_e(k)$:

$$A(k)=\begin{bmatrix} 1 & 0 & T & 0 & c_0' & c_1' \\ 0 & 1 & 0 & T & -s_0' & -s_1' \\ 0 & 0 & 1 & 0 & c_0 & c_1 \\ 0 & 0 & 0 & 1 & -s_0 & -s_1 \\ 0 & 0 & 0 & 0 & 1 & T \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B(k)=\begin{bmatrix} \tilde{s}_0' & \frac{1}{2}c_2' \\ \tilde{c}_0' & -\frac{1}{2}s_2' \\ \tilde{s}_0 & \frac{1}{2}c_2 \\ \tilde{c}_0 & -\frac{1}{2}s_2 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \end{bmatrix},$$

where $\mathbf{x}_e$ is a quadratic function of $\mathbf{x}_\delta,\mathbf{u}_\delta$, tilde implies division by $(u_1^0(kT)+g)$ and for $j=0,1,2$

$$c_j=(u_1^0(kT)+g)\int_{kT}^{(k+1)T}(\tau-kT)^j\cos\alpha^0(\tau)\,\mathrm{d}\tau,$$

$$c_j'=(u_1^0(kT)+g)\int_{kT}^{(k+1)T}\int_{kT}^{\sigma}(\tau-kT)^j\cos\alpha^0(\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma$$

and coefficients s_i,s_i', $i=0,1,2$ are defined analogously with $\cos\alpha^0$ replaced by $\sin\alpha^0$.

The simulations are based on $T=0.1$ and an equilibrium point $(\mathbf{x},\mathbf{u})=(0,0)$. For simplicity $F(i|k)$ and the bounds $\bar{\mathbf{z}}_\delta(i|k),\bar{\mathbf{u}}_\delta(i|k)$ in the definitions (9)–(10) of polyhedral sets $X(i|k),U(i|k)$ are constant over the prediction horizon and equal to their values within the terminal set X_r. The state transformation $z=Vx$ and F are chosen to minimize the Perron-Frobenius norm of $V(A_r+B_rF)W$, where V is taken to be the eigenvector matrix of A_r+B_rF. This results in $F=[0,12.60,0,8.61,0,0;0.32,0,5.60,0,35.54,10.92]$, for which the Perron–Frobenius norm of $|V(A_r+B_rF)W|+\Gamma_{\mathbf{z}r}$ is 0.96; $\bar{\mathbf{z}}_{\delta r}$ and $\bar{\mathbf{u}}_{\delta r}$ are determined so as to maximize the smallest element of $\bar{\mathbf{z}}_{\delta r}$ while satisfying (23) and (24). This is a convex problem since for this example $\Gamma_{\mathbf{z}r}$ is linear (and hence (23) is quadratic and convex) in $\bar{\mathbf{z}}_\delta,\bar{\mathbf{u}}_\delta$. Choosing the terminal weight $\mathbf{p}$ with

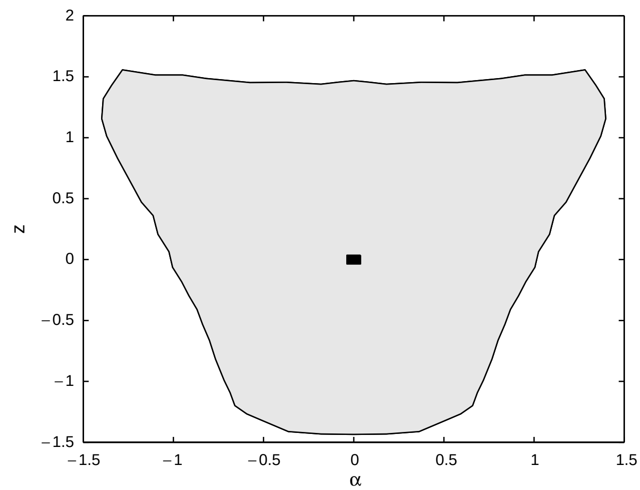

Fig. 2. Stabilizable set of initial conditions (grey) and terminal region (black) in the $z\alpha$-plane.

minimum 1-norm subject to the stability constraint (47) gives $\mathbf{p} = [128.6\ 188.6\ 3000.4\ 4326.9\ 30.3\ 31.5]^{\mathrm{T}}$.

Closed-loop responses of inputs and states for a horizon $N = 10$ were obtained for an initial $\mathbf{x}(0) = (0, -1, 0, 0, -0.5, 0)$. These were obtained by first using the procedure of Remark 9 to determine an initial feasible seed trajectory (six iterations are needed), then solving the LP (44) at sample instants $k = 0, 1, \ldots$. The computed perturbation $\mathbf{c}(0|k)$ are negligible for all $t \geqslant 0.6$, and, since the state does not enter the terminal region X_r until $t = 15$, the algorithm converges to a (possibly local) minimum of the cost index for the exact nonlinear dynamics at $t = 0.6$. Fig. 2 shows the extent of the stabilizable set of points on the $z\alpha$-plane for $N = 10$ free control moves. To demonstrate that the proposed NLRHPC strategy achieves an extremely favourable trade-off between computational efficiency and suboptimality, we compare the algorithm with a generic MPC approach based on the NLP:

$$\min_{\hat{\mathbf{u}}(k+j|k),\ j=0,1,\ldots,N-1} \sum_{i=1}^{N-1} \mathbf{1}^{\mathrm{T}}|\hat{\mathbf{e}}_z(i|k)| + \mathbf{p}^{\mathrm{T}}|\hat{\mathbf{e}}_z(N|k)| \tag{48}$$

subject to (1), the terminal constraint, and input constraints. This optimization is analogous to the LP of (44), but is based on an exact 1-norm cost and constraints instead of the upper bound (40) and sufficient constraints (26)–(31), (32)–(34), (35)–(36). The table below compares closed-loop costs (taken as the infinite sum of the 1-norms of the actual errors) and computational loads of NLRHPC based on the LP of (44) with those of the nonlinear programming (NLP) problem (48). The CPU times were obtained using the PCx LP solver for NLRHPC, and Matlab's SQP for the NLP, both implemented on a 440 MHz Sun workstation. The NLP solver was initialized at a feasible point; at $k = 0$ this was obtained from the seed trajectory computed for the NLRHPC. NLP provides better performance but is computationally very demanding. To reduce this one can terminate the computation before an optimal solution is reached; use of a feasible initial point would still guarantee stability despite suboptimality. However even when limited to a single iteration the NLP requires is 3 times more CPU time than the corresponding LP. Furthermore, limiting the number of iterations has a significant effect on performance: NLP with 1 iteration gives a 9% improvement on the cost of simply implementing the seed trajectory computed at $k = 0$, whereas NLRHPC achieves a cost improvement of 18% (Table 1).

Table 1
Closed-loop costs and computational load

	NLRHPC	NLP (optimal)	NLP (1 iter.)	Seed
J_{run}	24.7	19.3	27.4	30.1
CPU/iter.	1.04	285.5	3.16	0

6. Conclusions

A constrained RHC algorithm optimizing a tracking cost over a finite horizon for nonlinear discrete-time systems is proposed. Optimization is based on predictions via linearized incremental models at points defined by seed trajectories and linearization errors are taken into account. Feasibility is guaranteed given feasibility at start time and it is shown that the tracking error decreases asymptotically given an appropriate selection of terminal weights. The existence of a feasible invariant set and a stabilizing terminal weight is guaranteed by a condition on the Perron–Frobenius norm of a given matrix. The proposed algorithm is formulated so that it can be solved efficiently as an LP. An example illustrating the efficacy of the proposed algorithm is considered and it is shown that the algorithm can be deployed in the definition of an initial seed trajectory for the states and inputs.

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Appendix A. Proof of Theorem 4

Combining (11) and (25) and we have

$$|\hat{\mathbf{z}}_\delta(i|k)| \leqslant \left| \sum_{j=0}^{i-1} \prod_{l=j+1}^{i-1} \Phi^W(l|k) \cdot B^W(j|k)\mathbf{c}(j|k) \right| + \sum_{j=0}^{i-1} \left| \prod_{l=j+1}^{i-1} \Phi^W(l|k) \right| \{\Gamma_{\mathbf{z}}(j|k)\bar{\alpha}(j|k) + \Gamma_{\mathbf{u}}(j|k)\bar{\beta}(j|k)\} \tag{A.1}$$

so that (26)–(27) imply $|\hat{\mathbf{z}}_\delta(i|k)| \leqslant \mathbf{d}_1(i) + \mathbf{d}_2(i)$ which with (28) implies that $|\hat{\mathbf{z}}_\delta(i|k)| \leqslant \bar{\alpha}(i|k) \leqslant \bar{\mathbf{z}}_\delta(i|k)$ and thus $\hat{\mathbf{x}}(k+i|k) = \mathbf{x}^0(i|k) + \hat{\mathbf{x}}_\delta(i|k) \in X(i|k), i = 1, 2, \ldots, N-1$. Next from (7) we have $\hat{\mathbf{u}}_\delta(i|k) = F(i|k)W\hat{\mathbf{z}}_\delta(k+i|k) + \mathbf{c}(i|k)$, which, when combined with (11) and (25), shows that conditions (29)–(31) imply $|\hat{\mathbf{u}}_\delta(i|k)| \leqslant \bar{\beta}(i|k) \leqslant \bar{\mathbf{u}}_\delta(i|k)$. □

Appendix B. Proof of Theorem 5

From the definition (3) of X_r, $\hat{\mathbf{x}}(k+N|k) \in X_r$ requires that $|\hat{\mathbf{z}}(k+N|k) - \mathbf{z}_r \leqslant \bar{\mathbf{z}}_{\delta r}$. Let $\bar{\alpha}(\cdot|k)$ and $\bar{\beta}(\cdot|k)$ satisfy (26)–(31), then from the bounds

$$\begin{aligned}
&|\mathbf{z}_r - \hat{\mathbf{z}}(k+N|k)| \\
&\quad = |\mathbf{z}_r - \mathbf{z}^0(N|k) - \hat{\mathbf{z}}_\delta(N|k)| \\
&\quad \leqslant \Bigg| \mathbf{z}_r - \mathbf{z}^0(N|k) \\
&\qquad - \sum_{j=0}^{N-1} \left\{ \prod_{l=j+1}^{N-1} \Phi^W(l|k) \cdot B^W(j|k)\mathbf{c}(j|k) \right\} \Bigg| \\
&\qquad \sum_{j=0}^{N-1} \left| \prod_{l=j+1}^{N-1} \Phi^W(l|k) \right| \\
&\qquad \times \{ \Gamma_{\mathbf{z}}(j|k)\bar{\alpha}(j|k) + \Gamma_{\mathbf{u}}(j|k)\bar{\beta}(j|k) \}
\end{aligned}$$

it is clear that (32)–(34) ensures that $|\hat{\mathbf{z}}(k+N|k) - \mathbf{z}_r| \leqslant \bar{\mathbf{z}}_{\delta r}$. From (7) we have that the feasibility condition of the theorem can be written

$$|\mathbf{u}(i|k)| = |\mathbf{u}^0(i|k) + F(i|k)W\mathbf{z}_\delta(i|k) + \mathbf{c}(i|k)| \leqslant \bar{\mathbf{u}}. \quad \text{(B.1)}$$

Using (25) and working as in Appendix A, it is easy to show that (B.1) is ensured by (35)–(36). □

Appendix C. Proof of Theorem 10

Let $J^*(k)$ be the optimal cost of (40) with the optimal perturbations generated from Step 2 of NLRHPC and consider corresponding cost based on the actual prediction errors

$$\hat{J}(k) = \sum_{i=1}^{N-1} \mathbf{1}^{\mathrm{T}}|\mathbf{z}_r - \hat{\mathbf{z}}(k+i|k)| + \mathbf{p}^{\mathrm{T}}|\mathbf{z}_r - \hat{\mathbf{z}}(k+N|k)| \quad \text{(C.1)}$$

then we have $J^*(k) \geqslant \hat{J}(k)$ since along predicted trajectories $|\mathbf{z}_r - \hat{\mathbf{z}}(k+i|k)| \leqslant \mathbf{e}_{\max}(i|k)$. Also $\mathbf{c}(\cdot|k+1) = 0$, $\bar{\alpha}(\cdot|k+1) = 0$ and $\bar{\beta}(\cdot|k+1) = 0$ define a feasible solution of NLRHPC at time $k+1$ since $\mathbf{x}(k+1) = \mathbf{x}^0(0|k+1)$. In this case $\mathbf{e}_{\max}(\cdot|k+1) = |\mathbf{z}_r - \mathbf{z}^0(\cdot|k+1)|$ and therefore $J_{\max}(k+1) = \sum_{i=1}^{N-1} \mathbf{1}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}^0(i|k+1)| + \mathbf{p}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}^0(N|k+1)|$, which obviously is not less than $J^*(k+1)$. Thus we obtain the following relations:

$$J^*(k) \geqslant \hat{J}(k) \geqslant \mathbf{1}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}(k+1)| + J^*(k+1) + M_k,$$

$$\begin{aligned}
M_k = {} & \mathbf{p}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}^0(N-1|k+1)| - \mathbf{p}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}^0(N|k+1)| \\
& - \mathbf{1}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}^0(N-1|k+1)|.
\end{aligned}$$

If $M_k \geqslant 0$ for all $k \geqslant 0$, then summing this inequality over times $k, k+1, \ldots, L$ gives

$$J^*(k) \geqslant \sum_{i=1}^{L} \mathbf{1}^{\mathrm{T}}|\mathbf{z}_r - \mathbf{z}(k+i)| + J^*(k+L). \quad \text{(C.2)}$$

Thus $\mathbf{z}(k+L)$ tends to $\mathbf{z}_r$ as L grows. From (19) and the seed trajectory update in Step 3 of NLRHPC, we have

$$\begin{aligned}
\mathbf{z}^0(N|k+1) &= Vf(\mathbf{x}^0(N-1|k+1), \mathbf{u}^0(N-1|k+1)) \\
&= \mathbf{z}_r + V(A_r + B_rF_r)W\mathbf{z}^0_{\delta r}(N-1|k+1) \\
&\quad + \mathbf{z}_{\mathrm{er}}(N|k+1), \quad \text{(C.3)}
\end{aligned}$$

where $\mathbf{z}^0_{\delta r}(N-1|k+1) = \mathbf{z}^0(N-1|k+1) - \mathbf{z}_r$. From (C.3), $\mathbf{x}^0(N-1|k+1) \in X_r$, and $\mathbf{u}^0(N-1|k+1) \in U_r$, we get

$$M_k \geqslant \{(\mathbf{p}^{\mathrm{T}} - \mathbf{1}^{\mathrm{T}}) - \mathbf{p}^{\mathrm{T}}|\Phi_r^W| - \mathbf{p}^{\mathrm{T}}\Gamma_{\mathbf{zr}}\}|\mathbf{z}^0_{\delta r}(N-1|k+1)|.$$

Post-multiplying condition (47) of Theorem 10 by $|\mathbf{z}^0_{\delta r}(N-1|k+1)|$ and rearranging readily implies that $M_k \geqslant 0$. □

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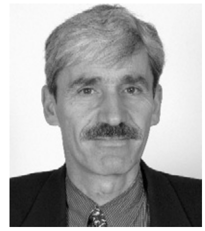

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