On optimality of Nonlinear Model Predictive Control*

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Abstract

In this note the Infinite Horizon (IH) optimality property of Nonlinear Model Predictive Control (MPC) is analysed. In particular it is shown with a contra example that the conjecture that the IH cost of the closed-loop system controlled with a stabilizing MPC controller is a monotonic decreasing function of the optimization horizon is fallacius.

1 Introduction

Model Predictive Control refers to a class of computer control algorithms that utilizes an explicit process model to predict the future response of a plant. At each control interval an MPC algorithm determines a sequence of manipulated variable adjustments that optimizes future plant behaviour. The first input in the optimal sequence is then sent into the plant, and the entire optimization is repeated at subsequent control intervals. In this way a closed-loop control law is obtained with an open-loop optimization. MPC technology was originally developed for petroleum refinery control applications, but can now be found in a wide variety of manufacturing environments including chemicals, food processing, automotive, aerospace, metallurgy and pulp and paper. In principle, the solution of these optimization based algorithms would be given by the optimal IH controller, obtained through the minimization of an IH cost function subject to the state and input constraints. Nevertheless, it is apparent that the practical implementation of such a control law poses formidable computational problems since it involves optimization in an infinite-dimensional decision space. For this reason, optimality is a topic of vivid importance in MPC beyond stability [9], [6], [5]. It is clear that it is ideally possible to approximate the optimal IH controller arbitrarily well by increasing the value of the finite horizon N. Otherwise

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the optimization horizon directly affect the computational complexity so that a shorter horizon is better than a longer one. In this respect one could assume that a trade-off between computational complexity and performance exists. In this paper it is shown with a contra example that IH cost is not a monotonic decreasing function of the optimization horizon N so that it is not generally true that increasing the optimization horizon is the correct solution to improve optimality.

2 Problem statement

Consider the nonlinear discrete-time dynamic system

$$x(k+1) = f(x(k), u(k)) \qquad x(t) = \bar{x} \qquad k \geq t \tag{1}$$

where k is the discrete time index, $x(k) \in R^n$, $u(k) \in R^m$, $f(\cdot,\cdot) \in C^2$ and $f(0,0) = 0$. The state and control variables are required to fulfill the following constraints:

$$x(k) \in X, \qquad u(k) \in U, \qquad k \geq t \tag{2}$$

where X and U are compact subsets of R^n and R^m, both containing the origin as an interior point. In order to design a state-feedback control law $u = \kappa(x)$ for (1), with optimal performance, one may consider the minimization with respect to $u(\cdot)$ of the IH cost function

$$J_{IH}(\bar{x}, u(\cdot)) = \sum_{k=t}^{\infty} x(k)'Qx(k) + u(k)'Ru(k) \tag{3}$$

subject to (1). In (3) Q and R are positive definite weighting matrices. In general, the IH nonlinear optimal control problem is computationally intractable since it involves an infinite number of decision variables. Nevertheless, it constitutes a touchstone for suboptimal approaches. An effective designing suboptimal controllers is to resort to the MPC control law.

3 Model predictive control law

The MPC control law is based on the solution of the following optimization problem.

Finite horizon optimal control problem (FHOCP). Given the terminal set X_f, a subset of R^n containing the origin as an interior point, the optimization horizon N, a positive integer, the terminal penalty $V_f(\cdot)$, a positive function, a control sequence $u_{t,t+N-1} := [u(t) \ u(t+1) \ \ldots \ u(t+N-1)]$, minimize, with respect to $u_{t,t+N-1}$, the performance index

$$J(\bar{x}, u_{t,t+N-1}, N) = \sum_{k=t}^{t+N-1} \{x(k)'Qx(k) + u(k)'Ru(k)\} + V_f(x(t+N))$$

subject to the state dynamics (1), the constraints (2), the terminal state constraint $x(t+N) \in X_f$.

Based on the solution of this open-loop optimization problem a state-feedback control law is obtained following the Receding Horizon (RH) paradigm: at every time instant t, define $\bar{x} = x(t)$ and find the optimal control sequence $u^o_{t,t+N-1}$ by solving the $FHOCP$. Then apply, the control

$$u(t) = \kappa^{MPC}(\bar{x}, N), \tag{4}$$

where $\kappa^{MPC}(\bar{x}, N) = u^o_{t,t}$ is the first column of $u^o_{t,t+N-1}$.

It is well known that most of the stabilizing MPC control algorithm satisfied the following monotonicity property:

$$J^o(\bar{x}, N+1) \leq J^o(\bar{x}, N) \tag{5}$$

where $J^o(\bar{x}, N)$ is the value of the $FHOCP$. The monotonicity property is exploited in order to analyze closed-loop stability and robustness properties of MPC control law inter alia by [1], [7], [9], [2], [8].

Otherwise $J^o(\bar{x}, N)$ is not the cost obtained applying the MPC control law because, according to the RH paradigm, at each time instant t we apply only the first element of the optimal sequence. The main goal of this note is to analyze the performance of the MPC control law with respect to the IH cost. First of all it is clear that it is ideally possible to approximate the optimal IH controller arbitrarily well by increasing the value of the horizon N. Otherwise the optimization horizon directly affect the computational complexity so that a shorter horizon is better than a longer one. In this way if the following conjecture would be true a trade-off between computational complexity and performance would be established.

Conjecture 1 *The IH cost (3) when the RH controller (4) is applied to (1) is a monotonically decreasing function of the optimization horizon N.*

In the following with a contra example it is shown that this conjecture is fallacious even when the model dynamic (1) is linear

$$x(k+1) = Ax(k) + Bu(k)$$

and there are no constraints (2).

Consider a stabilizing MPC control law obtained by solving the $FHOCP$ with the terminal penalty $V_f(x(t+N))$ given by

$$V_f(x) = x'Lx \tag{6}$$

where L is the solution of the following discrete-time Lyapunov equation

$$(A - BK)'L(A - BK) - L + Q + K'RK = 0$$

and $u = -Kx$ is a stabilizing control law [3]. The terminal constraint is not necessary in order to guarantee closed-loop stability and then we assume $X_f =$

R^n. With this choice of the terminal penalty and the terminal set, the MPC control law (4) is given by $u = -K^{MPC}(N)x$ with

$$K^{MPC}(N) = (R + B'P(N)B)^{-1}B'P(N)A$$

where $P(N)$ is the solution of the following Riccati equation

$$P(n+1) = A'P(n)A + Q - A'P(n)B(R + B'PB)^{-1}B'P(n)A$$

with $P(0) = L$. In this case $J^o(\bar{x}, N) = \bar{x}'P(N)\bar{x}$ and the monotonicity property (5) is satisfied [3].

The MPC IH cost, i.e. the IH cost (3) when the MPC control law $u = -K^{MPC}(N)x$ is applied to (1) is given by

$$J_{IH}^{K^{MPC}}(\bar{x}, N) = \bar{x}'\Pi(N)\bar{x} \tag{7}$$

where $\Pi(N)$ is the solution of the following discrete-time Lyapunov equation

$$(A - BK^{MPC}(N))'\Pi(N)(A - BK^{MPC}(N)) - \Pi(N) + Q + K^{MPC}(N)RK^{MPC}(N) = 0 \tag{8}$$

Example 1 *Given the linear system described by the matrices*

$$A = \begin{bmatrix} 0.3702 & -0.6395 & 0.1902 & 0.5375 \\ -0.0028 & 0.7295 & 0.0088 & -0.3928 \\ -0.0713 & -0.6123 & 0.6039 & 0.6683 \\ -0.0025 & -0.0217 & -0.0093 & 0.8903 \end{bmatrix} \qquad B = \begin{bmatrix} 1.4951 \\ 0.8895 \\ 1.1131 \\ 0.2302 \end{bmatrix}$$

and the stabilizing control law $u = -Kx$ where

$$K = \begin{bmatrix} 0.0583 & -0.2261 & 0.2719 & 0.4461 \end{bmatrix}$$

we consider the FHOCP with $R = 5$, $Q = 3 * I$, $X_f = R^n$ and the terminal penalty given by (6). In this example the closed-loop IH cost given by (7) is computed for the state $\bar{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}'$ and for several optimization horizons. In Figures 1 the MPC IH cost for different values of the optimization horizon N is reported, while in Figure 2 the difference between the MPC IH cost and the optimal IH cost $J_{IH}^{K^{IH}}$ is reported. It is clear that the Conjecture 1 is fallacious in fact, for example, $J_{IH}^{K^{MPC}}(\bar{x}, \cdot)$ is a growing function of the optimization horizon from 3 to 6 and from 23 to 26. Remarkably the IH cost with $N = 1$ is not reported in Figure 1 because it is significantly grater than the other ones (i.e. $J_{IH}^{K^{MPC}}(\bar{x}, 1) = 18415$).

4 Conclusion

In this paper it is shown that the IH cost (3) with the MPC controller (4) is not necessarily a monotonically decreasing function of the optimization horizon N.

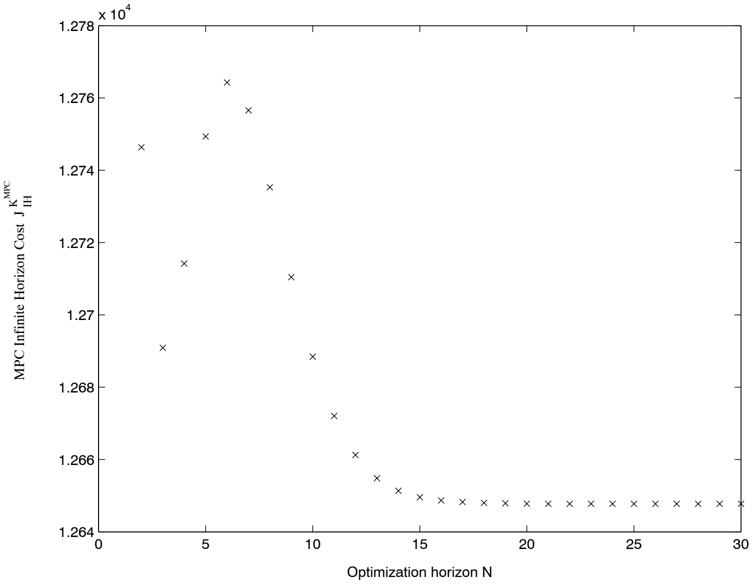

Figure 1: $MPC\ IH$ cost for different horizons N

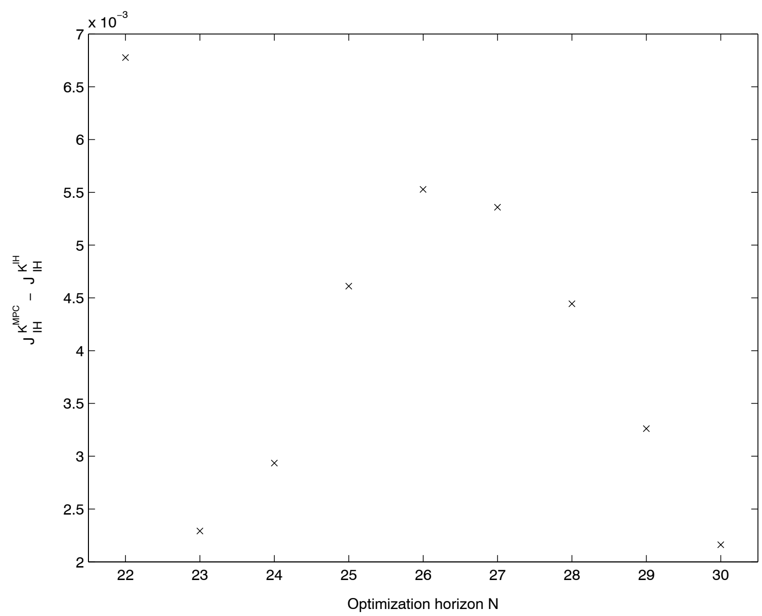

Figure 2: Difference between the $MPC\ IH$ cost for different horizons N and the optimal IH cost

In the linear case, one can choose the optimal IH control law as auxiliary control law in order to obtain optimality for any N. If state and control constraints are considered, optimality is in any case obtained if the terminal penalty (6) is considered with the optimal IH control law as auxiliary control law. In this case the longer the horizon N is the larger the domain of attraction is [10]. If a constrained nonlinear system is considered it is in general difficult to find an analytical value of the IH cost of the auxiliary control law so that it is necessary to compute it numerically as proposed in [4]. In any case only if the IH optimal control law for the nonlinear unconstrained is used as auxiliary control law optimality can be achieved for any value of the optimization horizon N.

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