Chapter 1: First Order ODE

Separable Equations

$$\int N(y)\,dy = \int M(x)\,dx$$

Consider change of variables

$y' = f\left(\frac{y}{x}\right)$ $y = \frac{v}{x}$ $y' = xv' + v$	$y' = f(ax + by + c)$ $u = ax + by + c$ $u' = a + by$

First Order Linear Equation

$$y' + P(x)y = Q(x)$$
$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)dx}$$

Bernoulli Equation

$$y' + P(x)y = Q(x)\,y^n$$
$$z = y^{1-n} \rightarrow z' = (1-n)y^{-n}y'$$
$$z' + (1-n)P(x)\,z = Q(x)$$

*solve z' with first order linear equation after that

Model Of Cooling

$$\frac{dT}{dt} = -k(T - T_{env})$$
$$T = T_{env} + Ae^{-kt}$$

Radioactive Decay

$$\frac{dx}{dt} = -kx$$
$$x = Ae^{-kt}, k = \frac{ln2}{t_{1/2}}$$

Chapter 2: Second Order ODE With Constant Coefficients

Homogenous Equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

$$\lambda_1, \lambda_2 \rightarrow y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$
$$\lambda \rightarrow y = C_1e^{\lambda x} + C_2xe^{\lambda x}$$
$$\lambda = a \pm bi \rightarrow y = e^{ax}\,(C_1\cos(bx) + C_2\sin(bx))$$

Non – Homogeneous Equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = R(x)$$

1) Find Homogeneous Equation → y_h
2) Find Particular Solution → y_g
3) General Solution = $y_h + y_g$

Method of Undetermined Coefficients

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = R(x)$$

Ensure that no term in a particular solution is a solution of the corresponding ODE.

$$e^{kix} = \cos(kx) + i\sin(kx)$$

Consider use of this to find real and imaginary part of solution to work with exponential

<table>
<tr><th>Case A: $r(x)$ is a polynomial</th><th>Case B: $r(x) = P(x)e^{kx}$</th><th>Case C: $r(x) \equiv P(x)e^{\alpha x}\sin\beta x$ or $r(x) \equiv P(x)e^{\alpha x}\cos\beta x$</th></tr>
<tr><td rowspan="2">Guess $y_p = Q(x)$.
1. If 0 is neither λ_1 nor λ_2, then guess $Q(x) = u(x)$.
2. If 0 is the same as either simple root λ_1 or λ_2, then guess $Q(x) = xu(x)$.
3. If 0 is the same as the repeated root λ, then guess $Q(x) = x^2u(x)$.

where u(x) is a polynomial with undetermined coefficients and with deg $(u(x)) = \deg(r(x))$.</td><td>Guess $y_p = Q(x)e^{kx}$.</td><td>Guess $z_p = Q(x)e^{(\alpha+i\beta)x} = Q(x)e^{kx}$, with $k = \alpha + i\beta$.
If $r(x)$ has $\sin\beta x$, then $y_p =$ Im $[z_p]$.
If $r(x)$ has $\cos\beta x$, then $y_p =$ Re $[z_p]$.</td></tr>
<tr><td colspan="2">1. If the value of k is neither λ_1 nor λ_2, then guess $Q(x) = u(x)$.
2. If the value of k is the same as either simple root λ_1 or λ_2, then guess $Q(x) = xu(x)$.
3. If the value of k is the same as the repeated root λ, then guess $Q(x) = x^2u(x)$.

where u(x) is a polynomial with undetermined coefficients and with deg $(u(x)) = \deg(P(x))$.</td></tr>
</table>

Method of Variation Of Parameters

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = r(x) \rightarrow y_h = C_1\,y_1(x) + C_2\,y_2(x)$$

Let $C_1 = U(x), C_2 = V(x)$

$$U = -\int \frac{y_2\,r}{y_1y_2' - y_1'y_2}\,dx$$

$$V = -\int \frac{y_1\,r}{y_1y_2' - y_1'y_2}\,dx$$

Working with Complex Numbers

1. Let $y'' + Ay' + B = f(x)e^{ikx}$, then find y where $y = (Ax + B)e^{ikx}$ then find y', y''
2. Convert y to the real or imaginary form depending on R(x)

Chapter 3 - Oscillations

Type 1 Simple Harmonic Oscillator

$$m\ddot{x} + kx = 0, \text{ where } m, k > 0$$

m is the mass and k is the spring constant.

General Solution:

$$\boxed{\begin{aligned} x(t) &= C\cos\omega t + D\sin\omega t \\ &= A\cos(\omega t - \delta) \end{aligned}}$$

Angular frequency: ω; Amplitude: A;

Phase Angle: δ

Period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$; Frequency, $f = \frac{1}{T}$;

$$w^2 = \frac{k}{m}$$

$$f = \frac{w}{2\pi} = \frac{1}{T}$$

Type 2 Forced Harmonic Oscillator

$$m\ddot{x} + kx = F(t) = F_0\cos\alpha t$$

(1) When $\alpha \neq \omega$, the general Solution is:

$$\boxed{x(t) = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)}$$

(2) **Resonance**: $\alpha = \omega$, the general Solution is:

$$\boxed{x(t) = A\cos(\omega t - \delta) + \frac{F_0/m}{2\omega}.t.\sin(\omega t)}$$

Assume initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$, we get the particular solution:

$$x(t) = \frac{F_0 t}{2m\omega}\sin(\omega t)$$

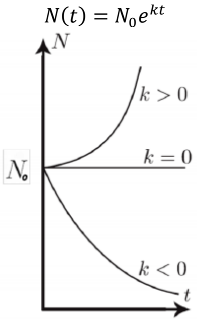
$$x'' = \frac{d}{dx}\left(\frac{(x')^2}{2}\right)$$

Chapter 3: Population

Malthus Model

$$\frac{dN}{dt} = (B - D)N$$

$$N(t) = N_0 e^{kt}$$



Logistic Growth Model

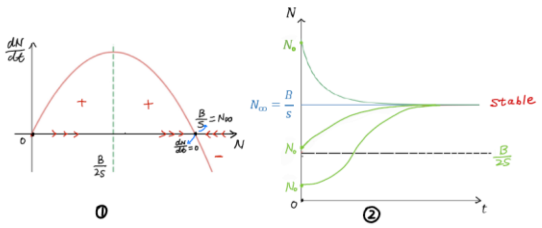
$$\frac{dN}{dt} = (B - SN)N = BN - SN^2$$

- Solution:

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{N_0} - 1\right)e^{-Bt}}$$

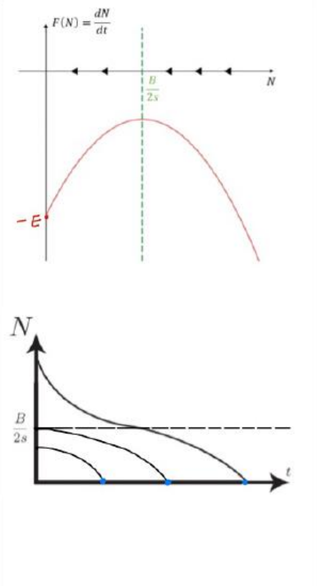
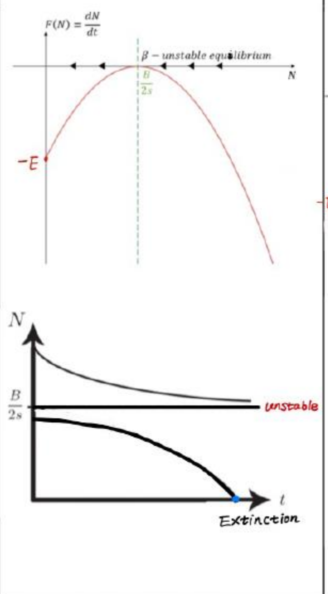
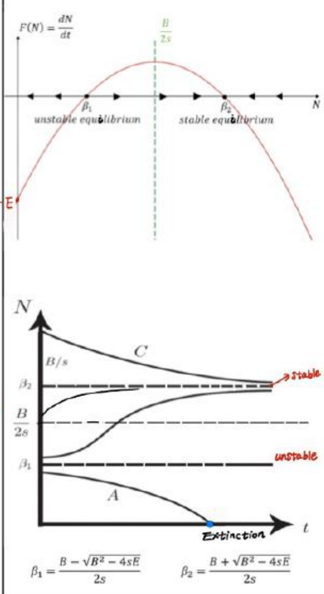
where $N_\infty = \frac{B}{s}$ (*carrying capacity/logistic population*) and $N(0) = N_0$.

- The graph of N against t is shown below, with different initial starting values of $N_0 = N(0)$.



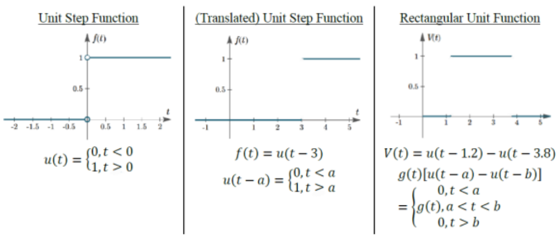
Harvesting Model with constant harvest E

$$\frac{dN}{dt} = BN - SN^2 - E \rightarrow \beta_{1,2} = \frac{B \pm \sqrt{B^2 - 4SE}}{2S}$$

Case 1: $B^2 - 4sE < 0 \Leftrightarrow E > \frac{B^2}{4s}$	Case 2: $B^2 - 4sE = 0 \Leftrightarrow E = \frac{B^2}{4s}$	Case 3: $B^2 - 4sE > 0 \Leftrightarrow E < \frac{B^2}{4s}$
		
Extinction!	$0 < x < \beta$: $\frac{dN}{dt} < 0, N\ decrease$ $x > \beta$: $\frac{dN}{dt} < 0, N\ decrease$	$0 < x < \beta_1$: $\frac{dN}{dt} < 0, N\ decrease$ $\beta_1 < x < \beta_2$: $\frac{dN}{dt} > 0, N\ increase$ $x > \beta_2$: $\frac{dN}{dt} < 0, N\ decrease$

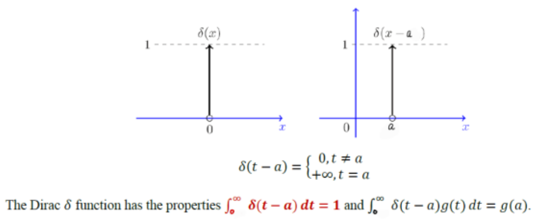
***Note that death rate = s, death rate per capita = sN**

Laplace Transform Concepts



Unit Step Function u(t-a) where a represents the start of the step

Dirac Delta



Dirac Delta – Sudden Spike to infinity

- Laplace transform $L(f)$ is a mapping L which maps a function $f(t)$ to a function $F(s)$, where $F(s)$ is given by

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

- The inverse Laplace transform is given by $L^{-1}(F(s)) = f(t)$.
- The Laplace transform and inverse Laplace's transform has the **linearity** property (α and β are constants):

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g) \qquad L^{-1}(\alpha f + \beta g) = \alpha L^{-1}(f) + \beta L^{-1}(g)$$

Laplace Transform Table

Laplace Transforms	Inverse Laplace Transforms
$L(k) = \frac{k}{s}, k \in \mathbb{R}$	$L^{-1}\left(\frac{k}{s}\right) = k, k \in \mathbb{R}$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
$L(e^{at}) = \frac{1}{s-a}, s > a$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(\cos at) = \frac{s}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L(\sin at) = \frac{a}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
$L(\cosh at) = \frac{s}{s^2-a^2}, s > \lvert a\rvert$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
$L(\sinh at) = \frac{a}{s^2-a^2}, s > \lvert a\rvert$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
$L(f(t-a)\cdot u(t-a)) = e^{-as}\cdot F(s)$ $L(u(t-a)) = \frac{e^{-as}}{s}$	$L^{-1}(e^{-as}F(s)) = f(t-a)\cdot u(t-a)$
$L(\delta(t-a)) = e^{-as}$	$L^{-1}(e^{-as}) = \delta(t-a)$
$L(y') = s\cdot L(y) - y(0)$ $L(y'') = s^2\cdot L(y) - s\cdot y(0) - y'(0)$	
$L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}L(f),\ s > 0$	
Frequency-Shifting (s-shifting): $L(e^{ct}f(t)) = F(s-c)$	Time-shifting (t-shifting): $L^{-1}(e^{-as}F(s)) = f(t-a)\cdot u(t-a)$
$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$	$L^{-1}\left(\frac{1}{(s-c)^n}\right) = \frac{e^{ct}t^{n-1}}{(n-1)!}$
$L(e^{ct}\cos\omega t) = \frac{s-c}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos\omega t$
$L(e^{ct}\sin\omega t) = \frac{\omega}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct}\sin\omega t$

Other Interesting Laplace Transforms

$$L\big(\delta(t-a)\ \times u(t-a)\big) = e^{-as}$$

Chapter 5 - Partial Differential Equations

Assume:

$$U(x,y) = X(x)Y(y)$$

Perform separation of variables, each side is equal to the same constant and integrate.

(a) $\quad yu_x - xu_y = 0$

Let $u(x,y) = X(x) \cdot Y(y)$.

Then,

$$yX'(x)Y(y) - xX(x)Y'(y) = 0$$

$$\frac{X'}{xX} = \frac{Y'}{yY} = \lambda$$

Thus, the first ODE we obtain is $\frac{X'}{X} = \lambda x$.

Integrating both sides with respect to x:

$$\ln|X| = \frac{\lambda}{2}x^2 + C_1$$

$$X = Ae^{\frac{\lambda}{2}x^2}.$$

The second ODE obtained is $\frac{Y'}{Y} = \lambda y$.

Similarly, we should get $Y = Be^{\frac{\lambda}{2}y^2}$.

Combining the two together, the solution is given by:

$$u(x,y) = XY = Ae^{\frac{\lambda}{2}x^2} \cdot Be^{\frac{\lambda}{2}y^2}$$

$$= Ce^{\frac{\lambda}{2}(x^2+y^2)} = \boxed{Ce^{d(x^2+y^2)}}.$$

Wave equation	**Heat equation**
$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \pi,\ \ t > 0$$ $$u(0,t) = 0,\ \ u(\pi,t) = 0,$$ $$u(x,0) = f(x),\ \ u_t(x,0) = 0.$$ Then, $\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$ is a solution, where **f is an odd extension with period 2π.**	$$u_t = c^2 u_{xx},$$ $$u(0,t) = 0 - (1),\ \ u(L,t) = 0, -(2)$$ $$u(x,0) = f(x). -(3)$$ Then the method of separation of variables gives the following solutions to the heat equation with the boundary conditions (1) and (2) is $$u(x,t) = B \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}$$ where B is a constant and n is an integer. One can then fit such solutions to Initial Condition (3), using the Superposition Principle if needed.

Superposition Principle

If u_1 , u_2 are solutions to a linear homogeneous DE, then so is

$$u = c_1 u_1 + c_2 u_2,$$

for any constants c_1 , c_2.

Ian's Note

Solution. The method of separation of variables gives the following solutions to the Heat Equation with Boundary Conditions 5.12(1):

$$u(x,t) = B \sin\left(\frac{n\pi}{L}x\right) \exp\left(-\left(\frac{n\pi}{L}\right)^2 c^2 t\right),$$

where B is a constant and n is an integer. One can then fit such solutions to Initial Condition 5.12(2), using Theorem 5.5 if needed.

Heat Equation

Summary

Solution of Heat equation

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{n\pi}{L}\right)x\right] e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Recall Solution of wave equation

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left[\left(\frac{n\pi}{L}\right)x\right] \cos\left[\left(\frac{n\pi}{L}c\right)t\right]$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Function	Integrand
$\frac{1}{x}$	$ln\lvert x\rvert$
a^x	$\frac{a^x}{\ln a}$
$\ln ax$	$x \ln ax - x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sec^2 x$	$\tan x$
$\csc^2 x$	$-\cot x$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$
$\csc x$	$-\ln\lvert\csc x + \cot x\rvert$
$\sec x \tan x$	$\sec x$
$\csc x \cot x$	$-\csc x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\lvert\cosh x\rvert$
$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1} \frac{x}{a}$
$\frac{1}{a^2-x^2}$	$\tanh^{-1} \frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a} \sec^{-1}\lvert\frac{x}{a}\rvert$
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$

Assorted Trig Identities

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$

$$\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$$

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + sin(x-y))$$

$$\cos x \sin y = \frac{1}{2}(\sin(x+y) - sin(x-y))$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b))$$

$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$$

$$\frac{N(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

$$\frac{N(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

$$\frac{N(x)}{(ax+b)(x^2+c)} = \frac{A}{(ax+b)} + \frac{Bx+c}{x^2+c}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$