

**Lecture Notes**  
**Differentiable Manifolds**  
**Ludwig-Maximilians-Universität München**  
**Winter Semester 2019/2020**

Flavio Rossetti

These notes are based on the content of the course *Differentiable Manifolds* held at LMU during the Winter Semester 2019/2020. Some additional content from books and from my own studies was also added. These notes are neither complete nor accurate. They might contain typos and mistakes. You can send an email to [flaviorossetti@outlook.com](mailto:flaviorossetti@outlook.com) to help me improve these notes. Any help would be greatly appreciated!

These notes were compiled on September 22, 2020. For the last version, check <https://github.com/fla-io/diff-manifolds>.

*Thanks to Janik Kruse and Raúl Morral for their help to improve the script!*

# Contents

<b>Contents</b>	<b>I</b>
1 Introduction . . . . .	1
2 Notations and Conventions . . . . .	1
3 Quick review: Basic Algebraic Structures . . . . .	2
4 Quick review: Morphisms . . . . .	3
5 Quick review: Equivalence Classes, Quotient Spaces . . . . .	4
6 Topology . . . . .	6
7 Multilinear Forms . . . . .	14
8 Exterior Product and Generalisation . . . . .	19
9 Differential Forms . . . . .	23
10 Integration of Differential Forms . . . . .	32
11 Tangent Bundles, Flows of Vector Fields . . . . .	33
12 Lie Derivative of a Vector Field . . . . .	37
13 Stokes' Theorem on $\mathbb{R}^n$ . . . . .	39
14 Closed and Exact $k$ -forms . . . . .	41
15 De Rham Cohomology . . . . .	49
16 Submanifolds of $\mathbb{R}^n$ . . . . .	55
17 Integration on Compact Manifolds . . . . .	65
18 Abstract Manifolds . . . . .	68
19 Lie Groups . . . . .	75
20 Compactly Supported Cohomology . . . . .	81
21 Poincaré Duality . . . . .	93
22 Riemannian Metric and the Hodge- $\star$ Operator . . . . .	94
<b>Bibliography</b>	<b>102</b>
Bibliography . . . . .	102

## 1 ■ Introduction

From the Lee book [Lee]: "The central idea of calculus is *linear approximation*". A function of one variable can be approximated by its tangent line, a curve by a tangent vector (i.e. velocity vector), a surface in  $\mathbb{R}^3$  can be approximated by its tangent plane, and a map from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  by its total derivative. Here it comes the importance of tangent spaces.

Main idea: in order to study tangent vectors, we identify them with "directional derivatives". In particular, there is a natural one-to-one correspondence between geometric tangent vectors and linear maps from  $C^\infty(\mathbb{R}^n)$  to  $\mathbb{R}$  satisfying the product rule. Such maps are called *derivations*.

*Remark 1.1. Points or vectors?* We can think of elements of  $\mathbb{R}^n$  either as points or vectors. As points, their only property is their location, given by the coordinates  $(x_1, \dots, x_n)$  on a chosen basis. As vectors, they are characterized by a direction and a magnitude, but their location is irrelevant (translational invariance). So given  $v \in \mathbb{R}^n, v = \sum_i v^i e_i = v^i e_i$ , it can be seen as an arrow with its initial point anywhere in  $\mathbb{R}^n$ . So, if we think about a vector tangent to the border of the sphere at a point  $a$ , we imagine the vector as living in a copy of  $\mathbb{R}^n$  with its origin translated to  $a$ .

## 2 ■ Notations and Conventions

In these notes the Einstein summation convention will be used. It means that the sum symbol will be omitted when it is clear with respect to which index we are summing. For instance we will write

$$v^i e_i$$

instead of

$$\sum_i v^i e_i.$$

Given a map  $f$  from a set  $X$  to a set  $Y$ , we will denote it by

$$f: X \rightarrow Y.$$

We will use the arrow " $\mapsto$ " to denote how each element of  $X$  is mapped into  $Y$  through  $f$ . For instance, given  $x_1, x_2 \in X, y \in Y$ , we will either use the notation:

$$x_1 \mapsto f(x_1)$$

or

$$x_2 \mapsto y.$$

We will use the symbol  $\equiv$  in order to define mathematical objects. We will use the latin abbreviation "i.e." (*id est*) to explain concepts in a more detailed way, and the abbreviation "e.g." (*exempli gratia*) to give examples.

### 3 ■ Quick review: Basic Algebraic Structures

**Definition 3.1** (Operation). Let  $G$  be a set. The map:

$$\begin{aligned} \cdot : G \times G &\longrightarrow G \\ (a, b) &\longmapsto a \cdot b \end{aligned}$$

is called a (binary) operation on  $G$ . Sometimes such a map is also denoted by the symbol  $+$ .

**Definition 3.2** (Group). Given a set  $G$  and an operation  $\cdot$  on such set, we will call such set with the operation (i.e. the couple  $(G, \cdot)$ ) a group if the following properties are satisfied for all  $a, b, c \in G$ :

- $a \cdot b \in G$ , *(closure property)*
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ , *(associativity)*
- $\exists \mathbf{1} \in G$  such that  $\mathbf{1} \cdot g = g, \forall g \in G$ , *(existence of the identity element)*
- $\forall g \in G, \exists g^{-1} \in G$  so that  $g \cdot g^{-1} = g^{-1} \cdot g = \mathbf{1}$ . *(existence of the inverse element)*

For the sake of simplicity, we will often call  $G$  a group, without referring to the operation on it.

*Example 3.3.*  $(\mathbb{Z}, +)$  is a group.

*Example 3.4.*  $(\mathbb{R}, +)$  is a group. Also:  $(\mathbb{R} \setminus \{0\}, \cdot)$  is a group.

*Example 3.5.*  $(\mathbb{N}, +)$  is not a group!

*Remark 3.6.* The identity element of a group is often denoted as  $\mathbf{1}$  if the operation is denoted by the symbol  $\cdot$ , whereas it is denoted as  $\mathbf{0}$  if the operation is denoted by the symbol  $+$ . In a similar way, the inverse element is often denoted as  $g^{-1}$  if the operation is denoted by the symbol  $\cdot$ , whereas it is denoted as  $-g$  if the operation is denoted by the symbol  $+$ .

**Definition 3.7** (Abelian group). A group  $(G, \cdot)$  is called abelian if its elements commute according to the operation  $\cdot$ , i.e.  $a \cdot b = b \cdot a, \forall a, b \in G$ .

Often we can consider sets with two operations, like  $(\mathbb{R}, +, \cdot)$ . If they satisfy some properties, they are called rings. If they satisfy even more properties, they are called fields. In particular:

**Definition 3.8** (Ring). Given a set  $R$  and two operations:  $+$  (usually called "additive operation") and  $\cdot$  (called "multiplicative operation") on it, we will call the set with the two operations, i.e.  $(R, +, \cdot)$ , a ring if the following properties are satisfied:

- $(R, +)$  is an abelian group,
- $\cdot$  is associative, i.e.  $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \forall a, b, c, \in R$ ,
- $\cdot$  is distributive with respect to  $+$ , i.e.  $a \cdot (b + c) = a \cdot b + a \cdot c$  and  $(b + c) \cdot a = b \cdot a + c \cdot a$ .

**Definition 3.9** (Commutative ring).  $(R, +, \cdot)$  is a commutative ring if the multiplication operation  $\cdot$  is commutative.

**Definition 3.10** (Unitary ring).  $(R, +, \cdot)$  is a unitary ring if it contains the multiplicative identity, i.e.  $\exists \mathbf{1} \in R$  such that  $\mathbf{1} \cdot r = r \cdot \mathbf{1} = r, \forall r \in R$ .

**Definition 3.11** (Invertible element). An element  $a$  in a ring  $(R, +, \cdot)$  is called invertible if there exists its inverse according to the multiplicative operation, i.e.  $\exists b \in R$  such that  $a \cdot b = b \cdot a = \mathbf{1}$ . We will also write  $a^{-1} \equiv b$ .

**Definition 3.12** (Field). A unitary, commutative ring such that every non-zero element is invertible is called a field.

*Remark 3.13.* When we talk about the zero element in a field or in a ring we always refer to the identity element according to the additive operation. See also remark 3.6.

*Example 3.14.* The set of  $2 \times 2$  matrices with real coefficients is a ring with the operations of sum between matrices and with the matrix multiplication. The ring is not commutative because the matrix multiplication is not a commutative operation. It is unitary, since the identity matrix is the multiplicative identity. But it's not a field because not every matrix is invertible (just take a matrix with zero determinant).

*Example 3.15.*  $(\mathbb{Z}, +, \cdot)$  is a unitary, commutative ring. It is not a field since the inverse element for  $\cdot$  is often in the rational numbers, i.e.  $3^{-1}$  is the multiplicative inverse of 3.

*Example 3.16.*  $(\mathbb{R}, +, \cdot)$  is a field.

## 4 ■ Quick review: Morphisms

**Definition 4.1** (Homomorphism). A homomorphism  $h$  between two sets endowed with structures (think about two groups, for instance) is a map which preserves the structure,

i.e. a map

$$h: (G, +) \longrightarrow (H, \circ)$$

such that  $h(a + b) = h(a) \circ h(b), \forall a, b \in G$ , where  $(G, +)$  and  $(H, \circ)$  are two groups. The definition for rings and fields is analogous. People from category theory also tend to call (group) homomorphisms just (group) morphisms since the morphisms in the category **Grp** of all groups are set to be group homomorphisms.

Now, we can consider some particular types of homomorphisms.

*Remark 4.2* (Monomorphism, epimorphisms and isomorphisms). Homomorphisms can be called in different ways depending whether they are injective, surjective or bijective. A monomorphism is an injective homomorphism. An epimorphism is a surjective homomorphism. An isomorphism is a bijective homomorphism (thus, it is both a monomorphism and an epimorphism).

*Remark 4.3* (Endomorphism, automorphism). An endomorphism is a homomorphism from one algebraic structure to itself. If such a morphism is bijective, it is called automorphism.

*Remark 4.4.* Don't get confused with homeomorphisms! A homeomorphism is a continuous and bijective map between two topological spaces, such that its inverse is also continuous. In general it is not a homomorphism (without the **e**).

## 5 ■ Quick review: Equivalence Classes, Quotient Spaces

**Definition 5.1** (Binary relation). Given a set  $E$ , a (binary) relation  $\sim$  on  $E$  is a set of couples  $(a, b) \in E \times E$ . In other words, a binary relation is a subset of  $E \times E$ . Moreover, if  $(a, b) \in \sim \subset E \times E$ , we use the following notation:  $a \sim b$ .

**Definition 5.2** (Equivalence relation). Given a set  $E$ , an equivalence relation  $\sim$  is a binary relation which satisfies the following properties:

- $a \sim a, \forall a \in E$ , *(reflexive property)*
- $a \sim b \Rightarrow b \sim a, \forall a, b \in E$ , *(symmetric property)*
- $a \sim b, b \sim c \Rightarrow a \sim c, \forall a, b, c \in E$ . *(transitive property)*

*Example 5.3.* The following ones are examples (or counter-examples) of equivalence relations:

- "Being equal to" (i.e. " $\sim$ " is " $=$ ") is an equivalence relation on  $\mathbb{R}$ :  $x = x \forall x \in \mathbb{R}$ ;  $x = y \Rightarrow y = x \forall x, y \in \mathbb{R}$ , etc.

- "Has the same birthday as" on the set of all people in the world is an equivalence relation.
- "Having mutual friends on Facebook" on the set of Facebook users is not an equivalence relation (there is some problem with transitive property).
- "Being greater or equal to" on  $\mathbb{R}$  is not an equivalence relation (it is not symmetric, however it is antisymmetric and so it is called "partial order relation").

**Definition 5.4** (Equivalence class). Given a set  $E$  and an equivalence relation  $\sim$  on  $E$ , the equivalence class of  $x \in E$  is the set  $[x]$ , where

$$[x] \equiv \{y \in E \mid y \sim x\} . \quad (5.1)$$

The equivalence class  $[x]$  always contains  $x$  itself, by definition of equivalence relation.

*Example 5.5.* We can examine the set of integers which differ (in absolute value) by a multiple of 4. If we consider the equivalence relation " $x \sim y$  if  $x - y = 4n$  for some  $n \in \mathbb{N} \cup \{0\}$ " on the set of  $\mathbb{Z}$ , the equivalence class of 1 is  $[1] = \{\dots, -7, -3, 1, 5, 9, \dots\}$ . We can also write " $x - y = 4n$  for some  $n$ " as  $x \equiv_4 y$  (congruence modulo 4).

**Definition 5.6** (Quotient space). Given a set  $E$  and an equivalence relation  $\sim$  on  $E$ , the quotient space with respect to  $\sim$  is denoted by  $E/\sim$  and it is the set of all equivalence classes:

$$E/\sim \equiv \{[x] \mid x \in E\} . \quad (5.2)$$

Intuitively, we can say that the quotient space  $E/\sim$  is a copy of  $E$  where all the elements equivalent to each other are reduced to one point.

*Example 5.7.* Let's consider two examples:

- If we consider the example 5.5,  $\mathbb{Z}/\sim = \{[0], [1], [2], [3]\}$ . We notice that all the integers that differ from 0 by a multiple of 4 (in absolute value) are identified with one point (which is the equivalence class  $[0]$ ). All the integers that differ from 1 by a multiple of 4 (in absolute value) are identified with one point (the equivalence class  $[1]$ ), and so on. In this case, we also write  $\mathbb{Z}/\sim = \mathbb{Z}/_4\mathbb{Z}$ .
- If  $V$  is a vector space and  $U$  is a vector subspace of  $V$ , then we can consider the equivalence relation " $x \sim y$  if  $x - y \in U$ ". We notice that  $x \sim x$  for each  $x \in V$  since  $x - x = 0 \in U$ , because  $U$  is a vector space as well. If  $V$  can be decomposed as a direct sum  $V = U \oplus W$ , where  $W$  is another vector space, then  $V/U$  is isomorphic to  $W$ .
- If  $V$  and  $U$  are just two groups instead of being vector spaces, finding the



quotient space requires a bit more effort.

## 6 ■ Topology

**Definition 6.1** (Topology, open and closed sets, neighbourhood). A topology on a set  $M$  is a subset  $\mathcal{O} \subset \mathcal{P}(M)$ , where  $\mathcal{P}(M)$  is the power set of  $M$ , such that:

1.  $\emptyset, M \in \mathcal{O}$ ,
2.  $\mathcal{O}$  is closed under arbitrary unions:  $U_i \in \mathcal{O}, i \in I$ , then  $\bigcup_{i \in I} U_i \in \mathcal{O}$ ,
3.  $\mathcal{O}$  is closed under finite intersections:  $U_1, \dots, U_k \in \mathcal{O}$ , then  $U_1 \cap \dots \cap U_k \in \mathcal{O}$ .

Sets in  $\mathcal{O}$  are called open sets. A set  $A \subset M$  is closed if  $M \setminus A$  is open. A subset  $V \subset M$  containing  $p$  is a neighbourhood of  $p$  if there is an open set  $U \subset V$  containing  $p$ . Sometimes we will denote the topology on a set  $X$  by the symbol  $\mathcal{O}_X$ .

*Remark 6.2* (Euclidean topology on  $\mathbb{R}$ ). The Euclidean topology is the one we implicitly use every time we want to talk about subsets of  $\mathbb{R}^n$ . Indeed, we use it whenever we talk about "open sets", "limit points" or "open balls" in  $\mathbb{R}^n$ . In  $\mathbb{R}$ , it can be defined by specifying which sets are open, in the following way: *in the Euclidean topology on  $\mathbb{R}$ , a subset  $U \subset \mathbb{R}$  is open if and only if it is union of open intervals*. It satisfies the definition 6.1 of topology, indeed:

1.  $\emptyset$  is the null union of open intervals, and  $\mathbb{R} = \bigcup_n ]-n, n[$ ,

2. if

$$A = \bigcup_{i \in I} ]a_i, b_i[$$

$$B = \bigcup_{j \in J} ]c_j, d_j[$$

are two open sets, then their union is still union of open intervals. Same result if we consider an arbitrary number of open sets  $A, B, C, \dots$

3. if  $A$  and  $B$  as before, then their intersection is

$$A \cap B = \bigcup_{\substack{i \in I \\ j \in J}} ]a_i, b_i[ \cap ]c_j, d_j[$$

which is still union of open intervals since the intersection of two open intervals is open.

In  $\mathbb{R}^n$ , a possible definition of this topology would be through the concept of basis. Alternatively, you could define it by explicitly considering  $n$  copies of  $\mathbb{R}$  and then considering the product topology. These concepts will be treated in the next pages.

*Remark 6.3.* From the definition above we can see that  $M$  and  $\emptyset$  are always both open and closed sets. Notice that a set in a topological space can be:

- *both open and closed*, e.g. the set  $\mathbb{R}$  (in the Euclidean topology on  $\mathbb{R}$ ) is open by definition of topology, and it is also closed because its complement is open,
- *neither open nor closed*, e.g. the interval  $[0, 2[$  in the Euclidean topology on  $\mathbb{R}$  (it cannot be written as union of open intervals, and neither can its complement),
- *open but not closed*, e.g.  $]0, 2[$  again in the Euclidean topology on  $\mathbb{R}$ ,
- *closed but not open*, e.g.  $[0, 2]$  in the same topological space.

**Definition 6.4** (Closure, interior). Let  $V \subset M$ , let  $\mathcal{O}$  be an arbitrary topology. The closure  $\bar{V}$  of  $V$  is the smallest closed subset of  $M$  containing  $V$ . The interior  $\overset{\circ}{V}$  of  $V$  is the largest open subset contained in  $V$ .

**Definition 6.5** (Continuous function, homeomorphism). A map  $f: X \rightarrow Y$  between topological spaces is continuous if  $f^{-1}(U) \in \mathcal{O}_X$  for all  $U \in \mathcal{O}_Y$  (i.e. if the pre-image of an open set through  $f$  is still an open set).  $f$  is a homeomorphism if it is bijective, continuous and  $f^{-1}$  is continuous.

*Remark 6.6* (Continuous maps vs. open maps). Requiring that the pre-image of an open set is open is different from requiring that the image of an open set is open! If  $f$  brings open sets to open sets, it is called an open map (and it might not be continuous).

*Example 6.7.* Given a set  $M$ , we can always consider two simple topologies on it:

1.  $\mathcal{O}_M = \mathcal{P}(M)$  is a topology, where  $\mathcal{P}(M)$  is the power set of  $M$  (i.e. the family of all subsets of  $M$ ). Thus, every subset of  $M$  is open, so it is easy to see that any function going from  $(M, \mathcal{O}_M)$  to any space  $(Y, \mathcal{O}_Y)$  is continuous, where  $\mathcal{O}_Y$  can be any topology on  $Y$ .
2.  $\mathcal{O}_M = \{\emptyset, M\}$  is a topology: the open sets are just the two ones required by the definition 6.1. In this case, every function going from any space  $(Y, \mathcal{O}_Y)$  to  $(M, \mathcal{O}_M)$  is continuous (the pre-image of the empty set is the empty set, and the pre-image of  $M$  is  $Y$ ).

What is more, we can consider topologies on metric spaces: if  $(M, d)$  is a metric space and  $B_\varepsilon(x)$  denotes the ball of radius  $\varepsilon$  around  $x$ , then:

$$\mathcal{O}_d = \{U \subset M \mid \forall x \in U \exists \varepsilon > 0: B_\varepsilon(x) \subset U\}$$

defines a topology. In particular, we say that the topology is induced by a metric if the open sets are the open balls of the metric space. If  $X, Y$  are metric spaces, the definition of continuity seen in the Analysis courses is:  $f: (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is continuous if and only if  $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0$  such that  $f(B_\delta(x)) \subset B_\varepsilon(f(x))$ . One can show that in metric spaces such a definition is equivalent to definition 6.5.

*Remark 6.8.* If  $\mathcal{O}_1, \mathcal{O}_2$  are topologies on  $M$ , then the intersection of the two topologies is again a topology. Therefore, given a family of subsets of  $M$  one can ask for the smallest topology on  $M$  which still contains one of those subsets (one of such topologies always exists and it is the power set topology).

**Definition 6.9** (Hausdorff space). A topological space  $(M, \mathcal{O}_M)$  is Hausdorff if for all  $p, q \in M$  there is a pair of open sets  $U_p, U_q$  so that  $U_p \cap U_q = \emptyset$ .

*Remark 6.10.* Points in Hausdorff spaces are closed sets. i.e. if  $M$  is Hausdorff and  $x \in X$ ,  $\{x\}$  is a closed set. In order to see that, we just need to prove that  $M \setminus \{x\}$  is open. Let's suppose  $M \setminus \{x\} \neq \emptyset$  (otherwise we are done, since  $M = \{x\}$  and  $M$  is both closed and open). Then we can find another point  $y \in M \setminus \{x\}$ . Since  $M$  is Hausdorff, we can find two open sets  $U_x, U_y$  containing  $x$  and  $y$  respectively, such that  $U_x \cap U_y = \emptyset$ . The proof is over if we prove that  $M \setminus \{x\} = \cup_{y \in M \setminus \{x\}} U_y$ , since it implies that  $M \setminus \{x\}$  is open (arbitrary union of open sets is open), and then  $\{x\}$  is closed. It is true because:

- " $M \setminus \{x\} \subseteq \cup_{y \in M \setminus \{x\}} U_y$ ": for each point  $z$  in  $M$  different from  $x$ , there exists an open set  $U_z$  entirely contained in  $M \setminus \{x\}$  by Hausdorff assumption. The point  $z$  is contained in  $U_z$  and  $U_z$  is an element of the union  $\cup_{y \in M \setminus \{x\}} U_y$ .
- " $M \setminus \{x\} \supseteq \cup_{y \in M \setminus \{x\}} U_y$ ": if  $z \in \cup_{y \in M \setminus \{x\}} U_y$ , then  $z \neq x$ , otherwise there would be an intersection between an open set  $U_{\bar{x}}$  for a fixed  $\bar{x}$  and every open set containing  $x$ . This is impossible because by Hausdorff assumption there exists at least one open sets of those which does not intersect  $U_{\bar{x}}$ .

Moreover:

- $(M, \mathcal{P}(M))$  is always Hausdorff (for  $x \neq y$ , the sets  $\{x\}$  and  $\{y\}$  are open and disjoint).
- If the topology is induced by a metric, then the topological space is Hausdorff (for instance,  $\mathbb{R}^n$  with the topology induced by Euclidean metric is a Hausdorff topological space).

**Definition 6.11** (Product topology). Let  $(M_i, \mathcal{O}_i), i \in I$  be topological spaces. Then the product topology on  $\prod_{i \in I} M_i$  is the smallest topology such that all projections

$$\pi_j: \prod_{i \in I} M_i \rightarrow M_j$$

are continuous maps for all  $j \in I$ .

**Definition 6.12** (Quotient topology). Let  $(M, \mathcal{O}_M)$  be a topological space and  $\sim$  an equivalence relation on  $M$ . Then the quotient topology on  $M/\sim$  is the largest topology so that the function

$$\begin{aligned} \pi: M &\rightarrow M/\sim \\ x &\mapsto [x] \end{aligned}$$

is continuous.  $\pi$  is also called canonical projection.

*Remark 6.13* (Quotient topology pt. 2). We can also use another useful characterization of the quotient topology: a set  $U \subset M/\sim$  is open if and only if  $\pi^{-1}(U) \in \mathcal{O}_M$ . It is easy to prove that this definition is equivalent to the one above. Let's call (1) the first definition and (2) the last one.

- "(1)  $\Rightarrow$  (2)":  $U$  open in the quotient topology, then  $\pi^{-1}(U)$  is open in  $M$  by (1), because  $\pi$  is a continuous map. If  $\pi^{-1}(U)$  is open in  $M$ , then  $U$  is open in  $M/\sim$  because in (1) we have defined the topology on  $M/\sim$  as the *largest* topology such that  $\pi$  is continuous!
- "(2)  $\Rightarrow$  (1)":  $\pi$  is obviously continuous. And the topology we get is the *largest* one such that  $\pi$  is continuous because if  $\pi^{-1}(U)$  is not open in  $M$ , then  $U$  is not open in the quotient space (read also below).

**Note that:** The "largest topology on  $Y$  such that  $f: X \rightarrow Y$  is continuous" means that the topology on  $Y$  contains all subsets of  $Y$  (cf. power set topology, remark 6.7) except for the sets that cannot be open because of the continuity assumption. Indeed, the continuity assumption says:  $U$  open in  $Y \Rightarrow f^{-1}(U)$  open in  $X$ . This is equivalent to:  $f^{-1}(U)$  **not** open in  $X \Rightarrow U$  **not** open in  $Y$ . So if we take the preimage of a set  $U$  and we notice that such a preimage is not an open set, then  $U$  cannot be open in  $Y$ : it has to be removed from the topology we are considering on  $Y$ .

*Example 6.14* (Cofinite topology). Let  $M$  be a set. We define the following topology: the open sets are  $\emptyset$  and the sets  $U \subset M$  such that  $U \neq \emptyset$  and that their complement in  $M$  is finite. Another way to define the same topology is using closed sets: the only closed sets in this topology are  $M$  and the finite sets. This topology is called cofinite topology.  $M$  with this topology is not Hausdorff if  $M$  is an infinite set. Why?

Because<sup>a</sup>: given  $x, y$  in  $M$ , with  $x \neq y$ , if  $U_a$  and  $U_b$  are open sets that contain  $a$  and  $b$  respectively, the complements of  $U_a$  and  $U_b$  must be finite (so,  $U_a$  and  $U_b$  must be infinite, because the bigger space  $M$  is infinite). But then  $U_a \cap U_b \neq \emptyset$ . Indeed, the complement of  $U_a$  in  $M$  is  $M \setminus U_a$  and it must be finite because  $U_a$  is open. If we assume  $U_a \cap U_b = \emptyset$ , then  $U_b$  must be contained in the complement of  $U_a$ . But then  $U_b$  (which is an infinite set, because it is open) would be contained in  $M \setminus U_a$  (which is a finite set). Contradiction!

<sup>a</sup>the proof is easier to follow if you think of the space  $M$  as  $\mathbb{R}$ .

**Lemma 6.15.** *A subspace of a Hausdorff space is Hausdorff.*

*Example 6.16* (Subspace topology). Let  $(M, \mathcal{O}_M)$  be a topological space and  $N \subset M$  a subset. Then a natural topology for  $N$  is the subspace (or subset) topology, defined in the following way. The subspace topology is the smallest topology so that the inclusion map  $N \rightarrow M$  is continuous, i.e.  $V \subset N$  is open if and only if there is an open set  $U \subset M$  such that  $U \cap N = V$ .

For instance, you may know that  $[0, 1)$  is neither open nor closed in  $\mathbb{R}$  with Euclidean topology  $\mathcal{O}_d$ . Actually, it is an open set in  $([0, 2], \mathcal{O})$ , where  $\mathcal{O}$  is the subset topology (with respect to  $(\mathbb{R}, \mathcal{O}_d)$ ). Indeed:

$$[0, 1) = (-1, 1) \cap [0, 2].$$

You should always pay attention to the topology and to where it is defined, for instance  $[0, 1]$  is closed (but not open) in  $(\mathbb{R}, \mathcal{O}_d)$ , whereas it is both open and closed in  $([0, 1], \mathcal{O})$ , for any topology  $\mathcal{O}$ .

**Definition 6.17** (Subbasis). Let  $M$  be a set and  $B \subset \mathcal{P}(M)$  some collection of subsets. Let  $\mathcal{O}_B$  be the smallest topology containing  $B$ , then we say that  $B$  is a subbasis of  $\mathcal{O}_B$ .

**Definition 6.18** (Basis). Let  $(M, \mathcal{O})$  be a topological space and  $\mathcal{B} \subset \mathcal{O}$ . Then  $\mathcal{B}$  is a basis for  $\mathcal{O}$  if every open set is a union of sets in  $\mathcal{B}$ .

**Definition 6.19** (Second countability). Let  $(M, \mathcal{O}_M)$  be a topological space. Then  $(M, \mathcal{O}_M)$  is second countable if it admits a countable basis.

*Example 6.20* (Euclidean topology in  $\mathbb{R}^n$ ). Now we can properly define the Euclidean topology on  $\mathbb{R}^n$ . In any metric space, the open balls form a topology for that space, and such a topology is also called metric topology. The Euclidean topology on  $\mathbb{R}^n$  is the one generated by the open balls (using Euclidean metric). In particular, let  $\mathcal{B} = \{B_\varepsilon(x) \mid \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}^n\}$ . This is a basis of the metric topology of  $\mathbb{R}^n$  (and it is a countable set, because we used the density of  $\mathbb{Q}$  in  $\mathbb{R}$ ).

**Lemma 6.21.** *If  $A \subset M$  and  $M$  has a countable basis, then the subspace topology on  $A$  has a countable basis.*

**Definition 6.22** (Cover). Let  $J$  be a set of indices, either finite or infinite. A family of sets  $\{U_i\}_{i \in J}$  is a cover for a topological space  $X$  if  $\bigcup_i U_i = X$  (the sets "cover" the entire space). It is an open cover if the  $U_i$  are open sets in the topology of  $X$ . A subcover is a subfamily of the initial cover which is still a cover.

**Definition 6.23** (Compact space). A topological space  $X$  is compact if every open cover  $\{U_i\}_{i \in J}$  admits a finite subcover  $\{U_i\}_{i=1}^k$ .

*Example 6.24.* If  $(X, d)$  is a metric space and is compact, then  $X$  is bounded (i.e. contained in a ball). Indeed:  $U_i = B_i(x) = \{y \in X \mid d(x, y) < i\}$  is an open set. And for a fixed  $x \in X$ ,  $\cup_i U_i$  is an open cover:  $\cup_{i \in \mathbb{N}} U_i = X$ . Since the space is compact  $\Rightarrow U_{i_1} \cup \dots \cup U_{i_k} = X$  for some indices  $i_1, \dots, i_k$ . These indices are the radii of the balls. Take  $c = 2 \max\{i_1, \dots, i_k\}$ , then we have that  $d(x, y) < c \forall y \in X$ , i.e.  $X$  is bounded because it is contained in the ball  $B_c(x)$ .

**Theorem 6.25** (Heine-Borel). *A subset of  $\mathbb{R}^n$  is compact if and only if it is closed and bounded.*

**Corollary 6.26** (Weierstrass). *Let  $X$  be a compact space,  $f: X \rightarrow \mathbb{R}$  continuous, then  $f$  attains its maximum (and minimum) in  $X$ .*

**Lemma 6.27** (Closed in a compact).  *$X$  is a compact topological space,  $A \subset X$  closed  $\Rightarrow A$  is compact (in subspace topology).*

*Proof.* Let  $\{U_i\}_{i \in I}$  be an open cover of  $A$ . There are open sets  $V_i \subset X$  with  $V_i \cap A = U_i$  by definition of open sets in the subspace topology. Moreover,  $V_A \equiv X \setminus A$  is open because  $A$  is closed. Since  $\{U_i\}_{i \in I}$  is an open cover of  $A$ , we have that  $\{U_i\}_{i \in I}$  and  $V_A$  form an open cover of  $X$ . Since  $X$  is compact, there is a finite number of indices  $i_1, \dots, i_k \in I$  such that  $V_{i_1}, \dots, V_{i_k}, V_A$  cover  $X$ . Since  $V_A = X \setminus A$ , it follows that the sets  $V_{i_1}, \dots, V_{i_k}$  cover  $A$ . Since  $U_i = V_i \cap A$ , also  $U_{i_1}, \dots, U_{i_k}$  cover  $A$ . Then  $A$  is compact because we proved that every open cover admits a finite subcover.  $\square$

**Lemma 6.28** (Cont. functions map compact sets to compact sets). *If  $A$  is compact and  $f: A \rightarrow Y$  is continuous, then  $f(A)$  is compact.*

*Proof.* Let  $\{U_i\}_{i \in I}$  be an open cover of  $f(A)$ . Then  $\exists \{V_i\}_{i \in I}, V_i \subset Y$  open, so that  $U_i = V_i \cap f(A)$  (by definition of open sets in subspace topology). Let  $V = \bigcup_i V_i$ . Since  $f$  is continuous,  $f^{-1}(V) \subset A$  is open, and  $f^{-1}(V)$  covers  $A$ . Since  $A$  is compact, there exist  $i_1, \dots, i_k$  such that  $f^{-1}(V_{i_1}), \dots, f^{-1}(V_{i_k})$  still cover  $A$ . Thus,  $V_{i_1}, \dots, V_{i_k}$  is a collection of open sets in  $Y$  covering  $f(A)$ . In the subspace topology,  $U_{i_1} = V_{i_1} \cap f(A), \dots, U_{i_k} = V_{i_k} \cap f(A)$  cover  $f(A)$ .  $\square$

**Lemma 6.29** (Compact set in a Hausdorff). *Let  $X$  be Hausdorff. Let  $A \subset X$ . Assume  $A$  is compact. Then  $A$  is closed.*

*Proof.* Let  $x \in X \setminus A$ . For every  $a \in A$ , there are  $U_a \subset X, V_x(a) \subset X$  open sets such that  $U_a \cap V_x(a) = \emptyset$  (because of the Hausdorff hypothesis). Here, we used the notation  $V_x(a)$  to stress out the dependence of  $V$  on  $x$  and  $a$ . The sets  $\{U_a \cap A\}_{a \in A}$  form an open cover of  $A$ . Since  $A$  is compact, there are finitely many  $a_1, \dots, a_k$  so that  $U_{a_1} \cap A, \dots, U_{a_k} \cap A$  cover  $A$ . Now,  $V_x = V_x(a_1) \cap \dots \cap V_x(a_k)$  contains  $x$  and it is open. Moreover,  $V_x \cap A = \emptyset$  because  $V_x$  is disjoint from each  $U_{a_i} \Rightarrow X \setminus A$  is open, because for each point we can find an open set that contains the point and that is contained in  $X \setminus A$ .  $\square$

**Theorem 6.30** (How to get a homeomorphism). *Let  $f: X \rightarrow Y$  be a continuous function,  $X$  compact,  $Y$  Hausdorff and  $f$  bijective. Then  $f$  is a homeomorphism.*

*Proof.* The only thing which is left to prove is that  $f^{-1}$  is a continuous function. First, we notice that an equivalent definition of continuous function is: " $f: X \rightarrow Y$  is continuous  $\Leftrightarrow f^{-1}(C)$  is closed  $\forall$  closed set  $C \subset Y$ ". In order to prove that  $f^{-1}$  is continuous, we need to show that  $f(C) \subset Y$  is closed for all  $C \subset X$  closed. This is true because:  $C$  closed in  $X \Rightarrow C$  compact because of lemma 6.27  $\Rightarrow f(C)$  is compact in  $Y$  because of lemma 6.28  $\Rightarrow f(C)$  is closed, using the lemma 6.29.  $\square$

*Remark 6.31.* We can consider the concept of sequential compactness:  $X$  is sequentially compact if every sequence in  $X$  contains a convergent subsequence in  $X$ . In general,  $X$  compact  $\not\Rightarrow X$  sequentially compact. But,  $X$  compact + first countable  $\Rightarrow X$  sequentially compact. Here,  $X$  is first countable if  $\forall x \in X$  there exists a countable collection  $N_1, N_2, \dots, N_i$  of open sets containing  $x$  such that every neighbourhood of  $x$  contains one of the  $N_i$ . Notice that manifolds are first countable! (not proved.)

**Definition 6.32** (Compact exhaustion). Let  $X$  be a topological space. A compact exhaustion of  $X$  is a countable collection of compact sets  $\{K_i\}_{i \in \mathbb{N}}$  such that  $\bar{K}_i \subset \overset{\circ}{K}_{i+1}$  and  $\bigcup_{i=1}^{\infty} K_i = X$ .

**Definition 6.33** (Paracompactness). A topological space is paracompact if every open cover admits a locally finite refinement. In other words: if  $\{U_i\}_{i \in I}$  is an open cover of  $X$ , there is another open cover  $\{V_j\}_{j \in J}$  of  $X$  such that  $V_j \subset U_{i(j)}$  for every  $j$  and for some index  $i(j)$  (refinement) and, moreover, every point  $x$  has a neighbourhood  $E_x$  such that  $E_x$  intersects only a finite number of  $V_j$ 's (locally finite).

**Definition 6.34** (Partition of unity). A partition of unity subordinate to an open cover  $\{U_i\}_{i \in I}$  of  $X$  is a collection of continuous functions  $f_j: X \rightarrow [0, 1]$  such that  $\forall x \exists V_x \ni x, V_x$  neighbourhood of  $x$  so that the support  $\text{supp}(f_j) = \overline{f_j^{-1}((0, 1])}$  is contained in one of the

$U_i$ , and  $V_x$  intersects only finitely many  $\text{supp}(f_j)$ , and

$$\sum_{j \in J} f_j(x) \equiv 1.$$

**Definition 6.35** (Connected space). A topological space  $X$  is connected if it cannot be written as a union of two non-empty, disjoint open sets.

*Remark 6.36* (Equivalent definitions of connected space). Sometimes it is useful to consider equivalent definitions of connection: a topological space  $X$  is connected if:

- It cannot be written as union of two non-empty, disjoint closed sets,
- $X = A \cup B$  with  $A, B$  disjoint and open sets implies that  $A = \emptyset$  or  $B = \emptyset$ ,
- The only sets in  $X$  that can be both open and closed are  $X$  and  $\emptyset$ .

**Definition 6.37** (Path-connected space). A topological space  $X$  is path-connected if given any two points in the space, they can be connected by a continuous path, i.e. if  $\forall x, y \in X$  there exists a continuous map  $\alpha: [0, 1] \rightarrow X$  such that  $\alpha(0) = x, \alpha(1) = y$ .

*Remark 6.38* (Path-connected implies connected). A path-connected space is always connected (the proof is by contradiction). The converse is not true in general, but we will see that it is true for smooth manifolds!

*Remark 6.39* (Topological invariants). In the first sections we analyzed the concept of homeomorphism (see remark 4.4). We will review its importance here. First of all, let's point out that the entire abstraction of topology was built by mathematicians in order to categorize objects in a more abstract way. Spaces which are homeomorphic one to another are considered to be the same in topology. It can be proved that several properties are left invariant by homeomorphisms. For instance if one compact space  $X$  is homeomorphic to  $Y$ ,  $Y$  must be compact as well. Some of these properties (called topological invariants) are:

1. Compactness,
2. Paracompactness,
3. Connectedness (and path-connectedness),
4. Cardinality,
5. Hausdorff property,
6. First countability property,
7. Second countability property,



8. Contractibility,
9. Simple connectedness,
10. etc...

Such topological invariants are also useful in order to verify if two topological spaces are **not** homeomorphic one to the other. For instance, a closed interval  $[a, b]$  in  $\mathbb{R}$  with Euclidean topology cannot be homeomorphic to  $\mathbb{R}$  because the former is a compact set, and the latter is not.

## 7 ■ Multilinear Forms

**Definition 7.1** (Derivation). If  $a$  is a point of  $\mathbb{R}^n$ , a map  $v: C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  is called a *derivation at  $a$*  if it is linear over  $\mathbb{R}$  and satisfies the following product rule:

$$v(fg)|_a = f(a)v(g)|_a + g(a)v(f)|_a.$$

*Remark 7.2.* Directional derivatives obviously satisfy the above definition, and in these cases such a rule is also called Leibnitz rule.

**Definition 7.3** (1-forms). Given a vector space  $V$  on a field  $\mathbb{K}$ , a 1-form (or linear form)  $\varphi: V \rightarrow \mathbb{K}$  is a linear function from  $V$  to  $\mathbb{K}$ .  $V^*$  (also denoted by  $\Lambda V^*$ ) is the set of all linear forms on  $V$  and is called the dual space.

*Remark 7.4.* The set of linear forms on  $V$  has the structure of a vector space. Indeed, if  $\mathbb{K}$  is the field of the vector space  $V$ ,  $\forall a, a_1, a_2 \in \mathbb{K}, \forall \varphi, \varphi_1, \varphi_2 \in V^*$ :

1.  $a(\varphi_1 + \varphi_2) = a\varphi_1 + a\varphi_2$  (because we define the map  $a_1\varphi_1 + a_2\varphi_2$  as  $a_1\varphi_1 + a_2\varphi_2(x) \equiv a_1\varphi_1(x) + a_2\varphi_2(x), \forall x \in V$ )
2.  $(a + b)\varphi(x) = (a\varphi + b\varphi)(x)$
3.  $\varphi = 0$  if  $\varphi(x) = 0 \forall x \in V$
4.  $-\varphi(x) = \varphi(-x)$
5.  $1\varphi(x) = \varphi(x)$
6.  $ab\varphi(x) = ba\varphi(x)$

**Definition 7.5** (Dual basis). If  $\{e_i\}_{i=1, \dots, n}$  is a basis of  $V$ , then  $\{e^{*i}\}_{i=1}^n \subseteq V^*$  is called the *dual basis* if  $e^{*i}(e_j) = \delta_j^i$ .

If the field of  $V$  is  $\mathbb{R}$ , we will denote the dual basis of  $V^*$  also by  $\{dx^i\}_{i \in \mathbb{N}}$  (see also the next examples).

*Remark 7.6.* We could prove that the dual basis is indeed a basis of the dual space, so  $\dim(V) = \dim(\Lambda V^*)$ . Check proposition 8.4.

*Example 7.7.* Let's consider three examples.

1.  $p \in \mathbb{R}^3$ , fixed. Let  $V_p$  be the vector space  $V_p \equiv \{q - p \mid q \in \mathbb{R}^3\}$ . We will also denote it by  $T_p\mathbb{R}^3$ . We notice that  $V_p = T_p\mathbb{R}^3 = \mathbb{R}^3$  (It is trivial to verify the inclusions " $\subseteq$ " and " $\supseteq$ "). Let  $\{(e_i)_p\}_{i=1,2,3}$  be a basis for  $V_p$ . If we consider the dual space  $V_p^*$ , we notice that  $\{(dx^i)_p\}_{i=1,2,3}$  is a dual basis, where  $(dx^i)_p(e_j) = \frac{\partial}{\partial x^j} x^i$ . The index position is just a matter of notation, for now. Then, we can show that  $V_p$  and  $V_p^*$  are isomorphic, and the isomorphism is:

$$\begin{aligned} g: V_p &\rightarrow V_p^* \\ x &\mapsto g(x, \cdot) \end{aligned} \tag{7.1}$$

where  $g$  is the Euclidean metric in  $\mathbb{R}^3$  (in the sense of General Relativity), which in  $\mathbb{R}^n$  coincides with the standard dot product:  $g(x, y) = \sum_i x^i y^i$ .

2.  $V = \{M \in M_2(\mathbb{C}) \mid M^\dagger = M, \text{tr}(M) = 0\}$  is a vector field on  $\mathbb{R}$ , where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, M^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}, \quad a, \dots, d \in \mathbb{C}.$$

A basis for  $V$  is  $\{\sigma_i\}_{i=1,2,3}$ , where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and a dual basis is given by  $e^{*i}(e_j) = \frac{1}{2} \text{tr}(e_i, e_j)$  (using the matrix product).

3. Quantum Mechanics:

$$V = \{f: \mathbb{R}^3 \rightarrow \mathbb{R} \mid \|f\| = \int_{\mathbb{R}^3} |f|^2 d^3x < \infty\} = L^2(\mathbb{R}^3)$$

is a vector space. If we consider the Laplacian operator  $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}$ , then a basis is given by  $\{e_n\} = \{\frac{f_n}{\|f_n\|}\}$ , where  $f_n$  eigenfunctions of  $\Delta$ :  $\Delta f_n = \lambda_n f_n$  (if we choose appropriate boundary conditions). A dual basis is given by  $e^{*n} = \int_{\mathbb{R}^3} e_n$  Notice that both  $V$  and  $V^*$  are infinite-dimensional spaces. Some notations:

$$e_n = |f_n\rangle, \quad e^{*n} = \langle f_n|, \quad e^{*n}(e_m) = \langle f_n | f_m \rangle$$

*Remark 7.8.* Every vector in  $\mathbb{R}^n$  about a point  $p \in \mathbb{R}^n$  (i.e. such that its origin is the point  $p$ ) "can be seen" as a derivation (cf. def. 7.1), i.e. as a directional derivative of a function evaluated at the point  $p$ . For the sake of simplicity, we think  $p = 0$  (but the following results are true  $\forall p \in \mathbb{R}^n$ ). The sentence "can be seen" means that there is an isomorphism  $\psi$  associating such vectors to such linear forms. Let's construct this isomorphism in the following steps:

1. Because of linearity, we just need to define the isomorphism for the basis vectors  $\{e_i\}_{i=1,\dots,n}$  of  $\mathbb{R}^n$ .
2. Given the vector  $e_j$  of the canonical basis, we associate it with the derivation  $\partial_{x_j}$  in 0:

$$\partial_{x_j}|_{p=0} \equiv \frac{\partial}{\partial x_j}|_{p=0} : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R} \quad (7.2)$$

$$f \mapsto \frac{\partial}{\partial x_j}|_0(f) \equiv \frac{\partial f}{\partial x_j}(0).$$

In particular, if  $\text{Der}(\mathbb{R}^n) = \{\text{derivations on } \mathbb{R}^n\} = \{v : C^\infty \rightarrow \mathbb{R}, \text{ satisfying Leibnitz rule}\}$ , then the map:

$$\mathbb{R}^n \xleftrightarrow{\psi} \text{Der}(\mathbb{R}^n) \quad (7.3)$$

such that  $\psi(e_j) = \partial_{x_j}|_{p=0}$ ,  $\forall e_j$  basis vector, and with  $\partial_{x_j}|_{p=0}$  partial derivative with respect to the  $j$ -th component, defines a linear map. You can read more on page 53 of [Lee]. In general we have  $\psi(v) = \partial_v|_{p=0}$ , where  $\partial_v|_{p=0}$  is the directional derivative with respect to  $v$ . Moreover,  $\text{Der}(\mathbb{R}^n)$  is a vector space and we used the double arrow above because  $\psi$  is an isomorphism, i.e. a bijective map which preserves operations from one space to the other. Here every derivative is evaluated at  $p = 0$ . We notice that the point  $p$  itself is not important for the directional derivative (the *direction* in which we differentiate is the same for every point of the space), but  $p$  is meaningful when we *evaluate* the derivative of the function at that point. Indeed,  $\partial_x(x^2)|_{x=0} \neq \partial_x(x^2)|_{x=1}$ , even if we are differentiating along the  $x$ -axis in both cases.

What is more: given  $V_0$ , the set of all the vectors about 0, we can consider its dual space  $V_0^*$ . What is a possible dual basis? We want to find linear forms

$$e^{*i} : V_0 \rightarrow \mathbb{R}$$

such that  $e^{*i}(e_j) = \delta_j^i$ . We have just seen that we can consider vectors as directional derivatives. So, given  $e_j$  vector of the canonical basis, we will call it  $\frac{\partial}{\partial x^j}|_0$  (because of the isomorphism, they are quite the same mathematical object). Now, we want that

$$e^{*i} \left( \frac{\partial}{\partial x^j} \right) = \delta_j^i. \quad (7.4)$$

First, let's consider the *coordinate function*:

$$\begin{aligned} x_j: \mathbb{R}^n &\rightarrow \mathbb{R} \\ v = (v_1, \dots, v_n) &\mapsto v_j \end{aligned} \quad (7.5)$$

where  $v_1, \dots, v_j$  are the coordinates of the vector  $v$  in the canonical basis. The linear form  $x_j$  returns the  $j$ -th coordinate of a vector. So, given a vector  $v \in \mathbb{R}^n$ , every coordinate  $v_j$  can be seen as  $v_j = x_j(v)$ . Now, let's just define

$$e^{*i} \left( \frac{\partial}{\partial x^j} \Big|_0 \right) \equiv \frac{\partial}{\partial x^j} \Big|_0 x_i = \frac{\partial}{\partial x^j} x_i \Big|_0 = \delta_j^i \quad (7.6)$$

where  $x_i$  is the coordinate function defined above (remember:  $\partial_{x_j}$  is a derivation, so it must be applied to functions!). Now, it might seem that  $e^{*i}$  does not take a vector as argument, but rather a function. Actually, this problem is solved by the isomorphism between vectors and directional derivatives proved above. If  $\psi$  is the name of such isomorphism, we could slightly change the definition (7.6) in order to solve this ambiguity:

$$e^{*i}(e_j) \equiv \psi(e_j)(x_i) \quad (7.7)$$

where  $x_i$  is the  $i$ -th coordinate function and

$$\psi(e_j) = \frac{\partial}{\partial x^j} \Big|_0 = \partial_{x_j} \Big|_0. \quad (7.8)$$

The vectors of the dual basis will also be called

$$dx^i \equiv e^{*i}. \quad (7.9)$$

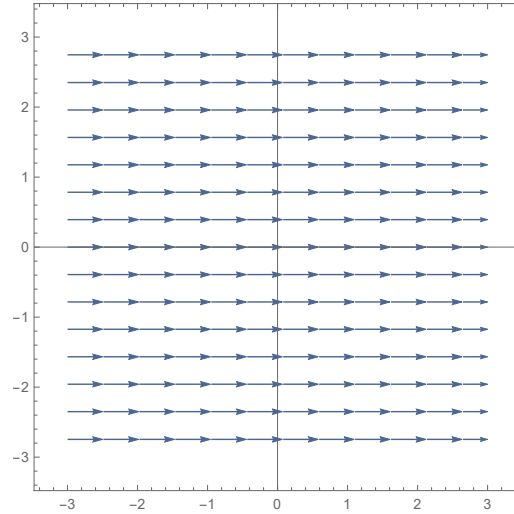
This will be important later: we will define exterior forms of degree  $k$  and we'll use both notations. The set of all these forms is  $\Lambda^k V^*$ , and its basis is given by products (in particular, exterior products) of  $e^{*i_1}, \dots, e^{*i_k}$  (i.e.  $dx^{i_1}, \dots, dx^{i_k}$ ).

**Definition 7.9** (Set of vector fields). We denote by  $\mathfrak{X}(\mathbb{R}^n)$  the set of all possible vector fields in  $\mathbb{R}^n$ , i.e.

$$\begin{aligned} \mathfrak{X}(\mathbb{R}^n) = \text{Der}\mathbb{F}(\mathbb{R}^n) &\equiv \{v: \mathbb{F}(\mathbb{R}^n) \rightarrow \mathbb{F}(\mathbb{R}^n) \text{ such that} \\ &v \text{ is } \mathbb{R}\text{-linear and } v(fg) = v(f)g + f v(g)\} \end{aligned} \quad (7.10)$$

where  $\mathbb{F} \equiv \{ \text{functions } f: \mathbb{R}^n \rightarrow \mathbb{R} \}$ .

*Remark 7.10* (Fields vs. Derivations). If  $v$  is a vector field in  $\mathbb{R}^n$ , then  $v$  assigns a vector to another vector of  $\mathbb{R}^n$ . So  $v: \mathbb{R}^n \rightarrow \mathbb{R}^n$ . So, the set of all vector fields should

Figure 1: Vector field  $v=e_1 = \partial_x$ 

be (we will use a different symbol to denote it):

$$X(\mathbb{R}^n) = \{v: \mathbb{R}^n \rightarrow \mathbb{R}^n\}$$

However, the definition 7.9 is a bit different. Why? The fact is, we can consider a vector of  $\mathbb{R}^n$  as a directional derivative, cf. remark 7.8 (we are not considering any fixed point here, but the results do not change). Now a derivation, as defined in def. 7.1, is a map  $v: C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ , i.e. we can say that a derivation is a very smooth element of  $\mathbb{F}(\mathbb{R}^n) = \{\text{functions } f: \mathbb{R}^n \rightarrow \mathbb{R}\}$ . So,  $\mathfrak{X}(\mathbb{R}^n) \cong X(\mathbb{R}^n)$  because we can associate a derivation of  $\mathfrak{X}(\mathbb{R}^n)$  to each vector of  $X(\mathbb{R}^n)$ , and vice versa. Then we also explained why in the definition of  $\mathfrak{X}(\mathbb{R}^n)$  every element must be  $\mathbb{R}$ -linear and satisfy the Leibnitz rule: it follows from the definition of derivations.

Now, a question arises: given  $v$  vector field, should we write  $v(p)$  (i.e. it takes vectors as argument) or should we write  $v(f)$  (i.e. it takes smooth functions as arguments)? The answer is: it depends on the case, since they are two different " $v$ "s. Which is, we will use vectors when we think of  $v$  as a function who takes elements of  $\mathbb{R}^n$ , and we will use functions in the other case. And we can choose which case to use, since we can identify every vector field with a derivation, and every derivation with a vector field (for more info, see pag. 181 of [Lee]). Now, let us analyze how  $v(f)$  is made in the latter case. Given  $v \in \mathfrak{X}(\mathbb{R}^n)$ ,  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , we define the function  $v(f)$  as

$$\begin{aligned} v(f): \mathbb{R}^n &\rightarrow \mathbb{R} \\ p &\mapsto v(f)(p) \equiv v_p f. \end{aligned} \tag{7.11}$$

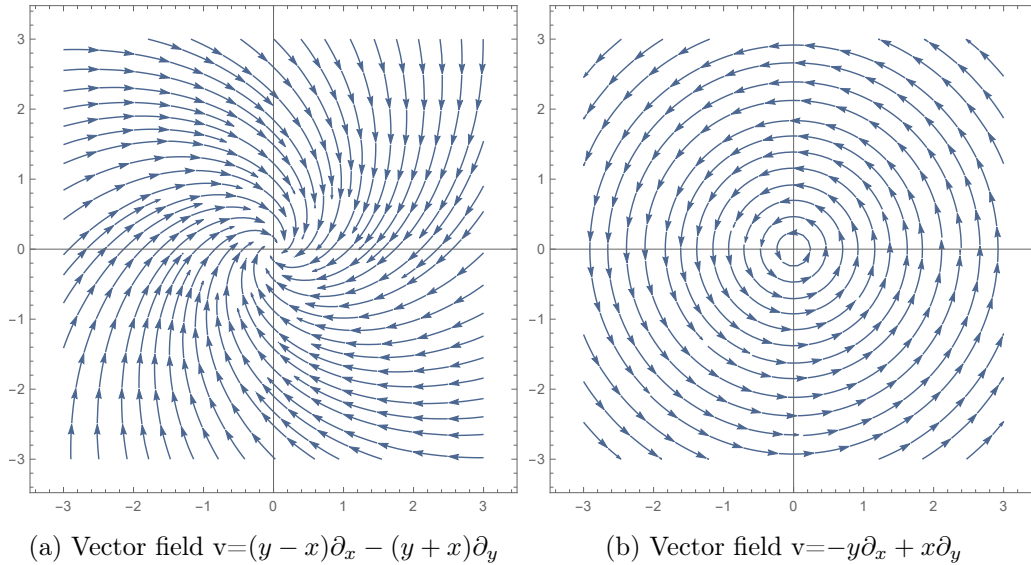
Now, in coordinates:

$$v(f)(p) = v_p f = v^i(p)(e_i)_p f = v^i(p) \frac{\partial}{\partial x^i} \Big|_p f = v^i(p) \frac{\partial f}{\partial x^i}(p) \tag{7.12}$$

where we used summation convention, and the fact that every basis vector  $e_i$  can be seen as  $\partial_{x_i}|_p$ . We defined it in the right way because, as expected, we found that  $v_p(f)$  is the directional derivative of  $f$  in the direction of  $v$ , evaluated at  $p$ . So, in brief:

- $v(f)(p)$  is a number
- $v(f)(\cdot)$  is a function from  $\mathbb{R}^n$  to  $\mathbb{R}$
- $v(\cdot)$  is a function from  $\mathbb{F}(\mathbb{R}^n)$  to  $\mathbb{F}(\mathbb{R}^n)$

We also notice that the mathematical object  $e_i$  is not much different from  $(e_i)|_p$  in this case: there is no difference if we think about them as directions, but it makes a difference if we think about them as directional derivatives, because the latter notation gives info about the point in which the derivative is evaluated. So, we add the pedix " $p$ " in order to make the isomorphism between vectors and directional derivatives more explicit. Check also the remark 1.1.



## 8 Exterior Product and Generalisation

**Definition 8.1** (Exterior form of degree  $k$ ). Given a vector space  $V$  on a field  $\mathbb{K}$ , with  $\dim(V) = n$ , and with  $k \leq n$ , an *exterior form of degree  $k$*  (or  $k$ -linear form, or  $k$ -form) is a multi- $\mathbb{K}$ -linear map

$$\omega: \underbrace{V \times \dots \times V}_{k \text{ times}} \rightarrow \mathbb{K}$$

such that

$$\omega(v_1, \dots, v_k) = \text{sgn}(\pi) \omega(v_{\pi(1)}, \dots, v_{\pi(k)}) \quad (8.1)$$

for all permutations  $\pi$  of  $k$  elements, i.e.  $\pi \in S_k$ , where  $\text{sgn}(\pi)$  is the sign of the permutation. We will also write  $\omega \in \Lambda^k V^*$ .

**Definition 8.2** (Exterior product between two 1-forms). Given a vector space  $V$  on a field  $\mathbb{K}$ ,  $\dim(V) \geq 2$ , and given  $\varphi^1, \varphi^2 \in \Lambda V^*$ , then we define the exterior product (or wedge product)  $\wedge$  as:

$$\begin{aligned} \wedge: \Lambda V^* \times \Lambda V^* &\rightarrow \Lambda^2 V^* \\ (\varphi^1, \varphi^2) &\mapsto \varphi^1 \wedge \varphi^2 \end{aligned}$$

where:

$$\varphi^1 \wedge \varphi^2(x_1, x_2) = \varphi^1(x_1)\varphi^2(x_2) - \varphi^2(x_1)\varphi^1(x_2) = \det(\varphi^i(x_j))$$

for  $i, j = 1, 2$ .

*Remark 8.3* (Exterior product between  $k$  1-forms). The exterior product  $\wedge$  that we defined for  $k = 2$  in def. 8.2 gives an exterior form of degree 2. We want to generalize it for  $k$  vector spaces. In order to extend the definition, we want it to give an exterior form of degree  $k$ , so:

$$\begin{aligned} \wedge: \underbrace{\Lambda V^* \times \dots \times \Lambda V^*}_{k \text{ times}} &\rightarrow \Lambda^k V^* \\ (\varphi^1, \dots, \varphi^k) &\mapsto \varphi^1 \wedge \dots \wedge \varphi^k \end{aligned}$$

where, given  $(x_1, \dots, x_k) \in \underbrace{V \times \dots \times V}_{k \text{ times}}$ :

$$\varphi^1 \wedge \dots \wedge \varphi^k(x_1, \dots, x_k) = \det(\varphi^i(x_j)) . \quad (8.2)$$

This is a particular case of an exterior  $k$ -form (because the sign of determinant changes if we swap two rows or two columns). Now let's consider an explicit computation using the determinant, in  $\mathbb{R}^3$ , with coordinates  $x, y, z$ , and with  $\varphi^1, \varphi^2, \varphi^3$  corresponding to the three elements of the dual basis  $dx \equiv e^{*1} = dx^1, dy \equiv e^{*2} = dx^2$  and  $dz \equiv e^{*3} = dx^3$ :

$$(dx \wedge dy \wedge dz)(x, y, z) = \det(dx^i(e_j)) = \det \begin{pmatrix} dx(x) & dx(y) & dx(z) \\ dy(x) & dy(y) & dy(z) \\ dz(x) & dz(y) & dz(z) \end{pmatrix}$$

and we can use that  $dx^i(e_j) = \delta_j^i$ . For instance, we can verify that the 3-form  $dx \wedge dy \wedge dz$  gives 0 if two coordinates of the input are repeated:

$$(dx \wedge dy \wedge dz)(x, x, z) = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 .$$

**Proposition 8.4.** *If  $\{e_i\}_{i=1,\dots,n}$  is a basis in  $V$ , then  $\{e^{*i_1} \wedge \dots \wedge e^{*i_k}\}_{i_1 < \dots < i_k, k \leq n}$  forms a basis of  $\Lambda^k V^*$ .*

*Proof.* In the statement we implicitly assumed that the elements of  $\{e^{*i_1} \wedge \dots \wedge e^{*i_k}\}$  are defined as usual, for instance  $e^{*i_1}(e_{j_1}) = \delta_{j_1}^{i_1}$ . In order to prove that it is a basis, we need to prove that the elements of the basis are linearly independent and that they span the entire space.

- Linear independence: We have to show that

$$0 = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} e^{*i_1} \wedge \dots \wedge e^{*i_k}$$

implies  $a_{i_1 \dots i_k} = 0, \forall i_1, \dots, i_k$ . Indeed, if we apply  $(e_{j_1}, \dots, e_{j_k})$  on the above equation, we obtain

$$\begin{aligned} 0 &= a_{j_1 \dots j_k} \underbrace{(e^{*j_1} \wedge \dots \wedge e^{*j_k})(e_{j_1}, \dots, e_{j_k})}_{=1} + \\ &+ \sum_{\substack{j_1 < \dots < j_k \\ i_1 \neq j_1, i_2 \neq j_2, \dots}} a_{i_1 \dots i_k} \underbrace{(e^{*i_1} \wedge \dots \wedge e^{*i_k})(e_{j_1}, \dots, e_{j_k})}_{=0} = a_{j_1 \dots j_k}. \end{aligned}$$

The second term is zero because  $(e^{*i_1} \wedge \dots \wedge e^{*i_k})(e_{j_1}, \dots, e_{j_k}) = \det(e^{*i_l}(e_{j_m}))$ . Indeed, the matrix  $e^{*i_l}(e_{j_m})$  has null coefficients on the main diagonal (because  $j_1 \neq i_1, j_2 \neq i_2, \dots$ ) and one of the rows must be 0 because of the ordering  $j_1 < \dots < j_k$  and  $i_1 < \dots < i_k$  (e.g. consider the case  $j_2 = i_1, \dots$ ). Thus, the determinant is 0. Therefore,  $a_{j_1 \dots j_k} = 0$  and the indices  $j_1, \dots, j_k$  were chosen arbitrarily.

- Completeness: Given  $\psi \in \Lambda^k V^*$ , we want to prove that there exists a coefficient  $a_{i_1 \dots i_k}$  such that  $\psi = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} (e^{*i_1} \wedge \dots \wedge e^{*i_k})$ , i.e. we want to prove that we can write any element of  $\Lambda^k V^*$  as a linear combination of elements of  $\{e^{*i_1} \wedge \dots \wedge e^{*i_k}\}$ . Let's consider  $(v_1, \dots, v_k) \in V \times \dots \times V$ . Then:

$$\begin{aligned} \psi(v_1, \dots, v_k) &\stackrel{(1)}{=} \sum_{i_1=1}^k \sum_{i_2=1}^k \dots \sum_{i_k=1}^k x_{i_1} \dots x_{i_k} \psi(e_{i_1}, \dots, e_{i_k}) \stackrel{(2)}{=} \\ &= \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k} \psi(e_{i_1}, \dots, e_{i_k}) \stackrel{(3)}{=} \\ &= \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k} \psi(e_{i_1}, \dots, e_{i_k}) (e^{*i_1} \wedge \dots \wedge e^{*i_k})(e_{i_1}, \dots, e_{i_k}) \stackrel{(4)}{=} \\ &= \sum_{i_1 < \dots < i_k} \psi(e_{i_1}, \dots, e_{i_k}) (e^{*i_1} \wedge \dots \wedge e^{*i_k})(x_{i_1} e_{i_1}, \dots, x_{i_k} e_{i_k}) \stackrel{(5)}{=} \\ &= \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} (e^{*i_1} \wedge \dots \wedge e^{*i_k})(v_1, \dots, v_k). \end{aligned}$$

Where we used: (1)  $v_1 = \sum_{i_1=1}^k x_{i_1} e_{i_1}$  in the  $\{e_i\}_{i=1}^n$  basis, (2) we can simplify the sum since  $\psi = 0$  if two entries are repeated (and any sign is absorbed in the coefficients



$x_i$ ), (3)  $1 = (e^{*i_1} \wedge \dots \wedge e^{*i_k})(e_{i_1}, \dots, e_{i_k})$ , (4) multilinearity, (5)  $0 = e^{*i}(e_j)$  if  $i \neq j$  and  $a_{i_1 \dots i_k} \equiv \psi(e_{i_1}, \dots, e_{i_k})$ .  $\square$

*Remark 8.5.* The above proposition proves that  $\dim(\Lambda^k V^*) = \binom{n}{k}$ , because through the expression " $i_1 < \dots < i_k, k \leq n$ " we are selecting subsets of  $k$  elements (subsets, not  $k$ -tuples, because the order of the elements is externally fixed) from a bigger set of  $n$  elements. This is exactly the definition of the binomial coefficient  $\binom{n}{k}$ . Moreover, it means that any  $\alpha \in \Lambda^k V^*$  can be written as:

$$\alpha = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} e^{*i_1} \wedge \dots \wedge e^{*i_k}$$

where  $a_{i_1 \dots i_k} \in \mathbb{K}$ ,  $\mathbb{K}$  field of the vector space.

Now, we want to define the exterior product between a  $k$ -form and a  $p$ -form (and it will return a  $(p+k)$ -form).

**Definition 8.6** (Exterior product between a  $k$ -form and a  $p$ -form). Given  $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*$ , the exterior product between them is defined as:

$$\begin{aligned} \wedge: \Lambda^k V^* \times \Lambda^p V^* &\rightarrow \Lambda^{k+p} V^* \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \end{aligned}$$

with:

$$\alpha \wedge \beta = \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_p}} \alpha_{i_1 \dots i_k} \beta_{j_1 \dots j_p} e^{*i_1} \wedge \dots \wedge e^{*i_k} \wedge e^{*j_1} \wedge \dots \wedge e^{*j_p}$$

where  $\alpha_{i_1 \dots i_k}, \beta_{j_1 \dots j_p} \in \mathbb{K}$ .

*Example 8.7* (Oriented area). Let's consider the following examples:

1. Let's suppose our space is  $V = \mathbb{R}^4$  with cartesian coordinates.

$$\varphi \equiv dx^1 \wedge dx^2 + dx^2 \wedge dx^4$$

where  $\varphi$  is a sum of two 2-forms. They are 2-forms (and not 1-forms, nor 3-forms, nor 4-forms, ...) because they are written as wedge product (" $\wedge$ ") of **two** elements of the dual basis. Remember that we denote the elements of the dual basis either by  $e^{*i}$  or  $dx^i$  for some  $i$ . Let's compute  $\varphi(e_i, e_j)$ . We can use the determinant formula (8.2):

$$\begin{aligned} \varphi(e_i, e_j) = & dx^1(e_i)dx^2(e_j) + dx^2(e_i)dx^4(e_j) - dx^2(e_i)dx^1(e_j) + \\ & - dx^4(e_i)dx^2(e_j). \end{aligned}$$

2. Let's consider  $V = \mathbb{R}^2$  with cartesian coordinates  $x^1, x^2$ . Let:

$$\varphi \equiv dx^1 \wedge dx^2.$$

Let's compute  $\varphi(ae_1, be_2)$ , where  $a, b \in \mathbb{R}$ :

$$\begin{aligned}\varphi(ae_1, be_2) &= dx^1 \wedge dx^2(ae_1, be_2) = ab \, dx^1 \wedge dx^2(e_1, e_2) = \\ &= ab (dx^1(e_1)dx^2(e_2) - \cancel{dx^2(e_1)}\cancel{dx^1(e_2)}) = ab = \\ &= \text{oriented area of the rectangle of sides } a \text{ and } b.\end{aligned}$$

There are some interesting properties about  $k$ -forms:

**Proposition 8.8.**  $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*, \gamma \in \Lambda^q V^*$ , then:

1.  $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ ,
2.  $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ ,
3.  $\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha$ .

*Sketch of the proof.*

1. Writing the forms in terms of the basis, we only need to prove that the property (1) holds for the elements of the dual basis. Moreover, by multilinearity, we just need to prove it for the elements of the dual basis applied to a basis  $\{e_i\}$  of  $V$ . Thus, the proof for (1) is over because this property is true for the determinant: when we compute it, we can start from any row or column.
2. By writing the forms in terms of the basis as before.
3. Again, by writing the forms in terms of the basis we have a sequence of wedge products like

$$e^{*i_1} \wedge \dots \wedge e^{*i_k} \wedge e^{*j_1} \wedge \dots \wedge e^{*j_p}.$$

If we swap two elements of the product we get a factor  $-1$ . In order to have the right hand side we need to "move  $e^{*j_1}$  on the left  $k$  times", so we get a factor  $(-1)^k$ . Same for  $e^{*j_2}$ : we need to move it on the left  $k$  times and so we get an extra factor  $(-1)^k$ . Again, we repeat the procedure for all the elements up to  $e^{*j_p}$ . At the end, we get a factor  $(-1)^{kp}$ .  $\square$

## 9 Differential Forms

**Definition 9.1** (Field of exterior forms, geometric definition). (A field of) exterior forms of degree  $k$ ,  $k \leq n$  is a map  $\omega$  that associates to each point  $p \in V$  an element  $\omega(p) \in \Lambda^k V_p^*$ . Choosing a basis, we have:

$$\omega(p) = \sum_{i_1 < \dots < i_k} \underbrace{a_{i_1 \dots i_k}(p)}_{\text{now it is a function!}} e^{*i_1} \wedge \dots \wedge e^{*i_k}. \quad (9.1)$$

The form  $\omega$  is a differential form if  $a_{i_1 \dots i_k}$  are differentiable functions. The set of differential  $k$ -forms with coefficients in  $\mathbb{R}^n$  is denoted by  $\Omega^k(\mathbb{R}^n)$ .

Another (equivalent) definition:

**Definition 9.2** (Algebraic definition of differential  $k$ -form). A differential  $k$ -form is a map:

$$\omega: \underbrace{\mathfrak{X}(\mathbb{R}^n) \times \dots \times \mathfrak{X}(\mathbb{R}^n)}_{k \text{ times}} \rightarrow \mathbb{F}(\mathbb{R}^n) \quad (9.2)$$

$C^\infty(\mathbb{R}^n)$ -linear and alternating.

*Remark 9.3.* To show the equivalence of the two definitions of differential  $k$ -forms we just need to show that:

$$\omega(p)(v_1, \dots, v_k) = \omega(v_1, \dots, v_k)(p). \quad (9.3)$$

See also the exercise 1, problem sheet 3.

We want to generalize the concept of a differential of a function.

**Definition 9.4** (Differential). Let  $f$  be a function  $f: U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $f$  differentiable. Let  $v \in \mathfrak{X}(\mathbb{R}^n) = \text{Der } \mathbb{F}(\mathbb{R}^n)$ . The exterior derivative of  $f$  is its differential  $d$ , defined as a 1-form such that:

$$df(v) = v(f). \quad (9.4)$$

*Remark 9.5* (Differential expression in coordinates). We want to verify that the above definition of differential is equivalent to our usual definition for  $C^1(\mathbb{R}^n)$  function, which is:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i = \frac{\partial f}{\partial x_i} dx^i. \quad (9.5)$$

In order to prove that, we first consider a pointwise definition. Given  $p \in \mathbb{R}^n$ :

$$df_p(v) = v(f), \forall v \in T_p \mathbb{R}^n \cong \mathbb{R}^n. \quad (9.6)$$

( $T_p \mathbb{R}^n$  is the tangent space to  $\mathbb{R}^n$  at  $p$ ). Now, we can write  $v(f)$  in coordinates (the gray part is the one we don't care about):

$$df_p = v(f) = v_i(p)(\lambda^i)_p \quad (9.7)$$

where  $(\lambda^i)_p$  is a dual basis at  $p$  (later, we will prove that  $(\lambda^i)_p = (dx^i)_p$ ). Now, applying  $df$  to a particular vector (i.e. directional derivative) at  $p$ :

$$df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) = v_i(p) \quad (9.8)$$

where we used the property of the dual basis

$$(\lambda^i)_p \frac{\partial}{\partial x^j} \Big|_p = \delta_j^i \quad (9.9)$$

and then:

$$df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) = v_i(p)(\lambda^i)_p \frac{\partial}{\partial x^i} \Big|_p = v_i(p). \quad (9.10)$$

On the other hand, by definition (9.6) we know that:

$$df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) = \frac{\partial}{\partial x^i} \Big|_p f = \frac{\partial f}{\partial x^i}(p). \quad (9.11)$$

Hence, using (9.8) and (9.11) we get:

$$v_i(p) = \frac{\partial f}{\partial x^i}(p). \quad (9.12)$$

Then, by the expression of differential in coordinates (9.7):

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p. \quad (9.13)$$

Applying the definition to  $f = x^j$  (coordinate function, as defined in (7.5)), we get:

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p = \frac{\partial f}{\partial x^i}(p)(dx^i)_p. \quad (9.14)$$

And then:

$$df = \frac{\partial f}{\partial x^i} dx^i. \quad (9.15)$$

Indeed, if  $f = x^j$  then, as before:

$$(dx^j)_p = \frac{\partial x^j}{\partial x^i}(p)(\lambda^i)_p = \delta_j^i(\lambda^i)_p = (\lambda^j)_p. \quad (9.16)$$

Pay attention: what we did here is a bit different from what we did for the definition 7.11 of a vector field applied to a function. In this case,  $p$  is the point where we fixed our vector, whereas in the other case  $p$  was the point where we wanted to evaluate the directional derivative of  $f$ .

In the above definition,  $f$  was a 0-form (i.e. a function). What is the generalization of the differential to  $k$ -forms?

**Definition 9.6** (Exterior derivative). If  $k > 0$ , then the exterior derivative (acting on  $k$ -forms) is a map

$$d: \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k+1}(\mathbb{R}^n) \\ \omega \mapsto d(\omega) \equiv d\omega$$

where

$$d\omega = \sum_{j_1 < \dots < j_k} (da_{j_1, \dots, j_k}) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

with  $da_{j_1, \dots, j_k}$  differential of the function  $a_{j_1, \dots, j_k}$ .

*Example 9.7* (Computation of  $d\omega$ ). Let's consider the following examples:

1. Let  $\omega$  be a 2-form on  $\mathbb{R}^3$  (coordinates  $x^1, x^2, x^3$ ):

$$\omega = dx^1 \wedge dx^2 + x^2 dx^1 \wedge dx^3.$$

Then:

$$d\omega = dx^2 \wedge dx^1 \wedge dx^3$$

where we used that " $d(dx^1 \wedge dx^2) = 0$ " because there is no non-zero 3-form on a 2-dimensional space. (otherwise, we can use that  $d^2 = 0$ , but we still have to prove it!)

2. In  $\mathbb{R}^n, n \geq 2$ , let's consider:

$$\omega = x^2 dx^1.$$

We have:

$$d\omega = dx^2 \wedge dx^1$$

where we used the definition 9.6. Moreover, if  $u, v$  are vector fields, then by definition of exterior product we have:

$$d\omega(u, v) = dx^2(u)dx^1(v) - dx^2(v)dx^1(u).$$

On the other hand, using that  $dg(v) = v(g)$  for a function  $g$  and a vector field  $v$ , we have:

$$\begin{aligned} dx^2(u) &= u(x^2), \\ v(x^2) &= dx^2(v) \end{aligned}$$

where  $x^2$  is a function, the coordinate function defined in (7.5). We also have:

$$d\omega(u, v) = u(x^2)dx^1(v) - v(x^2)dx^1(u).$$

Now, some properties:

**Proposition 9.8** (Properties of exterior derivatives).  $\omega_1 \in \Omega^k(\mathbb{R}^n), \omega_2 \in \Omega^p(\mathbb{R}^n)$ . Then:

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ ,
- $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$ ,
- $d(d\omega_1) = 0 = d(d\omega_2)$ .

*Sketch of the proof.*

- We have

$$d\omega_1 = \sum_{j_1 < \dots < j_k} \left( da_{j_1 \dots j_k}^{(1)} \right) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

and similarly for  $\omega_2$ . Then we use the linearity of the differential:  $d(a^{(1)} + a^{(2)}) = da^{(1)} + da^{(2)}$ .

- We have

$$d(\omega_1 \wedge \omega_2) = \sum_{\substack{j_1 < \dots < j_k \\ i_1 < \dots < i_p}} d \left( a_{j_1 \dots j_k}^{(1)} a_{i_1 \dots i_p}^{(2)} \right) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}.$$

Using the product rule we have a term  $(da^{(1)}) a^{(2)} + a^{(1)} (da^{(2)})$  inside the sum. Now,  $(da^{(1)}) a^{(2)}$  gives the term  $d\omega_1 \wedge \omega_2$ . In order to get the term  $(-1)^k \omega_1 \wedge d\omega_2$  we consider  $a^{(1)} (da^{(2)}) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$  inside the sum and we move the factor  $da^{(2)}$  to the right for  $k$  times. So we get the factor  $(-1)^k$  and the wedge product  $\omega_1 \wedge d\omega_2$ .

- Let's consider

$$d(d\omega) = \sum_{j_1 < \dots < j_k} d(da_{j_1 \dots j_k}) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}.$$

Now,

$$da_{j_1 \dots j_k} = \left( \frac{\partial}{\partial x^l} a_{j_1 \dots j_k} \right) dx^l$$

and:

$$\left[ d \left( \frac{\partial}{\partial x^l} a_{j_1 \dots j_k} \right) \right] dx^l = \underbrace{\frac{\partial^2}{\partial x^m \partial x^l} a_{j_1 \dots j_k}}_{\text{symmetric in } l \leftrightarrow m} \underbrace{dx^m \wedge dx^l}_{\text{antisymmetric in } l \leftrightarrow m}.$$

Then we get a minus sign if we exchange  $l$  and  $m$ . Moreover we can compute  $\left[ d \left( \frac{\partial}{\partial x^m} a_{j_1 \dots j_k} \right) \right] dx^m$  in the same way exchanging the roles of  $l$  and  $m$ , and we wouldn't get any minus sign. Then we have that  $d(d\omega) = -d(d\omega) \Rightarrow d(d\omega) = 0$ .  $\square$

*Remark 9.9.* In the above proposition, we claimed that  $d(d\omega) = 0$  if  $\omega \in \Omega^k(\mathbb{R}^n)$ . The notation here is not the most precise, since the inner "d" is acting on a  $k$ -form, whereas the outer "d" is acting on a  $(k+1)$ -form (so, even if they share the same name, they are different maps). However the behaviour of both "d"s is clear, so we will continue with this abuse of notation.

*Remark 9.10.* The exterior derivative increases the degree of a  $k$ -form by 1 (the  $k$ -form becomes a  $(k+1)$ -form). Can we get backwards, which is, can we decrease the degree of a  $k$ -form? Yes, using the interior derivative.

**Definition 9.11** (Interior derivative). Let  $z$  be a vector field on  $\mathbb{R}^n$ , i.e.  $z \in \mathfrak{X}(\mathbb{R}^n)$ , then we define the *interior derivative*  $i_z$  (acting on differential  $k$ -forms) as the map:

$$i_z: \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k-1}(\mathbb{R}^n) \tag{9.17}$$

$$\omega \mapsto i_z(\omega) \equiv i_z \omega$$

such that

$$(i_z \omega)(v_1, \dots, v_{k-1}) = \omega(z, v_1, \dots, v_{k-1}), \forall v_i \in \mathfrak{X}(\mathbb{R}^n).$$

$i_z \omega$  is also called *contraction* or *interior multiplication*. Another notation for  $i_z \omega$  is  $z \lrcorner \omega$ .

*Example 9.12* (Some computations). In  $\mathbb{R}^2$ , Let  $e_x, e_y$  be the basis vectors (that can be seen as vector fields). Then we have:

$$i_{e_x}(dx \wedge dy) = dy,$$

$$i_{e_y}(dx \wedge dy) = -dx.$$

*Remark 9.13.* In the definition 9.11 above, we used the algebraic definition of differential  $k$ -forms, i.e. definition 9.2.

Now some properties for interior derivatives.

**Proposition 9.14.** *Let  $\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^p(\mathbb{R}^n), z \in \mathfrak{X}(\mathbb{R}^n)$ , then:*

- $i_z(\omega \wedge \eta) = (i_z \omega) \wedge \eta + (-1)^k \omega \wedge (i_z \eta),$
- $i_z^2 \omega = i_z(i_z \omega) = 0.$

*Sketch of the proof.*

- First, we assume  $k + p \leq n$ , otherwise both sides of the equation are 0 and we would be already done. Then, let's consider a basis  $\mathcal{B} = \{e_1, \dots, e_n\}$  in  $\mathbb{R}^n$  which is positively oriented (i.e. if  $A$  is the matrix that we use to change coordinates from the canonical basis to  $\mathcal{B}$ , we have  $\det A > 0$ ). Let's choose  $\mathcal{B}$  such that the vector field  $e_1$  is the tangent vector to the vector field  $z$  in one point (we will make a pointwise reasoning). Moreover,  $\{dx^l\}_{l=1}^n$  is the dual basis with respect to the canonical basis, and  $\{e^{*l}\}_{l=1}^n$  is the dual basis with respect to  $\mathcal{B}$ . We have

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and

$$\omega \wedge \eta = \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_p}} a_{i_1 \dots i_k} b_{j_1 \dots j_p} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}.$$

Then either:

1.  $dx^{i_r} = e^{*1}$  for some  $i_r, r = 1, \dots, k$ , or
2.  $dx^{j_r} = e^{*1}$  for some  $j_r, r = 1, \dots, p$ , or
3. there are no  $i_r, j_r$  such that " $dx^{i_r} = e^{*1}$  or  $dx^{j_r} = e^{*1}$ ".

So,  $i_z(\omega \wedge \eta) = (\omega \wedge \eta)(z, \dots)$  is either equal to:

1.  $\sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_p}} a_{i_1 \dots i_k} b_{j_1 \dots j_p} i_z (dx^{i_1} \wedge \dots \wedge dx^{i_k}) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_p}$ , because the only non-zero term comes from  $i_z(e^{*1}) = e^{*1}(z) = z_1 e^{*1}(e_1) = z_1$  (locally), and we know that the non-zero term must be hidden in the first  $k$  wedge products, or
2.  $\sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_p}} (-1)^k a_{i_1 \dots i_k} b_{j_1 \dots j_p} dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge i_z (dx^{j_1} \wedge \dots \wedge dx^{j_p})$ , for the same reason above, and we have the factor  $(-1)^k$  because we moved the interior derivative  $k$  times (it is like moving an entry of a  $(k+p)$ -form  $k$  times), or
3. 0.

Then,  $i_z(\omega \wedge \eta)$  is the sum of these possible terms.

- Using the fact that a differential form gives 0 if two or more entries are repeated:

$$i_z(i_z(\omega))(v_1, \dots, v_{k-2}) = i_z\omega(z, v_1, \dots, v_{k-2}) = \omega(z, z, v_1, \dots, v_{k-2}) = 0.$$

□

*Remark 9.15.* In the above proposition there is a little abuse of notation when we claimed  $i_z(i_z\omega) = 0$ , see also the remark 9.9.

**Definition 9.16** (Nondegenerate 2-form). An exterior form  $\omega$  of degree 2 on a vector space  $V$ ,  $\omega \in \Lambda^2 V^*$ , is said to be nondegenerate if the map from  $V$  to  $V^*$  defined as  $z \mapsto i_z\omega$  is invertible.

*Remark 9.17* (Alternative definitions of nondegenerate form). The following ones are equivalent:

1.  $\omega$  is non degenerate,
2. for each nonzero  $v \in V$ , there exists  $w \in V$  such that  $\omega(v, w) \neq 0$ ,
3. in terms of some basis, the matrix  $(\omega_{ij})$  representing  $\omega$  is nonsingular.

Now, let's talk about *pullbacks* and *pushforwards* for functions and  $k$ -forms.

**Definition 9.18** (Pullback). Let  $f: U \rightarrow V$  (with  $U, V \subseteq \mathbb{R}^n$ ) be a differentiable map. Let's suppose that  $\dim(U) = \dim(V) = n$  (just for the sake of simplicity, since it is not necessary). Then the *pullback* of a  $k$ -form (from  $V$ ) to  $U$  is the map:

$$\begin{aligned} f^*: \Omega^k(V) &\rightarrow \Omega^k(U) \\ \omega &\mapsto f^*\omega \end{aligned}$$

such that

$$(f^*\omega)(p)(u_1, \dots, u_k) = \omega(f(p))(df_p(u_1), \dots, df_p(u_k)), \quad \forall p \in \mathbb{R}^n, \forall u_i \in \mathfrak{X}(U).$$



Now, we want to give another name to the differential of a function.

**Definition 9.19** (Pushforward). Given  $f: U \rightarrow V$  as before, we will also call the differential of  $f$  at  $p \in \mathbb{R}^n$ , i.e.  $df_p = df(p)$ , as the *pushforward* of  $f$  at  $p$ , and it will be denoted by the symbol  $(f_*)_p$ . So, the pushforward map is:

$$\begin{aligned} df_p &\equiv (f_*)_p: U \subset \mathbb{R}^n \rightarrow V \subset \mathbb{R}^m \\ v &\mapsto (f_*)_p(v). \end{aligned}$$

*Remark 9.20* (Pushforward of composition of functions). In our mind, we'll think of the pushforward as the differential of a map:  $df_p = (f_*)_p$ , at least until this concept is generalized. By definition of differential:  $df_p(v) = v_p(f)$ , where  $v_p$  is a vector tangent to  $\mathbb{R}^n$  at  $p$ . Since vectors are like directional derivatives,  $v_p(f)$  is the directional derivative of  $f$  with respect to  $v_p$  (not evaluated at any point, for now!). In particular, if we apply the definition to function  $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ , we have (cf. also p. 55 [Lee]):

$$(f_*)_p(v)(h) \stackrel{(1)}{=} v(h \circ f) = v(h(f)) \stackrel{(2)}{=} v(f^*h)$$

where we used: (1)  $(f_*)_p(v)(h) = v_p(h \circ f)$ , it can be proved that it is the right expression in a similar way to what we did to prove that  $(f_*)_p(v) = v_p(f)$  is the right expression (cf. remark 9.5), (2) the expression of the pullback for a differentiable function, which is completely legal since we defined it for differentiable  $k$ -forms, and a differentiable function is just a 0-form.

*Remark 9.21.* Using the pushforward, we can define the pullback of a differential form using a different notation (i.e. using  $f_*$  instead of  $df$ ):

$$(f^*\omega)(p)(u_1, \dots, u_k) = \omega(f(p))(f_*(u_1), \dots, f_*(u_k)), \forall p \in \mathbb{R}^n, \forall u_i \in \mathfrak{X}(U) \quad (9.18)$$

Now, some properties of the pullback.

**Proposition 9.22.** Let  $g, f \in C^1(\mathbb{R}^n, \mathbb{R})$ ,  $\omega, \varphi \in \Omega^k(\mathbb{R}^n)$ ,  $h: \mathbb{R}^n \rightarrow \mathbb{R}$ . Then:

1.  $f^*(\omega + \varphi) = f^*(\omega) + f^*(\varphi)$ ,
2.  $f^*(h\omega) = f^*(h)f^*(\omega)$ ,
3.  $(f \circ g)^* = g^*(f^*(\omega))$ ,
4. If  $\varphi^1, \dots, \varphi^k \in \Omega^1(\mathbb{R}^n)$ , then  $f^*(\varphi^1 \wedge \dots \wedge \varphi^k) = f^*(\varphi^1) \wedge \dots \wedge f^*(\varphi^k)$ ,
5.  $df^*(\omega) = f^*(d\omega)$ .

From property (4) also follows that  $f^*(\omega \wedge \phi) = (f^*\omega) \wedge (f^*\phi)$

*Sketch of the proof.*

1.

$$\begin{aligned} f^*(\omega + \varphi) &= (\omega + \varphi)(f_*(u_1), \dots, f_*(u_k)) = \\ &= \omega(f_*(u_1), \dots, f_*(u_k)) + \varphi(f_*(u_1), \dots, f_*(u_k)) = \\ &= f^*(\omega) + f^*(\varphi). \end{aligned}$$

$$2. \quad f^*(h\omega)(z_1, \dots, z_k)(p) = h(f(p)) \omega(f_*z_1, \dots, f_*z_k)(f(p)) = f^*(h)f^*(\omega).$$

3. We will use:  $(f \circ g)_*v(h) = v((f \circ g)^*h) = v(g^* \circ f^*(h)) \stackrel{(1)}{=} g_*v(f^*h) = f_*\tilde{v}(h) = f_* \circ g_*v$ , where  $\tilde{v} \equiv g_*v$ . In (1) we used  $v(h \circ f) = f_*(v \circ h)$  with  $f, h$  differentiable functions,  $v$  smooth vector field (see remark 9.20). Then:

$$\begin{aligned} (f \circ g)^*\omega(z_1, \dots, z_k) &= \omega((f \circ g)_*z_1, \dots, (f \circ g)_*z_k) = \omega(f_* \circ g_*z_1, \dots, f_* \circ g_*z_k) = \\ &= g^*(f^*\omega)(z_1, \dots, z_k). \end{aligned}$$

4. We have:

$$\begin{aligned} f^*(\varphi^1 \wedge \dots \wedge \varphi^k)(z_1, \dots, z_k) &= (\varphi^1 \wedge \dots \wedge \varphi^k)(f_*z_1, \dots, f_*z_k) = \det(\varphi^i(f_*z_j)) = \\ &= \det(f^*\varphi^i(z_j)) = (f^*\varphi^1) \wedge \dots \wedge (f^*\varphi^k)(z_1, \dots, z_k). \end{aligned}$$

5. If  $z$  is a vector field and  $dx^i$  is an element of the dual basis (in particular,  $dx^i(e_j) = \frac{\partial}{\partial x^j}x^i$ ), we have

$$f^*dx^i(z) = dx^i(f_*z) \stackrel{(A)}{=} z(f^i) = df^i(z)$$

where in (A) we used:

$$dx^i(f_*z) = f_*z(x^i) \text{ because of the definition } df(v) = v(f).$$

Moreover,

$$f_*z(x^i) = f_*(z \circ x^i) = z(x^i \circ f) \text{ by the property } f_*(v \circ h) = v(h \circ f).$$

Finally,

$$z(x^i \circ f) = z(f^i).$$

In the last equality we used the def. 9.4 of differential. Now,

$$\begin{aligned} df^*(\omega) &= df^* \left( \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \right) = \\ &= d \sum_{i_1 < \dots < i_k} f^*(a_{i_1 \dots i_k}) f^*(dx^{i_1}) \wedge \dots \wedge f^*(dx^{i_k}) \stackrel{(1)}{=} \\ &= \sum_{i_1 < \dots < i_k} df^*(a_{i_1 \dots i_k}) \wedge df^{i_1} \wedge \dots \wedge df^{i_k} \stackrel{(2)}{=} \\ &= f^*d\omega \end{aligned}$$

Where we used: (1)  $f^*dx^i = df^i$  ( $i$ -th coordinate), the product rule and the fact that  $d^2 = 0$ , (2) given  $h \equiv a_{i_1 \dots i_k} \in C^1(\mathbb{R}^n)$ , then

$d(f^*h)(z) = z(f^*h) = z(h \circ f) = f_*(z \circ h)$ , using  $f_*(v \circ h) = v(h \circ f)$  in the last step.

And

$$f_*(z \circ h) = (f_*z)(h) = dh(f_*z) = f^*dh(z).$$

□

*Remark 9.23.* We can express the pullback of a differential form in the following way:

$$\begin{aligned} (f^*\omega)(p) &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (f^*a_{i_1, \dots, i_k}(p)) f^*dy^{i_1} \wedge f^*dy^{i_2} \wedge \dots \wedge f^*dy^{i_k} = \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(f(p)) df^{i_1} \wedge df^{i_2} \wedge \dots \wedge df^{i_k} \end{aligned}$$

where  $f^i = y^i(f)$ . We used properties (2) and (4) of proposition 9.22

*Remark 9.24.* From our definition of pullback, it is not necessary that  $f_*$  is invertible.

*Example 9.25* (Example of a pullback). Let's consider

$$U = \{r > 0, 0 < \theta \leq 2\pi\}$$

$$V = \mathbb{R}^2 \setminus \{(0, 0)\}$$

Let:

$$f: U \rightarrow V$$

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta) \equiv (x, y)$$

Let's consider the 1-form  $\Omega^1(V) \ni \omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$  on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ . Then:

$$f^*\omega = -\frac{r \sin \theta}{r^2} \cdot (\cancel{dr \cos \theta} - r \sin \theta d\theta) + \frac{r \cos \theta}{r^2} (\cancel{dr \sin \theta} + r \cos \theta d\theta) = d\theta$$

where we used  $f^*dx^i = df^i$ .

## 10 ■ Integration of Differential Forms

Let  $\omega$  be a differential form of degree  $n$  in  $\mathbb{R}^n$ . Then  $\omega$  is necessarily of the form

$$\omega = \underbrace{a(p)}_{\text{it's a function}} dx^1 \wedge \dots \wedge dx^n. \quad (10.1)$$

Such a form can be integrated:

$$\int_{f(D)} \omega = \int_D f^* \omega. \quad (10.2)$$

*Example 10.1* (Example of  $\int$  of a  $k$ -form). Let's consider

$$D = [0, 1] \times [0, 1]$$

$$f(D) = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1], 0 \leq y \leq x\}$$

i.e.  $f(D)$  is the lower-right triangle of the square  $D$ . In particular, we can write  $f$  as  $f(x^1, x^2) = (x^1, x^1 x^2)$ . Let  $\omega = dy^1 \wedge dy^2$  on  $\mathbb{R}^2$ , thus:

$$\begin{aligned} \int_{f(D)} dy^1 \wedge dy^2 &= \int_D f^*(dy^1 \wedge dy^2) = \int_D df^1 \wedge df^2 = \int_D dx^1 \wedge (x^2 dx^1 + x^1 dx^2) = \\ &= \int_D x^1 dx^1 \wedge dx^2 = \int_D x^1 dx^1 dx^2 = \frac{1}{2}. \end{aligned}$$

## 11 ■ Tangent Bundles, Flows of Vector Fields

**Definition 11.1** (Tangent space). Let  $U \subset \mathbb{R}^n$ ,  $U$  open set. Given  $p \in U$  the set of all derivations of  $C^\infty(U)$  at  $p$  (cf. def 7.1) is called tangent space to  $U$  at  $p$  and is denoted by  $T_p U$ . An element of  $T_p U$  is called a tangent vector at  $p$ , and it is often denoted by  $(p, v)$ ,  $v_p$ , or just  $v$ .

There are alternative equivalent definitions of the tangent space. Some of them will be analyzed in the next sections.

**Definition 11.2** (Tangent bundle). The tangent bundle over an open subset  $U \subset \mathbb{R}^n$  is defined as

$$TU \equiv \bigsqcup_{p \in U} T_p U \quad (11.1)$$

where  $T_p U$  is the tangent space of  $U$  at  $p$ . Every element of the disjoint union is represented by an ordered pair  $(p, v)$  where  $p \in U$ ,  $v \in T_p U$ . So, elements of the tangent bundle are couples that consist of a tangent vector at a point of  $U$ , and the point itself. The tangent bundle comes equipped with the projection map

$$\begin{aligned} \text{pr}: TU &\rightarrow U \\ (p, v) &\mapsto p \end{aligned} \quad (11.2)$$

which sends each vector in  $T_p U$  to the point  $p$  at which it is tangent.

*Remark 11.3.* In the previous definition, the " $\sqcup$ " symbol denotes a disjoint union. The disjoint union of two elements  $x, y$  can be written as  $\{(1, x), (2, y)\}$ , where we introduced an artificial index in order to distinguish  $(1, x)$  from  $(2, y)$  even if  $x = y$ .

**Definition 11.4** (Alternative definition of vector field). A smooth vector field  $v$  on  $U \subset \mathbb{R}^n, U$  open, is a smooth map

$$v: U \rightarrow TU \quad (11.3)$$

such that  $\text{pr}(v_p) = p, \forall p \in U$ .

*Remark 11.5* (Space of sections). The set of all vector fields  $\mathfrak{X}(U) \equiv \{v \in C^\infty(U, TU) \mid \text{pr}(v_p) = p\}$  is also called the space of sections in  $TU$ . Thus, a section of  $TU$  is a vector field.

**Definition 11.6** (Cotangent space). Let  $U \subset \mathbb{R}^n, U$  open set. Given  $p \in U$ , then the cotangent space at  $p$  is denoted by  $T_p^*U$  and it is the dual space of  $T_pU$ :

$$T_p^*U = (T_pU)^*.$$

**Definition 11.7** (Cotangent bundle). We define

$$T^*U \equiv \bigsqcup_{p \in U} T_p^*U \quad (11.4)$$

as the cotangent bundle. We also associate a projection  $\text{pr}: T^*U \rightarrow U$  with it.

Now, let's talk about Lie algebras.

**Definition 11.8** (Abstract Lie algebra). A Lie algebra  $(V, [\cdot, \cdot])$  is a vector space  $V$  over  $\mathbb{R}$  endowed with a map

$$[\cdot, \cdot]: V \times V \rightarrow V$$

with the following properties:

- $[\cdot, \cdot]$  is bilinear,
- $[\cdot, \cdot]$  is antisymmetric ( $[u, v] = -[v, u], \forall u, v \in V$ ),
- $[\cdot, \cdot]$  satisfies the *Jacobi identity*:

$$[[u, v], z] + [[z, u], v] + [[v, z], u] = 0 \forall u, v, w \in V.$$

*Remark 11.9* (Jacobi). How to remember Jacobi identity: remember  $[[u, v], z]$  and then permute cyclically.

*Example 11.10.*  $\mathbb{R}^3$  equipped with the usual vector product and the vector space of matrices with the matrix commutator  $[A, B] = AB - BA$  are typical Lie algebras.

*Remark 11.11* (Lie bracket for vector fields). The vector space  $V = \mathfrak{X}(\mathbb{R}^n)$  can be equipped with a canonical Lie bracket. For  $u, v \in \mathfrak{X}(\mathbb{R}^n)$  we set for  $f \in C^\infty(\mathbb{R}^n)$

$$[u, v](f) = uv(f) - vu(f).$$

In fact,  $[u, v]$  is a vector field as it satisfies the Leibniz rule:

$$\begin{aligned} [u, v](fg) &= uv(fg) - vu(fg) \\ &= u(gv(f) + fv(g)) - v(gu(f) + fu(g)) \\ &= guv(f) + fuv(g) - gvu(f) - fvu(g) \\ &= g[u, v](f) + f[u, v](g) \end{aligned}$$

It is easily verified that  $(\mathfrak{X}(\mathbb{R}^n), [\cdot, \cdot])$  is indeed a Lie algebra.

**Definition 11.12** (Integral curve). An integral curve for a vector field  $v$  is a smooth curve  $\phi: (a, b) \rightarrow \mathbb{R}^n$  satisfying  $\dot{\phi}(t) = v_{\phi(t)}$  ( $v_{\phi(t)}$  is the vector tangent at  $\phi(t)$  for  $t$  fixed, remember the previous notation!). Let us suppose  $0 \in (a, b)$ . Then,  $\phi(0)$  is called the starting point of  $\phi$ .

**Lemma 11.13** (Translation lemma). If  $\phi: (a, b) \rightarrow \mathbb{R}^n$  is an integral curve of the vector field  $v$ , then for all  $c \in \mathbb{R}$  the curve  $\tilde{\phi}: (a + c, b + c) \rightarrow \mathbb{R}^n$  defined by  $\tilde{\phi}(t) = \phi(t + c)$  is also an integral curve of  $v$ .

*Proof.* We simply check the defining property of an integral curve

$$\dot{\tilde{\phi}}(t) = \dot{\phi}(t + c) = v_{\phi(t+c)} = v_{\tilde{\phi}(t)}$$

where in the second step we used that  $\phi$  is an integral curve for  $v$ . □

We can also visualize the family of integral curves in the following way.

**Definition 11.14** (Flow). The map

$$\begin{aligned} \theta: \mathbb{R} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (t, p) &\mapsto \theta_t(p) \end{aligned} \tag{11.5}$$

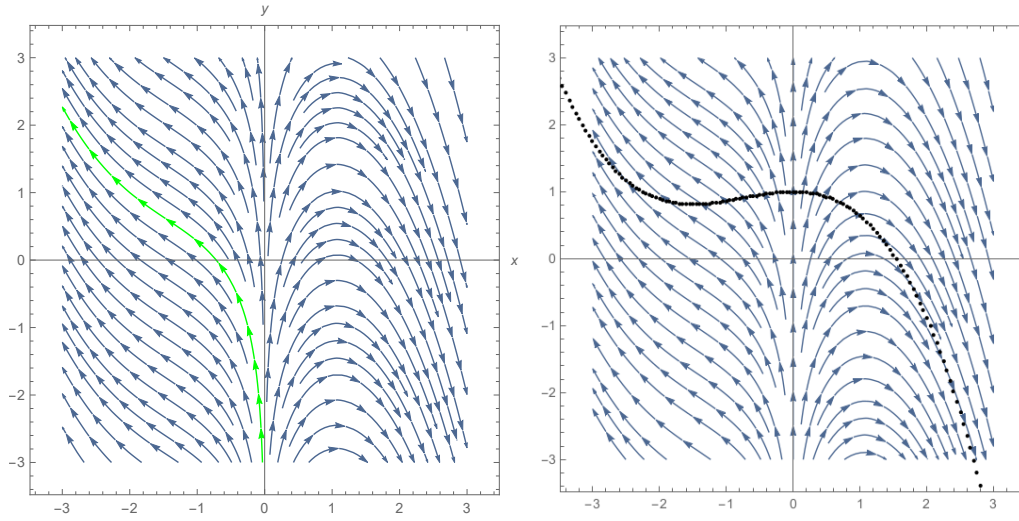
such that  $\dot{\theta}_t(p) = v_{\theta_t(p)}$  is the flow of the vector field  $v$  where, if we fix  $p$ ,  $\theta_t(p)$  is the integral curve which passes through  $p$  at  $t = 0 \in (a, b)$ , i.e. for all  $p \in \mathbb{R}^n$  the flow  $\theta$  satisfies

$$\dot{\theta}_t(p) = v_{\theta_t(p)} \tag{11.6}$$

$$\theta_0(p) = p. \tag{11.7}$$

Under the right hypotheses (e.g. Lipschitz hypothesis and smoothness of  $v$ ) we can prove existence and uniqueness of the solution of such ODEs ( $\forall p \in \mathbb{R}^n$ ). By fixing either the time or the starting point of the flow, we can consider two maps:

- $p \mapsto \theta_t(p)$ , for each fixed  $t$  (we are observing several integral curves at the same time  $t$ ),
- $t \mapsto \theta_t(p)$ , for each fixed  $p$  (we are observing the integral curve starting from  $p$ , for all times).



(a) the map  $t \mapsto \theta_t(p)$  selects just one integral curve

(b) the map  $p \mapsto \theta_t(p)$  selects all the curves evaluated at the same time  $t$

*Remark 11.15.* The translation lemma above implies that  $t \mapsto \theta_{t+s}(p)$  is an integral curve starting at  $q = \theta_s(p)$ . By uniqueness of the integral curve we obtain  $\theta_t(q) = \theta_{t+s}(p)$ , i.e.  $\theta_t \circ \theta_s = \theta_{t+s}$ . This property together with  $\theta_0(p) = p$  for all  $p$  can be used as a definition of the flow.

*Remark 11.16.* Each flow defines locally a vector field  $v$  given pointwise by

$$v_p = \left. \frac{d}{dt} \right|_{t=0} \theta_t(p).$$

The vector field  $v$  is called the infinitesimal generator of  $\theta$ . For fixed  $p$  each curve  $\theta_t(p)$  is an integral curve for  $v$ .

**Definition 11.17** (Lie derivative of a  $k$ -form). Let  $z \in \mathfrak{X}(\mathbb{R}^n)$  be a differentiable vector field,  $\theta_t$  its flow and  $\omega \in \Omega^k(\mathbb{R}^n)$ . Then the Lie derivative of  $\omega$  is defined as

$$L_z \omega = \left. \frac{d}{dt} (\theta_t^* \omega) \right|_{t=0}. \quad (11.8)$$

*Remark 11.18.* We denoted the flow by the symbol  $\theta_t$  and not  $\theta$ . What we are doing here is not caring about  $p$ :  $\theta_t^* \omega(\cdot) = \omega(\theta_t(\cdot))$ . Useful formula (Cartan's formula):  $L_z \omega = (di_z + i_z d)\omega$ .

## 12 ■ Lie Derivative of a Vector Field

**Definition 12.1** (Pullback of a vector field through a diffeomorphism). Let  $\varphi$  be a diffeomorphism of  $\mathbb{R}^n$  (i.e. a differentiable and invertible map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ , such that its inverse is differentiable as well). Let  $v \in \mathfrak{X}(\mathbb{R}^n)$ . Then:

$$\varphi^* v \equiv \varphi_*^{-1} v \quad (12.1)$$

is the pullback of  $v$  with  $\varphi$ . In particular, given a flow  $\phi$  and  $t$  fixed,  $\phi(t, \cdot) = \phi_t(\cdot)$  is a diffeomorphism on  $\mathbb{R}^n$  with  $\phi^{-1}(t, \cdot) = \phi(-t, \cdot) = \phi_{-t}(\cdot)$ .

**Definition 12.2** (Lie derivative of a vector field). Let  $u, v \in \mathfrak{X}(\mathbb{R}^n)$ . Let  $\phi$  be the flow of  $u$ . The Lie derivative of  $v$  in direction  $u$  is

$$L_u v \equiv \frac{d}{dt}(\phi_t^* v)|_{t=0} \quad (12.2)$$

(Remember:  $\phi: (t, p) \mapsto \phi(t, p)$ , whereas  $\phi^* v: (t, p) \mapsto v(\phi(t, p))$ ).

**Lemma 12.3.** Let  $u, v$  be smooth vector fields on  $\mathbb{R}^n$  and  $\varphi \in \text{Diff}(\mathbb{R}^n)$ . Let  $\phi$  be the flow of  $u$  and let  $\psi$  be the flow of  $v$ . Then

- $\varphi^* v = \frac{d}{ds}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi$ ,
- $\varphi^* v = v \Leftrightarrow \varphi \circ \psi_s = \psi_s \circ \varphi$  for all  $s$ ,
- $L_u v = 0 \Leftrightarrow \phi_t \circ \psi_s = \psi_s \circ \phi_t$  for all  $s, t$ .

*Sketch of the proof.*

- We have:

$$\frac{d}{ds}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi(p) \stackrel{(1)}{=} \varphi_*^{-1}(v_{\varphi(p)}) \stackrel{(2)}{=} (\varphi_*^{-1} v)_p \stackrel{(3)}{=} (\varphi^* v)_p$$

where we used: (1) the chain rule and the fact that  $\psi_s$  is the flow of  $v$ . We applied the definitions to the point  $\varphi(p)$  instead of  $p$  as usual. (2)  $(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$ , see also page 183 [Lee], (3)  $\varphi_*^{-1} = \varphi^*$  by definition.

- ( $\Leftarrow$ ): if  $\varphi \circ \psi_s = \psi_s \circ \varphi$  then, by using the previous point:

$$\varphi^* v = \frac{\partial}{\partial s}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi = \frac{\partial}{\partial s}|_{s=0} \psi_s = v.$$

- ( $\Rightarrow$ ): if  $\varphi^* v = v$ , we use the previous point again:

$$\frac{\partial}{\partial s}|_{s=0} \psi_s = v = \frac{\partial}{\partial s}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi.$$



Now, the solution of such differential equation is unique because of the regularity assumptions on the flows. Then:

$$\psi_s = \varphi^{-1} \circ \psi_s \circ \varphi$$

which implies the thesis.

- ( $\Leftarrow$ ): follows from the previous point, and using the definition of Lie derivative for a vector field. ( $\Rightarrow$ ):

$$\frac{d}{dt}\phi_t^*v = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\phi_{t+\varepsilon}^*v \stackrel{(1)}{=} \phi_t^*\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\phi_\varepsilon^*v = \phi_t^*L_uv = 0, \forall t$$

where in (1) we used that  $\phi_{t+\varepsilon}^* = \phi_t^* \circ \phi_\varepsilon^*$ . Thus  $\phi_t^*v = v \forall t$ , and by the previous point we have the thesis.  $\square$

**Lemma 12.4.** *Let  $u, v$  be smooth vector fields on  $\mathbb{R}^n$  and let  $\phi_t$  (respectively  $\psi_s$ ) be the flow of  $u$  (respectively  $v$ ). Then:*

- $L_uv = \frac{\partial^2}{\partial s \partial t}\phi_{-t} \circ \psi_s \circ \phi_t|_{t=0, s=0}$ ,
- $(L_uv)(f) = [u, v](f) = u(v(f)) - v(u(f))$  for all smooth functions  $f$  on  $\mathbb{R}^n$ .

*Sketch of the proof.*

- $\frac{d}{dt}\phi_t^*v = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\phi_{-t} \circ \psi_s \circ \phi_t$  by the first point of the lemma 12.3, where we put  $\varphi = \phi_t$ .
- See problem sheet 4, exercise 4.  $\square$

*Remark 12.5.* The above lemmata show that two vector fields  $u, v$  commute, i.e.  $[u, v] = 0$ , if and only if their corresponding flows  $\phi, \psi$  commute, i.e.  $\phi_t \circ \psi_s = \psi_s \circ \phi_t$  for all  $s, t \in \mathbb{R}$ .

**Lemma 12.6.** *Let  $u, v$  be smooth vector fields on  $\mathbb{R}^n$  and  $\varphi \in \text{Diff}(\mathbb{R}^n)$ . Then:*

1.  $[u, v]$  is  $\mathbb{R}$ -bilinear (i.e. bilinear for a parameter  $\lambda \in \mathbb{R}$ ),
2.  $[u, v] = -[v, u]$ ,
3. The Jacobi identity holds,
4.  $[u, fv] = f[u, v] + u(f)v$ ,
5.  $\varphi_*[u, v] = [\varphi_*u, \varphi_*v]$ .

*Proof.* See problem sheet 4, exercise 4.  $\square$

*Remark 12.7* (Naturality of the Lie bracket). The property (5) of lemma 12.6 is called the **naturality** of the commutator. It implies that if  $u, v$  are vector fields tangent to some space, then  $[u, v]$  is tangent to the same space. See also problem sheet 12, ex. 3.

*Remark 12.8.* From lemma 12.6 and the correspondence  $L_u v = [u, v]$  one can easily obtain non-trivial properties for the Lie derivative, e.g. (notation as above)

1.  $L_u v = -L_v u$ ,
2.  $L_u(fv) = fL_u v + u(f)v$ ,
3.  $\varphi^* L_u v = L_{\varphi^* u}(\varphi^* v)$ .

### 13 ■ Stokes' Theorem on $\mathbb{R}^n$

- For a function  $f$  (i.e. a 0-form) on  $[a, b] \subset \mathbb{R}$  we have  $\int_a^b df = \int_a^b \partial_x f dx = f(b) - f(a)$  (fundamental theorem of calculus).
- for  $\omega = a_i dx^i$ , a 1-form on  $U = [0, 1] \times [0, 1] \subset \mathbb{R}^2$  we have:

$$\int_U d\omega = \int_U (\partial_{x^1} a_2) dx^1 \wedge dx^2 + (\partial_{x^2} a_1) dx^2 \wedge dx^1 = \int_{\partial U} \omega$$

where we used the fundamental theorem of calculus.

- More generally, if  $U$  is a compact subset of  $\mathbb{R}^2$  with piecewise regular boundary  $\partial U$  (piecewise homeomorphic to intervals in  $\mathbb{R}$ ) then, by decomposing  $S$  in terms of little squares and interpreting  $\int_U d\omega$  as a Riemann sum over the square, we obtain the result

$$\int_U d\omega = \int_{\partial U} \omega.$$

This result generalizes immediately to compact co-dimension zero subsets of  $\mathbb{R}^n$  with piecewise regular boundary.

- If  $M$  is a compact subset of dimension  $m \leq n$  in  $\mathbb{R}^n$  (with piecewise regular boundary  $\partial M$ ), diffeomorphic to a compact subset of  $U \subset \mathbb{R}^m$  (i.e.  $M = f(U)$ ) and  $\omega \in \Omega^{m-1}(M)$ , then:

$$\int_M d\omega = \int_U f^* d\omega = \int_U df^* \omega = \int_{\partial U} f^* \omega = \int_{\partial M} \omega.$$

- More generally, the parametrization of  $\partial M$  may be different from the one induced by  $M$ . Then we have:

$$\int_M d\omega = \int_{\partial M} i^* \omega$$

where  $i: \partial M \rightarrow M$  is the inclusion map of  $\partial M$  into  $M$ .

So, the most general result that we achieved is the following:

**Theorem 13.1** (Stokes). *Let  $\omega \in \Omega^{m-1}(\mathbb{R}^n)$ . Let  $M$  be a compact subset of  $\mathbb{R}^n$ ,  $\dim(M) = m \leq n$ , such that  $M$  is homeomorphic to a closed subset  $U \subset \mathbb{R}^m$ .  $\partial M$  is the boundary of  $M$  and  $i: \partial M \rightarrow M$  is the inclusion map of  $\partial M$  into  $M$ . Then:*

$$\int_{\partial M} i^* \omega = \int_M d\omega. \quad (13.1)$$

*Sketch of the proof.* First, we notice that  $U$  is compact as well because it is homeomorphic to a compact set. Then, the proof uses the naturality of the pullback " $f^*d = df^*$ " and the results for open subsets of  $\mathbb{R}^n$ .  $\square$

**Corollary 13.2** (Fundamental theorem of line integrals). *Let  $f$  be a smooth function defined near an oriented curve  $C$  in  $\mathbb{R}^n$ , with endpoints  $A$  and  $B$ . Then:*

$$\int_C \nabla f \cdot dx = f(B) - f(A). \quad (13.2)$$

*Proof.*  $\int df = \int \nabla f \cdot dx$  because  $\nabla f \cdot dx = \frac{\partial f}{\partial x^i} dx^i = df$ . Moreover, thanks to the Stokes' theorem, we have  $\int df = f(B) - f(A)$ .  $\square$

**Corollary 13.3** (Curl theorem or Classical Stokes theorem). *Let  $v$  be a differentiable vector field defined near a surface  $S \subset \mathbb{R}^3$  with boundary  $\partial S$ .*

$$\int_S n \cdot (\nabla \times v) dS = \int_{\partial S} v \cdot dx \quad (13.3)$$

where  $n$  is the normal vector on the surface at each point.

*Proof.* Let  $\alpha$  be a 1-form in  $\mathbb{R}^3$ . Since the Euclidean metric gives an isomorphism between  $\mathbb{R}^3$  and its dual space (see also eq. (7.1)), we can always write  $\alpha$  as  $\alpha = g(v, \cdot)$ , for a certain  $v \in \mathbb{R}^3$ , where  $g$  is the Euclidean metric, i.e.  $g(x, y)$  is the standard dot product between  $x$  and  $y$ . So,  $\alpha = g(v, \cdot) = \delta_{ij} v^j dx^i = v_i dx^i$  in the dual basis  $\{dx^i\}$ . Then:  $d\alpha = (\partial_{x^j} v_i) dx^j \wedge dx^i \stackrel{(1)}{=} \partial_{x^j} v_i \varepsilon_k^{ji} n^k dS$ . Where in (1) we used that  $dx^j \wedge dx^i$  is the oriented surface generated by  $dx^j$  and  $dx^i$  (see also the second point of the example 8.7). Then we used that the oriented surface area can be expressed using the Levi-Civita tensor  $\varepsilon$ , the normal vector  $n$  and the unoriented area element  $dS$ . Hence, using that  $\nabla \times v = \partial_{x^j} v_i \varepsilon_k^{ji}$ :

$$\int_S n \cdot (\nabla \times v) dS = \int_S d\alpha \stackrel{(2)}{=} \int_{\partial S} \alpha = \int_{\partial S} v_i dx^i = \int_{\partial S} v \cdot dx$$

using the Stokes' theorem in (2).  $\square$

**Corollary 13.4** (Divergence theorem). *For a smooth vector field  $v$  defined on a solid  $T \subset \mathbb{R}^3$  with boundary  $\partial T$ :*

$$\int_T \nabla \cdot v dV = \int_{\partial T} v \cdot n dS \quad (13.4)$$

where  $dV$  is the unoriented volume element.

*Proof.* See problem sheet 5, exercise 3.  $\square$

## 14 ■ Closed and Exact $k$ -forms

**Definition 14.1** (Closed and exact forms). If  $\omega \in \Omega^k(U)$  such that  $d\omega = 0$ , then  $\omega$  is closed. If there exists  $\alpha \in \Omega^{k-1}(V)$ ,  $V \subset U$  such that  $\omega = d\alpha$  in  $V$  then  $\omega$  is exact.

**Proposition 14.2.** *The following are equivalent (note that we are considering just 1-forms!):*

1.  $\omega \in \Omega^1(U)$  is exact in a connected open subset  $V \subset U$ ,
2. For any curve  $\gamma: (a, b) \rightarrow U$ ,  $\int_\gamma \omega$  depends only on the endpoints  $\gamma(a)$  and  $\gamma(b)$ ,
3.  $\int_\gamma \omega = 0$  for any closed curve  $\gamma$  in  $V$ .

*Sketch of the proof.* First, (1)  $\Rightarrow$  (2) because of Stokes' theorem. Moreover, (2)  $\Rightarrow$  (3) using Stokes' theorem again. Now, (3)  $\Rightarrow$  (2) : the curve made of the union of  $\gamma_1$  and  $\gamma_2$  is a closed curve. Then, if  $\gamma$  is the name of such a closed curve:

$$0 = \int_\gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega$$

where the minus sign comes from the orientation of the curves. Then:  $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$  where  $\gamma_1$  and  $\gamma_2$  are any two curves with the same endpoints. It means that we can consider the integral of  $\omega$  on any curve and the result wouldn't change, **if** the new curve has the same endpoints of the initial curve.

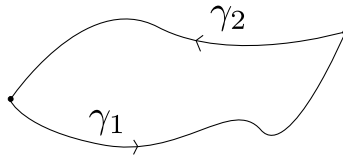


Figure 4: Union of  $\gamma_1$  and  $\gamma_2$

(2)  $\Rightarrow$  (1): Let's consider the 1-form  $\omega = a_i dx^i$  (when  $\omega$  has this form, we call it a *standard form*). Let's fix a point  $p \in V$  and define a function

$$f: V \rightarrow \mathbb{R}$$

$$x \mapsto \int_\gamma \omega$$

where  $x = x(p')$  is the coordinate given by a point  $p'$  (i.e.  $x$  gives a parametrization) and  $\gamma$  is a curve joining  $p$  to  $p'$ . By (2),  $f$  is well defined (i.e. the definition does not depend on the  $\gamma$  chosen, so we don't need to specify it). Furthermore,  $df = \frac{\partial f}{\partial x^i} dx^i$ . Since  $\omega = a_i dx^i$ , we only need to prove that  $\frac{\partial f}{\partial x^i} = a_i$ . Let's consider the curve  $\gamma$ . We extend it from  $p'$  with a straight segment  $\Delta\gamma$ . In particular,  $\Delta\gamma = x + te_i$ ,  $t \in (-\varepsilon, \varepsilon)$  with  $e_i$  any vector of the canonical basis of  $\mathbb{R}^n$ . We choose  $\varepsilon$  small such that  $\gamma + \Delta\gamma \subset V$ . Then:

$$\begin{aligned} \frac{\partial f}{\partial x^i} \Big|_x &= \lim_{t \rightarrow 0} \frac{1}{t} [f(x + te_i) - f(x)] = \lim_{t \rightarrow 0} \frac{1}{t} \left[ \int_{\gamma + \Delta\gamma} \omega - \int_\gamma \omega \right] = \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_{\Delta\gamma} \omega = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t a_i(x(t)) dt = a_i(x) \end{aligned}$$

where we used  $\int \omega = \int a_i \frac{dx^i}{dt} dt$  and the fact that every point belonging to  $\Delta\gamma$  can be written as  $x + te_i$ , so we have  $\frac{dx^i}{dt} = 1$  on  $\Delta\gamma$ . Moreover, in the last equality we Taylor-expanded  $a_i(x(t))$  with respect to  $t$ . Only the first-order term  $a_i(x)$  matters because we are considering the limit as  $t \rightarrow 0$ . Then we have the thesis for standard forms. Then, the result holds for any form (in general we could have  $a_i = 0$  for some  $i$ ).



Figure 5: A possible curve  $\gamma$  from  $p$  to  $p'$

□

*Remark 14.3* (A closed form is not always exact). If  $\omega$  is exact, then it is closed (because  $d^2 = 0$ ). But not every closed form in  $\Omega^1(U)$ ,  $U$  open subset of  $\mathbb{R}^n$  is exact. Cf.  $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$  in  $\mathbb{R}^2$  minus the origin. If  $\gamma$  is a closed curve around the origin of  $\mathbb{R}^2$ , we have:

$$\int_{\gamma} \omega = \int_{\gamma} d\theta = 2\pi$$

and therefore  $\omega$  cannot be exact by the previous proposition. However, we notice that we have problems only with the origin of  $\mathbb{R}^2$ . If we consider a subset of  $\mathbb{R}^2$  which is far enough from the origin, the form would be an exact form in such subset. Indeed, we say that  $\omega$  is locally exact, and the general result follows from the next theorem.

**Theorem 14.4** (Poincaré theorem for 1-forms on  $\mathbb{R}^n$ ). *Let  $\omega \in \Omega^1(U)$ ,  $U \subset \mathbb{R}^n$ ,  $U$  open. Then  $d\omega = 0$  if and only if for each  $p \in U$  there is a neighbourhood  $V \subset U$  of  $p$  and a differentiable function  $f: V \rightarrow \mathbb{R}$  such that  $\omega = df$ , i.e.  $\omega$  is closed if and only if  $\omega$  is locally exact.*

*Remark 14.5.* We will not prove this theorem, because it follows from the more general case for  $k$ -forms, that we will analyze in the next pages. Using the Poincaré theorem for 1-forms, we can extend the definition of the integral of a closed 1-form along a **continuous** path (until now, we have always assumed the our paths were piecewise differentiable). In fact, assume that  $\omega \in \Omega^1(U)$ ,  $d\omega = 0$ , and  $\gamma$  such that:

$$\gamma: [0, 1] \rightarrow U$$

is a differentiable map. Now, we choose a partition of  $[0, 1]$ , i.e. a collection of points  $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$  such that the restriction of  $\gamma$  to the interval  $(t_i, t_{i+1})$  is contained in a ball  $B_i$  where  $\omega$  is exact. In particular:

$$\omega = df_i, \text{ for } f_i: B_i \rightarrow \mathbb{R}.$$

Then:

$$\int_{\gamma} \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)] .$$

If  $\gamma$  is only continuous, we could still consider such a partition, and still define

$$\int_{\gamma} \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)] .$$

The integral of  $\gamma$  is well defined because the definition is independent from the choice of our partition: if  $P$  is one partition and  $P'$  is a refinement of  $P$  (i.e. it is the same partition plus an extra point  $t' \in (t_i, t_{i+1})$  for some  $i$ ), then:

$$[f_i(t_{i+1}) - \cancel{f_i(t')}] + [\cancel{f_i(t')} - f_i(t_i)] = [f_i(t_{i+1}) - f_i(t_i)] .$$

Then the integral does not change if we consider a refinement. If we consider a general partition  $P'$ , we can add every point of the partition  $P$  to  $P'$ , so that we get a refinement of  $P'$  that we will call  $P''$ . The integral on the partition  $P'$  has the same value of the integral on the partition  $P''$  by the above argument. Now, we can add every point of the partition  $P'$  to  $P$ , so to get the partition  $P''$  again, but now we can see  $P''$  as a refinement of  $P$ . Then the integral on  $P$  and on  $P''$  are the same. Then also the integrals on  $P$  and  $P'$  are the same.

Now, we want to extend the above theorem to  $k$ -forms.

**Definition 14.6** (Contractible set). An open subset  $U \subset \mathbb{R}^n$  is contractible to some point  $p_0 \in U$  if there exists a differentiable map

$$\begin{aligned} H: U \times [0, 1] &\rightarrow U \\ (p, t) &\mapsto H(p, t) \end{aligned} \tag{14.1}$$

such that  $H(p, 1) = p, H(p, 0) = p_0, \forall p \in U$ .

*Example 14.7.*  $\mathbb{R}^n$  is contractible to any point  $p_0 \in \mathbb{R}^n$ . The homotopy is given by:

$$H(p, t) = p_0 + t(p - p_0) .$$

*Remark 14.8.* If  $U$  is contractible, we can associate a  $k$ -form  $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$  to every  $\omega \in \Omega^k(U)$ .  $\bar{\omega}$  is defined as

$$\bar{\omega} = H^* \omega . \tag{14.2}$$

Conversely, we can associate a  $k$ -form  $\omega \in \Omega^k(U)$  to each  $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$  with the help of the inclusion map

$$\begin{aligned} i_t: U &\rightarrow U \times \mathbb{R} \\ p &\mapsto i_t(p) = (p, t) \end{aligned} \tag{14.3}$$

Then,  $i_t^* \bar{\omega} \in \Omega^k(U)$  if  $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ .

**Lemma 14.9.** *Any  $k$ -form on  $U \times \mathbb{R}$  has a unique decomposition of the form*

$$\bar{\omega} = \omega_1 + dt \wedge \eta \quad (14.4)$$

where  $\omega_1 \in \Omega^k(U \times \mathbb{R})$ ,  $\eta \in \Omega^{k-1}(U \times \mathbb{R})$ , and with  $i_{\partial_t} \omega_1 = 0$  and  $i_{\partial_t} \eta = 0$ .

*Proof.* Let's choose coordinates  $\{x^1, \dots, x^n, t\}$  for  $U \times \mathbb{R}$ . Then we write  $\bar{\omega}$  in the basis:

$$\bar{\omega} = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} + \sum_{i_1 < \dots < i_{k-1}} b_{i_1 \dots i_{k-1}} dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}.$$

(We can always do this, the coefficients could also be trivial). We set  $\omega_1$  to be the first and  $dt \wedge \eta$  to be the second summand. Obviously, we have  $i_{\partial_t} \omega_1 = 0$  and  $i_{\partial_t} \eta = 0$  by the definition of the dual basis:  $dx^j(\partial_t) = 0, \forall j$ ,  $dt(\partial_t) = 1$ . This decomposition is unique since it is independent of the coordinates chosen on  $U$ .  $\square$

*Remark 14.10.* Now, we want to integrate the form  $\bar{\omega} = \omega_1 + dt \wedge \eta$  of lemma 14.9. Let's define the map

$$\begin{aligned} I: \Omega^k(U \times \mathbb{R}) &\rightarrow \Omega^{k-1}(U) \\ \eta &\mapsto I\eta \end{aligned} \quad (14.5)$$

such that

$$(I\eta)_p(z_1, \dots, z_{k-1}) = \int_0^1 \eta(p, t)(\partial_t, i_t^* z_1, \dots, i_t^* z_{k-1}) dt$$

where we used the inclusion map:

$$\begin{aligned} i_t: U &\rightarrow U \times \mathbb{R} \\ p &\mapsto (p, t) \end{aligned} \quad (14.6)$$

and here  $i_t$  "includes"  $U$  into  $U \times \mathbb{R}$  at  $t$  (it is **not** the interior derivative!)

**Lemma 14.11.** *The following holds:*

$$i_1^* \bar{\omega} - i_0^* \bar{\omega} = d(I\bar{\omega}) + I(d\bar{\omega}) \quad (14.7)$$

where  $i_1$  is  $i_t$  for  $t = 1$ , and  $I$  is the integration defined in remark 14.10.  $i_t^*: \Omega(U \times \mathbb{R}) \rightarrow \Omega(U)$  is the pullback of  $i_t$ .

*Proof.* Let's consider  $p \in U$  with coordinates  $x^1, \dots, x^n$ . It suffices to check the claim for the summands in the decomposition  $\bar{\omega} = \omega_1 + dt \wedge \eta$  due to linearity.

**case a)** Assume:

$$\bar{\omega} = f dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad (14.8)$$

Then:

$$d\bar{\omega} = df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{\partial f}{\partial t} dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} + \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

So:

$$I(d\bar{\omega}) = \int_0^1 \frac{\partial f}{\partial t}(p, t) \underbrace{i_t^* dx^{i_1}}_{=dx^{i_1}} \wedge \dots \wedge i_t^* dx^{i_k} dt = (f(p, 1) - f(p, 0)) dx^{i_1} \wedge \dots \wedge dx^{i_k} = i_1^* \bar{\omega} - i_0^* \bar{\omega}.$$

Moreover,  $I\bar{\omega} = 0$  because  $\bar{\omega}$  does not contain any  $dt$  in this case.

**case b)** Assume:

$$\bar{\omega} = f dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}. \quad (14.9)$$

Then:

$$d\bar{\omega} = \frac{\partial f}{\partial x^i} dx^i \wedge dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}.$$

So, changing the order of the basis and getting a minus sign:

$$I(d\bar{\omega}) = - \int_0^1 \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} dt.$$

And also:

$$d(I\bar{\omega}) = d \int_0^1 f(p, t) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} dt = \int_0^1 \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} dt.$$

Thus, we have  $I(d\bar{\omega}) + d(I\bar{\omega}) = 0$ . In this case (cf. (14.9)), we also have  $i_t^* dt = 0$  (recall that  $i_t: p \mapsto (p, t)$ ). Therefore:  $i_t^* \bar{\omega} = 0, \forall t$ .  $\square$

Then we can extend Poincaré lemma to  $k$ -forms:

**Theorem 14.12** (Poincaré Lemma). *Let  $k \geq 1$ . Let  $U$  be a contractible, open subset of  $\mathbb{R}^n$  and  $\omega \in \Omega^k(U)$  with  $d\omega = 0$ . Then there exists a  $(k-1)$ -form  $\alpha \in \Omega^{k-1}(U)$  such that  $\omega = d\alpha$ .*

*Proof.* The proof follows from the previous lemmas 14.9 and 14.11. In fact, in the hypotheses of the theorem, the  $\alpha$  that we need is  $\alpha = I\bar{\omega}$ . Indeed: given  $\omega \in \Omega^k(U)$ , we can define  $\bar{\omega} = H^* \omega \in \Omega^k(U \times \mathbb{R})$ , where  $H$  is the map that can contract  $U$  to a point. If we consider the map  $i_t$  defined in (14.6), we have:

$$H \circ i_1 = \text{id} \quad \text{and} \quad H \circ i_0 = \text{const.}$$

So:

$$\omega = (H \circ i_1)^* \omega = i_1^* (H^* \omega) = i_1^* \bar{\omega}$$

and by remark 9.23 we know that the pullback of a  $k$ -form ( $k \geq 1$ ) through a constant map is 0, so we also have:

$$0 = (H \circ i_0)^* \omega = i_0^* (H^* \omega) = i_0^* \bar{\omega}.$$



From the lemma 14.11, we know that  $i_1^* \bar{\omega} - i_0^* \bar{\omega} = d(I\bar{\omega}) + I(d\bar{\omega})$ . From previous computations, it means:

$$\omega = d(I\bar{\omega}) + I(d\bar{\omega}).$$

Actually,  $I(d\bar{\omega}) = 0$  because  $d\bar{\omega} = 0$ . Indeed:

$$d\bar{\omega} = d(H^* \omega) \stackrel{(1)}{=} H^* d\omega \stackrel{(2)}{=} 0$$

where we used: (1) naturality of the pullback, cf. proposition 9.22, point 5, (2)  $d\omega = 0$  by assumption. Then,  $\omega = d(I\bar{\omega})$ .  $\square$

Question: For  $\omega \in \Omega^1(U)$ , when is  $\int_\gamma \omega$  independent of the choice of  $\gamma$ ?

**Definition 14.13** (Homotopy between curves). Two continuous curves  $\gamma_1$  and  $\gamma_2$ ,  $\gamma_i: [a, b] \rightarrow U, i = 1, 2, U \subset \mathbb{R}^n$  are freely homotopic if there exists a continuous map  $H$

$$\begin{aligned} H: [a, b] \times [0, 1] &\rightarrow U && \text{such that:} \\ H(s, 0) &= \gamma_1(s), && \forall s \in [a, b] \\ H(s, 1) &= \gamma_2(s), && \forall s \in [a, b] \end{aligned}$$

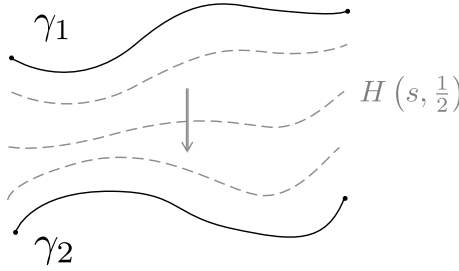


Figure 6: Homotopy between curves

**Definition 14.14** (Homotopy between curves with same endpoints). Two continuous curves  $\gamma_1$  and  $\gamma_2$ ,  $\gamma_i: [a, b] \rightarrow U, i = 1, 2, U \subset \mathbb{R}^n$ , with  $\gamma_1(a) = \gamma_2(a)$  and  $\gamma_1(b) = \gamma_2(b)$  are homotopic relatively to  $\{\gamma_1(a), \gamma_2(b)\}$  if there exists a continuous map  $H$

$$\begin{aligned} H: [a, b] \times [0, 1] &\rightarrow U && \text{such that:} \\ H(s, 0) &= \gamma_1(s), && \forall s \in [a, b] \\ H(s, 1) &= \gamma_2(s), && \forall s \in [a, b] \\ H(a, t) &= \gamma_1(a) = \gamma_2(a), && \forall t \in [0, 1] \\ H(b, t) &= \gamma_1(b) = \gamma_2(b), && \forall t \in [0, 1] \end{aligned}$$

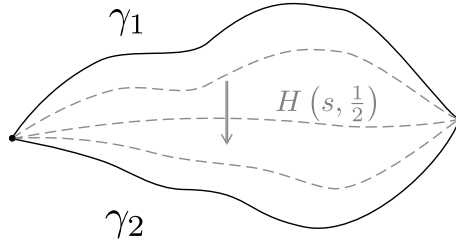
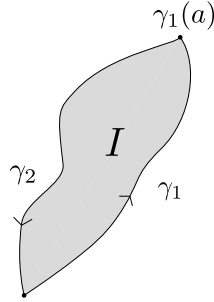


Figure 7: Homotopy between curves with same endpoints

**Theorem 14.15.** *Let  $U$  be an open, contractible set. Let  $\omega \in \Omega^1(U)$ , with  $d\omega = 0$  (closed), and  $\gamma_1, \gamma_2$  be two homotopic curves in  $U$  with the same endpoints. then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega. \quad (14.10)$$

*Proof.* Let's suppose the two curves do not intersect each other. We can go through  $\gamma_1$  backwards. Let  $I$  be the interior of the space bounded by the two curves.



$I$  is contractible, then by Poincaré lemma 14.12,  $\omega$  is exact in  $I$ . Using Stokes's theorem and denoting the curve  $\gamma_1$  with reversed orientation by  $\hat{\gamma}_1$ :

$$\int_{\hat{\gamma}_1 + \gamma_2} \omega = \int_I d\omega = 0$$

where last equality follows from the exactness of  $\omega$ :  $d\omega = d^2\alpha = 0$  for some 0-form  $\alpha$ . On the other hand, if the two curves intersect each other, we split the curves for each intersection and apply the previous proof for each one of them.  $\square$

What if  $\gamma_1(a) \neq \gamma_2(a), \gamma_1(b) \neq \gamma_2(b)$ ?

**Definition 14.16** (Homotopy between closed curves).  $\gamma_1, \gamma_2: [a, b] \rightarrow U$ ,  $\gamma_i$  closed curves, are freely homotopic if there exists a continuous map

$$\begin{aligned} H: [a, b] \times [0, 1] &\rightarrow U && \text{such that:} \\ H(s, 0) &= \gamma_1(s), && \forall s \in [a, b] \\ H(s, 1) &= \gamma_2(s), && \forall s \in [a, b] \\ H(a, t) &= H(b, t), && \forall t \in [0, 1] \end{aligned}$$

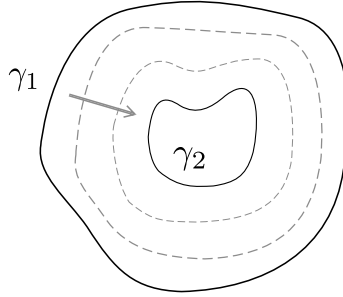


Figure 8: Homotopy between closed curves

**Proposition 14.17.** *Let  $U$  be an open, contractible set. If  $\omega$  is a closed 1-form on  $U$ ,  $\gamma_1$  and  $\gamma_2$  two closed curves, freely homotopic in  $U$ , then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega. \quad (14.11)$$

*In particular, if  $\gamma_1$  is freely homotopic to a point, then  $\int_{\gamma_1} \omega = 0$*

*Idea of the proof.* We can decompose the space in small squares and use the Stokes' theorem for each of them.  $\square$

**Definition 14.18** (Simply connected set). A connected open set  $U \subset \mathbb{R}^n$  is simply connected if every continuous closed curve in  $U$  is freely homotopic to a point in  $U$ .

*Example 14.19.*  $\mathbb{R}^n$ , the unitary ball in  $\mathbb{R}^n$  and its homeomorphic images are simply connected.

*Remark 14.20* (Contractible vs. simply connected). "Contractible  $\implies$  simply connected" (every single point of a closed curve can be brought to a point), but "simply connected  $\not\implies$  contractible" (cf.  $S^2$ , from intuition we can say that contracting the sphere to a point implies "cutting" the sphere somewhere, and it can't be done in a continuous way).

*Remark 14.21.* Every closed 1-form on a simply connected subset  $U$  of  $\mathbb{R}^n$  is exact. Indeed,  $\int_{\gamma} \omega = 0$  for every closed  $\gamma$  in the space, where  $\omega$  is a closed 1-form, because every closed curve can be shrunk to a point (by def. 14.18). By proposition 14.2,  $\omega$  is exact. The result is not true for  $k$ -forms (in such cases, we need more advanced tools).

We can limit ourselves to consider continuous curves, thanks to the two following results:

**Theorem 14.22** (Whitney approximation on  $\mathbb{R}^n$ ). *If  $\gamma$  is a continuous map between  $U, V \subseteq \mathbb{R}^n$ , then  $\gamma$  is homotopic to a smooth map  $\tilde{\gamma}$ . If  $\gamma$  is smooth on a closed subset  $A$  of  $U$ , then the homotopy can be taken relatively to  $A$ .*

**Theorem 14.23.** *If  $\gamma_1$  and  $\gamma_2$  are homotopic maps between  $U$  and  $V$ , then they are smoothly homotopic.*

## 15 ■ De Rham Cohomology

Let  $U$  be an open subset of  $\mathbb{R}^n$ . We can think of  $\Omega^k(U)$  as a vector space over  $\mathbb{R}$ . Indeed, if  $\alpha, \beta \in \Omega^k(U)$ , then  $a\alpha + b\beta \in \Omega^k(U)$ ,  $\forall a, b \in \mathbb{R}$ . Also the remaining properties of a vector space are satisfied.

*Remark 15.1.* We say that  $\Omega^k(U, \mathbb{Z}) = \{k\text{-forms on } U \text{ with coefficients in } \mathbb{Z}\}$  forms a group (and not a vector space) since  $\mathbb{Z}$  is not a field. In contrast,  $\Omega^k(U, \mathbb{R})$  is a vector space. We will often denote  $\Omega^k(U, \mathbb{R})$  just by  $\Omega^k(U)$ .

Now, we want to introduce some concepts of cohomology theory (see also page 460, [Lee]).

**Definition 15.2** (Complex). Let's consider the sequence

$$\dots \longrightarrow A^{k-1} \xrightarrow{d} A^k \xrightarrow{d} A^{k+1} \longrightarrow \dots$$

where for every  $k$ ,  $A^k$  is a vector space on  $\mathbb{R}$  and the maps  $d$  are linear maps. Such a sequence is said to be a complex if the composition of any two successive applications of  $d$  is the zero map:

$$d \circ d = 0: A^k \rightarrow A^{k+2} \text{ for each } k.$$

Sometimes it is also called a cochain complex, whereas it is called a chain complex if the arrows go in the opposite direction. Sometimes we use the notation  $d_k: A^{k-1} \rightarrow A^k$  to denote the position of the map in the sequence. The property  $d_{k+1} \circ d_k = 0$  implies  $\text{im}(d_k) \subset \ker(d_{k+1})$ .

**Definition 15.3** (Exact sequence). Let  $A^*$  be a complex. It is called an exact sequence if  $\text{im}(d_k) = \ker(d_{k+1})$  for each  $k$ .

**Definition 15.4** ( $k$ -th cohomology group). Let  $A^*$  be a complex. We define the  $k$ -th cohomology group of  $A^*$  to be the quotient module

$$H^k(A^*) \equiv \ker(d_{k+1}) / \text{im}(d_k).$$

It is well defined since  $\text{im}(d_k) \subset \ker(d_{k+1})$  by definition of cochain complex. If we are dealing with a chain complex, it is called homology group instead of cohomology group. The  $k$ -th cohomology group is a quantitative measure for the failure of exactness at  $A^k$ .

**Definition 15.5** (de Rham complex). The de Rham complex is the sequence

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \xrightarrow{d} 0$$

where  $d$  is the exterior derivative now. It is well defined since we know that  $d^2 = d \circ d = 0$  from the properties of the exterior derivative. Thus,  $\text{im}(d_k) \subset \ker(d_{k+1})$ . Notice that we wrote  $\Omega^k$  instead of  $\Omega^k(U)$  for the sake of simplicity, and that the sequence stops at  $\Omega^n$  because  $U \subset \mathbb{R}^n$ .

**Definition 15.6** (de Rham cohomology group). Let  $U \subset \mathbb{R}^n$ ,  $U$  open,  $\dim(U) = m \leq n$ . Then:

- The set of closed  $k$ -forms is the  $k$ -th *cocycle group*  $Z^k(U, \mathbb{R})$  (it is a group with respect to addition),
- The set of exact  $k$ -forms is the  $k$ -th *coboundary group*  $B^k(U, \mathbb{R})$ ,
- The  $k$ -th *de Rham cohomology group*  $H^k(U, \mathbb{R})$  is defined as:

$$H^k(U, \mathbb{R}) \equiv Z^k(U, \mathbb{R}) / B^k(U, \mathbb{R}) = \ker(d_{k+1}) / \text{im}(d_k).$$

*Remark 15.7* (Cohomologous forms). The set  $\ker(d_{k+1})$  contains all the  $k$  forms that are closed. The set  $\text{im}(d_k)$  contains the exact forms. As usual when we deal with vector spaces, the equivalence relation is:

$$\omega \sim \omega' \Leftrightarrow \omega - \omega' \in \text{im}(d_k) \Leftrightarrow \omega - \omega' = d\alpha \text{ for some } (k-1)\text{-form } \alpha. \quad (15.1)$$

From the above definition, we see that two closed forms  $\omega, \omega'$  are in the same equivalence class (so they are said to be **cohomologous**) if they differ by an exact form. Obviously, if  $\omega$  and  $\omega'$  are exact, we have  $\omega = dF$  and  $\omega' = dG$  for some  $(k-1)$ -forms  $F$  and  $G$ . So, the two forms are cohomologous because the  $\alpha$  that we need in the definition (15.1) is  $\alpha = F - G$ . All the exact forms are identified with the 0 element of  $H^k(U, \mathbb{R})$ , because  $\omega \sim 0$  if  $\omega$  is an exact  $(k-1)$ -form (the difference between  $\omega$  and 0 is an exact form). To read more about quotient spaces, see also the example 5.7.

*Example 15.8.* Let's consider the following examples:

- If  $U$  is contractible, then  $H^k(U, \mathbb{R}) = \{0\}$  by Poincaré lemma,
- $U = \mathbb{R}^2 \setminus \{0\}$ ,  $H^0(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \mathbb{R}$  (constant functions),
- $H^1(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \mathbb{R}$ ,
- $H^2(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \{0\}$ ,
- Torus  $T$ :  $H^0(T) = \mathbb{R}$ ,  $H^1(T) = \mathbb{R} \oplus \mathbb{R}$ ,  $H^2(T) = \mathbb{R}$ .

**Definition 15.9.** If  $M \subset \mathbb{R}^n$ , then we define:

$$\Omega^*(M, \mathbb{R}) \equiv \bigoplus_{k=0}^n \Omega^k(M, \mathbb{R}).$$

*Remark 15.10.*  $\wedge: \Omega^* \times \Omega^* \rightarrow \Omega^*$  endows  $\Omega^*$  with the structure of a ring.

*Remark 15.11* (Exact sequence). If  $H^k(M, \mathbb{R}) = 0$  for  $k = 1, \dots, n$ , then

$$\dots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \dots$$

is such that  $\ker(d_{k+1}) = \text{im}(d_k)$  for every  $k = 1, \dots, n$ . Notice that for  $k = 0$ ,  $\Omega^0 = \{\text{functions}\}$ , we have  $H^0 \neq 0$ . So, it doesn't hold for  $k = 0$ .

**Definition 15.12** (de Rham cohomology ring). The de Rham cohomology ring is defined as

$$H^*(M, \mathbb{R}) = \bigoplus_{k=0}^n H^k(M, \mathbb{R}).$$

*Example 15.13.*  $H^*(T^2, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$ , where  $T^2$  is the torus.

*Remark 15.14.* If  $\phi: U \rightarrow V$ ,  $U, V$  open subsets of  $\mathbb{R}^n$ ,  $\phi$  diffeomorphism, then  $\phi^*: \Omega^k(V) \rightarrow \Omega^k(U)$  is the pullback of  $\phi$  on  $k$ -forms. By the construction above, we can also define such pullbacks on  $H^k$ , as  $\phi^*: H^k(V) \rightarrow H^k(U)$ , which maps closed-but-not-exact forms to closed-but-not-exact forms due to the naturality of the pullback. If  $\phi$  is a diffeomorphism, then the pushforward  $\phi_*$  is an isomorphism and hence also  $\phi^*$  due to  $\phi^* = \phi_*^{-1}$ . Question: what if  $U$  and  $V$  are not diffeomorphic, but only homotopy equivalent? We want to show that the final result is the same.

**Definition 15.15** (Homotopy between maps). Two maps  $\phi, \psi: U \rightarrow V$  are homotopy equivalent if there exists a continuous map

$$\begin{aligned} H: [0, 1] \times U &\rightarrow V \text{ such that:} \\ H(0, \cdot): U &\rightarrow V, \quad \text{with } H(0, \cdot) = \phi(\cdot) \\ H(1, \cdot): U &\rightarrow V, \quad \text{with } H(1, \cdot) = \psi(\cdot) \end{aligned}$$

**Definition 15.16** (Homotopy equivalence). Two subsets  $U$  and  $V$  of  $\mathbb{R}^n$  are homotopy equivalent if there exist continuous maps  $f: U \rightarrow V$  and  $g: V \rightarrow U$  such that the compositions  $g \circ f$  and  $f \circ g$  are homotopic to the identity in  $U$  and  $V$ , respectively.  $f$  and  $g$  are called homotopy equivalences.

*Example 15.17.* Let's consider the following examples:

- We call "homotopy inverse" the inverse of a function up to a homotopy. Any homeomorphism  $\phi: U \rightarrow V$  with homotopy inverse  $\phi^{-1}$  is a homotopy equivalence, but the converse is not always true (a disk is homotopy equivalent to a point, but it's not homeomorphic to a point). So, the concept of "homeomorphism" between sets is stronger than the concept of "homotopy equivalence",
- $S^1$  is homotopy equivalent to  $\mathbb{R}^2 \setminus \{0\}$ ,
- $S^{n-1}$  is homotopy equivalent to  $\mathbb{R}^n \setminus \{0\}$ ,
- A solid torus is homotopy equivalent to a tea cup.

**Lemma 15.18.** *Let  $\phi, \text{id}: V \rightarrow V$  be two smoothly homotopic maps. Then  $\phi^*|_{H^*(V, \mathbb{R})} = \text{id}^*|_{H^*(V, \mathbb{R})}$ .*

*Proof.* We want to show that  $(\phi^* - \text{id}^*)(\omega) = d\alpha$ . If it is true,  $\phi^*$  and  $\text{id}^*$  map to the same equivalence class in the cohomology group, because two forms are in the same equivalence class if they differ by an exact form. Let  $H: \mathbb{R} \times V \rightarrow V$  be the smooth homotopy between  $\phi$  and  $\text{id}$  in  $V$ . We can consider it smooth thanks to Whitney approximation (cf. theorem 14.22). Let  $\omega \in H^*(V)$ . From the proof of Poincaré lemma 14.12 we have:

$$d(IH^*\omega) + I(dH^*\omega) = i_1^*H^*\omega - i_0^*H^*\omega$$

with:  $i_0(p) = (0, p), i_1(p) = (1, p)$  (recall the definition of the inclusion (14.6)). Here,  $H \circ i_0 = \text{id}|_V, H \circ i_1 = \phi|_V$ . Thus, using that

$$i_1^*H^*\omega = (H \circ i_1)^*\omega = \phi^*\omega \text{ and } i_0^*H^*\omega = (H \circ i_0)^*\omega = \text{id}^*\omega,$$

we have:

$$(\phi^* - \text{id}^*)(\omega) = dI(H^*\omega) + I(\underbrace{dH^*\omega}_{=H^*d\omega=0}).$$

Where we used the naturality of the pullback and the fact that  $\omega \in H^*(V) \Rightarrow \omega$  is closed.  $\square$

**Theorem 15.19.** *Let  $\phi: U \rightarrow V$  be a homotopy equivalence between  $U$  and  $V$  with homotopy inverse  $\psi: V \rightarrow U$ . then  $\phi^*$  induces an isomorphism  $\hat{\phi}^*$  such that:*

$$H^n(V, \mathbb{R}) \cong H^n(U, \mathbb{R}), \forall n. \quad (15.2)$$

*Proof.* Let  $\varphi \equiv \tilde{\phi} \circ \tilde{\psi}$ , where  $\tilde{\phi}$  is smooth and smoothly homotopy equivalent to  $\phi$ .  $\tilde{\psi}$  is smooth and smoothly homotopy equivalent to  $\psi$ . So,  $\varphi$  is homotopy equivalent to the identity, i.e.  $\varphi \simeq \text{id}_V$ . Thus:

$$\hat{\text{id}}_{H^n(V, \mathbb{R})}^* = \hat{\varphi}_{H^n(V, \mathbb{R})}^* = \left( \hat{\psi}^* \circ \hat{\phi}^* \right)_{H^n(V, \mathbb{R})}$$

by lemma 15.18. In the same way:

$$\left(\hat{\phi}^* \circ \hat{\psi}^*\right)_{H^n(U, \mathbb{R})} = \hat{\text{id}}_{H^n(U, \mathbb{R})}^*.$$

Thus,  $\hat{\phi}$  is a bijection. □

The following definitions will be given for cochain complexes, but the respective definitions for chain complexes are analogous (you just need to replace "cochain" with "chain" and "cohomology" with "homology").

**Definition 15.20** (Cochain map). Let  $A^*$  and  $B^*$  be two cochain complexes. A cochain map  $\hat{\phi}^*: A^* \rightarrow B^*$  is a collection of linear maps  $\phi^*: A^n \rightarrow B^n$  s.t.  $d \circ \phi^* = \phi^* \circ d: A^n \rightarrow B^{n+1}$ . We often denote  $\hat{\phi}^*$  simply by  $\phi^*$ .

*Example 15.21.* Let's consider:

$$\begin{aligned} A^* &= \Omega^*(U, \mathbb{R}) = \Omega^0(U, \mathbb{R}) \oplus \Omega^1(U, \mathbb{R}) \oplus \dots \\ B^* &= \Omega^*(V, \mathbb{R}) = \Omega^0(V, \mathbb{R}) \oplus \Omega^1(V, \mathbb{R}) \oplus \dots \end{aligned}$$

Then some cochain maps might be given by  $\hat{\phi}^*: A^* \rightarrow B^*$ ,  $\phi^*: \Omega^0(U, \mathbb{R}) \rightarrow \Omega^0(V, \mathbb{R})$ ,  $\phi^*: \Omega^1(U, \mathbb{R}) \rightarrow \Omega^1(V, \mathbb{R})$ , etc.

**Definition 15.22.** A short exact sequence (SES) is a collection of cochain complexes  $A^*, B^*, C^*$  and cochain maps  $\phi^*: A^n \rightarrow B^n, \psi^*: B^n \rightarrow C^n$  such that for each  $n$ :

$$0 \longrightarrow A^n \xrightarrow{\phi^*} B^n \xrightarrow{\psi^*} C^n \longrightarrow 0 \quad (15.3)$$

is exact.

*Remark 15.23.* Remember the definition of exact sequence: it gives a condition on the kernel and the range of the maps (cf. definition 15.11). By this condition, we have that  $\phi^*$  must be an injective map, and  $\psi^*$  must be a surjective map in the above definition.

*Example 15.24.* Let's consider the following example:



$$\begin{array}{ccccccc}
\Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \longrightarrow & \dots & \} \text{ sequence} \\
\phi^* \downarrow & & \phi^* \downarrow & & \phi^* \downarrow & & & \\
\Omega^0(V) & \xrightarrow{d} & \Omega^1(V) & \xrightarrow{d} & \Omega^2(V) & \longrightarrow & \dots & \} \text{ sequence} \\
\psi^* \downarrow & & \psi^* \downarrow & & \psi^* \downarrow & & & \\
\Omega^0(W) & \xrightarrow{d} & \Omega^1(W) & \xrightarrow{d} & \Omega^2(W) & \longrightarrow & \dots & \} \text{ sequence}
\end{array}$$

$\underbrace{\hspace{1.5cm}}$  sequence       $\underbrace{\hspace{1.5cm}}$  sequence       $\underbrace{\hspace{1.5cm}}$  sequence

Since the cochain maps commute with  $d$ , every small square is a commuting diagram.

**Lemma 15.25** (Zig-zag lemma). *Let's consider  $\phi^*, \psi^*$  as in the previous definitions. Then there exists a linear map  $\delta$ , called the connecting homomorphism, such that:*

$$\begin{array}{ccccccc}
\dots & \longrightarrow & H^{n-1}(C^*) & \xrightarrow{\delta} & H^n(A^*) & \xrightarrow{\phi^*} & H^n(B^*) & \xrightarrow{\psi^*} \\
& & \xrightarrow{\psi^*} & H^n(C^*) & \xrightarrow{\delta} & H^{n+1}(A^*) & \xrightarrow{\phi^*} & \dots
\end{array}$$

is an exact sequence.

*Example 15.26.* We have:

$$\begin{array}{ccccccc}
H^0(U) & \xrightarrow{d} & H^1(U) & \xrightarrow{d} & H^2(U) & \longrightarrow & \dots \\
\phi^* \downarrow & & \phi^* \downarrow & & \phi^* \downarrow & & \\
H^0(V) & \xrightarrow{d} & H^1(V) & \xrightarrow{d} & H^2(V) & \longrightarrow & \dots \\
\psi^* \downarrow & & \psi^* \downarrow & & \psi^* \downarrow & & \\
H^0(W) & \xrightarrow{d} & H^1(W) & \xrightarrow{d} & H^2(W) & \longrightarrow & \dots
\end{array}$$

$\delta$        $\delta$        $\delta$

See also example 15.24.

**Theorem 15.27** (Mayer-Vietoris). *Let  $M \subset \mathbb{R}^n$  such that  $M = f(U) \cup g(V)$ ,  $U, V$  open subsets in  $\mathbb{R}^m$ ,  $m \leq n$ .  $f, g$  homeomorphisms. Let:*

$$i: U \cap V \rightarrow U \text{ (inclusion)}$$

$$j: U \cap V \rightarrow V \text{ (inclusion)}$$

Let:

$$\begin{aligned}
(f^* \oplus g^*): \Omega^k(M) &\rightarrow \Omega^k(U) \oplus \Omega^k(V) \\
\omega &\mapsto (f^*(\omega), g^*(\omega))
\end{aligned}$$

$$(i^* - j^*): \Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V)$$

$$(\omega, \eta) \mapsto i^*\omega - j^*\eta$$

Then, for each  $k$  there exists a linear map  $\delta$  such that:

$$\dots \rightarrow H^k(M) \xrightarrow{f^* \oplus g^*} H^k(U) \oplus H^k(V) \xrightarrow{i^* - j^*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \rightarrow \dots$$

is exact.

## 16 ■ Submanifolds of $\mathbb{R}^n$

**Premise:** We want to go outside  $\mathbb{R}^n$ , and analyze more general topologies. For now, we can think about them as generalizations of curves and surfaces of  $\mathbb{R}^n$ , a proper definition will come later. Note that what we studied until now can often be extended to manifolds: with sufficient conditions, manifolds can be "embedded" in  $\mathbb{R}^n$ . We will give four equivalent definitions of a submanifold  $M$  of dimensions  $m$  in  $\mathbb{R}^n$  ( $m \leq n$ ).

**Definition 16.1** (Submanifold - Local parametrization).  $M \subset \mathbb{R}^n$  is a submanifold of  $\mathbb{R}^n$  if  $\forall p \in M$  there exists an open neighbourhood  $V \subset \mathbb{R}^n$  of  $p$  and an open set  $U \subset \mathbb{R}^m$ , where  $m \leq n$ , and there exists a smooth map  $\phi: U \rightarrow \mathbb{R}^n$  such that:

- $\phi: U \rightarrow M \cap V$  is a homeomorphism,
- $\phi_*(x): \mathbb{R}^m \rightarrow \mathbb{R}^n$  is injective  $\forall x \in U$ .

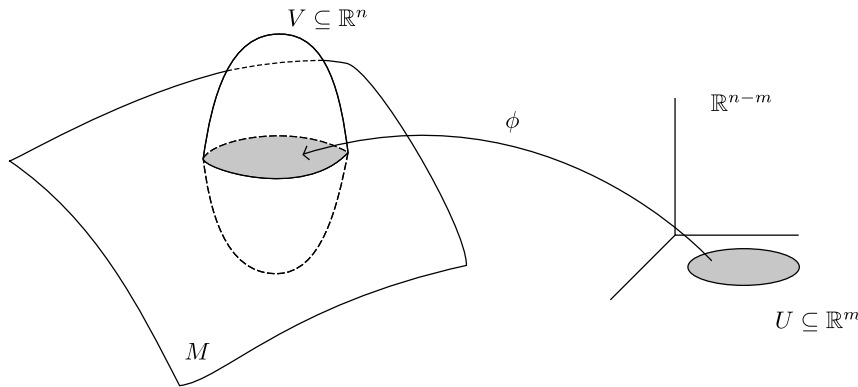


Figure 9: Submanifold - local parametrization

**Definition 16.2** (Submanifold - Locally flat).  $M \subset \mathbb{R}^n$  is a submanifold of  $\mathbb{R}^n$  if  $\forall p \in M$ , there exists a neighbourhood  $V \subset \mathbb{R}^n$  of  $p$ , a neighbourhood  $W \subset \mathbb{R}^n$  of 0, and a diffeomorphism  $\Phi: V \rightarrow W$  such that:

$$\Phi(p) = 0 \text{ and } \Phi(V \cap M) = W \cap (\mathbb{R}^m \times \{0\}^{n-m}) .$$

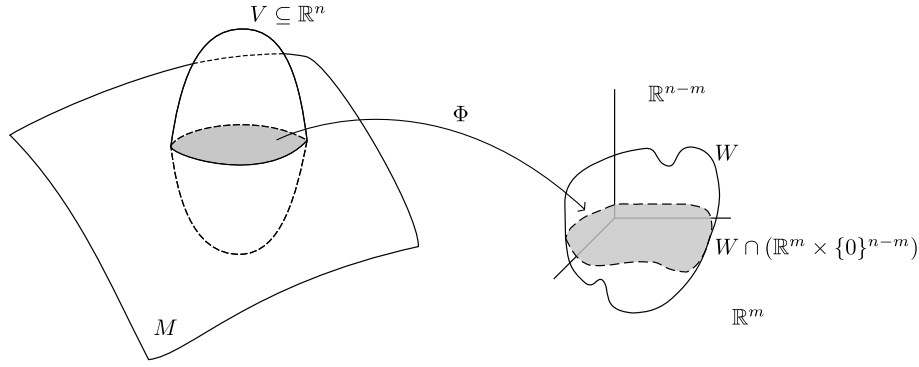


Figure 10: Submanifold - locally flat

**Definition 16.3** (Submanifold - Locally a zero set).  $M \subset \mathbb{R}^n$  is a submanifold of  $\mathbb{R}^n$  if  $\forall p \in M$  there exists an open neighbourhood  $V \subset \mathbb{R}^n$  of  $p$ , and a smooth map  $F: V \rightarrow \mathbb{R}^{n-m}$  such that:

- $V \cap M = \{x \in V \mid F(x) = 0\}$ ,
- $F_*: V \rightarrow \mathbb{R}^{n-m}$  is surjective.

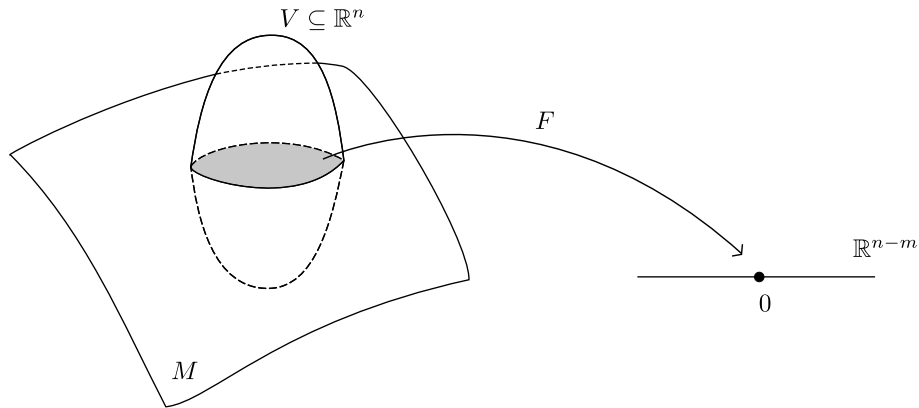


Figure 11: Submanifold - locally a zero set

**Definition 16.4** (Submanifold - Locally a graph).  $M \subset \mathbb{R}^n$  is a submanifold of  $\mathbb{R}^n$  if  $\forall p \in M$  there exists a neighbourhood  $V \subset \mathbb{R}^n$  of  $p$ , a permutation  $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  and an open set  $U \subset \mathbb{R}^m$ ,  $U$ , together with a smooth map  $g: U \rightarrow \mathbb{R}^{n-m}$  such that:

$$V \cap M = \{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \mid (x_1, \dots, x_m) \in U \text{ and } (x_{m+1}, \dots, x_n) = g(x_1, \dots, x_m)\}.$$

The map  $g$  is called a *graph*.

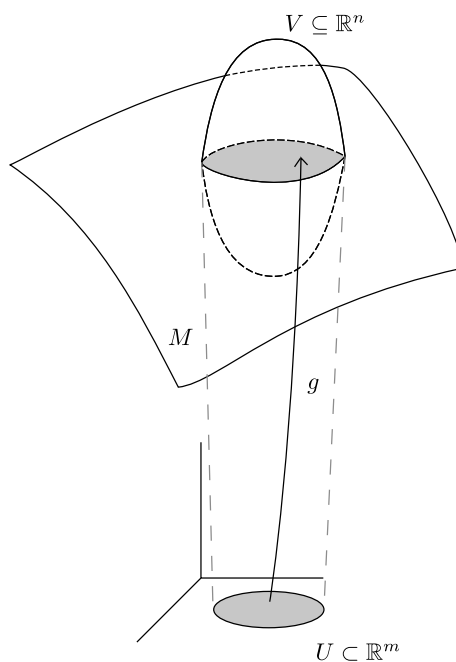


Figure 12: Submanifold - locally a graph

*Remark 16.5* (4 definitions). Let's analyze two of the four definitions of a submanifold of  $\mathbb{R}^n$ .

**16.1)** The homeomorphism hypothesis is quite intuitive: we want our space to be locally similar to  $\mathbb{R}^n$ . In fact, the fact that  $\phi$  is an isomorphism make the vector field given by the differential of  $\phi$  well-defined. Indeed, the map  $\phi$  can define a vector field on  $M$  through its differential  $d\phi = \phi_*$ , if  $\phi_*$  is a bijective map.

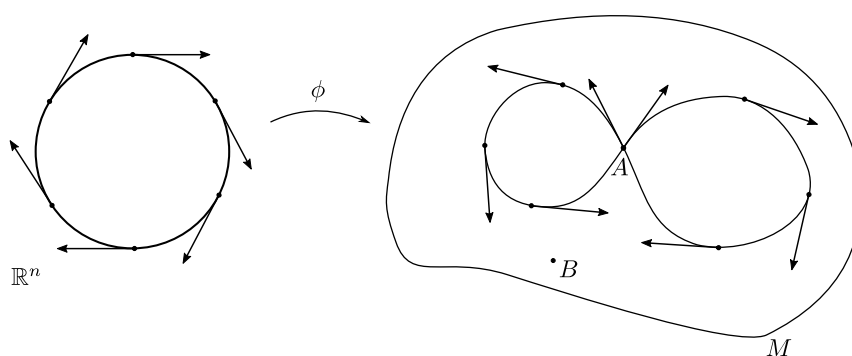


Figure 13: If  $\phi$  is not injective, we have problems in  $A$ . If not surjective, we have problems in  $B$  (see also page 182, [Lee]).

If  $\phi$  is not injective, we would have some points on  $M$  such that the tangent vector about these points is not unique: the physical meaning would be lost,

and also the vector field given by the differential would not be well defined since it assigns more than one element to the initial vector. If  $\phi$  is not surjective, we would have some points on  $M$  that have no tangent vector at all. Moreover, we have the hypothesis of injectivity of  $\phi_*$ . It means that  $\phi$  is a smooth embedding (see also the def. 18.33 that will be given in the next sections). Being a smooth embedding means that  $U$  is being included in  $V \cap M$ . In fact, the Rank theorem (page 81 [Lee]) says that a such a map behaves like an inclusion ( $U$  is "injected" in  $V \cap M$ ).

**16.3)** The hypothesis of surjectivity is required because of what we previously said about the Rank theorem. In particular, we require  $F$  to be a smooth submersion (see also the def. 18.31 that will be given in the next sections). Using the Rank theorem, it means that the map  $F$  behaves like a projection ( $V \cap M$  is projected into the zero set). Notice that in this case we don't care about vector fields, so  $F$  doesn't have to be bijective!

**Theorem 16.6.** *The four definitions above are equivalent.*

*Proof.*

**(b)  $\Rightarrow$  (c):**  $\Phi: W \rightarrow V$  is a diffeomorphism. Let  $F(x) = \pi \circ \Phi^{-1}$ , where  $x \in M \cap V$ ,  $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$  ( $\pi$  is the projection on the last  $n - m$  coordinates). Then  $F$  satisfies (c) by construction.

**(c)  $\Rightarrow$  (d):** We have:

$$F: V \subset \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n-m}$$

$$(a, b) \mapsto F(a, b)$$

Let's consider the Jacobian matrix:

$$(F_*)_p = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \boxed{\frac{\partial F^1}{\partial x^{m+1}} & \cdots & \frac{\partial F^1}{\partial x^n}} \\ \vdots & & & & \\ \frac{\partial F^{n-m}}{\partial x^1} & \cdots & \boxed{\frac{\partial F^{n-m}}{\partial x^{m+1}} & \cdots & \frac{\partial F^{n-m}}{\partial x^n}} \end{pmatrix}$$

Let  $A \in \mathbb{R}^{(n-m) \times (n-m)}$  be the submatrix denoted by the rectangle.  $A$  is invertible because  $F$  is surjective. Then, by Dini theorem (or implicit function theorem):  $\exists U' \subset \mathbb{R}^m$ ,  $U'$  open,  $a \in U'$  and a map  $g: U' \rightarrow \mathbb{R}^{n-m}$  such that  $F(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = 0$  implies  $(x_{m+1}, \dots, x_n) = g(x_1, \dots, x_m)$ , where  $(x_1, \dots, x_m) \in U'$ . Moreover, different orderings of columns of  $A$  give rise to different permutations. Then,  $g$  is the graph of definition (d).

(d)  $\Rightarrow$  (a): From (d), we know that  $g: \mathbb{R}^m \supset U \rightarrow \mathbb{R}^{n-m}$ . Then:

$$\begin{aligned}\phi: U &\rightarrow \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \\ x &\mapsto (x, g(x))\end{aligned}$$

is a local parametrization: it is a homeomorphism onto the image, and it is differentiable because  $g$  is differentiable.

(a)  $\Rightarrow$  (b): Let  $\phi: \mathbb{R}^m \supset U \rightarrow \mathbb{R}^n$  be a local parametrization with  $\phi_*: \mathbb{R}^m \rightarrow \mathbb{R}^n$  injective. Let

$$B \equiv \left( \left( \frac{\partial \phi^i}{\partial x^j} \right)_{x=q} \right)_{i,j=1,\dots,m}, \quad q = \phi^{-1}(p), p \in M \cap V.$$

In general, it is not true that  $\phi^1$  corresponds to the first component. But we can always reorder the coordinates so that it is true.  $B$  is invertible. Let's define:

$$\begin{aligned}\Phi: U \times \mathbb{R}^{n-m} &\rightarrow \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \\ (x, y) &\mapsto \phi(x) + (0, y) = \\ &= (\phi_1(x_1, \dots, x_m), \dots, \phi_m(x_1, \dots, x_m), \\ &\quad \phi_{m+1}(x_1, \dots, x_m) + y_1, \dots, \phi_n(x_1, \dots, x_m) + y_{n-m}).\end{aligned}$$

Then:

$$\Phi_*(q, 0) = \left( \begin{array}{c|c} B & 0 \\ \hline \text{anything} & \text{Id} \end{array} \right)$$

is invertible. Thus, by the implicit function theorem, there exists a neighbourhood  $W \subset (U \times \mathbb{R}^{n-m})$  of  $(q, 0)$  and  $V' \subset \mathbb{R}^n$  of  $p$  such that  $\Phi: W \rightarrow V'$  is a diffeomorphism. Furthermore:

$$\Phi(W \cap (\mathbb{R}^m \times \{0\}^{n-m})) = \phi(W \cap (\mathbb{R}^m \times \{0\}^{n-m})) \subset M \cap V$$

because  $\Phi$  acts like  $\phi$  for  $y = 0$ . The local inverse of  $\Phi$  satisfies the definition (b). □

**Corollary 16.7.** *Let:*

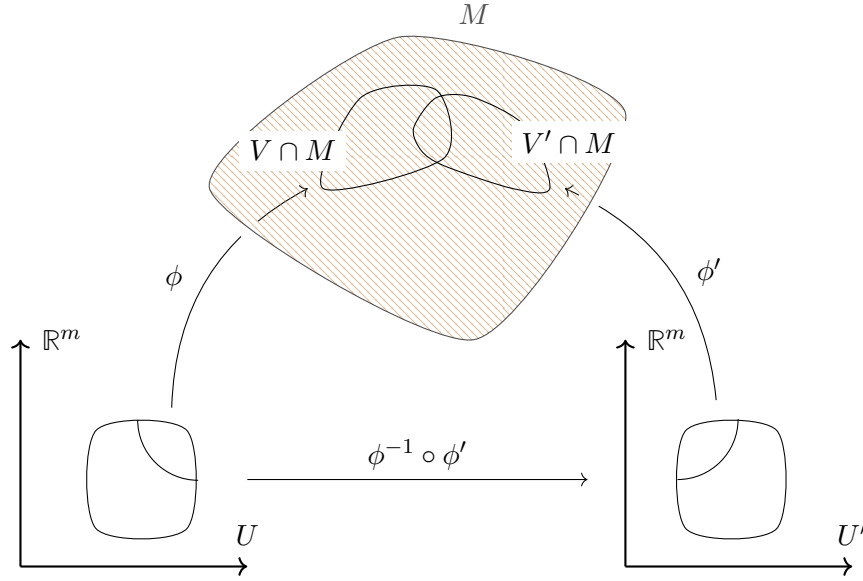
$$\phi: \mathbb{R}^m \supset U \rightarrow V \cap M$$

$$\phi': \mathbb{R}^m \supset U' \rightarrow V' \cap M$$

*be local parametrizations. Then:*

$$\phi^{-1} \circ \phi': (\phi')^{-1}(\phi(U) \cap \phi'(U')) \rightarrow \phi^{-1}(\phi(U) \cap \phi'(U'))$$

*is a diffeomorphism.*



*Proof.*  $\phi^{-1} \circ \phi'$  is a homeomorphism because  $\phi$  and  $\phi'$  are homeomorphisms. We want to show that they are smooth. Let:

$$\begin{aligned} q' &\in (\phi')^{-1}(\phi(U) \cap \phi'(U')) \subset U' \\ q &\in \phi^{-1} \circ \phi'(q') \in U. \end{aligned}$$

By the proof of (a)  $\Rightarrow$  (b) of the theorem 16.6, we know that there exists a diffeomorphism  $\Phi: \mathbb{R}^n \supset W \rightarrow V'' \subset \mathbb{R}^n$ , with  $(q, 0) \in W$ , such that:

$$\Phi|_{W \cap \mathbb{R}^m} = \phi|_{W \cap \mathbb{R}^m}.$$

Thus:

$$\phi^{-1} \circ \phi'|_{(\phi')^{-1} \circ \phi(W \cap \mathbb{R}^m)} = \Phi^{-1} \circ \phi'|_{(\phi')^{-1} \circ \phi(W \cap \mathbb{R}^m)}$$

where the right-hand side is smooth. We can prove the same result the other way round.  $\square$

*Example 16.8* (Submanifolds of  $\mathbb{R}^n$ ). Let's consider some examples.

1. An open subset of dimension  $n$  in  $\mathbb{R}^n$  is a submanifold (by definition (a), we just take  $\phi = \text{id}$ ). For instance  $B^n \subset \mathbb{R}^n$  is a submanifold of  $\mathbb{R}^n$ .
2.  $S^{n-1} = \partial B^n$  is a submanifold of  $\mathbb{R}^n$ . Indeed, by definition (c), it is the zero set of the function:

$$F(\underline{x}) = (x^1)^2 + (x^2)^2 + \dots + (x^n)^2 - 1 = \underline{x}^2 - 1, \underline{x} \in \mathbb{R}^n$$

and  $F_*$  is surjective:  $F_*(\underline{x}) = 2\underline{x}$ .

3.  $O(n) = \{A \mid AA^t = E\} \subset \text{Mat}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ , with  $E$  identity matrix.  $O(n)$  is a submanifold of dimension  $\frac{n(n-1)}{2}$ . Indeed, using definition (c), it is the

zero set of the function:

$$\begin{aligned} F: \text{Mat}(n \times n, \mathbb{R}) &\rightarrow \text{Symm}(n, \mathbb{R}) \\ A &\mapsto AA^t - E. \end{aligned}$$

What is more,  $F_*$  is surjective. In order to prove that, we prove that " $F_*|_A(X) = 0, \forall A \Rightarrow X = 0$ " (then we know that for a linear map  $L$  on a vector space  $V$ ,  $\dim(V) = \dim \text{Ker}(L) + \dim \text{Ran}(L)$ , so the dimension of the range of  $F_*|_A$  must be  $\dim \text{Mat}(n \times n, \mathbb{R})$ , so the map is surjective). In fact:

$$\begin{aligned} F_*|_A &= \frac{d}{dt}\bigg|_{t=0} F(A + tX) = \frac{d}{dt}\bigg|_{t=0} [AA^t + tAX^t + tXA^t + t^2XX^t] = \\ &= AX^t + XA^t. \end{aligned}$$

And the solution of:  $AX^t + XA^t = S \in \text{Symm}(n, \mathbb{R})$  is  $X = \frac{SA}{2}$  because  $\frac{AA^t}{2}S + S\frac{AA^t}{2} = S$ . Then:

$$AX^t + XA^t = 0 \implies X = 0.$$

Now, we want to use diffeomorphisms like those in the corollary 16.7 in order to introduce the concept of manifold.

**Definition 16.9** (Atlas). An ( $n$ -dimensional, smooth) atlas  $\mathcal{A}$  on a set  $M$  is a collection of maps (called charts)

$$\begin{aligned} \phi_\alpha: \mathbb{R}^n &\xrightarrow{\sim} M \\ U_\alpha &\mapsto W_\alpha \end{aligned} \tag{16.1}$$

such that:

- $\bigcup_{\alpha \in I} W_\alpha = M$ ,
- $\forall \alpha, \beta \in I$  with  $W_\alpha \cap W_\beta \neq \emptyset$ ,

$$\phi_\beta^{-1} \circ \phi_\alpha: \phi_\alpha^{-1}(W_\alpha \cap W_\beta) \rightarrow \phi_\beta^{-1}(W_\alpha \cap W_\beta)$$

is a diffeomorphism.

where  $\alpha \in I$ ,  $I$  index set, and the  $\sim$  above the arrow means that  $\phi_\alpha$  is bijective.

**Definition 16.10** (Equivalence relation on atlases). Two atlases  $\mathcal{A}$  and  $\mathcal{A}'$  are equivalent  $\Leftrightarrow \mathcal{A} \cup \mathcal{A}'$  is an atlas ( $\Leftrightarrow \phi_\beta^{-1} \circ \phi'_\alpha$  is a diffeomorphism,  $\forall \phi'_\alpha \in \mathcal{A}', \phi_\beta \in \mathcal{A}$ ).

**Definition 16.11** (Preliminary definition of manifold). A manifold is a set  $M$  with an equivalence class of atlases.



*Remark 16.12.* The definition 16.11 above is "preliminary" because we have not specified anything about the topology yet (we are still working on Euclidean topology). Moreover, in every equivalence class there exists a unique maximal atlas (i.e. such that, if combined with an other atlas, it can't get any bigger). We also notice that if the charts are smooth enough, we can pullback "everything" (e.g. all our vector fields, differential forms, etc. defined for subset of  $\mathbb{R}^n$ ). However, in that case, everything is defined *locally*. In order to patch them all together, we need some other result (like the partition of unity, coming soon).

*Example 16.13* (Projective space). Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  ( $\mathbb{K}$  is some field).

$$\mathbb{K}P^n \equiv (\mathbb{K}^{n+1} \setminus \{0\}) / \sim$$

where

$$(x_0, \dots, x_n) \sim (x'_0, \dots, x'_n) \iff \exists \lambda \in \mathbb{K} \setminus \{0\} \text{ such that } (\lambda x_0, \dots, \lambda x_n) = (x'_0, \dots, x'_n).$$

For instance,  $\mathbb{C}P^1 = S^2$  (every point on a line passing through the origin is identified with the point on such line at distance 1 from the origin). We use the notation  $[x_0, \dots, x_n]$  for one equivalence class. Let's consider the atlas  $\mathcal{A} = \{\phi_i: \mathbb{K}^n \rightarrow \mathbb{K}P^n\}$ . Where

$$\begin{aligned} \phi_i: \mathbb{K}^n &\rightarrow \mathbb{K}P^n \\ (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n) &\mapsto [x_0, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n] \end{aligned}$$

And  $\phi_i(\mathbb{K}^n) = \{[x_0, \dots, x_n] \in \mathbb{K}P^n \mid \text{i-th entry is } \neq 0\}$ . We notice that  $\mathcal{A}$  is an atlas because  $\phi_i$  satisfies the properties of the definition 16.9:

- $\phi$  is a bijection and its inverse is

$$\begin{aligned} \phi_i^{-1}: \phi_i(\mathbb{K}^n) &\rightarrow \mathbb{K}^n \\ [x_0, \dots, x_{i-1}, \underbrace{x_i}_{\neq 0}, \dots, x_n] &\mapsto \left( \frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right) \end{aligned}$$

- $\phi$  is also a diffeomorphism, because  $\phi_j^{-1} \circ \phi_i$ , defined as

$$\begin{aligned} \phi_j^{-1} \circ \phi_i: \phi_i^{-1}(\phi_i(\mathbb{K}^n) \cap \phi_j(\mathbb{K}^n)) &\rightarrow \phi_j^{-1}(\phi_i(\mathbb{K}^n) \cap \phi_j(\mathbb{K}^n)) \\ (x_1, \dots, x_n) &\mapsto \left( \frac{x_1}{x_j}, \dots, \underbrace{\frac{1}{x_j}}_{i\text{-th}}, \dots, \frac{x_n}{x_j} \right) \end{aligned}$$

is a diffeomorphism (where  $x_i \neq 0, x_j \neq 0$ ) because it is a differentiable map and the inverse is again differentiable.

*Remark 16.14* (Extension of analysis to manifolds). The existence of an atlas allows us to define concepts from analysis to manifolds. For instance, we can define continuous and smooth functions on manifolds using the concepts of continuity and smoothness that we use in  $\mathbb{R}^n$ .

**Definition 16.15** (Continuous/smooth function on a manifold). Let  $M$  be a manifold, a function  $f: M \rightarrow \mathbb{R}$  is a continuous (respectively, smooth) function if and only if  $f \circ \phi_\alpha: U_\alpha \rightarrow \mathbb{R}$  is continuous (smooth) function  $\forall \alpha$ . Such a function is well-defined, since the definition is independent on the local parametrization chosen, thanks to the definition of atlas.

*Remark 16.16* (cut-off functions). We can construct functions on a manifold in the following way: we consider a function defined on an open set in  $M$  which is homeomorphic to an open set in  $\mathbb{R}^n$ . Then we extend it to zero outside such open set, but in order to have a differentiable function we smoothly bring it to zero using the so-called cut-off functions. For instance, a smooth function from  $\mathbb{R}$  to  $[0, 1] \subset \mathbb{R}$  such that:

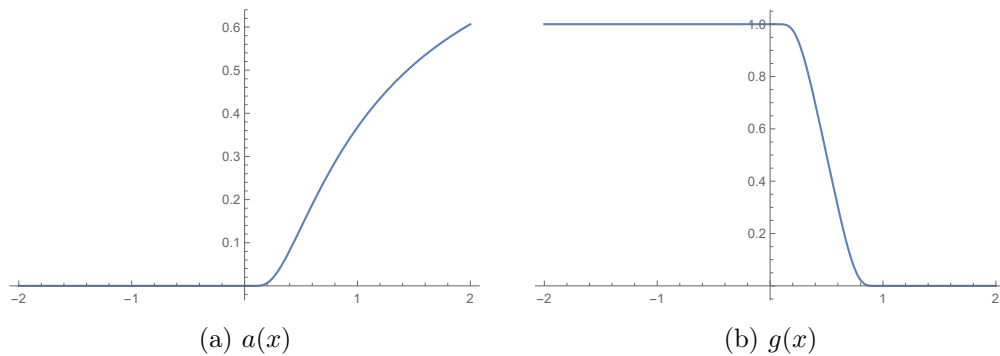
$$h(x) = \begin{cases} 1, & x \leq 1 \\ \text{anything}, & 1 \leq x \leq 2 \\ 0, & x \geq 2 \end{cases}$$

is a cut-off function. For example, given

$$a(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x \geq 0 \end{cases}$$

we can consider

$$g(x) = 1 - \frac{a(x)}{a(x) + a(1-x)}.$$



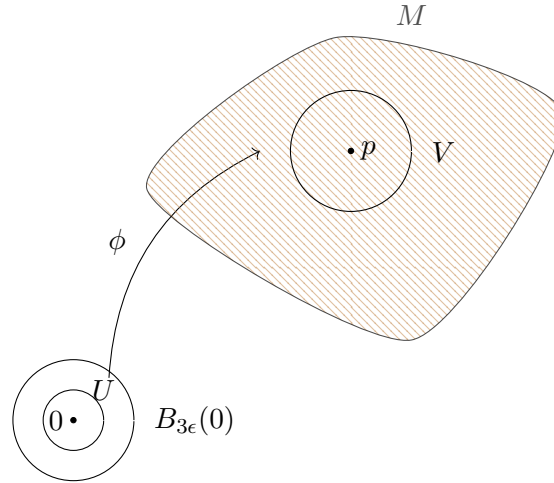
Then we use the function

$$h_\epsilon(x) \equiv h\left(\frac{x}{\epsilon}\right) = \begin{cases} 1, & x \leq \epsilon \\ 0, & x \geq 2\epsilon \end{cases}$$

Now, for  $p \in M$ , let

$$\phi: \mathbb{R}^n \supset U \rightarrow V \subset M$$

such that  $\phi(0) = p$  (where  $0 \in U$ ) and  $\epsilon > 0$  such that  $B_{3\epsilon} \supset U$ .



Then  $g(x) = h_\epsilon(|x|)$  defines a smooth function  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\text{supp}(g) = \overline{\{x \mid g(x) \neq 0\}} \subset U$ . Moreover,  $g \circ \phi^{-1}: V \rightarrow \mathbb{R}$  extends (by zero) to a function  $f \in \mathbb{F}(M)$  with  $\text{supp} f \subset V$  and  $f \equiv 1$  in a neighbourhood of  $p$ . In particular, this shows that  $\mathbb{F}(M)$  is infinite-dimensional.

**Definition 16.17** (Vector fields on manifolds). Vector fields are defined as before (we just replace  $\mathbb{F}(\mathbb{R}^n)$  with  $\mathbb{F}(M)$ ). They are maps

$$\text{Der}|_p \mathbb{F}(M) = T_p M \ni v_p: \mathbb{F}(M) \rightarrow \mathbb{R}$$

which are  $\mathbb{R}$ -linear and satisfying Leibnitz rule. The set of all vector fields is:

$$\mathfrak{X}(M) = \text{Der } \mathbb{F}(M) = TM \ni v: \mathbb{F}(M) \rightarrow \mathbb{F}(M)$$

*Remark 16.18* (Representation of  $v$  in  $U_\alpha$ , local coordinates). Everything is done in  $\mathbb{R}^n$ . Indeed, if  $(\phi_\alpha, U_\alpha)$  is a chart, with  $U_\alpha \subset \mathbb{R}^n$ ,  $\phi_\alpha$  is called a local coordinate map. Then, the representation of a vector field  $v$  in  $U_\alpha$  is:

$$v_{(\alpha)} = v_{(\alpha)}^i \partial_{x_i}$$

where

$$v_{(\alpha)}^i(x) = v^i \circ \phi_\alpha(x).$$

We point out that given a chart  $(\phi_\alpha, U_\alpha)$  and a point  $x$  in the manifold, the local coordinates of  $x$  with respect to the chart are the coordinates of  $\phi_\alpha(x)$  as a vector in  $U_\alpha \subset \mathbb{R}^n$ . (Here, we are denoting the chart by using the subset of  $\mathbb{R}^n$ , and not the subset of the manifold. Since  $\phi$  is a diffeomorphism we can do that.)

Notice that cut-off functions allow us to extend any smooth function defined on  $V_\alpha \subset M$  (for some  $\alpha \in I$ ) to all of  $M$  through extension by zero outside  $V_\alpha$ . Suppose we are given a function on  $M$ . How do we decide if it is continuous (or smooth)?

**Proposition 16.19** (Partition of unity). *Let  $M$  be a compact manifold and let  $\{V_\alpha\}$  be a covering of  $M$ . Then there exists a family of differentiable functions  $\varphi_1, \dots, \varphi_m$  such that:*

- $\sum_{i=1}^m \varphi_i \equiv 1$ ,
- $0 \leq \varphi_i \leq 1$  and  $\text{supp}(\varphi_i) \subset V_\alpha$  for some  $\alpha \in I$ .

**Definition 16.20** (Partition of unity). The family  $\{\varphi_i\}$  defined above is said to be a partition of unity subordinate to the covering  $\{V_\alpha\}$ .

Now, if  $f: M \rightarrow \mathbb{R}$ , we can consider  $f_i = \varphi_i f: v_\alpha \rightarrow \mathbb{R}$  for some  $\alpha \in I$ .

**Definition 16.21** (Differential forms on  $M$ ). A differential form on  $M$  is a map

$$\begin{aligned} \Omega^k(M) \ni \omega: \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) &\rightarrow \mathbb{F}(M) \\ (v_1, \dots, v_k) &\mapsto \omega(v_1, \dots, v_k) \end{aligned} \quad (16.2)$$

which is linear, skew-symmetric (i.e. alternating) as already defined.

*Remark 16.22* (Representation of  $\omega$  in  $U_\alpha$ ). We represent  $\omega$  in  $U_\alpha$  as:

$$\omega_{(\alpha)} = \sum_{i_1 < \dots < i_k} (a_{i_1 \dots i_k})_{(\alpha)} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where  $\{x^i\}$  are the coordinates on  $U_\alpha$ . Moreover:

$$\omega_{(\alpha)}((v_1)_\alpha, \dots, (v_k)_\alpha) = \pm \sum_{i_1 < \dots < i_k} (a_{i_1 \dots i_k})_\alpha (v^{i_1})_\alpha \dots (v^{i_k})_\alpha$$

where the  $\pm$  sign depends on the orientation used.

**Definition 16.23** (Curve on a manifold). A curve  $\gamma: (a, b) \rightarrow M$  is continuous (respectively, differentiable) if  $\phi_\alpha^{-1} \circ \gamma: (a, b) \rightarrow U_\alpha$  is continuous (differentiable).

## 17 ■ Integration on Compact Manifolds

**Definition 17.1** (Orientable manifold).  $M$  is orientable if there exists an atlas  $\mathcal{A} = \{\phi_\alpha, U_\alpha\}$  such that for each pair  $\alpha, \beta$  with  $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$ , the differential

$$(\phi_\beta^{-1} \circ \phi_\alpha)_*: U_\alpha \cap U_\beta \rightarrow U_\alpha \cap U_\beta$$

has positive determinant. Example of a non-orientable manifold: Möbius strip.

*Remark 17.2* (How to integrate). Let  $\omega \in \Omega^n(M)$ ,  $M$  compact manifold,  $\dim M = n$ . Pick a partition of unity subordinate to a covering  $\{V_\alpha\}$  (i.e. pick some maps  $\varphi_i, i = 1, \dots, m, 0 \leq \varphi_i \leq 1, \sum_{i=1}^m \varphi_i \equiv 1, \text{supp}(\varphi_i) \subset V_{\alpha(i)}$  for some  $\alpha = \alpha(i)$ ). Then:

$$\begin{aligned} \int_M \omega &= \int_M \sum_{i=1}^m \varphi_i \omega = \sum_i \int_M (\varphi_i \omega) = \sum_i \int_{V_{\alpha(i)}} (\varphi_i \omega) = \\ &= \sum_i \int_{U_{\alpha(i)}} \phi_{\alpha(i)}^* (\varphi_i \omega) \end{aligned} \quad (17.1)$$

where  $V_\alpha = \phi_\alpha(U_\alpha)$ . Notice that **the integral is well-defined if  $M$  is orientable**: if it's not, when we make a change of coordinates to swap to another chart we get a different sign. Indeed, the formula of the change of coordinate contains the determinant of the Jacobian, and we would find a chart that gives us a different final sign.

**Lemma 17.3.**  $\int_M \omega$  is independent of the choice of  $(V_\alpha, \varphi_i)$ .

*Proof.* Let  $\{W_\alpha\}$  be another covering of  $M$  and  $\{\psi_j\}_{j=1, \dots, m'}$  a partition of unity subordinate to  $\{W_\alpha\}$ , then  $\{W_\alpha \cap V_\beta\}_{\alpha, \beta}$  is also a covering of  $M$  and

$$(\varphi_i \psi_j)_{\substack{i=1, \dots, m \\ j=1, \dots, m'}}$$

is a partition of unity subordinate to  $\{W_\alpha \cap V_\beta\}$ . Thus:

$$\sum_{i=1}^m \int_M \varphi_i \omega = \sum_{i=1}^m \sum_{j=1}^{m'} \int_M \varphi_i \psi_j \omega = \sum_{j=1}^{m'} \int_M \psi_j \left( \sum_{i=1}^m \varphi_i \omega \right) = \sum_{j=1}^{m'} \int_M \psi_j \omega$$

where we could swap the sums since they are finite (they are finite because  $M$  is compact by assumption).  $\square$

*Remark 17.4.* What happens with boundaries? If  $p \in \partial M$ , there is no neighbourhood of  $p$  homeomorphic to an open set  $U \subset \mathbb{R}^n$ . Let's define the following set:

$$H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^1 \geq 0\}.$$

**Note** that on [Lee],  $H^n$  is defined as

$$H^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

(everything is the same, but with  $x^1$  replaced by  $x^n$ ).

**Definition 17.5** (subset topology in  $H^n$ ). An open set in  $H^n$  is the intersection between  $H^n$  and an open set in  $\mathbb{R}^n$ .

**Definition 17.6** (Functions on  $H^n$ ). A function  $f: V \rightarrow \mathbb{R}, V$  open,  $V \subset H^n$  is differentiable if  $\exists$  an open set  $U, \mathbb{R}^n \supset U \supset V$  and a differentiable function  $\bar{f}: U \rightarrow \mathbb{R}$  such that  $\bar{f}|_V = f|_V$ . Furthermore,  $(f_*)_p = (\bar{f}_*)_p, p \in V$ .

**Definition 17.7** (Preliminary definition of diff. manifold with boundary). An  $n$ -dimensional differentiable manifold with a regular boundary is a set  $M$  with an equivalence class of atlases, as usual, but with the difference that  $\mathbb{R}^n$  in the definition 16.9 is replaced by  $H^n$  everywhere. The boundary is "regular" if it is described by a regular curve (no intersection, etc...).

When is a point on the boundary of a manifold?

**Definition 17.8** (Point on the boundary). A point  $p \in M$  is on the boundary of  $M$  if for some parametrisation  $\phi: U \subset H^n \rightarrow M$  around  $p$ , we have  $\phi(0, x^2, \dots, x^n) = p$  for some  $x^2, \dots, x^n$ .

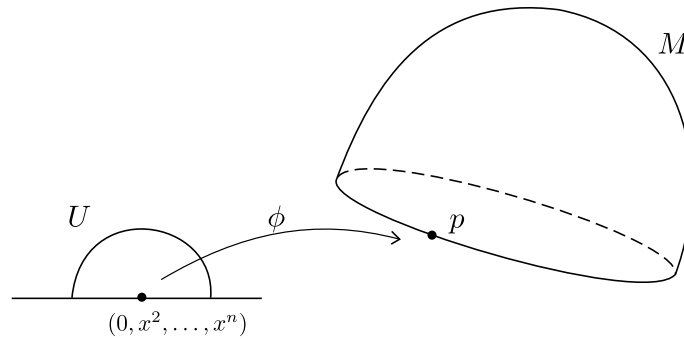


Figure 15: Point on the boundary of a manifold

**Lemma 17.9.** *The definition 17.8 does not depend on the choice of parametrisation.*

*Proof.* The proof is by contradiction. Let  $\phi_1: U_1 \rightarrow M, U_1 \subset H^n$  such that  $\phi_1(0, x^2, \dots, x^n) = p$  for some  $x^2, \dots, x^n$ . Assume there exists  $\phi_2: U_2 \rightarrow M$  parametrization around  $p$  such that  $\phi_2^{-1}(p) = q = (x^1, \dots, x^n)$  with  $x^1 \neq 0$ . Let  $\Omega = \phi_1(U_1) \cap \phi_2(U_2)$ . Then, the map:

$$(\phi_1^{-1} \circ \phi_2): \phi_2^{-1}(\Omega) \rightarrow \phi_1^{-1}(\Omega)$$

is a diffeomorphism (by definition of manifold). Since  $x^1 \neq 0$ , there exists a neighbourhood  $U$  of  $q$  such that  $U \subset \phi_2^{-1}(\Omega)$  and such that  $U$  does not intersect with:

$$\{(x^1, \dots, x^n) \in H^n \mid x^1 = 0\}.$$

The restriction of  $\phi_1^{-1} \circ \phi_2$  to  $U$  gives a differentiable map  $U \rightarrow H^n$  such that  $\det(\phi_1^{-1} \circ \phi_2)_* \neq 0$  (invertible map). Thus,  $H^n$  is diffeomorphic to a set that doesn't contain  $\{(x^1, \dots, x^n) \mid x^1 = 0\}$ , contradiction.  $\square$

**Proposition 17.10.**

- The boundary  $\partial M$  of a  $n$ -dimensional differentiable manifold with boundary is a  $(n - 1)$ -dimensional differentiable manifold,

- The orientation on  $M$  induces an orientation on  $\partial M$ .

**Theorem 17.11** (Stokes theorem on manifolds).  $M$  orientable, then:

$$\int_M d\omega = \int_{\partial M} i^* \omega. \quad (17.2)$$

*Idea of the proof.* First, we prove the theorem for a rectangle and then, for a generic set, we fill the set itself with rectangles.  $\square$

## 18 ■ Abstract Manifolds

*Remark 18.1* (Non-Example of a manifold). The set

$$M = (-\infty, 0) \cup (0, \infty) \cup \{a, b\}$$

(endowed with a particular topology that we will see later) is not a manifold. Here,  $a, b$  can be seen as points of  $\mathbb{R}^2$ : just imagine  $M$  as a subset of  $\mathbb{R}^2$ , consisting of a real line without the origin plus any two points in  $\mathbb{R}^2$ . Even if we have not defined a topology on  $M$ , we can feel that it doesn't look right: there are two maps  $\varphi_a, \varphi_b$  identifying the subsets  $U_a = M \setminus \{b\}$  and  $U_b = M \setminus \{a\}$  with  $\mathbb{R}$ . Indeed, in the first case the 0 element is given by the point  $a$ , whereas in the second case the 0 is given by  $b$ . The transition function

$$\varphi_b \circ \varphi_a^{-1}: \varphi_a(U_a \cap U_b) = \mathbb{R} \setminus \{0\} \rightarrow \varphi_b(U_b \cap U_a) = \mathbb{R} \setminus \{0\}$$

is the identity, and is smooth. **But** a smooth function on  $M$  must have the same value when evaluated at  $a$  and  $b$ , and this does not look right (why?).

**Definition 18.2** (Manifold). A manifold  $M$  of dimension  $n$  is a topological space which is Hausdorff, second countable and admits an (equivalence class of) atlas(es)  $\varphi_i: U_i \rightarrow V_i$ , where the  $\varphi_i$  are homeomorphisms,  $U_i \subset M$  is open and  $V_i \subset \mathbb{R}^n$  is also open, so that the transition functions

$$\varphi_i \circ \varphi_j^{-1}: \varphi_j(U_i \cap U_j) \rightarrow \varphi_i(U_i \cap U_j)$$

are smooth (i.e.  $C^\infty$ ). Sometimes we consider  $\varphi$  going from  $\mathbb{R}^n$  to  $M$  instead that the other way around, without any loss of generality. Notice that we are considering equivalence classes of atlases. We recall that the equivalence relation is the following one: two atlases are equivalent if their union is still a smooth atlas.

*Remark 18.3* (2nd countable hypothesis for a manifold). We require the hypothesis of second countability for manifolds in order to apply several theorems or lemmas, e.g. Sard's theorem.

**Fact 18.4.** *Manifolds are metrizable, i.e. there exists a metric  $d$  such that the topology on the manifold is the one induced by  $d$ .*

**Theorem 18.5.** *A manifold admits a compact exhaustion.*

*Proof.* The proof uses the second countability. We will skip it.  $\square$

**Theorem 18.6.** *Manifolds are paracompact.*

*Proof.* The proof uses the second countability property. We will skip it here.  $\square$

**Theorem 18.7.** *Every open cover of a manifold admits a subordinate partition of unity which is smooth.*

**Definition 18.8** (Smooth function from a manifold). Let  $M$  be a smooth manifold, then  $f: M \rightarrow \mathbb{R}$  is smooth at  $p$  if for a chart  $\varphi: U \rightarrow \mathbb{R}^n$ , with  $U \ni p$ , we have that

$$f \circ \varphi^{-1}: \varphi(U) \rightarrow \mathbb{R}$$

is smooth near  $\varphi(p)$ .

*Remark 18.9.* The definition does not depend on the chart chosen. In fact, let's choose another chart  $(V, \psi)$ , then:

$$f \circ \psi^{-1} = \underbrace{f \circ \varphi^{-1}}_{\text{smooth}} \circ \underbrace{\varphi \circ \psi^{-1}}_{\text{smooth}}$$

is smooth around  $p$  because the composition of smooth functions is smooth.

**Definition 18.10** (Smooth function between manifolds). Let  $M, N$  be smooth manifolds. Then  $f: M \rightarrow N$  is smooth around  $p$  if  $f$  is continuous and for charts  $(U, \varphi)$  around  $p$  and  $(V, \psi)$  around  $f(p)$  we have that  $\psi \circ f \circ \varphi^{-1}$  is smooth on a neighbourhood of  $\varphi(p)$ .

Now that we know when a map is smooth, we want to differentiate. For this, we want the concept of tangent vector. We will use three equivalent definitions.

**Definition 18.11** (Equivalent curves). Two curves  $\gamma_0, \gamma_1: (-\varepsilon, \varepsilon) \rightarrow M$  with  $\gamma_0(0) = \gamma_1(0) = p$  are equivalent if for a chart  $(U, \varphi)$  around  $p$ :

$$\frac{d}{dt}\bigg|_{t=0} \varphi \circ \gamma_0(t) = \frac{d}{dt}\bigg|_{t=0} \varphi \circ \gamma_1(t).$$

And we notice that such definition does not depend on the chart chosen.

**Definition 18.12** (Geom. tangent space). We define the geometric tangent space at  $p$  as:

$$T_p^{\text{geom}} M = \{ \text{smooth curves } \gamma: (-\varepsilon, \varepsilon) \rightarrow M \mid \gamma(0) = p, \varepsilon > 0 \} / \sim \quad (18.1)$$

where  $\sim$  is the equivalence relation described in def. 18.11. The elements of  $T_p^{\text{geom}} M$  are called tangent vectors.



*Remark 18.13.* Let's analyze def. 18.12: it is an intuitive definition of the tangent space but from it it's not obvious that  $T_p^{\text{geom}}M$  is a vector space. The following definition of tangent space is more abstract, but on the other hand it will be evident that the tangent space is a vector space.

**Definition 18.14** (Derivation - pt.2). A derivation  $v_p$  on smooth functions  $C^\infty(M)$  at  $p$  is a  $\mathbb{R}$ -linear map  $v_p: C^\infty(M) \rightarrow \mathbb{R}$  which satisfies the Leibnitz rule  $v(fg)|_p = v(f)|_p g(p) + f(p)v(g)|_p$ .

**Definition 18.15** (Alg. tangent space). We define the algebraic tangent space at  $p$  as:

$$T_p^{\text{alg}}M = \{\text{derivations of } p\}.$$

*Remark 18.16.*

- Instead of using  $C^\infty(M)$  (which is a ring) we could have used the space  $\mathcal{E}_p^\infty(M) = C^\infty / \sim$  of germs of functions, where the equivalence relation is the following one:

$$f \sim g \Leftrightarrow f \equiv g \text{ on a neighbourhood of } p.$$

In such cases,  $v(f) = v(g)$  for  $v \in T_p^{\text{alg}}M$ . In fact, let's consider a smooth function  $h$  such that  $h(0) = 0$  and  $h \equiv 1$  near  $p$ . Then:

$$0 \stackrel{(1)}{=} v(h(f-g)) \stackrel{(2)}{=} v(h) \underbrace{(f(p) - g(p))}_{=0} + \underbrace{h(p)}_{=1} (v(f) - v(g))$$

where we used: (1)  $h(f-g) = h(0) = 0$  near  $p$  and the derivative of a constant function is 0 by the Leibnitz rule:  $v(1) = v(1 \cdot 1) = v(1) + v(1) \Rightarrow v(1) = 0$ . (2) Leibnitz rule again. We also say that the derivation  $v$  is local.

- As we already pointed out,  $T_p^{\text{alg}}M$  is a vector space. The only non-trivial thing to prove is that  $\dim T_p^{\text{alg}}M = n$  if the dimension of  $M$  is  $n$ .

**Lemma 18.17.** Let  $f: \mathbb{R}^N \supset B_\varepsilon(0) \rightarrow \mathbb{R}$  be smooth, then there are smooth functions  $f_i$  on  $B_\varepsilon(0)$  so that:

$$f(x) = f(0) + \sum_i x^i f_i(x^1, \dots, x^n). \quad (18.2)$$

*Proof.*

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx^1, \dots, tx^n) dt = \int_0^1 \sum_i x^i \frac{\partial}{\partial x^i} f(tx^1, \dots, tx^n) dt.$$

So, the  $f_i$  we are looking for are:

$$f_i = \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \dots, tx^n) dt.$$

□

*Remark 18.18.* Now, we can prove that the  $\dim T_p^{\text{alg}} M = n$ . Let  $v$  be a derivation at  $p$  and  $(U, \varphi)$  a local chart around  $p$ . Let  $x^1, \dots, x^n$  be the local coordinates, such that  $p$  coincides with the origin ( $x^i(p) = 0$ ). From lemma 18.17:

$$f(x) = f(0) + \sum_i x^i f_i(x)$$

and

$$v(f) = \underbrace{v(f(0))}_{=0} + \sum_i \left( \underbrace{v(x^i)}_{=f_i(0)} \underbrace{f_i(x^i)}_{=f_i(0)} + \underbrace{x^i}_{=0} v(f_i) \right).$$

Then,  $v$  is defined by the way it reacts to coordinate functions around  $p$ . Thus,  $\dim T_p^{\text{alg}} M \leq n$ . Since a particular derivation is  $v_i = \frac{\partial}{\partial x^i}$  and since it "reacts" in  $n$  different ways, we have  $\dim T_p^{\text{alg}} M = n$ .

**Definition 18.19** (Physicists' tangent space). Let  $M$  be a manifold of dimension  $n$ , then a tangent vector is a map

$$v: \{ \text{charts } (U, \varphi) \text{ around } p \} \rightarrow \mathbb{R}^n \quad (18.3)$$

where  $v$  is defined such that if we change charts we have:

$$v(V, \psi) = D_p(\psi \circ \varphi^{-1}) v(U, \varphi). \quad (18.4)$$

The space of such maps is called tangent space (according to the physicists' definition) and it is denoted by  $T_p^{\text{phys}} M$ .

*Remark 18.20.* The three previous definitions of tangent space are all equivalent, indeed we can find isomorphisms between such tangent spaces.

1.

$$T_p^{\text{geom}} M \rightarrow T_p^{\text{alg}} M \quad (18.5)$$

$$[\gamma] \mapsto v_\gamma: C^\infty(M) \rightarrow \mathbb{R} \quad (18.6)$$

$$f \mapsto \left. \frac{d}{dt} \right|_{t=0} (f \circ \gamma)(t) \quad (18.7)$$

2.

$$T_p^{\text{alg}} M \rightarrow T_p^{\text{phys}} M \quad (18.8)$$

$$v \mapsto \{(U, \varphi) \mapsto \{v(x^i)\}_{i=1, \dots, n}\} \quad (18.9)$$

where  $x^i$  are the coordinates around  $p$  from  $\varphi$ .

3.

$$T_p^{\text{phys}} M \rightarrow T_p^{\text{geom}} M \quad (18.10)$$

$$v \mapsto \{\gamma: t \mapsto \varphi(\varphi^{-1}(p) + tv((U, \varphi)))\} \quad (18.11)$$

with  $|t| < \varepsilon$ , where we picked up a chart  $(U, \varphi)$  around  $p$  with coordinate  $x^i$ .

Accordingly, we also have three different definitions for differentiating, i.e. three definitions for the map

$$D_p F: T_p M \rightarrow T_{F(p)} N \quad (18.12)$$

(sometimes we denote it by  $d$  instead of  $D$ ) where  $p \in M, F: M \rightarrow N$  smooth:

1.  $D_p^{\text{geom}} F([\gamma]) = [F \circ \gamma]$ , where the right-hand side is an equivalence class of curves on  $N$ ,
2.  $D_p^{\text{alg}} F(v) = \{C^\infty(N) \ni f \mapsto v(f \circ F)\}$ ,
3.  $D_p^{\text{phys}} F(v) = \{(V, \psi) \rightarrow D_p^{\text{phys}}(\psi \circ F \circ \varphi^{-1})v(U, \varphi)\}$ , where  $(V, \psi)$  is any chart of  $N$  near  $F(p)$ .

These definitions are compatible with the maps above, e.g. the following diagram

$$\begin{array}{ccc} T^{\text{geom}} M & \xrightarrow{D^{\text{geom}}} & T^{\text{geom}} N \\ \downarrow & & \downarrow \\ T^{\text{alg}} M & \xrightarrow{D^{\text{alg}}} & T^{\text{alg}} N \end{array}$$

commutes.

**Lemma 18.21** (Chain rule). *Let  $M_0, M_1, M_2$  be smooth manifolds, and  $f_0: M_0 \rightarrow M_1, f_1: M_1 \rightarrow M_2$  smooth. Then:*

$$D(f_1 \circ f_0) = Df_1 \circ Df_0.$$

*Proof.* Using the geometric definition:

$$D(f_1 \circ f_0)[\gamma] = [(f_1 \circ f_0) \circ \gamma] = [f_1 \circ (f_0 \circ \gamma)] = (Df_1 \circ Df_0)[\gamma].$$

□

**Definition 18.22.** For a manifold  $M$  of dimension  $n$ , let

$$TM = \bigsqcup_{p \in M} T_p^{\text{phys}} M \xrightarrow{\text{pr}} M \quad (18.13)$$

$$v \longmapsto p.$$

This map is the bundle projection of the tangent bundle  $TM$ . Notice that we are using the notation  $v$  instead of  $(p, v)$  to denote an element of the tangent bundle, for the sake of simplicity (see also remark 11.3).

*Remark 18.23.*  $TM$  is a manifold, if equipped with the following structure. Pick an atlas  $(U_i, \varphi_i)_{i \in I}$  of  $M$  (remember:  $\varphi: M \supset U_i \rightarrow \mathbb{R}^n, U_i$  open). Thanks to the second-countability property, we can take  $I$  as a countable set without loss of generality. Then, one defines local coordinates (i.e. a local parametrization) on  $\text{pr}^{-1}(U_i)$  via

$$\begin{aligned} \hat{\varphi}_i: TM \supset \text{pr}^{-1}(U_i) &\longrightarrow \varphi_i(U_i) \times \mathbb{R}^n \\ v &\longmapsto (\varphi_i(\text{pr}(v)), v(U_i, \varphi_i)). \end{aligned} \quad (18.14)$$

This is a bijective map, and we define the topology on  $TM$  as the topology so that  $\hat{\varphi}_i$  is a homeomorphism for all  $i$ . This topology is Hausdorff, indeed: let  $v_1, v_2 \in TM$ . If  $p_1 = \text{pr } v_1 \neq \text{pr } v_2 = p_2$ , then there exist open sets  $V_j \ni p_j, j = 1, 2$  which are disjoint and open (because  $M$  is Hausdorff). Let  $U_{i(p_1)}, U_{i(p_2)}$  be charts around  $p_1, p_2$ . Then:

$$\text{pr}^{-1}(U_{i(p_1)} \cap V_1) \cap \text{pr}^{-1}(U_{i(p_2)} \cap V_2) = \emptyset.$$

On the other hand, if  $\text{pr } v_1 = \text{pr } v_2$ , let  $(U_{i(v_1)}, \varphi_{i(v_1)})$  be a chart containing  $\text{pr } v_1$ . Then  $\hat{\varphi}_i(v_1), \hat{\varphi}_i(v_2)$  differ in the second entry in  $\mathbb{R}^n$ : pick disjoint open sets  $W_1, W_2$  of  $\mathbb{R}^n$  separating those entries and consider the open sets  $\hat{\varphi}_i^{-1}(\varphi_i(U_i) \times W_j), j = 1, 2$ . So, we get an empty intersection as in the previous case.

Moreover, the topology is second countable. Finally, we need to check the smoothness of the transition functions  $\hat{\varphi}_j \circ \hat{\varphi}_i^{-1}$ . Note that  $\hat{\varphi}_i(\text{pr}^{-1}(U_i) \cap \text{pr}^{-1}(U_j)) = \varphi_i(U_i \cap U_j) \times \mathbb{R}^n$ . By the transformation rule (18.4), and using (18.14):

$$\hat{\varphi}_j \circ \hat{\varphi}_i^{-1}(x, w = (w^1, \dots, w^n)) = (\varphi_j \circ \varphi_i^{-1}(x), D_x(\varphi_j \circ \varphi_i^{-1})(w))$$

This map is smooth. Note that  $D_x(\dots)$  is a linear transformation for a fixed base point  $x$ . Thus  $T_p M = \text{pr}^{-1}(p)$  is a vector space.

**Definition 18.24** (Vector field). A vector field  $X$  on a manifold  $M$  is a smooth map

$$X: M \xrightarrow[\text{pr}]{\quad} TM$$

so that  $\text{pr} \circ X = \text{id}_M$ . A map that satisfies these properties is also called a section of the tangent bundle.

*Remark 18.25* (Commutator of vector fields). Let  $X, Y$  be two vector fields on  $M$ . Then the assignment

$$C^\infty(M) \ni f \mapsto [X, Y](f) = X(Y(f)) - Y(X(f))$$

is independent of the local coordinates and satisfies the Leibniz rule (it can be easily verified):  $[X, Y](fg) = g[X, Y](f) + f[X, Y](g)$ . Moreover, the commutator satisfies the properties already seen in lemma 12.6. And in local coordinates:

$$X = \sum_i a^i \partial_i, \quad Y = \sum_j b^j \partial_j.$$

we have:

$$[X, Y] = \sum_{i,j} \left[ \left( a^i \frac{\partial}{\partial x^i} b^j \right) \frac{\partial}{\partial x^j} - \left( b^j \frac{\partial}{\partial x^j} a^i \right) \frac{\partial}{\partial x^i} \right].$$

*Remark 18.26* (Tangent space to a  $\mathbb{R}$ -vector space). Let  $V$  be a  $\mathbb{R}$ -(vector space) (i.e. a vector space on the field  $\mathbb{R}$ ).  $V$  is a manifold of dimension  $\dim V$ . Then, the tangent space to  $V$  is  $V$  itself (or, better, the tangent space to  $V$  is isomorphic to  $V$ ). Indeed, given  $v \in V$ , there exists an isomorphism:

$$\begin{aligned} V &\cong T_v V \\ w &\mapsto [t \mapsto (v + tw)] \end{aligned}$$

where the map  $t \mapsto (v + tw)$  goes from  $(-\varepsilon, \varepsilon) \subset \mathbb{R}$  to  $V$ .

**Definition 18.27.** In a similar manner to  $TM$  becoming a manifold, one can define the manifold given by the cotangent bundle:

$$T^*M = \bigsqcup_{p \in M} T_p^*M \xrightarrow{\text{pr}} M \quad (18.15)$$

where  $T_p^*M = (T_p M)^*$  is the cotangent space, defined similarly to def. 11.6.

**Definition 18.28** (1-form). A 1-form is a smooth map

$$\alpha: M \xrightarrow[\text{pr}]{\quad} T^*M$$

so that  $\text{pr} \circ \alpha = \text{id}_M$  (it is a section of the cotangent bundle).  $k$ -forms are defined in an analogous fashion, and so are  $\Omega^k(M)$ ,  $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ ,  $f^*$  and  $f_*$ .

*Remark 18.29.* As a short summary: given two manifolds  $M$  and  $N$  and a map  $f: M \rightarrow N$ , we have:

- The pushforward of  $f$  is a map  $f_*: TM \rightarrow TN$ . When evaluated at a point

$p \in M$ , it is a map  $T_p M \rightarrow T_{f(p)} N$ ,

- The pullback of  $f$  is a map  $f^*: \Omega^k(N) \rightarrow \Omega^k(M)$ .

In the following definitions,  $M$  and  $N$  are two topological spaces, and  $\text{rank } F$  is the rank of the Jacobian matrix of  $F$  at a fixed point. If we don't specify any point, we refer to the Jacobian at any point.

**Definition 18.30** (Topological embedding). A map  $F: M \rightarrow N$  is a topological embedding if it is a homeomorphism onto its image, i.e. if  $F: M \rightarrow F(M)$  is a homeomorphism.

**Definition 18.31** (Smooth submersion). A smooth map  $F: M \rightarrow N$  is a smooth submersion if its differential is surjective at each point. If  $M, N$  are smooth manifolds, it means  $\text{rank } F = \dim N$ .

**Definition 18.32** (Smooth immersion). A smooth map  $F: M \rightarrow N$  is a smooth immersion if its differential is injective at each point. If  $M, N$  are smooth manifolds, it means  $\text{rank } F = \dim M$ .

**Definition 18.33** (Smooth embedding). A smooth map  $F: M \rightarrow N$  is a smooth embedding if it is both a smooth immersion and a topological embedding.

*Remark 18.34.* A topological embedding which is smooth is not necessarily a smooth embedding!

*Example 18.35.*

- Let  $M$  be a smooth manifold (with or without boundary). Let  $U \subset M$  be an open submanifold, then the inclusion map  $i: U \hookrightarrow M$  is a smooth embedding,
- $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$  given by  $\gamma(t) = (t^3, 0)$  is a smooth map and a topological embedding, but it's not a smooth embedding because  $\gamma'(0) = (0, 0)$ , so  $\text{rank } \gamma'(0) = 0$  (there exists at least one point such that the rank is different from  $1 = \dim M = \dim \mathbb{R}$ ).

## 19 ■ Lie Groups

**Definition 19.1.** A Lie group  $G$  is a group and a smooth manifold, such that the operation of the group and the inverse of the group are smooth. i.e. the maps

$$\mu: G \times G \rightarrow G \tag{19.1}$$

$$(g, h) \mapsto g \cdot h = gh$$

$$\text{inv}: G \rightarrow G \tag{19.2}$$

$$g \mapsto g^{-1}$$

are smooth.

*Remark 19.2* ( $\text{Aut}(T_e G)$  is a Lie group). Let  $G$  be a Lie group and  $g \in G$ . Let's consider the map

$$\begin{aligned} c_g: G &\rightarrow G \\ h &\mapsto ghg^{-1} \end{aligned} \quad (19.3)$$

$c_g$  is a smooth group homomorphism (i.e.  $c_g(h)c_g(k) = c_g(hk)$ ) and has the property

$$c_g \circ c_{g'} = c_{gg'} \quad (19.4)$$

It also preserves the identity element:  $c_g(e) = e$ . If we differentiate at  $e$ :

$$\text{Ad}_g = D_e c_g: T_e G \rightarrow T_e G \quad (19.5)$$

is an isomorphism with inverse  $\text{Ad}_{g^{-1}}$ . Using the property (19.4) we have:

$$\text{Ad}_g \circ \text{Ad}_{g'} = \text{Ad}_{gg'} \quad (19.6)$$

Therefore, we obtain a smooth group homomorphism:

$$\begin{aligned} \text{Ad}: G &\rightarrow \text{Aut}(T_e G) \\ g &\mapsto \text{Ad}_g \end{aligned} \quad (19.7)$$

with  $\text{Ad}_e = \text{id}_{T_e G}$ . Here,  $\text{Aut}(T_e G)$  is the set of automorphisms of  $T_e G$  (see also remark 4.3). Differentiating again, we get:

$$\begin{aligned} \text{ad}: T_e G &\rightarrow T_{\text{id}}(\text{Aut}(T_e G)) \cong \text{End}(T_e G) \\ X &\mapsto \{Y \mapsto \text{ad}(X)(Y)\} \end{aligned} \quad (19.8)$$

where  $\text{End}(T_e G)$  is the space of endomorphisms of  $T_e G$  (see also the remark (4.3)). So, since the map given by 19.7 is bijective, we have that  $G \cong \text{Aut}(T_e G)$ , i.e.  $\text{Aut}(T_e G)$  is a Lie group. The map given by (19.8) is also bijective. In the next remark we will see that the tangent space at the identity element of a Lie group is isomorphic to a Lie algebra (in particular, it is the Lie algebra generated by the group). So, from what we found it follows that  $\text{End}(T_e G)$  is a Lie algebra.

**Definition 19.3** (Left-invariant vector field). A vector field  $X$  on a Lie group is left-invariant if  $l_{g*}X = X \forall g \in G$ , where  $l_g$  is the homomorphism given by

$$\begin{aligned} l_g: G &\rightarrow G \\ h &\mapsto gh. \end{aligned} \quad (19.9)$$

The space of left-invariant vector fields is denoted by  $\mathfrak{g}$ .

*Remark 19.4* ( $\mathfrak{g}$  is a Lie algebra and  $\mathfrak{g} \cong T_e G$ ). Let's consider:

$$\begin{aligned}\mathfrak{g} &\rightarrow T_e G \\ X &\mapsto X(e) = X_e.\end{aligned}\tag{19.10}$$

It is an isomorphism since it is linear, i.e.  $(X + Y)_e = X_e + Y_e$ , and we can get the vector field back from the vector field itself evaluated at  $e$ . Indeed, the inverse is:

$$X_h = (l_{h*})_e X_e = (Dl_h)_e(X_e) \quad (*)$$

by left-invariance. In fact, we can analyze and prove  $(*)$  in the following way: the left-invariance condition  $Dl_g X = l_{g*} X = X$  means that

$$D(l_g)_{g'}(X_{g'}) = X_{gg'}, \quad \forall g, g' \in G$$

where  $D(l_g)_{g'}$  means that the derivation is computed at  $g'$  (we use this notation instead of  $D_{g'}(l_g)$ ) and  $g'$  is also the point at which we consider our vector field  $X$ . In the right-hand side the product  $gg'$  must be interpreted using the operation of the group  $G$ . Then, by left-invariance, we have that:

$$D(l_h)_e(X_e) = X_{eh} = X_h$$

which is what we wanted to prove. So, we have  $\mathfrak{g} \cong T_e G$ . In order to prove that  $\mathfrak{g}$  is a Lie algebra, we need Lie brackets that satisfy Lie algebra properties. In order to solve the problem, we notice that  $l_g$  is not only a homomorphism, but also a diffeomorphism. By lemma 12.6, point 5, and by recalling that the commutator of vector fields is still a vector field:

$$Dl_g[V, W] = [Dl_g V, Dl_g W] \quad \text{for vector fields } V, W$$

which means that the commutator of left-invariant vector fields is still left-invariant. All of this makes  $\mathfrak{g}$  a Lie algebra, i.e. a  $\mathbb{R}$ -(vector space) with a bilinear pairing  $[\cdot, \cdot]$  which is antisymmetric and satisfies the Jacobi identity.

*Remark 19.5.* Left-invariance provides even more properties. Given a left-invariant vector field  $X$ , let's consider the curve

$$\begin{aligned}\alpha^X: \mathbb{R} &\rightarrow G \\ t &\mapsto \alpha^X(t)\end{aligned}\tag{19.11}$$

such that  $\frac{d}{dt}|_{t=t_0} \alpha^X(t) = X(\alpha^X(t_0))$  and  $\alpha^X(0) = e$ . Such a curve locally exists by the theorems about ordinary differential equations. The global existence is assured by the left-invariance property (it lets us extend the solution on larger sets).

Then, we can completely describe the flow of  $X$  using the curve  $\alpha^X$ . In fact, we can



write the flow  $\varphi$  of  $X$  as:

$$\begin{aligned}\varphi: \mathbb{R} \times G &\rightarrow G \\ (t, h) &\mapsto h\alpha^X(t) = l_h(\alpha^X(t)).\end{aligned}\tag{19.12}$$

It satisfies  $\frac{d}{dt}\varphi(t, h) = X(h\alpha^X(t))$  by the chain rule and by left-invariance. Moreover,  $\alpha^X(s+t) = \alpha^X(s)\alpha^X(t)$  holds.

*Example 19.6.* Assume that  $G \subset GL(n, \mathbb{R})$ . Then  $\mathfrak{g} \cong T_e G \subset \text{End}(n, \mathbb{R})$  and  $X \in \mathfrak{g}$  is given by a matrix  $X$ . The curve  $\alpha^X$  in this case is given by:

$$\alpha^X(t) = \sum_{j=0}^{\infty} \frac{t^j X^j}{j!}\tag{19.13}$$

What if we want to compute the commutator of two left-invariant vector fields? We have that  $[X, Y] = \text{ad}(X)(Y)$ , where "ad" is the map defined in (19.8). This is true in general, not only for  $G \subset GL(n, \mathbb{R})$ . In fact, let's recall the maps defined in the remark 19.2:

1.

$$\begin{aligned}c_g: G &\rightarrow G \\ h &\mapsto ghg^{-1}\end{aligned}\tag{19.14}$$

2.

$$\begin{aligned}G &\rightarrow \text{Aut} G \\ g &\mapsto c_g\end{aligned}\tag{19.15}$$

3.

$$\begin{aligned}\text{Ad}: G &\rightarrow \text{End}(T_e G) \\ g &\mapsto D_e c_g = D(c_g)(e)\end{aligned}\tag{19.16}$$

$D_e c_g \in \text{End}(T_e G)$  because the differential of the map  $c_g$  is itself a map from  $T_e G$  to  $T_e G$ .

4.

$$\text{ad}: T_e G \rightarrow T_e \text{End}(T_e G) \cong \text{End}(T_e G)\tag{19.17}$$

Then, if we consider:

$$\text{Ad}(g)(Y) = (D_e c_g)(Y) \stackrel{(1)}{=} \frac{d}{dt}\bigg|_{t=0} c_g(\alpha^Y(t))$$

where we used: (1) the geometric definition of differential, see remark 18.20. Thus:

$$\begin{aligned} \operatorname{ad}(X)(Y) &= \frac{d}{ds}\bigg|_{s=0} \left( \frac{d}{dt}\bigg|_{t=0} c_{\alpha^X(s)} \alpha^Y(t) \right) \stackrel{(2)}{=} \frac{d}{ds}\bigg|_{s=0} \left( \frac{d}{dt}\bigg|_{t=0} \alpha^X(s) \alpha^Y(t) \alpha^X(-s) \right) \stackrel{(3)}{=} \\ &= \frac{d}{ds}\bigg|_{s=0} \left( \frac{d}{dt}\bigg|_{t=0} \exp(sX) \exp(tY) \exp(-sX) \right) = \dots = XY - YX \end{aligned}$$

where we used: (2) definition of  $c_g$  and the fact the  $\varphi^{-1}(t) = \varphi(-t)$  if  $\varphi$  is the flow of a vector field, (3) definition (19.13) for the case  $G \subset GL(n, \mathbb{R})$ . It is common to use the following notation for Lie groups:

$$\begin{aligned} \exp: \mathfrak{g} &\rightarrow G \\ X &\mapsto \alpha^X(1) \end{aligned} \tag{19.18}$$

where, in general,  $\exp(X + Y) \neq \exp(X) \exp(Y)$ . The map "exp" is smooth and

$$D_e \exp \equiv \operatorname{id}_{\mathfrak{g}} \tag{19.19}$$

(recall that  $T_e \mathfrak{g} \equiv \mathfrak{g}$  because  $\mathfrak{g}$  is a vector field and  $T_e G \cong \mathfrak{g}$ ).

**Lemma 19.7.** *Let  $f: G \rightarrow H$  be a smooth group homomorphism. then*

$$\exp^H(D_e f(X)) = f(\exp^G(X)) \tag{19.20}$$

where  $D_e f: \mathfrak{g} \rightarrow \mathfrak{h}$  with  $\mathfrak{g} \cong T_e G, \mathfrak{h} \cong T_e H$ , as usual.

*Proof.* Here and in the following, we will use the notation  $Df$  instead of  $D_e f$  in order to avoid too many symbols in the notations. Given  $Df(X)$ ,  $\alpha^{Df(X)}: \mathbb{R} \rightarrow H$  is a 1-parameter subgroup (i.e. it satisfies the same properties of  $\alpha^X$ , see also remark 19.5).  $\alpha^{Df(x)}$  coincides with  $f \circ \alpha^X$ . i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{Df} & \mathfrak{h} \\ \exp^G \downarrow & & \downarrow \exp^H \\ G & \xrightarrow{f} & H \end{array}$$

□

**Lemma 19.8.** *Let  $G$  be a connected Lie group. Then, if  $f: G \rightarrow H$  is a smooth homomorphism, it is determined by its differential at the identity element  $D_e f$ .*

*Proof.* Let's take a look at the diagram of the previous proof. It would be nice if we could go back from  $G$  to  $\mathfrak{g}$  like this:

$$\begin{array}{c} \mathfrak{g} \\ \exp^G \downarrow \nearrow \\ G \end{array}$$

If it was true, then the thesis would be satisfied because the diagram of the previous proof commutes (we could write  $f$  using  $Df$ ,  $\exp^H$  and  $(\exp^G)^{-1}$ ). Unfortunately, this is not always true. It is *locally* true because  $D_e \exp^G \equiv \text{id}_{\mathfrak{g}}$  (think about the Taylor expansion, see (19.19)). Since it is locally true, there exists a neighbourhood  $\mathfrak{U} \subset \mathfrak{g}$  of 0 and a neighbourhood  $\mathcal{U} \subset G$  of  $e$  such that

$$\exp^G|_{\mathfrak{U}}: \mathfrak{U} \rightarrow \mathcal{U}$$

is a diffeomorphism. Thus, on  $\mathcal{U}$  the  $f$  is defined by:

$$f = \exp^H \circ Df \circ (\exp^G|_{\mathfrak{U}})^{-1}.$$

Since the Lie group is connected, this is enough to prove the thesis. Indeed every neighbourhood  $\mathcal{U}$  of  $e$  in  $G$  generates  $G$  itself if it is connected. This fact can be proved in the following way, let's consider:

$$V \equiv \mathcal{U} \cap \mathcal{U}^{-1}.$$

It is an open neighbourhood, because intersection of open sets, with  $\mathcal{U}^{-1} = \{u^{-1} \mid u \in \mathcal{U}\}$ . Moreover,

$$G^V \equiv \bigcup_{j=0}^{\infty} V^j, \quad V^j \equiv \{v_1 \cdot \dots \cdot v_j \mid v_i \in V\}.$$

$G^V$  is a subgroup of  $G$ , and it is open because every element of the union is open:  $V$  is open,  $V^2 = \bigcup_{v \in V} vV$  is open, etc... We will now show that also the complement is open. First, we have that:

$$G = \bigcup_{[g] \in G/G^V} gG^V$$

because:

- " $\supseteq$ " is trivial since we are considering union of elements of the group,
- " $\subseteq$ " follows from the fact that " $[g] \in G/G^V$ " means that if  $g \in G^V$ , then " $gG^V$ " yields the entire subgroup  $G^V$ . If  $g \notin G^V$ , then we are considering the elements in  $G \setminus G^V$ . Since " $gG^V$ " means that we are multiplying  $g$  by all possible element of  $G^V$ , and since  $G^V$  contains the identity element (because it is a subgroup), we can get all the possible elements of  $G \setminus G^V$  from the right-hand side. Then  $G \subseteq \bigcup_{[g] \in G/G^V} gG^V$ .

Thus, we can obtain  $G \setminus G^V$  if  $g \notin G^V$  in the union:

$$G \setminus G^V = \bigcup_{[g] \neq [e]} gG^V \text{ (it is an open set).}$$

Now, given that  $G$  is connected (see definition 6.35), either  $G^V$  or  $G \setminus G^V$  must be empty.  $G^V$  contains at least the identity element (obviously we are assuming that  $G$  is non-empty), so  $G \setminus G^V$  must be the empty one. So,  $G = G^V$ .  $\square$

*Remark 19.9.* The hypothesis of connection is necessary for theorem 19.8.

## 20 ■ Compactly Supported Cohomology

Let  $M$  be a manifold with or without boundary. We have already seen (remark 15.6) that the  $k$ -th de Rham cohomology of  $M$  is

$$H^k(M) \equiv H_{dR}^k(M) = \frac{\ker(d_{k+1}: \Omega^k(M) \rightarrow \Omega^{k+1}(M))}{\operatorname{im}(d_k: \Omega^{k-1}(M) \rightarrow \Omega^k(M))} \quad (20.1)$$

And we have already considered the Poincaré lemma 14.12:

$$H^k(\mathbb{R}^n) \cong \begin{cases} 0, & k \neq 0 \\ \mathbb{R}, & k = 0 \end{cases} \quad (20.2)$$

Indeed, in (20.2) we used that  $\mathbb{R}^n$  is contractible (case  $k \neq 0$ ) and that the only exact 0-forms (i.e. functions) are the constant ones (case  $k = 0$ ). Now, we want to discuss about  $k$ -forms with compact support. We define the support of  $\omega$  as  $\operatorname{supp}(\omega) \equiv \overline{\{p \in M \mid \omega(p) \neq 0\}}$ . So, a  $k$ -form  $\omega$  has compact support if  $\operatorname{supp}(\omega)$  is a compact set. First, we notice that the exterior derivative  $d$  maps forms with compact support to forms with compact support. Then, it is natural to define the following.

**Definition 20.1** (Compactly supported de Rham cohomology). We define the compactly supported de Rham cohomology of a manifold  $M$  as:

$$H_c^k(M) = \frac{\ker(d_{k+1}|_{\text{compact supp forms}})}{\operatorname{im}(d_k|_{\text{compact supp forms}})} \quad (20.3)$$

*Remark 20.2.* The compactly supported de Rham cohomology is somehow different from the standard de Rham cohomology. Indeed, for the latter, given a map  $f: U \rightarrow V$ , we could consider the pullback:

$$\begin{aligned} f^*: H^k(V) &\rightarrow H^k(U) \\ [\omega] &\mapsto [f^*\omega]. \end{aligned}$$

For the compactly supported case, we can't do that anymore, because if  $\omega$  is compactly supported, in general  $f^*\omega$  might not be compactly supported (preimages of compact sets are not compact in general).

Now, we want to prove a compactly supported version of the Poincaré lemma. In order to do that, we need some general result on manifolds.

*Remark 20.3* (Integration over the fiber). Let  $M$  be a smooth manifold, oriented. Let's consider the projection map

$$\pi: M \times \mathbb{R} \rightarrow M. \quad (20.4)$$

Given  $\omega \in \Omega_c^k(M)$ ,  $\pi^*\omega$  is not compactly supported in general. However, we can define a map

$$\pi_*: \Omega_c^k(M \times \mathbb{R}) \rightarrow \Omega_c^{k-1}(M) \quad (20.5)$$

which actually maps compactly supported forms to compactly supported forms. So, if now  $\omega \in \Omega_c^k(M \times \mathbb{R})$ , then  $\pi_*\omega$  has compact support as a  $(k-1)$ -form in  $M$ . Such a map is called *integration along the fiber*. We used the notation of the pushforward because it is indeed another way to write the pushforward of a  $k$ -form. We also notice that, because of a generalized version of lemma 14.9, every  $k$ -form on  $M \times \mathbb{R}$  can be written as a sum (with more than two summands in general) of two types of  $k$ -forms:

- **Type A**  $\omega = f(x, t) \pi^*\eta$ ,  $\eta \in \Omega^k(M)$ ,
- **Type B**  $\omega = f(x, t) \pi^*\eta \wedge dt$ ,  $\eta \in \Omega^{k-1}(M)$ .

where  $f(x, t)$  is a compactly supported function,  $dt \in \Omega^1(M \times \mathbb{R})$ .

Now, we define  $\pi_*$  in the way it acts on (type A) and (type B) forms:

$$\text{(type A)} \quad f(x, t) \pi^*\eta \xrightarrow{\pi_*} 0 \quad (20.6)$$

$$\text{(type B)} \quad f(x, t) \pi^*\eta \wedge dt \xrightarrow{\pi_*} \eta \underbrace{\int_{-\infty}^{\infty} f(x, s) ds}_{\equiv F(x)} \quad (20.7)$$

Formally speaking,  $f$  is the pullback of  $F$ .

**Lemma 20.4.**  $\pi_*$  is a chain map, i.e.

$$\pi_* \circ d = d \circ \pi_* \quad (20.8)$$

where the differential in the left-hand side acts on  $M \times \mathbb{R}$ , whereas the one in the right-hand side acts on  $M$ .

*Proof.* We prove the lemma for (type A) and (type B) forms, thus it is proved for any form on  $M \times \mathbb{R}$  (see remark 20.3).

- (Type A):

$$\begin{aligned} \pi_* \circ d(f(x, t) \pi^* \eta) &= \pi_* \left( \underbrace{\frac{\partial f}{\partial x}(x, t) dx \wedge \pi^* \eta}_{\text{type A}} + \underbrace{\frac{\partial f}{\partial t}(x, t) dt \wedge \pi^* \eta}_{\text{type B}} + \underbrace{f(x, t) d\pi^* \eta}_{\substack{\text{type A because} \\ d\pi^* = \pi^* d}} \right) \stackrel{(1)}{=} \\ &= \int_{-\infty}^{\infty} \frac{\partial f}{\partial s}(x, s) ds \pi^* \eta = 0 \end{aligned}$$

where (1) is true up to a sign (but in the end the sign doesn't matter), and we used the definition of  $\pi_*$  for (type B) forms (20.7) and the fact that  $\pi_*(\text{type A}) = 0$ . Finally, we used the fundamental theorem of integral calculus with the fact that  $f$  has compact support. Moreover,  $d \circ \pi_*(f(x, t) \pi^* \eta) = 0$ .

- (Type B):

$$\begin{aligned} \pi_* \circ d(f(x, t) \pi^* \eta \wedge dt) &= \pi_* \left( \frac{\partial f}{\partial x}(x, t) dx \wedge \pi^* \eta \wedge dt + f(x, t) \pi^* d\eta \wedge dt \right) = \\ &= \pi_* \left( dx \wedge \frac{\partial}{\partial x} (f(x, t) \pi^* \eta \wedge dt) + f(x, t) \pi^* d\eta \wedge dt \right) \\ &= \frac{\partial F}{\partial x}(x, t) dx \wedge \eta + F(x) d\eta \end{aligned}$$

where we used the  $F$  defined in (20.7). Moreover:

$$d \circ \pi_*(f(x, t) \pi^* \eta \wedge dt) = d(F(x) \eta) = \frac{\partial F}{\partial x}(x, t) dx \wedge \eta + F(x) d\eta.$$

□

*Remark 20.5.* Consequence of lemma 20.4:  $\pi_*$  defines a map

$$\pi_*: H_c^k(M \times \mathbb{R}) \rightarrow H_c^{k-1}(M).$$

We want to prove that such a map is an isomorphism. Let's construct the inverse in the following way: let  $e: \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function with compact support such that  $\int_{-\infty}^{\infty} e(s) ds = 1$ . Then, we define  $e_*$  as:

$$\begin{aligned} e_*: \Omega_c^{k-1}(M) &\rightarrow \Omega_c^k(M \times \mathbb{R}) \\ \eta &\mapsto e(t) \pi^* \eta \wedge dt. \end{aligned} \tag{20.9}$$

We have  $\pi_* \circ e_* = \text{id}_{\Omega_c^{k-1}(M)}$  **but**  $e_* \circ \pi_* \neq \text{id}_{\Omega_c^k(M \times \mathbb{R})}$ . Nevertheless, we just need  $e_*$  to be the inverse of  $\pi_*$  on  $H_c^k(M \times \mathbb{R})$ , not on all  $\Omega_c^k(M \times \mathbb{R})$ . This is true thanks to the two following results.

**Lemma 20.6.** *The map  $e_*$  defined in (20.9) is a chain map, i.e.*

$$d \circ e_* = e_* \circ d \tag{20.10}$$

where, again, the differential in the left-hand side acts on  $M \times \mathbb{R}$  and the one in the right-hand side acts on  $M$ .

*Proof.* Let  $\eta \in \Omega_c^{k-1}(M)$ .

$$d \circ e_*(\eta) = d(e(t)\pi^*\eta \wedge dt) = e(t)d^{M \times \mathbb{R}}\pi^*\eta \wedge dt = e(t)\pi^*(d^M\eta) \wedge dt = e_*d\eta. \quad \square$$

**Proposition 20.7.** *The maps  $e_*, \pi_*$  are mutually inverse isomorphisms on  $H_c$ , since there is an operator  $K: \Omega_c^k(M \times \mathbb{R}) \rightarrow \Omega_c^{k-1}(M)$  such that:*

$$\mathbb{1} - e_* \circ \pi_* = (-1)^{k-1}(d \circ K - K \circ d) \quad (20.11)$$

where:

- $K$  is defined as:

$$K(f(x, t)\pi^*\eta) = 0 \quad \text{on (type A) forms,} \quad (20.12)$$

$$K(f(x, t)\pi^*\eta \wedge dt) = \int_{-\infty}^t f(x, s)ds \pi^*\eta - F(x) \left( \int_{-\infty}^t e(s)ds \right) \pi^*\eta \quad \text{on (type B) forms.} \quad (20.13)$$

where  $\eta$  is a  $k$ -form in the first case, and it is a  $(k-1)$ -form in the second case.

- $K$  is not a chain map (otherwise the right-hand side would be 0).

*Proof.* We will prove it for (type B) forms. Then, for (type A) forms it is analogous. By recalling the definition of the integration along the fiber  $F$  contained in (20.7):

1.

$$\begin{aligned} (\mathbb{1} - e_* \circ \pi_*)(f(x, t)\pi^*\eta \wedge dt) &= f\pi^*\eta \wedge dt - e_*(F(x)\eta) = \\ &= \underbrace{f\pi^*\eta \wedge dt}_{(a)} - \underbrace{F(x)e(t)\pi^*\eta \wedge dt}_{(b)} \end{aligned}$$

2.

$$\begin{aligned} (d \circ K)(f(x, t)\pi^*\eta \wedge dt) &= \underbrace{f\pi^*\eta \wedge dt}_{(a)}(-1)^{k-1} + \underbrace{\left( \int_{-\infty}^t \frac{\partial f}{\partial x}(x, s)ds \right) dx \wedge \pi^*\eta}_{(c)} + \\ &+ \underbrace{\int_{-\infty}^t f(x, s)ds \pi^*d\eta}_{(d)} - \underbrace{\left( \frac{\partial F}{\partial x} \int_{-\infty}^t e(s)ds \right) dx \wedge \pi^*\eta}_{(e)} + \\ &- \underbrace{e(t)F(x)\pi^*\eta \wedge dt}_{(b)}(-1)^{k-1} - \underbrace{F(x) \left( \int_{-\infty}^t e(s)ds \right) \pi^*d\eta}_{(f)} \end{aligned}$$

3.

$$\begin{aligned}
(K \circ d)(f(x, t) \pi^* \eta \wedge dt) &= K \left( \frac{\partial f}{\partial x}(x, t) dx \wedge \pi^* \eta \wedge dt + f(x, t) \pi^* d\eta \wedge dt \right) \stackrel{(1)}{=} \\
&= \underbrace{\left( \int_{-\infty}^t \frac{\partial f}{\partial x}(x, s) ds \right) dx \wedge \pi^* \eta}_{(c)} - \underbrace{\frac{\partial F}{\partial x} \int_{-\infty}^t e(s) ds dx \wedge \pi^* \eta}_{(e)} + \\
&\quad + \underbrace{\int_{-\infty}^t f(x, s) ds \pi^* d\eta}_{(d)} - \underbrace{F(x) \int_{-\infty}^t e(s) ds \pi^* d\eta}_{(f)}
\end{aligned}$$

Using that: (1) we can exchange integral and derivative, and remember that:  $F(x) = \int_{-\infty}^{\infty} f(x, s) ds$ .

By comparing the three results, we have the thesis.  $\square$

*Remark 20.8.* We proved the isomorphism in  $H_c^k(M)$ . Indeed, if  $\omega$  is closed, then the (20.11) becomes:

$$(\mathbb{1} - e_* \circ \pi_*)(\omega) = d \left( (-1)^{k-1} K\omega \right).$$

So,  $\omega$  and  $e_* \circ \pi_*(\omega)$  differ by an exact form, i.e. they live in the same equivalence class.

Now, we are finally ready to prove the compactly supported Poincaré lemma.

**Lemma 20.9** (Compactly supported Poincaré lemma).

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} 0, & k \neq n \\ \mathbb{R}, & k = n \end{cases} \quad (20.14)$$

*Proof.* The proof is by induction. For  $\boxed{n=1}$ , the case  $k \neq 1$  would be the case  $k = 0$ : we are considering closed and compactly supported 0-forms, i.e. functions which are 0 everywhere. Thus, the only closed and compactly supported 0-form which is not exact is the trivial one:  $H_c^0(\mathbb{R}) = 0$ . As regards the case  $k = 1$ , we prove that the following map is an isomorphism:

$$\begin{aligned}
I: H_c^1(\mathbb{R}) &\rightarrow \mathbb{R} \\
[\omega] &\mapsto \int_{\mathbb{R}} \omega.
\end{aligned} \quad (20.15)$$

First of all,  $I$  is well defined, i.e. the integral does not depend on the element of the chosen equivalence class. In fact, if we consider another 1-form  $\omega'$  such that  $[\omega'] = [\omega]$ , then it means that  $\omega'$  and  $\omega$  differ by an exact form:  $\omega' = \omega + d\alpha, \alpha \in \Omega_c^0(\mathbb{R})$ . Let's suppose, without loss of generality, that  $\text{supp } \alpha, \text{supp } \omega, \text{supp } \omega' \subset B_{2R}(0)$ . Then:

$$\int_{\mathbb{R}} \omega' = \int_{B_{2R}(0)} \omega' = \int_{B_{2R}(0)} (\omega + d\alpha) = \int_{B_{2R}(0)} \omega + \int_{B_{2R}(0)} d\alpha \stackrel{\text{Stokes}}{=} \int_{\mathbb{R}} \omega + \underbrace{\int_{\partial B_{2R}(0)} \alpha}_{=0}.$$



Now, we prove that  $I$  is surjective. Indeed, given a function  $f > 0$  on  $\mathbb{R}$  with compact support, let's consider  $\eta = f(x)dx \in H_c^1(\mathbb{R})$ . We have that  $\int_{\mathbb{R}} \eta = d > 0$ . So, for every  $c \in \mathbb{R}$  the form  $\omega \equiv \frac{c}{d} \eta$  is such that  $\int_{\mathbb{R}} \omega = c$ . Thus,  $I$  is surjective.

In order to prove that  $I$  is injective, we just need to prove that  $\ker I = \{0\}$  (since  $I$  is linear). By the standard Poincaré lemma (14.12) we know that every closed 1-form on  $\mathbb{R}$  is exact, i.e.  $\omega = d\alpha$ , but  $\alpha$  might not be compactly supported! In order to have an exact form in  $H_c^1(\mathbb{R})$ ,  $\alpha$  should have compact support as well. But, we notice that  $\omega$  is exact if  $\int_{\mathbb{R}} \omega = 0$  (i.e.  $\ker I = \{0\}$ ). Indeed let's write  $\omega$  as  $\omega = f(x)dx$ ,  $f$  compactly supported function. If  $\int_{\mathbb{R}} \omega = \int_{\mathbb{R}} f(x)dx = 0$ , then the primitive  $F(x) = \int_{-\infty}^x f(t)dt$  has compact support (for  $x$  small we are outside of the support of  $f$ , whereas for  $x$  big we integrate over all the support, i.e. it's like considering  $\int_{\mathbb{R}} f(x)dx = 0$ ). Then  $\omega$  is exact:  $\omega = dF$  and  $F$  has compact support  $\Rightarrow [\omega] = 0$ . So,  $H_c^1(\mathbb{R}) \cong \mathbb{R}$ . For  $[n > 1]$ : from proposition 20.7, we know that there is an isomorphism

$$H_c^k(M \times \mathbb{R}) \xrightleftharpoons[\pi_*]{e_*} H_c^{k-1}(M) \quad (20.16)$$

Let's suppose the claim is true for  $n-1$ , then we can consider  $M = \mathbb{R}^{n-1}$  (notice that  $\mathbb{R}^n$  is an oriented manifold) and thanks to the isomorphism (20.16) the claim is true also for  $n$ .  $\square$

*Remark 20.10.* The Poincaré lemma 20.9 is valid for  $\mathbb{R}^n$ , but it also holds for open subsets of manifolds which are diffeomorphic to  $\mathbb{R}^n$ . For instance, we can consider the convex subsets  $U$  contained in a manifold. The form  $\eta \in \Omega_c^{n-1}(U)$  with  $d\eta = \omega$  produced in the proof of the Poincaré lemma for a compactly supported form  $\omega$  can be extended by zero to a smooth form on the entire manifold. We will not make any distinction between the extended form and the form on  $U$ .

We will now analyze some consequences of the compactly supported Poincaré lemma.

**Definition 20.11** (Bump form). A bump form on  $M^n$  (manifold of dimension  $n$ ) is a compactly supported  $n$ -form  $\omega = \rho dx^1 \wedge \dots \wedge dx^n$  supported in a coordinate domain  $(U, \varphi)$ ,  $\varphi: U \rightarrow \varphi(U) \subseteq \mathbb{R}^n$  so that  $|\int_U \omega| = 1$ .

*Remark 20.12* (Cohomologous bump forms). We know that two forms are cohomologous in  $H^k(U)$  if and only if they differ by an exact form (see remark 15.1). The Poincaré lemma tells us that bump forms are cohomologous on  $U$  if their integral on  $U$  is the same. Given  $\omega, \omega'$  bump forms in  $U$ , then:

$$\omega = \omega' + d\eta, \eta \in \Omega_c^{n-1}(U) \iff \int_U \omega = \int_U \omega' \quad (20.17)$$

Indeed, the " $\Rightarrow$ " is always true thanks to Stokes's theorem, whereas the " $\Leftarrow$ " follows from the proof of the Poincaré lemma 20.9, because bump forms are  $n$ -forms in a  $n$ -dimensional manifold and the equality between the integrals implies that  $[\omega] = [\omega']$

(because the isomorphism used in the proof is given by the integral map (20.15)).  
Thus, the two forms differ by an exact form.

**Proposition 20.13.**  *$M$  connected manifold, then any two bump forms are cohomologous (on  $M$ ) up to a sign.*

*Proof.* Let's consider two bump forms  $\omega, \omega'$ , compactly supported on  $(U, \varphi), (U', \varphi')$ , respectively. If  $(U, \varphi) = (U', \varphi')$ , then we have that  $\int_U \omega = \int_U \omega'$  up to a sign (by definition 20.11). Then, by the previous remark 20.12 we have that  $\omega$  and  $\omega'$  are cohomologous up to a sign. On the other hand, if  $(U, \varphi) \neq (U', \varphi')$ , we do it in the following way: since  $M$  is a connected manifold, it is also path-connected (see problem set 12, ex. 1.(e)). Thus, we can find a path between  $U$  and  $U'$ , and we can cover it with open sets  $U_1, U_2, \dots$ . Then, we can consider a bump form  $\omega_1$  in  $U \cap U_1$ , and we can extend it by zero in  $U$ . For the same reason seen in the case  $U = U'$ , we have that  $\omega$  is cohomologous to  $\omega_1$  up to a sign. Again, we can consider a bump form  $\omega_2$  in  $U_1 \cap U_2$  and extend it by zero in  $U_1$ . As before,  $\omega_2$  is cohomologous to  $\omega_1$  up to a sign, and so on. We continue in this way up to  $U'$ , and using the transitive property of the equivalence relation we find that  $\omega$  is cohomologous to  $\omega'$  up to a sign.

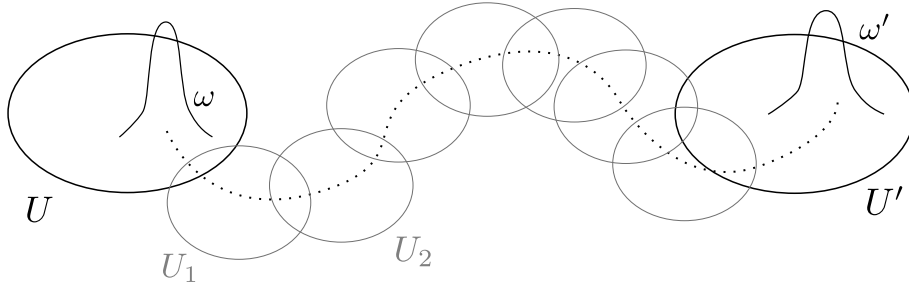


Figure 16: We can cover the path between  $U$  and  $U'$

□

**Lemma 20.14.** *Every  $n$ -form on  $M$  is cohomologous to a locally finite sum of bump forms.*

*Proof.* Pick a cover  $(U_i, \varphi_i), \varphi_i: U_i \rightarrow \varphi_i(U_i) \subseteq \mathbb{R}^n$ , and a locally finite refinement  $V_j$  (recall the definition contained in def. 6.33) with a subordinate partition of unity  $\lambda_j$  (i.e.  $\sum_j \lambda_j = 1$ ). Then:

$$\omega = \sum_j (\lambda_j \omega) \sim \sum_{\substack{j \text{ s.t.} \\ \int \lambda_j \omega \neq 0}} \underbrace{\left( \frac{\int \lambda_j \omega}{\int \lambda_j \omega} \right)}_{\text{bump form}} \cdot \underbrace{\left( \int \lambda_j \omega \right)}_{\text{coefficients}}.$$

And the sum is locally finite because we chose a locally finite refinement, which is always possible since manifolds are paracompact (thm 18.6). □

**Theorem 20.15.** *If  $M$  is a connected, smooth manifold of dimension  $n$ , such that  $\partial M = \emptyset$  and  $M$  is oriented, then the map:*

$$H^n(M) \rightarrow \mathbb{R} \quad (20.18)$$

$$[\omega] \mapsto \int_M \omega$$

*is well-defined and surjective.*

*Proof.* Since  $\partial M = \emptyset$ , we have  $\int_M(\omega + d\eta) = \int_M \omega$  by Stokes's theorem. So, the definition does not depend on the element of the equivalence class chosen. Now, let  $c$  be a real number. Moreover, let  $(U_i, \varphi_i)$  be a chart from the oriented atlas, and let's consider  $\alpha = f(x_1, \dots, x_n) dx^1 \wedge \dots \wedge dx^n$  in  $U_i$ , with  $f$  smooth function,  $f > 0$ . Then,  $\int_M \alpha = d > 0$ . Now,  $\omega = \frac{c}{d} \alpha \in H^n(M)$  and  $\int_M \omega = c$ . So, for every  $c \in \mathbb{R}$  we can find a form  $\omega$  such that  $\int_M \omega = c$ , i.e. the map is surjective.  $\square$

**Definition 20.16.** A manifold  $M$  is closed if it is compact and  $\partial M = \emptyset$ .

*Example 20.17.*  $S^1$  is a closed manifold. A torus is closed as well.

**Theorem 20.18.** *If  $M$  is a connected, smooth manifold of dimension  $n$ ,  $\dim(H^n(M)) \leq 1$ . If  $M$  is also closed and oriented,  $H^n(M) \cong \mathbb{R}$ .*

*Proof.* Using the previous theorem 20.15, we just need to verify how many values  $\int_U \omega$  can have, where  $U$  open subset of  $M$  diffeomorphic to  $\mathbb{R}^n$  (Notice:  $\partial U = \emptyset$  because  $U$  is open). Now, every element of  $H^n(M)$  is cohomologous to a (real) multiple of a bump form in a fixed chart  $(U, \varphi)$ . Bump forms have integral  $= \pm 1$ . Therefore, the value of the integral of a closed-but-not-exact  $n$ -form in  $H^n(M)$  can be any real number. Then the first part is proved. If  $M$  is also closed, then the map (20.18) given by integration is both surjective and injective. The proof of this is on page 454, [Lee], using that  $H^k(M) = H_c^k(M)$  if  $M$  is compact.  $\square$

**Theorem 20.19.** *Let  $M$  be a connected, compact, orientable manifold with  $\partial M \neq \emptyset$  and of dimension  $n$ . Then  $H^n(M) = 0$ .*

*Proof.* It suffices to find a bump form which is cohomologous to zero in  $M$ . Once we find it, we have the thesis because  $M$  is connected, so any other bump form in the manifold is zero by proposition 20.13. Also, by lemma 20.14, every  $n$ -form is cohomologous to a locally finite sum of zero-forms, i.e. it is a zero form and we would have the thesis.

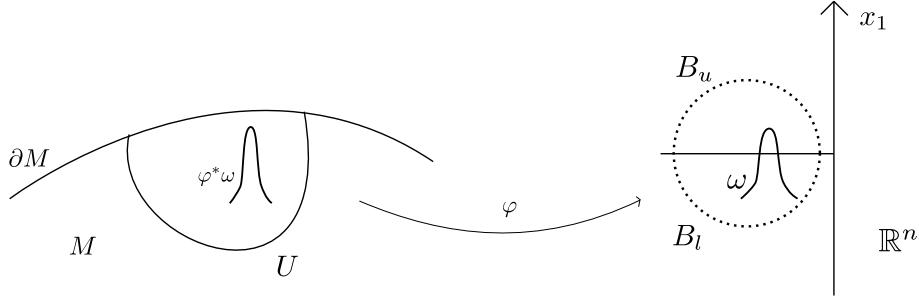


Figure 17: We can find a bump form cohomologous to a zero form

Let's find such a bump form. First, let's consider a boundary chart  $(U, \varphi), \varphi: U \rightarrow \mathbb{R}^n \cap \{x_1 < 0\}$  (such that  $\varphi(U)$  is a half-ball) and a bump form  $\omega$  compactly supported in the interior of the half-ball. We will also consider the other half-ball. We will call  $B_l$  the lower half-ball ( $B_l = \varphi(U)$ ) and we will call  $B_u$  the other half. We consider another bump form  $\omega_u$  contained in  $B_u$  such that:

$$\int_{B_u} \omega_u = \int_{B_l} -\omega.$$

It is always possible, because bump forms can only have integral  $= \pm 1$ . Now, we consider the form  $\omega_u + \omega$  and we extend it by zero. It is compactly supported in the open ball  $B \equiv B_u \cup B_l$ . By remark 20.12 we have that  $\omega_u$  and  $-\omega$  are cohomologous, so  $d\eta = \omega_u + \omega$ . Thus:

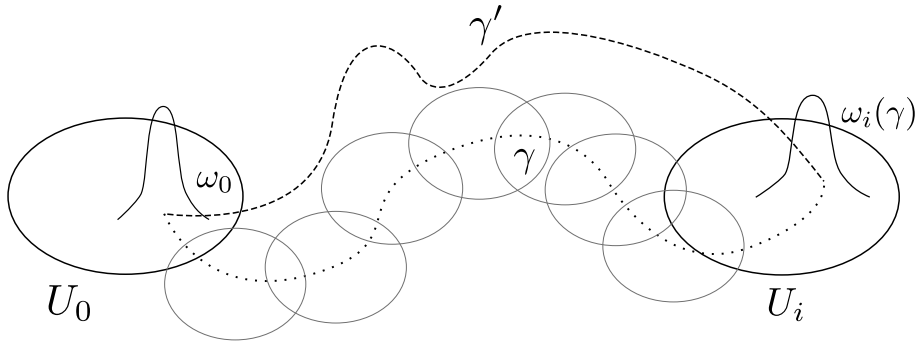
$$d\eta|_{B_l} = \omega_u|_{B_l} + \omega|_{B_l} = \omega|_{B_l}.$$

Then,  $\varphi^*\eta$  is a primitive of the bump form  $\varphi^*\omega$ . So,  $\varphi^*\omega$  is the zero form we were looking for.  $\square$

What if we remove orientation? Recall that we can't compute integrals without orientation.

**Theorem 20.20.** *Let  $M$  be a connected but not orientable manifold of dimension  $n$ . Then:  $H^n(M) = 0$ .*

*Proof.* Let's cover  $M$  by an atlas  $(U_i, \varphi_i), \varphi_i: U_i \rightarrow \varphi_i(U_i) = \mathbb{R}^n$  (we choose the atlas such that  $\varphi(U_i) = \mathbb{R}^n$ ). Let's consider an open set  $U_0$ .

Figure 18: We can consider two paths connecting  $U_0$  and  $U_i$

Let's consider another set  $U_i$  from the cover,  $U_0 \neq U_i$ . Since  $M$  is a connected manifold, it is also path-connected, so there exists a path  $\gamma: [0, 1] \rightarrow M$  that connects  $U_0$  to  $U_i$ . By the same reasoning of the proof of proposition 20.13, we get that a bump form  $\omega_0$  contained in  $U_0$  is cohomologous to a bump form  $\omega_i(\gamma)$  contained in  $U_i$ . But, since  $M$  is not orientable, there also exists a second path  $\gamma'$  connecting  $U_0$  and  $U_i$ , and a second chain of charts covering  $\gamma'$ , such that, repeating the same procedure,  $\omega_0$  is cohomologous to  $\omega_{i'}(\gamma')$  with the opposite sign of before (i.e. the integrals have opposite signs). Notice that even if we cannot integrate on  $M$ , we can integrate on the charts. It means that:

$$\omega_i(\gamma) \sim \omega_0 \sim \omega_{i'}(\gamma') \sim -\omega_0.$$

So,  $\omega \sim -\omega \Leftrightarrow 2\omega \sim 0$ . □

*Remark 20.21.* The proofs of the following facts are similar to the previous ones: given a connected manifold  $M$ :

- $M$  not compact,  $\partial M = \emptyset \Rightarrow H^n(M) = 0$ ,
- $M$  compact  $\Rightarrow H_c^*(M) = H^*(M)$ ,
- $M$  not compact, oriented,  $\partial M = \emptyset \Rightarrow H_c^n(M) = \mathbb{R}$ .

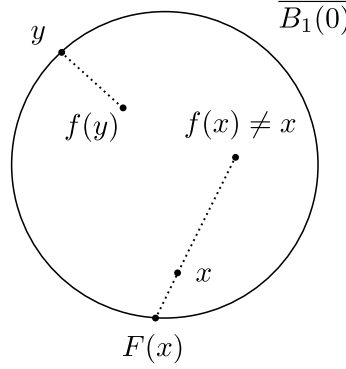
**Theorem 20.22** (Brouwer). *Let  $f: \overline{B_1(0)} \rightarrow \overline{B_1(0)} \subset \mathbb{R}^n$  be smooth. Then,  $f$  has a fixed point, i.e.  $\exists x \in \overline{B_1(0)}$  such that  $f(x) = x$ .*

*Proof.* Let's prove it for  $f$  smooth. If  $f$  is continuous but not smooth, we can approximate it using smooth maps (Weierstrass approximation). Moreover, we will prove it for  $n \geq 2$  (for  $n = 0, 1$  it is a consequence of the intermediate value theorem). By contradiction: let's assume  $f$  has no fixed point. Then we consider the map

$$F: \overline{B_1(0)} \rightarrow \partial \overline{B_1(0)} = S^{n-1}$$

$$x \mapsto \begin{array}{c} \text{intersection point with } S^{n-1} \\ \text{of the ray passing through } x \text{ and } f(x) \end{array}$$

(Notice that: when both  $x$  and  $f(x)$  belong to the border of the sphere the intersection is not unique. In that case we choose the point  $x$ ).  $F$  is a smooth function: we will not verify it because it is not interesting for the proof. Moreover, we notice that if  $y \in \partial \overline{B_1(0)} \Rightarrow F(y) = y$ .

Figure 19:  $F$  sends every point of the ball on its border

Let's now consider the map:  $\iota: S^{n-1} \rightarrow \overline{B_1(0)}$  such that  $F \circ \iota = \text{id}_{S^{n-1}}$ . What happens here?  $\iota$  sends any point  $z$  on the surface of the ball to the ball itself.  $\iota$  is defined in such a way that the position of  $\iota(z)$  must be on the line connecting  $z$  and  $f(\iota(z))$ . This way, when we apply  $F$  to  $\iota(z)$  we go back to  $z \in S^{n-1}$ . Now, we can consider the pullback:

$$\text{id}_{S^{n-1}}^* = \iota^* \circ F^* \quad (*)$$

where we consider  $F^*$  acting on closed-but-not-exact forms:  $F^*: H^*(S^{n-1}) \rightarrow H^*(\overline{B_1(0)})$ . If  $* = n - 1$ , the map:

$$F^*: \underbrace{H^{n-1}(S^{n-1})}_{\cong \mathbb{R}} \longrightarrow \underbrace{H^{n-1}(\overline{B_1(0)})}_{\cong 0}$$

has to be injective because of  $(*)$  (if the composition of maps is injective, the first map acting is injective). But this is impossible because  $H^{n-1}(S^{n-1}) \cong \mathbb{R}$  (maybe it was proved in the exercise sheets?) and  $H^{n-1}(\overline{B_1(0)}) \cong 0$ , with  $B_1(0)$   $n$ -dimensional ball. In order to prove the latter, we can't use the Poincaré lemma because  $\overline{B_1(0)}$  is not homeomorphic to  $\mathbb{R}^n$  (one is compact and the other is not, and compactness is left invariant under homeomorphisms). But we can prove that  $\text{id}: \overline{B_1(0)} \rightarrow \overline{B_1(0)}$  is homotopy equivalent to the map zero:  $\overline{B_1(0)} \rightarrow \overline{B_1(0)}$  that sends every element of  $\overline{B_1(0)}$  to  $0 \in \overline{B_1(0)}$ . Once we do this, we use the fact that the space itself is homotopy equivalent to the trivial set (for  $n \geq 2$ ) and then the de Rham cohomology must be trivial as well.  $\square$

*Remark 20.23.* The important assumption of the Brouwer theorem 20.22 is that the entire space is  $\overline{B_1(0)}$ . We also notice that the theorem can be seen as a corollary of the Borsuk-Ulam theorem. It says that there exists at least a couple of antipodal points on the Earth surface with same temperature and barometric pressure (if we assume temperature and pressure to be two continuous functions).

**Definition 20.24** (degree of a map). Let  $f: M \rightarrow N$  be smooth, let  $N, M$  be closed, connected and orientable manifolds of the same dimension. Then, the degree of  $f$  is

$$\deg(f) = \frac{\int_M f^* \omega}{\int_N \omega}, \quad \text{for a form } \omega \in \Omega^n(N) \text{ s.t. } \int_N \omega \neq 0. \quad (20.19)$$

*Remark 20.25.* The definition 20.24 does not depend on the  $\omega$  chosen: the proof is in the first lines of page 458 of [Lee]. It is proved that the number  $\deg f$  exists and the proof does not depend on  $\omega$ . Moreover, the degree of a map is homotopy invariant: if  $f_0, f_1: M \rightarrow N$  are two homotopic maps, then  $f_0^* = f_1^*$ , so  $f_0$  and  $f_1$  have the same degree.

*Example 20.26.*

- $\deg(\text{id}_M) = 1$ ,
- $\deg(\text{constant map}) = 0$ , because the definition of pullback uses the pushforward:  $f^*\omega = \omega(f_*, \dots)$ ,
- If  $M, N$  are closed, orientable and of the same dimensions, from the previous two points we know that the identity map cannot be homotopy equivalent to a constant map,
- If  $M = N = S^1$ , let's consider

$$\begin{aligned} f_k: S^1 &\rightarrow S^1 \subset \mathbb{C} \\ e^{i\varphi} = z &\mapsto z^k \end{aligned}$$

Then we have  $\deg(f_k) = k$ , indeed we can consider  $\omega = d\varphi$ ,  $\varphi$  angular coordinate on  $S^1$  (cf. example 9.25). We stress out that  $d\varphi$  is not exact (it is just a bad notation) because  $\varphi$  is not a well defined function (it is well defined up to a  $2\pi$  angle). Then, we have:

$$\int_{S^1} \omega = 2\pi, \quad \int_{S^1} f_k^* d\varphi = \int_{S^1} k d\varphi = 2\pi k \Rightarrow \deg(f_k) = k$$

where we used that  $\varphi$  is the angular coordinate, so  $f(e^{i\varphi}) = e^{ik\varphi}$ ,

- For the next theorem, we introduce the concept of critical value and regular value of a function  $f$ . A critical value is the image of a critical point. A point  $p$  is a regular value for  $f$  if  $D_q f$  is a surjective map for every  $q \in f^{-1}(p)$ : in a simpler way,  $p$  is a regular value if it is not a critical value.

**Theorem 20.27.** *For all  $f: M \rightarrow N$ , in the hypotheses of def. 20.24, the degree of  $f$  is an integer. In particular, it is the signed count of preimages of a regular value.*

*Proof.* Let us consider a regular value  $p \in N$  of  $f$ . The value  $p$  exists thanks to Sard's theorem. Sard's theorem states that the set of the critical values of a smooth function  $f$  from a manifold to another one is a null-measure set. We can apply this theorem to our case (and whenever  $f$  is only continuous we can approximate it by smooth functions). Indeed, if the set of critical values has zero measure, its complementary set (the regular values) has

dimension  $n$ . In particular,  $f^{-1}(p)$  is a submanifold of dimension 0 by definition 16.3. So,  $f^{-1}(p)$  consists of a finite collection of points  $q_1, \dots, q_m$ . By the inverse function theorem, there exists a neighbourhood  $U_i$  of  $q_i \in f^{-1}(p)$  such that  $f|_{U_i}$  is a diffeomorphism onto its image. Let  $\omega$  be a bump form supported in  $\bigcap_i f(U_i) = V$ . Then:

$$\int_M f^* \omega = \sum_i \underbrace{\int_{U_i} f^* \omega}_{\substack{= \pm \int_{f(U_i)} \omega \\ \text{depending} \\ \text{on orientation}}} = \left( \sum_i \pm 1 \right) \int_V \omega$$

So, the degree is an integer: the signed count of preimages of a regular value. The sign depends on the orientation (i.e. the sign of the determinant of the Jacobian at  $p_i, \forall i$ ).  $\square$

## 21 ■ Poincaré Duality

In the last section, we studied the degree of a map. It is not always possible to find a map of non-vanishing degree between manifolds of the same dimension. For instance, every smooth map  $S^2 \rightarrow T^2 = S^1 \times S^1$  has degree 0.

*Example 21.1.* Let's consider a smooth map  $f: S^2 \rightarrow T^2 = S^1 \times S^1$ .

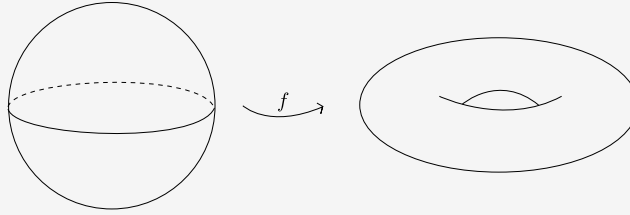


Figure 20:  $f: S^2 \rightarrow T^2$

The map  $f$  has degree 0, indeed the pullback  $f^*: H^2(T^2) \rightarrow H^2(S^2)$  is the zero map. In order to show this, we consider two projection maps,  $\text{pr}_1$  and  $\text{pr}_2$ , defined as follows:

$$\begin{array}{ccc} & T^2 = S^1 \times S^1 & \\ \text{pr}_1 \swarrow & & \searrow \text{pr}_2 \\ (S^1, d\varphi_1) & & (S^1, d\varphi_2) \end{array}$$

where  $\varphi_i$  are angular functions and  $d\varphi_1$  and  $d\varphi_2$  are not exact, as already seen in the example 20.26. We choose  $\omega \equiv (\text{pr}_1^* d\varphi_1) \wedge (\text{pr}_2^* d\varphi_2)$ . We have:

$$\int_{S^1 \times S^1} \omega = \int_{S^1} d\varphi_1 \int_{S^1} d\varphi_2 = 4\pi^2$$

Now, we have:

$$f^* \omega = \underbrace{((\text{pr}_1 \circ f)^* d\varphi_1)}_{\text{closed 1-form on } S^2} \wedge \underbrace{((\text{pr}_2 \circ f)^* d\varphi_2)}_{\text{also closed}}$$

because we used the proposition 9.22, point 3, and because of the naturality of the pullback. Now,  $H^1(S^2) = 0$  (not proved, but it can be computed using the Mayer-



Vietoris theorem 15.27). So, every closed form on  $S^2$  is exact. Thus, the two closed 1-forms above are actually exact forms. So,  $(\text{pr}_1 \circ f)^* d\varphi_1 = dg$  and we have that

$$f^* \omega = dg \wedge ((\text{pr}_2 \circ f)^* d\varphi_2)$$

is exact because the product between an exact form and a closed one is an exact form. Finally:

$$\int_{S^2} f^* \omega = 0$$

using Stokes's theorem (we get  $\int_{B^3} 0 = 0$ ).

**Theorem 21.2.** *Let  $M$  be a closed, oriented, connected  $n$ -dimensional manifold. Then, if we consider maps to  $S^n$ , not only the degree is homotopy invariant but the converse is also true:*

$$f_0 \sim f_1: M \rightarrow S^n \Leftrightarrow \deg(f_0) = \deg(f_1) \quad (21.1)$$

**Theorem 21.3** (Poincaré Duality). *Let  $M$  be an oriented, connected,  $n$ -dimensional manifold with  $\partial M = \emptyset$ . We define the map:*

$$\begin{aligned} PD: H^k(M) \times H_c^{n-k}(M) &\rightarrow \mathbb{R} \\ ([\omega], [\eta]) &\mapsto \int_M (\omega \wedge \eta) \end{aligned} \quad (21.2)$$

where  $[\omega]$  and  $[\eta]$  are equivalence classes for different equivalence relations. This map is non-degenerate<sup>1</sup> and establishes a natural isomorphism:

$$H^k(M) \cong (H_c^{n-k}(M))^* \quad (21.3)$$

$$(H^k(M))^* \cong H_c^{n-k}(M) \quad (21.4)$$

Moreover, if  $M$  is closed, then  $H^k(M) \cong (H^{n-k}(M))^*$  and these vector spaces have finite dimension.

*Idea of the proof.* The proof is not hard, but it is quite long, under the assumption that exists a finite "good" cover by charts. Here, "good" means that any non-empty intersection of charts is diffeomorphic to a ball. Then, the proof is by induction on the cardinality of a finite good cover. Then, we also need to use the zig-zag lemma for  $H^*$  and  $H_c^*$  and the so-called 5-lemma.  $\square$

## 22 ■ Riemannian Metric and the Hodge- $\star$ Operator

**Definition 22.1** (Riemannian metric). Let  $M$  be a smooth manifold. A Riemannian metric is a map

$$g: (TM \times TM)|_{\Delta} \rightarrow \mathbb{R} \quad (22.1)$$

<sup>1</sup>PD is non-degenerate: if  $[\omega]$  is non-zero, we can find a  $[\eta]$  such that  $PD([\omega], [\eta]) \neq 0$ .

fiberwise bilinear, symmetric and positive definite. Here,  $\Delta = \{(x, x) \mid x \in M\} \subset M \times M$  is the diagonal set<sup>2</sup> of  $M$ .

*Remark 22.2.* Let's analyze the definition 22.1. The term "fiber" denotes the preimage of a point. The restriction of the metric  $g$  to  $\Delta$  means that the first and the second arguments of the map  $g$  are in the same space  $T_x M$ , for some  $x \in M$ . Moreover, we can generalize the definition to consider metrics with the same hypotheses, but which are not positive definite and non-degenerate: this how a Lorentzian metric is defined.

**Theorem 22.3.** *Every smooth manifold admits a Riemannian metric,*

*Proof.* Let's cover  $M$  with charts  $(U_i, \varphi_i)$  and pick a subordinate partition of unity  $\phi_j: V_j \subset U_i(\dots) \rightarrow \mathbb{R}$ . We can do that because our manifold is second-countable and paracompact (the latter one is actually a necessary hypothesis). On each set  $V_j$  there is a metric  $g_j$  coming from the coordinates. Then,  $g = \sum_j \lambda_j g_j$  is a Riemannian metric (easy to verify). The key point is that a convex combination of positive definite is still positive definite).  $\square$

*Remark 22.4* (Counter-example for Lorentzian metric). It is not true that every manifold admits a Lorentzian metric. For instance,  $S^{2n}$  does not.

A Riemannian metric assigns lengths to vectors, angles between vectors...we want to extend this (or, analogously, a Lorentzian metric) to alternating forms.

*Remark 22.5.* A Riemannian metric provides an isomorphism:

$$\flat: T_p M \rightarrow (T_p M)^*$$

$$v \mapsto (w \mapsto g(v, w)).$$

And the inverse is the map  $\sharp \equiv \flat^{-1}$ . This is called musical isomorphism and it is related to the action of "raising an index" and "lowering an index" in physics. For more info, see page 342 of [Lee]. Such a map can be extended as  $\flat: TM \rightarrow T^*M$  and so associate a 1-form to each vector field (since vector fields are sections of the tangent bundle and 1-forms are sections of the cotangent bundle). We can then further extend the isomorphism to alternating forms: there exists a canonical isomorphism:

$$\text{Alt}^k(T_p^* M) \xrightarrow{\sim} (\text{Alt}^k(T_p M))^*.$$

So, we can now define a Lorentzian metric on forms (in particular, we can extend the concept from  $T_p^* M$  to  $\Lambda^k T_p^* M \equiv \text{Alt}^k(T_p^* M)$ ). A precise definition is the following

<sup>2</sup> $\Delta$  is a submanifold of  $M \times M$  and it is diffeomorphic to  $M$ .

one: if  $e_1, \dots, e_n$  is an orthonormal basis of  $T_p M$  and  $\delta^1, \dots, \delta^n$  is the dual basis, then:

$$\langle \delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k}, \delta^{\lambda_1} \wedge \dots \wedge \delta^{\lambda_k} \rangle = \begin{cases} 0, & \text{if } (\lambda_1, \dots, \lambda_k) \neq (\mu_1, \dots, \mu_k) \\ \varepsilon_{\lambda_1} \dots \varepsilon_{\lambda_k}, & \text{if } (\lambda_1, \dots, \lambda_k) = (\mu_1, \dots, \mu_k) \end{cases}$$

where:  $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$  is the Lorentzian/Riemannian metric defined on forms,  $\mu_1 < \dots < \mu_k$ ,  $\lambda_1 < \dots < \lambda_k$ , and  $\varepsilon_{\lambda_i} = \langle e_i, e_i \rangle = \pm 1$  (we always have the "+" sign for the Riemannian metric, whereas we have some "-" sign for a Lorentzian metric). Thus,  $\langle \cdot, \cdot \rangle$  for alternating forms is symmetric.

**Definition 22.6** (Volume form). If  $M$  is an oriented manifold, then a volume form  $\omega$  on  $T_p M$  is a differential  $n$ -form such that:

$$\omega(e_1, \dots, e_n) = 1, \quad \forall \{e_1, \dots, e_n\} \text{ orthonormal basis of } T_p M \quad (22.2)$$

*Remark 22.7.* If  $e'_1, \dots, e'_n$  is another oriented orthonormal basis, then  $\exists$  a  $f$  such that:

$$\begin{aligned} f: T_p M &\rightarrow T_p M \\ e_i &\mapsto e'_i \end{aligned}$$

is a linear map, orthogonal and positive definite (i.e.  $\det f = 1$ ). Moreover:

$$\omega(e'_1, \dots, e'_n) = \omega(f(e_1), \dots, f(e_n)) = (\det f) \omega(e_1, \dots, e_n) = 1$$

where we used the alternating property of forms:

$$\omega(Te_1, \dots, Te_n) = (\det T) \omega(e_1, \dots, e_n)$$

when  $T$  is a linear map (see page 354 of [Lee] for more details).

We saw how a volume form reacts to an orthonormal basis. How about a general basis?

**Lemma 22.8.** Let  $f_1, \dots, f_n$  be an oriented basis (not orthonormal) of  $T_p M$ . Let  $\omega$  be a volume form. Then:

$$\omega(f_1, \dots, f_n) = \sqrt{\det g_{ij}}, \quad g_{ij} \equiv \langle f_i, f_j \rangle. \quad (22.3)$$

*Proof.* Let  $e_1, \dots, e_n$  be an oriented orthonormal basis and let's consider the map:

$$\begin{aligned} F: T_p M &\rightarrow T_p M \\ e_i &\mapsto f_i \end{aligned}$$

which has positive determinant. With respect to the basis  $e_1, \dots, e_n$ ,  $F$  is represented by the matrix  $A = (\alpha_{ij})$ , where  $\alpha_{ij}$  such that  $f_i = \sum \alpha_{ij} e_j$ . Now,

$$\langle f_i, f_j \rangle = \left\langle \sum_l \alpha_{il} e_l, \sum_m \alpha_{jm} e_m \right\rangle = \dots = \sum_l \alpha_{il} \alpha_{jl} = (A^t A)_{ij}.$$

Thus,

$$\begin{aligned} \omega(f_1, \dots, f_n) &= \omega(F(e_1), \dots, F(e_n)) = (F^* \omega)(e_1, \dots, e_n) = \det A \omega(e_1, \dots, e_n) = \\ &= \det A = \sqrt{\det g_{ij}}. \end{aligned}$$

where we used again the definition of pullback for linear maps (see remark 22.7), the formula  $\det(AA^t) = \det(A)^2$ , and the fact that  $\omega$  is a volume form.  $\square$

**Lemma 22.9** (Hodge- $\star$  operator). *Let  $V$  be an oriented<sup>3</sup>  $\mathbb{R}$ -(vector space),  $\dim V = n$ . Let  $\langle \cdot, \cdot \rangle$  be a non-degenerate, symmetric bilinear form. Let  $\omega$  be the canonical volume form. Then there is a unique linear map*

$$\star: \Lambda^k V^* \rightarrow \Lambda^{n-k} V^*, \quad \forall k = 0, 1, \dots, n \quad (22.4)$$

such that:

$$\langle \eta, \zeta \rangle \omega = \eta \wedge \star \zeta, \quad \forall \eta, \zeta \in \Lambda^k V^*.$$

The map  $\star$  is the Hodge- $\star$  (Hodge star) operator.

*Proof.*

**Uniqueness:** Assume  $\star, \tilde{\star}$  satisfy the assumption. Then:

$$\underbrace{\eta}_{\in \Lambda^k V^*} \wedge \underbrace{(\star \zeta - \tilde{\star} \zeta)}_{\in \Lambda^{n-k} V^*} = 0, \quad \forall \eta, \zeta \in \Lambda^k V^*.$$

If  $\star \neq \tilde{\star} \Rightarrow \exists \zeta : \forall \eta, \eta \wedge \underbrace{(\star \zeta - \tilde{\star} \zeta)}_{\neq 0} = 0$ , but it is easy to find an  $\eta$  violating this equality, writing  $\star \zeta - \tilde{\star} \zeta \in \Lambda^{n-k} V^*$  in terms of a basis.

**Existence:** To prove existence, we use an orthonormal basis  $e_1, \dots, e_n$  of  $V$  and the dual basis  $\delta^1, \dots, \delta^n$  of  $V^*$ . Recall that for  $1 \leq \mu_1 < \dots < \mu_k \leq n, 1 \leq \lambda_1 < \dots < \lambda_k \leq n$ , we have (see also remark 22.5):

$$\circledast = \langle \delta^{\lambda_1} \wedge \dots \wedge \delta^{\lambda_k}, \delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k} \rangle \omega = \begin{cases} \varepsilon_{\mu_1} \dots \varepsilon_{\mu_k} \omega, & \text{if } (\lambda_1, \dots, \lambda_k) = (\mu_1, \dots, \mu_k) \\ 0, & \text{otherwise} \end{cases}$$

And in order to have the thesis, we need:

$$\circledast \stackrel{!}{=} \delta^{\lambda_1} \wedge \dots \wedge \delta^{\lambda_k} \wedge \star(\delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k})$$

---

<sup>3</sup>In 3 dimensions, a vector space is "right-oriented" if its basis is oriented according to the right hand. In general dimension there is not such a thing as a preferred orientation. In this case we just want to say that we arbitrarily chose an orientation for our basis.

where the symbol  $\stackrel{!}{=}$  means "must be equal", as usual. It follows that  $\star(\delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k})$  is a multiple of  $\delta^{\nu_1} \wedge \dots \wedge \delta^{\nu_{n-k}}$ , where  $\nu_1, \dots, \nu_{n-k}$  such that  $\{\mu_1, \dots, \mu_k, \nu_1, \dots, \nu_{n-k}\} = \{1, \dots, n\}$  (because we want to get a multiple of  $\omega$ ). More precisely, using the permutations, only one coefficient of  $\star(\delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k})$  is non-zero, and it is the one for  $(\lambda_1, \dots, \lambda_k) = (\mu_1, \dots, \mu_k)$ :

$$\star(\delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k}) = \varepsilon_{\mu_1} \cdots \varepsilon_{\mu_k} \operatorname{sgn} \begin{pmatrix} 1 \dots k & k+1 \dots n \\ \mu_1 \dots \mu_k & \nu_1 \dots \nu_{n-k} \end{pmatrix} \delta^{\nu_1} \wedge \dots \wedge \delta^{\nu_{n-k}}$$

This holds without any assumption on the order of the  $\lambda_i$ , and it defines  $\star$  on a basis. The resulting operator has the desired property:

$$\langle \eta, \zeta \rangle \omega = \eta \wedge \star \zeta, \quad \forall \eta, \zeta \in \Lambda^k V^*.$$

□

*Remark 22.10* (Properties of  $\star$ ). The Hodge- $\star$  operator has the property:

$$\star\star = \varepsilon_1 \cdots \varepsilon_n (-1)^{k(n-k)}, \quad \text{on } \Lambda^k V^* \quad (22.5)$$

where  $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$ , so  $\operatorname{Index}(V) \equiv \varepsilon_1 \cdots \varepsilon_n$  is the number of "-" signs that we get. Let's prove the property (22.5) on a basis:

$$\begin{aligned} \star\star(\delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k}) &= \star \left( \varepsilon_{\mu_1} \cdots \varepsilon_{\mu_k} \operatorname{sgn} \begin{pmatrix} 1 \dots k & k+1 \dots n \\ \mu_1 \dots \mu_k & \nu_1 \dots \nu_{n-k} \end{pmatrix} \delta^{\nu_1} \wedge \dots \wedge \delta^{\nu_{n-k}} \right) = \\ &= \underbrace{\varepsilon_{\mu_1} \cdots \varepsilon_{\mu_k} \varepsilon_{\nu_1} \cdots \varepsilon_{\nu_k}}_{=(-1)^{\operatorname{Index}}} \operatorname{sgn} \begin{pmatrix} 1 \dots k & k+1 \dots n \\ \mu_1 \dots \mu_k & \nu_1 \dots \nu_{n-k} \end{pmatrix} \underbrace{\operatorname{sgn} \begin{pmatrix} 1 \dots n-k & n-k+1 \dots n \\ \nu_1 \dots \nu_{n-k} & \mu_1 \dots \mu_k \end{pmatrix}}_{\sigma} \\ &\stackrel{(1)}{=} \varepsilon_1 \cdots \varepsilon_n \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma^{-1}) \delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k} = \\ &= \varepsilon_1 \cdots \varepsilon_n \operatorname{sgn} \begin{pmatrix} 1 \dots k & k+1 \dots k+n-k \\ k-n+1 \dots n & \dots n-k \end{pmatrix} \delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k} = \\ &\quad \underbrace{\hspace{10em}}_{\sigma^{-1} \circ \tau} \\ &= \varepsilon_1 \cdots \varepsilon_n (-1)^{k(n-k)} \delta^{\mu_1} \wedge \dots \wedge \delta^{\mu_k} \end{aligned}$$

where we used: (1)  $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$ . So, up to a sign,  $\star\star$  is an involution. Moreover, up to a sign,  $\star$  is an isometry. Indeed:

$$\star\eta \wedge \star\star\zeta = \langle \star\eta, \star\zeta \rangle \omega = (-1)^{\operatorname{Index} V} \langle \eta, \zeta \rangle \omega.$$

**Lemma 22.11.** *Given a symmetric, bilinear, non-degenerate pairing  $\langle \cdot, \cdot \rangle$  on a manifold  $M$  (oriented, smooth, compact), there is a induced non-degenerate, symmetric, bilinear pairing*

$$\Omega^k(M) \times \Omega^k(M) \rightarrow \mathbb{R} \quad (22.6)$$

$$(\eta, \zeta) \mapsto \int_M \langle \eta, \zeta \rangle \omega = \int_M (\eta \wedge \star \zeta) \equiv \langle \eta, \zeta \rangle_M$$

**Definition 22.12** (Codifferential). The map  $\delta = (-1)^k \star d \star^{-1}: \Omega^{n-k}(M) \rightarrow \Omega^{n-k-1}(M)$  is called codifferential (it decreases the degree of a form).

*Remark 22.13.* We notice that  $\delta^2 = 0$ . Moreover, we want to compare  $\star\delta$  and  $d\star$ . So, let  $\zeta \in \Omega^{k+1}(M)$ , we compute the following:

$$\begin{aligned} \underbrace{d \star \zeta}_{n-k-1} &\stackrel{(22.5)}{=} (-1)^{k(n-k)+\text{Index}(M)} \star \star d \star \zeta \stackrel{(1)}{=} (-1)^{k(n-k)+\text{Index}(M)} \star \star d \star \star \underbrace{\star^{-1} \zeta}_{n-(k+1)} \stackrel{(22.5)}{=} \\ &= (-1)^{k(n-k)+2\text{Index}(M)+(n-k-1)(k+1)} \star \star d \star^{-1} \zeta = \dots = (-1)^k \star \delta \zeta \end{aligned}$$

where we used:  $(1) \ 1 = \star \star^{-1}$ .

**Lemma 22.14.** *The following holds:*

$$d(\eta \wedge \star \zeta) = (d\eta) \wedge \star \zeta + (-1)^{\deg \eta} \eta \wedge d \star \zeta = \dots = d\eta \wedge \star \zeta + \eta \wedge \star \delta \zeta \quad (22.7)$$

where  $\eta \in \Omega^k(M), \zeta \in \Omega^{k+1}(M)$ . If  $M$  is closed, then:

$$0 = \int_M d(\eta \wedge \star \zeta) = \langle d\eta, \zeta \rangle_M + \langle \eta, \delta \zeta \rangle_M. \quad (22.8)$$

Using the notations of mathematical analysis (where an operator  $A^*$  is the adjoint of  $A$  if  $\langle A^*x, y \rangle = \langle x, Ay \rangle, \forall x, y$  in the domain of  $A$  and  $A^*$ ), we say that  $-\delta$  is formally the adjoint of  $d$ .

*Remark 22.15* (Notation). We will write:

$$d_k: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

$$\delta_k: \Omega^{n-k}(M) \rightarrow \Omega^{n-k-1}(M)$$

**Notice that** in the first part of the course the first map above was called  $d_{k+1}$  and not  $d_k$ ! We are changing notation here with respect to the notations used in the course. Then:

$$(-1)^k \delta_k \star = \star d_k$$

and  $-\delta_k$  is the formal adjoint of  $d_{n-k-1}$  (via  $\langle \cdot, \cdot \rangle_M$ ). In addition,  $(-1)^k \delta_k$  is conjugate to  $d_k$  (via  $\star$ ). Let's look again at the adjointness condition:

$$(*) \quad \langle d_k \eta, \zeta \rangle_M + \langle \eta, \delta_{n-k-1} \zeta \rangle_M = 0. \quad (22.9)$$

**Fact:** From  $(*)$ , it follows that  $\ker d_k \subset (\text{im } \delta_{n-k-1})^\perp$ . Actually, also  $\supset$  holds. Furthermore:  $\ker \delta_{n-k} = (\text{im } d_{k-1})^\perp$ . For finite-dimensional vector spaces, this would imply:

$$\odot \quad \Omega^k(M) = (\ker d_k) \oplus (\text{im } \delta_{n-k-1}) = (\ker \delta_{n-k}) \oplus (\text{im } d_{k-1}) \quad (22.10)$$

Actually,  $\odot$  holds also for infinite-dimensional spaces, thanks to the following theorem.

**Theorem 22.16** (Fundamental theorem of Hodge-de Rham theory). *The expression  $\odot$  above is an orthogonal direct sum decomposition.*

**Corollary 22.17.** *The following holds:*

$$\begin{aligned} \ker d_k &= \Omega^k(M) \cap \ker d_k = (\ker(\delta_{n-k}) \oplus \operatorname{im}(d_{k-1})) \cap \ker d_k = \\ &= ((\ker \delta_{n-k}) \cap \ker d_k) \oplus \operatorname{im} d_{k-1}. \end{aligned} \quad (22.11)$$

Comparing first and last term, and remembering that the de Rham cohomology of  $M$  is  $H^k(M) = \ker d_k / \operatorname{im} d_{k-1}$ , we have that  $H^k(M) \cong ((\ker \delta_{n-k}) \cap \ker d_k)$ . In a similar way:

$$\ker \delta_{n-k} = \operatorname{im} \delta_{n-k-1} \oplus (\ker d_k \cap \ker \delta_{n-k}). \quad (22.12)$$

**Lemma 22.18** (Harmonic forms). *Elements of  $\mathcal{H}^k(M) \equiv \ker d_k \cap \ker \delta_{n-k}$  are closed and coclosed (i.e. closed with respect to  $\delta$ ), and they are said to be harmonic. The Laplace operator is  $\Delta = d \circ \delta + \delta \circ d$ , and*

$$\mathcal{H}^k(M) = \ker \left( \Delta|_{\Omega^k(M)} \right). \quad (22.13)$$

*Proof of 22.13.*  $\eta, \zeta \in \Omega^k(M)$ .

" $\subseteq$ ": trivial by definition of  $\Delta$ ,

" $\supseteq$ ": We have:

$$0 = \langle \Delta \eta, \zeta \rangle_M = \langle d \circ \delta \eta + \delta \circ d \eta, \zeta \rangle_M = -\langle d \eta, d \zeta \rangle_M - \langle \delta \eta, \delta \zeta \rangle_M, \quad \forall \zeta.$$

where we used the adjointness (22.9). Since it holds for every  $\zeta$ , it is also true for  $\zeta = \eta$ :

$$0 = -\underbrace{\|d \eta\|^2}_{\geq 0} - \underbrace{\|\delta \eta\|^2}_{\geq 0}$$

and the equality holds  $\Leftrightarrow \eta \in \ker d_k, \eta \in \ker \delta_{n-k}$ . □

**Corollary 22.19** (Hodge theorem). *There is an isomorphism*

$$\mathcal{H}^k(M) \rightarrow H^k(M). \quad (22.14)$$

Moreover, we have:

$$\begin{array}{ccc} \mathcal{H}^k(M) & \longrightarrow & H^k(M) \\ \star^{-1} \downarrow \uparrow \star & & \downarrow \uparrow \sim PD \\ \mathcal{H}^{n-k}(M) & \longrightarrow & H^{n-k}(M) \end{array}$$

*Remark 22.20.*

**Fact:**  $\mathcal{H}^k(M)$  has finite dimension ( $\Delta$  is an elliptic operator).

**Fact** (Hodge decomposition):

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d_{k-1}\Omega^{k-1}(M) \oplus \delta_{n-k-1}\Omega^{k+1}$$

is an orthogonal sum decomposition.



## Bibliography

- [1] J. M. Lee, Introduction to Smooth Manifolds, 2nd edition, Springer
- [2] Lecture notes Differentiable Manifolds Sachs, Vogel WS 19