Notes on Quantum Information: Physical Control of Qubits

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I made extensive use of IBM's Qiskit Quantum SDK and Python's QuTip package to help me generate results and animations.

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1 Introduction and Motivation

Information is stored in the physical state of a system. This is no exception in quantum computers; quantum information algorithms require reliable methods of controlling the physical state of qubits. The word "qubit" is a portmanteau of "quantum binary digit"; this is a revealing name, because it suggests that we can implement a qubit within just about any quantum two-state system. Some examples include an atom with a ground and excited state, or a quantum harmonic oscillator with two selected energy eigenstates. Although such realizations of a qubit are all described by the formulation for a two-level quantum mechanical system, they differ physically. Hence, while they share some common fundamental characteristics, there are unique properties of every physical implementation that are worthy of examination. They also offer various advantages and disadvantages when it comes to performance metrics such as speed or coherence time, which is of utmost importance to scientists building an actual quantum computer.

The reader may be familiar with some quantum communication protocols (such as quantum dense coding), or classes of powerful quantum algorithms (such as variational quantum algorithms). However, these procedures take for granted that any operation we perform on qubits will result in exactly what we intended. Moreover, we have only considered "snapshots" of qubits in the limit as time approaches zero, whereby they remain frozen in the same state. Unsurprisingly, in practice, this is not the case. Physical implementation of quantum gates will never be perfect, and so the operated qubit will almost never result in exactly the state that we hoped for. Moreover, even isolated quantum systems evolve over time, and thus the state of a qubit will also vary with time.

These set of notes serve as a comprehensive, step-by-step introduction to the physical implementation of qubits and quantum gates. Researchers are still trying to find new ways to increase the fidelity of qubit operations to the point where fault-tolerant quantum computing can become a reality. On the other hand, fault-tolerance and error correction in classical computing is so well-developed that we don't even think about (e.g when we accidentally drop our laptops). Clearly, there is still a lot of work left to be done in the quantum case. Here I attempt to offer a thorough examination of this exciting field.

2 Dynamical Evolution of a Single Qubit

2.1 Constructing a Qubit

Naturally, we begin our discussion with the simplest scenario: a single qubit. One such construction is as follows: place a silver atom with a magnetic moment $\hat{\mu}$ (provided by the one unpaired electron in the atom's 5s orbital) inside a region permeated by an applied magnetic field $\hat{B} = |B|\hat{z}$. Suppose that, upon measurement with a Stern-Gerlach Magnet, the atom's magnetic moment can only be aligned parallel or anti-parallel to the applied magnetic field. In effect, we are measuring the direction of the spin angular momentum of the atom, which is essentially the same as measuring the direction of the spin of the unpaired electron, which can only have a spin of $\frac{1}{2}\hbar$ (up) or $-\frac{1}{2}\hbar$ (down). This is very similar to the first qubits constructed by Stern and Gerlach in their world-renowned experiment, except there is no need to use an imhomogeneous magnetic field on our case (in our case, a uniform field works just fine).

In practice, more sophisticated constructions of qubit are needed for reasons such as maximizing stability and minimizing decoherence[1]. Transmon superconducting qubits, for example, are a prime example and were invented at Yale in 2007.

However, for now we are only interested in how our qubit state evolves over time, which will shed light on how all two-level quantum systems.

2.2 Brief Discussion on Schrödinger's Equation

How exactly does the state of the qubit evolve over time? The time evolution of a quantum mechanical system is governed by the Schrödinger equation, which states:

$$i\hbar\frac{\partial\left|\psi(t,x)\right\rangle}{\partial t} = \hat{H}\left|\psi(t,x)\right\rangle \tag{1}$$

where i is the imaginary unit, $\hbar$ is the reduced plank's constant ($\hbar = \frac{h}{2\pi} \approx 6.626 \cdot 10^{-34} J \cdot s$), $\left|\psi(t,x)\right\rangle$ is the wave-function of the quantum system, and $\hat{H}$ is the Hamiltonian operator, which may itself be a function of time. Here, "Hamiltonian" is synonymous with "total energy". As per the usual mathematical formalism of quantum mechanics, the eigenvalues of $\hat{H}$ give the possible measurable values corresponding to the operator (here, the possible measurable values for total energy), and the corresponding eigenvectors represent the possible energy states to which the system may collapse upon measurement in the energy basis. For a single particle, at position x during time t, the wave-function is a complex number whose norm squared gives the probability of finding the quantum system in a certain state. The wave-function is the most complete possible description of a quantum description.

One may wonder where the Schrödinger Equation came from. In a strict sense it is not "derived" but rather an experimental law, much like $F = ma$ in classical physics. But we can still try to capture its essence. The Schrödinger equation relies on two important facts: 1) the wavefunction evolving "continuously" and 2) unitarily, both with respect to time.

The "continuous" evolution of the wavefunction means that at any time we should be able to predict what the wavefunction is a small time step later. Now the evolution is not continuous, in a strict sense, since it is actually discrete; that is why I use quotation marks. The discrete quality of the time evolution is granted by $\hbar$, which has a fixed finite value. In another universe, if we were to hypothetically shrink $\hbar$ to an arbitrarily small value, then we would achieve truly continuous evolution. However, even with the value of $\hbar$ in our world, the evolution is still continuous in the sense that there are no abrupt jumps: the wavefunction at any time should be able to be inferred from the value that it possessed one discrete time step ago based on the Hamiltonian. At every point in space, the Schrödinger equation relates the small change of the wavefunction during an infinitesimal time step to the Hamiltonian operated on the wavefunction; and we can figure out the value of the wavefunction at the end of the time step by simply adding the value of the wavefunction at the beginning of the time step to the small change during the time step. This is why the Shrödinger equation is a differential equation. Furthermore, $\hbar$ is an incredibly small value, and we don't need to zoom out on the time scale much before the discrete nature of the evolution appears continuous to us (one pertinent analogy is that the sand at the beach looks smooth at a distance even though each individual grain is clearly discrete).

Unitary evolution is granted by the mathematical fact that the expression e^{iHt} (for any real parameter t, where e is Euler's constant, i is the imaginary unit, and H is a hermitian matrix) is a unitary matrix. In the Shrödinger equation, $\hat{H}$ must be hermitian since it represents a physical observable, and thus has real

[1]For example, see DiVincenzo's criteria, a set of conditions proposed in 2000 by David P. DiVincenzo that are necessary for the construction of a quantum computer.

eigenvalues. Notice also that the Shrödinger equation must have solutions that involve exponential functions, since their first derivative is related to the function itself up to a constant. Putting all these facts together and we see that solutions to the Shrödinger equation must be unitary, which is what we would want as well as expect.

2.3 Solving for Time-Evolution of the Qubit

Let us solve for $|\psi(t,x)\rangle$ for the qubit above. First, since the atom can be assumed to be a point-like particle fixed in a certain position in space, we can remove the wave-function's dependency on position and write $|\psi(t)\rangle$ instead. Essentially, we have resolved a partial differential equation into an ordinary differential equation. Secondly, we can use the usual computational basis of $|\downarrow\rangle$ and $|\uparrow\rangle$ to represent the ground and excited energy eigenstates, respectively. Because these two states are orthogonal (and thus linearly independent) and our Hilbert space only has two dimensions, these two eigenstates form a basis for all of the superposition energy states. In other words, we can write the wave-function as $|\psi(t)\rangle = \alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle = \begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}$, where $\alpha(t)$ and $\beta(t)$ are complex numbers. According to the Born rule then, given any time t, the probability of finding the atom in the excited state is given by $|\alpha(t)|^2$ and the probability of finding the atom in the ground state is given by $|\beta(t)|^2$.

Suppose that the ground state has energy $E_0 = 0$ and the excited state has energy $E_1 = \hbar\omega_0$. Note that it is only the difference $E_1 - E_0$ that is physically significant, and that is why without loss of generality we can set E_0 to be zero.

Then $\hat{H} = \frac{E_0+E_1}{2}\hat{I} + \frac{E_1-E_0}{2}\hat{\sigma_z} = \begin{pmatrix}\frac{E_0+E_1}{2} & 0\\ 0 & \frac{E_0+E_1}{2}\end{pmatrix} + \begin{pmatrix}\frac{E_1-E_0}{2} & 0\\ 0 & \frac{-E_0-E_1}{2}\end{pmatrix} = \begin{pmatrix}E_1 & 0\\ 0 & 0\end{pmatrix} = E_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \hbar\omega_0\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.

It is easy to see that the eigenvectors and eigenvalues are indeed what we want.

Plugging this matrix into the Shrödinger equation, we have:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hbar\omega_0\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}|\psi(t)\rangle \tag{2}$$

To solve for $|\psi(t)\rangle$, we can evaluate the right-hand side and left-hand side of equation (2), and then equate the $|\uparrow\rangle$ and $|\downarrow\rangle$ components separately (since $|\uparrow\rangle$ and $|\downarrow\rangle$ are orthogonal). This will reduce a matrix differential equation to a system of two ordinary differential equations.

The left-hand side of equation (2) gives:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = i\hbar\dot{\alpha}|\uparrow\rangle + i\hbar\dot{\beta}|\downarrow\rangle \tag{3}$$

Along with simplifying the notation for $\alpha(t)$ and $\beta(t)$ by removing the explicit time dependency, the right-hand side of equation (2) gives:

$$\hbar\omega_0\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}|\psi(t)\rangle = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \hbar\omega_0\alpha|\uparrow\rangle \tag{4}$$

Equating like components, we get the two differential equations:

$$\begin{cases} i\hbar\dot{\alpha} = \hbar\omega_0\alpha & (5)\\ i\hbar\dot{\beta} = 0 & (6)\end{cases}$$

Equation (5) can be rewritten as $\dot{\alpha} = \frac{\omega_0}{i}\alpha$. Differential Equations of the form $\dot{f} = kf$, where k is a constant, have solutions of the form $f(t) = f(0)e^{kt}$ (exponential). Therefore, we obtain: $\alpha(t) = \alpha(0)e^{\frac{\omega_0}{i}t} = \alpha(0)e^{-i\omega_0 t}$. Moreover, $\dot{\beta} = 0$, which implies that $\beta(t) = \beta(0)$.

Let's relabel $\alpha(0) \to \alpha_0$ and $\beta(0) \to \beta_0$. So the state $|\psi(t)\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle$ of our qubit evolves accordingly:

$$\begin{cases} \alpha(t) = \alpha_0 e^{-i\omega_0 t} & (7) \\ \beta(t) = \beta_0 & (8) \end{cases}$$

2.4 Discussion

Note that $|\alpha(t)|^2 = |\alpha_0 e^{-i\omega_0 t}|^2$ and $|\beta(t)|^2 = |\beta_0|^2$. That is, the probability of finding the qubit in either the ground or excited state remain the same over time.

However, notice that the relative phase between α and β change over time by a factor $e^{-i\omega_0 t}$. What does this mean physically? The pure state of any two-level quantum mechanical system can be represented geometrically by a vector lying on the surface of the Bloch sphere, with the azimuthal angle (angle from the positive z-axis down to the state vector) denoted by θ and the polar angle (angle from the positive-x axis to the state vector) denoted by ϕ.

Then an arbitary qubit state can be parameterized as follows: $|\hat{n}\rangle = \cos\frac{\theta}{2} |\uparrow\rangle + e^{i\phi} \sin\frac{\theta}{2} |\downarrow\rangle$.

Review: All single-qubit operations are unitary, and therefore can be considered to be rotations in a 2D Hilbert space. To rotate the state of a qubit on the Bloch Sphere an angle χ, about an axis pointing in the $\hat{w}$ direction, we multiply the qubit state by the operator:

$$U_{\hat{w}}(\chi) = e^{-i\frac{\chi}{2}\hat{w}\cdot\vec{\sigma}} \tag{9}$$

where $\vec{\sigma} = \begin{pmatrix} \hat{\sigma_x} & \hat{\sigma_y} & \hat{\sigma_z} \end{pmatrix}$.
For a (counterclockwise) rotation of angle χ about the z-axis, this reduces to:

$$U_{\hat{z}}(\chi) = e^{-i\frac{\chi}{2}\hat{\sigma_z}} \tag{10}$$

What happens when we operate on the state $|\hat{n}\rangle$? Using the facts that $\sigma_z |\uparrow\rangle = (+1) |\uparrow\rangle$ and $\sigma_z |\downarrow\rangle = (-1) |\downarrow\rangle$, as well as the relation $e^{-i\frac{\chi}{2}\hat{\sigma_z}} = \cos\frac{\chi}{2}\hat{I} - i\sin\frac{\chi}{2}\sigma^z$, we get:

$$U_{\hat{z}}(\chi) |\hat{n}\rangle = e^{-i\frac{\chi}{2}\hat{\sigma_z}} |\hat{n}\rangle = (\cos\frac{\chi}{2}\hat{I} - i\sin\frac{\chi}{2}\sigma^z)(\cos\frac{\theta}{2} |\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2} |\downarrow\rangle)$$

$= (\ \cos\frac{\chi}{2} - isin\frac{\chi}{2})(\cos\frac{\theta}{2}) |\uparrow\rangle + (\cos\frac{\chi}{2} + isin\frac{\chi}{2})(e^{i\phi}\sin\frac{\theta}{2}) |\downarrow\rangle$
$= (\mathrm{e}^{-i\frac{\chi}{2}}) \cos\frac{\theta}{2} |\uparrow\rangle + (e^{i\frac{\chi}{2}})(e^{i\phi}\sin\frac{\theta}{2}) |\downarrow\rangle$
$= (\mathrm{e}^{-i\frac{\chi}{2}})[\cos\frac{\theta}{2} |\uparrow\rangle + (e^{i\chi})(e^{i\phi}\sin\frac{\theta}{2}) |\downarrow\rangle]$
$= (\mathrm{e}^{-i\frac{\chi}{2}})[\cos\frac{\theta}{2} |\uparrow\rangle + (e^{i(\chi+\phi)}\sin\frac{\theta}{2}) |\downarrow\rangle]$
The global phase shift due to a gauge change, corresponding to the $(e^{-i\frac{\chi}{2}})$ factor, is purely a mathematical consequence and is physically meaningless and irrelevant. Hence, the overall effect of the operator is to introduce a relative phase of χ.

Returning to our qubit, this means that its Bloch sphere vector rotates at a constant rate ω_0 about the z-axis. Also note that in this case (because of our Bloch sphere parameterization) the rotation is counterclockwise, but if the coefficients α and β had been swapped in front of the computational basis states, the rotation would have been clockwise.

This constant-rate rotation means that the probability of the qubit to be measured in one of the antipodal states on the bloch sphere $|\pm X\rangle$, or $|\pm Y\rangle$, or any pair of states not equal to $|\pm Z\rangle$, will vary as a function of time. This can be significant depending on the circumstance. For instance, if our qubit was created using an atom placed within an external magnetic field and we are performing a Stern-Gerlach measurement to measure the spin angular momentum vector of the atom, then the probability that the qubit is in $|\pm\hat{n}\rangle \neq |\pm Z\rangle$ will vary with time, and therefore should be taken into account when quantum algorithms are implemented with such qubits. On the other hand, if we are simply interested in the probabilities of the qubit in the ground or excited computational basis state, then we will not be concerned so much because as we have shown, the norm-squared values of α and β are invariant with respect to time.

The phrase "Larmor precession" is used to denote the rotation of the magnetic moment of an object about an external magnetic field–in both the classical and quantum cases. It is clear then that Larmor precession will be relevant in many of the most common constructions of a qubit, namely those involving magnetic moments and applied magnetic fields.

2.5 A Real Example

In quantum computing, artificial atoms are commonly fashioned out of superconducting circuit elements called Josephson junctions. These artificial atoms are much more sophisticated versions of our silver atom qubits, but behave similarly in many ways. Most notably, they also possess a certain transition frequency between its excited and ground states, and evolves in time as prescribed by the Shrödinger Equation. These artificial atoms typically have a transition frequency (denoted ω_0) given by the relation $\frac{\omega_0}{2\pi} = 5\,\text{GHz}$, or $\omega_0 = 2\pi \cdot 5\,\text{GHz} = 31.41\,\text{GHz}$. On the electromagnetic spectrum, this frequency lies in the microwave range. On the Bloch sphere this gives a rotational period of $\frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi \cdot 5\,\text{GHz}} = \frac{1}{5\,\text{GHz}} = 2\,\text{ns}$. Indeed, these atoms precess extremely quickly. In general, the higher the atom's transition frequency, the faster its Bloch vector precesses.

Here is a link to an animation that visualizes the Bloch vector of a typical artifical atom: (click here to view). The animation was generated using QuTip, a software that uses Python to simulate the dynamics of quantum systems. I created a code that computed the values of α and β for a range of time values, and plotted each pair of these values on the Bloch sphere. From these images I was able to create a gif file which animated the bloch vector, much like a stop-frame animation movie. I also made the animations using Qiskit, by using plt.blochvector and joining a series of saved images together.

Note that QuTip points $|0\rangle$ in the positive z-direction (which I will take to be the excited state) and $|1\rangle$ in the negative z-direction (which I will take to be the ground state). Here I asummed initial conditions where $\alpha_0 = \frac{\sqrt{3}}{2}$ and $\beta_0 = \frac{1}{2}$. Here is the same rotation but viewed from the top down, which even more clearly depicts the rotation: (click here to view).

3 Resonant Rabi Oscillations and the Rotating Wave Approximation

3.1 Solving for the Time Evolution of a Driven Qubit

Consider the same construction of the qubit from before. In addition, we also apply an oscillatory magnetic field in the x-direction. Suppose this oscillatory magnetic field has a frequency ω_0 and amplitude f. Note that ω_0 is the natural transition frequency of the qubit. In other words, the qubit is driven exactly at resonance by this oscillatory field.

Then the Hamiltonian $\hat{H}$ of the qubit is given by $\hat{H} = \hat{H}_0 + \hat{V}(t)$, where $\hat{H}_0 = \hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (this is same as the last section, corresponding to the constant magnetic field in the z-direction); and $\hat{V}(t) = f \cos \omega_0 t \cdot \sigma^x$ (corresponding to the oscillating magnetic field in the x-direction), where f is a real constant with units of energy. Namely, f is the amplitude of the driven oscillations, where I can assume without loss of generality that $f > 0$. To clearly distinguish between the two different contributions to the overall Hamiltonian operator, We refer to $\hat{H}_0$ as the "system Hamiltonian", and $\hat{V}(t)$ as the "control field Hamiltonian".

Again we use the Schrödinger equation to solve for the qubit's time evolution. But let us not plug in the arbitrary state $|\psi\rangle = \alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle$ as we previously did, and instead try the ansatz $|\psi\rangle = e^{-i\omega_0 t}\alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle$.

Ansatz is a german word for "educated guess". Indeed, our ansatz is a very well-informed guess, as we did in fact obtain a constant-rate rotation about the z-axis (i.e a constant relative phase shift) in the last section where there was no driving oscillatory magnetic field. Moreover, by using this ansatz, we will have transformed from the laboratory frame to the frame rotating at frequency ω_0. It will become clear later why this transformation is beneficial.

The Schrödinger equation here is:

$$i\hbar\frac{\partial\left|\psi(t)\right\rangle}{\partial t}=\hat{H}\left|\psi(t)\right\rangle$$

Plugging our ansatz into the Schrödinger equation, and evaluating the left-hand side:

$$i\hbar\tfrac{\partial|\psi(t)\rangle}{\partial t}=i\hbar[-i\omega_0 e^{i\omega_0 t}\alpha\left|\uparrow\right\rangle+e^{-i\omega_0 t}\dot{\alpha}\left|\uparrow\right\rangle+\dot{\beta}\left|\downarrow\right\rangle]$$
$$=\hbar\omega_0 e^{-i\omega_0 t}\alpha\left|\uparrow\right\rangle+i\hbar e^{-i\omega_0 t}\dot{\alpha}\left|\uparrow\right\rangle+i\hbar\dot{\beta}\left|\downarrow\right\rangle$$
$$=[\hbar\omega_0 e^{-i\omega_0 t}\alpha+i\hbar e^{-i\omega_0 t}\dot{\alpha}]\left|\uparrow\right\rangle+i\hbar\dot{\beta}\left|\downarrow\right\rangle$$

Evaluating the right hand-side:

$$\hat{H}\left|\psi(t)\right\rangle=\hat{H}_0\left|\psi(t)\right\rangle+\hat{V}(t)\left|\psi(t)\right\rangle$$
$$=\hat{H}_0[e^{-i\omega_0 t}\alpha(t)\left|\uparrow\right\rangle+\beta(t)\left|\downarrow\right\rangle]+\hat{V}(t)[e^{-i\omega_0 t}\alpha(t)\left|\uparrow\right\rangle+\beta(t)\left|\downarrow\right\rangle]$$
$$=\hbar\omega_0 e^{-i\omega_0 t}\alpha\left|\uparrow\right\rangle+(f\cos\omega_0 t\sigma^x)[e^{-i\omega_0 t}\alpha(t)\left|\uparrow\right\rangle+\beta(t)\left|\downarrow\right\rangle]$$
$$=\hbar\omega_0 e^{-i\omega_0 t}\alpha\left|\uparrow\right\rangle+(f\cos\omega_0 t)[e^{-i\omega_0 t}\alpha(t)\left|\downarrow\right\rangle+\beta(t)\left|\uparrow\right\rangle]$$
$$=[\hbar\omega_0 e^{-i\omega_0 t}\alpha+f\cos\omega_0 t\beta]\left|\uparrow\right\rangle+[f\cos\omega_0 t e^{-i\omega_0 t}\alpha]\left|\downarrow\right\rangle$$

Equating like components, we have:

$$\begin{cases}\hbar\omega_0 e^{-i\omega_0 t}\alpha+i\hbar e^{-i\omega_0 t}\dot{\alpha}=\hbar\omega_0 e^{-i\omega_0 t}\alpha+f\cos\omega_0 t\beta, & (11)\\ i\hbar\dot{\beta}=f\cos\omega_0 t e^{-i\omega_0 t}\alpha & (12)\end{cases}$$

This system reduces to:

$$\begin{cases}i\hbar e^{-i\omega_0 t}\dot{\alpha}=f\cos\omega_0 t\beta & (13)\\ i\hbar\dot{\beta}=f\cos\omega_0 t e^{-i\omega_0 t}\alpha & (14)\end{cases}$$

multiplying equation (13) on both sides by $e^{i\omega_0 t}$, we get:

$$\begin{cases}i\hbar\dot{\alpha}=f\cos\omega_0 t e^{i\omega_0 t}\beta & (15)\\ i\hbar\dot{\beta}=f\cos\omega_0 t e^{-i\omega_0 t}\alpha & (16)\end{cases}$$

This is a coupled system of two differential equations. This system is quite difficult to solve analytically. In order to make further progress, we need to make use of a relatively simple yet elegant complex trigonometric identity, derived below.

Euler's Formula states:

$$e^{ix}=\cos x+i\sin x \tag{17}$$

Moreover,

$$e^{-ix}=e^{i(-x)}=\cos(-x)+i\sin(-x)=\cos x-i\sin x \tag{18}$$

from the fact that cosine is even and sine is odd in their arguments.
Adding (17) and (18) we get:

$$e^{ix}+e^{-ix}=2\cos(x) \tag{19}$$

which leads to the desired identity

$$\cos(x)=\frac{e^{ix}+e^{-ix}}{2} \tag{20}$$

Therefore,

$$\cos(\omega_0 t)=\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}. \tag{21}$$

.

We can roughly interpret this identity as the decomposition of a linear oscillation into a sum of two opposing angular rotations.

This allows us to rewrite our system of euqations as:

$$\begin{cases} i\hbar\dot{\alpha} = f(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2})e^{i\omega_0 t}\beta = \frac{fe^{2i\omega_0 t}}{2}\beta + \frac{f}{2}\beta & (22) \\ i\hbar\dot{\beta} = f(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2})e^{-i\omega_0 t}\alpha = \frac{fe^{-2i\omega_0 t}}{2}\alpha + \frac{f}{2}\alpha & (23) \end{cases}$$

The beauty of using the previous identity is that now we can use what is the called the rotating wave approximation (RWA). In this approximation, terms in the equations of motion of the system which oscillate at "rapid" frequencies can be neglected. What does "rapid" mean exactly? As we discussed previously, the linearly oscillating cosine term can be viewed as the sum of two opposite circularly oscillating terms. Moreover, in using our ansatz, we have transformed from the lab frame to a frame rotating with one of these circularly oscillating terms with an angular frequency ω_0. In this frame, the co-rotating term appears to be stationary, while the other circularly oscillating term appears to be rotating with an angular velocity $\omega_0 + \omega_0 = 2\omega_0$. Then we can say informally that, from our reference frame, rapidly rotating terms appear so fast that they appear "blurry", and therefore don't contribute much to the evolution of the qubit. More rigorously, we can say that the contributions to the unitary evolution of the system due to relatively rapidly rotating terms average out to zero quickly over small time intervals, and therefore don't significantly affect the overall behavior of the system. Some might wonder exactly how significant the RWA changes our solution. The answer is that the solution even when using the RWA is very close to the exact solution.

Applying the RWA to our equations of motion, we can discard the term $\frac{fe^{2i\omega_0 t}}{2}\beta$ from equation (22) as well as the term $\frac{fe^{-2i\omega_0 t}}{2}\alpha$ from equation (23). This is because both of these terms correspond to relatively rapid rotations in our transformed frame of reference. This results in much simpler equations:

$$\begin{cases} i\hbar\dot{\alpha} = \frac{f}{2}\beta & (24) \\ i\hbar\dot{\beta} = \frac{f}{2}\alpha & (25) \end{cases}$$

Now define $f \equiv \hbar\Omega$, where Ω is called the Rabi frequency, named after the American phycisist Isidor Isaac Rabi. So $\Omega = \frac{f}{\hbar}$.

Then

$$\begin{cases} \dot{\alpha} = \frac{f}{2i\hbar}\beta = -\frac{if}{2\hbar} = -\frac{i\Omega}{2}\beta & (26) \\ \dot{\beta} = -\frac{i\Omega}{2}\alpha & (27) \end{cases}$$

Taking the time derivative of equation (26), we get:

$$\ddot{\alpha} = -\frac{i\Omega}{2}\dot{\beta} \tag{28}$$

Then substituting equation (27) into equation (28), eliminating $\dot{\beta}$, we have:

$$\ddot{\alpha} = -\frac{i\Omega}{2}(-\frac{i\Omega}{2})\alpha = -(\frac{\Omega}{2})^2\alpha \tag{29}$$

Finally, we have cast the equations into a familiar form. Equation (29) exactly resembles the differential equation of a simple harmonic oscillator. A quick reminder: differential equations of the form $\ddot{f} = -k^2 f$, where k is a constant, have solutions of the form $f(t) = A\cos(kt + \phi)$, where A and ϕ are constants chosen to satisfy boundary conditions of the particular situation. But in the most general case, there is a nuance that we have not considered before: while ϕ has to be real because it is enclosed within the argument of a trigonometric function, A can be complex (in the case of an simple spring, A was simply the amplitude of the oscillations and therefore had to be real).

So from equation 29 we can deduce that:

$$\alpha(t) = A\cos\left(\frac{\Omega}{2}t + \phi\right) \tag{30}$$

where $A \in \mathbb{C}$ and $\phi \in \mathbb{R}$ are constants to be determined.

Rearranging equation (26) gives:

$$\beta = \frac{\dot{\alpha}}{-\frac{i\Omega}{2}} = (\frac{2i}{\Omega})\dot{\alpha} \tag{31}$$

Taking the time derivative of equation 30, we have:

$$\dot{\alpha} = -\frac{A\Omega}{2}\sin\left(\frac{\Omega}{2}t + \phi\right) \tag{32}$$

Finally, substituting equation (32) into equation (31) gives:

$$\beta(t) = (\frac{2i}{\Omega})[\frac{-A\Omega}{2}\sin\left(\frac{\Omega}{2}t + \phi\right)] = -iA\sin\left(\frac{\Omega}{2}t + \phi\right) \tag{33}$$

We need to satisfy the normalization requirement that $|\alpha(t)|^2 + |\beta(t)|^2 = 1$ for all times t. That is, $|A|^2(\cos\big(\frac{\Omega}{2}t + \phi\big))^2 + |A|^2(\sin\frac{\Omega}{2}t + \phi))^2 = |A|^2[(\cos\big(\frac{\Omega}{2}t + \phi\big))^2 + (\sin\big(\frac{\Omega}{2}t + \phi\big))^2] = 1.$

Since $(\cos(\Theta))^2 + (\sin(\Theta))^2 = 1$ for all arguments Θ, we have: $|A|^2(1) = 1$, or $|A|^2 = 1$. So A is a complex number with norm 1, lying on the complex unit circle. While in principle this complex number should be specified exactly by initial conditions, in this case we are only interested in the probabilities that the qubit is in the ground or excited states as a function of time, and therefore without loss of generality we can take A=1 since only its norm-squared is physically relevant.

So our qubit evolves as follows:

$$\begin{cases} \alpha(t) = \cos\left(\dfrac{\Omega}{2}t + \phi\right) & (34) \\ \beta(t) = -i\sin\left(\dfrac{\Omega}{2}t + \phi\right) & (35) \end{cases}$$

If we assume initial conditions such that the qubit starts in the ground state, and so where $|\alpha(0)|^2 = 0$ and $|\beta(0)|^2 = 1$, then without loss of generality these conditions can be satisfied if we take $\phi = -\frac{\pi}{2}$. Using the identities $\cos\big(\Theta - \frac{\pi}{2}\big) = \sin(\Theta)$ and $\sin\big(\Theta - \frac{\pi}{2}\big) = \cos(\Theta)$ for all Θ, equations (34) and (35) become:

$$\begin{cases} \alpha(t) = \sin\left(\dfrac{\Omega}{2}t\right) & (36) \\ \beta(t) = -i\cos\left(\dfrac{\Omega}{2}t\right) & (37) \end{cases}$$

Corresponding to:

$$\begin{cases} |\alpha(t)|^2 = \sin^2(\dfrac{\Omega}{2}t) & (38) \\ |\beta(t)|^2 = \cos^2(\dfrac{\Omega}{2}t) & (39) \end{cases}$$

where $|\alpha(t)|^2$ is the probability of finding the qubit in the excited state, and $|\beta(t)|^2$ is the probability of finding the qubit in the ground state.

Furthermore, using the identities $\cos^2(x) = \frac{1+\cos(2x)}{2}$ and $\sin^2(x) = \frac{1-\cos(2x)}{2}$, we can rewrite equations (36) and (37) in a more intuitive form. This gives the final solution set:

$$\begin{cases} |\alpha(t)|^2 = \frac{1}{2} - \frac{1}{2}\cos(\Omega t) & (40) \\ |\beta(t)|^2 = \frac{1}{2} + \frac{1}{2}\cos(\Omega t) & (41) \end{cases}$$

3.2 Discussion

We see the probability amplitudes oscillate out of phase with each other by π, and exhibit simple harmonic motion. Furthermore, they oscillate at precisely the Rabi frequency, Ω.

Despite of the long series of steps that we carried out to order to arrive at a final solution, we must not lose sight of the bigger picture. The takeaway is this: if we drive the qubit with an external oscillating magnetic field orthogonal to the z-axis, we can continually rotate the qubit between the ground and excited states. Equivalently, to carry out a rotation on the Bloch Sphere we can apply a resonant driving field[2] parallel to the axis of rotation. Furthermore, suppose we prepare a fiducial state of $|{\downarrow}\rangle$, and turn on the oscillatory magnetic field for a certain amount of time before we turn it off. Then we can rotate and stop the at exactly the state that we desire, if the duration is correct. This is precisely how scientists actually prepare qubits to be in arbitrary states on the bloch sphere.

Our computation also shows that even the most simple quantum systems (i.e. a two-level quantum system) behave in complex ways that evade exact solutions. The use of approximation techniques such as the RWA proves to be very convenient, and oftentimes necessary, in order for us to obtain analytical solutions.

3.3 A Real Example

Here is an animation of the qubit state (in the rotating frame) when a resonant pulse is used to drive it (NOTE: this animation isn't that smooth so I will create another one soon): (click here to view). I used the same parameters as I did in the Larmor Precession example. We see that in this frame the qubit state always lies in the y-z plane.

But remember that the transformation to the rotating frame eliminates the effect of the rotation. If we jump back to the lab frame we shall observe Larmor Precession again, and the qubit would look like this: (click here to view). As we see it follows sort of a helical path along the surface of the Bloch Sphere as the x, y, and z components along the Bloch Vector oscillate sinusoidally. This is the same animation from the top down, which shows this more clearly [NOTE: gif doesn't loop perfectly so I might create another one]: (click here to view).

The corresponding values of $|\alpha(t)|^2$ and $|\beta(t)|^2$ are visualized in Figure 1.

4 Near-Resonant Rabi Oscillations

In practice, it is hard to apply a microwave pulse exactly on resonance. Even if we get the signal very close to resonance, it is impossible to keep the microwave signal at a single frequency due to experimental imperfections as well as unavoidable random errors. While random errors are harder to correct, systematic errors are reproducible and so should be able to be mitigated. It should be illuminating to solve for the qubit evolution for driving frequencies in a range near the qubit's natural transition frequency.

Suppose that the qubit's natural transition frequency is still ω_0, but now the drive frequency is ω_1. ω_1 is not necessarily the same as ω_0. Then the Hamiltonian $\hat{H}$ of the qubit is given by $\hat{H} = \hat{H}_0 + \hat{V}(t)$, where $\hat{H}_0 = \hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the term due to the system's own contribution to the Hamiltonian; and $\hat{V}(t) = f\cos\omega_1 t \cdot \sigma^x$ is the control field's contribution to the Hamiltonian.

[2] In practice, scientists often use resonant microwave or radio-frequency pulses (through channels coupled to the qubit).

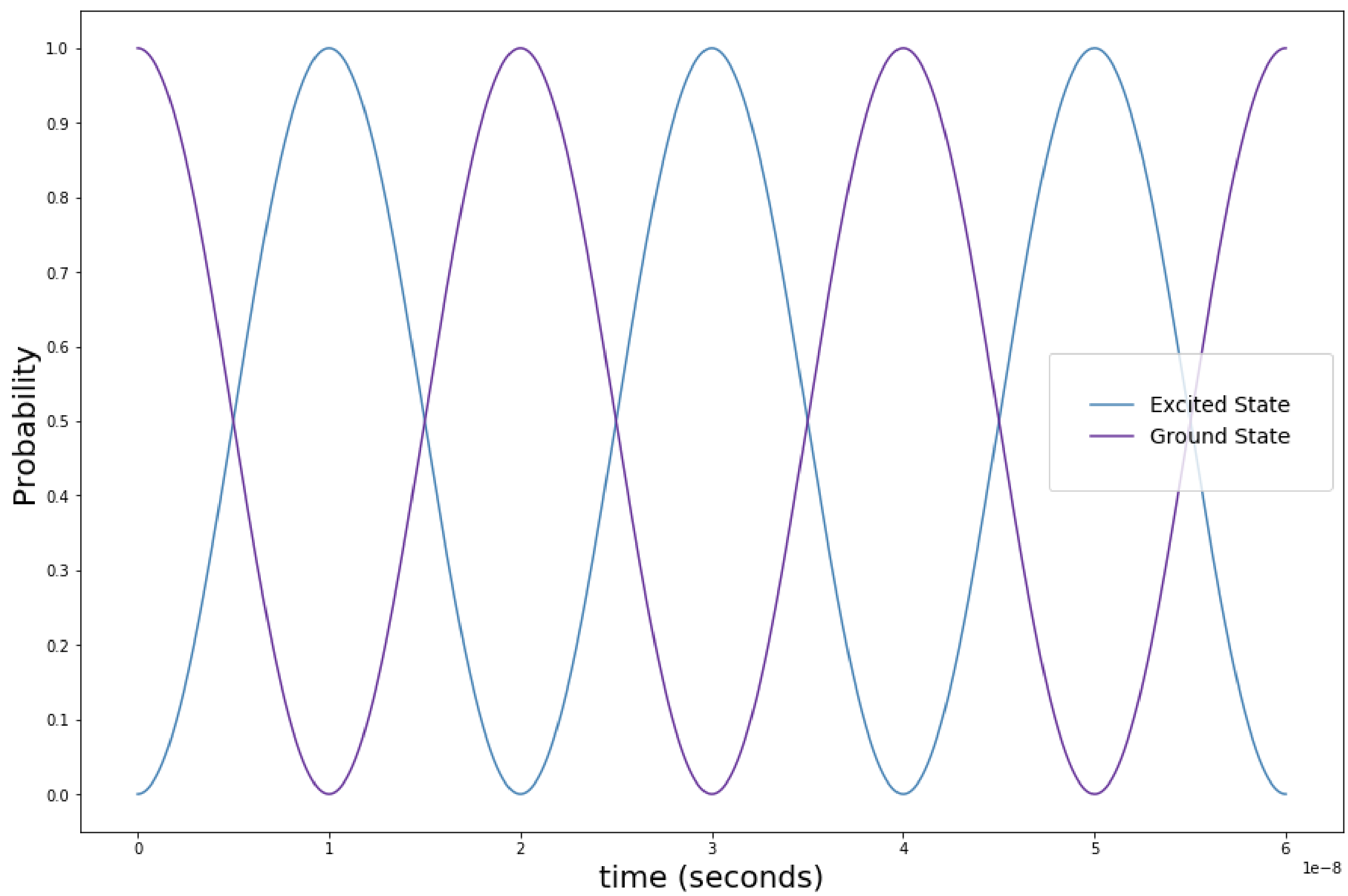

Figure 1: A plot of the probabilities of finding the qubit in the excited state and ground states as a function of time. Here I used $\Omega = 314\,\mathrm{MHz}$, which gives a period of $t = \frac{2\pi}{\Omega} \approx= 20\,\mathrm{ns}$. The probabilities each oscillate around an equilibrium probability of 0.5, with an amplitude of 0.5, and are exactly out of phase with each other. So we see that if we want to achieve a rotation from the ground to excited state, we should apply the pulse for half of the period, or $10\,\mathrm{ns}$. The duration of the pulse is referred to as the pulse length.

Again, we enter a frame rotating at the drive frequency ω_1. This can be done by using the ansatz $|\psi\rangle = e^{-i\omega_1 t}\alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle$.

The Schrödinger equation here is:

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle$$

Plugging our ansatz into the Schrödinger equation, and evaluating the left-hand side:

$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = i\hbar[-i\omega_1 e^{i\omega_1 t}\alpha|\uparrow\rangle + e^{-i\omega_1 t}\dot{\alpha}|\uparrow\rangle + \dot{\beta}|\downarrow\rangle]$
$= \hbar\omega_1 e^{-i\omega_1 t}\alpha|\uparrow\rangle + i\hbar e^{-i\omega_1 t}\dot{\alpha}|\uparrow\rangle + i\hbar\dot{\beta}|\downarrow\rangle$
$= [\hbar\omega_1 e^{-i\omega_1 t}\alpha + i\hbar e^{-i\omega_1 t}\dot{\alpha}]|\uparrow\rangle + i\hbar\dot{\beta}|\downarrow\rangle$

Evaluating the right hand-side:

$\hat{H}|\psi(t)\rangle = \hat{H}_0|\psi(t)\rangle + \hat{V}(t)|\psi(t)\rangle$
$= \hat{H}_0[e^{-i\omega_1 t}\alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle] + \hat{V}(t)[e^{-i\omega_1 t}\alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle]$
$= \hbar\omega_0 e^{-i\omega_1 t}\alpha|\uparrow\rangle + (f\cos\omega_1 t\sigma^x)[e^{-i\omega_1 t}\alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle]$
$= \hbar\omega_0 e^{-i\omega_1 t}\alpha|\uparrow\rangle + (f\cos\omega_1 t)[e^{-i\omega_1 t}\alpha(t)|\downarrow\rangle + \beta(t)|\uparrow\rangle]$
$= [\hbar\omega_0 e^{-i\omega_1 t}\alpha + f\cos\omega_1 t\beta]|\uparrow\rangle + [f\cos\omega_1 t e^{-i\omega_1 t}\alpha]|\downarrow\rangle$

Equating like components, we have:

$$\begin{cases} \hbar\omega_1 e^{-i\omega_1 t}\alpha + i\hbar e^{-i\omega_1 t}\dot{\alpha} = \hbar\omega_0 e^{-i\omega_1 t}\alpha + f\cos\omega_1 t\beta & (42) \\ i\hbar\dot{\beta} = f\cos\omega_1 t e^{-i\omega_1 t}\alpha & (43) \end{cases}$$

Subtracting $\hbar\omega_1 e^{-i\omega_1 t}\alpha$ from both sides of equation (40):

$$i\hbar e^{-i\omega_1 t}\dot{\alpha} = \hbar\omega_0 e^{-i\omega_1 t}\alpha - \hbar\omega_1 e^{-i\omega_1 t}\alpha + f\cos\omega_1 t\beta = \hbar(\omega_0 - \omega_1)e^{-i\omega_1 t}\alpha + f\cos\omega_1 t\beta \tag{44}$$

Multiplying both sides of equation (44) by $e^{i\omega_1 t}$:

$$i\hbar\dot{\alpha} = \hbar(\omega_0 - \omega_1)\alpha + f\cos\omega_1 t\beta e^{i\omega_1 t} \tag{45}$$

So now we have the system:

$$\begin{cases} i\hbar\dot{\alpha} = \hbar(\omega_0 - \omega_1)\alpha + f\cos\omega_1 t\beta e^{i\omega_1 t} & (46) \\ i\hbar\dot{\beta} = f\cos\omega_1 t e^{-i\omega_1 t}\alpha & (47) \end{cases}$$

Now define $\Delta \equiv \omega_0 - \omega_1$, called the detuning frequency. This is an appropriate name since it measures how far off the driving frequency is from resonance. Equivalently, $\omega_1 = \omega_0 - \Delta$. So:

$$\begin{cases} i\hbar\dot{\alpha} = \hbar\Delta\alpha + f\cos\omega_1 t\beta e^{i\omega_1 t} & (48) \\ i\hbar\dot{\beta} = f\cos\omega_1 t e^{-i\omega_1 t}\alpha & (49) \end{cases}$$

.

Using the identity $\cos(\omega_1 t) = \frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2}$ to rewrite the cosine terms:

$$\begin{cases} i\hbar\dot{\alpha} = \hbar\Delta\alpha + f(\frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2})\beta e^{i\omega_1 t} = \hbar\Delta\alpha + \frac{f\beta}{2}e^{2i\omega_1 t} + \frac{f\beta}{2} & (50) \\ i\hbar\dot{\beta} = (f\frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2})e^{-i\omega_1 t}\alpha = \frac{f\alpha}{2} + \frac{f\alpha}{2}e^{-2i\omega_1 t} & (51) \end{cases}$$

.

Then applying the RWA to eliminate the rapidly oscillating terms, we arrive at the desired system of differential equations:

$$\begin{cases} i\hbar\dot{\alpha} = \hbar\Delta\alpha + \dfrac{f\beta}{2} & (52) \\ i\hbar\dot{\beta} = \dfrac{f\alpha}{2} & (53) \end{cases}$$

.

As a sanity check, let's see what happens when $\Delta \to 0$. Indeed, The system reduces to equations (24) and (25), which were the differential equations in the case of a pulse with an exactly on-resonance driving frequency.

Note that this system is now harder to solve than the on-resonance case as the extra term in equation (50) causes α to be related to both its first and second time derivatives.

Differentiating equation (52) gives us:

$$i\hbar\ddot{\alpha} = \hbar\Delta\dot{\alpha} + \frac{f\dot{\beta}}{2} \tag{54}$$

Equation (53) can be rewritten as:

$$\dot{\beta} = \frac{f\alpha}{2i\hbar} = -\frac{if\alpha}{2\hbar} \tag{55}$$

Substituting equation (55) into equation (54), we have:

$$i\hbar\ddot{\alpha} = \hbar\Delta\dot{\alpha} + \frac{f}{2}(\frac{-if\alpha}{2\hbar}) = \hbar\Delta\dot{\alpha} + \frac{-if^2\alpha}{4\hbar} \tag{56}$$

This can be rewritten as:

$$\ddot{\alpha} = \frac{\hbar\Delta\dot{\alpha} + \frac{-if^2\alpha}{4\hbar}}{i\hbar} = \frac{\hbar\Delta\dot{\alpha}}{i\hbar} + \frac{(\frac{-if^2\alpha}{4\hbar})}{i\hbar} = -i\Delta\dot{\alpha} - \frac{f^2\alpha}{4\hbar^2} \tag{57}$$

Moving everything to one side gives a differential equation entirely in terms of one variable:

$$\ddot{\alpha} + (i\Delta)\dot{\alpha} + (\frac{f^2}{4\hbar^2})\alpha = 0 \tag{58}$$

In the real case, this is exactly the differential equation for a damped harmonic oscillator: differential Equations of the form $\ddot{f} + \gamma\dot{f} + \omega_0^2 f = 0$ have solutions of the form $f(t) = f_0 e^{-\frac{\gamma}{2}t}\cos\Big((\sqrt{\omega_0^2 - (\frac{\gamma}{2})^2})t + \phi\Big)$, where f_0 and ϕ are constants determined by boundary conditions, and provided that $\omega_0^2 - (\frac{\gamma}{2})^2 > 0$.

Then we have $\alpha(t) = \alpha_0 e^{\frac{-i\Delta}{2}t}\cos\Big(\sqrt{(\frac{f}{2\hbar})^2 - (\frac{i\Delta}{2})^2}t + \phi\Big)$, or:

$$\alpha(t) = \alpha_0 e^{\frac{-i\Delta}{2}t}\cos\left(\sqrt{(\frac{f}{2\hbar})^2 + (\frac{\Delta}{2})^2}t + \phi\right) \tag{59}$$

where α_0 may be complex but ϕ is real, and provided that the condition $(\frac{f}{2\hbar})^2 + (\frac{\Delta}{2})^2 > 0$ is satisfied. Since Δ^2 is always equal to or greater than zero, the above condition is always satisfied and hence equation (59) is valid for all values of Δ near zero.[3] However, if Δ gets too big, the RWA breaks down. This is because for large enough Δ the supposedly "slowly" rapidly rotating terms actually start to rotate rapidly, nearly as rapidly as the formerly "rapidly rotating terms" that are neglected when using the RWA. Therefore, equation (59) is only valid when Δ is reasonably close to zero.

Let us define $k \equiv \sqrt{(\frac{f}{2\hbar})^2 + (\frac{\Delta}{2})^2}$. This parameter will simplify the notation immensely. Rewriting equation (57):

$$\alpha(t) = \alpha_0 e^{\frac{-i\Delta}{2}t}\cos(kt + \phi) \tag{60}$$

[3] This solution also assumes that $f \neq 0$, which is simply stating that there has to be a driving pulse.

Our initial conditions stipulate that $|\alpha(t)|^2 = 0$. Then $\alpha(0) = |\alpha_0|^2 \cos^2(\phi) = 0$. $|\alpha_0|^2$ cannot be zero since this will force $\alpha(t)$ and $\beta(t) = 0$ for all time t, which violates the normalization condition). Hence $\cos^2(\phi) = 0$, meaning $\cos(\phi) = 0$. This means that $\phi = \frac{\pi}{2} + n\pi$, where n is an integer. In theory we could take any integer for n and our result for $|\alpha(t)|^2$ would be physically equivalent, but let us take $n = -1$. Equation (58) then becomes:

$$\alpha(t) = \alpha_0 e^{\frac{-i\Delta}{2}t} \cos\Big(kt - \frac{\pi}{2}\Big) \tag{61}$$

Now it becomes clear why we chose $n = -1$. Because a sine curve can be interpreted merely as a cosine curve lagging behind by a phase of $\frac{\pi}{2}$, expressed by the identity $\cos\big(\Theta - \frac{\pi}{2}\big) = \sin(\Theta)$ for any angle Θ, we can further rewrite equation (59) as:

$$\alpha(t) = \alpha_0 e^{\frac{-i\Delta}{2}t} \sin(kt) \tag{62}$$

.

The advantage of using the previous identity relatively early on is that now we have also disposed of the phase angle ϕ, which will simplify further calculations.[4]

Next we need equation for $\beta(t)$, which will then further allow us to solve for the final undetermined constant α_0. From equation (50), we can first express β in terms of α and $\dot{\alpha}$:

$$\beta = (\frac{2i\hbar}{f})\dot{\alpha} - (\frac{2\hbar\Delta}{f})\alpha \tag{63}$$

We already have an expression for α, given by equation (62). So we can take its derivative (using the product and chain rule carefully):

$$\dot{\alpha} = \alpha_0 e^{\frac{-i\Delta}{2}t}[k][\cos(kt)] + \alpha_0[\frac{-i\Delta}{2}]e^{\frac{-i\Delta}{2}t}(-\sin(kt)) \tag{64}$$

Factoring, we get:

$$\dot{\alpha} = \alpha_0 e^{\frac{-i\Delta}{2}t}([k]\cos(kt) + [\frac{-i\Delta}{2}]\sin(kt)]) \tag{65}$$

Plugging equation (65) and equation (62) into equation (63), we have:

$$\beta = (\frac{2i\hbar}{f})\alpha_0 e^{\frac{-i\Delta}{2}t}([k]\cos(kt) + [\frac{-i\Delta}{2}]\sin(kt)]) - (\frac{2\hbar\Delta}{f})\alpha_0 e^{\frac{-i\Delta}{2}t}\sin(kt) \tag{66}$$

Factoring and rearranging gives us:

$$\beta(t) = (\frac{2\hbar}{f})\alpha_0 e^{\frac{-i\Delta}{2}t}[ik\cos(kt) + \frac{\Delta}{2}\sin(kt) - \Delta\sin(kt)] \tag{67}$$

Grouping the like sine terms, we finally get:

$$\beta(t) = (\frac{2\hbar}{f})\alpha_0 e^{\frac{-i\Delta}{2}t}[ik\cos(kt) - [\frac{\Delta}{2}]\sin(kt)] \tag{68}$$

Initial conditions furgther stipulate that $|\beta(0)|^2 = 1$. We need to be careful and remind ourselves that $|\beta(0)|^2 = \beta(0) * \overline{\beta(0)}$, where the overline denotes the complex conjugate.

Since $\beta(0) = \frac{2\hbar}{f}\alpha_0(-ik) = -\frac{2\hbar}{f}\alpha_0(ik)$, $\overline{\beta(0)} = \frac{2\hbar}{f}\alpha_0(ik)$. Therefore, $|\beta(0)|^2 = (\frac{4\hbar^2}{f^2})|\alpha_0|^2(k)^2 = 1$. This means that $|\alpha_0|^2 = (\frac{f^2}{4\hbar^2 k^2})$. As in the last section, in principle we should specify this number further using

[4] Some might argue that eliminating ϕ at this stage reduces the generality of the final solution set. While this is somewhat true, we take the fiducial state of $|\downarrow\rangle$ throughout these notes since it serves as a insightful benchmark to use to compare the resonant and off-resonant cases. Furthermore, if we were not to eliminate ϕ right now, the resulting equations become very cumbersome to examine, even though the extra ϕ's will not reveal anything particularly novel in the most general case.

boundary conditions, but for simplicity we suppose that $|\alpha_0|$ lies purely along the real axis of the complex plane and therefore has no imaginary component, since it is only the norm-squared of α_0 which will actually be physically valuable. Then $\alpha_0 = \pm\sqrt{\frac{f^2}{4h^2k^2}}$.

This gives the system:

$$\begin{cases} \alpha(t) = (\pm\sqrt{\dfrac{f^2}{4h^2k^2}})e^{\frac{-i\Delta}{2}t}\sin(kt) & (69) \\ \beta(t) = (\dfrac{2\hbar}{f})(\pm\sqrt{\dfrac{f^2}{4h^2k^2}})e^{\frac{-i\Delta}{2}t}[ik\cos(kt) - [\dfrac{\Delta}{2}]\sin(kt)] & (70) \end{cases}$$

where $k = \sqrt{(\frac{f}{2\hbar})^2 + (\frac{\Delta}{2})^2}$.

Let us again use the Rabi Frequency $\Omega \equiv \frac{f}{\hbar}$, defined identically as in the last section.

Then equations (69) and (70) become:

$$\begin{cases} \alpha(t) = (\pm\sqrt{\dfrac{\Omega^2}{4k^2}})e^{\frac{-i\Delta}{2}t}\sin(kt) & (71) \\ \beta(t) = \dfrac{2}{\Omega}(\pm\sqrt{\dfrac{\Omega^2}{4k^2}})e^{\frac{-i\Delta}{2}t}[ik\cos(kt) - [\dfrac{\Delta}{2}]\sin(kt)] & (72) \end{cases}$$

where $k = \sqrt{(\frac{\Omega}{2})^2 + (\frac{\Delta}{2})^2} = \sqrt{\frac{\Omega^2}{4} + \frac{\Delta^2}{4}} = \sqrt{\frac{\Omega^2+\Delta^2}{4}} = \frac{1}{2}\sqrt{\Omega^2+\Delta^2}$. Then $k^2 = \frac{\Omega^2+\Delta^2}{4}$. Plugging this expression for k^2 into equations (71) and (72) gives:

$$\begin{cases} \alpha(t) = (\pm\sqrt{\dfrac{\Omega^2}{\Omega^2+\Delta^2}})e^{\frac{-i\Delta}{2}t}\sin(kt) & (73) \\ \beta(t) = \dfrac{2}{\Omega}(\pm\sqrt{\dfrac{\Omega^2}{\Omega^2+\Delta^2}})e^{\frac{-i\Delta}{2}t}[ik\cos(kt) - \dfrac{\Delta}{2}\sin(kt)] & (74) \end{cases}$$

The norm squared of these expressions correspond to the probabilities of finding the qubit in the ground and excited state as a function of time, given by:

$$\begin{cases} |\alpha(t)|^2 = \dfrac{\Omega^2}{\Omega^2+\Delta^2}\sin^2(kt) & (75) \\ |\beta(t)|^2 = \dfrac{4}{\Omega^2}(\dfrac{\Omega^2}{\Omega^2+\Delta^2})[k^2\cos^2(kt) + \dfrac{\Delta^2}{4}\sin^2(kt)] & (76) \end{cases}$$

Using the identities $\cos^2(x) = \frac{1+\cos(2x)}{2}$ and $\sin^2(x) = \frac{1-\cos(2x)}{2}$, we can write:

$$\begin{cases} |\alpha(t)|^2 = \dfrac{\Omega^2}{\Omega^2+\Delta^2}\dfrac{1-\cos(2kt)}{2} & (77) \\ |\beta(t)|^2 = \dfrac{4}{\Omega^2}(\dfrac{\Omega^2}{\Omega^2+\Delta^2})[k^2\dfrac{1+\cos(2kt)}{2} + \dfrac{\Delta^2}{4}\dfrac{1-\cos(2kt)}{2}] & (78) \end{cases}$$

Cleaning up the expressions, bringing the $\frac{1}{2}$ factor in equation (77) to the front and substituting in $k^2 = \frac{\Omega^2+\Delta^2}{4}$ into equation (78):

$$\begin{cases} |\alpha(t)|^2 = \dfrac{1}{2}\dfrac{\Omega^2}{\Omega^2+\Delta^2}(1-\cos(2kt)) & (79) \\ |\beta(t)|^2 = \dfrac{4}{\Omega^2+\Delta^2}[\dfrac{\Omega^2+\Delta^2}{4}\dfrac{1+\cos(2kt)}{2} + \dfrac{\Delta^2}{4}\dfrac{1-\cos(2kt)}{2}] & (80) \end{cases}$$

Simplifying equation (80) by distributing out the $\frac{4}{\Omega^2+\Delta^2}$ term:

$$\begin{cases} |\alpha(t)|^2 = \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2}(1-\cos(2kt)) & (81) \\ |\beta(t)|^2 = \frac{1+\cos(2kt)}{2} + \frac{\Delta^2}{\Omega^2+\Delta^2}\frac{1-\cos(2kt)}{2}] & (82) \end{cases}$$

Factoring out the $\frac{1}{2}$ factor in equation (82), and bringing it to the front:

$$\begin{cases} |\alpha(t)|^2 = \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2}(1-\cos(2kt)) & (83) \\ |\beta(t)|^2 = \frac{1}{2}[1+\cos(2kt) + \frac{\Delta^2}{\Omega^2+\Delta^2}(1-\cos(2kt)] & (84) \end{cases}$$

Where (emphasizing again for clarity) $k = \frac{1}{2}\sqrt{\Omega^2+\Delta^2}$. To verify the validity of these equations, let us check that equations (81) and (82) satisfy the normalization condition:

$|\alpha(t)|^2 + |\beta(t)|^2$
$= \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2}(1-\cos(2kt)) + \frac{1}{2}[1+\cos(2kt) + \frac{\Delta^2}{\Omega^2+\Delta^2}(1-\cos(2kt)]$
$= \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2} - \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2}(\cos(2kt)) + \frac{1}{2} + \frac{1}{2}(\cos(2kt)) + \frac{1}{2}\frac{\Delta^2}{\Omega^2+\Delta^2} - \frac{1}{2}\frac{\Delta^2}{\Omega^2+\Delta^2}\cos(2kt))$
$= \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2} + \frac{1}{2}\frac{\Delta^2}{\Omega^2+\Delta^2} - \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2}\cos(2kt) - \frac{1}{2}\frac{\Delta^2}{\Omega^2+\Delta^2}\cos(2kt) + \frac{1}{2}\cos(2kt) + \frac{1}{2}$
$= \frac{1}{2}\frac{\Omega^2+\Delta^2}{\Omega^2+\Delta^2} - \frac{1}{2}\frac{\Omega^2+\Delta^2}{\Omega^2+\Delta^2}\cos(2kt) + \frac{1}{2}\cos(2kt) + \frac{1}{2}$
$= \frac{1}{2} - \frac{1}{2}\cos(2kt) + \frac{1}{2}\cos(2kt) + \frac{1}{2}$
$= \frac{1}{2} + \frac{1}{2}$
$= 1$
as desired.

4.1 Summary of Solution and Discussion

4.1.1 Incomplete Bloch Sphere Rotation

The last section showed that even the calculations for the evolution of a single qubit can involve extremely long calculations and we must be very careful in our algebra every step of the way.

There is a lot of information to be unpacked and analyzed in our solution. To start off, let us summarize our results so far. For a qubit with a transition frequency ω_o driven by an oscillatory pulse of frequency ω_1 and amplitude f along the x-axis, its state $|\psi(t)\rangle = \alpha(t)|\uparrow\rangle + \beta(t)|\downarrow\rangle$ evolves in time according to the following equations:

$$\begin{cases} \alpha(t) = (\pm\sqrt{\frac{\Omega^2}{\Omega^2+\Delta^2}})e^{\frac{-i\Delta}{2}t}\sin(kt) & (85) \\ \beta(t) = \frac{2}{\Omega}(\pm\sqrt{\frac{\Omega^2}{\Omega^2+\Delta^2}})e^{\frac{-i\Delta}{2}t}[ik\cos(kt) - \frac{\Delta}{2}\sin(kt)] & (86) \end{cases}$$

Where $\Omega = \frac{f}{\hbar}$ is the Rabi frequency, $\Delta = \omega_0 - \omega_1$ is the detuning, and $k = \frac{1}{2}\sqrt{\Omega^2+\Delta^2}$. We know that the evolution is unitary and hence physically valid since we checked that $|\alpha(t)|^2 + |\beta(t)|^2 = 1$ for all times t. For the case when $\Delta = 0$, our equations reduce to equations (36) and (37), which are precisely the equations governing the evolution of the qubit of the resonant case. On the other hand, when $\Delta \neq 0$, the qubit no longer rotates about the x-axis, but instead about an axis that is slightly tilted from the x-axis. Nevertheless, the evolution is still a constant rate-rotation.

Interestingly, notice that the parameter k characterizes the rate of the rotation and is a function of both the drive amplitude as well as the detuning: the qubit rotates quicker for higher drive amplitudes and higher detuning values. In summary, both off-resonance errors (affects axis of rotation as well as rotation frequency) and drive amplitude errors (affects rotation frequency) will negatively affect the intended qubit rotation.

Notice that if the intended rotation was a π rotation to take the qubit from the ground state to the excited state, for off-resonant driving pulses the qubit will never reach the excited state fully since the rotation axis is tilted from the x-axis.

4.2 Introduction to Fidelity

The amplitude of $\alpha(t)$ measures the maximum extent to which the qubit can rotate to the excited state, given by $\sqrt{\frac{\Omega^2}{\Omega^2+\Delta^2}}$. I will name this expression α_{max}. If we divide both the numerator and the denominator of the fraction inside the square root by Ω^2, $\alpha_{max} = \sqrt{\frac{1}{1+\frac{\Delta^2}{\Omega^2}}}$. Define $\gamma \equiv \frac{\Delta}{\Omega}$, which is the ratio between the detuning value and the Rabi frequency. Then $\alpha_{max} = \sqrt{\frac{1}{1+\gamma^2}}$.

Therefore, the extent to which the qubit rotates to the excited state is dependent on the ratio between the detuning and the Rabi frequency (remember that the Rabi frequency is in turn a function of the drive amplitude). As the magnitude of γ increases, A decreases and therefore the maximum extent to which the qubit can rotate to the excited state decreases. This also gives us another insight: we can mitigate the negative effects of detuning if we decrease the size of γ by increasing Ω. We can achieve this by simply increasing our drive amplitude. Note that this will also increases the speed of rotation, however, and we may need to recalibrate the duration for our π pulse.

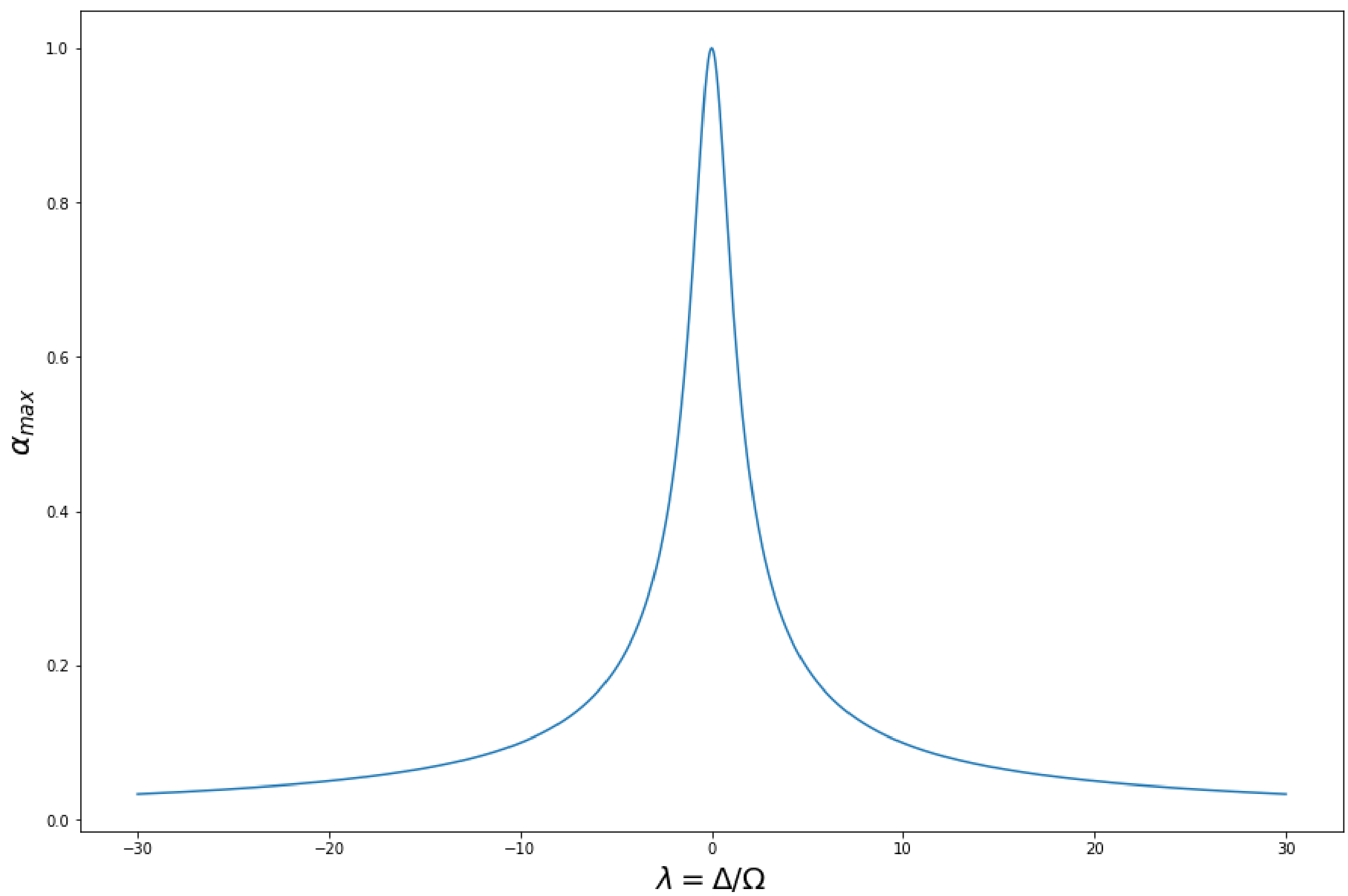

Figure 2: A plot of α_{max} as we vary γ. α_{max} achieves a maximum of 1 when $\gamma = 0$ and falls asymptotically and symmetrically on both sides as $|\gamma|$ increases.

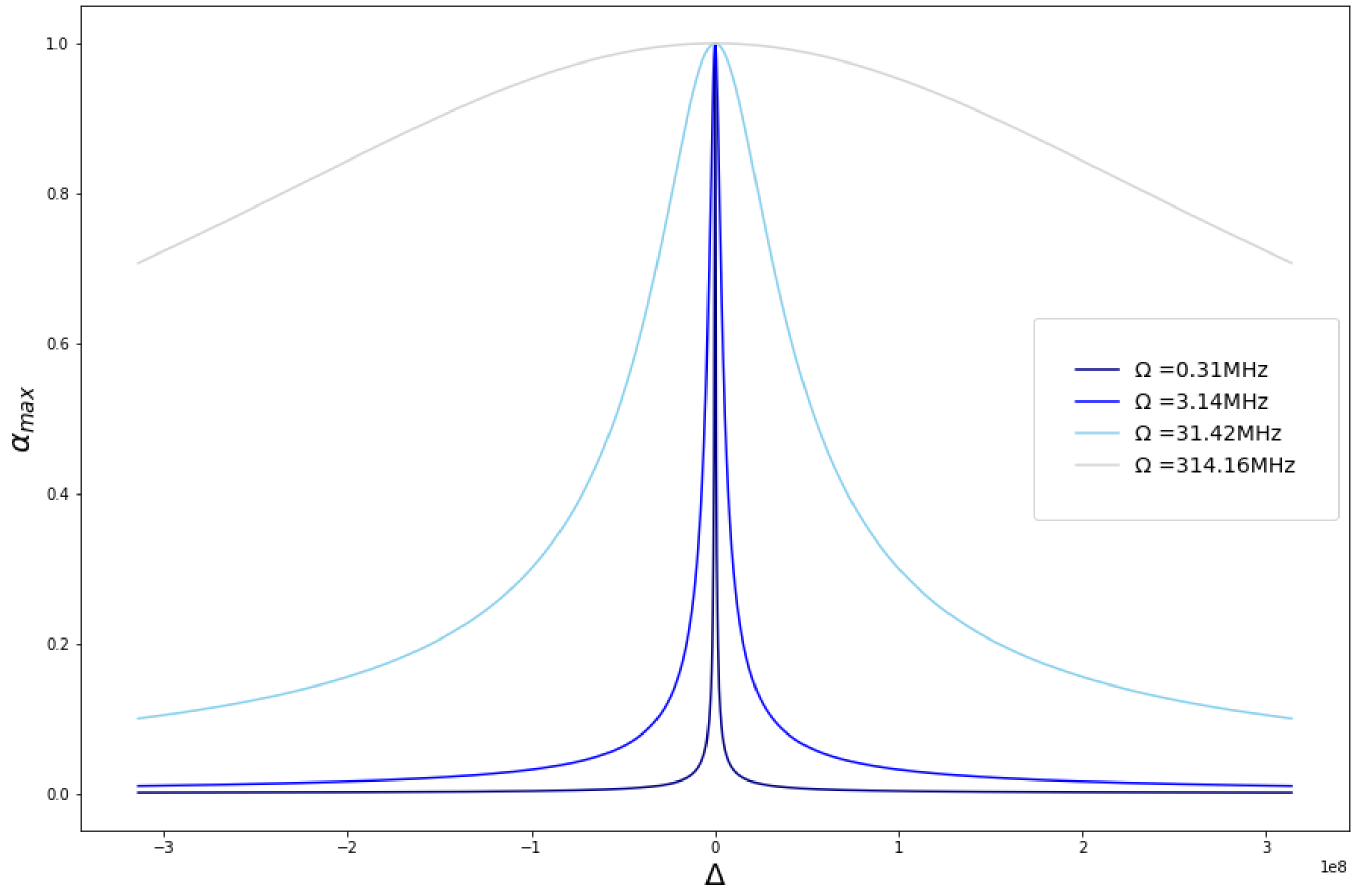

Figure 3: A plot of α_{max} as a function of Δ, for four different values of Ω. This graph was generated using real numerical values, where I have assumed the qubit's natural transition frequency is again 31.41 GHz, as in all the previous examples. The x-axis stretches a mere 1 percent in either direction relative to the resonant frequency (range of $\pm 3.141 \times 10^8$ Hz). Typically the largest Rabi frequencies we can achieve are around 314 MHz, which is equivalent to 314 MHz $= \Omega = \frac{f}{\hbar}$. This corresponds to $f \approx 3.3115 \times 10^{-26}$ J. The graph shows that α_{max} decreases at a slower rate for higher Rabi frequencies, indicated by progressively lighter shades. Therefore, for higher Rabi frequencies, the qubit is more robust against off-resonance errors. However, in order to achieve the extremely high fidelity rates that will make quantum computing a reality, we require near-perfect fidelity where $\alpha_{max} = 0.999999$, or much greater. It is easy to see from the graph that even for the highest Rabi frequency (in grey) such a fidelity requires almost perfectly resonant pulses. To be more exact, in order to achieve $\alpha_{max} = 0.999999$ for the driving pulse with $\Omega = 314$ MHz, we must be within $\pm 4.440\,63 \times 10^5$ Hz of the qubit's natural frequency, which corresponds to only roughly 0.1414% off resonance. What's more, very high Rabi frequencies require more power from cooling units, and beyond a certain frequency threshold the qubit may even be forced into higher excited states other than the desired two states. Clearly just maxing out Ω is not sufficient for us to successfully mitigate off-resonance errors.

4.3 Geometrical Interpretation

Let us now examine our results from a geometric perspective. We return to equations (52) and (53), which represented the differential equations that contained information on the qubit evolution after applying the RWA.

First, the Hamiltonian in the rotating wave approximation H_{RWA} is $H_{RWA} = \frac{\Delta}{2}\hat{I} + \frac{\Delta}{2}\sigma^z + \frac{\Omega}{2}\sigma^x = \lambda \cdot \hat{I} + \hat{B} \cdot \hat{\sigma}$, where $\lambda = \frac{\Delta}{2}$ and $\hat{B} = (\frac{\Omega}{2}, 0, \frac{\Delta}{2})$. The identity matrix is simply a relative potential energy shift (physically insignificant) as we discussed before. This Hamiltonian corresponds to the right-hand sides of equations (52) and (53), condensed into matrix form.

Furthermore, while H_{RWA} does not give us exactly the rotation matrix that represents the operator that takes the initial qubit state to the final qubit state, we can find this desired rotation matrix by resolving to the fact that $U = e^{-iH_{RWA}t}$, where U is the desired represents the desired (unitary) rotation matrix, and t is some real parameter.

So we have that $U = e^{-\frac{i}{2}[\Delta\hat{I}+\Omega\sigma^x+\Delta\sigma^z]t}$. We can rewrite this expression by diving all the terms in the square brackets by Ω and then multiplying the entire exponent by Ω, which gives us $U = e^{-\frac{i}{2}[\frac{\Delta}{\Omega}\hat{I}+\sigma^x+\frac{\Delta}{\Omega}\sigma^z]\Omega t}$. Using $\gamma = \frac{\Delta}{\Omega}$ as we defined before and now defining $\theta = \Omega t$, we get: $U = e^{-\frac{i}{2}[\gamma\hat{I}+\sigma^x+\gamma\sigma^z]\theta}$. If we choose to dispense of the physically irrelevant potential energy shift, then we can dispense of the identity matrices and write $U = e^{-\frac{i}{2}[\sigma^x+\gamma\sigma^z]\theta} = e^{-i\frac{\theta}{2}[\sigma^x+\gamma\sigma^z]}$.

From the review on page 5, we know that a rotation of an angle χ, about an axis pointing in the $\hat{w}$ direction, is given by: $U_{\hat{w}}(\chi) = e^{i\frac{\chi}{2}\hat{w}\cdot\vec{\sigma}}$. Therefore in our case, the rotation axis is $\hat{\omega} = (1, 0, \gamma)$.

In the ideal case we would like to achieve $\hat{\omega} = (1, 0, 0)$ (a rotation about the x-axis only) but instead we have a rotation about some axis tilted with both x and z components.

Let us denote the angle between the x-axis and the actual rotation axis as Θ. We can interpret Θ to be a measure of how far off the actual rotation axis is from the ideal rotation axis. From simple trigonometry we see that $\tan(\Theta) = \frac{Opp}{Adj} = \frac{\gamma}{1} = \gamma$, and therefore $\Theta = \arctan(\gamma)$. As $\gamma \longrightarrow \infty$, $\Theta \longrightarrow \frac{\pi}{2}$, and so the rotation axis approaches the z-axis. A graph of Θ is provided below:

While the fiducial state of $|\downarrow\rangle$ is trivially achieved in all cases (mimimum component along the $-z$ axis is $z = -1$), the imperfect rotation means that the maximal component achieved along the $+z$ axis (i.e. the maximal projection along the z-axis) , is $z = \cos(\Theta)$. Again from inspecting the geometry, $\cos(\theta) = \frac{Adj}{Hypo} = \frac{1}{\sqrt{1+\gamma^2}}$. Equivalently, $\cos(\theta) = \frac{Adj}{Hypo} = \frac{\frac{\Omega}{2}}{\sqrt{(\frac{\Omega}{2})^2+(\frac{\Delta}{2})^2}} = \sqrt{(\frac{(\frac{\Omega}{2})^2}{(\frac{\Omega}{2})^2+(\frac{\Delta}{2})^2}} = \sqrt{\frac{\frac{\Omega^2}{4}}{\frac{\Omega^2}{4}+\frac{\Delta^2}{4}}} = \sqrt{\frac{\Omega^2}{\Omega^2+\Delta^2}} = \sqrt{\frac{1}{1+\gamma^2}}$. Both turn out to be exactly what we defined α_{max} as earlier.

4.3.1 Population Transfer

The norm-squared of equations for $\alpha(t)$ and $\beta(t)$ correspond to the actual probabilities of finding the qubit in the excited and ground states, respectively:

$$\begin{cases} |\alpha(t)|^2 = \frac{1}{2}\frac{\Omega^2}{\Omega^2+\Delta^2}(1-\cos(2kt)) & (87) \\ |\beta(t)|^2 = \frac{1}{2}[1+\cos(2kt)+\frac{\Delta^2}{\Omega^2+\Delta^2}(1-\cos(2kt)] & (88) \end{cases}$$

Again, in the case where $\Delta = 0$, these equations reduce to equations (38) and (39), as expected. Let us gather some more insight by introducing γ as we defined previously. We can rewrite equations (85) and (86) as:

$$\begin{cases} |\alpha(t)|^2 = \frac{1}{2}\frac{1}{1+\gamma^2}(1-\cos(2kt)) & (89) \\ |\beta(t)|^2 = \frac{1}{2}[1+\cos(2kt)+\frac{1}{(\frac{1}{\gamma})^2+1}(1-\cos(2kt)] & (90) \end{cases}$$

Substituting in $k = \frac{1}{2}\sqrt{\Omega^2+\Delta^2}$, we get:

$$\begin{cases} |\alpha(t)|^2 = \frac{1}{2}\frac{1}{1+\gamma^2}(1-\cos\Big(\sqrt{\Omega^2+\Delta^2}t\Big)) & (91) \\ |\beta(t)|^2 = \frac{1}{2}[1+\cos\Big(\sqrt{\Omega^2+\Delta^2}t\Big)+\frac{1}{(\frac{1}{\gamma})^2+1}(1-\cos\Big(\sqrt{\Omega^2+\Delta^2}t\Big)] & (92) \end{cases}$$

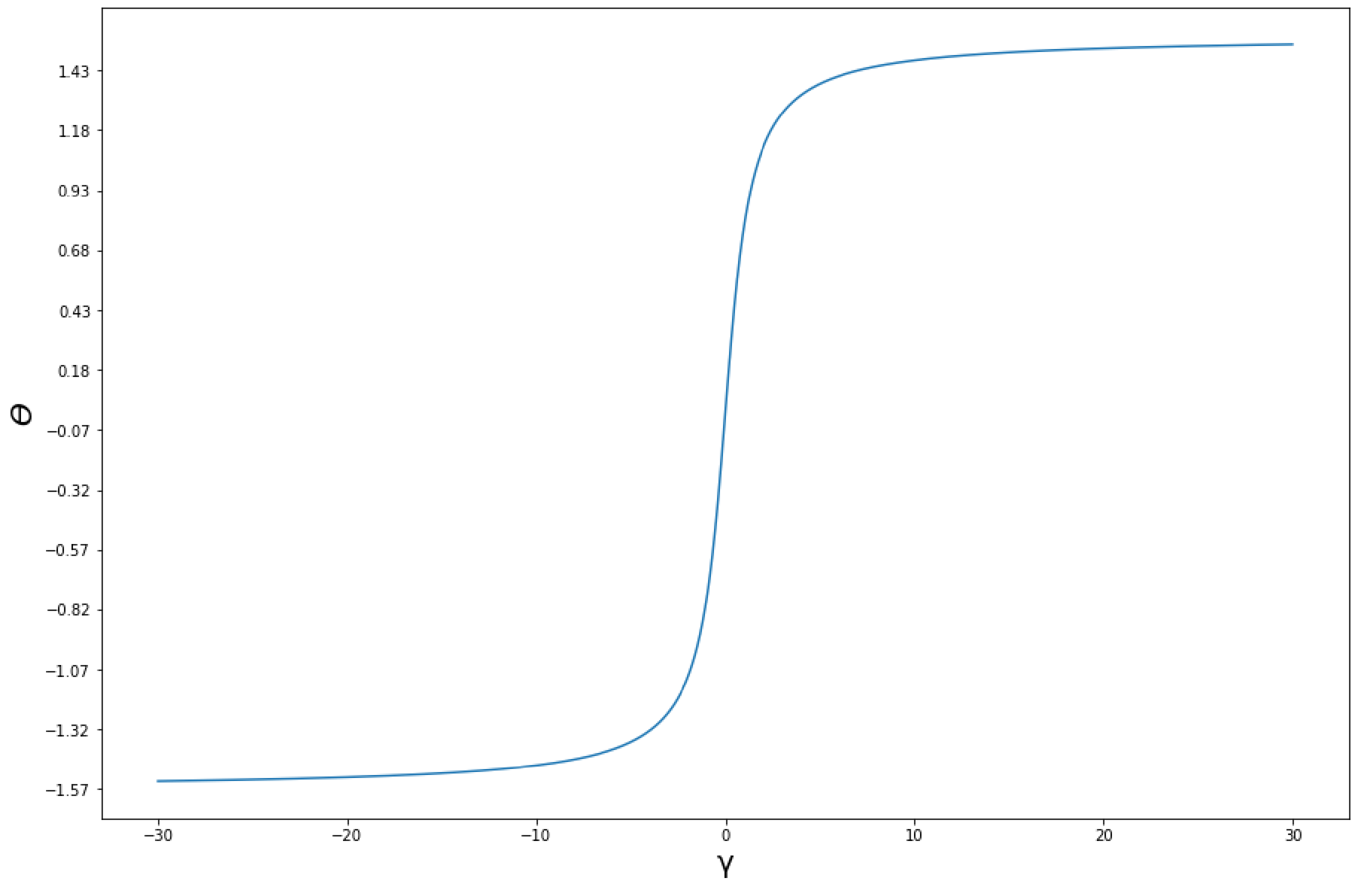

Figure 4: A plot of Θ as we vary γ. When γ approaches positive infinity, Θ approaches $\frac{\pi}{2}$ and the true rotation axis approaches the +z axis; when γ is very small, γ approaches negative infinity and the true rotation axis approaches the -z axis. What differs in the two extreme detuning cases is that the qubit state rotates in opposite directions, but otherwise the true rotation axis is identically faulty.

So we see again that the one-half factor in k vanishes when we consider the probabilities, and the rate of oscillations is given by $\sqrt{\Omega^2+\Delta^2}$. Clearly when $\Delta = 0$, the probabilities oscillate at precisely the Rabi frequency Ω and increases as Δ increases.

Again, the probabilities of finding the qubit in the ground and excited states are exchanged back and forth. The formal name for this exchange is called "population transfer". With an ideal π pulse, we would like to achieve full population transfer (that is, we would like to achieve $|\alpha(t)|^2 = 1$ for some time(s) t when starting off with $|\alpha(0)|^2 = 0$). But if $\Delta \neq 0$, then this population transfer does not occur completely. Analogous to the case of the Bloch sphere rotation where the amplitude of $\alpha(t)$ measured the extent to which the qubit can rotate to the excited state, the amplitude of $|\alpha(t)|^2$ measures the extent to which population transfer can be achieved, which is given by the expression $\frac{1}{2}\frac{1}{1+\gamma^2}$, which I will name A_{pop}. In the case where $\Delta = 0$ and hence $\gamma = 0$, $A_{\text{pop}} = \frac{1}{2}$, which is its ideal value and signifies population transfer to the fullest extent (the oscillation is centered around$\frac{1}{2}$, and an amplitude of also $\frac{1}{2}$ allow $|\alpha(t)|^2$ to be mapped onto every value between 0 and 1. As γ increases, A_{pop} decreases and the degree of population transfer decreases.

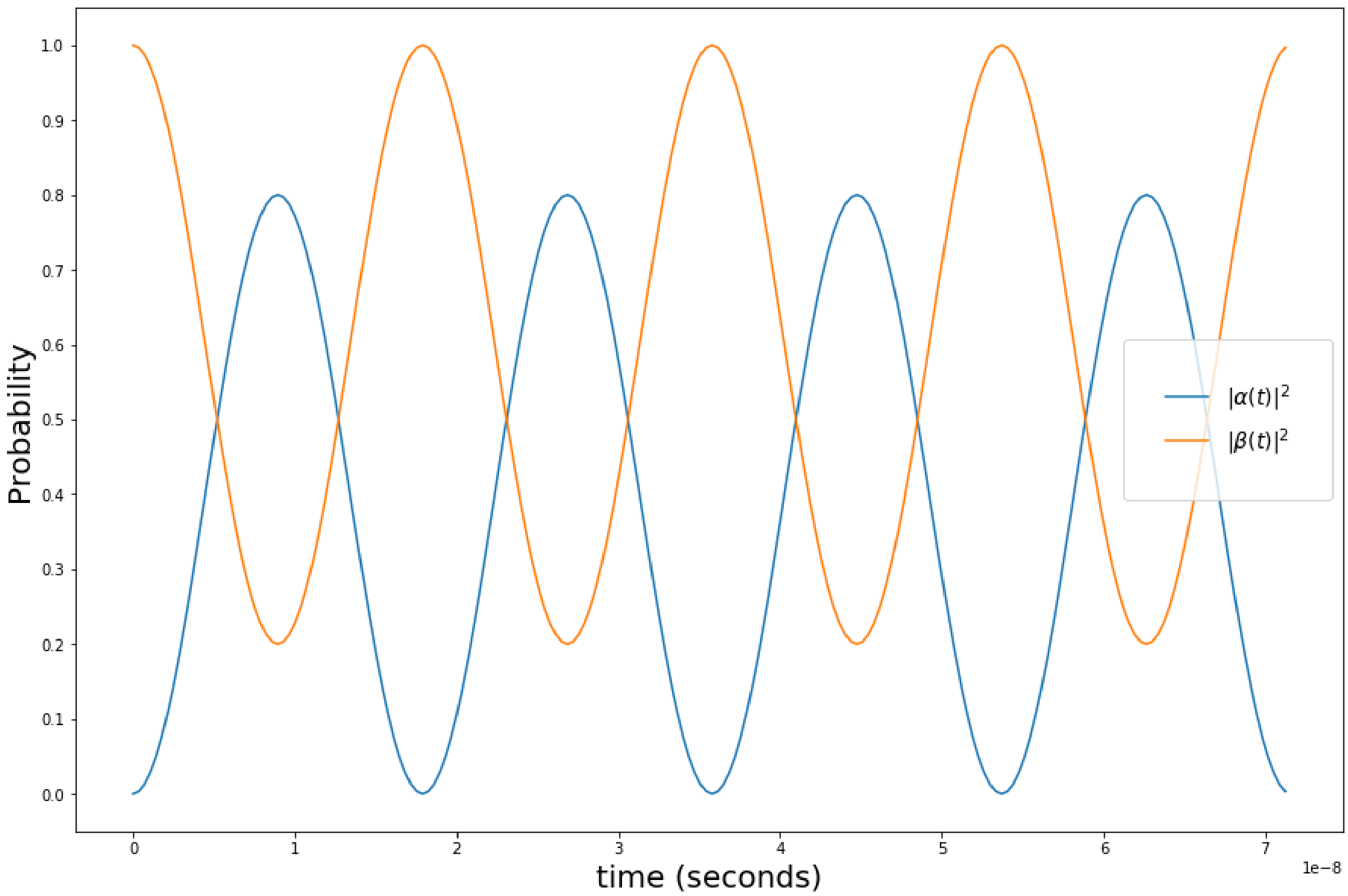

Figure 5: A plot of $|\alpha(t)|^2$ and $|\beta(t)|^2$ versus time. Here I again took that $\Omega = 314\,\text{MHz}$, but that $\Delta = 0.5 \cdot \Omega$. In practice, the detuning will hopefully not be so extreme as to be fifty percent of the Rabi Frequency, but nevertheless this example is illustrative of how the center of oscillations for the ground and excited states are shifted up and down respectively, the amplitude of each oscillation decreases, and that the average probability of finding the qubit in the ground state is now higher than the average probability of finding the qubit in the excited state, over time. Note that the two probabilities still however oscillate with the same frequency as each other, and the normalization condition is satisfied for all time.

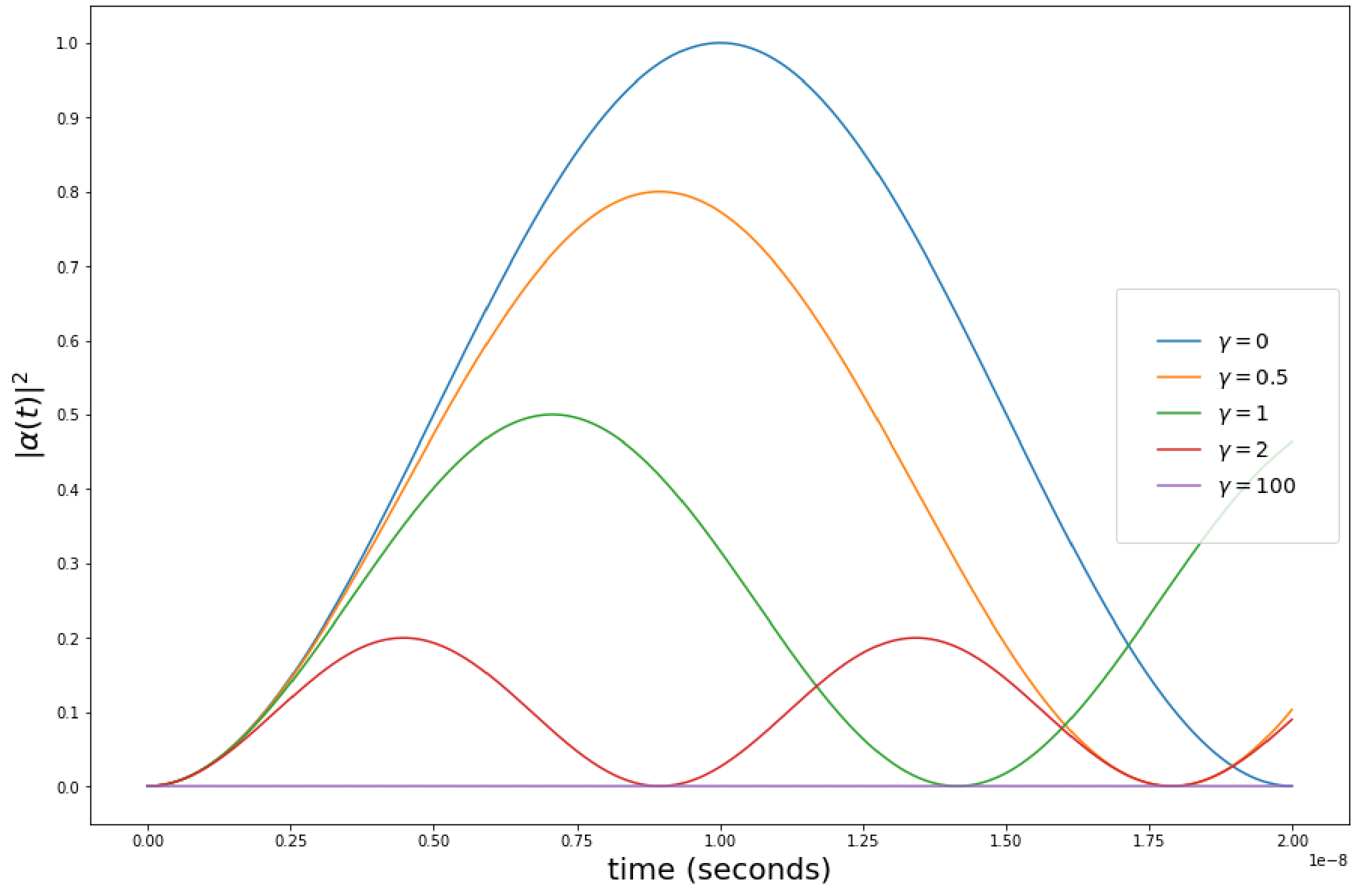

Figure 6: $|\alpha(t)|^2$ versus time, for different values of γ. The x-axis covers a range of exactly one period where $\gamma = 0$. As γ increases, we see that the maximum value that $|\alpha(t)|^2$ decreases and the frequency of oscillation increases. The purple curve, corresponding to the case where $\gamma = 100$, demonstrates a severely detuned driving frequency whereby the probability of finding the qubit in the excited state is near zero for all time.

5 More on Fidelity

We can also measure the fidelity of the driven oscillations by computing the overlap of the actual qubit state with the qubit state under an ideal rotation at every moment in time (using the inner product). This is called Point-to-Point Fidelity.

6 CORPSE and Composite Pulses

6.1 What is CORPSE?

In the last few sections we explored how a constant amplitude electromagnetic pulse can drive a qubit to rotate. In the case of a π rotation that takes the qubit from the ground state to the excited state, we can apply one pulse, appropriate called a π pulse, that is ideally on resonance and calibrated for a specific time period. However, for off-resonance pulses we saw that the full rotation is not achieved since the axis becomes titled.

In a seminal 2003 paper titled ”Tackling systematic errors in quantum logic gates with composite rotations”, researchers at the University of Oxford showed that it can be more favorable to apply a sequence of three constant amplitude pulses in place of a singular pulse, in the sense that their method was more robust to small off-resonant frequency errors (a.k.a for the same amount of detuning a better rotation was achieved). For different nominal angles of rotation, the three component rotation angles change accordingly.How these

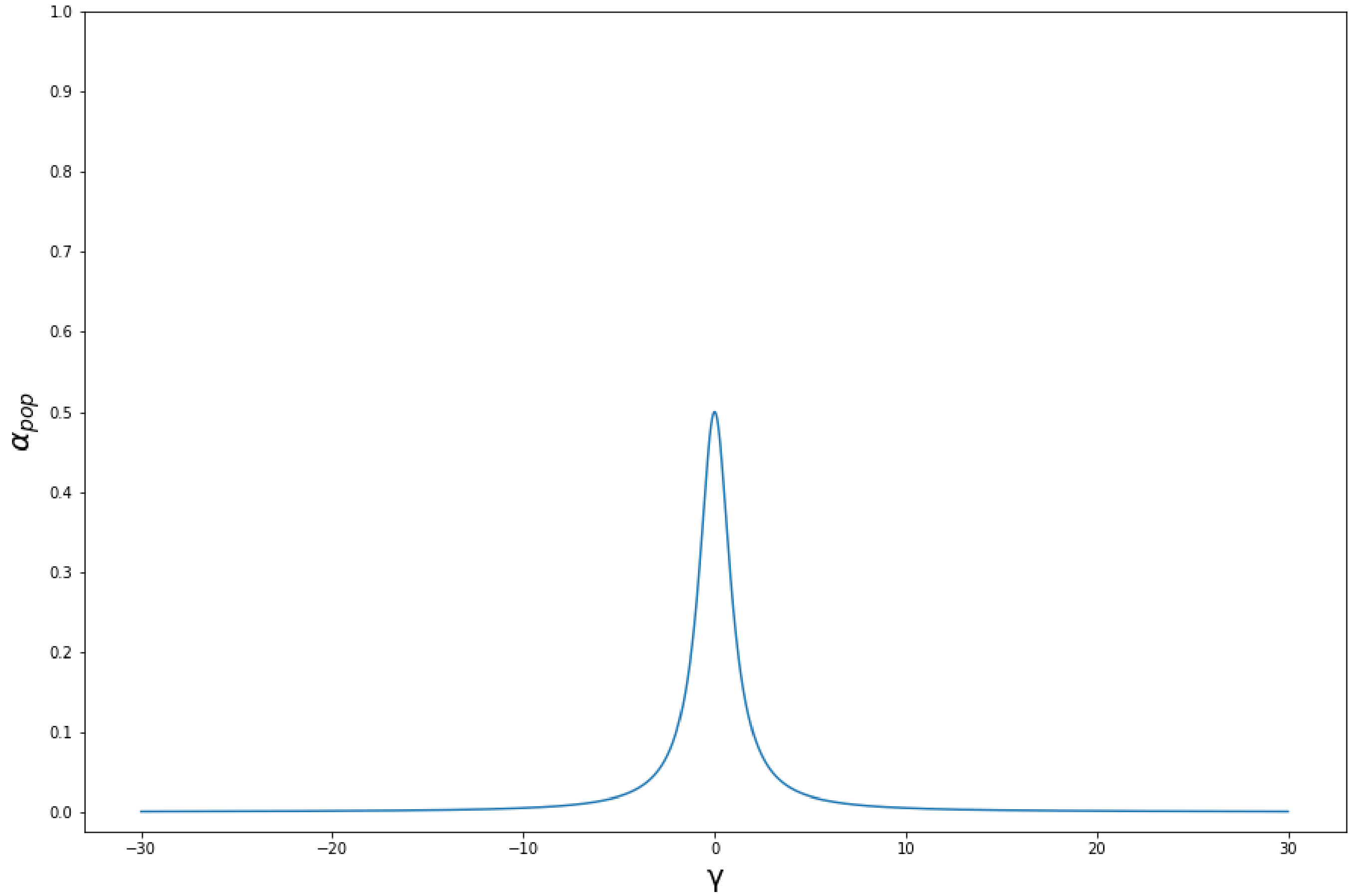

Figure 7: A plot of a_{pop} vs. γ. a_{pop} achieves a maximum of 1/2 when $\gamma = 0$ and decreases asympotically and symmetrically towards zero when $|\gamma|$ increases. Qualitatively, this graph appears similiar to that of a_{max}, but there are some differences: 1) a_{pop} has half the maximum value of a_{max}'s (which doesn't really mean anything physically significant); 2) a_{pop} decreases faster than a_{max}.

angles are determined will be discussed in short notice.

In the paper mentioned above, the Oxford researchers use the mathematical formulation of quaternions to represent qubit rotations, or equivalently to represent the pulses that drive the qubit rotations. A quaternion is a vector-like object whose properties that will not be discussed here. The important thing to know is that each individual rotation is represented by a single quaternion; the information about the rotation angle and axis is contained entirely within the quaternion. The overall rotation of the CORPSE pulse is simply represented by the product of all of the quaternions in the right succession. There are two points with regards to the original Oxford paper that I would like to expound on here: 1) their derivation is quick, but the rule for multiplying quaternions together is cumbersome; and 2) the researchers do not shed intuition on why their result works, but simply resorted to a rather mechanical derivation whereby they equated the overall CORPSE quaternion with the ideal (nominal) quaternion. In my presentation of CORPSE pulses here, I will use matrices to represent rotations which will be much more straightforward and illuminative than quaternions. I will also talk a little bit about why CORPSE works as a general procedure.

6.2 Why CORPSE works

If one off-resonant pulse achieves an imperfect rotation, how can three off-resonant pulses achieve a better (albeit still imperfect) rotation? Here is a short explanation for now: Because the three individual pulses are applied along the x,-x, and x axes respectively, the deleterious effects of each individual off-resonant pulses somehow each other out–to some extent.

As discussed earlier, a driving pulse characterized by the detuning ratio of $\gamma = \frac{\Delta}{\Omega}$, where Δ is the detuning and Ω is the Rabi frequency can be described as operating on the qubit via the matrix $U_\theta = e^{-i\frac{\theta}{2}[\sigma^x + \gamma\sigma^z]}$.

Firstly, consider the expression in the square brackets: $\sigma^x + \gamma\sigma^z$. This seems rather arbitary right now, but lets divide and multiply this expression with the factor $\sqrt{1+\gamma^2}$, giving: $\sqrt{1+\gamma^2}(\frac{\gamma}{\sqrt{1+\gamma^2}}\sigma^z + \frac{1}{\sqrt{1+\gamma^2}}\sigma^x)$. Now define $\tilde{\sigma} \equiv \frac{\gamma}{\sqrt{1+\gamma^2}}\sigma^z + \frac{1}{\sqrt{1+\gamma^2}}\sigma^x$. Much like how all of the Pauli matrices can each represent rotations along a principal axis, $\tilde{\sigma}$ encapsulates the intended component of the rotation. Note that it also squares to identity.

Then

$$U_\theta = e^{-i\frac{\theta}{2}\sqrt{1+\gamma^2}\tilde{\sigma}} \tag{93}$$

The CORPSE sequence is the product of each individual rotation in succession:

$$U_{CORPSE} = U_{\theta_3} U_{\theta_2} U_{\theta_1} \tag{94}$$

where U_{θ_i} represents the i-th rotation in the CORPSE sequence. That is, if the qubit state again begins in the fiducial state of $|\downarrow\rangle$, the state of the qubit after being driven by a CORPSE sequence is simply:

$$|\psi_{CORPSE}\rangle = U_{\theta_3} U_{\theta_2} U_{\theta_1} |\downarrow\rangle \tag{95}$$

The analysis of CORPSE pulse sequences is continued in the jupyter notebook contained in the Github Repo.