Notes
Differentiable Manifolds
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Don't trust these notes!

Contents

1 Intro

From the Lee book [1]: "The central idea of calculus is *linear approximation*". A function of one variable can be approximated by its tangent line, a curve by a tangent vector (i.e. velocity vector), a surface in $\mathbb{R}^3$ can be approximated by its tangent plane, and a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by its total derivative. Here it comes the importance of tangent spaces.

Main idea: in order to study tangent vectors, we identify them with "directional derivatives". In particular, there is a natural one-to-one correspondence between geometric tangent vectors and linear maps from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$ satisfying the product rule. Such maps are called *derivations*.

Remark 1.1. **Points or vectors?** We can think of elements of $\mathbb{R}^n$ either as points or vectors. As points, their only property is their location, given by the coordinates $(x_1, \dots, x_n)$ on a chosen basis. As vectors, they are characterized by a direction and a magnitude, but their location is irrelevant (translational invariance). So given $v \in \mathbb{R}^n, v = v^i e_i$, it can be seen as an arrow with its initial point anywhere in $\mathbb{R}^n$. So, if we think about a vector tangent to the border of the sphere at a point a, we imagine the vector as living in a copy of $\mathbb{R}^n$ with its origin translated to a.

2 Derivations

Definition 2.1 (Derivation)**.** If a is a point of $\mathbb{R}^n$, a map $v\colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at a* if it is linear over $\mathbb{R}$ and satisfies the following product rule:

$$v(fg)\big|_a = f\big|_a v(g)\big|_a + g\big|_a v(f)\big|_a$$

Remark 2.1. Directional derivatives obviously satisfy the above definition, and in these cases such a rule is also called Leibnitz rule.

3 Multilinear Forms

Definition 3.1 (1-forms)**.** Given a vector space V on a field K, a 1-form (or linear form) $\varphi\colon V \to K$ is a linear function from V to K. V^* (also denoted by ΛV^*) is the set of all linear forms on V.

Definition 3.2 (Dual basis)**.** If $\{e_i\}_{i=1,\dots,n}$ is a basis of V, then $\{e^{*i}\}_{i=1}^n \subseteq V^*$ is called the *dual basis* if $e^{*i}(e_j) = \delta^i_j$.

Remark 3.1. We could prove that the dual basis is indeed a basis of the dual space, so $\dim(V) = \dim(\Lambda V^*)$. Check proposition 4.1.

Remark 3.2. Every vector in $\mathbb{R}^n$ about a point $p \in \mathbb{R}^n$ (i.e. such that its origin is the point p) "can be seen" as a derivation (cf. def. 2.1), i.e. as a directional derivative of a function evaluated at the point p. For the sake of simplicity, we think $p = 0$ (but the following results are true $\forall\, p \in \mathbb{R}^n$). The sentence "can be seen" means that there is an isomorphism ψ associating such vectors to such linear forms. Let's construct this isomorphism in the following steps:

1. Because of linearity, we just need to define the isomorphism for the basis vectors $\{e_i\}_{i=1,\dots,n}$ of $\mathbb{R}^n$.

2. Given the vector e_j of the canonical basis, we associate it with the derivation ∂_{x_j} in 0:

$$\begin{aligned} \partial_{x_j}\big|_{p=0} \equiv \frac{\partial}{\partial x_j}\big|_{p=0} \colon C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial}{\partial x_j}\big|_0(f) \equiv \frac{\partial f}{\partial x_j}(0) \end{aligned} \tag{3.1}$$

   In particular, if $\mathrm{Der}(\mathbb{R}^n) = \{\text{derivations on } \mathbb{R}^n\} = \{v \colon C^\infty \to \mathbb{R}$, satisfying Leibnitz rule$\}$, then the map:

$$\mathbb{R}^n \overset{\psi}{\longleftrightarrow} \mathrm{Der}(\mathbb{R}^n) \tag{3.2}$$

   such that $\psi(e_j) = \partial_{x_j}\big|_{p=0}, \forall\, e_j$ basis vector, and with $\partial_{x_j}\big|_{p=0}$ partial derivative with respect to the j-th component, defines a linear map. Indeed, it is linear because of linearity of derivations, and since we defined its behaviour on the basis vectors, it is also defined for every vector of $\mathbb{R}^n$. In general we have $\psi(v) = \partial_v\big|_{p=0}$, where $\partial_v\big|_{p=0}$ is the directional derivative with respect to v. Moreover, $\mathrm{Der}(\mathbb{R}^n)$ is a vector space and we used the double arrow above because ψ is an isomorphism, i.e. a bijective map which preserves operations from one space to the other. Here every derivative is evaluated at $p = 0$. We notice that the point p itself is not important for the directional derivative (the *direction* in which we differentiate is the same for every point of the space), but p is meaningful when we *evaluate* the derivative of the function at that point. Indeed, $\partial_x(x^2)\big|_{x=0} \neq \partial_x(x^2)\big|_{x=1}$, even if we are differentiating along the x-axis in both cases.

What is more: given V_0, the set of all the vectors about 0, we can consider its dual space V_0^*. What is a possible dual basis? We want to find linear forms

$$e^{*i}\colon V_0 \to \mathbb{R}$$

such that $e^{*i}(e_j) = \delta^i_j$. We have just seen that we can consider vectors as directional derivatives. So, given e_j vector of the canonical basis, we will call it $\frac{\partial}{\partial x^j}|_0$ (because of the isomorphism, they are quite the same mathematical object). Now, we want that

$$e^{*i}\left(\frac{\partial}{\partial x^j}|_0\right) = \delta^i_j \tag{3.3}$$

First, let's consider the *coordinate function*:

$$\begin{aligned} x_j\colon \mathbb{R}^n &\to \mathbb{R} \\ v = (v_1, \dots, v_n) &\mapsto v_j \end{aligned} \tag{3.4}$$

where $v_1, \dots, v_j$ are the coordinates of the vector v in the canonical basis. The linear form x_j returns the j-th coordinate of a vector. So, given a vector $v \in \mathbb{R}^n$, every coordinate v_j can be seen as $v_j = x_j(v)$. Now, let's just define

$$e^{*i}\left(\frac{\partial}{\partial x^j}|_0\right) \equiv \frac{\partial}{\partial x^j}|_0 x_i = \frac{\partial}{\partial x^j} x_i|_0 = \delta^i_j \tag{3.5}$$

Where x_i is the coordinate function defined above (remember: ∂_{x_j} is a derivation, so it must be applied to functions!). Now, it might seem that e^{*i} does not take a vector as argument, but rather a function. Actually, this problem is solved by the isomorphism between vectors and directional derivatives proved above. If ψ is the name of such isomorphism, we could slightly change the definition (3.5) in order to solve this ambiguity:

$$e^{*i}(e_j) \equiv \psi(e_j)(x_i) \tag{3.6}$$

where x_i is the i-th coordinate function and

$$\psi(e_j) = \frac{\partial}{\partial x^j}|_0 = \partial_{x_j}|_0 \tag{3.7}$$

The vectors of the dual basis will also be called

$$dx^i \equiv e^{*i} \tag{3.8}$$

This will be important later: we will define exterior forms of degree k and we'll use both notations. The set of all these forms is $\Lambda^k V^*$, and its basis is given by products (in particular, exterior products) of $e^{*i_1}, \ldots, e^{*i_k}$ (i.e. $dx^{i_1}, \ldots, dx^{i_k}$).

Definition 3.3 (Set of vector fields)**.** We denote by $\mathfrak{X}(\mathbb{R}^n)$ the set of al possible vector fields in $\mathbb{R}^n$, i.e.

$$\begin{aligned}\mathfrak{X}(\mathbb{R}^n) = \mathrm{Der}\mathbb{F}(\mathbb{R}^n) \equiv \{v \colon \mathbb{F}(\mathbb{R}^n) \to \mathbb{F}(\mathbb{R}^n) \text{ such that}\\ v \text{ is } \mathbb{R}\text{-linear and } v(fg) = v(f)g + fv(g)\}\end{aligned} \tag{3.9}$$

where $\mathbb{F} \equiv \{$ functions $f \colon \mathbb{R}^n \to \mathbb{R}\}$.

Remark 3.3 (Fields vs. Derivations). If v is a vector field in $\mathbb{R}^n$, then v assigns a vector to another vector of $\mathbb{R}^n$. So $v \colon \mathbb{R}^n \to \mathbb{R}^n$. So, the set of all vector fields should be (we will use a different symbol to denote it):

$$X(\mathbb{R}^n) = \{v \colon \mathbb{R}^n \to \mathbb{R}^n\}$$

However, the definition 3.3 is a bit different. Why? The fact is, we can consider a vector of $\mathbb{R}^n$ as a directional derivative, cf. remark 3.2 (we are not considering any fixed point here, but the results do not change). Now a derivation, as defined in def. 2.1, is a map $v \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$, i.e. we can say that a derivation is a very smooth element of $\mathbb{F}(\mathbb{R}^n) = \{$functions $f \colon \mathbb{R}^n \to \mathbb{R}\}$. So, $\mathfrak{X}(\mathbb{R}^n) \cong X(\mathbb{R}^n)$ because we can associate a derivation of $\mathfrak{X}(\mathbb{R}^n)$ to each vector of $X(\mathbb{R}^n)$, and vice versa. Then we also explained why in the definition of $\mathfrak{X}(\mathbb{R}^n)$ every element must be $\mathbb{R}$-linear and satisfy the Leibnitz rule: it follows from the definition of derivations.

Now, a question arises: given v vector field, should we write $v(p)$ (i.e. it takes vectors as argument) or should we write $v(f)$ (i.e. it takes smooth functions as arguments)? The answer is: it depends on the case, since they are two different "v"s. Which is, we will use vectors when we think of v as a function who takes elements of $\mathbb{R}^n$, and we will use functions in the other case. And we can choose which case to use, since we can identify every vector field with a derivation, and every derivation with a vector field (for more info, see pag. 181 of [1]). Now, let us analyze how $v(f)$ is made in the latter case. Given $v \in \mathfrak{X}(\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}$, we define the function $v(f)$ as

$$\begin{aligned} v(f) \colon \mathbb{R}^n &\to \mathbb{R}\\ p &\mapsto v(f)(p) \equiv v_p f\end{aligned} \tag{3.10}$$

Now, in coordinates:

$$v(f)(p) = v_p f = v^i(p)(e_i)_p f = v^i(p)\frac{\partial}{\partial x^i}|_p f = v^i(p)\frac{\partial f}{\partial x^i}(p) \tag{3.11}$$

where we used summation convention, and the fact that every vector basis e_i can be seen as $\partial_{x_i}|_p$. We defined it in the right way because, as expected, we found that $v_p(f)$ is the directional derivative of f in the direction of v, evaluated at p. So, in brief:

- $v(f)(p)$ is a number
- $v(f)(\cdot)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$
- $v(\cdot)$ is a function from $\mathbb{F}(\mathbb{R}^n)$ to $\mathbb{F}(\mathbb{R}^n)$

We also notice that the mathematical object e_i is not much different from $(e_i)|_p$ in this case: there is no difference if we think about them as directions, but it makes a difference if we think about them as directional derivatives, because the latter notation gives info about the point in which the derivative is evaluated. So, we add the pedix "p" in order to make the isomorphism between vectors and directional derivatives more explicit. Check also the remark 1.1.

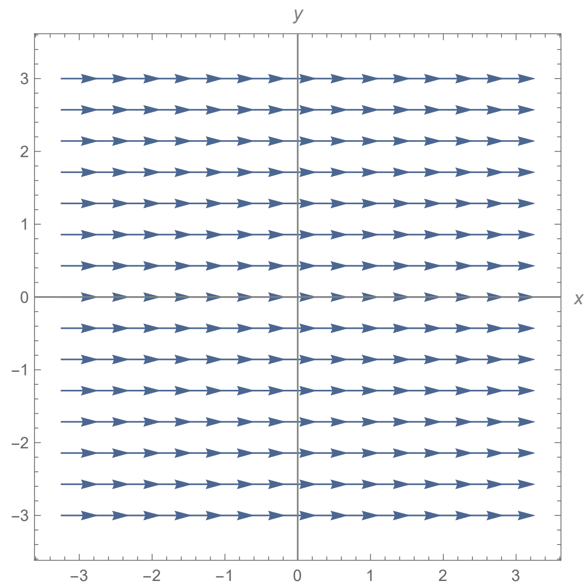

Figure 1: Vector field v=$e_1 = \partial_x$

4 Exterior Product and Generalisation

Definition 4.1 (Exterior form of degree k)**.** Given a vector space V on a field $\mathbb{K}$, with $\dim(V) = n$, and with $k \leq n$, an *exterior form of degree* k (or k-linear form, or k-form) is a map ω:

$$\omega \colon \underbrace{V \times \ldots \times V}_{\text{k times}} \to \mathbb{K}$$

such that

$$\omega(v_1, \ldots, v_k) = \operatorname{sgn}(\pi)\omega(v_{\pi(1)}, \ldots, v_{\pi(k)})$$

and such that ω is multilinear. Where π is a permutation of k elements, i.e. $\pi \in S_k$, and $\operatorname{sgn}(\pi)$ is the sign of the permutation. We will also write $\omega \in \Lambda^k V^*$.

Definition 4.2 (Exterior product between two 1-forms)**.** Given a vector space V on a field $\mathbb{K}, \dim(V) \geq 2$, and given $\varphi^1, \varphi^2 \in \Lambda V^*$, then we define the exterior product (or wedge product) $\wedge$ as:

$$\begin{aligned} \wedge \colon \Lambda V^* \times \Lambda V^* &\to \Lambda^2 V^* \\ (\varphi^1, \varphi^2) &\mapsto \varphi^1 \wedge \varphi^2 \end{aligned}$$

where:

$$\varphi^1 \wedge \varphi^2(x_1, x_2) = \varphi^1(x_1)\varphi^2(x_2) - \varphi^2(x_1)\varphi^1(x_2) = \det(\varphi^i(x_j))$$

for $i, j = 1, 2$.

Remark 4.1 (Exterior product between k 1-forms). The exterior product $\wedge$ that we defined for $k = 2$ in 4.2 is an exterior form of degree 2. We want to generalize it for k vector spaces. In order to extend the definition, we want it to be an exterior form of degree k, so:

$$\begin{aligned} \wedge \colon \underbrace{\Lambda V^* \times \ldots \times \Lambda V^*}_{k \text{ times}} &\to \Lambda^k V^* \\ (\varphi^1, \ldots, \varphi^k) &\mapsto \varphi^1 \wedge \ldots \wedge \varphi^k \end{aligned}$$

where, given $(x_1, \ldots, x_k) \in \underbrace{V \times \ldots \times V}_{k \text{ times}}$:

$$\varphi^1 \wedge \ldots \wedge \varphi^k(x_1, \ldots, x_k) = \det(\varphi^i(x_j))$$

This is a particular case of an exterior k-form (because the sign of determinant changes if we swap two rows or two columns).

Proposition 4.1. *If* $\{e_i\}_{i=1,\ldots,n}$ *is a basis in* V*, then* $\{e^{*i_1} \wedge \ldots \wedge e^{*i_k}\}_{i_1<\ldots<i_k, k\leq n}$ *forms a basis of* $\Lambda^k V^*$

Remark 4.2. The above proposition proves that $\dim(\Lambda^k V^*) = \binom{n}{k}$. Moreover, it means that any $\alpha \in \Lambda^k V^*$ can be written as:

$$\alpha = \sum_{i_1<\ldots<i_k} a_{i_1\cdots i_k} e^{*i_1} \wedge \cdots \wedge e^{*i_k}$$

where $a_{i_1\cdots i_k} \in \mathbb{K}$, $\mathbb{K}$ field of the vector space.

Now, we want to define the exterior product between a k-form and a p-form (and it will return a $(p+k)$-form).

Definition 4.3 (Exterior product between a k-form and a p-form)**.** Given $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*$, the exterior product between them is defined as:

$$\begin{aligned} \wedge \colon \Lambda^k V^* \times \Lambda^p V^* &\to \Lambda^{k+p} V^* \\ (\alpha, \beta) &\mapsto \alpha \wedge \beta \end{aligned}$$

with:

$$\alpha \wedge \beta = \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_p}} \alpha_{i_1\cdots i_k} \beta_{j_1\cdots j_k} e^{*i_1} \wedge \cdots \wedge e^{*i_k} \wedge e^{*j_1} \wedge \cdots \wedge e^{*j_k}$$

where $\alpha_{i_1\cdots i_k}, \beta_{j_1\cdots j_k} \in \mathbb{K}$

Let's check some properties about k-forms:

Proposition 4.2. $\alpha \in \Lambda^k V^*, \beta \in \Lambda^p V^*, \gamma \in \Lambda^q V^*$*, then:*

1. $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$
2. $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$
3. $\alpha \wedge \beta = (-1)^{kp} \beta \wedge \alpha$

5 Differential Forms

Definition 5.1 (Field of exterior forms, geometric definition)**.** (A field of) exterior forms of degree k, $k \leq n$ is a map ω that associates to each point $p \in \mathbb{R}^n$ an element $\omega(p) \in \Lambda^k V_p^*$. Choosing a basis, we have:

$$\omega(p) = \sum_{i_1<\ldots<i_k} \underbrace{a_{i_1\cdots i_k}(p)}_{\text{now it is a function!}} \quad e^{*i_1} \wedge \cdots \wedge e^{*i_k} \tag{5.1}$$

ω is a differential form if $a_{i_1\cdots i_k}$ are differentiable. The set of differential k-forms is denoted by $\Omega^k(\mathbb{R}^n)$

Another (equivalent) definition:

Definition 5.2 (Algebraic definition of differential k-form)**.** A differential k-form is a map:

$$\underbrace{\mathfrak{X}(\mathbb{R}^n) \times \ldots \times \mathfrak{X}(\mathbb{R}^n)}_{k \text{ times}} \to \mathbb{F}(\mathbb{R}^n) \tag{5.2}$$

$C^\infty(\mathbb{R}^n)$ linear and alternating.

Remark 5.1. To show the equivalence of the two definition of differential k-forms we just need to show that:

$$\omega(p)(v_1, \ldots, v_k) = \omega(v_1, \ldots, v_k)(p) \tag{5.3}$$

We want to generalize the concept of differential of a function.

Definition 5.3 (Differential)**.** Let f be a function $f\colon U \subseteq \mathbb{R}^n \to \mathbb{R}$, f differentiable. Let $v \in \mathfrak{X}(\mathbb{R}^n) = \operatorname{Der}\mathbb{F}(\mathbb{R}^n)$. The exterior derivative of f is its differential d, defined as a 1-form such that:

$$df(v) = v(f) \tag{5.4}$$

Remark 5.2 (differential expression in coordinates)*.* We want to verify that the above definition of differential is equivalent to our usual definition for $C^1(\mathbb{R}^n)$ function, which is:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i = \frac{\partial f}{\partial x_i} dx^i \tag{5.5}$$

In order to prove that, we first consider a pointwise definition. Given $p \in \mathbb{R}^n$:

$$df_p(v) = v(f), \forall\ v \in T_p\mathbb{R}^n \cong \mathbb{R}^n \tag{5.6}$$

($T_p\mathbb{R}^n$ is the tangent space to $\mathbb{R}^n$ at p). Now, we can write $v(f)$ in coordinates (the gray part is the one we don't care about):

$$df_p = v(f) = \ v_i(p)(\lambda^i)_p \tag{5.7}$$

where $(\lambda^i)_p$ is a dual basis at p (later, we will prove that $(\lambda^i)_p = (dx^i)_p$). Now, applying df to a particular vector (i.e. directional derivative) at p:

$$df_p\left(\frac{\partial}{\partial x^i}|_p\right) = v_i(p) \tag{5.8}$$

where we used the property of the dual basis

$$(\lambda^i)_p \frac{\partial}{\partial x^j}|_p = \delta^i_j \tag{5.9}$$

and then:

$$df_p\left(\frac{\partial}{\partial x^i}|_p\right) = v_i(p)(\lambda^i)_p \frac{\partial}{\partial x^i}|_p = v_i(p) \tag{5.10}$$

On the other hand, by definition (5.6) we know that:

$$df_p\left(\frac{\partial}{\partial x^i}|_p\right) = \frac{\partial}{\partial x^i}|_p f = \frac{\partial f}{\partial x^i}(p) \tag{5.11}$$

Hence, using (5.8) and (5.11) we get:

$$v_i(p) = \frac{\partial f}{\partial x^i}(p) \tag{5.12}$$

Then, by the expression of differential in coordinates (5.7):

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p \tag{5.13}$$

Applying the definition to $f = x^j$ (coordinate function, as defined in (3.4)), we get:

$$df_p = \frac{\partial f}{\partial x^i}(p)(\lambda^i)_p = \frac{\partial f}{\partial x^i}(p)(dx^i)_p \tag{5.14}$$

And then:

$$df = \frac{\partial f}{\partial x^i}dx^i \tag{5.15}$$

Indeed, if $f = x^j$ then, as before:

$$(dx^j)_p = \frac{\partial x^j}{\partial x^i}(p)(\lambda^i)\big|_p = \delta^i_j(\lambda^i)\big|_p = (\lambda^j)\big|_p \tag{5.16}$$

Pay attention: what we did here is a bit different from what we did for the definition 3.10 of a vector field applied to a function. In this case, p is the point where we fixed our vector, whereas in the other case p was the point where we wanted to evaluate the directional derivative of f.

In the above definition, f was a 0-form (i.e. a function). What is the generalization of the differential to k-forms?

Definition 5.4 (Exterior derivative)**.** If $k > 0$, then the exterior derivative (acting on k-forms) is a map

$$\begin{aligned} \mathrm{d}\colon \Omega^k(\mathbb{R}^n) &\to \Omega^{k+1}(\mathbb{R}^n) \\ \omega &\mapsto \mathrm{d}(\omega) \equiv \mathrm{d}\,\omega \end{aligned}$$

where

$$\mathrm{d}\,\omega = \sum_{j_1<\ldots<j_k} (\mathrm{d}\,a_{j_1,\ldots,j_k}) \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_k}$$

With $\mathrm{d}\,a_{j_1,\ldots,j_k}$ differential of the function $a_{j_1,\ldots,j_k}$.

Some properties:

Proposition 5.1 (Properties of exterior derivatives)**.** $\omega_1 \in \Omega^k(\mathbb{R}^n), \omega_2 \in \Omega^p(\mathbb{R}^n)$. *Then:*

- $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$
- $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$
- $d(d\omega_1) = 0 = d(d\omega_2)$

Remark 5.3. In the above proposition, we claimed that $d(d\omega) = 0$ if $\omega \in \Omega^k(\mathbb{R}^n)$. The notation here is not the most precise, since the inner "d" is acting on a k-form, whereas the outer "d" is acting on a $(k+1)$-form (so, even if they share the same name, they are different maps). However the behaviour of both "d"s is clear, so we will continue with this abuse of notation.

Remark 5.4. The exterior derivative increases the degree of a k-form by 1 (the k-form becomes a $(k+1)$-form). Can we get backwards, which is, can we decrease the degree of a k-form? Answer: yes.

Definition 5.5 (Interior derivative)**.** $z \in \mathfrak{X}(\mathbb{R}^n)$ (i.e. z is a vector field), then we define the *interior derivative* i_z (acting on differential k-forms) as:

$$\begin{aligned} \mathrm{i}_z \colon \Omega^k(\mathbb{R}^n) &\to \Omega^{k-1}(\mathbb{R}^n) \\ \omega &\mapsto \mathrm{i}_z(\omega) \equiv \mathrm{i}_z\,\omega \end{aligned} \tag{5.17}$$

where

$$\mathrm{i}_z\,\omega(v_1,\ldots,v_{k-1}) = \omega(z, v_1, \ldots, v_{k-1}), \forall\, v_i \in \mathfrak{X}(\mathbb{R}^n)$$

$\mathrm{i}_z\,\omega$ is also called the *contraction* of ω.

Remark 5.5. In the definition 5.5 above, we used the algebraic definition of differential k-forms, i.e. definition 5.2

Now some properties for interior derivatives.

Proposition 5.2. $\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^p(\mathbb{R}^n), z \in \mathfrak{X}(\mathbb{R}^n)$, *then:*

- $\mathrm{i}_z(\omega \wedge \eta) = (\mathrm{i}_z\,\omega) \wedge \eta + (-1)^k \omega \wedge (\mathrm{i}_z\,\eta)$
- $\mathrm{i}_z^2\,w = \mathrm{i}_z(\mathrm{i}_z\,\omega) = 0$

Remark 5.6. In the above proposition there is a little abuse of notation when we claimed $\mathrm{i}_z(\mathrm{i}_z\,\omega) = 0$, see also remark 5.3.

Now, let's talk about *pullbacks* and *pushforwards* for functions and k-forms.

Definition 5.6 (Pullback)**.** Let $f \colon U \to V$ (with $U, V \subseteq \mathbb{R}^n$) be a differentiable map. Let us suppose that $\dim(U) = \dim(V) = n$ (just for the sake of simplicity, since it is not necessary). Then the *pullback* of a k-form (from V) to U is the map:

$$\begin{aligned} f^* \colon \Omega^k(V) &\to \Omega^k(U) \\ \omega &\mapsto f^* w \end{aligned}$$

such that

$$(f^*\omega)(p)(u_1, \ldots, u_k) = \omega(f(p))(df(u_1), \ldots, df(u_k)), \forall\; p \in \mathbb{R}^n, \forall\; u_i \in \mathfrak{X}(U)$$

Now, we want to give another name to the differential of a function.

Definition 5.7 (Pushforward)**.** Given $f \colon U \to V$ as before, we will also call the differential of f at $p \in \mathbb{R}^n$, i.e. $df_p = df(p)$, as the *pushforward* of f at p, and it will be denoted by the symbol $(f_*)_p$.

In our mind, we'll think of $df_p = (f_*)_p$, at least until this concept is generalized.

In particular, using the pullback definition above, we can write the pushforward map as:

$$\begin{aligned} df_p \equiv (f_*)_p \colon U \subset \mathbb{R}^n &\to V \subset \mathbb{R}^m \\ v &\mapsto (f_*)_p(v) \end{aligned}$$

By definition of differential, $df_p(v) = v(f)$, where v is a vector tangent to $\mathbb{R}^n$ at p. Since vector are like directional derivatives, $v(f)$ is the directional derivative of f with respect to v (not evaluated at any point, for now!). In particular, if we apply the definition to a point $h(q)$, where $h \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ (**CHECK**) and $q \in \mathbb{R}^n$, we have:

$$(f_*)_p(v)(h)(q) = (f_*)_p(v)(h(q)) = v(h(f(q))) = v(h \circ f)(q) = v(f^* h)(q)$$

In the last passage, we used the pullback for a differentiable function, which is completely legal since we defined it for differentiable k-forms, and a differentiable function is just a 0-form.

Remark 5.7. Using the pushforward, we can define the pullback of a differential form using a different notation (i.e. using f_* instead of df):

$$(f^*\omega)(p)(u_1,\ldots,u_k) = \omega(f(p))(f_*(u_1),\ldots,f_*(u_k)), \forall\ p \in \mathbb{R}^n, \forall\ u_i \in \mathfrak{X}(U) \tag{5.18}$$

Now, some properties of the pullback.

Proposition 5.3. *$g, f \in C^1(\mathbb{R}^n,\mathbb{R})$, $\omega, \varphi \in \Omega^k(\mathbb{R}^n)$, $h\colon \mathbb{R}^n \to \mathbb{R}$. Then:*

1. $f^*(\omega + \varphi) = f^*(\omega) + f^*(\varphi)$
2. $f^*(h\omega) = f^*(h)f^*(\omega)$
3. $(f \circ g)^* = g^*(f^*(\omega))$
4. *If* $\varphi^1,\ldots,\varphi^k \in \Omega^1(\mathbb{R}^n)$*, then* $f^*(\varphi^1 \wedge \ldots \wedge \varphi^k) = f^*(\varphi^1) \wedge \ldots \wedge f^*(\varphi^k)$
5. $df^*(\omega) = f^*(d\omega)$

From property (4) also follows that $f^*(\omega \wedge \phi) = (f^*\omega) \wedge (f^*\phi)$

Remark 5.8. We can express the pullback of a differential form in the following way:

$$\begin{aligned}(f^*\omega)(p) = &\sum_{1\le i_1<i_2<\cdots<i_k\le n} (f^*a_{i_1,\ldots i_k}(p))f^*dy^{i_1} \wedge f^*dy^{i_2} \wedge \cdots \wedge f^*dy^{i_k} = \\ = &\sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1,\ldots,i_k}(f(p))df^{i_1} \wedge df^{i_2} \wedge \cdots \wedge df^{i_k}\end{aligned}$$

where $f^i = y^i(f)$. We used properties (2) and (4) of proposition 5.3

Remark 5.9. From our definition of pullback, it is not necessary that f_* is invertible.

6 Integration of differential forms

Let ω be a differential form of degree n in $\mathbb{R}^n$. Then ω is necessarily of the form

$$\omega = \underbrace{a(p)}_{\text{it's a function}} dx^1 \wedge \ldots \wedge dx^n \tag{6.1}$$

Such a form can be integrated:

$$\int_{f(D)} \omega = \int_D f^* \omega \tag{6.2}$$

7 More on Vector Fields

Definition 7.1 (Tangent space)**.** $U \subset \mathbb{R}^n, U$ is an open set. $p \in U$, then the set of all derivations of $C^\infty(U)$ (cf. def 2.1) is called tangent space to U at p and is denoted by T_pU. An element of T_pU is called a tangent vector at p, and it is often denoted by v_p.

Definition 7.2 (Tangent bundle)**.** The tangent bundle over an open subset $U \subset \mathbb{R}^n$ is defined as

$$TU \equiv \bigsqcup_{p \in U} T_pU \tag{7.1}$$

where T_pU is the tangent space of U at p. Every element of the disjoint union is represented by an ordered pair (v, p) where $p \in U, v \in T_pU$. The tangent bundle comes equipped with the projection map

$$\begin{aligned} \mathrm{pr}\colon TU &\to U \\ (p, v) &\mapsto p \end{aligned} \tag{7.2}$$

So, every element of the tangent bundle is a couple made of a tangent space to a point, and the point itself.

Remark 7.1. In the previous definition, the "$\sqcup$" symbol denotes a disjoint union. "Disjoint" here means that, if we consider the disjoint union of two elements x and y such that $x = y$, the union is the set $\{x, y\}$ and not $\{x\} = \{y\}$ as in normal unions. The mathematical operator doesn't know if two elements are equal. Since we are not mathematical operators, we can enumerate the elements like: $\{(1, x), (2, y)\} = \{(1, x), (2, x)\} = \{(1, y), (2, y)\}$ in order to distinguish them.

Definition 7.3 (Alternative definition of vector field)**.** A smooth vector field v on $U \subset \mathbb{R}^n, U$ open, is a smooth map

$$v\colon U \to TU \tag{7.3}$$

such that $\mathrm{pr}(v_p) = p, \forall\, p \in U$

Remark 7.2 (Space of sections). The set of all vector fields $\mathfrak{X}(U) \equiv \{C^\infty(U, TU) \mid \mathrm{pr}(v_p) = p\}$ is also called the space of sections in TU.

Definition 7.4 (Cotangent bundle)**.** By duality we define

$$T^*U \equiv \bigsqcup_{p \in U} T_p^*U \tag{7.4}$$

as the cotangent bundle. Where T_p^*, the dual space of the tangent space, is called cotangent space. We also associate a projection $\mathrm{pr}\colon T^*U \to U$ with it.

Now, let's talk about about Lie algebras.

Definition 7.5 (Lie algebra)**.** A Lie algebra $(V, [\cdot, \cdot])$ is a vector space V over $\mathbb{R}$ endowed with a map

$$[\cdot, \cdot]\colon V \times V \to V$$

with the following properties:

- $[\cdot, \cdot]$ is bilinear
- $[\cdot, \cdot]$ is antisymmetric ($[u, v] = -[v, u], \forall\, u, v \in V$)
- $[\cdot, \cdot]$ satisfies the *Jacobi identity*:

$$[[u, v], z] + [[z, u], v] + [[v, z], u] = 0$$

Remark 7.3 (Jacobi). How to remember Jacobi identity: remember $[[u, v], z]$ and then permute cyclically.

Proposition 7.1. $\mathfrak{X}(\mathbb{R}^n)$ *is an (infinite dimensional) Lie algebra with* $[u, v](f) = u(v(f)) - v(u(f))$*, for* $u, v \in \mathfrak{X}(\mathbb{R}^n), f \in C^\infty(\mathbb{R}^n)$*. (Note that* u *and* v *are vector fields and* $[u, v]$ *is still a vector field).*

Definition 7.6 (Integral curve)**.** An integral curve for a vector field v is a smooth curve $\phi\colon (a, b) \to \mathbb{R}^n$ satisfying $\dot{\phi}(t) = v_{\phi(t)}$ ($v_{\phi(t)}$ is the vector tangent at $\phi(t)$ for t fixed, remember the previous notation!). Let us suppose $0 \in (a, b)$. Then, $\phi(0)$ is called the starting point of ϕ.

We can also visualize the family of integral curves in the following way.

Definition 7.7 (Flow)**.** The map

$$\begin{aligned} \theta\colon \mathbb{R} \times \mathbb{R}^n &\to \mathbb{R}^n \\ (t, p) &\mapsto \theta_t(p) \end{aligned} \tag{7.5}$$

such that $\dot{\theta}_t(p) = v_{\theta_t(p)}$ is the flow of the vector field v where, if we fix p, $\theta_t(p)$ is the integral curve which passes through p at $t = 0 \in (a, b)$. So the flow satisfies two conditions:

$$\dot{\theta}_t(p) = v_{\theta_t(p)}, \qquad \forall\, p \in \mathbb{R}^n \tag{7.6}$$

$$\theta_0(p) = p, \qquad \forall\, p \in \mathbb{R}^n \tag{7.7}$$

Under the right hypothesis (e.g. Lipschitz hypothesis and smoothness of v) we can prove existence and uniqueness of the solution of such ODEs ($\forall p \in \mathbb{R}^n$).

By fixing either the time or the starting point of the flow, we can consider two maps:

- $p \mapsto \theta_t(p)$, for each fixed t (we are observing several integral curves at the same time t)
- $t \mapsto \theta_t(p)$, for each fixed p (we are observing the integral curve starting from p, for all times)

Definition 7.8 (Lie derivative)**.** Let $z \in \mathfrak{X}(\mathbb{R}^n)$ be a differentiable vector field, ϕ_t its flow and $\omega \in \Omega^k(\mathbb{R}^n)$. Then the Lie derivative of ω is defined as

$$L_z\omega = \frac{d}{dt}(\phi_t^*\omega)|_{t=0} \tag{7.8}$$

Remark 7.4. We denoted the flow by the symbol ϕ_t and not ϕ. What we are doing here is not caring about p: $\phi_t^*\omega(\cdot) = \omega(\phi_t(\cdot))$ Useful formula: $L_z\omega = (\mathrm{d}\,\mathrm{i}_z + \mathrm{i}_z\,\mathrm{d})\omega$

8 Lie derivative of a vector field

Definition 8.1 (pullback of a vector field)**.** Let φ be a diffeomorphism of $\mathbb{R}^n$ (i.e. a differentiable and invertible map from $\mathbb{R}^n$ to $\mathbb{R}^n$, such that its inverse is differentiable as well). Let $v \in \mathfrak{X}(\mathbb{R}^n)$. Then:

$$\varphi^* v \equiv \varphi_*^{-1} v \tag{8.1}$$

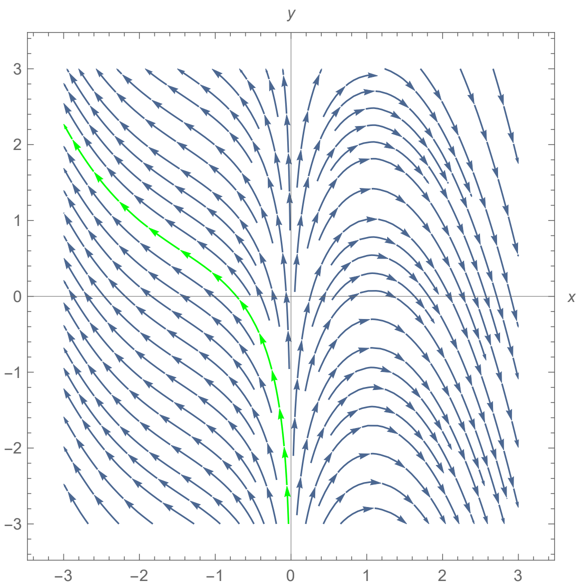

Figure 2: the map $t \mapsto \theta_t(p)$ selects just one integral curve

is the pullback of v with φ. In particular, given a flow ϕ and t fixed, $\phi(t, \cdot) = \phi_t(\cdot)$ is a diffeomorphism on $\mathbb{R}^n$ with $\phi^{-1} = \phi(-t, \cdot) = \phi_{-t}(\cdot)$.

Definition 8.2 (Lie derivative of a vector field)**.** Let $u, v \in \mathfrak{X}(\mathbb{R}^n)$. The Lie derivative of v in direction u is

$$L_u v \equiv \frac{d}{dt}(\phi_t^* v)|_{t=0} \tag{8.2}$$

(Remember: $\phi\colon (t,p) \mapsto \phi(t,p), \phi_t^* v\colon (t,p) \mapsto v(\phi(t,p))$).

Lemma 8.1. *Let u, v be smooth vector fields on $\mathbb{R}^n$ and $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$. Let ϕ_t be the flow of u and let ψ_s be the flow of v. Then*

- $\varphi^* v = \frac{d}{ds}|_{s=0} \varphi^{-1} \circ \psi_s \circ \varphi$
- $\varphi^* v = v \Leftrightarrow \varphi \circ \psi_s = \psi_s \circ \varphi$ *for all* s.
- $L_u v = 0 \Leftrightarrow \phi_t \circ \psi_s = \psi_s \circ \phi_t$ *for all* s, t

Lemma 8.2. *Let u, v be smooth vector fields on $\mathbb{R}^n$ and and let ϕ_t (respectively ψ_s) be the flow of u (respectively v). Then:*

- $L_u v = \frac{\partial^2}{\partial s \partial t} \phi_{-t} \circ \psi_s \circ \phi_t|_{t=0,s=0}$

- $(L_u v)(f) = [u, v](f) = u(v(f)) - v(u(f))$ *for all smooth functions* f *on* $\mathbb{R}^n$

Lemma 8.3. *Let* u, v *be smooth vector fields on* $\mathbb{R}^n$ *and* $\varphi \in \mathrm{Diff}(\mathbb{R}^n)$. *Then:*

1. $[u, v]$ *is* $\mathbb{R}-$*bilinear (i.e. bilinear for a parameter* $\lambda \in \mathbb{R}$*)*
2. $[u, v] = -[v, u]$
3. *The Jacobi identity holds*
4. $[u, fv] = f[u, v] + u(f)v$
5. $\varphi_*[u, v] = [\varphi_* v, \varphi_* u]$

9 Stokes' Theorem on $\mathbb{R}^n$

- For a function f (i.e. a 0-form) on $[a, b] \subset \mathbb{R}$ we have $\int_a^b df = \int_a^b \partial_x f dx = f(b) - f(a)$ (fundamental theorem of calculus).

- for $\omega = a_i dx^i$, a 1-form on $U = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ we have:

$$\int_S d\omega = \int_S (\partial_{x^1} a_2) dx^1 \wedge dx^2 + (\partial_x^2 a_1) dx^2 \wedge dx^1 = \int_{\partial S} w$$

  where we used the fundamental theorem of calculus.

- More generally, if S is a compact subset of $\mathbb{R}^2$ with piecewise regular boundary ∂S (piecewise homeomorphic to intervals in $\mathbb{R}$) then we obtain by decomposing S in terms of little squares and interpreting $\int_U d\omega$ as a Riemann sum over the square the result

$$\int_U d\omega = \int_{\partial U} \omega$$

  This result generalizes immediately to compact co-dimension zero subsets of $\mathbb{R}^n$ with piecewise regulary boundary

- If M is a compact subset of dimension $m \leq n$ in $\mathbb{R}^n$ (with piecewise regular boundary ∂M), diffeomorphic to a compact subset of $U \subset \mathbb{R}^m$ (i.e. $M = f(U)$) and $\omega \in \Omega^{m-1}(M)$, then:

$$\int_M d\omega = \int_U f^* d\omega = \int_U df^* \omega = \int_{\partial U} f^* \omega = \int_{\partial M} \omega$$

- More generally, the parametrization of ∂M may be different from that induced by M. Then we have:

$$\int_M d\omega = \int_{\partial M} i^*\omega$$

where $i\colon \partial M \to M$ is the inclusion map of ∂M into M.

So, the most general result that we achieved is the following:

Theorem 9.1 (Stokes)**.** $\omega \in \Omega^{m-1}(\mathbb{R}^n)$. *Let M be a closed compact subset of $\mathbb{R}^n$, dim$(M) = m \leq n$, such that M is homeomorphic to a closed subset $U \subset \mathbb{R}^m$. ∂M is the boundary of M and $i\colon \partial M \to M$ is the inclusion map of ∂M into M. Then:*

$$\int_{\partial M} i^*\omega = \int_M d\omega \tag{9.1}$$

Corollary 9.1 (Fundamental theorem of line integrals)**.** *Let f be a smooth function defined near an oriented curve C in $\mathbb{R}^n$, with endpoints A and B. Then:*

$$\int df = \int \nabla f \cdot dx = f(B) - f(A) \tag{9.2}$$

Corollary 9.2 (Curl theorem or Classical Stokes theorem)**.** *Let v be a differentiable vector field defined near a surface $S \subset \mathbb{R}^3$ with boundary ∂S.*

$$\int_S n \cdot (\nabla \times v) dS = \int_{\partial S} v \cdot dx \tag{9.3}$$

where n is the normal vector on the surface at each point.

Corollary 9.3 (Divergence theorem)**.** *For a smooth vector field v defined on a solid $T \subset \mathbb{R}^3$ with boundary ∂T:*

$$\int_T \nabla \cdot v \, dV = \int_{\partial T} v \cdot n \, dS \tag{9.4}$$

where dV is the unoriented volume element.

10 Poincarè Theorem of 1-forms

Definition 10.1 (Closed and exact forms)**.** If $\omega \in \Omega^k(U)$ such that $\mathrm{d}\,\omega = 0$, then ω is closed. If there exists $\alpha \in \Omega^{k-1}(V), V \subset U$ such that $\omega = \mathrm{d}\,\alpha$ in V then ω is exact.

Proposition 10.1. *The following are equivalent:*

1. *$\omega \in \Omega^1(U)$ is exact in a connected open subset $V \subset U$*

2. *For any curve $\gamma\colon (a,b) \to U, \int_\gamma \omega$ depends only on the endpoints $\gamma(a)$ and $\gamma(b)$.*

3. *$\int_\gamma \omega = 0$ for any closed curve γ in V*

Remark 10.1 (A closed form is not always exact). If ω is exact, then it is closed (because $d^2 = 0$). But not every closed form in $\Omega^1(U), U$ open subset of $\mathbb{R}^n$ is exact. Cf. $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ in $\mathbb{R}^2$ minus the non-negative x-axis. If γ is a closed curve around the origin of $\mathbb{R}^2$, we have:

$$\int_\gamma \omega = \int_\gamma d\theta = 2\pi$$

and therefore ω cannot be exact by the previous proposition. However, we notice that we have problems only with the origin of $\mathbb{R}^2$. If we consider a subset of $\mathbb{R}^2$ which is enough far from the origin, the form would be an exact form in such subset. Indeed, we say that ω is locally exact, and the general result follows from the next theorem.

Theorem 10.1 (Poincarè theorem for 1-forms on $\mathbb{R}^n$)**.** *Let $\omega \in \Omega^1(U), U \subset \mathbb{R}^n, U$ open. Then $d\omega = 0$ if and only if for each $p \in U$ there is a neighbourhood $V \subset U$ of p and a differentiable function $f\colon V \to \mathbb{R}$ such that $\omega = df$.*

Remark 10.2. Using the Poincarè theorem for 1-forms, we can extend the definition of the integral of a closed 1-form along a **continuous** path (until now, we have always assumed the our paths were piecewise differentiable). In fact, assume that $\omega \in \Omega^1(U), \mathrm{d}\,\omega = 0$, and γ such that:

$$\gamma\colon [0,1] \to U$$

is a differentiable map. Now, we choose a partition of $[0,1]$, i.e. a collection of points $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ such that the restriction of γ to the interval (t_i, t_{i+1}) is contained in a ball B_i where ω is exact. In particular:

$$\omega = df_i, \text{ for } f_i\colon B_i \to \mathbb{R}$$

Then:

$$\int_\gamma \omega = \sum_i \left[f_i(t_{i+1}) - f_i(t_i)\right]$$

If γ is only continuous, we could still consider such a partition, and still define

$$\int_\gamma \omega = \sum_i [f_i(t_{i+1}) - f_i(t_i)]$$

The integral of γ is well defined because the definition is independent from the choice of our partition: if P is one partition and P' is a refinement of P (i.e. it is the same partition plus an extra point $t' \in (t_i, t_{i+1})$ for some i), then:

$$[f_i(t_{i+1}) - \cancel{f_i(t')}] + [\cancel{f_i(t')} - f_i(t_i)] = [f_i(t_{i+1}) - f_i(t_i)]$$

Then the integral does not change if we consider a refinement. If we consider a general partition P', we can add every point of the partition P to P', so that we get a refinement of P' that we will be called P''. The integral on the partition P' has the same value of the integral on the partition P'' by the above argument. Now, we can add every point of the partition P' to P, so to get the partition P'' again, but now we can see P'' as a refinement of P. Then the integral on P and on P'' are the same. Then also the integrals on P and P' are the same.

Now, we want to extend the above theorem to k-forms.

Definition 10.2 (Contractible set)**.** An open subset $U \subset \mathbb{R}^n$ is contractible to some point $p_0 \in U$ if there exists a differentiable map

$$\begin{aligned} H\colon U \times R &\to U \\ (p,t) &\mapsto H(p,t) \end{aligned} \tag{10.1}$$

such that $H(p,1) = p, H(p,0) = p_0, \forall\, p \in U$

Remark 10.3. To every $\omega \in \Omega^k(U)$ we can associate a k-form $\bar\omega \in \Omega^k(U \times \mathbb{R})$ defined as

$$\bar\omega = H^*\omega \tag{10.2}$$

On the other hand, any $\bar\omega \in \Omega^k(U \times \mathbb{R})$ has a unique decomposition of the form

$$\bar\omega = \omega_1 + dt \wedge \eta \tag{10.3}$$

with $i_{\partial_t}\omega_1 = 0$ and $i_{\partial_t}\eta_1 = 0$. Conversely, we can associate a k-form $\omega \in \Omega^k(U)$ to each $\bar\omega \in \Omega^k(U \times \mathbb{R})$ with the help of the inclusion map

$$\begin{aligned} i_t\colon U &\to U \times \mathbb{R} \\ p &\mapsto i_t(p) = (p,t) \end{aligned} \tag{10.4}$$

Then, $i_t^*\bar{\omega} \in \Omega^k(U)$ if $\bar{\omega} \in \Omega^k(U \times \mathbb{R})$

Furthermore, let's define the map

$$\begin{aligned} I\colon \Omega^k(U \times \mathbb{R}) &\to \Omega^{k-1}(U) \\ \eta &\mapsto I\eta \end{aligned} \tag{10.5}$$

such that

$$(I\eta)(z_1, \ldots, z_{k-1}) = \int_0^1 \eta(p,t)(\partial_t, i_{t^*}z_1, \ldots, i_{t^*}z_{k-1})dt$$

Proof. Let's choose coordinates $\{x^1, \ldots, x^n, t\}$ for $U \times \mathbb{R}$. Then we write $\bar{\omega}$ on the basis:

$$\bar{\omega} = \sum_{i_1<\ldots<i_k} a_{i_1\cdots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \sum_{i_1<\ldots<i_{k-1}} b_{i_1\ldots i_{k-1}} dt \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$$

(We can always do this, the coefficients could also be trivial). Now, we want to integrate this form. Let:

$$\begin{aligned} i_t\colon U &\to U \times \mathbb{R} \\ p &\mapsto (p,t) \end{aligned}$$

i_t is the inclusion map (it "includes" U into $U \times \mathbb{R}$ at t)...TO DO ☐

Lemma 10.1.

$$i_1^*\bar{\omega} - i_0^*\bar{\omega} = d(I\bar{\omega}) + I(d\bar{\omega}) \tag{10.6}$$

Indeed, since $H \circ i_1 = Id$ and $H \circ i_0 = p_0$, $\forall\, p \in U$ we have

$$\omega = (H \circ i_1)^*\omega = i_1^*\bar{\omega}$$

and

$$0 = (H \circ i_0)^*\omega = i_0^*\bar{\omega}$$

Then we can extend Poincarè lemma to k-forms:

Theorem 10.2. *Let U be a contractible, open subset of $\mathbb{R}^n$ and $\omega \in \Omega^k(U)$ with $d\omega = 0$. Then there exists a $(k-1)$-form $\alpha \in \Omega^{k-1}(U)$ such that $\omega = d\alpha$.*

Question: For $\omega \in \Omega^1(U)$, when is $\int_\gamma \omega$ independent of the choice of γ?

Definition 10.3 (Homotopic curves 1)**.** Two continuous curves γ_1 and γ_2, $\gamma_i\colon [a,b] \to U, i = 1,2, U \subset \mathbb{R}^n$, with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$ are homotopic relatively to $\{\gamma_1(a), \gamma_2(b)\}$ if there exists a continuous map H

$$\begin{aligned}
&H\colon [a,b]\times[0,1] \to U && \text{such that:}\\
&H(s,0) = \gamma_1(s), && \forall\, s \in [a,b]\\
&H(s,1) = \gamma_2(s), && \forall\, s \in [a,b]\\
&H(a,t) = \gamma_1(a) = \gamma_2(a), && \forall\, t \in [0,1]\\
&H(b,t) = \gamma_1(b) = \gamma_2(b), && \forall\, t \in [0,1]
\end{aligned}$$

Theorem 10.3. *Let $\omega \in \Omega^1(U)$, with $\mathrm{d}\,\omega = 0$ (closed), and γ_1, γ_2 be two homotopic curves (as in the previous definition). then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{10.7}$$

What if $\gamma_1(a) \neq \gamma_2(a), \gamma_1(b) \neq \gamma_2(b)$

Definition 10.4 (Homotopic curves 2)**.** $\gamma_1, \gamma_2\colon [a,b] \to U$, γ_i closed curves, are freely homotopic if there exists a continuous map

$$\begin{aligned}
&H\colon [a,b]\times[0,1] \to U && \text{such that:}\\
&H(s,0) = \gamma_1(s) && \forall\, s \in [a,b]\\
&H(s,1) = \gamma_2(s) && \forall\, s \in [a,b]
\end{aligned}$$

Proposition 10.2. *If ω is a closed 1-form on U, γ_1 and γ_2 two closed curves, freely homotopic in U, then:*

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega \tag{10.8}$$

In particular, if γ_1 is freely homotopic to a point, then $\int_{\gamma_1} \omega = 0$

Definition 10.5 (Simply connected set)**.** A connected open set $U \subset \mathbb{R}^n$ is simply connected if every continuous closed curve in U is freely homotopic to a point in U.

Example 10.1. $\mathbb{R}^n$, the unitary ball in $\mathbb{R}^n$ and its homeomorphic images are simply connected

Remark 10.4 (Contractible vs. simply connected). "Contractible $\Longrightarrow$ simply connected" (why?), but "Simply connected $\not\Longrightarrow$ contractible" (cf. S^2).

Remark 10.5. Every closed form on a simply connected subset U of $\mathbb{R}^n$ is exact.

11 de Rham Cohomology

We can think of $\Omega^k(U)$ as a vector space over $\mathbb{R}$

Remark 11.1. We say that $\Omega^k(U, \mathbb{Z})$ forms a group (and not a vector space) since $\mathbb{Z}$ is not a field. In contrast, $\Omega^k(U, \mathbb{R})$ is a vector space.

Definition 11.1. Let $U \subset \mathbb{R}^n$, U open, $\dim(U) = m \leq n$. Then:

- The set of closed k-forms is the k-th *cocycle group* $Z^k(U, \mathbb{R})$ (it is a group with respect to addition).
- The set of exact k-forms is the k-th *coboundary group* $B^k(u, \mathbb{R})$
- The k-th *de Rham cohomology group* $H^k(U, \mathbb{R})$ is defined as:

$$H^k(U, \mathbb{R}) = {Z^k(U, \mathbb{R})} \big/ {B^k(U, \mathbb{R})}$$

References

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