Notes
Differentiable Manifolds
LMU Munich WS 19

flaviorossetti@outlook.com

Last version: January 12, 2020

Don't trust these notes!

Many thanks to Raul Morral for his help to improve the script

Contents

1   Intro

From the Lee book [1]: "The central idea of calculus is *linear approximation*". A function of one variable can be approximated by its tangent line, a curve by a tangent vector (i.e. velocity vector), a surface in $\mathbb{R}^3$ can be approximated by its tangent plane, and a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by its total derivative. Here it comes the importance of tangent spaces.

Main idea: in order to study tangent vectors, we identify them with "directional derivatives". In particular, there is a natural one-to-one correspondence between geometric tangent vectors and linear maps from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$ satisfying the product rule. Such maps are called *derivations*.

Remark 1.1. **Points or vectors?** We can think of elements of $\mathbb{R}^n$ either as points or vectors. As points, their only property is their location, given by the coordinates $(x_1, \ldots, x_n)$ on a chosen basis. As vectors, they are characterized by a direction and a magnitude, but their location is irrelevant (translational invariance). So given $v \in \mathbb{R}^n, v = v^i e_i$, it can be seen as an arrow with its initial point anywhere in $\mathbb{R}^n$. So, if we think about a vector tangent to the border of the sphere at a point a, we imagine the vector as living in a copy of $\mathbb{R}^n$ with its origin translated to a.

2   Quick review: Basic Algebraic Structures

Definition 2.1 (Operation). Let G be a set. $\cdot$ is called a (binary) operation on G if it is a map

$$\cdot : G \times G \longrightarrow G$$
$$(a,b) \longmapsto a \cdot b$$

Such a map is usually denoted by the symbol $\cdot$ or, analogously, with the symbol $+$.

Definition 2.2 (Group). Given a set G and an operation $\cdot$ on such set, we will call such set with the operation (i.e. the couple $(G, \cdot)$) a group if the following properties are satisfied (for $a, b, c \in G$):

- $a \cdot b \in G$ (closure property, which often follows from the definition of our operation)

- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity)

- $\exists \, \mathbf{1} \in G$ such that $\mathbf{1} \cdot g = g$, $\forall \, g \in G$ (existence of the identity element)

- $\forall \, g \in G, \exists \, g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = \mathbf{1}$ (existence of the inverse element)

For the sake of simplicity, we will often call G a group, without referring to the operation on it.

Example 2.3. $(\mathbb{Z}, +)$ is a group.

Example 2.4. $(\mathbb{R}, +)$ is a group. Also: $(\mathbb{R}, \cdot)$ is a group.

Example 2.5. $(\mathbb{N}, +)$ is not a group!

Remark 2.6. The identity element of a group is often denoted as $\mathbf{1}$ if the operation is denoted by the symbol $\cdot$, whereas it is denoted as $\mathbf{0}$ if the operation is denoted by the symbol $+$. In a similar way, the inverse element is often denoted as g^{-1} if the operation is denoted by the symbol $\cdot$, whereas it is denoted as $-g$ if the operation is denoted by the symbol $+$.

Definition 2.7 (Abelian group). A group $(G, \cdot)$ is called abelian if its elements commute according to the operation $\cdot$, i.e. $a \cdot b = b \cdot a, \forall \, a, b \in G$.

Often we can consider sets with two operations, like $(\mathbb{R}, +, \cdot)$. If they satisfy some properties, they are called rings. If they satisfy even more properties, they are called fields. In particular:

Definition 2.8 (Ring). Given a set R and two operations: $+$ (usually called "additive operation") and $\cdot$ (called "multiplicative operation") on it, we will call the set with the two operations, i.e. $(R, +, \cdot)$, a ring if the following properties are satisfied:

- $(R, +)$ is an abelian group

- $\cdot$ is associative, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \, \forall a, b, c, \in R$

- the multiplicative identity $\mathbf{1}$ exists, i.e. $\exists \mathbf{1} \in R$ such that $\mathbf{1} \cdot r = r, \, \forall r \in R$

- $\cdot$ is distributive with respect to $+$, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$

Definition 2.9 (Commutative ring). $(R, +, \cdot)$ is a commutative ring if the multiplication operation $\cdot$ is commutative

Definition 2.10 (Unitary ring). $(R, +, \cdot)$ is a unitary ring if it contains the multiplication inverse, i.e. the inverse element according to the operation $\cdot$.

Definition 2.11 (Field). A unitary, commutative ring is called a field.

Example 2.12. The set of 2×2 matrices made by real coefficients is a ring with the operation of sum between matrices and with the matrix multiplication. The ring is not commutative because the matrix multiplication is not a commutative operation. Nor it is unitary, since not all the matrices are invertible.

Example 2.13. The set of 2×2 matrices made by real coefficients is a ring with the operation of sum between matrices and with the matrix multiplication. The ring is not commutative because the matrix multiplication is not a commutative operation. Nor it is unitary, since not all the matrices are invertible.

Example 2.14. $(\mathbb{Z}, +, \cdot)$ is a commutative ring. It is not unitary since the inverse element for $\cdot$ is often in the rational numbers, i.e. 3^{-1} is the multiplicative inverse of 3.

Example 2.15. $(\mathbb{R}, +, \cdot)$ is a field

3   Quick review: Morphisms

Definition 3.1 (Homomorphism)**.** A homomorphism h between two sets
endowed with operations (think about two groups, for instance) is a map
which preserves the operations, i.e.:

$$h\colon (G, +) \longrightarrow (H, \circ)$$

such that $h(a + b) = h(a) \circ h(b), \forall\, a, b, \in G$, where $(G, +)$ and $(H, \circ)$ are two
groups. The definition for rings and fields is analogous. Sometimes these
maps are just called morphisms (there is some difference between morphisms
and homomorphisms but it usually matters only if you are dealing with more
abstract algebraic structures).

And now, some particular types of homomorphisms.

Remark 3.2 (Monomorphism, epimorphisms and isomorphisms)*.* A monomor-
phism is an injective homomorphism. An epimorphism is a surjective homo-
morphism. An isomorphism is a bijective homomorphism (thus, it is both a
monomorphism and an epimorphism).

Remark 3.3 (Endomorphism, automorphism)*.* An endomorphism is a ho-
momorphism from one algebraic structure to itself. If such morphism is
bijective, it is called automorphism.

Remark 3.4*.* Don't get confused with hom**e**omorphisms! A homeomorphism
is a continuous and bijective map between two topological space, such that
its inverse is also continuous. In general it is not a homomorphism (without
the **e**), because a topological space is not necessarily associated with an
operation on it. (However if it has an operation, the continuity of the map
implies that it is also a homomorphism).

4   Quick review: Equivalence Classes, Quotient Spaces

Definition 4.1 (Binary relation)**.** Given a set E, a (binary) relation $\sim$ on E is a set of couples $(a,b) \in E \times E$. In other words, a binary relation is a subset of $E \times E$. Moreover, if $(a,b) \in \sim \subset E \times E$, we use the following notation: $a \sim b$.

Definition 4.2 (Equivalence relation)**.** Given a set E, an equivalence relation $\sim$ is a binary relation which satisfies the following properties:

- $a \sim a, \forall\, a \in E$ (reflexive property)

- $a \sim b \Rightarrow b \sim a, \forall\, a,b \in E$ (symmetric property)

- $a \sim b, b \sim c \Rightarrow a \sim c, \forall\, a,b,c \in E$ (transitive property)

Example 4.3. The following are examples (or counter-examples) of equivalence relations:

- "*Being equal to*" (i.e. "$\sim$" is "$=$") is an equivalence relation on $\mathbb{R}$: $x = x \,\forall\, x \in \mathbb{R}; x = y \Rightarrow y = x \,\forall\, x,y \in \mathbb{R}$, etc.

- "*Has the same birthday as*" on the set of all people in the world is an equivalence relation.

- "*Having mutual friends on Facebook*" on the set of Facebook users is not an equivalence relation (there is some problem with transitive property)

- "*Being greater or equal to*" on $\mathbb{R}$ is not an equivalence relation (it is not symmetric, however it is antisymmetric and so it is called "partial order relation")

Definition 4.4 (Equivalence class)**.** Given a set E and an equivalence relation $\sim$ on E, the equivalence class of $x \in E$ is the set $[x]$, where

$$[x] \equiv \{y \in E \,|\, y \sim x\} \tag{4.1}$$

The equivalence class $[x]$ always contains x itself, by definition of equivalence relation.

Example 4.5. If we consider the equivalence relation "$x \sim y$ if $x - y = 4n$ for some $n \in \mathbb{N} \cup \{0\}$" on the set of $\mathbb{Z}$, the equivalence class of 1 is $[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$. We can also write "$x - y = 4n$ for some n" as $x \equiv_4 y$ (congruence modulo 4)

Definition 4.6 (Quotient space). Given a set E, an equivalence relation $\sim$ on E, the quotient space with respect to $\sim$ is denoted by $E\big/\!\!\sim$ and it is the set of all equivalence classes:

$$E\big/\!\!\sim \;\equiv\; \{[x] \,|\, x \in E\} \tag{4.2}$$

Example 4.7. Let's consider two examples:

- If we consider the example 4.5, $\mathbb{Z}\big/\!\!\sim \;=\; \{[0],[1],[2],[3]\}$. In this case, we also write $\mathbb{Z}\big/\!\!\sim \;=\; \mathbb{Z}\big/4\mathbb{Z}$

- If V is a vector space and U is a vector subspace of V, then we can consider the equivalence relation "$x \sim y$ if $x - y \in U$". We notice that $x \sim x$ for each $x \in V$ since $x - x = 0 \in U$, because U is a vector space as well. Then, $V\big/\!\!\sim$ is isomorphic to V', where V' is V without its subspace U (every element of U is identified with 0 in the quotient space, because every element of U is in the same equivalence class of 0).

- If we consider two groups instead of vector spaces, finding the quotient space requires a bit more effort.

5 Derivations

Definition 5.1 (Derivation)**.** If a is a point of $\mathbb{R}^n$, a map $v\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at a* if it is linear over $\mathbb{R}$ and satisfies the following product rule:

$$v(fg)|_a = f|_a\, v(g)|_a + g|_a\, v(f)|_a$$

Remark 5.2. Directional derivatives obviously satisfy the above definition, and in these cases such a rule is also called Leibnitz rule.

6 Multilinear Forms

Definition 6.1 (1-forms). Given a vector space V on a field K, a 1-form (or linear form) $\varphi \colon V \to K$ is a linear function from V to K. V^* (also denoted by ΛV^*) is the set of all linear forms on V.

Remark 6.2. The set of linear forms on V has the structure of a vector space. Indeed, if $\mathbb{K}$ is the field of the vector space V, $\forall\, a, a_1, a_2 \in \mathbb{K}, \forall\, \varphi, \varphi_1, \varphi_2 \in V^*$:

1. $a(\varphi_1 + \varphi_2) = a\varphi_1 + a\varphi_2$ (because we define the map $a_1\varphi_1 + a_2\varphi_2$ as $a_1\varphi_1 + a_2\varphi_2(x) \equiv a_1\varphi_1(x) + a_2\varphi_2(x), \forall\, x \in V$)

2. $(a + b)\varphi(x) = (a\varphi + b\varphi)(x)$

3. $\varphi = 0$ if $\varphi(x) = 0 \forall\, x \in V$

4. $-\varphi(x) = \varphi(-x)$

5. $\mathbb{1}\varphi(x) = \varphi(x)$

6. $ab\varphi(x) = ba\varphi(x)$

Definition 6.3 (Dual basis). If $\{e_i\}_{i=1,\ldots,n}$ is a basis of V, then $\{e^{*i}\}_{i=1}^n \subseteq V^*$ is called the *dual basis* if $e^{*i}(e_j) = \delta_j^i$.

Remark 6.4. We could prove that the dual basis is indeed a basis of the dual space, so $\dim(V) = \dim(\Lambda V^*)$. Check proposition 7.4.

Example 6.5. Let's consider three examples.

1. $p \in \mathbb{R}^3$, fixed. Let V_p be the vector space $V_p \equiv \{q - p \,|\, q \in \mathbb{R}^3\}$. We will also denote it by $T_p\mathbb{R}^3$. We notice that $V_p = T_p\mathbb{R}^3 = \mathbb{R}^3$ (It is trivial to verify the inclusions "$\subseteq$" and "$\supseteq$"). Let $\{(e_i)_p\}_{i=1,2,3}$ be a basis for V_p. If we consider the dual space V_p^*, we notice that $\{(dx^i)_p\}_{i=1,2,3}$ is a dual basis, where $(dx^i)_p(e_j) = \frac{\partial}{\partial x^j}x^i$. The index position is just a matter of notation, for now. Then, we can show that V_p and V_p^* are isomorphic, and the isomorphism is:

$$g \colon V_p \to V_p^* \tag{6.1}$$
$$x \mapsto g(x, \cdot)$$

where g is the euclidean metric in $\mathbb{R}^3$, i.e. $g(x, y) = d(x, y) = $
$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$

2. $V = \{M \in M_2(\mathbb{C}) \,|\, M^\dagger = M, \operatorname{tr}(M) = 0\}$ is a vector field on $\mathbb{R}$, where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, M^\dagger = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}, a, \ldots, d \in \mathbb{C}$$

A basis for V is $\{\sigma_i\}_{i=1,2,3}$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and a dual basis is given by $e^{*i}(e_j) = \frac{1}{2}\operatorname{tr}(e_i, e_j)$ (using the matrix product).

3. Quantum Mechanics:

$$V = \{f\colon \mathbb{R}^3 \to \mathbb{R} \,|\, ||f|| = \int_{\mathbb{R}^3} |f|^2 d^3x < \infty\} = L^2(\mathbb{R}^3)$$

is a vector space. If we consider the Laplacian operator $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2}$, then a basis is given by $\{e_n\} = \{\frac{f_n}{||f_n||}\}$, where f_n eigenfunctions of Δ: $\Delta f_n = \lambda_n f_n$. A dual basis is given by $e^{*n} = \int_{\mathbb{R}^3} e_n$. Notice that both V and V^* are infinite-dimensional spaces. Some notations:

$$e_n = |f_n\rangle, e^{*n} = \langle f_n|, e^{*n}(e_m) = \langle f_n|f_m\rangle$$

Remark 6.6. Every vector in $\mathbb{R}^n$ about a point $p \in \mathbb{R}^n$ (i.e. such that its origin is the point p) "can be seen" as a derivation (cf. def. 5.1), i.e. as a directional derivative of a function evaluated at the point p. For the sake of simplicity, we think $p = 0$ (but the following results are true $\forall p \in \mathbb{R}^n$). The sentence "can be seen" means that there is an isomorphism ψ associating such vectors to such linear forms. Let's construct this isomorphism in the following steps:

1. Because of linearity, we just need to define the isomorphism for the basis vectors $\{e_i\}_{i=1,\ldots,n}$ of $\mathbb{R}^n$.

2. Given the vector e_j of the canonical basis, we associate it with the derivation ∂_{x_j} in 0:

$$\partial_{x_j}|_{p=0} \equiv \frac{\partial}{\partial x_j}|_{p=0}\colon C^\infty(\mathbb{R}^n) \to \mathbb{R} \tag{6.2}$$

$$f \mapsto \frac{\partial}{\partial x_j}|_0(f) \equiv \frac{\partial f}{\partial x_j}(0)$$

9

In particular, if $\mathrm{Der}(\mathbb{R}^n) = \{\text{derivations on } \mathbb{R}^n\} = \{v\colon C^\infty \to \mathbb{R},$ satisfying Leibnitz rule$\}$, then the map:

$$\mathbb{R}^n \xleftrightarrow{\psi} \mathrm{Der}(\mathbb{R}^n) \tag{6.3}$$

such that $\psi(e_j) = \partial_{x_j}\big|_{p=0}, \forall\, e_j$ basis vector, and with $\partial_{x_j}\big|_{p=0}$ partial derivative with respect to the j-th component, defines a linear map. Indeed, it is linear because of linearity of derivations, and since we defined its behaviour on the basis vectors, it is also defined for every vector of $\mathbb{R}^n$. In general we have $\psi(v) = \partial_v\big|_{p=0}$, where $\partial_v\big|_{p=0}$ is the directional derivative with respect to v. Moreover, $\mathrm{Der}(\mathbb{R}^n)$ is a vector space and we used the double arrow above because ψ is an isomorphism, i.e. a bijective map which preserves operations from one space to the other. Here every derivative is evaluated at $p = 0$. We notice that the point p itself is not important for the directional derivative (the *direction* in which we differentiate is the same for every point of the space), but p is meaningful when we *evaluate* the derivative of the function at that point. Indeed, $\partial_x(x^2)\big|_{x=0} \neq \partial_x(x^2)\big|_{x=1}$, even if we are differentiating along the x-axis in both cases.

What is more: given V_0, the set of all the vectors about 0, we can consider its dual space V_0^*. What is a possible dual basis? We want to find linear forms

$$e^{*i}\colon V_0 \to \mathbb{R}$$

such that $e^{*i}(e_j) = \delta_j^i$. We have just seen that we can consider vectors as directional derivatives. So, given e_j vector of the canonical basis, we will call it $\frac{\partial}{\partial x^j}\big|_0$ (because of the isomorphism, they are quite the same mathematical object). Now, we want that

$$e^{*i}\left(\frac{\partial}{\partial x^j}\big|_0\right) = \delta_j^i \tag{6.4}$$

First, let's consider the *coordinate function*:

$$x_j\colon \mathbb{R}^n \to \mathbb{R} \tag{6.5}$$
$$v = (v_1, \ldots, v_n) \mapsto v_j$$

where $v_1, \ldots, v_j$ are the coordinates of the vector v in the canonical basis. The linear form x_j returns the j-th coordinate of a vector. So, given a vector

$v \in \mathbb{R}^n$, every coordinate v_j can be seen as $v_j = x_j(v)$. Now, let's just define

$$e^{*i}\left(\frac{\partial}{\partial x^j}|_0\right) \equiv \frac{\partial}{\partial x^j}|_0 x_i = \frac{\partial}{\partial x^j}x_i|_0 = \delta^i_j \qquad (6.6)$$

Where x_i is the coordinate function defined above (remember: ∂_{x_j} is a derivation, so it must be applied to functions!). Now, it might seem that e^{*i} does not take a vector as argument, but rather a function. Actually, this problem is solved by the isomorphism between vectors and directional derivatives proved above. If ψ is the name of such isomorphism, we could slightly change the definition (6.6) in order to solve this ambiguity:

$$e^{*i}(e_j) \equiv \psi(e_j)(x_i) \qquad (6.7)$$

where x_i is the i-th coordinate function and

$$\psi(e_j) = \frac{\partial}{\partial x^j}|_0 = \partial_{x_j}|_0 \qquad (6.8)$$

The vectors of the dual basis will also be called

$$dx^i \equiv e^{*i} \qquad (6.9)$$

This will be important later: we will define exterior forms of degree k and we'll use both notations. The set of all these forms is $\Lambda^k V^*$, and its basis is given by products (in particular, exterior products) of $e^{*i_1}, \ldots, e^{*i_k}$ (i.e. $dx^{i_1}, \ldots, dx^{i_k}$).

Definition 6.7 (Set of vector fields)**.** We denote by $\mathfrak{X}(\mathbb{R}^n)$ the set of al possible vector fields in $\mathbb{R}^n$, i.e.

$$\mathfrak{X}(\mathbb{R}^n) = \mathrm{Der}\mathbb{F}(\mathbb{R}^n) \equiv \{v \colon \mathbb{F}(\mathbb{R}^n) \to \mathbb{F}(\mathbb{R}^n) \text{ such that}$$
$$v \text{ is } \mathbb{R}\text{-linear and } v(fg) = v(f)g + fv(g)\} \qquad (6.10)$$

where $\mathbb{F} \equiv \{ \text{ functions } f \colon \mathbb{R}^n \to \mathbb{R}\}$.

Remark 6.8 (Fields vs. Derivations)*.* If v is a vector field in $\mathbb{R}^n$, then v assigns a vector to another vector of $\mathbb{R}^n$. So $v \colon \mathbb{R}^n \to \mathbb{R}^n$. So, the set of all vector fields should be (we will use a different symbol to denote it):

$$X(\mathbb{R}^n) = \{v \colon \mathbb{R}^n \to \mathbb{R}^n\}$$

However, the definition 6.7 is a bit different. Why? The fact is, we can consider a vector of $\mathbb{R}^n$ as a directional derivative, cf. remark 6.6 (we are

not considering any fixed point here, but the results do not change). Now a derivation, as defined in def. 5.1, is a map $v\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$, i.e. we can say that a derivation is a very smooth element of $\mathbb{F}(\mathbb{R}^n) = \{$functions $f\colon \mathbb{R}^n \to \mathbb{R}\}$. So, $\mathfrak{X}(\mathbb{R}^n) \cong X(\mathbb{R}^n)$ because we can associate a derivation of $\mathfrak{X}(\mathbb{R}^n)$ to each vector of $X(\mathbb{R}^n)$, and vice versa. Then we also explained why in the definition of $\mathfrak{X}(\mathbb{R}^n)$ every element must be $\mathbb{R}$-linear and satisfy the Leibnitz rule: it follows from the definition of derivations.

Now, a question arises: given v vector field, should we write $v(p)$ (i.e. it takes vectors as argument) or should we write $v(f)$ (i.e. it takes smooth functions as arguments)? The answer is: it depends on the case, since they are two different "v"s. Which is, we will use vectors when we think of v as a function who takes elements of $\mathbb{R}^n$, and we will use functions in the other case. And we can choose which case to use, since we can identify every vector field with a derivation, and every derivation with a vector field (for more info, see pag. 181 of [1]). Now, let us analyze how $v(f)$ is made in the latter case. Given $v \in \mathfrak{X}(\mathbb{R}^n), f\colon \mathbb{R}^n \to \mathbb{R}$, we define the function $v(f)$ as

$$v(f)\colon \mathbb{R}^n \to \mathbb{R}$$
$$p \mapsto v(f)(p) \equiv v_p f \tag{6.11}$$

Now, in coordinates:

$$v(f)(p) = v_p f = v^i(p)(e_i)_p f = v^i(p)\frac{\partial}{\partial x^i}\big|_p f = v^i(p)\frac{\partial f}{\partial x^i}(p) \tag{6.12}$$

where we used summation convention, and the fact that every vector basis e_i can be seen as $\partial_{x_i}\big|_p$. We defined it in the right way because, as expected, we found that $v_p(f)$ is the directional derivative of f in the direction of v, evaluated at p. So, in brief:

- $v(f)(p)$ is a number

- $v(f)(\cdot)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$

- $v(\cdot)$ is a function from $\mathbb{F}(\mathbb{R}^n)$ to $\mathbb{F}(\mathbb{R}^n)$

We also notice that the mathematical object e_i is not much different from $(e_i)\big|_p$ in this case: there is no difference if we think about them as directions, but it makes a difference if we think about them as directional derivatives, because the latter notation gives info about the point in which the derivative is evaluated. So, we add the pedix "p" in order to make the isomorphism

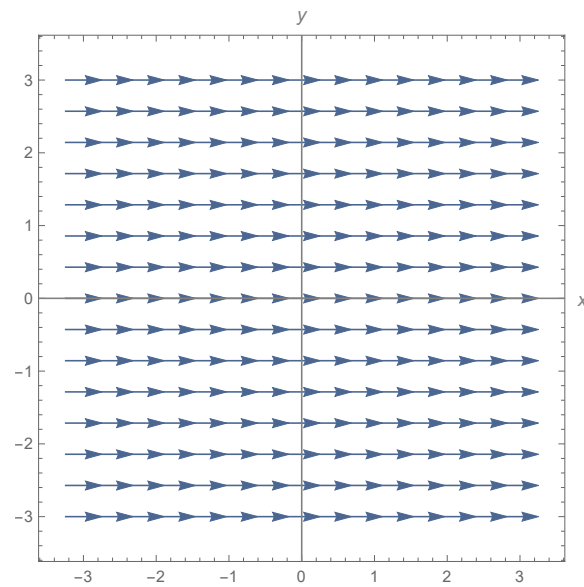between vectors and directional derivatives more explicit. Check also the remark 1.1.

Figure 1: Vector field $v = e_1 = \partial_x$

7 Exterior Product and Generalisation

Definition 7.1 (Exterior form of degree k)**.** Given a vector space V on a field $\mathbb{K}$, with $\dim(V) = n$, and with $k \leq n$, an *exterior form of degree k* (or k-linear form, or k-form) is a map ω:

$$\omega \colon \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{K}$$

such that

$$\omega(v_1, \ldots, v_k) = \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, \ldots, v_{\pi(k)})$$

and such that ω is multilinear. Where π is a permutation of k elements, i.e. $\pi \in S_k$, and $\operatorname{sgn}(\pi)$ is the sign of the permutation. We will also write $\omega \in \Lambda^k V^*$.

Definition 7.2 (Exterior product between two 1-forms)**.** Given a vector space V on a field $\mathbb{K}, \dim(V) \geq 2$, and given $\varphi^1, \varphi^2 \in \Lambda V^*$, then we define the exterior product (or wedge product) $\wedge$ as:

$$\wedge \colon \Lambda V^* \times \Lambda V^* \to \Lambda^2 V^*$$
$$(\varphi^1, \varphi^2) \mapsto \varphi^1 \wedge \varphi^2$$

where:

$$\varphi^1 \wedge \varphi^2(x_1, x_2) = \varphi^1(x_1)\varphi^2(x_2) - \varphi^2(x_1)\varphi^1(x_2) = \det(\varphi^i(x_j))$$

for $i, j = 1, 2$.

Remark 7.3 (Exterior product between k 1-forms)*.* The exterior product $\wedge$ that we defined for $k = 2$ in 7.2 is an exterior form of degree 2. We want to generalize it for k vector spaces. In order to extend the definition, we want it to be an exterior form of degree k, so:

$$\wedge \colon \underbrace{\Lambda V^* \times \ldots \times \Lambda V^*}_{k \text{ times}} \to \Lambda^k V^*$$
$$(\varphi^1, \ldots, \varphi^k) \mapsto \varphi^1 \wedge \ldots \wedge \varphi^k$$

where, given $(x_1, \ldots, x_k) \in \underbrace{V \times \ldots \times V}_{k \text{ times}}$:

$$\varphi^1 \wedge \ldots \wedge \varphi^k(x_1, \ldots, x_k) = \det(\varphi^i(x_j))$$

This is a particular case of an exterior k-form (because the sign of determinant changes if we swap two rows or two columns).

Proposition 7.4. *If* $\{e_i\}_{i=1,\ldots,n}$ *is a basis in* V, *then* $\{e^{*i_1} \wedge \ldots \wedge e^{*i_k}\}_{i_1 < \ldots < i_k, k \le n}$ *forms a basis of* $\Lambda^k V^*$

Remark 7.5. The above proposition proves that $\dim(\Lambda^k V^*) = \binom{n}{k}$. Moreover, it means that any $\alpha \in \Lambda^k V^*$ can be written as:

$$\alpha = \sum_{i_1 < \ldots < i_k} a_{i_1 \cdots i_k} e^{*i_1} \wedge \cdots \wedge e^{*i_k}$$

where $a_{i_1 \cdots i_k} \in \mathbb{K}$, $\mathbb{K}$ field of the vector space.

Now, we want to define the exterior product between a k-form and a p-form (and it will return a $(p+k)$-form).

Definition 7.6 (Exterior product between a k-form and a p-form). Given $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*$, the exterior product between them is defined as:

$$\wedge \colon \Lambda^k V^* \times \Lambda^p V^* \to \Lambda^{k+p} V^*$$

$$(\alpha, \beta) \mapsto \alpha \wedge \beta$$

with:

$$\alpha \wedge \beta = \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_p}} \alpha_{i_1 \cdots i_k} \beta_{j_1 \cdots j_k} e^{*i_1} \wedge \cdots \wedge e^{*i_k} \wedge e^{*j_1} \wedge \cdots \wedge e^{*j_k}$$

where $\alpha_{i_1 \cdots i_k}, \beta_{j_1 \cdots j_k} \in \mathbb{K}$

Example 7.7 (Oriented area). Let's consider the following examples:

1. $V = \mathbb{R}^3 \times \mathbb{R}^3$ with cartesian coordinates.

$$\varphi \equiv dx^1 \wedge dx^2 + dx^2 \wedge dx^4$$

Let's compute: $\varphi(e_i, e_j) = dx^1(e_i)dx^2(e_j) + dx^2(e_i)dx^4(e_j) + $
$- dx^2(e_i)dx^1(e_j) - dx^4(e_i)dx^2(e_j)$

2. $V = \mathbb{R}^2$ with cartesian coordinates x^1, x^2.

$$\varphi \equiv dx^1 \wedge dx^2$$

Let's compute $\varphi(ae_1, be_2)$, where $a, b \in \mathbb{R}$:

$$\varphi(ae_1, be_2) = dx^1 \wedge dx^2(ae_1, be_2) = ab\, dx^1 \wedge dx^2(e_1, e_2) =$$
$$= ab(dx^1(e_1)dx^2(e_2) - \cancel{dx^2(e_1)dx^1(e^2)}) = ab =$$
$$= \text{oriented area of the rectangle of sides } a \text{ and } b$$

15

Let's check some properties about k-forms:

Proposition 7.8. $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*, \gamma \in \Lambda^q V^*$, *then:*

1. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

2. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$

3. $\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha$

8 Differential Forms

Definition 8.1 (Field of exterior forms, geometric definition)**.** (A field of) exterior forms of degree k, $k \leq n$ is a map ω that associates to each point $p \in \mathbb{R}^n$ an element $\omega(p) \in \Lambda^k V_p^*$. Choosing a basis, we have:

$$\omega(p) = \sum_{i_1 < \ldots < i_k} \underbrace{a_{i_1 \cdots i_k}(p)}_{\text{now it is a function!}} e^{*i_1} \wedge \cdots \wedge e^{*i_k} \tag{8.1}$$

ω is a differential form if $a_{i_1 \cdots i_k}$ are differentiable. The set of differential k-forms is denoted by $\Omega^k(\mathbb{R}^n)$

Another (equivalent) definition:

Definition 8.2 (Algebraic definition of differential k-form)**.** A differential k-form is a map:

$$\underbrace{\mathfrak{X}(\mathbb{R}^n) \times \ldots \times \mathfrak{X}(\mathbb{R}^n)}_{k \text{ times}} \to \mathbb{F}(\mathbb{R}^n) \tag{8.2}$$

$C^\infty(\mathbb{R}^n)$ linear and alternating.

Remark 8.3. To show the equivalence of the two definition of differential k-forms we just need to show that:

$$\omega(p)(v_1, \ldots, v_k) = \omega(v_1, \ldots, v_k)(p) \tag{8.3}$$

We want to generalize the concept of differential of a function.

Definition 8.4 (Differential)**.** Let f be a function $f \colon U \subseteq \mathbb{R}^n \to \mathbb{R}$, f differentiable. Let $v \in \mathfrak{X}(\mathbb{R}^n) = \operatorname{Der} \mathbb{F}(\mathbb{R}^n)$. The exterior derivative of f is its differential d, defined as a 1-form such that:

$$df(v) = v(f) \tag{8.4}$$

Remark 8.5 (differential expression in coordinates). We want to verify that the above definition of differential is equivalent to our usual definition for $C^1(\mathbb{R}^n)$ function, which is:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx^i = \frac{\partial f}{\partial x_i} dx^i \tag{8.5}$$

In order to prove that, we first consider a pointwise definition. Given $p \in \mathbb{R}^n$:

$$df_p(v) = v(f), \forall \, v \in T_p \mathbb{R}^n \cong \mathbb{R}^n \tag{8.6}$$

($T_p\mathbb{R}^n$ is the tangent space to $\mathbb{R}^n$ at p). Now, we can write $v(f)$ in coordinates (the gray part is the one we don't care about):

$$df_p = v(f) = v_i(p)(\lambda^i)_p \tag{8.7}$$

where $(\lambda^i)_p$ is a dual basis at p (later, we will prove that $(\lambda^i)_p = (dx^i)_p$). Now, applying df to a particular vector (i.e. directional derivative) at p:

$$df_p\left(\frac{\partial}{\partial x^i}\big|_p\right) = v_i(p) \tag{8.8}$$

where we used the property of the dual basis

$$(\lambda^i)_p\frac{\partial}{\partial x^j}\big|_p = \delta^i_j \tag{8.9}$$

and then:

$$df_p\left(\frac{\partial}{\partial x^i}\big|_p\right) = v_i(p)(\lambda^i)_p\frac{\partial}{\partial x^i}\big|_p = v_i(p) \tag{8.10}$$

On the other hand, by definition (8.6) we know that:

$$df_p\left(\frac{\partial}{\partial x^i}\big|_p\right) = \frac{\partial}{\partial x^i}\big|_p f = \frac{\partial f}{\partial x^i}(p) \tag{8.11}$$

Hence, using (8.8) and (8.11) we get:

$$v_i(p) = \frac{\partial f}{\partial x^i}(p) \tag{8.12}$$

Then, by the expression of differential in coordinates (8.7):

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p \tag{8.13}$$

Applying the definition to $f = x^j$ (coordinate function, as defined in (6.5)), we get:

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p = \frac{\partial f}{\partial x^i}(p)(dx^i)_p \tag{8.14}$$

And then:

$$df = \frac{\partial f}{\partial x^i}dx^i \tag{8.15}$$

Indeed, if $f = x^j$ then, as before:

$$(dx^j)_p = \frac{\partial x^j}{\partial x^i}(p)(\lambda^i)\big|_p = \delta^i_j(\lambda^i)\big|_p = (\lambda^j)\big|_p \tag{8.16}$$

Pay attention: what we did here is a bit different from what we did for the definition 6.11 of a vector field applied to a function. In this case, p is the point where we fixed our vector, whereas in the other case p was the point where we wanted to evaluate the directional derivative of f.

18

In the above definition, f was a 0-form (i.e. a function). What is the generalization of the differential to k-forms?

Definition 8.6 (Exterior derivative)**.** If $k > 0$, then the exterior derivative (acting on k-forms) is a map

$$\mathrm{d}\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$$
$$\omega \mapsto \mathrm{d}(\omega) \equiv \mathrm{d}\,\omega$$

where

$$\mathrm{d}\,\omega = \sum_{j_1 < \ldots < j_k} (\mathrm{d}\,a_{j_1,\ldots,j_k}) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

With $\mathrm{d}\,a_{j_1,\ldots,j_k}$ differential of the function $a_{j_1,\ldots,j_k}$.

Example 8.7 (Computation of $d\omega$)*.* Let's consider the following examples:

1. Let ω be a 2-form on $\mathbb{R}^3$ (coordinates x^1, x^2, x^3):

$$\omega = dx^1 \wedge dx^2 + x^2 dx^1 \wedge dx^3$$

   Then

$$\mathrm{d}\,\omega = dx^2 \wedge dx^1 \wedge dx^3$$

   where we used that $\mathrm{d}(dx^1 \wedge dx^2) = 0$ because there is no 3-form on a 2-dimensional space (otherwise, we can use that $d^2 = 0$, but we still have to prove it!)

2. In $\mathbb{R}^n$, let's consider:

$$\omega = x^2 dx^1, \mathrm{d}\,\omega = dx^2 \wedge dx^1$$

   where we computed $\mathrm{d}\,\omega$ by using $\mathrm{d}\,g(v) = v(g)$ for a function g and a vector field v. In fact, if $u, v \in \mathbb{R}^n$, then by definition of exterior product we have:

$$\mathrm{d}\,\omega(u, v) = dx^2(u)dx^1(v) - dx^2(v)dx^1(u)$$

   On the other hand, using $dx^2(u) = u(x^2), v(x^2) = dx^2(v)$ (where x^2 is a function, the coordinate function defined in (6.5)) we also have:

$$\mathrm{d}\,\omega(u, v) = u(x^2)dx^1(v) - v(x^2)dx^2(u)$$

   Some properties:

Proposition 8.8 (Properties of exterior derivatives). $\omega_1 \in \Omega^k(\mathbb{R}^n), \omega_2 \in \Omega^p(\mathbb{R}^n)$. *Then:*

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

- $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$

- $d(d\omega_1) = 0 = d(d\omega_2)$

Remark 8.9. In the above proposition, we claimed that $d(d\omega) = 0$ if $\omega \in \Omega^k(\mathbb{R}^n)$. The notation here is not the most precise, since the inner "d" is acting on a k-form, whereas the outer "d" is acting on a $(k+1)$-form (so, even if they share the same name, they are different maps). However the behaviour of both "d"s is clear, so we will continue with this abuse of notation.

Remark 8.10. The exterior derivative increases the degree of a k-form by 1 (the k-form becomes a $(k+1)$-form). Can we get backwards, which is, can we decrease the degree of a k-form? Answer: yes.

Definition 8.11 (Interior derivative). $z \in \mathfrak{X}(\mathbb{R}^n)$ (i.e. z is a vector field), then we define the *interior derivative* i_z (acting on differential k-forms) as:

$$i_z: \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n) \tag{8.17}$$
$$\omega \mapsto i_z(\omega) \equiv i_z\,\omega$$

where

$$i_z\,\omega(v_1, \ldots, v_{k-1}) = \omega(z, v_1, \ldots, v_{k-1}), \forall\, v_i \in \mathfrak{X}(\mathbb{R}^n)$$

$i_z\,\omega$ is also called the *contraction* of ω.

Example 8.12 (Some computations). In $\mathbb{R}^2$, e_x, e_y basis vectors (that can be seen as vector fields):

$$i_{e_x}(dx \wedge dy) = dy$$
$$i_{e_y}(dx \wedge dy) = -dx$$

Remark 8.13. In the definition 8.11 above, we used the algebraic definition of differential k-forms, i.e. definition 8.2

Now some properties for interior derivatives.

Proposition 8.14. $\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^p(\mathbb{R}^n), z \in \mathfrak{X}(\mathbb{R}^n)$, *then:*

- $i_z(\omega \wedge \eta) = (i_z\,\omega) \wedge \eta + (-1)^k \omega \wedge (i_z\,\eta)$

- $i_z^2 w = i_z(i_z \omega) = 0$

Remark 8.15. In the above proposition there is a little abuse of notation when we claimed $i_z(i_z \omega) = 0$, see also remark 8.9.

Now, let's talk about *pullbacks* and *pushforwards* for functions and k-forms.

Definition 8.16 (Pullback)**.** Let $f \colon U \to V$ (with $U, V \subseteq \mathbb{R}^n$) be a differentiable map. Let us suppose that $\dim(U) = \dim(V) = n$ (just for the sake of simplicity, since it is not necessary). Then the *pullback* of a k-form (from V) to U is the map:

$$f^* \colon \Omega^k(V) \to \Omega^k(U)$$
$$\omega \mapsto f^* w$$

such that

$$(f^*\omega)(p)(u_1, \ldots, u_k) = \omega(f(p))(df(u_1), \ldots, df(u_k)), \forall\, p \in \mathbb{R}^n, \forall\, u_i \in \mathfrak{X}(U)$$

Now, we want to give another name to the differential of a function.

Definition 8.17 (Pushforward)**.** Given $f \colon U \to V$ as before, we will also call the differential of f at $p \in \mathbb{R}^n$, i.e. $df_p = df(p)$, as the *pushforward* of f at p, and it will be denoted by the symbol $(f_*)_p$.

In our mind, we'll think of $df_p = (f_*)_p$, at least until this concept is generalized.

In particular, using the pullback definition above, we can write the pushforward map as:

$$df_p \equiv (f_*)_p \colon U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$$
$$v \mapsto (f_*)_p(v)$$

By definition of differential, $df_p(v) = v(f)$, where v is a vector tangent to $\mathbb{R}^n$ at p. Since vector are like directional derivatives, $v(f)$ is the directional derivative of f with respect to v (not evaluated at any point, for now!). In particular, if we apply the definition to a point $h(q)$, where $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ (**CHECK**) and $q \in \mathbb{R}^n$, we have:

$$(f_*)_p(v)(h)(q) = (f_*)_p(v)(h(q)) = v(h(f(q))) = v(h \circ f)(q) = v(f^* h)(q)$$

In the last passage, we used the pullback for a differentiable function, which is completely legal since we defined it for differentiable k-forms, and a differentiable function is just a 0-form.

Remark 8.18. Using the pushforward, we can define the pullback of a differential form using a different notation (i.e. using f_* instead of df):

$$(f^*\omega)(p)(u_1,\ldots,u_k) = \omega(f(p))(f_*(u_1),\ldots,f_*(u_k)), \forall\, p \in \mathbb{R}^n, \forall\, u_i \in \mathfrak{X}(U) \tag{8.18}$$

Now, some properties of the pullback.

Proposition 8.19. $g, f \in C^1(\mathbb{R}^n, \mathbb{R})$, $\omega, \varphi \in \Omega^k(\mathbb{R}^n)$, $h \colon \mathbb{R}^n \to \mathbb{R}$. *Then:*

1. $f^*(\omega + \varphi) = f^*(\omega) + f^*(\varphi)$

2. $f^*(h\omega) = f^*(h)f^*(\omega)$

3. $(f \circ g)^* = g^*(f^*(\omega))$

4. *If* $\varphi^1, \ldots, \varphi^k \in \Omega^1(\mathbb{R}^n)$, *then* $f^*(\varphi^1 \wedge \ldots \wedge \varphi^k) = f^*(\varphi^1) \wedge \ldots \wedge f^*(\varphi^k)$

5. $df^*(\omega) = f^*(d\omega)$

From property (4) also follows that $f^*(\omega \wedge \phi) = (f^*\omega) \wedge (f^*\phi)$

Remark 8.20. We can express the pullback of a differential form in the following way:

$$(f^*\omega)(p) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (f^* a_{i_1,\ldots i_k}(p)) f^* dy^{i_1} \wedge f^* dy^{i_2} \wedge \cdots \wedge f^* dy^{i_k} =$$
$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1,\ldots,i_k}(f(p)) df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k}$$

where $f^i = y^i(f)$. We used properties (2) and (4) of proposition 8.19

Remark 8.21. From our definition of pullback, it is not necessary that f_* is invertible.

9   Integration of differential forms

Let ω be a differential form of degree n in $\mathbb{R}^n$. Then ω is necessarily of the form

$$\omega = \underbrace{a(p)}_{\text{it's a function}} dx^1 \wedge \ldots \wedge dx^n \tag{9.1}$$

Such a form can be integrated:

$$\int_{f(D)} \omega = \int_D f^* \omega \tag{9.2}$$

10 More on Vector Fields

Definition 10.1 (Tangent space)**.** $U \subset \mathbb{R}^n, U$ is an open set. $p \in U$, then the set of all derivations of $C^\infty(U)$ (cf. def 5.1) is called tangent space to U at p and is denoted by T_pU. An element of T_pU is called a tangent vector at p, and it is often denoted by v_p.

Definition 10.2 (Tangent bundle)**.** The tangent bundle over an open subset $U \subset \mathbb{R}^n$ is defined as

$$TU \equiv \bigsqcup_{p \in U} T_pU \tag{10.1}$$

where T_pU is the tangent space of U at p. Every element of the disjoint union is represented by an ordered pair (v, p) where $p \in U, v \in T_pU$. The tangent bundle comes equipped with the projection map

$$\mathrm{pr}\colon TU \to U \tag{10.2}$$
$$(p, v) \mapsto p$$

So, every element of the tangent bundle is a couple made of a tangent space to a point, and the point itself.

Remark 10.3. In the previous definition, the "$\sqcup$" symbol denotes a disjoint union. "Disjoint" here means that, if we consider the disjoint union of two elements x and y such that $x = y$, the union is the set $\{x, y\}$ and not $\{x\} = \{y\}$ as in normal unions. The mathematical operator doesn't know if two elements are equal. Since we are not mathematical operators, we can enumerate the elements like: $\{(1, x), (2, y)\} = \{(1, x), (2, x)\} = \{(1, y), (2, y)\}$ in order to distinguish them.

Definition 10.4 (Alternative definition of vector field)**.** A smooth vector field v on $U \subset \mathbb{R}^n, U$ open, is a smooth map

$$v\colon U \to TU \tag{10.3}$$

such that $\mathrm{pr}(v_p) = p, \forall\, p \in U$

Remark 10.5 (Space of sections). The set of all vector fields $\mathfrak{X}(U) \equiv \{C^\infty(U, TU) \mid \mathrm{pr}(v_p) = p\}$ is also called the space of sections in TU.

Definition 10.6 (Cotangent bundle)**.** By duality we define

$$T^*U \equiv \underset{p \in U}{\sqcup} T_p^* U \qquad (10.4)$$

as the cotangent bundle. Where T_p^*, the dual space of the tangent space, is called cotangent space. We also associate a projection $\mathrm{pr}\colon T^*U \to U$ with it.

Now, let's talk about about Lie algebras.

Definition 10.7 (Lie algebra)**.** A Lie algebra $(V, [\cdot, \cdot])$ is a vector space V over $\mathbb{R}$ endowed with a map

$$[\cdot, \cdot]\colon V \times V \to V$$

with the following properties:

- $[\cdot, \cdot]$ is bilinear

- $[\cdot, \cdot]$ is antisymmetric $([u, v] = -[v, u], \forall\, u, v \in V)$

- $[\cdot, \cdot]$ satisfies the *Jacobi identity*:

$$[[u, v], z] + [[z, u], v] + [[v, z], u] = 0$$

Remark 10.8 (Jacobi). How to remember Jacobi identity: remember $[[u, v], z]$ and then permute cyclically.

Proposition 10.9. *$\mathfrak{X}(\mathbb{R}^n)$ is an (infinite dimensional) Lie algebra with $[u, v](f) = u(v(f)) - v(u(f))$, for $u, v \in \mathfrak{X}(\mathbb{R}^n), f \in C^\infty(\mathbb{R}^n)$. (Note that u and v are vector fields and $[u, v]$ is still a vector field).*

Definition 10.10 (Integral curve)**.** An integral curve for a vector field v is a smooth curve $\phi\colon (a, b) \to \mathbb{R}^n$ satisfying $\dot{\phi}(t) = v_{\phi(t)}$ ($v_{\phi(t)}$ is the vector tangent at $\phi(t)$ for t fixed, remember the previous notation!). Let us suppose $0 \in (a, b)$. Then, $\phi(0)$ is called the starting point of ϕ.

We can also visualize the family of integral curves in the following way.

Definition 10.11 (Flow). The map

$$\theta \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \tag{10.5}$$
$$(t, p) \mapsto \theta_t(p)$$

such that $\dot{\theta}_t(p) = v_{\theta_t(p)}$ is the flow of the vector field v where, if we fix p, $\theta_t(p)$ is the integral curve which passes through p at $t = 0 \in (a, b)$. So the flow satisfies two conditions:

$$\dot{\theta}_t(p) = v_{\theta_t(p)}, \qquad\qquad \forall\, p \in \mathbb{R}^n \tag{10.6}$$
$$\theta_0(p) = p, \qquad\qquad \forall\, p \in \mathbb{R}^n \tag{10.7}$$

Under the right hypothesis (e.g. Lipschitz hypothesis and smoothness of v) we can prove existence and uniqueness of the solution of such ODEs ($\forall p \in \mathbb{R}^n$).

By fixing either the time or the starting point of the flow, we can consider two maps:

- $p \mapsto \theta_t(p)$, for each fixed t (we are observing several integral curves at the same time t)

- $t \mapsto \theta_t(p)$, for each fixed p (we are observing the integral curve starting from p, for all times)

Definition 10.12 (Lie derivative). Let $z \in \mathfrak{X}(\mathbb{R}^n)$ be a differentiable vector field, ϕ_t its flow and $\omega \in \Omega^k(\mathbb{R}^n)$. Then the Lie derivative of ω is defined as

$$L_z\omega = \frac{d}{dt}(\phi_t^*\omega)|_{t=0} \tag{10.8}$$

Remark 10.13. We denoted the flow by the symbol ϕ_t and not ϕ. What we are doing here is not caring about p: $\phi_t^*\omega(\cdot) = \omega(\phi_t(\cdot))$ Useful formula: $L_z\omega = (\mathrm{d}\,\mathrm{i}_z + \mathrm{i}_z\,\mathrm{d})\omega$
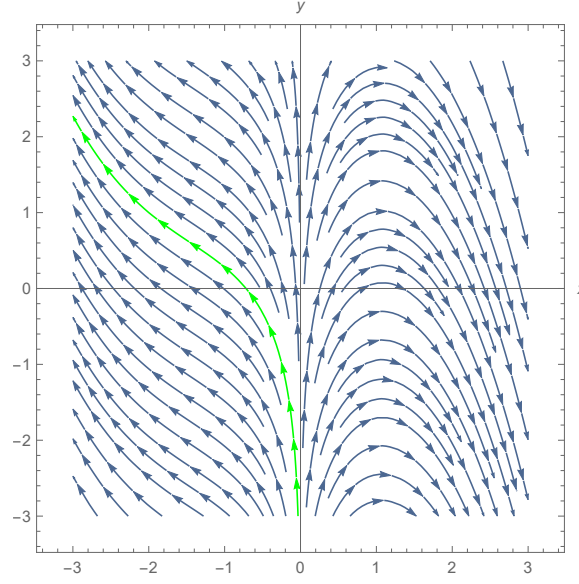
Figure 2: the map $t \mapsto \theta_t(p)$ selects just one integral curve

11 Lie derivative of a vector field

Definition 11.1 (pullback of a vector field)**.** Let φ be a diffeomorphism of $\mathbb{R}^n$ (i.e. a differentiable and invertible map from $\mathbb{R}^n$ to $\mathbb{R}^n$, such that its inverse is differentiable as well). Let $v \in \mathfrak{X}(\mathbb{R}^n)$. Then:

$$\varphi^* v \equiv \varphi_*^{-1} v \tag{11.1}$$

is the pullback of v with φ. In particular, given a flow ϕ and t fixed, $\phi(t, \cdot) = \phi_t(\cdot)$ is a diffeomorphism on $\mathbb{R}^n$ with $\phi^{-1} = \phi(-t, \cdot) = \phi_{-t}(\cdot)$.

Definition 11.2 (Lie derivative of a vector field)**.** Let $u, v \in \mathfrak{X}(\mathbb{R}^n)$. The Lie derivative of v in direction u is

$$L_u v \equiv \frac{d}{dt}(\phi_t^* v)|_{t=0} \tag{11.2}$$

(Remember: $\phi \colon (t, p) \mapsto \phi(t, p), \phi_t^* v \colon (t, p) \mapsto v(\phi(t, p))$).

Lemma 11.3. *Let u, v be smooth vector fields on $\mathbb{R}^n$ and $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$. Let ϕ_t be the flow of u and let ψ_s be the flow of v. Then*

- $\varphi^* v = \frac{d}{ds}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi$

- $\varphi^* v = v \Leftrightarrow \varphi \circ \psi_s = \psi_s \circ \varphi$ *for all s.*

27

- $L_u v = 0 \Leftrightarrow \phi_t \circ \psi_s = \psi_s \circ \phi_t$ *for all* s, t

Lemma 11.4. *Let u, v be smooth vector fields on $\mathbb{R}^n$ and and let ϕ_t (respectively ψ_s) be the flow of u (respectively v). Then:*

- $L_u v = \frac{\partial^2}{\partial s \partial t} \phi_{-t} \circ \psi_s \circ \phi_t |_{t=0, s=0}$

- $(L_u v)(f) = [u, v](f) = u(v(f)) - v(u(f))$ *for all smooth functions f on* $\mathbb{R}^n$

Lemma 11.5. *Let u, v be smooth vector fields on $\mathbb{R}^n$ and $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$. Then:*

1. $[u, v]$ *is* $\mathbb{R}-$*bilinear (i.e. bilinear for a parameter $\lambda \in \mathbb{R}$)*

2. $[u, v] = -[v, u]$

3. *The Jacobi identity holds*

4. $[u, fv] = f[u, v] + u(f)v$

5. $\varphi_*[u, v] = [\varphi_* v, \varphi_* u]$

12   Stokes' Theorem on $\mathbb{R}^n$

- For a function f (i.e. a 0-form) on $[a, b] \subset \mathbb{R}$ we have $\int_a^b df = \int_a^b \partial_x f dx = f(b) - f(a)$ (fundamental theorem of calculus).

- for $\omega = a_i dx^i$, a 1-form on $U = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ we have:

$$\int_S d\omega = \int_S (\partial_{x^1} a_2) dx^1 \wedge dx^2 + (\partial_x^2 a_1) dx^2 \wedge dx^1 = \int_{\partial S} w$$

  where we used the fundamental theorem of calculus.

- More generally, if S is a compact subset of $\mathbb{R}^2$ with piecewise regular boundary ∂S (piecewise homeomorphic to intervals in $\mathbb{R}$) then we obtain by decomposing S in terms of little squares and interpreting $\int_U d\omega$ as a Riemann sum over the square the result

$$\int_U d\omega = \int_{\partial U} \omega$$

  This result generalizes immediately to compact co-dimension zero subsets of $\mathbb{R}^n$ with piecewise regulary boundary

- If M is a compact subset of dimension $m \leq n$ in $\mathbb{R}^n$ (with piecewise regular boundary ∂M), diffeomorphic to a compact subset of $U \subset \mathbb{R}^m$ (i.e. $M = f(U)$) and $\omega \in \Omega^{m-1}(M)$, then:

$$\int_M d\omega = \int_U f^* d\omega = \int_U df^* \omega = \int_{\partial U} f^* \omega = \int_{\partial M} \omega$$

- More generally, the parametrization of ∂M may be different from that induced by M. Then we have:

$$\int_M d\omega = \int_{\partial M} i^* \omega$$

  where $i \colon \partial M \to M$ is the inclusion map of ∂M into M.

So, the most general result that we achieved is the following:

Theorem 12.1 (Stokes). $\omega \in \Omega^{m-1}(\mathbb{R}^n)$. *Let M be a closed compact subset of $\mathbb{R}^n$, $dim(M) = m \leq n$, such that M is homeomorphic to a closed subset $U \subset \mathbb{R}^m$. ∂M is the boundary of M and $i \colon \partial M \to M$ is the inclusion map of ∂M into M. Then:*

$$\int_{\partial M} i^* \omega = \int_M d\omega \tag{12.1}$$

Corollary 12.2 (Fundamental theorem of line integrals)**.** *Let f be a smooth function defined near an oriented curve C in $\mathbb{R}^n$, with endpoints A and B. Then:*

$$\int df = \int \nabla f \cdot dx = f(B) - f(A) \tag{12.2}$$

Corollary 12.3 (Curl theorem or Classical Stokes theorem)**.** *Let v be a differentiable vector field defined near a surface $S \subset \mathbb{R}^3$ with boundary ∂S.*

$$\int_S n \cdot (\nabla \times v) dS = \int_{\partial S} v \cdot dx \tag{12.3}$$

where n is the normal vector on the surface at each point.

Corollary 12.4 (Divergence theorem)**.** *For a smooth vector field v defined on a solid $T \subset \mathbb{R}^3$ with boundary ∂T:*

$$\int_T \nabla \cdot v \, dV = \int_{\partial T} v \cdot n \, dS \tag{12.4}$$

where dV is the unoriented volume element.

13 Poincarè Theorem of 1-forms

Definition 13.1 (Closed and exact forms)**.** If $\omega \in \Omega^k(U)$ such that $\mathrm{d}\,\omega = 0$, then ω is closed. If there exists $\alpha \in \Omega^{k-1}(V), V \subset U$ such that $\omega = \mathrm{d}\,\alpha$ in V then ω is exact.

Proposition 13.2. *The following are equivalent:*

1. *$\omega \in \Omega^1(U)$ is exact in a connected open subset $V \subset U$*

2. *For any curve $\gamma\colon (a,b) \to U, \int_\gamma \omega$ depends only on the endpoints $\gamma(a)$ and $\gamma(b)$.*

3. *$\int_\gamma \omega = 0$ for any closed curve γ in V*

Remark 13.3 (A closed form is not always exact)*.* If ω is exact, then it is closed (because $d^2 = 0$). But not every closed form in $\Omega^1(U), U$ open subset of $\mathbb{R}^n$ is exact. Cf. $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ in $\mathbb{R}^2$ minus the non-negative x-axis. If γ is a closed curve around the origin of $\mathbb{R}^2$, we have:

$$\int_\gamma \omega = \int_\gamma d\theta = 2\pi$$

and therefore ω cannot be exact by the previous proposition. However, we notice that we have problems only with the origin of $\mathbb{R}^2$. If we consider a subset of $\mathbb{R}^2$ which is enough far from the origin, the form would be an exact form in such subset. Indeed, we say that ω is locally exact, and the general result follows from the next theorem.

Theorem 13.4 (Poincarè theorem for 1-forms on $\mathbb{R}^n$)**.** *Let $\omega \in \Omega^1(U), U \subset \mathbb{R}^n, U$ open. Then $d\omega = 0$ if and only if for each $p \in U$ there is a neighbourhood $V \subset U$ of p and a differentiable function $f\colon V \to \mathbb{R}$ such that $\omega = df$.*

Remark 13.5*.* Using the Poincarè theorem for 1-forms, we can extend the definition of the integral of a closed 1-form along a **continuous** path (until now, we have always assumed the our paths were piecewise differentiable). In fact, assume that $\omega \in \Omega^1(U), \mathrm{d}\,\omega = 0$, and γ such that:

$$\gamma\colon [0,1] \to U$$

is a differentiable map. Now, we choose a partition of $[0,1]$, i.e. a collection of points $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ such that the restriction of γ to

the interval (t_i, t_{i+1}) is contained in a ball B_i where ω is exact. In particular:

$$\omega = df_i, \text{ for } f_i \colon B_i \to \mathbb{R}$$

Then:

$$\int_\gamma \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)]$$

If γ is only continuous, we could still consider such a partition, and still define

$$\int_\gamma \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)]$$

The integral of γ is well defined because the definition is independent from the choice of our partition: if P is one partition and P' is a refinement of P (i.e. it is the same partition plus an extra point $t' \in (t_i, t_{i+1})$ for some i), then:

$$[f_i(t_{i+1}) - f_i(t')] + [f_i(t') - f_i(t_i)] = [f_i(t_{i+1}) - f_i(t_i)]$$

Then the integral does not change if we consider a refinement. If we consider a general partition P', we can add every point of the partition P to P', so that we get a refinement of P' that we will be called P''. The integral on the partition P' has the same value of the integral on the partition P'' by the above argument. Now, we can add every point of the partition P' to P, so to get the partition P'' again, but now we can see P'' as a refinement of P. Then the integral on P and on P'' are the same. Then also the integrals on P and P' are the same.

Now, we want to extend the above theorem to k-forms.

Definition 13.6 (Contractible set)**.** An open subset $U \subset \mathbb{R}^n$ is contractible to some point $p_0 \in U$ if there exists a differentiable map

$$H \colon U \times R \to U \tag{13.1}$$
$$(p, t) \mapsto H(p, t)$$

such that $H(p, 1) = p, H(p, 0) = p_0, \forall\, p \in U$

Remark 13.7. To every $\omega \in \Omega^k(U)$ we can associate a k-form $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ defined as

$$\bar{\omega} = H^*\omega \tag{13.2}$$

On the other hand, any $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ has a unique decomposition of the form

$$\bar{\omega} = \omega_1 + dt \wedge \eta \qquad (13.3)$$

with $i_{\partial_t}\omega_1 = 0$ and $i_{\partial_t}\eta_1 = 0$. Conversely, we can associate a k-form $\omega \in \Omega^k(U)$ to each $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$ with the help of the inclusion map

$$i_t : U \to U \times \mathbb{R} \qquad (13.4)$$
$$p \mapsto i_t(p) = (p, t)$$

Then, $i_t^*\bar{\omega} \in \Omega^k(U)$ if $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$

Furthermore, let's define the map

$$I : \Omega^k(U \times \mathbb{R}) \to \Omega^{k-1}(U) \qquad (13.5)$$
$$\eta \mapsto I\eta$$

such that

$$(I\eta)(z_1, \ldots, z_{k-1}) = \int_0^1 \eta(p, t)(\partial_t, i_{t*}z_1, \ldots, i_{t*}z_{k-1})dt$$

Proof. Let's choose coordinates $\{x^1, \ldots, x^n, t\}$ for $U \times \mathbb{R}$. Then we write $\bar{\omega}$ on the basis:

$$\bar{\omega} = \sum_{i_1 < \ldots < i_k} a_{i_1 \cdots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \sum_{i_1 < \ldots < i_{k-1}} b_{i_1 \ldots i_{k-1}} dt \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$$

(We can always do this, the coefficients could also be trivial). Now, we want to integrate this form. Let:

$$i_t : U \to U \times \mathbb{R}$$
$$p \mapsto (p, t)$$

i_t is the inclusion map (it "includes" U into $U \times \mathbb{R}$ at t)...TO DO $\qquad \square$

Lemma 13.8.

$$i_1^*\bar{\omega} - i_0^*\bar{\omega} = d(I\bar{\omega}) + I(d\bar{\omega}) \qquad (13.6)$$

Indeed, since $H \circ i_1 = Id$ and $H \circ i_0 = p_0$, $\forall p \in U$ we have

$$\omega = (H \circ i_1)^*\omega = i_1^*\bar{\omega}$$

and

$$0 = (H \circ i_0)^*\omega = i_0^*\bar{\omega}$$

Then we can extend Poincarè lemma to k-forms:

Theorem 13.9. *Let U be a contractible, open subset of $\mathbb{R}^n$ and $\omega \in \Omega^k(U)$ with $d\omega = 0$. Then there exists a $(k-1)$-form $\alpha \in \Omega^{k-1}(U)$ such that $\omega = d\alpha$.*

Question: For $\omega \in \Omega^1(U)$, when is $\int_\gamma \omega$ independent of the choice of γ?

Definition 13.10 (Homotopy between curves). Two continuous curves γ_1 and γ_2, $\gamma_i \colon [a,b] \to U, i = 1, 2, U \subset \mathbb{R}^n$ are freely homotopic if there exists a continuous map H

$$H \colon [a,b] \times [0,1] \to U \qquad \text{such that:}$$
$$H(s,0) = \gamma_1(s), \qquad \forall \, s \in [a,b]$$
$$H(s,1) = \gamma_2(s), \qquad \forall \, s \in [a,b]$$

Definition 13.11 (Homotopy between closed curves with same endpoints). Two continuous curves γ_1 and γ_2, $\gamma_i \colon [a,b] \to U, i = 1, 2, U \subset \mathbb{R}^n$, with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$ are homotopic relatively to $\{\gamma_1(a), \gamma_2(b)\}$ if there exists a continuous map H

$$H \colon [a,b] \times [0,1] \to U \qquad \text{such that:}$$
$$H(s,0) = \gamma_1(s), \qquad \forall \, s \in [a,b]$$
$$H(s,1) = \gamma_2(s), \qquad \forall \, s \in [a,b]$$
$$H(a,t) = \gamma_1(a) = \gamma_2(a), \qquad \forall \, t \in [0,1]$$
$$H(b,t) = \gamma_1(b) = \gamma_2(b), \qquad \forall \, t \in [0,1]$$

Theorem 13.12. *Let $\omega \in \Omega^1(U)$, with $\mathrm{d}\,\omega = 0$ (closed), and γ_1, γ_2 be two homotopic curves (as in the previous definition). then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{13.7}$$

What if $\gamma_1(a) \neq \gamma_2(a), \gamma_1(b) \neq \gamma_2(b)$

Definition 13.13 (Homotopy between closed curves). $\gamma_1, \gamma_2 \colon [a,b] \to U$, γ_i closed curves, are freely homotopic if there exists a continuous map

$$H \colon [a,b] \times [0,1] \to U \qquad \text{such that:}$$
$$H(s,0) = \gamma_1(s) \qquad \forall \, s \in [a,b]$$
$$H(s,1) = \gamma_2(s) \qquad \forall \, s \in [a,b]$$
$$H(a,t) = H(b,t) \qquad \forall \, t \in [0,1]$$

Proposition 13.14. *If ω is a closed 1-form on U, γ_1 and γ_2 two closed curves, freely homotopic in U, then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{13.8}$$

In particular, if γ_1 is freely homotopic to a point, then $\int_{\gamma_1} \omega = 0$

Definition 13.15 (Simply connected set)**.** A connected open set $U \subset \mathbb{R}^n$ is simply connected if every continuous closed curve in U is freely homotopic to a point in U.

Example 13.16. $\mathbb{R}^n$, the unitary ball in $\mathbb{R}^n$ and its homeomorphic images are simply connected

Remark 13.17 (Contractible vs. simply connected). "Contractible $\Longrightarrow$ simply connected" (why?), but "Simply connected $\not\Longrightarrow$ contractible" (cf. S^2).

Remark 13.18. Every closed form on a simply connected subset U of $\mathbb{R}^n$ is exact.

Lemma 13.19. *A connected open subset U of $\mathbb{R}^n$ is simply connected if every closed curve in U is homotopic to a point in U.*

We can limit ourselves to consider continuous curves, thanks to the two following results:

Theorem 13.20 (Whitney approximation on $\mathbb{R}^n$)**.** *If γ is a continuous map between $U, V \subsetneq \mathbb{R}^n$, then γ is homotopic to a smooth map $\tilde{\gamma}$. If γ is smooth on a closed subset A of U, then the homotopy can be taken relatively to A.*

Theorem 13.21. *If γ_1 and γ_2 are homotopic maps between U and V then they are smoothly homotopic.*

14 de Rham Cohomology

We can think of $\Omega^k(U)$ as a vector space over $\mathbb{R}$

Remark 14.1. We say that $\Omega^k(U, \mathbb{Z})$ forms a group (and not a vector space) since $\mathbb{Z}$ is not a field. In contrast, $\Omega^k(U, \mathbb{R})$ is a vector space.

Definition 14.2. Let $U \subset \mathbb{R}^n$, U open, $\dim(U) = m \leq n$. Then:

- The set of closed k-forms is the k-th *cocycle group* $Z^k(U, \mathbb{R})$ (it is a group with respect to addition).

- The set of exact k-forms is the k-th *coboundary group* $B^k(u, \mathbb{R})$

- The k-th *de Rham cohomology group* $H^k(U, \mathbb{R})$ is defined as:

$$H^k(U, \mathbb{R}) = Z^k(U, \mathbb{R}) \Big/ B^k(U, \mathbb{R})$$

Remark 14.3. $H^k(U, \mathbb{R})$ contains the closed k-forms defined on U which are not exact. See also the example 4.7.

Example 14.4. Let's consider the following examples:

- If U is contractible, then $H^k(U, \mathbb{R}) = \{0\}$ by Poincarè lemma.

- $U = \mathbb{R}^2 \setminus \{0\}$, $H^0(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \mathbb{R}$ (constant functions)

- $H^1(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \mathbb{R}$

- $H^2(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = \{0\}$

- Torus T: $H^0(T) = \mathbb{R}$, $H^1(T) = \mathbb{R} \oplus \mathbb{R}$, $H^2(T) = \mathbb{R}$

Definition 14.5. $M \subset \mathbb{R}^n, \Omega^*(M, \mathbb{R}) = \oplus_{k=0}^n \Omega^k(M, \mathbb{R})$

Remark 14.6. $\wedge \colon \Omega^* \times \Omega^* \to \Omega^*$ endows Ω^* with the structure of a ring.

Definition 14.7 (de Rham Complex). The de Rham Complex is defined by $\Omega^*(M, \mathbb{R})$ together with the sequence

$$\xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} 0$$

with $Im(d_k) \subset Ker(d_{k+1})$ since $d^2 = d \circ d = 0$

Remark 14.8. $H^k(M) = Ker(d_{k+1}) \Big/ Im(d_k)$

Definition 14.9 (exact sequence). If $H^k(M, \mathbb{R}) = 0, k = 1, \ldots, n$, then

$$\xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \cdots$$

is an exact sequence if and only if $Ker(d_{k+1}) = Im(d_k)$

Remark 14.10. For $k = 0$, $\Omega^0 = \{\text{functions}\}$, then $H^0 = 0$

Definition 14.11 (Cohomology ring). The cohomology ring is defined as

$$H^*(M, \mathbb{R}) = \oplus_{k=0}^n H^k(M, \mathbb{R})$$

Example 14.12. $H^*(T^2, \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$, where T^2 is the torus.

Remark 14.13. If $\phi \colon U \to V$, U, V subsets of $\mathbb{R}^n$, ϕ diffeomorphism, then $\phi^* \colon \Omega^k(V) \to \Omega^k(U)$ is the pullback of ϕ on k forms. By the construction above, we can also define such pullback on H^K, as $\phi^* \colon H^k(V) \to H^k(U)$, which maps closed-but-not-exact forms to closed-but-not-exact forms. If ϕ is a diffeomorphism, then ϕ^* is an isomorphism. Thus, $H^k(V) \cong H^k(U)$. Question: what if U and V are not diffeomorphic, but only homotopic equivalent? We want to show that the final result is the same.

Definition 14.14 (Homotopy between maps). Two maps $\phi, \psi \colon U \to V$ are homotopic if there exists a continuous map H

$$H \colon [0, 1] \times U \to V \qquad \text{such that:}$$
$$H(0, \cdot) \colon U \to V \qquad \text{with } H(0, \cdot) = \phi(\cdot)$$
$$H(1, \cdot) \colon U \to V \qquad \text{with } H(1, \cdot) = \psi(\cdot)$$

Definition 14.15 (Homotopy equivalence between sets). Two subsets U and V of $\mathbb{R}^n$ are homotopic equivalent if there exist continuous maps $f \colon U \to V$ and $g \colon V \to U$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity in U and V, respectively. f and g are called homotopy equivalences.

Example 14.16. Let's consider the following examples:

- Any homeomorphism $\phi \colon U \to V$ with homotopy inverse (i.e. inverse up to a homotopy) ϕ^{-1} is a homotopy equivalence, but the converse is not always true (a disk is homotopy equivalent to a point, but it's not homeomorphic to a point)

- A circle is homotopy equivalent to $\mathbb{R}^2 \setminus \{0\}$

- S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$

- A solid torus is homotopy equivalent to a tea cup

Lemma 14.17. *Let $\phi, id\colon V \to V$ be two smoothly homotopic maps. Then $\phi^*|_{H^*(V,\mathbb{R})} = id^*|_{H^*(V,\mathbb{R})}$*

Theorem 14.18. *Let $\phi\colon U \to V$ be a homotopy equivalence between U and V with homotopy inverse ψ. Then ϕ^* is an isomorphism between $H^k(U)$ and $H^k(V)$*

Theorem 14.19. *Let $\phi\colon U \to V$ be a homotopy equivalence between U and V with homotopy inverse $\psi\colon V \to U$. then ϕ^* induces an isomorphism $\hat{\phi}^*$ such that:*

$$H^n(V,\mathbb{R}) \cong H^n(U,\mathbb{R}), \forall\, n \tag{14.1}$$

Definition 14.20 ((co)chain map)**.** Let A^* and B^* be two (co)chain complexes (a (co)chain complex is a generalization of a de Rham complex, e.g. think about $A^* = \Omega^*$). A (co)chain map $\hat{\phi}^*\colon A^* \to B^*$ is a collection of maps $\phi^*\colon A^n \to B^n$ s.t. $d \circ \phi^* = \phi^* \circ d\colon A^n \to B^{n+1}$. We often denote $\hat{\phi}^*$ by ϕ^*.

Example 14.21. Let's consider:

$$A^* = \Omega^*(U,\mathbb{R}) = \Omega^0(U,\mathbb{R}) \oplus \Omega^1(U,\mathbb{R}) \oplus \ldots$$
$$B^* = \Omega^*(V,\mathbb{R}) = \Omega^0(V,\mathbb{R}) \oplus \Omega^1(V,\mathbb{R}) \oplus \ldots$$

And we consider $\hat{\phi}^*\colon A^* \to B^*$, $\phi^*\colon \Omega^0(U,\mathbb{R}) \to \Omega^0(V,\mathbb{R})$, $\phi^*\colon \Omega^1(U,\mathbb{R}) \to \Omega^1(V,\mathbb{R})$, etc.

Definition 14.22. A short exact sequence (SES) is a collection of (co)chain complexes A^*, B^*, C^* and (co)chain maps $\phi^*\colon A^n \to B^n, \psi^*\colon B^n \to C^n$ such that for each n:

$$0 \longrightarrow A^n \xrightarrow{\phi^*} B^n \xrightarrow{\psi^*} C^n \longrightarrow 0 \tag{14.2}$$

is exact

Remark 14.23. Remember the definition of exact sequence: it gives a condition on the kernel and the range of the maps (cf. definition 14.9). By this condition, we have that ϕ^* must be an injective map, and ψ^* mjust be a surjective map in the above definition.

Example 14.24. Let's consider the following example:

$$\Omega^0(U) \xrightarrow{\;d\;} \Omega^1(U) \xrightarrow{\;d\;} \Omega^2(U) \longrightarrow \ldots \qquad \} \; sequence$$

$$\phi^* \Big\downarrow \qquad\qquad \phi^* \Big\downarrow \qquad\qquad \phi^* \Big\downarrow$$

$$\Omega^0(V) \xrightarrow{\;d\;} \Omega^1(V) \xrightarrow{\;d\;} \Omega^2(V) \longrightarrow \ldots \qquad \} \; sequence$$

$$\psi^* \Big\downarrow \qquad\qquad \psi^* \Big\downarrow \qquad\qquad \psi^* \Big\downarrow$$

$$\Omega^0(W) \xrightarrow{\;d\;} \Omega^1(W) \xrightarrow{\;d\;} \Omega^2(W) \longrightarrow \ldots \qquad \} \; sequence$$

$$\underbrace{\qquad}_{sequence} \qquad \underbrace{\qquad}_{sequence} \qquad \underbrace{\qquad}_{sequence}$$

Lemma 14.25 (zig-zag lemma). ϕ^*, ψ^* *as in the previous definitions. Then there exists a linear map δ such that:*

$$\ldots \longrightarrow H^{n-1}(C^*) \xrightarrow{\;\delta\;} H^n(A^*) \xrightarrow{\;\phi^*\;} H^n(B^*) \xrightarrow{\;\psi^*\;}$$

$$\xrightarrow{\;\psi^*\;} H^n(C^*) \xrightarrow{\;\delta\;} H^{n+1}(A^*) \xrightarrow{\;\phi^*\;} \ldots$$

is an exact sequence.

Example 14.26. See also example 14.24.

$$H^0(U) \xrightarrow{\;d\;} H^1(U) \xrightarrow{\;d\;} H^2(U) \longrightarrow \ldots$$

$$\phi^* \Big\downarrow \qquad\qquad \phi^* \Big\downarrow \qquad\qquad \phi^* \Big\downarrow$$

$$H^0(V) \xrightarrow{\;d\;} H^1(V) \xrightarrow{\;d\;} H^2(V) \longrightarrow \ldots$$

$$\psi^* \Big\downarrow \qquad\qquad \psi^* \Big\downarrow \qquad\qquad \psi^* \Big\downarrow$$

$$H^0(W) \xrightarrow{\;d\;} H^1(W) \xrightarrow{\;d\;} H^2(W) \longrightarrow \ldots$$

Theorem 14.27 (Mayer-Vietoris). *Let $M \subset \mathbb{R}^n$ such that $M = f(U) \cup g(V), U, V$ open subsets in $\mathbb{R}^m, m \leq n$. f, g homeomorphisms. Let:*

$$i \colon U \cap V \to U \quad (inclusion)$$

$$j \colon U \cap V \to V \quad (inclusion)$$

Let:

$$(f^* \oplus g^*) \colon \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V)$$

$$\omega \mapsto (f^*(\omega), g^*(\omega))$$

$$(i^* - j^*) \colon \Omega^k(U) \oplus \Omega^k(V) \to \Omega(U \cap V)$$

$$(\omega, \eta) \mapsto i^*\omega - j^*\eta$$

39

Then, for each k there exists a linear map δ such that:

$$\ldots \longrightarrow H^k(M) \xrightarrow{f^*\oplus g^*} H^k(U) \oplus H^k(V) \xrightarrow{i^*-j^*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \longrightarrow \ldots$$

is exact.

15 Submanifolds of $\mathbb{R}^n$

Premise: We want to go outside $\mathbb{R}^n$, and analyze the so called manifolds. For now, we can think about them as generalizations of curves and surface of $\mathbb{R}^n$, a proper definition will come later. Note that what we studied until now can often be extended to manifolds: with sufficient conditions, manifolds can be "embedded" in $\mathbb{R}^n$. We will give four equivalent definitions of a submanifold M of dimensions m in $\mathbb{R}^n$ ($m \leq n$).

Definition 15.1 ((a) Submanifold - Local parametrization)**.** $\forall\, p \in M \subset \mathbb{R}^n, \exists$ a neighbourhood V, $p \in V$, $V \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m, U, V$ open, $m \leq n$, and a smooth map $\phi \colon U \to \mathbb{R}^n$ such that:

- $\phi \colon U \to M \cap V$ is a homeomorphism,

- $\phi_*(x) \colon \mathbb{R}^m \to \mathbb{R}^n$ is injective $\forall\, x \in U$

Definition 15.2 ((b) Submanifold - Locally flat)**.** $\forall\, p \in M, \exists$ a neighbourhood $V \subset \mathbb{R}^n, p \in V$ and a neighbourhood $W \subset \mathbb{R}^n$, $0 \in W$, and a diffeomorphism $\Phi, \Phi \colon W \to V$ such that

$$\Phi\left(W \cap (\mathbb{R}^m \times \{0\}^{n-m})\right) = V \cap M$$

Definition 15.3 ((c) Submanifold - Locally a zero set)**.** $\forall\, p \in M, \exists$ a neighbourhood $V \subset \mathbb{R}^n, V$ open, and a smooth map $F \colon V \to \mathbb{R}^{n-m}$ such that

$$V \cap M = \{x \in V \mid F(x) = 0\}$$

and $F_* \colon \mathbb{R}^n \to \mathbb{R}^{n-m}$ is surjective.

Definition 15.4 ((d) Submanifold - Locally a graph)**.** $\forall\, p \in M, \exists$ a neighbourhood $V \subset \mathbb{R}^n$ and a permutation $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ and $U \subset \mathbb{R}^m, U$ open, together with a smooth map $g \colon U \to \mathbb{R}^{n-m}$ such that

$$V \cap M = \{(x_{\sigma(1)} \cdots x_{\sigma(n)}) \mid (x_1, \ldots, x_m) \in U$$
$$\text{and } (x_{m+1}, \ldots, x_n) = g(x_1, \ldots, x_m)\}$$

(g is called a *graph*).

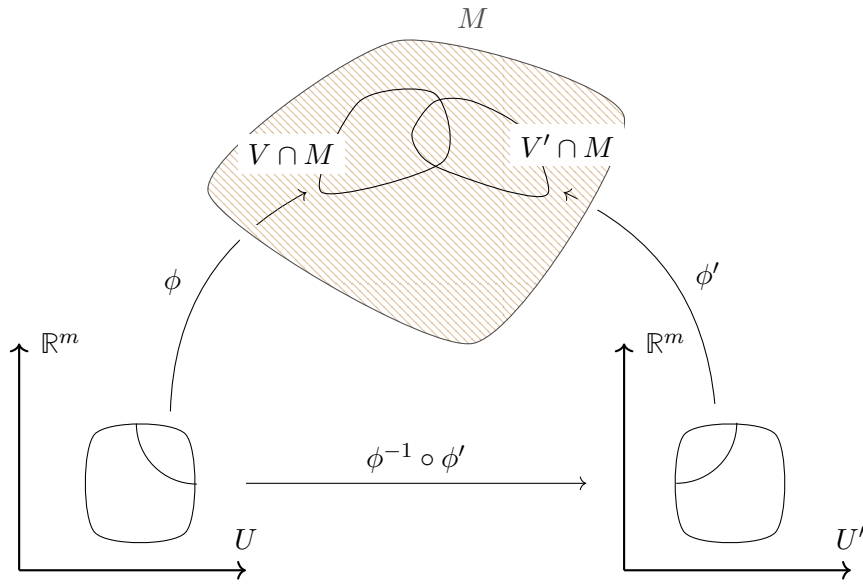**Theorem 15.5.** *The four definitions above are equivalent.*

Corollary 15.6. *Let:*

$$\phi \colon \mathbb{R}^m \supset U \to V \cap M$$

$$\phi' \colon \mathbb{R}^m \supset \to V' \cap M$$

be local parametrizations. then

$$\phi^{-1} \circ \phi' \colon (\phi')^{-1}(V \cap V' \cap M) \to \phi^{-1}(V \cap V' \cap M)$$

is a diffeomorphism.

Example 15.7 (Submanifolds of $\mathbb{R}^n$). Let's consider some examples.

1. An open subset of dimension n in $\mathbb{R}^n$ is a submanifold (by definition (a), we just take $\phi = id$). For instance $B^n \subset \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$.

2. $S^{n-1} = \partial B^n$ is a submanifold of $\mathbb{R}^n$. Indeed, by definition (c), it is the zero set of the function:

$$F(\underline{x}) = (x^1)^2 + (x^2)^2 + \ldots + (x^n)^2 - 1 = \underline{x}^2 - 1, \underline{x} \in \mathbb{R}^n$$

and F_* is surjective: $F_*(\underline{x}) = 2\underline{x}$

3. $O(n) = \{A \,|\, AA^t = E\} \subset Mat(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, with E identity matrix. $O(n)$ is a submanifold of dimension $\frac{n(n-1)}{2}$. Indeed, using definition

(c), it is the zero set of the function:

$$F\colon Mat(n \times n, \mathbb{R}) \to Symm(n, \mathbb{R})$$
$$A \mapsto AA^t - E$$

What is more, F_* is surjective. In order to prove that, we prove that
"$F_*|_A(X) = 0, \forall A \Rightarrow X = 0$" (then we know that for a linear map
L on a vector space V, $\dim(V) = \dim Ker(L) + \dim Ran(L)$, so the
dimension of the range of $F_*|_A$ must be $\dim Mat(n \times n, \mathbb{R})$, so the map
is surjective). In fact:

$$F_*|_A = \frac{d}{dt}\big|_{t=0} F(A + tX) = \frac{d}{dt}\big|_{t=0} \left[AA^t + tAX^t + tXA^t + t^2 XX^t\right] =$$
$$= AX^t + XA^t$$

And the solution of: $AX^t + XA^t = S \in Symm(n, \mathbb{R})$ is $X = \frac{SA}{2}$
because $\frac{AA^t}{2}S + S\frac{AA^t}{2} = S$. Then:

$$AX^t + XA^t = 0 \Longrightarrow X = 0$$

Now, we want to use diffeomorphisms like those in the corollary 15.6 in
order to introduce the concept of manifold.

Definition 15.8 (Atlas). An (n-dimensional, smooth) atlas $\mathcal{A}$ on a set M
is a collection of maps (called charts)

$$\phi_\alpha\colon \mathbb{R}^n \overset{\sim}{\to} M \qquad\qquad (15.1)$$
$$U_\alpha \mapsto W_\alpha$$

such that:

- $\cup_{\alpha \in I} W_\alpha = M$

- $\forall \alpha, \beta \in I$ with $W_\alpha \cap W_\beta \neq \emptyset$,

$$\phi_\beta^{-1} \circ \phi_\alpha\colon \phi_\alpha^{-1}(W_\alpha \cap W_\beta) \to \phi_\beta^{-1}(W_\alpha \cap W_\beta)$$

  is a diffeomorphism.

where $\alpha \in I, I$ index set, and the $\sim$ above the arrow means that ϕ_α is
bijective.

Definition 15.9 (Equivalence relation on atlases). Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are equivalent $\Leftrightarrow \mathcal{A} \cap \mathcal{A}'$ is an atlas ($\Leftrightarrow \phi_\beta^{-1} \circ \phi_\alpha'$ is a diffeomorphism, $\forall \phi_\alpha' \in \mathcal{A}', \phi_\beta \in \mathcal{A}$).

Definition 15.10 (**Preliminary** definition of manifold). A manifold is a set M with an equivalence class of atlases.

Remark 15.11. The definition 15.10 above is "*preliminary*" because we have not specified anything about the topology yet (we are still working on Euclidean topology). Moreover, in every equivalence class $\exists!$ maximal atlas (i.e. such that, if combined with an other atlas, it can't get any bigger). We also notice that, if the charts are smooth enough, we can pullback "everything" (e.g. all our vector fields, differential forms, etc. defined for subset of $\mathbb{R}^n$). However, in that case, everything is defined *locally*. In order to patch them all together, we need some other result (like the partition of unity, coming soon).

Example 15.12 (Projective space). $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ($\mathbb{K}$ is some field).

$$\mathbb{K}P^n \equiv \left(\mathbb{K}^{n+1} \setminus \{0\}\right)\big/_\sim$$

where

$$(x_0, \ldots, x_n) \sim (x_0', \ldots, x_n') \Longleftrightarrow$$
$$\exists \lambda \in \mathbb{K} \setminus \{0\} \text{ such that } (\lambda x_0, \ldots, \lambda x_n) = (x_0', \ldots, x_n')$$

For instance, $\mathbb{C}P^1 = S^2$ (every point on a line passing through the origin is identified with the point on such line at distance 1 from the origin). We use the notation $[x_0, \ldots, x_n]$ for one equivalence class. Let's consider the atlas $\mathcal{A} = \{\phi_i \colon \mathbb{K}^n \to \mathbb{K}P^n\}$. Where

$$\phi_i \colon \mathbb{K}^n \to \mathbb{K}P^n$$
$$(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \mapsto [x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n]$$

And $\phi_i(\mathbb{K}^n) = \{[x_0, \ldots, x_n] \in \mathbb{K}P^n \mid$ i-th entry is $\neq 0\}$. We notice that $\mathcal{A}$ is an atlas because ϕ_i satisfies the properties of the definition 15.8:

- ϕ is a bijection and its inverse is

$$\phi_i^{-1} \colon \phi_i(\mathbb{K}^n) \to \mathbb{K}^n$$
$$[x_0, \ldots, x_{i-1}, \underbrace{x_i}_{\neq 0}, \ldots, x_n] \mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

- ϕ is also a diffeomorphism, because $\phi_j^{-1} \circ \phi_i$, defined as

$$\phi_j^{-1} \circ \phi_i \colon \phi_i^{-1}\left(\phi_i(\mathbb{K}^n) \cap \phi_j(\mathbb{K}^n)\right) \to \phi_j^{-1}\left(\phi_i(\mathbb{K}^n) \cap \phi_j(\mathbb{K}^n)\right)$$
$$(x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{x_j}, \ldots, \underbrace{\frac{1}{x_j}}_{i-th}, \ldots, \frac{x_n}{x_j}\right)$$

  is a diffeomorphism (where $x_i \neq 0, x_j \neq 0$) because it is a differentiable map and the inverse is again differentiable.

Remark 15.13 (Extension of analysis to manifolds). The existence of an atlas allows us to define concepts from analysis to manifolds. For instance, we can define continuous and smooth functions on manifolds using the concepts of continuity and smoothness that we use in $\mathbb{R}^n$.

Definition 15.14 (Continuous/smooth function on a manifold). Let M be a manifold, a function $f \colon M \to \mathbb{R}$ is a continuous [smooth] function if and only if $f \circ \phi_\alpha \colon U_\alpha \to \mathbb{R}$ is continuous [smooth] function $\forall \, \alpha$. Such a function is well-defined, since the definition is independent on the local parametrization chosen, thanks to the definition of atlas.

Remark 15.15 (cut-off functions). We can construct functions on a manifold in the following way: we consider a function defined on an open set in M which is homeomorphic to an open set in $\mathbb{R}^n$. Then we extend it to zero outside such open set, but in order to have a differentiable function we smoothly bring it to zero using the so-called cut-off functions. For instance, a smooth function from $\mathbb{R}$ to $[0, 1] \subset \mathbb{R}$ such that:
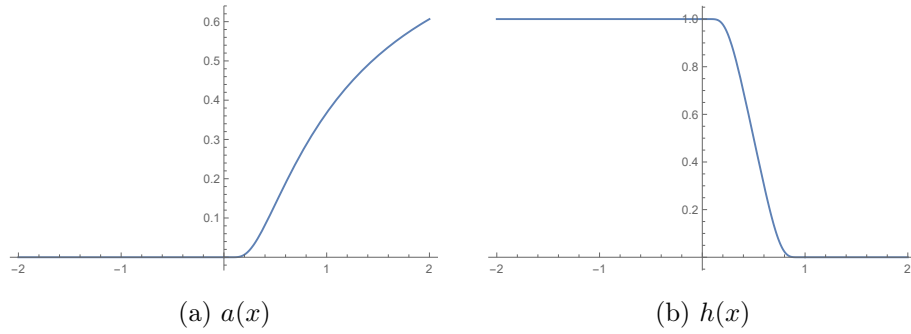
$$h(x) = \begin{cases} 1, & x \leq 1 \\ \text{anything}, & 1 \leq x \leq 2 \\ 0, & x \geq 2 \end{cases}$$

is a cut-off function. For example, given

$$a(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x \geq 0 \end{cases}$$

we can consider

$$h(x) = 1 - \frac{a(x)}{a(x) + a(1 - x)}$$
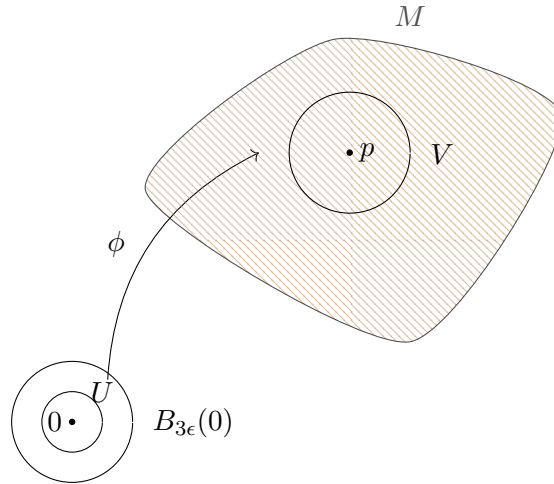
(a) $a(x)$

(b) $h(x)$

Then we use the function

$$h_\epsilon(x) \equiv h\left(\frac{x}{\epsilon}\right) = \begin{cases} 1, & x \leq \epsilon \\ 0, & x \geq 2\epsilon \end{cases}$$

Now, for $p \in M$, let

$$\phi\colon \mathbb{R}^n \supset U \to V \subset M$$

such that $\phi(0) = p$ (where $0 \in U$) and $\epsilon > 0$ such that $B_{3\epsilon} \supset U$.

Then $g(x) = h_\epsilon(|x|)$ defines a smooth function $g\colon \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{supp}(g) = \overline{\{x \mid g(x) \neq 0\}} \subset U$ and $g \circ \phi^{-1}\colon V \to \mathbb{R}$ extends (by zero) to a function $f \in \mathbb{F}(M)$ with $\mathrm{supp} f \subset V$ and $f \equiv 1$ in a neighbourhood of p. In particular, this shows that $\mathbb{F}(M)$ is infinite-dimensional.

Definition 15.16 (Vector fields on manifolds)**.** Vector fields are defined as before (we just replace $\mathbb{F}(\mathbb{R}^n)$ with $\mathbb{F}(M)$):

$$\mathrm{Der}\big|_p \mathbb{F}(M) = T_p M \ni v_p\colon \mathbb{F}(M) \to \mathbb{R}$$

46

And

$$\mathfrak{X}(M) = \operatorname{Der} \mathbb{F}(M) = TM \ni v \colon \mathbb{F}(M) \to \mathbb{F}(M)$$

$\mathbb{R}$-linear and satisfying Leibnitz rule.

Remark 15.17 (Representation of v in U_α). Everything is done in $U_\alpha \subset \mathbb{R}^n$ (chart). Representation of v in U_α:

$$v_{(\alpha)} = v^i_{(\alpha)} \partial_{x_i}$$

where

$$v^i_{(\alpha)}(x) = v^i \circ \phi_\alpha(x)$$

Notice that cut-off functions allow us to extend any smooth function on V_α (for some $\alpha \in I$) to all of M through extension by zero outside V_α. Suppose we are given a function on M. How do we decide if it is continuous (or smooth)?

Proposition 15.18 (Partition of unity). *Let M be a compact manifold and let $\{V_\alpha\}$ be a covering of M. Then there exists a family of differentiable functions $\varphi_1, \ldots, \varphi_m$ such that:*

- $\sum\limits_{i=1}^m \varphi_i \equiv 1$

- $0 \leq \varphi_i \leq 1$ *and* $supp(\varphi_i) \subset V_\alpha$ *for some* $\alpha \in I$

(without proof)

Remark 15.19. Definition of compactness for a manifold: coming soon.

Definition 15.20 (Partition of unity)**.** The family $\{\varphi_i\}$ defined above is said to be a partition of unity subordinate to the covering $\{V_\alpha\}$.

Now, if $f \colon M \to \mathbb{R}$, we can consider $f_i = \varphi_i f \colon v_\alpha \to \mathbb{R}$ for some $\alpha \in I$.

Definition 15.21 (Differential forms on M)**.**

$$\Omega^k(M) \ni \omega \colon \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathbb{F}(M) \tag{15.2}$$
$$(v_1, \ldots, v_k) \mapsto \omega(v_1, \ldots, v_k)$$

linear, skew-symmetric (i.e. alternating) as already defined.

Remark 15.22 (Representation of ω in U_α). Representation of ω in U_α:

$$\omega_{(\alpha)} = \sum_{i_1 < \cdots < i_k} (a_{i_1 \cdots i_k})_{(\alpha)} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

where $\{x^i\}$ are the coordinates on U_α. Moreover:

$$\omega_{(\alpha)}((v_1)_\alpha, \ldots, (v_k)_\alpha) = \pm \sum_{i_1 < \cdots < i_k} (a_{i_1 \cdots i_k})_\alpha (v^{i_1})_\alpha \cdots (v^{i_k})_\alpha$$

where the $\pm$ sign depends on the orientation used.

Definition 15.23 (Curve on a manifold). A curve $\gamma \colon (a, b) \to M$ is continuous [differentiable] if $\phi_\alpha^{-1} \circ \gamma \colon (a, b) \to U_\alpha$ is continuous [differentiable]

16 Integration of differential forms on a compact manifold

$\omega \in \Omega^n(M), M$ compact manifold $, \dim M = n$. Pick a partition of unity subordinate to a covering $\{V_\alpha\}$ (i.e. pick some maps $\varphi_i, i = 1, \ldots, n, 0 \leq \varphi_i \leq 1, \sum_{i=1}^m \varphi_i \equiv 1,$ supp$(\varphi_i) \subset V_{\alpha(i)}$ for some $\alpha = \alpha(i)$). Then:

$$\int_M \omega = \int_M \sum_{i=1}^m \varphi_i \omega = \sum_i \int_M (\varphi_i \omega) = \sum_i \int_{V_{\alpha(i)}} (\varphi_i \omega) =$$

$$= \sum_i \int_{U_{\alpha(i)}} \phi_{\alpha(i)}^*(\varphi_i \omega) \tag{16.1}$$

where $V_\alpha = \phi_\alpha(U_\alpha)$ Notice that the integral is well-defined if M is orientable (definition of orientable manifolds in the next pages).

Lemma 16.1. $\int_M \omega$ is independent of the choice (V_α, φ_i)

Remark 16.2. What happens with boundaries? If $p \in \partial M$, there is no neighbourhood of p homeomorphic to an open set $U \subset \mathbb{R}^n$. Let's define the following set:

$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^1 \geq 0\}$$

Note that on [1] H^n is defined as

$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

(everything is the same, but with x^1 replaced by x^n).

Definition 16.3 (subset topology in H^n)**.** An open set in H^n is the intersection between H^n and an open set in $\mathbb{R}^n$.

Definition 16.4 (Functions on H^n)**.** A function $f \colon V \to \mathbb{R}, V$ open, $V \subset H^n$ is differentiable if $\exists$ an open set $U, \mathbb{R}^n \supset U \supset V$ and a differentiable function $\bar{f} \colon U \to \mathbb{R}$ such that $\bar{f}|_V = f|_V$. Furthermore, $(f_*)_p = (\bar{f}_*)_p, p \in V$.

Definition 16.5 (**Preliminary** definition of diff. manifold with boundary)**.** An n-dimensional differentiable manifold with a regular boundary is a set M with an equivalence class of atlases, as usual, but with the difference that $\mathbb{R}^n$ in the definition 15.8 is replaced by H^n everywhere. The boundary is "regular" if it is described by a regular curve (no intersection, etc...)

When is a point on the boundary of a manifold?

Definition 16.6 (Point on the boundary). A point $p \in M$ is on the boundary of M if for some parametrisation $\phi \colon U \subset H^n \to M$ around p, we have $\phi(0, x^2, \ldots, x^n) = p$ for some $x^2, \ldots, x^n$.

Lemma 16.7. *The definition 16.6 does not depend on the choice of parametrisation.*

Definition 16.8 (Orientable manifold). M is orientable if there exists an atlas $\mathcal{A} = \{\phi_\alpha, U_\alpha\}$ such that for each pair α, β with $\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset$, the differential

$$(\phi_\beta^{-1} \circ \phi_\alpha)_* \colon U_\beta \to U_\alpha$$

has positive determinant. Example of a non-orientable manifold: Möbius strip.

Proposition 16.9. • *The boundary ∂M of a n-dimensional differentiable manifold with boundary is a $(n-1)$-dimensional differentiable manifold.*

• *The orientation on M induces an orientation on ∂M*

Theorem 16.10 (Stokes theorem on manifolds). *M orientable, then:*

$$\int_M d\omega = \int_{\partial M} i^*\omega \tag{16.2}$$

50

17   Abstract Manifolds and Topology

Remark 17.1 (Non-Example of a manifold)*.* The set

$$M = (-\infty, 0) \cup (0, \infty) \cup \{a, b\}$$

(endowed with a particular topology that we will see later) is not a manifold. Here, a, b are just two points (not necessary real numbers): just imagine M as a subset of $\mathbb{R}^2$, consisting of a real line without the origin plus any two points in $\mathbb{R}^2$. Even if we have not defined a topology on M, we can feel that it doesn't look right: there are two maps φ_a, φ_b identifying the subsets $U_a = M \setminus \{b\}$ and $U_b = M \setminus \{a\}$ with $\mathbb{R}$ (we are just calling the 0 element in other way). The transition function

$$\varphi_b \circ \varphi_a^{-1} \colon \varphi(U_a \cap U_b) = \mathbb{R} \setminus \{0\} \to \varphi_b(U_b \cap U_a) = \mathbb{R} \setminus \{0\}$$

is the identity, and is smooth. **But** a smooth function on M must have the same value when evaluated at a and b, and this does not look right (why?).

Definition 17.2 (Topology)**.** A topology on a set M is a subset $\mathcal{O} \subset \mathcal{P}(M)$, where $\mathcal{P}(M)$ is the power set of M, such that:

1. $\emptyset, M \in \mathcal{O}$

2. $\mathcal{O}$ is closed under arbitrary unions: $U_i \in \mathcal{O}, i \in I$, then $\cup_{i \in I} U_i \in \mathcal{O}$

3. $\mathcal{O}$ is closed under finite intersections: $U_1, \dots, U_k \in \mathcal{O}$, then $U_1 \cap \dots \cap U_k \in \mathcal{O}$.

Sets in $\mathcal{O}$ are called open, $A \subset M$ is closed if $M \setminus A$ is open. A subset $V \subset M$ containing p is a neighbourhood of p if there is an open set $U \subset V$ containing p.

Remark 17.3*.* From the definition above we can see that M and $\emptyset$ are always both open and closed sets. Notice that a set in a topological space can be:

- both open and closed

- neither open nor closed

- open but not closed

- closed but not open

(more examples later).

Definition 17.4 (Closure, interior). Let $V \subset M$, let $\mathcal{O}$ be an arbitrary topology. The closure $\bar{V}$ of V is the smallest closed subset of M containing V. The interior $\mathring{V}$ of V is the largest open subset contained in V.

Definition 17.5 (Continuous function, homeomorphism). A map $f \colon X \to Y$ between topological spaces is continuous if $f^{-1}(U) \in \mathcal{O}_X$ for all $U \in \mathcal{O}_Y$ (i.e. if the pre-image of an open set through f is still an open set). f is a homeomorphism if it is bijective, continuous and f^{-1} is continuous.

Remark 17.6 (Continuous maps vs. open maps). Requiring that the pre-image of an open set is open is different from requiring that the image of an open set is open! If f brings open sets to open sets, it is called open map (and it might not be continuous).

Example 17.7. Given a set M, we can always consider two simple topologies on it:

1. $\mathcal{O}_M = \mathcal{P}(M)$ is a topology, where $\mathcal{P}(M)$ is the power set of M (i.e. the family of all subsets of M). Everything is open in M, so it is easy to see that any function going from $(M, \mathcal{O}_M)$ to any space $(Y, \mathcal{O}_Y)$ must be continuous.

2. $\mathcal{O}_M = \{\emptyset, M\}$ is a topology: the open sets are just the two ones required by the definition 17.2. In this case, every function going from any space $(Y, \mathcal{O}_Y)$ to $(M, \mathcal{O}_M)$ must be continuous (the pre-image of the empty set is the empty set, and the pre-image of M is Y).

What is more, we can consider topologies on metric spaces: if (M, d) is a metric space and $B_\varepsilon(x)$ denotes the ball of radius ε around x, then

$$\mathcal{O}_d = \{U \subset M \,|\, \forall\, x \in U \,\exists\, \varepsilon > 0 \colon B_\varepsilon(x) \subset U\}$$

defines a topology. In particular, we say that the topology is induced by a metric if the open sets are the open balls of the metric space. If X, Y are metric spaces, the definition of continuity seen in the Analysis courses is: $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is continuous if and only if $\forall\, x \in X, \forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. One can show that in metric spaces such a definition is equivalent to definition 17.5.

Remark 17.8. If $\mathcal{O}_1, \mathcal{O}_2$ are topologies on M, then the intersection of the two topologies is again a topology. Therefore, given a family of subsets of M one can ask for the smallest topology on M which contains a subset.

Definition 17.9 (Hausdorff). A topological space $(M, \mathcal{O})$ is Hausdorff if for all $p, q \in M$ there is a pair of open sets $p \in U_p, q \in U_q$ so that $U_p \cap U_q = \emptyset$.

Remark 17.10. Points in Hausdorff spaces are closed sets. i.e. if M is Hausdorff and $x \in X$, $\{x\}$ is a closed set. In order to see that, we just need to prove that $M \setminus \{x\}$ is open. Let's suppose $M \setminus \{x\} \neq \emptyset$ (otherwise we are done, since $M = \{x\}$ and M is both closed and open). Then we can find another point $y \in M \setminus \{x\}$. Since M is Hausdorff, we can find two open sets U_x, U_y containing x and y respectively, such that $U_x \cap U_y = \emptyset$. The proof is over if we prove that $M \setminus \{x\} = \cup_{y \in M \setminus \{x\}} U_y$, since it implies that $M \setminus \{x\}$ is open (arbitrary union of open sets is open), and then $\{x\}$ is closed. It is true because:

- "$M \setminus \{x\} \subseteq \cup_{y \in M \setminus \{x\}} U_y$": for each point z in M different from x, there exists an open set U_z entirely contained in $M \setminus \{x\}$ by Hausdorff assumption. The point is contained in a set which is contained in $\cup_{y \in M \setminus \{x\}} U_y$ so, in particular, the point is in $\cup_{y \in M \setminus \{x\}} U_y$.

- "$M \setminus \{x\} \supseteq \cup_{y \in M \setminus \{x\}} U_y$": if $z \in \cup_{y \in M \setminus \{x\}} U_y$, then $z \neq x$, otherwise there would be an intersection between an open set $U_{\bar{x}}$ for a fixed $\bar{x}$ and every open set containing x. This is impossible because by Hausdorff assumption there exists at least one open sets of those which does not intersect $U_{\bar{x}}$.

Moreover:

- $(M, \mathcal{P}(M))$ is always Hausdorff (there are "so many" open sets that you can always find two of them containing the right points and so that they do not intersect each other).

- If the topology is induced by a metric, then the topological space is Hausdorff (for instance, $\mathbb{R}^n$ with the topology induced by Euclidean metric is a Hausdorff topological space).

Definition 17.11 (Product topology). Let $(M_i, \mathcal{O}_i), i \in I$ be topological spaces. Then the product topology on $\prod_{i \in I} M_i$ is the smallest topology such

that all projections

$$\prod_{i \in I} M_i \to M_j$$

are continuous maps for all $j \in I$.

Definition 17.12 (Quotient topology). Let $(M, \mathcal{O}_M)$ be a topological space and $\sim$ an equivalence relation on M. Then the quotient topology on $M\big/_\sim$ is the largest topology so that the function

$$\pi\colon M \to M\big/_\sim$$
$$x \mapsto [x]$$

is continuous. π is also called canonical projection.

Example 17.13 (Cofinite topology). Let M be a set. We define the following topology: the open sets are $\emptyset$ and the sets $U \neq \emptyset, U \subset M$ such that their complement in M is finite. Another way to define the same topology is using closed sets: the only closed sets in this topology are M and the finite sets. This topology is called cofinite topology. M with this topology is not Hausdorff if M is an infinite set. Why? Because [1]: given x, y in $M, x \neq y$, if U_a and U_b are open sets that contain a and b respectively, the complements of U_a and U_b must be finite (so, U_a and U_b must be infinite, because the bigger space M is infinite). But then $U_a \cap U_b \neq \emptyset$. Indeed, the complement of U_a in M is $M \setminus U_a$ and it must be finite because U_a is open. If we assume $U_a \cap U_b = \emptyset$, then U_b must be contained in the complement of U_a. But then U_b (which is an infinite set, because it is open) would be contained in $M \setminus U_a$ (which is a finite set). Contradiction!

Lemma 17.14. *A subspace of a Hausdorff space is Hausdorff.*

Example 17.15 (Subspace topology). Let $(M, \mathcal{O}_M)$ be a topological space and $N \subset M$ a subset. Then a natural topology for N is the subspace (or subset) topology, defined in the following way. The subspace topology is the smallest topology so that the inclusion map $N \to M$ is continuous, i.e. $V \subset N$ is open if and only if there is an open set $U \subset M$ such that $U \cap N = V$.

For instance, you may know that $[0, 1)$ is neither open nor closed in $\mathbb{R}$ with Euclidean topology (i.e. the topology induced by the standard metric).

---

[1] the proof is easier to follow if you think of M with cofinite topology as $\mathbb{R}$ endowed with cofinite topology

Actually, it is an open set in $([0, 2], \mathcal{O})$, where $\mathcal{O}$ is the subset topology (with respect to $(\mathbb{R}, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the Euclidean topology). Indeed

$$[0, 1) = (-1, 1) \cap [0, 2]$$

You should always pay attention to the topology and to where it is defined, for instance $[0, 1]$ is closed (but not open) in $(\mathbb{R}, \mathcal{O}_d)$, whereas it is both open and closed in $([0, 1], \mathcal{O})$, for any topology $\mathcal{O}$.

Definition 17.16 (subbasis). Let M be a set and $B \subset \mathcal{P}(M)$ some collection of subsets. Let $\mathcal{O}_B$ be the smallest topology containing B, then we say that B is a subbasis of $\mathcal{O}_B$.

Definition 17.17 (Basis). Let $(M, \mathcal{O})$ be a topological space and $\mathcal{B} \subset \mathcal{O}$. Then $\mathcal{B}$ is a basis for $\mathcal{O}$ if every open set is a union of sets in $\mathcal{B}$.

Definition 17.18 (Second countability). Let $(M, \mathcal{O}_M)$ be a topological space. Then $(M, \mathcal{O}_M)$ is second countable if it admits a countable basis.

Example 17.19. Let $\mathcal{B} = \{B_\varepsilon(x) \mid \varepsilon \in \mathbb{Q}, x \in \mathbb{Q}^n\}$. This is a basis of the metric topology of $\mathbb{R}^n$ (and it is a countable set, because we used the density of $\mathbb{Q}$ in $\mathbb{R}$).

Lemma 17.20. *If $A \subset M$ and M has a countable basis, then the subspace topology on A has a countable basis.*

Definition 17.21 (Manifold). A manifold M of dimension n is a topological space which is Hausdorff, second countable and admits an (equivalence class of) atlas(es) $\varphi_i \colon U_i \to V_i$, where V_i is an open subset of $\mathbb{R}^n$ and $U_i \subset M$ is open so that the transition functions

$$\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are smooth (i.e. C^∞). We are considering equivalence classes. we recall that the equivalence relation is the following one: two atlases are equivalent if their union is still a smooth atlas.

References

[1] J. M. Lee, Introduction to Smooth Manifolds, Springer

[2] Lecture notes (Differentiable Manifolds Saachs, Vogel WS 19)