These notes are based on the content of the course *Differentiable Manifolds* held at LMU during the Winter Semester 2019/2020. Some additional content from books and from my own studies was also added. These notes are neither complete nor accurate. They might contain typos and mistakes. You can send an email to `flaviorossetti@outlook.com` to help me improve these notes. Any help would be greatly appreciated!

These notes were compiled on March 6, 2020. For the last version, check `https://github.com/fla-io/diff-manifolds`.

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Contents

1 Introduction

From the Lee book [1]: "The central idea of calculus is *linear approximation*". A function of one variable can be approximated by its tangent line, a curve by a tangent vector (i.e. velocity vector), a surface in $\mathbb{R}^3$ can be approximated by its tangent plane, and a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by its total derivative. Here it comes the importance of tangent spaces.

Main idea: in order to study tangent vectors, we identify them with "directional derivatives". In particular, there is a natural one-to-one correspondence between geometric tangent vectors and linear maps from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$ satisfying the product rule. Such maps are called *derivations*.

Remark 1.1. **Points or vectors?** We can think of elements of $\mathbb{R}^n$ either as points or vectors. As points, their only property is their location, given by the coordinates $(x_1, \ldots, x_n)$ on a chosen basis. As vectors, they are characterized by a direction and a magnitude, but their location is irrelevant (translational invariance). So given $v \in \mathbb{R}^n, v = \sum_i v^i e_i = v^i e_i$, it can be seen as an arrow with its initial point anywhere in $\mathbb{R}^n$. So, if we think about a vector tangent to the border of the sphere at a point a, we imagine the vector as living in a copy of $\mathbb{R}^n$ with its origin translated to a.

2 Notations and Conventions

In these notes the Einstein summation convention will be used. It means that the sum symbol will be omitted when it is clear with respect to which index we are summing. For instance we will write

$$v^i e_i$$

instead of

$$\sum_i v^i e_i$$

Given a map f from a set X to a set Y, we will denote it by

$$f\colon X \to Y$$

We will use the arrow "$\mapsto$" to denote how each element of X is mapped into Y through f. For instance, given $x_1, x_2 \in X, y \in Y$:

$$x_1 \mapsto f(x)$$

or

$$x_2 \mapsto y$$

3 Quick review: Basic Algebraic Structures

Definition 3.1 (Operation)**.** Let G be a set. $\cdot$ is called a (binary) operation on G if it is a map

$$\begin{aligned} \cdot : G \times G &\longrightarrow G \\ (a, b) &\longmapsto a \cdot b \end{aligned}$$

Such a map is usually denoted by the symbol $\cdot$ or, analogously, with the symbol $+$.

Definition 3.2 (Group)**.** Given a set G and an operation $\cdot$ on such set, we will call such set with the operation (i.e. the couple $(G, \cdot)$) a group if the following properties are satisfied (for $a, b, c \in G$):

- $a \cdot b \in G$ (closure property, which often follows from the definition of our operation)
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity)
- $\exists\, \mathbf{1} \in G$ such that $\mathbf{1} \cdot g = g, \forall\, g \in G$ (existence of the identity element)
- $\forall\, g \in G, \exists\, g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = \mathbf{1}$ (existence of the inverse element)

For the sake of simplicity, we will often call G a group, without referring to the operation on it.

Example 3.3. $(\mathbb{Z}, +)$ is a group.

Example 3.4. $(\mathbb{R}, +)$ is a group. Also: $(\mathbb{R}, \cdot)$ is a group.

Example 3.5. $(\mathbb{N}, +)$ is not a group!

Remark 3.6. The identity element of a group is often denoted as $\mathbf{1}$ if the operation is denoted by the symbol $\cdot$, whereas it is denoted as $\mathbf{0}$ if the operation is denoted by the symbol $+$. In a similar way, the inverse element is often denoted as g^{-1} if the operation is denoted by the symbol $\cdot$, whereas it is denoted as $-g$ if the operation is denoted by the symbol $+$.

Definition 3.7 (Abelian group)**.** A group $(G, \cdot)$ is called abelian if its elements commute according to the operation $\cdot$, i.e. $a \cdot b = b \cdot a, \forall\, a, b \in G$.

Often we can consider sets with two operations, like $(\mathbb{R}, +, \cdot)$. If they satisfy some properties, they are called rings. If they satisfy even more properties, they are called fields. In particular:

Definition 3.8 (Ring)**.** Given a set R and two operations: $+$ (usually called "additive operation") and $\cdot$ (called "multiplicative operation") on it, we will call the set with the two operations, i.e. $(R, +, \cdot)$, a ring if the following properties are satisfied:

- $(R, +)$ is an abelian group
- $\cdot$ is associative, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall\, a, b, c, \in R$
- the multiplicative identity $\mathbf{1}$ exists, i.e. $\exists\, \mathbf{1} \in R$ such that $\mathbf{1} \cdot r = r, \forall\, r \in R$
- $\cdot$ is distributive with respect to $+$, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 3.9 (Commutative ring)**.** $(R, +, \cdot)$ is a commutative ring if the multiplication operation $\cdot$ is commutative

Definition 3.10 (Unitary ring)**.** $(R, +, \cdot)$ is a unitary ring if it contains the multiplication inverse, i.e. the inverse element according to the operation $\cdot$.

Definition 3.11 (Field)**.** A unitary, commutative ring is called a field.

Example 3.12. The set of 2×2 matrices made by real coefficients is a ring with the operation of sum between matrices and with the matrix multiplication. The ring is not commutative because the matrix multiplication is not a commutative operation. Nor it is unitary, since not all the matrices are invertible.

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Example 3.14. $(\mathbb{Z}, +, \cdot)$ is a commutative ring. It is not unitary since the inverse element for $\cdot$ is often in the rational numbers, i.e. 3^{-1} is the multiplicative inverse of 3.

Example 3.15. $(\mathbb{R}, +, \cdot)$ is a field

4 Quick review: Morphisms

Definition 4.1 (Homomorphism)**.** A homomorphism h between two sets endowed with operations (think about two groups, for instance) is a map which preserves the operations, i.e. a map

$$h\colon (G, +) \longrightarrow (H, \circ)$$

such that $h(a+b) = h(a) \circ h(b), \forall\, a, b, \in G$, where $(G, +)$ and $(H, \circ)$ are two groups. The definition for rings and fields is analogous. Sometimes these maps are just called morphisms (there is some difference between morphisms and homomorphisms but it usually matters only if you are dealing with more abstract algebraic structures).

Now, we can consider some particular types of homomorphisms.

Remark 4.2 (Monomorphism, epimorphisms and isomorphisms)*.* Homomorphisms can be called in different ways depending whether they are injective, surjective or bijective. A monomorphism is an injective homomorphism. An epimorphism is a surjective homomorphism. An isomorphism is a bijective homomorphism (thus, it is both a monomorphism and an epimorphism).

Remark 4.3. [Endomorphism, automorphism] An endomorphism is a homomorphism from one algebraic structure to itself. If such morphism is bijective, it is called automorphism.

Remark 4.4. Don't get confused with hom**e**omorphisms! A homeomorphism is a continuous and bijective map between two topological space, such that its inverse is also continuous. In general it is not a homomorphism (without the **e**), because a topological space is not necessarily associated with an operation on it. (However if it has an operation, the continuity of the map implies that it is also a homomorphism).

5 Quick review: Equivalence Classes, Quotient Spaces

Definition 5.1 (Binary relation)**.** Given a set E, a (binary) relation $\sim$ on E is a set of couples $(a, b) \in E \times E$. In other words, a binary relation is a subset of $E \times E$. Moreover, if $(a, b) \in \sim \subset E \times E$, we use the following notation: $a \sim b$.

The mathematical definition of binary relation is usually not very helpful in the applications. Often it is more convenient not to think about $\sim$, but instead thinking about what is in relation with what. In particular, we want to consider the following:

Definition 5.2 (Equivalence relation)**.** Given a set E, an equivalence relation $\sim$ is a binary relation which satisfies the following properties:

- $a \sim a, \forall\, a \in E$ (reflexive property)
- $a \sim b \Rightarrow b \sim a, \forall\, a, b \in E$ (symmetric property)
- $a \sim b, b \sim c \Rightarrow a \sim c, \forall\, a, b, c \in E$ (transitive property)

Example 5.3. The following are examples (or counter-examples) of equivalence relations:

- "*Being equal to*" (i.e. "$\sim$" is "$=$") is an equivalence relation on $\mathbb{R}$: $x = x \,\forall\, x \in \mathbb{R}; x = y \Rightarrow y = x \,\forall\, x, y \in \mathbb{R}$, etc.
- "*Has the same birthday as*" on the set of all people in the world is an equivalence relation.
- "*Having mutual friends on Facebook*" on the set of Facebook users is not an equivalence relation (there is some problem with transitive property)
- "*Being greater or equal to*" on $\mathbb{R}$ is not an equivalence relation (it is not symmetric, however it is antisymmetric and so it is called "partial order relation")

Definition 5.4 (Equivalence class)**.** Given a set E and an equivalence relation $\sim$ on E, the equivalence class of $x \in E$ is the set $[x]$, where

$$[x] \equiv \{y \in E \,|\, y \sim x\} \tag{5.1}$$

The equivalence class $[x]$ always contains x itself, by definition of equivalence relation.

Example 5.5. We can examine the integers which differ in absolute value by a multiple of 4. If we consider the equivalence relation "$x \sim y$ if $x - y = 4n$ for some $n \in \mathbb{N} \cup \{0\}$" on the set of $\mathbb{Z}$, the equivalence class of 1 is $[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$. We can also write "$x - y = 4n$ for some n" as $x \equiv_4 y$ (congruence modulo 4)

Definition 5.6 (Quotient space)**.** Given a set E, an equivalence relation $\sim$ on E, the quotient space with respect to $\sim$ is denoted by ${}^{E}\!/\!_{\sim}$ and it is the set of all equivalence classes:

$${}^{E}\!/\!_{\sim} \equiv \{\, [x] \,|\, x \in E\} \tag{5.2}$$

Intuitively, we can say that the quotient space ${}^{E}\!/\!_{\sim}$ is a copy of E where all the elements equivalent to each other are reduced to one point.

Example 5.7. Let's consider two examples:

- If we consider the example 5.5, ${}^{\mathbb{Z}}\!/\!_{\sim} = \{[0], [1], [2], [3]\}$. We notice that all the integers that differ from 0 by a multiple of 4 (in absolute value) are identified with one point (the equivalence class $[0]$). All the integers that differ from 1 by a multiple of 4 (in absolute value) are identified with one point (the equivalence class $[1]$), and so on. In this case, we also write ${}^{\mathbb{Z}}\!/\!_{\sim} = {}^{\mathbb{Z}}\!/\!_{4\mathbb{Z}}$
- If V is a vector space and U is a vector subspace of V, then we can consider

the equivalence relation "$x \sim y$ if $x - y \in U$". We notice that $x \sim x$ for each $x \in V$ since $x - x = 0 \in U$, because U is a vector space as well. Then, $V/\sim$ is isomorphic to V', where V' is V without its subspace U (every element of U is identified with 0 in the quotient space, because every element of U is in the same equivalence class of 0).

- If we consider two groups instead of vector spaces, finding the quotient space requires a bit more effort.

6 Derivations

Definition 6.1 (Derivation)**.** If a is a point of $\mathbb{R}^n$, a map $v\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at a* if it is linear over $\mathbb{R}$ and satisfies the following product rule:

$$v(fg)\big|_a = f\big|_a v(g)\big|_a + g\big|_a v(f)\big|_a$$

Remark 6.2. Directional derivatives obviously satisfy the above definition, and in these cases such a rule is also called Leibnitz rule.

7 Multilinear Forms

Definition 7.1 (1-forms)**.** Given a vector space V on a field K, a 1-form (or linear form) $\varphi\colon V \to K$ is a linear function from V to K. V^* (also denoted by ΛV^*) is the set of all linear forms on V.

Remark 7.2. The set of linear forms on V has the structure of a vector space. Indeed, if $\mathbb{K}$ is the field of the vector space V, $\forall\, a, a_1, a_2 \in \mathbb{K}, \forall\, \varphi, \varphi_1, \varphi_2 \in V^*$:

1. $a(\varphi_1 + \varphi_2) = a\varphi_1 + a\varphi_2$ (because we define the map $a_1\varphi_1 + a_2\varphi_2$ as $a_1\varphi_1 + a_2\varphi_2(x) \equiv a_1\varphi_1(x) + a_2\varphi_2(x), \forall\, x \in V$)
2. $(a + b)\varphi(x) = (a\varphi + b\varphi)(x)$
3. $\varphi = 0$ if $\varphi(x) = 0\,\forall\, x \in V$
4. $-\varphi(x) = \varphi(-x)$
5. $\mathbb{1}\varphi(x) = \varphi(x)$
6. $ab\varphi(x) = ba\varphi(x)$

Definition 7.3 (Dual basis)**.** If $\{e_i\}_{i=1,\ldots,n}$ is a basis of V, then $\{e^{*i}\}_{i=1}^n \subseteq V^*$ is called the *dual basis* if $e^{*i}(e_j) = \delta^i_j$.

If the field of V is $\mathbb{R}$, we will denote the dual basis of V^* also by $\{dx^i\}_{i\in\mathbb{N}}$ (see also the next examples).

Remark 7.4. We could prove that the dual basis is indeed a basis of the dual space, so $\dim(V) = \dim(\Lambda V^*)$. Check proposition 8.4.

Example 7.5. Let's consider three examples.

1. $p \in \mathbb{R}^3$, fixed. Let V_p be the vector space $V_p \equiv \{q - p \,|\, q \in \mathbb{R}^3\}$. We will also denote it by $T_p\mathbb{R}^3$. We notice that $V_p = T_p\mathbb{R}^3 = \mathbb{R}^3$ (It is trivial to verify the inclusions "$\subseteq$" and "$\supseteq$"). Let $\{(e_i)_p\}_{i=1,2,3}$ be a basis for V_p. If we consider the dual space V_p^*, we notice that $\left\{(dx^i)_p\right\}_{i=1,2,3}$ is a dual basis, where $(dx^i)_p(e_j) = \frac{\partial}{\partial x^j}x^i$. The index position is just a matter of notation, for now. Then, we can show that V_p and V_p^* are isomorphic, and the isomorphism is:

$$\begin{aligned} g\colon V_p &\to V_p^* \\ x &\mapsto g(x,\cdot) \end{aligned} \tag{7.1}$$

   where g is the Euclidean metric in $\mathbb{R}^3$ (in the sense of General Relativity), i.e. the standard dot product: $g(x,y) = x^i y^i$

2. $V = \left\{M \in M_2(\mathbb{C}) \,|\, M^\dagger = M, \mathrm{tr}(M) = 0\right\}$ is a vector field on $\mathbb{R}$, where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, M^\dagger = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}, a, \ldots, d \in \mathbb{C}$$

   A basis for V is $\{\sigma_i\}_{i=1,2,3}$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   and a dual basis is given by $e^{*i}(e_j) = \frac{1}{2}\,\mathrm{tr}(e_i, e_j)$ (using the matrix product).

3. Quantum Mechanics:

$$V = \{f\colon \mathbb{R}^3 \to \mathbb{R} \,|\, ||f|| = \int_{\mathbb{R}^3} |f|^2 d^3x < \infty\} = L^2(\mathbb{R}^3)$$

   is a vector space. If we consider the Laplacian operator $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}$, then a basis is given by $\{e_n\} = \{\frac{f_n}{||f_n||}\}$, where f_n eigenfunctions of Δ: $\Delta f_n = \lambda_n fn$. A dual basis is given by $e^{*n} = \int_{\mathbb{R}^3} e_n$ Notice that both V and V^* are infinite-dimensional spaces. Some notations:

$$e_n = |f_n\rangle, \quad e^{*n} = \langle f_n|, \quad e^{*n}(e_m) = \langle f_n|f_m\rangle$$

Remark 7.6. Every vector in $\mathbb{R}^n$ about a point $p \in \mathbb{R}^n$ (i.e. such that its origin is the point p) "can be seen" as a derivation (cf. def. 6.1), i.e. as a directional derivative of a function evaluated at the point p. For the sake of simplicity, we think $p = 0$ (but the following results are true $\forall\, p \in \mathbb{R}^n$). The sentence "can be seen" means that there is an isomorphism ψ associating such vectors to such linear forms. Let's construct this isomorphism in the following steps:

1. Because of linearity, we just need to define the isomorphism for the basis vectors $\{e_i\}_{i=1,\dots,n}$ of $\mathbb{R}^n$.

2. Given the vector e_j of the canonical basis, we associate it with the derivation ∂_{x_j} in 0:

$$\begin{aligned} \partial_{x_j}\big|_{p=0} \equiv \frac{\partial}{\partial x_j}\big|_{p=0} \colon C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial}{\partial x_j}\big|_0(f) \equiv \frac{\partial f}{\partial x_j}(0) \end{aligned} \tag{7.2}$$

   In particular, if $\mathrm{Der}(\mathbb{R}^n) = \{$derivations on $\mathbb{R}^n\} = \{v \colon C^\infty \to \mathbb{R}$, satisfying Leibnitz rule$\}$, then the map:

$$\mathbb{R}^n \overset{\psi}{\longleftrightarrow} \mathrm{Der}(\mathbb{R}^n) \tag{7.3}$$

   such that $\psi(e_j) = \partial_{x_j}\big|_{p=0}$, $\forall\, e_j$ basis vector, and with $\partial_{x_j}\big|_{p=0}$ partial derivative with respect to the j-th component, defines a linear map. Indeed, it is linear because of linearity of derivations, and since we defined its behaviour on the basis vectors, it is also defined for every vector of $\mathbb{R}^n$. In general we have $\psi(v) = \partial_v\big|_{p=0}$, where $\partial_v\big|_{p=0}$ is the directional derivative with respect to v. Moreover, $\mathrm{Der}(\mathbb{R}^n)$ is a vector space and we used the double arrow above because ψ is an isomorphism, i.e. a bijective map which preserves operations from one space to the other. Here every derivative is evaluated at $p = 0$. We notice that the point p itself is not important for the directional derivative (the *direction* in which we differentiate is the same for every point of the space), but p is meaningful when we *evaluate* the derivative of the function at that point. Indeed, $\partial_x(x^2)\big|_{x=0} \neq \partial_x(x^2)\big|_{x=1}$, even if we are differentiating along the x-axis in both cases.

What is more: given V_0, the set of all the vectors about 0, we can consider its dual space V_0^*. What is a possible dual basis? We want to find linear forms

$$e^{*i} \colon V_0 \to \mathbb{R}$$

such that $e^{*i}(e_j) = \delta^i_j$. We have just seen that we can consider vectors as directional derivatives. So, given e_j vector of the canonical basis, we will call it $\frac{\partial}{\partial x^j}\big|_0$ (because of the isomorphism, they are quite the same mathematical object). Now, we want that

$$e^{*i}\left(\frac{\partial}{\partial x^j}|_0\right) = \delta^i_j \tag{7.4}$$

First, let's consider the *coordinate function*:

$$\begin{aligned} x_j \colon \mathbb{R}^n &\to \mathbb{R} \\ v = (v_1, \dots, v_n) &\mapsto v_j \end{aligned} \tag{7.5}$$

where $v_1, \dots, v_j$ are the coordinates of the vector v in the canonical basis. The linear form x_j returns the j-th coordinate of a vector. So, given a vector $v \in \mathbb{R}^n$, every coordinate v_j can be seen as $v_j = x_j(v)$. Now, let's just define

$$e^{*i}\left(\frac{\partial}{\partial x^j}|_0\right) \equiv \frac{\partial}{\partial x^j}|_0 x_i = \frac{\partial}{\partial x^j} x_i|_0 = \delta^i_j \tag{7.6}$$

Where x_i is the coordinate function defined above (remember: ∂_{x_j} is a derivation, so it must be applied to functions!). Now, it might seem that e^{*i} does not take a vector as argument, but rather a function. Actually, this problem is solved by the isomorphism between vectors and directional derivatives proved above. If ψ is the name of such isomorphism, we could slightly change the definition (7.6) in order to solve this ambiguity:

$$e^{*i}(e_j) \equiv \psi(e_j)(x_i) \tag{7.7}$$

where x_i is the i-th coordinate function and

$$\psi(e_j) = \frac{\partial}{\partial x^j}|_0 = \partial_{x_j}|_0 \tag{7.8}$$

The vectors of the dual basis will also be called

$$dx^i \equiv e^{*i} \tag{7.9}$$

This will be important later: we will define exterior forms of degree k and we'll use both notations. The set of all these forms is $\Lambda^k V^*$, and its basis is given by products (in particular, exterior products) of $e^{*i_1}, \dots, e^{*i_k}$ (i.e. $dx^{i_1}, \dots, dx^{i_k}$).

Definition 7.7 (Set of vector fields)**.** We denote by $\mathfrak{X}(\mathbb{R}^n)$ the set of all possible vector fields in $\mathbb{R}^n$, i.e.

$$\begin{aligned} \mathfrak{X}(\mathbb{R}^n) = \mathrm{Der}\mathbb{F}(\mathbb{R}^n) \equiv \{ &v \colon \mathbb{F}(\mathbb{R}^n) \to \mathbb{F}(\mathbb{R}^n) \text{ such that} \\ &v \text{ is } \mathbb{R}\text{-linear and } v(fg) = v(f)g + fv(g)\} \end{aligned} \tag{7.10}$$

where $\mathbb{F} \equiv \{$ functions $f \colon \mathbb{R}^n \to \mathbb{R}\}$.

Remark 7.8 (Fields vs. Derivations). If v is a vector field in $\mathbb{R}^n$, then v assigns a vector to another vector of $\mathbb{R}^n$. So $v\colon \mathbb{R}^n \to \mathbb{R}^n$. So, the set of all vector fields should be (we will use a different symbol to denote it):

$$X(\mathbb{R}^n) = \{v\colon \mathbb{R}^n \to \mathbb{R}^n\}$$

However, the definition 7.7 is a bit different. Why? The fact is, we can consider a vector of $\mathbb{R}^n$ as a directional derivative, cf. remark 7.6 (we are not considering any fixed point here, but the results do not change). Now a derivation, as defined in def. 6.1, is a map $v\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$, i.e. we can say that a derivation is a very smooth element of $\mathbb{F}(\mathbb{R}^n) = \{\text{functions } f\colon \mathbb{R}^n \to \mathbb{R}\}$. So, $\mathfrak{X}(\mathbb{R}^n) \cong X(\mathbb{R}^n)$ because we can associate a derivation of $\mathfrak{X}(\mathbb{R}^n)$ to each vector of $X(\mathbb{R}^n)$, and vice versa. Then we also explained why in the definition of $\mathfrak{X}(\mathbb{R}^n)$ every element must be $\mathbb{R}$-linear and satisfy the Leibnitz rule: it follows from the definition of derivations.
Now, a question arises: given v vector field, should we write $v(p)$ (i.e. it takes vectors as argument) or should we write $v(f)$ (i.e. it takes smooth functions as arguments)? The answer is: it depends on the case, since they are two different "v"s. Which is, we will use vectors when we think of v as a function who takes elements of $\mathbb{R}^n$, and we will use functions in the other case. And we can choose which case to use, since we can identify every vector field with a derivation, and every derivation with a vector field (for more info, see pag. 181 of [1]). Now, let us analyze how $v(f)$ is made in the latter case. Given $v \in \mathfrak{X}(\mathbb{R}^n), f : \mathbb{R}^n \to \mathbb{R}$, we define the function $v(f)$ as

$$\begin{aligned} v(f)\colon \mathbb{R}^n &\to \mathbb{R} \\ p &\mapsto v(f)(p) \equiv v_p f \end{aligned} \tag{7.11}$$

Now, in coordinates:

$$v(f)(p) = v_p f = v^i(p)(e_i)_p f = v^i(p)\frac{\partial}{\partial x^i}\Big|_p f = v^i(p)\frac{\partial f}{\partial x^i}(p) \tag{7.12}$$

where we used summation convention, and the fact that every vector basis e_i can be seen as $\partial_{x_i}\big|_p$. We defined it in the right way because, as expected, we found that $v_p(f)$ is the directional derivative of f in the direction of v, evaluated at p. So, in brief:

- $v(f)(p)$ is a number
- $v(f)(\cdot)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$
- $v(\cdot)$ is a function from $\mathbb{F}(\mathbb{R}^n)$ to $\mathbb{F}(\mathbb{R}^n)$

We also notice that the mathematical object e_i is not much different from $(e_i)\big|_p$ in this case: there is no difference if we think about them as directions, but it makes a difference if we think about them as directional derivatives, because the latter notation gives info about the point in which the derivative is evaluated. So, we add the pedix "p" in order to make the isomorphism between vectors and directional derivatives more explicit. Check also the remark 1.1.

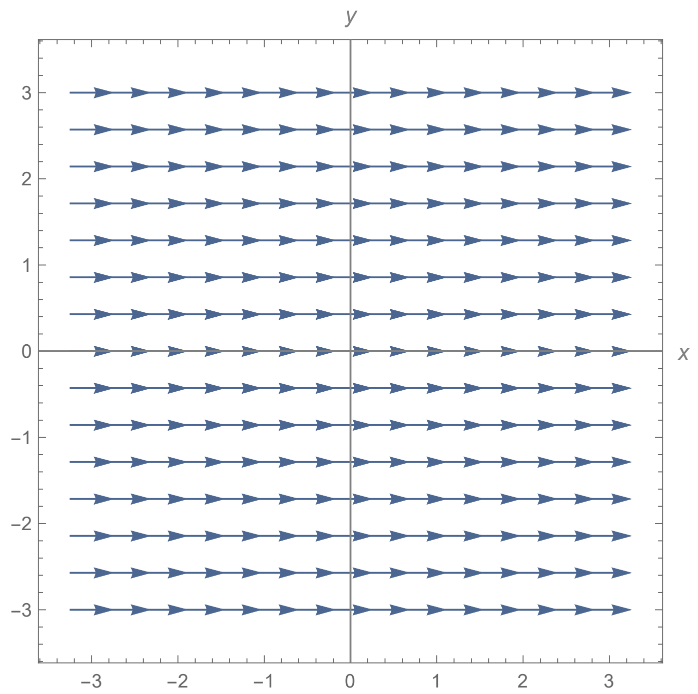


Figure 1: Vector field v=$e_1 = \partial_x$

8 Exterior Product and Generalisation

Definition 8.1 (Exterior form of degree k)**.** Given a vector space V on a field $\mathbb{K}$, with $\dim(V) = n$, and with $k \leq n$, an *exterior form of degree k* (or k-linear form, or k-form) is a map ω:

$$\omega\colon \underbrace{V \times \ldots \times V}_{\text{k times}} \to \mathbb{K}$$

such that

$$\omega(v_1, \ldots, v_k) = \text{sgn}(\pi)\, \omega(v_{\pi(1)}, \ldots, v_{\pi(k)}) \tag{8.1}$$

and such that ω is multilinear. Where π is a permutation of k elements, i.e. $\pi \in S_k$, and $\text{sgn}(\pi)$ is the sign of the permutation. We will also write $\omega \in \Lambda^k V^*$.

Definition 8.2 (Exterior product between two 1-forms)**.** Given a vector space V on a field $\mathbb{K}, \dim(V) \geq 2$, and given $\varphi^1, \varphi^2 \in \Lambda V^*$, then we define the exterior product (or wedge product) $\wedge$ as:

$$\begin{aligned}\wedge\colon \Lambda V^* \times \Lambda V^* &\to \Lambda^2 V^* \\ (\varphi^1, \varphi^2) &\mapsto \varphi^1 \wedge \varphi^2\end{aligned}$$

where:

$$\varphi^1 \wedge \varphi^2(x_1, x_2) = \varphi^1(x_1)\varphi^2(x_2) - \varphi^2(x_1)\varphi^1(x_2) = \det(\varphi^i(x_j))$$

for $i, j = 1, 2$.

Remark 8.3 (Exterior product between k 1-forms). The exterior product $\wedge$ that we defined for $k = 2$ in def. 8.2 gives an exterior form of degree 2. We want to generalize it for k vector spaces. In order to extend the definition, we want it to give an exterior form of degree k, so:

$$\begin{aligned}\wedge\colon \underbrace{\Lambda V^* \times \ldots \times \Lambda V^*}_{k \text{ times}} &\to \Lambda^k V^* \\ (\varphi^1, \ldots, \varphi^k) &\mapsto \varphi^1 \wedge \ldots \wedge \varphi^k\end{aligned}$$

where, given $(x_1, \ldots, x_k) \in \underbrace{V \times \ldots \times V}_{k \text{ times}}$:

$$\varphi^1 \wedge \ldots \wedge \varphi^k(x_1, \ldots, x_k) = \det\left(\varphi^i(x_j)\right) \tag{8.2}$$

This is a particular case of an exterior k-form (because the sign of determinant changes if we swap two rows or two columns). Now let's consider and explicit computation using the determinant, in $\mathbb{R}^3$, with coordinates x, y, z, and with $\varphi^1, \varphi^2, \varphi^3$ corresponding to the three elements of the dual basis $dx \equiv e^{*1} = dx^1, dy \equiv e^{*2} = dx^2$ and $dz \equiv e^{*3} = dx^3$:

$$(dx \wedge dy \wedge dz)(x, y, z) = \det\left(dx^i(e_j)\right) = \det\begin{pmatrix} dx(x) & dx(y) & dx(z) \\ dy(x) & dy(y) & dy(z) \\ dz(x) & dz(y) & dz(z)\end{pmatrix}$$

and we can use that $dx^i(e_j) = \delta^i_j$. For instance, we can verify that the 3-form $dx \wedge dy \wedge dz$ gives 0 if two coordinates of the input are repeated:

$$(dx \wedge dy \wedge dz)(x, x, z) = \det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} = 0$$

Proposition 8.4. *If $\{e_i\}_{i=1,\ldots,n}$ is a basis in V, then $\{e^{*i_1} \wedge \ldots \wedge e^{*i_k}\}_{i_1 < \ldots < i_k, k \leq n}$ forms a basis of $\Lambda^k V^*$*

Proof. In the statement we implicitly assumed that the elements of $\{e^{*i_1} \wedge \ldots \wedge e^{*i_k}\}$ are defined as usual, for instance $e^{*i_1}(e_{j_1}) = \delta^{i_1}_{j_1}$. In order to prove that it is a basis, we need to prove that the element of the basis are linearly independent and that they span the entire space.

- Linear independence: If

$$0 = \sum_{i_1<\ldots<i_k} a_{i_1\cdots i_k}(e^{*i_1}\wedge\ldots\wedge e^{*i_k})(e_{j_1},\ldots,e_{j_k}), \quad \forall\,(e_{j_1},\ldots,e_{j_k})$$

then $a_{i_1\cdots i_k}=0, \forall\, i_1,\ldots,i_k$. Indeed, if it is true $\forall\,(e_{j_1},\ldots,e_{j_k})$, then we can choose $e_{j_1},\ldots,e_{j_k}$ such that $a_{i_1\cdots i_k}=0, \forall\, i_1,\ldots,i_k$. e.g.:

$$\begin{aligned} 0 = a_{i_1\cdots i_k}&\underbrace{(e^{*i_1}\wedge\ldots\wedge e^{*i_k})(e_{i_1},\ldots,e_{i_k})}_{=1}+\\ +\sum_{\substack{j_1<\ldots<j_k\\ j_1\neq i_1, j_2\neq i_2,\ldots}} &a_{j_1\cdots j_k}\underbrace{(e^{*j_1}\wedge\ldots\wedge e^{*j_k})(e_{i_1},\ldots,e_{i_k})}_{=0} = a_{i_1\cdots i_k}\end{aligned}$$

The second term is zero because $(e^{*j_1}\wedge\ldots\wedge e^{*j_k})(e_{i_1},\ldots,e_{i_k}) = \det\big(e^{*j_l}(e_{i_l})\big)$ and the matrix $e^{*j_l}(e_{i_l})$ has null coefficients on the main diagonal (because $j_1\neq i_1, j_2\neq i_2,\ldots$) and null coefficients on the lower triangle (because we have the ordering $j_1<\ldots<j_k$). Thus, we are computing the determinant of an upper triangular matrix with null elements on the diagonal, i.e. the determinant is 0. Therefore, $a_{i_1\cdots i_k}=0$. So, we proved the linear independence for the case in which we consider the basis vectors $e_{j_1},\ldots,e_{j_k}$. Because of multilinearity, the result holds for all vectors $(x_1,\ldots,x_k)$ as well.

- Completeness: given $\psi\in\Lambda^k V^*$, we want to prove that there exists a coefficient $a_{i_1\cdots i_k}$ such that $\psi = \sum\limits_{i_1<\ldots<i_k} a_{i_1\cdots i_k}(e^{*i_1}\wedge\ldots\wedge e^{*i_k})$, i.e. we want to prove that we can write any element of $\Lambda^k V^*$ as a linear combination of elements of $\{e^{*i_1}\wedge\ldots\wedge e^{*i_k}\}$. Let's consider $(v_1,\ldots v_k)\in V\times\ldots\times V$. Then:

$$\begin{aligned}\psi(v_1,\ldots,v_k) &\overset{(1)}{=} \sum_{i_1=1}^{k}\sum_{i_2=1}^{k}\cdots\sum_{i_k=1}^{k} x_{i_1}\cdot\ldots\cdot x_{i_k}\psi(e_{i_1},\ldots,e_{i_k}) \overset{(2)}{=}\\ &= \sum_{i_1<\ldots<i_k} x_{i_1}\cdot\ldots\cdot x_{i_k}\psi(e_{i_1},\ldots,e_{i_k}) \overset{(3)}{=}\\ &= \sum_{i_1<\ldots<i_k} x_{i_1}\cdot\ldots\cdot x_{i_k}\psi(e_{i_1},\ldots,e_{i_k})(e^{*i_1}\wedge\ldots\wedge e^{*i_k})(e_{i_1},\ldots,e_{i_k}) \overset{(4)}{=}\\ &= \sum_{i_1<\ldots<i_k} \psi(e_{i_1},\ldots,e_{i_k})(e^{*i_1}\wedge\ldots\wedge e^{*i_k})(x_{i_1}e_{i_1},\ldots,x_{i_k}e_{i_k}) \overset{(5)}{=}\\ &= \sum_{i_1<\ldots<i_k} a_{i_1\cdots i_k}(e^{*i_1}\wedge\ldots\wedge e^{*i_k})(v_1,\ldots,v_k)\end{aligned}$$

Where we used: (1) $v_1=\sum\limits_{i_1=1}^{k} x_{i_1}e_{i_1}$ in the $\{e_i\}_{i=1}^n$ basis, (2) we can simplify the sum since $\psi=0$ if two entries are repeated, (3) $1=(e^{*i_1}\wedge\ldots\wedge e^{*i_k})(e_{i_1},\ldots,e_{i_k})$, (4) multilinearity, (5) $0=e^{*i}(e_j)$ if $i\neq j$ and $a_{i_1\cdots i_k}\equiv\psi(e_{i_1},\ldots e_{i_k})$. $\square$

Remark 8.5. The above proposition proves that $\dim(\Lambda^k V^*) = \binom{n}{k}$, because through the expression "$i_1 < \ldots < i_k$, $k \leq n$" we are selecting subsets of k elements (subsets, not k-tuples, because the order of the elements is externally fixed) from a bigger set of n elements. This is exactly the definition of the binomial coefficient $\binom{n}{k}$. Moreover, it means that any $\alpha \in \Lambda^k V^*$ can be written as:

$$\alpha = \sum_{i_1 < \ldots < i_k} a_{i_1 \cdots i_k} e^{*i_1} \wedge \cdots \wedge e^{*i_k}$$

where $a_{i_1 \cdots i_k} \in \mathbb{K}$, $\mathbb{K}$ field of the vector space.

Now, we want to define the exterior product between a k-form and a p-form (and it will return a $(p+k)$-form).

Definition 8.6 (Exterior product between a k-form and a p-form)**.** Given $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*$, the exterior product between them is defined as:

$$\begin{aligned} \wedge \colon \Lambda^k V^* \times \Lambda^p V^* &\to \Lambda^{k+p} V^* \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \end{aligned}$$

with:

$$\alpha \wedge \beta = \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_p}} \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_k} e^{*i_1} \wedge \cdots \wedge e^{*i_k} \wedge e^{*j_1} \wedge \cdots \wedge e^{*j_k}$$

where $\alpha_{i_1 \cdots i_k}, \beta_{j_1 \cdots j_k} \in \mathbb{K}$

Example 8.7 (Oriented area)*.* Let's consider the following examples:

1. $V = \mathbb{R}^3 \times \mathbb{R}^3$ with cartesian coordinates.

$$\varphi \equiv dx^1 \wedge dx^2 + dx^2 \wedge dx^4$$

φ is a sum of two 2-forms. They are 2-forms (and not 1-forms, nor 3-forms, nor 4-forms, ...) because they are written as wedge product ("$\wedge$") of **two** elements of the dual basis. Remember that we denote the elements of the dual basis either by e^{*i} or dx^i for some i. Let's compute $\varphi(e_i, e_j)$. We can use the determinant formula (8.2):

$$\begin{aligned} \varphi(e_i, e_j) =& dx^1(e_i) dx^2(e_j) + dx^2(e_i) dx^4(e_j) - dx^2(e_i) dx^1(e_j) + \\ & - dx^4(e_i) dx^2(e_j) \end{aligned}$$

2. $V = \mathbb{R}^2$ with cartesian coordinates x^1, x^2.

$$\varphi \equiv dx^1 \wedge dx^2$$

Let's compute $\varphi(ae_1, be_2)$, where $a, b \in \mathbb{R}$:

$$\begin{aligned}\varphi(ae_1, be_2) &= dx^1 \wedge dx^2(ae_1, be_2) = ab\, dx^1 \wedge dx^2(e_1, e_2) = \\ &= ab\,(dx^1(e_1)dx^2(e_2) - \cancel{dx^2(e_1)\,dx^1(e^2)}) = ab = \\ &= \text{oriented area of the rectangle of sides } a \text{ and } b\end{aligned}$$

There are some interesting properties about k-forms:

Proposition 8.8. *$\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*, \gamma \in \Lambda^q V^*$, then:*

1. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$
2. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$
3. $\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha$

Sketch of the proof.

1. Writing the forms in terms of the basis, we only need to prove that the property (1) holds for the elements of the dual basis. Moreover, by multilinearity, we just need to prove it for the elements of the dual basis applied to a basis $\{e_i\}$ of V. Thus, the proof for (1) is over because this property is true for the determinant: when we compute it, we can start from any row or column.

2. By writing the forms in terms of the basis as before.

3. Again, by writing the forms in terms of the basis we have a sequence of wedge products like

$$e^{*i_1} \wedge \ldots \wedge e^{*i_k} \wedge e^{*j_1} \wedge \ldots \wedge e^{*j_p}$$

If we swap two elements of the product we get a factor "-1". In order to have the right hand side we need to "move e^{*j_1} on the left k times", so we get a factor $(-1)^k$. Same for e^{*j_2}: we need to move it on the left k times and so we get an extra factor $(-1)^k$. Again, we repeat the procedure for all the elements up to e^{*j_p}. At the end, we get a factor $(-1)^{kp}$.

□

9 Differential Forms

Definition 9.1 (Field of exterior forms, geometric definition)**.** (A field of) exterior forms of degree k, $k \leq n$ is a map ω that associates to each point $p \in V$ an element $\omega(p) \in \Lambda^k V_p^*$. Choosing a basis, we have:

$$\omega(p) = \sum_{i_1 < \ldots < i_k} \underbrace{a_{i_1 \cdots i_k}(p)}_{\text{now it is a function!}} e^{*i_1} \wedge \cdots \wedge e^{*i_k} \tag{9.1}$$

ω is a differential form if $a_{i_1 \cdots i_k}$ are differentiable. The set of differential k-forms is denoted by $\Omega^k(\mathbb{R}^n)$.

Another (equivalent) definition:

Definition 9.2 (Algebraic definition of differential k-form)**.** A differential k-form is a map:

$$\omega\colon \underbrace{\mathfrak{X}(\mathbb{R}^n)\times\ldots\times\mathfrak{X}(\mathbb{R}^n)}_{k\text{ times}} \to \mathbb{F}(\mathbb{R}^n) \tag{9.2}$$

$C^\infty(\mathbb{R}^n)$-linear and alternating.

Remark 9.3. To show the equivalence of the two definition of differential k-forms we just need to show that:

$$\omega(p)(v_1,\ldots,v_k) = \omega(v_1,\ldots,v_k)(p) \tag{9.3}$$

See also the exercise 1, problem sheet 3.

We want to generalize the concept of differential of a function.

Definition 9.4 (Differential)**.** Let f be a function $f\colon U\subseteq\mathbb{R}^n\to\mathbb{R}$, f differentiable. Let $v\in\mathfrak{X}(\mathbb{R}^n) = \operatorname{Der}\mathbb{F}(\mathbb{R}^n)$. The exterior derivative of f is its differential d, defined as a 1-form such that:

$$df(v) = v(f) \tag{9.4}$$

Remark 9.5 (differential expression in coordinates)*.* We want to verify that the above definition of differential is equivalent to our usual definition for $C^1(\mathbb{R}^n)$ function, which is:

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx^i = \frac{\partial f}{\partial x_i}dx^i \tag{9.5}$$

In order to prove that, we first consider a pointwise definition. Given $p\in\mathbb{R}^n$:

$$df_p(v) = v(f), \forall\ v\in T_p\mathbb{R}^n \cong \mathbb{R}^n \tag{9.6}$$

($T_p\mathbb{R}^n$ is the tangent space to $\mathbb{R}^n$ at p). Now, we can write $v(f)$ in coordinates (the gray part is the one we don't care about):

$$df_p = v(f) = \ v_i(p)(\lambda^i)_p \tag{9.7}$$

where $(\lambda^i)_p$ is a dual basis at p (later, we will prove that $(\lambda^i)_p = (dx^i)_p)$. Now, applying df to a particular vector (i.e. directional derivative) at p:

$$df_p\left(\frac{\partial}{\partial x^i}|_p\right) = v_i(p) \tag{9.8}$$

where we used the property of the dual basis

$$(\lambda^i)_p\frac{\partial}{\partial x^j}|_p = \delta^i_j \tag{9.9}$$

and then:

$$df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = v_i(p)(\lambda^i)_p \frac{\partial}{\partial x^i}\Big|_p = v_i(p) \tag{9.10}$$

On the other hand, by definition (9.6) we know that:

$$df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial f}{\partial x^i}(p) \tag{9.11}$$

Hence, using (9.8) and (9.11) we get:

$$v_i(p) = \frac{\partial f}{\partial x^i}(p) \tag{9.12}$$

Then, by the expression of differential in coordinates (9.7):

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p \tag{9.13}$$

Applying the definition to $f = x^j$ (coordinate function, as defined in (7.5)), we get:

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p = \frac{\partial f}{\partial x^i}(p)(dx^i)_p \tag{9.14}$$

And then:

$$df = \frac{\partial f}{\partial x^i} dx^i \tag{9.15}$$

Indeed, if $f = x^j$ then, as before:

$$(dx^j)_p = \frac{\partial x^j}{\partial x^i}(p)(\lambda^i)\big|_p = \delta^i_j(\lambda^i)\big|_p = (\lambda^j)\big|_p \tag{9.16}$$

Pay attention: what we did here is a bit different from what we did for the definition 7.11 of a vector field applied to a function. In this case, p is the point where we fixed our vector, whereas in the other case p was the point where we wanted to evaluate the directional derivative of f.

In the above definition, f was a 0-form (i.e. a function). What is the generalization of the differential to k-forms?

Definition 9.6 (Exterior derivative)**.** If $k > 0$, then the exterior derivative (acting on k-forms) is a map

$$\begin{aligned} d\colon \Omega^k(\mathbb{R}^n) &\to \Omega^{k+1}(\mathbb{R}^n) \\ \omega &\mapsto d(\omega) \equiv d\omega \end{aligned}$$

where

$$d\omega = \sum_{j_1<\ldots<j_k} (da_{j_1,\ldots,j_k}) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

With $da_{j_1,\ldots,j_k}$ differential of the function $a_{j_1,\ldots,j_k}$.

Example 9.7 (Computation of $d\omega$)*.* Let's consider the following examples:

1. Let ω be a 2-form on $\mathbb{R}^3$ (coordinates x^1, x^2, x^3):

$$\omega = dx^1 \wedge dx^2 + x^2 dx^1 \wedge dx^3$$

   Then

$$d\omega = dx^2 \wedge dx^1 \wedge dx^3$$

   where we used that $d(dx^1 \wedge dx^2) = 0$ because there is no 3-form on a 2-dimensional space (otherwise, we can use that $d^2 = 0$, but we still have to prove it!)

2. In $\mathbb{R}^n$, let's consider:

$$\omega = x^2 dx^1, \quad d\omega = dx^2 \wedge dx^1$$

   where we computed $d\omega$ by using $dg(v) = v(g)$ for a function g and a vector field v. In fact, if $u, v \in \mathbb{R}^n$, then by definition of exterior product we have:

$$d\omega(u, v) = dx^2(u)dx^1(v) - dx^2(v)dx^1(u)$$

   On the other hand, using $dx^2(u) = u(x^2), v(x^2) = dx^2(v)$ (where x^2 is a function, the coordinate function defined in (7.5)) we also have:

$$d\omega(u, v) = u(x^2)dx^1(v) - v(x^2)dx^2(u)$$

Now, some properties:

Proposition 9.8 (Properties of exterior derivatives)**.** $\omega_1 \in \Omega^k(\mathbb{R}^n), \omega_2 \in \Omega^p(\mathbb{R}^n)$*. Then:*

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
- $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$
- $d(d\omega_1) = 0 = d(d\omega_2)$

Sketch of the proof.

- We have

$$d\omega_1 = \sum_{j_1 < \ldots < j_k} \left(da^{(1)}_{j_1 \cdots j_k} \right) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

  and similarly for ω_2. Then we use the linearity of the differential: $d(a^{(1)} + a^{(2)}) = da^{(1)} + da^{(2)}$.

- We have

$$d(\omega_1 \wedge \omega_2) = \sum_{\substack{j_1 < \ldots < j_k \\ i_1 < \ldots < i_k}} d\left(a^{(1)}_{j_1 \cdots j_k} a^{(2)}_{i_1 \cdots i_k} \right) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Using the product rule we have a term $\left(da^{(1)}\right)a^{(2)} + a^{(1)}\left(da^{(2)}\right)$ inside the sum. Now, $\left(da^{(1)}\right)a^{(2)}$ gives the term $d\omega_1 \wedge \omega_2$. In order to get the term $(-1)^k\omega_1 \wedge d\omega_2$ we consider $a^{(1)}\left(da^{(2)}\right) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ inside the sum and we move the factor $da^{(2)}$ to the right for k times. So we get the factor $(-1)^k$) and the wedge product $\omega_1 \wedge d\omega_2$.

- Let's consider

$$d(d\omega) = \sum_{j_1<\ldots<j_k} d\left(da_{j_1\cdots j_k}\right) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

Now,

$$da_{j_1\cdots j_k} = \left(\frac{\partial}{\partial x^l}a_{j_1\cdots j_k}\right)dx^l$$

and:

$$\left[d\left(\frac{\partial}{\partial x^l}a_{j_1\cdots j_k}\right)\right]dx^l = \underbrace{\frac{\partial^2}{\partial x^m \partial x^l}a_{j_1\cdots j_k}}_{\text{symmetric in } l\leftrightarrow m} \quad \underbrace{dx^m \wedge dx^l}_{\text{antisymmetric in } l\leftrightarrow m}$$

Then we get a minus sign if we exchange l and m. Moreover we can compute $\left[d\left(\frac{\partial}{\partial x^m}a_{j_1\cdots j_k}\right)\right]dx^m$ in the same way exchanging the roles of l and m, and we wouldn't get any minus sign. Then we have that $d(d\omega) = -d(d\omega) \Rightarrow d(d\omega) = 0$.

□

Remark 9.9. In the above proposition, we claimed that $d(d\omega) = 0$ if $\omega \in \Omega^k(\mathbb{R}^n)$. The notation here is not the most precise, since the inner "d" is acting on a k-form, whereas the outer "d" is acting on a $(k+1)$-form (so, even if they share the same name, they are different maps). However the behaviour of both "d"s is clear, so we will continue with this abuse of notation.

Remark 9.10. The exterior derivative increases the degree of a k-form by 1 (the k-form becomes a $(k+1)$-form). Can we get backwards, which is, can we decrease the degree of a k-form? Yes, using the interior derivative.

Definition 9.11 (Interior derivative)**.** Let z be a vector field on $\mathbb{R}^n$, i.e. $z \in \mathfrak{X}(\mathbb{R}^n)$, then we define the *interior derivative* i_z (acting on differential k-forms) as the map:

$$\begin{aligned} i_z \colon \Omega^k(\mathbb{R}^n) &\to \Omega^{k-1}(\mathbb{R}^n) \\ \omega &\mapsto i_z(\omega) \equiv i_z\omega \end{aligned} \tag{9.17}$$

such that

$$(i_z\omega)(v_1,\ldots,v_{k-1}) = \omega(z,v_1,\ldots,v_{k-1}), \forall\, v_i \in \mathfrak{X}(\mathbb{R}^n)$$

$i_z\omega$ is also called *contraction* or *interior multiplication*. Another notation for $i_z\omega$ is $z \lrcorner\, \omega$.

Example 9.12 (Some computations). In $\mathbb{R}^2$, e_x, e_y basis vectors (that can be seen as vector fields):

$$i_{e_x}(dx \wedge dy) = dy$$

$$i_{e_y}(dx \wedge dy) = -dx$$

Remark 9.13. In the definition 9.11 above, we used the algebraic definition of differential k-forms, i.e. definition 9.2

Now some properties for interior derivatives.

Proposition 9.14. *$\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^p(\mathbb{R}^n), z \in \mathfrak{X}(\mathbb{R}^n)$, then:*

- $i_z(\omega \wedge \eta) = (i_z\omega) \wedge \eta + (-1)^k \omega \wedge (i_z\eta)$
- $i_z^2 w = i_z(i_z\omega) = 0$

Sketch of the proof.

- First, we assume $k + p \leq n$, otherwise $\omega \wedge \eta$ would be 0. Then, let's consider a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ in $\mathbb{R}^n$ which is positively oriented (i.e. if A is the matrix that we use to change coordinates from the canonical basis to $\mathcal{B}$, we have $\det A > 0$). Let's choose $\mathcal{B}$ such that the vector field e_1 is the tangent vector to the vector field z in one point (we will make a pointwise reasoning). Moreover, $\{dx^l\}_{l=1}^n$ is the dual basis with respect to the canonical basis, and $\{e^{*l}\}_{l=1}^n$ is the dual basis with respect to $\mathcal{B}$. We have

$$\omega = \sum_{i_1 < \dots i_k} a_{i_1 \cdots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and

$$\omega \wedge \eta = \sum_{\substack{i_1 < \dots i_k \\ j_1 < \dots j_p}} a_{i_1 \cdots i_k} b_{j_1 \cdots j_p} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}$$

Then either:

1. $dx^{i_r} = e^{*1}$ for some $i_r, r = 1, \dots, k$, or
2. $dx^{j_r} = e^{*1}$ for some $j_r, r = 1, \dots, p$, or
3. there are no i_r, j_r such that "$dx^{i_r} = e^{*1}$ or $dx^{j_r} = e^{*1}$"

So, $i_z(\omega \wedge \eta) = (\omega \wedge \eta)(z, \dots)$ is either equal to:

1. $\sum_{\substack{i_1 < \dots i_k \\ j_1 < \dots j_p}} a_{i_1 \cdots i_k} b_{j_1 \cdots j_p} i_z \left(dx^{i_1} \wedge \dots \wedge dx^{i_k}\right) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}$,because the only non-zero term comes from $i_z(e^{*1}) = e^{*1}(z) = e^{*1}(e_1) = 1$ (locally), and we know that the non-zero term must be hidden in the first k wedge products,

2. $\sum\limits_{\substack{i_1<\dots i_k \\ j_1<\dots j_p}} (-1)^k a_{i_1\cdots i_k} b_{j_1\cdots j_p} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge i_z \left(dx^{j_1} \wedge \ldots \wedge dx^{j_p}\right)$, for the same reason above, and we have the factor $(-1)^k$ because we moved the interior derivative k times (it is like moving an entry of a $(k+p)$-form k times),
3. 0

Then, $i_z(\omega \wedge \eta)$ is the sum of these non-zero terms.

- Using the fact that a differential form gives 0 if two or more entries are repeated:

$$i_z\left(i_z(\omega)\right)(v_1,\ldots,v_{k-2}) = i_z\omega(z,v_1,\ldots,v_{k-2}) = \omega(z,z,v_1,\ldots,v_{k-2}) = 0$$

□

Remark 9.15. In the above proposition there is a little abuse of notation when we claimed $i_z(i_z\omega) = 0$, see also the remark 9.9.

Now, let's talk about *pullbacks* and *pushforwards* for functions and k-forms.

Definition 9.16 (Pullback)**.** Let $f\colon U \to V$ (with $U, V \subseteq \mathbb{R}^n$) be a differentiable map. Let's suppose that $\dim(U) = \dim(V) = n$ (just for the sake of simplicity, since it is not necessary). Then the *pullback* of a k-form (from V) to U is the map:

$$\begin{aligned} f^*\colon \Omega^k(V) &\to \Omega^k(U) \\ \omega &\mapsto f^*w \end{aligned}$$

such that

$$(f^*\omega)(p)(u_1,\ldots,u_k) = \omega(f(p))(df(u_1),\ldots,df(u_k)), \forall\, p \in \mathbb{R}^n, \forall\, u_i \in \mathfrak{X}(U)$$

Now, we want to give another name to the differential of a function.

Definition 9.17 (Pushforward)**.** Given $f\colon U \to V$ as before, we will also call the differential of f at $p \in \mathbb{R}^n$, i.e. $df_p = df(p)$, as the *pushforward* of f at p, and it will be denoted by the symbol $(f_*)_p$.

In our mind, we'll think of $df_p = (f_*)_p$, at least until this concept is generalized. Using the pullback definition above, we can write the pushforward map as:

$$\begin{aligned} df_p \equiv (f_*)_p\colon U \subset \mathbb{R}^n &\to V \subset \mathbb{R}^m \\ v &\mapsto (f_*)_p(v) \end{aligned}$$

By definition of differential, $df_p(v) = v(f)$, where v is a vector tangent to $\mathbb{R}^n$ at p. Since vector are like directional derivatives, $v(f)$ is the directional derivative of f with respect to v (not evaluated at any point, for now!). In particular, if we apply the definition to a point $h(q)$, where $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ (**CHECK**) and $q \in \mathbb{R}^n$, we have:

$$(f_*)_p(v)(h)(q) = (f_*)_p(v)(h(q)) = v(h(f(q))) = v(h \circ f)(q) = v(f^*h)(q)$$

In the last passage, we used the pullback for a differentiable function, which is completely legal since we defined it for differentiable k-forms, and a differentiable function is just a 0-form.

Remark 9.18. Using the pushforward, we can define the pullback of a differential form using a different notation (i.e. using f_* instead of df):

$$(f^*\omega)(p)(u_1, \dots, u_k) = \omega(f(p))(f_*(u_1), \dots, f_*(u_k)), \forall\, p \in \mathbb{R}^n, \forall\, u_i \in \mathfrak{X}(U) \tag{9.18}$$

Now, some properties of the pullback.

Proposition 9.19. *$g, f \in C^1(\mathbb{R}^n, \mathbb{R})$, $\omega, \varphi \in \Omega^k(\mathbb{R}^n)$, $h \colon \mathbb{R}^n \to \mathbb{R}$. Then:*

1. $f^*(\omega + \varphi) = f^*(\omega) + f^*(\varphi)$
2. $f^*(h\omega) = f^*(h) f^*(\omega)$
3. $(f \circ g)^* = g^*(f^*(\omega))$
4. *If* $\varphi^1, \dots, \varphi^k \in \Omega^1(\mathbb{R}^n)$, *then* $f^*(\varphi^1 \wedge \dots \wedge \varphi^k) = f^*(\varphi^1) \wedge \dots \wedge f^*(\varphi^k)$
5. $df^*(\omega) = f^*(d\omega)$

From property (4) also follows that $f^*(\omega \wedge \phi) = (f^*\omega) \wedge (f^*\phi)$

Sketch of the proof.

1.

$$\begin{aligned} f^*(\omega + \varphi) &= (\omega + \varphi)(f(p))(f_*(u_1), \dots, f_*(u_k)) = \\ &= \omega(f(p))(f_*(u_1), \dots, f_*(u_k)) + \varphi(f(p))(f_*(u_1), \dots, f_*(u_k)) = \\ &= f^*(\omega) + f^*(\varphi) \end{aligned}$$

2. $f^*(h\omega)(z_1, \dots, z_k)(p) = h(f(p))\, \omega(f_* z_1, \dots, f_* z_k)(f(p)) = f^*(h) f^*(\omega)$
3. We will use: $(f \circ g)_* v(h) = v\,(f \circ g)^* h) = v(g^* \circ f^*(h)) = g_* v(f^* h) = f_* \tilde{v}(h) = f_* \circ g_* v$, where $\tilde{v} \equiv g_* v$. Then:

$$\begin{aligned} (f \circ g)^* \omega(z_1, \dots, z_k) = \omega((f \circ g)_* z_1, \dots, (f \circ g)_* z_k)) = \omega(f_* \circ g_* z_1, \dots, f_* \circ g_* z_k) = \\ = g^*(f^*\omega)(z_1, \dots, z_k) \end{aligned}$$

4.

$$\begin{aligned} f^*(\varphi^1 \wedge \varphi^k)(z_1, \dots, z_k) = (\varphi^1 \wedge \dots \wedge \varphi^k)(f_* z_1, \dots, f_* z_k) = \det\left(\varphi^i(f_* z_j)\right) = \\ = \det\left(f^* \varphi^i(z_j)\right) = (f^*\varphi^1) \wedge \dots \wedge (f^*\varphi^k)\,(z_1, \dots, z_k) \end{aligned}$$

5. If z is a vector field and dx^i is an element of the dual basis (in particular, $dx^i(e_j) = \frac{\partial}{\partial x^j}x^i$), we have

$$f^*dx^i(z) = dx^i(f_*z) = z(f^*x^i) = df^i(z)$$

where in the last inequality we used the def. 9.4 of differential. Now,

$$\begin{aligned} df^*(\omega) = df^*\left(\sum_{i_1<\ldots i_k} a_{i_1\cdots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right) &= \\ = d\sum_{i_1<\ldots i_k} f^*(a_{i_1\cdots i_k})f^*(dx^{i_1})\wedge\ldots\wedge f^*(dx^{i_k}) &\overset{(1)}{=} \\ = \sum_{i_1<\ldots i_k} df^*(a_{i_1\cdots i_k})df^{i_1}\wedge\ldots\wedge df^{i_k} &\overset{(2)}{=} \\ = f^*d\omega & \end{aligned}$$

Where we used: (1) $f^*dx^i = df^i$ (i-th coordinate), the product rule and the fact that $d^2 = 0$, (2) given $h \equiv a_{i_1\cdots i_k} \in C^1(\mathbb{R}^n)$, then $df^*h(z) = z(f^*h) = dh(f_*z) = f_*z(h) = f^*dh(z)$.

□

Remark 9.20. We can express the pullback of a differential form in the following way:

$$\begin{aligned}(f^*\omega)(p) &= \sum_{1\le i_1<i_2<\cdots<i_k\le n}(f^*a_{i_1,\ldots i_k}(p))f^*dy^{i_1}\wedge f^*dy^{i_2}\wedge\cdots\wedge f^*dy^{i_k} = \\ &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1,\ldots,i_k}(f(p))df^{i_1}\wedge df^{i_2}\wedge\cdots\wedge df^{i_k}\end{aligned}$$

where $f^i = y^i(f)$. We used properties (2) and (4) of proposition 9.19

Remark 9.21. From our definition of pullback, it is not necessary that f_* is invertible.

Example 9.22 (Example of a pullback). Let's consider

$$U = \{r > 0, 0 < \theta \le 2\pi\}$$

$$V = \mathbb{R}^2 \setminus \{(0,0)\}$$

Let:

$$\begin{aligned} f &: U \to V \\ (r,\theta) &\mapsto (r\cos\theta, r\sin\theta) \equiv (x,y)\end{aligned}$$

Let's consider the 1-form $\Omega^1(V) \ni \omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $\mathbb{R}^2 \setminus \{(0,0\}$. Then:

$$f^*\omega = -\frac{r\sin\theta}{r^2}\cdot(\cancel{dr\cos\theta} - r\sin\theta\, d\theta) + \frac{r\cos\theta}{r^2}(\cancel{dr\sin\theta} + r\cos\theta\, d\theta) = d\theta$$

where we used $f^* dx^i = df^i$.

10 Integration of differential forms

Let ω be a differential form of degree n in $\mathbb{R}^n$. Then ω is necessarily of the form

$$\omega = \underbrace{a(p)}_{\text{it's a function}} \quad dx^1 \wedge \ldots \wedge dx^n \tag{10.1}$$

Such a form can be integrated:

$$\int_{f(D)} \omega = \int_D f^* \omega \tag{10.2}$$

Example 10.1 (Example of $\int$ of a k-form). Let's consider

$$D = [0,1] \times [0,1]$$

$$f(D) = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1], 0 \leq y \leq x\}$$

i.e. $f(D)$ is the lower-right triangle of the square D. In particular, we can write f as $f(x^1, x^2) = (x^1, x^1 x^2)$. Let $\omega = dy^1 \wedge dy^2$ on $\mathbb{R}^2$, thus:

$$\int_{f(D)} dy^1 \wedge dy^2 = \int_D f^*(dy^1 \wedge dy^2) = \int_D df^1 \wedge df^2 = \int_D dx^1 \wedge (x^2 dx^1 + x^1 dx^2) =$$
$$= \int_D x^1 dx^1 \wedge dx^2 = \int_D x^1 dx^1 dx^2 = \frac{1}{2}$$

11 More on Vector Fields

Definition 11.1 (Tangent space). $U \subset \mathbb{R}^n, U$ is an open set. $p \in U$, then the set of all derivations of $C^\infty(U)$ (cf. def 6.1) is called tangent space to U at p and is denoted by T_pU. An element of T_pU is called a tangent vector at p, and it is often denoted by v_p.

Definition 11.2 (Tangent bundle). The tangent bundle over an open subset $U \subset \mathbb{R}^n$ is defined as

$$TU \equiv \bigsqcup_{p \in U} T_pU \tag{11.1}$$

where T_pU is the tangent space of U at p. Every element of the disjoint union is represented by an ordered pair (p, v) where $p \in U, v \in T_pU$. So, elements of the tangent bundle are couples that consist of a tangent vector at a point of U, and the point itself. The tangent bundle comes equipped with the projection map

$$\begin{aligned} \mathrm{pr} \colon TU &\to U \\ (p,v) &\mapsto p \end{aligned} \tag{11.2}$$

which sends each vector in T_pU to the point p at which it is tangent.

Remark 11.3. In the previous definition, the "$\sqcup$" symbol denotes a disjoint union. "Disjoint" here means that, if we consider the disjoint union of two elements x and y such that $x = y$, the union is the set $\{x, y\}$ and not $\{x\} = \{y\}$ as in normal unions. The mathematical operator doesn't know if two elements are equal. Since we are not mathematical operators, we can enumerate the elements like: $\{(1, x), (2, y)\} = \{(1, x), (2, x)\} = \{(1, y), (2, y)\}$ in order to distinguish them.

Definition 11.4 (Alternative definition of vector field). A smooth vector field v on $U \subset \mathbb{R}^n, U$ open, is a smooth map

$$v\colon U \to TU \tag{11.3}$$

such that $\mathrm{pr}(v_p) = p, \forall\, p \in U$

Remark 11.5 (Space of sections). The set of all vector fields $\mathfrak{X}(U) \equiv \{C^\infty(U, TU) \mid \mathrm{pr}(v_p) = p\}$ is also called the space of sections in TU.

Definition 11.6 (Cotangent bundle). We define

$$T^*U \equiv \bigsqcup_{p \in U} T_p^*U \tag{11.4}$$

as the cotangent bundle. Where T_p^*, the dual space of the tangent space, is called cotangent space. We also associate a projection $\mathrm{pr}\colon T^*U \to U$ with it.

Now, let's talk about about Lie algebras.

Definition 11.7 (Abstract Lie algebra). A Lie algebra $(V, [\cdot, \cdot])$ is a vector space V over $\mathbb{R}$ endowed with a map

$$[\cdot, \cdot]\colon V \times V \to V$$

with the following properties:

- $[\cdot, \cdot]$ is bilinear
- $[\cdot, \cdot]$ is antisymmetric ($[u, v] = -[v, u], \forall\, u, v \in V$)
- $[\cdot, \cdot]$ satisfies the *Jacobi identity*:

$$[[u, v], z] + [[z, u], v] + [[v, z], u] = 0$$

Remark 11.8 (Jacobi). How to remember Jacobi identity: remember $[[u, v], z]$ and then permute cyclically.

Proposition 11.9. *$\mathfrak{X}(\mathbb{R}^n)$ is an (infinite dimensional) Lie algebra with $[u,v](f) = u(v(f)) - v(u(f))$, for $u, v \in \mathfrak{X}(\mathbb{R}^n), f \in C^\infty(\mathbb{R}^n)$. (Note that u and v are vector fields and $[u,v]$ is still a vector field).*

Definition 11.10 (Integral curve)**.** An integral curve for a vector field v is a smooth curve $\phi\colon (a,b) \to \mathbb{R}^n$ satisfying $\dot{\phi}(t) = v_{\phi(t)}$ ($v_{\phi(t)}$ is the vector tangent at $\phi(t)$ for t fixed, remember the previous notation!). Let us suppose $0 \in (a,b)$. Then, $\phi(0)$ is called the starting point of ϕ.

We can also visualize the family of integral curves in the following way.

Definition 11.11 (Flow)**.** The map

$$\begin{aligned} \theta\colon \mathbb{R} \times \mathbb{R}^n &\to \mathbb{R}^n \\ (t,p) &\mapsto \theta_t(p) \end{aligned} \tag{11.5}$$

such that $\dot{\theta}_t(p) = v_{\theta_t(p)}$ is the flow of the vector field v where, if we fix p, $\theta_t(p)$ is the integral curve which passes through p at $t = 0 \in (a,b)$. So the flow satisfies two conditions:

$$\dot{\theta}_t(p) = v_{\theta_t(p)}, \qquad \forall\, p \in \mathbb{R}^n \tag{11.6}$$

$$\theta_0(p) = p, \qquad \forall\, p \in \mathbb{R}^n \tag{11.7}$$

Under the right hypothesis (e.g. Lipschitz hypothesis and smoothness of v) we can prove existence and uniqueness of the solution of such ODEs ($\forall\, p \in \mathbb{R}^n$). By fixing either the time or the starting point of the flow, we can consider two maps:

- $p \mapsto \theta_t(p)$, for each fixed t (we are observing several integral curves at the same time t)
- $t \mapsto \theta_t(p)$, for each fixed p (we are observing the integral curve starting from p, for all times)

Definition 11.12 (Lie derivative of a k-form)**.** Let $z \in \mathfrak{X}(\mathbb{R}^n)$ be a differentiable vector field, ϕ_t its flow and $\omega \in \Omega^k(\mathbb{R}^n)$. Then the Lie derivative of ω is defined as

$$L_z\omega = \frac{d}{dt}(\phi_t^*\omega)\big|_{t=0} \tag{11.8}$$

Remark 11.13. We denoted the flow by the symbol ϕ_t and not ϕ. What we are doing here is not caring about p: $\phi_t^*\omega(\cdot) = \omega(\phi_t(\cdot))$. Useful formula (Cartan's formula): $L_z\omega = (di_z + i_z d)\omega$

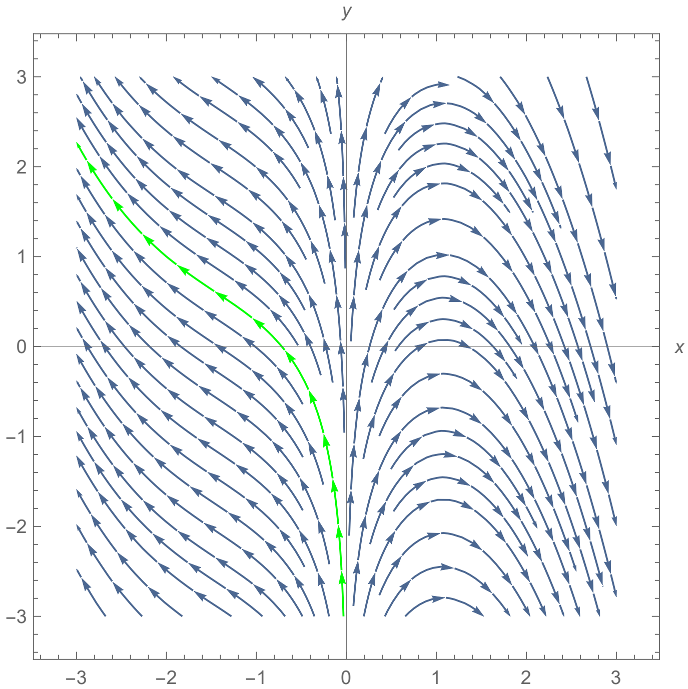


Figure 2: the map $t \mapsto \theta_t(p)$ selects just one integral curve

12 Lie derivative of a vector field

Definition 12.1 (pullback of a vector field)**.** Let φ be a diffeomorphism of $\mathbb{R}^n$ (i.e. a differentiable and invertible map from $\mathbb{R}^n$ to $\mathbb{R}^n$, such that its inverse is differentiable as well). Let $v \in \mathfrak{X}(\mathbb{R}^n)$. Then:

$$\varphi^* v \equiv \varphi_*^{-1} v \tag{12.1}$$

is the pullback of v with φ. In particular, given a flow ϕ and t fixed, $\phi(t, \cdot) = \phi_t(\cdot)$ is a diffeomorphism on $\mathbb{R}^n$ with $\phi^{-1} = \phi(-t, \cdot) = \phi_{-t}(\cdot)$.

Definition 12.2 (Lie derivative of a vector field)**.** Let $u, v \in \mathfrak{X}(\mathbb{R}^n)$. The Lie derivative of v in direction u is

$$L_u v \equiv \frac{d}{dt}(\phi_t^* v)|_{t=0} \tag{12.2}$$

(Remember: $\phi\colon (t,p) \mapsto \phi(t,p), \phi_t^* v\colon (t,p) \mapsto v(\phi(t,p)))$.

Lemma 12.3. *Let u, v be smooth vector fields on $\mathbb{R}^n$ and $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$. Let ϕ_t be the flow of u and let ψ_s be the flow of v. Then*

- $\varphi^* v = \frac{d}{ds}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi$
- $\varphi^* v = v \Leftrightarrow \varphi \circ \psi_s = \psi_s \circ \varphi$ *for all s.*
- $L_u v = 0 \Leftrightarrow \phi_t \circ \psi_s = \psi_s \circ \phi_t$ *for all s, t*

Sketch of the proof.

- We have
$$\frac{d}{ds}|_{s=0}\,\varphi^{-1}\circ\psi_s\circ\varphi(p)\stackrel{(1)}{=}\varphi_*^{-1}v_{\varphi(p)}=(\varphi_*^{-1}v)_p\stackrel{(2)}{=}(\varphi^*v)_p$$
where we used: (1) the definition of flow for the flow φ^{-1} of the vector field u and the fact that ψ_s is the flow of v. And we applied the definitions to the point $\varphi(p)$ instead of p as usual. (2) $\varphi_*^{-1}=\varphi^*$ by definition.

- $$\frac{d}{ds}(\varphi\circ\psi_s)(p)=\varphi_*v_{\psi_s(p)}=(\varphi_*v)_(\varphi\circ\psi_s(p))$$
and so, in a similar way:
$$\frac{d}{ds}(\psi_s\circ\varphi)=v_{\psi_s\circ\varphi(p)}=\frac{d}{ds}(\varphi\circ\psi_s)+O(s)$$
where in the last step we Taylor-expanded with respect to s, using the fact that for $s=0$ we have $\psi_0=\mathbb{1}$, so the thesis is verified. Then we can proceed in the same way for each order of the expansion.

- ($\Leftarrow$): follows from the previous point, and using the definition of Lie derivative for a vector field. ($\Rightarrow$):
$$\frac{d}{dt}\phi_t^*v=\frac{d}{d\varepsilon}|_{\varepsilon=0}\phi_{t+\varepsilon}^*v\stackrel{(1)}{=}\phi_t^*\frac{d}{d\varepsilon}|_{\varepsilon=0}\phi_\varepsilon^*v=\phi_t^*L_uv=0,\forall\, t$$
where in (1) we used that $\phi_{t+\varepsilon}^*=\phi_t^*\circ\phi_\varepsilon^*$. Thus $\phi_t^*v=v\forall\, t$, and by the previous point we have the thesis.

□

Lemma 12.4. *Let u,v be smooth vector fields on $\mathbb{R}^n$ and and let ϕ_t (respectively ψ_s) be the flow of u (respectively v). Then:*

- $L_uv=\frac{\partial^2}{\partial s\partial t}\phi_{-t}\circ\psi_s\circ\phi_t|_{t=0,s=0}$
- $(L_uv)(f)=[u,v](f)=u(v(f))-v(u(f))$ *for all smooth functions f on $\mathbb{R}^n$*

Sketch of the proof.

- $\frac{d}{dt}\phi_t^*v=\frac{\partial}{\partial t}\frac{\partial}{\partial s}\phi_{-t}\circ\psi_s\circ\phi_t$ by the first point of the lemma 12.3, where we put $\varphi^*=\phi_t$.
- See problem sheet 4, exercise 4.

□

Lemma 12.5. *Let u,v be smooth vector fields on $\mathbb{R}^n$ and $\varphi\in\mathrm{Diff}(\mathbb{R}^n)$. Then:*

1. *$[u,v]$ is $\mathbb{R}-$bilinear (i.e. bilinear for a parameter $\lambda\in\mathbb{R}$)*
2. *$[u,v]=-[v,u]$*
3. *The Jacobi identity holds*

4. $[u, fv] = f[u,v] + u(f)v$

5. $\varphi_*[u,v] = [\varphi_* v, \varphi_* u]$

Proof. See problem sheet 4, exercise 4. □

13 Stokes' Theorem on $\mathbb{R}^n$

- For a function f (i.e. a 0-form) on $[a,b] \subset \mathbb{R}$ we have $\int_a^b df = \int_a^b \partial_x f dx = f(b) - f(a)$ (fundamental theorem of calculus).

- for $\omega = a_i dx^i$, a 1-form on $U = [0,1] \times [0,1] \subset \mathbb{R}^2$ we have:

$$\int_S d\omega = \int_S (\partial_{x^1} a_2) dx^1 \wedge dx^2 + (\partial_{x^2} a_1) dx^2 \wedge dx^1 = \int_{\partial S} \omega$$

  where we used the fundamental theorem of calculus.

- More generally, if S is a compact subset of $\mathbb{R}^2$ with piecewise regular boundary ∂S (piecewise homeomorphic to intervals in $\mathbb{R}$) then we obtain by decomposing S in terms of little squares and interpreting $\int_U d\omega$ as a Riemann sum over the square the result

$$\int_U d\omega = \int_{\partial U} \omega$$

  This result generalizes immediately to compact co-dimension zero subsets of $\mathbb{R}^n$ with piecewise regulary boundary

- If M is a compact subset of dimension $m \leq n$ in $\mathbb{R}^n$ (with piecewise regular boundary ∂M), diffeomorphic to a compact subset of $U \subset \mathbb{R}^m$ (i.e. $M = f(U)$) and $\omega \in \Omega^{m-1}(M)$, then:

$$\int_M d\omega = \int_U f^* d\omega = \int_U d f^* \omega = \int_{\partial U} f^* \omega = \int_{\partial M} \omega$$

- More generally, the parametrization of ∂M may be different from that induced by M. Then we have:

$$\int_M d\omega = \int_{\partial M} i^* \omega$$

  where $i \colon \partial M \to M$ is the inclusion map of ∂M into M.

So, the most general result that we achieved is the following:

Theorem 13.1 (Stokes). $\omega \in \Omega^{m-1}(\mathbb{R}^n)$. *Let M be a compact subset of $\mathbb{R}^n$, $dim(M) = m \leq n$, such that M is homeomorphic to a closed subset $U \subset \mathbb{R}^m$. ∂M is the boundary of M and $i \colon \partial M \to M$ is the inclusion map of ∂M into M. Then:*

$$\int_{\partial M} i^* \omega = \int_M d\omega \tag{13.1}$$

Sketch of the proof. First, we notice that U is compact as well because it is homeomorphic to a compact set. Then, the proof uses the naturality of the pullback "$f^*d = df^*$" and the results for open subsets of $\mathbb{R}^n$. [**TO DO**] □

Corollary 13.2 (Fundamental theorem of line integrals). *Let f be a smooth function defined near an oriented curve C in $\mathbb{R}^n$, with endpoints A and B. Then:*

$$\int \nabla f \cdot dx = f(B) - f(A) \tag{13.2}$$

Proof. $\int df = \int \nabla f \cdot dx$ because $\nabla f = \frac{df}{dt}dt = \frac{\partial f}{\partial x^i}$ and $dx^i = \frac{dx^i}{dt}dt$. Moreover, thanks to the Stokes' theorem, we have $\int df = f(B) - f(A)$. □

Corollary 13.3 (Curl theorem or Classical Stokes theorem). *Let v be a differentiable vector field defined near a surface $S \subset \mathbb{R}^3$ with boundary ∂S.*

$$\int_S n \cdot (\nabla \times v)\, dS = \int_{\partial S} v \cdot dx \tag{13.3}$$

where n is the normal vector on the surface at each point.

Proof. Let α be a 1-form in $\mathbb{R}^3$. Since the euclidean metric gives an isomorphism between $\mathbb{R}^3$ and its dual space (see also eq. (7.1)), we can always write α as $\alpha = g(v, \cdot)$, for a certain $v \in \mathbb{R}^3$, where g is the Euclidean metric, i.e. $g(x, y)$ is the standard dot product between x and y. So, $\alpha = g(v, \cdot) = \delta_{ij} v^j dx^i = v_i dx^i$. Then: $d\alpha = (\partial_{x^j} v_i) dx^j \wedge dx^i \overset{(1)}{=} \partial_{x^j} v_i \varepsilon^{ji}_k n^k dS$. Where in (1) we used that $dx^j \wedge dx^i$ is the oriented surface generated by dx^j and dx^i (see also the second point of the example 8.7). Then we used that the oriented surface area can be expressed using the Levi-Civita tensor ε, the normal vector n and the unoriented area element dS. Hence, using that $\nabla \times v = \partial_{x^j} v_i \varepsilon^{ji}_k$:

$$\int_S n \cdot (\nabla \times v)\, dS = \int_S d\alpha \overset{(2)}{=} \int_{\partial S} \alpha = \int_{\partial S} v_i dx^i = \int_{\partial S} v \cdot dx$$

Using the Stokes' theorem in (2). □

Corollary 13.4 (Divergence theorem). *For a smooth vector field v defined on a solid $T \subset \mathbb{R}^3$ with boundary ∂T:*

$$\int_T \nabla \cdot v\, dV = \int_{\partial T} v \cdot n\, dS \tag{13.4}$$

where dV is the unoriented volume element.

Proof. See problem sheet 5, exercise 3. □

14 Poincarè Theorem of 1-forms

Definition 14.1 (Closed and exact forms). If $\omega \in \Omega^k(U)$ such that $d\omega = 0$, then ω is closed. If there exists $\alpha \in \Omega^{k-1}(V), V \subset U$ such that $\omega = d\alpha$ in V then ω is exact.

Proposition 14.2. *The following are equivalent (**note that we are considering just 1-forms!**):*

1. $\omega \in \Omega^1(U)$ *is exact in a connected open subset* $V \subset U$
2. *For any curve* $\gamma\colon (a,b) \to U$, $\int_\gamma \omega$ *depends only on the endpoints* $\gamma(a)$ *and* $\gamma(b)$.
3. $\int_\gamma \omega = 0$ *for any closed curve* γ *in* V

Sketch of the proof. First, (1) $\Rightarrow$ (2) because of Stokes' theorem. Moreover, (2) $\Rightarrow$ (3) using Stokes' theorem again. Now, (3) $\Rightarrow$ (2) : the curve made of the union of γ_1 and γ_2 is a closed curve. Then, if γ is the name of such a closed curve:

$$0 = \int_\gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega$$

where the minus sign comes from the orientation of the curves. Then: $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ where γ_1 and γ_2 are any two curves with the same endpoints. It means that we can consider the integral of ω on any curve and the result wouldn't change, **if** the new curve has the same endpoints of the initial curve. (2) $\Rightarrow$ (1): Let's consider the 1-form $\omega = a_i dx^i$ (when ω has

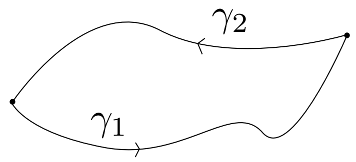


Figure 3: Union of γ_1 and γ_2

this form, we call it a *standard form*). Let's fix a point $p \in V$ and define a function

$$\begin{aligned} f\colon V &\to \mathbb{R} \\ x &\mapsto \int_\gamma \omega \end{aligned}$$

where $x = x(p')$ is the coordinate given by a point p' (i.e. x gives a parametrization) and γ is a curve joining p to p'. By (2), f is well defined (i.e. the definition does not depend on the γ chosen, so we don't need to specify it). Furthermore, $df = \frac{\partial f}{\partial x^i} dx^i$. Since $\omega = a_i dx^i$, we only need to prove that $\frac{\partial f}{\partial x^i} = a_i$. Let's consider the curve γ. We extend it from p' with a straight segment $\Delta\gamma$. In particular, $\Delta\gamma = x + te_i, t \in (-\varepsilon, \varepsilon)$ with e_i any vector of the canonical basis of $\mathbb{R}^n$. We choose ε small such that $\gamma + \Delta\gamma \subset V$. Then:

$$\begin{aligned} \frac{\partial f}{\partial x^i}|_x &= \lim_{t\to 0} \frac{1}{t}\left[f(x+te_i) - f(x)\right] = \lim_{t\to 0}\frac{1}{t}\left[\int_{\gamma+\Delta\gamma} \omega - \int_\gamma \omega\right] = \\ &= \lim_{t\to 0}\frac{1}{t}\int_{\Delta\gamma} \omega = \lim_{t\to 0}\frac{1}{t}\int_0^t a_i(x(t))dt = a_i(x) \end{aligned}$$

where we used $\int \omega = \int a_i \frac{dx^i}{dt} dt$ and the fact that every point belonging to $\Delta\gamma$ can be written as $x + te_i$, so we have $\frac{dx^i}{dt} = 1$ on $\Delta\gamma$. Moreover, in the last equality we Taylor-expanded $a_i(x(t))$ with respect to t. Only the first-order term $a_i(x)$ matters because we

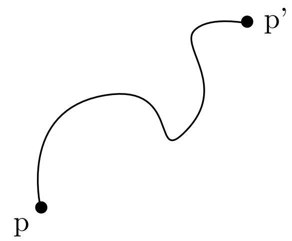


Figure 4: A possible curve γ from p to p'

are considering the limit as $t \to 0$. Then we have the thesis for standard forms. Then, the result holds for any form (in general we could have $a_i = 0$ for some i). □

Remark 14.3 (A closed form is not always exact). If ω is exact, then it is closed (because $d^2 = 0$). But not every closed form in $\Omega^1(U), U$ open subset of $\mathbb{R}^n$ is exact. Cf. $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ in $\mathbb{R}^2$ minus the non-negative x-axis. If γ is a closed curve around the origin of $\mathbb{R}^2$, we have:

$$\int_\gamma \omega = \int_\gamma d\theta = 2\pi$$

and therefore ω cannot be exact by the previous proposition. However, we notice that we have problems only with the origin of $\mathbb{R}^2$. If we consider a subset of $\mathbb{R}^2$ which is far enough from the origin, the form would be an exact form in such subset. Indeed, we say that ω is locally exact, and the general result follows from the next theorem.

Theorem 14.4 (Poincarè theorem for 1-forms on $\mathbb{R}^n$). *Let $\omega \in \Omega^1(U), U \subset \mathbb{R}^n, U$ open. Then $d\omega = 0$ if and only if for each $p \in U$ there is a neighbourhood $V \subset U$ of p and a differentiable function $f\colon V \to \mathbb{R}$ such that $\omega = df$.*

Remark 14.5. Using the Poincarè theorem for 1-forms, we can extend the definition of the integral of a closed 1-form along a **continuous** path (until now, we have always assumed the our paths were piecewise differentiable). In fact, assume that $\omega \in \Omega^1(U), d\omega = 0$, and γ such that:

$$\gamma\colon [0,1] \to U$$

is a differentiable map. Now, we choose a partition of $[0,1]$, i.e. a collection of points $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ such that the restriction of γ to the interval (t_i, t_{i+1}) is contained in a ball B_i where ω is exact. In particular:

$$\omega = df_i, \text{ for } f_i\colon B_i \to \mathbb{R}$$

Then:

$$\int_\gamma \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)]$$

If γ is only continuous, we could still consider such a partition, and still define

$$\int_\gamma \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)]$$

The integral of γ is well defined because the definition is independent from the choice of our partition: if P is one partition and P' is a refinement of P (i.e. it is the same partition plus an extra point $t' \in (t_i, t_{i+1})$ for some i), then:

$$[f_i(t_{i+1}) - \cancel{f_i(t')}] + [\cancel{f_i(t')} - f_i(t_i)] = [f_i(t_{i+1}) - f_i(t_i)]$$

Then the integral does not change if we consider a refinement. If we consider a general partition P', we can add every point of the partition P to P', so that we get a refinement of P' that we will be called P''. The integral on the partition P' has the same value of the integral on the partition P'' by the above argument. Now, we can add every point of the partition P' to P, so to get the partition P'' again, but now we can see P'' as a refinement of P. Then the integral on P and on P'' are the same. Then also the integrals on P and P' are the same.

Now, we want to extend the above theorem to k-forms.

Definition 14.6 (Contractible set)**.** An open subset $U \subset \mathbb{R}^n$ is contractible to some point $p_0 \in U$ if there exists a differentiable map

$$\begin{aligned} H\colon U \times [0,1] &\to U \\ (p,t) &\mapsto H(p,t) \end{aligned} \tag{14.1}$$

such that $H(p,1) = p, H(p,0) = p_0, \forall\, p \in U$

Remark 14.7. If U is contractible, we can associate a k-form $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ to every $\omega \in \Omega^k(U)$. $\bar{\omega}$ is defined as

$$\bar{\omega} = H^*\omega \tag{14.2}$$

On the other hand, any $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ has a unique decomposition of the form

$$\bar{\omega} = \omega_1 + dt \wedge \eta \tag{14.3}$$

with $i_{\partial_t}\omega_1 = 0$ and $i_{\partial_t}\eta = 0$. Conversely, we can associate a k-form $\omega \in \Omega^k(U)$ to each $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ with the help of the inclusion map

$$\begin{aligned} i_t\colon U &\to U \times \mathbb{R} \\ p &\mapsto i_t(p) = (p,t) \end{aligned} \tag{14.4}$$

Then, $i_t^*\bar{\omega} \in \Omega^k(U)$ if $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$

Furthermore, let's define the map

$$I\colon \Omega^k(U\times\mathbb{R}) \to \Omega^{k-1}(U) \qquad (14.5)$$
$$\eta \mapsto I\eta$$

such that

$$(I\eta)(z_1,\dots,z_{k-1}) = \int_0^1 \eta(p,t)(\partial_t, i_{t^*}z_1,\dots,i_{t^*}z_{k-1})dt$$

Proof. Let's choose coordinates $\{x^1,\dots,x^n,t\}$ for $U\times\mathbb{R}$. Then we write $\bar\omega$ on the basis:

$$\bar\omega = \sum_{i_1<\ldots<i_k} a_{i_1\cdots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k} + \sum_{i_1<\ldots<i_{k-1}} b_{i_1\ldots i_{k-1}}dt\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_{k-1}}$$

(We can always do this, the coefficients could also be trivial). Now, we want to integrate this form. Let:

$$i_t\colon U \to U\times\mathbb{R}$$
$$p \mapsto (p,t)$$

i_t is the inclusion map (it "includes" U into $U\times\mathbb{R}$ at t)...TO DO ☐

Lemma 14.8.

$$i_1^*\bar\omega - i_0^*\bar\omega = d(I\bar\omega) + I(d\bar\omega) \qquad (14.6)$$

Indeed, since $H\circ i_1 = Id$ and $H\circ i_0 = p_0$, $\forall\, p\in U$ we have

$$\omega = (H\circ i_1)^*\omega = i_1^*\bar\omega$$

and

$$0 = (H\circ i_0)^*\omega = i_0^*\bar\omega$$

Then we can extend Poincarè lemma to k-forms:

Theorem 14.9 (Poincarè Lemma)**.** *Let $k\geq 1$. Let U be a contractible, open subset of $\mathbb{R}^n$ and $\omega\in\Omega^k(U)$ with $d\omega = 0$. Then there exists a $(k-1)$-form $\alpha\in\Omega^{k-1}(U)$ such that $\omega = d\alpha$.*

Question: For $\omega\in\Omega^1(U)$, when is $\int_\gamma\omega$ independent of the choice of γ?

Definition 14.10 (Homotopy between curves)**.** Two continuous curves γ_1 and γ_2, $\gamma_i\colon[a,b]\to U, i=1,2, U\subset\mathbb{R}^n$ are freely homotopic if there exists a continuous map H

$$\begin{aligned} &H\colon [a,b]\times[0,1]\to U && \text{such that:}\\ &H(s,0) = \gamma_1(s), && \forall\, s\in[a,b]\\ &H(s,1) = \gamma_2(s), && \forall\, s\in[a,b]\end{aligned}$$

Definition 14.11 (Homotopy between closed curves with same endpoints)**.** Two continuous curves γ_1 and γ_2, $\gamma_i\colon [a,b] \to U, i = 1,2, U \subset \mathbb{R}^n$, with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$ are homotopic relatively to $\{\gamma_1(a), \gamma_2(b)\}$ if there exists a continuous map H

$$
\begin{aligned}
&H\colon [a,b] \times [0,1] \to U && \text{such that:}\\
&H(s,0) = \gamma_1(s), && \forall\, s \in [a,b]\\
&H(s,1) = \gamma_2(s), && \forall\, s \in [a,b]\\
&H(a,t) = \gamma_1(a) = \gamma_2(a), && \forall\, t \in [0,1]\\
&H(b,t) = \gamma_1(b) = \gamma_2(b), && \forall\, t \in [0,1]
\end{aligned}
$$

Theorem 14.12. *Let $\omega \in \Omega^1(U)$, with $d\omega = 0$ (closed), and γ_1, γ_2 be two homotopic curves (as in the previous definition). then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{14.7}$$

What if $\gamma_1(a) \neq \gamma_2(a), \gamma_1(b) \neq \gamma_2(b)$

Definition 14.13 (Homotopy between closed curves)**.** $\gamma_1, \gamma_2\colon [a,b] \to U$, γ_i closed curves, are freely homotopic if there exists a continuous map

$$
\begin{aligned}
&H\colon [a,b] \times [0,1] \to U && \text{such that:}\\
&H(s,0) = \gamma_1(s) && \forall\, s \in [a,b]\\
&H(s,1) = \gamma_2(s) && \forall\, s \in [a,b]\\
&H(a,t) = H(b,t) && \forall\, t \in [0,1]
\end{aligned}
$$

Proposition 14.14. *If ω is a closed 1-form on U, γ_1 and γ_2 two closed curves, freely homotopic in U, then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{14.8}$$

In particular, if γ_1 is freely homotopic to a point, then $\int_{\gamma_1} \omega = 0$

Definition 14.15 (Simply connected set)**.** A connected open set $U \subset \mathbb{R}^n$ is simply connected if every continuous closed curve in U is freely homotopic to a point in U.

Example 14.16. $\mathbb{R}^n$, the unitary ball in $\mathbb{R}^n$ and its homeomorphic images are simply connected

Remark 14.17 (Contractible vs. simply connected). "Contractible $\Longrightarrow$ simply connected" (why?), but "Simply connected $\not\Longrightarrow$ contractible" (cf. S^2).

Remark 14.18. Every closed form on a simply connected subset U of $\mathbb{R}^n$ is exact.

Lemma 14.19. *A connected open subset U of $\mathbb{R}^n$ is simply connected if every closed curve in U is homotopic to a point in U.*

We can limit ourselves to consider continuous curves, thanks to the two following results:

Theorem 14.20 (Whitney approximation on $\mathbb{R}^n$)**.** *If γ is a continuous map between $U, V \subsetneq \mathbb{R}^n$, then γ is homotopic to a smooth map $\tilde{\gamma}$. If γ is smooth on a closed subset A of U, then the homotopy can be taken relatively to A.*

Theorem 14.21. *If γ_1 and γ_2 are homotopic maps between U and V then they are smoothly homotopic.*

15 de Rham Cohomology

Let U be an open subset of $\mathbb{R}^n$. We can think of $\Omega^k(U)$ as a vector space over $\mathbb{R}$. Indeed, if $\alpha, \beta \in \Omega^k(U)$, then $a\alpha + b\beta \in \Omega^k(U), \forall\, a, b, \in \mathbb{R}$. And also the remaining properties of a vector space are satisfied.

Remark 15.1. We say that $\Omega^k(U, \mathbb{Z}) = \{k\text{-forms on } U \text{ with coefficients in } \mathbb{Z}\}$ forms a group (and not a vector space) since $\mathbb{Z}$ is not a field. In contrast, $\Omega^k(U, \mathbb{R})$ is a vector space.

Definition 15.2. Let $U \subset \mathbb{R}^n$, U open, $\dim(U) = m \leq n$. Then:

- The set of closed k-forms is the k-th *cocycle group* $Z^k(U, \mathbb{R})$ (it is a group with respect to addition).
- The set of exact k-forms is the k-th *coboundary group* $B^k(u, \mathbb{R})$
- The k-th *de Rham cohomology group* $H^k(U, \mathbb{R})$ is defined as:

$$H^k(U, \mathbb{R}) = {}^{Z^k(U,\mathbb{R})}\!/_{B^k(U,\mathbb{R})}$$

Remark 15.3. $H^k(U, \mathbb{R})$ contains the closed k-forms defined on U which are not exact. See also the example 5.7.

Example 15.4. Let's consider the following examples:

- If U is contractible, then $H^k(U, \mathbb{R}) = \{0\}$ by Poincarè lemma.
- $U = \mathbb{R}^2 \setminus \{0\}$, $H^0(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \mathbb{R}$ (constant functions)
- $H^1(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \mathbb{R}$

- $H^2(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \{0\}$
- Torus T: $H^0(T) = \mathbb{R}$, $H^1(T) = \mathbb{R} \oplus \mathbb{R}$, $H^2(T) = \mathbb{R}$

Definition 15.5. $M \subset \mathbb{R}^n, \Omega^*(M, \mathbb{R}) = \oplus_{k=0}^n \Omega^k(M, \mathbb{R})$

Remark 15.6. $\wedge \colon \Omega^* \times \Omega^* \to \Omega^*$ endows Ω^* with the structure of a ring.

Definition 15.7 (de Rham Complex)**.** The de Rham Complex is defined by $\Omega^*(M, \mathbb{R})$ together with the sequence

$$\xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} 0$$

with $Im(d_k) \subset Ker(d_{k+1})$ since $d^2 = d \circ d = 0$

Remark 15.8. $H^k(M) = {}^{Ker(d_{k+1})}\!/_{Im(d_k)}$

Definition 15.9 (exact sequence)**.** If $H^k(M, \mathbb{R}) = 0, k = 1, \ldots, n$, then

$$\xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \cdots$$

is an exact sequence if and only if $Ker(d_{k+1}) = Im(d_k)$

Remark 15.10. For $k = 0$, $\Omega^0 = \{\text{functions}\}$, then $H^0 = 0$

Definition 15.11 (Cohomology ring)**.** The cohomology ring is defined as

$$H^*(M, \mathbb{R}) = \oplus_{k=0}^n H^k(M, \mathbb{R})$$

Example 15.12. $H^*(T^2, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$, where T^2 is the torus.

Remark 15.13. If $\phi \colon U \to V$, U, V subsets of $\mathbb{R}^n$, ϕ diffeomorphism, then $\phi^* \colon \Omega^k(V) \to \Omega^k(U)$ is the pullback of ϕ on k forms. By the construction above, we can also define such pullback on H^k, as $\phi^* \colon H^k(V) \to H^k(U)$, which maps closed-but-not-exact forms to closed-but-not-exact forms. If ϕ is a diffeomorphism, then ϕ^* is an isomorphism. Thus, $H^k(V) \cong H^k(U)$. Question: what if U and V are not diffeomorphic, but only homotopic equivalent? We want to show that the final result is the same.

Definition 15.14 (Homotopy between maps)**.** Two maps $\phi, \psi \colon U \to V$ are homotopic if there exists a continuous map H

$$\begin{aligned} H\colon [0,1]\times U &\to V && \text{such that:}\\ H(0,\cdot)\colon U &\to V && \text{with } H(0,\cdot)=\phi(\cdot)\\ H(1,\cdot)\colon U &\to V && \text{with } H(1,\cdot)=\psi(\cdot) \end{aligned}$$

Definition 15.15 (Homotopy equivalence between sets)**.** Two subsets U and V of $\mathbb{R}^n$ are homotopic equivalent if there exist continuous maps $f\colon U \to V$ and $g\colon V \to U$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity in U and V, respectively. f and g are called homotopy equivalences.

Example 15.16. Let's consider the following examples:

- Any homeomorphism $\phi\colon U \to V$ with homotopy inverse (i.e. inverse up to a homotopy) ϕ^{-1} is a homotopy equivalence, but the converse is not always true (a disk is homotopy equivalent to a point, but it's not homeomorphic to a point)
- A circle is homotopy equivalent to $\mathbb{R}^2 \setminus \{0\}$
- S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$
- A solid torus is homotopy equivalent to a tea cup

Lemma 15.17. *Let $\phi, id\colon V \to V$ be two smoothly homotopic maps. Then $\phi^*|_{H^*(V,\mathbb{R})} = id^*|_{H^*(V,\mathbb{R})}$*

Theorem 15.18. *Let $\phi\colon U \to V$ be a homotopy equivalence between U and V with homotopy inverse ψ. Then ϕ^* is an isomorphism between $H^k(U)$ and $H^k(V)$*

Theorem 15.19. *Let $\phi\colon U \to V$ be a homotopy equivalence between U and V with homotopy inverse $\psi\colon V \to U$. then ϕ^* induces an isomorphism $\hat{\phi}^*$ such that:*

$$H^n(V,\mathbb{R}) \cong H^n(U,\mathbb{R}), \forall\, n \tag{15.1}$$

Definition 15.20 ((co)chain map)**.** Let A^* and B^* be two (co)chain complexes (a (co)chain complex is a generalization of a de Rham complex, e.g. think about $A^* = \Omega^*$). A (co)chain map $\hat{\phi}^*\colon A^* \to B^*$ is a collection of maps $\phi^*\colon A^n \to B^n$ s.t. $d \circ \phi^* = \phi^* \circ d\colon A^n \to B^{n+1}$. We often denote $\hat{\phi}^*$ by ϕ^*.

Example 15.21. Let's consider:

$$\begin{aligned} A^* &= \Omega^*(U,\mathbb{R}) = \Omega^0(U,\mathbb{R}) \oplus \Omega^1(U,\mathbb{R}) \oplus \dots\\ B^* &= \Omega^*(V,\mathbb{R}) = \Omega^0(V,\mathbb{R}) \oplus \Omega^1(V,\mathbb{R}) \oplus \dots \end{aligned}$$

And we consider $\hat{\phi}^*\colon A^* \to B^*$, $\phi^*\colon \Omega^0(U,\mathbb{R}) \to \Omega^0(V,\mathbb{R})$, $\phi^*\colon \Omega^1(U,\mathbb{R}) \to \Omega^1(V,\mathbb{R})$, etc.

Definition 15.22. A short exact sequence (*SES*) is a collection of (co)chain complexes A^*, B^*, C^* and (co)chain maps $\phi^*\colon A^n \to B^n, \psi^*\colon B^n \to C^n$ such that for each n:

$$0 \longrightarrow A^n \xrightarrow{\phi^*} B^n \xrightarrow{\psi^*} C^n \longrightarrow 0 \tag{15.2}$$

is exact

Remark 15.23. Remember the definition of exact sequence: it gives a condition on the kernel and the range of the maps (cf. definition 15.9). By this condition, we have that ϕ^* must be an injective map, and ψ^* must be a surjective map in the above definition.

Example 15.24. Let's consider the following example:

$$\begin{array}{ccccccl}
\Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \longrightarrow \dots & \} \textit{ sequence} \\
{\scriptstyle\phi^*}\big\downarrow & & {\scriptstyle\phi^*}\big\downarrow & & {\scriptstyle\phi^*}\big\downarrow & & \\
\Omega^0(V) & \xrightarrow{d} & \Omega^1(V) & \xrightarrow{d} & \Omega^2(V) & \longrightarrow \dots & \} \textit{ sequence} \\
{\scriptstyle\psi^*}\big\downarrow & & {\scriptstyle\psi^*}\big\downarrow & & {\scriptstyle\psi^*}\big\downarrow & & \\
\Omega^0(W) & \xrightarrow{d} & \Omega^1(W) & \xrightarrow{d} & \Omega^2(W) & \longrightarrow \dots & \} \textit{ sequence} \\
\underbrace{\quad}_{\textit{sequence}} & & \underbrace{\quad}_{\textit{sequence}} & & \underbrace{\quad}_{\textit{sequence}} & &
\end{array}$$

Lemma 15.25 (zig-zag lemma)**.** *ϕ^*, ψ^* as in the previous definitions. Then there exists a linear map δ such that:*

$$\begin{aligned}\dots \longrightarrow H^{n-1}(C^*) \xrightarrow{\delta} H^n(A^*) \xrightarrow{\phi^*} H^n(B^*) \xrightarrow{\psi^*} \\ \xrightarrow{\psi^*} H^n(C^*) \xrightarrow{\delta} H^{n+1}(A^*) \xrightarrow{\phi^*} \dots\end{aligned}$$

is an exact sequence.

Example 15.26. See also example 15.24.

$$\begin{tikzcd}
H^0(U) \arrow[r, "d"] \arrow[d, "\phi^*"'] & H^1(U) \arrow[r, "d"] \arrow[d, "\phi^*"'] & H^2(U) \arrow[r] \arrow[d, "\phi^*"'] & \dots \\
H^0(V) \arrow[r, "d"] \arrow[d, "\psi^*"'] & H^1(V) \arrow[r, "d"] \arrow[d, "\psi^*"'] & H^2(V) \arrow[r] \arrow[d, "\psi^*"'] & \dots \\
H^0(W) \arrow[r, "d"] \arrow[uur, dotted, bend left, "\delta"] & H^1(W) \arrow[r, "d"] \arrow[uur, dotted, bend left, "\delta"] & H^2(W) \arrow[r] \arrow[uur, dotted, bend left, "\delta"] & \dots
\end{tikzcd}$$

Theorem 15.27 (Mayer-Vietoris). *Let $M \subset \mathbb{R}^n$ such that $M = f(U) \cup g(V), U, V$ open subsets in $\mathbb{R}^m, m \leq n$. f, g homeomorphisms. Let:*

$$i \colon U \cap V \to U \ \textit{(inclusion)}$$

$$j \colon U \cap V \to V \ \textit{(inclusion)}$$

Let:

$$\begin{aligned}(f^* \oplus g^*) \colon \Omega^k(M) &\to \Omega^k(U) \oplus \Omega^k(V) \\ \omega &\mapsto (f^*(\omega), g^*(\omega))\end{aligned}$$

$$\begin{aligned}(i^* - j^*) \colon \Omega^k(U) \oplus \Omega^k(V) &\to \Omega(U \cap V) \\ (\omega, \eta) &\mapsto i^*\omega - j^*\eta\end{aligned}$$

Then, for each k there exists a linear map δ such that:

$$\dots \longrightarrow H^k(M) \overset{f^* \oplus g^*}{\longrightarrow} H^k(U) \oplus H^k(V) \overset{i^* - j^*}{\longrightarrow} H^k(U \cap V) \overset{\delta}{\longrightarrow} H^{k+1}(M) \longrightarrow \dots$$

is exact.

16 Submanifolds of $\mathbb{R}^n$

Premise: We want to go outside $\mathbb{R}^n$, and analyze more general topologies. For now, we can think about them as generalizations of curves and surface of $\mathbb{R}^n$, a proper definition will come later. Note that what we studied until now can often be extended to manifolds: with sufficient conditions, manifolds can be "embedded" in $\mathbb{R}^n$. We will give four equivalent definitions of a submanifold M of dimensions m in $\mathbb{R}^n$ $(m \leq n)$.

Definition 16.1 ((a) Submanifold - Local parametrization). $\forall p \in M \subset \mathbb{R}^n, \exists$ a neighbourhood V, $p \in V$, $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m, U, V$ open, $m \leq n$, and a smooth map $\phi \colon U \to \mathbb{R}^n$ such that:

- $\phi \colon U \to M \cap V$ is a homeomorphism,
- $\phi_*(x) \colon \mathbb{R}^m \to \mathbb{R}^n$ is injective $\forall x \in U$

Definition 16.2 ((b) Submanifold - Locally flat). $\forall p \in M, \exists$ a neighbourhood $V \subset \mathbb{R}^n, p \in V$ and a neighbourhood $W \subset \mathbb{R}^n$, $0 \in W$, and a diffeomorphism $\Phi, \Phi \colon W \to V$ such that

$$\Phi\left(W \cap (\mathbb{R}^m \times \{0\}^{n-m})\right) = V \cap M$$

Definition 16.3 ((c) Submanifold - Locally a zero set). $\forall p \in M, \exists$ a neighbourhood $V \subset \mathbb{R}^n, V$ open, and a smooth map $F \colon V \to \mathbb{R}^{n-m}$ such that

$$V \cap M = \{x \in V \mid F(x) = 0\}$$

and $F_* \colon V \to \mathbb{R}^{n-m}$ is surjective.

Definition 16.4 ((d) Submanifold - Locally a graph)**.** $\forall p \in M, \exists$ a neighbourhood $V \subset \mathbb{R}^n$ and a permutation $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ and $U \subset \mathbb{R}^m, U$ open, together with a smooth map $g\colon U \to \mathbb{R}^{n-m}$ such that

$$\begin{aligned} V \cap M = \{(x_{\sigma(1)} \cdots x_{\sigma(n)}) \,|\, &(x_1,\dots,x_m) \in U \\ &\text{and } (x_{m+1},\dots,x_n) = g(x_1,\dots,x_m)\} \end{aligned}$$

(g is called a *graph*).

Theorem 16.5. *The four definitions above are equivalent.*

Corollary 16.6. *Let:*

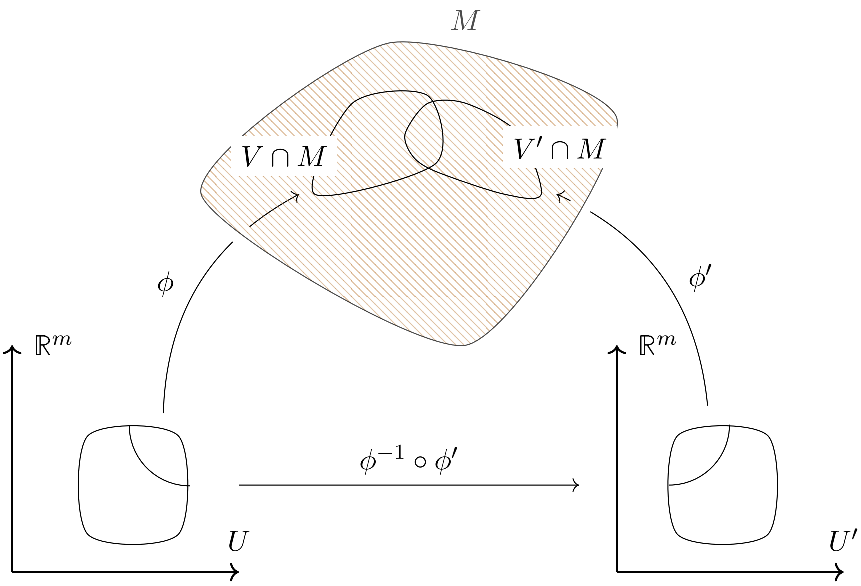
$$\phi\colon \mathbb{R}^m \supset U \to V \cap M$$

$$\phi'\colon \mathbb{R}^m \supset U' \to V' \cap M$$

be local parametrizations. then

$$\phi^{-1} \circ \phi'\colon (\phi')^{-1}(V \cap V' \cap M) \to \phi^{-1}(V \cap V' \cap M)$$

is a diffeomorphism.



Example 16.7 (Submanifolds of $\mathbb{R}^n$)*.* Let's consider some examples.

1. An open subset of dimension n in $\mathbb{R}^n$ is a submanifold (by definition (a), we just take $\phi = id$). For instance $B^n \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$.

2. $S^{n-1} = \partial B^n$ is a submanifold of $\mathbb{R}^n$. Indeed, by definition (c), it is the zero set of the function:

$$F(\underline{x}) = (x^1)^2 + (x^2)^2 + \ldots + (x^n)^2 - 1 = \underline{x}^2 - 1, \underline{x} \in \mathbb{R}^n$$

   and F_* is surjective: $F_*(\underline{x}) = 2\underline{x}$

3. $O(n) = \{A \mid AA^t = E\} \subset Mat(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, with E identity matrix. $O(n)$ is a submanifold of dimension $\frac{n(n-1)}{2}$. Indeed, using definition (c), it is the zero set of the function:

$$\begin{aligned} F \colon Mat(n \times n, \mathbb{R}) &\to Symm(n, \mathbb{R}) \\ A &\mapsto AA^t - E \end{aligned}$$

What is more, F_* is surjective. In order to prove that, we prove that "$F_*|_A(X) = 0, \forall A \Rightarrow X = 0$" (then we know that for a linear map L on a vector space V, $\dim(V) = \dim Ker(L) + \dim Ran(L)$, so the dimension of the range of $F_*|_A$ must be $\dim Mat(n \times n, \mathbb{R})$, so the map is surjective). In fact:

$$\begin{aligned} F_*|_A &= \frac{d}{dt}|_{t=0} F(A + tX) = \frac{d}{dt}|_{t=0} \left[AA^t + tAX^t + tXA^t + t^2XX^t\right] = \\ &= AX^t + XA^t \end{aligned}$$

And the solution of: $AX^t + XA^t = S \in Symm(n, \mathbb{R})$ is $X = \frac{SA}{2}$ because $\frac{AA^t}{2}S + S\frac{AA^t}{2} = S$. Then:

$$AX^t + XA^t = 0 \Longrightarrow X = 0$$

Now, we want to use diffeomorphisms like those in the corollary 16.6 in order to introduce the concept of manifold.

Definition 16.8 (Atlas)**.** An (n-dimensional, smooth) atlas $\mathcal{A}$ on a set M is a collection of maps (called charts)

$$\begin{aligned} \phi_\alpha \colon \mathbb{R}^n &\overset{\sim}{\to} M \\ U_\alpha &\mapsto W_\alpha \end{aligned} \tag{16.1}$$

such that:

- $\cup_{\alpha \in I} W_\alpha = M$
- $\forall\, \alpha, \beta \in I$ with $W_\alpha \cap W_\beta \neq \emptyset$,

  $$\phi_\beta^{-1} \circ \phi_\alpha \colon \phi_\alpha^{-1}(W_\alpha \cap W_\beta) \to \phi_\beta^{-1}(W_\alpha \cap W_\beta)$$

  is a diffeomorphism.

where $\alpha \in I, I$ index set, and the $\sim$ above the arrow means that ϕ_α is bijective.

Definition 16.9 (Equivalence relation on atlases)**.** Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are equivalent $\Leftrightarrow \mathcal{A} \cup \mathcal{A}'$ is an atlas ($\Leftrightarrow \phi_\beta^{-1} \circ \phi'_\alpha$ is a diffeomorphism, $\forall\, \phi'_\alpha \in \mathcal{A}', \phi_\beta \in \mathcal{A}$).

Definition 16.10 (**Preliminary** definition of manifold)**.** A manifold is a set M with an equivalence class of atlases.

Remark 16.11. The definition 16.10 above is "*preliminary*" because we have not specified anything about the topology yet (we are still working on Euclidean topology). Moreover, in every equivalence class $\exists!$ maximal atlas (i.e. such that, if combined with an other atlas, it can't get any bigger). We also notice that, if the charts are smooth enough, we can pullback "everything" (e.g. all our vector fields, differential forms, etc. defined for subset of $\mathbb{R}^n$). However, in that case, everything is defined *locally*. In order to patch them all together, we need some other result (like the partition of unity, coming soon).

Example 16.12 (Projective space). $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ($\mathbb{K}$ is some field).

$$\mathbb{K}P^n \equiv {}^{(\mathbb{K}^{n+1} \setminus \{0\})}\big/_{\sim}$$

where

$$(x_0, \dots, x_n) \sim (x_0', \dots, x_n') \iff$$
$$\exists\, \lambda \in \mathbb{K} \setminus \{0\} \text{ such that } (\lambda x_0, \dots, \lambda x_n) = (x_0', \dots, x_n')$$

For instance, $\mathbb{C}P^1 = S^2$ (every point on a line passing through the origin is identified with the point on such line at distance 1 from the origin). We use the notation $[x_0, \dots, x_n]$ for one equivalence class. Let's consider the atlas $\mathcal{A} = \{\phi_i \colon \mathbb{K}^n \to \mathbb{K}P^n\}$. Where

$$\phi_i \colon \mathbb{K}^n \to \mathbb{K}P^n$$
$$(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto [x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n]$$

And $\phi_i(\mathbb{K}^n) = \{[x_0, \dots, x_n] \in \mathbb{K}P^n \mid \text{ i-th entry is } \neq 0\}$. We notice that $\mathcal{A}$ is an atlas because ϕ_i satisfies the properties of the definition 16.8:

- ϕ is a bijection and its inverse is

$$\phi_i^{-1} \colon \phi_i(\mathbb{K}^n) \to \mathbb{K}^n$$
$$[x_0, \dots, x_{i-1}, \underbrace{x_i}_{\neq 0}, \dots, x_n] \mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

- ϕ is also a diffeomorphism, because $\phi_j^{-1} \circ \phi_i$, defined as

$$\phi_j^{-1} \circ \phi_i \colon \phi_i^{-1}\left(\phi_i(\mathbb{K}^n) \cap \phi_j(\mathbb{K}^n)\right) \to \phi_j^{-1}\left(\phi_i(\mathbb{K}^n) \cap \phi_j(\mathbb{K}^n)\right)$$
$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{x_j}, \dots, \underbrace{\frac{1}{x_j}}_{i-th}, \dots, \frac{x_n}{x_j}\right)$$

  is a diffeomorphism (where $x_i \neq 0, x_j \neq 0$) because it is a differentiable map and the inverse is again differentiable.

Remark 16.13 (Extension of analysis to manifolds)*.* The existence of an atlas allows us to define concepts from analysis to manifolds. For instance, we can define continuous and smooth functions on manifolds using the concepts of continuity and smoothness that we use in $\mathbb{R}^n$.

Definition 16.14 (Continuous/smooth function on a manifold)**.** Let M be a manifold, a function $f\colon M \to \mathbb{R}$ is a continuous [smooth] function if and only if $f \circ \phi_\alpha\colon U_\alpha \to \mathbb{R}$ is continuous [smooth] function $\forall\, \alpha$. Such a function is well-defined, since the definition is independent on the local parametrization chosen, thanks to the definition of atlas.

Remark 16.15 (cut-off functions)*.* We can construct functions on a manifold in the following way: we consider a function defined on an open set in M which is homeomorphic to an open set in $\mathbb{R}^n$. Then we extend it to zero outside such open set, but in order to have a differentiable function we smoothly bring it to zero using the so-called cut-off functions. For instance, a smooth function from $\mathbb{R}$ to $[0,1] \subset \mathbb{R}$ such that:

$$h(x) = \begin{cases} 1, & x \leq 1 \\ \text{anything}, & 1 \leq x \leq 2 \\ 0, & x \geq 2 \end{cases}$$

is a cut-off function. For example, given

$$a(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x \geq 0 \end{cases}$$

we can consider

$$h(x) = 1 - \frac{a(x)}{a(x) + a(1-x)}$$

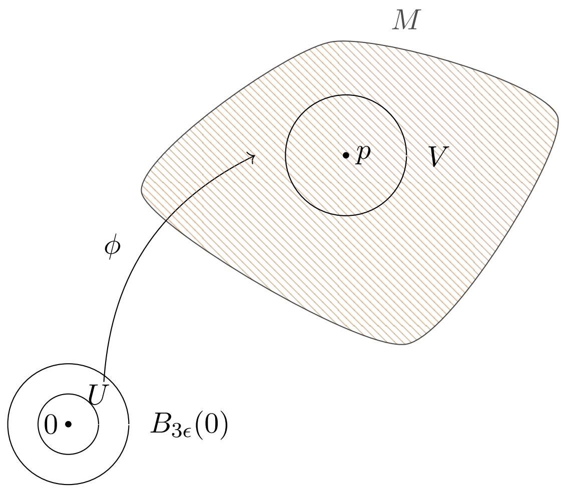
Then we use the function

$$h_\epsilon(x) \equiv h\left(\frac{x}{\epsilon}\right) = \begin{cases} 1, & x \leq \epsilon \\ 0, & x \geq 2\epsilon \end{cases}$$

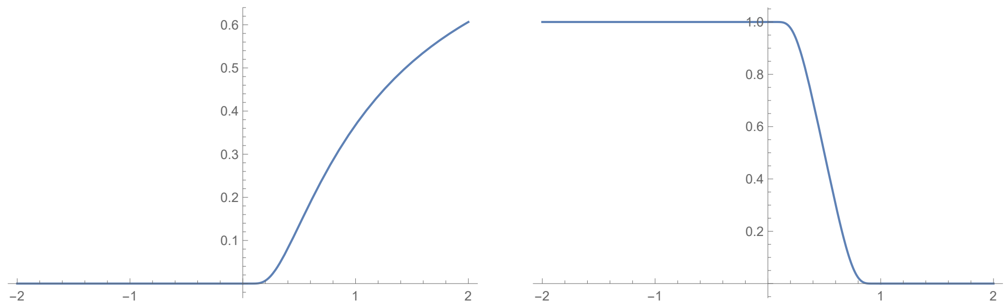
Now, for $p \in M$, let

$$\phi\colon \mathbb{R}^n \supset U \to V \subset M$$

such that $\phi(0) = p$ (where $0 \in U$) and $\epsilon > 0$ such that $B_{3\epsilon} \supset U$.



Then $g(x) = h_\epsilon(|x|)$ defines a smooth function $g\colon \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{supp}(g) = \overline{\{x \mid g(x) \neq 0\}} \subset U$ and $g \circ \phi^{-1}\colon V \to \mathbb{R}$ extends (by zero) to a function $f \in \mathbb{F}(M)$ with $\mathrm{supp} f \subset V$ and $f \equiv 1$ in a neighbourhood of p. In particular, this shows that $\mathbb{F}(M)$ is infinite-dimensional.



(a) $a(x)$ (b) $h(x)$

Definition 16.16 (Vector fields on manifolds)**.** Vector fields are defined as before (we just replace $\mathbb{F}(\mathbb{R}^n)$ with $\mathbb{F}(M)$):

$$\mathrm{Der}\big|_p \mathbb{F}(M) = T_pM \ni v_p\colon \mathbb{F}(M) \to \mathbb{R}$$

And

$$\mathfrak{X}(M) = \mathrm{Der}\,\mathbb{F}(M) = TM \ni v\colon \mathbb{F}(M) \to \mathbb{F}(M)$$

$\mathbb{R}$-linear and satisfying Leibnitz rule.

Remark 16.17 (Representation of v in U_α). Everything is done in $U_\alpha \subset \mathbb{R}^n$ (chart). Representation of v in U_α:

$$v_{(\alpha)} = v^i_{(\alpha)} \partial_{x_i}$$

where

$$v^i_{(\alpha)}(x) = v^i \circ \phi_\alpha(x)$$

Notice that cut-off functions allow us to extend any smooth function on V_α (for some $\alpha \in I$) to all of M through extension by zero outside V_α. Suppose we are given a function on M. How do we decide if it is continuous (or smooth)?

Proposition 16.18 (Partition of unity)**.** *Let M be a compact manifold and let $\{V_\alpha\}$ be a covering of M. Then there exists a family of differentiable functions $\varphi_1, \dots, \varphi_m$ such that:*

- $\sum_{i=1}^{m} \varphi_i \equiv 1$
- $0 \le \varphi_i \le 1$ *and supp*$(\varphi_i) \subset V_\alpha$ *for some* $\alpha \in I$

(without proof)

Remark 16.19. Definition of compactness for a manifold: coming soon.

Definition 16.20 (Partition of unity)**.** The family $\{\varphi_i\}$ defined above is said to be a partition of unity subordinate to the covering $\{V_\alpha\}$.

Now, if $f\colon M \to \mathbb{R}$, we can consider $f_i = \varphi_i f\colon v_\alpha \to \mathbb{R}$ for some $\alpha \in I$.

Definition 16.21 (Differential forms on M)**.**

$$\begin{aligned} \Omega^k(M) \ni \omega\colon \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) &\to \mathbb{F}(M) \\ (v_1, \dots, v_k) &\mapsto \omega(v_1, \dots, v_k) \end{aligned} \tag{16.2}$$

linear, skew-symmetric (i.e. alternating) as already defined.

Remark 16.22 (Representation of ω in U_α)*.* Representation of ω in U_α:

$$\omega_{(\alpha)} = \sum_{i_1 < \dots < i_k} (a_{i_1 \cdots i_k})_{(\alpha)} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

where $\{x^i\}$ are the coordinates on U_α. Moreover:

$$\omega_{(\alpha)}((v_1)_\alpha, \dots, (v_k)_\alpha) = \pm \sum_{i_1 < \dots < i_k} (a_{i_1 \cdots i_k})_\alpha (v^{i_1})_\alpha \cdots (v^{i_k})_\alpha$$

where the $\pm$ sign depends on the orientation used.

Definition 16.23 (Curve on a manifold)**.** A curve $\gamma\colon (a,b) \to M$ is continuous [differentiable] if $\phi_\alpha^{-1} \circ \gamma\colon (a,b) \to U_\alpha$ is continuous [differentiable]

17 Integration of differential forms on a compact manifold

$\omega \in \Omega^n(M), M$ compact manifold , $\dim M = n$. Pick a partition of unity subordinate to a covering $\{V_\alpha\}$ (i.e. pick some maps $\varphi_i, i = 1, \dots, n, 0 \leq \varphi_i \leq 1, \sum_{i=1}^m \varphi_i \equiv 1,$ $\operatorname{supp}(\varphi_i) \subset V_{\alpha(i)}$ for some $\alpha = \alpha(i)$). Then:

$$\begin{aligned}\int_M \omega = \int_M \sum_{i=1}^m \varphi_i \omega = \sum_i \int_M (\varphi_i \omega) = \sum_i \int_{V_{\alpha(i)}} (\varphi_i \omega) = \\ = \sum_i \int_{U_{\alpha(i)}} \phi^*_{\alpha(i)}(\varphi_i \omega)\end{aligned} \tag{17.1}$$

where $V_\alpha = \phi_\alpha(U_\alpha)$. Notice that **the integral is well-defined if M is orientable** (definition of orientable manifolds in the next pages).

Lemma 17.1. *$\int_M \omega$ is independent of the choice (V_α, φ_i)*

Remark 17.2. What happens with boundaries? If $p \in \partial M$, there is no neighbourhood of p homeomorphic to an open set $U \subset \mathbb{R}^n$. Let's define the following set:

$$H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \,|\, x^1 \geq 0\}$$

Note that on [1] H^n is defined as

$$H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \,|\, x^n \geq 0\}$$

(everything is the same, but with x^1 replaced by x^n).

Definition 17.3 (subset topology in H^n)**.** An open set in H^n is the intersection between H^n and an open set in $\mathbb{R}^n$.

Definition 17.4 (Functions on H^n)**.** A function $f \colon V \to \mathbb{R}, V$ open, $V \subset H^n$ is differentiable if $\exists$ an open set $U, \mathbb{R}^n \supset U \supset V$ and a differentiable function $\bar{f} \colon U \to \mathbb{R}$ such that $\bar{f}|_V = f|_V$. Furthermore, $(f_*)_p = (\bar{f}_*)_p, p \in V$.

Definition 17.5 (**Preliminary** definition of diff. manifold with boundary)**.** An n-dimensional differentiable manifold with a regular boundary is a set M with an equivalence class of atlases, as usual, but with the difference that $\mathbb{R}^n$ in the definition 16.8 is replaced by H^n everywhere. The boundary is "regular" if it is described by a regular curve (no intersection, etc...)

When is a point on the boundary of a manifold?

Definition 17.6 (Point on the boundary)**.** A point $p \in M$ is on the boundary of M if for some parametrisation $\phi \colon U \subset H^n \to M$ around p, we have $\phi(0, x^2, \dots, x^n) = p$ for some $x^2, \dots, x^n$.

Lemma 17.7. *The definition 17.6 does not depend on the choice of parametrisation.*

Definition 17.8 (Orientable manifold)**.** M is orientable if there exists an atlas $\mathcal{A} = \{\phi_\alpha, U_\alpha\}$ such that for each pair α, β with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$, the differential

$$(\phi_\beta^{-1} \circ \phi_\alpha)_* \colon U_\beta \to U_\alpha$$

has positive determinant. Example of a non-orientable manifold: Möbius strip.

Proposition 17.9.
- *The boundary ∂M of a n-dimensional differentiable manifold with boundary is a $(n-1)$-dimensional differentiable manifold.*
- *The orientation on M induces an orientation on ∂M*

Theorem 17.10 (Stokes theorem on manifolds)**.** *M orientable, then:*

$$\int_M d\omega = \int_{\partial M} i^*\omega \tag{17.2}$$

18 Abstract Manifolds and Topology

Remark 18.1 (Non-Example of a manifold). The set

$$M = (-\infty, 0) \cup (0, \infty) \cup \{a, b\}$$

(endowed with a particular topology that we will see later) is not a manifold. Here, a, b are just two points (not necessary real numbers): just imagine M as a subset of $\mathbb{R}^2$, consisting of a real line without the origin plus any two points in $\mathbb{R}^2$. Even if we have not defined a topology on M, we can feel that it doesn't look right: there are two maps φ_a, φ_b identifying the subsets $U_a = M \setminus \{b\}$ and $U_b = M \setminus \{a\}$ with $\mathbb{R}$ (we are just calling the 0 element in other way). The transition function

$$\varphi_b \circ \varphi_a^{-1} \colon \varphi(U_a \cap U_b) = \mathbb{R} \setminus \{0\} \to \varphi_b(U_b \cap U_a) = \mathbb{R} \setminus \{0\}$$

is the identity, and is smooth. **But** a smooth function on M must have the same value when evaluated at a and b, and this does not look right (why?).

Definition 18.2 (Topology)**.** A topology on a set M is a subset $\mathcal{O} \subset \mathcal{P}(M)$, where $\mathcal{P}(M)$ is the power set of M, such that:

1. $\emptyset, M \in \mathcal{O}$
2. $\mathcal{O}$ is closed under arbitrary unions: $U_i \in \mathcal{O}, i \in I$, then $\cup_{i \in I} U_i \in \mathcal{O}$
3. $\mathcal{O}$ is closed under finite intersections: $U_1, \ldots, U_k \in \mathcal{O}$, then $U_1 \cap \ldots \cap U_k \in \mathcal{O}$.

Sets in $\mathcal{O}$ are called open, $A \subset M$ is closed if $M \setminus A$ is open. A subset $V \subset M$ containing p is a neighbourhood of p if there is an open set $U \subset V$ containing p.

Remark 18.3. From the definition above we can see that M and $\emptyset$ are always both open and closed sets. Notice that a set in a topological space can be:

- both open and closed
- neither open nor closed
- open but not closed
- closed but not open

(more examples later).

Definition 18.4 (Closure, interior)**.** Let $V \subset M$, let $\mathcal{O}$ be an arbitrary topology. The closure $\bar{V}$ of V is the smallest closed subset of M containing V. The interior $\mathring{V}$ of V is the largest open subset contained in V.

Definition 18.5 (Continuous function, homeomorphism)**.** A map $f\colon X \to Y$ between topological spaces is continuous if $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}_Y$ (i.e. if the pre-image of an open set through f is still an open set). f is a homeomorphism if it is bijective, continuous and f^{-1} is continuous.

Remark 18.6 (Continuous maps vs. open maps)*.* Requiring that the pre-image of an open set is open is different from requiring that the image of an open set is open! If f brings open sets to open sets, it is called open map (and it might not be continuous).

Example 18.7. Given a set M, we can always consider two simple topologies on it:

1. $\mathcal{O}_M = \mathcal{P}(M)$ is a topology, where $\mathcal{P}(M)$ is the power set of M (i.e. the family of all subsets of M). Everything is open in M, so it is easy to see that any function going from $(M, \mathcal{O}_M)$ to any space $(Y, \mathcal{O}_Y)$ must be continuous.

2. $\mathcal{O}_M = \{\emptyset, M\}$ is a topology: the open sets are just the two ones required by the definition 18.2. In this case, every function going from any space $(Y, \mathcal{O}_Y)$ to $(M, \mathcal{O}_M)$ must be continuous (the pre-image of the empty set is the empty set, and the pre-image of M is Y).

What is more, we can consider topologies on metric spaces: if (M, d) is a metric space and $B_\varepsilon(x)$ denotes the ball of radius ε around x, then

$$\mathcal{O}_d = \{U \subset M \,|\, \forall\, x \in U \,\exists\, \varepsilon > 0\colon B_\varepsilon(x) \subset U\}$$

defines a topology. In particular, we say that the topology is induced by a metric if the open sets are the open balls of the metric space. If X, Y are metric spaces, the definition of continuity seen in the Analysis courses is: $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is continuous if and only if $\forall\, x \in X, \forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $f\left(B_\delta(x)\right) \subset B_\varepsilon\left(f(x)\right)$. One can show that in metric spaces such a definition is equivalent to definition 18.5.

Remark 18.8. If $\mathcal{O}_1, \mathcal{O}_2$ are topologies on M, then the intersection of the two topologies is again a topology. Therefore, given a family of subsets of M one can ask for the smallest topology on M which contains a subset.

Definition 18.9 (Hausdorff)**.** A topological space $(M, \mathcal{O})$ is Hausdorff if for all $p, q \in M$ there is a pair of open sets $p \in U_p, q \in U_q$ so that $U_p \cap U_q = \emptyset$.

Remark 18.10. Points in Hausdorff spaces are closed sets. i.e. if M is Hausdorff and $x \in X$, $\{x\}$ is a closed set. In order to see that, we just need to prove that $M \setminus \{x\}$ is open. Let's suppose $M \setminus \{x\} \neq \emptyset$ (otherwise we are done, since $M = \{x\}$ and M is both closed and open). Then we can find another point $y \in M \setminus \{x\}$. Since M is Hausdorff, we can find two open sets U_x, U_y containing x and y respectively, such that $U_x \cap U_y = \emptyset$. The proof is over if we prove that $M \setminus \{x\} = \cup_{y \in M\setminus\{x\}} U_y$, since it implies that $M \setminus \{x\}$ is open (arbitrary union of open sets is open), and then $\{x\}$ is closed. It is true because:

- "$M \setminus \{x\} \subseteq \cup_{y\in M\setminus\{x\}} U_y$": for each point z in M different from x, there exists an open set U_z entirely contained in $M \setminus \{x\}$ by Hausdorff assumption. The point is contained in a set which is contained in $\cup_{y\in M\setminus\{x\}} U_y$ so, in particular, the point is in $\cup_{y\in M\setminus\{x\}} U_y$.
- "$M \setminus \{x\} \supseteq \cup_{y\in M\setminus\{x\}} U_y$": if $z \in \cup_{y\in M\setminus\{x\}} U_y$, then $z \neq x$, otherwise there would be an intersection between an open set $U_{\bar{x}}$ for a fixed $\bar{x}$ and every open set containing x. This is impossible because by Hausdorff assumption there exists at least one open sets of those which does not intersect $U_{\bar{x}}$.

Moreover:

- $(M, \mathcal{P}(M))$ is always Hausdorff (there are "so many" open sets that you can always find two of them containing the right points and so that they do not intersect each other).
- If the topology is induced by a metric, then the topological space is Hausdorff (for instance, $\mathbb{R}^n$ with the topology induced by Euclidean metric is a Hausdorff topological space).

Definition 18.11 (Product topology)**.** Let $(M_i, \mathcal{O}_i), i \in I$ be topological spaces. Then the product topology on $\prod_{i\in I} M_i$ is the smallest topology such that all projections

$$\prod_{i\in I} M_i \to M_j$$

are continuous maps for all $j \in I$.

Definition 18.12 (Quotient topology)**.** Let $(M, \mathcal{O}_M)$ be a topological space and $\sim$ an equivalence relation on M. Then the quotient topology on ${}^{M}\!/\!_{\sim}$ is the largest topology so that the function

$$\begin{aligned} \pi\colon M &\to {}^{M}\!/\!_{\sim} \\ x &\mapsto [x] \end{aligned}$$

is continuous. π is also called canonical projection.

Remark 18.13 (Quotient topology pt. 2). We can also use another useful characterization of the quotient topology: a set $U \subset {}^{M}\!/\!_{\sim}$ is open if and only if $\pi^{-1}(U) \in \mathcal{O}_M$. It is easy to prove that this definition is equivalent to the one above. Let's call (1) the first definition and (2) the last one.

- "(1) $\Rightarrow$ (2)": U open in the quotient topology, then $\pi^{-1}(U)$ is open in M by (1), because π is a continuous map. If $\pi^{-1}(U)$ is open in M, then U is open in ${}^{M}\!/\!_{\sim}$ because in (1) we have defined the topology on ${}^{M}\!/\!_{\sim}$ as the *largest* topology such that π is continuous!
- "(2) $\Rightarrow$ (1): π is obviously continuous. And the topology we get is the *largest* one such that π is continuous because if $\pi^{-1}(U)$ is not open in M, then U is not open in the quotient space (read also below).

Note that: $f\colon X \to Y$ is continuous if the preimage of every open set of Y is an open set of X. But if A is an open set in X, in general it is not true that it exists an open set U in Y such that $A = f^{-1}(U)$. In the "(1) $\Rightarrow$ (2)" above it was true because in Y we had the largest topology such that $f\colon X \to Y$ was continuous. It means that the topology on Y had the same open sets of the "power set topology" (i.e. everything is open, cf. remark 18.7) except for some sets. Which sets did we remove? We were considering the "largest" topology on Y such that "$f\colon X \to Y$ is continuous". In order to have the continuity assumption hold true, we need: U open in $Y \Rightarrow f^{-1}(U)$ open in X. Which is equivalent to: $f^{-1}(U)$ **not** open in $X \Rightarrow U$ **not** open in Y. So if we take the preimage of any set and we notice that such preimage is not open, than the set itself cannot be open in Y: it has to be removed from the topology we are considering on Y.

Example 18.14 (Cofinite topology). Let M be a set. We define the following topology: the open sets are $\emptyset$ and the sets $U \neq \emptyset, U \subset M$ such that their complement in M is finite. Another way to define the same topology is using closed sets: the only closed sets in this topology are M and the finite sets. This topology is called cofinite topology. M with this topology is not Hausdorff if M is an infinite set. Why? Because [a]: given x, y in $M, x \neq y$, if U_a and U_b are open sets that contain a and b respectively, the complements of U_a and U_b must be finite (so, U_a and U_b must be infinite, because the bigger space M is infinite). But then $U_a \cap U_b \neq \emptyset$. Indeed, the complement of U_a in M is $M \setminus U_a$ and it must be finite because U_a is open. If we assume $U_a \cap U_b = \emptyset$, then U_b must be contained in the complement of U_a. But then U_b (which is an infinite set, because it is open) would be contained in $M \setminus U_a$ (which is a finite set). Contradiction!

[a] the proof is easier to follow if you think of M with cofinite topology as $\mathbb{R}$ endowed with cofinite topology

Lemma 18.15. *A subspace of a Hausdorff space is Hausdorff.*

Example 18.16 (Subspace topology). Let $(M, \mathcal{O}_M)$ be a topological space and $N \subset M$ a subset. Then a natural topology for N is the subspace (or subset) topology, defined in the following way. The subspace topology is the smallest topology so that the inclusion map $N \to M$ is continuous, i.e. $V \subset N$ is open if and only if there is an open set $U \subset M$ such that $U \cap N = V$.
For instance, you may know that $[0, 1)$ is neither open nor closed in $\mathbb{R}$ with Euclidean topology (i.e. the topology induced by the standard metric). Actually, it is an open set in $([0, 2], \mathcal{O})$, where $\mathcal{O}$ is the subset topology (with respect to $(\mathbb{R}, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the Euclidean topology). Indeed

$$[0, 1) = (-1, 1) \cap [0, 2]$$

You should always pay attention to the topology and to where it is defined, for instance $[0, 1]$ is closed (but not open) in $(\mathbb{R}, \mathcal{O}_d)$, whereas it is both open and closed in $([0, 1], \mathcal{O})$, for any topology $\mathcal{O}$.

Definition 18.17 (subbasis)**.** Let M be a set and $B \subset \mathcal{P}(M)$ some collection of subsets. Let $\mathcal{O}_B$ be the smallest topology containing B, then we say that B is a subbasis of $\mathcal{O}_B$.

Definition 18.18 (Basis)**.** Let $(M, \mathcal{O})$ be a topological space and $\mathcal{B} \subset \mathcal{O}$. Then $\mathcal{B}$ is a basis for $\mathcal{O}$ if every open set is a union of sets in $\mathcal{B}$.

Definition 18.19 (Second countability)**.** Let $(M, \mathcal{O}_M)$ be a topological space. Then $(M, \mathcal{O}_M)$ is second countable if it admits a countable basis.

Example 18.20. Let $\mathcal{B} = \{B_\varepsilon(x) \,|\, \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}^n\}$. This is a basis of the metric topology of $\mathbb{R}^n$ (and it is a countable set, because we used the density of $\mathbb{Q}$ in $\mathbb{R}$).

Lemma 18.21. *If $A \subset M$ and M has a countable basis, then the subspace topology on A has a countable basis.*

Definition 18.22 (Manifold)**.** A manifold M of dimension n is a topological space which is Hausdorff, second countable and admits an (equivalence class of) atlas(es) $\varphi_i \colon U_i \to V_i$, where the φ_i are homeomorphisms, V_i is an open subset of $\mathbb{R}^n$ and $U_i \subset M$ is open so that the transition functions

$$\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are smooth (i.e. C^∞). We are considering equivalence classes. we recall that the equivalence relation is the following one: two atlases are equivalent if their union is still a smooth atlas.

Remark 18.23 (2nd Countable hypothesis for a manifold)*.* We require the hypothesis of second countability for manifolds in order to apply several theorems or lemmas, e.g. Sard's theorem.

Fact 18.24. *Manifolds are metrizable, i.e. there exists a metric d such that the topology on the manifold is the one induced by d.*

Definition 18.25 (Cover)**.** Let J be a set of indices, either finite or infinite. A family of sets $\{U_i\}_{i\in J}$ is a cover for a topological space X if $\cup_i U_i = X$ (the sets "cover" the entire space). It is an open cover if the U_i are open sets in the topopolgy of X. A subcover is a subfamily of the initial cover which is still a cover.

Definition 18.26 (Compact space)**.** A topological space X is compact if every open cover $\{U_i\}_{i\in J}$ admits a finite subcover $\{U_i\}_{i=1}^k$.

Example 18.27. If (X, d) is a metric space and is compact, then X is bounded (i.e. contained in a ball). Indeed: $U_i = B_i(x) = \{y \in X \,|\, d(x, y) < i\}$ is an open set. And for a fixed $x \in X$, $\cup_i U_i$ is an open cover: $\cup_{i\in\mathbb{N}} U_i = X$. Since the space is compact $\Rightarrow U_{i_1} \cup \ldots \cup U_{i_k} = X$ for some indices $i_1, \ldots, i_k$. These indices are the radii of the balls. Take $c = 2\max\{i_1, \ldots, i_k\}$, then we have that $d(x, y) < c\,\forall\, y, \in X$, i.e. X is bounded because it is contained in the ball $B_c(x)$.

Theorem 18.28 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Corollary 18.29 (Weierstrass)**.** *Let X be a compact space, $f\colon X \to \mathbb{R}$ continuous, then f attains its maximum (and minimum) in X.*

Lemma 18.30 (Closed in a compact)**.** *X is a compact topological space, $A \subset X$ closed $\Rightarrow A$ is compact (in subspace topology).*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of A. There are open sets $V_i \subset X$ with $V_i \cap A = U_i$ by definition of open sets in the subspace topology. Moreover, $V_A \equiv X \setminus A$ is open because A is closed. Since $\{U_i\}_{i\in I}$ is an open cover of A, we have that $\{U_i\}_{i\in I}$ and V_A form an open cover of X. Since X is compact, there is a finite number of indices $i_1, \ldots, i_k \in I$ such that $V_{i_1}, \ldots, V_{i_k}, V_A$ cover X. Since $V_A = X \setminus A$, it follows that the sets $V_{i_1}, \ldots V_{i_k}$ cover A. Since $U_i = V_i \cap A$, also $U_{i_1}, \ldots, U_{i_k}$ cover A. Then A is compact because we proved that every open cover admits a finite subcover. □

Lemma 18.31 (Cont. functions map compact sets to compact sets)**.** *If A is compact and $f\colon A \to Y$ is continuous, then $f(A)$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(A)$. Then $\exists \{V_i\}_{i\in I}, V_i \subset Y$ open, so that $U_i = V_i \cap f(A)$ (by definition of open sets in subspace topology). Since f is continuous, $f^{-1}(V) \subset A$ is open, and $f^{-1}(V)$ covers A. Since A is compact, there exist $i_1, \ldots, i_k$ such that $f^{-1}(V_{i_1}), \ldots, f^{-1}(V_{i_k})$ still cover A. Thus, $V_{i_1}, \ldots, V_{i_k}$ is a collection of open sets in Y covering $f(A)$. In the subspace toplogy, $U_{i_1} = V_{i_1} \cap f(A), \ldots, U_{i_k} = V_{i_k} \cap f(A)$ cover $f(A)$. □

Lemma 18.32 (Compact set in a Hausdorff)**.** *Let X be Hausdorff. Let $A \subset X$. Assume A is compact. Then A is closed.*

Proof. Let $x \in X \setminus A$. For every $a \in A$, there are $U_a \subset X, V_x(a) \subset X$ open sets such that $U_a \cap V_x(a) = \emptyset$ (because of the Hausdorff hypothesis). Here, w used the notation $V_x(a)$ to stress out the dependence of V on x and a. The sets $\{U_a \cap A\}_{a\in A}$ are form an open cover of A. Since A is compact, there are finitely many $a_1, \ldots a_k$ so that $U_{a_1} \cap A, \ldots, U_{a_k} \cap A$ cover A. Now, $V_x = V_x(a_1) \cap \ldots \cap V_x(a_k)$ contains x and it is open. Moreover, $V_x \cap A = \emptyset$ because V_x is disjoint from each $U_{a_i} \Rightarrow X \setminus A$ is open, becuase for each point we can find an open set that contains the point and that is contained in $X \setminus A$. □

Theorem 18.33 (How to get a homeomorphism)**.** *Let $f\colon X \to Y$ be a continuous function, X compact, Y Hausdorff and f bijective. Then f is a homeomorphism.*

Proof. The only thing which is left to prove is that f^{-1} is a continuous function. First, we notice that an equivalent definition of continuous function is: "$f\colon X \to Y$ is continuous $\Leftrightarrow f^{-1}(C)$ is closed $\forall$ closed set $C \subset Y$". In order to prove that f^{-1} is continuous, we need to show that $f(C) \subset Y$ is closed for all $C \subset X$ closed. This is true because: C closed in $X \Rightarrow C$ compact because of lemma 18.30 $\Rightarrow f(C)$ is compact in Y because of lemma 18.31 $\Rightarrow f(C)$ is closed, using the lemma 18.32. □

Remark 18.34. We can consider the concept of sequential compactness: X is sequentially compact if every sequence in X contians a convergent subsequence in X. In general, X compact $\not\Rightarrow X$ sequentially compact. But, X compact + first countable $\Rightarrow X$ sequentially compact. Here, X is first countable if $\forall\, x \in X \,\exists\, N_1, N_2, \ldots, N_i$ (where $i \in I, I$) countable collection of neighbourhoods of x such that every neighbourhood contains one of the N_i. Notice that manifolds are first countable! (CHECK)

Definition 18.35 (Compact exhaustion)**.** Let X be a topological space. A compact exhaustion $\{K_i\}_{i\in\mathbb{N}}$ of X is a countable collection of compact sets K so that $\bar{K}_i \subset \mathring{K}_{i+1}$

Theorem 18.36. *A manifold admits a compact exhaustion.*

Proof. The proof uses the second countability. We will skip it. □

Definition 18.37 (Paracompactness)**.** A topological space is paracompact if every open cover admits a locally finite refinement. i.e. if $\{U_i\}_{i\in I}$ is an open cover of X, there is another open cover $\{V_j\}_{j\in J}$ of X such that $V_j \subset U_i(j)$ (refinement) and every point x has a neighbourhood V_x such that V_x intersects only a finite number of V_j (locally finite).

Theorem 18.38. *Manifolds are paracompact.*

Proof. The proof uses the second countability property. We will skip it here. □

Definition 18.39 (Partition of unity)**.** A partition of unity subordinate to an open cover $\{U_i\}_{i\in I}$ of X is a collection of continuous functions $f_j\colon X \to [0,1]$ such that $\forall\, x \exists\, V_x \ni x, V_x$ neighbourhood of x so that the support $\mathrm{supp}(f_j) = \overline{f_j^{-1}((0,1])}$ is contained in one of the U_i, and V_x intersects only finitely many $\mathrm{supp}(f_j)$, and

$$\sum_{j\in J} f_j(x) \equiv 1$$

Theorem 18.40. *Every open cover of a manifold admits a subordinate partition of unity which is smooth.*

Definition 18.41 (Connected space)**.** A topological space X is connected if it cannot be written as union of two non-empty, disjoint open sets.

Remark 18.42 (Equivalent definitions of connected space)*.* Sometimes it is useful to consider equivalent definitions of connection: a topological space X is connected if:

- It cannot be written as union of two non-empty, disjoint closed sets,
- $X = A \cup B$ with A, B disjoint and open sets implies that $A = \emptyset$ or $B = \emptyset$,
- The only sets in X that can be both open and closed are X and $\emptyset$.

Definition 18.43 (Smooth function from a manifold)**.** Let M be a smooth manifold, then $f\colon M \to \mathbb{R}$ is smooth at p if for a chart $\varphi\colon U \to \mathbb{R}^n$, with $U \ni p$, we have that

$$f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}$$

is smooth near $\varphi(p)$

Remark 18.44. The definition does not depend on the chart chosen. In fact, let's choose another chart (V, ψ), then:

$$f \circ \psi^{-1} = \underbrace{f \circ \varphi^{-1}}_{\text{smooth}} \circ \underbrace{\varphi \circ \psi^{-1}}_{\text{smooth}}$$

is smooth around p because the composition of smooth functions is smooth.

Definition 18.45 (Smooth function between manifolds)**.** Let M, N be smooth manifolds. Then $f\colon M \to N$ is smooth around p if f is continuous and for charts (U, φ) around p and (V, ψ) around $f(p)$ we have that $\psi \circ f \circ \varphi^{-1}$ is smooth on a neighbourhood of $\varphi(p)$

Now that we know when a map is smooth, we want to differentiate. For this, we want the concept of tangent vector. We will use three equivalent definitions.

Definition 18.46 (Equivalent curves)**.** Two curves $\gamma_0, \gamma_1\colon (-\varepsilon, \varepsilon) \to M$ with $\gamma_0(0) = \gamma_1(0) = p$ are equivalent if for a chart (U, φ) around p:

$$\frac{d}{dt}\Big|_{t=0} \varphi \circ \gamma_0(t) = \frac{d}{dt}\Big|_{t=0} \varphi \circ \gamma_1(t)$$

And we notice that such definition does not depend on the chart chosen.

Definition 18.47 (Geom. tangent space)**.** We define the geometric tangent space at p as:

$$T_p^{\text{geom}} M = \left\{ \text{ smooth curves } \gamma\colon (-\varepsilon, \varepsilon) \to M \mid \gamma(0) = p, \varepsilon > 0 \right\} \big/ \sim \tag{18.1}$$

where $\sim$ is the equivalence relation described in def. 18.46. The elements of $T_p^{\text{geom}} M$ are called tangent vectors.

Remark 18.48. Let's analyze def. 18.47: it is an intuitive definition of the tangent space but from it it's not obvious that $T_p^{\text{geom}} M$ is a vector space. The following definition of tangent space is more abstract, but on the other hand it will be evident that the tangent space is a vector space.

Definition 18.49 (Derivation - pt.2)**.** A derivation v on smooth functions $C^\infty(M)$ at p is a $\mathbb{R}-$linear map $v\colon C^\infty(M) \to \mathbb{R}$ which satisfies the Leibniz rule $v(fg) = v(f)|_p g|_p + f|_p v(g)|_p$

Definition 18.50 (Alg. tangent space)**.** We define the algebraic tangent space at p as:

$$T_M^{\text{alg}} = \{\text{derivations of } p\}$$

Remark 18.51.

- Instead of using $C^\infty(M)$ (which is a ring) we could have used the space $\mathcal{E}_p^\infty(M) = {C^\infty}/{\sim}$ of germs of functions, where the equivalence relation is the following one:
$$f \sim g \Leftrightarrow f \equiv g \text{ on a neighbourhood of } p$$
In such cases, $v(f) = v(g)$ for $v \in T_p^{\mathrm{alg}}M$. In fact, let's consider a smooth function h with support $\overline{h^{-1}(\mathbb{R} \setminus 0)}$ (so, $h(0) = 0$) and $h \equiv 1$ near p. Then:
$$0 \overset{(1)}{=} v\left(h(f-g)\right) = v(h)\underbrace{(f(p) - g(p))}_{=0} + \underbrace{h(p)}_{=1}(v(f) - v(g))$$
where we used: (1) $h(f-g) = h(0) = 0$ near p and the derivative of a constant function is 0 by the Leibnitz rule: $v(1) = v(1 \cdot 1) = v(1) + v(1) \Rightarrow v(1) = 0$. (2) Leibnitz rule again. We also say that the derivation v is local.
- As we already pointed out, $T_p^{\mathrm{alg}}M$ is a vector space. The only non-trivial thing to prove is that $\dim T_p^{\mathrm{alg}}M = n$ if the dimension of M is n.

Lemma 18.52. *Let $f\colon \mathbb{R}^N \supset B_\varepsilon(0) \to \mathbb{R}$ be smooth, then there are smooth functions f_i on $B_\varepsilon(0)$ so that:*
$$f(x) = f(0) + \sum_i x^i f_i(x^1, \dots, x^n) \tag{18.2}$$

Proof.
$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx^1, \dots, x^n)\, dt = \int_0^1 \sum_i x^i \frac{\partial}{\partial x^i} f(tx^1, \dots, tx^n)\, dt$$
So, the f_i we are looking for are:
$$f_i = \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \dots, tx^n)\, dt$$
□

Remark 18.53. Now, we can prove that the $\dim T_p^{\mathrm{alg}}M = n$. Let v be a derivation at p and (U, φ) a local chart around p. $x^1, \dots, x^n$ are the local coordinates, such that p coincides with the origin ($x^i(p) = 0$). From lemma 18.52:
$$f(x) = f(0) + \sum_i x^i f_i(x)$$

and

$$v(f) = \underbrace{v(f(0))}_{=0} + \sum_i \left(v(x^i)\underbrace{f_i(x^i)}_{=f_i(0)} + \underbrace{x^i}_{=0} v(f_i) \right)$$

Then, v is defined by the way it reacts to coordinate functions around p. Thus, $\dim T_p^{\text{alg}} M \leq n$. Since $v_i = \frac{\partial}{\partial x^i}$ is a derivation, we have $\dim T_p^{\text{alg}} M = n$.

Definition 18.54 (Physicists's tangent space)**.** Let M be a manifold of dimension n, then a tangent vector is a map

$$v \colon \{ \text{ charts } (U, \varphi) \text{ around } p\} \to \mathbb{R}^n \tag{18.3}$$

where v is defined such that if we change charts we have:

$$v(V, \psi) = D_p(\psi \circ \varphi^{-1})\, v((U, \varphi)) \tag{18.4}$$

The space of such maps is called tangent space (according to the physicists's definition) and it is denoted by $T_p^{\text{phys}} M$.

Remark 18.55. The three previous definitions of tangent space are all equivalent, indeed we can find isomorphisms between such tangent spaces.

1.

$$T_p^{\text{geom}} M \to T_p^{\text{alg}} M \tag{18.5}$$

$$[\gamma] \mapsto v_\gamma \colon C^\infty(M) \to \mathbb{R} \tag{18.6}$$

$$f \mapsto \frac{d}{dt}|_{t=0} (f \circ \gamma)(t) \tag{18.7}$$

2.

$$T_p^{\text{alg}} M \to T_p^{\text{phys}} M \tag{18.8}$$

$$v \mapsto \left\{ (U, \varphi) \mapsto \{ v(x^i) \}_{i=1,\dots,n} \right\} \tag{18.9}$$

where x^i are the coordinates around p from φ.

3.

$$T_p^{\text{phys}} M \to T_p^{\text{geom}} M \tag{18.10}$$

$$v \mapsto \left\{ \gamma_i \colon t \mapsto \varphi(\varphi^{-1}(p) + t v((U, \varphi))) \right\} \tag{18.11}$$

with $|t| < \varepsilon$, where we picked up a chart (U, φ) around p with coordinate x^i.

Accordingly, we also have three different definitions for differentiating, i.e. three definitions for the map

$$D_p f \colon T_p M \to T_{F(p)} N \tag{18.12}$$

where $p \in M, F \colon M \to N$ smooth:

1. $D_p^{\mathrm{geom}} F([\gamma]) = [F \circ \gamma]$, where the right-hand side is an equivalence class of equivalence curves on N,
2. $D_p^{\mathrm{alg}} F(v) = \{C^\infty(N) \ni f \mapsto v(f \circ F)\}$,
3. $D_p^{\mathrm{phys}} F(v) = \{(V, \psi) \to D_p^{\mathrm{phys}}(\psi \circ F \circ \varphi^{-1})\, v(U, \varphi)\}$, where (V, ψ) is any chart of N near $F(p)$.

These definitions are compatible with the maps above, e.g. the following diagram

$$\begin{array}{ccc} T^{\mathrm{geom}}M & \xrightarrow{D^{\mathrm{geom}}} & T^{\mathrm{geom}}N \\ \downarrow & & \downarrow \\ T^{\mathrm{alg}}M & \xrightarrow{D^{\mathrm{alg}}} & T^{\mathrm{alg}}N \end{array}$$

commutes.

Lemma 18.56 (Chain rule)**.** *Let M_0, M_1, M_2 be smooth manifolds, and $f_0 \colon M_0 \to M_1, f_1 \colon M_1 \to M_2$ smooth. Then:*

$$D(f_1 \circ f_0) = Df_1 \circ Df_0$$

Proof. Using the geometric definition:

$$[D(f_1 \circ f_0)][\gamma] = [(f_1 \circ f_0) \circ \gamma] = [f_1 \circ (f_0 \circ \gamma)] = (Df_1 \circ Df_0)[\gamma]$$

□

Definition 18.57. For a manifold M of dimension n, let

$$\begin{aligned} TM = \bigcup_{p \in M} T_p^{\mathrm{phys}} M &\xrightarrow{\mathrm{pr}} M \\ v &\longmapsto p \end{aligned} \tag{18.13}$$

This map is the bundle projection of the tangent bundle TM.

Remark 18.58. TM is a manifold, if equipped with the following structure. Pick an atlas $(U_i, \varphi_i)_{i \in I}$ of M (remember: $\varphi \colon M \supset U_i \to \mathbb{R}^n, U_i$ open). Thanks to the second-countability property, we can take I as a countable set without loss of generality. Then, one defines local coordinates (i.e. a local parametrization) on $\mathrm{pr}^{-1}(U_i)$ via

$$\begin{aligned} \hat{\varphi}_i \colon TM \supset \mathrm{pr}^{-1}(U_i) &\longrightarrow \varphi_i(U_i) \times \mathbb{R}^n \\ v &\longmapsto (\varphi(\mathrm{pr}(v)),\, v((U_i, \varphi_i))) \end{aligned} \tag{18.14}$$

This is a bijective map, and we define the topology on TM as the topology so that $\hat{\varphi}_i$ is a homeomorphism for all i. This topology is Hausdorff, indeed: let $v_1, v_2 \in TM$. If $p_1 = \mathrm{pr}\, v_1 \neq \mathrm{pr}\, v_2 = p_2$, then there exist open sets $V_j \ni p_j, j = 1, 2$ which are disjoint and open (because M is Hausdorff). Let $U_{i(p_1)}, U_{i(p_2)}$ be charts around p_1, p_2. Then:

$$\mathrm{pr}^{-1}\left(U_{i(p_1)} \cap V_1\right) \cap \mathrm{pr}^{-1}\left(U_{i(p_2)} \cap V_2\right) = \emptyset$$

On the other hand, if $\mathrm{pr}\, v_1 = \mathrm{pr}\, v_2$, let $\left(U_{i(v_1)}, \varphi_{i(v_1)}\right)$ be a chart containing $\mathrm{pr}\, v_1$. Then $\hat{\varphi}_i(v_1), \hat{\varphi}_i(v_2)$ differ in the second entry in $\mathbb{R}^n$: pick disjoint open sets W_1, W_2 of $\mathbb{R}^n$ separating those entries and consider the open sets $\hat{\varphi}_i{}^{-1}\left(\varphi_i(U_i) \times W_j\right), j = 1, 2$. Moreover, the topology is second countable. Finally, we need to check the smoothness of the transition functions $\hat{\varphi}_j \circ \hat{\varphi}_i^{-1}$. Note that $\hat{\varphi}_i\left(\mathrm{pr}^{-1}(U_i) \cap \mathrm{pr}^{-1}(U_j)\right) = \varphi_i(U_i \cap U_j) \times \mathbb{R}^n$. By the transformation rule (18.4), and using (18.14):

$$\hat{\varphi}_j \circ \hat{\varphi}_i^{-1}\left(x, w = (w^1, \ldots, w^n)\right) = (\varphi_j \circ \varphi_i^{-1}(x), D_x(\varphi_j \circ \varphi_i^{-1})(w))$$

This map is smooth. Note that $D_x(\cdots)$ is a linear transformation for fixed base point x. Thus $T_pM = \mathrm{pr}^{-1}(p)$ is a vector space

Definition 18.59 (Vector field). A vector field X on a manifold M is a smooth map

$$X\colon \quad M \underset{\curvearrowleft\ \mathrm{pr}}{\longrightarrow} TM$$

so that $\mathrm{pr} \circ X = \mathrm{id}_M$.

Remark 18.60 (Commutator of vector fields). Let X, Y be two vector fields on M. Then the assignment

$$C^\infty(M) \ni f \longmapsto [X,Y](f) = X\left(Y(f)\right) - Y\left(X(f)\right)$$

is independent of the local coordinates and satisfies the Leibniz rule (it can be easily verified): $[X,Y](fg) = g\,[X,Y](f) + f\,[X,Y](g)$. Moreover, the commutator satisfies the properties already seen in lemma 12.5. And in local coordinates:

$$X = \sum_i a^i \partial_i, \qquad Y = \sum_j b^j \partial_j$$

we have:

$$[X,Y] = \sum_{i,j}\left[\left(a^i \frac{\partial}{\partial x^i} b^j\right)\frac{\partial}{\partial x^j} - \left(b^j \frac{\partial}{\partial x^j} a^i\right)\frac{\partial}{\partial x^i}\right]$$

Remark 18.61 (Tangent space to a $\mathbb{R}$-vector space). Let V be a an $\mathbb{R}$-(vector space) (i.e. a vector space on the field $\mathbb{R}$). V is a manifold of dimension $\dim V$. Then, the tangent space to V is V itself (or, better, the tangent space to V is isomorphic to

V). Indeed, given $v \in V$, there exists an isomorphism:

$$\begin{aligned} V &\cong T_v V \\ w &\mapsto [t \mapsto (v + tw)] \end{aligned}$$

where the map $t \mapsto (v + tw)$ goes from $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ to V.

Definition 18.62. In a manner similar to TM becoming a manifold, one can define the manifold given by the cotangent bundle

$$T^*M = \bigcup_{p \in M} T_p^* M \xrightarrow{\text{pr}} M \tag{18.15}$$

Definition 18.63 (1-form)**.** A 1-form is a smooth map

$$\alpha\colon \quad \begin{tikzcd} M \arrow[r] & T^*M \arrow[l, bend left, "\text{pr}"] \end{tikzcd}$$

so that $\text{pr} \circ \alpha = \text{id}_M$ (it is a section of the bundle). k-forms are defined in an analogous fashion, and so are $\Omega^k(M), d\colon \Omega^k(M) \to \Omega^{k+1}(M), f^*$ and f_*.

19 Lie Groups

Definition 19.1. A Lie group G is a group and a smooth manifold, such that the operation of the group and the inverse of the group (seen as maps) are smooth. i.e. the maps

$$\begin{aligned} \mu\colon G \times G &\to G \\ (g, h) &\mapsto g \cdot h \end{aligned} \tag{19.1}$$

$$\begin{aligned} \text{inv}\colon G &\to G \\ g &\mapsto g^{-1} \end{aligned} \tag{19.2}$$

are smooth.

Remark 19.2 (Aut(T_eG) is a Lie group). Let G be a Lie group and $g \in G$. Let's consider the map

$$\begin{aligned} c_g\colon G &\to G \\ h &\mapsto ghg^{-1} \end{aligned} \tag{19.3}$$

c_g is a smooth group homomorphism (i.e. $c_g(h)c_g(k) = c_g(hk)$) and has the property

$$c_g \circ c_{g'} = c_{g \circ g'} \tag{19.4}$$

It also preserves the identity element: $c_g(e) = e$. If we differentiate at e:

$$\text{Ad}_g = D_e c_g\colon T_eG \to T_eG \tag{19.5}$$

is an isomorphism with inverse $\mathrm{Ad}_{g^{-1}}$. Using the property (19.4) we have:

$$\mathrm{Ad}_g \circ \mathrm{Ad}_{g'} = \mathrm{Ad}_{gg'} \tag{19.6}$$

Therefore, we obtain a smooth group homomorphism:

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{Aut}(T_eG) \\ g &\mapsto \mathrm{Ad}_g \end{aligned} \tag{19.7}$$

with $\mathrm{Ad}_e = \mathrm{id}_{T_eG}$. Here, $\mathrm{Aut}(T_eG)$ is the set of automorphisms of T_eG (see also remark 4.3). Differentiating again, we get:

$$\begin{aligned} \mathrm{ad}\colon T_eG &\to T_{\mathrm{id}}(\mathrm{Aut}(T_eG)) \cong \mathrm{End}(T_eG) \\ X &\mapsto \{Y \mapsto \mathrm{ad}(X)(Y)\} \end{aligned} \tag{19.8}$$

where $\mathrm{End}(T_eG)$ is the space of endomorphisms of T_eG (see also the remark (4.3)). So, since the map given by 19.7 is bijective, we have that $G \cong \mathrm{Aut}(T_eG)$, i.e. $\mathrm{Aut}(T_eG)$ is a Lie group. For the same reason, since the map given by (19.8) is bijective, we have that $\mathrm{End}(T_eG)$ is a Lie algebra.

Definition 19.3 (Left-invariant vector field)**.** A vector field X on a Lie group is left-invariant if $l_{g*}X = X$, where l_g is the homeomorphism given by

$$\begin{aligned} G &\to G \\ h &\mapsto gh \end{aligned} \tag{19.9}$$

The space of left-invariant vector fields is denoted by $\mathfrak{g}$.

Remark 19.4 ($\mathfrak{g}$ is a Lie algebra). Let's consider:

$$\begin{aligned} \mathfrak{g} &\to T_eG \\ X &\mapsto X(e) = X_e \end{aligned} \tag{19.10}$$

It is an isomorphism since it is linear, i.e. $(X+Y)_e = X_e + Y_e$, and we can get the vector field back from the vector field itself evaluated at e. Indeed, the inverse is:

$$X_h = l_{h*}X_e = (Dl_h)(X_e) \quad \circledast$$

by left-invariance. In fact, we can prove $\circledast$ in the following way: the left-invariance condition $Dl_gX = l_{g*}X = X$ means that:

$$D(l_g)'_g\,(X'_g) = X_{gg'},\ \forall\, g, g' \in G$$

where we wrote explicitly the element g' about which we compute the derivation and the vector field. In the right-hand side the product gg' is well defined by the operation of the group G. Then, by left-invariance, we have that:

$$D_e(l_h)\,(X_e) = X_{eh} = X_h$$

which is what we wanted to prove. So, we have $\mathfrak{g} \cong T_eG$. In order to prove that $\mathfrak{g}$ is a Lie algebra, we need Lie brackets that satisfy Lie algebra properties. In order to solve the problem, we notice that l_g is not only a homomorphism, but also a diffeomorphism. By lemma 12.5, point 5, and by recalling that the commutator of vector fields is still a vector field:

$$Dl_g[V, W] = [Dl_gV, Dl_gW] \quad \text{for vector fields } V, W$$

which means that the commutator of left-invariant vector fields is still left-invariant. All of this makes $\mathfrak{g}$ a Lie algebra, i.e. a $\mathbb{R}$-(vector space) with a pairing $[\cdot, \cdot]$ which is antisymmetric and satisfy Jacobi identity.

Remark 19.5. Left-invariance provides even more properties. Given a left-invariant vector field X, let's consider the curve

$$\begin{aligned} \alpha^X \colon \mathbb{R} &\to G \\ t &\mapsto \alpha^X(t) \end{aligned} \tag{19.11}$$

such that $\frac{d}{dt}|_{t=t_0}\alpha^X(t) = X(\alpha^X(t_0))$ and $\alpha^X(0) = e$. Such a curve locally exists by the theorems about ordinary differential equations. The global existence is assured by the left-invariance property (it lets us extend the solution on larger sets).
Then, we can completely describe the flow of X using the curve α^X. In fact, we can write the flow φ of X as:

$$\begin{aligned} \varphi \colon \mathbb{R} \times G &\to G \\ (t, h) &\mapsto h\alpha^X(t) = l_h(\alpha^X(t)) \end{aligned} \tag{19.12}$$

It satisfies $\frac{d}{dt}\varphi(t, h) = X(h\alpha^X(t))$ by the chain rule and by left-invariance. Moreover, $\alpha^X(s + t) = \alpha^X(s)\alpha^X(t)$ holds.

Example 19.6. Assume that $G \subset GL(n, \mathbb{R})$. Then $\mathfrak{g} \cong T_eG \subset \mathrm{End}(n, \mathbb{R})$ and $X \in G$ is given by a matrix X. The curve α^X in this case is given by:

$$\alpha^X(t) = \sum_{j=0}^{\infty} \frac{t^j X^j}{j!} \tag{19.13}$$

What if we want to compute the commutator of two left-invariant vector fields? We have that $[X, Y] = \mathrm{ad}(X)(Y)$, where "ad" is the map defined in (19.8). This is true in general, not only for $G \subset GL(n, \mathbb{R})$. In fact, let's recall the maps defined in the remark 19.2:

1.

$$c_g\colon G \to G \qquad (19.14)$$
$$h \mapsto ghg^{-1}$$

2.

$$G \to \mathrm{Aut}G \qquad (19.15)$$
$$g \mapsto c_g$$

3.

$$\mathrm{Ad}\colon G \to \mathrm{End}(T_eG) \qquad (19.16)$$
$$g \mapsto D_e c_g = D(c_g)(e)$$

$D_e c_g \in \mathrm{End}(T_eG)$ because the differential of the map c_g is itself a map from T_eG to T_eG

4.

$$\mathrm{ad}\colon T_eG \to T_e\mathrm{End}(T_eG) \cong \mathrm{End}(T_eG) \qquad (19.17)$$

Then, if we consider:

$$\mathrm{Ad}(g)(Y) = (D_e c_g)(Y) \overset{(1)}{=} \frac{d}{dt}|_{t=0} c_g(\alpha^Y(t))$$

where we used: (1) the geometric definition of differential, see remark 18.55. Thus:

$$\mathrm{ad}(X)(Y) = \frac{d}{ds}|_{s=0}\left(\frac{d}{dt}|_{t=0} c_{\alpha^X(s)}\alpha^Y(t)\right) \overset{(2)}{=} \frac{d}{ds}|_{s=0}\left(\frac{d}{dt}|_{t=0} \alpha^X(s)\alpha^Y(t)\alpha^X(-s)\right) \overset{(3)}{=}$$
$$= \frac{d}{ds}|_{s=0}\left(\frac{d}{dt}|_{t=0} \exp(sX)\exp(tY)\exp(-sX)\right) = \ldots = XY - YX$$

where we used: (2) definition of c_g and the fact the $\varphi^{-1}(t) = \varphi(-t)$ if φ is the flow of a vector field, (3) definition (19.13) for the case $G \subset GL(n, \mathbb{R})$. It is common to use the following notation for Lie groups:

$$\exp\colon \mathfrak{g} \to G \qquad (19.18)$$
$$X \mapsto \alpha^X(1)$$

where, in general, $\exp(X+Y) \neq \exp(X)\exp(Y)$. The map "exp" is smooth and

$$D_e \exp \equiv \mathrm{id}_{\mathfrak{g}} \qquad (19.19)$$

(recall that $T_e\mathfrak{g} \equiv \mathfrak{g}$ because $\mathfrak{g}$ is a vector field and $T_eG \cong \mathfrak{g}$).

Lemma 19.7. *Let $f\colon G \to H$ be a smooth group homomorphism. then*

$$\exp^H\left(Df(X)\right) = f\left(\exp^G(X)\right) \tag{19.20}$$

where $Df\colon \mathfrak{g} \to \mathfrak{h}$ with $\mathfrak{g} \cong T_eG, \mathfrak{h} \cong T_eH$, as usual.

Proof. Given $Df(X)$, $\alpha^{Df(X)}\colon \mathbb{R} \to H$ is a 1-parameter subgroup (i.e. it satisfies the same properties of α^X, see also remark 19.5). $\alpha^{Df(x)}$ coincides with $f \circ \alpha^X$. i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "Df"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

□

Lemma 19.8. *Let G be a connected Lie group. Then, if $f\colon G \to H$ is a smooth homomorphism, it is determined by its differential at the identity element D_ef.*

Proof. Let's take a lok at the diagram of the previous proof. It would be nice if we could go back from G to $\mathfrak{g}$ like this:

$$\begin{tikzcd}
\mathfrak{g} \arrow[d, "\exp^G"', shift right] \\
G \arrow[u, bend right]
\end{tikzcd}$$

If it was true, then the thesis would be satisfied because the diagram of the previous proof commutes. Unfortunately, this is not always true. It is *locally* true because $D_e \exp^G \equiv \mathrm{id}_{\mathfrak{g}}$ (think about the Taylor expansion, see (19.19)). Since it is locally true, there exists a neighbourhood $\mathfrak{U} \subset \mathfrak{g}$ of 0 and a neighbourhood $\mathcal{U} \subset G$ of e such that

$$\exp^G|_{\mathfrak{U}}\colon \mathfrak{U} \to \mathcal{U}$$

is a diffeomorphism. Thus, on $\mathcal{U}$ the f is defined by:

$$f = \exp^H \circ Df \circ \left(\exp^G|_{\mathfrak{U}}\right)^{-1}$$

Since the Lie group is connected, this is enough to prove the thesis. Indeed every neighbourhood $\mathcal{U}$ of e in G generates G itself if it is connected. This fact can be proved in the following way, let's consider:

$$V \equiv \mathcal{U} \cap \mathcal{U}^{-1}$$

It is an open neighbourhood, because intersection of open sets, with $\mathcal{U}^{-1} = \left\{u^{-1} \,|\, u \in \mathcal{U}\right\}$. Moreover,

$$G^V \equiv \bigcup_{j=0}^{\infty} V^j, \qquad V^j \equiv \{v_1 \cdot \ldots \cdot v_j \,|\, v_i \in V\}$$

G^V is a subgroup of G, and it is open because every element of the union is open: V is open, $V^2 = \bigcup_{v\in V} vV$ is open, etc... We will now show that also the complement is open:

$$G = \bigcup_{[g]\in {}^{G}\!/_{G^V}} gG^V$$

because:

- "$\supseteq$" is trivial since we are considering union of elements of the group,
- "$\subseteq$" follows from the fact that "$[g] \in {}^{G}\!/_{G^V}$" means that if $g \in G^V$, then "gG^V" yields the entire subgroup G^V. If $g \notin G^V$, then we are considering the elements in $G \setminus G^V$. Since "gG^V" means that we are multiplying g by all possible element of G^V, and since G^V contains the identity element (because it is a subgroup), we can get all the possible elements of $G \setminus G^V$ from the right-hand side. Then $G \subseteq \bigcup_{[g]\in {}^{G}\!/_{G^V}} gG^V$.

Thus, we can obtain $G \setminus G^V$ if $g \notin G^V$ in the union:

$$G \setminus G^V = \bigcup_{[g]\neq[e]} gG^V \text{ is open}$$

Now, given that G is connected (see definition 18.41), either G^V or $G \setminus G^V$ must be empty. G^V contains at least the identity element if G is non-empty, so $G \setminus G^V$ must be the empty one. So, $G = G^V$. □

Remark 19.9. The hypothesis of connection is necessary for theorem 19.8.

20 Compactly Supported Cohomology and Poincarè Lemma

Let M be a manifold with or without boundary. We have already seen (remark 15.8) that the k-th de Rham cohomology of M is

$$H^k(M) \equiv H^k_{dR}(M) = \frac{\ker(d\colon \Omega^k(M) \to \Omega^{k+1}(M)))}{\operatorname{im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M)))} \tag{20.1}$$

And we have already considered the Poincarè lemma 14.9:

$$H^k(\mathbb{R}^n) \cong \begin{cases} 0, & k \neq 0 \\ \mathbb{R}, & k = 0 \end{cases} \tag{20.2}$$

Indeed, in (20.2) we used that $\mathbb{R}^n$ is contractible (case $k \neq 0$) and that the only exact 0-forms (i.e. functions) are the constant ones (case $k = 0$). Now, we want to discuss about k-forms with compact support (i.e. when the function a of the definition (9.1) has compact support). First, we notice that the exterior derivative d maps forms with compact support to forms with compact support. Then, it is natural to define the following.

Definition 20.1 (Compactly supported de Rham cohomology)**.** We define the compactly supported de Rham cohomology of a manifold M as:

$$H_c^k(M) = \frac{\ker\left(d|_{\text{compact supp forms}}\right)}{\text{im}\left(d|_{\text{compact supp forms}}\right)} \tag{20.3}$$

Remark 20.2. The compactly supported de Rham cohomology is somehow different from the standard de Rham cohomology. Indeed, for the latter, given a map $f\colon U \to V$, we could consider the pullback:

$$\begin{aligned} f^*\colon H^k(V) &\to H^k(U) \\ [\omega] &\mapsto [f^*\omega] \end{aligned}$$

For the compactly supported case, we can't do that anymore, because if ω is compactly supported, in general $f^*\omega$ might not be compactly supported (preimages of compact sets are not compact in general).

Now, we want to prove a compactly supported version of the Poincarè lemma. In order to do that, we need some general result on manifolds.

Remark 20.3 (Integration over the fiber). Let M be a smooth manifold, oriented. Let's consider the projection map

$$\pi\colon M \times \mathbb{R} \to M \tag{20.4}$$

Given $\omega \in \Omega^k(M), \pi^*\omega$ is not compactly supported in general. However, we can define a map

$$\pi_*\colon \Omega_c^k(M \times \mathbb{R}) \to \Omega_c^{k-1}(M) \tag{20.5}$$

defined in such a way that $\pi_*\omega$ still has compact support. Such a map is called *integration along the fiber*. We used the notation of the pushforward because it is indeed another way to write the pushforward of a k-form. We also notice that every k-form on $M \times \mathbb{R}$ can be written as sum (more than two summands in general) of two types of k-forms:

- **Type A)** $\omega = f(x,t)\,\pi^*\eta, \quad \eta \in \Omega^k(M)$,
- **Type B)** $\omega = f(x,t)\,\pi^*\eta \wedge dt, \quad \eta \in \Omega^{k-1}(M)$.

where $f(x,t)$ is a compactly supported function, $dt \in \Omega^1(\mathbb{R})$.
Now, we define π_* in the way it acts on (type A) and (type B) forms:

$$\text{(type A)} \qquad f(x,t)\,\pi^*\eta \xmapsto{\pi_*} 0 \tag{20.6}$$

$$\text{(type B)} \qquad f(x,t)\,\pi^*\eta \wedge dt \xmapsto{\pi_*} \eta \underbrace{\int_{-\infty}^{\infty} f(x,s)ds}_{\equiv F(x)} \tag{20.7}$$

Formally speaking, f is the pullback of F.

Lemma 20.4. *π_* is a chain map, i.e.*

$$\pi_* \circ d = d \circ \pi_* \tag{20.8}$$

where the differential in the left-hand side acts on $M \times \mathbb{R}$, whereas the one in the right-hand side acts on M.

Proof. We prove the lemma for (type A) and (type B) forms, thus it is proved for any form on $M \times \mathbb{R}$ (see remark 20.3).

- (Type A):

$$\pi_* \circ d\,(f(x,t)\,\pi^*\eta) = \pi_* \left(\underbrace{\frac{\partial f}{\partial x}(t,x)dx \wedge \pi^*\eta}_{\text{type A}} + \underbrace{\frac{\partial f}{\partial t}(x,t)\,dt \wedge \pi^*\eta}_{\text{type B}} + \underbrace{f(x,t)\,d\pi^*\eta}_{\substack{\text{type A because} \\ d\pi^* = \pi^* d}} \right) =$$
$$= \int_{-\infty}^{\infty} \frac{\partial f}{\partial s}(x,s)ds\,\pi^*\eta = 0$$

  where we used the definition of π^* for (type B) forms (20.7) and the fact that $\pi_*(\text{type A}) = 0$. Finally, we used the fundamental theorem of integral calculus with the fact that f has compact support. Moreover, $d \circ \pi_*(f(x,t)\,\pi^*\eta) = 0$.

- (Type B):

$$\pi_* \circ d(f(x,t)\,\pi^*\eta \wedge dt) = \pi_* \left(\frac{\partial f}{\partial x}(x,t)dx \wedge \pi^*\eta \wedge dt + f(x,t)\pi^* d\eta \wedge dt \right) =$$
$$= \frac{\partial F}{\partial x}(x,t)dx \wedge \pi^*\eta + F(x)\,\pi^* d\eta$$

  where we used the F defined in (20.7). Moreover,

$$d \circ \pi_*(f(x,t)\,\pi^*\eta \wedge dt) = d(F(x)\,\pi^*\eta) = \frac{\partial F}{\partial x}(x,t)dx \wedge \pi^*\eta + F(x)\,\pi^* d\eta$$

□

Remark 20.5. Consequence of lemma 20.4: π_* defines a map

$$\pi_* \colon H_c^k(M \times \mathbb{R}) \to H_c^{k-1}(M)$$

We want to prove that such a map is an isomorphism. Let's construct the inverse in the following way: let $e \colon \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support such that $\int_{-\infty}^{\infty} e(s)ds = 1$. Then, we define e_* as:

$$\begin{aligned} e_* \colon \Omega_c^{k-1}(M) &\to \Omega^k(M \times \mathbb{R}) \\ \eta &\mapsto e(t)\,\pi^*\eta \wedge dt \end{aligned} \tag{20.9}$$

We have $\pi_* \circ e_* = \mathrm{id}_{\Omega_c^{k-1}(M)}$ **but** $e_* \circ \pi_* \neq \mathrm{id}_{\Omega_c^k(M\times\mathbb{R})}$. Nevertheless, we just need e_* to be the inverse of π_* on $H_c^k(M\times\mathbb{R})$, not on all $\Omega_c^k(M\times\mathbb{R})$. This is true thanks to the two following results.

Lemma 20.6. *The map e_* defined in* (20.9) *is a chain map, i.e.*

$$d \circ e_* = e_* \circ d \tag{20.10}$$

where, again, the differential in the left-hand side acts on $M\times\mathbb{R}$ and the one in the right-hand side acts on M.

Proof. Let $\eta \in \Omega_c^{k-1}(M)$.

$$d \circ e_*(\eta) = d\left(e(t)\pi^*\eta \wedge dt\right) = e(t)d^{M\times\mathbb{R}}\pi^*\eta \wedge dt = e(t)\pi^*(d^M\eta)\wedge dt = e_* d\eta$$

□

Proposition 20.7. *e_*, π_* are mutually inverse isomorphisms on H_c, since there is an operator $K\colon \Omega_c^k(M\times\mathbb{R}) \to \Omega_c^{k-1}(M)$ such that:*

$$\mathbb{1} - e_* \circ \pi_* = (-1)^{k-1}(d \circ K - K \circ d) \tag{20.11}$$

where:

- *K is defined as:*

$$K(f(x,t)\,\pi^*\eta) = 0 \quad \text{on (type A) forms} \tag{20.12}$$

$$K(f(x,t)\,\pi^*\eta\wedge dt) = \int_{-\infty}^{t} f(x,s)ds\,\pi^*\eta - F(x)\left(\int_{-\infty}^{t} e(s)ds\right)\pi^*\eta \text{ on (type B) forms} \tag{20.13}$$

  where η is a k-form in the first case, and it is a $(k-1)$-form in the second case.

- *K is not a chain map (otherwise the right-hand side would be 0)*

Proof. We will prove it for (type B) forms. Then, for (type A) forms it is analogous. Then, recalling the definition of the integration along the fiber F contained in (20.7):

1.

$$\begin{aligned}(\mathbb{1} - e_* \circ \pi_*)(f(x,t)\,\pi^*\eta\wedge dt) &= f\,\pi^*\eta\wedge dt - e_*(F(x)\eta) = \\ &= \underbrace{f\,\pi^*\eta\wedge dt}_{(a)} - \underbrace{F(x)e(t)\pi^*\eta\wedge dt}_{(b)}\end{aligned}$$

2.

$$(d \circ K)(f(x,t)\,\pi^*\eta \wedge dt) = \underbrace{f\,\pi^*\eta \wedge dt}_{(a)}(-1)^{k-1} + \underbrace{\left(\int_{-\infty}^{t} \frac{\partial f}{\partial x}(x,s)ds\right) dx \wedge \pi^*\eta}_{(c)} +$$
$$+ \underbrace{\int_{-\infty}^{t} f(x,s)ds\,\pi^* d\eta}_{(d)} - \underbrace{\left(\frac{\partial F}{\partial x}\int_{-\infty}^{t} e(s)ds\right) dx \wedge \pi^*\eta}_{(e)} +$$
$$- \underbrace{e(t)F(x)\,\pi^*\eta \wedge dt}_{(b)}(-1)^{k-1} - \underbrace{F(x)\left(\int_{-\infty}^{t} e(s)ds\right) \pi^* d\eta}_{(f)}$$

3.

$$(K \circ d)(f(x,t)\,\pi^*\eta \wedge dt) = K\left(\frac{\partial f}{\partial x}(x,t)dx \wedge \pi^*\eta \wedge dt + f(x,t)\pi^* d\eta \wedge dt\right) \overset{(1)}{=}$$
$$= \underbrace{\left(\int_{-\infty}^{t} \frac{\partial f}{\partial x}(x,s)ds\right) dx \wedge \pi^*\eta}_{(c)} - \underbrace{\frac{\partial F}{\partial x}\int_{-\infty}^{t} e(s)ds\,dx \wedge \pi^*\eta}_{(e)} +$$
$$+ \underbrace{\int_{-\infty}^{t} f(x,s)ds\,\pi^* d\eta}_{(d)} - \underbrace{F(x)\int_{-\infty}^{t} e(s)ds\,\pi^* d\eta}_{(f)}$$

Using that: (1) we can exchange integral and derivative, and remember that: $F(x) = \int_{-\infty}^{\infty} f(x,s)ds$.

By comparing the three results, we have the thesis. □

Now, we are finally ready to prove the compactly supported Poincarè lemma.

Lemma 20.8 (Compactly supported Poincarè lemma)**.**

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} 0, & k \neq n \\ \mathbb{R}, & k = n \end{cases} \tag{20.14}$$

Proof. The proof is by induction. The map used to prove the isomorphism for the case $k = n$ is:

$$[\omega] \mapsto \int_{\mathbb{R}^n} \omega \tag{20.15}$$

We notice that the integral is well defined because ω is compactly supported, and the integral does not depend on the element of the equivalence class chosen. In fact, if we consider another n-form ω' such that $[\omega'] = [\omega]$, then it means that ω' and ω differ by an exact form: $\omega' = \omega + d\alpha, \alpha \in \Omega_c^{n-1}(\mathbb{R}^n)$. Let's suppose, without loss of generality, that $\operatorname{supp}\alpha, \operatorname{supp}\omega, \operatorname{supp}\omega' \subset B_{2R}(0)$. Then:

$$\int_{\mathbb{R}^n} \omega' = \int_{B_{2R}(0)} \omega' = \int_{B_{2R}(0)} (\omega + d\alpha) = \int_{B_{2R}(0)} \omega + \int_{B_{2R}(0)} d\alpha \overset{\text{Stokes}}{=} \int_{\mathbb{R}^n} \omega + \underbrace{\int_{\partial B_{2R}(0)} \alpha}_{=0}$$

For $\boxed{\text{n=1}}$, the case $k \neq 1$ coincides with the case $k = 0$: we are considering compactly supported closed 0-forms, i.e. functions which are 0 everywhere. Thus, the only compactly supported closed 0-form which is not exact is the trivial one: $H_c^0(\mathbb{R}) = 0$. As regards the case $k = 1$, by the standard Poincarè lemma (14.9) we know that every closed 1-form on $\mathbb{R}^n$ is exact, i.e. $\omega = d\alpha$, but α might not be compactly supported! Indeed let's write ω as $\omega = f(x)dx$, f compactly supported function. If $\int_{\mathbb{R}} \omega = \int_{\mathbb{R}} f(x)dx = 0$, then the primitive $F(x) = \int_{-\infty}^{x} f(t)dt$ has compact support (for x small we are outside of the support of f, whereas for x big we integrate over all the support, i.e. it's like considering $\int_{\mathbb{R}} f(x)dx = 0$). Then ω is exact: $\omega = dF$ and F has compact support $\Rightarrow [\omega] = 0$. Indeed, this explains why we chose the map (20.15): if the integral of the n-form is 0, the form is exact and it is the differential of a compactly supported $(n-1)$-form. In a similar way, we can choose another closed n-form $\eta = g(x)dx$, g with compact support such that $\int_{\mathbb{R}} \eta = \int_{\mathbb{R}} g(x)dx = c \neq 0$. In this case, the primitive is $G(x) = \int_{-\infty}^{x} g(t)dt$ and $\eta = dG$ (we already knew that such a G existed, thanks to the standard Poincarè lemma), but G is not compactly supported anymore (we have problems for x big). Moreover, for each value of c we get a different primitive. Since we can choose $g(x)$ such that we get any value of $c \in \mathbb{R}$, we have that $H_c^1(\mathbb{R}) \cong \mathbb{R}$. For $\boxed{\text{n} > 1}$: from proposition 20.7, we know that there is an isomorphism

$$H_c^k(M \times \mathbb{R}) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^{k-1}(M) \tag{20.16}$$

Let's suppose the thesis is true for $n-1$, then we can consider $M = \mathbb{R}^{n-1}$ (notice that $\mathbb{R}^n$ is an oriented manifold) and thanks to the isomorphism (20.16) the thesis is true also for n. □

Remark 20.9. The Poincarè lemma 20.8 is valid for $\mathbb{R}^n$, but it also holds for open subsets of manifolds which are diffeomorphic to $\mathbb{R}^n$. For instance, we can consider the convex subsets U of $\mathbb{R}^n$ contained in a manifold. The form $\eta \in \Omega_c^{n-1}(U)$ with $d\eta = \omega$ produced in the proof of the Poincarè lemma for a compactly supported form ω can be extended by zero to a smooth form on the entire manifold. We will not make any distinction between the extended form and the form on U.

We will now analyze some consequences of the compactly supported Poincarè lemma.

Definition 20.10 (Bump form)**.** A bump form on M^n (manifold of dimension n) is a compactly supported n-form $\omega = \rho\, dx^1 \wedge \ldots \wedge dx^n$ in a coordinate domain $(U, \varphi), \varphi\colon U \to \mathbb{R}^n \supseteq \varphi(U)$ so that $\left|\int_U \omega\right| = 1$.

Remark 20.11. The Poincarè lemma tells us that: given ω, ω' bump forms in U, then:

$$\omega = \omega' + d\eta,\ \eta \in \Omega_c^{n-1}(U) \Leftrightarrow \int_U \omega = \int_U \omega' \tag{20.17}$$

Indeed, the "$\Rightarrow$" is always true thanks to Stokes's theorem, whereas the "$\Leftarrow$" follows from the proof of the Poincarè lemma 20.8, because bump forms are n-forms in a n-

dimensional manifold and the equality between the integrals implies that $[\omega] = [\omega']$ (because the isomorphism used in the proof is given by the integral map (20.15)). Thus, the two forms differ by an exact form. If $[\omega] = [\omega']$, the two forms are called *cohomologous*, and we will also write $\omega \sim \omega'$.

Proposition 20.12. *M connected manifold, then any two bump forms are cohomologous (on M) up to a sign.*

Proof. Let's consider two bump forms ω, ω', compactly supported on $(U, \varphi), (U', \varphi')$, respectively. If $(U, \varphi) = (U', \varphi')$, then we have that $\int_U \omega = \int_U \omega'$ up to a sign (by definition 20.10). Then, by the previous remark 20.11 we have that ω and ω' are cohomologous up to a sign. On the other hand, if $(U, \varphi) \neq (U', \varphi')$, we do in the following way: since M is a connected manifold, it is also path-connected (see problem set 12, ex. 1.(e)). Thus, we can find a path between U and U', and we can cover it with open sets $U_1, U_2, \ldots$. Then,

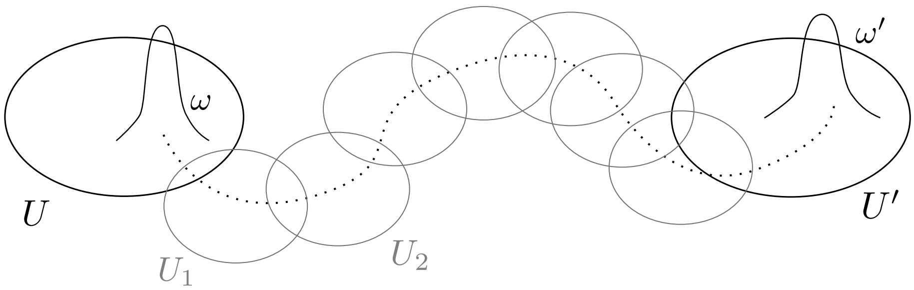


Figure 6: We can cover the path between U and U'

we can consider a bump form ω_1 in $U \cap U_1$, and we can extend it by zero in U. For the same reason seen in the case $U = U'$, we have that ω is cohomologous to ω_1 up to a sign. Again, we can consider a bump form ω_2 in $U_1 \cap U_2$ and extend it by zero in U_1. As before, ω_2 is cohomologous to ω_1 up to a sign, and so on. We continue in this way up to U', and using the transitive property of the equivalence relation we find that ω is cohomologous to ω' up to a sign. $\square$

Lemma 20.13. *Every n-form on M is cohomologous to a locally finite sum of bump forms.*

Proof. Pick a cover $(U_i, \varphi_i), \varphi_i \colon U_i \to \varphi_i(U_i) \subseteq \mathbb{R}^n$, and a locally finite refinement V_j (recall the definition contained in def. 18.37) with a subordinate partition of unity λ_j (i.e. $\sum_j \lambda_j = 1$). Then:

$$\omega = \sum_j (\lambda_j \omega) \sim \sum_{\substack{j \text{ s.t.} \\ \int \lambda_j \omega \neq 0}} \underbrace{\left(\frac{\lambda_j \omega}{\int \lambda_j \omega} \right)}_{\text{bump form}} \cdot \underbrace{\left(\int \lambda_j \omega \right)}_{\text{coefficients}}$$

And the sum if locally finite because we chose a locally finite refinement, which is always possible since manifolds are paracompact (thm 18.38). $\square$

Theorem 20.14. *If M is a connected, smooth manifold of dimension n, such that $\partial M = \emptyset$ and M is oriented, then the map:*

$$\begin{aligned} H^n(M) &\to \mathbb{R} \\ [\omega] &\mapsto \int_M \omega \end{aligned} \tag{20.18}$$

is well defined and surjective.

Proof. Since $\partial M = \emptyset$, we have $\int_M (\omega + d\eta) = \int_M \omega$ by Stokes's theorem. So, the definition does not depend on the element of the equivalence class chosen. Moreover, let (U_i, φ_i) be a chart from the oriented atlas, and let $\omega = f(x_1, \dots, x_n)dx^1 \wedge \dots dx^n$ in U_i, with f smooth function, $f > 0$. Then, $\int_M \omega > 0$. Since ω and $c\omega$ are cohomologous $\forall\, c \in \mathbb{R}$, we have that for every real number λ we can consider a n-form $c\omega$ such that $\int_M c\omega = \lambda$. So, the map is surjective. □

Theorem 20.15. *If M is a connected, smooth manifold of dimension n,* $\dim(H^n(M)) \leq 1$.

Proof. Using the previous theorem 20.14, we just need to verify how many values $\int_U \omega$ can have, where U open subset of M diffeomorphic to $\mathbb{R}^n$ (Notice: $\partial U = \emptyset$ because U is open). Now, every element of $H^n(M)$ is cohomologous to a (real) multiple of a bump form in a fixed chart (U, φ). Bump forms have integral $= \pm 1$. Therefore, the value of the integral of a closed-but-not-exact n-form in $H^n(M)$ can be any real number. Since the map (20.18) is surjective, we have the thesis. □

Definition 20.16. A manifold M is closed if it is compact and $\partial M = \emptyset$.

Theorem 20.17. *Let M be a connected, compact, orientable manifold with $\partial M \neq \emptyset$ and of dimension n. Then $H^n(M) = 0$.*

Proof. It suffices to find a bump form which is cohomologous to zero in M. Once we find it, we have the thesis because M is connected, so any other bump form in the manifold is zero by proposition 20.12. Also, by lemma 20.13, every n-form is cohomologous to a locally finite sum of zero-forms, i.e. it is a zero form and we would have the thesis. Let's find such

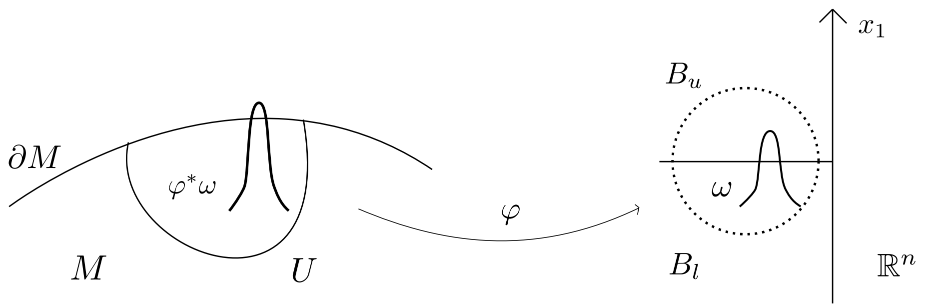


Figure 7: We can find a bump form cohomologous to a zero form

a bump form. First, let's consider a boundary chart $(U, \varphi), \varphi \colon U \to \mathbb{R}^n \cap x_1 < 0$ (such that $\varphi(U)$ is a half-ball) and a bump form ω compactly supported in the interior of the half-ball.

We will also consider the other half-ball. We will call B_l the lower half-ball ($B_l = \varphi(U)$) and we will call B_u the other half. We consider another bump form ω_u contained in B_u such that:

$$\int_{B_u} \omega_u = \int_{B_l} -\omega$$

It is always possible, because bump forms can only have integral $= \pm 1$. Now, we consider the form $\omega_u + \omega$ and we extend it by zero. It is compactly supported in the open ball $B \equiv B_u \cup B_l$. We notice that $\int_B (\omega_u + \omega) = \int_B d\eta$, $\eta \in \Omega_c(B)$ by Stokes's theorem and using that $\partial B = \emptyset$. Then, by remark 20.11 we have that $d\eta = \omega_u + \omega$. Thus:

$$d\eta|_{B_l} = \omega_u|_{B_l} + \omega|_{B_l} = \omega|_{B_l}$$

Then, if $\varphi^*\eta$ is a primitive of the bump form $\varphi^*\omega$. So, $\varphi^*\omega$ is the zero form we were looking for. □

What if we remove orientation? Recall that we can't compute integrals without orientation.

Theorem 20.18. *Let M be a connected but not orientable manifold of dimension n. Then: $H^n(M) = 0$.*

Proof. Let's cover M by an atlas $(U_i, \varphi_i), \varphi_i \colon U_i \to \varphi_i(U_i) = \mathbb{R}^n$ (we choose the atlas such that $\varphi(U_i) = \mathbb{R}^n$). Let's consider an open set U_0. Let's consider another set U_i from

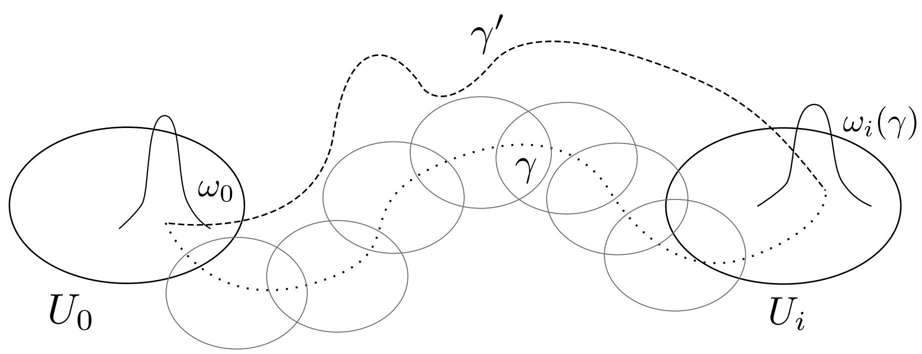


Figure 8: We can consider two paths connecting U_0 and U_i

the cover, $U_0 \neq U_i$. Since M is a connected manifold, it is also path-connected, so there exists a path $\gamma \colon [0,1] \to M$ that connects U_0 to U_i. By the same reasoning of the proof of proposition 20.12, we get that a bump form ω_0 contained in U_0 is cohomologous to a bump form $\omega_i(\gamma)$ contained in U_i. But, since M is not orientable, there also exists a second path γ' connecting U_0 and U_i, and a second chain of charts covering γ', such that, repeating the same procedure, ω_0 is cohomologous to $\omega_{i'}(\gamma')$ with the opposite sign of before (i.e. the integrals have opposite signs). Notice that even if we cannot integrate on M, we can integrate on the charts. It means that:

$$\omega_i(\gamma) \sim \omega_0 \sim \omega_{i'}(\gamma') \sim -\omega_0$$

So, $\omega \sim -\omega \Leftrightarrow 2\omega \sim 0$. □

Remark 20.19. The proofs of the following facts are similar to the previous ones: given a connected manifold M:

- M not compact, $\partial M = \emptyset \Rightarrow H^n(M) = 0$
- M compact $\Rightarrow H_c^*(M) = H^*(M)$
- M not compact, oriented, $\partial M = \emptyset \Rightarrow H_c^n(M) = \mathbb{R}$

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