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Topic :- OR Assignment Unit - 5

Q.1 :- What is dynamic programming ? State the Bellman's "principle of optimality" in dynamic prgramming & give mathematical formulation of D.P.

Solⁿ 1 → Dynamic Programming is a mathematical technique which is often useful for making a sequence of inter-related decisions. It provides a dynamic procedure for determing the combination of decisions which maximize overall effectiveness. Classical mathematics can not handle problem with large no. of decision variable, such problem can be solved by dynamic programming. The technique decomposes the original problem in n-variables into n-sub-problems, ~~including allocation~~, each in one variable.

Dynamic Programming method is applicable in solving a wide variety of problems, including allocation, inventory control & replacement.

Bellman's Principle of Optimality:-

It states that "An optimal policy (set of decisions) has the property that whatever be the initial state & initial decisions, the remaining decision must constitute an optimal policy for the state resulting from the first decision.

The problem which does not satisfy the principle of optimality cannot be solved by using dynamic programming.

The solution of a problem (multi-stage problem) by dynamic programming involves following steps:-

(i) The mathematical formulation of the problem & the development of the functional eqn i.e, to develop a recurrence reln connecting the optimal decision function for n stage problem with the optimal decision function for the (n-1) stage sub-problem $n = 1, 2, --- n$.

(ii) To solve the functional eqn for determining the optimal policy.

first we write the optimal decision function for one stage sub-problem & solve it. Then we solve the optimal decision function for 2-stage, 3 stage --- n-stage problem.

Q.2. Define the following Dynamic programming terms :-

(i) Stage

(ii) State variable

(iii) Decision variable

(iv) Return function

(v) Recursive rel^n ship:

Sol^n:- 2:-

(i) Stage :-

→ We break a big problem into the sub-problem and each sub-problem is called as a stage.

~~(ii) State va~~

→ The point at which decisions called for, are refferred as stage.

→ Each stage can be thought of having a beginning & an end. The different stage come in sequence.

(ii) State variable:-

→ The variable that link up two stages is called the state variable.

→ At any stage, the status of the problem can be decided by the values. That value are refferred as state.

(iii) Decision Variable :-

→ A multistage decision system, in which decision & state variable can take only finite number of values can be represented graphically by a decision tree.

(iv) Return function :-

→ At each stage, a decision is made which ~~are~~ can affect the state of the system at the next state of the system ~~at the~~ & help in arriving at the optimal solⁿ at the current stage. Every decision has its merit which can be represented in an algebric eqⁿ form. This is called a return function.

(v) Recurrsive Relationship :-

Dynamic programming is both a mathematical optimization method & computer programming method. If sub problem can be nested recursively inside larger problem so that dynamic problem methods are applicable, then there is a relation b/w value of the larger problem & the value of the sub problem.

Q.3. Use Dynamic programming to solve

$\min Z = y_1^2 + y_2^2 + y_3^2$

s.t. $y_1 + y_2 + y_3 = 10$, $y_1, y_2, y_3 \geq 0$.

Solⁿ :- let $y_1 + y_2 + y_3 = c$

so, let $f_n(c)$ be the minization of z

$\Rightarrow$ for $n = 1$

$f_1(c) = y_1^2$ & $y_1 = c$

$f_1(c) = c^2$

for $n = 2$

$y_1 + y_2 = c$ let $y_1 = z$

$y_2 = c - z$

$$f_2(c) = \min_{0 \leq z \leq c} (z^2 + (c-z)^2)$$

& for $n = 3$

$y_1 + y_2 + y_3 = c$ let $y_1 = c$

$y_2 + y_3 = c - z$

$$f_3(c) = \min_{0 \leq z \leq c} \{z^2 + f_2(c-z)\}$$

Now,

$$\frac{d}{dz} f_2(c) = 0 \Rightarrow 2z + 2(c-z)(-1) = 0$$

$\Rightarrow 2z + 2(z-c) = 0$

$2z + 2z = 2c$

$$\boxed{z = c/2}$$

$y_1 + y_2 = c$

$$\boxed{y_1 = \frac{c}{2}} \; \& \; \boxed{y_2 = \frac{c}{2}}$$

$\min f_2(c-z) = z^2 + (c-z)^2 \Rightarrow \left(\frac{z}{2}\right)^2 + \left(\frac{c}{2}\right)^2 = \frac{c^2}{2}$

$$f_3(c) = \min_{0 \le z \le c} \left\{ z^2 + \frac{(c-z)^2}{2} \right\}$$

$$\frac{df_3}{dz} = 0$$

$2z + (c-z)(-1) = 0$

$$\boxed{z = c/3}$$

$y_1 = \frac{c}{3}, \quad y_2 + y_3 = \frac{2c}{3}$ — ①

from eqn ①

for 2 division, $z = \frac{c}{2}$

so, $y_2 = \left(\frac{2c}{3}\right) \times \frac{1}{2} = \frac{c}{3}$

$y_3 = \frac{c}{3}$

now we get,

$\boxed{y_1 = \frac{10}{3}}$ $\boxed{y_2 = \frac{10}{3}}$ $\boxed{y_3 = \frac{10}{3}}$

$f_3(c) = y_1^2 + y_2^2 + y_3^2$

$= \frac{100}{3}$ Ans.

Q.4. Use D.P to find max value of

Maximize $Z = x_1 \cdot x_2 \cdot \ldots \cdot x_n$

s.t. $x_1 + x_2 + x_3 \ldots x_n = c$

$x_1, x_2, x_3 \ldots x_n \geq 0.$

Sol^n-4:- Max $Z = x_1 \cdot x_2 \cdot x_3 \ldots x_{nth}$

let $y_1 + y_2 + \ldots y_n = C$

$f_n(c)$ = max attainable product

for $n = 1$ only one division

$y_1 = c$ & $f_1(c) = y_1 = c$

for $n = 2$

$y_1 + y_2 = c$, let $y_1 = z$

$y_2 = c - z$

$f_2(c) = \{z(c-z)\}_{max}$

$$f_2(c) = \max_{0 \le z \le c} \{z f_1(c-z)\}$$

for $n = 3$

$y_1 + y_2 + y_3 = c$, $y_1 = z$

$y_2 + y_3 = z$

$f_3(c) = \max(y_1 y_2 y_3)$

$$\Rightarrow \max_{0 \le z \le c} \{z \cdot f_2(c-z)\}$$

Now, $f_1(c) = c$

$$f_2(c) = \max_{0 \le z \le c} \{z \cdot f_2(c-z)\}$$

$$\Rightarrow \frac{d}{dz}(z \cdot (c-z)) = 0$$

$c - 2z = 0$

$$\boxed{z = \frac{c}{2}}$$

$$f_2(c) = z(z-c) = \frac{c}{2}\left(\frac{c}{2}\right) = \left(\frac{c}{2}\right)^2$$

$$f_3(c) = \max_{0 \le z \le c} \{ z \cdot f_2(c-z) \}$$

$$\frac{d}{dz}\left\{ z \cdot \left(\frac{c-z}{2}\right)^2 \right\} = 0$$

$$\Rightarrow \frac{(c-z)^2}{4} - z \frac{2(c-z)}{2} = 0$$

$$\Rightarrow \frac{c-z}{4} - \frac{z}{2} = 0$$

$$\boxed{z = \frac{c}{3}}$$

& $y_1 + y_2 + y_3 = c$

$y_1 + y_2 = c - z$

$y_1 + y_2 = \frac{2c}{3}$

& $y_1 = \frac{1}{2}\left(\frac{2c}{3}\right) = \frac{c}{3}$

$y_2 = \frac{1}{2}\left(\frac{2c}{3}\right) = \frac{c}{3}$

$$f_3(c) = \frac{c}{3} \times \frac{c}{3} \times \frac{c}{3} = \left(\frac{c}{3}\right)^3$$

$$\boxed{f_n(c) = \left(\frac{c}{n}\right)^3}$$

finally,

$$\boxed{y_1 = \frac{c}{n},\ y_2 = \frac{c}{n},\ y_3 = \frac{c}{n} \ \cdots\ y_n = \frac{c}{n}}$$

Q.5. Use D.P. to solve the following problem

$$\text{Min. } Z = y_1^2 + y_2^2 + y_3^2$$

s.t. $y_1 + y_2 + y_3 \geq 15$

$y_1, y_2, y_3 \geq 0$

Same procedure as Question-3.

Q.7. Use D.P. to solve the following LPP:-

$$\text{Max } Z = 3x_1 + 5x_2$$

s.t. $x_1 \leq 4$, $x_2 \leq 6$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0.$$

Sol^n-7:- Max $Z = 3x_1 + 5x_2$.

$x_1 \leq 4$, $x_2 \leq 6$ $3x_1 + 2x_2 \leq 18$

$j = 1, 2$ are states.

B_{1j}, B_{2j}, B_{3j} are state variable.

$$f_2(B_{12}, B_{22}, B_{32}) = \text{Max}(5x_2)$$

& $0 \leq x_2 \leq B_{22}$, $0 \leq x_2 \leq B_{33}/2$

so $\text{Max}(5x_2) = 5 \max(x_2)$

so $\max\{x_2\} = \min\{B_{22}, B_{32}/2\}$

$= \min(6, 18)$

$\max\{x_2\} = 6$

$$\boxed{x_2 \Rightarrow 6}$$

$$f_1(B_{11}, B_{21}, B_{31}) = \text{Max}\left\{3x_1 + f_2\left(6, 9 - \frac{3}{2}x_1\right)\right\}$$

$$= \max_{0 \le x_1 \le 4}\left\{3x_1 + 5\min\left(6, \frac{18 - 3x_1}{2}\right)\right\}$$

$$\min\left[6, \frac{18 - 3x_1}{2}\right] = \begin{cases} 6, & 0 \le x_1 \le 2 \\ \frac{18 - 3x_1}{2} & 2 \le x_1 \le 4 \end{cases}$$

$$f_1(B_{11}, B_{21}, B_{31}) = \max\begin{cases} 3x_1 + 6 \times 5, & 0 \le x_1 \le 2 \\ 3x_1 + 5\left(\frac{18 - 3x_1}{2}\right), & 2 \le x_2 \le 4 \end{cases}$$

max values of $f_1(B_{11}, B_{21}, B_{31})$ will be at $x_1 = 2$

so, $\boxed{x_1 = 2}$

$\boxed{x_2 = 6}$ &

$\max z = 3 \times 2 + 5 \times 6$

$= 36$

$\boxed{\max z = 36}$

Q.8. Use D.P to solve LPP

$$\max\ Z = x_1 + 9x_2$$

s.t. $2x_1 + x_2 \le 25$

$x_2 \le 11$

$x_1, x_2 \ge 0.$

Solⁿ:-8:- x_1, x_2 are decision var.

$j = 1, 2$ is state & B_{1j}, B_{2j} are states

for $j = 2$ $f_2(B_{12}, B_{22}) = \max(9x_2)$

now $0 \le x_2 \le 25$ & $0 \le x_2 \le 11$

so, $f_2(B_{12}, B_{22})$ will be max value of x_2

now, max value of $x_2 = 11$, which will satisfy both condition.

$x_2 = 11$ (max)

so, $f_2(B_{12}, B_{22}) = 9\max(x_2)$

$= 9 \cdot 11$

$= 99$

for $j = 1$

$$f_1(B_{11}, B_{21}) = \max\{x_1 + 9x_2\}$$

& $2x_1 + x_2 \leq 25 \Rightarrow x_2 \leq 25 - 2x_1$

$$B_{12} = B_{11} - 2x_1$$

$$x_2 \leq 11$$

$$B_{22} = B_{21} - 0$$

$$f_1(B_{11}, B_{21}) = \max\{x_1 + f_2(B_{12}, B_{22})\}$$

$$= \max\{x_1 + f_2(B_{11} - 2x_1, B_{21} - 0)\}$$

now,

$$0 \leq x_1 \leq \frac{25}{2}$$

$$f(25, 11) = \max\{x_1 + 9\min(25 - 2x_1, 11)\}$$

&

$$\min(25 - 2x_1, 11) = \begin{cases} 11 & ; \ 0 \leq x_1 \leq 7 \\ 25 - 2x_1, & 7 \leq x_1 \leq 25/2 \end{cases}$$

$$x_1 + 9\min(25 - 2x_1, 11) = \begin{cases} 99 + x_1 & 0 \leq x_1 \leq 7 \\ x_1 + 9(25 - 2x_1) & 7 \leq x_1 \leq \frac{25}{2} \end{cases}$$

so, $f_1(25,11)$ will be max at $x_1 = 7$

at $x_1 = 7$, $f_1(B_{11}, B_{22}) = 106$ (max).

so, $\boxed{x_2 = 11}$ & $\boxed{x_1 = 7}$

& max $Z = x_1 + 9x_2$

$= 7 + 11 \times 9$

$\boxed{\text{Max } Z = 106}$

Q.9. State the various steps involved for solving the multistage problem by D.P.

Solⁿ 9:- The solⁿ of a multi-stage problem by D.P. involve following steps:-

(i) The mathematical formulation of problem & the development of the functional eqⁿ i.e. to develop a recurrence relⁿ connecting the optimal decision function for n-stage problem with the optimal decision function for the (n-1) stage sub-problem $n = 1, 2, \ldots n$.

(ii) To solve the functional eqn for determining the optimal policy.

first we write the optimal decision funtion for one stage problem is solve it. Then we solve the optimal decision function for 2-stage, 3-stage - - - - n stage problem